Abstract and Applied Analysis

Optimal Control: Theory and Application to Science, Engineering, and Social Sciences

Guest Editors: Davide La Torre, Herb Kunze, Manuel Ruiz-Galan, Tufail Malik, and Simone Marsiglio

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 890527, 2 pages
http://dx.doi.org/10.1155/2015/890527

Editorial

Optimal Control: Theory and Application to Science, Engineering, and Social Sciences

Davide La Torre,[1,2] Herb Kunze,[3] Manuel Ruiz-Galan,[4] Tufail Malik,[1] and Simone Marsiglio[5]

[1] *Department of Applied Mathematics and Sciences, Khalifa University, Abu Dhabi 127788, UAE*
[2] *Department of Economics, Management, and Quantitative Methods, University of Milan, 20122 Milan, Italy*
[3] *Department of Mathematics and Statistics, University of Guelph, Guelph, ON, Canada N1G 2W1*
[4] *Department of Applied Mathematics, University of Granada, 18071 Granada, Spain*
[5] *School of Accounting, Economics and Finance, University of Wollongong, Wollongong, NSW 2522, Australia*

Correspondence should be addressed to Davide La Torre; davide.latorre@kustar.ac.ae

Received 12 March 2015; Accepted 12 March 2015

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An optimal control problem entails the identification of a feasible scheme, policy, program, strategy, or campaign, in order to achieve the optimal possible outcome of a system. More formally, an optimal control problem means endogenously controlling a parameter in a mathematical model to produce an optimal output, using some optimization technique. The problem comprises an objective (or cost) functional, which is a function of the state and control variables, and a set of constraints. The problem seeks to optimize the objective function subject to the constraints construed by the model describing the evolution of the underlying system. The two popular solution techniques of an optimal control problem are Pontryagin's maximum principle and the Hamilton-Jacobi-Bellman equation [1].

Optimal control has become a highly established research front in recent years with numerous contributions to the theory, in both deterministic and stochastic contexts. Its application to diverse fields such as biology, economics, ecology, engineering, finance, management, and medicine cannot be overlooked (see, e.g., [1–12]). The associated mathematical models are formulated, for example, as systems of ordinary, partial, or stochastic differential equations or discrete dynamical systems, for both scalar and multicriteria decision-making contexts.

This special issue is aimed at creating a multidisciplinary forum of discussion on recent advances in theory as well as applications. In response to the call for papers, 58 papers were received. After a rigorous refereeing process, 7 papers were accepted for publication in this special issue. The articles included in the issue cover novel contributions to optimal control theory as well as a broad spectrum of applications of optimal control from finance to resource management to engineering to marketing.

The paper by Z. Lu and X. Huang investigates discretization of general linear hyperbolic optimal control problems and derives a priori error estimates for mixed finite element approximation of such optimal control problems.

The paper by R. Wu et al. proposes, analyzes, and solves an optimal control exploiting model of renewable resources based on effective utilization rate and illustrates its solution numerically.

The paper by N. Zulkarnain et al. compares the performance of an active antiroll bar system in an automobile using two control models, namely, the linear quadratic Gaussian and the composite nonlinear feedback controller.

There are three works focusing on applications in economics and finance. The paper by J. Ma et al. studies the dynamics of an oligopoly model considering the output of an upstream monopoly and prices of two downstream oligopolies; it is shown that the long-run average profits of the three firms are optimal in the region of stability of the Nash equilibrium point. The paper by D.-L. Sheng et al. uses the

constant elasticity of variance model to describe the dynamics of the risk asset and establish a Hamilton-Jacobi-Bellman equation to solve the associated optimal control problem. An asset and liability management problem with stochastic interest rate is examined in the paper by H. Chang et al., who investigate an optimal investment strategy to maximize the expected utility of the terminal surplus.

Finally, the paper by C. Colapinto and E. Benecchi uses a dynamic diffusion model of a TV series to analyze the evolution of electronic media transmissions and propose an optimal control model to assist the decision-maker with an optimal solution that balances the investment in a program's popularity sustenance and profit maximization.

Davide La Torre
Herb Kunze
Manuel Ruiz-Galan
Tufail Malik
Simone Marsiglio

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 369493, 7 pages
http://dx.doi.org/10.1155/2015/369493

Research Article

Optimal Control of Renewable Resources Based on the Effective Utilization Rate

Rui Wu, Zhengwei Shen, and Fucheng Liao

Department of Mathematics of University of Science and Technology, 30th Xueyuan Road, Haidian District, Beijing 100083, China

Correspondence should be addressed to Zhengwei Shen; ijwmip@ustb.edu.cn

Received 14 May 2014; Revised 10 August 2014; Accepted 26 August 2014

Academic Editor: Simone Marsiglio

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The effective utilization rate of exploited renewable resources affects the final total revenue and the further exploitation of renewable resources. Considering the effective utilization rate, we propose an optimal control model for the exploitation of the renewable resources in this study. Firstly, we can prove that the novel model is nonsingular compared with the singular basic model. Secondly, we solve the novel model and obtain the optimal solution by Bang-Bang theory. Furthermore, we can determine the optimal total resources and the maximal total revenue. Finally, a numerical example is provided to verify the obtained theoretical results.

1. Introduction

Renewable resources (such as fisheries resources) are considered to be "inexhaustible" at all times, but excessive exploitation will actually exhaust them. Thus, it is useful to study the reasonable exploitation of renewable resources and their effective utilization to obtain the maximum revenue.

Various optimum exploitation schemes for renewable resources have been studied in recent decades. In a pioneering study, Gordon [1, 2] proposed the Gordon-Schaefer bioeconomic model by introducing the concept of economic efficiency and cost management. Although this model included a large number of unrealistic assumptions, it exhibited a certain degree of concordance with the histories of empirical fisheries [3]. Subsequently, further economic control models of renewable resources [4–10] were proposed based on the work of Gordon. Clark and Munro [11] extended the static version of the fisheries economics model to nonautonomous and nonlinear cases. Indeed, the models proposed by Gordon and other researchers were all static until the 1980s when Clark [12] established a dynamic bioeconomic model based on the Gordon-Schaefer biological model. In addition to the classical model of Clark, another study [13] provided a nonlinear optimal control bioeconomic model that used the variation in the fishing effort rate as the control. In the 1990s, Defeo and Seijo [14] developed a bioeconomic model using a yield-mortality model. Later, based on the Beverton-Holt age structure model [15], Beverton and Holt [16] constructed a dynamic bioeconomic model that considered interactions among populations, which was not based on the Gordon-Schaefer bioeconomic model.

In recent years, various bioeconomic models of fisheries resources have also been proposed. Smith [17] established a Bayesian bioeconomic dynamic model by introducing the Bayesian statistical method. Domíguez-Torreiro and Surís-Regueiro [18] introduced game theory into a bioeconomic model of fishery resources and proposed a management strategy that addressed fishery issues. Das et al. [19] proposed a new bioeconomic model that combined a predator-prey ecological model with marine environmental factors. Other studies [20–22] introduced the fundamentals of control parameterization, which is a popular numerical technique for solving optimal control problems. At the same time, a switched autonomous system was proposed to formulate a fed-batch culture process where the switching instants bet ween the feed and batch processes were used as control variables, which is similar to the method proposed in the present study. The Food and Agriculture Organization also produced a series of bioeconomic models [23, 24] to provide a theoretical basis for policy to facilitate the sustainable use of fisheries resources.

However, all of these bioeconomic models and optimal exploitation schemes did not consider the effective utilization rate, especially for renewable resources. Indeed, if the

exploited renewable resources are not utilized fully, they will be wasted, but they may also pollute the environment. Given the impact on the expected revenue of the effective utilization rate for renewable resources, we introduce the concept of effective utilization into a renewable resources development model and we propose an optimal control model to ensure that it approximates the actual situation.

In Section 2, we introduce the basic bioeconomic model and propose an optimal control bioeconomic model based on the effective utilization rate. In Section 3, the singularity of this novel model is analyzed based on the maximal principle. Using the Bang-Bang theory, we obtain the optimal exploitation scheme and the optimal total renewable resources subject to the maximum total revenue. In Section 4, a numerical example is given that verifies the results.

2. Optimal Control Model Based on the Effective Utilization Rate

2.1. The Basic Model. It is well known that renewable resources have their own life cycles, even if they are not exploited or consumed. In general, we expect that the natural growth rate is greater than the natural mortality rate for renewable resources. However, the maximum amount of resources cannot exceed the environmental carrying capacity. In general, it is considered that the growth of resources satisfies the following logistic equation [4]:

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{N}\right), \tag{1}$$

where $x(t)$ denotes the resource biomass, N is the carrying capacity of the ecosystem, and r is the intrinsic growth rate of resources. If we let $u(t)$ be the exploitation amount, then model (1) becomes

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{N}\right) - u(t). \tag{2}$$

Let the total revenue from the exploited resources in unit time be expressed as

$$(p - c)\, u(t)\, e^{-\rho t}, \tag{3}$$

where p is the revenue per unit, c is the cost per unit, and ρ is the instantaneous social rate of discount. In this case, the objective function of the total revenue can be stated as follows:

$$J = \int_0^T (p - c)\, u(t)\, e^{-\rho t} dt. \tag{4}$$

Thus, the optimal control problem involving the basic model of renewable resources can be expressed as follows:

$$\begin{gathered} \max J = \int_0^T (p - c)\, u(t)\, e^{-\rho t} dt, \\ \dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{N}\right) - u(t), \\ 0 \le u(t) \le \overline{u}, \\ x(0) = x_0, \qquad x(T) = x_T. \end{gathered} \tag{5}$$

2.2. The Proposed Model Based on the Effective Utilization Rate. Let $s(t)$ be the effective utilization rate at time t; then $s(t)$ should satisfy the following three assumptions.

(A1) The effective utilization rate $s(t)$ will increase gradually with respect to t (i.e., with the development of technology); that is, $ds(t)/dt > 0$.

(A2) The increase in $s(t)$ will become more difficult after it reaches a certain level; that is, $s(t)$ must satisfy $d^2 s(t)/dt^2 < 0$.

(A3) The ideal or the best utilization of resources is achieved completely; that is, $\lim_{t\to\infty} s(t) = 1$.

By these assumptions, we take $s(t) = 1 - ae^{-bt}$ $(a, b > 0)$ as our effective utilization rate, which satisfies the preceding assumptions. Furthermore, let the initial effective utilization rate be $s(0) = s_0$ and the ultimate effective utilization rate be $s(T) = s_T$. Then, parameter a, b in $s(t)$ can be obtained as follows:

$$\begin{gathered} a = 1 - s_0, \\ b = \frac{1}{T} \ln \frac{1 - s_0}{1 - s_T}. \end{gathered} \tag{6}$$

Thus, our improved objective function for optimal control associated with the effective utilization rate can be written as

$$J = \int_0^T (p - c)\, u(t)\, e^{-\rho t} \left(1 - ae^{-bt}\right) dt \tag{7}$$

and our proposed optimal control model can be established as follows:

$$\begin{gathered} \max J = \int_0^T (p - c)\, u(t)\, e^{-\rho t} \left(1 - ae^{-bt}\right) dt, \\ \dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{N}\right) - u(t), \\ 0 \le u(t) \le \overline{u}, \\ x(0) = x_0, \qquad x(T) = x_T, \end{gathered} \tag{8}$$

where J represents the management objective, $\overline{u}$ is the maximum amount of exploitation, and the initial and the terminal populations of renewable resources x_0, x_T are assumed to be known. In model (8), the meanings of N, r, p, c, ρ are similar to those in model (5).

3. Solutions to the Model

In this section, we first analyze the singularity of the proposed model (8) and we then apply the Bang-Bang approach to obtain its solution.

3.1. Solution of the Hamiltonian Formulation. The Hamiltonian formulation to the optimal control problem (8) can be written as follows:

$$\begin{aligned} H(x, \lambda, u, t) = {} & (p - c)\, u(t)\, e^{-\rho t} \left(1 - ae^{-bt}\right) \\ & + \lambda(t)\left(rx(t)\left(1 - \frac{x(t)}{N}\right) - u(t)\right). \end{aligned} \tag{9}$$

Then, we have

$$H(x,\lambda,u,t) = \left[(p-c)e^{-\rho t}\left(1-ae^{-bt}\right)-\lambda(t)\right]u(t) + \lambda(t)\left(rx(t)\left(1-\frac{x(t)}{N}\right)\right). \tag{10}$$

By the maximal principle, the Hamiltonian $H(x,\lambda,u,t)$ will obtain the maximal value with respect to $u(t)$ if the objective function obtains the maximal value. The linear relationship between the Hamiltonian function and control $u(t)$ changes the computation of the maximization problem (10) into a Bang-Bang optimal control problem, as follows:

$$u^*(t) = \begin{cases} 0 & \lambda(t) > (p-c)e^{-\rho t}\left(1-ae^{-bt}\right) \\ \overline{u} & \lambda(t) < (p-c)e^{-\rho t}\left(1-ae^{-bt}\right). \end{cases} \tag{11}$$

However, from the solution of (11), we cannot obtain any information about the optimal control $u^*(t)$ by $\lambda(t) = (p-c)e^{-\rho t}(1-ae^{-bt})$. Indeed, there are two possible solutions for $\lambda(t) = (p-c)e^{-\rho t}(1-ae^{-bt})$. Firstly, if there is only a countable time set $t_j = \{t_1, t_2, t_3, \ldots\} \in [t_0, t_f]$ that satisfies $\lambda(t) = (p-c)e^{-\rho t}(1-ae^{-bt})$, we can still use the Bang-Bang theory to solve this model. Secondly, if there is an interval $I \subset [0,T]$ that satisfies $\lambda(t) = (p-c)e^{-\rho t}(1-ae^{-bt})$ for $\forall t \in I$, the problem will become more complicated and difficult to solve. Thus, to obtain a better solution to the model, we first need to analyze whether such an interval I exists or not. Thus, we need to discuss the singularity [25, 26] of the optimal control model (8).

3.2. Singularity Analysis of the Model. Based on the preceding discussion, we can see that the singularity of model (8) will affect the choice of method used to solve this model. In this subsection, we discuss the singularity of model (8) by reduction to absurdity.

Assuming that the optimal control is a singularity, that is, there is an interval I for all $t \in I \subset [0,T]$ that satisfies

$$\lambda(t) = (p-c)e^{-\rho t}\left(1-ae^{-bt}\right), \tag{12}$$

the derivative of $\lambda(t)$ with respect to t can be expressed as

$$\dot{\lambda}(t) = (p-c)\left[-\rho e^{-\rho t}\left(1-ae^{-bt}\right)+abe^{-\rho t}e^{-bt}\right], \quad \forall t \in I; \tag{13}$$

by the costate equation of model (8)

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -\lambda(t)\left(r-\frac{2rx(t)}{N}\right) \tag{14}$$

we obtain the following equation:

$$(p-c)\left[-\rho e^{-\rho t}\left(1-ae^{-bt}\right)+abe^{-bt}\right] = -\lambda(t)\left(r-\frac{2rx(t)}{N}\right), \quad \forall t \in I. \tag{15}$$

By substituting (12) into (15), we obtain

$$\left[-\rho\left(1-ae^{-bt}\right)+abe^{-bt}\right] = -\left(1-ae^{-bt}\right)\left(r-\frac{2rx(t)}{N}\right), \quad \forall t \in I. \tag{16}$$

Equation (16) admits the positive root given by

$$x(t) = \frac{N}{2r}\left(r-\rho+\frac{abe^{-bt}}{1-ae^{-bt}}\right) \tag{17}$$

which indicates that if we want (16) to hold at time $t \in I$, then $x(t)$ must satisfy (17). By taking the derivative of (17) with respect to t, we obtain

$$\dot{x}(t) = \frac{N}{2r}\frac{-ab^2e^{-bt}}{\left(1-ae^{-bt}\right)^2}. \tag{18}$$

In fact, (18) does not equal (2). In other words, $x(t)$ that satisfies the condition given above does not exist. Compared with the basic model (5), which is a singularity, our proposed model (8) involving the effective utilization rate is a normal model. Therefore, the optimal control of model (8) can be solved using the Bang-Bang approach.

3.3. Existence of the Switching Time t_s. In this subsection, we discuss the existence of the switching time t_s. First, we consider $\lambda(t) \le 0$, which indicates that the inequality

$$\lambda(t) < (p-c)e^{-\rho t}\left(1-ae^{-bt}\right) \tag{19}$$

will hold. In this case, the optimal strategy is $u^* \equiv \overline{u}$ by the Bang-Bang method.

However, this is not a reasonable method for exploiting renewable resources because the renewable resources will be destroyed. Therefore, we make the following assumption: $\lambda(t) > 0$. Then, (14) is obtained if the condition that $N > 2x(t)$ is satisfied and $\lambda(t)$ is monotonically decreasing for $t \in [0,T]$. In addition, we discuss the monotonicity of the following function:

$$f(t) = (p-c)e^{-\rho t}\left(1-ae^{-bt}\right). \tag{20}$$

By taking the derivative of (20) with respect to t

$$\frac{df(t)}{dt} = (p-c)\left[-\rho e^{-\rho t}\left(1-ae^{-bt}\right)+abe^{-\rho t}e^{-bt}\right]. \tag{21}$$

Then,

$$\frac{df(t)}{dt} = (p-c)e^{-\rho t}\left(\rho ae^{-bt}-\rho+abe^{-bt}\right). \tag{22}$$

It can be seen that $df(t)/dt > 0$ if and only if t satisfies the following condition:

$$t < -\frac{1}{b}\ln\frac{1}{a}\frac{\rho}{\rho+b}. \tag{23}$$

However, this condition cannot be obtained from (22). Thus, (20) is monotonically decreasing in time period $[0, T]$, which indicates that there will be an appropriate intersection between $f(t)$ and $\lambda(t)$; that is, the switching time t_s exists. Then we can obtain that

$$\begin{aligned} \lambda(t) &> (p-c)e^{-\rho t}\left(1-ae^{-bt}\right), \quad t \in [0, t_s), \\ \lambda(t) &< (p-c)e^{-\rho t}\left(1-ae^{-bt}\right), \quad t \in [t_s, T]. \end{aligned} \tag{24}$$

3.4. Solution to Model (8). After obtaining the switching time t_s and proving the nonsingularity of model (8), the optimal control strategy to (8) can be expressed by the Bang-Bang approach as follows:

$$u^*(t) = \begin{cases} 0 & 0 \le t < t_s \\ \overline{u} & t_s \le t \le T. \end{cases} \tag{25}$$

Next, we find the expressions of t_s and the optimal renewable resources function $x^*(t)$. First, if we let $0 \le t < t_s$ and $u^*(t) = 0$, we obtain

$$\begin{aligned} \dot{x}(t) &= rx(t)\left(1-\frac{x(t)}{N}\right), \\ x(0) &= x_0, \quad 0 \le t < t_s. \end{aligned} \tag{26}$$

By solving the ordinary differential equation (26), we obtain

$$x^*(t) = \frac{N}{1+Nc_1e^{-rt}}, \tag{27}$$

where c_1 is expressed as follows:

$$c_1 = \frac{1}{x_0} - \frac{1}{N}. \tag{28}$$

Second, if we let $t_s \le t \le T$ and $u^*(t) = \overline{u}$, we can obtain

$$\begin{aligned} \dot{x}(t) &= rx(t)\left(1-\frac{x(t)}{N}\right) - \overline{u}, \\ x(T) &= x_T \quad t_s \le t \le T. \end{aligned} \tag{29}$$

Unlike (26), we cannot solve the ordinary differential equation (29) directly using the Bernoulli equation. However, if there are suitable parameters p, q, the following formula holds; that is,

$$\frac{d[x(t)-p]}{dt} = -\frac{r}{N}[x(t)-p]^2 + q[x(t)-p]. \tag{30}$$

Then, we can solve (29) by the Bernoulli equation. In fact, by (29) and (30), we have

$$\begin{aligned} \frac{2r}{N}p + q - r &= 0, \\ -\frac{r}{N}p^2 - qp + \overline{u} &= 0; \end{aligned} \tag{31}$$

that is, the following quadratic equation holds:

$$\overline{u} + \frac{r}{N}p^2 - pr = 0. \tag{32}$$

Irrespective of whether (32) is positive or 0, that is,

$$\Delta = r^2 - 4\frac{r}{N}\overline{u} = r^2\left(1 - 4\frac{\overline{u}}{Nr}\right) \ge 0, \tag{33}$$

the suitable parameters p, q will be obtained, which indicates that (29) can be solved by the Bernoulli equation. It is known that N is the environmental carrying capacity, which is a very large number; thus the formula $1 - 4(\overline{u}/Nr) > 0$ holds. Then, we can obtain

$$\Delta \ge 0. \tag{34}$$

Therefore, we can solve (31), where the parameters p, q are given as follows:

$$\begin{aligned} p &= \frac{N}{2}\left(1 - \sqrt{1 - \frac{4\overline{u}}{Nr}}\right), \\ q &= r - r\left(1 - \sqrt{1 - \frac{4\overline{u}}{Nr}}\right). \end{aligned} \tag{35}$$

Moreover, we can obtain the solution of (29) by the Bernoulli equation:

$$(x(t)-p)^{-1} = \frac{r}{Nq} + c_2e^{-qt}. \tag{36}$$

That is,

$$x^*(t) = \frac{Nq}{r + Nqc_2e^{-qt}} + p, \tag{37}$$

where c_2 is expressed as

$$c_2 = \left(\frac{1}{x_T - p} - \frac{r}{Nq}\right)e^{qT}. \tag{38}$$

To summarize, in the interval $[0, T]$, the optimal renewable resources function can be expressed as

$$x^*(t) = \begin{cases} \dfrac{N}{1+Nc_1e^{-rt}} & 0 \le t < t_s \\ \dfrac{Nq}{r+Nqc_2e^{-qt}} + p & t_s \le t \le T. \end{cases} \tag{39}$$

The total amount of resources is continuous with respect to time t, even with $t = t_s$; that is,

$$\frac{1}{1+Nc_1e^{-rt_s}} = \frac{Nq}{r+Nqc_2e^{-qt_s}} + p. \tag{40}$$

We can determine the time t_s by (40). Thus, we obtain the optimal exploitation and the optimal level of total resources under the maximum revenue:

$$\begin{aligned} u^*(t) &= \begin{cases} 0 & 0 \le t < t_s \\ \overline{u} & t_s \le t \le T, \end{cases} \\ x^*(t) &= \begin{cases} \dfrac{N}{1+Nc_1e^{-rt}} & 0 \le t < t_s \\ \dfrac{Nq}{r+Nqc_2e^{-qt}} + p & t_s \le t \le T. \end{cases} \end{aligned} \tag{41}$$

Table 1: Initial parameters for model (8).

Parameter	Parameter value	Unit
N	100,000	Tons
r	4.4	/
p	38	Dollars per ton
c	20	Dollars per ton
ρ	0.1	/
$\overline{u}$	20000	Tons
T	1	Year
$x(T)$	70000	Tons
S_0	0.6	/
S_T	0.62	/

In summary, the optimal management strategy can be interpreted as follows. At the beginning of a time period $[0, t_s)$, we should not exploit the renewable resources to reach a certain number. At time $[t_s, T]$, we can exploit the renewable resources at the maximum capacity. Finally, we obtain the total benefit as follows:

$$J^* = \int_0^{t_s} (p-c) \times 0 \times e^{-\rho t} \left(1 - ae^{-bt}\right) dt + \int_{t_s}^{T} (p-c)\, \overline{u} e^{-\rho t} \left(1 - ae^{-bt}\right) dt. \tag{42}$$

4. Numerical Example

Based on the solution to the proposed model (8), we now present an example that verifies the model. The initial parameters for the fisheries resources are specified in Table 1, that is, the carrying capacity of the ecosystem N, the intrinsic growth rate of the renewable resources r, the revenue per unit p, the cost per unit c, the instantaneous social rate of discount ρ, the maximum amount of exploitation $\overline{u}$, the time interval $[0, T]$, the ultimate amount of renewable resources $x(T)$, the initial effective utilization rate s_0, and the ultimate effective utilization rate s_T. Thus, by (6), (28), (35), and (38), we can obtain the remaining parameters: $a = 0.4$, $b = 0.05$, $c_1 = 1.9\times10^{-4}$, $p = 4775$, $q = 3.9799$, and $c_2 = 2.288\times10^{-4}$, respectively. By substituting parameters N, r, c_1, c_2, p, q into (38), the switching time $t_s = 0.33999 \approx 0.34$ is obtained (note that in another example $t_s = 0.80487$ for $\overline{u} = 70000$ tons based on the same calculation used for $\overline{u} = 20000$). Finally, we obtain the expressions for optimal exploitation $u^*(t)$ and the optimal level $x^*(t)$ of total resources under the maximum revenue as follows (we omit the case where $\overline{u} = 70000$):

$$u^*(t) = \begin{cases} 0 & 0 \le t < 0.34 \\ 20000 & 0.34 \le t \le 1, \end{cases}$$
$$x^*(t) = \begin{cases} \dfrac{100000}{1 + 19e^{-4.4t}} & 0 \le t < 0.34 \\ \dfrac{90450}{1 + 20.6969e^{-3.9799t}} + 4775 & 0.34 \le t \le 1, \end{cases} \tag{43}$$

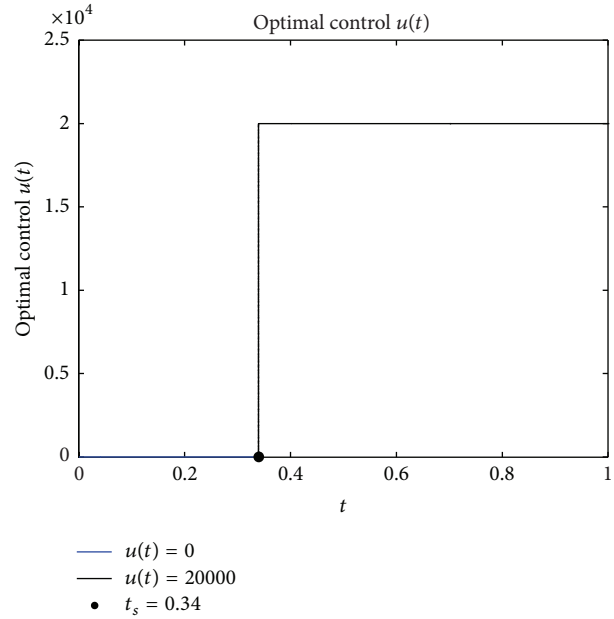

Figure 1: Optimal control $u(t) = 20000$.

which are shown in Figures 1 and 2, respectively. The total maximum revenue for the cases where $\overline{u} = 20000$ and $\overline{u} = 70000$, respectively, can also be obtained from (42) as follows:

$$\begin{aligned} J^*_{\overline{u}=20000} &= \int_0^{t_s} (p-c) \times 0 \times e^{-\rho t} \left(1 - ae^{-bt}\right) dt + \int_{t_s}^{T} (p-c)\, \overline{u} e^{-\rho t} \left(1 - ae^{-bt}\right) dt \\ &\approx 136254.85, \\ J^*_{\overline{u}=70000} &= \int_0^{t_s} (p-c) \times 0 \times e^{-\rho t} \left(1 - ae^{-bt}\right) dt + \int_{t_s}^{T} (p-c)\, \overline{u} e^{-\rho t} \left(1 - ae^{-bt}\right) dt \\ &\approx 138753.7. \end{aligned} \tag{44}$$

In Figure 2, for the case where $\overline{u} = 20000$, the switching time is $t_s = 0.34$, which indicates that we should allow the renewable resources to grow naturally for about 4 months of 1 year. Subsequently, the renewable resources can be exploited optimally where $\overline{u} = 20000$ and the final total maximum revenue is 136,254 dollars. Figure 2 also shows that the renewable resources retain approximately the same growth rate as before. However, when the optimal exploitation is $\overline{u} = 70000$ with the development of exploitation technology, we should delay the exploitation time; that is, $t_s = 0.80487$, which is about 10 months of 1 year. In this case, the total maximum revenue is about 138,753 dollars, which is greater than that for $\overline{u} = 20000$ and the renewable resources continue to grow, but at a slower growth rate.

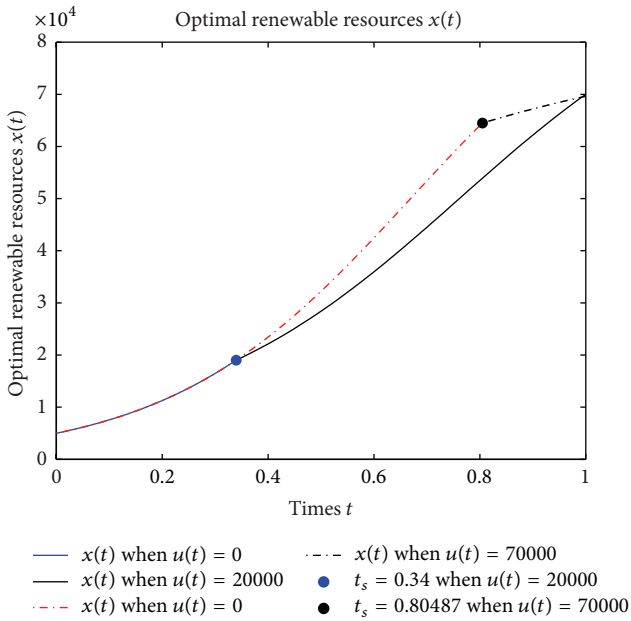

Figure 2: Optimal renewable resources $x(t)$.

5. Conclusion

In this study, we considered a model for the optimal control of renewable resources, which is affected by the effective utilization rate. We analyzed the singularity of the model. If the model is a singularity, the problem is difficult because the maximal principle cannot identify an optimal candidate explicitly. Indeed, we proved that our proposed optimal control model involving the effective utilization rate is normal and we solved the model with the aid of the maximal principle. Finally, we determined the optimal exploitation rate and the optimal level of the total renewable resources under the maximum revenue. This study may provide reference values to facilitate the development of renewable resources. However, it is clear that $u(t)$ is not related directly to $x(t)$ according to the present study. Thus, further research is required to address the case where $u(t)$ and $x(t)$ have a linear relationship.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the reviewers for many helpful suggestions which improved the presentation of the paper. The authors acknowledge supports by the National Science Foundation of China (no. 61174209) and the Oriented Award Foundation for Science and Technological Innovation, Inner Mongolia Autonomous Region, China (2012).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 698195, 14 pages
http://dx.doi.org/10.1155/2014/698195

Research Article

Improving Vehicle Ride and Handling Using LQG CNF Fusion Control Strategy for an Active Antiroll Bar System

N. Zulkarnain,[1] H. Zamzuri,[1] Y. M. Sam,[2] S. A. Mazlan,[3] and S. M. H. F. Zainal[1]

[1] *Vehicle System Engineering Research Laboratory, Universiti Teknologi Malaysia, Jalan Semarak, 54100 Kuala Lumpur, Malaysia*
[2] *Faculty of Electrical Engineering, Universiti Teknologi Malaysia, 81310 Skudai, Johor, Malaysia*
[3] *Malaysia-Japan International Institute of Technology, Universiti Teknologi Malaysia, Jalan Semarak, 54100 Kuala Lumpur, Malaysia*

Correspondence should be addressed to N. Zulkarnain; noraishikin2@live.utm.my

Received 7 May 2014; Revised 6 August 2014; Accepted 11 August 2014; Published 23 October 2014

Academic Editor: Manuel Ruiz Galán

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This paper analyses a comparison of performance for an active antiroll bar (ARB) system using two types of control strategy. First of all, the LQG control strategy is investigated and then a novel LQG CNF fusion control method is developed to improve the performances on vehicle ride and handling for an active antiroll bar system. However, the ARB system has to balance the trade-off between ride and handling performance, where the CNF consists of a linear feedback law and a nonlinear feedback law. Typically, the linear feedback is designed to yield a quick response at the initial stage, while the nonlinear feedback law is used to smooth out overshoots in the system output when it approaches the target reference. The half car model is combined with a linear single track model with roll dynamics which are used for the analysis and simulation of ride and handling. The performances of the control strategies are compared and the simulation results show the LQG CNF fusion improves the performances in vehicle ride and handling.

1. Introduction

The automobile is composed of many systems. One of these is the antiroll bar system. The main functions of the automotive antiroll bar system are to provide vehicle support, stability, and directional control during handling maneuvers and to provide effective isolation from road disturbances. Besides that, antiroll bar is a suspension element that is used at the front, rear, or at both ends of a car that reduces body roll by resisting any unequal vertical motion between the pair of wheels to which it is connected [1]. Figure 1 shows the illustration of antiroll bar that connects with left and right wheels.

A ground vehicle design typically represents a trade-off between ride comfort performance and also safety criteria. Ride and handling are one of the key attributes in the vehicle, which communicate directly to the customer perception of satisfaction [2]. The ride comfort is defined as the level of comfort experienced by the passenger in a form of numerical values, that is, weighted root mean square (RMS) of acceleration [3, 4]. Meanwhile, handling performance is defined by quality of handling, which relates to subjective feelings of human driver and also objective measurement of the vehicle characteristics. Unfortunately, there are trade-offs between these attributes and it is a challenge for automotive engineer to make a vehicle with a good ride and a good handling at the same time. Many studies have been done on active system to tackle this trade-off. Some prefer to develop active suspension while some others prefer to focus on advancing the antiroll bar. However, from the manufacturer's point of view, cost is the key factor. In this factor, antiroll bar system has the advantage since it can provide solution to ride and handling trade-off with lower cost compared to active suspension.

Basically, the passive antiroll bar (ARB) has the advantages to reduce the body roll acceleration and roll angle during single wheel lifting and cornering maneuver. By reducing body roll motion, the driving safety and handling stability will be highly improved [5]. However, the passive antiroll bar also has disadvantages. During cornering maneuver, antiroll bar will transfer vertical forces of one side of the suspension to the other side and therefore create moments against lateral force. Unfortunately, during straight line driving, lateral forces

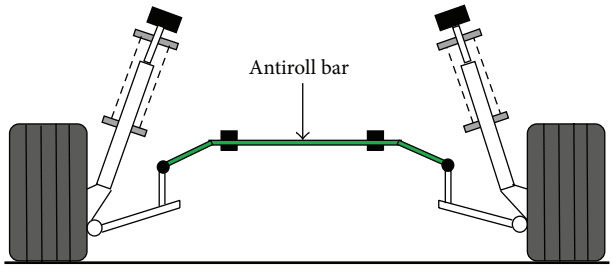


Figure 1: Schematic presentation of an antiroll in action [1].

induced by road irregularities will also get the same effect as the one induced by cornering maneuver and therefore deteriorate ride comfort. In a conventional passive suspension system, which comprises springs and dampers, a trade-off is needed to resolve the conflicted requirements of ride comfort and better handling performance. The reason is that stiff suspension is required to support the weight of the vehicle and to follow the track. On the other hand, soft suspension is needed to isolate the disturbance from the road. Hence there exists a significantly growing interest in the design and control of active ARB suspension systems from automotive engineers and researchers in the past three decades [4]. For that reason, an active antiroll bar is developed to improve the disadvantages of passive antiroll bar while at the same time augmenting ride comfort and handling stability.

A variety of solutions have been studied to compromise the trade-off between ride and handling. These solutions include passive suspension systems, semiactive suspension systems, active suspension systems, and antiroll bar. But antiroll bar recently has become very popular among researchers to tackle the issues of trade-off between ride and handling [6, 7]. Cronjé and Els had studied the effect of the active antiroll bar on ride comfort and handling of an off-road vehicle [8]. They also analysed the use of an active antiroll bar as a means of improving the handling of an off-road vehicle without sacrificing the ride comfort.

In a ground vehicle, ride comfort and handling stability conflict with each other making it hard for a vehicle suspension system to pursue them simultaneously. However, Danesin et al. have successfully designed an active roll control system to increase handling and ride comfort [9]. In order to improve the vehicle performance relating in this area, a lot of control schemes have been proposed in the framework of the computer controlled suspension system such as active or semiactive suspension system [4, 5]. Some studies investigating the use of preview control have predicted potential benefits.

Moreover, a design strategy for the control system of the electric power steering (EPS) in consideration of the active ARB to avoid side effects to the steering system by the reaction force of the front tire due to roll stiffness control for the active ARB was proposed by Yamamoto and Nishimura [10]. Gain-scheduled (GS) control is used in designing the EPS method to assist properly driver in the steering according to the vehicle speed and realize the hydraulic power steering (HPS) like steering feeling. The active ARB is obtained using the linear quadratic control theory in order to balance competing goals for the roll reduction on cornering and the suppression of the rolling vibration on the rough road [10]. An active ARB has an actuator which can actively generate the torque at the centre of an ARB. It is verified that the control system designed can settle the rolling vibration on the rough road more quickly than the vehicle model of a normal suspension while reducing the steady roll angle during cornering. Besides, an electric ARB system has been developed as a technology for controlling vehicle roll angle performance by [11, 12]. Various sensors that detect the vehicle's running state, an active ARB that used electric motors and reduction gears to control roll, are included in the system.

However, the CNF control law so far is not yet implemented in the active ARB suspension system. This motivated us to apply the CNF control technique to improve the performances of the active ARB system as the one contribution of this study. The CNF control technique is a nonlinear method applicable to control design for the linear time-invariant systems with input saturation. The CNF control technique was proposed in [13] and was effectively implemented in [14, 15]. There are two parts of the CNF control law such as the linear control law and the nonlinear feedback part. Such CNF control laws can significantly improve the tracking and regulating performances of the resulting closed-loop systems as the CNF control technique makes use of the advantages of the low and high damping-ratios by smoothly switching.

By using LQR, Gosselin-Brisson et al. designed an active antiroll bar using full state feedback optimal control strategy. For this control method all the state variables have to be measured by electronic sensors. A controller using four different measurements proves to be more efficient than any passive system for the vehicle studied. In selection of ARB stiffness based on selecting appropriate parameters, a graphical method has been presented. The active ARB controller has been designed by solving the ARE and a proposed simplification of the controller. The results show a superior performance in ride comfort and handling or stability for the active controller in the frequency range of interest instead of the time range [16].

In this paper, the CNF control technique is successfully applied to design the ARB controller. The designed control law is verified in simulation. The simulation results demonstrate that the improvement of the active ARB system performances using CNF with LQG compared to LQG technique. The outline of this paper is divided into four sections. In the next section, the dynamic mathematical modelling of an active ARB system is introduced. The control technique and design are applied to design the control laws for the active ARB system in Section 3. The simulation and comparison results will be presented in Section 4 and, finally, the concluding remarks are drawn.

2. Mathematical Modeling

The model represents the front view of a vehicle suspension system as shown in Figure 2. It includes four degrees of freedom (DOF) to model an independent suspension instead

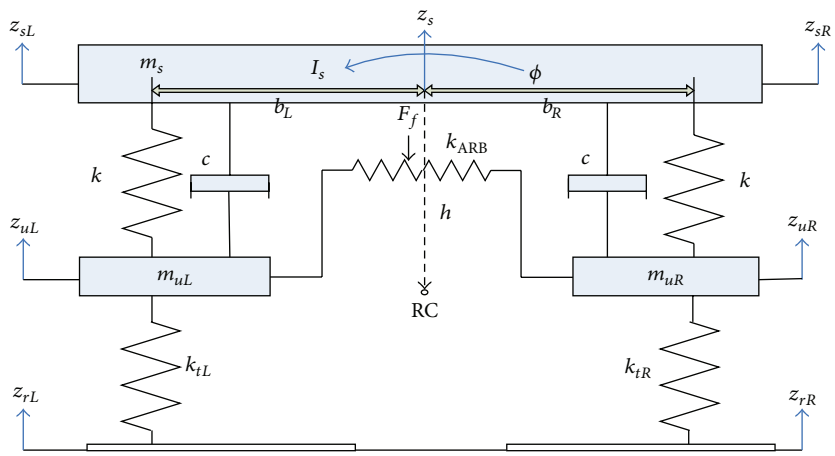


Figure 2: Half car model of a vehicle from front view.

of three DOF commonly used in the literature where the right and left unsprung masses are modelled by one axle only. For this work, the vehicle assumed has the same parameters on the right and left sides, which is generally true. Tire damping is ignored in this model. The sprung mass gravity centre (CG) is in the middle of the track.

During roll motion, the vehicle rotates around the sprung mass roll centre (RC). The position of this point is a function of suspension geometry, vehicle ride height and roll angle. As the sprung mass moves in roll and heave, the RC movement amplitude depends on the suspension geometry. As a general rule, large RC movement could compromise stability and should be avoided. In this study, RC position is kept and fixed to simplify the model. Parameter h corresponds to the distance between the sprung mass CG and the sprung mass RC. F_f is the force applied to control the vehicle sprung mass roll.

The vehicle suspension model is based on a half car model from front view and a single track model with roll dynamics developed from the bicycle model.

2.1. Half Car Model. The half car model explains the relation between body bounce, body roll angle, left and right wheels hop, and road excitations. Then, the equations of motions for this model are combined with the single track model with roll dynamics to design the four DOF vehicle dynamic model as follows.

Body vertical acceleration is

$$\begin{aligned}\ddot{z}_s = &-\frac{2c\dot{z}_s}{m_s}+\frac{c\dot{z}_{uL}}{m_s}+\frac{c\dot{z}_{uR}}{m_s}+\frac{c\left(b_L-b_R\right)}{m_s}\dot{\phi}\\ &-\frac{2kz_s}{m_s}+\frac{kz_{uL}}{m_s}+\frac{kz_{uR}}{m_s}-\frac{k\left(b_L-b_R\right)}{m_s}\phi.\end{aligned} \tag{1}$$

Left wheel vertical acceleration is

$$\begin{aligned}\ddot{z}_{uL} = &\frac{kz_s}{m_{uL}}-\frac{\left(k+k_{tL}\right)}{m_{uL}}z_{uL}+\frac{c\dot{z}_s}{m_{uL}}-\frac{c\dot{z}_{uL}}{m_{uL}}+\frac{kb_L}{m_{uL}}\phi\\ &+\frac{cb_L}{m_{uL}}\dot{\phi}+\frac{k_{tL}}{m_{uL}}z_{rL}+F_L.\end{aligned} \tag{2}$$

Right wheel vertical acceleration is

$$\begin{aligned}\ddot{z}_{uR} = &\frac{kz_s}{m_{uR}}-\frac{\left(k+k_{tR}\right)}{m_{uR}}z_{uR}+\frac{c\dot{z}_s}{m_{uR}}-\frac{c\dot{z}_{uR}}{m_{uR}}-\frac{kb_R}{m_{uR}}\phi\\ &-\frac{cb_R}{m_{uR}}\dot{\phi}+\frac{k_{tR}}{m_{uR}}z_{rR}-F_R.\end{aligned} \tag{3}$$

Roll acceleration of half car model is

$$\begin{aligned}\ddot{\phi} = &-\frac{\left(b_L-b_R\right)kz_s}{I_s}+\frac{\left(b_Lkz_{uL}\right)}{I_s}-\frac{\left(b_Rkz_{uR}\right)}{I_s}-\frac{c\left(b_L-b_R\right)\dot{z}_s}{I_s}\\ &+\frac{b_Lc\dot{z}_{uL}}{I_s}-\frac{b_Rc\dot{z}_{uR}}{I_s}-\frac{\left(kb_L^2+kb_R^2\right)}{I_s}\phi\\ &-\frac{c\left(b_L^2+b_R^2\right)}{I_s}\dot{\phi}+eF_f.\end{aligned} \tag{4}$$

2.2. The Linear Single Track Model with Roll Dynamics. The simplest vehicle model is the linear single track model, also known as the bicycle model, which is obtained by approximating the front and rear pairs of wheels as single wheels. The model is illustrated in Figure 3. Assuming that the steering angle is small, the equations of motion are given by [17]. The equations of motion are augmented by a torque balance around the x-axis. The equations are the following.

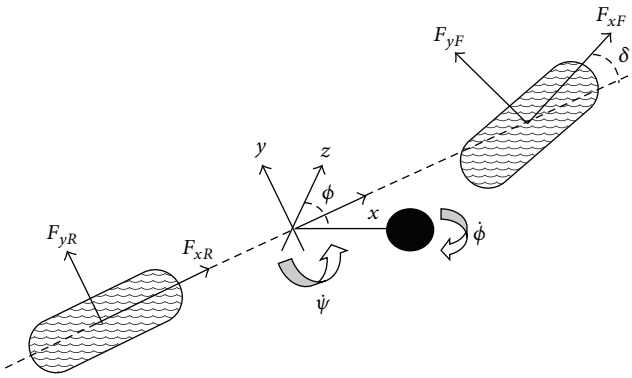


Figure 3: Single track model showing the combined front and rear tire forces, the steering angle, the yaw rate, and the vehicle sideslip angle [17].

Lateral motion is

$$m\left(\dot{V}_y + \dot{V}_x\dot{\psi}\right) = F_{yF} + F_{yR}. \tag{5}$$

Yaw rate is

$$\begin{gathered} I_{zz}\ddot{\psi} = aF_{yF} - bF_{yR}, \\ I_{xx}\ddot{\phi} + C_q\dot{\phi} + K_q\phi = mh\left(\dot{V}_y + V_x\dot{\psi}\right), \\ F_{yF} \approx C_f\alpha_F, \\ F_{yR} \approx C_r\alpha_R, \end{gathered} \tag{6}$$

where F_{yF} and F_{yR} are the combined front and rear lateral tire forces, I_{zz} is the moment of inertia around the z-axis, a and b are the distances from the front and rear wheels to the centre of gravity, and $\dot{\psi}$ is the yaw rate. The slip angles of the front and rear wheels α_F and α_R can be approximated as

$$\begin{gathered} \alpha_F \approx \delta - \frac{1}{V_x}\left(V_y + \alpha\dot{\psi}\right), \\ \alpha_R \approx -\frac{1}{V_x}\left(V_y - b\dot{\psi}\right). \end{gathered} \tag{7}$$

The input to the system is steering angle while the outputs of the system are lateral velocity, yaw rate, and roll rate. The transfer function equations single track model with roll dynamics system is written as follows.

Lateral acceleration is

$$\begin{aligned} \dot{V}_y = {} & \frac{C_f}{m}\left[\delta - \frac{1}{V_x}\left(V_y + a\dot{\psi}\right)\right] + \frac{C_r}{m}\left[-\frac{1}{V_x}\left(V_y - b\dot{\psi}\right)\right] \\ & - \dot{V}_x\dot{\psi}. \end{aligned} \tag{8}$$

Yaw acceleration is

$$\ddot{\psi} = \frac{aC_f}{I_{zz}}\left[\delta - \frac{1}{V_x}\left(V_y + a\dot{\psi}\right)\right] - \frac{bC_r}{I_{zz}}\left[-\frac{1}{V_x}\left(V_y - b\dot{\psi}\right)\right]. \tag{9}$$

Roll acceleration of single track model is

$$\ddot{\phi} = \frac{mh}{I_{xx}}\left(\dot{V}y + V_x\dot{\psi}\right) - \frac{C_q\dot{\phi}}{I_{xx}} - \frac{K_q\phi}{I_{xx}}. \tag{10}$$

2.3. Suspension Model. Three different cases are analysed which are passive suspension without ARB, passive suspension with passive ARB, and passive suspension with active ARB. Each case is modelled by different roll control force equations. The suspension is composed of a spring and a damper for each side of the vehicle. In this case the roll control forces are eliminated, because the only roll stiffness is provided by the suspension spring and damper [16]. Here

$$\begin{gathered} F_R = 0, \\ F_L = 0. \end{gathered} \tag{11}$$

The passive suspension with the passive ARB model is representative of the most common vehicles in use today. To reduce roll motion, an ARB is added to the suspension system, increasing the total roll stiffness. The ARB is usually made of steel and acts like a spring connected to the right and left unsprung masses. The ARB force is a function of the difference between right and left suspension deflection. The force is applied by the bar on each side of the vehicle so that the left force has the same magnitude and the opposite direction as the right one. For an ARB stiffness, k_{ARB}, the force corresponds to

$$F_{\text{ARB}} = -k_{\text{ARB}}\left[\left(z_{sR} - z_{uR}\right) - \left(z_{sL} - z_{uL}\right)\right]. \tag{12}$$

To model the passive ARB the forces F_R and F_L are given as follows:

$$\begin{gathered} F_R = F_{\text{ARB}} = -k_{\text{ARB}}\left[\left(z_{sR} - z_{uR}\right) - \left(z_{sL} - z_{uL}\right)\right], \\ F_L = -F_{\text{ARB}} = k_{\text{ARB}}\left[\left(z_{sR} - z_{uR}\right) - \left(z_{sL} - z_{uL}\right)\right]. \end{gathered} \tag{13}$$

For a passive suspension with an active ARB, generally a pneumatic or hydraulic piston [18] is used; therefore it is modelled by a controllable force. This actuator is placed in series with a stiff ARB. The force is applied with the same magnitude but in opposite directions as for the passive ARB. However, the force magnitude F_f is calculated by the controller. As a result the forces are applied as follows:

$$\begin{gathered} F_R = F_f, \\ F_L = -F_f. \end{gathered} \tag{14}$$

Three different inputs can be applied to the models which are lateral force, left road motion, and right road motion. The application of left and right road motion with equal phase and amplitude results in pure vertical motion similar to the quarter model behaviour. Since this model is extensively studied in the literature, it is not detailed in this paper. On the other hand, the excitation of a single wheel gives valuable indication on how the system transfers the road perturbation from one side to the other. Since the ARB has a noticeable impact on this transfer, the first case studied is the application of a road motion input to a single wheel. As the side of the excitation application does not change the response amplitude, the input is applied at the right tire or road interface.

In ground vehicle real life use, cornering maneuvers are unavoidable. The change in trajectory is accomplished

by creating a lateral acceleration. Since the response of the vehicle's suspension during these maneuvers is critical for stability, lateral acceleration input is the second case studied in this work.

The roll motion is caused by the vehicle's lateral acceleration, which depends on the speed and corner radius. In a manner similar to the road perturbation, the direction of the force only changes the direction of the response since the vehicle is symmetrical. For this work a right turn is simulated resulting in a left acceleration creating a force at the sprung mass CG. Since the force is applied at a distance h from the rotation centre, a moment M_s is created on the sprung mass with a magnitude given as follows:

$$M_s = a_y m_s h. \tag{15}$$

3. Controller Design

3.1. Linear Quadratic Gaussian. The theory of LQG control design is well known, and it is summarised below.

Consider the state space model of the process in the form of

$$\begin{aligned} \dot{x} &= Ax + Bu + \Gamma w, \\ y &= Cx + v, \end{aligned} \tag{16}$$

where $x \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{m\times n}$, and $y \in \mathbb{R}^{q\times 1}$, while w and v are white uncorrelated process and measurement noises, respectively. In the state feedback version of the LQR, it assumes that the whole state x can be measured and therefore is available for control. The state feedback controller is given by

$$\begin{aligned} u &= -K_r x, \\ K_r &= R^{-1}B^T P, \end{aligned} \tag{17}$$

where K_r is gain matrix and is subject to minimize the quadratic cost function given by

$$J = \int_0^\infty \left[x^T(t) Q_r x(t) + u^T(t) R_r u(t)\right] dt, \tag{18}$$

where state weighting matrix Q is $n \times n$ symmetric positive definite matrix and control weighting matrix R is $m \times m$ symmetric positive definite matrix.

The gain matrix K is the solution matrix Riccati differential equation which is subject to given values of A, B, C, Q, and R given by

$$A^T P + PA + \dot{P} + Q = PBR_k^{-1}B^T P. \tag{19}$$

If P is constant then $\dot{P} = 0$; the Riccati equation can be simplified to

$$A^T P + PA + \dot{P} + Q - PBR_k^{-1}B^T P = 0, \tag{20}$$

and the solution of the gain matrix is given by

$$K_r = R_k^{-1}B^T P \tag{21}$$

subject to (A, B) being stabilizable R_k and $Q \geq 0$ is positive definite and positive semidefinite. (Q, A) has no unobservable modes on the imaginary axis.

The LQR part guarantees the closed loop stability with appropriate choice of weighting matrices Q and R variables [19–21]. Since not all states parameters can be measured directly, an estimator is used for the entire state vector with respect to the plant's output. The estimator is given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + K_f\left(y - C\hat{x}\right) \\ &= \left(A - K_f C\right)\hat{x} + K_f y, \end{aligned} \tag{22}$$

where K_f is the estimator filter gain matrix given as

$$K_f = P_f C^T R_f^{-1}. \tag{23}$$

The covariance vector P_f of the estimated state variable is obtained by solving the algebraic Riccati equation:

$$AP_f + PA^T - P_f C^T R_f^{-1} C P_f + BQ_f B^T = 0, \tag{24}$$

where $Q_f = E(\omega\omega^T)$ and $R_f = E(vv^T)$ are process and sensor noise covariances.

