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Lead Guest Editor: Nawab Hussain
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Research Article

Best Proximity Point Theorems for Single and Multivalued Mappings in Fuzzy Multiplicative Metric Space

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In this paper, we introduce fuzzy multiplicative metric space and prove some best proximity point theorems for single-valued and multivalued proximal contractions on the newly introduced space. As corollaries of our results, we prove some fixed-point theorems. Also, we present best proximity point theorems for Feng-Liu-type multivalued proximal contraction in fuzzy metric space. Moreover, we illustrate our results with some interesting examples.

1. Introduction and Preliminaries

Best proximity point is the generalization of fixed point and is useful when contraction map is not a self-map that is $T : A \longrightarrow B$ where $A \cap B = \phi$. A point $\mu \in A$ is known as best proximity point if $d(\mu, T\mu) = d(A, B)$. Fan [1] presented best approximation theorem which is stated as follows: "If K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \longrightarrow E$ is a continuous non-self-mapping, then there exists an element μ in K such a way that $d(\mu, T\mu) = d(T\mu, K)$." A best proximity point theorem is more applicable than best approximation theorem, as it provides optimal approximate solution. Therefore, best proximity point theory seeks attention of authors such as [2–7]. Many research works done on multivalued non-self-maps use Nadler's approach [8]. Nadler's theorem is stated as follows: "Let (M, d) be a complete metric space and T be a mapping from M into $CB(M)$, where $CB(M)$ is the collection of all closed and bounded subsets of M, such that for all $\mu, \nu \in M$, $H(T\mu, T\nu) \leq \lambda d(\mu, \nu)$ where $0 < \lambda < 1$. Then, T has a fixed point." Another way of defining multivalued contraction is approached by Feng and Liu [9]. They proved a fixed-point theorem for newly defined multivalued contraction which is stated as follows: "Let (M, d) be a complete metric space, $T : M \longrightarrow C(M)$, where $C(M)$ is the collection of all closed subsets of M, be a multivalued mapping. If there exists a constant $c \in (0, 1)$ such that for any $\mu \in M$, there is $\nu \in I_b^\mu$ (where $I_b^\mu = \{\nu \in T\mu \mid bd(\mu, \nu) \leq d(\mu, T\mu)\} \subset M$ for some $b \in (0, 1)$) satisfying $d(\nu, T\nu) \leq c\, d(\mu, \nu)$. Then, T has a fixed point in M provided that $c < b$ and $f(\mu) = d(\mu, T\mu)$ is lower semicontinuous". With the help of example, in the same article, they also have shown that Feng-Liu-type multivalued contraction is more general than Nadler's multivalued contraction. Recently, Sahin et al. [10] proved best proximity point theorem for Feng-Liu-type multivalued map.

On the other hand, fuzzy metric space was firstly defined by Kramosil and Michalek [11] and then modified by George and Veeramani [12]. The modified definition is given as follows.

Definition 1 (see [12]). A 3-tuple $(M, F_M, \star)$ is called fuzzy metric space if M is an arbitrary set, $\star$ is continuous t-norm, and F_M is a fuzzy set on $M \times M \times (0,\infty)$ satisfying the following conditions for all $\mu, \nu, \rho \in M$ and $t, s > 0$:

FM1: $F_M(\mu, \nu, t) > 0$
FM2: $F_M(\mu, \nu, t) = 1$ if and only if $\mu = \nu$
FM3: $F_M(\mu, \nu, t) = F_M(\nu, \mu, t)$
FM4: $F_M(\mu, \rho, t + s) \geq F_M(\mu, \nu, t) \star F_M(\nu, \rho, s)$
FM5: $F_M(\mu, \nu, .)$: $(0,\infty) \longrightarrow [0, 1]$ is continuous

The $t-$norm is defined as follows.

Definition 2 (see [12]). A continuous t-norm is a binary operation $\star : [0,1]^2 \longrightarrow [0,1]$ if the pair $([0,1],\star)$ is a topological monoid, that is,

(1) $\star$ satisfies associative and commutative laws

(2) $\star$ is continuous

(3) $a\star 1 = a$, $\forall a \in [0,1]$

(4) for every $a, b, c, d \in [0,1]$, $a\star b \le c\star d$ whenever $a \le c$ and $b \le d$

Some known examples of a continuous t-norm are $a\star_1 b = \min\{a, b\}$, $a\star_2 b = (ab/\max\{a, b, \lambda\})$ for $0 < \lambda < 1$, $a\star_3 b = ab$, $a\star_4 b = \max\{a + b - 1, 0\}$.

Many researches have been produced on fixed-point theory in fuzzy metric spaces [4, 13–19]. Vetro and Salimi [20] proved best proximity point theorem in fuzzy metric spaces. Due to the development of new calculus by Grossman and Katz [21], known as multiplicative calculus, a metric was introduced by Bashirov et al. [22] called multiplicative metric defined as follows.

Definition 3 (see [22]). Assume a nonempty set M. Regard multiplicative metric as a mapping $d : M \times M \longrightarrow \mathbb{R}$ obeying the following assertions:

M1: $d(\mu, \nu) > 1$ for all $\mu, \nu \in M$ and $d(\mu, \nu) = 1$ if and only if $\mu = \nu$

M2: $d(\mu, \nu) = d(\nu, \mu)$

M3: $d(\mu, \rho) \le d(\mu, \nu) \cdot d(\nu, \mu)$ for all $\mu, \nu, \rho \in M$

Getting inspiration from all the work mentioned above, we firstly introduce fuzzy multiplicative metric space and prove some of its topological properties. Moreover, we obtain some best proximity point theorems for Feng-Liu-type multivalued non-self-maps on fuzzy multiplicative metric space.

2. Fuzzy Multiplicative Metric Spaces

This section introduces a new type of metric space which is fuzzy analogy of multiplicative metric space. We give an example to show the existence of such space.

Definition 4. A triplet $(M, F_{MM},\star)$ is termed as fuzzy multiplicative metric space if $\star$ is continuous t -norm, M is arbitrary set, and F_{MM} is fuzzy set on $M \times M \times (1,\infty)$ fulfilling the accompanying conditions for all $\mu, \nu, \rho \in M$ and $t, s > 1$.

FMM1: $F_{MM}(\mu, \nu, t) > 0$

FMM2: $F_{MM}(\mu, \nu, t) = 1$ if and only if $\mu = \nu$

FMM3: $F_{MM}(\mu, \nu, t) = F_{MM}(\nu, \mu, t)$

FMM4: $F_{MM}(\mu, \rho, t.s) \ge F_{MM}(\mu, \nu, t)\star F_{MM}(\nu, \rho, s)$

FMM5: $F_{MM}(\mu, \nu, .)$: $(1,\infty) \longrightarrow [0,1]$ is continuous

Here, we have an example of fuzzy multiplicative metric which cannot be fuzzy metric.

Example 5. Let $M = \mathbb{R}^+$ and $F_{MM}(\mu, \nu, t) = ((t+1)/(t + |\mu/\nu|^*))$, consider a continuous t-norm $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ as $\mu\star\nu = \mu\nu$. Then, M is fuzzy multiplicative metric space.

Remark 6.

(1) Let $(M, F_{MM},\star)$ be a fuzzy multiplicative metric space. Whenever $F_{MM}(\mu, \nu, t) > 1 - \varepsilon$ for $\mu, \nu \in M$ and $t > 1$, $0 < \varepsilon < 1$, we can find t_0, $1 < t_0 < t$ such that $F_{MM}(\mu, \nu, t_0) > 1 - \varepsilon$

(2) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in (0,1)$. For any $\varepsilon_1 > \varepsilon_2$, we can find ε_3 such that $\varepsilon_1\star\varepsilon_3 \ge \varepsilon_2$, and for any ε_4, we can find ε_5 such that $\varepsilon_5\star\varepsilon_5 \ge \varepsilon_4$

We now discuss some topological properties of fuzzy multiplicative metric space.

Definition 7. Let $(M, F_{MM},\star)$ be a fuzzy multiplicative metric space and $0 < \varepsilon < 1$, $t > 1$; then, an open ball having center μ and radius ε is defined as

$$B(\mu, \varepsilon, t) = \{\nu \in M : F_{MM}(\mu, \nu, t) > 1 - \varepsilon\}. \tag{1}$$

Proposition 8. *Every open ball is an open set in fuzzy multiplicative metric space.*

Proof. Consider an open ball $B(\mu, \varepsilon, t)$ and let $\nu \in B(\mu, \varepsilon, t)$. This implies that $F_{MM}(\mu, \nu, t) > 1 - \varepsilon$. Since $F_{MM}(\mu, \nu, t) > 1 - \varepsilon$, using Remark 6, we can find t_0, $1 < t_0 < t$, such that $F_{MM}(\mu, \nu, t_0) > 1 - \varepsilon$. Let $\varepsilon_0 = F_{MM}(\mu, \nu, t_0) > 1 - \varepsilon$. Since $\varepsilon_0 > 1 - \varepsilon$, therefore by using Remark 6, we can find ε_1, $0 < \varepsilon_1 < 1$, such that $\varepsilon_0 > 1 - \varepsilon_1 > 1 - \varepsilon$. Now, for a given ε_0 and ε_1 such that $\varepsilon_0 > 1 - \varepsilon_1$, we can find ε_2, $0 < \varepsilon_2 < 1$ such that $\varepsilon_0\star\varepsilon_2 \ge 1 - \varepsilon_1$. Now, consider the ball $B(\nu, 1 - \varepsilon_2, t/t_0)$. We claim that $B(\nu, 1 - \varepsilon_2, t/t_0) \subset B(\mu, \varepsilon, t)$.

Now, $\rho \in B(\nu, 1 - \varepsilon_2, t/t_0)$ implies that $F_{MM}(\nu, \rho, t/t_0) > \varepsilon_2$. Therefore,

$$F_{MM}(\mu, \rho, t) \ge F_{MM}(\mu, \nu, t_0)\star F_{MM}\left(\nu, \rho, \frac{t}{t_0}\right) \ge \varepsilon_0\star\varepsilon_2 \ge 1 - \varepsilon_1 > 1 - \varepsilon. \tag{2}$$

Therefore, $\rho \in B(\mu, \varepsilon, t)$, and hence,

$$B\left(\nu, 1 - \varepsilon_2, \frac{t}{t_0}\right) \subset B(\mu, \varepsilon, t). \tag{3}$$

□ □

Proposition 9. *Let $(M, F_{MM},\star)$ be a fuzzy multiplicative metric space. Define $\tau = \{A \subset M : \mu \in A$ if and only if there exist $t > 1$ and ε, $0 < \varepsilon < 1$ such that $B(\mu, \varepsilon, t) \subset A\}$.*

Then, τ is a topology on M.

Theorem 10. *Every fuzzy multiplicative metric space is Hausdorff.*

Proof. Assume that $(M, F_{MM}, \star)$ is a given fuzzy multiplicative metric space. Let μ, ν be two distinct points of M, and then, $0 < F_{MM}(\mu, \nu, t) < 1$. Let $F_{MM}(\mu, \nu, t) = \varepsilon$, $0 < \varepsilon < 1$. For each ε_0, $\varepsilon < \varepsilon_0 < 1$, using Remark 6, we can find ε_1 such that $\varepsilon_1 \star \varepsilon_1 \geq \varepsilon_0$. Now, consider the open balls $B(\mu, 1-\varepsilon_1, t^{1/2})$ and $B(\nu, 1-\varepsilon_1, t^{1/2})$. Clearly,

$$B\left(\mu, 1-\varepsilon_1, t^{1/2}\right) \cap B\left(\nu, 1-\varepsilon_1, t^{1/2}\right) = \phi. \quad (4)$$

For if there exists

$$\rho \in B\left(\mu, 1-\varepsilon_1, t^{1/2}\right) \cap B\left(\nu, 1-\varepsilon_1, t^{1/2}\right). \quad (5)$$

Then,

$$\varepsilon = F_{MM}(\mu, \nu, t) \geq F_{MM}\left(\mu, \rho, t^{1/2}\right) \star F_{MM}\left(\rho, \nu, t^{1/2}\right) \geq \varepsilon_1 \star \varepsilon_1 \geq \varepsilon_0 > \varepsilon, \quad (6)$$

which is a contradiction. Therefore, $(M, F_{MM}, \star)$ is Hausdorff.□ □

Definition 11. In a fuzzy multiplicative metric space $(M, F_{MM}, \star)$, a sequence $\{\mu_a\}$ is a convergent sequence which converges to μ if and only if there exist $a_1 \in \mathbb{N}$ with $F_{MM}(\mu_a, \mu, t) > 1 - \varepsilon$, for all $a \geq a_1$ and for each $\varepsilon > 0, t > 1$.

Theorem 12. *Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space, $\mu \in M$ and $\{\mu_a\}$ be a sequence in M. Then, $\{\mu_a\}$ converges to μ if and only if $F_{MM}(\mu_a, \mu, t) \longrightarrow 1$ as $a \longrightarrow \infty$ for each $t > 1$.*

Proof. Suppose that $\mu_a \longrightarrow \mu$. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exists a natural number a_1 such that $F_{MM}(\mu_a, \mu, t) > 1 - \varepsilon$ for all $a \geq a_1$. We have $1 - F_{MM}(\mu_a, \mu, t) < \varepsilon$. Hence, $F_{MM}(\mu_a, \mu, t) \longrightarrow 1$ as $a \longrightarrow \infty$.

Conversely, suppose that $F_{MM}(\mu_a, \mu, t) \longrightarrow 1$ as $a \longrightarrow \infty$. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exist a natural number a_1 such that $1 - F_{MM}(\mu_a, \mu, t) < \varepsilon$ for all $a \geq a_1$. In that case, $F_{MM}(\mu_a, \mu, t) > 1 - \varepsilon$. Hence, $\mu_a \longrightarrow \mu$ as $a \longrightarrow \infty$.□ □

Definition 13. Consider a sequence $\{\mu_a\}$ in a fuzzy multiplicative metric space $(M, F_{MM}, \star)$. If for each $\varepsilon > 0, t > 1$, there exist $a_1 \in \mathbb{N}$ such that $F_{MM}(\mu_a, \mu_b, t) > 1 - \varepsilon$ for all $a, b \geq a_1$, and then, $\{\mu_a\}$ is termed as Cauchy sequence in M.

Theorem 14. *Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space, $\mu \in M$ and $\{\mu_a\}$ be a sequence in M. Then, $\{\mu_a\}$ is Cauchy if and only if $F_{MM}(\mu_a, \mu_b, t) \longrightarrow 1$ as $a, b \longrightarrow \infty$ for each $t > 1$.*

Proof. Suppose that μ_a is a Cauchy sequence in M. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exists a natural number a_1 such that $F_{MM}(\mu_a, \mu_b, t) > 1 - \varepsilon$ for all $a, b \geq a_1$. We have $1 - F_{MM}(\mu_a, \mu_b, t) < \varepsilon$. Hence, $F_{MM}(\mu_a, \mu_b, t) \longrightarrow 1$ as $a, b \longrightarrow \infty$.