3.2. Composite Nonlinear Feedback Controller. Composite nonlinear control feedback is a nonlinear control technique, which contains composition of a linear feedback law and a nonlinear feedback law without any switching element. In other words, linear and nonlinear laws in this type of algorithm are working simultaneously all the time. In this method, the linear feedback objective is to get a small damping ratio for quick response and at the same time without exceeding the nature of actuator limits. Meanwhile, for nonlinear feedback, it is designed to increase the damping ratio as the system output approaches the target reference to avoid overshoot from occurrence [19, 20].

There are some efforts by researchers to come out more generalize version that can be implemented in higher order in multiple-input and multiple-output (MIMO) configuration since exploration of CNF for linear 2nd order system single-input and single-output (SISO) plant as intensively discussed in literature review part. Furthermore, CNF algorithm has been enhanced to cater external disturbance as reported in several papers. Although the CNF technique is fairly mature to solve general linear system, it is not robust enough to solve parameter variation so-called uncertainties that may occur in most typical controlled plant [21].

For this classical CNF technique, controlled plant is considered without external disturbance. A linear continuous time system with actuator saturation can be expressed in general state space as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\,\mathrm{sat}(u(t)), \qquad x(0) = x_0, \\ y(t) &= C_1 x(t), \\ h(t) &= C_2 x(t), \end{aligned} \tag{25}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}^p$, and $h \in \mathbb{R}$ are the state vector, control input, measurement output, and controlled

output vector, respectively. A, B, C_1, and C_2 are appropriate dimensional constant matrices, and sat: $\mathbb{R} \rightarrow \mathbb{R}$ represents actuator saturation which can be defined as

$$\operatorname{sat}(u) = \operatorname{sgn}(u) \min\{u_{\max}, |u|\}. \tag{26}$$

In (26), $u_{\max}$ is a saturation level of actuator input. To apply the CNF, the following assumptions of system matrices are considered:

(1) (A, B) is controllable (stabilizable);

(2) (A, C_1) is observable (detectable);

(3) (A, B, C_2) is invertible and there is no invariant zero at $s = 0$.

In this research, the controller is designed based on a state feedback case ($y = x$) where the assumption is made that all states of plant are available to measure even though in most cases observer is needed to estimate immeasurable states. There are three steps involved in the CNF design which are linear feedback design to get optimal respond, nonlinear feedback design to prevent overshoot, and finally merging both feedbacks to work together. Details of these steps are as follows.

Step 1. Design a linear feedback law:

$$u_L = Fx + Gr, \tag{27}$$

where r is zero and F is chosen such that $A+BF$ asymptotically stable and closed loop systems $C_2(sI - A - BF)^{-1}$ have certain desired properties such as low damping ratio. Selection of matrix F is not unique and can be found using optimization method such H_2 and H_∞ approaches. Besides, matrix G is scalar and is given by

$$G = -\left[C_2(A + BF)^{-1}B\right]^{-1}. \tag{28}$$

Step 2. Design a nonlinear feedback law:

$$u_N = \rho(y, r) B^T P (x - x_e), \tag{29}$$

where $\rho(y, r)$ is any nonpositive local Lipschitz in x such as $-\beta e^{-\alpha\alpha_0|y-r|}$. This function is used to change damping ratio of the closed loop system when output approaches the reference given. This nonlinear function is not unique and there are solutions to obtain optimal value of tuning the parameter that will be explained later. Matrix P exists since $A + BF$ is asymptotically stable and can be obtained from following Lyapunov equation:

$$(A + BF)^T P + (A + BF) P = -W. \tag{30}$$

There are proper ways to find suitable value of W which is positive definite as proposed by author Chen et al. [13]; however in this research, W is simplifying as matrix identity. In other terms, equilibrium point x_e is obtained by

$$x_e = G_e r, \tag{31}$$

where

$$G_e = -(A + BF)^{-1}BG. \tag{32}$$

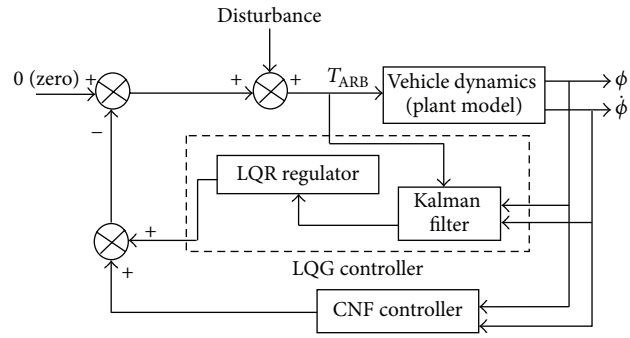


Figure 4: LQG CNF fusion block diagram configuration.

Step 3. Combine linear and nonlinear feedback laws:

$$u = u_L + u_N = Fx + Gr + \rho(y, r) B^T P (x - x_e). \tag{33}$$

In this research, tuning parameters α and β as proposed by Lan et al. [22] will be used. The first parameter β can be found by setting a desired damping ratio of closed loop system ξ_∞:

$$\dot{\tilde{x}} = \left(A + BF - \beta BB^T P\right)\tilde{x}, \tag{34}$$

whereas optimal value α is obtained by solving minimization of some appreciable criterions, such as integral of absolute error (IAE) and integral of time-multiplied absolute value of error (ITAE) which can be written as follows:

$$\min_\alpha \int_0^\infty |e|\, dt \quad \text{or} \quad \min_\alpha \int_0^\infty t\,|e|\, dt, \tag{35}$$

where

$$e = y - r. \tag{36}$$

4. Results and Discussions

4.1. Design of Composite Nonlinear Feedback Based Active Antiroll Bar System. This section will present a design process of proposed control strategy. The objective of the active ARB controller is to reject disturbance and to improve the roll angle and roll rate responses as close to zero. Three types of disturbances are used in this work, which are fishhook, single sine, and speed bump.

The controllers used in this work are linear quadratic Gaussian (LQG) and composite nonlinear feedback (CNF) controller. These controllers are designed and tuned to generate the input of torque to control the active ARB system. Figure 4 shows a block diagram configuration that has been used about the combination of CNF controller, LQR controller, and Kalman filter estimator to form a compensator. This solution in designing LQG controller is based on the separation principle where the LQR controller and Kalman filter are designed independently and then combined to form the LQG compensator. A Kalman filter is a typical application and feasible estimation approach that can fuse multiple sensory measurements to provide an accurate position estimation [23].

Table 1: Numerical parameters for CNF controller.

Parameter	Value	Description
F	[0.2398 − 46.7288 49.3638 0.0890 0.3439 −0.3579 − 0.6518 23.3893 0.7640 2.3462]	Obtained based on damping ratio set as $\xi = 0.67$
α	0.5826	Nonlinear 1st parameter, obtained by the integral of absolute error (IAE) method
β	1000	Nonlinear 2nd parameter, obtained by setting desired damping ratio of closed loop system
P	Diagonal 10×10	Obtained based on Lyapunov as in equation

While, for CNF controller, it has two parts which are a linear feedback part and nonlinear feedback part. The objective of linear feedback is to get a small damping ratio for quick response and nonlinear feedback is designed to increase the damping ratio as the system output approaches the target reference to avoid overshoot [24, 25]. In this study, the target reference is zero.

The linear control law for torque of antiroll bar is defined as

$$\left| T_{\text{ARB}} = \text{Disturbance} - \left(K_{\text{lqr}} K_{\text{kalman}} + Fx + Gr + \rho(y,r) B^T P (x - x_e) \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} \right) \right|, \tag{37}$$

where

$$K_{\text{lqr}} = RB^T P_{\text{lqr}}, \qquad K_{\text{kalman}} = P_f C^T R_f^{-1}, \qquad r = 0, \quad \rho(y,r) = -\beta e^{-\alpha\alpha_0 |y-r|}, \qquad x_e = G_r r. \tag{38}$$

The value of P is obtained from Lyapunov method in (30).

Then, the states are

$$x = \left[z_s \;\; z_{uR} \;\; z_{uL} \;\; \dot{z}_s \;\; \dot{z}_{uR} \;\; \dot{z}_{uL} \;\; \phi \;\; \dot{\phi} \;\; V_y \;\; \dot{\psi} \right]^T. \tag{39}$$

4.2. Parameters Tuning for LQG and CNF Controller

(1) Gain for LQR regulator: using Bryson's rule method

$$Q = \text{diag}\left[\frac{1}{(0.09)^2} \;\; \frac{1}{(0.09)^2} \right], \qquad R = \frac{1}{3^2}. \tag{40}$$

(2) Gain for LQG: using intuitive method

$$Q_f = 500, \qquad R_f = \text{diag}\left[0.001 \;\; 0.001\right]. \tag{41}$$

(3) State feedback gain matrix F gain in linear part of CNF controller obtained based to damping ratio, set at 0.67 and using the LQR tuning method. Firstly, the gain of LQR is tuned using intuitive method. Then, the variation of tuning is designed until the damping ratio is set at 0.67 in finding the gain value of a matrix F. The parameter α can be found by the integral of absolute error (IAE) method. Meanwhile, an optimal β is obtained by setting the desired damping ratio of closed loop system. Table 1 shows numerical parameters CNF controller obtained during these processes.

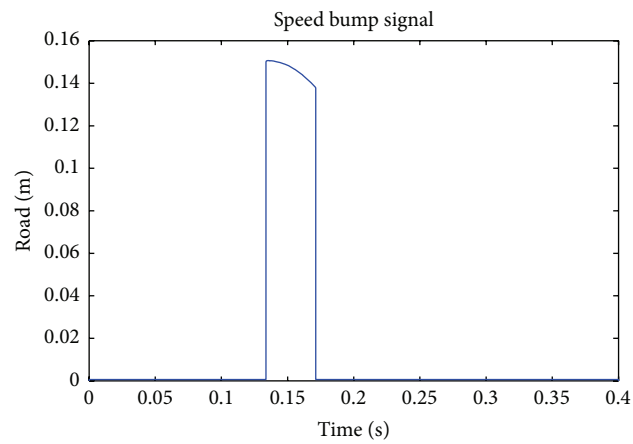


Figure 5: Speed bump signal.

4.3. Simulation Results. Two patterns and level of cornering manoeuvres and road holding disturbance is considered to evaluate the performance of the designed controller. Besides, the comparison is performed by comparing LQG with LQG CNF fusion results. Three types of input disturbance patterns are used which are single sine, fishhook, and speed bump tests. Figures 5, 6, and 7 show three types of tests used.

The controllers are designed and tuned to realize their performance corresponding to a torque to reduce the body roll angle and body roll rate. Therefore, from these simulations, the vehicle that is using active ARB with LQG CNF fusion control strategy has a good ride and handling at the same time.

Moreover, to verify and investigate the effectiveness of the proposed control method, computer simulations based on Matlab software were conducted and the output responses of roll angle and roll rate are compared with output signal between the different control strategies.

4.4. Road Profiles. The characteristic of road disturbance that is made for the system is shown in Figure 5 and the road profile is speed bump signal with speed of 40 km/h and frequency of 14.815 Hz, and time delay required is 0.18 s.

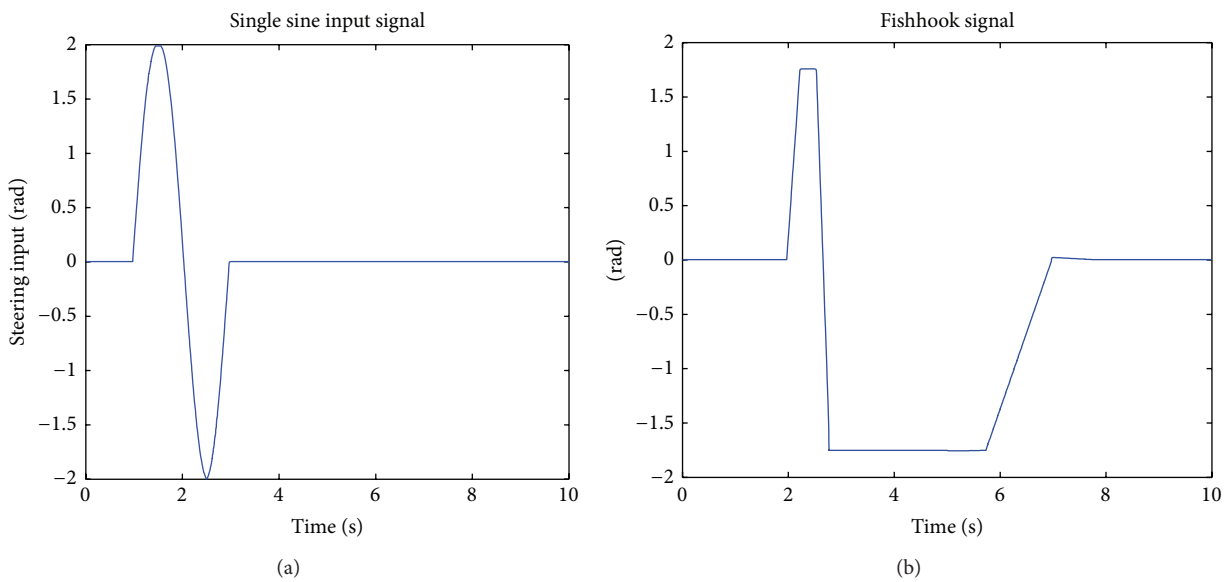


Figure 6: (a) Single sine input signal and (b) fishhook signal.

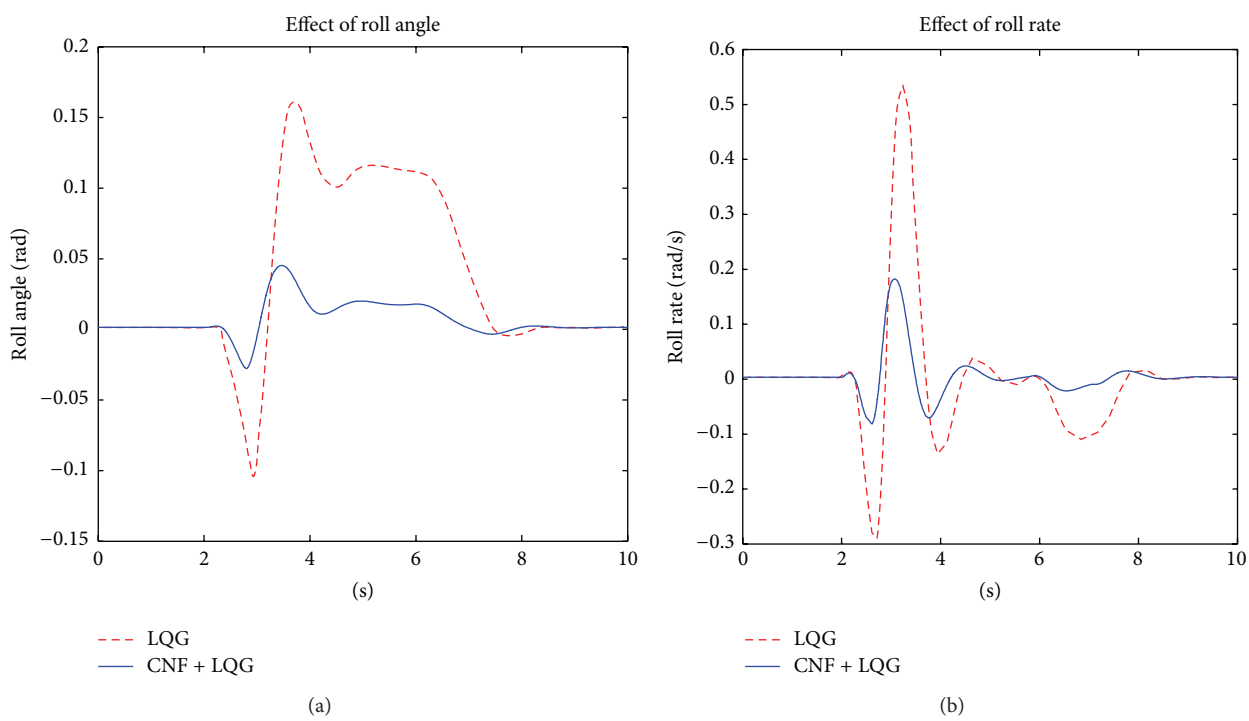


Figure 7: Fishhook test (a) roll angle response and (b) roll rate response.

4.5. Manoeuvres. Various patterns and levels of cornering manoeuvres are considered to evaluate the performances of roll angle and roll rate as well as the robustness of the controller design. Two types of pattern inputs are used, single sine input with an amplitude of 2 rad and fishhook signal with amplitude of 1.755 rad. Figure 6 shows both types of steering inputs.

The trend indicates that further improvements in ride comfort and handling performance bring a similar pattern but the value of overshoot is dissimilar by using the different types of controller. Figures 7 and 8 show the signal responses for handling case while Figure 9 shows the signal performances for ride case. In handling case, it can be shown that an active ARB system using LQG controller is a very inferior quality while for the ride test also it is not as good as using a combination of LQG and CNF controllers.

Figures 7, 8, and 9 show the vehicle roll angle and roll rate response for the different tests which are fishhook, speed

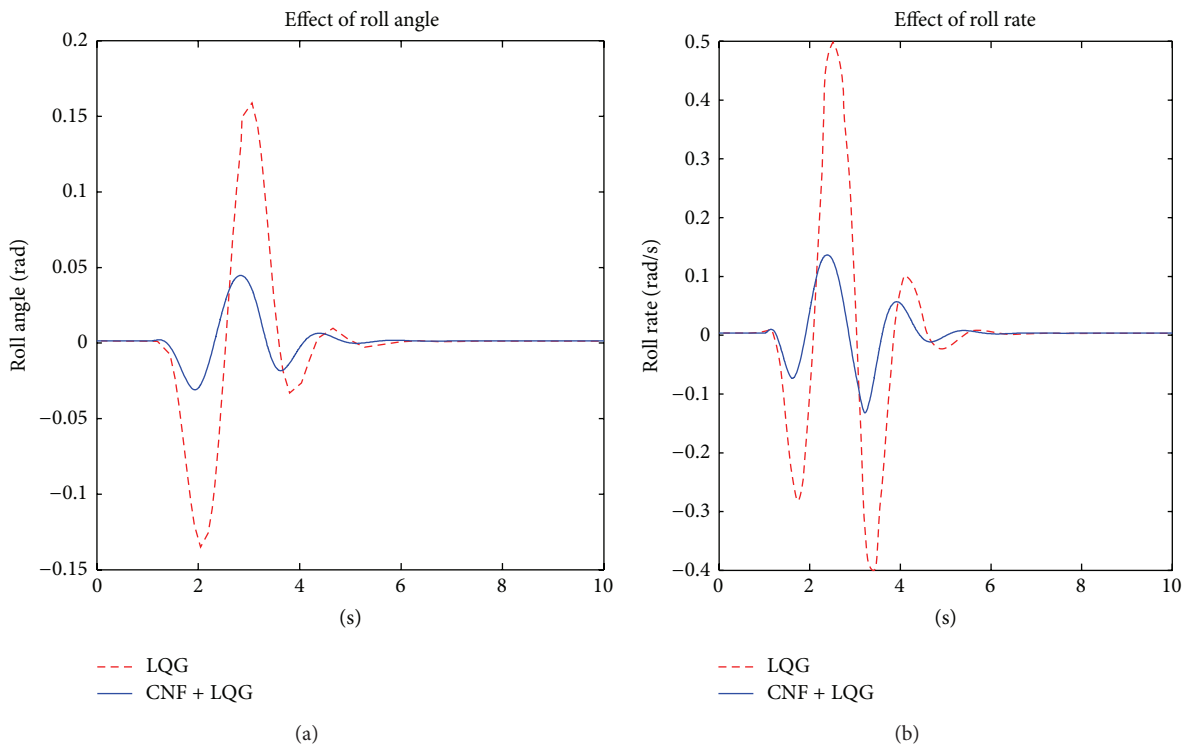


Figure 8: Single sine input test (a) roll angle response and (b) roll rate response.

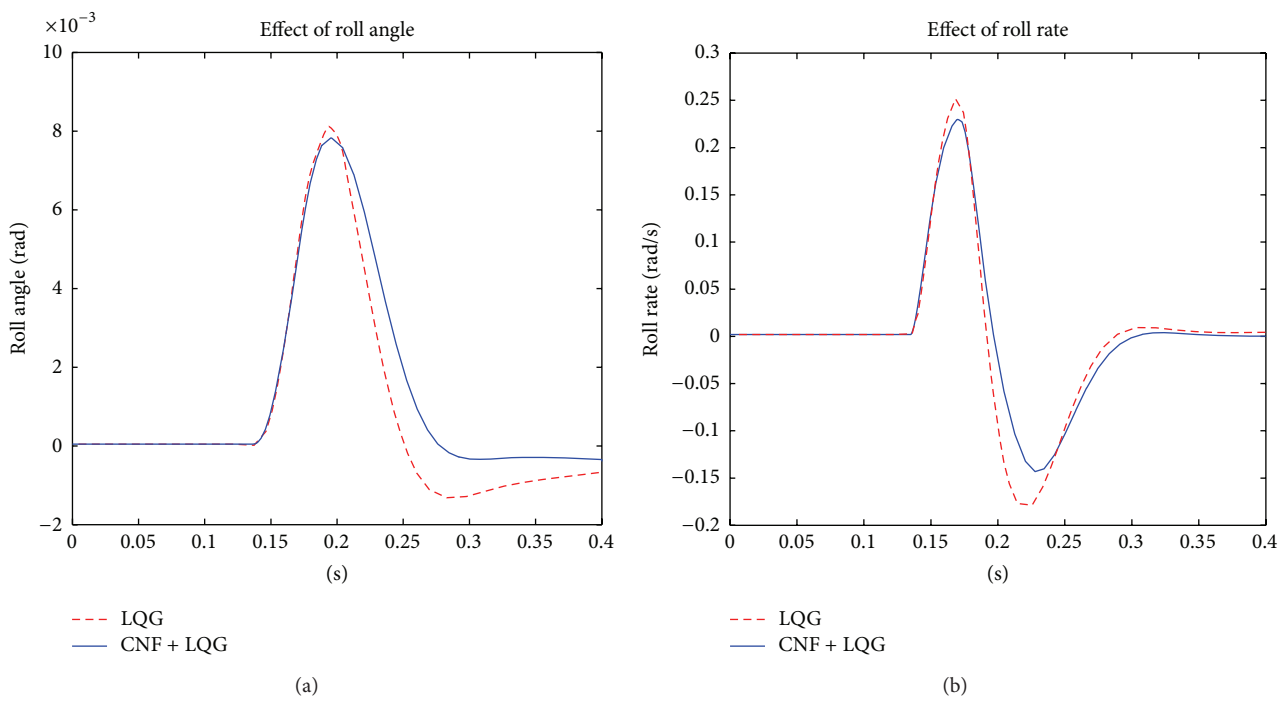


Figure 9: Speed bump test (a) roll angle response and (b) roll rate response.

bump, and single sine inputs. It can be seen in Figure 7 that the roll angle and roll rate response is highly reduced for an active ARB system that used LQG CNF fusion compared with LQG control strategy. The reduction of the body roll angle for LQG CNF fusion compared with LQG is about 74.2%, whereas the roll rate improvement in this figure is approximately 66.7%. For Figure 8, there was improvement in the maximum body roll angle between the systems with

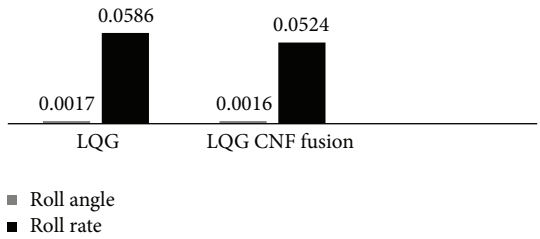


Figure 10: Speed bump test.

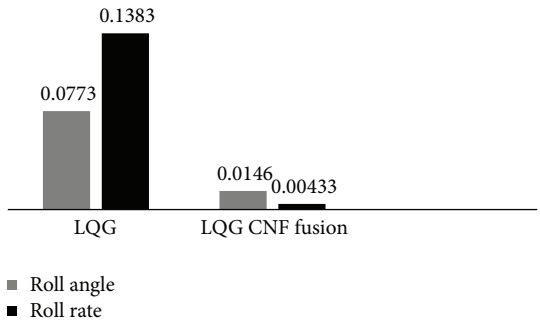


Figure 11: Fishhook test.

LQG CNF fusion control scheme and with LQG of 75%, while for the body roll rate an improvement is around 72%. Moreover, for Figure 9, the roll angle and roll rate responses are slightly reduced for the active ARB system that is used for LQG CNF fusion controller compared with LQG. The body roll angle is reduced about 4.9% while roll rate showed 11.54% improvement compared between LQG CNF fusion and LQG control strategy. Therefore, by using LQG CNF fusion control strategy, the performances of roll angle and roll rate in handling and ride test are better and the rollover of the vehicle is more decreased compared with the system connected to LQG. From these simulation results, the vehicles that used active ARB with CNF with LQG controller will reduce the trade-off gap between ride comforts and improve handling at the same time. From three tests conducted in this study, it can be shown that the performance by using an active ARB system with LQG combined with CNF controller seems better. The settling time is quicker and reduces the overshoot. The improvement of response signals is calculated by using the percentage reduction formula. The formula for percentage reduction is calculated as follows (see [3]):

$$\%\text{reduction} = \frac{(\text{baseline_value} - \text{new_value})}{\text{baseline_value}} \times 100. \quad (42)$$

Figures 10, 11, and 12 show the numerical results of three different tests that compared between LQG and LQG CNF fusion control strategies. The RMS value for CNF with LQG controller is more decreased than LQG.

This section has presented simulation results of the proposed controllers and evaluation on the controller's performance in handling and ride performance. In order to verify the effectiveness and performance of designing controllers, various simulation tests have been carried out. Simulation result shows that LQG CNF fusion control method is the best control strategy with fast response, lower settling time, reduced overshoot, and improved response of roll angle and roll rate.

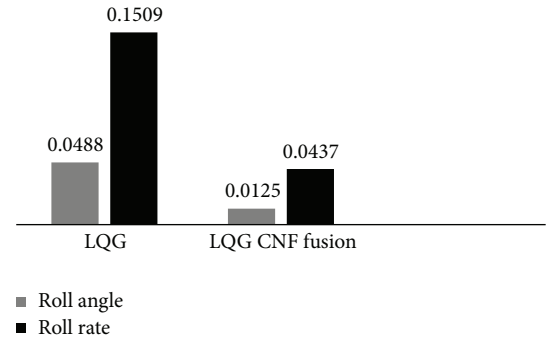


Figure 12: Single sine input test.

Table 2: Nominal CNF parameters.

Parameter	Value
α	1
β	1000

4.6. Effect of Variation of CNF Parameters. The nominal parameters of the CNF controller are set as in Table 2 to investigate the influence of each parameter and then varied as follows. The test inputs in analysis of variation of CNF parameters are fishhook, single sine, and speed bump tests. The variation analysis for fishhook test is shown in Figures 13 and 14, whereas the variation analysis for single sine test is shown in Figures 15 and 16. In addition, the variation analysis for speed bump test is shown in Figures 17 and 18.

The value of α is varied from 1 to 10 in increment of 1. The response for roll angle and roll rate has large overshoot as α is higher in magnitude. The variations in the responses as α varies are shown in Figure 13.

β is varied from 1000 to 10000 in increment of 1000. As shown in Figure 14, the responses get a lesser amount of overshoot as β is larger in magnitude. As the nonlinear part controls how well the response is in terms of overshoots, it is expected that larger β means better control and reduces the overshoot of the responses to fishhook test. The variations analysis for single sine test also has the same results as fishhook test. The results are shown in Figures 15 and 16. The settling times for roll angle and roll rate are decreased as β is enlarged and the results are shown in Figures 14 and 16.

Moreover, the results of variation analysis for speed bump test are shown in Figures 17 and 18. As α increased, the roll angle and roll rate have a large magnitude of overshoot, while, as the value of β increased, the responses to speed bump test also get larger overshoot and the settling time is decreased.

It can be concluded that the resulting performances have difference variations analysis between the ride comfort test and handling test. The difference of variation is for the value of β only. For ride comfort test, β must have lower value to

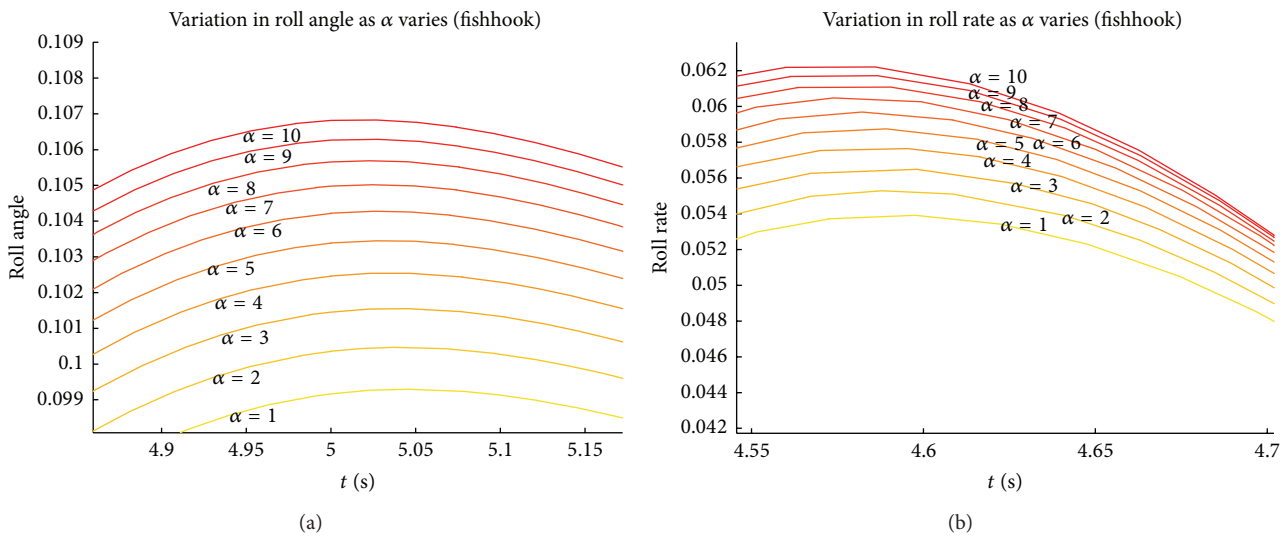


Figure 13: Fishhook: (a) variation in ϕ as α varies; (b) variation in $\dot{\phi}$ as α varies.

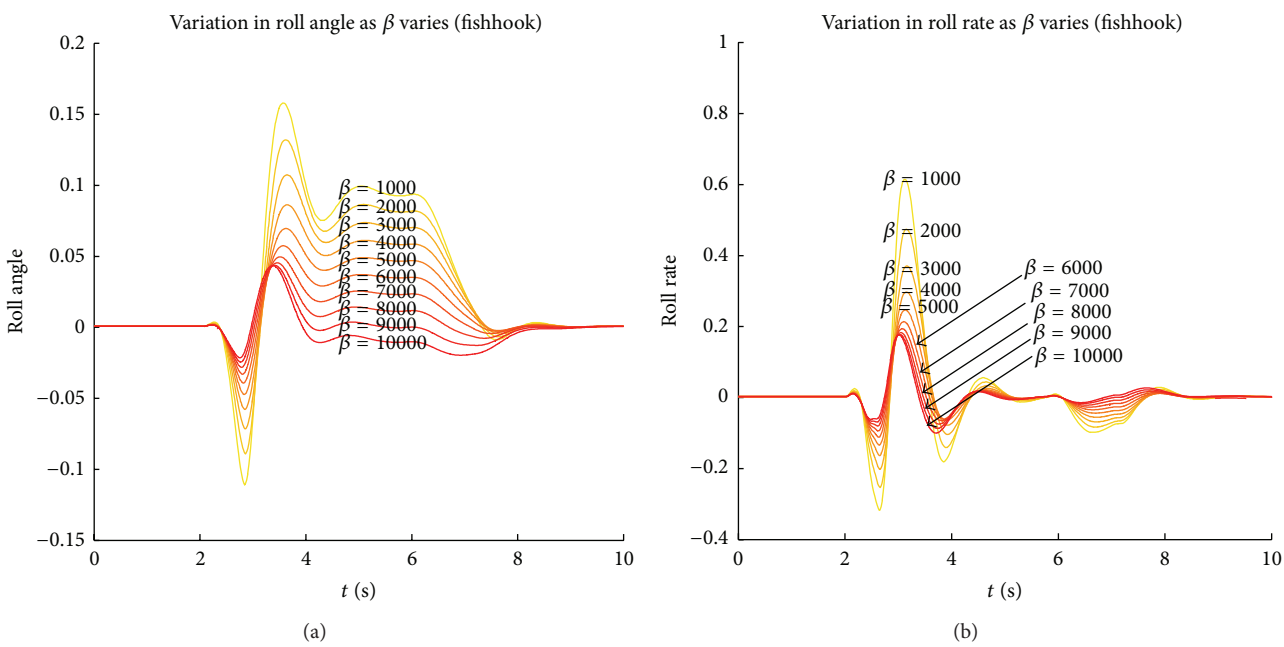
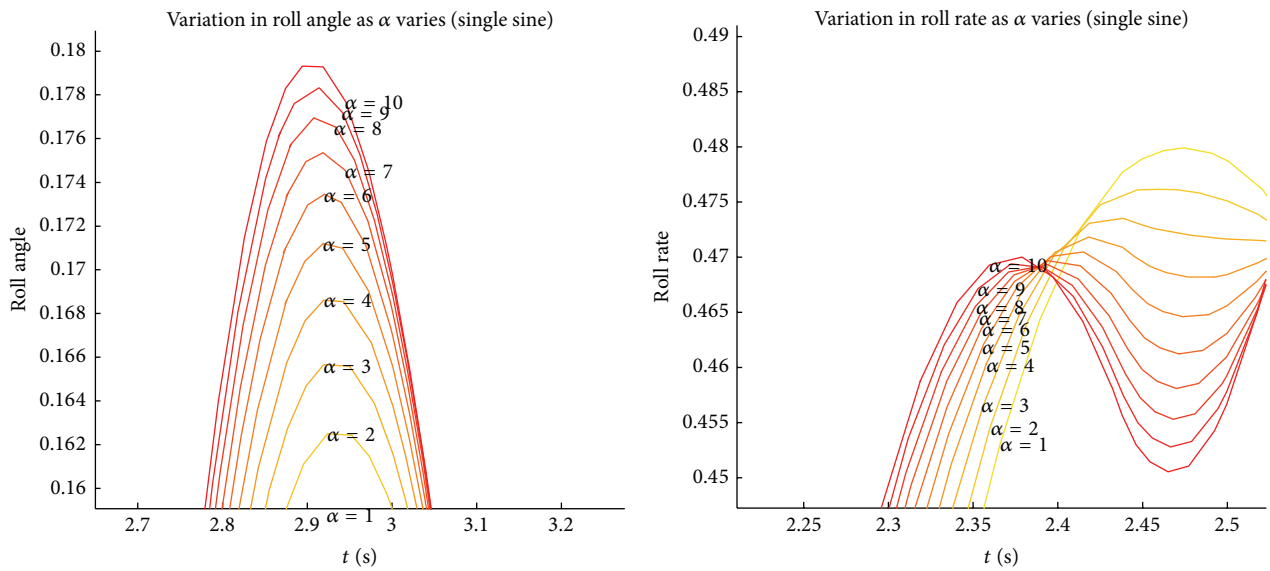


Figure 14: Fishhook: (a) variation in ϕ as β varies; (b) variation in $\dot{\phi}$ as β varies.

get less overshoot and fast settling time for roll angle and roll rate. At the same time for handling test, the value of β needs to increase to get a reduced amount of overshoot and quick settling time for responses of roll angle and roll rate.

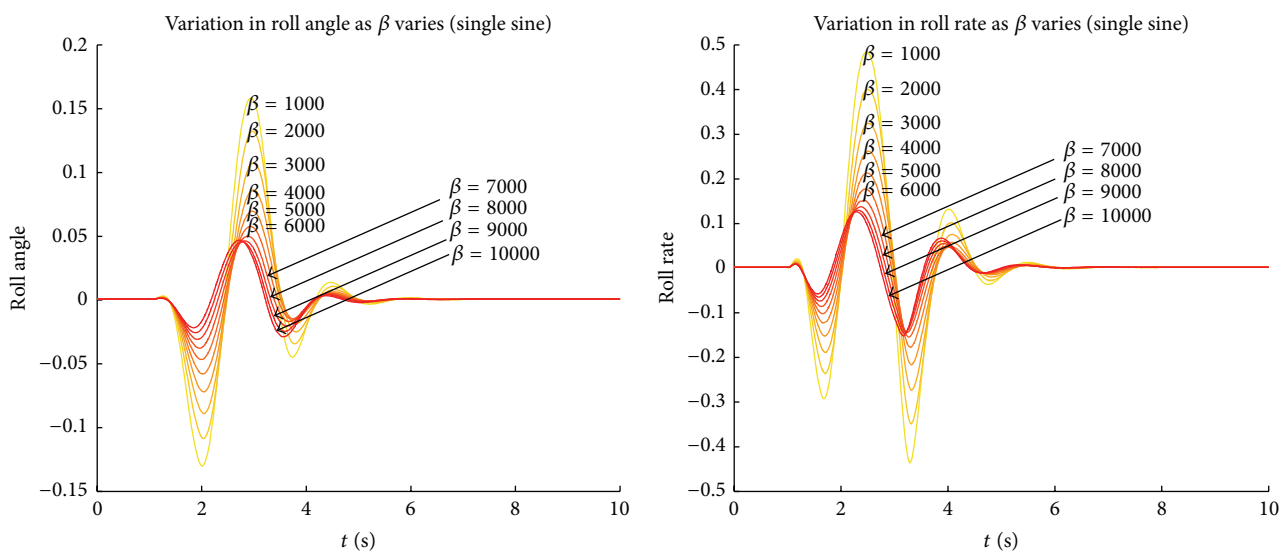
5. Conclusions

This paper compares the performance of different types of controllers which are the system with LQG and LQG CNF fusion control strategy. A four DOF vehicle model has been used to model the vehicle suspension system. The Matlab/SIMULINK environment of the simulation model on the antiroll designs for four DOF vehicle dynamic models is used to simulate the proposed control system of an active ARB and its counterparts. Based on the results, the performance of the proposed system is capable of achieving better performance than its counterparts in terms of roll angle and roll rate reduction during roll induced manoeuvre. For future work, more advanced control structure and control design will be investigated under various manoeuvres in order to improve the vehicle stability and this would not only benefit handling aspect of the vehicle but also contribute to vehicle ride comfort. The techniques of optimization also will be further studied to gain better performances.



(a) (b)

Figure 15: Single sine: (a) variation in ϕ as α varies; (b) variation in $\dot{\phi}$ as α varies.



(a) (b)

Figure 16: Single sine: (a) variation in ϕ as β varies; (b) variation in $\dot{\phi}$ as β varies.

Nomenclatures of the Vehicle System

m_s: Sprung mass weight
m_{uL}: Left unsprung mass weight
m_{uR}: Right unsprung mass weight
z_s: Body displacement
$\dot{z}_s$: Body deflection
z_{uL}: Left wheel displacement
$\dot{z}_{uL}$: Left wheel deflection
z_{uR}: Right wheel displacement
$\dot{z}_{uR}$: Right wheel deflection
ϕ: Roll angle
$\dot{\phi}$: Roll rate
V_y: Lateral velocity
b_L: Length for CG to the left wheel suspension
b_R: Length for CG to the right wheel suspension
c: Damping stiffness
k: Spring stiffness
k_{tL}: Left tire stiffness
k_{tR}: Right tire stiffness
δ: Steering angle
K_q: Spring coefficient

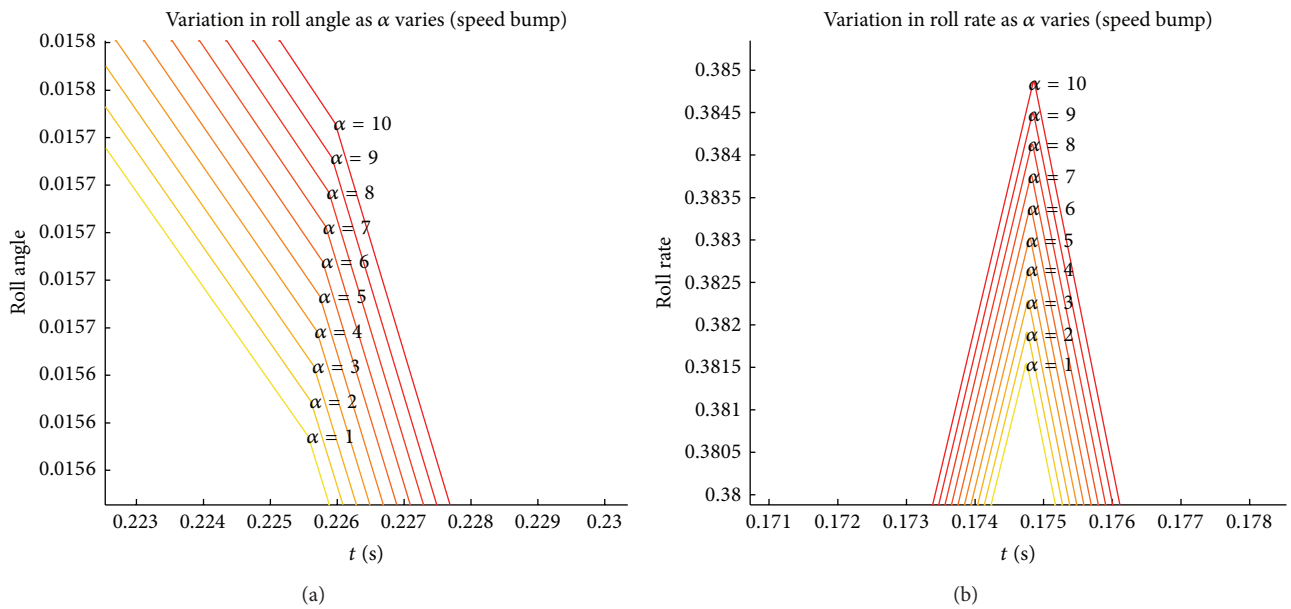


FIGURE 17: Speed bump: (a) variation in ϕ as α varies; (b) variation in $\dot{\phi}$ as α varies.

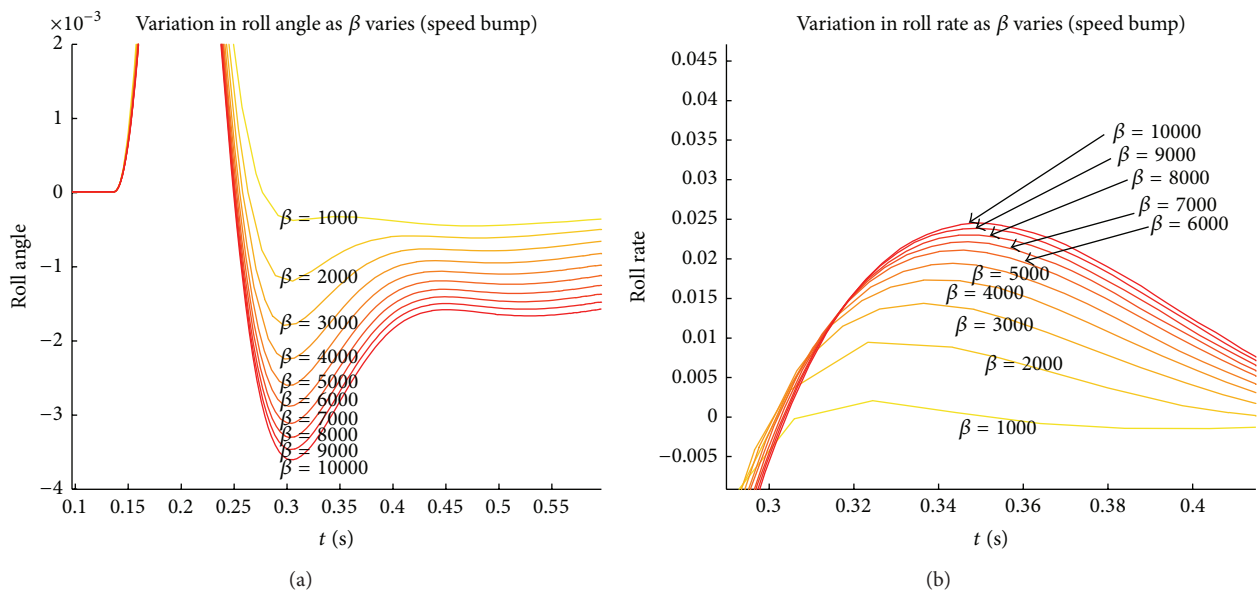


FIGURE 18: Speed bump: (a) variation in ϕ as β varies; (b) variation in $\dot{\phi}$ as β varies.

K_{lqr}: Gain of LQR regulator
I_s: Roll axis moment inertia
F_L: Left suspension force
F_R: Right suspension force
k_{ARB}: Antiroll bar stiffness
RC: Roll centre
m: Mass of vehicle
V_x: Longitudinal velocity
C_f: Front cornering stiffness
C_r: Rear cornering stiffness
F_{yF}: Front lateral tire force
F_{yR}: Rear lateral tire force
I_{zz}: Moment of inertia around the z-axis
a: Distance from the front wheel to the centre of gravity
b: Distance from the rear wheel to the centre of gravity
$\dot{\psi}$: Yaw rate
α_F: Slip angle of the front wheel
α_R: Slip angle of the rear wheel
h: Height
C_q: Damping coefficient
T_{ARB}: Torque of antiroll bar
K_{kalman}: Gain of Kalman filter.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is fully supported by Ministry of Higher Education Malaysia and Universiti Teknologi Malaysia under FRGS research Grant (Vote no. 4F370) and research university Grant (Vote no. 00G64). The work is also supported by Proton Sdn. Bhd.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 970205, 13 pages
http://dx.doi.org/10.1155/2014/970205

Research Article

Complexity Analysis of a Master-Slave Oligopoly Model and Chaos Control

Junhai Ma,[1] Fang Zhang,[1,2] and Yanyan He[2]

[1] *College of Management and Economics, Tianjin University, Tianjin 300072, China*
[2] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China*

Correspondence should be addressed to Fang Zhang; zhangfangsx@163.com

Received 11 April 2014; Revised 11 June 2014; Accepted 18 June 2014; Published 13 August 2014

Academic Editor: Simone Marsiglio

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We establish a master-slave oligopoly game model with an upstream monopoly whose output is considered and two downstream oligopolies whose prices are considered. The existence and the local stable region of the Nash equilibrium point are investigated. The complex dynamic properties, such as bifurcation and chaos, are analyzed using bifurcation diagrams, the largest Lyapunov exponent diagrams, and the strange attractor graph. We further analyze the long-run average profit of the three firms and find that they are all optimal in the stable region. In addition, delay feedback control method and limiter control method are used in nondelayed model to control chaos. Furthermore, a delayed master-slave oligopoly game model is considered, and the three firms' profit in various conditions is analyzed. We find that suitable delayed parameters are important for eliminating chaos and maximizing the profit of the players.

1. Introduction

Oligopoly is a market structure between monopoly and perfect competition. It is characterized by a domination of several firms, which completely control trade. These firms manufacture the same or homogeneous products. They have to consider both the market demand, that is, the behavior of consumers, and the strategies of their competitors; that is, they form expectations concerning how their rivals will act. The most widely used and simultaneously the first formal model of oligopoly market were proposed by Cournot. Cournot model assumed that each company adjusts its quantity of production to that of its rivals and there is no retaliation at all, so that in every step the player perceives the latest move made by the competitors to remain his last. Besides Cournot model, there is another important model: the Bertrand model. The former is under the assumption that producers in an oligopoly decide their policy assuming that other producers will maintain their output at its existing level, while Bertrand model is based on the assumption that producers act on the belief that competitors will maintain their price to maximize profits rather than their output.

Works on Cournot or Bertrand model showed that it has an ample dynamical behavior under different expectations [1–6]. A large number of literatures have been published. Puu considered a triopoly Cournot competition model [7]. Agiza and Elsadany studied the dynamics of a Cournot duopoly with heterogeneous players [8]. Zhang and Ma analyzed Bertrand competition model of four oligopolists with heterogeneous expectations [9]. And there are some articles about Cournot-Bertrand competition [10, 11]. For instance, Ma and Pu studied a Cournot-Bertrand duopoly model with bounded rational expectations [12]. The master-slave Cournot or Bertrand model exists in realistic economy. Xin and Chen studied a master-slave duopoly Bertrand game model in the setting that the upstream firm might be the monopolistic supplier of fresh water and the downstream firm might be the monopolistic supplier of pure distilled water [13]. And the upstream monopolistic firm's output is used as the main factor of production by the downstream monopolistic firm who is a negligible purchaser of the upstream monopolist's output.

In general situations, a system with nonlinear term will not always be stable and sometimes can even be chaotic.

However, the appearance of chaos in the economic system is not expected and even is harmful. Thus, people hope to find some methods to control the chaos of economic system. By controlling the chaotic phenomenon occurring on the market, bifurcation and chaos are delayed or eliminated, and the system is stabilized to the Nash equilibrium; that is, the market goes back to orderly competition. In recent years, scholars put forward many control methods to different chaotic systems such as OGY control, adaptive control, and feedback control.

As the game player makes decisions at time t that depend on past observed variables by means of a prediction feedback and the functional relationships describing the dynamics of the model may depend on both the current state and the past states, a delayed structure in economics models emerges. Yassen and Agiza considered a delayed duopoly game and got some important conclusions [14]. Since then, many experts and scholars also extensively studied delay oligopoly models, such as Peng et al. [15] and Ma and Wu [16]. These studies focus on the changes of stability domain of system in the case of delay or the bifurcation of system with the parameters changing, but the research to the players' profits is less.

Based on the research of experts and scholars on these models, this paper builds a master-slave oligopoly game model in which upstream monopolistic firm competes in output and two downstream oligopolistic firms compete in price. We use two control methods: the delayed feedback control method and the limiter control method to control chaos. In addition, we study the delayed game model. And we have given thorough discussion on profits of players.

The paper is organized as follows. In Section 2, a non-delayed oligopoly model with bounded rational expectations is presented. Equilibrium points and stability are analyzed. And numerical simulations which confirm analytical results are given. Two chaos control methods, the delayed feedback control method and the limiter control method, are shown in Section 3. Delayed system is investigated in Section 4. Section 5 gives the conclusion.

2. The Nondelayed Master-Slave Model

2.1. Model. In an area, there are three firms in the market and firm i produces goods x_i, $i = 1, 2, 3$. Firm 1 represents the upstream monopolistic firm and firms 2 and 3 represent two downstream oligopolistic firms. The output and price of firm i are represented, respectively, as q_i and p_i.

This model is based on the following assumptions.

(1) The upstream monopolist supplies output (e.g., fresh water) to two downstream firms that compete in a final goods market (e.g., distilled bottled water). The upstream market is monopoly (a single firm: firm 1), and the downstream market is duopoly (two firms: 2 and 3). In addition, the two downstream oligopolistic firms do not cooperate. Then we have $q_1 > q_2 + q_3$, $p_1 < p_2$ and $p_1 < p_3$.

(2) The cost of the two upstream monopolistic firms is a quadratic function. Because firm 1's price will affect the cost of the downstream firms (assuming 1 unit of fresh water is used to make 1 unit spring water), we assume the two firms' marginal cost is p_1.

(3) In the vertically connected market, the three firms make decisions at the same time. The upstream monopolistic firm's decision variable is its output q_1 while the two downstream firms' decision variables are their prices p_2 and p_3.

Assume that the inverse demand function of firm 1 and demand functions of firms 2 and 3, respectively, are

$$\begin{aligned} p_1(t) &= a_1 - b_1 q_1(t), \\ q_2(t) &= a_2 - b_2 p_2(t) + d_2 p_3(t), \\ q_3(t) &= a_3 - b_3 p_3(t) + d_3 p_2(t), \end{aligned} \tag{1}$$

where $a_i, b_i > 0$, $i = 1, 2, 3$, and $d_i > 0$, $i = 2, 3$. d_2 and d_3 mean the competition parameters between firms 2 and 3.