Conversely, suppose that $F_{MM}(\mu_a, \mu_b, t) \longrightarrow 1$ as $a, b \longrightarrow \infty$. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exists a natural number a_1 such that $1 - F_{MM}(\mu_a, \mu_b, t) < \varepsilon$ for all $a, b \geq a_1$. In that case, $F_{MM}(\mu_a, \mu_b, t) > 1 - \varepsilon$. Hence, μ_a is a Cauchy sequence.□ □

Proposition 15. *In a fuzzy multiplicative metric space $(M, F_{MM}, \star)$, if a sequence $\{\mu_a\}$ converges in M, then $\{\mu_a\}$ is Cauchy.*

Proof. Let ε and t be real numbers with $\varepsilon \in (0, 1)$, $t > 1$. Since $\varepsilon \in (0, 1)$, there is some $\varepsilon_0 \in (0, 1)$ such that $(1 - \varepsilon_0) \star (1 - \varepsilon_0) > 1 - \varepsilon$. Also, suppose that $\{\mu_a\}$ converges in M, say it converges to $\mu \in M$. Then, there exists $a_0 \in \mathbb{N}$ such that for each $a \geq a_0$,

$$F_{MM}\left(\mu_a, \mu, t^{1/2}\right) > 1 - \varepsilon_0. \quad (7)$$

Thus, for $a > b \geq a_0$, we have

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_a, \mu, t^{1/2}\right) \star F_{MM}\left(\mu_b, \mu, t^{1/2}\right) > (1 - \varepsilon_0) \star (1 - \varepsilon_0) > 1 - \varepsilon. \quad (8)$$

□ □

That is $\{\mu_a\}$ is a Cauchy sequence.

Definition 16. A fuzzy multiplicative metric space $(M, F_{MM}, \star)$ is termed as complete if and only if every sequence in M which is Cauchy must converge in M.

Definition 17. Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space. A subset A of M is closed if for each sequence $\{\mu_a\}$ in A which is convergent with $\mu_a \longrightarrow \mu$, we have $\mu \in A$.

Remark 18. Let $(M, F_{MM}, \star)$ be a complete fuzzy multiplicative metric space. A subset A of M is closed if and only if $(A, F_{MM}, \star)$ is complete.

The following lemma is the analogue of Kiany's lemma [16] in the setting of newly defined space.

Lemma 19. *Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space such that for $\mu, \nu \in M$, $t > 1$ and $h > 1$*

$$\lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_{MM}\left(\mu, \nu, t^{h^i}\right) = 1. \quad (9)$$

Suppose $\{\mu_a\}$ is a sequence in M such that for all $a \in \mathbb{N}$

$$F_{MM}(\mu_a, \mu_{a+1}, t^{\alpha}) \geq F_{MM}(\mu_{a-1}, \mu_a, t), \quad (10)$$

where $0 < \alpha < 1$. Then, $\{\mu_a\}$ is a Cauchy sequence.

Proof. For each $a \in \mathbb{N}$ and $t > 1$, we have

$$F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}(\mu_{a-1}, \mu_a, t^{1/\alpha}) \geq F_{MM}\left(\mu_{a-2}, \mu_{a-1}, t^{1/\alpha^2}\right) \geq \cdots \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}}\right). \tag{11}$$

Thus, for each $a \in \mathbb{N}$, we get

$$F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}}\right). \tag{12}$$

□ □

Choosing constants $h > 1$ and $l \in \mathbb{N}$ such that $h\alpha < 1$ and $\sum_{i=l}^{\infty} 1/h^i = (1/h^l)/(1-(1/h)) < 1$. Therefore, for $b \geq a$,

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_a, \mu_b, t^{\left(\left(1/h^l\right)+\left(1/h^{l+1}\right)+\cdots+\left(1/h^{l+b}\right)\right)}\right) \geq F_{MM}\left(\mu_a, \mu_{a+1}, t^{1/h^l}\right) \star F_{MM}\left(\mu_{a+1}, \mu_{a+2}, t^{1/h^{l+1}}\right) \text{no number,}$$

$$\star \cdots \star F_{MM}\left(\mu_{b-1}, \mu_b, t^{1/h^{l+b}}\right). \tag{13}$$

Using (12) in above inequality, we have

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}h^l}\right) \star F_{MM} \cdot \left(\mu_0, \mu_1, t^{1/\alpha^a h^{l+1}}\right) \star \cdots \star F_{MM}\left(\mu_0, \mu_1, t^{1/\left(\alpha^{b-2}h^{l+b-a-2}\right)}\right). \tag{14}$$

That is

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{a-1}}\right) \star F_{MM} \cdot \left(\mu_0, \mu_1, t^{1/(\alpha h)^a}\right) \star \cdots \star F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{b-2}}\right). \tag{15}$$

The above expression can be simplified as

$$F_{MM}(\mu_a, \mu_b, t) \geq \star_{i=a}^{\infty} F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{i-1}}\right). \tag{16}$$

Then, from the above, we have

$$\lim_{a,b \longrightarrow \infty} F_{MM}(\mu_a, \mu_b, t) \geq \lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{i-1}}\right) = 1, \tag{17}$$

for each $t > 1$. Hence, for each $t > 1$,

$$\lim_{a,b \longrightarrow \infty} F_{MM}(\mu_a, \mu_b, t) = 1, \tag{18}$$

which shows that $\{\mu_a\}$ is a Cauchy sequence.

Definition 20. Consider a fuzzy multiplicative metric space $(M, F_{MM}, \star)$ and $A, B \subset M$; then, for all $t > 1$,

$$A_0 = \{\mu \in A : F_{MM}(\mu, \nu, t) = F_{MM}(A, B, t) \text{ for some } \nu \in B\},$$
$$B_0 = \{\nu \in B : F_{MM}(\mu, \nu, t) = F_{MM}(A, B, t) \text{ for some } \mu \in A\}, \tag{19}$$

where

$$F_{MM}(A, B, t) = \text{Sup}\{F_{MM}(\mu, \nu, t), \mu \in A, \nu \in B\}, \tag{20}$$

for all $t > 1$.

Definition 21. Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space and $A, B \subset M$. If every sequence $\{\mu_a\}$ of A, fulfilling the condition that $F_{MM}(\nu, \mu_a, t) \longrightarrow F_{MM}(\nu, A, t)$ for some ν in B and for all $t > 1$, has a convergent subsequence, then A is termed as approximatively compact with respect to B.

3. Best Proximity Point Theorems in Fuzzy Multiplicative Metric Spaces

In the present section, we prove some best proximity point theorems for single-valued and multivalued proximal contractions. First, we define the analogous of proximal contractions in the setting of fuzzy multiplicative metric space and then proceed to the main results.

Definition 22. Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space and $A, B \subset M$. A mapping $T : A \longrightarrow B$ is named as multiplicative contraction of first kind if there exists $\alpha \in [0, 1)$, such that for all $u, v, \mu, \nu \in A$

$$F_{MM}(u, T\mu, t) = F_{MM}(A, B, t), \tag{21}$$

$$F_{MM}(\nu, T\nu, t) = F_{MM}(A, B, t) \Rightarrow F_{MM}(u, v, t^{\alpha}) \geq F_{MM}(\mu, \nu, t). \tag{22}$$

Theorem 23. *Let $(M, F_{MM}, \star)$ be a complete fuzzy multiplicative metric space and $A, B \subset M$ such that B is approximatively compact with respect to A. Assume that $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$, $T : A \longrightarrow B$ be multiplicative contraction of first kind and $T(A_0) \subset B_0$. Then, T possesses best proximity point.*

Proof. Let $\mu_0 \in A_0$ then for $T\mu_0 \in TA_0 \subset B_0$, there exist $\mu_1 \in A_0$ such that

$$F_{MM}(\mu_1, T\mu_0, t) = F_{MM}(A, B, t). \tag{23}$$

Further, since $T\mu_1 \in TA_0 \subset B_0$, there exist $\mu_2 \in A_0$ such that

$$F_{MM}(\mu_2, T\mu_1, t) = F_{MM}(A, B, t). \tag{24}$$

Similarly, for $T\mu_2 \in TA_0 \subset B_0$, there exist $\mu_3 \in A_0$ such that

$$F_{MM}(\mu_3, T\mu_2, t) = F_{MM}(A, B, t). \tag{25}$$

By continuing the similar steps, we get

$$F_{MM}(\mu_{a+1}, T\mu_a, t) = F_{MM}(A, B, t) \text{ for all } a \in \mathbb{N}. \tag{26}$$

By successive application of fuzzy multiplicative contraction, we have for all $a \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} F_{MM}(\mu_a, \mu_{a+1}, t^{\alpha}) &\geq F_{MM}(\mu_{a-1}, \mu_a, t) \geq F_{MM}\left(\mu_{a-2}, \mu_{a-1}, t^{1/\alpha}\right) \\ &\geq F_{MM}\left(\mu_{a-3}, \mu_{a-2}, t^{1/\alpha^2}\right) \geq \cdots \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}}\right). \end{aligned} \tag{27}$$

For any $q \in \mathbb{N}$,

$$\begin{aligned} F_{MM}\left(\mu_a, \mu_{a+q}, t\right) &\geq F_{MM}\left(\mu_a, \mu_{a+1}, t^{1/q}\right) \star F_{MM} \\ &\cdot \left(\mu_{a+1}, \mu_{a+2}, t^{1/q}\right) \star \cdots \star F_{MM}\left(\mu_{a+q-1}, \mu_{a+q}, t^{1/q}\right). \end{aligned} \tag{28}$$

Using (27) in above inequality, we obtain

$$\begin{aligned} F_{MM}\left(\mu_a, \mu_{a+q}, t\right) &\geq F_{MM}\left(\mu_0, \mu_1, t^{1/q\alpha^a}\right) \star F_{MM} \\ &\cdot \left(\mu_0, \mu_1, t^{1/q\alpha^{a+1}}\right) \star \cdots \star F_{MM}\left(\mu_0, \mu_1, t^{1/q\alpha^{a+q-1}}\right). \end{aligned} \tag{29}$$

By assumption, $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$, we get that

$$\lim_{a \longrightarrow \infty} F_{MM}\left(\mu_a, \mu_{a+q}, t\right) = 1 \star 1 \star \cdots \star 1 = 1. \tag{30}$$

Hence, $\{\mu_a\}$ is a Cauchy sequence. The completeness of fuzzy multiplicative metric space $(M, F_{MM}, \star)$ implies that $\{\mu_a\}$ converges to $\mu^* \in A$, that is,

$$\lim_{a \longrightarrow \infty} F_{MM}(\mu_a, \mu^*, t) = 1, \tag{31}$$

for all $t > 1$. Notice that

$$\begin{aligned} F_{MM}(\mu, B, t) &\geq F_{MM}(\mu, T\mu_a, t) \geq F_{MM}\left(\mu, \mu_{a+1}, t^{1/2}\right) \star F_{MM}\left(\mu_{a+1}, T\mu_a, t^{1/2}\right) \\ &= F_{MM}\left(\mu, \mu_{a+1}, t^{1/2}\right) \star F_{MM}(A, B, t) \\ &\geq F_{MM}\left(\mu, \mu_{a+1}, t^{1/2}\right) \star F_{MM}(\mu, B, t). \end{aligned} \tag{32}$$

□ □

Therefore, $F_{MM}(\mu, T\mu_a, t) \longrightarrow F_{MM}(\mu, B, t)$ as $a \longrightarrow \infty$. Since B is approximatively compact with respect to A, so $\{T\mu_a\}$ has a convergent subsequence $\{T\mu_{a_k}\}$ converging to some $\rho \in B$. Further, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} F_{MM}(A, B, t) &\geq F_{MM}(\mu, \rho, t) \geq F_{MM}\left(\mu, \mu_{a_{k+1}}, t^{1/3}\right) \star F_{MM} \\ &\cdot \left(\mu_{a_{k+1}}, T\mu_{a_k}, t^{1/3}\right) \star F_{MM}\left(T\mu_{a_k}, \rho, t^{1/3}\right) \\ &= F_{MM}\left(\mu, \mu_{a_{k+1}}, t^{1/3}\right) \star F_{MM}\left(A, B, t^{1/3}\right) \star F_{MM}\left(T\mu_{a_k}, \rho, t^{1/3}\right). \end{aligned} \tag{33}$$

Letting $k \longrightarrow \infty$, we get $F_{MM}(\mu, \rho, t) = F_{MM}(A, B, t)$, which implies that $\mu \in A_0$ and $T(A_0) \subseteq B_0$ implies that $T\mu \in B_0$, there exist $\mu^* \in A$, such that $F_{MM}(\mu^*, T\mu, t) = F_{MM}(A, B, t)$. From this and equation (26) implies that

$$F_{MM}(\mu_{a+1}, \mu^*, t) \geq F_{MM}\left(\mu_a, \mu, t^{1/\alpha}\right). \tag{34}$$

Applying limit $a \longrightarrow \infty$ to above inequality gives $F_{MM}(\mu, \mu^*, t) = 1$ which implies that $\mu = \mu^*$. Hence, $F_{MM}(\mu, T\mu, t) = F_{MM}(A, B, t)$, which shows that T possesses best proximity point μ.

Example 24. Let $M = \mathbb{R}^+ \times \mathbb{R}^+$ and $F_{MM}(\mu, \nu, t) = (t+1)/(t + d(\mu, \nu))$ where $d(\mu, \nu) = |\mu_1/\nu_1|^* . |\mu_2/\nu_2|^*$ for $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$. Then, $(M, F_{MM}, \star)$ is complete fuzzy multiplicative metric space with $\star : [0, 1]^2 \longrightarrow [0, 1]$ defined as $a \star b = ab$. Let $A = \{(1, \mu) : \mu \in \mathbb{R}^+\}$ and $B = \{(2, \nu) : \nu \in \mathbb{R}^+\}$ then A and B are closed subsets of M and $F_{MM}(A, B, t) = (t+1)/(t+2)$, $A_0 = A$, $B_0 = B$. Define $T : A \longrightarrow B$ as

$$T(1, \mu) = \left(2, \frac{\mu^2}{2}\right). \tag{35}$$

Let $\mu = (1, \mu), \nu = (1, \nu) \in A$ and then $u = (1, \mu^2/2)$ and $v = (1, \nu^2/2) \in A$ such that $F_{MM}(u, T\mu, t) = F_{MM}(A, B, t) = F_{MM}(v, T\nu, t)$. It can be easily checked that T is proximal contraction in fuzzy multiplicative metric space M with $\alpha = 2/3$. Also, the condition $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$ holds.

Since all statements of Theorem 23 hold, therefore T possesses best proximity point. We can see that $(1, 2)$ is best proximity point of T.

If $A = B = M$ in Theorem 23, then we obtain the following corollary which is the fixed-point theorem for fuzzy multiplicative contraction in fuzzy multiplicative metric space.

Corollary 25. *Let $(M, F_{MM}, \star)$ be a complete fuzzy multiplicative metric space. A mapping $T : M \longrightarrow M$ satisfying $F_{MM}(\mu, \nu, t^{\alpha}) \geq F_{MM}(\mu, \nu, t)$ has fixed point provided that $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$.*

Now, we prove a best proximity theorem for Feng-Liu-type multivalued contraction in fuzzy multiplicative metric space.

Theorem 26. *Let $(M, F_{MM}, \star)$ be complete fuzzy multiplicative metric space. $A, B \subseteq M$ be two nonempty closed subsets of M having P-property and $A_0 \neq \phi$. Let $T : A \longrightarrow C(B)$ be a mapping such that $T(A_0) \subseteq B_0$ and for all $\mu \in A_0$ and $\nu \in$*

$T\mu$, there exist $\rho \in A_0$ satisfying

$$F_{MM}(\nu, \rho, t) = F_{MM}(A, B, t) \text{ and } F_{MM}(\nu, T\rho, t^c) \geq F_{MM}(\mu, \rho, t), \tag{36}$$

for some $c \in (0, 1)$ and $t > 1$. Assume that $(M, F_{MM}, \star)$ satisfy

$$\lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_{MM}\left(\mu, \nu, t^{h^i}\right) = 1, \tag{37}$$

for every $\mu, \nu \in M$, $t > 1$ and $h > 1$. Then, T has best proximity point in A provided that $f(\mu, \nu) = F_{MM}(\nu, T\mu, t)$ is upper semicontinuous.