The cost functions of the first firm has the following form [17]:

$$C_1(t) = c_1 q_1^2(t), \tag{2}$$

where $2c_1$ is marginal cost and c_1 is positive for any $t > 0$. Firm 1's price will affect the cost of the two downstream firms. That is, $C_j = f(p_1)$, $j = 2, 3$. We assume firm j's cost function is

$$C_j(t) = p_1(t) q_j(t), \tag{3}$$

where $j = 2, 3$.

The profit functions of the three firms are

$$\begin{aligned} \Pi_1(t) &= q_1(t) p_1(t) - C_1(t) \\ &= q_1(t)\left(a_1 - b_1 q_1(t)\right) - c_1 q_1^2(t) \\ &= -b_1 q_1^2(t) - c_1 q_1^2(t) + a_1 q_1(t), \\ \Pi_2(t) &= q_2(t) p_2(t) - C_2(t) \\ &= \left(a_2 - b_2 p_2(t) + d_2 p_3(t)\right) p_2(t) - p_1(t) q_2(t) \\ &= a_2 p_2(t) - b_2 p_2^2(t) + d_2 p_2(t) p_3(t) \\ &\quad - \left(a_1 - b_1 q_1(t)\right)\left(a_2 - b_2 p_2(t) + d_2 p_3(t)\right), \\ \Pi_3(t) &= q_3(t) p_3(t) - C_3(t) \\ &= \left(a_3 - b_3 p_3(t) + d_3 p_2(t)\right) p_3(t) - p_1(t) q_3(t) \\ &= a_3 p_3(t) - b_3 p_3^2(t) + d_3 p_2(t) p_3(t) \\ &\quad - \left(a_1 - b_1 q_1(t)\right)\left(a_3 - b_3 p_3(t) + d_3 p_2(t)\right). \end{aligned} \tag{4}$$

We assume that the three firms do not have a complete knowledge of the market and the other players. In games, players behave adaptively, following a bounded rational adjustment process and they build decisions on the basis of the expected marginal profit; that is, if the marginal profit is positive (negative), they increase (decrease) their production

or price in the next period. The marginal profit functions of the three firms are as follows:

$$\begin{aligned}
\frac{\partial \Pi_1(t)}{\partial q_1(t)} &= -2b_1q_1(t) - 2c_1q_1(t) + a_1,\\
\frac{\partial \Pi_2(t)}{\partial p_2(t)} &= a_2 - 2b_2p_2(t) + d_2p_3(t) + b_2(a_1 - b_1q_1(t))\\
&= a_2 + a_1b_2 - b_1b_2q_1(t) - 2b_2p_2(t) + d_2p_3(t),\\
\frac{\partial \Pi_3(t)}{\partial p_3(t)} &= a_3 - 2b_3p_3(t) + d_3p_2(t) + b_3(a_1 - b_1q_1(t))\\
&= a_3 + a_1b_3 - b_1b_3q_1(t) + d_3p_2(t) - 2b_3p_3(t).
\end{aligned} \tag{5}$$

Under the above assumptions, the dynamic adjustment equation of the master-slave game is

$$\begin{aligned}
q_1(t+1) &= q_1(t) + \alpha q_1(t)\frac{\partial \Pi_1(t)}{\partial q_1(t)},\\
p_2(t+1) &= p_2(t) + \beta p_2(t)\frac{\partial \Pi_2(t)}{\partial p_2(t)},\\
p_3(t+1) &= p_3(t) + \gamma p_3(t)\frac{\partial \Pi_3(t)}{\partial p_3(t)},
\end{aligned} \tag{6}$$

where α, β, γ are the adjustment speeds of the three firms, respectively. And $\alpha, \beta, \gamma > 0$.

Substituting (5) into (6), then the model has the following form:

$$\begin{aligned}
q_1(t+1) &= q_1(t) + \alpha q_1(t)\\
&\quad\times(-2b_1q_1(t) - 2c_1q_1(t) + a_1),\\
p_2(t+1) &= p_2(t) + \beta p_2(t)\\
&\quad\times[a_2 + a_1b_2 - b_1b_2q_1(t) - 2b_2p_2(t) + d_2p_3(t)],\\
p_3(t+1) &= p_3(t) + \gamma p_3(t)\\
&\quad\times[a_3 + a_1b_3 - b_1b_3q_1(t) + d_3p_2(t) - 2b_3p_3(t)].
\end{aligned} \tag{7}$$

2.2. Equilibrium Points and Local Stability. The system (7) has eight equilibrium points:

$$\begin{aligned}
E_0 &= (0,0,0),\\
E_1 &= \left(0, 0, \frac{(a_3 + a_1b_3)}{2b_3}\right),\\
E_2 &= \left(0, \frac{(a_2 + a_1b_2)}{2b_2}, 0\right),\\
E_3 &= \left(\frac{a_1}{2(b_1 + c_1)}, 0, 0\right)\\
E_4 &= \left(0, \frac{(2a_2b_3 + 2a_1b_2b_3 + a_3d_2 + a_1b_3d_2)}{(4b_2b_3 - d_2d_3)},\right.\\
&\qquad\left.\frac{(2a_3b_2 + 2a_1b_2b_3 + a_2d_3 + a_1b_2d_3)}{(4b_2b_3 - d_2d_3)}\right),\\
E_5 &= \left(\frac{a_1}{2(b_1 + c_1)}, 0, \frac{(2a_2b_1 + 2a_3c_1 + a_1b_1b_3 + 2a_1b_3c_1)}{4b_3(b_1 + c_1)}\right),\\
E_6 &= \left(\frac{a_1}{2(b_1 + c_1)}, \frac{(2a_2b_1 + 2a_2c_1 + a_1b_1b_2 + 2a_1b_2c_1)}{4b_2(b_1 + c_1)}, 0\right),\\
E^* &= \left(\frac{a_1}{2(b_1 + c_1)}, \frac{h_2}{g}, \frac{h_3}{g}\right) = (q_1^*, p_2^*, p_3^*),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
h_2 &= 4a_2b_1b_3 + 4a_2b_3c_1 + 2a_1b_1b_2b_3 + 4a_1b_2b_3c_1\\
&\quad + 2a_3b_1d_2 + 2a_3c_1d_2 + a_1b_1b_3d_2 + 2a_1b_3c_1d_2,\\
h_3 &= 4a_3b_1b_2 + 4a_3b_2c_1 + 2a_2b_1d_3 + 2a_2c_1d_3\\
&\quad + 2a_1b_1b_2b_3 + 4a_1b_2b_3c_1 + a_1b_1b_2d_3 + 2a_1b_2c_1d_3,\\
g &= 2(b_1 + c_1)(4b_2b_3 - d_2d_3).
\end{aligned} \tag{9}$$

Since all the equilibrium points should be nonnegative, the parameters satisfy $d_1d_2 < 4b_2b_3$. Since $q_1^* > q_2^* + q_3^*$, we should have

$$\begin{aligned}
&2[2b_2b_3(a_2 + a_3) + (a_2b_3d_3 + a_3b_2d_3)](b_1 + c_1)\\
&\quad + a_1b_2b_3[(d_2 + d_3)(b_1 + 2c_1) - 2]\\
&\quad < a_1[1 + (b_1 + 2c_1)(b_2 + b_3)](2b_2b_3 - d_2d_3).
\end{aligned} \tag{10}$$

The local stability of equilibrium points can be determined by the nature of the eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points. The Jacobian matrix of the system (7) corresponding to the state variables (q_1, p_2, p_3) is

$$J(E) = \begin{pmatrix} w_{11} & 0 & 0\\ w_{21} & w_{22} & w_{23}\\ w_{31} & w_{32} & w_{33}\end{pmatrix}, \tag{11}$$

where

$$\begin{aligned}
w_{11} &= 1 + \alpha(-4b_1q_1 + a_1 - 4c_1q_1),\\
w_{21} &= -b_1b_2\beta p_2,\\
w_{22} &= 1 + \beta(a_2 + a_1b_2 - b_1b_2q_1 - 4b_2p_2 + d_2p_3),\\
w_{23} &= d_2\beta p_2,\\
w_{31} &= -b_1b_3\gamma p_3,\\
w_{32} &= d_3\gamma p_3,\\
w_{33} &= 1 + \gamma(a_3 + a_1b_3 - b_1b_3q_1 + d_3p_2 - 4b_3p_3).
\end{aligned} \tag{12}$$

Theorem 1. *All the boundary equilibrium points $E_0, E_1, \ldots E_6$ are unstable.*

Proof. E_0, E_1, E_2, and E_4 all have the eigenvalue $\lambda_1 = 1 + a_1\alpha$. Since $a_1, \alpha > 0$, then $\lambda_1 > 1$. Hence the equilibrium points E_0, E_1, E_2, and E_4 are unstable equilibrium points [18, 19]. Similarly, E_3 has one eigenvalue $\lambda = 1 + a_2\beta + a_1b_2\beta(b_1 + 2c_1)/2(b_1 + c_1) > 1$; $\lambda = 1 + a_2\beta + \beta(2a_3b_1d_1 + 2a_3c_1d_2 + 2a_1b_1b_2b_3 + 4a_1b_2b_3c_1 + a_1b_1b_3d_2 + 2a_1b_3c_1d_2)/4b_3(b_1 + c_1) > 1$ for E_5 and $\lambda = 1 + a_3\gamma + \gamma(2a_2b_1d_3 + 2a_3c_1d_3 + 2a_2b_1b_2b_3 + 4a_1b_2b_3c_1 + a_1b_1b_2d_3 + 2a_1b_2c_1d_3)/4b_2(b_1 + c_1) > 1$ for E_6. Then all the boundary equilibrium points are unstable. □

Now we investigate the local stability of Nash equilibrium point E^*. The Jacobian matrix $J(E^*)$ is

$$J(E^*) = \begin{pmatrix} w_{11}^* & 0 & 0 \\ w_{21}^* & w_{22}^* & w_{23}^* \\ w_{31}^* & w_{32}^* & w_{33}^* \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} w_{11}^* &= 1 + \alpha\left(-4b_1q_1^* + a_1 - 4c_1q_1^*\right), \\ w_{21}^* &= -b_1b_2\beta p_2^*, \\ w_{22}^* &= 1 + \beta\left(a_2 + a_1b_2 - b_1b_2q_1^* - 4b_2p_2^* + d_2p_3^*\right), \\ w_{23}^* &= d_2\beta p_2^*, \\ w_{31}^* &= -b_1b_3\gamma p_3^*, \\ w_{32}^* &= d_3\gamma p_3^*, \\ w_{33}^* &= 1 + \gamma\left(a_3 + a_1b_3 - b_1b_3q_1^* + d_3p_2^* - 4b_3p_3^*\right). \end{aligned} \tag{14}$$

The characteristic polynomial of the Jacobian matrix $J(E^*)$ is

$$f(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C. \tag{15}$$

And its local stability is given by the Jury conditions [20]:

$$\begin{aligned} &\text{(i)}: f(1) = A + B + C + 1 > 0, \\ &\text{(ii)}: f(-1) = A - B + C - 1 < 0, \\ &\text{(iii)}: C^2 - 1 < 0, \\ &\text{(iv)}: \left(1 - C^2\right)^2 - (B - AC)^2 > 0. \end{aligned} \tag{16}$$

In order to analyze the stability of Nash equilibrium point, we perform some numerical simulations.

2.3. Numerical Simulations. In this section, we will show the complex behaviors of the system (7) including bifurcation and strange attractor. In order to further analyze long-run profit of the three firms with parameters changing, the long-run average profit figures are given. It is convenient to take the parameters values as follows: $a_1 = 2$, $a_2 = 4$, $a_3 = 3$, $b_1 = 0.2$, $b_2 = 2$, $b_3 = 1.5$, $c_1 = 0.3$, $d_2 = 0.5$, and $d_3 = 0.6$. The initial values are chosen as $(q_1(0), p_2(0), p_3(0)) = (1.8, 2.1, 2.2)$. Through (7), the Nash equilibrium point is $(2, 2.0769, 2.2154)$. Then its Jacobian matrix is

$$J(E^*) = \begin{pmatrix} 1 - 2\alpha & 0 & 0 \\ -0.8308\beta & 1 - 8.3077\beta & 1.0385\beta \\ -0.6646\gamma & 1.3292\gamma & 1 - 6.6462\gamma \end{pmatrix}, \tag{17}$$

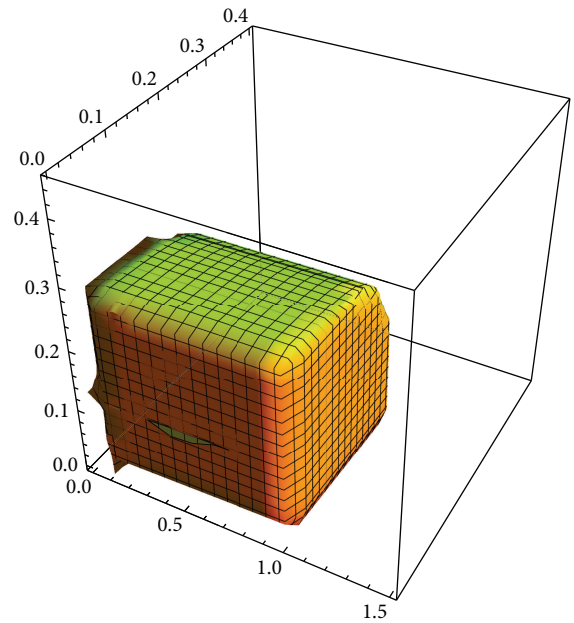


Figure 1: The stable region of the Nash equilibrium point E^*.

where $A = 2\alpha + 8.3077\beta + 6.6462\gamma - 3$, $B = (6.6462\gamma - 1)(2\alpha + 8.3077\beta - 2) - 1.3804\beta\gamma + (2\alpha - 1)(8.3077\beta - 1)$, and $C = -1.3804\beta\gamma(2\alpha + 8.3077\beta - 2) + 1.3804\beta\gamma(8.3077\beta - 1) + (2\alpha - 1)(8.3077\beta - 1)(6.6462\gamma - 1)$.

Figure 1 gives the stable region of the Nash equilibrium point E^*. We can see that the stable region is $\alpha < 1$, $\beta < 0.24$, $\gamma < 0.3$ approximately. From the figure, we can conclude that the stability region is asymmetric and the higher adjustment speeds will push the system out of the stable region.

Figure 2 displays the bifurcation diagram and the largest Lyapunov exponent with respect to the parameter α which is the adjustment speed of the upstream monopoly when $\beta = 0.15$ and $\gamma = 0.2$. By comparing the largest Lyapunov exponent diagram, one can have a better understanding of the particular properties of the system. In Figure 2, the system (7) converges to the Nash equilibrium point for $0 < \alpha < 0.99$. If α increases, that is, $\alpha > 0.99$, the system turns unstable and complex dynamic behavior is observed. At $\alpha = 0.99$, a flip bifurcation arises, which is followed by further flips and the largest Lyapunov exponent increases to zero for the first time; hence the system enters a period doubling routes to chaos. When $\alpha \geq 1.29$, the largest Lyapunov exponent is positive and chaos emerges.

Figure 3 is the bifurcation diagram with respect to the parameter β, when given $\alpha = 0.5$ and $\gamma = 0.2$. In Figure 3, the output $q_1(t)$ of upstream firm is always stable which illustrates that the adjustment speed β has little effect on the output $q_1(t)$ of the upstream firm, while the prices $p_2(t)$, $p_3(t)$ of the two downstream firms generate bifurcation behaviors at $\beta = 0.2312$. And when $\beta \geq 0.306$, the largest Lyapunov exponent is positive; then the system is in a state of chaos.

Similar to Figure 3, Figure 4 gives the bifurcation diagram with respect to the parameter γ, when given $\alpha = 0.5$ and $\beta = 0.15$. In Figure 4, the output $q_1(t)$ is also stable. And when

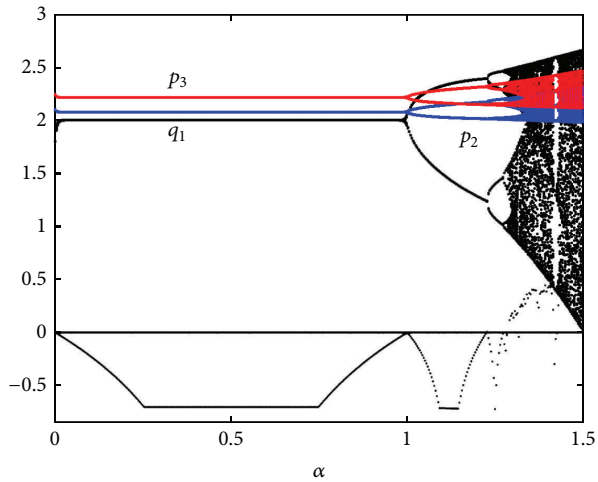


FIGURE 2: Bifurcation diagram and the Largest Lyapunov exponent of the discrete dynamic system (7) for $\beta = 0.15$, $\gamma = 0.2$, and α varying from 0 to 1.5.

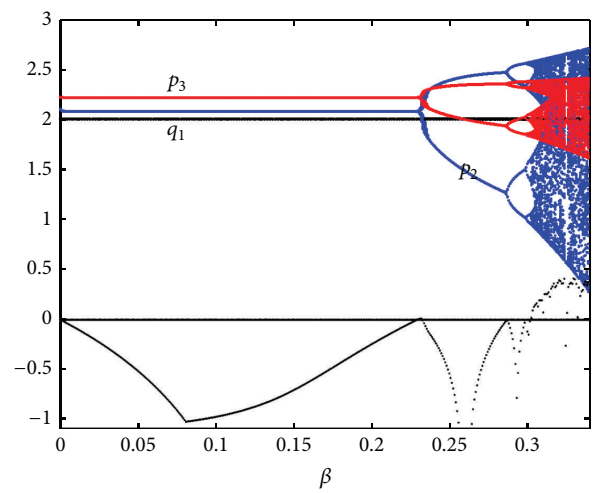


FIGURE 3: Bifurcation diagram and the Largest Lyapunov exponent of the discrete dynamic system (7) for $\alpha = 0.5$, $\gamma = 0.2$, and β varying from 0 to 0.34.

$\gamma > 0.2856$ the system turns unstable and enters chaos when $\gamma \geq 0.3822$.

From the above analysis, it can be seen that the adjustment parameter α has an important influence on system (7). That is to say the behavior of upstream monopolist has a decisive influence on the market in economics. And it is harmful for the development of the two downstream firms if the changes of adjustment parameters α, β, γ are too big.

Figure 5 represents the graph of a strange attractors of the dynamical system (7) for the adjustment parameter values $\beta = 0.3$, $\gamma = 0.35$, and $\alpha = 1.35$, which exhibits fractal structure of the system.

Then we analyse the long-run average profit of the three firms. The results are shown in Figures 6, 7, and 8.

From these figures, we can see that the long-run average profit of firm 1 is larger than the other firms. Figure 6 shows

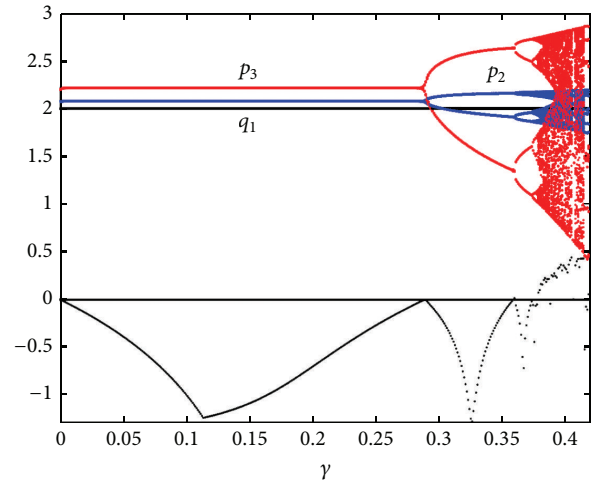


FIGURE 4: Bifurcation diagram and the Largest Lyapunov exponent of the discrete dynamic system (7) for $\alpha = 0.5$, $\beta = 0.15$, and γ varying from 0 to 0.44.

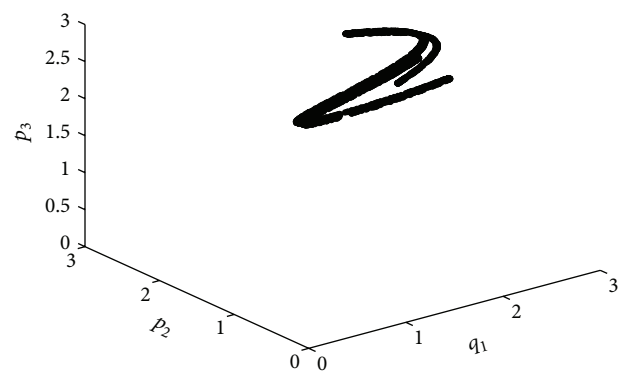


FIGURE 5: Strange attractors for $\alpha = 1.35$, $\beta = 0.3$, and $\gamma = 0.35$.

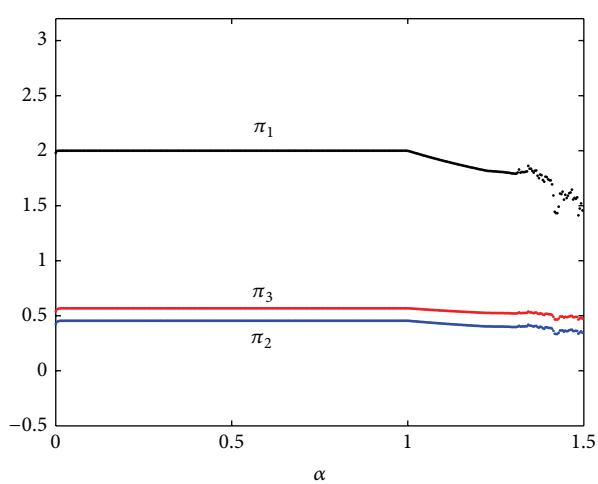


FIGURE 6: The long-run average profits of the players with α for system (7).

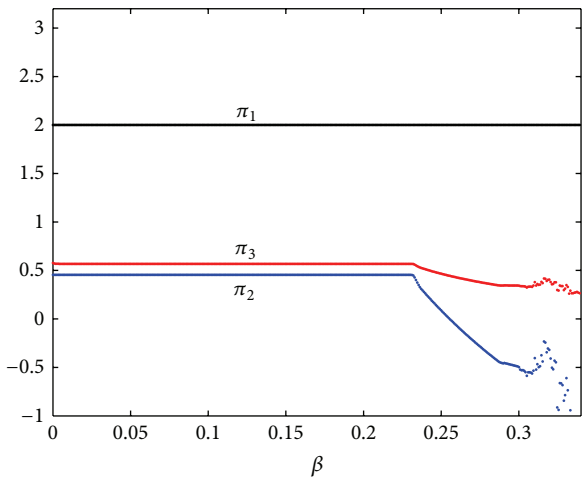


FIGURE 7: The long-run average profits of the players with β for system (7).

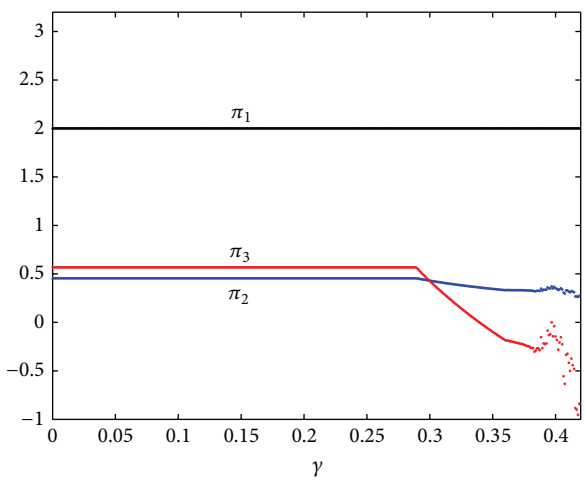


FIGURE 8: The long-run average profits of the players with γ for system (7).

that the long-run average profit of the three firms fluctuates and is lower than that of the stable state with the production adjustment speed increasing when the system (7) enters into bifurcation and chaotic states. When $\alpha \geq 0.99$, the profit of the three firms begins to decrease. From Figures 7 and 8, we can see that the average profit of the upstream firm is a constant value, so we deduce that the adjustment speeds β and γ have little effect on the profit of firm 1. In Figure 7, the average profit of firms 1 and 3 is always higher than firm 2 and the long-run average profit of the downstream firms decreases when the bifurcation begins. When $\beta = 0.255$, the profit of firm 2 is negative. In Figure 8, the long-run average profit of the downstream firms decreases when the bifurcation begins and when $\gamma = 0.3024$, they are the same. From then on, the profit of firm 2 is higher than firm 3 and when $\gamma = 0.336$, the profit of firm 3 turns to negative.

Our study finds that the overall profit of the system will decrease in bifurcation and chaotic states with adjustment speeds increasing.

3. Chaos Control of Nondelayed Master-Slave Model

3.1. Delay Feedback Control Method. In this part, we first take the delay feedback control method [21, 22] to control chaos. The method is based on the difference between the T-time delayed state and the current state, where T denotes a period of the stabilized orbits. The controlled system is

$$x(t+1) = f(x(t), u(t)), \tag{18}$$

where $u(t)$ is the input signal and $x(t)$ is the state. Consider

$$u(t) = k(x(t+1-T) - x(t+1)), \quad t > T, \tag{19}$$

where T is the time delay and k is the controlling factor.

Because the behavior of firm 1 has great effect on the system, we add the function in the first equation from the system (7). Then the controlled system is

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha q_1(t)(-2b_1 q_1(t) - 2c_1 q_1(t) + a_1) \\ &\quad + k(q_1(t+1-T) - q_1(t+1)), \\ p_2(t+1) &= p_2(t) + \beta p_2(t) \\ &\quad \times [a_2 + a_1 b_2 - b_1 b_2 q_1(t) - 2b_2 p_2(t) + d_2 p_3(t)], \\ p_3(t+1) &= p_3(t) + \gamma p_3(t) \\ &\quad \times [a_3 + a_1 b_3 - b_1 b_3 q_1(t) + d_3 p_2(t) - 2b_3 p_3(t)]. \end{aligned} \tag{20}$$

By choosing $T = 1$, the system becomes

$$\begin{aligned} q_1(t+1) &= q_1(t) + \frac{\alpha q_1(t)(-2b_1 q_1(t) - 2c_1 q_1(t) + a_1)}{(1+k)} \\ p_2(t+1) &= p_2(t) + \beta p_2(t) \\ &\quad \times [a_2 + a_1 b_2 - b_1 b_2 q_1(t) - 2b_2 p_2(t) + d_2 p_3(t)], \\ p_3(t+1) &= p_3(t) + \gamma p_3(t) \\ &\quad \times [a_3 + a_1 b_3 - b_1 b_3 q_1(t) + d_3 p_2(t) - 2b_3 p_3(t)]. \end{aligned} \tag{21}$$

It can be seen from Figure 9, ($\alpha = 1.35$, $\beta = 0.15$, $\gamma = 0.2$), that the chaotic system was gradually controlled with controlling parameter k increasing. When $k = 0$, it turns uncontrolled system (7) which is at chaotic state. And when $k \geq 0.36$, the system is controlled. Taking $k = 0.8$, we can see that the stable region of α expands to 1.775 from Figure 10, which indicates that chaos is delayed or eliminated completely. Figure 11 gives the long-run average profit of the players for the controlled system (21). Comparing Figure 11 with Figure 6, we see that in Figure 6 the three firms' profit starts to fall in $\alpha = 0.99$ while in Figure 11 this phenomenon does not happen.

3.2. Limiter Control Method. We use limiter control method [23, 24] which is better for firms 2 and 3. This method only

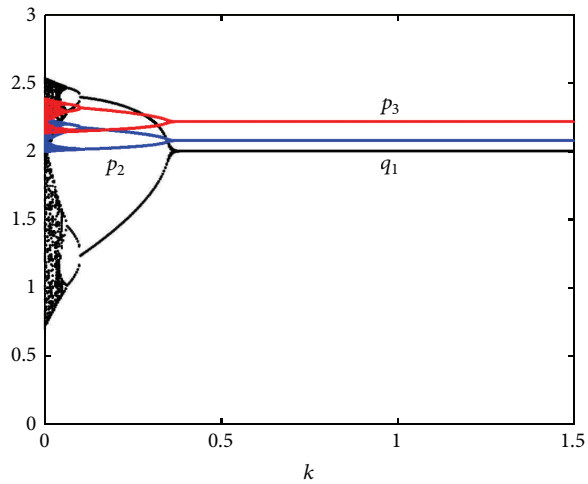


FIGURE 9: The trend of system with k increasing for $\alpha = 1.35$, $\beta = 0.15$, and $\gamma = 0.2$.

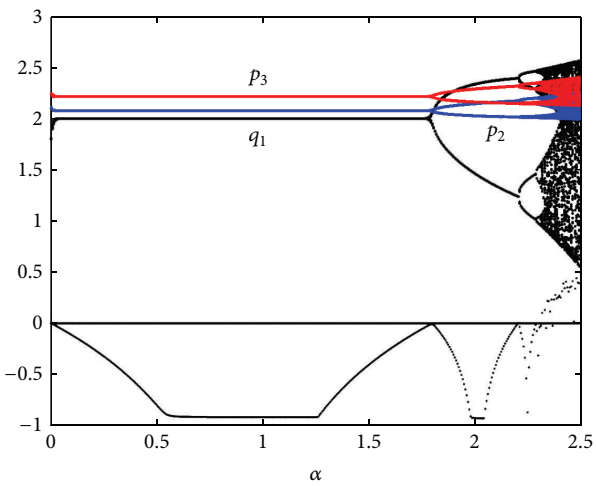


FIGURE 10: The bifurcation diagram of the controlled system (21) for $k = 0.8$, $\beta = 0.15$, $\gamma = 0.2$, and α varying from 0 to 2.5.

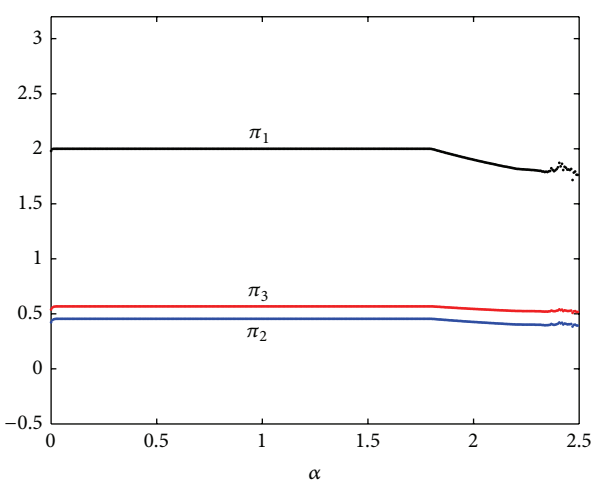


FIGURE 11: The long-run average profits of the players with α for the controlled system (21).

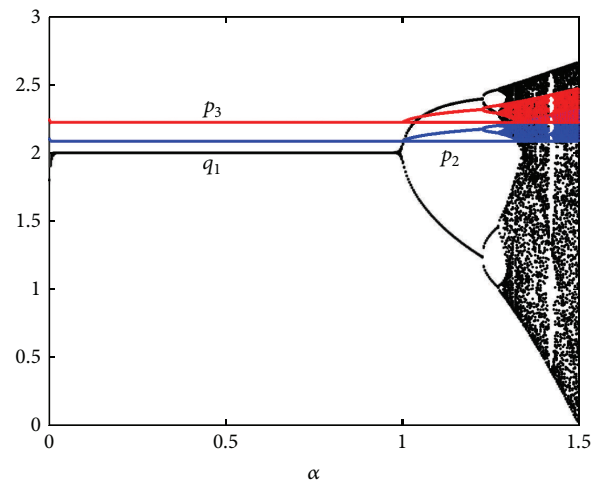


FIGURE 12: The bifurcation diagram of the controlled system (22) for $\beta = 0.15$ and $\gamma = 0.2$.

requires the player who wants to improve his performance to take measures without the other players' cooperation. We impose lower limiters on the price of firms 2 and 3, noted $p_2^{\min}$, $p_3^{\min}$, and it has no effect on the behavior of firm 1.

The controlled system is

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha q_1(t)\left(-2b_1 q_1(t) - 2c_1 q_1(t) + a_1\right), \\ p_2(t+1) &= \operatorname{Max}\left[f_2\left(q_1(t), p_2(t), p_3(t)\right), p_2^{\min}\right], \\ p_3(t+1) &= \operatorname{Max}\left[f_3\left(q_1(t), p_2(t), p_3(t)\right), p_3^{\min}\right], \end{aligned} \tag{22}$$

where $p_2^{\min}, p_3^{\min} > 0$, and

$$\begin{aligned} &f_2\left(q_1(t), p_2(t), p_3(t)\right) \\ &= p_2(t) + \beta p_2(t) \\ &\quad \times \left[a_2 + a_1 b_2 - b_1 b_2 q_1(t) - 2b_2 p_2(t) + d_2 p_3(t)\right], \\ &f_3\left(q_1(t), p_2(t), p_3(t)\right) \\ &= p_3(t) + \gamma p_3(t) \\ &\quad \times \left[a_3 + a_1 b_3 - b_1 b_3 q_1(t) + d_3 p_2(t) - 2b_3 p_3(t)\right]. \end{aligned} \tag{23}$$

By choosing $p_2^{\min} = 2.0907$ and $p_3^{\min} = 2.2264$, Figures 12 and 13 give bifurcation diagram and long-run average profit of the three firms with the α changing. They show that the behaviors of firm 1 are the same with the original system (7). Figure 12 shows that the bifurcation and chaotic behaviors of firms 2 and 3 have been controlled. In Figure 13, the decreasing speed of π_2 and π_3 is under control.

4. The Delayed Master-Slave Model

4.1. Model. The primary reason for the occurrence of such a delayed structure in economic models is that (a) decisions

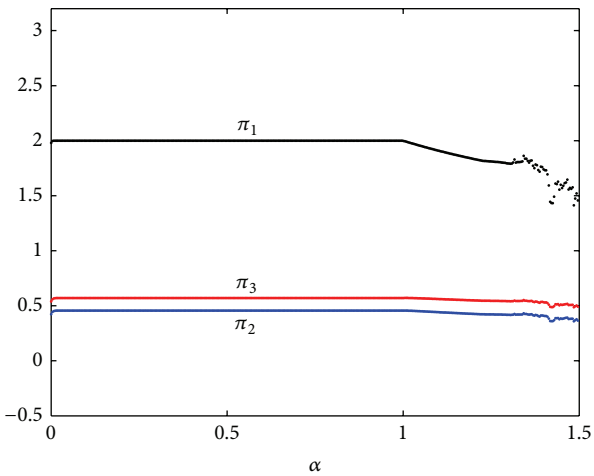


FIGURE 13: The long-run average profits of the players with α for the controlled system (22).

made by economic agents at time t depend on past observed variables by means of a prediction feedback and (b) the functional relationships describing the dynamics of the model may not only depend on the current state of the firm but also, in a nontrivial manner, on past states. Considering these reasons, we introduce the delayed model and compare the three firms' profits in various cases.

Then the bounded rationality dynamical model evolved from system (7) with one step delayed is given by

$$\begin{aligned}
&q_1(t+1)\\
&= q_1(t) + \alpha q_1(t)\left(-2b_1\left(\left(1-w_1\right)q_1(t) + w_1 q_1(t-1)\right)\right.\\
&\qquad \left. - 2c_1\left(\left(1-w_1\right)q_1(t) + w_1 q_1(t-1)\right) + a_1\right),\\
&p_2(t+1)\\
&= p_2(t) + \beta p_2(t)\left[a_2 + a_1 b_2 - b_1 b_2\right.\\
&\qquad \times\left(\left(1-w_1\right)q_1(t) + w_1 q_1(t-1)\right)\\
&\qquad - 2b_2\left(\left(1-w_2\right)p_2(t) + w_2 p_2(t-1)\right)\\
&\qquad \left. + d_2\left(\left(1-w_3\right)p_3(t) + w_3 p_3(t-1)\right)\right],\\
&p_3(t+1)\\
&= p_3(t) + \gamma p_3(t)\left[a_3 + a_1 b_3 - b_1 b_3\right.\\
&\qquad \times\left(\left(1-w_1\right)q_1(t) + w_1 q_1(t-1)\right)\\
&\qquad + d_3\left(\left(1-w_2\right)p_2(t) + w_2 p_2(t-1)\right)\\
&\qquad \left. - 2b_3\left(\left(1-w_3\right)p_3(t) + w_3 p_3(t-1)\right)\right].
\end{aligned} \tag{24}$$

w_i represent the weights given to previous production and prices, and $0 \le w_1, w_2, w_3 \le 1$.

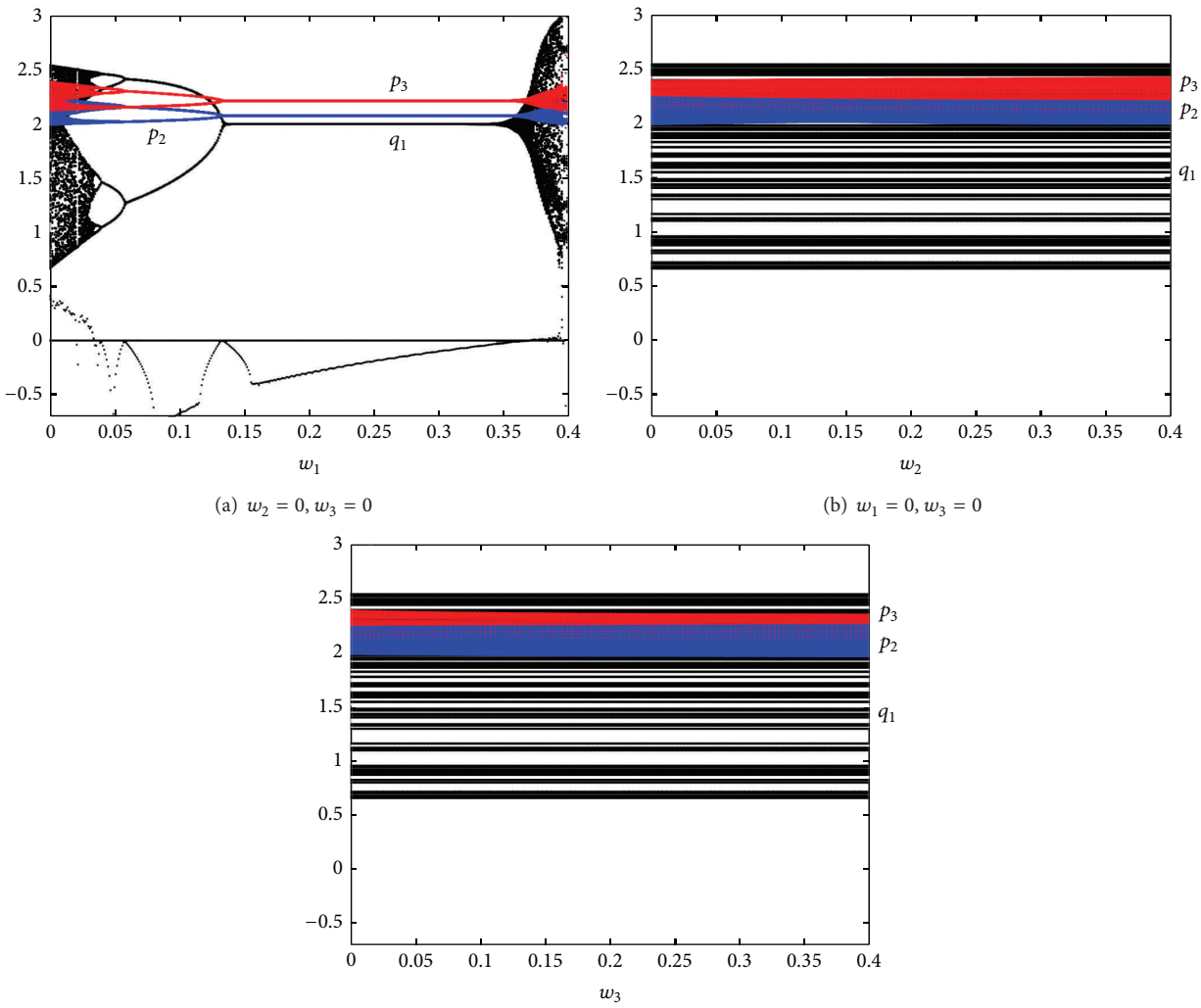
It is convenient to take $x(t) = q_1(t-1)$, $y(t) = p_2(t-1)$, and $z(t) = p_3(t-1)$; then system (24) becomes

$$\begin{aligned}
&q_1(t+1)\\
&= q_1(t) + \alpha q_1(t)\left(-2b_1\left(\left(1-w_1\right)q_1(t) + w_1 x(t)\right)\right.\\
&\qquad \left. - 2c_1\left(\left(1-w_1\right)q_1(t) + w_1 x(t)\right) + a_1\right),\\
&p_2(t+1)\\
&= p_2(t) + \beta p_2(t)\left[a_2 + a_1 b_2 - b_1 b_2\right.\\
&\qquad \times\left(\left(1-w_1\right)q_1(t) + w_1 x(t)\right)\\
&\qquad - 2b_2\left(\left(1-w_2\right)p_2(t) + w_2 y(t)\right)\\
&\qquad \left. + d_2\left(\left(1-w_3\right)p_3(t) + w_3 z(t)\right)\right],\\
&p_3(t+1)\\
&= p_3(t) + \gamma p_3(t)\left[a_3 + a_1 b_3 - b_1 b_3\right.\\
&\qquad \times\left(\left(1-w_1\right)q_1(t) + w_1 x(t)\right)\\
&\qquad + d_3\left(\left(1-w_2\right)p_2(t) + w_2 y(t)\right)\\
&\qquad \left. - 2b_3\left(\left(1-w_3\right)p_3(t) + w_3 z(t)\right)\right],\\
&x(t+1) = q_1(t),\\
&y(t+1) = p_2(t),\\
&z(t+1) = p_3(t).
\end{aligned} \tag{25}$$

We can get eight equilibrium points, denoted by $E_i = (q_1^i, p_2^i, p_3^i, x^i, y^i, z^i)$. Consider $(q_1^i, p_2^i, p_3^i) = (x^i, y^i, z^i)$, and (q_1^i, p_2^i, p_3^i) $(i = 0, 1, \ldots, 7)$ have the same values as equilibrium points of system (7). The Jacobi matrix for the system (25) is

$$J(E) = \begin{pmatrix} J_{11} & 0 & 0 & J_{14} & 0 & 0\\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26}\\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36}\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

where $J_{11} = w_{11} + 2\alpha q_1 w_1(b_1 + c_1)$, $J_{14} = -2w_1\alpha q_1(b_1 + c_1)$, $J_{21} = w_{31}(1 - w_1)$, $J_{22} = w_{22} + 2\beta w_2 p_2 b_2$, $J_{23} = w_{23}(1 - w_3)$, $J_{24} = w_1 w_{21}$, $J_{25} = -2b_2 w_2 \beta p_2$, $J_{26} = w_3 w_{23}$, $J_{31} = w_{31}(1-w_1)$, $J_{32} = w_{32}(1 - w_2)$, $J_{33} = w_{33} + 2\gamma w_3 p_3 b_3$, $J_{34} = w_2 w_{32}$, $J_{35} = w_2 w_{32}$, and $J_{36} = -2b_3\gamma p_3 w_3$.



(a) $w_2 = 0, w_3 = 0$

(b) $w_1 = 0, w_3 = 0$

(c) $w_1 = 0, w_2 = 0$

Figure 14: The bifurcation diagram and the corresponding largest Lyapunov exponent of the delayed system (25).

$E_0, \ldots, E_6$ are boundary equilibrium points; E_7 is the unique Nash equilibrium point. The Jury conditions are

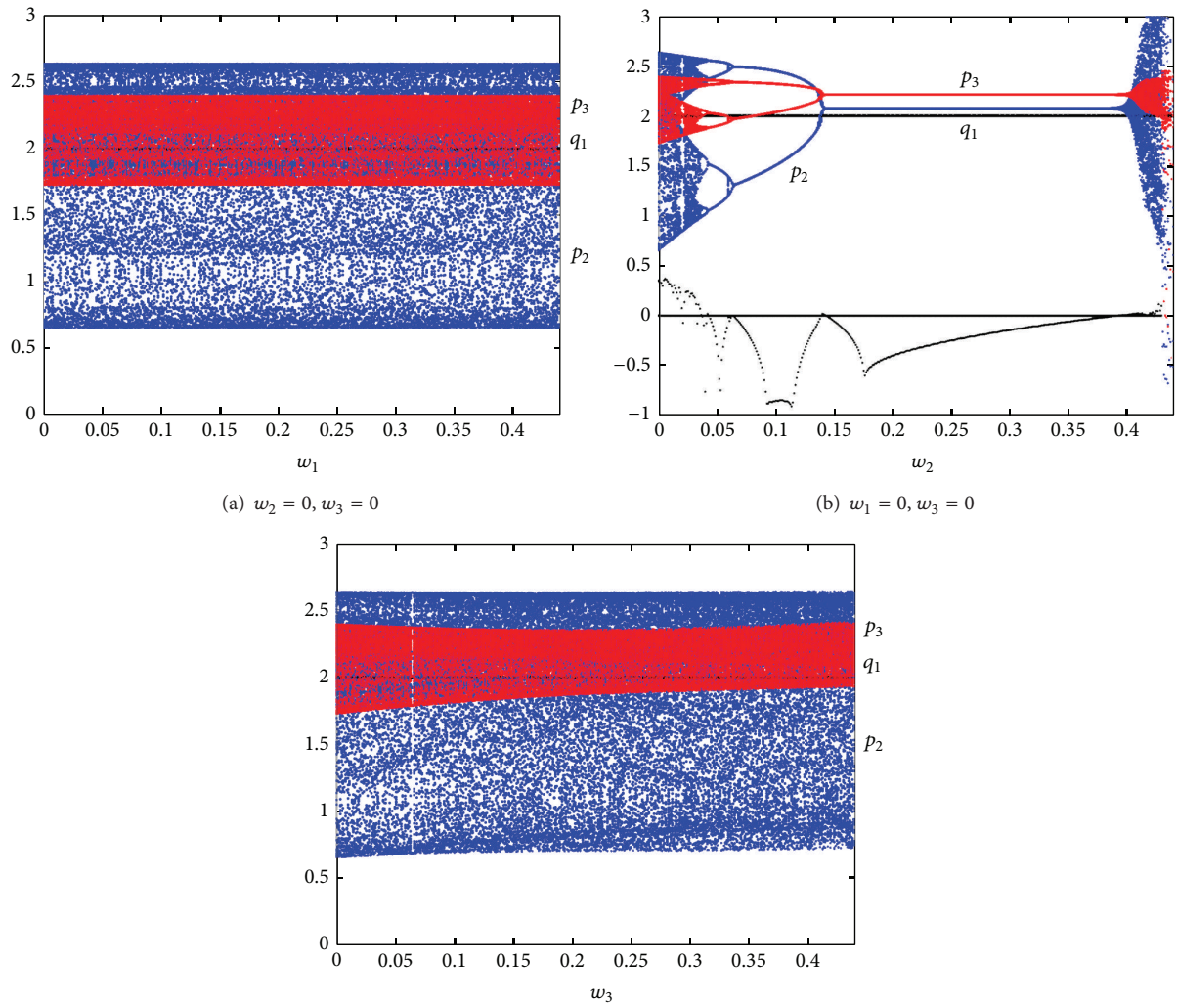
$$\begin{aligned}
&(\text{i}) : F(1) = \delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 > 0,\\
&(\text{ii}) : F(-1) = \delta_0 - \delta_1 + \delta_2 - \delta_3 + \delta_4 - \delta_5 + \delta_6 > 0,\\
&(\text{iii}) : |\delta_0| < \delta_6,\\
&(\text{iv}) : |f_1(\delta_0, \ldots, \delta_6)| > |\varphi_1(\delta_0, \ldots, \delta_6)|,\\
&(\text{v}) : |f_2(\delta_0, \ldots, \delta_6)| > |\varphi_2(\delta_0, \ldots, \delta_6)|,\\
&(\text{vi}) : |f_3(\delta_0, \ldots, \delta_6)| > |\varphi_3(\delta_0, \ldots, \delta_6)|,\\
&(\text{vii}) : |f_4(\delta_0, \ldots, \delta_6)| > |\varphi_4(\delta_0, \ldots, \delta_6)|,
\end{aligned} \tag{27}$$

where $F(\lambda) = \delta_0 + \delta_1\lambda + \delta_2\lambda^2 + \cdots + \delta_6\lambda^6$ is the characteristic polynomial, and f_i, φ_i $(i = 1, 2, 3, 4)$ are the functions of δ_i $(i = 1, 2, 3, 4, 5, 6)$.

4.2. Numerical Simulations. In order to discuss the complexity of delayed system (25), we first take the α, β, γ as constant values and then analyze behaviors with w_1, w_2, w_3 changing.

Figure 14 gives the bifurcation diagrams with respect to the parameters w_i, $(i = 1, 2, 3)$ when $\alpha = 1.36$, $\beta = 0.15$, $\gamma = 0.2$ in (a), (b), and (c). In the nondelayed system, when α, β, γ take the values as above, as we see from Figure 2, system (7) is at the state of chaos while in Figure 14(a), when $w_2 = w_3 = 0$, system (25) from chaotic state changes to stable state, then from stable state to chaotic state with the changing of w_1. And when $w_1 = 0$, $w_i = 0$, as w_j $(i, j = 2, 3, i \neq j)$ changing, the system (25) is always in the chaotic state. We conclude that w_1 plays an important role in changing system (7) from chaos to stable state with the rest parameters fixed.

Similarly, taking $\alpha = 0.5$, $\beta = 0.32$, and $\gamma = 0.2$, we know from Figure 4 that the system is in chaotic state. In Figure 15, the bifurcation diagram (b) shows that when $w_1 = w_3 = 0$, the system (25) is also from the chaotic state to stable state, then from stable state to unstable state with w_2 changing.



(a) $w_2 = 0, w_3 = 0$

(b) $w_1 = 0, w_3 = 0$

(c) $w_1 = 0, w_2 = 0$

Figure 15: The bifurcation diagram and the corresponding largest Lyapunov exponent of the delayed system (25).

Figure 16 shows the bifurcation diagrams and the largest Lyapunov exponent diagram taking $\alpha = 0.5$, $\beta = 0.15$, $\gamma = 0.41$. When $w_1 = w_2 = 0$, system (25) also experiences the process from chaos to stability then again to the chaos as the changes of the w_3.

From the analysis of the Figures 14–16, we conclude that the appropriate delay parameters play an important role for controlling the system from chaos to stability in system (7).

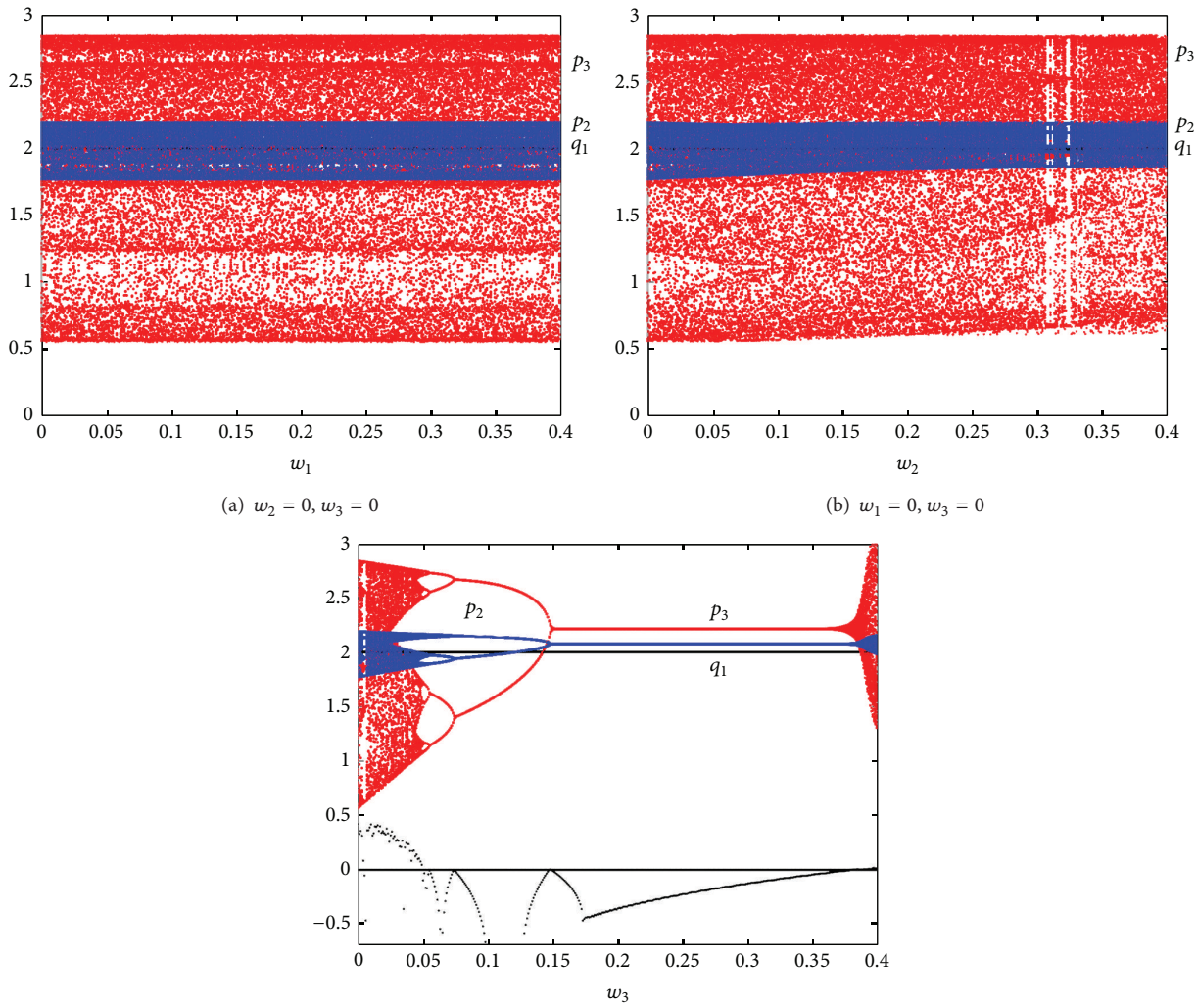
Tables 1, 2, and 3 show three firms' profit and the total profit when the systems are nondelayed, delay feedback controlled, limiter controlled, and delayed. We conclude the following.

(a) The profit of upstream firm is apparently higher than the other two downstream firms'.

(b) In Table 1, although the value of α goes beyond the stable region of system (7), when $w_1 = 0.2$, the three firms achieve maximal profit: $\pi_1 = 60$, $\pi_2 = 13.6473$, $\pi_3 = 17.0414$, total profit $\sum \pi = 90.6888$. In Table 2, when the value of β goes beyond the stable region, the three firms achieve maximal profit with $w_2 = 0.2$. While in Table 3, when the value of γ goes beyond the stable region, the three firms achieve maximal profit with $w_3 = 0.2$. It can be seen that the effect of selecting appropriate delay parameters is the same as applying a control on the system, which can make profit of the system maximization.

(c) Taking the appropriate value of k can reach the best control effect for this case where α goes beyond the stable region. This proves that the delay feedback control method is advantageous to the upstream firm.

(d) According to the comparison of the three tables, limiter control method is also effective in controlling chaos. And it has great significance when requiring



(a) $w_2 = 0, w_3 = 0$

(b) $w_1 = 0, w_3 = 0$

(c) $w_1 = 0, w_2 = 0$

Figure 16: The bifurcation diagram and the corresponding largest Lyapunov exponent of the delayed system (25).

the player who wants to improve his performance to take measures without the other players' cooperation.

5. Conclusions

In this paper, we investigate the dynamics of a bounded rational oligopoly model which contains three players: one is the upstream monopoly who chooses his production to compete and the other two downstream oligopolies adjust their prices to maximize their expected profit. The existence and stability of equilibria, bifurcation, and chaotic behavior are analyzed in this game. In addition, the largest Lyapunov exponent and strange attractors are also applied to display the bifurcation and chaotic behavior of this system. We show that if the adjustment speeds of players are too high, then they will change the stability of equilibrium and cause a market structure to behave chaotically. Furthermore, we give long-run average profit of the three firms, which demonstrates that the equilibrium state is satisfactory to the three firms.

We adopt two kinds of control methods and consider the delayed system. Then the following conclusions are obtained.

(1) Delay feedback control method and limiter control method both can control chaos and make profit increase.

(2) From the perspective of profits, the delay feedback control method is advantageous to the upstream firm. The limiter control method is effective in preventing and controlling the two downstream firms' profit decline.

(3) The effect of selecting appropriate delay parameters is the same as applying a control on the system.

Table 1: Comparative analysis of oligarchs' profits when $\alpha = 1.36$, $\beta = 0.15$, and $\gamma = 0.2$.

Parameters	Nondelayed	Delay feedback control	Limiter control	Delayed	Delayed	Delayed
w_1	0	0	0	0.2	0	0
w_2	0	0	0	0	0.2	0
w_3	0	0	0	0	0	0.2
k	0	0.8	0	0	0	0
$p_2^{\min}$	0	0	2.0842	0	0	0
$p_3^{\min}$	0	0	2.2213	0	0	0
π_1	53.9601	60	53.9601	60	53.9601	53.9601
π_2	12.1514	13.6473	12.6415	13.6473	12.4463	12.7958
π_3	15.8208	17.0414	16.3254	17.0414	15.5874	15.3587
$\sum \pi_i$	81.9322	90.6888	82.9271	90.6888	82.9938	82.1146

Table 2: Comparative analysis of oligarchs' profits when $\alpha = 0.5$, $\beta = 0.32$, and $\gamma = 0.2$.

Parameters	Nondelayed	Limiter control	Delayed	Delayed	Delayed
w_1	0	0	0.2	0	0
w_2	0	0	0	0.2	0
w_3	0	0	0	0	0.2
$p_2^{\min}$	0	2.0842	0	0	0
$p_3^{\min}$	0	2.2213	0	0	0
π_1	60	60	60	60	60
π_2	−16.7885	9.9633	−15.6247	13.6473	−13.0483
π_3	10.9234	16.5478	10.6893	17.0414	12.4387
$\sum \pi_i$	54.1349	86.5111	55.0646	90.6888	59.3904

Table 3: Comparative analysis of oligarchs' profits when $\alpha = 0.5$, $\beta = 0.15$, and $\gamma = 0.41$.

Parameters	Nondelayed	Limiter control	Delayed	Delayed	Delayed
w_1	0	0	0.2	0	0
w_2	0	0	0	0.2	0
w_3	0	0	0	0	0.2
$p_2^{\min}$	0	2.0842	0	0	0
$p_3^{\min}$	0	2.2213	0	0	0
π_1	60	60	60	60	60
π_2	9.4274	12.6782	8.8646	10.2317	13.6473
π_3	−13.2909	13.9479	−19.2747	−10.3930	17.0414
$\sum \pi_i$	56.1364	86.6260	49.5898	59.8387	90.6888

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the reviewers for their careful reading and for providing some pertinent suggestions. The research was supported by the National Natural Science Foundation of China (no. 61273231) and Doctoral Fund of Ministry of Education of China (Grant no. 20130032110073), and it was supported by Tianjin University Innovation Fund.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 194962, 19 pages
http://dx.doi.org/10.1155/2014/194962

Research Article

Optimal Control of Investment-Reinsurance Problem for an Insurer with Jump-Diffusion Risk Process: Independence of Brownian Motions

De-Lei Sheng,[1] Ximin Rong,[1,2] and Hui Zhao[1]

[1] *Department of Mathematics, Tianjin University, Tianjin 300072, China*
[2] *Center for Applied Mathematics, Tianjin University, Tianjin 300072, China*

Correspondence should be addressed to De-Lei Sheng; tjhbsdl@126.com

Received 18 March 2014; Revised 20 June 2014; Accepted 25 June 2014; Published 24 July 2014

Academic Editor: Simone Marsiglio

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This paper investigates the excess-of-loss reinsurance and investment problem for a compound Poisson jump-diffusion risk process, with the risk asset price modeled by a constant elasticity of variance (CEV) model. It aims at obtaining the explicit optimal control strategy and the optimal value function. Applying stochastic control technique of jump diffusion, a Hamilton-Jacobi-Bellman (HJB) equation is established. Moreover, we show that a closed-form solution for the HJB equation can be found by maximizing the insurer's exponential utility of terminal wealth with the independence of two Brownian motions $W(t)$ and $W_1(t)$. A verification theorem is also proved to verify that the solution of HJB equation is indeed a solution of this optimal control problem. Then, we quantitatively analyze the effect of different parameter impacts on optimal control strategy and the optimal value function, which show that optimal control strategy is decreasing with the initial wealth x and decreasing with the volatility rate of risk asset price. However, the optimal value function $V(t; x; s)$ is increasing with the appreciation rate μ of risk asset.

1. Introduction

By means of investment and reinsurance, insurers can protect themselves against potentially large losses or ensure their earnings remain relatively stable. Therefore, many optimal investment and reinsurance problems have arisen in insurance risk management and have been extensively studied in the literature.

In the older forms, reinsurance was often referred to as "proportional" reinsurance; few studies pay attention to reinsurance. Since Borch [1] studied the safety loading of reinsurance premiums, a vast amount of literature is particularly concerned about reinsurance. However, the excess-of-loss reinsurance is a tool commonly employed in risk management in the recent thirty years. Tapiero and Zuckerman [2] gave the optimum excess-loss reinsurance under a dynamic framework. Taylor [3] studied reserving consecutive layers of inwards excess-of-loss reinsurance. Cao and Xu [4] investigated both proportional and excess-of-loss reinsurance under investment gains. Gu et al. [5] investigated optimal control of excess-of-loss reinsurance and investment for insurers under a constant elasticity of variance model but without compound Poisson jump in their research. Zhao et al. [6] studied optimal excess-of-loss reinsurance and investment problem for an insurer with jump-diffusion risk process under the Heston model.

It is well known that the compound Poisson process is the most popular and useful model to describe claims process ever since the classical Cramér-Lundberg model in risk theory. However, as a compound Poisson process perturbed by a standard Brownian motion, jump diffusion has been researched extensively in the recent ten years. Jump diffusion can give more practical description of claims than continuous models, which widely used to describe dynamics of surplus process. Yang and Zhang [7] investigated the problem of optimal investment for insurer with jump-diffusion risk process. Li et al. [8] studied the threshold dividend strategy for renewal jump-diffusion process. Ruan et al. [9] studied the optimal portfolio and consumption with habit formation in a jump-diffusion market.

Not only does the insurer cede part of premiums for reinsurance, but the insurer also invests in a financial market consisting of one risk-free asset and a risk asset. The constant elasticity of variance (CEV) model describes the volatility of risk asset that received widespread interests after being proposed by Cox and Ross [10] for European option pricing, partly because the CEV model can capture the implied volatility skew and can explain the volatility smile as well. Hsu et al. [11] gave the integration and detailed derivation for constant elasticity of variance (CEV) option pricing model. Campi et al. [12] investigated systematic equity-based credit risk for a CEV model with jump to default. Applying Legendre transform and dual theory, Xiao et al. [13] investigated analytical strategies for annuity contracts under constant elasticity of variance model. The CEV model was applied to the optimal investment and reinsurance problems in Gu et al. [14]. By maximizing the expected exponential utility of terminal wealth, Lin and Li [15] studied the optimal reinsurance and investment for a jump-diffusion risk process under the CEV model.

The goal of this paper is therefore to investigate the optimal control strategy of excess-of-loss reinsurance and investment problem for an insurer with compound Poisson jump-diffusion risk process from the stochastic control point of view. Under the hypothesis that two Brownian motions $W(t)$ and $W_1(t)$ are mutually independent, we present three main results. We first give the optimal excess-of-loss reinsurance strategy and investment strategy, respectively, according to the first-order necessary condition of maximum. We also obtain the optimal value function by plugging a reasonable conjecture into the HJB equation and solving linear second-order partial differential equations. In the third part, we specialize in a verification theorem. We also prove the verification theorem in detail inspired by the results of Taksar and Zeng [16], Gu et al. [5], and Zhao et al. [6]. At last, we present some numerical examples to illustrate the impacts of model parameters.

The paper is organized as follows. Section 2 introduces the risk asset price model under the CEV model and the wealth process with jump diffusion. In Section 3, we consider the optimal excess-of-loss reinsurance and investment strategy and a verification theorem as well. Section 4 contains some numerical simulations and theoretical results. Section 5 brings this paper to an end.

2. The Model

Throughout this paper, $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}_{0\le t\le T})$ denotes a complete probability space satisfying the usual condition, where a finite constant $T > 0$ represents the investment time horizon; two standard Brownian motions $W_1(t)$ and $W(t)$ are independent of each other; γ is a Poisson random measure; $\mathscr{F}_t := \mathscr{F}_t^W \vee \mathscr{F}_t^{W_1} \vee \mathscr{F}_t^{\gamma}$ represents the minimal σ-field generated by $\mathscr{F}_t^W$, $\mathscr{F}_t^{W_1}$, and $\mathscr{F}_t^{\gamma}$, which contains all the information available until time t. All stochastic processes involved in this paper are supposed to be $\mathscr{F}_t$ adapted. In addition, let $x \wedge y = \min\{x, y\}$ below.

Following the same formulation of Zhao et al. [6], the surplus process of an insurer is described by the following jump-diffusion model:

$$\mathrm{d}R(t) = c\mathrm{d}t + \sigma \mathrm{d}W(t) - \mathrm{d}C(t), \tag{1}$$

where c and σ are positive constants, $W(t)$ is a standard Brownian motion, and $\sigma W(t)$ describes the uncertainty associated with the surplus of the insurer at time t. Assume $C(t) = \sum_{i=1}^{N(t)} Z_i$ which represents the cumulative claims until time t, where $N(t)$ is a homogeneous Poisson process with intensity λ and the claim sizes $\{Z_i, i \ge 1\}$ are independent and identically distributed positive random variables with common distribution $F(z)$. Denote the mean value $E[Z_i] = \mu_\infty$ and

$$D := \sup\{z : F(z) \le 1\} < +\infty. \tag{2}$$

Suppose that $F(0) = 0$, $0 < F(z) < 1$ if $z \in (0, D)$ and $F(z) = 1$ if $z \ge D$. The premium according to the expected value principle is $c = (1+\eta)\lambda\mu_\infty$, where $\eta > 0$ is the safety loading of the insurer.

According to the theory of Poisson random measure (referring to Øksendal and Sulem [17]), we can rewrite the compound Poisson process $C(t) = \sum_{i=1}^{N(t)} Z_i$. γ denotes the Poisson random measure; then

$$\begin{aligned} \sum_{i=1}^{N(t)} Z_i &= \int_0^t \int_{R^+} z\gamma(\mathrm{d}z, \mathrm{d}u), \\ C^a(t) = \sum_{i=1}^{N(t)} Z_i &= \int_0^t \int_{R^+} (z \wedge a)\gamma(\mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{3}$$

The compensator ν of the random measure γ is

$$\nu(\mathrm{d}z, \mathrm{d}t) = \lambda \mathrm{d}F(z)\mathrm{d}t, \tag{4}$$

so the compensated Poisson random measure of γ is

$$\tilde{\gamma}(\mathrm{d}z, \mathrm{d}t) = \gamma(\mathrm{d}z, \mathrm{d}t) - \lambda \mathrm{d}F(z)\mathrm{d}t. \tag{5}$$

Insurance company can purchase excess-of-loss reinsurance to reduce the risk. Supposing the insurer's (fixed) retention level denoted by a, the corresponding reserve process is

$$\mathrm{d}R(t) = c^{(a)}\mathrm{d}t + \sigma \mathrm{d}W(t) - \mathrm{d}C^{(a)}(t) \tag{6}$$

and the reinsurer's premium rate is

$$\begin{aligned} c^{(a)} &= (1+\eta)\lambda\mu_\infty - (1+\theta)\lambda\{\mu_\infty - E[Z_i \wedge a]\} \\ &= (\eta - \theta)\lambda\mu_\infty + (1+\theta)\lambda E[Z_i \wedge a], \\ E[Z_i \wedge a] &= \int_0^a z\mathrm{d}F(z) + \int_a^\infty a\mathrm{d}F(z) \\ &= \int_0^a (1 - F(z))\mathrm{d}z \triangleq \int_0^a \overline{F(z)}\mathrm{d}z \end{aligned} \tag{7}$$

and $C^{(a)} = \sum_{i=1}^{N(t)} (Z_i \wedge a)$ satisfying

$$e^{\int_0^t e^{-rs} \mathrm{d}C^{(a)}(s)} < \infty, \quad \forall t < \infty. \tag{8}$$

In particular, if the retention level $a = D$, then

$$c^{(a)} = (1+\eta)\lambda\mu_\infty = c^{(D)} = c. \tag{9}$$

Without loss of generality, we always assume the reinsurance is not cheap; that is, $\theta > \eta$ and θ denotes the safety loading of the reinsurer.

Moreover, the insurer also can invest in a financial market consisting of one risk-free asset and one risky asset. The price process $S_0(t)$ of the risk-free asset is given by

$$\mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t, \tag{10}$$

where $r > 0$ is the risk-free interest rate. The price process $S(t)$ of the risky asset is described by the constant elasticity of variance (CEV) model:

$$\mathrm{d}S(t) = S(t)\left[\mu \mathrm{d}t + kS^{\beta}(t)\,\mathrm{d}W_1(t)\right], \tag{11}$$

where μ and k are positive constants, $\mu > r$ is the expected instantaneous return rate of the risky asset, $kS^{\beta}(t)$ is the instantaneous volatility, β is the elasticity parameter, and $W_1(t)$ is a standard Brownian motion. When $\beta = 0$, a CEV model degenerates into a GBM. In this paper, we assume the standard Brownian motion $W_1(t)$ is independent of $W(t)$.

The control $\alpha = \{(a(t), \pi(t)) : t \in [0,T]\}$ is a two-dimensional $\{\mathscr{F}(t)\}$ adapted stochastic process, where $a(t) \in [0,D]$ is the retention level of the excess-of-loss at time t, in which $a(t) \equiv D$ means "no reinsurance" and $a(t) \equiv 0$ means "full reinsurance," and $\pi(t)$ represents the proportion invested in the risky asset. Here, short-selling is not allowed; that is, $\pi(t) \geq 0$. The amount invested in the risk-free asset is $X(t)(1-\pi(t))$, where $X(t)$ is the wealth process for the insurer under the strategy α and the dynamics of $X(t)$ is given by

$$\begin{aligned}
\mathrm{d}X(t) = {} & \Big[(\eta-\theta)\lambda\mu_\infty + (1+\theta)\lambda\int_0^a \overline{F(x)}\mathrm{d}x \\
& + \mu\pi(t)X(t) + r(1-\pi(t))X(t)\Big]\mathrm{d}t \\
& + \sigma \mathrm{d}W(t) + \pi(t)X(t)kS^{\beta}(t)\,\mathrm{d}W_1(t) - \mathrm{d}C^{(a)}(t) \\
X(0) = {} & x_0.
\end{aligned} \tag{12}$$

A strategy α is said to be admissible, if $\forall t \in [0,T]$, $(a(t), \pi(t))$ is $\mathscr{F}(t)$ progressively measurable, and $E[\int_0^\infty \pi^2(t)X^2(t)S^{2\beta}(t)] < \infty$, $a(t) \in [0,D]$, and $\pi(t) \in [0,+\infty)$. The set of all admissible strategies denoted by Λ and (12) has a unique (strong) solution. Suppose the insurer has a utility function $U(x)$ which is strictly concave and continuously differentiable on $(-\infty,+\infty)$ and aims to maximize the expected utility of his/her terminal wealth; that is,

$$\max_{\alpha\in\Lambda} E[U(X(T))]. \tag{13}$$

3. Main Results

In this section, we solve the excess-of-loss reinsurance and investment problem with independence of two Brownian motions $W(t)$ and $W_1(t)$. By maximizing the expected utility of terminal wealth, the optimal strategy and value function are given. At the beginning, let us give the following exponential utility function of an risk aversion insurer:

$$U(x) = -\frac{1}{q}e^{-qx}, \quad q > 0. \tag{14}$$

This utility function has a constant absolute risk aversion parameter q and is the only utility function under the principle of "zero utility" giving a fair premium that is independent of the level of reserves of insurers. For an admissible strategy $\alpha = (a(t), \pi(t))$, we define the value function as

$$H^{\alpha}(t,x,s) = E[U(X(T)) \mid X(t) = x, S(t) = s] \tag{15}$$

and the optimal value function is

$$H(t,x,s) = \sup_{\alpha\in\Lambda} H^{\alpha}(t,x,s) \tag{16}$$

with the boundary condition

$$H(T,x,s) = -\frac{1}{q}e^{-qx}. \tag{17}$$

The objective of the insurer is to find an optimal strategy $\alpha^* = (a^*(t), \pi^*(t))$ such that $H(t,x,s) = H^{\alpha^*}(t,x,s)$, where $a^*(t)$ is called the optimal reinsurance strategy and $\pi^*(t)$ is called the optimal investment strategy.

For any $H^{\alpha} \in C^{1,2,2}([0,T]\times R_+\times R_+)$, define the generator

$$\begin{aligned}
& \mathscr{A}H^{\alpha}(t,x,s) \\
& \quad = H_t^{\alpha} + [(\eta-\theta)\lambda\mu_\infty + rx]H_x^{\alpha} \\
& \qquad + \mu s H_s^{\alpha} + \frac{1}{2}\sigma^2 H_{xx}^{\alpha} + \frac{1}{2}k^2 s^{2\beta+2} H_{ss}^{\alpha} \\
& \qquad + \left\{(\mu-r)\pi x H_x^{\alpha} + \frac{1}{2}\pi^2 x^2 k^2 s^{2\beta} H_{xx}^{\alpha} + k^2 \pi x s^{2\beta+1} H_{xs}^{\alpha}\right\} \\
& \qquad + \Big\{H_x^{\alpha}(1+\theta)\lambda\int_0^a \overline{F(x)}\mathrm{d}x \\
& \qquad\quad + \lambda E\left[H^{\alpha}(t, x-(Z_i\wedge a), s) - H^{\alpha}(t,x,s)\right]\Big\},
\end{aligned} \tag{18}$$

where H_t^{α}, H_x^{α}, H_s^{α}, H_{xx}^{α}, and H_{ss}^{α} denote the corresponding first- and second-order partial derivatives of $H^{\alpha}(t,x,s)$ with respect to (w.r.t.) the corresponding variables, respectively.

Applying the classical tools of stochastic optimal control, we can derive the following Hamilton-Jacobi-Bellman (HJB) equation for problem (16):

$$\sup_{\alpha\in\Lambda}\{\mathscr{A}H^{\alpha}(t,x,s)\} = 0 \tag{19}$$

with the boundary condition (17).

Standard results (e.g., Fleming and Soner [18]) tell us (19) admits the unique strong solution, which, together with the fact that the value function is twice-continuously differentiable, gives the following theorem.

Theorem 1. *For the optimal excess-of-loss reinsurance and investment problem with jump-diffusion risk process under the CEV model, a solution to HJB equation* (19) *with boundary condition* (17) *is given by* $V(t,x,s)$ *and the corresponding maximizer is given by* $\alpha^* = (a^*, \pi^*)$ *in feedback form, where one has the following.*

(1) If $D \geq \ln(1+\theta)/q$, *the optimal retention level on the whole interval* $[0,T]$ *always is*

$$a_*^0(t) = \frac{\ln(1+\theta)}{q} e^{-r(T-t)} \tag{20}$$

and the optimal investment strategy is given by

$$\pi^*(t) = \left[\frac{\mu - r}{xqk^2} - \frac{(\mu - r)^2 \left(e^{2r\beta(t-T)} - 1\right)}{2qxrk^2}\right] e^{-r(T-t)} s^{-2\beta}. \tag{21}$$

The optimal value function is

$$\begin{aligned}
V(t,x,s) &= -\frac{1}{q}\exp\Bigg\{-q \\
&\quad \times \Bigg[xe^{r(T-t)} + \frac{(\eta-\theta)\lambda\mu_\infty e^{r(T-t)}}{r} - \frac{\sigma^2 q e^{2r(T-t)}}{4r} \\
&\quad + \int_t^T \Bigg(e^{r(T-y)}(1+\theta)\lambda \\
&\qquad \times \int_0^{(\ln(1+\theta)/q)e^{-r(T-y)}} \overline{F(x)}\,\mathrm{d}x\Bigg)\mathrm{d}y \\
&\quad - \int_t^T \Bigg(\lambda e^{r(T-y)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-y)}} e^{qxe^{r(T-y)}} \\
&\qquad \times \overline{F(x)}\,\mathrm{d}x\Bigg)\mathrm{d}y \\
&\quad - \frac{(\eta-\theta)\lambda\mu_\infty}{r} + \frac{\sigma^2 q}{4r} \\
&\quad + \frac{e^{-2r\beta T}\left(-e^{2r\beta t} + e^{2r\beta T}\right)(\mu-r)^2}{4qr\beta k^2} s^{-2\beta} \\
&\quad + \Big(e^{-2r\beta T}(1+2\beta) \\
&\qquad \times \left(e^{2r\beta t} + e^{2r\beta T}(-1 + 2r(T-t)\beta)\right) \\
&\qquad \times (\mu-r)^2\Big) \times \left(8q\beta r^2\right)^{-1}\Bigg]\Bigg\}.
\end{aligned} \tag{22}$$

(2) If $D < \ln(1+\theta)/q$, *the optimal retention level is*

$$a^*(t) = \begin{cases} \dfrac{\ln(1+\theta)}{q} e^{-r(T-t)}, & t \in [0, t^0), \\ D, & t \in [t^0, T]. \end{cases} \tag{23}$$

And the optimal investment strategy is given by

$$\pi^*(t) = \left[\frac{\mu - r}{xqk^2} - \frac{(\mu - r)^2 \left(e^{2r\beta(t-T)} - 1\right)}{2qxrk^2}\right] e^{-r(T-t)} s^{-2\beta}. \tag{24}$$

The optimal value function is given by

$$\begin{aligned}
V(t,x,s) &= -\frac{1}{q}\exp\left\{-q\left[xe^{r(T-t)} + f^*(t) + g^*(t,s)\right]\right\} \\
&= \begin{cases} -\dfrac{1}{q}\exp\left\{-q\left[xe^{r(T-t)} + \widetilde{g(t,s)} + \widetilde{f(t)}\right]\right\}, & t \in [0, t^0), \\ -\dfrac{1}{q}\exp\left\{-q\left[xe^{r(T-t)} + \widehat{g(t,s)} + \widehat{f(t)}\right]\right\}, & t \in [t^0, T], \end{cases}
\end{aligned}$$

$$\begin{aligned}
\widetilde{g(t,s)} &= \left[\left(-\frac{(\mu-r)^2 e^{-2rT\beta}}{4qrk^2\beta}\right) e^{2\beta rt} + \frac{(\mu-r)^2}{4qk^2\beta r}\right] s^{-2\beta} \\
&\quad + \left(\frac{(\mu-r)^2 e^{-2rT\beta}}{4qrk^2\beta}\right)\frac{k^2(2\beta+1)}{2r} e^{2\beta rt} \\
&\quad - \frac{(2\beta+1)}{qr}(\mu-r)^2 + (2\beta+1)(\mu-r)^2 \\
&\quad \times \left(\frac{4+T-t^0}{4qr} - \frac{1}{8qr^2\beta}\right), \quad t \in [0, t^0),
\end{aligned}$$

$$\begin{aligned}
\widehat{g(t,s)} &= \frac{\left(1 - e^{2r\beta(t-T)}\right)(\mu-r)^2}{4qk^2 r\beta} s^{-2\beta} \\
&\quad + \frac{(1+2\beta)\left[e^{2r\beta(t-T)} - 1 + 2r(T-t)\beta\right](\mu-r)^2}{8qr^2\beta}, \\
&\qquad t \in [t^0, T],
\end{aligned}$$

$$\begin{aligned}
\widehat{f(t)} &= \frac{(1+\eta)\lambda\mu_\infty e^{r(T-t)}}{r} - \frac{\sigma^2 q e^{2r(T-t)}}{4r} \\
&\quad - \int_t^T \left[\frac{\lambda}{q}\int_0^D e^{qxe^{r(T-\tau)}}\mathrm{d}F(x)\right]\mathrm{d}\tau \\
&\quad - \frac{\lambda(t-T)}{q} - \frac{(1+\eta)\lambda\mu_\infty}{r} + \frac{\sigma^2 q}{4r}, \quad t \in [t^0, T],
\end{aligned}$$

$$\begin{aligned}
&\widetilde{f(t)} \\
&= \frac{(\eta-\theta)\lambda\mu_\infty}{r}e^{r(T-t)} - \frac{\sigma^2 q}{4r}e^{2r(T-t)} \\
&\quad - \int_0^t \left[(1+\theta)\lambda e^{r(T-\tau)}\int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} \overline{F(x)}\,\mathrm{d}x\right]\mathrm{d}\tau \\
&\quad + \lambda\int_0^t \left[e^{r(T-\tau)}\int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} e^{qxe^{r(T-\tau)}}\overline{F(x)}\,\mathrm{d}x\right]\mathrm{d}\tau \\
&\quad - \frac{(\eta-\theta)\lambda\mu_\infty}{r}e^{r(T-t^0)} \\
&\quad + (1+\theta)\lambda\int_0^{t^0} \left[e^{r(T-\tau)}\int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} \overline{F(x)}\,\mathrm{d}x\right]\mathrm{d}\tau \\
&\quad - \lambda\int_0^{t^0} \left[e^{r(T-\tau)}\int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} e^{qxe^{r(T-\tau)}}\overline{F(x)}\,\mathrm{d}x\right]\mathrm{d}\tau \\
&\quad + \frac{(1+\eta)\lambda\mu_\infty}{r}e^{r(T-t^0)} \\
&\quad - \frac{\lambda}{q}\int_{t^0}^T \left[\int_0^D e^{qxe^{r(T-\tau)}}\mathrm{d}F(x)\right]\mathrm{d}\tau \\
&\quad - \frac{\lambda\left(t^0-T\right)}{q} - \frac{(1+\eta)\lambda\mu_\infty}{r} + \frac{\sigma^2 q}{4r}, \quad t\in\left[0,t^0\right).
\end{aligned} \tag{25}$$

Proof. See Appendix A. □

Remark 2. From Theorem 1, we find that the optimal investment strategy is a function of $(\mu - r)$, q, x, ks^β, r, and t, which fully reflects the influence of various factors on the investment strategy. x is the initial wealth, ks^β represents the volatility of risk asset price, and $\mu - r$ is the profit of risk asset appreciation higher than risk-free interest rate. By simple deformation,

$$\begin{aligned}
\pi^*(t) &= \left[\frac{\mu-r}{xqk^2} - \frac{(\mu-r)^2\left(e^{2r\beta(t-T)}-1\right)}{2qxrk^2}\right]e^{-r(T-t)}s^{-2\beta} \\
&= (\mu-r)\left[\frac{1}{q} - \frac{(\mu-r)\left(1-e^{-2r\beta(T-t)}\right)}{2qr}\right] \\
&\quad \cdot e^{-r(T-t)}\cdot\frac{1}{x}\cdot\frac{1}{\left(ks^\beta\right)^2}.
\end{aligned} \tag{26}$$

Obviously, the optimal investment strategy decreases with respect to the initial wealth x and the volatility of risk asset price ks^β. In particular, if $\beta = 0$, then $\pi^*(t) = ((\mu - r)/xqk^2)e^{-r(T-t)}$ which is the same as that in Theorem 1 in Gu et al. [5].

A surprising finding is that the optimal reinsurance strategy has nothing to do with the initial wealth and risk asset. However, insurer's safe load θ and the parameter q in the utility function play an important role in determining reinsurance strategy.

Motivated by the results of Taksar and Zeng [16], Gu et al. [5], and Zhao et al. [6], we obtain the following verification theorem.

Theorem 3 (verification theorem). *For the optimal excess-of-loss reinsurance and investment problem with jump-diffusion risk process under the CEV model, the wealth process $X(t)$ is associated with an admissible strategy $(a(t), \pi(t))$. If the solution to HJB equation* (19) *with boundary condition* (17) *is given by $V(t, x, s)$ and the parameters $\mu > r > 0$ satisfy either of the following conditions:*

(I) $r > (1 - 1/\sqrt{6})\mu$;

(II) $r < (1 - 1/\sqrt{6})\mu$; $T < (1/\beta\sqrt{6(\mu-r)^2-\mu^2})\arctan(-\sqrt{6(\mu-r)^2-\mu^2}/\mu)$,

then the optimal value function is $H(t, x, s) = V(t, x, s)$, and the optimal strategy is $\alpha^* = (a^*(t), \pi^*(t))$, given in Theorem 1.

Proof. See Appendix B. □

Remark 4. In contrast with Gu et al. [5], they also considered the excess-of-loss with a compound Poisson jump under the CEV model; however, they simplified the compound Poisson jump by diffusion approximation process which leads the problem back to the case without jump.

4. Numerical Results

In this section, we analyze the impacts of some parameters on the optimal strategies and the value function. Theoretical analysis and some corresponding numerical examples are given to illustrate the influences of model parameters on the optimal strategy and the optimal value function when $D \geq \ln(1 + \theta)/q$. The analysis for the case of $D < \ln(1 + \theta)/q$ is similar.

Throughout this section, we always assume $\beta \geq 0$; the claim sizes follow exponential distribution $F(x) = 1 - e^{-x/m}$, $x > 0$, and if $\beta < 0$, some of the conclusions will be different. The parameters are given by $m = 2$, $q = 0.5$, $T = 10$, $\sigma = 1$, $\mu_\infty = 0.5$, $\theta = 0.2$, $r = 0.05$, $\mu = 0.12$, $k = 0.2$, $\lambda = 2$, $x = 1$, $\beta = 0.5$, and $s = 1$.

(1) First, let us pay attention to the expression of the optimal strategy when $\beta \geq 0$:

$$\begin{aligned}
\pi^*(t) &= \left[\frac{\mu-r}{xqk^2} - \frac{(\mu-r)^2\left(e^{2r\beta(t-T)}-1\right)}{2qxrk^2}\right]e^{-r(T-t)}s^{-2\beta} \\
&= \left[\frac{\mu-r}{q} - \frac{(\mu-r)^2\left(e^{2r\beta(t-T)}-1\right)}{2qr}\right] \\
&\quad \cdot e^{-r(T-t)}\cdot\frac{1}{x}\cdot\frac{1}{\left(ks^\beta\right)^2},
\end{aligned} \tag{27}$$

where x is the initial wealth and ks^β represents the volatility of risk asset price.

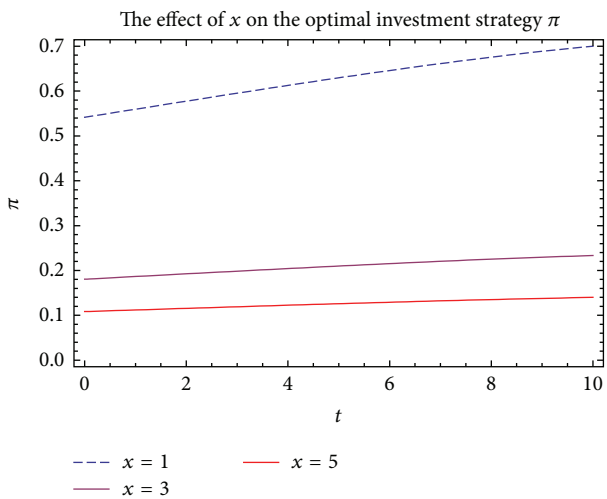

Figure 1: The optimal investment strategy π decreases with respect to x.

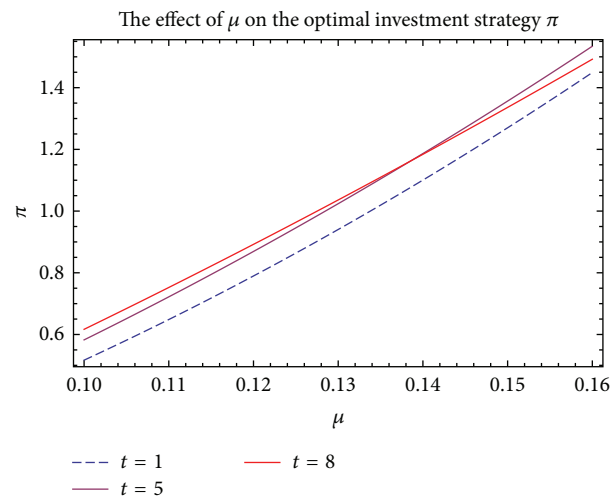

Figure 2: The optimal investment strategy increases with the appreciation rate μ of risky asset.

Deriving π^* with respect to x, we have

$$\begin{aligned}\frac{\partial \pi^*}{\partial x} &= \left(e^{r(t-T(1+2\beta))}s^{-2\beta}(r-\mu)\right.\\ &\quad\times\left.\left[\left(e^{2rt\beta}(r-\mu)+e^{2rT\beta}(r+\mu)\right)\right]\right)\times\left(2k^2qrx^2\right)^{-1}\\ &= -\left((\mu-r)s^{-2\beta}e^{r(t-T(1+2\beta))}\right.\\ &\quad\times\left.\left[r\left(e^{2rT\beta}+e^{2rt\beta}\right)+\mu\left(e^{2rT\beta}-e^{2rt\beta}\right)\right]\right)\\ &\quad\times\left(2k^2qrx^2\right)^{-1}<0,\end{aligned} \tag{28}$$

because $\mu > r > 0$, $T > t$, $s^{-2\beta} > 0$, $e^{r(t-T(1+2\beta))} > 0$, and $\mu(e^{2rT\beta} - e^{2rt\beta}) > 0$.

Remark 5. Obviously, the optimal investment strategy π^* decreases with respect to the initial wealth x, which tells us that, for a risk aversion insurer with the initial fund bigger and bigger, the investment proportion on risk asset must be reduced to make the actual amount invested on risk asset stay at an appropriate level to avoid the potential risk of being unbearable. See Figure 1.

However, the optimal strategy increases with respect to appreciation rate μ of the risky asset.

Since $T > t$, $\mu > r > 0$ so that $e^{2rT\beta} > e^{2rt\beta}$ and $\mu > (\mu - r)$, we have

$$\frac{\partial \pi^*}{\partial \mu} = \frac{e^{r(t-T(1+2\beta))}s^{-2\beta}\left[e^{2rT\beta}\mu - e^{2rt\beta}(\mu-r)\right]}{k^2qrx} > 0. \tag{29}$$

Figure 2 shows that the optimal strategy increases with respect to the return rate of the risky asset μ, which is obviously established. The higher appreciation rate μ of risky asset will attract more money invested on the risk asset and force the investment proportion π increase. See Figure 2.

Proposition 6. *Suppose $\beta \geq 0$ is the volatility parameter in CEV model; π^* is the optimal investment strategy. If $s^2 > e^{(\mu-r)(T-t)}$, $s > 1$, $\mu > r > 0$, $T > t$, then the optimal strategy π^* decreases with respect to k, β, and s. In fact, the optimal investment strategy decreases with the volatility rate of risk asset price denoted by $ks^{\beta}(t)$.*

Proof. The optimal strategy with respect to β may raise various possible cases under different conditions:

$$\begin{aligned}\frac{\partial \pi^*}{\partial \beta} &= -\left(e^{r(t-T(1+2\beta))}s^{-2\beta}(r-\mu)\right.\\ &\quad\times\left[e^{2rt\beta}(r-\mu)(r(t-T)-\ln[s])\right.\\ &\quad\left.\left.-e^{2rT\beta}(r+\mu)\ln[s]\right]\right)\times\left(k^2qrx\right)^{-1}\\ &= \left(e^{r(t-T(1+2\beta))}s^{-2\beta}(\mu-r)\right.\\ &\quad\times\left[e^{2rt\beta}(\mu-r)(r(T-t)+\ln[s])\right.\\ &\quad\left.\left.-e^{2rT\beta}(r+\mu)\ln[s]\right]\right)\times\left(k^2qrx\right)^{-1}.\end{aligned} \tag{30}$$

Since $s > 1$, $\mu > r$, and $T > t$, then $\mu - r > 0$, $T - t > 0$, $e^{2rT\beta} > e^{2rt\beta} > 0$, and $\ln[s] > 0$.

If $s^2 > e^{(\mu-r)(T-t)}$, take logarithm on both sides

$$\ln s^2 > (\mu-r)(T-t), \quad \text{i.e.,}\ 2\ln s > (\mu-r)(T-t). \tag{31}$$

We have

$$\begin{aligned}&e^{2rT\beta}2r\ln[s] - e^{2rt\beta}r(\mu-r)(T-t)\\ &\quad > e^{2rt\beta}r\left(2\ln[s]-(\mu-r)(T-t)\right) > 0,\end{aligned} \tag{32}$$

so that

$$\begin{aligned}\frac{\partial \pi^*}{\partial \beta} &= \left(k^2 qrx\right)^{-1} e^{r(t-T(1+2\beta))} s^{-2\beta} (\mu - r) \\ &\quad \times \left[e^{2rt\beta} (\mu - r) \ln [s] \right. \\ &\quad + e^{2rt\beta} (\mu - r) r (T - t) - e^{2rT\beta} \\ &\quad \left. \times (\mu - r) \ln [s] - 2r \ln [s] e^{2rT\beta}\right] \\ &= \left(k^2 qrx\right)^{-1} e^{r(t-T(1+2\beta))} s^{-2\beta} (\mu - r) \\ &\quad \times \left[- \left(e^{2rT\beta} - e^{2rt\beta}\right) (\mu - r) \ln [s] \right. \\ &\quad \left. - \left(e^{2rT\beta} 2r \ln [s] - e^{2rt\beta} r (\mu - r) (T - t)\right)\right] < 0,\end{aligned} \tag{33}$$

which means that the optimal strategy decreases with respect to β (see Figure 3(c)):

$$\begin{aligned}\frac{\partial \pi^*}{\partial s} &= \left(e^{r(t-T(1+2\beta))} s^{-2\beta} (r - \mu) \right. \\ &\quad \left. \times \left[\left(e^{2rt\beta} (r - \mu) + e^{2rT\beta} (r + \mu)\right)\right]\right) \times \left(2k^2 qrx^2\right)^{-1} \\ &= -\left((\mu - r) s^{-2\beta} e^{r(t-T(1+2\beta))} \right. \\ &\quad \left. \times \left[r \left(e^{2rT\beta} + e^{2rt\beta}\right) + \mu \left(e^{2rT\beta} - e^{2rt\beta}\right)\right]\right) \\ &\quad \times \left(2k^2 qrx^2\right)^{-1} < 0,\end{aligned} \tag{34}$$

because $\mu - r > 0$, $\beta > 0$, $s^{-2\beta} > 0$, $e^{r(t-T(1+2\beta))} > 0$, and $\mu(e^{2rT\beta} - e^{2rt\beta}) > 0$.

Consider

$$\frac{\partial \pi^*}{\partial k} = -\frac{e^{-r(T-t)} s^{-2\beta} (\mu - r) \left(\mu + r - e^{-2r(T-t)\beta} (\mu - r)\right)}{k^3 qrx} < 0. \tag{35}$$

Thus the optimal strategy decreases with respect to s and k, respectively. See Figures 3(a) and 3(b). □

All the calculations above show that the optimal strategy decreases with respect to k, s, and β simultaneously. So it comes to a conclusion that the optimal investment strategy decreases with the volatility rate $ks^{\beta}(t)$ of risk asset price.

Remark 7. The conclusion of this proposition is very natural. Risk averse investors always adopt a relatively conservative investment strategy to avoid risk. With the increase of risk asset price volatility rate $ks^{\beta}(t)$ which means risk increase, the risk averse insurer must reduce the investment proportion π on risk asset to keep the company's risk at an appropriate level. So the optimal investment strategy π^* decreases with the risk asset price volatility rate $ks^{\beta}(t)$. See Figure 3.

(2) Next, we are concerned with the expression of the value function for the special case when $\eta = \theta = 0$, which makes the value function simplified completely. Without loss of generality, we also suppose $\beta \geq 0$; the case of $\beta < 0$ is similar. Hence,

$$\begin{aligned}&V(t, x, s) \\ &= -\frac{1}{q} \exp \\ &\quad \times \left\{-q \left[x e^{r(T-t)} - \frac{\sigma^2 q e^{2r(T-t)}}{4r} \right.\right. \\ &\quad + \frac{e^{-2r\beta T} \left(-e^{2r\beta t} + e^{2r\beta T}\right) (\mu - r)^2}{4qr\beta k^2} s^{-2\beta} \\ &\quad + \left(e^{-2r\beta T} (1 + 2\beta) \right. \\ &\quad \times \left(e^{2r\beta t} + e^{2r\beta T} (-1 + 2r (T - t) \beta)\right) (\mu - r)^2\Big) \\ &\quad \left.\left. \times \left(8q\beta r^2\right)^{-1} + \frac{\sigma^2 q}{4r}\right]\right\}.\end{aligned} \tag{36}$$

Proposition 8. *$V(t, x, s)$ is the optimal value function under the optimal strategy (a^*, π^*) and k is the volatility coefficient of risk asset price in CEV model. If $k > 0$, $s > 0$, $\mu > r > 0$, and $T > t$, then the optimal value function $V(t, x, s)$ decreases with respect to k and s, respectively.*

Proof. If $k > 0$, $s > 0$, $\mu > r > 0$, and $T > t$, then

$$\left(2k^3 qr\beta\right)^{-1} \left(e^{2rT\beta} - e^{2rt\beta}\right) s^{-2\beta} (r - \mu)^2 > 0, \tag{37}$$

so that

$$\begin{aligned}\frac{\partial V}{\partial k} &= -\left(2k^3 qr\beta\right)^{-1} \left(e^{2rT\beta} - e^{2rt\beta}\right) s^{-2\beta} (r - \mu)^2 \\ &\quad \cdot \exp \left\{ - 2rT\beta - \frac{1}{8} q \right. \\ &\quad \times \left[8 e^{r(-t+T)} x - \frac{2\left(-1 + e^{2r(-t+T)}\right) q\sigma^2}{r} \right. \\ &\quad - \left(k^2 qr^2 \beta\right)^{-1} e^{-2rT\beta} s^{-2\beta} (r - \mu)^2 \\ &\quad \times \left[2 \left(e^{2rt\beta} - e^{2rT\beta}\right) r + k^2 s^{2\beta} (1 + 2\beta) \right. \\ &\quad \left.\left.\left. \times \left(-e^{2rt\beta} + e^{2rT\beta} (1 + 2r (t - T) \beta)\right)\right]\right]\right\} \\ &< 0,\end{aligned} \tag{38}$$

which means the value function decreases with respect to k. See Figure 4(b).

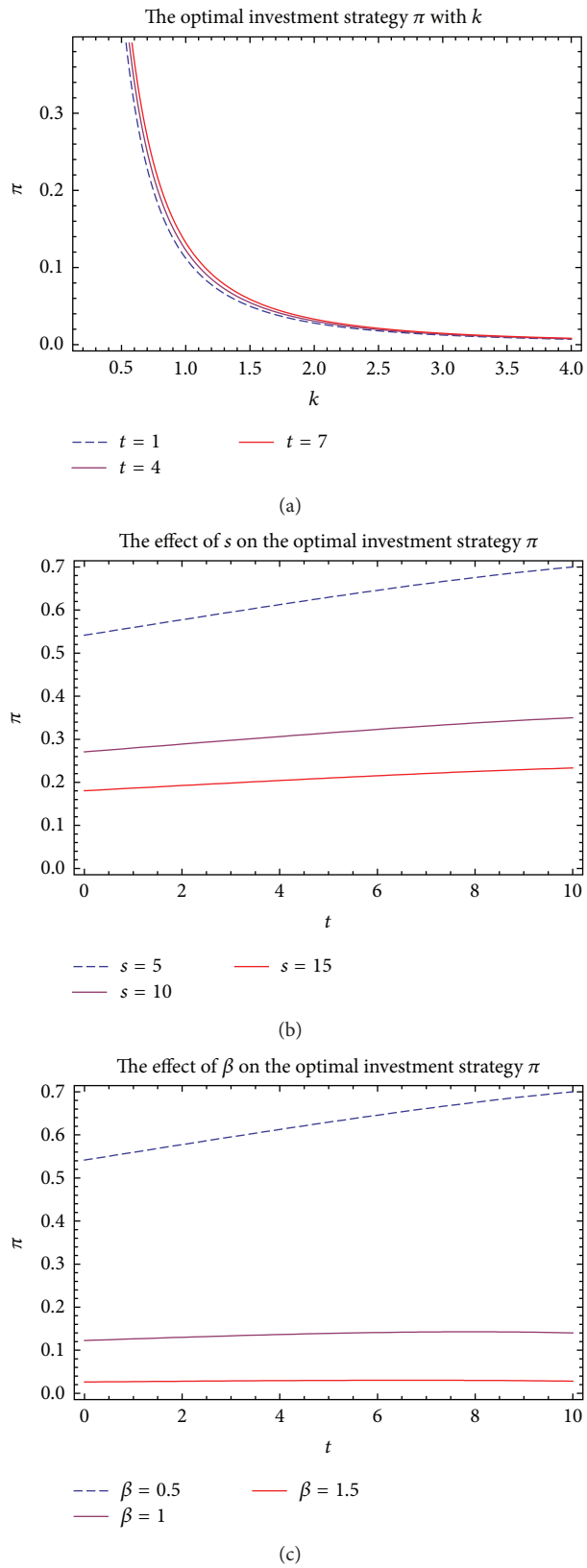

Figure 3: (a) The optimal investment strategy decreases with respect to k. (b) The optimal investment strategy π decreases with respect to s. (c) The optimal investment strategy π decreases with respect to β.

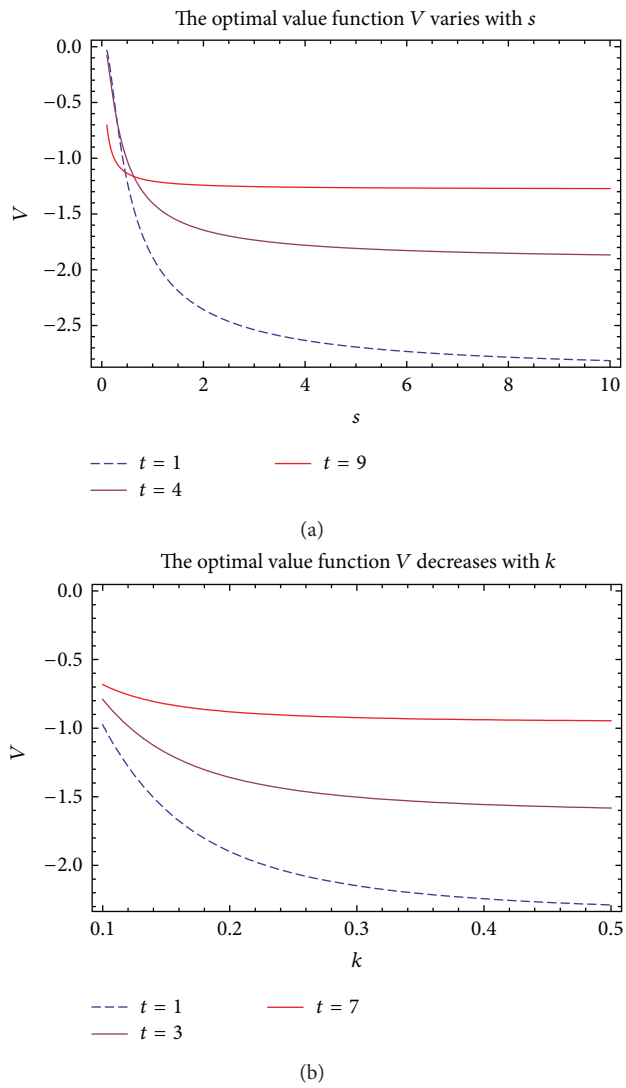

FIGURE 4: (a) The optimal value function V decreases with s. (b) The value function V decreases with k.