Proof. Let $\mu_0 \in A_0$ be arbitrary point. Choose $\nu_0 \in T\mu_0$. Then, by assumption, there exist $\mu_1 \in A_0$ such that

$$\begin{gathered} F_{MM}(\nu_0, \mu_1, t) = F_{MM}(A, B, t), \\ F_{MM}(\nu_0, T\mu_1, t^c) \geq F_{MM}(\mu_0, \mu_1, t). \end{gathered} \tag{38}$$

Presently, let $b \in (c, 1)$, and then, we can discover $\nu_1 \in T\mu_1$ such that

$$F_{MM}(\nu_0, \nu_1, t) \geq F_{MM}\left(\nu_0, T\mu_1, t^b\right). \tag{39}$$

Again by assumption, there exist $\mu_2 \in A_0$ such that

$$\begin{gathered} F_{MM}(\nu_1, \mu_2, t) = F_{MM}(A, B, t), \\ F_{MM}(\nu_1, T\mu_2, t^c) \geq F_{MM}(\mu_1, \mu_2, t). \end{gathered} \tag{40}$$

Also, we can find $\nu_2 \in T\mu_2$ such that

$$F_{MM}(\nu_1, \nu_2, t) \geq F_{MM}\left(\nu_1, T\mu_2, t^b\right). \tag{41}$$

□ □

Proceeding in similar manner, we develop two sequences $\{\mu_a\}$ and $\{\nu_a\}$ in A and B, respectively, with $\mu_a \in A_0$, $\nu_a \in T\mu_a$ and

$$F_{MM}(\nu_a, \mu_{a+1}, t) = F_{MM}(A, B, t), \tag{42}$$

$$F_{MM}(\nu_a, T\mu_{a+1}, t^c) \geq F_{MM}(\mu_a, \mu_{a+1}, t), \tag{43}$$

$$F_{MM}(\nu_a, \nu_{a+1}, t) \geq F_{MM}(\nu_a, T\mu_{a+1}, t), \tag{44}$$

for all $a \in \mathbb{N}$ and $t > 1$. Then again, since A and B have P -property, so from inequality (43), we get

$$F_{MM}(\mu_a, \mu_{a+1}, t) = F_{MM}(\nu_{a-1}, \nu_a, t). \tag{45}$$

Therefore, from inequality (44), we have

$$F_{MM}(\mu_a, \mu_{a+1}, t) = F_{MM}(\nu_{a-1}, \nu_a, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^b\right). \tag{46}$$

From inequality (44), we have

$$F_{MM}(\nu_{a-1}, T\mu_a, t) \geq F_{MM}\left(\mu_{a-1}, \mu_a, t^{1/c}\right). \tag{47}$$

Combining inequalities (46) and (47), we get

$$F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}\left(\mu_{a-1}, \mu_a, t^{b/c}\right), \tag{48}$$

for all $a \geq 1$ and $t > 1$.
Let $k = c/b$ and then $0 < k < 1$. The inequality (48) gives

$$F_{MM}\left(\mu_a, \mu_{a+1}, t^k\right) \geq F_{MM}(\mu_{a-1}, \mu_a, t), \tag{49}$$

for $0 < k < 1$ and $t > 1$. By our assumption (37) and Lemma 19, $\{\mu_a\}$ is Cauchy sequence.
Now, from inequalities (44) and (46), we have

$$\begin{aligned} F_{MM}(\nu_a, T\mu_{a+1}, t^c) \geq F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^b\right) \\ \Rightarrow F_{MM}(\nu_a, T\mu_{a+1}, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^{b/c}\right). \end{aligned} \tag{50}$$

Also, from inequalities (44) and (56), we have

$$\begin{aligned} F_{MM}\left(\nu_a, \nu_{a+1}, t^{1/b}\right) \geq F_{MM}(\nu_a, T\mu_{a+1}, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^{b/c}\right) \\ \Rightarrow F_{MM}(\nu_a, \nu_{a+1}, t^c) \geq F_{MM}(\nu_{a-1}, T\mu_a, t), \end{aligned} \tag{51}$$

for $0 < c < 1$ and $t > 1$. Hence, $\{\nu_a\}$ is Cauchy sequence.
As subsets A and B are closed in M, therefore $\{\mu_a\}$, $\{\nu_a\}$ converges to points of A and B, respectively. Thus, there exist $\mu^* \in A$ and $\nu^* \in B$ such that $\mu_a \longrightarrow \mu^*$ and $\nu_a \longrightarrow \nu^*$ as $a \longrightarrow \infty$.
Letting $a \longrightarrow \infty$ in inequality (43), we have

$$F_{MM}(\mu^*, \nu^*, t) = F_{MM}(A, B, t), \tag{52}$$

for $t > 1$. The inequality (56) shows that the sequence $f(\mu_a, \nu_a) = F_{MM}(\nu_a, T\mu_a, t)$ is increasing and it converges to 1. Since $f(\mu, \nu)$ is upper semicontinuous, so

$$1 = \limsup_{a \longrightarrow \infty} f(\mu_a, \nu_a) \leq f(\mu^*, \nu^*) \leq 1 \tag{53}$$

implies to the fact that $f(\mu^*, \nu^*) = 1$, that is, $F_{MM}(\nu^*, T\mu^*, t) = 1$, and hence, $\nu^* \in T\mu^*$. Therefore,

(FM1)

$$\begin{aligned} F_{MM}(A, B, t) \geq F_{MM}(\mu^*, T\mu^*, t) \geq F_{MM}(\mu^*, \nu^*, t) \\ = F_{MM}(A, B, t), \end{aligned} \tag{54}$$

that is, $F_{MM}(\mu^*, T\mu^*, t) = F_{MM}(A, B, t)$. This shows that T possesses best proximity point μ^*.

If $A = B = M$ in Theorem 26, then we obtain the following corollary which is the fixed-point theorem for Feng-Liu-type contraction in fuzzy multiplicative metric space.

Corollary 27. *Let $(M, F_{MM}, \star)$ be complete fuzzy multiplicative metric space. Let $T : M \longrightarrow C(M)$ be a mapping, for all $\mu \in M$ and $\nu \in I_b^\mu$ (where $I_b^\mu = \{\nu \in T\mu \mid F_{MM}(\mu, \nu, t) \geq F_{MM}(\mu, T\mu, t^b)\} \subset M$ for some $b \in (0, 1)$) satisfying*

$$F_{MM}(\nu, T\nu, t^c) \geq F_{MM}(\mu, \nu, t), \tag{55}$$

for some $c \in (0, 1)$ and $t > 1$. Then, T possesses fixed point provided that $c < b$ and $f(\mu) = F_{MM}(\mu, T\mu, t)$ is upper semicontinuous.

4. Best Proximity Point Theorems of Feng-Liu-Type Mappings in Fuzzy Metric Space

Getting motivation from the work of Sahin et al. [10], we prove the following result.

Theorem 28. *Let $(M, F_M, \star)$ be complete fuzzy metric space. $A, B \subseteq M$ be closed and nonempty having P-property and $A_0 \neq \phi$. Let $T : A \longrightarrow C(B)$ be a mapping such that $T(A_0) \subseteq B_0$ and for all $\mu \in A_0$ and $\nu \in T\mu$ there exist $\rho \in A_0$ satisfying*

$$\begin{aligned} &F_M(\nu, \rho, t) = F_M(A, B, t), \\ &F_M(\nu, T\rho, ct) \geq F_M(\mu, \rho, t), \end{aligned} \tag{56}$$

for some $c \in (0, 1)$ and $t > 0$. Assume that $(M, F_M, \star)$ satisfy

$$\lim_{a \longrightarrow \infty} *_{i=a}^{\infty} F_M\left(\mu, \nu, th^i\right) = 1, \tag{57}$$

for every $t > 0, h > 1$ and $\mu, \nu \in M$. Then, T possesses best proximity point in A provided that $f(\mu, \nu) = F_M(\nu, T\mu, t)$ is upper semicontinuous.

Proof. Let $\mu_0 \in A_0$ be arbitrary point. Choose $\nu_0 \in T\mu_0$. Then, by assumption, there exist $\mu_1 \in A_0$ such that $F_M(\nu_0, \mu_1, t) = F_M(A, B, t)$ and $F_M(\nu_0, T\mu_1, ct) \geq F_M(\mu_0, \mu_1, t)$.

Presently, let $b \in (c, 1)$, then we can discover $\nu_1 \in T\mu_1$ such that

$$F_M(\nu_0, \nu_1, t) \geq F_M(\nu_0, T\mu_1, bt). \tag{58}$$

Again by assumption, there exist $\mu_2 \in A_0$ such that $F_M(\nu_1, \mu_2, t) = F_M(A, B, t)$ and $F_M(\nu_1, T\mu_2, ct) \geq F_M(\mu_1, \mu_2, t)$.

Also, we can find $\nu_2 \in T\mu_2$ such that

$$F_M(\nu_1, \nu_2, t) \geq F_M(\nu_1, T\mu_2, bt). \tag{59}$$

□ □

Proceeding in a similar manner, we develop two sequences $\{\mu_a\}$ and $\{\nu_a\}$ in A and B, respectively, with $\mu_a \in A_0$, $\nu_a \in T\mu_a$ and

$$F_M(\nu_a, \mu_{a+1}, t) = F_M(A, B, t), \tag{60}$$

$$F_M(\nu_a, T\mu_{a+1}, ct) \geq F_M(\mu_a, \mu_{a+1}, t), \tag{61}$$

$$F_M(\nu_a, \nu_{a+1}, t) \geq F_M(\nu_a, T\mu_{a+1}, t), \tag{62}$$

for all $a \in \mathbb{N}$ and $t > 0$. Then again, since A and B have P-property, so from inequality (61), we get

$$F_M(\mu_a, \mu_{a+1}, t) = F_M(\nu_{a-1}, \nu_a, t). \tag{63}$$

Therefore, from inequality (62), we have

$$F_M(\mu_a, \mu_{a+1}, t) = F_M(\nu_{a-1}, \nu_a, t) \geq F_M(\nu_{a-1}, T\mu_a, bt). \tag{64}$$

From inequality (62), we have

$$F_M(\nu_{a-1}, T\mu_a, t) \geq F_M\left(\mu_{a-1}, \mu_a, \frac{1}{c}t\right). \tag{65}$$

Combining inequalities (64) and (65), we get

$$F_M(\mu_a, \mu_{a+1}, t) \geq F_M\left(\mu_{a-1}, \mu_a, \frac{b}{c}t\right), \tag{66}$$

for all $a \geq 1$ and $t > 0$.

Let $k = c/b$ and then $0 < k < 1$. The inequality (66) gives

$$F_M(\mu_a, \mu_{a+1}, kt) \geq F_M(\mu_{a-1}, \mu_a, t), \tag{67}$$

for $0 < k < 1$ and $t > 0$. By our assumption (57) and Lemma 19, $\{\mu_a\}$ is Cauchy sequence.

Now, from inequalities (62) and (64), we have

$$\begin{aligned} &F_M(\nu_a, T\mu_{a+1}, ct) \geq F_M(\mu_a, \mu_{a+1}, t) \geq F_M(\nu_{a-1}, T\mu_a, bt) \\ &\qquad \Rightarrow F_M(\nu_a, T\mu_{a+1}, t) \geq F_M\left(\nu_{a-1}, T\mu_a, \frac{b}{c}t\right). \end{aligned} \tag{68}$$

Also, from inequalities (62) and (68), we have

$$\begin{aligned} &F_M\left(\nu_a, \nu_{a+1}, \frac{1}{b}t\right) \geq F_M(\nu_a, T\mu_{a+1}, t) \geq F_M\left(\nu_{a-1}, T\mu_a, \frac{b}{c}t\right) \\ &\qquad \Rightarrow F_M(\nu_a, \nu_{a+1}, ct) \geq F_M(\nu_{a-1}, T\mu_a, t), \end{aligned} \tag{69}$$

for $0 < c < 1$ and $t > 0$. Hence, $\{\nu_a\}$ is Cauchy sequence.

As subsets A and B are closed in M, so $\{\mu_a\}, \{\nu_a\}$ converges to points of A and B, respectively. Thus, there is some $\mu^* \in A$ and $\nu^* \in B$ such that $\mu_a \longrightarrow \mu^*$ and $\nu_a \longrightarrow \nu^*$ as $a \longrightarrow \infty$.

Letting $a \longrightarrow \infty$ in equation (61), we have

$$F_M(\mu^*, \nu^*, t) = F_M(A, B, t), \tag{70}$$

for $t > 0$. The inequality (68) shows that the sequence $f(\mu_a$

$, \nu_a) = F_M(\nu_a, T\mu_a, t)$ is increasing sequence, so it converges to 1. Since $f(\mu, \nu)$ is upper semicontinuous, so

$$1 = \limsup_{a \longrightarrow \infty} f(\mu_a, \nu_a) \le f(\mu^*, \nu^*) \le 1 \tag{71}$$

implies to the fact that $f(\mu^*, \nu^*) = 1$, that is, $F_M(\nu^*, T\mu^*, t) = 1$, and hence, $\nu^* \in T\mu^*$.

Therefore,

$$F_M(A, B, t) \ge F_M(\mu^*, T\mu^*, t) \ge F_M(\mu^*, \nu^*, t) = F_M(A, B, t), \tag{72}$$

that is, $F_M(\mu^*, T\mu^*, t) = F_M(A, B, t)$. This shows that T possesses best proximity point μ^*.

Example 29. Let $J = \{0, 1\} \cup \{1/2^a : a \in \mathbb{N}\}$ and $M = J \times J$, $F_M(\mu, \nu, t) = t/(t + d(\mu, \nu))$ and $d(\mu, \nu) = |\mu_1 - \nu_1| + |\mu_2 - \nu_2|$ for $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2) \in M$. Then, $(M, F_M, \star)$ is complete fuzzy metric space where $\star : [0, 1]^2 \longrightarrow [0, 1]$ defined by $a \star b = ab$. Let $A = \{(0, 1/2^a) : a \in \mathbb{N}\} \cup \{(0, 0), (0, 1)\}$ and $B = \{(1, 1/2^a) : a \in \mathbb{N}\} \cup \{(1, 0), (1, 1)\}$. Then, $A_0 = A$, $B_0 = B$ and $F_M(A, B, t) = t/(t + 1)$. Define $T : A \longrightarrow C(B)$ as

$$T(1, \mu) = \begin{cases} \left\{ \left(0, \dfrac{1}{2^{a+1}}\right), (0, 1) \right\} \text{ if } \mu = \dfrac{1}{2^a}, & a = 0, 1, 2, \cdots, \\ \left\{ (0, 0), \left(0, \dfrac{1}{2}\right) \right\} \text{ if } \mu = 0. \end{cases} \tag{73}$$

For all $\mu, \nu \in M, \lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_M(\mu, \nu, th^i) = 1$ which implies that M satisfies 16. Let $\mu = (1, 1/2^a) \in A_0$ and $\nu = (0, 1/2^{a_1}) \in T\mu = (1, 1/2^a)$; then, for $\rho = (1, 1/2^{a_1}) \in A_0$, we have $F_M(\nu, \rho, t) = F_M(A, B, t)$ and $F_M(\nu, T\rho, t) = 1 \ge F_M(\mu, \nu, t)$.

Also,

$$f(\mu, \nu) = F_M(\nu, T\mu, t) = \frac{t}{t + d(\nu, T\mu)} = \begin{cases} \dfrac{t}{t + \left(1/2^{a+1}\right)} \text{ for } \mu = \left(1, \dfrac{1}{2^a}\right) \\ 1 \text{ for } \mu = (1, 0), (1, 1) \end{cases} \tag{74}$$

is continuous. Since all conditions of Theorem 28 are satisfied, so best proximity point for T exists. Furthermore, for $u = (1, 1/2^a), \nu = (1, 0) \in A_0$ $H_{F_M}(T(1, 1/2^a), T(1, 0), ct) = ct/(ct + (1/2))$ and $F_M((1, 1/2^a), (1, 0), t) = t/(t + (1/2^a))$.

Assume that for $c \in (0, 1)$, $H_{F_M}(T(1, 1/2^a), T(1, 0), ct) \ge F_M((1, 1/2^a), (1, 0), t)$. That is

$$\frac{ct}{ct + (1/2)} \ge \frac{t}{t + (1/2^a)}, \tag{75}$$

which implies that $c \ge 2^{a-1}$ for $a \in \mathbb{N}$ which is a contradiction. This shows that T does not satisfy the contraction condition of Nadler's multivalued mapping.

As corollary of Theorem 28, we obtain a result which was proved in [23]. We get the corollary by taking $A = B = M$.

Corollary 30. *Let $(M, F_M, \star)$ be complete fuzzy metric space. Let $T : M \longrightarrow C(M)$ be a mapping, for all $\mu \in M$ and $\nu \in I_b^\mu$ (where $I_b^\mu = \{\nu \in T\mu \mid F_M(\mu, \nu, t) \ge F_M(\mu, T\mu, bt)\} \subset M$ for some $b \in (0, 1)$) satisfying*

$$F_M(\nu, T\nu, ct) \ge F_M(\mu, \nu, t), \tag{76}$$

for some $c \in (0, 1)$ and $t > 1$. Then, T possesses fixed point provided that $c < b$ and $f(\mu) = F_M(\mu, T\mu, t)$ is upper semicontinuous.

5. Conclusion

Zadeh [24] introduced the notion of fuzzy logic to cope with the problem of uncertainty that occurs essentially while studying real-life problem. Many researchers found easiness to study the phenomenon of different fields that were too complex to be analyzed by conventional techniques. Fuzzy metric introduced by Kaleva and Seikkala [25] measures the imprecision of distance between elements. Fuzzy metric has been applied in variety of applications like color image filtering [26] and in engineering methods [15]. Multiplicative calculus has its great applications in various fields, few of which are in biomedical image analysis [27] and contour detection in images [28]. In this paper, we introduced fuzzy multiplicative metric space and proved some best proximity point and fixed-point results in this new framework. The above discussion shows the possible applications in this framework in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

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Research Article

Characterization and Stability of Multimixed Additive-Quartic Mappings: A Fixed Point Application

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In this article, we introduce the multi-additive-quartic and the multimixed additive-quartic mappings. We also describe and characterize the structure of such mappings. In other words, we unify the system of functional equations defining a multi-additive-quartic or a multimixed additive-quartic mapping to a single equation. We also show that under what conditions, a multimixed additive-quartic mapping can be multiadditive, multiquartic, and multi-additive-quartic. Moreover, by using a fixed point technique, we prove the Hyers-Ulam stability of multimixed additive-quartic functional equations thus generalizing some known results.

1. Introduction

Let V be a commutative group, W be a linear space over rational numbers, and n be an integer with $n \geq 2$. A mapping $f : V^n \longrightarrow W$ is called

(i) *Multiadditive* if it satisfies the Cauchy's functional equation $A(x+y)=A(x)+A(y)$ in each variable [1]

(ii) *Multiquadratic* if it fulfills quadratic functional equation $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ in each variable [2, 3]

(iii) *Multicubic* if it satisfies the cubic equation $C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x)$ in each variable [4, 5]

(iv) *Multiquartic* if it satisfies the quartic equation

$$\mathfrak{Q}(x+2y)+\mathfrak{Q}(x-2y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)-6\mathfrak{Q}(x)+24\mathfrak{Q}(y), \tag{1}$$

in each variable [6, 7].

We have the following observations about a several variables mapping $f : V^n \longrightarrow W$.

(i) f is multiadditive [1] if and only if it satisfies

$$f(x_1+x_2)=\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right). \tag{2}$$

(ii) f is multiquadratic [8] if and only if it satisfies

$$\sum_{s\in\{-1,1\}^n} f(x_1+sx_2)=2^n \sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right), \quad (3)$$

where $x_j=(x_{1j},x_{2j},\cdots,x_{nj})\in V^n$ with $j\in\{1,2\}$. More information about the structure of multiadditive and multiquadratic mappings, we refer for instance to [9, 10].

Bodaghi et al. [4] (resp., [6]) provided a characterization of multicubic (resp., multiquartic) mappings, and they showed that every multicubic (resp., multiquartic) mapping can be shown a single functional equation and vice versa.

Lee et al. [11] introduced and obtained the general solution of the quartic functional equation which somewhat different from (1) as follows:

$$\mathfrak{Q}(2x+y)+\mathfrak{Q}(2x-y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)+24\mathfrak{Q}(x)-6\mathfrak{Q}(y). \quad (4)$$

For the generalized forms of the quartic functional, equations (1) and (4) refer to [12, 13]. Recently, in [14] and motivated by (4), a new form of multiquartic mappings was introduced, and the structure of such mappings was described.

Speaking of the stability of a functional equation, we follow the question raised in 1940 by Ulam [15] for group homomorphisms. Hyers [16] presented a partial solution to the problem of Ulam. Later, Hyers' theorem was extended and generalized in various forms by many mathematicians such as Aoki [17] and Rassias [18]. Recall that a functional equation $\mathfrak{F}$ is said to be *stable* if any mapping ϕ fulfilling $\mathfrak{F}$ approximately; then, it is near to an exact solution of $\mathfrak{F}$. Next, several stability problems of various functional equations and mappings have been investigated by many mathematicians which can be found in literatures.

In the last two decades, the stability problem for several variable mappings such as multiadditive, multi-Jensen, multiquadratic, multicubic, and multiquartic mappings by applying direct and fixed point methods has been studied by a number of authors which are available for example in [1, 2, 4, 8, 9, 19–26].

In [27], Eshaghi Gordji introduced and obtained the general solution of the following mixed type additive and quartic functional equation

$$f(2x+y)+f(2x-y)=4[(f(x+y)+f(x-y)]-\frac{3}{7}\cdot(f(2y)-2f(y))+2f(2x)-8f(x). \quad (5)$$

He also established the Hyers-Ulam Rassias stability of the above functional equation in real normed spaces. The stability of (5) in non-Archimedean orthogonality spaces is studied in [28]. A different and equivalent form of mixed type additive and quartic functional equation from (5) was introduced by the first author in [29] as follows:

$$\begin{aligned} f(x+2y)-4f(x+y)-4f(x-y)+f(x-2y) \\ =\frac{12}{7}(f(2y)-2f(y))-6f(x). \end{aligned} \quad (6)$$

It is easily verified that the function $f(x)=\alpha x^4+\beta x$ is a solution of equations (5) and (6); the generalized version of equation (6) can be found in [30].

This paper is organized as follows: In the second section, we firstly define multi-additive-quartic mappings and include a characterization of such mappings. In fact, we prove that every multi-additive-quartic mapping can be shown a single functional equation and vice versa (under some extra conditions). Section 3 is devoted to the study of stricture of multimixed additive-quartic mappings. In other words, motivated by equation (6), we introduce the multimixed additive-quartic mappings and reduce the system of n equations defining the multimixed additive-quartic mappings to a single equation, namely, the multimixed additive-quartic functional equation. In Section 4, we prove the Hyers-Ulam stability for the multi-additive-quartic and the multimixed additive-quartic mappings in the setting of Banach spaces by applying a fixed point method [31]. As an application of this result, we establish the stability of multi-additive-quartic mappings. Finally, we show that under some mild conditions every multiadditive and multiquartic functional equations are δ-stable for a small positive number δ.

2. Characterization of Multi-Additive-Quartic Mappings

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ stand for the set of all positive integers and the rational numbers, respectively, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathbb{R}_+:=[0,\infty)$. For any $l\in\mathbb{N}_0$, $m\in\mathbb{N}$, $t=(t_1,\cdots,t_m)\in\{-1,1\}^m$, and $x=(x_1,\cdots,x_m)\in V^m$, we write $lx:=(lx_1,\cdots,lx_m)$ and $tx:=(t_1x_1,\cdots,t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the commutative group V.

Let V and W be linear spaces, $n\in\mathbb{N}$ and $k\in\{0,\cdots,n\}$. A mapping $f:V^n\longrightarrow W$ is called k-additive and $n-k$-quartic (briefly, multi-additive-quartic) if f is additive in each of some k variables and satisfies (4) in each of the other variables. In what follows, for simplicity, it is assume that f is additive in each of the first k variables. Moreover, for $k=n$ ($k=0$), the above definition leads to the so-called multiadditive (multiquartic) mappings.

In the sequel, we assume that V and W are vector spaces over $\mathbb{Q}$. Moreover, we identify $x=(x_1,\cdots,x_n)\in V^n$ with $(x^k,x^{n-k})\in V^k\times V^{n-k}$, where $x^k:=(x_1,\cdots,x_k)$ and $x^{n-k}:=(x_{k+1},\cdots,x_n)$. Let $n\in\mathbb{N}$ with $n\geq 2$ and $x_i^n=(x_{i1},x_{i2},\cdots,x_{in})\in V^n$, where $i\in\{1,2\}$. Throughout, we shall denote x_i^n by x_i if there is no risk of mistake. Put also $x_i^k=(x_{i1},\cdots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1}\cdots,x_{in})\in V^{n-k}$. For $x_1,x_2\in V^n$ and $p_i\in\mathbb{N}_0$ with $0\leq p_i\leq n$ and $0\leq k\leq n-1$, set $\mathcal{N}^{n-k}=\{\mathfrak{N}_{n-k}=(N_{k+1},\cdots,N_n)\mid N_j\in\{x_{1j}\pm x_{2j},x_{1j},x_{2j}\}\}$, where $j\in\{k+1,\cdots,n\}$. Consider the subset $\mathcal{N}^{n-k}_{(p_1,p_2)}$ of $\mathcal{N}$ as follows:

$$\mathcal{N}^{n-k}_{(p_1,p_2)} := \left\{\mathfrak{N}_{n-k} \in \mathcal{N}^{n-k} \mid \mathrm{Card}\{N_j : N_j = x_{ij}\} = p_i(i \in \{1,2\})\right\}. \tag{7}$$

To achieve our aims, for the multi-additive-quartic mappings, we use the oncoming notations:

$$f\left(\mathcal{N}^{n-k}_{(p_1,p_2)}\right) := \sum_{\mathfrak{N}_{n-k}\in\mathcal{N}^{n-k}_{(p_1,p_2)}} f(\mathfrak{N}_n), \tag{8}$$

$$f\left(z, \mathcal{N}^{n-k}_{(p_1,p_2)}\right) := \sum_{\mathfrak{N}_{n-k}\in\mathcal{N}^{n-k}_{(p_1,p_2)}} f(z, \mathfrak{N}_{n-k})(z \in V). \tag{9}$$

For each $x_1, x_2 \in V^n$, we consider the equation

$$\begin{aligned} &\sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + tx_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{i\in\{1,2\}} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(x_i^k, \mathcal{N}^{n-k}_{(p_1,p_2)}\right), \end{aligned} \tag{10}$$

for all $x_i^k = (x_{i1},\cdots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}\cdots,x_{\mathrm{in}}) \in V^{n-k}$ where $i \in \{1,2\}$.

It is shown in Proposition 2.2 in [14] that if a mapping $f : V^n \longrightarrow W$ is multiquartic, then it satisfies the equation

$$\sum_{t\in\{-1,1\}^n} f(2x_1 + tx_2) = \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(\mathcal{N}^{n-k}_{(p_1,p_2)}\right). \tag{11}$$

The next proposition shows that the system of n equations defining a multi-additive-quartic mapping can be reduced to (10).

Proposition 1. *Let $n \in \mathbb{N}$ and $k \in \{0,\cdots,n\}$. Suppose that a mapping $f : V^n \longrightarrow W$ is k-additive and $n-k$-quartic (multi-additive-quartic) mapping. Then, f fulfills equation (10).*

Proof. For $k \in \{0, n\}$, the result follows from Proposition 2.2 in [14] and Theorem 2 in [1], and so we prove the assertion for the case that $k \in \{1,\cdots,n-1\}$. For any $x^{n-k} \in V^{n-k}$, consider the mapping $g_{x^{n-k}} : V^k \longrightarrow W$ defined by $g_{x^{n-k}}(x^k) := f(x^k, x^{n-k})$ for $x^k \in V^k$. The assumption shows that $g_{x^{n-k}}$ is k-additive, and thus, we can obtain from Theorem 2 in [1] that

$$g_{x^{n-k}}\left(x_1^k + x_2^k\right) = \sum_{j_1,j_2,\cdots,j_k\in\{1,2\}} g_{x^{n-k}}\left(x_{j_1 1}, x_{j_2 2},\cdots,x_{j_k k}\right), \left(x_1^k, x_2^k \in V^k\right). \tag{12}$$

The above equality implies that

$$f\left(x_1^k + x_2^k, x^{n-k}\right) = \sum_{j_1,\cdots,j_k\in\{1,2\}} f\left(x_{j_1 1},\cdots,x_{j_k k}, x^{n-k}\right), \tag{13}$$

for all $x_1^k, x_2^k \in V^k$ and $x^{n-k} \in V^{n-k}$. Repeat the above method, and for any $x^k \in V^k$, define the mapping $h_{x^k} : V^{n-k} \longrightarrow W$ via $h_{x^k}(x^{n-k}) := f(x^k, x^{n-k}), x^{n-k} \in V^{n-k}$. This mapping is $n-k$-quartic, and hence, by Proposition 2.2 from [14], we have

$$\begin{aligned} &\sum_{t\in\{-1,1\}^{n-k}} h_{x^k}\left(2x_1^{n-k} + tx_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} h_{x^k}\left(\mathcal{N}^{n-k}_{(p_1,p_2)}\right), \end{aligned} \tag{14}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$. On the other hand, by the definition of h_{x^k}, relation (14) converts to

$$\begin{aligned} &\sum_{t\in\{-1,1\}^{n-k}} f\left(x^k, 2x_1^{n-k} + tx_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(x^k, \mathcal{N}^{n-k}_{(p_1,p_2)}\right), \end{aligned} \tag{15}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. It now follows between (13) and (15) that

$$\begin{aligned} &\sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + tx_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(x_1^k + x_2^k, \mathcal{N}^{n-k}_{(p_1,p_2)}\right) \\ &= \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{j_1,j_2,\cdots,j_n\in\{1,2\}} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f \\ &\quad\cdot\left(x_{j_1 1}, x_{j_2 2},\cdots,x_{j_n n}, \mathcal{N}^{n-k}_{(p_1,p_2)}\right) \\ &= \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{i\in\{1,2\}} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(x_i^k, \mathcal{N}^{n-k}_{(p_1,p_2)}\right), \end{aligned} \tag{16}$$

for all $x_i^k = (x_{i1},\cdots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}\cdots,x_{\mathrm{in}}) \in V^{n-k}$. This finishes the proof. □

By Proposition 6, it is easily verified that the mapping $f(z_1,\cdots,z_n) = c\prod_{i=1}^k z_i \prod_{j=k+1}^n z_j^4$ satisfies (10), and so this equation is said to be *multi-additive-quartic functional equation*.