Completely similar to the analysis in (1), we also have

$$\begin{aligned}\frac{\partial V}{\partial s} &= -\left(2k^2qr\right)^{-1}\left(e^{2rT\beta}-e^{2rt\beta}\right)s^{-1-2\beta}(r-\mu)^2 \\ &\quad\cdot\exp\left\{-2rT\beta-\frac{1}{8}q\right. \\ &\quad\times\left[8e^{r(-t+T)}x-\frac{2\left(-1+e^{2r(-t+T)}\right)q\sigma^2}{r}\right. \\ &\quad-\left(k^2qr^2\beta\right)^{-1}e^{-2rT\beta}s^{-2\beta}(r-\mu)^2 \\ &\quad\cdot\left[2\left(e^{2rt\beta}-e^{2rT\beta}\right)r+k^2s^{2\beta}(1+2\beta)\right. \\ &\quad\left.\left.\left.\times\left(-e^{2rt\beta}+e^{2rT\beta}(1+2r(t-T)\beta)\right)\right]\right]\right\} \\ &< 0,\end{aligned} \tag{39}$$

which shows that the value function decreases with respect to s. See Figure 4(a). □

Let

$$\begin{aligned}D^{\beta}(t) &\triangleq k^2s^{2\beta} \\ &\quad\times\left(e^{2rT\beta}\left(1+4r(T-t)\beta^2\right)\right. \\ &\quad\left.-e^{2rt\beta}\left(1+2r(T-t)\beta(1+2\beta)\right)\right) \\ &\quad+2e^{2rt\beta}r\left(1+2r(T-t)\beta+2\beta\ln[s]\right) \\ &\quad-2e^{2rT\beta}r\left(1+2\beta\ln[s]\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial V}{\partial \beta} &= \left(8k^2qr^2\beta^2\right)^{-1}s^{-2\beta}(r-\mu)^2 \\ &\quad\cdot\left\{k^2s^{2\beta}\left(e^{2rT\beta}\left(1+4r(T-t)\beta^2\right)-e^{2rt\beta}\right.\right. \\ &\quad\left.\times\left(1+2r(T-t)\beta(1+2\beta)\right)\right) \\ &\quad+2e^{2rt\beta}r\left(1+2r(T-t)\beta+2\beta\ln[s]\right) \\ &\quad\left.-2e^{2rT\beta}r\left(1+2\beta\ln[s]\right)\right\}\end{aligned}$$

$$\cdot \exp\left\{-2rT\beta - \frac{1}{8}q \times \left[8e^{r(-t+T)}x - \frac{2\left(-1+e^{2r(-t+T)}\right)q\sigma^2}{r} - \left(k^2qr^2\beta\right)^{-1}e^{-2rT\beta}s^{-2\beta}(r-\mu)^2 \times \left[2\left(e^{2rt\beta}-e^{2rT\beta}\right)r + k^2s^{2\beta}(1+2\beta) \times \left(-e^{2rt\beta}+e^{2rT\beta} \times (1+2r(t-T)\beta)\right)\right]\right]\right\}. \tag{40}$$

Obviously, $D^\beta(T) = 0$. Deriving D^β defined above with respect to t, we have

$$\frac{\partial D^\beta}{\partial t} = 4r\beta^2\left\{-e^{2rT\beta}k^2s^{2\beta} + 2e^{2rt\beta}r\ln[s] + e^{2rt\beta}\left(2r^2(-t+T) + k^2s^{2\beta} \times (1+r(t-T)(1+2\beta))\right)\right\}. \tag{41}$$

Remark 9. If t is big enough relative to T, then $\partial D^\beta/\partial t > 0$, which tells us that $D^\beta(t) \to 0$ increases as time t closes to T. So $D^\beta(t) < 0$ as $t \to T$. The sign of $\partial V/\partial\beta$ is the same as $D^\beta(t)$; that is, $\partial V/\partial\beta < 0$ when t is large enough relative to T, which means the value function decreases with respect to β when t is large enough. Else if t is small enough relative to T, the value function may increase with respect to β.

In conclusion, the volatility ks^β of risk asset has a significant negative influence on the expected utility of insurance company; through varying parameters k, s, and β some approximate rule can be found.

Proposition 10. *$V(t,x,s)$ is the optimal value function under the optimal strategy (a^*,π^*) and μ is the expected return of risk asset price in CEV model. If $\beta \geq 0$, $s > 0$, $\mu > r > 0$, and $T > t$, then the optimal value function $V(t,x,s)$ increases with respect to the appreciation rate of risk asset μ.*

Proof. Letting

$$D^\mu(t) \triangleq \left[k^2s^{2\beta}(1+2\beta) - 2r\right]\left(e^{2rT\beta} - e^{2rt\beta}\right) - 2r(T-t)\beta k^2s^{2\beta}(1+2\beta)e^{2rT\beta}, \tag{42}$$

obviously, $D^\mu(T) = 0$ and

$$\begin{aligned}\frac{\partial D^\mu}{\partial t} &= \left[k^2s^{2\beta}(1+2\beta) - 2r\right]\left(-2r\beta e^{2rt\beta}\right) + 2r\beta k^2s^{2\beta}(1+2\beta)e^{2rT\beta} \\ &= -2r\beta e^{2rt\beta}k^2s^{2\beta}(1+2\beta) + 4r^2\beta e^{2rt\beta} + 2r\beta k^2s^{2\beta}(1+2\beta)e^{2rT\beta} \\ &= 2r\beta k^2s^{2\beta}(1+2\beta)\left(e^{2rT\beta} - e^{2rt\beta}\right) + 4r^2\beta e^{2rt\beta} > 0,\end{aligned} \tag{43}$$

which shows that $D^\mu(t)$ increases to $D^\mu(T) = 0$ with respect to time t. Thus

$$\left[k^2s^{2\beta}(1+2\beta) - 2r\right]\left(e^{2rT\beta} - e^{2rt\beta}\right) - 2r(T-t)\beta k^2s^{2\beta}(1+2\beta)e^{2rT\beta} \leq 0. \tag{44}$$

So

$$\frac{\partial V}{\partial \mu} = -\left(4k^2qr^2\beta\right)^{-1}s^{-2\beta}(\mu - r) \cdot \left\{\left[k^2s^{2\beta}(1+2\beta) - 2r\right]\left(e^{2rT\beta} - e^{2rt\beta}\right) - 2r(T-t)\beta k^2s^{2\beta}(1+2\beta)e^{2rT\beta}\right\} \cdot \exp\left\{-2rT\beta - \frac{1}{8}q \times \left[8e^{r(-t+T)}x - \frac{2\left(-1+e^{2r(-t+T)}\right)q\sigma^2}{r} - \left(k^2qr^2\beta\right)^{-1}e^{-2rT\beta}s^{-2\beta}(r-\mu)^2 \cdot \left[2\left(e^{2rt\beta}-e^{2rT\beta}\right)r + k^2s^{2\beta}(1+2\beta) \times \left(-e^{2rt\beta}+e^{2rT\beta}(1+2r(t-T)\beta)\right)\right]\right]\right\} \geq 0. \tag{45}$$

□

The value function V increases with μ referring to Figure 5, which shows that the higher appreciation rate of risk asset will lead to relatively more greater expected wealth utility. As a matter of fact, appreciation rate always is the main driving factor to make the wealth increase. Of course, the insurer's utility maximization can be better realized for a larger appreciation rate.

5. Conclusion

In this paper, we describe the dynamics of the risky assets' prices with the CEV model, under which the optimal excess-of-loss reinsurance and investment problem with jump-diffusion risk process is investigated by maximizing the insurer's exponential utility of terminal wealth. Applying

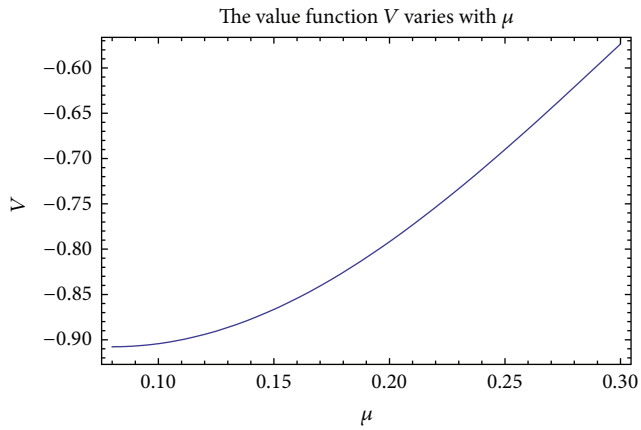

FIGURE 5: The value function V varies with μ.

stochastic control of jump diffusion, a Hamilton-Jacobi-Bellman (HJB) equation associated with a compound Poisson jump risk process is established. The explicit solution for the exponential utility function is given. In the last part of the text, some numerical examples are given to illustrate the effects of model parameters on the optimal strategy and the optimal value function. Meanwhile, we give some propositions and remarks which enriched the innovation of the paper.

Appendices

A. Proof of Theorem 1

This appendix collects the proofs of the main results stated in Section 3. Other necessary lemmas are also presented and proved in this appendix.

Proof. Differentiating (19) with respect to π gives the optimal investment policy

$$\pi^* = -\frac{(\mu - r) H_x^\alpha + k^2 s^{2\beta+1} H_{xs}^\alpha}{x k^2 s^{2\beta} H_{xx}^\alpha}. \tag{A.1}$$

Putting (A.1) into (19), after simplification, we have

$$\begin{aligned} &H_t^\alpha + [(\eta - \theta)\lambda\mu_\infty + rx] H_x^\alpha + \mu s H_s^\alpha \\ &\quad + \frac{1}{2}\sigma^2 H_{xx}^\alpha + \frac{1}{2}k^2 s^{2\beta+2} H_{ss}^\alpha \\ &\quad - \frac{\left[(\mu - r) H_x^\alpha + k^2 s^{2\beta+1} H_{xs}^\alpha\right]^2}{2k^2 s^{2\beta} H_{xx}^\alpha} \\ &\quad + \max_a \left\{ H_x^\alpha (1+\theta)\lambda \int_0^a \overline{F(x)}\mathrm{d}x \right. \\ &\qquad \left. + \lambda E\left[H^\alpha(t, x - (Z_i \wedge a), s) - H^\alpha(t, x, s)\right] \right\} \\ &= 0. \end{aligned} \tag{A.2}$$

That is,

$$\begin{aligned} &H_t^\alpha + [(\eta - \theta)\lambda\mu_\infty + rx] H_x^\alpha + \mu s H_s^\alpha + \frac{1}{2}\sigma^2 H_{xx}^\alpha \\ &\quad + \frac{1}{2}k^2 s^{2\beta+2} H_{ss}^\alpha - \frac{(\mu - r)^2 (H_x^\alpha)^2}{2k^2 s^{2\beta} H_{xx}^\alpha} \\ &\quad - \frac{k^2 s^{2\beta+2} (H_{xs}^\alpha)^2}{2H_{xx}^\alpha} - \frac{(\mu - r) s H_x^\alpha H_{xs}^\alpha}{H_{xx}^\alpha} \\ &\quad + \max_a \left\{ H_x^\alpha (1+\theta)\lambda \int_0^a \overline{F(x)}\mathrm{d}x \right. \\ &\qquad \left. + \lambda E\left[H^\alpha(t, x - (Z_i \wedge a), s) - H^\alpha(t, x, s)\right] \right\} \\ &= 0. \end{aligned} \tag{A.3}$$

We try to conjecture (A.2) having a solution $V(t, x, s)$ of this form

$$\begin{aligned} &V(t, x, s) = -\frac{1}{q} e^{\{-q[xe^{r(T-t)} + f(t) + g(t,s)]\}}, \quad t \in [0, T] \\ &f(T) = 0, \\ &g(T, s) = 0, \\ &V(T, x, s) = -\frac{1}{q} e^{-qx}. \end{aligned} \tag{A.4}$$

Differentiating the solution (A.4) with respect to the corresponding variables,

$$\begin{aligned} &V_t = V(-q)\left[-rxe^{r(T-t)} + f'(t) + g_t\right], \\ &V_x = V(-q) e^{r(T-t)}, \qquad V_{xx} = V(-q)^2 e^{2r(T-t)}, \\ &V_s = V(-q) g_s, \qquad V_{ss} = V\left[(-q)^2 g_s^2 + (-q) g_{ss}\right], \\ &V_{xs} = V(-q)^2 e^{r(T-t)} g_s, \\ &E\left[H^\alpha(t, x - (Z_i \wedge a), s) - H^\alpha(t, x, s)\right] \\ &\quad = Vqe^{r(T-t)} \int_0^a e^{qxe^{r(T-t)}} \overline{F(x)}\mathrm{d}x. \end{aligned} \tag{A.5}$$

Plugging the above derivatives (A.5) into (A.1) gives

$$\pi^* = \frac{\mu - r}{xqk^2 s^{2\beta} e^{r(T-t)}} - \frac{s g_s}{x e^{r(T-t)}} \tag{A.6}$$

and substituting the above derivatives into (A.2) yields

$$\begin{aligned} & f'(t) + g_t + (\eta - \theta)\lambda\mu_\infty e^{r(T-t)} + rsg_s \\ & \quad - \frac{1}{2}\sigma^2 q e^{2r(T-t)} + \frac{1}{2}k^2 s^{2\beta+2} g_{ss} + \frac{(\mu - r)^2}{2qk^2} s^{-2\beta} \\ & \quad + \max_a \left\{ (1+\theta)\lambda e^{r(T-t)} \int_0^a \overline{F(x)}\mathrm{d}x \right. \\ & \qquad \left. - \lambda e^{r(T-t)} \int_0^a e^{qxe^{r(T-t)}} \overline{F(x)}\mathrm{d}x \right\} = 0. \end{aligned} \tag{A.7}$$

Differentiating with respect to a in (A.7), we find that the minimizer $a_*^0(t)$ satisfies

$$\left[(1+\theta) - e^{qa^0(t)e^{r(T-t)}} \right] \overline{F(a^0(t))} = 0. \tag{A.8}$$

If $a^0(t) \in [0, D)$, then $0 < \overline{F(a^0(t))} \leq 1$ and $(1+\theta) - e^{qa^0(t)e^{r(T-t)}} = 0$, which gives

$$a_*^0(t) = \frac{\ln(1+\theta)}{q} e^{-r(T-t)}. \tag{A.9}$$

To ensure that $0 \leq a_*^0(t) < D$, that is, $0 \leq (\ln(1+\theta)/q)e^{-r(T-t)} < D$, it must be ensured that

$$t < T + \frac{1}{r} \ln \left[\frac{qD}{\ln(1+\theta)} \right] \triangleq t^0, \tag{A.10}$$

which leads to two different cases.

Case 1. If $D \geq (\ln(1+\theta)/q)$, then $t^0 \geq T$ and

$$a_*^0(t) = \frac{\ln(1+\theta)}{q} e^{-r(T-t)} \tag{A.11}$$

always is optimal retention level on the whole interval $[0, T]$.

Inserting (A.5) into (A.1), the optimal investment policy is given by

$$\pi^*(t) = \frac{\mu - r}{xqk^2 s^{2\beta} e^{r(T-t)}} - \frac{sg_s}{xe^{r(T-t)}} \tag{A.12}$$

and $a^* = a^0(t) = (\ln(1+\theta)/q)e^{-r(T-t)}$. The corresponding HJB equation (A.7) becomes

$$\begin{aligned} & f'(t) + g_t + rsg_s + (\eta - \theta)\lambda\mu_\infty e^{r(T-t)} \\ & \quad - \frac{1}{2}\sigma^2 q e^{2r(T-t)} + \frac{1}{2}k^2 s^{2\beta+2} g_{ss} \\ & \quad + \frac{(\mu - r)^2}{2qk^2 s^{2\beta}} + e^{r(T-t)}(1+\theta)\lambda \\ & \quad \times \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} \overline{F(x)}\mathrm{d}x - \lambda e^{r(T-t)} \\ & \quad \times \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} e^{qxe^{r(T-t)}} \overline{F(x)}\mathrm{d}x = 0. \end{aligned} \tag{A.13}$$

To solve (A.13), we decompose (A.13) into two equations:

$$\begin{aligned} & f'(t) + (\eta - \theta)\lambda\mu_\infty e^{r(T-t)} - \frac{1}{2}\sigma^2 q e^{2r(T-t)} \\ & \quad + e^{r(T-t)}(1+\theta)\lambda \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} \overline{F(x)}\mathrm{d}x \\ & \quad - \lambda e^{r(T-t)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} e^{qxe^{r(T-t)}} \overline{F(x)}\mathrm{d}x \\ & = 0, \end{aligned} \tag{A.14}$$

$$g_t + rsg_s + \frac{1}{2}k^2 s^{2\beta+2} g_{ss} + \frac{(\mu - r)^2}{2qk^2 s^{2\beta}} = 0. \tag{A.15}$$

By simple transposition and integration for (A.14) with $f(T) = 0$, we have

$$\begin{aligned} f(t) = & \frac{(\eta - \theta)\lambda\mu_\infty e^{r(T-t)}}{r} - \frac{\sigma^2 q e^{2r(T-t)}}{4r} \\ & + \int_t^T \left(e^{r(T-y)}(1+\theta)\lambda \int_0^{(\ln(1+\theta)/q)e^{-r(T-y)}} \overline{F(x)}\mathrm{d}x \right) \mathrm{d}y \\ & - \int_t^T \left(\lambda e^{r(T-y)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-y)}} e^{qxe^{r(T-y)}} \overline{F(x)}\mathrm{d}x \right) \mathrm{d}y \\ & - \frac{(\eta - \theta)\lambda\mu_\infty}{r} + \frac{\sigma^2 q}{4r}. \end{aligned} \tag{A.16}$$

To solve (A.20), we conjecture a solution with the following form with $A(T) = 0$ and $B(T) = 0$:

$$g(t, s) = A(t) s^{-2\beta} + B(t). \tag{A.17}$$

Differentiating (A.17) with respect to the corresponding variables, respectively,

$$\begin{aligned} & g_t = A'(t) s^{-2\beta} + B'(t), \qquad g_s = -2\beta s^{-2\beta-1} A(t), \\ & g_{ss} = 2\beta(2\beta+1) s^{-2\beta-2} A(t). \end{aligned} \tag{A.18}$$

Substituting the above derivatives into (A.20) yields

$$\begin{aligned} & A'(t) s^{-2\beta} + B'(t) - 2r\beta s^{-2\beta} A(t) \\ & \quad + k^2\beta(2\beta+1) A(t) + \frac{(\mu - r)^2}{2qk^2} s^{-2\beta} = 0. \end{aligned} \tag{A.19}$$

To solve (A.20), we decompose (A.20) into two equations:

$$\left[A'(t) - 2r\beta A(t) + \frac{(\mu - r)^2}{2qk^2} \right] s^{-2\beta} = 0, \tag{A.20}$$

$$B'(t) + k^2\beta(2\beta+1) A(t) = 0. \tag{A.21}$$

Obviously, $s^{-2\beta} > 0$, (A.20) is equivalent to

$$A'(t) - 2r\beta A(t) + \frac{(\mu - r)^2}{2qk^2} = 0. \tag{A.22}$$

Equation (A.22) with $A(T) = 0$ is easy to be solved and the solution is

$$A(t) = \frac{e^{-2r\beta T}\left(-e^{2r\beta t} + e^{2r\beta T}\right)(\mu - r)^2}{4qr\beta k^2}. \tag{A.23}$$

Inserting (A.23) into (A.21) with $B(T) = 0$, we have

$$\begin{aligned} B(t) = {} & \Big(e^{-2r\beta T}(1+2\beta) \\ & \times \left(e^{2r\beta t} + e^{2r\beta T}(-1 + 2r(T-t)\beta)\right) \\ & \times (\mu - r)^2\Big) \times \left(8q\beta r^2\right)^{-1}. \end{aligned} \tag{A.24}$$

Substituting (A.23) and (A.24) back into (A.17), we obtain

$$\begin{aligned} g(t,s) = {} & \frac{e^{-2r\beta T}\left(-e^{2r\beta t} + e^{2r\beta T}\right)(\mu - r)^2}{4qr\beta k^2} s^{-2\beta} \\ & + \Big(e^{-2r\beta T}(1+2\beta) \\ & \times \left(e^{2r\beta t} + e^{2r\beta T}(-1 + 2r(T-t)\beta)\right) \\ & \times (\mu - r)^2\Big) \times \left(8q\beta r^2\right)^{-1}. \end{aligned} \tag{A.25}$$

Substituting (A.16) and (A.25) back into (A.4), the optimal value function is given by

$$\begin{aligned} & V(t,x,s) \\ & = -\frac{1}{q}\exp \\ & \times \Bigg\{ -q \\ & \times \Bigg[x e^{r(T-t)} + \frac{(\eta - \theta)\lambda\mu_\infty e^{r(T-t)}}{r} - \frac{\sigma^2 q e^{2r(T-t)}}{4r} \\ & + \int_t^T \Bigg(e^{r(T-y)}(1+\theta)\lambda \\ & \times \int_0^{(\ln(1+\theta)/q)e^{-r(T-y)}} \overline{F(x)}\,\mathrm{d}x \Bigg)\mathrm{d}y \\ & - \int_t^T \Bigg(\lambda e^{r(T-y)} \\ & \times \int_0^{(\ln(1+\theta)/q)e^{-r(T-y)}} e^{qxe^{r(T-y)}}\overline{F(x)}\,\mathrm{d}x \Bigg)\mathrm{d}y \\ & - \frac{(\eta - \theta)\lambda\mu_\infty}{r} + \frac{\sigma^2 q}{4r} \\ & + \Big(e^{-2r\beta T}(1+2\beta) \\ & \times \left(e^{2r\beta t} + e^{2r\beta T}(-1 + 2r(T-t)\beta)\right) \\ & \times (\mu - r)^2\Big) \times \left(8q\beta r^2\right)^{-1} \\ & + \frac{e^{-2r\beta T}\left(-e^{2r\beta t} + e^{2r\beta T}\right)(\mu - r)^2}{4qr\beta k^2} s^{-2\beta} \Bigg] \Bigg\}, \\ & t \in [0,T]. \end{aligned} \tag{A.26}$$

Case 2. If $D < \ln(1+\theta)/q$, then $t^0 < T$; the whole interval $[0,T]$ is divided into two parts, $[0,t^0)$ and $[t^0,T]$. $a^0(t) = (\ln(1+\theta)/q)e^{-r(T-t)}$ is the optimal retention level only on $[0,t^0)$; however, $a^0(t) \geq D$ on $[t^0,T]$; we can only take D as the retention level naturally. Therefore, the optimal retention level is denoted by

$$a^*(t) = \begin{cases} \dfrac{\ln(1+\theta)}{q} e^{-r(T-t)}, & t \in \left[0,t^0\right), \\ D, & t \in \left[t^0,T\right]. \end{cases} \tag{A.27}$$

The solving process depends on two different time intervals corresponding to the different optimal retention level, respectively.

(1) For

$$a^*(t) = D, \quad t \in \left[t^0,T\right], \tag{A.28}$$

the corresponding HJB equation is

$$\begin{aligned} & H_t^\alpha + \left[(1+\eta)\lambda\mu_\infty + rx\right]H_x^\alpha + \mu s H_s^\alpha \\ & + \frac{1}{2}\sigma^2 H_{xx}^\alpha + \frac{1}{2}k^2 s^{2\beta+2} H_{ss}^\alpha \\ & + \max_\pi \Big\{ (\mu - r)\pi x H_x^\alpha + \frac{1}{2}\pi^2 x^2 k^2 s^{2\beta} H_{xx}^\alpha \\ & \quad + k^2 \pi x s^{2\beta+1} H_{xs}^\alpha \Big\} \\ & + \lambda E\left[H^\alpha\left(t, x - (Z_i \wedge D), s\right) - H^\alpha(t,x,s)\right] = 0. \end{aligned} \tag{A.29}$$

The first-order maximizing condition for the optimal investment strategy is

$$\pi^*(t) = -\frac{(\mu - r)H_x^\alpha + k^2 s^{2\beta+1} H_{xs}^\alpha}{x k^2 s^{2\beta} H_{xx}^\alpha}. \tag{A.30}$$

Inserting (A.30) into (A.29) and simplifying, the HJB equation becomes

$$\begin{aligned} &H_t^\alpha + \left[(1+\eta)\lambda\mu_\infty + rx\right]H_x^\alpha + \mu s H_s^\alpha \\ &\quad + \frac{1}{2}\sigma^2 H_{xx}^\alpha + \frac{1}{2}k^2 s^{2\beta+2} H_{ss}^\alpha \\ &\quad - \frac{\left[(\mu-r)H_x^\alpha + k^2 s^{2\beta+1} H_{xs}^\alpha\right]^2}{2k^2 s^{2\beta} H_{xx}^\alpha} \\ &\quad + \lambda E\left[H^\alpha\left(t, x-(Z_i\wedge D), s\right) - H^\alpha(t,x,s)\right] = 0. \end{aligned} \tag{A.31}$$

We conjecture (A.31) having a solution $V(t,x,s)$ as follows:

$$\begin{aligned} V(t,x,s) &= -\frac{1}{q}\exp\left\{-q\left[xe^{r(T-t)} + \widehat{f(t)} + \widehat{g(t,s)}\right]\right\}, \\ &\qquad t\in\left[t^0, T\right], \\ V(T,x,s) &= U(x), \\ \widehat{f(T)} &= 0, \\ \widehat{g(T,s)} &= 0. \end{aligned} \tag{A.32}$$

Differentiating the conjecture (A.32) with respect to the corresponding variables,

$$\begin{aligned} V_t &= V(-q)\left[-rxe^{r(T-t)} + \widehat{f'(t)} + \widehat{g_t}\right], \\ V_x &= V(-q)e^{r(T-t)}, \qquad V_{xx} = V(-q)^2 e^{2r(T-t)}, \\ V_s &= V(-q)\widehat{g_s}, \qquad V_{ss} = V\left[(-q)^2\widehat{g_s}^2 + (-q)\widehat{g_{ss}}\right], \\ V_{xs} &= V(-q)^2 e^{r(T-t)}\widehat{g_s}, \\ &E\left[H^\alpha\left(t, x-(Z_i\wedge D), s\right) - H^\alpha(t,x,s)\right] \\ &= V\int_0^D e^{qxe^{r(T-t)}}\,\mathrm{d}F(x) - V. \end{aligned} \tag{A.33}$$

Substituting the above derivatives (A.33) into (A.31) and (A.30), after simplifying it gives

$$\widehat{\pi^*}(t) = \frac{(\mu-r)}{qk^2 x s^{2\beta} e^{r(T-t)}} - \frac{s\widehat{g_s}}{xe^{r(T-t)}}, \tag{A.34}$$

$$\begin{aligned} &\widehat{f'(t)} + \widehat{g_t} + (1+\eta)\lambda\mu_\infty e^{r(T-t)} - \frac{1}{2}\sigma^2 q e^{2r(T-t)} \\ &\quad + rs\widehat{g_s} + \frac{1}{2}k^2 s^{2\beta+2}\widehat{g}_{ss} + \frac{(\mu-r)^2}{2qk^2}s^{-2\beta} \\ &\quad - \frac{\lambda}{q}\int_0^D e^{qxe^{r(T-t)}}\,\mathrm{d}F(x) + \frac{\lambda}{q} = 0. \end{aligned} \tag{A.35}$$

Decompose (A.35) into two equations

$$\begin{aligned} &\widehat{f'(t)} + (1+\eta)\lambda\mu_\infty e^{r(T-t)} - \frac{1}{2}\sigma^2 q e^{2r(T-t)} \\ &\quad - \frac{\lambda}{q}\int_0^D e^{qxe^{r(T-t)}}\,\mathrm{d}F(x) + \frac{\lambda}{q} = 0, \end{aligned} \tag{A.36}$$

$$\widehat{g_t} + rs\widehat{g_s} + \frac{1}{2}k^2 s^{2\beta+2}\widehat{g}_{ss} + \frac{(\mu-r)^2}{2qk^2}s^{-2\beta} = 0. \tag{A.37}$$

By transposition and integration for (A.36) with $\widehat{f(T)} = 0$,

$$\begin{aligned} \widehat{f(t)} &= \frac{(1+\eta)\lambda\mu_\infty e^{r(T-t)}}{r} - \frac{\sigma^2 q e^{2r(T-t)}}{4r} \\ &\quad - \int_t^T\left[\frac{\lambda}{q}\int_0^D e^{qxe^{r(T-\tau)}}\,\mathrm{d}F(x)\right]\mathrm{d}\tau \\ &\quad - \frac{\lambda(t-T)}{q} - \frac{(1+\eta)\lambda\mu_\infty}{r} + \frac{\sigma^2 q}{4r}. \end{aligned} \tag{A.38}$$

For (A.37), we try to find a solution as follows:

$$\widehat{g(t,s)} = \widehat{A(t)}s^{-2\beta} + \widehat{B(t)} \tag{A.39}$$

satisfying $\widehat{A(T)} = 0$ and $\widehat{B(T)} = 0$. Equation (A.37) turns into

$$\begin{aligned} &\widehat{A'(t)}s^{-2\beta} + \widehat{B'(t)} - 2r\beta\widehat{A(t)}s^{-2\beta} + k^2\beta(2\beta+1)\widehat{A(t)} \\ &\quad + \frac{(\mu-r)^2}{2qk^2}s^{-2\beta} = 0. \end{aligned} \tag{A.40}$$

Decompose (A.40) into two equations

$$\begin{aligned} &\left[\widehat{A'(t)} - 2r\beta\widehat{A(t)} + \frac{(\mu-r)^2}{2qk^2}\right]s^{-2\beta} = 0, \\ &\widehat{B'(t)} + k^2\beta(2\beta+1)\widehat{A(t)} = 0. \end{aligned} \tag{A.41}$$

Solving the above simple ordinary differential equations, we get

$$\begin{aligned} \widehat{A(t)} &= \frac{\left(1-e^{2r\beta(t-T)}\right)(\mu-r)^2}{4qk^2 r\beta}, \\ \widehat{B(t)} &= \frac{(1+2\beta)\left[e^{2r\beta(t-T)} - 1 + 2r(T-t)\beta\right](\mu-r)^2}{8qr^2\beta}. \end{aligned} \tag{A.42}$$

So

$$\begin{aligned} \widehat{g(t,s)} &= \frac{\left(1-e^{2r\beta(t-T)}\right)(\mu-r)^2}{4qk^2 r\beta}s^{-2\beta} \\ &\quad + \frac{(1+2\beta)\left[e^{2r\beta(t-T)} - 1 + 2r(T-t)\beta\right](\mu-r)^2}{8qr^2\beta}, \\ \widehat{\pi^*}(t) &= \frac{2r(\mu-r) + (\mu-r)^2\left(1-e^{2r\beta(t-T)}\right)}{2qk^2 rxs^{2\beta}e^{r(T-t)}}. \end{aligned} \tag{A.43}$$

(2) For

$$a^*(t) = \frac{\ln(1+\theta)}{q} e^{-r(T-t)}, \quad t \in \left[0, t^0\right), \tag{A.44}$$

the corresponding HJB equation is the same as (A.29), which corresponds to $[0, t^0)$ having a solution $V(t, x, s)$ as follows:

$$\begin{aligned} V(t,x,s) &= -\frac{1}{q} \exp\left\{-q\left[xe^{r(T-t)} + \widetilde{f(t)} + \widetilde{g(t,s)}\right]\right\}, \\ &\qquad t \in \left[0, t^0\right], \\ \widetilde{f(t^0)} &= \widehat{f(t^0)}, \\ \widetilde{g(t^0,s)} &= \widehat{g(t^0,s)}. \end{aligned} \tag{A.45}$$

After the same calculation as (1), we obtain the following results:

$$\begin{aligned} &\widetilde{f'(t)} + \widetilde{g_t} + (\eta-\theta)\lambda\mu_\infty e^{r(T-t)} - \frac{1}{2}\sigma^2 q e^{2r(T-t)} \\ &\quad + \frac{1}{2}k^2 s^{2\beta+2}\widetilde{g_{ss}} + rs\widetilde{g_s} + \frac{(\mu-r)^2}{2qk^2}s^{-2\beta} \\ &\quad + (1+\theta)\lambda e^{r(T-t)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} \overline{F(x)}\mathrm{d}x \\ &\quad - \lambda e^{r(T-t)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} e^{qxe^{r(T-t)}}\overline{F(x)}\mathrm{d}x = 0. \end{aligned} \tag{A.46}$$

Splitting (A.46) into two equations,

$$\widetilde{g_t} + \frac{1}{2}k^2 s^{2\beta+2}\widetilde{g_{ss}} + rs\widetilde{g_s} + \frac{(\mu-r)^2}{2qk^2}s^{-2\beta} = 0. \tag{A.47}$$

The equation (A.47) is a linear second-order partial differential equation with variable coefficients, which exhibits a unique solution according to the appendix of Badaoui and Fernandez [19] or is ensured by the theorem of Friedman [20]:

$$\begin{aligned} &\widetilde{f'(t)} + (\eta-\theta)\lambda\mu_\infty e^{r(T-t)} - \frac{1}{2}\sigma^2 q e^{2r(T-t)} \\ &\quad + (1+\theta)\lambda e^{r(T-t)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} \overline{F(x)}\mathrm{d}x \\ &\quad - \lambda e^{r(T-t)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-t)}} e^{qxe^{r(T-t)}}\overline{F(x)}\mathrm{d}x = 0. \end{aligned} \tag{A.48}$$

By transposition and integration to (A.48), the result is given by

$$\begin{aligned} \widetilde{f(t)} &= \frac{(\eta-\theta)\lambda\mu_\infty}{r} e^{r(T-t)} - \frac{\sigma^2 q}{4r} e^{2r(T-t)} \\ &\quad - \int_0^t \left[(1+\theta)\lambda e^{r(T-\tau)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} \overline{F(x)}\mathrm{d}x\right]\mathrm{d}\tau \\ &\quad + \lambda \int_0^t \left[e^{r(T-\tau)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} e^{qxe^{r(T-\tau)}}\overline{F(x)}\mathrm{d}x\right]\mathrm{d}\tau \\ &\quad + C_3, \end{aligned} \tag{A.49}$$

where w can be determined by $\widetilde{f(t^0)} = \widehat{f(t^0)}$. Consider

$$\begin{aligned} C_3 &= -\frac{(\eta-\theta)\lambda\mu_\infty}{r} e^{r(T-t^0)} + (1+\theta)\lambda \\ &\quad \times \int_0^{t^0} \left[e^{r(T-\tau)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} \overline{F(x)}\mathrm{d}x\right]\mathrm{d}\tau \\ &\quad - \lambda \int_0^{t^0} \left[e^{r(T-\tau)} \int_0^{(\ln(1+\theta)/q)e^{-r(T-\tau)}} e^{qxe^{r(T-\tau)}}\overline{F(x)}\mathrm{d}x\right]\mathrm{d}\tau \\ &\quad + \frac{(1+\eta)\lambda\mu_\infty}{r} e^{r(T-t^0)} \\ &\quad - \frac{\lambda}{q}\int_{t^0}^T \left[\int_0^D e^{qxe^{r(T-\tau)}}\mathrm{d}F(x)\right]\mathrm{d}\tau \\ &\quad - \frac{\lambda\left(t^0-T\right)}{q} - \frac{(1+\eta)\lambda\mu_\infty}{r} + \frac{\sigma^2 q}{4r}. \end{aligned} \tag{A.50}$$

So

$$f^*(t) = \begin{cases} \widetilde{f(t)}, & t \in \left[0, t^0\right), \\ \widehat{f(t)}, & t \in \left[t^0, T\right]. \end{cases} \tag{A.51}$$

With the same calculations as before, we obtain

$$\begin{aligned} \widetilde{g(t,s)} &= \left[C_1 e^{2\beta rt} + \frac{(\mu-r)^2}{4qk^2\beta r}\right]s^{-2\beta} - C_1\frac{k^2(2\beta+1)}{2r}e^{2\beta rt} \\ &\quad - \frac{(2\beta+1)}{qr}(\mu-r)^2 + C_2; \end{aligned} \tag{A.52}$$

here, the undetermined constants C_1 and C_2 will be determined by using the continuity of $V(t, x, s)$.

By the continuity of $V(t, x, s)$ at $t = t^0$, we have

$$\overline{g(t^0,s)} = \widehat{g(t^0,s)}. \tag{A.53}$$

Comparing the items and the coefficients on both sides of (A.53), we obtain the undetermined constants C_1 and C_2:

$$C_1 = -\frac{(\mu - r)^2 e^{-2rT\beta}}{4qrk^2\beta},$$
$$C_2 = (2\beta + 1)(\mu - r)^2 \left(\frac{4 + T - t^0}{4qr} - \frac{1}{8qr^2\beta}\right). \tag{A.54}$$

So

$$g^*(t,s) = \begin{cases} \widetilde{g(t,s)}, & t \in [0, t^0), \\ \widehat{g(t,s)}, & t \in [t^0, T]. \end{cases} \tag{A.55}$$

And the optimal value function is

$$V(t,x,s) = -\frac{1}{q}\exp\left\{-q\left[xe^{r(T-t)} + f^*(t) + g^*(t,s)\right]\right\}. \tag{A.56}$$

□

B. Proof of Theorem 3

A new lemma is necessary to prove Theorem 3. For convenience, denote $Q := [0, +\infty) \times [0, +\infty)$.

Lemma B.1. *Take a sequence of bounded open sets $Q_1, Q_2, Q_3, \ldots$, with $Q_i \subset Q_{i+1} \subset Q$, $i = 1, 2, \ldots$ and $Q = \cup_i Q_i$. Let τ_i denote the exit time of $(X^{\alpha^*}(t), S(t))$ from Q_i. If the conditions in Theorem 3 hold, then one has $E[V^2(\tau_i \wedge T, X^{\alpha^*}(\tau_i \wedge T), S(\tau_i \wedge T)) \mid X(t) = x, S(t) = s] < \infty$ for $i = 1, 2, \ldots$.*

Proof. According to Øksendal and Sulem [17], we rewrite the compound Poisson process $C(t) = \sum_{i=1}^{N(t)} Z_i$ in terms of a Poisson random measure. Suppose γ is the Poisson random measure; then

$$\sum_{i=1}^{N(t)} Z_i = \int_0^t \int_{R^+} z\gamma(\mathrm{d}z, \mathrm{d}u),$$
$$C^a(t) = \sum_{i=1}^{N(t)} Z_i = \int_0^t \int_{R^+} (z \wedge a)\gamma(\mathrm{d}z, \mathrm{d}u). \tag{B.1}$$

The compensator ν of the random measure γ is

$$\nu(\mathrm{d}z, \mathrm{d}t) = \lambda \mathrm{d}F(z)\,\mathrm{d}t, \tag{B.2}$$

so the compensated Poisson random measure of $C(t)$ is

$$\widetilde{\gamma}(\mathrm{d}z, \mathrm{d}t) = \gamma(\mathrm{d}z, \mathrm{d}t) - \lambda \mathrm{d}F(z)\,\mathrm{d}t. \tag{B.3}$$

Set $\widetilde{V}(t) = V(t, X^{\alpha^*}(t), S(t))$. Applying Itô's formula to $\widetilde{V}^2(t)$,

$$\begin{aligned}
\mathrm{d}\widetilde{V}^2(t) &= 2\widetilde{V}(t)\Big\{V_x\sigma \mathrm{d}W(t) + kS^\beta(t) \\
&\quad \times \left[V_x\pi(t)X(t) + V_sS(t)\right]\mathrm{d}W_1(t) + \mathscr{A}\widetilde{V}(t)\Big\} \\
&\quad + \Big[V_x^2\sigma^2 + V_x^2\pi^2X(t)^2k^2S^{2\beta}(t) + V_s^2k^2S^{2\beta+2}(t) \\
&\quad + 2V_xV_s\pi(t)X(t)k^2S^{2\beta+1}(t)\Big]\mathrm{d}t \\
&\quad + \int_{R^+}\Big[V^2(t, X(t-) + (z \wedge a), S(t)) \\
&\quad - V^2(t, X(t-), S(t)) - 2V(t, X(t-), S(t)) \\
&\quad \cdot V_x \cdot (z \wedge a)\Big]\lambda \mathrm{d}F(z)\,\mathrm{d}t \\
&\quad + \int_{R^+}\Big[V^2(t, X(t-) + (z \wedge a), S(t)) \\
&\quad - V^2(t, X(t-), S(t))\Big]\widetilde{\gamma}(\mathrm{d}z, \mathrm{d}t).
\end{aligned} \tag{B.4}$$

Since α^* is the optimal strategy for (19), then $\mathscr{A}^{\alpha^*}\widetilde{V}(t) = 0$. Plugging the expression of V_x, V_s, and $a*(t)$ into (B.4), we obtain the following equation by simple calculations:

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{V}^2(t)}{\widetilde{V}^2(t)} &= \Bigg\{-2q\sigma e^{r(T-t)}\mathrm{d}W(t) - \frac{2(\mu - r)s^{-\beta}}{k} \\
&\quad \times \left[1 + \frac{\left(1 - e^{-2r\beta(T-t)}\right)(\mu - r)}{2r}\right]\mathrm{d}W_1(t) \\
&\quad + \frac{\left(1 - e^{-2r\beta(T-t)}\right)(\mu - r)^2 s^{-\beta}}{rk}\mathrm{d}W_1(t)\Bigg\} \\
&\quad + \Bigg\{q^2\sigma^2e^{2r(T-t)} \\
&\quad + \frac{(\mu - r)^2 s^{-2\beta}}{k^2}\left[1 + \frac{\left(1 - e^{-2r\beta(T-t)}\right)(\mu - r)}{2r}\right]^2 \\
&\quad + \frac{\left(1 - e^{-2r\beta(T-t)}\right)^2(\mu - r)^4 s^{-2\beta}}{4r^2k^2} \\
&\quad - \frac{\left(1 - e^{-2r\beta(T-t)}\right)(\mu - r)^3 s^{-2\beta}}{rk^2} \\
&\quad \times \left[1 + \frac{\left(1 - e^{-2r\beta(T-t)}\right)(\mu - r)}{2r}\right]\Bigg\}\mathrm{d}t
\end{aligned}$$

$$+\int_{R^+}\left[e^{-2q(z\wedge a^*)e^{r(T-t)}}+2q\left(z\wedge a^*\right)e^{r(T-t)}-1\right]\lambda \mathrm{d}F(z)\,\mathrm{d}t+\int_{R^+}\left[e^{-2q(z\wedge a^*)e^{r(T-t)}}-1\right]\widetilde{\gamma}(\mathrm{d}z,\mathrm{d}t). \tag{B.5}$$

The solution of (B.5) is

$$\begin{aligned}\frac{\widetilde{V}^2(t)}{\widetilde{V}^2(0)} &= \exp\left\{\int_0^t -2q\sigma e^{r(T-u)}\mathrm{d}W(u)\right.\\ &\quad -\frac{1}{2}\int_0^t 4q^2\sigma^2 e^{2r(T-u)}\mathrm{d}u\\ &\quad +\int_0^t -\frac{2(\mu-r)s^{-\beta}}{k}\times\left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]\mathrm{d}W_1(u)\\ &\quad -\frac{1}{2}\int_0^t \frac{4(\mu-r)^2 s^{-2\beta}}{k^2}\times\left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]^2\mathrm{d}u\\ &\quad +\int_0^t \frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)^2 s^{-\beta}}{rk}\mathrm{d}W_1(u)\\ &\quad -\frac{1}{2}\int_0^t \frac{\left(1-e^{-2r\beta(T-u)}\right)^2(\mu-r)^4 s^{-2\beta}}{r^2k^2}\mathrm{d}u\\ &\quad +\int_0^t 3q^2\sigma^2 e^{2r(T-u)}\mathrm{d}u\\ &\quad +\int_0^t \frac{3\left(1-e^{-2r\beta(T-u)}\right)^2(\mu-r)^4 s^{-2\beta}}{4r^2k^2}\mathrm{d}u\\ &\quad +\int_0^t \frac{3(\mu-r)^2 s^{-2\beta}}{k^2}\times\left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]^2\mathrm{d}u\\ &\quad +\int_0^t -\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)^3 s^{-2\beta}}{rk^2}\times\left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]\mathrm{d}u\\ &\quad +\int_0^t\int_{R^+}\left[e^{-2q(z\wedge a^*)e^{r(T-u)}}+2q\left(z\wedge a^*\right)e^{r(T-u)}-1\right]\lambda \mathrm{d}F(z)\,\mathrm{d}u\\ &\quad \left.+\int_0^t\int_{R^+}\left[e^{-2q(z\wedge a^*)e^{r(T-u)}}-1\right]\widetilde{\gamma}(\mathrm{d}z,\mathrm{d}u)\right\},\end{aligned} \tag{B.6}$$

where (x_0, s_0) is the state at time 0. Applying Itô's formula to (11), we can derive

$$\begin{aligned}\mathrm{d}S^{-2\beta}(t) &= \left(\beta(2\beta+1)k^2-2\beta r S^{-2\beta}(t)\right)\mathrm{d}t\\ &\quad -2\beta k\sqrt{S^{-2\beta}(t)}\mathrm{d}W_1(t).\end{aligned} \tag{B.7}$$

According to the results of Taksar and Zeng [16] and Gu et al. [5] and other existing literature, $\mu > r > 0$, we know that

$$\begin{aligned}&\exp\left\{\int_0^t -2q\sigma e^{r(T-u)}\mathrm{d}W(u)-\frac{1}{2}\int_0^t 4q^2\sigma^2 e^{2r(T-u)}\mathrm{d}u\right\}\\ &\exp\left\{\int_0^t -\frac{2(\mu-r)s^{-\beta}}{k}\times\left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]\mathrm{d}W_1(u)\right.\\ &\quad \left.-\frac{1}{2}\int_0^t \frac{4(\mu-r)^2 s^{-2\beta}}{k^2}\times\left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]^2\mathrm{d}u\right\},\\ &\exp\left\{\int_0^t \frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)^2 s^{-\beta}}{rk}\mathrm{d}W_1(u)\right.\\ &\quad \left.-\frac{1}{2}\int_0^t \frac{\left(1-e^{-2r\beta(T-u)}\right)^2(\mu-r)^4 s^{-2\beta}}{r^2k^2}\mathrm{d}u\right\}\end{aligned} \tag{B.8}$$

are martingales. Since $z \in [0, D]$, $D = \sup\{z : F(z) \le 1\} < +\infty$, and

$$E\left[\int_0^T\int_{R^+}\left[e^{-2q(z\wedge a^*)e^{r(T-u)}}-1\right]^2\lambda \mathrm{d}F(z)\,\mathrm{d}u\right] < \infty, \tag{B.9}$$

then the process

$$\int_0^t\int_{R^+}\left[e^{-2q(z\wedge a^*)e^{r(T-u)}}-1\right]\widetilde{\gamma}(\mathrm{d}z,\mathrm{d}u) \tag{B.10}$$

is also a martingale.

According to Taksar and Zeng [16],

$$E\left[\exp\left\{\int_0^t \frac{3(\mu-r)^2 s^{-2\beta}(u)}{k^2}\mathrm{d}u\right\}\right] < \infty \tag{B.11}$$

when either of conditions (I) and (II) in Theorem 3 is satisfied. Taking expectation from both sides of (B.6) yields

$$\begin{aligned}
&E\left[\widetilde{V}^2(t)\right] \\
&\quad = \widetilde{V}^2(0)\, E \\
&\qquad \times \left[\exp\left\{\int_0^t 3q^2\sigma^2 e^{2r(T-u)}\mathrm{d}u\right.\right. \\
&\qquad\qquad + \int_0^t \frac{3\left(1-e^{-2r\beta(T-u)}\right)^2(\mu-r)^4 s^{-2\beta}(u)}{4r^2k^2}\mathrm{d}u \\
&\qquad\qquad + \int_0^t \frac{3(\mu-r)^2 s^{-2\beta}(u)}{k^2} \\
&\qquad\qquad\qquad \times \left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]^2\mathrm{d}u \\
&\qquad\qquad + \int_0^t -\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)^3 s^{-2\beta}(u)}{rk^2} \\
&\qquad\qquad\qquad \times \left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]\mathrm{d}u \\
&\qquad\qquad + \int_0^t \int_{R^+} \left[e^{-2q(z\wedge a^*)e^{r(T-u)}}\right. \\
&\qquad\qquad\qquad \left. + 2q(z\wedge a^*)e^{r(T-u)} - 1\right] \\
&\qquad\qquad\qquad \left.\left. \times \lambda \mathrm{d}F(z)\,\mathrm{d}u\right\}\right].
\end{aligned} \tag{B.12}$$

If $\beta \geq 0$, then $0 \leq (1-e^{-2r\beta(T-u)}) < 1$. For $\mu > r > 0$ and $T > 0$ are constants, we have

$$\begin{aligned}
&\exp\left\{\int_0^t \frac{3(\mu-r)^2 s^{-2\beta}(u)}{k^2}\right. \\
&\qquad \left. \times \left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]^2\mathrm{d}u\right\} \\
&\quad < \exp\left\{\int_0^t \frac{3(\mu-r)^2 s^{-2\beta}(u)}{k^2}\left[1+\frac{(\mu-r)}{2r}\right]^2\mathrm{d}u\right\} \\
&\quad = \exp\left\{\left[1+\frac{(\mu-r)}{2r}\right]^2\int_0^t \frac{3(\mu-r)^2 s^{-2\beta}(u)}{k^2}\mathrm{d}u\right\}, \\
&\exp\left\{\int_0^t \frac{3\left(1-e^{-2r\beta(T-u)}\right)^2(\mu-r)^4 s^{-2\beta}(u)}{4r^2k^2}\mathrm{d}u\right\} \\
&\quad < \exp\left\{\int_0^t \frac{3(\mu-r)^4 s^{-2\beta}(u)}{4r^2k^2}\mathrm{d}u\right\}, \\
&\exp\left\{\int_0^t \left| -\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)^3 s^{-2\beta}(u)}{rk^2}\right.\right. \\
&\qquad \left.\left. \times \left[1+\frac{\left(1-e^{-2r\beta(T-u)}\right)(\mu-r)}{2r}\right]\right|\mathrm{d}u\right\} \\
&\quad < \exp\left\{\int_0^t \frac{(\mu-r)^3 s^{-2\beta}(u)}{rk^2}\left[1+\frac{(\mu-r)}{2r}\right]\mathrm{d}u\right\}.
\end{aligned} \tag{B.13}$$

For the case of $\beta < 0$, $|1-e^{-2r\beta(T-u)}| < e^{-2r\beta T}$, which is similar. After simple calculation and analysis, we obtain

$$E\left[V^2\left(t, X^{\alpha^*}(t), S(t)\right)\right] = E\left[\widetilde{V}^2(t)\right] < \infty. \tag{B.14}$$

So $E[V^2(\tau_i\wedge T, X^{\alpha^*}(\tau_i\wedge T), S(\tau_i\wedge T))] < \infty$ for $i = 1, 2, \ldots$. □

We use Lemma B.1 to prove the verification Theorem 3 as follows.

Proof. By Itô's formula for Itô-Lévy process, we have

$$\begin{aligned}
&\mathrm{d}V\left(t, X^\alpha(t), S(t)\right) \\
&\quad = \mathscr{A}^\alpha V\left(t, X^\alpha(t), S(t)\right) + V_x\sigma\mathrm{d}W(t) \\
&\qquad + kS^\beta(t)\left[V_x\pi(t)X(t) + V_sS(t)\right]\mathrm{d}W_1(t) \\
&\qquad + \int_{R^+}\left[V(t, X(t-) + (z\wedge a), S(t))\right. \\
&\qquad\qquad \left. - V(t, X(t-), S(t))\right]\widetilde{\gamma}(\mathrm{d}z, \mathrm{d}t).
\end{aligned} \tag{B.15}$$

Take a sequence of bounded open sets $Q_1, Q_2, Q_3, \ldots$, with $Q_i \subset Q_{i+1} \subset Q$, $i = 1, 2, \ldots$. For $(x, s) \in Q_1$, let τ_i denote the exit time of (x, s) from Q_i. Then $\tau_i \wedge T \to T$ when $i \to \infty$. Integrating both sides of the above equation,

$$\begin{aligned}
&V\left(\tau_i\wedge T, X^\alpha(\tau_i\wedge T), S(\tau_i\wedge T)\right) \\
&\quad = V(t, x, s) + \int_t^{\tau_i\wedge T}\mathscr{A}^\alpha V\left(u, X^\alpha(u), S(u)\right)\mathrm{d}u \\
&\qquad + \int_t^{\tau_i\wedge T} V_x\sigma\mathrm{d}W(u) \\
&\qquad + \int_t^{\tau_i\wedge T} kS^\beta(u)\left[V_x\pi x + V_sS(u)\right]\mathrm{d}W_1(u) \\
&\qquad + \int_t^{\tau_i\wedge T}\int_{R^+}\left[V(u, X(u-) + (z\wedge a), S(u))\right. \\
&\qquad\qquad \left. - V(u, X(u-), S(u))\right]\widetilde{\gamma}(\mathrm{d}z, \mathrm{d}u).
\end{aligned} \tag{B.16}$$

Since $\sup_{\alpha\in\Lambda}\{\mathscr{A}^{\alpha}V(t,x,s)\} = 0$, then $\mathscr{A}^{\alpha}V(t,x,s) < 0$. The last three terms of (B.16) are square-integrable martingales with zero expectation, taking conditional expectation given (t,x,s) on two sides of the above formula:

$$\begin{aligned} &E\left[V\left(\tau_i\wedge T, X^{\alpha}\left(\tau_i\wedge T\right), S\left(\tau_i\wedge T\right)\right)\right. \\ &\quad \left.\mid X^{\alpha}(t)=x, S(t)=s\right] \\ &\quad = V(t,x,s) \\ &\qquad + E\left[\int_t^{\tau_i\wedge T}\mathscr{A}^{\alpha}V\left(u, X^{\alpha}(u), S(u)\right)\mathrm{d}u\right. \\ &\qquad\qquad \left.\mid X^{\alpha}(t)=x, S(t)=s\right] \\ &\quad \le V(t,x,s). \end{aligned} \tag{B.17}$$

From Lemma B.1, $V(\tau_i\wedge T, X^{\alpha}(\tau_i\wedge T), S(\tau_i\wedge T))$, $i = 1,2,\ldots$, are uniformly integrable. Thus we have

$$\begin{aligned} &H(t,x,s) \\ &\quad = \sup_{\alpha\in\Lambda} E\left[U(X(T)) \mid X^{\alpha^*}(t)=x, S(t)=s\right] \\ &\quad = \lim_{t\to\infty} E\left[V\left(\tau_i\wedge T, X^{\alpha}\left(\tau_i\wedge T\right), S\left(\tau_i\wedge T\right)\right)\right] \\ &\quad \le V(t,x,s). \end{aligned} \tag{B.18}$$

When $\alpha = \alpha^*$, the inequality in the above formula becomes an equality; that is, $H(t,x,s) = V(t,x,s)$. The proof is finished. □

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant nos. 11201335 and 11301376) and Humanity and Social Science Foundation of Ministry of Education of China (Grant no. 11YJC910007).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 547490, 10 pages
http://dx.doi.org/10.1155/2014/547490

Research Article

A Priori Error Estimates of Mixed Finite Element Methods for General Linear Hyperbolic Convex Optimal Control Problems

Zuliang Lu[1,2] and Xiao Huang[3]

[1] *School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing 404000, China*
[2] *Laboratory for Applied Mathematics, Beijing Computational Science Research Center, Beijing 100084, China*
[3] *College of Electronic and Information Engineering, Chongqing Three Gorges University, Chongqing 404000, China*

Correspondence should be addressed to Zuliang Lu; zulianglux@126.com

Received 21 March 2014; Accepted 16 June 2014; Published 21 July 2014

Academic Editor: Manuel Ruiz-Galan

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The aim of this work is to investigate the discretization of general linear hyperbolic convex optimal control problems by using the mixed finite element methods. The state and costate are approximated by the k order ($k \geq 0$) Raviart-Thomas mixed finite elements and the control is approximated by piecewise polynomials of order k. By applying the elliptic projection operators and Gronwall's lemma, we derive a priori error estimates of optimal order for both the coupled state and the control approximation.

1. Introduction

With the advances of scientific computing, optimal control problems are now widely used in multidisciplinary applications such as physics, biology, medicine, engineering design, fluid mechanics, and social-economic systems. The finite element method is undoubtedly the most widely used numerical method in computing optimal control problems. Finite element approximation of a class of elliptic optimal control problems has been studied by Falk in [1]. Then, Alt and Mackenroth in [2] established a priori error estimates for the finite element approximations to state constrained convex parabolic boundary control problems. Finite element approximation of optimal control problems was developed in [3–16], but there are very less published results on this topic for hyperbolic optimal control problems.

Since the pioneering work of Brezzi and Fortin [17], the mixed finite element methods to second order elliptic problems have drawn the attention of many specialists in partial differential equations. Mixed finite elements are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy. In finite element methods, mixed finite element methods were widely used to approximate flux variables, although there was only very limited research work on analyzing such elements for optimal control problems. More recently, in [9], the authors derived a priori error estimates and superconvergence for bilinear quadratic optimal control problems using mixed finite element methods. A posteriori error analysis of mixed finite element methods for some optimal control problems was addressed in [18, 19]. In [20], the author discussed the semidiscrete mixed finite element methods for quadratic hyperbolic optimal control problems. By using mixed elliptic reconstruction methods, he obtained a posteriori $L^\infty(L^2)$-error estimates for both the state and the control approximation.

The purpose of this work is to obtain a priori error estimates of mixed finite element methods for general convex optimal control problems governed by linear hyperbolic partial differential equations. Analogous a priori error estimates of mixed finite element solutions for optimal control problems governed by linear parabolic equations can be found in [21]. However, it does not seem to be straightforward to extend the existing techniques to general optimal control problems involving hyperbolic equations.

For $1 \leq p < \infty$ and m any nonnegative integer, let $W^{m,p}(\Omega) = \{v \in L^p(\Omega); D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \leq m\}$ denote the Sobolev spaces endowed with the norm $\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$ and the seminorm $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$,

$H_0^m(\Omega) = W_0^{m,2}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We denote by $L^s(J; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J to $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = (\int_0^T \|v\|^s_{W^{m,p}(\Omega)} dt)^{1/s}$ for $s \in [1,\infty)$, $\|v\|_{L^\infty(J;W^{m,p}(\Omega))} = \sup_{t\in J}\|v\|_{W^{m,p}(\Omega)}$, and the standard modification for $s = \infty$.

In this paper, we focus our attention on the following general linear hyperbolic convex optimal control problems:

$$\min_{u\in K\subset U} \left\{ \int_0^T \left(g_1(\mathbf{p}) + g_2(y) + h(u)\right) dt \right\}, \tag{1}$$

subject to the state equations

$$y_{tt} + \operatorname{div}\mathbf{p} = f + u, \quad x \in \Omega,\ t \in J, \tag{2}$$

$$\mathbf{p} = -A\nabla y, \quad x \in \Omega,\ t \in J, \tag{3}$$

$$y|_{\partial\Omega} = 0, \quad t \in J, \tag{4}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{5}$$

$$y_t(x,0) = y_1(x), \quad x \in \Omega, \tag{6}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$ is a convex polygon with the boundary $\partial\Omega$, Ω_U is a bounded open set in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega_U$, g_1, g_2, and h are convex functionals, and $J = [0,T]$. We assume that K is a closed convex set in $U = L^2(J; L^2(\Omega_U))$, $f \in L^2(J; L^2(\Omega))$, and $y_0, y_1 \in H^1(\Omega)$. Furthermore, we assume the coefficient matrix $A(x) = (a_{i,j}(x))_{2\times 2} \in (W^{1,\infty}(\Omega))^{2\times 2}$ is a symmetric 2×2-matrix and there is a constant $c > 0$ satisfying for any vector $\mathbf{X} \in \mathbb{R}^2$, $\mathbf{X}'A\mathbf{X} \geq c\|\mathbf{X}\|^2_{\mathbb{R}^2}$. The set of admissible controls K is defined by

$$K = \left\{ u \in U = L^2\left(J; L^2\left(\Omega_U\right)\right) : \int_0^T \int_{\Omega_U} u\, dx\, dt \geq 0 \right\}. \tag{7}$$

The remainder of the paper is organized as follows. In Section 2, we construct the k order Raviart-Thomas mixed finite element approximation for general convex optimal control problems governed by linear hyperbolic equations and briefly state the definitions and properties of some interpolation operators. In Section 3, we derive a priori error estimates of the mixed finite element solutions for the general hyperbolic optimal control problems. Finally, we give the conclusion and the future work in Section 4.

2. Mixed Methods of Hyperbolic Optimal Control

We will now describe the mixed finite element discretization of general linear hyperbolic convex optimal control problems (1)–(6). Firstly, we introduce the costate hyperbolic equation,

$$z_{tt}(x,t) - \operatorname{div}\left(A\left(\nabla z(x,t) + g_1'(\mathbf{p}(x,t))\right)\right) = g_2'(y(x,t)), \quad x \in \Omega, \tag{8}$$

with the conditions,

$$z|_{\partial\Omega} = 0, \quad t \in J; \quad z(x,T) = 0, \quad x \in \Omega; \quad z_t(x,T) = 0, \quad x \in \Omega. \tag{9}$$

Next, we need the following regularity assumptions for the hyperbolic equations (2) and (8): there exists a constant C such that

$$\begin{gathered} \|y\|_{L^\infty(J;H^{k+2}(\Omega))} + \|y_t\|_{L^\infty(J;H^{k+2}(\Omega))} + \|y_{tt}\|_{L^2(J;H^{k+2}(\Omega))} \leq C, \\ \|z\|_{L^\infty(J;H^{k+2}(\Omega))} + \|z_t\|_{L^\infty(J;H^{k+2}(\Omega))} \leq C, \\ \|\mathbf{p}\|_{L^\infty(J;(H^{k+2}(\Omega))^2)} + \|\mathbf{q}\|_{L^\infty(J;(H^{k+2}(\Omega))^2)} \\ + \|\mathbf{q}_t\|_{L^\infty(J;(H^{k+1}(\Omega))^2)} \leq C. \end{gathered} \tag{10}$$

We will take the state spaces $L^2(\mathbf{V}) = L^2(J;\mathbf{V})$ and $L^2(W) = L^2(J;W)$, where $\mathbf{V}$ and W are defined as follows:

$$\begin{aligned} \mathbf{V} &= H(\operatorname{div};\Omega) = \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^2, \operatorname{div}\mathbf{v} \in L^2(\Omega)\right\}, \\ W &= L^2(\Omega). \end{aligned} \tag{11}$$

The Hilbert space $\mathbf{V}$ is equipped with the following norm:

$$\|\mathbf{v}\|_{H(\operatorname{div};\Omega)} = \left(\|\mathbf{v}\|^2_{0,\Omega} + \|\operatorname{div}\mathbf{v}\|^2_{0,\Omega}\right)^{1/2}. \tag{12}$$

We recast (1)–(5) as the following weak form: find $(\mathbf{p}, y, u) \in L^2(\mathbf{V}) \times L^2(W) \times K$ such that

$$\begin{gathered} \min_{u\in K} \left\{ \int_0^T \left(g_1(\mathbf{p}) + g_2(y) + h(u)\right) dt \right\}, \\ \left(A^{-1}\mathbf{p}, \mathbf{v}\right) - (y, \operatorname{div}\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \\ (y_{tt}, w) + (\operatorname{div}\mathbf{p}, w) = (f + u, w), \quad \forall w \in W, \\ y(x,0) = y_0(x), \quad \forall x \in \Omega, \\ y_t(x,0) = y_1(x), \quad \forall x \in \Omega. \end{gathered} \tag{13}$$

Hereafter, we assume that $h(u) = \int_{\Omega_U} j(u)dx$, where $j(\cdot)$ is a convex continuously differentiable function on $\mathbb{R}$. Then, it is easy to see that $(h'(u), \mathbf{v})_U = (j'(u), \mathbf{v})_U = \int_{\Omega_U} j'(u)\mathbf{v}dx$.

Taking into account the precious result in [20, 22], the optimal control problem (13) has a unique solution $(\mathbf{p}, y, u)$, and a triplet $(\mathbf{p}, y, u)$ is the solution of (13) if and only if there

is a costate $(\mathbf{q}, z) \in L^2(\mathbf{V}) \times L^2(W)$ such that $(\mathbf{p}, y, \mathbf{q}, z, u)$ satisfies the following optimality conditions:

$$\left(A^{-1}\mathbf{p}, \mathbf{v}\right) - \left(y, \operatorname{div}\mathbf{v}\right) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{14}$$

$$\left(y_{tt}, w\right) + \left(\operatorname{div}\mathbf{p}, w\right) = \left(f + u, w\right), \quad \forall w \in W, \tag{15}$$

$$y(x, 0) = y_0(x), \quad \forall x \in \Omega, \tag{16}$$

$$y_t(x, 0) = y_1(x), \quad \forall x \in \Omega, \tag{17}$$

$$\left(A^{-1}\mathbf{q}, \mathbf{v}\right) - \left(z, \operatorname{div}\mathbf{v}\right) = -\left(g_1'(\mathbf{p}), \mathbf{v}\right), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{18}$$

$$\left(z_{tt}, w\right) + \left(\operatorname{div}\mathbf{q}, w\right) = \left(g_2'(y), w\right), \quad \forall w \in W, \tag{19}$$

$$z(x, T) = 0, \quad \forall x \in \Omega, \tag{20}$$

$$z_t(x, T) = 0, \quad \forall x \in \Omega, \tag{21}$$

$$\int_0^T \left(j'(u) + z, \tilde{u} - u\right)_U dt \geq 0, \quad \forall \tilde{u} \in K, \tag{22}$$

where $(\cdot,\cdot)_U$ is the inner product of U and g_1', g_2', and j' are the derivatives of g_1, g_2, and j. For simplification, the product $(\cdot,\cdot)_U$ will be denoted by $(\cdot,\cdot)$.

For ease of exposition, we will assume that Ω and Ω_U are both polygons. Let $\mathcal{T}_h$ and $\mathcal{T}_h(\Omega_U)$ be regular triangulations or rectangulations of Ω and Ω_U, respectively. They are assumed to satisfy the angle condition which means that there is a positive constant C such that, for all $\tau \in \mathcal{T}_h(\tau_U \in \mathcal{T}_h(\Omega_U))$, $C^{-1}h_\tau^2 \leq |\tau| \leq Ch_\tau^2$, $C^{-1}h_{\tau_U}^2 \leq |\tau_U| \leq Ch_{\tau_U}^2$, where $|\tau|$ is the area of τ, $|\tau_U|$ is the area of τ_U, h_τ is the diameter of τ, and h_{τ_U} is the diameter of τ_U. Let $h = \max h_\tau (h_U = \max h_{\tau_U})$. In addition, C or c denotes a general positive constant independent of h.

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the order k Raviart-Thomas space [23] associated with the triangulations or rectangulations $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree at most k and $Q_{m,n}$ indicates the space of polynomials of degree no more than m and n in x and y, respectively. If τ is a triangle, $\mathbf{V}(\tau) = \{\mathbf{v} \in P_k^2(\tau) + x \cdot P_k(\tau)\}$, and if τ is a rectangle, $\mathbf{V}(\tau) = \{\mathbf{v} \in Q_{k+1,k}(\tau) \times Q_{k,k+1}(\tau)\}$, $W(\tau) = P_k(\tau)$. We define

$$\begin{aligned} \mathbf{V}_h &:= \left\{\mathbf{v}_h \in \mathbf{V} : \forall \tau \in \mathcal{T}_h, \mathbf{v}_h|_\tau \in \mathbf{V}(\tau)\right\}, \\ W_h &:= \left\{w_h \in W : \forall \tau \in \mathcal{T}_h, w_h|_\tau \in W(\tau)\right\}, \\ U_h &:= \left\{\tilde{u}_h \in U : \forall \tau \in \mathcal{T}_h(\Omega_U), \tilde{u}_h|_\tau \in W(\tau)\right\}. \end{aligned} \tag{23}$$

By the definition of finite element subspace, the mixed finite element discretization of (13) is as follows: compute $(\mathbf{p}_h, y_h, u_h) \in L^2(\mathbf{V}_h) \times L^2(W_h) \times K_h$ such that

$$\begin{gathered} \min_{u_h \in K_h \subset U_h} \left\{\int_0^T \left(g_1(\mathbf{p}_h) + g_2(y_h) + h(u_h)\right) dt\right\}, \\ \left(A^{-1}\mathbf{p}_h, \mathbf{v}_h\right) - \left(y_h, \operatorname{div}\mathbf{v}_h\right) = 0, \quad \forall v_h \in V_h, \\ \left(y_{htt}, w_h\right) + \left(\operatorname{div}\mathbf{p}_h, w_h\right) = \left(f + u_h, w_h\right), \quad \forall w_h \in W_h, \\ y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \\ y_{ht}(x, 0) = y_1^h(x), \quad \forall x \in \Omega, \end{gathered} \tag{24}$$

where $K_h = U_h \cap K$ and $y_0^h(x)$ and $y_1^h(x) \in W_h$ are two finite element approximations of $y_0(x)$ and $y_1(x)$.

It is well known (see, e.g., [7, 20]) that the optimal control problem (24) again has a unique solution $(\mathbf{p}_h, y_h, u_h)$ and that a triplet $(\mathbf{p}_h, y_h, u_h)$ is the solution of (24) if and only if there is a costate $(\mathbf{q}_h, z_h) \in L^2(\mathbf{V}_h) \times L^2(W_h)$ such that $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$\left(A^{-1}\mathbf{p}_h, \mathbf{v}\right) - \left(y_h, \operatorname{div} v\right) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{25}$$

$$\left(y_{htt}, w\right) + \left(\operatorname{div}\mathbf{p}_h, w\right) = \left(f + u_h, w\right), \quad \forall w \in W_h, \tag{26}$$

$$y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{27}$$

$$y_{ht}(x, 0) = y_1^h(x), \quad \forall x \in \Omega, \tag{28}$$

$$\left(A^{-1}\mathbf{q}_h, \mathbf{v}\right) - \left(z_h, \operatorname{div}\mathbf{v}\right) = -\left(g_1'(\mathbf{p}_h), \mathbf{v}\right), \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{29}$$

$$\left(z_{htt}, w\right) + \left(\operatorname{div}\mathbf{q}_h, w\right) = \left(g_2'(y_h), w\right), \quad \forall w \in W_h, \tag{30}$$

$$z_h(x, T) = 0, \quad \forall x \in \Omega, \tag{31}$$

$$z_{ht}(x, T) = 0, \quad \forall x \in \Omega, \tag{32}$$

$$\int_0^T \left(j'(u_h) + z_h, \tilde{u} - u_h\right)_U dt \geq 0, \quad \forall \tilde{u} \in K_h. \tag{33}$$

Let $P_h : W \to W_h$ be the orthogonal $L^2(\Omega)$-projection into W_h defined by

$$\left(P_h w - w, \chi\right) = 0, \quad w \in W, \ \chi \in W_h, \tag{34}$$

which satisfies

$$\begin{gathered} \left\|P_h w - w\right\|_{0,q} \leq C\|w\|_{s,q} h^s, \\ 0 \leq s \leq k+1, \quad \text{if } w \in W \cap W^{s,q}(\Omega), \end{gathered} \tag{35}$$

$$\begin{gathered} \left\|P_h w - w\right\|_{-r} \leq C\|w\|_s h^{r+s}, \\ 0 \leq r, \quad s \leq k+1, \quad \text{if } w \in H^s(\Omega), \end{gathered} \tag{36}$$

$$\left(\operatorname{div}\mathbf{v}, w - P_h w\right) = 0, \quad w \in W, \ \mathbf{v} \in \mathbf{V}_h. \tag{37}$$

Let $\pi_h : \mathbf{V} \to \mathbf{V}_h$ be the Raviart-Thomas projection [24], which satisfies

$$\left(\operatorname{div}\left(\pi_h \mathbf{v} - \mathbf{v}\right), w_h\right) = 0, \quad \mathbf{v} \in \mathbf{V}, \ w_h \in W_h, \tag{38}$$

$$\begin{gathered} \left\|\pi_h \mathbf{v} - \mathbf{v}\right\|_{0,q} \leq C\|\mathbf{v}\|_{s,q} h^s, \\ \frac{1}{q} < s \leq k+1, \quad \text{if } \mathbf{v} \in \mathbf{V} \cap W^{s,q}(\Omega)^2, \end{gathered} \tag{39}$$

$$\begin{gathered} \left\|\operatorname{div}\left(\pi_h \mathbf{v} - \mathbf{v}\right)\right\|_0 \leq C\|\operatorname{div}\mathbf{v}\|_s h^s, \\ 0 \leq s \leq k+1, \quad \text{if } \mathbf{v} \in \mathbf{V} \cap H^s(\operatorname{div}; \Omega). \end{gathered} \tag{40}$$

We have the commuting diagram property

$$\operatorname{div}\circ\pi_h = P_h\circ\operatorname{div}:\mathbf{V}\longrightarrow W_h,\quad \operatorname{div}(I-\pi_h)\mathbf{V}\perp W_h, \tag{41}$$

where I denotes identity matrix. We point out $(\pi_h\mathbf{v})_t = \pi_h\mathbf{v}_t$ and $(P_h w)_t = P_h w_t$.

In the rest of the paper, we will use some intermediate variables. For any control function $\tilde{u}\in K$, we first define the state solution $(\mathbf{p}(\tilde{u}), y(\tilde{u}), \mathbf{q}(\tilde{u}), z(\tilde{u}))$ associated with $\tilde{u}$ that satisfies

$$\left(A^{-1}\mathbf{p}(\tilde{u}),\mathbf{v}\right)-\left(y(\tilde{u}),\operatorname{div}\mathbf{v}\right)=0,\quad \forall\mathbf{v}\in\mathbf{V}, \tag{42}$$

$$\left(y_{tt}(\tilde{u}),w\right)+\left(\operatorname{div}\mathbf{p}(\tilde{u}),w\right)=\left(f+\tilde{u},w\right),\quad \forall w\in W, \tag{43}$$

$$y(\tilde{u})(x,0)=y_0(x),\quad \forall x\in\Omega, \tag{44}$$

$$y_t(\tilde{u})(x,0)=y_1(x),\quad \forall x\in\Omega, \tag{45}$$

$$\left(A^{-1}\mathbf{q}(\tilde{u}),\mathbf{v}\right)-\left(z(\tilde{u}),\operatorname{div}\mathbf{v}\right)=-\left(g_1'(\mathbf{p}(\tilde{u})),\mathbf{v}\right),\quad \forall\mathbf{v}\in\mathbf{V}, \tag{46}$$

$$\left(z_{tt}(\tilde{u}),w\right)+\left(\operatorname{div}\mathbf{q}(\tilde{u}),w\right)=\left(g_2'(y(\tilde{u})),w\right),\quad \forall w\in W, \tag{47}$$

$$z(\tilde{u})(x,T)=0,\quad \forall x\in\Omega, \tag{48}$$

$$z_t(\tilde{u})(x,T)=0,\quad \forall x\in\Omega. \tag{49}$$

Correspondingly, we define the discrete state solution $(\mathbf{p}_h(\tilde{u}), y_h(\tilde{u}), \mathbf{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u}\in K$ that satisfies

$$\left(A^{-1}\mathbf{p}_h(\tilde{u}),\mathbf{v}\right)-\left(y_h(\tilde{u}),\operatorname{div}\mathbf{v}\right)=0,\quad \forall\mathbf{v}\in\mathbf{V}_h, \tag{50}$$

$$\left(y_{htt}(\tilde{u}),w\right)+\left(\operatorname{div}\mathbf{p}_h(\tilde{u}),w\right)=\left(f+\tilde{u},w\right),\quad \forall w\in W_h, \tag{51}$$

$$y_h(\tilde{u})(x,0)=y_0^h(x),\quad \forall x\in\Omega, \tag{52}$$

$$y_{ht}(\tilde{u})(x,0)=y_1^h(x),\quad \forall x\in\Omega, \tag{53}$$

$$\left(A^{-1}\mathbf{q}_h(\tilde{u}),\mathbf{v}\right)-\left(z_h(\tilde{u}),\operatorname{div}\mathbf{v}\right)=-\left(g_1'(\mathbf{p}_h(\tilde{u})),\mathbf{v}\right),\quad \forall\mathbf{v}\in\mathbf{V}_h, \tag{54}$$

$$\left(z_{htt}(\tilde{u}),w\right)+\left(\operatorname{div}\mathbf{q}_h(\tilde{u}),w\right)=\left(g_2'(y_h(\tilde{u})),w\right),\quad \forall w\in W_h, \tag{55}$$

$$z_h(\tilde{u})(x,T)=0,\quad \forall x\in\Omega, \tag{56}$$

$$z_{ht}(\tilde{u})(x,T)=0,\quad \forall x\in\Omega. \tag{57}$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$\begin{aligned}(\mathbf{p},y,\mathbf{q},z)&=\left(\mathbf{p}(u),y(u),\mathbf{q}(u),z(u)\right),\\(\mathbf{p}_h,y_h,\mathbf{q}_h,z_h)&=\left(\mathbf{p}_h(u_h),y_h(u_h),\mathbf{q}_h(u_h),z_h(u_h)\right).\end{aligned} \tag{58}$$

In the following, we further assume that g_1', g_2', and j' are locally Lipschitz continuous, that $g_1''(\cdot)$ and $g_2''(\cdot)$ are bound functions on $(L^2(\Omega))^2$ and $L^2(\Omega)$, and that there is a $c>0$ such that

$$\left(j'(u)-j'(v),u-v\right)\geq c\|u-v\|^2_{L^2(\Omega_U)},\quad \forall u,v\in L^2(\Omega_U). \tag{59}$$

For $\varphi\in W_h$, we will write

$$\phi(\varphi)-\phi(\rho)=\tilde{\phi}'(\varphi)(\varphi-\rho), \tag{60}$$

where $\tilde{\phi}'(\varphi)=\int_0^1\phi'(\varphi+s(\rho-\varphi))ds$ is bounded function in $\overline{\Omega}$ [25].

3. A Priori Error Estimates

Now we will construct an analogue of the family of elliptic projection operators defined by Wheeler [26] in her thesis. Let $(\mathbf{p},y,\mathbf{q},z)$ be the solution of (14)–(18). Then, define the elliptic projection of $(\mathbf{p},y,\mathbf{q},z)$ to be (P,Y,Q,Z) by the following relations:

$$\left(A^{-1}P,\mathbf{v}_h\right)-\left(Y,\operatorname{div}\mathbf{v}_h\right)=0,\quad \forall\mathbf{v}_h\in\mathbf{V}_h, \tag{61}$$

$$\left(\operatorname{div}P,w_h\right)=\left(f+u-y_{tt},w_h\right),\quad \forall w_h\in W_h, \tag{62}$$

$$\left(A^{-1}Q,\mathbf{v}_h\right)-\left(Z,\operatorname{div}\mathbf{v}_h\right)=-\left(g_1'(\mathbf{p}),\mathbf{v}_h\right),\quad \forall\mathbf{v}_h\in\mathbf{V}_h, \tag{63}$$

$$\left(\operatorname{div}Q,w_h\right)=\left(g_2'(y)-z_{tt},w_h\right),\quad \forall w_h\in W_h, \tag{64}$$

where we assume that $Z(x,T)=Z_t(x,T)=0$.

Let

$$\begin{aligned}\tau_1&=y-Y,\qquad \sigma_1=\mathbf{p}-P,\\ \tau_2&=z-Z,\qquad \sigma_2=\mathbf{q}-Q.\end{aligned} \tag{65}$$

From (14)–(18) and (61)–(64), we can easily derive the following error equations:

$$\begin{aligned}\left(A^{-1}\sigma_1,\mathbf{v}_h\right)-\left(\tau_1,\operatorname{div}\mathbf{v}_h\right)&=0,\quad \forall\mathbf{v}_h\in\mathbf{V}_h,\\ \left(\operatorname{div}\sigma_1,w_h\right)&=0,\quad \forall w_h\in W_h,\\ \left(A^{-1}\sigma_2,\mathbf{v}_h\right)-\left(\tau_2,\operatorname{div}\mathbf{v}_h\right)&=0,\quad \forall\mathbf{v}_h\in\mathbf{V}_h,\\ \left(\operatorname{div}\sigma_2,w_h\right)&=0,\quad \forall w_h\in W_h.\end{aligned} \tag{66}$$

Estimates for τ_1, τ_2, σ_1, σ_2 are given in [27] and are presented in Lemma 1 without proof.

Lemma 1. *Assume that the optimal control problems* (1)–(5) *have a unique solution* $(\mathbf{p}, y, u)$. *For* $t \in J$ *and for* h *sufficiently small, there is a positive constant* C *independent of* h *such that*

$$\begin{aligned} \|\sigma_1\|_0 + \|\tau_1\|_0 &\le Ch^{k+1}\|y\|_{k+2}, \quad \text{if } y \in H^{k+2}(\Omega), \\ \|\sigma_2\|_0 + \|\tau_2\|_0 &\le Ch^{k+1}\|y\|_{k+2}, \quad \text{if } y \in H^{k+2}(\Omega). \end{aligned} \tag{67}$$

By using Lemma 3 in [22], we can obtain the following technical results.

Lemma 2. *For* $t \in J$ *and for* h *sufficiently small, if* $y_t, y_{tt}, y_{ttt} \in H^{k+2}(\Omega)$, *there is a positive constant* C *independent of* h *such that*

$$\begin{aligned} \|\sigma_{1t}\|_0 + \|\tau_{1t}\|_0 &\le Ch^{k+1}\|y_t\|_{k+2}, \\ \|\sigma_{1tt}\|_0 + \|\tau_{1tt}\|_0 &\le Ch^{k+1}\|y_{tt}\|_{k+2}, \\ \|\sigma_{1ttt}\|_0 + \|\tau_{1ttt}\|_0 &\le Ch^{k+1}\|y_{ttt}\|_{k+2}. \end{aligned} \tag{68}$$

By Theorem 3 in [28], we can establish the following useful result.

Lemma 3. *Suppose* $v \in L^2(J; H^1(\Omega)) \cap H^1(J; H^1(\Omega)^*)$. *Then,*

$$v \in C\left(J; L^2(\Omega)\right), \quad \frac{d}{dt}\|v(t)\|_0^2 = 2\left(v'(t), v(t)\right), \quad \forall t \in J. \tag{69}$$

Now, we investigate the intermediate error estimates between $(\mathbf{p}, y, \mathbf{q}, z)$ and the intermediate solution $(\mathbf{p}_h(u), y_h(u), \mathbf{q}_h(u), z_h(u))$. Benefit from the previous results in this section, we only need to estimate $\|P - \mathbf{p}_h(u)\|$, $\|Y - y_h(u)\|$ and $\|P_h\mathbf{q} - \mathbf{q}_h(u)\|$, $\|\pi_h z - z_h(u)\|$.

Let

$$\begin{aligned} \alpha_1 &= Y - y_h(u), \qquad \beta_1 = P - \mathbf{p}_h(u), \\ \alpha_2 &= P_h z - z_h(u), \qquad \beta_2 = \pi_h\mathbf{q} - \mathbf{q}_h(u). \end{aligned} \tag{70}$$

Lemma 4. *Assume that the optimal control problems* (1)–(5) *have a unique solution* $(\mathbf{p}, y, u)$ *and that* Ω *is 2-regular. Assume that the regularity assumptions* (10) *are valid. There is a positive constant* $C > 0$, *independent of* h, *such that*

$$\|P - \mathbf{p}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)} + \|Y - y_h(u)\|_{L^\infty(J;L^2(\Omega))} \le Ch^{k+1}, \tag{71}$$

$$\|\pi_h\mathbf{q} - \mathbf{q}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)} + \|P_h z - z_h(u)\|_{L^\infty(J;L^2(\Omega))} \le Ch^{k+1}. \tag{72}$$

Proof. Firstly, we prove the first inequality (71). From (61)-(62) and (50)-(51), we can derive the following error equations:

$$\left(A^{-1}\beta_1, \mathbf{v}\right) - \left(\alpha_1, \operatorname{div}\mathbf{v}\right) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{73}$$

$$\left(\alpha_{1tt}, w\right) + \left(\operatorname{div}\beta_1, w\right) = -\left((y - Y)_{tt}, w\right), \quad \forall w \in W_h. \tag{74}$$

Differentiating (73) with respect to t, we obtain

$$\left(A^{-1}\beta_{1t}, \mathbf{v}\right) - \left(\alpha_{1t}, \operatorname{div}\mathbf{v}\right) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h. \tag{75}$$

Taking $t = 0$ and $\mathbf{v} = \beta_1(0)$ in (75) and choosing $y_0^h = Y(x, 0)$ and $y_1^h = Y_t(x, 0)$, we can derive that

$$\alpha_1(0) = \alpha_{1t}(0) = \beta_1(0) = \operatorname{div}\beta_1(0) = 0. \tag{76}$$

Next, taking $t = 0$ and $\mathbf{v} = \beta_{1t}(0)$ in (75) and choosing $y_0^h = Y(x, 0)$ and $y_1^h = Y_t(x, 0)$, we also find that $\beta_{1t}(0) = 0$. Now, choosing $w = \alpha_1$ and $\mathbf{v} = \beta_1$ as test functions in (73) and (74), we have

$$\|\alpha_{1t}\|_{L^\infty(J;L^2(\Omega))} + \|\beta_1\|_{L^\infty(J;(L^2(\Omega))^2)} \le Ch^{k+1}\|y_{tt}\|_{L^2(J;H^{k+2}(\Omega))}. \tag{77}$$

From (76), we find that $\alpha_1(0) = 0$, and then we have

$$\|\alpha_1\|_{L^\infty(J;L^2(\Omega))} \le C\|\alpha_{1t}\|_{L^2(J;L^2(\Omega))}. \tag{78}$$

Then we obtain (71) from (77), (78), and the triangle inequality.

Furthermore, we prove the second inequality (72). By using (34), subtract (18)-(19) and (46)-(47) to get the following error equations:

$$\begin{aligned} &\left(A^{-1}\beta_2, \mathbf{v}\right) - \left(\alpha_2, \operatorname{div}\mathbf{v}\right) \\ &\quad = -\left(A^{-1}(\mathbf{q} - \pi_h\mathbf{q}) + g_1'(\mathbf{p}) - g_1'(\mathbf{p}_h(u)), \mathbf{v}\right), \\ &\qquad\qquad \forall \mathbf{v} \in \mathbf{V}_h, \end{aligned} \tag{79}$$

$$\begin{aligned} &\left(\alpha_{2tt}, w\right) + \left(\operatorname{div}\beta_2, w\right) = \left(g_2'(y) - g_2'(y_h(u)), w\right), \\ &\qquad\qquad \forall w \in W_h. \end{aligned} \tag{80}$$

Noting that $\alpha_2(T) = P_h z(T) - z_h(u)(T) = 0$ and taking $t = T$ in (79), we find that

$$\begin{aligned} &\left(A^{-1}\beta_2(T), \mathbf{v}\right) \\ &\quad = -\left(A^{-1}(\mathbf{q} - \pi_h\mathbf{q})(T) + g_1'(\mathbf{p}(T)) - g_1'(\mathbf{p}_h(u)(T)), \mathbf{v}\right), \\ &\qquad\qquad \forall \mathbf{v} \in \mathbf{V}_h. \end{aligned} \tag{81}$$

By using Lemma 1 and (77), we can obtain that

$$\begin{aligned}
&\|\beta_2(T)\| \\
&\quad\le C\|(\mathbf{q}-\pi_h\mathbf{q})(T)\| \\
&\qquad + C\|g_1'(\mathbf{p}(T)) \\
&\qquad\qquad -g_1'(\mathbf{p}_h(u)(T))\| \\
&\quad\le C\|(\mathbf{q}-\pi_h\mathbf{q})(T)\| \\
&\qquad + C\|\tilde{g}_1''(\mathbf{p}(T))((\mathbf{p}-\mathbf{p}_h(u))(T))\| \\
&\quad\le Ch^{k+1}\|\mathbf{q}\|_{L^\infty(J;(H^{k+2}(\Omega))^2)} \\
&\qquad + C\|\mathbf{p}-P\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\qquad + C\|P-\mathbf{p}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\quad\le Ch^{k+1}.
\end{aligned} \tag{82}$$

Taking $t=0$ and $w=\alpha_{1tt}(0)$ in (74), since $\operatorname{div}\beta_1(0)=0$, we have

$$\begin{aligned}
\|\alpha_{1tt}(0)\|_{L^\infty(J;L^2(\Omega))} &\le C\|(y-Y)_{tt}(0)\|_{L^\infty(J;L^2(\Omega))} \\
&\le Ch^{k+1}\|y_{tt}\|_{L^\infty(J;H^{k+2}(\Omega))}.
\end{aligned} \tag{83}$$

Differentiating (75) and (74) with respect to t, we obtain

$$\left(A^{-1}\beta_{1tt},\mathbf{v}\right)-(\alpha_{1tt},\operatorname{div}\mathbf{v})=0,\quad \forall\mathbf{v}\in\mathbf{V}_h, \tag{84}$$

$$(\alpha_{1ttt},w)+(\operatorname{div}\beta_{1t},w)=-((y-Y)_{ttt},w),\quad \forall w\in W_h. \tag{85}$$

Selecting $\mathbf{v}=\beta_{1t}$ and $w=\alpha_{1tt}$ as test functions in (84) and (85), respectively, we get

$$(\alpha_{1ttt},\alpha_{1tt})+\left(A^{-1}\beta_{1tt},\beta_{1t}\right)=-((y-Y)_{ttt},\alpha_{1tt}),\quad \forall w\in W_h. \tag{86}$$

Integrating (86) from 0 to t, using (83) and the Gronwall's Lemma, we obtain

$$\begin{aligned}
&\|\alpha_{1tt}\|_{L^\infty(J;L^2(\Omega))}+\|\beta_{1t}\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\quad\le Ch^{k+1}\left(\|y_{tt}\|_{L^\infty(J;H^{k+2}(\Omega))}+\|y_{ttt}\|_{L^2(J;H^{k+2}(\Omega))}\right).
\end{aligned} \tag{87}$$

Differentiating (79) with respect to t, we obtain

$$\begin{aligned}
&\left(A^{-1}\beta_{2t},\mathbf{v}\right)-(\alpha_{2t},\operatorname{div}\mathbf{v}) \\
&\quad=-\left(A^{-1}(\mathbf{q}-\pi_h\mathbf{q})_t+g_1'(\mathbf{p}_t)-g_1'(\mathbf{p}_{ht}(u)),\mathbf{v}\right), \\
&\qquad\qquad \forall\mathbf{v}\in\mathbf{V}_h.
\end{aligned} \tag{88}$$

Now we choose $w=-\alpha_{2t}$ and $\mathbf{v}=-\beta_2$ as test function in (80) and (88), and we have

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\left(\|\alpha_{2t}\|+\left\|A^{-1/2}\beta_2\right\|\right)^2 \\
&\quad=-\left(g_2'(y)-g_2'(y_h(u)),\alpha_{2t}\right) \\
&\qquad+\left(A^{-1}(\mathbf{q}-\pi_h\mathbf{q})_t+g_1'(\mathbf{p}_t)-g_1'(\mathbf{p}_{ht}(u)),\beta_2\right) \\
&\quad=-\left(\tilde{g}_2''(y)(y-y_h(u)),\alpha_{2t}\right) \\
&\qquad+\left(A^{-1}(\mathbf{q}-\pi_h\mathbf{q})_t+\tilde{g}_1''(\mathbf{p}_t)(\mathbf{p}-\mathbf{p}_h(u))_t,\beta_2\right).
\end{aligned} \tag{89}$$

Then, integrating (89) from t into T, using (83) and (87), we obtain

$$\begin{aligned}
&\|\alpha_{2t}\|_{L^\infty(J;L^2(\Omega))}+\|\beta_2\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\quad\le C\left(\|y-y_h(u)\|_{L^\infty(J;L^2(\Omega))}+\|\mathbf{q}_t-\pi_h\mathbf{q}_t\|_{L^\infty(J;(L^2(\Omega))^2)}\right. \\
&\qquad\left.+\|\mathbf{p}-\mathbf{p}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)}\right) \\
&\quad\le C\left(\|y-Y\|_{L^\infty(J;L^2(\Omega))}+\|Y-y_h(u)\|_{L^\infty(J;L^2(\Omega))}\right. \\
&\qquad+h^{k+1}\|\mathbf{q}_t\|_{L^\infty(J;(H^{k+1}(\Omega))^2)}+\|\mathbf{p}-P\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\qquad\left.+\|P-\mathbf{p}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)}\right) \\
&\quad\le Ch^{k+1}.
\end{aligned} \tag{90}$$

Note that $Z(x,T)=Z_t(x,T)=0$; then $\alpha_2(T)=0$. Since $\alpha_2(t)=\alpha_2(t)-\alpha_2(T)=-\int_t^T\alpha_{2s}ds$, we have

$$\|\alpha_2(t)\|\le C\|\alpha_{2t}\|_{L^\infty(J;L^2(\Omega))}. \tag{91}$$

Then we complete the proof by combining (90), (91), and the triangle inequality. □

Using the Lemmas 1 and 4, we can also derive the following error estimates.

Theorem 5. *Assume that the optimal control problems* (1)–(5) *have a unique solution* $(\mathbf{p},y,u)$ *and that* Ω *is 2-regular. Assume that the regularity assumptions* (10) *are valid. There is a positive constant* $C>0$*, independent of* h*, such that*

$$\begin{aligned}
&\|\mathbf{p}-\mathbf{p}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)}+\|y-y_h(u)\|_{L^\infty(J;L^2(\Omega))}\le Ch^{k+1}, \\
&\|\mathbf{q}-\mathbf{q}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)}+\|z-z_h(u)\|_{L^\infty(J;L^2(\Omega))}\le Ch^{k+1}.
\end{aligned} \tag{92}$$

Proof. Combining Lemmas 1 and 4, (35), (39), and the triangle inequality, we obtain that

$$\begin{aligned}
&\left\|\mathbf{p}-\mathbf{p}_h(u)\right\|_{L^\infty(J;(L^2(\Omega))^2)}\\
&\quad\le \|\mathbf{p}-P\|_{L^\infty(J;(L^2(\Omega))^2)}\\
&\qquad+\left\|P-\mathbf{p}_h(u)\right\|_{L^\infty(J;(L^2(\Omega))^2)}\\
&\quad\le Ch^{k+1}+Ch^{k+1}\\
&\quad= Ch^{k+1},
\end{aligned}\tag{93}$$

$$\begin{aligned}
&\left\|y-y_h(u)\right\|_{L^\infty(J;L^2(\Omega))}\\
&\quad\le \|y-Y\|_{L^\infty(J;L^2(\Omega))}+\left\|Y-y_h(u)\right\|_{L^\infty(J;L^2(\Omega))}\\
&\quad\le Ch^{k+1}+Ch^{k+1}\\
&\quad= Ch^{k+1}.
\end{aligned}$$

Similarly, we can also obtain that

$$\begin{aligned}
&\left\|\mathbf{q}-\mathbf{q}_h(u)\right\|_{L^\infty(J;(L^2(\Omega))^2)}\\
&\quad\le \|\mathbf{q}-\pi_h\mathbf{q}\|_{L^\infty(J;(L^2(\Omega))^2)}+\left\|\pi_h\mathbf{q}-\mathbf{q}_h(u)\right\|_{L^\infty(J;(L^2(\Omega))^2)}\\
&\quad\le Ch^{k+1}+Ch^{k+1}\\
&\quad= Ch^{k+1},\\
&\left\|z-z_h(u)\right\|_{L^\infty(J;L^2(\Omega))}\\
&\quad\le \|z-P_hz\|_{L^\infty(J;L^2(\Omega))}+\left\|P_hz-z_h(u)\right\|_{L^\infty(J;L^2(\Omega))}\\
&\quad\le Ch^{k+1}+Ch^{k+1}\\
&\quad= Ch^{k+1}.
\end{aligned}\tag{94}$$

This proves (92). □

By applying the results we have proved above, we only need to estimate $\|\mathbf{p}_h(u)-\mathbf{p}_h\|_{L^\infty(J;(L^2(\Omega))^2)}$, $\|y_h(u)-y_h\|_{L^\infty(J;L^2(\Omega))}$ and $\|\mathbf{q}_h(u)-\mathbf{q}_h\|_{L^\infty(J;(L^2(\Omega))^2)}$, $\|z_h(u)-z_h\|_{L^\infty(J;L^2(\Omega))}$. For convenience, let

$$\begin{aligned}
e_1&=y_h(u)-y_h, & r_1&=\mathbf{p}_h(u)-\mathbf{p}_h,\\
e_2&=z_h(u)-z_h, & r_2&=\mathbf{q}_h(u)-\mathbf{q}_h.
\end{aligned}\tag{95}$$

Theorem 6. *Let* $(\mathbf{p}_h,y_h,\mathbf{q}_h,z_h,u_h)\in(\mathbf{V}_h\times W_h)^2\times U_h$ *be the solution of* (25)–(33) *and* $(\mathbf{p}_h(u),y_h(u),\mathbf{q}_h(u),z_h(u))\in(\mathbf{V}\times W)^2$ *the solution of* (50)–(57) *with* $\tilde{u}=u$. *There is a constant* $C>0$, *independent of* h, *such that*

$$\begin{aligned}
&\left\|\mathbf{p}_h(u)-\mathbf{p}_h\right\|_{L^\infty(J;(L^2(\Omega))^2)}+\left\|y_h(u)-y_h\right\|_{L^\infty(J;L^2(\Omega))}\\
&\quad\le C\|u-u_h\|_{L^2(J;L^2(\Omega_U))},
\end{aligned}\tag{96}$$

$$\begin{aligned}
&\left\|\mathbf{q}_h(u)-\mathbf{q}_h\right\|_{L^\infty(J;(L^2(\Omega))^2)}+\left\|z_h(u)-z_h\right\|_{L^\infty(J;L^2(\Omega))}\\
&\quad\le C\|u-u_h\|_{L^2(J;L^2(\Omega_U))}.
\end{aligned}\tag{97}$$

Proof. From (25)-(26) and (50)-(51), we obtain the following error equations:

$$\left(A^{-1}r_1,\mathbf{v}\right)-\left(e_1,\operatorname{div}\mathbf{v}\right)=0,\quad\forall\mathbf{v}\in\mathbf{V}_h,\tag{98}$$

$$\left(e_{1tt},w\right)+\left(\operatorname{div}r_1,w\right)=\left(u-u_h,w\right),\quad\forall w\in W_h.\tag{99}$$

Let $t=0$ and $\mathbf{v}=r_1(0)$ in (98); since $e_1(0)=0$, we have $r_1(0)=0$. We differentiate (98) with respect to t, and we derive

$$\left(A^{-1}r_{1t},\mathbf{v}\right)-\left(e_{1t},\operatorname{div}v\right)=0,\quad\forall\mathbf{v}\in\mathbf{V}_h.\tag{100}$$

Choose $w=e_{1t}$ and $\mathbf{v}=r_1$ as test functions and add the two relations of (99) and (100); using the Cauchy inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left\|A^{1/2}r_1\right\|^2+\left\|e_{1t}\right\|^2\right)\le\left\|u-u_h\right\|^2+\left\|e_{1t}\right\|^2.\tag{101}$$

Integrating (101) with respect to time from 0 to t, we derive

$$\left\|r_1\right\|^2+\left\|e_{1t}\right\|^2\le C\int_0^t\left\|u-u_h\right\|^2ds+C\int_0^t\left\|e_{1t}\right\|^2ds.\tag{102}$$

By using Gronwall's lemma to (102), we obtain

$$\left\|r_1\right\|_{L^\infty(J;(L^2(\Omega))^2)}+\left\|e_{1t}\right\|_{L^\infty(J;L^2(\Omega))}\le C\left\|u-u_h\right\|_{L^2(J;L^2(\Omega_U))}.\tag{103}$$

Since $e_1(t)=e_1(t)-e_1(0)=\int_0^t e_{1,s}\,ds$, using (103), we have

$$\left\|e_1\right\|\le C\left\|e_{1t}\right\|_{L^\infty(J;L^2(\Omega))}\le C\left\|u-u_h\right\|_{L^2(J;L^2(\Omega_U))}.\tag{104}$$

Then we derive (96).

From (29)-(30) and (54)-(55), we obtain the following error equations:

$$\left(A^{-1}r_2,\mathbf{v}\right)-\left(e_2,\operatorname{div}v\right)=-\left(\widetilde{g}_1'\left(\mathbf{p}_h(u)\right)r_1,\mathbf{v}\right),\quad\forall\mathbf{v}\in\mathbf{V}_h,\tag{105}$$

$$\left(e_{2tt},w\right)+\left(\operatorname{div}r_2,w\right)=\left(\widetilde{g}_2'\left(y_h(u)\right)e_1,w\right),\quad\forall w\in W_h.\tag{106}$$

Let $t=T$ and $\mathbf{v}=r_2(T)$ in (105); since $e_2(T)=0$, we have

$$\left\|A^{1/2}r_2(T)\right\|\le C\left\|\widetilde{g}_1'\left(\mathbf{p}_h(u)\right)\right\|\cdot\left\|r_1(T)\right\|\le C\left\|r_1(T)\right\|.\tag{107}$$

Introduce the symbol $\widehat{\varphi}:=\int_t^T\varphi(s)\,ds$, let $\widehat{\widetilde{g}_2'e_1}=\int_t^T\widetilde{g}_2'(\mathbf{p}_h(u))e_1ds$, and integrate (105) with respect to time from t to T, and we obtain

$$-\left(e_{2t},w\right)+\left(\operatorname{div}\widehat{r}_2,w\right)=\left(\widehat{\widetilde{g}_2'e_1},w\right),\quad\forall w\in W_h.\tag{108}$$

Set $w=e_2$ in (108) and $\mathbf{v}=\widehat{r}_2$ in (105), note that $r_2=-(d/dt)\widehat{r}_2$, and then add those equations to derive

$$-\frac{1}{2}\frac{d}{dt}\left(\left\|e_2\right\|^2+\left\|A^{1/2}\widehat{r}_2\right\|^2\right)=\left(\widehat{\widetilde{g}_2'e_1},e_2\right)-\left(r_1,\widehat{r}_2\right),\quad\forall w\in W_h.\tag{109}$$

Integrating (109) with respect to time from t to T, using (107) and Yong's inequalities, we get

$$\begin{aligned}
&\|e_2\|_{L^\infty(J;L^2(\Omega))} \\
&\quad \le C\left(\|r_2(T)\| + \|e_1\|_{L^2(J;L^2(\Omega))} + \|r_1\|_{L^2(J;(L^2(\Omega))^2)}\right) \\
&\quad \le C\left(\|r_1(T)\| + \|e_1\|_{L^2(J;L^2(\Omega))} + \|r_1\|_{L^2(J;(L^2(\Omega))^2)}\right) \\
&\quad \le C\left(\|e_1\|_{L^2(J;L^2(\Omega))} + \|r_1\|_{L^2(J;(L^2(\Omega))^2)}\right).
\end{aligned} \tag{110}$$

Choosing $\mathbf{v} = r_2$ and $w = e_2$ as test functions in (105) and (106), it is easy to get

$$\begin{aligned}
&\|e_{2t}\|^2_{L^\infty(J;L^2(\Omega))} + \|r_2\|^2_{L^\infty\left(J;(L^2(\Omega))^2\right)} \\
&\quad \le C\|e_1\|^2_{L^\infty(J;L^2(\Omega))} + C\|r_1\|^2_{L^\infty\left(J;(L^2(\Omega))^2\right)} \\
&\qquad + C\delta\left(\|e_2\|^2_{L^\infty(J;L^2(\Omega))} + \|r_2\|^2_{L^\infty\left(J;(L^2(\Omega))^2\right)}\right),
\end{aligned} \tag{111}$$

where δ is an arbitrary small positive constant. Namely,

$$\begin{aligned}
&\|e_{2t}\|^2_{L^\infty(J;L^2(\Omega))} + \|r_2\|^2_{L^\infty\left(J;(L^2(\Omega))^2\right)} \\
&\quad \le C\|e_1\|^2_{L^\infty(J;L^2(\Omega))} + C\|r_1\|^2_{L^\infty\left(J;(L^2(\Omega))^2\right)} \\
&\quad \le C\|u - u_h\|_{L^2(J;L^2(\Omega_U))}.
\end{aligned} \tag{112}$$

Combining (103)-(104) and (110)-(112), we derive (97). □

In the following, we estimate $\|u - u_h\|_{L^2(J;L^2(\Omega_U))}$ and then obtain the following main result.