Definition 2. Let $r \in \mathbb{N}$. Consider a mapping $f : V^n \longrightarrow W$. We say f

(i) Satisfies (has) the *r -power condition* in the jth variable if

$$f(z_1, \cdots, z_{j-1}, 2z_j, z_{j+1}, \cdots, z_n) = 2^r f(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \cdots, z_n), \tag{17}$$

for all $z_1, \cdots, z_n \in V^n$. Sometimes 4-power condition is called *quartic condition*.

(ii) Has *zero condition* if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero

We remember that the binomial coefficient for all $n, r \in \mathbb{N}_0$ with $n \geq r$ is defined and denoted by $\binom{n}{r} := n!/r!(n-r)!$.

We wish to show that if a mapping satisfies equation (10), then it is multi-additive-quartic. For doing it, we need the upcoming lemma. The method of the proof of Lemma 3 is similar to the proof of ([14], Lemma 2.5) and so we include lemma without the proof.

Lemma 3. *Suppose that a mapping $f : V^n \longrightarrow W$ satisfies equation (10). Under one of the following assumptions, f satisfying zero condition.*

(i) f satisfies the quartic condition in the last $n-k$ variables

(ii) f is even in the last $n-k$ variables

Theorem 4. *Suppose that a mapping $f : V^n \longrightarrow W$ fulfilling equation (10). Under one of the hypothesis of Lemma 3, f is multi-additive-quartic.*

Proof. It follows from Lemma 3; f satisfies zero condition. Putting $x_2^{n-k} = (0, \cdots, 0)$ in the left side of (10) and applying the hypothesis, we obtain

$$2^{n-k} \times 2^{4(n-k)} f\left(x_1^k + x_2^k, x_1^{n-k}\right) = 2^{5(n-k)} f\left(x_1^k + x_2^k, 2x_1^{n-k}\right). \tag{18}$$

On the other hand, by using Lemma 3, the right side of (10) converts to

$$\begin{aligned} &\sum_{p_1=0}^{n-k} \binom{n-k}{p_1} 4^{n-k-p_1} 24^{p_1} 2^{n-k-p_1} f\left(x_{j_1 1}, x_{j_2 2}, \cdots, x_{j_k k}, x_1^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \binom{n-k}{p_1} 8^{n-k-p_1} 24^{p_1} f\left(x_{j_1 1}, x_{j_2 2}, \cdots, x_{j_k k}, x_1^{n-k}\right) \\ &= 2^{5(n-k)} \sum_{j_1, j_2, \cdots, j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \cdots, x_{j_k k}, x_1^{n-k}\right). \end{aligned} \tag{19}$$

Now, relations (18) and (19) necessitate that

$$f\left(x_1^k + x_2^k, x_1^{n-k}\right) = \sum_{j_1, j_2, \cdots, j_n \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \cdots, x_{j_n n}, x_1^{n-k}\right), \tag{20}$$

for all $x_1^k, x_2^k \in V^n$ and $x_1^{n-k} \in V^{n-k}$. In light of Theorem 2 in [1], we see that f is additive in each of the k first variables. In addition, by considering $x_2^k = (0, \cdots, 0)$ in (10) and applying again Lemma 3, we have

$$\begin{aligned} &\sum_{t \in \{-1,1\}^{n-k}} f\left(x_1^k, 2x_1^{n-k} + tx_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(x_1^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right), \end{aligned} \tag{21}$$

for all $x_1^k \in V^k$ and $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$, and thus, by Theorem 2.6 in [14], f is quartic in each of the last $n-k$ variables. The proof of second part is similar. □

3. Characterization of Multimixed Additive-Quartic Mappings

In this section, we introduce the multimixed additive-quartic mappings and then characterize them as an equation. We start this section with the definition of such mappings.

Definition 5. Let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$. A mapping $f : V^n \longrightarrow W$ is called *n -multimixed additive-quartic* or briefly *multimixed additive-quartic* if f satisfies mixed additive-quartic equation (6) in each variable.

Let $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \cdots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. For $x_1, x_2 \in V^n$ and $q \in \mathbb{N}_0$ with $0 \leq q \leq n$, put

$$\mathscr{M} = \left\{\mathfrak{M}_n = (M_1, \cdots, M_n) \mid M_j \in \left\{x_{1j} \pm 2x_{2j}, 2x_{2j}\right\}, j \in \{1, \cdots, n\}\right\}. \tag{22}$$

Consider the subset $\mathscr{M}_q^n$ of $\mathscr{M}$ as follows:

$$\mathscr{M}_q^n := \left\{\mathfrak{M}_n \in \mathscr{M} \mid \operatorname{Card}\left\{M_j : M_j = 2x_{2j}\right\} = q\right\}. \tag{23}$$

Hereafter, for the multimixed additive-quartic mappings, we use the following notations:

$$f\left(\mathscr{M}_q^n\right) := \sum_{\mathfrak{M}_n \in \mathscr{M}_q^n} f(\mathfrak{M}_n), \tag{24}$$

$$f\left(\mathscr{M}_q^n, z\right) := \sum_{\mathfrak{M}_n \in \mathscr{M}_q^n} f(\mathfrak{M}_n, z)(z \in V). \tag{25}$$

Next, we reduce the system of n equations defining the multimixed additive-quartic mapping to obtain a single functional equation.

Proposition 6. *If a mapping $f : V^n \longrightarrow W$ is multimixed additive-quartic, it satisfies the equation*

$$\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^{q} f\left(\mathscr{M}_q^n\right) = \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}^n_{(p_1,p_2)}\right), \tag{26}$$

where $f(\mathscr{M}_q^n)$ and $f(\mathscr{N}^n_{(p_1,p_2)})$ are defined in (24) and (8), respectively.

Proof. The proof is based on induction for n. For $n = 1$, it is obvious that f satisfies (6). Assume that (26) holds for some positive integer $n > 1$. Then

$$\begin{aligned}
&\sum_{q=0}^{n+1}\left(-\frac{12}{7}\right)^{q+1} f\left(\mathscr{M}_q^{n+1}\right)\\
&= \sum_{q=0}^{n}\sum_{t\in\{1,-1\}}\left(-\frac{12}{7}\right)^{q} f\left(\mathscr{M}_q^n, x_{1n+1} + 2tx_{2,n+1}\right)\\
&\quad - \frac{12}{7}\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^{q} f\left(\mathscr{M}_q^n, 2x_{2,n+1}\right)\\
&= \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{t\in\{1,-1\}} 4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2} f\\
&\quad \cdot\left(\mathscr{N}^n_{(p_1,p_2)}, x_{1,n+1} + 2tx_{2,n+1}\right) - \frac{12}{7}\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1}\\
&\quad \cdot\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}^n_{(p_1,p_2)}, 2x_{2,n+1}\right)\\
&= 4\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{s\in\{1,-1\}} 4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2} f\\
&\quad \cdot\left(\mathscr{N}^n_{(p_1,p_2)}, x_{1,n+1} + sx_{1,n+1}\right) - 6\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1}\\
&\quad \cdot\left(-\frac{24}{7}\right)^{p_2}\left(f\left(\mathscr{N}^n_{(p_1,p_2)}, x_{1,n+1}\right) - \frac{24}{7}\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1}\right.\\
&\quad \cdot\left.\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}^n_{(p_1,p_2)}, x_{2,n+1}\right)\right) = \sum_{p_1=0}^{n+1}\sum_{p_2=0}^{n+1-p_1} 4^{n+1-p_1-p_2}(-6)^{p_1}\\
&\quad \cdot\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}^{n+1}_{(p_1,p_2)}\right).
\end{aligned} \tag{27}$$

The assertion is now proved. □

Since the mapping $f(z_1, \cdots, z_n) = \prod_{j=1}^{n}(a_j z_j^4 + b_j z_j)$ is multimixed additive-quartic, it satisfies (26) by proposition above, and so this equation is called *multimixed additive-quartic functional equation*.

Here, we bring an elementary lemma without proof.

Lemma 7. *Let $n, k, p_l \in \mathbb{N}_0$, such that $k + \sum_{l=1}^{m} p_l \le n$, where $l \in \{1, \cdots, m\}$. Then*

$$\begin{aligned}
&\binom{n-k}{n-k-\sum_{l=1}^{m} p_l}\binom{\sum_{l=1}^{m} p_l}{\sum_{l=1}^{m-1} p_l}\cdots\binom{p_1+p_2}{p_1}\\
&= \binom{n-k}{p_1}\binom{n-k-p_1}{p_2}\cdots\binom{n-k-\sum_{l=1}^{m-1} p_l}{p_m}.
\end{aligned} \tag{28}$$

Similar to Lemma 2.1 from [6], we need the following lemma in obtaining our goal in this section. The proof is similar, but we include some parts for the sake of completeness.

Lemma 8. *If a mapping $f : V^n \longrightarrow W$ satisfies equation (26), then it has zero condition.*

Proof. Putting $x_1 = x_2 = (0,\cdots,0)$ in (26), we have

$$\begin{aligned}
&\left[\sum_{q=0}^{n}\binom{n}{n-q}\left(-\frac{12}{7}\right)^{q} 2^{n-q}\right] f(0, \cdots, 0)\\
&= \left[\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1} 2^{n-p_1-p_2}4^{n-p_1-p_2}(-6)^{p_1}\right.\\
&\quad \cdot\left.\left(-\frac{24}{7}\right)^{p_2}\right] f(0, \cdots, 0).
\end{aligned} \tag{29}$$

Using Lemma 7 for $k = 0$ and p_1, p_2, the right side of (29) will be as follows:

$$\begin{aligned}
&\left[\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1} 2^{n-p_1-p_2}4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2}\right] f(0, \cdots, 0)\\
&= 2^n\left[\sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{12}{7}\right)^{p_2}\sum_{p_1=0}^{n-p_2}\binom{n-p_2}{p_1} 4^{n-p_1-p_2}(-3)^{p_1}\right] f(0, \cdots, 0)\\
&= 2^n\left[\sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{12}{7}\right)^{p_2}(4-3)^{n-p_2}\right] f(0, \cdots, 0) = 2^n\left(-\frac{12}{7}+1\right)^{n} f(0,\cdots,0)\\
&= \left(-\frac{10}{7}\right)^{n} f(0, \cdots, 0).
\end{aligned} \tag{30}$$

On the other hand, by a simple computation, the left side of (29) is

$$\left(\frac{2}{7}\right)^{n} f(0, \cdots, 0). \tag{31}$$

It follows from relations (29), (30), and (31) that $f(0, \cdots, 0) = 0$. One can continue this method to show that f has zero condition. □

Definition 9. A mapping $f : V^n \longrightarrow W$ is

(iii) Odd in the jth variable if

$$f(z_1, \cdots, z_{j-1}, -z_j, z_{j+1}, \cdots, z_n) = -f(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \cdots, z_n). \tag{32}$$

(iv) Even in the jth variable if

$$f(z_1, \cdots, z_{j-1}, -z_j, z_{j+1}, \cdots, z_n) = f(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \cdots, z_n). \tag{33}$$

Proposition 10. *Suppose that a mapping $f : V^n \longrightarrow W$ satisfies equation (26). Then, it is multimixed additive-quartic. Moreover,*

(i) If f is odd in a variable, then it is additive in the same variable

(ii) If f is even in a variable, then it is quartic in the same variable

Proof. Let $j \in \{1, \cdots, n\}$ be arbitrary and fixed. Set

$$f_j^*(z) := f(z_1, \cdots, z_{j-1}, z, z_{j+1}, \cdots, z_n). \tag{34}$$

Putting $x_{2k} = 0$ for all $k \in \{1,\cdots,n\} \setminus \{j\}$ in (26) and using Lemma 8, we get

$$\begin{aligned}
&2^{n-1}\left[f_j^*(z+2w) + f_j^*(z-2w) - \frac{12}{7} f_j^*(2w)\right] \\
&= \left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-p_1}(-6)^{p_1} 2^{n-p_1-1}\right]\left(f_j^*(z+w) + f_j^*(z-w)\right) \\
&\quad + \left[\sum_{p_1=1}^{n} \binom{n-1}{p_1-1} 4^{n-p_1}(-6)^{p_1} 2^{n-p_1}\right] f_j^*(z) \\
&\quad + \left[\sum_{p_1=0}^{n} \binom{n-1}{p_1} 4^{n-p_1-1}(-6)^{p_1}\left(-\frac{24}{7}\right) 2^{n-p_1-1}\right] f_j^*(w) \\
&= 4\left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right]\left(f_j^*(z+w) + f_j^*(z-w)\right) \\
&\quad - 6\left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right] f_j^*(z) - \frac{24}{7} \\
&\quad \cdot \left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right] f_j^*(w) \\
&= 4 \times 2^{n-1}\left(f_j^*(z+w) + f_j^*(z-w)\right) - 6 \times 2^{n-1} f_j^*(z) \\
&\quad - \frac{24}{7} \times 2^{n-1} f_j^*(w).
\end{aligned} \tag{35}$$

The above equalities show that

$$\begin{aligned}
&f_j^*(z+2w) + f_j^*(z-2w) - \frac{12}{7} f_j^*(2w) \\
&\quad = 4\left[f_j^*(z+w) + f_j^*(z-w)\right] - 6 f_j^*(z) - \frac{24}{7} f_j^*(w).
\end{aligned} \tag{36}$$

In other words, (6) is true for f_j^*. Since j is arbitrary, f is a multimixed additive-quartic mapping.

(i) Repeating the proof of Lemma 2.1 (i) from [29] for f_j^*, we see that $f_j^*(z+w) = f_j^*(z) + f_j^*(w)$. This means that f is additive in the jth variable

(ii) Similar to the previous part, it follows from the proof of part (ii) of Lemma 2.1 in [29] that

$$\begin{aligned}
f_j^*(2z+w) + f_j^*(2z-w) &= 4\left[f_j^*(z+w) + f_j^*(z-w)\right] \\
&\quad + 24 f_j^*(z) - 6 f_j^*(w).
\end{aligned} \tag{37}$$

Therefore, f is quartic in the jth variable. □

Corollary 11. *Suppose a mapping $f : V^n \longrightarrow W$ satisfies equation (26).*

(i) If f is odd in each variable, then it is multiadditive. Moreover, it satisfies (2)

(ii) If f is even in each variable, then it is multiquartic. In particular, it fulfills (11)

(iii) If f is odd in each of some k variables and is even in each of the other variables, then it is multi-additive-quartic. In addition, (10) is valid for f

4. Various Stability Results

In this section, we prove some Hyers-Ulam stability results by a fixed point method in the setting of Banach spaces. In what follows, we denote the set of all mappings from E to F by F^E. We remember the following theorem which is an essential result in fixed point theory ([23], Theorem 1). This achievement is a key tool in obtaining our aim in this section.

Theorem 12. *Let the hypotheses*

(A1) Y is a Banach space, E is a nonempty set, $j \in \mathbb{N}$, $g_1, \cdots, g_j : E \longrightarrow E$, and $L_1, \cdots, L_j : E \longrightarrow \mathbb{R}_+$

(A2) $\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{j} L_i(x)\|\lambda(g_i(x)) - \mu(g_i(x))\|, \lambda, \mu \in Y^E, x \in E, \tag{38}$$

(A3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ *is an operator defined through*

$$\Lambda\delta(x) \coloneqq \sum_{i=1}^{j} L_i(x)\delta(g_i(x))\delta \in \mathbb{R}_+^E, x \in E, \quad (39)$$

hold, and a function $\theta : E \longrightarrow \mathbb{R}_+$ *and a mapping* $\phi : E \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\phi(x) - \phi(x)\| \le \theta(x), \theta^*(x) \coloneqq \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty (x \in E). \quad (40)$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *such that*

$$\|\phi(x) - \psi(x)\| \le \theta^*(x)(x \in E). \quad (41)$$

Moreover, $\psi(x) = \lim_{l\longrightarrow\infty}\mathcal{T}^l\phi(x)$ *for all* $x \in E$.