Theorem 7. *Let* $(\mathbf{p}, y, \mathbf{q}, z, u) \in (\mathbf{V} \times W)^2 \times U$ *and* $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h) \in (\mathbf{V}_h \times W_h)^2 \times U_h$ *be the solutions of* (14)–(22) *and* (25)–(33), *respectively. Assume that the regularity assumptions* (10) *and* (59) *are valid. Furthermore, one assumes that*

$$u \in H^{k+1}(\Omega_U), \quad j'(u) + z \in H^{k+1}(\Omega_U). \tag{113}$$

Then, one has

$$\begin{gathered}
\|u - u_h\|_{L^2(J;L^2(\Omega_U))} \le C\left(h^{k+1} + h_U^{k+1}\right), \\
\|\mathbf{p} - \mathbf{p}_h\|_{L^\infty(J;(L^2(\Omega))^2)} + \|y - y_h\|_{L^\infty(J;L^2(\Omega))} \le C\left(h^{k+1} + h_U^{k+1}\right), \\
\|\mathbf{q} - \mathbf{q}_h\|_{L^\infty(J;(L^2(\Omega))^2)} + \|z - z_h\|_{L^\infty(J;L^2(\Omega))} \le C\left(h^{k+1} + h_U^{k+1}\right).
\end{gathered} \tag{114}$$

Proof. First, in (34), let $\chi = 1$, and we have

$$\int_\Omega P_h u\, dx = \int_\Omega u\, dx. \tag{115}$$

Integrating (115) from 0 to T, we can obtain that

$$\int_0^T \int_\Omega P_h u\, dx\, dt = \int_0^T \int_\Omega u\, dx\, dt \ge 0. \tag{116}$$

Therefore, we know that $P_h u \in K_h$. Now we choose $\tilde{u} = u_h$ in (22) and $\tilde{u}_h = P_h u$ in (33) to get that

$$\begin{gathered}
\int_0^T \left(j'(u) + z, u_h - u\right)_U dt \ge 0, \\
\int_0^T \left(j'(u_h) + z_h, P_h u - u_h\right)_U dt \ge 0.
\end{gathered} \tag{117}$$

By using (117) and the assumption (59), we have

$$\begin{aligned}
&c\|u - u_h\|^2_{L^2(J;L^2(\Omega_U))} \\
&\quad \le \int_0^T \left(j'(u) - j'(u_h), u - u_h\right)_U dt \\
&\quad = \int_0^T \left(j'(u) + z, u - u_h\right)_U dt \\
&\qquad + \int_0^T \left(z_h(u) - z, u - u_h\right)_U dt \\
&\qquad - \int_0^T \left(z_h(u) - z_h, u - u_h\right)_U dt \\
&\qquad - \int_0^T \left(j'(u_h) + z_h, u - u_h\right)_U dt \\
&\quad \le \int_0^T \left(z_h(u) - z, u - u_h\right)_U dt \\
&\qquad - \int_0^T \left(z_h(u) - z_h, u - u_h\right)_U dt \\
&\qquad + \int_0^T \left(j'(u_h) + z_h, P_h u - u\right)_U dt \\
&\quad = \int_0^T \left(z_h(u) - z, u - u_h\right)_U dt \\
&\qquad - \int_0^T \left(z_h(u) - z_h, u - u_h\right)_U dt \\
&\qquad + \int_0^T \left(j'(u_h) - j'(u), P_h u - u\right)_U dt \\
&\qquad + \int_0^T \left(j'(u) + z, P_h u - u\right)_U dt \\
&\qquad + \int_0^T \left(z_h - z, P_h u - u\right)_U dt.
\end{aligned} \tag{118}$$

From (25)–(33) and (50)–(57), we have

$$\begin{aligned}
&-\int_0^T \left(z_h(u) - z_h, u - u_h\right)_U dt \\
&\quad = -\int_0^T \left(u - u_h, e_2\right)_U dt
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^T (e_{1tt}, e_2)\,dt - \int_0^T (\operatorname{div} r_1, e_2)\,dt \\
&= -\int_0^T (e_{1tt}, e_2)\,dt - \int_0^T (\operatorname{div} r_1, e_2)\,dt \\
&\quad + \int_0^T \left(A^{-1} r_1, r_2\right) dt - \int_0^T (e_1, \operatorname{div} r_2)\,dt \\
&= -\int_0^T (e_{2tt}, e_1)\,dt - \int_0^T (\operatorname{div} r_2, e_1)\,dt \\
&\quad + \int_0^T \left(A^{-1} r_2, r_1\right) dt - \int_0^T (e_2, \operatorname{div} r_1)\,dt \\
&= -\int_0^T \Big(\left(g_2'(y_h(u)) - g_2'(y_h), e_1\right) \\
&\qquad + \left(g_1'(\mathbf{p}_h(u)) - g_1'(\mathbf{p}_h), r_1\right)\Big)\,dt \\
&= -\int_0^T \Big(\left(g_2'(y_h(u)) - g_2'(y_h), y_h(u) - y_h\right) \\
&\qquad + \left(g_1'(\mathbf{p}_h(u)) - g_1'(\mathbf{p}_h), \mathbf{p}_h(u) - \mathbf{p}_h\right)\Big)\,dt \\
&\le 0,
\end{aligned}
\tag{119}
$$

where we used the fact that g_1 and g_2 are convex functionals. By using (119) and ε-Caunchy inequality,

$$
\begin{aligned}
&c\|u - u_h\|^2_{L^2(J;L^2(\Omega_U))} \\
&\quad \le \int_0^T (z_h(u) - z, u - u_h)_U dt \\
&\qquad + \int_0^T \left(j'(u_h) - j'(u), P_h u - u\right)_U dt \\
&\qquad + \int_0^T \left(j'(u) + z, P_h u - u\right)_U dt \\
&\qquad + \int_0^T (z_h(u) - z, P_h u - u)_U dt \\
&\quad \le C\|z_h(u) - z\|^2_{L^2(J;L^2(\Omega))} \\
&\qquad + C\varepsilon\|u - u_h\|^2_{L^2(J;L^2(\Omega_U))} \\
&\qquad + C\int_0^T \|j'(u) + z\|_{k+1,\Omega_U} \|P_h u - u\|_{-k-1,\Omega_U} dt \\
&\qquad + C\|P_h u - u\|^2_{L^2(J;L^2(\Omega_U))} \\
&\quad \le C\|z_h(u) - z\|^2_{L^2(J;L^2(\Omega))} \\
&\qquad + C\varepsilon\|u - u_h\|^2_{L^2(J;L^2(\Omega_U))} + Ch_U^{2(k+1)},
\end{aligned}
\tag{120}
$$

for any small $\varepsilon > 0$, where $\|P_h u - u\|_{-k-1,\Omega_U} \le Ch^{2(k+1)}\|u\|_{k+1,\Omega_U}$ has been used. It is easy to see that

$$
\begin{aligned}
&c\|u - u_h\|_{L^2(J;L^2(\Omega_U))} \\
&\quad \le C\|z_h(u) - z\|_{L^2(J;L^2(\Omega))} + Ch_U^{k+1} \le C\left(h^{k+1} + h_U^{k+1}\right).
\end{aligned}
\tag{121}
$$

From Theorems 5 and 6 and (122), we can obtain that

$$
\begin{aligned}
&\|y - y_h\|_{L^\infty(J;L^2(\Omega))} + \|\mathbf{p} - \mathbf{p}_h\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\quad + \|z - z_h\|_{L^\infty(J;L^2(\Omega))} + \|\mathbf{q} - \mathbf{q}_h\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\quad \le \|y - y_h(u)\|_{L^\infty(J;L^2(\Omega))} + \|\mathbf{p} - \mathbf{p}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\qquad + \|z - z_h(u)\|_{L^\infty(J;L^2(\Omega))} \\
&\qquad + \|\mathbf{q} - \mathbf{q}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)} + \|y_h(u) - y_h\|_{L^\infty(J;L^2(\Omega))} \\
&\qquad + \|\mathbf{p}_h(u) - \mathbf{p}_h\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\qquad + \|z_h(u) - z_h\|_{L^\infty(J;L^2(\Omega))} + \|\mathbf{q}_h(u) - \mathbf{q}_h\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\quad \le \|y - y_h(u)\|_{L^\infty(J;L^2(\Omega))} + \|\mathbf{p} - \mathbf{p}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)} \\
&\qquad + \|\mathbf{q} - \mathbf{q}_h(u)\|_{L^\infty(J;(L^2(\Omega))^2)} + \|z - z_h(u)\|_{L^\infty(J;L^2(\Omega))} \\
&\qquad + C\|u - u_h\|_{L^2(J;L^2(\Omega_U))} \\
&\quad \le C\left(h^{k+1} + h_U^{k+1}\right).
\end{aligned}
\tag{122}
$$

Then we complete the proof. □

4. Conclusion and Future Works

In this paper we presented a priori error estimate for mixed finite element approximation of the general linear hyperbolic optimal control problems (1)–(5). Using the elliptic projection operators and Gronwall's Lemma, we have established some error estimate results for both the state and the costate discrete solutions and the control approximation. To the best of our knowledge in the context of optimal control problems, these a priori error estimates for the general hyperbolic optimal control problems are new. In our future work, we will use the fully discrete mixed finite element method to deal with nonlinear hyperbolic optimal control problems. Furthermore, we will consider a priori error estimates and superconvergence of these optimal control problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors express their thanks to the referees for their helpful suggestions, which led to improvements of the presentation. This work is supported by National Science Foundation of China (11201510), Chongqing Research Program of Basic Research and Frontier Technology (cstc2012jjA00003), Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJ121113), and Science and Technology Project of Wanzhou District of Chongqing (2013030050).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 312640, 12 pages
http://dx.doi.org/10.1155/2014/312640

Research Article

Portfolio Selection with Liability and Affine Interest Rate in the HARA Utility Framework

Hao Chang,[1] Kai Chang,[2] and Ji-mei Lu[3]

[1] *School of Science, Tianjin Polytechnic University, Tianjin 300387, China*
[2] *School of Finance, Zhejiang University of Finance & Economics, Hangzhou 310018, China*
[3] *School of Electrical Engineering and Automation, Tianjin Polytechnic University, Tianjin 300387, China*

Correspondence should be addressed to Hao Chang; ch8683897@126.com

Received 15 March 2014; Revised 18 May 2014; Accepted 27 May 2014; Published 17 June 2014

Academic Editor: Simone Marsiglio

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This paper studied an asset and liability management problem with stochastic interest rate, where interest rate is assumed to be governed by an affine interest rate model, while liability process is driven by the drifted Brownian motion. The investors wish to look for an optimal investment strategy to maximize the expected utility of the terminal surplus under hyperbolic absolute risk aversion (HARA) utility function, which consists of power utility, exponential utility, and logarithm utility as special cases. By applying dynamic programming principle and Legendre transform, the explicit solutions for HARA utility are achieved successfully and some special cases are also discussed. Finally, a numerical example is provided to illustrate our results.

1. Introduction

In practice of investments, it is all well known that some investors or financial institutions are often confronted with liability factor. It is very clear that considering liability factor in the portfolio selection problems will be more practical. In recent years, portfolio selection problems with liability have inspired literally hundreds of extensions and applications. For example, Chiu and Li [1] supposed liability process to be driven by a geometric Brownian motion and studied the efficient strategy and the efficient frontier under mean-variance criterion. Xie et al. [2] studied a mean-variance model with random liability, which is assumed to be governed by the drifted Brownian motion. Zeng and Li [3] investigated a mean-variance model with a jump diffusion liability process. Chen et al. [4] studied the asset and liability management (ALM) problems with regime switching, which described the effect of the changes in macroeconomic conditions. Chiu and Wong [5] investigated an ALM problem under the assumption that risky assets were cointegrated. Those models studied the ALM problems with exogenous liabilities, which cannot be controlled. Along another line, the ALM problems with endogenous liabilities have been paid more and more attention nowadays. Being different from exogenous liabilities, endogenous liabilities can be controlled by various financial instruments and investors' decisions. The interested readers can refer to the works of Leippold et al. [6] and Yao et al. [7].

However, the above mentioned research results were generally achieved under the assumption of constant interest rate. As a matter of fact, interest rate is always changing with time and can be delineated by some term structure models, for example, the Vasicek model [8] or the CIR model [9]. Therefore, some scholars began to be concerned with the portfolio selection problems with stochastic interest rate. For example, Korn and Kraft [10] investigated the portfolio selection problems with stochastic interest rates and proved the verification theorem. Deelstra et al. [11] studied the optimal investment strategies in the affine interest rate environment. Gao [12] applied dynamic programming principle and Legendre transform to study the DC pension problems with affine interest rate. Liu [13] and Chang and Rong [14] considered the optimal investment and consumption strategy with stochastic interest rate and stochastic volatility. These models were all studied in the utility maximization framework and

the optimal investment strategies were established under power utility or logarithm utility.

Power utility, logarithm utility, and exponential utility are all special cases of HARA utility, which used to be studied by Grasselli [15], Çanakoğlu and Özekici [16], and Jung and Kim [17]. Due to the complexity of HARA utility, there is little work on HARA utility in the existing literature. In recent years, some scholars found that Legendre transform is an effective method in solving sophisticated portfolio section problems. One can refer to the works of Jonsson and Sircar [18], Gao [12], and Jung and Kim [17].

This paper introduces liability process into a continuous-time dynamic portfolio selection problem and assumes interest rate to be driven by affine interest rate model. The objective of the investor is to look for an optimal investment strategy to maximize the expected utility of the terminal surplus. Considering the generality of HARA utility, we take HARA utility for our analysis. Applying dynamic programming principle, we obtain the Hamilton-Jacobi-Bellman (HJB) equation for the value function. Due to the nonlinearity of HARA utility, it is very difficult for us to conjecture the form of the solution for the HJB equation. Therefore, we apply Legendre transform to change the nonlinear HJB equation into a linear dual one. What is more, the form of the solution for the dual equation is linear and is easy to construct. Finally, we establish the explicit expression of the optimal investment strategy for HARA utility. Some special cases are also discussed. There are three highlights: (i) we study an asset and liability management problem under the affine interest rate model; (ii) interest rate is supposed to be governed by the affine interest rate model, while liability is assumed to be driven by the drifted Brownian motion; (iii) the optimal investment strategies for HARA utility are obtained explicitly.

This paper proceeds as follows. In Section 2, an asset and liability management problem with affine interest rate is formulated. In Section 3, we use dynamic programming principle to obtain the HJB equation and apply Legendre transform to change the HJB equation into a linear dual one. In Section 4, we derive the optimal investment strategy for HARA utility explicitly. Section 5 discusses some special cases of the optimal policies. Section 6 provides a numerical example to illustrate our results. Some interesting results are concluded in Section 7.

2. Model and Assumptions

In this paper, we assume that the financial market consists of three assets: one risk-free asset, one risky asset, and one zero-coupon bond. $[0,T]$ represents certain investment horizon, where $T < \infty$. An investor is confronted with two key sources of risk, that is, interest rate risk and stock information risk, which are described by two one-dimensional standard and adapted independent Brownian motion $W_r(t)$ and $W_S(t)$. $W_r(t)$ and $W_S(t)$ are all defined on complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$.

The price process of the risk-free asset (i.e., cash), denoted by $S_0(t)$, is given by

$$dS_0(t) = r(t) S_0(t)\, dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t)$ is the risk-free interest rate. In this paper, $r(t)$ is supposed to be stochastic process and is governed by

$$\begin{aligned} dr(t) &= (a - br(t))\, dt - \sqrt{k_1 r(t) + k_2}\, dW_r(t), \\ r(0) &= r_0 > 0, \end{aligned} \tag{2}$$

where the parameters a, b, k_1, and k_2 are positive real constants.

Remark 1. Notice that (2) consists of the Vasicek model (resp., the CIR model) as special cases, when k_1 (resp., k_2) is equal to zero. Under these dynamics, the term structure of interest rate is affine.

The price process of the risky asset (i.e., the stock), denoted by $S_1(t)$, can be described by the following stochastic differential equation (SDE) (referring to Deelstra et al. [11]):

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= r(t)\, dt + \sigma_1 \left(dW_S(t) + \lambda_1 dt\right) \\ &\quad + \sigma_2 \sqrt{k_1 r(t) + k_2} \\ &\quad \times \left(dW_r(t) + \lambda_2 \sqrt{k_1 r(t) + k_2}\, dt\right), \\ S_1(0) &= 1, \end{aligned} \tag{3}$$

where λ_1, λ_2 (resp., σ_1, σ_2) are constants (resp., positive constants).

The third asset is one zero-coupon bond with maturity T, whose price process is denoted by $B(t,T)$. Then $B(t,T)$ evolves (referring to Deelstra et al. [11])

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= r(t)\, dt + \sigma_B(t, r(t)) \\ &\quad \times \left(dW_r(t) + \lambda_2 \sqrt{k_1 r(t) + k_2}\, dt\right), \\ B(T,T) &= 1, \end{aligned} \tag{4}$$

where $\sigma_B(t, r(t)) = h(t)\sqrt{k_1 r(t) + k_2}$ and $h(t)$ is given by

$$\begin{aligned} h(t) &= \frac{2\left(e^{m(T-t)} - 1\right)}{m - (b - k_1\lambda_2) + e^{m(T-t)}(m + b - k_1\lambda_2)}, \\ m &= \sqrt{(b - k_1\lambda_2)^2 + 2k_1}. \end{aligned} \tag{5}$$

Remark 2. Equations (3) and (4) have been investigated by Deelstra et al. [11] and Gao [12], but their results were only achieved under power utility or logarithm utility. From (3), we can see that the price process of the stock is affected by uncertainty of interest rate, which is consistent with reality. In addition, we find that the price process of zero-coupon bond and interest rate is driven by the same Brownian motion. It implies that the zero-coupon bond is reduced to the risk-free asset when interest rate is reduced to a constant.

Assume that the liability process satisfies the following SDE:

$$dL(t) = u dt + v dW_S(t), \quad L(0) = l_0 > 0, \tag{6}$$

where u and v are positive constants.

Assume that the initial capital of an investor is given by w_0 and the initial liability is denoted by l_0; then the initial surplus is $w_0 - l_0$, which is denoted by x_0. The investor wishes to invest the surplus in the financial assets to maximize the expected utility of the terminal surplus. The amount invested in the stock and zero-coupon bond is denoted by $\pi_1(t)$ and $\pi_2(t)$, respectively. Therefore, the surplus process $X(t)$ at time t under the strategy $\pi(t) = (\pi_1(t), \pi_2(t))$ satisfies the following SDE:

$$\begin{aligned} dX(t) = \Big(& X(t) r(t) + \pi_1(t) \sigma_1 \lambda_1 \\ & + \pi_1(t) \sigma_2 \lambda_2 (k_1 r(t) + k_2) \\ & + \pi_2(t) \sigma_B \lambda_2 \sqrt{k_1 r(t) + k_2} - u \Big) dt \\ & + (\pi_1(t) \sigma_1 - \nu) dW_S(t) \\ & + \Big(\pi_1(t) \sigma_2 \sqrt{k_1 r(t) + k_2} + \pi_2(t) \sigma_B \Big) dW_r(t), \end{aligned} \tag{7}$$

with the initial surplus $X(0) = x_0 > 0$.

Definition 3 (admissible strategy). An investment strategy $\pi(t) = (\pi_1(t), \pi_2(t))$ is said to be admissible if the following conditions are satisfied:

(i) $\pi_1(t)$ and $\pi_2(t)$ are all $\mathscr{F}_t$-measurable;

(ii) $E(\int_0^T ((\pi_1(t)\sigma_1 - \nu)^2 + (\pi_1(t)\sigma_2 \sqrt{k_1 r(t) + k_2} + \pi_2(t)\sigma_B)^2) dt) < +\infty$;

(iii) SDE (7) has a unique solution for all $\pi(t) = (\pi_1(t), \pi_2(t))$.

Assume that the set of all admissible strategies is denoted by Π. The investor wishes to look for an optimal strategy to maximize the expected utility of the terminal surplus; that is,

$$\underset{\pi(t)\in\Pi}{\text{Maximize}}\ \mathbb{E}(U(X(T))), \tag{8}$$

where $U(\cdot)$ represents utility function, which is strictly concave and satisfies the Inada conditions: $\dot{U}(0) = +\infty$ and $\dot{U}(+\infty) = 0$.

In this paper, we choose hyperbolic absolute risk aversion (HARA) utility function for our analysis. The HARA utility function with parameters η, p, and q is given by

$$U(x) = U(\eta, p, q, x) = \frac{1-p}{qp}\left(\frac{q}{1-p}x + \eta\right)^p, \quad q > 0,\ p < 1,\ p \neq 0. \tag{9}$$

As a matter of fact, HARA utility function recovers power utility and exponential utility as special cases.

(i) If we choose $\eta = 0$ and $q = 1 - p$, then we have

$$U(0, p, 1-p, x) = \frac{x^p}{p} = U_{\text{power}}(x). \tag{10}$$

(ii) If we choose $\eta = 1$ and $p \to -\infty$, then we have

$$U(1, p, q, x) = -\frac{e^{-qx}}{q} = U_{\exp}(x). \tag{11}$$

3. HJB Equation and Legendre Transform

According to dynamic programming principle, we define the value function as

$$V(t, r, x) = \sup_{\pi(t)\in\Pi} \mathbb{E}(U(X(T)) \mid X(t) = x, r(t) = r) \tag{12}$$

with boundary condition given by $V(T, r, x) = U(x)$.

By applying the principle of optimality, we obtain the following proposition.

Proposition 4. *If the value function $V(t, r, x) \in C^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R})$, then $V(t, r, x)$ satisfies the following HJB equation:*

$$\sup_{\pi(t)\in\Pi} \mathscr{A}V(t, r, x) = 0, \tag{13}$$

where $\mathscr{A}$ is a variational operator, and by letting $\sigma_r = \sqrt{k_1 r(t) + k_2}$ we obtain

$$\begin{aligned} & \mathscr{A}V(t, r, x) \\ & = V_t + \Big(rx + \pi_1(t)\sigma_1\lambda_1 + \pi_1(t)\sigma_2\lambda_2\sigma_r^2 \\ & \quad + \pi_2(t)\sigma_B\lambda_2\sigma_r - u \Big) V_x \\ & + \frac{1}{2}\Big((\pi_1(t)\sigma_1 - \nu)^2 \\ & \quad + (\pi_1(t)\sigma_2\sigma_r + \pi_2(t)\sigma_B)^2 \Big) V_{xx} \\ & + (a - br) V_r + \frac{1}{2}\sigma_r^2 V_{rr} \\ & - \Big(\pi_1(t)\sigma_2\sigma_r^2 + \pi_2(t)\sigma_B\sigma_r \Big) V_{rx}. \end{aligned} \tag{14}$$

Here, V_t, V_x, V_{xx}, V_r, V_{rr}, and V_{rx} represent first-order and second-order partial derivatives with respect to the variables t, r, and x.

Proof. The proof is standard. The interested readers can refer to the work of Korn and Kraft [10].

Assume that $H(t, r, x)$ is a solution of the HJB equation (13); then we get

$$\begin{aligned} \pi_1(t) &= -\frac{\lambda_1}{\sigma_1} \cdot \frac{H_x}{H_{xx}} + \frac{\nu}{\sigma_1}, \\ \pi_2(t) &= \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_2\lambda_1 - \sigma_1\lambda_2}{\sigma_1} \cdot \frac{H_x}{H_{xx}} + \frac{\sigma_r}{\sigma_B} \cdot \frac{H_{rx}}{H_{xx}} - \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_r}{\sigma_B}\nu. \end{aligned} \tag{15}$$

Putting (15) into (13), we derive

$$\begin{aligned} & H_t + (rx + \lambda_1\nu - u) H_x + (a - br) H_r + \frac{1}{2}\sigma_r^2 H_{rr} \\ & - \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\sigma_r^2\right)\frac{H_x^2}{H_{xx}} + \lambda_2\sigma_r^2\frac{H_x H_{rx}}{H_{xx}} - \frac{1}{2}\sigma_r^2\frac{H_{rx}^2}{H_{xx}} = 0, \end{aligned} \tag{16}$$

with boundary condition given by $H(T, r, x) = ((1-p)/qp)((q/(1-p))x + \eta)^p$.

Noting that (16) is a nonlinear second-order partial differential equation and it is not easy for us to conjecture the structure of a solution to (16) for HARA utility, we introduce the following Legendre transform to change (16) into a linear second-order partial differential equation such that we can obtain the explicit solution to (16). □

Definition 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. For all $z > 0$, Legendre transform can be defined as follows:

$$L(z) = \max_x \{f(x) - zx\}, \tag{17}$$

and then the function $L(z)$ is called the Legendre dual function of $f(x)$ (cf. [12, 17, 18]).

If $f(x)$ is strictly concave, the maximum in (17) will be attained at just one point, which we denoted by x_0. We can attain at the unique solution by the first-order condition

$$\frac{df(x)}{dx} - z = 0. \tag{18}$$

So we have

$$L(z) = f(x_0) - zx_0. \tag{19}$$

Following Jonsson and Sircar [18], Gao [12], and Jung and Kim [17], Legendre transform can be defined by

$$\widehat{H}(t,r,z) = \sup_{x>0} \{H(t,r,x) - zx\}, \tag{20}$$

where $z > 0$ denotes the dual variable to x. The value of x where this optimum is attained is denoted by $g(t,r,z)$, so we have

$$g(t,r,z) = \inf_{x>0} \left\{x \mid H(t,r,x) \geqslant zx + \widehat{H}(t,r,z)\right\}. \tag{21}$$

The relationship between $\widehat{H}(t,r,z)$ and $g(t,r,z)$ is given by

$$g(t,r,z) = -\widehat{H}_z(t,r,z). \tag{22}$$

Hence, we can choose either one of two functions $g(t,r,z)$ and $\widehat{H}(t,r,z)$ as the dual function of $H(t,r,x)$. In this paper, we choose $g(t,r,z)$. Moreover, we have

$$H_x = z, \qquad \widehat{H}(t,r,z) = H(t,r,g) - zg, \qquad g(t,r,z) = x. \tag{23}$$

Differentiating (23) with respect to t, r, and x, we get

$$\begin{gathered} H_t = \widehat{H}_t, \qquad H_x = z, \qquad H_{xx} = -\frac{1}{\widehat{H}_{zz}}, \\ H_r = \widehat{H}_r, \qquad H_{rr} = \widehat{H}_{rr} - \frac{\widehat{H}_{rz}^2}{\widehat{H}_{zz}}, \qquad H_{xr} = -\frac{\widehat{H}_{rz}}{\widehat{H}_{zz}}. \end{gathered} \tag{24}$$

Notice that $H(T,r,x) = U(x)$; then at the terminal time T, we can define

$$\begin{gathered} \widehat{H}(T,r,z) = \sup_{x>0} \{U(x) - zx\}, \\ g(T,r,z) = \inf_{x>0} \left\{x \mid U(x) \geqslant zx + \widehat{H}(T,r,z)\right\}. \end{gathered} \tag{25}$$

So we have $g(T,r,z) = (\dot{U})^{-1}(z)$, where $(\dot{U})^{-1}(z)$ is taken as the inverse of marginal utility.

Substituting (24) back into (16) yields

$$\begin{aligned} &\widehat{H}_t + rzg + (\lambda_1 v - u) z + (a - br) \widehat{H}_r + \frac{1}{2}\sigma_r^2 \widehat{H}_{rr} \\ &\quad + \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\sigma_r^2\right) z^2 \widehat{H}_{zz} + \lambda_2 \sigma_r^2 z \widehat{H}_{rz} = 0. \end{aligned} \tag{26}$$

Differentiating (26) with respect to z and using (22), we obtain

$$\begin{aligned} &g_t - rg + \left(\lambda_1^2 + \lambda_2^2\sigma_r^2 - r\right) z g_z \\ &\quad + \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\sigma_r^2\right) z^2 g_{zz} + \left(a - br + \lambda_2\sigma_r^2\right) g_r \\ &\quad + \frac{1}{2}\sigma_r^2 g_{rr} + \lambda_2 \sigma_r^2 z g_{rz} + u - \lambda_1 v = 0, \end{aligned} \tag{27}$$

with boundary condition given by $g(T,r,z) = (\dot{U})^{-1}(z)$.

Under HARA utility function (9), we have

$$g(T,r,z) = \frac{1-p}{q}\left(z^{1/(p-1)} - \eta\right). \tag{28}$$

4. The Optimal Portfolio

Assume that a solution of (27) is conjectured as follows:

$$g(t,r,z) = \frac{1-p}{q} z^{1/(p-1)} f(t,r) - \frac{1-p}{q}\eta h(t,r) + J(t,r), \tag{29}$$

with boundary conditions given by $f(T,r) = 1$, $h(T,r) = 1$, and $J(T,r) = 0$.

Further, the partial derivatives of $g(t,r,z)$ with respect to t, r, and z are as follows:

$$\begin{aligned} g_t &= \frac{1-p}{q} z^{1/(p-1)} f_t - \frac{1-p}{q}\eta h_t + J_t, \\ g_z &= -\frac{1}{q} z^{(2-p)/(p-1)} f, \\ g_r &= \frac{1-p}{q} z^{1/(p-1)} f_r - \frac{1-p}{q}\eta h_r + J_r, \\ g_{rz} &= -\frac{1}{q} z^{(2-p)/(p-1)} f_r, \\ g_{rr} &= \frac{1-p}{q} z^{1/(p-1)} f_{rr} - \frac{1-p}{q}\eta h_{rr} + J_{rr}, \\ g_{zz} &= -\frac{1}{q} \cdot \frac{2-p}{p-1} \cdot z^{(3-2p)/(p-1)} f. \end{aligned} \tag{30}$$

Plugging (30) into (27), after some simple calculations, we derive

$$\begin{aligned} &\frac{1-p}{q} z^{1/(p-1)} \left(f_t + \frac{p}{2(1-p)^2}\left(\lambda_1^2 + \lambda_2^2\sigma_r^2\right) f + \frac{p}{1-p} r f \right. \\ &\qquad \left. + \left(a - br - \frac{p}{1-p}\lambda_2\sigma_r^2\right) f_r + \frac{1}{2}\sigma_r^2 f_{rr} \right) \end{aligned}$$

$$-\frac{1-p}{q}\eta\left(h_t - rh + \left(a - br + \lambda_2\sigma_r^2\right)h_r + \frac{1}{2}\sigma_r^2 h_{rr}\right) + J_t - rJ + \left(a - br + \lambda_2\sigma_r^2\right)J_r + \frac{1}{2}\sigma_r^2 J_{rr} + u - \lambda_1 v = 0. \tag{31}$$

Equation (31) can be decomposed into the following three equations:

$$f_t + \frac{p}{2(1-p)^2}\left(\lambda_1^2 + \lambda_2^2\sigma_r^2\right)f + \frac{p}{1-p}rf + \left(a - br - \frac{p}{1-p}\lambda_2\sigma_r^2\right)f_r + \frac{1}{2}\sigma_r^2 f_{rr} = 0, \quad f(T,r) = 1; \tag{32}$$

$$J_t - rJ + \left(a - br + \lambda_2\sigma_r^2\right)J_r + \frac{1}{2}\sigma_r^2 J_{rr} + u - \lambda_1 v = 0, \quad J(T,r) = 0; \tag{33}$$

$$h_t - rh + \left(a - br + \lambda_2\sigma_r^2\right)h_r + \frac{1}{2}\sigma_r^2 h_{rr} = 0, \quad h(T,r) = 1. \tag{34}$$

Lemma 6. *Assume that a solution of* (32) *is conjectured as* $f(t,r) = e^{D_1(t)+D_2(t)r}$, *with boundary conditions given by* $D_1(T) = 0$ *and* $D_2(T) = 0$; *then under the condition of* $p < \min\{1, b^2/(2k_1 + (b-\lambda_2 k_1)^2)\}$ *and* $p \neq 0$, $D_1(t)$ *and* $D_2(t)$ *are given by* (44) *and* (42), *respectively.*

Proof. Putting $f(t,r) = e^{D_1(t)+D_2(t)r}$ into (32) and taking $\sigma_r = \sqrt{k_1 r(t) + k_2}$ into consideration, we obtain

$$e^{D_1(t)+D_2(t)r}\left(\dot{D}_1(t) + \frac{p}{2(1-p)^2}\left(\lambda_1^2 + \lambda_2^2 k_2\right) + \left(a - \frac{p}{1-p}\lambda_2 k_2\right)D_2(t) + \frac{1}{2}k_2 D_2^2(t) + r\left(\dot{D}_2(t) + \frac{p}{2(1-p)^2}\lambda_2^2 k_1 + \frac{p}{1-p} - \left(b + \frac{p}{1-p}\lambda_2 k_1\right)D_2(t) + \frac{1}{2}k_1 D_2^2(t)\right)\right) = 0. \tag{35}$$

Comparing the coefficients yields

$$\dot{D}_2(t) + \frac{p}{2(1-p)^2}\lambda_2^2 k_1 + \frac{p}{1-p} - \left(b + \frac{p}{1-p}\lambda_2 k_1\right)D_2(t) + \frac{1}{2}k_1 D_2^2(t) = 0, \tag{36}$$

$$\dot{D}_1(t) + \frac{p}{2(1-p)^2}\left(\lambda_1^2 + \lambda_2^2 k_2\right) + \left(a - \frac{p}{1-p}\lambda_2 k_2\right)D_2(t) + \frac{1}{2}k_2 D_2^2(t) = 0. \tag{37}$$

For the quadratic equation

$$-\frac{1}{2}k_1 D_2^2(t) + \left(b + \frac{p}{1-p}\lambda_2 k_1\right)D_2(t) - \left(\frac{p}{2(1-p)^2}\lambda_2^2 k_1 + \frac{p}{1-p}\right) = 0, \tag{38}$$

it is easy to calculate its discriminant, which is given by

$$\Delta_1 = \frac{p}{1-p}\left(\frac{1}{p}b^2 - 2k_1 - \left(b - \lambda_2 k_1\right)^2\right). \tag{39}$$

(i) If $0 < p < 1$, then when $p < b^2/(2k_1 + (b-\lambda_2 k_1)^2)$, that is, $0 < p < \min\{1, b^2/(2k_1 + (b-\lambda_2 k_1)^2)\}$, we have $\Delta_1 > 0$.

(ii) If $p < 0$, we find that $\Delta_1 > 0$ always holds.

Hence, under the condition of $p < \min\{1, b^2/(2k_1 + (b-\lambda_2 k_1)^2)\}$ and $p \neq 0$, we have $\Delta_1 > 0$. In addition, two different roots of (38) are given by

$$m_{1,2} = \frac{b + (p/(1-p))\lambda_2 k_1}{k_1} \pm \frac{\sqrt{\Delta_1}}{k_1}. \tag{40}$$

Further, (36) can be rewritten as

$$\frac{1}{m_1 - m_2}\int_t^T \left(\frac{1}{D_2(s) - m_1} - \frac{1}{D_2(s) - m_2}\right)dD_2(s) = -\frac{1}{2}k_1(T-t). \tag{41}$$

After easy calculations, we have

$$D_2(t) = \frac{m_1 m_2\left(1 - \exp\{-(1/2)k_1(m_1 - m_2)(T-t)\}\right)}{m_1 - m_2 \cdot \exp\{-(1/2)k_1(m_1 - m_2)(T-t)\}}. \tag{42}$$

By using (37) $\times$ k_1 – (36) $\times$ k_2, we get

$$\dot{D}_1(t) = \frac{k_2}{k_1}\dot{D}_2(t) + \frac{ak_1 + bk_2}{k_1}D_2(t) + \frac{p}{2(1-p)^2}\cdot\frac{\lambda_1^2}{k_1} - \frac{p}{1-p}\cdot\frac{k_2}{k_1}. \tag{43}$$

Integrating (43) from t to T, we derive

$$D_1(t) = \frac{k_2}{k_1}D_2(t) - \frac{ak_1 + bk_2}{k_1}\int_t^T D_2(s)\,ds - \left(\frac{p}{2(1-p)^2}\cdot\frac{\lambda_1^2}{k_1} - \frac{p}{1-p}\cdot\frac{k_2}{k_1}\right)(T-t). \tag{44}$$

As a result, Lemma 6 is completed. □

Lemma 7. *Assume that a solution of* (33) *is given by the structure* $J(t,r) = (u-\lambda_1 v)\int_t^T \widehat{J}(s,r)ds$; *then* $\widehat{J}(t,r)$ *satisfies the following PDE:*

$$\widehat{J}_t - r\widehat{J} + \left(a - br + \lambda_2\sigma_r^2\right)\widehat{J}_r + \frac{1}{2}\sigma_r^2\widehat{J}_{rr} = 0, \quad \widehat{J}(T,r) = 1. \tag{45}$$

Proof. Introducing the following variational operator on any function $J(t,r)$:

$$\nabla J(t,r) = -rJ + \left(a - br + \lambda_2\sigma_r^2\right)J_r + \frac{1}{2}\sigma_r^2 J_{rr}, \tag{46}$$

(33) can be rewritten as

$$\frac{\partial J(t,r)}{\partial t} + \nabla J(t,r) + u - \lambda_1 v = 0, \quad J(T,r) = 0. \tag{47}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial J(t,r)}{\partial t} &= -(u-\lambda_1 v)\widehat{J}(t,r)\\ &= (u-\lambda_1 v)\left(\int_t^T \frac{\partial \widehat{J}(s,r)}{\partial s}ds - \widehat{J}(T,r)\right),\\ \nabla J(t,r) &= (u-\lambda_1 v)\int_t^T \nabla\widehat{J}(s,r)\,ds.\end{aligned} \tag{48}$$

Putting (48) into (47), we get

$$(u-\lambda_1 v)\left(\int_t^T\left(\frac{\partial\widehat{J}(s,r)}{\partial s} + \nabla\widehat{J}(s,r)\right)ds - \widehat{J}(T,r) + 1\right) = 0. \tag{49}$$

Therefore, we obtain

$$\frac{\partial\widehat{J}(s,r)}{\partial s} + \nabla\widehat{J}(s,r) = 0, \quad \widehat{J}(T,r) = 1. \tag{50}$$

That is, (45) holds. □

Lemma 8. *Assume that a solution of* (45) *is of the form* $\widehat{J}(t,r) = e^{D_3(t)+D_4(t)r}$, *with terminal conditions* $D_3(T) = 0$ *and* $D_4(T) = 0$; *then* $D_4(t)$ *and* $D_3(t)$ *are given by* (56) *and* (58), *respectively.*

Proof. Plugging $\widehat{J}(t,r) = e^{D_3(t)+D_4(t)r}$ into (45), we derive

$$\begin{aligned}e^{D_3(t)+D_4(t)r}\Big(&\dot{D}_3(t) + (\lambda_2 k_2 + a)D_4(t) + \frac{1}{2}k_2 D_4^2(t)\\ &+ r\Big(\dot{D}_4(t) - 1 + (\lambda_2 k_1 - b)D_4(t)\\ &+ \frac{1}{2}k_1 D_4^2(t)\Big)\Big) = 0.\end{aligned} \tag{51}$$

Comparing the coefficients on both sides of (51), we get the following two equations:

$$\begin{aligned}\dot{D}_4(t) - 1 + (\lambda_2 k_1 - b)D_4(t) + \frac{1}{2}k_1 D_4^2(t) = 0,\\ D_4(T) = 0;\end{aligned} \tag{52}$$

$$\begin{aligned}\dot{D}_3(t) + (\lambda_2 k_2 + a)D_4(t) + \frac{1}{2}k_2 D_4^2(t) = 0,\\ D_3(T) = 0.\end{aligned} \tag{53}$$

For (52), we find that the discriminant of the quadratic equation

$$-\frac{1}{2}k_1 D_4^2(t) - (\lambda_2 k_1 - b)D_4(t) + 1 = 0 \tag{54}$$

is given by $\Delta_2 = (\lambda_2 k_1 - b)^2 + 2k_1 > 0$. Therefore, two different roots of (54) are given by

$$m_{3,4} = \frac{b - \lambda_2 k_1}{k_1} \pm \frac{\sqrt{\Delta_2}}{k_1}. \tag{55}$$

Using the same approach as (36), we obtain

$$D_4(t) = \frac{m_3 m_4\left(1 - \exp\{-(1/2)k_1(m_3 - m_4)(T-t)\}\right)}{m_3 - m_4\cdot\exp\{-(1/2)k_1(m_3 - m_4)(T-t)\}}. \tag{56}$$

By applying (53) × k_1− (52) × k_2, we get

$$\dot{D}_3(t) = \frac{k_2}{k_1}\dot{D}_4(t) - \frac{ak_1 + bk_2}{k_1}D_4(t) - k_2. \tag{57}$$

Further, after easy integration, we obtain

$$\begin{aligned}D_3(t) &= \frac{k_2}{k_1}D_4(t) + k_2(T-t) + \frac{ak_1 + bk_2}{k_1}\\ &\times\Bigg(m_4(T-t) + \frac{2}{k_1}\\ &\times \ln\frac{m_3 - m_4}{m_3 - m_4\cdot\exp\{-(1/2)k_1(m_3 - m_4)(T-t)\}}\Bigg).\end{aligned} \tag{58}$$

Therefore, the proof is completed. □

Lemma 9. *Suppose that* $h(t,r) = e^{D_5(t)+D_6(t)r}$ *is a solution of* (34), *with boundary conditions* $D_5(T) = 0$ *and* $D_6(T) = 0$; *then one has* $D_5(t) = D_3(t)$ *and* $D_6(t) = D_4(t)$.

Proof. Investigating (34), we find that (34) and (45) have the same solutions. Hence, the conclusion holds. □

Applying (24), (29), (30), and Lemmas 6–9, we have

$$\begin{aligned}\frac{H_x}{H_{xx}} &= -z\widehat{H}_{zz} = zg_z = -\frac{1}{q}z^{1/(p-1)}f\\ &= -\frac{1}{1-p}\left(g + \frac{1-p}{q}\eta h - J\right),\end{aligned}$$

$$\begin{aligned}\frac{H_{rx}}{H_{xx}} = \widehat{H}_{rz} = -g_r &= -\left(\frac{1-p}{q}z^{1/(p-1)}f_r - \frac{1-p}{q}\eta h_r + J_r\right)\\ &= -\frac{1-p}{q}z^{1/(p-1)}D_2(t)f + \frac{1-p}{q}\eta D_6(t)h - J_r\\ &= -D_2(t)\left(g + \frac{1-p}{q}\eta h - J\right) + \frac{1-p}{q}\eta D_6(t)h - J_r.\end{aligned} \tag{59}$$

Namely, we obtain

$$\begin{aligned}\frac{H_x}{H_{xx}} &= -\frac{1}{1-p}(X(t) - J(t,r)) - \frac{1}{q}\eta h(t,r),\\ \frac{H_{rx}}{H_{xx}} &= -D_2(t)(X(t) - J(t,r))\\ &\quad - \frac{1-p}{q}\eta h(t,r)(D_2(t) - D_6(t)) - \frac{\partial J(t,r)}{\partial r}.\end{aligned} \tag{60}$$

Meantime, considering (29) and letting $g(t,r,z) = x$, we derive

$$z = \left(\frac{q}{1-p}(x - J(t,r)) + \eta h(t,r)\right)^{p-1} f^{1-p}(t,r). \tag{61}$$

By using $H_x = z$, we obtain the following optimal value function:

$$\begin{aligned}&H^*(t,r,x)\\ &\quad = \frac{1-p}{qp}\left(\frac{q}{1-p}(x - J(t,r)) + \eta h(t,r)\right)^p f^{1-p}(t,r).\end{aligned} \tag{62}$$

In conclusion, we can summarize the above results in the following theorem.

Theorem 10. *For HARA utility* (9)*, if a solution to the HJB equation* (16) *is given by $H(t,r,x)$, then, under the condition of $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ and $p \neq 0$, the optimal investment strategies for the problem* (8) *are given by*

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\left(\frac{1}{1-p}(X(t) - J(t,r)) + \frac{1}{q}\eta h(t,r)\right) + \frac{\nu}{\sigma_1}, \tag{63}$$

$$\begin{aligned}&\pi_2^*(t)\\ &\quad = \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\\ &\qquad \cdot\left(\frac{1}{1-p}(X(t) - J(t,r)) + \frac{1}{q}\eta h(t,r)\right)\\ &\qquad - \frac{\sigma_r}{\sigma_B}\cdot\left(D_2(t)(X(t) - J(t,r)) + \frac{1-p}{q}\eta h(t,r)\right.\\ &\qquad\qquad \left.\times(D_2(t) - D_6(t)) + \frac{\partial J(t,r)}{\partial r}\right) - \frac{\sigma_2}{\sigma_1}\cdot\frac{\sigma_r}{\sigma_B}\nu,\end{aligned} \tag{64}$$

where $J(t,r) = (u - \lambda_1\nu)\int_t^T e^{D_3(s)+D_4(s)r}ds$, $h(t,r) = e^{D_5(t)+D_6(t)r}$, and $D_2(t)$ are given by Lemmas 8, 9, and 6, respectively. Moreover, $\partial J(t,r)/\partial r$ in (64) *is given by*

$$\frac{\partial J(t,r)}{\partial r} = (u - \lambda_1\nu)\int_t^T D_4(s)e^{D_3(s)+D_4(s)r}ds. \tag{65}$$

The following theorem verifies that the optimal strategy given by Theorem 10 is optimal for the problem (8).

Theorem 11 (verification theorem). *If $H(t,r,x)$ is a solution to* (13)*, that is, $H(t,r,x)$ satisfies*

$$\sup_{\pi(t)\in\Pi}\mathscr{A}H(t,r,x) = 0, \tag{66}$$

then for all admissible strategies $\pi(t) = (\pi_1(t), \pi_2(t)) \in \Pi$, one has $V(t,r,x) \leqslant H(t,r,x)$; if $\pi^*(t) = (\pi_1^*(t), \pi_2^*(t))$ satisfies

$$\pi^*(t) \in \arg\sup_{\pi(t)\in\Pi}\mathscr{A}H(t,r,x), \tag{67}$$

then one has $V(t,r,x) = H(t,r,x)$, and it implies that $\pi^*(t) = (\pi_1^*(t), \pi_2^*(t)) \in \Pi$ is the optimal investment strategy of the problem (8).

Proof. Considering $H(t,r,x) \in C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ and using Itô formula from t to T for $H(t,r,x)$, we obtain

$$\begin{aligned}&H(T, r(T), X(T))\\ &\quad = H(t,r,x) + \int_t^T \mathscr{A}^{\pi(t)}H(s, r(s), X(s))\,ds\\ &\qquad + \int_t^T (\pi_1(s)\sigma_1 - \nu)H_x(s, r(s), X(s))\,dW_S(s)\\ &\qquad + \int_t^T \left(\pi_1(s)\sigma_2\sqrt{k_1 r(s) + k_2} + \pi_2(s)\sigma_B\right)\\ &\qquad\qquad \times H_x(s, r(s), X(s))\,dW_r(s)\\ &\qquad - \int_t^T H_r(s, r(s), X(s))\sqrt{k_1 r(s) + k_2}\,dW_r(s).\end{aligned} \tag{68}$$

For $H(t,r,x)$ is a solution to (13); that is, we have $\sup_{\pi(t)\in\Pi}\mathscr{A}^{\pi(t)}H(s,r(s),X(s)) = 0$. So we obtain

$$\begin{aligned}&H(T, r(T), X(T))\\ &\quad \leqslant H(t,r,x)\\ &\qquad + \int_t^T (\pi_1(s)\sigma_1 - \nu)H_x(s, r(s), X(s))\,dW_S(s)\\ &\qquad + \int_t^T \left(\pi_1(s)\sigma_2\sqrt{k_1 r(s) + k_2} + \pi_2(s)\sigma_B\right)\\ &\qquad\qquad \times H_x(s, r(s), X(s))\,dW_r(s)\\ &\qquad - \int_t^T H_r(s, r(s), X(s))\sqrt{k_1 r(s) + k_2}\,dW_r(s).\end{aligned} \tag{69}$$

The last three terms in (69) are local martingales and their expected values are equal to zero. Thus, we get

$$\mathbb{E}\left(H\left(T, r\left(T\right), X\left(T\right)\right) \mid r\left(t\right) = r, X\left(t\right) = x\right) \leqslant H\left(t, r, x\right). \tag{70}$$

Maximizing (70) for all admissible strategies $\pi(t) = (\pi_1(t), \pi_2(t)) \in \Pi$, we derive

$$V\left(t, r, x\right) \leqslant H\left(t, r, x\right). \tag{71}$$

When $\pi(t) = \pi^*(t)$, all inequalities become equalities; that is, $H(t, r, x) = V(t, r, x)$, and $\pi^*(t) = (\pi_1^*(t), \pi_2^*(t))$ is the optimal investment strategy of the problem (8).

The proof is completed. □

Remark 12. Some interesting conclusions can be seen from (63) and (64). (i) The optimal amount $\pi_1^*(t)$ invested in the stock is affected by the parameters a, b, k_1, k_2, λ_1, σ_1, λ_2, u, and v but does not depend on the parameter σ_2. In fact, σ_2 has an influence on the dynamics of stock price, which can be observed from (3). (ii) The optimal amount $\pi_2^*(t)$ invested in the zero-coupon bond depends on all the model parameters $a, b, k_1, k_2, \lambda_1, \sigma_1, \lambda_2, u, v$, and σ_2; however, the price dynamics of zero-coupon bond is only impacted by b, k_1, k_2, and λ_2.

According to Theorem 10, we derive the following three corollaries.