For the rest of this paper and for each mapping $f : V^n \longrightarrow W$, we consider the difference operator $\Gamma_{AQ}f : V^n \times V^n \longrightarrow W$ defined via

$$\Gamma_{AQ}f(x_1, x_2) \coloneqq \sum_{q=0}^{n}\left(-\frac{12}{7}\right)^q f\left(\mathscr{M}_q^n\right) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} \cdot \left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}_{(p_1,p_2)}^n\right), \quad (42)$$

where $f(\mathscr{M}_q^n)$ and $f(\mathscr{N}_{(p_1,p_2)}^n)$ are defined in (24) and (8), respectively. In the sequel, all mappings $f : V^n \longrightarrow W$ are assumed that satisfy zero condition. With this assumption, we have the next stability result for functional equation (26) in the odd case.

Theorem 13. *Let* $\beta \in \{-1, 1\}$ *be fixed,* V *be a linear space, and* W *be a Banach space. Suppose that* $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a mapping satisfying*

$$\lim_{l\longrightarrow\infty}\left(\frac{1}{2^{n\beta}}\right)^l \phi\left(2^{\beta l}x_1, 2^{\beta l}x_2\right) = 0, \quad (43)$$

for all $x_1, x_2 \in V^n$ *and*

$$\Phi(x) =: \left(\frac{7}{12}\right)^n \frac{1}{2^{n((\beta+1)/2)}}\sum_{l=0}^{\infty}\left(\frac{1}{2^{n\beta}}\right)^l \phi\left(0, 2^{\beta l+((\beta-1)/2)}x\right) < \infty, \quad (44)$$

for all $x \in V^n$. *Assume also* $f : V^n \longrightarrow W$ *is a mapping fulfilling the inequality*

$$\|\Gamma_{AQ}f(x_1, x_2)\| \le \phi(x_1, x_2), \quad (45)$$

for all $x_1, x_2 \in V^n$. *If* f *is odd in each variable, then there exists a unique multiadditive mapping* $\mathscr{A} : V^n \longrightarrow W$ *such that*

$$\|f(x) - \mathscr{A}(x)\| \le \Phi(x), \quad (46)$$

for all $x \in V^n$.

Proof. Replacing (x_1, x_2) by $(0, x_1)$ in (45) and using the assumptions, we have

$$\left\|\left(-\frac{12}{7}\right)^n f(2x) - \left(-\frac{24}{7}\right)^n f(x)\right\| \le \phi(0, x), \quad (47)$$

for all $x = x_1 \in V^n$ (here and the rest of the proof) and so

$$\left\|f(x) - \frac{1}{2^n}f(2x)\right\| \le \left(\frac{7}{24}\right)^n \phi(0, x). \quad (48)$$

Set

$$\theta(x) \coloneqq \left(\frac{7}{12}\right)^n \frac{1}{2^{n((\beta+1)/2)}}\phi\left(0, 2^{(\beta-1)/2}x\right), \text{and } \mathcal{T}\theta(x) \coloneqq \frac{1}{2^{n\beta}}\theta\left(2^\beta x\right)\left(\theta \in W^{V^n}\right). \quad (49)$$

Then, relation (48) can be modified as

$$\|f(x) - \mathcal{T}f(x)\| \le \theta(x)(x \in V^n). \quad (50)$$

Define $\Lambda\eta(x) \coloneqq (1/2^{n\beta})\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$. It is seen that Λ has the form (A3) of Theorem 12 for which $E = V^n$, $g_1(x) = 2^\beta x$, and $L_1(x) = 1/2^{n\beta}$. Furthermore, for each $\lambda, \mu \in W^{V^n}$, we get

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\|\frac{1}{2^{n\beta}}\left[\lambda\left(2^\beta x\right) - \mu\left(2^\beta x\right)\right]\right\| \le L_1(x)\|\lambda(g_1(x)) - \mu(g_1(x))\|. \quad (51)$$

The above relation portrays that the hypothesis (A2) holds. By induction on l, one can check that for any $l \in \mathbb{N}_0$, we have

$$\Lambda^l\theta(x) \coloneqq \left(\frac{1}{2^{n\beta}}\right)^l \theta\left(2^{\beta l}x\right) = \left(\frac{7}{12}\right)^n\left(\frac{1}{2^{n((\beta+1)/2)}}\right)^l \phi\left(0, 2^{\beta l+((\beta-1)/2)}x\right). \quad (52)$$

Now, relations (44) and (52) necessitate that all assumptions of Theorem 12 are satisfied. Hence, there exists a unique mapping $\mathscr{A} : V^n \longrightarrow W$ such that

$$\mathscr{A}(x) = \lim_{l\longrightarrow\infty}\left(\mathcal{T}^l f\right)(x) = \frac{1}{2^{n\beta}}\mathscr{A}\left(2^\beta x\right)(x \in V^n), \quad (53)$$

and (46) holds. In continuation, we prove that

$$\left\|\Gamma_{AQ}\left(\mathcal{T}^l f\right)(x_1,x_2)\right\| \le \left(\frac{1}{2^{n\beta}}\right)^l \phi\left(2^{\beta l}x_1, 2^{\beta l}x_2\right), \tag{54}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on l. Clearly, inequality (54) is valid for $l=0$ by (45). Assume that (54) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
&\|\Gamma_{AQ}\left(\mathcal{T}^{l+1}f\right)(x_1,x_2)\| \\
&= \left\|\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^q\left(\mathcal{T}^{l+1}f\right)\left(\mathcal{M}_q^n\right) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}4^{n-p_1-p_2}(-6)^{p_1} \cdot\left(-\frac{24}{7}\right)^{p_2}\left(\mathcal{T}^{l+1}f\right)\left(\mathcal{N}_{(p_1,p_2)}^n\right)\right\| \\
&= \frac{1}{2^{n\beta}}\left\|\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^q\left(\mathcal{T}^{l+1}f\right)\left(2^\beta\mathcal{M}_q^n\right) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}4^{n-p_1-p_2}(-6)^{p_1} \cdot\left(-\frac{24}{7}\right)^{p_2}\left(\mathcal{T}^{l+1}f\right)\left(2^\beta\mathcal{N}_{(p_1,p_2)}^n\right)\right\| \\
&= \frac{1}{2^{n\beta}}\left\|\Gamma_{AQ}\left(\mathcal{T}^l f\right)\left(2^\beta x_1, 2^\beta x_2\right)\right\| \\
&\le \left(\frac{1}{2^{n\beta}}\right)^{l+1}\phi\left(2^{\beta(l+1)}x_1, 2^{\beta(l+1)}x_2\right),
\end{aligned} \tag{55}$$

for all $x_1, x_2 \in V^n$. Letting $l \longrightarrow \infty$ in (54) and applying (43), we arrive at $\Gamma_{AQ}\mathcal{A}(x_1,x_2)=0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathcal{A}$ satisfies (26), and so it is multi-additive by Corollary 11. This finishes the proof. □

Here, in analogy with Theorem 13, we bring the next stability result for functional equation (26) in the even case.

Theorem 14. *Let $\beta \in \{-1, 1\}$ be fixed, V be a linear space, and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying*

$$\lim_{l\longrightarrow\infty}\left(\frac{1}{2^{4n\beta}}\right)^l \phi\left(2^{\beta l}x_1, 2^{\beta l}x_2\right) = 0, \tag{56}$$

for all $x_1, x_2 \in V^n$ and

$$\Psi(x) =: \left(\frac{7}{2}\right)^n \frac{1}{2^{4n((\beta+1)/2)}}\sum_{l=0}^{\infty}\left(\frac{1}{2^{4n\beta}}\right)^l \phi\left(2^{\beta l+((\beta+1)/2)}x, 0\right) < \infty, \tag{57}$$

for all $x \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality

$$\left\|\Gamma_{AQ}f(x_1,x_2)\right\| \le \phi(x_1,x_2), \tag{58}$$

for all $x_1, x_2 \in V^n$. If f is even in each variable, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (26) such that

$$\|f(x) - \mathcal{Q}(x)\| \le \Psi(x), \tag{59}$$

for all $x \in V^n$. In particular, if $\mathcal{Q}$ is even mapping in each variable, then it is multiquartic.

Proof. Replacing (x_1, x_2) by $(0, x_1)$ in (58) and applying the hypotheses, we obtain

$$\begin{aligned}
&\left\|\sum_{q=0}^{n}\binom{n}{q}\left(-\frac{12}{7}\right)^q 2^{n-q}f(2x) - \sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{24}{7}\right)^{p_2}4^{n-p_2}\times 2^{n-p_2}f(x)\right\| \\
&\quad\le \phi(0,x),
\end{aligned} \tag{60}$$

where $x = x_1 \in V^n$ (here and the rest of the proof). On the other hand

$$\begin{aligned}
\sum_{q=0}^{n}\binom{n}{q}\left(-\frac{12}{7}\right)^q 2^{n-q} &= \left(-\frac{12}{7}+2\right)^n = \left(\frac{2}{7}\right)^n, \\
\sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{24}{7}\right)^{p_2}4^{n-p_2}\times 2^{n-p_2} &= \left(-\frac{24}{7}+8\right)^n = \left(\frac{32}{7}\right)^n.
\end{aligned} \tag{61}$$

By the relations above (60) will be

$$\left\|\left(\frac{2}{7}\right)^n f(2x) - \left(\frac{32}{7}\right)^n f(x)\right\| \le \phi(0,x), \tag{62}$$

and so

$$\left\|f(x) - \frac{1}{2^{4n}}f(2x)\right\| \le \left(\frac{7}{32}\right)^n \phi(0,x). \tag{63}$$

One can rewrite (63) as

$$\|f(x) - \mathcal{T}f(x)\| \le \theta(x)(x \in V^n), \tag{64}$$

where

$$\begin{aligned}
\theta(x) &:= \left(\frac{7}{2}\right)^n \frac{1}{2^{4n((\beta+1)/2)}}\phi\left(0, 2^{(\beta-1)/2}x\right), \\
\mathcal{T}\theta(x) &:= \frac{1}{2^{4n\beta}}\theta\left(2^\beta x\right)\left(\theta \in W^{V^n}\right).
\end{aligned} \tag{65}$$

Similar to the proof of Theorem 13, consider $\Lambda\eta(x) := (1/2^{4n\beta})\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$, and hence, Λ satisfies (A3) of Theorem 12 with $E = V^n$, $g_1(x) = 2^\beta x$, and $L_1(x) = 1/2^{4n\beta}$. Moreover, for each $\lambda, \mu \in W^{V^n}$, we obtain

$$\begin{aligned}
\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| &= \left\|\frac{1}{2^{4n\beta}}\left[\lambda\left(2^\beta x\right) - \mu\left(2^\beta x\right)\right]\right\| \\
&\le L_1(x)\|\lambda(g_1(x)) - \mu(g_1(x))\|.
\end{aligned} \tag{66}$$

The last relation implies that the hypothesis (A2) is true. It is easily checked by induction on l that for any $l \in \mathbb{N}_0$ and $x \in V^n$

$$\Lambda^l \theta(x) := \left(\frac{1}{2^{4n\beta}}\right)^l \theta\left(2^{\beta l} x\right) = \left(\frac{7}{32}\right)^n \left(\frac{1}{2^{4n((\beta+1)/2)}}\right)^l \psi \cdot \left(0, 2^{\beta l+((\beta-1)/2)} x\right). \tag{67}$$

It now follows between (57) and (67) that all assumptions of Theorem 12 hold, and thus, there exists a unique mapping $\mathcal{Q} : V^n \longrightarrow W$ such that

$$\mathcal{Q}(x) = \lim_{l\longrightarrow\infty} \left(\mathcal{T}^l f\right)(x) = \frac{1}{2^{4n\beta}} \mathcal{Q}\left(2^{\beta} x\right) (x \in V^n), \tag{68}$$

and (46) is valid. Similar to the proof of Theorem 13, one can show that

$$\left\| \Gamma_{AQ}\left(\mathcal{T}^l f\right)(x_1, x_2) \right\| \le \left(\frac{1}{2^{4n\beta}}\right)^l \psi\left(2^{\beta l} x_1, 2^{\beta l} x_2\right), \tag{69}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. Letting $l \longrightarrow \infty$ in (69) and applying (56), we arrive at $\Gamma_{AQ}\mathcal{Q}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$, and therefore, the mapping $\mathcal{Q}$ satisfies (26). The last part follows from part (ii) of Corollary 11. □

Here and subsequently, it is assumed that V is a normed space and W is a Banach space unless otherwise stated explicitly. In the following corollary, we show that the multiadditive and multiquartic mappings are stable. Since the proof is routine, we include it without proof.

Corollary 15. *Given $\alpha \in \mathbb{R}$. Suppose that $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma_{AQ} f(x_1, x_2)\| \le \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_{ij}\|^{\alpha}, \tag{70}$$

for all $x_1, x_2 \in V^n$.

(i) If $\alpha \neq n$ and f is odd in each variable, then there exists a unique multiadditive mapping $\mathcal{A} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{A}(x)\| \le \left(\frac{7}{12}\right)^n \frac{1}{|2^{\alpha} - 2^n|} \sum_{j=1}^{n} \|x_{1j}\|^{\alpha}. \tag{71}$$

(ii) If $\alpha \neq 4n$ and f is even in each variable, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (26) such that

$$\|f(x) - \mathcal{Q}(x)\| \le \left(\frac{7}{2}\right)^n \frac{1}{|2^{\alpha} - 2^{4n}|} \sum_{j=1}^{n} \|x_{1j}\|^{\alpha}, \tag{72}$$

for all $x = x_1 \in V^n$. Moreover, if $\mathcal{Q}$ is even mapping in each variable, then it is multiquartic.

The upcoming corollaries are direct consequences of Theorems 13 and 14 when the functional equation (26) is controlled by a small positive number δ.

Corollary 16. *Let $\delta > 0$ and $f : V^n \longrightarrow W$ be a mapping satisfying the inequality*

$$\|\Gamma_{AQ} f(x_1, x_2)\| \le \delta, \tag{73}$$

for all $x_1, x_2 \in V^n$.

(i) If f is odd in each variable, then there exists a unique multiadditive mapping $\mathcal{A} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{A}(x)\| \le \left(\frac{7}{12}\right)^n \frac{\delta}{2^n - 1}, \tag{74}$$

for all $x \in V^n$

(ii) If f is even in each variable, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (26) such that

$$\|f(x) - \mathcal{Q}(x)\| \le \left(\frac{7}{2}\right)^n \frac{\delta}{2^{4n} - 1}, \tag{75}$$

for all $x \in V^n$. Furthemore, if $\mathcal{Q}$ is even mapping in each variable, then it is multiquartic.

Proof. Letting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$ and using Theorem 13 and Theorem 14 in the case $\beta = 1$, one can obtain the desired result. □

Given the mapping $f : V^n \longrightarrow W$, we define the operator $\Gamma f : V^n \times V^n \longrightarrow W$ through

$$\Gamma f(x_1, x_2) := \sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + t x_2^{n-k}\right) - \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \sum_{i\in\{1,2\}} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f \cdot \left(x_i^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right), \tag{76}$$

for all $x_i^k = (x_{i1}, \cdots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \cdots, x_{\text{in}}) \in V^{n-k}$ where $i \in \{1, 2\}$ and $f(\mathcal{N}_{(p_1,p_2)}^{n-k})$ are defined in (8).

In the next result, we show that the functional equation (10) can be stable.