Corollary 13. *If $\eta = 0$ and $q = 1 - p$, HARA utility* (9) *is degenerated into power utility* (10). *Therefore, under the condition of $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ and $p \neq 0$, the optimal investment strategies for power utility are given by*

$$\pi_1^*\left(t\right) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right) + \frac{v}{\sigma_1}, \tag{72}$$

$$\begin{aligned}\pi_2^*\left(t\right) = {} & \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right) \\ & - \frac{\sigma_r}{\sigma_B} \cdot \left(D_2\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) + \frac{\partial J\left(t, r\right)}{\partial r}\right) \\ & - \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_r}{\sigma_B} v,\end{aligned} \tag{73}$$

where $J(t, r) = (u - \lambda_1 v)\int_t^T e^{D_3(s)+D_4(s)r} ds$ and $D_2(t)$ are given by Lemmas 8 and 6, respectively. Moreover, $\partial J(t, r)/\partial r$ in (73) *is still given by*

$$\frac{\partial J\left(t, r\right)}{\partial r} = \left(u - \lambda_1 v\right)\int_t^T D_4\left(s\right) e^{D_3(s)+D_4(s)r} ds. \tag{74}$$

Proof. It is very easy to derive that

$$\begin{aligned}\frac{H_x}{H_{xx}} &= -\frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right), \\ \frac{H_{rx}}{H_{xx}} &= -D_2\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) - \frac{\partial J\left(t, r\right)}{\partial r}.\end{aligned} \tag{75}$$

Therefore, (72) and (73) hold. □

Corollary 14. *If $\eta = 1$ and $p \to -\infty$, HARA utility* (9) *is reduced to exponential utility* (11); *then the optimal policies for exponential utility are determined by*

$$\pi_1^*\left(t\right) = \frac{\lambda_1}{\sigma_1} \cdot \frac{1}{q} h\left(t, r\right) + \frac{v}{\sigma_1}, \tag{76}$$

$$\begin{aligned}&\pi_2^*\left(t\right) \\ &= \frac{\sigma_r}{\sigma_B} \cdot \frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1} \cdot \frac{1}{q} h\left(t, r\right) - \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_r}{\sigma_B} v \\ &\quad - \frac{\sigma_r}{\sigma_B} \cdot \left(D_4\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) + \frac{\partial J\left(t, r\right)}{\partial r}\right. \\ &\qquad + \frac{1}{q} h\left(t, r\right)\int_t^T \left(-\frac{1}{2}\lambda_2^2 k_1 + 1 - \lambda_2 k_1 D_4\left(s\right)\right) \\ &\qquad \left.\cdot \exp\left\{\int_t^s \left(k_1 D_4\left(z\right) - b + \lambda_2 k_1\right) dz\right\} ds\right),\end{aligned} \tag{77}$$

where $J(t, r) = (u - \lambda_1 v)\int_t^T e^{D_3(s)+D_4(s)r} ds$ and $h(t, r) = e^{D_5(t)+D_6(t)r}$ are given by Lemmas 8 and 9, respectively. Moreover, $\partial J(t, r)/\partial r$ in (77) *is still given by*

$$\frac{\partial J\left(t, r\right)}{\partial r} = \left(u - \lambda_1 v\right)\int_t^T D_4\left(s\right) e^{D_3(s)+D_4(s)r} ds. \tag{78}$$

Proof. When $p \to -\infty$, then we have $D_2(t) \to D_4(t)$ and $D_6(t) = D_4(t)$. Moreover, we derive that

$$\begin{aligned}&\lim_{p\to-\infty} \frac{H_x}{H_{xx}} \\ &\quad = \lim_{p\to-\infty}\left(-\frac{1}{1-p}\left(X\left(t\right) - J\left(t, r\right)\right) - \frac{1}{q}\eta h\left(t, r\right)\right) \\ &\quad = -\frac{1}{q}\eta h\left(t, r\right), \\ &\lim_{p\to-\infty} \frac{H_{rx}}{H_{xx}} \\ &\quad = \lim_{p\to-\infty}\left(-D_2\left(t\right)\left(X\left(t\right) - J\left(t, r\right)\right) - \frac{1-p}{q}\eta h\left(t, r\right)\right. \\ &\qquad \left.\times \left(D_2\left(t\right) - D_6\left(t\right)\right) - \frac{\partial J\left(t, r\right)}{\partial r}\right).\end{aligned} \tag{79}$$

On the other hand, we have

$$\begin{aligned}&\lim_{p\to-\infty} \frac{1-p}{q}\eta h\left(t, r\right)\left(D_2\left(t\right) - D_6\left(t\right)\right) \\ &\quad = \frac{1}{q}\eta h\left(t, r\right) \lim_{p\to-\infty} \frac{D_2\left(t\right) - D_6\left(t\right)}{1/\left(1-p\right)}. \quad \left(\frac{0}{0}\right)\end{aligned} \tag{80}$$

Further, we obtain $\partial D_6(t)/\partial p = \partial D_4(t)/\partial p = 0$ and

$$\frac{\partial D_2(t)}{\partial p} = \int_t^T \left(\frac{1+p}{2(1-p)^3}\lambda_2^2 k_1 + \frac{1}{(1-p)^2}\left(1-\lambda_2 k_1 D_2(s)\right)\right) \times \exp\left\{\int_t^s \left(k_1 D_2(z) - b - \frac{p}{1-p}\lambda_2 k_1\right) dz\right\} ds. \tag{81}$$

Using L'Hopital's rule, (80) is equal to

$$\frac{1}{q}\eta h(t,r)\int_t^T \left(-\frac{1}{2}\lambda_2^2 k_1 + 1 - \lambda_2 k_1 D_4(s)\right) \times \exp\left\{\int_t^s \left(k_1 D_4(z) - b + \lambda_2 k_1\right) dz\right\} ds. \tag{82}$$

Let $\eta = 1$; we obtain that (76) and (77) hold. □

Corollary 15. *If $\eta = 0$, $p \to 0$, and $q \to 1$, HARA utility is reduced to logarithm utility $U_{\log}(x) = \ln x$; then the optimal policies for logarithm utility are given by*

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\left(X(t) - J(t,r)\right) + \frac{v}{\sigma_1}, \tag{83}$$

$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\cdot\left(X(t) - J(t,r)\right) - \frac{\sigma_r}{\sigma_B}\cdot\frac{\partial J(t,r)}{\partial r} - \frac{\sigma_2}{\sigma_1}\cdot\frac{\sigma_r}{\sigma_B}v, \tag{84}$$

where $J(t,r) = (u-\lambda_1 v)\int_t^T e^{D_3(s)+D_4(s)r} ds$ is given by Lemma 8. Moreover, $\partial J(t,r)/\partial r$ in (83) is still given by

$$\frac{\partial J(t,r)}{\partial r} = (u-\lambda_1 v)\int_t^T D_4(s)\, e^{D_3(s)+D_4(s)r} ds. \tag{85}$$

Proof. If $p \to 0$, then we have $D_2(t) \to 0$. Therefore, we derive

$$\frac{H_x}{H_{xx}} = -\left(X(t) - J(t,r)\right), \qquad \frac{H_{rx}}{H_{xx}} = -\frac{\partial J(t,r)}{\partial r}. \tag{86}$$

Therefore, (83) and (84) hold. □

5. Special Cases

In this section, we give several special cases for HARA utility, power utility, exponential utility, and logarithm utility, respectively.

In our model, if we do not consider liability factor, that is, $u = v = 0$, then we have $J(t,r) = 0$. Therefore, we obtain the following special cases.

Special Case 1. Under HARA utility (9), if there is no liability and p satisfies the conditions $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ and $p \neq 0$, then the optimal policies for the problem (8) are given by

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\left(\frac{1}{1-p}X(t) + \frac{1}{q}\eta h(t,r)\right),$$
$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\cdot\left(\frac{1}{1-p}X(t) + \frac{1}{q}\eta h(t,r)\right) - \frac{\sigma_r}{\sigma_B}\cdot\left(D_2(t)X(t) + \frac{1-p}{q}\eta h(t,r)\left(D_2(t) - D_6(t)\right)\right), \tag{87}$$

where $h(t,r) = e^{D_5(t)+D_6(t)r}$ and $D_2(t)$ are given by Lemmas 9 and 6, respectively.

Special Case 2. If utility function is given by $U_{\text{power}}(x) = x^p/p$, $p < 1$ and $p \neq 0$, and p satisfies the conditions $p < \min\{1, b^2/(2k_1 + (b - \lambda_2 k_1)^2)\}$ and $p \neq 0$, then the optimal policies for the problem (8) under the condition of no liability are given by

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\frac{1}{1-p}X(t),$$
$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\cdot\frac{1}{1-p}X(t) - \frac{\sigma_r}{\sigma_B}\cdot D_2(t)X(t), \tag{88}$$

where $D_2(t)$ is given by Lemma 6.

Special Case 3. If utility function is given by $U_{\exp}(x) = -e^{-qx}/q$, $q > 0$, then the optimal policies for the problem (8) under the condition of no liability are

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\frac{1}{q}h(t,r),$$
$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\cdot\frac{1}{q}h(t,r) - \frac{\sigma_r}{\sigma_B}\cdot D_4(t)X(t),$$
$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\cdot\frac{1}{q}h(t,r) - \frac{\sigma_r}{\sigma_B}\cdot\left(D_4(t)X(t) + \frac{1}{q}h(t,r)\int_t^T\left(-\frac{1}{2}\lambda_2^2 k_1 + 1 - \lambda_2 k_1 D_4(s)\right)\cdot\exp\left\{\int_t^s\left(k_1 D_4(z) - b + \lambda_2 k_1\right)dz\right\}ds\right), \tag{89}$$

where $D_4(t)$ and $h(t,r) = e^{D_5(t)+D_6(t)r}$ are given by Lemmas 8 and 9, respectively.

Special Case 4. If utility function is given by $U_{\log}(x) = \ln x$, then the optimal policies for the problem (8) with no liability are

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot X(t),$$
$$\pi_2^*(t) = \frac{\sigma_r}{\sigma_B}\cdot\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1}\cdot X(t) - \frac{\sigma_r}{\sigma_B}\cdot D_2(t)X(t), \tag{90}$$

where $D_2(t)$ are given by Lemma 6.

On the other hand, if we consider the problem (8) in the constant interest rate environments, that is, $a = b = k_1 = k_2 = 0$, then the price dynamics of zero-coupon bond is degenerated into that of risk-free asset. As a result, the optimal amount invested in the zero-coupon bond is zero; that is, $\pi_2^*(t) = 0$. And it leads to some new expressions:

$$D_1(t) = \frac{p}{2(1-p)^2}\lambda_1^2(T-t),$$
$$D_2(t) = \frac{p}{1-p}(T-t), \tag{91}$$
$$D_3(t) = 0, \qquad D_4(t) = -(T-t).$$

Therefore, we can derive the optimal policy of the problem (8) with liability in the following special cases.

Special Case 5. Under constant interest rate model, if utility function is given by HARA function (9), then the optimal policy of the problem (8) with liability is given by

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\left(\frac{1}{1-p}\left(X(t) - (u-\lambda_1 v)\int_t^T e^{-(T-s)r}ds\right) + \frac{1}{q}\eta e^{-(T-t)r}\right) + \frac{v}{\sigma_1}. \tag{92}$$

Special Case 6. If utility function is given by $U_{\text{power}}(x) = x^p/p$, $p < 1$ and $p \neq 0$, then the optimal policy of the problem (8) with liability and constant interest rate is

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\frac{1}{1-p}\left(X(t) - (u-\lambda_1 v)\int_t^T e^{-(T-s)r}ds\right) + \frac{v}{\sigma_1}. \tag{93}$$

Special Case 7. If utility function is given by $U_{\exp}(x) = -e^{-qx}/q$, $q > 0$, then the optimal policy of the problem (8) with liability and constant interest rate is

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\frac{1}{q}e^{-(T-t)r} + \frac{v}{\sigma_1}. \tag{94}$$

Special Case 8. If utility function is given by $U_{\log}(x) = \ln x$, then the optimal policy of the problem (8) with liability and constant interest rate is

$$\pi_1^*(t) = \frac{\lambda_1}{\sigma_1}\cdot\left(X(t) - (u-\lambda_1 v)\int_t^T e^{-(T-s)r}ds\right) + \frac{v}{\sigma_1}. \tag{95}$$

6. Numerical Analysis

In this section, we provide a numerical example to illustrate the effect of market parameters on the optimal investment strategy and compare our results with those in the existing literature. Throughout this section, unless otherwise stated, the basic parameters are given by $a = 0.18$, $b = 0.23$, $k_1 = 0.7$, $k_2 = 0.9$, $r(0) = 0.05$, $\lambda_1 = 0.6$, $\lambda_2 = 0.8$, $\sigma_1 = 1.9$, $\sigma_2 = 1.7$, $u = 0.6$, $v = 0.6$, $p = -0.2$, $q = 2$, $\eta = 0.02$, $t = 0$, $T = 1$, and $x_0 = 100$.

Table 1: The optimal policy with liability and affine interest rate.

	HARA utility	Power utility	Exponential utility	Logarithm utility
Stock	26.6064	26.6049	0.390775	31.8627
Bond	37.8308	37.8343	99.4978	24.7162
Cash	35.5629	35.5608	0.111445	43.4211

Table 2: The optimal policy only with affine interest rate.

	HARA utility	Power utility	Exponential utility	Logarithm utility
Stock	26.3173	26.3158	0.0749854	31.5789
Bond	38.3303	38.3338	100.06	25.2023
Cash	35.3525	35.3504	−0.134829	43.2187

Table 3: The optimal policy with liability and constant interest rate.

	HARA utility	Power utility	Exponential utility	Logarithm utility
Stock	26.573	26.57	0.465984	31.8208
Cash	73.427	73.43	99.534	68.1792

By applying the above conclusions obtained, some optimal investment policies are calculated in Tables 1, 2, and 3. From Tables 1–3, we draw some conclusions as follows.

(a1) The optimal policy under HARA utility is roughly equivalent to that under power utility but is markedly different from that under exponential utility or logarithm utility.

(a2) In the stochastic interest rate environments, the optimal amount invested in the stock and zero-coupon bond under HARA utility is less than that under exponential utility but is more than that under logarithm utility. This situation in the constant interest rate environments is contrary to that in the stochastic interest rate environments.

(a3) The optimal amount with liability and affine interest rate invested in the risky assets (including a stock and a zero-coupon bond) is less than that only with affine interest rate but is more than that with liability and constant interest rate.

In order to illustrate the impact of model parameters on the optimal investment strategy with liability and affine interest rate, we depict the following graphs. In Figures 1 and 2, the optimal amount invested in the stock is depicted by the dashed line and is denoted by $\pi_1^*(t)$; the optimal amount invested in the zero-coupon bond is depicted by the orange line and is denoted by $\pi_2^*(t)$; the optimal amount invested in the cash is depicted by the thick line and is denoted by $\pi_0^*(t)$.

Form Figures 1 and 2, we can draw some conclusions as follows.

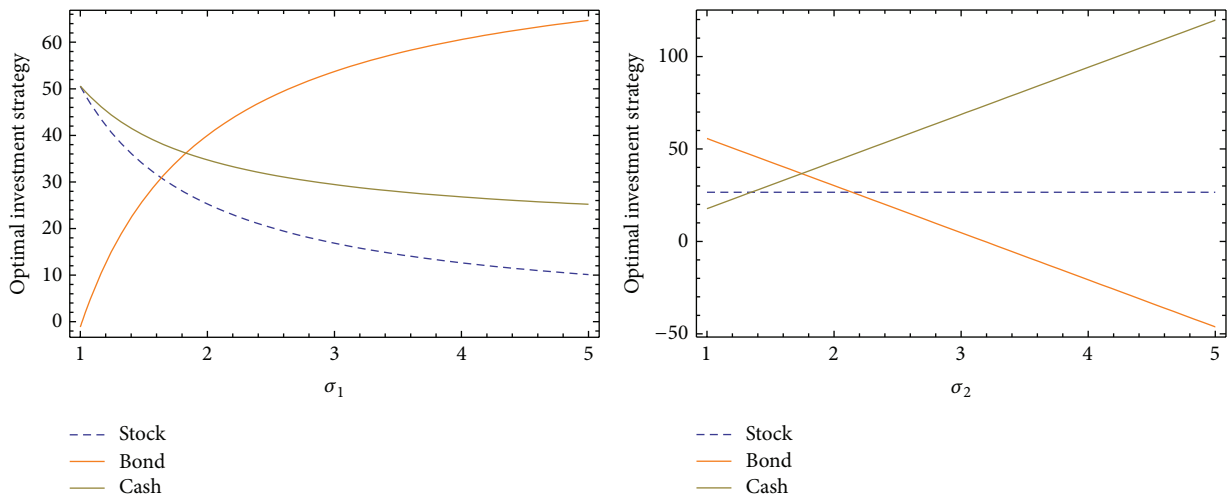

Figure 1: The effect of σ_1 and σ_2 on the optimal strategy.

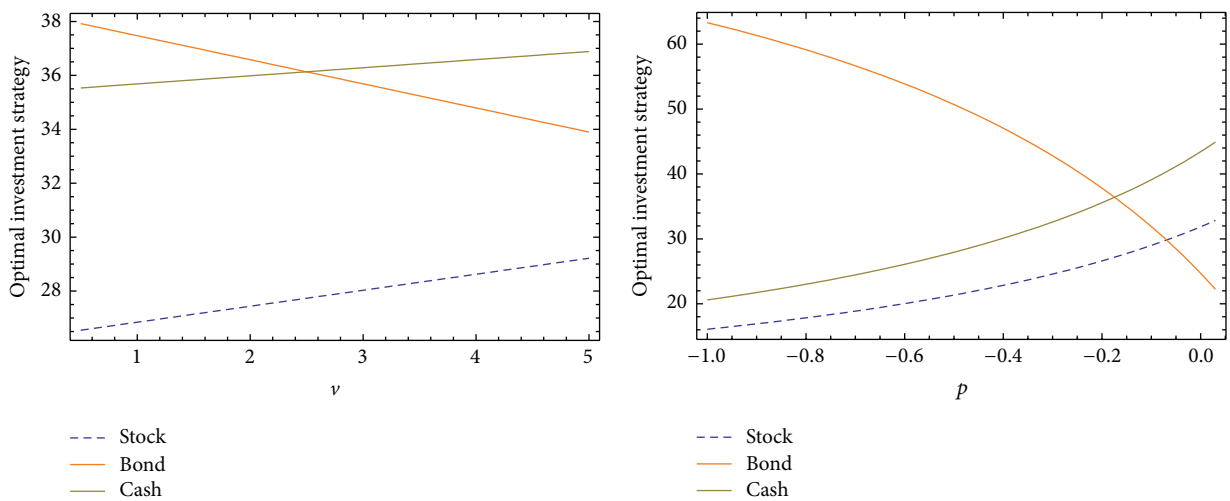

Figure 2: The effect of v and p on the optimal strategy.

(b1) $\pi_0^*(t)$ and $\pi_1^*(t)$ decrease with respect to the parameter σ_1, while $\pi_2^*(t)$ increases with respect to σ_1. In fact, as the value of σ_1 is increasing, the volatility risk of the stock is also increasing. It leads to that the investors would invest less money in the stock. In order to hedge interest rate risk, the investors need to invest much more money in the zero-coupon bond and less money in the cash.

(b2) $\pi_1^*(t)$ does not depend on the parameter σ_2, while $\pi_2^*(t)$ decreases and $\pi_0^*(t)$ increases with respect to σ_2. It shows that the investors need to invest less money in the zero-coupon bond and more money in the cash when the value of σ_2 increases.

(b3) $\pi_0^*(t)$ and $\pi_1^*(t)$ increase with respect to the parameter v, while $\pi_2^*(t)$ decreases with respect to v. As a matter of fact, as the value of v becomes larger, the volatility of liability is increasing. This means that the investors need to invest more money in the stock and cash in order to hedge the risk that resulted from liability.

(b4) $\pi_0^*(t)$ and $\pi_1^*(t)$ increase with respect to the parameter p, while $\pi_2^*(t)$ decreases with respect to p. This shows that the larger the value of p, the more the amount invested in the stock and cash. It leads to the fact that less money is invested in the zero-coupon bond.

7. Conclusions

This paper investigates the optimal investment strategy for an ALM problem in the HARA utility framework, where interest rate is supposed to be driven by an affine interest rate model, while liability process follows Brownian motion with drift. By applying dynamic programming principle and Legendre transform, we obtain the explicit expressions of the optimal investment strategies. Some special cases are discussed. Finally, we illustrate the impact of model parameters on

the optimal policy by providing a numerical example. Some interesting conclusions are found as follows: (i) although the parameter σ_2 has an influence on the dynamics of stock price, the optimal amount invested in the stock does not depend on the value of σ_2; (ii) the optimal amount under HARA utility is roughly equivalent to that under power utility but is markedly different from that under exponential utility and logarithm utility.

As far as we know, there is little work on the ALM problems with stochastic interest rate in the existing literature. However, our work has also some limits: (i) we only consider the liability process driven by the drifted Brownian motion, which is the simplest stochastic process; (ii) we study an ALM problem in the utility framework and do not consider it in a continuous-time mean-variance framework. We would leave those problems for future research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is supported by Humanities and Social Science Research Youth Foundation of Ministry of Education of China (no. 11YJC790006), Center for Research of Regulation and Policy of Zhejiang Province of China (no. 13JDGZ03YB), the Project of National Statistical Science (no. 2013LY125), and Higher School Science and Technology Development Foundation of Tianjin (no. 20100821).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 393168, 9 pages
http://dx.doi.org/10.1155/2014/393168

Research Article

Dynamics and Motivations of Media Marketing: The Role of Globalization and Empowerment

Cinzia Colapinto[1] and Eleonora Benecchi[2]

[1] *Department of Management, Ca' Foscari University of Venice, San Giobbe, Cannaregio, 873-30121 Venice, Italy*
[2] *Institute of Media and Journalism, Università della Svizzera Italiana, Via Buffi 13, 6900 Lugano, Switzerland*

Correspondence should be addressed to Cinzia Colapinto; cinzia.colapinto@unive.it

Received 27 February 2014; Accepted 9 May 2014; Published 3 June 2014

Academic Editor: Simone Marsiglio

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Audience fragmentation, convergence, and technology development have changed consumption patterns and brand communication. Media companies should consider TV series and movies as brands to be exploiting at national/international level. Using a dynamic diffusion model, we analyze the evolution of popularity in a specific country under different hypotheses. Then we propose an optimal decision model which describes the decision maker's point of view in a two-sided market.

1. Introduction

Many changes (such as media convergence, transition to digital format, and audience fragmentation) are affecting media companies' business model and have given to media content a primary role. Indeed, media environment has changed and in order to better satisfy audience's new tastes and consumption paths, most broadcasters have been transforming into multichannel and multiservices companies, whose supply is characterized by interactivity [1]. Companies, actors, or sportsmen rely more and more on the power of new media and of a viral spread, which is heightened by online communities and the cross-pollination of content on other social media sites [2–4]. The real value of social media consists in rebuilding the buzz and reinforcing the attitude that distinguishes the brand from its rivals. Media companies need to include consumers in their value chain (a real value constellation) recalling Toffler's [5] concept of *prosumer*: this term identified consumer involvement in product design; nowadays consumers are involved in value creation. Content is now being produced and consumed in innovative ways; consequently, internet and new media have become both a viewing site and a site of fan engagement. Technological advances lead towards new media content models, mainly based on participatory media [6]. Such a new strategy deeply relies on consumer behaviours. Thus, focusing on the concept of brand, we believe TV series are an interesting case study.

Television programs are a public good, and they are sold through many different channels over different points in time to maximize profits (windowing). As people are doing their best to avoid commercials, companies need to aggregate and to encourage customers to come back for more; as a result, Internet and new and social media are becoming very important marketing tools. The synergistic marketing communication [7] implies the use of different channels (often through synergistic ownership strategies) and approaches to persuasion: companies use multiple promotional outlets to reach desirable audiences. In particular, media marketing has moved toward the Internet discovering the importance of viral marketing, word of mouth, and online communities in all the phases of the life cycle of a product.

In a complex evolving system, TV programs' and marketing investment choices by media managers are getting more difficult and complex; they need to pursue data-driven decisions. Most analysis tolls consider TV programs ratings and we propose a decisional dynamic model which utilizes a dynamic model explaining the path of program popularity over time.

The paper is organized as follows. Section 2 introduces the concept of connected marketing and recalls the fundamental literature. Section 3 is dedicated to the diffusion model based on the literature of biological field and the related examples: in this way we intend to shed a light on the responses of TV series fans and their behaviour. We point out media managers' choices as well as viewers' actions. In Section 4 using real data we test our model and provide the long run behaviour of an imported TV series. Section 5 introduces a model to sustain the decision maker (DM) in the context of management operational policy. Section 6 concludes.

2. Connected Marketing: The Consumer-Led Future

It is increasingly clear that Marketing Departments for TV Shows behave according to the rule that the more endorsements that trickle through blogs, forums, Facebook, and Twitter, the better it is. Whether online activity, especially social media activity, could make or break a TV show is still unclear, however it is a fact that networks are increasingly trying to monitor Internet discussions to gain insight into whether a show will move from hype to hit. Recent deals between the major American networks and research organizations such as Networked Insight are proofs of that.

Marketing is crucial for media companies to survive and prosper in the current media ecosystem, characterized on one hand by an increasing competition for scarce audience and convergence and on the other by the empowerment of consumers. Media companies are special because they can be utilized as communication tools for self-branding [8]. In particular, considering TV series. it is evident that fans are being exposed to a greater number of products than ever before. That is why developing direct relationship with fans is vital for media companies in the wake of the growing exposure fans have to new products all competing for their time and attention. In this scenario, the role of peers and of direct marketing tools (websites, social media, ...) is getting important and Internet shakes up the long-standing balances of power between fans and the media content makers.

The connected marketing approach—word of mouth, buzz, and viral marketing, that is all different techniques originating from the sociological theory of the two-step flow [9]—dominates the information society. The main characteristic is the active role of actual and potential customers and the viral diffusion of messages, mimicking the idea of *prosumer*.

Fandom studies confirm that fans are becoming a relevant if not central part of the new television ecosystem: on one side the crucial role of fans in the new media ecosystem is enlightened by studies portraying fans as a selected but strategic audience for TV producers (fan as super-consumers) and therefore focusing on the incorporation of fan practices by the TV industry [10–12]; on the other side the power of fandom is recognized in its growing influence on creative and productive choices made by TV shows' creators (fans as creative subjects) and therefore the focus is on changing relationship between producers and consumers [13–15].

Considering fans either super-consumers or creative subjects, recent field studies demonstrated that media conglomerates have become more aware of how fan creations can add to and affect media product development, marketing, and distribution. Therefore they seek to harness fan activities for low-cost and effective advertisements [14, 16]. Nevertheless, it is also true that the monetization of fan creations is not an easy aim to accomplish: fans show an anticommercial nature and embrace a gift economy model [17], often refusing to be incorporated in the commercial production process [11].

All techniques are exploited in the digital ecosystem. Feedback opportunities provided by Web 2.0 technologies represent both new chances for audience engagement and new challenges for the television industry [13, 18]. Instead of replacing television, the Web seems to have embraced it and, with regard to TV series, the Internet seems to have turned television into a shared event for the most passionate viewers, who have become plugged-in, mainly sharing the show with the other viewers [19–21]. Indeed, through online communities passionate viewers find and share a privileged knowledge which is derived both from the viewing experience and the extratextual elements of the TV program [21]. Fans try to influence TV series production, promotion, and distribution in two main ways: (a) actively petitioning a TV network to keep a show on air and (b) participating online in official and unofficial promotional activities in order to keep communication about the show alive.

As we said, it is interesting to go beyond the opposition between producer and consumer, focusing on the role of fan communities in promoting TV series within the framework of the gift economy [14, 22–24]. The media product is a gift which the receiver can reciprocate through attention, feedback, fandom, or even purchasing advertised products. In particular, digital environments embrace the reply instead of reciprocation: it means giving a feedback to the author or passing along to others. Media fandom is a battleground where the territory between online gift economies and commodity culture will be negotiated.

In this paper we focus on American TV series, which are built in order to be "distributed" to an international audience [15] and use transmedia storytelling [25] as a promotional strategy to engage a wide and dispersed audience. Indeed, many TV series have been distributed with success in European countries, giving birth to strong fan communities [26, 27].

There exists a correlation between TV programs' ratings and choices (even the choice of doing nothing) by the DM: ratings act as a proxy of the product quality. However, the DM cannot completely control the program popularity because of the consumer empowerment. Media marketing managers usually pursue all possible actions to back a TV program and limit the risk of failure. However, this is a costly behaviour and understanding the impacts of specific actions is crucial to reduce investments and increase returns. To conclude, it is evident that popularity is affected and sustained not only by media managers' decisions, as viewers are empowered. Indeed, in Section 3 the model will consider both agents.

3. Popularity of TV Series: A Dynamic Model

Connected marketing is used to underpin the popularity of a brand or a product. The success is affected by the capability of generating engagement with customers and the engagement is relevant if we consider media content.

From now on, let $P(t)$ be a time dependent variable which describes the degree of popularity of a given product at the time t. We now formulate a model (similar models have been used in different contexts such as economic growth, population dynamics, and technology diffusion.) which describes the behaviour of P over time and allows us to draw long run analysis. We consider a local analysis, in other words, the diffusion path occurring in a specific country. We assume that popularity is driven by the following differential equation:

$$\frac{dP(t)}{dt} = (\eta - \delta) P(t) + \tau, \tag{1}$$

where

(i) δ is the natural depreciation rate ($0 < \delta < 1$) of popularity. Every product or service has a life cycle and when the market is mature, it enters the decline stage and may die. This term is also linked to the cultural discount [28] that is a particular program is rooted in one culture and thus is attractive in that environment, whilst it shall have diminished appeal elsewhere. The higher the cultural discount is, the higher the depreciation rate is;

(ii) η describes the actions made by the decision maker to support the campaign and by the other customers to spread information concerning the product;

(iii) τ is an exogenous factor, which takes into account the influence of the popularity of the product abroad. A negative value means that the product failed in a foreign market and had a bad reputation which can affect the campaign in the country under analysis. A positive value represents the success of the product or service abroad which can be translated in positive network externalities in the country under analysis. Even if it is a local model, we have to consider the effects derived from information flows and the track records of the product elsewhere.

It is possible to show that model (1) admits a solution of the form ($\eta \neq \delta$)

$$P(t) = \left(P(0) + \frac{\tau}{\eta - \delta}\right) e^{(-\delta+\eta)t} - \frac{\tau}{\eta - \delta}. \tag{2}$$

This kind of dynamic model formulated in terms of differential equation has been widely used in the economic and social science literature to describe the evolution of variables such as innovation, population, and human capital (see for instance [29–32]). If $\eta = \delta$ the above model (1) becomes trivial and it admits either a constant solution or a linear behavior. ($\tau = 0$ or $\tau \neq 0$, resp).

4. Model Discussion

In the following discussion we present for each possible solution a set of cases that could represent a concrete occurrence of the solution itself. The examples we are providing are only explanatory and limited to the field of American TV series imported in Italy. We decided to focus on American TV series for different reasons. First, foreign TV series and especially American ones have historically dominated the Italian television market. This gave us a sample which was internally coherent but provided also a wide set of cases to test our model with. Since we are taking into consideration not only marketing actions undertaken by the local networks in order to support specific TV series, but also cultural practices and fan activities that helped even unsupported TV series to gain strength in the local market, the fact that a remarkable number of American TV series had successfully conquered the Italian public, despite the inherent cultural differences between those shows and the local audience strategic [33], was also an influential factor in choosing our sample.

(A) $\tau = 0$ and $\eta = 0$. This is the case in which there is no influence from abroad and no actions are undertaken. The solution to (1) is given by $P(t) = P(0)e^{-\delta t}$. As a result P goes to zero in the long run (Figure 1): the popularity tends naturally to fade away. If the decision maker does not intervene, the attention of consumers will focus on something else.

This was the typical case of American TV series bought through an *output deal*, where the Italian broadcasters agreed to buy everything a specific major produces for an established period of time. This kind of blind agreement, obviously, leads to the purchase of TV series that are not strategic or even adapted to the broadcasting network entitled to air them. In many cases Italian TV channels used those "residual" products to fill up the night schedule or the many "void" spaces of a full day schedule. This meant that the popularity and ratings of the TV series abroad were not influential in choosing the time slot in which the products were aired, nor the broadcasting network felt the need to promote or push the product inside the local market.

One classic example of the case in which there is no influence from abroad and no actions is the prime time drama *Angel* (1999), a spin-off of the more popular *Buffy the Vampire Slayers*, produced by Warner Bros and purchased in Italy by Mediaset. The TV series was broadcasted in Italy by the private channel Italia 1, which had been airing Buffy as well, in the overnight programming and with an erratic schedule. It must be said that the purchase of *Angel* was a direct consequence of the previous purchase of *Buffy* and not a strategic choice and this explains the lack of "respect" for the product, both in its airing strategy and in its adaptation. In this case the popularity of the series abroad had no influence in the marketing and distribution in Italy and the degree of popularity of the TV series in the local market faded away quickly, despite the huge popularity of its mother series, *Buffy*, in the same market. The fact that Italia 1 never invested in the promotion of *Angel* and did not intend to profit from the popularity of the series abroad,

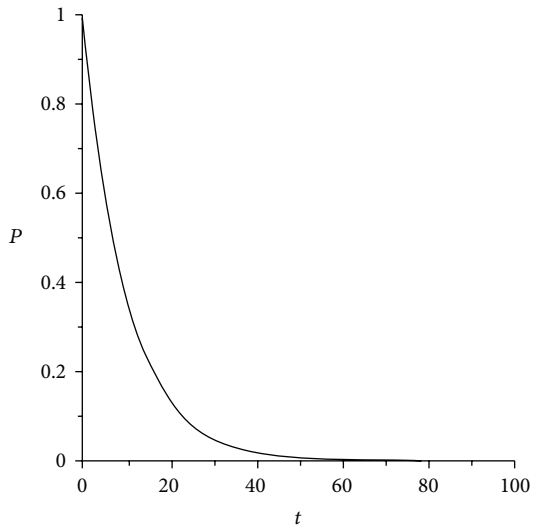

Figure 1: Popularity fades away.

ignoring it instead, was crucial to the lack of success of the series: the airing of which was stopped after the third season despite that there were other two seasons to go before the production was stopped. This is even clearer now, since the same TV series has been aired by the digital channel Rai4 with discrete success. Indeed the airing of American TV series, in a new uncensored version and in day-time, was crucial to the success of the digital channel which reached a share of 1.30% on the national level in 2009, ranking third among the most viewed national channels digitally broadcasted (Auditel data, season 2009, 34). The promotion of Angel proposed by Rai4 was aimed at profiting from its international "cult status" and underling strategic importance of the TV series for the digital channel itself. Headlines such as "for the first time in prime time" and "the cult series in a new version only on RAI4" were used, among others, to sustain those promotional ideas. The launch was backed by serial fans on their blogs (see f.i. http://antoniogenna.wordpress.com/2009/05/28/tv-)dig-angel-arriva-su-rai-4/). Moreover Rai4 bets on interactivity and digital communication and has also Facebook and Twitter accounts, in order to discuss with viewers.

Another example is the American TV drama Supernatural. Since its first airing, on 13th September 2005, the CW prime time TV show became a fan favourite, as acknowledged by TV critics and fandom experts [34–36]. Despite a "rabid fan following" and the identification of "cult show," Supernatural was never a hit show. What is relevant to this study is that the CW has never highly promoted Supernatural, since the WB and UPN merged into the new network back in 2006, in comparison with its other prime time series. The low promotion was combined, during the 7 years of airing, with a schedule hopping from Tuesday to Thursday to Friday. This fact was admitted in official statements by the CW communication executives who justified the lack of promotion with the acknowledgment that Supernatural was considered a "proven veteran" that had already established itself and should allow newcomers to "gain additional exposure" [37]. Nevertheless the lack of promotion by the CW could be, according to our model, one of the reasons for the deceasing ratings for the show. Interestingly the same treatment, in term of lack of promotion and erratic scheduling, has been given to Supernatural in its Italian adaptation (see Section 4.1).

(B) $\tau = 0$; and $\eta \neq 0$. We assume that the actions depend on the degree of popularity and that the trend is linear. This approach is quite similar to the viral equation used by many researchers. In this case the solution is $P(t) = P(0)e^{(\eta-\delta)t}$ and two scenarios can happen.

(a) *If* $\eta > \delta$, then P *will increase providing a successful campaign.*

The first five years of *The Bold and the Beautiful* (1987) in US were not a good start for the CBS soap; it was in fact the network's lowest-rated soap. In this case though, the aim of the producers was to create an international success and not a national one. In fact since its first airing the soap has become the most-watched soap in the world with an audience of an estimated 26.2 million viewers [37]. In Italy the investment made on the soap, simply known as *Beautiful*, was huge in terms of promotion and distribution since the beginning of its airing in 1990. The soap was scheduled in a crucial daytime timeslot (13.45) on Raidue and later on even in prime time with a rhythm of 3 episodes per night. In 1994 the soap was bought by Mediaset but the daytime slot remained the same even when the series started its airing on Canale5. This was a crucial point in the promotional campaign which stated that "the lunch time is *always* beautiful" stating the happy marriage between Beautiful and the 13.40 time slot. The faithfulness of the Italian audience toward the soap was built over time and based on the strategic scheduling, the collaboration between the American producers and Mediaset group which lead to the setting of many storylines in Italy (f. i. Venezia 1997, Como 2000, and Portofino 2007), the investments in the promotional area that lead to localized promos with the leading actors and frequent visits from the cast in Italian talk shows and entertainment programs (last of which was the participation of Ronn Moss, the leading male actor, in the *Dancing with the stars* Italian edition) and the investments in the distribution area, with special edition DVD sold in kiosks, specialized volumes published by Mediaset Group and a reair of the daytime episodes in prime time on Mediaset digital platforms (La5 channel). The ratings for the American soap reflected its popularity in Italy with episodes reaching the record of 7 million viewers and a 37% share and more than 400 episodes with 6 million of viewers each (Auditel data, 34). Interesting enough the international success in this case reverberated on the original country of production in two ways: for once the soap was kept under production because of the selling power registered abroad; secondly, the popularity of the show abroad seemed to affect the national audiences and by the mid-to-late 1990s the numbers of the soap begun to climb and in the 2000s constantly held the second position in the Nielsen weekly ratings for daytime dramas.

(b) *If $\eta < \delta$ the product is not able to gain attention and in the long run people will forget it.*

Women Murders Club, a legal drama which ran on ABC from October 12, 2007, to May 13, 2008, was affected by the 2007/2008 Writers Guild of America and by the subsequent erratic programming: after the first 10 episodes were aired, people had to wait 4 months to see the last 3 episodes, the show was also put in the Friday night prime time slot which is considered a "death slot," due to the concept that many shows scheduled on or moved to Friday nights would not last long before cancellation due to low ratings. From a reception point of view not only the ratings were low (first season ranked 57th with a medium of 8.964 m viewers), but also the critical response was tepid with Goodman [38], the San Francisco Chronicle TV critic, publishing a highly critical review assigning the paper's lowest possible rating and the New York Times TV critic Stanley describing the show as "all right, but not good enough" [39]. In fact, *Women Murders Club* was cancelled on May 12, 2008. Mediaset had purchased the drama series before the strike, judging just from the pilot, and could not of course forecast the damages the strike would have had on the development and even the ratings of many American TV shows. *Women Murders Club* suddenly became an unwanted product because of the fast cancellation and the lack of popularity in the original country. Mediaset ran the TV series first on the digital channel Joy, part of its Premium Gallery, and later, because of the general lack of attention, decides to use it to fill up the prime time slot of Canale5, during the summer of 2010. The choice of programming a new drama series outside the guarantee period (11 January–30 June) is unusual but wise in this case: the tepid reception of the TV series first run both in US and in Italy, the lack of a previous successful campaign and of previous good results to build from and the small number of episodes available, convinced the network that the series could not be considered strategic, despite the fact that it was consistent to the channel identity and target. *Women Murders Club* is then offered to the summer audience as a gift, with a promotion built around its unusual scheduling ("Canale5 offers you something new, even if it is Summer") and its resemblance to more popular dramas ("A mix between *CSI, Sex and The City*, and *Grey's Anatomy*"). The series is then used up as fast as possible: in the first week 4 episodes are aired in two nights (Tuesday and Thursday), and from the second week the series is aired with a rhythm of 2 episodes per night on Thursday. After a good start (14.72% share and winner of the prime time) the ratings drop down week after week and the TV series is constantly surpassed by old episodes of the scientific program Quark, aired by RAI in the same timeslot.

(C) $\tau \neq 0$ Under these hypotheses the solution is $P(t) = (P(0) + \tau/(\eta - \delta))e^{(\eta-\delta)t} - \tau/(\eta - \delta)$. Different scenarios can occur:

(a) *If $\eta > \delta$ then P will increase providing a successful campaign, whatever is the influence from abroad; great investments in creating a fandom would overturn the performance and decrease the effects of bad reputation.*

The TV series *Tequila and Bonetti* (1992), inspired by the more popular movie *Turner and Hooch* (1989) starring Tom Hanks, was aired by CBS from January 17th to April 18th 1992. The detective comedy series was not a very successful one and was cancelled after only 12 episodes. Despite that, it proved to be a perfect match for the Italian channel Italia1 which aired the series repeatedly during the 1990s. The popularity of the TV series remained stable even after many years had passed since its last airing so much that Rete4 (Mediaset Group) started airing the TV series again in November 2009. What is even more interesting is that after 8 years from its cancellation in US, a new season of the TV series was produced in Italy, under the new title of *Tequila & Bonetti* (Mastrofilm/MediaTrade/Columbia TriStar International Television, 2000). Promotion and localization of the series (Nick Bonetti was "transferred" in Rome and partnered with an Italian celebrity in her first acting role) were independent from the American production and distribution, so much that the original producer Don Bellisario only discovered the existence of this second series years after its first airing in Italy, with obvious consequent legal issues. The "localized" second season of the original American series was gifted with similar success and the 22 episodes aired repeatedly on Italia1 and later on the satellite channels Sky Show (2008) and Sky Uno (2009). In this case the investment in the promotion and adaptation of the TV series was huge, despite the lack of success of the product in the original country of production.

(b) If $\eta < \delta$ the popularity will tend to $-\tau/(\eta - \delta)$. In this case the reputation and popularity abroad really drive the internal performance.

(b1) *If τ is positive, then the long run results of the campaign will end up in a success case, whose intensity is proportional to τ.*

In this case the popularity of the TV series in the local market is connected with the popularity of the TV series abroad. *CSI Miami* (2002), the first spin-off of the forensic drama *CSI*, arrived in Italy at the end of 2003, after the huge success registered by the original *CSI* on the channel Italia1 and the even more huge success of the show in the worldwide market. It was not necessary to build up a big promotional campaign for the TV series because the international success of the *CSI* franchise was speaking for itself. Later on, *CSI Miami* was even named the most popular program in the world because it featured in more countries' top 10 rankings for 2005 than any other [40] and this data was used in the Italian promos for the show. Building from the international success of the franchise and of the TV series itself *CSI Miami* soon became one of the most viewed programs of Wednesday night and the most viewed series between the *CSI* franchises in Italy. The confidence in the Italian success of *CSI Miami* is so strong that in 2006 the series was even used to counterbalance the flop of the third season of *The OC*, replacing the teen drama on Sunday night and airing twice a week for months.

(b2) *If τ is negative, the fail in the foreign countries will imply a decreasing popularity and even worse the creation of people spreading negative comments on the product.*

This was the case of *Defying Gravity*, a multinationally produced space travel drama, which first aired on ABC in August 2009. The low ratings convinced the American network to suspend the series, even if only 8 episodes out of the 13 produced had been aired. When the debacle of the series abroad was clear, the Italian satellite channel Sky had already purchased the full first season and decided to air the product anyway on Fox Life starting 12 June 2010 in the late prime time-slot (21:55). The promotion and distribution of the show could not, in this case, profit from the international success and had instead to fight against the bad popularity of the TV series, which was even more evident because of the high profile promotion made in the original country. *Defying Gravity* was in fact supposed to have 6 seasons, 6 years was in fact the length of the interplanetary voyage which planned to visit most of the planets of the Solar System, and was marketed as "a *Grey's anatomy* set in space" establishing a direct link with the most successful medical drama of ABC and playing with the fact that the two series had the same producing team. This kind of promotion made the series' flop even more problematic for the international broadcasters. Sky reused the ABC reference to *Grey's Anatomy*, a huge success in the Italian market too, but tried to obscure any reference to sci-fi, selling the show as a soap and adding to the main title the subtitle "the galaxy of hearts." The choice of scheduling *Defying Gravity* on Fox Life—dedicated to family drama and medicals—and the late night summer slot-time, though, were clear indications that the product was not considered strategic anymore because of the bad reputation it had gained abroad.

If $\eta = 0$ then the popularity will tend to τ/δ. This is the case in Italy of the *Law and Order* franchise (1990), the success ($\tau > 0$) of which is obviously disconnected from the amount of local investments: the dedication of the broadcasting networks in the promotion and distribution area for the single series, in fact, has always tended to zero in Italy. This is shown by the small number of promos realized for each series all through their airing, by the scheduling of all the series in the late night or even in the night-time slots and by the fact that they have been running both on RAI and Mediaset channels, with no intention of connecting the single series to the specific channels. *Law and Order SVU*, for instance, has been used by Rete4 to fill up void spaces in the late night schedule (between October and November 2007, 10 out of the 22 episodes of the first season were aired, in no particular order), before becoming the regular fill for the late night time slot of the channel. *Law and Order* timeslot on Raidue often changed from daytime to late night and back again, depending on the nature of the single episode, while the 17th season (last one of the original series) has been moved to the prime time of Raitre. Despite all the changes and lack of promotion the constant popularity of the TV series abroad and the strong niche following in Italy has made of the *Law and Order* franchise a reliable show, with special regards to the ratings which remain constant.

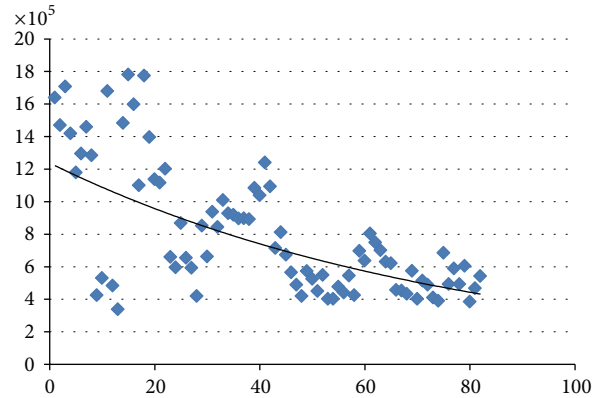

Figure 2: Popularity of Supernatural fades away.

4.1. An Empirical Evidence. As mention above, in Italy, Supernatural's performance has been affected by the lack of (or scarce) promotion and erratic scheduling (in particular shifting from springtime to summer, a season characterized by lower audience data).

Figure 2 shows the Auditel data [41] of the first four series of Supernatural broadcast on Rai2 channel in the period 2007–2010: in detail the first season was broadcast during February-July 2007, the second June-September 2008, the third July-September 2009, and the fourth July-August 2010. The x axis shows the sequence of episodes, whilst the y axis lists the corresponding audience data.

Figure 2 suggests that there are fluctuations of the data around the black trend line, whose intensity is decreasing over time. However, Supernatural's long-run behavior can be well approximated by the function $P(t) = 10^6 e^{-0.013t}$ that shows an exponential decay of popularity. In particular this empirical evidence shows that $\eta - \delta = -0.013$, that is, $\eta < \delta$, implying that either $\eta = 0$ (lack of promotion) or nonzero $\eta < \delta$ (scarse promotion).

The parametric estimation has been done using the technique presented in [42, 43] and based on the solution of an inverse problem for a differential equation.

5. An Optimal Decision Model: The Decision Maker View Point

In a media company the decision maker faces the problem of maximizing profits subject to some constraints. Media companies must cover their costs (license fees, production costs of programs, ...) and create returns just as any other business [44]. The above formulated model is purely dynamic and that does not provide any optimal decision rule. The purpose of this section is indeed to formulate a dynamic programming model that can fully describe the complexity of a decision making process [45, 46]. We assume we have a closed market in which there is no influence of the popularity of the product abroad and then τ is assumed to be

zero. With respect to the purely dynamic model previously described in which the term η was supposed to be constant, in this context we assume that $\eta(P(t)) = \Omega P(t) - \pi(t)$ where $\Omega P(t)$ represents the revenues (depending on the level of popularity) that a TV company receives from the sale of ad spaces to advertisers (Ω can be interpreted as the price of ad space) while $\pi(t)$ are the profits to be maximized by the DM.

The difference $\Omega P(t) - \pi(t)$ is the investment in a marketing campaign. The DM tries to maximize his/her intertemporal utility subject to a dynamic constraint that describes the evolution of popularity and it can be formulated as follows:

$$\begin{aligned} &\text{Max} \quad \int_0^{+\infty} U(\pi(t))\, e^{-\rho t} dt \\ &\text{s.t.} \quad \frac{dP(t)}{dt} = (-\delta + \Omega P(t) - \pi(t))\, P(t) \\ &\qquad P(0) = P_0, \end{aligned} \tag{3}$$

where ρ is the discount rate.

The variable $P(t)$ is the state variable while $\pi(t)$ is the control. For simplicity we assume a logarithmic utility U, that is, $U(\pi(t)) = \ln(\pi(t))$, even if other utility expressions could be considered analogously. The following result states the existence of a nontrivial equilibrium for the above dynamic model.

Proposition 1. *Suppose $\rho > \delta$. Then the above model shows the existence of a nonzero equilibrium (P^*, π^*) where*

$$\begin{aligned} P^* &= \frac{\rho}{\Omega}, \\ \pi^* &= \rho - \delta. \end{aligned} \tag{4}$$

Proof. It is provided in the Appendix. □

In other words, if the technical condition $\rho > \delta$ is satisfied, this dynamic model shows the existence of a nontrivial equilibrium (P^*, π^*). The equilibrium of popularity P^* depends on the price Ω and the higher will be the cost of a media product for a customer the lower will be the level of P^*. However, as the analytical discussion in the Appendix shows, this equilibrium is unstable; this practically means that only under strict conditions the popularity will not vanish in the long run.

6. Conclusions

Internet provides tools that can alter the whole experience of watching television, giving access to the process, not just the results. Through online communities, in fact, passionate viewers find and share a privileged knowledge which is derived both from the viewing experience and the extratextual elements of the show [21]. According to recent studies, fans are making a new kind of television scene that transcends place and shakes up long-standing balances of power between fans and the television makers. In general the most recent literature published in the field of fan studies suggests that the line between consumption and production is blurring into a new paradigm, with fans growing up to be creators and creators trying to connect and engage with fan communities when not labeling themselves as fans [13, 21, 47].

The most successful TV series analyzed in this study have, in fact, been created, promoted, and distributed not only taking into account the existence of an active and internet-based fan audience but also relying on this same audience in order to make the series an international success. Compared with the ones that did not profit from the presence of a fan audience or did not use connected marketing strategies, these TV series have indeed succeeded in generating deep, perpetual audience engagement.

From a previous desk research and exploratory analysis [48], it emerged in fact that (a) there is a willingness to consume texts offered by producers in different formats outside of the narrative space of the show itself and (b) part of the reason for this extratextual success lies in the initial construction and marketing of the show itself.

Building from these results, the model we propose here tries to identify and explain the main factors on which the popularity of American TV series in Italy is based. All the examples confirm in fact the relevance of collateral actions, specifically web orientated, to underpin the success of television's content as well as the influence of foreign popularity (globalization), which can be considered the explaining variable involved in the success of a media product.

The paper also introduces a decisional dynamic model able to sustain the DM in his/her decisions. In the Italian scenario, commercial television plays a predominant role and competes to attract viewers' attention and then sells that attention to advertisers [49]. In other words a broadcasting media platform can only succeed if it has viewers; otherwise, its revenues from advertising would be zero. As it is an advertisement-supported media, from the two-sided markets literature we know that ad prices reflect both the value of reaching a given audience and the marginal effect of the advertisement sale on total audience size. Thus popular TV programs indirectly generate returns for a media company attracting viewers. Our model provides the DM the optimal solution balancing the investment in sustaining program popularity and the maximization of profits.

For future works we aim at modifying this model by introducing a stochastic term able to catch the natural fluctuations arising from audience data.

Appendix

In an optimal decision making process, the DM tries to solve the following model:

$$\begin{aligned} &\text{Max} \quad \int_0^{+\infty} \ln(\pi(t))\, e^{-\rho t} dt \\ &\text{s.t.} \quad \frac{dP(t)}{dt} = (-\delta + \Omega P(t) - \pi(t))\, P(t) \\ &\qquad P(0) = P_0. \end{aligned} \tag{.5}$$

The Hamiltonian associated with model (.5) is

$$H(P(t),\pi(t)) = \ln(\pi(t))\,e^{-\rho t} + \lambda(t) \times (-\delta P(t) + (\Omega P(t) - \pi(t))\,P(t)). \tag{.6}$$

Optimality conditions for (.6) read as

$$\begin{aligned} 0 &= \frac{\partial H}{\partial \pi} = \frac{e^{-\rho t}}{\pi(t)} - \lambda(t)\,P(t), \\ -\dot{\lambda}(t) &= \frac{\partial H}{\partial P} = \lambda(t)\,(-\delta + 2\Omega P(t) - \pi(t)), \\ \dot{P}(t) &= \frac{\partial H}{\partial \lambda} = -\delta P(t) + (\Omega P(t) - \pi(t))\,P(t). \end{aligned} \tag{.7}$$

The system (.7) is equivalent to

$$\begin{aligned} &\pi(t) = \frac{e^{-\rho t}}{\lambda(t)\,P(t)} \Longrightarrow \frac{\dot{\pi}(t)}{\pi(t)} = -\rho - \frac{\dot{\lambda}(t)}{\lambda(t)} - \frac{\dot{P}(t)}{P(t)}, \\ &-\dot{\lambda}(t) = \lambda(t)\,(-\delta + 2\Omega P(t) - \pi(t)) \\ &\qquad \Longrightarrow -\frac{\dot{\lambda}(t)}{\lambda(t)} = -\delta + 2\Omega P(t) - \pi(t), \\ &\dot{P}(t) = -\delta P(t) + (\Omega P(t) - \pi(t))\,P(t) \\ &\qquad \Longrightarrow \frac{\dot{P}(t)}{P(t)} = -\delta + \Omega P(t) - \pi(t) \end{aligned} \tag{.8}$$

which implies

$$\begin{aligned} \frac{\dot{\pi}(t)}{\pi(t)} &= -\rho + \Omega P(t), \\ \frac{\dot{P}(t)}{P(t)} &= -\delta + \Omega P(t) - \pi(t). \end{aligned} \tag{.9}$$

The only nontrivial equilibrium is

$$\begin{aligned} P^* &= \frac{\rho}{\Omega}, \\ \pi^* &= \rho - \delta > 0. \end{aligned} \tag{.10}$$

The Jacobian of the linearized system at (P^*, π^*) is equal to

$$J = \begin{bmatrix} 0 & \Omega(\rho - \delta) \\ -\dfrac{\rho}{\Omega} & \rho \end{bmatrix}. \tag{.11}$$

The determinant of J is equal to

$$\det(J) = \rho(\rho - \delta) > 0, \tag{.12}$$

while the trace of J is equal to

$$\operatorname{trace}(J) = \rho > 0, \tag{.13}$$

which implies that (P^*, π^*) is an unstable equilibrium. It is possible to show that this system admits three more equilibria, namely $(0,0)$, $(P^*, 0)$ and $(0, \pi^*)$. Among them, $(0,0)$ and $(0, \pi^*)$, are locally asymptotically stable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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