Theorem 17. *Let $\beta \in \{-1, 1\}$ be fixed, V be a linear space, and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the inequality*

$$\sum_{l=0}^{\infty} \left(\frac{1}{2^{(4n-3k)\beta}}\right)^l \phi\left(2^{\beta l-(|\beta-1|/2)} x_1, 2^{\beta l-(|\beta-1|/2)} x_2\right) < \infty, \quad (77)$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\|\Gamma f(x_1, x_2)\| \le \phi(x_1, x_2), \quad (78)$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathscr{F} : V^n \longrightarrow W$ of (10) such that

$$\|f(x) - \mathscr{F}(x)\| \le \Phi(x), \quad (79)$$

for all $x = (x^k, x^{n-k}) \in V^n$, where

$$\Phi(x) = \frac{1}{2^{(4n-3k)(|\beta+1|/2)+n-k}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{(4n-3k)\beta}}\right)^l \phi \cdot \left(2^{\beta l-(|\beta-1|/2)} x, \left(2^{\beta l-(|\beta-1|/2)} x^k, 0\right)\right). \quad (80)$$

Proof. Putting $x_1^k = x_2^k = x^k$ and $x_1^{n-k} = x^{n-k}, x_2^{n-k} = 0$ in (79), we have

$$\left\| 2^{n-k} f(2x) - \sum_{p_1=0}^{n} \binom{n-k}{p_1} 2^k 4^{n-k-p_1} 24^{p_1} 2^{n-k-p_1} f(x) \right\| \le \phi\left(x, \left(x^k, 0\right)\right), \quad (81)$$

in which $x = (x^k, x^{n-k})$. A computation shows that (81) can be rewritten as follows:

$$\left\| 2^{n-k} f(2x) - 2^{5n-4k} f(x) \right\| \le \phi\left(x, \left(x^k, 0\right)\right), \quad (82)$$

and so

$$\left\| f(2x) - 2^{4n-3k} f(x) \right\| \le \frac{1}{2^{n-k}} \phi\left(x, \left(x^k, 0\right)\right). \quad (83)$$

Set

$$\xi(x) \coloneqq \frac{1}{2^{(4n-3k)(|\beta+1|/2)+n-k}} \phi\left(\frac{x}{2^{(|\beta-1|/2)}}, \left(\frac{x^k}{2^{(|\beta-1|/2)}}, 0\right)\right), \quad (84)$$

and $\mathscr{T}\xi(x) \coloneqq (1/2^{(4n-3k)\beta})\xi(2^\beta x)$ where $\xi \in W^{V^n}$. Then, relation (83) can be modified as

$$\|f(x) - \mathscr{T} f(x)\| \le \xi(x)(x \in V^n). \quad (85)$$

Define $\Lambda\eta(x) \coloneqq (1/2^{(4n-3k)\beta})\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}, x = (x^k, x^{n-k}) \in V^n$. The rest of the proof is similar to the proof of Theorem 13. □

Corollary 18. *Let $\delta > 0$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \le \delta, \quad (86)$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathscr{F} : V^n \longrightarrow W$ of (10) such that

$$\|f(x) - \mathscr{F}(x)\| \le \frac{\delta}{2^{n-k}\left(2^{4n-3k} - 1\right)}, \quad (87)$$

for all $x \in V^n$.

Proof. Setting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$ and applying Theorem 17 in the case $\beta = 1$, the result can be found. □

5. Conclusion

In the present paper, we introduced the multi-additive-quartic and multimixed additive-quartic mappings. Indeed, we characterized the mentioned mappings and then unified the system of functional equations defining a multi-additive-quartic or a multimixed additive-quartic mapping to a single equation. We also showed that under which conditions a multimixed additive-quartic mapping is multiadditive, multiquartic, and multi-additive-quartic. Finally, we applied a fixed point theorem to establish the Hyers-Ulam stability of multi-additive-quartic mappings and multimixed additive-quartic functional equations.

Data Availability

All results are obtained without any software and found by manual computations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Research Article

Fixed Points and Stability for Integral-Type Multivalued Contractive Mappings

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The existence and iterative approximations of fixed points concerning two classes of integral-type multivalued contractive mappings in complete metric spaces are proved, and the stability of fixed point sets relative to these multivalued contractive mappings is established. The results obtained in this article generalize and improve some known results in the literature. An illustrative example is given.

1. Introduction

The famous Banach fixed point theorem has both various extensions and valuable applications in a mass of differential equations, difference equations, functional equations, matrix equations, and integral equations ([1–26]). In 2002, Branciari [3] obtained an interesting integral-type fixed point theorem for the contractive mapping of integral type, which is an integral version of the Banach contraction mapping.

Theorem 1 (see [3]). *Let f be a mapping from a complete metric space (X, ρ) into itself satisfying*

$$\int_0^{\rho(fx,fy)} p(s)ds \le c\int_0^{\rho(x,y)} p(s)ds, \forall x, y \in X, \tag{1}$$

where $c \in (0, 1)$ is a constant and $p : [0,+\infty) \longrightarrow [0,+\infty)$ is Lebesgue integrable, summable on each compact subset of $[0,+\infty)$ and $\int_0^{\varepsilon} p(s)ds > 0$ for each $\varepsilon > 0$. Then, f has a unique fixed point $u \in X$ such that $\lim_{n\longrightarrow\infty} f^n x = u$ for each $x \in X$.

Later, the researchers [1, 2, 6, 7, 9, 10, 12, 14, 17–21, 26] generalized Theorem 1 from different directions and got a lot of fixed point results for various contractive mappings of integral type.

In 1969, Nadler [15] gave a multivalued analog of the Banach fixed point theorem by using the Hausdorff metric and introducing the multivalued contraction mapping, that is, he presented a nice fixed point theorem for the multivalued contraction mapping.

Theorem 2 (see [10]). *Let (X, ρ) be a complete metric space and $T : X \longrightarrow CB(X)$ be a multivalued contraction mapping, that is, there exists a constant $r \in [0, 1)$ satisfying*

$$H(Tx, Ty) \le r\rho(x, y), \forall x, y \in X. \tag{2}$$

Then, T has a fixed point in X.

Czerwik [5] and Gordji et al. [8] extended Theorem 2 and proved fixed point theorems for some multivalued contractive mappings, which include (2) as special cases. The researchers [4, 11, 13, 22–24] gained fixed point theorems for several multivalued contractive mappings and studied also the stability of fixed point sets with respect to the multivalued contractive mappings. Lim [11] established the stability of fixed point sets associated with the multivalued contraction mappings in Theorem 2. Choudhury et al. [24] proved that a uniformly convergent sequence of $\alpha_* - \psi$ multivalued contractions has stable fixed point sets.

By combining the ideas of Nadler, Branciari, and Lim, in this article, we study the existence and iterative approximations of fixed points concerning two classes of integral-type multivalued contractive mappings in complete metric spaces and present stability of fixed point sets relative to a sequence of integral-type multivalued contractive mappings. Our results generalize and unify a few results in [5, 8, 11, 15]. An example is also presented to illustrate the efficiency of our results.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and

$$\Phi_1 = \left\{ p \mid p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ is Lebesgue integrable, summable on eachcompact subset of } \mathbb{R}^+ \text{ and } \int_0^\varepsilon p(s)ds > 0 \text{ for each } \varepsilon > 0 \right\},$$
$$\Phi_2 = \left\{ p \mid p \text{ is in } \Phi_1 \text{ and } \int_0^{a+b} p(s)ds \le \int_0^a p(s)ds + \int_0^b p(s)ds \text{ for each } a, b \in \mathbb{R}^+ \right\}. \tag{3}$$

Assume that (X, ρ) is a metric space, $CL(X)$ stands for the family of all nonempty closed subsets of X, and $CB(X)$ denotes the family of all nonempty closed bounded subsets of X. For $C, D \in CL(X)$ and T, $\{T_i\}_{i\in N_0} : X \longrightarrow CL(X)$, define

$$F(T) = \{x \in X : x \in Tx\}, \rho(x, D) = \inf_{y\in D} \rho(x, y), \forall x \in X,$$

$$H(C, D) = \begin{cases} \text{ax}\left\{ \sup_{x\in C} \rho(x, D), \sup_{y\in D} \rho(y, C) \right\}, & m \text{ if the maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

$$N(x, y) = \max\left\{ \rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)] \right\}, \forall x, y \in X,$$

$$\begin{aligned} M(x, y) = \max \Bigg\{ & \rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)], \\ & \cdot \frac{\rho(x, Tx)\rho(y, Ty)}{1 + \rho(x, y)}, \frac{\rho(x, Ty)\rho(y, Tx)}{1 + \rho(x, y)}, \frac{\rho(x, Tx)\rho(y, Ty)}{1 + H(Tx, Ty)}, \\ & \cdot \frac{\rho(x, Ty)\rho(y, Tx)}{1 + H(Tx, Ty)}, \frac{1 + \rho(x, Tx) + \rho(y, Ty)}{1 + \rho(x, y) + H(Tx, Ty)} \rho(x, y), \\ & \cdot \frac{1 + \rho(x, Ty) + \rho(y, Tx)}{1 + \rho(x, y) + H(Tx, Ty)} \rho(x, y) \Bigg\}, \forall x, y \in X, \end{aligned}$$

$$N_i(x, y) = \max\left\{ \rho(x, y), \rho(x, T_i x), \rho(y, T_i y), \frac{1}{2}[\rho(x, T_i y) + \rho(y, T_i x)] \right\}, \forall x, y \in X, i \in N_0,$$

$$\begin{aligned} M_i(x, y) = \max \Bigg\{ & \rho(x, y), \rho(x, T_i x), \rho(y, T_i y), \frac{1}{2}[\rho(x, T_i y) + \rho(y, T_i x)], \\ & \cdot \frac{\rho(x, T_i x)\rho(y, T_i y)}{1 + \rho(x, y)}, \frac{\rho(x, T_i y)\rho(y, T_i x)}{1 + \rho(x, y)}, \frac{\rho(x, T_i x)\rho(y, T_i y)}{1 + H(T_i x, T_i y)}, \\ & \cdot \frac{\rho(x, T_i y)\rho(y, T_i x)}{1 + H(T_i x, T_i y)}, \frac{1 + \rho(x, T_i x) + \rho(y, T_i y)}{1 + \rho(x, y) + H(T_i x, T_i y)} \rho(x, y), \\ & \cdot \frac{1 + \rho(x, T_i y) + \rho(y, T_i x)}{1 + \rho(x, y) + H(T_i x, T_i y)} \rho(x, y) \Bigg\}, \forall x, y \in X, i \in N_0. \end{aligned} \tag{4}$$

A sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X is called an orbit of T at x_0 if $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$.

Lemma 3 (see [12]). *Let $p \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence and $\lim_{n\longrightarrow\infty} r_n = a$. Then*

$$\lim_{n\longrightarrow\infty} \int_0^{r_n} p(s)ds = \int_0^a p(s)ds. \tag{5}$$

Lemma 4 (see [12]). *Let $p \in \Phi_1$ and $\{r_n\}_{n\in N}$ be a nonnegative sequence. Then*

$$\lim_{n\longrightarrow\infty} \int_0^{r_n} p(s)ds = 0, \tag{6}$$

if and only if $\lim_{n\longrightarrow\infty} r_n = 0$.

It follows from [13] the following.

Lemma 5. *Assume that (X, ρ) is a metric space and $C, D \in CL(X)$. Then, for any $r > 1$ and $a \in C$, there exists $b \in D$ such that*

$$\rho(a, b) \le rH(C, D). \tag{7}$$

Lemma 6 (see [13]). *Assume that (X, ρ) is a metric space. Then*

$$|\rho(x, C) - \rho(y, C)| \le \rho(x, y), \forall x, y \in X, C \in CL(X). \tag{8}$$

Lemma 7. *Assume that (X, ρ) is a metric space, $C \in CL(X)$, and $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $a \in X$. Then*

$$\lim_{n\longrightarrow\infty} \rho(x_n, C) = \rho(a, C). \tag{9}$$

Proof. It follows from Lemma 6 that

$$|\rho(x_n, C) - \rho(a, C)| \le \rho(x_n, a) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \quad (10)$$

that is

$$\lim_{n\longrightarrow\infty} \rho(x_n, C) = \rho(a, C). \quad (11)$$

This completes the proof.□ □

Lemma 8. *Let $C \subseteq \mathbb{R}^+$ and $p \in \Phi_1$. Then*

$$\sup_{a\in C} \int_0^a p(s)ds = \int_0^{\sup C} p(s)ds, \quad (12)$$

$$\inf_{a\in C} \int_0^a p(s)ds = \int_0^{\inf C} p(s)ds. \quad (13)$$

Proof. (see (12)). Let $\sup C = c$. It follows that

$$a \le c, \forall a \in C, \quad (14)$$

and there exist a sequence $\{a_n\}_{n\in\mathbb{N}}$ in C satisfying

$$\lim_{n\longrightarrow\infty} a_n = c. \quad (15)$$

Thus, (14) and $p \in \Phi_1$ mean that

$$\int_0^a p(s)ds \le \int_0^c p(s)ds, \forall a \in C, \quad (16)$$

which yields that

$$\sup_{a\in C} \int_0^a p(s)ds \le \int_0^c p(s)ds. \quad (17)$$

On account of (15) and Lemma 3, we infer that

$$\lim_{n\longrightarrow\infty} \int_0^{a_n} p(s)ds = \int_0^c p(s)ds. \quad (18)$$

Clearly, (12) follows from (17) and (18). The proof of (13) is similar to that of (12) and is omitted. This completes the proof.□ □

3. Fixed Point Theorems and an Example

Now, we investigate the existence and iterative approximations of fixed points for the integral-type multivalued contractive mappings (19) and (42), respectively.

Theorem 9. *Assume that (X, ρ) is a complete metric space and $T : X \longrightarrow CL(X)$ satisfies that*

$$\int_0^{\frac{1}{q}H(Tx,Ty)} p(s)ds \le q \int_0^{M(x,y)} p(s)ds, \forall x, y \in X, \quad (19)$$

where q is a constant in $(0, 1)$ and $p \in \Phi_2$. Then, for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T at x_0 such that it converges to some fixed point $a \in X$ of T and

$$\int_0^{\rho(x_n,a)} p(s)ds \le \frac{q^n}{1-q} \int_0^{\rho(x_0,x_1)} p(s)ds, \forall n \in \mathbb{N}. \quad (20)$$

Proof. For any x_0 in X and x_1 in Tx_0, Lemma 5 guarantees that

$$\rho(x_1, x_2) \le \frac{1}{q} H(Tx_0, Tx_1) \text{ for some } x_2 \in Tx_1. \quad (21)$$

Note that

$$\begin{aligned}
\rho(x_0, x_1) &\le M(x_0, x_1) \\
&= \max \Bigg\{ \rho(x_0, x_1), \rho(x_0, Tx_0), \rho(x_1, Tx_1), \frac{1}{2}[\rho(x_0, Tx_1) + \rho(x_1, Tx_0)], \\
&\cdot \frac{\rho(x_0, Tx_0)\rho(x_1, Tx_1)}{1 + \rho(x_0, x_1)}, \frac{\rho(x_0, Tx_1)\rho(x_1, Tx_0)}{1 + \rho(x_0, x_1)}, \\
&\cdot \frac{\rho(x_0, Tx_0)\rho(x_1, Tx_1)}{1 + H(Tx_0, Tx_1)}, \frac{\rho(x_0, Tx_1)\rho(x_1, Tx_0)}{1 + H(Tx_0, Tx_1)}, \\
&\cdot \frac{1 + \rho(x_0, Tx_0) + \rho(x_1, Tx_1)}{1 + \rho(x_0, x_1) + H(Tx_0, Tx_1)} \rho(x_0, x_1), \\
&\cdot \frac{1 + \rho(x_0, Tx_1) + \rho(x_1, Tx_0)}{1 + \rho(x_0, x_1) + H(Tx_0, Tx_1)} \rho(x_0, x_1) \Bigg\} \\
&\le \max \Bigg\{ \rho(x_0, x_1), \rho(x_0, x_1), \rho(x_1, x_2), \frac{1}{2}[\rho(x_0, x_1) + \rho(x_1, Tx_1)], \\
&\cdot \frac{\rho(x_0, x_1)\rho(x_1, x_2)}{1 + \rho(x_0, x_1)}, 0, \frac{\rho(x_0, x_1)\rho(x_1, Tx_1)}{1 + \rho(x_1, Tx_1)}, 0, \\
&\cdot \frac{1 + \rho(x_0, x_1) + \rho(x_1, Tx_1)}{1 + \rho(x_0, x_1) + \rho(x_1, Tx_1)} \rho(x_0, x_1), \\
&\cdot \frac{1 + \rho(x_0, x_1) + \rho(x_1, Tx_1)}{1 + \rho(x_0, x_1) + \rho(x_1, Tx_1)} \rho(x_0, x_1) \Bigg\} \\
&\le \max \Bigg\{ \rho(x_0, x_1), \rho(x_1, x_2), \frac{1}{2}[\rho(x_0, x_1) + \rho(x_1, x_2)], \rho(x_1, x_2), \\
&\quad \rho(x_0, x_1), \rho(x_0, x_1), \rho(x_0, x_1) \Bigg\} = \max \{\rho(x_0, x_1), \rho(x_1, x_2)\},
\end{aligned} \quad (22)$$

which together with (19), (21), and $p \in \Phi_2$ yields.

$$\begin{aligned}
\int_0^{\rho(x_1,x_2)} p(s)ds &\le \int_0^{\frac{1}{q}H(Tx_0,Tx_1)} p(s)ds \le q \int_0^{M(x_0,x_1)} p(s)ds \\
&\le q \int_0^{\max\{\rho(x_0,x_1),\rho(x_1,x_2)\}} p(s)ds = q \int_0^{\rho(x_0,x_1)} p(s)ds,
\end{aligned} \quad (23)$$

and

$$\rho(x_1, x_2) \le \rho(x_0, x_1) = M(x_0, x_1) \text{and} \int_0^{\rho(x_1,x_2)} p(s)ds \le q \int_0^{\rho(x_0,x_1)} p(s)ds. \quad (24)$$

Lemma 5 reveals that

$$\rho(x_2, x_3) \le \frac{1}{q} H(Tx_1, Tx_2) \text{ for some } x_3 \in Tx_2. \quad (25)$$

Notice that

$$
\begin{aligned}
\rho(x_1,x_2) &\le M(x_1,x_2)\\
&= \max\Big\{\rho(x_1,x_2), \rho(x_1,Tx_1), \rho(x_2,Tx_2), \frac{1}{2}[\rho(x_1,Tx_2)+\rho(x_2,Tx_1)],\\
&\cdot \frac{\rho(x_1,Tx_1)\rho(x_2,Tx_2)}{1+\rho(x_1,x_2)}, \frac{\rho(x_1,Tx_2)\rho(x_2,Tx_1)}{1+\rho(x_1,x_2)}, \frac{\rho(x_1,Tx_1)\rho(x_2,Tx_2)}{1+H(Tx_1,Tx_2)},\\
&\cdot \frac{\rho(x_1,Tx_2)\rho(x_2,Tx_1)}{1+H(Tx_1,Tx_2)}, \frac{1+\rho(x_1,Tx_1)+\rho(x_2,Tx_2)}{1+\rho(x_1,x_2)+H(Tx_1,Tx_2)}\rho(x_1,x_2),\\
&\cdot \frac{1+\rho(x_1,Tx_2)+\rho(x_2,Tx_1)}{1+\rho(x_1,x_2)+H(Tx_1,Tx_2)}\rho(x_1,x_2)\Big\}\\
&\le \max\Big\{\rho(x_1,x_2), \rho(x_1,x_2), \rho(x_2,x_3), \frac{1}{2}[\rho(x_1,x_2)+\rho(x_2,Tx_2)],\\
&\cdot \frac{\rho(x_1,x_2)\rho(x_2,x_3)}{1+\rho(x_1,x_2)}, 0, \frac{\rho(x_1,x_2)\rho(x_2,Tx_2)}{1+\rho(x_2,Tx_2)}, 0,\\
&\cdot \frac{1+\rho(x_1,x_2)+\rho(x_2,Tx_2)}{1+\rho(x_1,x_2)+\rho(x_2,Tx_2)}\rho(x_1,x_2), \frac{1+\rho(x_1,x_2)+\rho(x_2,Tx_2)}{1+\rho(x_1,x_2)+\rho(x_2,Tx_2)}\rho(x_1,x_2)\Big\}\\
&\le \max\{\rho(x_1,x_2), \rho(x_2,x_3),\\
&\cdot \frac{1}{2}[\rho(x_1,x_2)+\rho(x_2,x_3)], \rho(x_2,x_3), \rho(x_1,x_2), \rho(x_1,x_2), \rho(x_1,x_2)\Big\}\\
&= \max\{\rho(x_1,x_2), \rho(x_2,x_3)\},
\end{aligned}
\tag{26}
$$

which together with (19), (25), and $p \in \Phi_2$ infers

$$
\begin{aligned}
\int_0^{\rho(x_2,x_3)} p(s)ds &\le \int_0^{\frac{1}{q}H(Tx_1,Tx_2)} p(s)ds \le q\int_0^{M(x_1,x_2)} p(s)ds\\
&\le q\int_0^{\max\{\rho(x_1,x_2),\rho(x_2,x_3)\}} p(s)ds = q\int_0^{\rho(x_1,x_2)} p(s)ds,
\end{aligned}
\tag{27}
$$

and

$$
\rho(x_2,x_3) \le \rho(x_1,x_2) = M(x_1,x_2) \text{and} \int_0^{\rho(x_2,x_3)} p(s)ds \le q\int_0^{\rho(x_1,x_2)} p(s)ds.
\tag{28}
$$

Making use of (24) and (28), we deduce

$$
\int_0^{\rho(x_2,x_3)} p(s)ds \le q\int_0^{\rho(x_1,x_2)} p(s)ds \le q^2\int_0^{\rho(x_0,x_1)} p(s)ds.
\tag{29}
$$

Continuing the process, we obtain an order $\{x_n\}_{n\in N_0}$ of T at x_0 satisfying

$$
\begin{aligned}
x_n \in Tx_{n-1}, \rho(x_n,x_{n+1}) \le \rho(x_{n-1},x_n) = M(x_{n-1},x_n), \forall n \in N, \int_0^{\rho(x_n,x_{n+1})} p(s)ds\\
\le q\int_0^{\rho(x_{n-1},x_n)} p(s)ds, \forall n \in N.
\end{aligned}
\tag{30}
$$

Thus, (30), $q \in (0,1)$, and $p \in \Phi_2$ mean

$$
0 \le \int_0^{\rho(x_n,x_{n+1})} p(s)ds \le q^n\int_0^{\rho(x_0,x_1)} p(s)ds, \forall n \in N.
\tag{31}
$$

Lemma 4 gives

$$
\lim_{n\longrightarrow\infty} \rho(x_n,x_{n+1}) = 0.
\tag{32}
$$

By (31) and $p \in \Phi_2$, we obtain

$$
\begin{aligned}
\int_0^{\rho(x_n,x_m)} p(s)ds &\le \int_0^{\rho(x_n,x_{n+1})+\rho(x_{n+1},x_{n+2})+\cdots+\rho(x_{m-1},x_m)} p(s)ds\\
&\le \int_0^{\rho(x_n,x_{n+1})} p(s)ds + \int_0^{\rho(x_{n+1},x_{n+2})} p(s)ds + \cdots + \int_0^{\rho(x_{m-1},x_m)} p(s)ds\\
&\le q^n\int_0^{\rho(x_0,x_1)} p(s)ds + q^{n+1}\int_0^{\rho(x_0,x_1)} p(s)ds + \cdots + q^{m-1}\int_0^{\rho(x_0,x_1)} p(s)ds\\
&= q^n\left(1+q+\cdots+q^{m-n-1}\right)\int_0^{\rho(x_0,x_1)} p(s)ds\\
&\le \frac{q^n}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds, \forall m, n \in N \text{ with } m > n.
\end{aligned}
\tag{33}
$$

It is clear that (33), $q \in (0,1)$, $p \in \Phi_2$, and Lemma 8 guarantee

$$
\begin{aligned}
0 &\le \int_0^{\sup\{\rho(x_n,x_m):\, m>n\}} p(s)ds = \sup\left\{\int_0^{\rho(x_n,x_m)} p(s)ds : m > n\right\}\\
&\le \frac{q^n}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds \longrightarrow 0 \text{ as} n \longrightarrow \infty,
\end{aligned}
\tag{34}
$$

that is

$$
\lim_{n\longrightarrow\infty} \int_0^{\sup\{\rho(x_n,x_m):\, m>n\}} p(s)ds = 0.
\tag{35}
$$

Lemma 4 ensures

$$
\lim_{n\longrightarrow\infty} \sup\{\rho(x_n,x_m):\, m > n\} = 0.
\tag{36}
$$

Hence, $\{x_n\}_{n\in N_0}$ is a Cauchy sequence.

Completeness of (X,ρ) means that there exists a point a in X with

$$
\lim_{n\longrightarrow\infty} x_n = a.
\tag{37}
$$

Letting $m \longrightarrow \infty$ in (33) and using (37) and Lemma 3, we arrive at

$$
\int_0^{\rho(x_n,a)} p(s)ds \le \frac{q^n}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds, \forall n \in N,
\tag{38}
$$

that is, (20) holds.

Next, we prove that $a = Ta$. From (32), (37), and Lemmas 6 and 7, we get immediately

$$\begin{aligned}
\rho(a, Ta) \le{}& M(x_{n-1}, a) \\
={}& \max\left\{\rho(x_{n-1}, a), \rho(x_{n-1}, Tx_{n-1}), \rho(a, Ta), \frac{1}{2}[\rho(x_{n-1}, Ta) + \rho(a, Tx_{n-1})],\right. \\
&\cdot \frac{\rho(x_{n-1}, Tx_{n-1})\rho(a, Ta)}{1+\rho(x_{n-1}, a)}, \frac{\rho(x_{n-1}, Ta)\rho(a, Tx_{n-1})}{1+\rho(x_{n-1}, a)}, \\
&\cdot \frac{\rho(x_{n-1}, Tx_{n-1})\rho(a, Ta)}{1+H(Tx_{n-1}, Ta)}, \frac{\rho(x_{n-1}, Ta)\rho(a, Tx_{n-1})}{1+H(Tx_{n-1}, Ta)}, \\
&\cdot \frac{1+\rho(x_{n-1}, Tx_{n-1})+\rho(a, Ta)}{1+\rho(x_{n-1}, a)+H(Tx_{n-1}, Ta)}\rho(x_{n-1}, a), \\
&\left.\cdot \frac{1+\rho(x_{n-1}, Ta)+\rho(a, Tx_{n-1})}{1+\rho(x_{n-1}, a)+H(Tx_{n-1}, Ta)}\rho(x_{n-1}, a)\right\} \\
\le{}& \max\left\{\rho(x_{n-1}, a), \rho(x_{n-1}, x_n), \rho(a, Ta), \frac{1}{2}[\rho(x_{n-1}, Ta) + \rho(a, x_n)],\right. \\
&\cdot \frac{\rho(x_{n-1}, x_n)\rho(a, Ta)}{1+\rho(x_{n-1}, a)}, \frac{\rho(x_{n-1}, Ta)\rho(a, x_n)}{1+\rho(x_{n-1}, a)}, \frac{\rho(x_{n-1}, x_n)\rho(a, Ta)}{1+\rho(x_n, Ta)}, \\
&\cdot \frac{\rho(x_{n-1}, Ta)\rho(a, x_n)}{1+\rho(x_n, Ta)}, \frac{1+\rho(x_{n-1}, x_n)+\rho(a, Ta)}{1+\rho(x_{n-1}, a)+\rho(x_n, Ta)}\rho(x_{n-1}, a), \\
&\left.\cdot \frac{1+\rho(x_{n-1}, Ta)+\rho(a, x_n)}{1+\rho(x_{n-1}, a)+\rho(x_n, Ta)}\rho(x_{n-1}, a)\right\} \longrightarrow \max \\
&\cdot \left\{0, 0, \rho(a, Ta), \frac{1}{2}\rho(a, Ta), 0, 0, 0, 0, 0, 0\right\} = \rho(a, Ta) \text{as} n \longrightarrow \infty,
\end{aligned} \tag{39}$$

that is

$$\lim_{n\longrightarrow\infty} M(x_{n-1}, a) = \rho(a, Ta). \tag{40}$$

Taking into account (19), (37), (40), and Lemmas 3 and 7, we gain

$$\begin{aligned}
\int_0^{\rho(a,Ta)} p(s)ds ={}& \limsup_{n\longrightarrow\infty} \int_0^{\rho(x_n,Ta)} p(s)ds \le \limsup_{n\longrightarrow\infty} \int_0^{H(Tx_{n-1},Ta)} p(s)ds \\
\le{}& \limsup_{n\longrightarrow\infty} \int_0^{\frac{1}{q}H(Tx_{n-1},Ta)} p(s)ds \le q \limsup_{n\longrightarrow\infty} \int_0^{M(x_{n-1},a)} p(s)ds \\
={}& q\int_0^{\rho(a,Ta)} p(s)ds,
\end{aligned} \tag{41}$$

which yields $\rho(a, Ta) = 0$ because $q \in (0, 1)$. That is, $a = Ta$. This completes the proof.□ □

Note that $N(x, y) \le M(x, y), \forall x, y \in X$. It follows from Theorem 9 the following.

Theorem 10. *Assume that (X, ρ) is a complete metric space and $T : X \longrightarrow CL(X)$ satisfies*

$$\int_0^{\frac{1}{q}H(Tx,Ty)} p(s)ds \le q\int_0^{N(x,y)} p(s)ds, \forall x, y \in X, \tag{42}$$

where q is a constant in $(0, 1)$ and $p \in \Phi_2$. Then, for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n\in N_0}$ of T at x_0 such that it converges to some fixed point $a \in X$ of T and (20) holds.

Remark 11. In case $p(t) = t, \forall t \in R^+$ and $q = \sqrt{\alpha}$, where α is a constant in $(0, 1)$, then Theorem 12 reduces to a result, which generalizes Theorems 1 and 2 in [5], Theorem 2.1 in [8], and Theorem 5 in [15]. The example below shows that Theorem 10 extends properly Theorem 5 in [15].

Example 1. Let $X = [0, 1]$ be endowed with the Euclidean metric $\rho = |\cdot|$. Let $q = 9/10$, $T : X \longrightarrow CL(X)$, and $p : R^+ \longrightarrow R^+$ be defined by

$$\begin{aligned}
Tx &= \left\{\frac{1}{10}\right\}, \forall x \in [0, 1), \left\{0, \frac{1}{10}\right\}, x = 1, \\
p(t) &= \frac{1}{2\sqrt{t+1/10}}, \forall t \in R^+.
\end{aligned} \tag{43}$$

It is clear that $p \in \Phi_2$. Let $x, y \in X$ with $y < x$. In order to verify (19), we consider below two possible cases:

Case 1. $0 \le y < x < 1$. Clearly,

$$\int_0^{(1/q)(H)(Tx,Ty)} p(s)ds = 0 \le q\int_0^{M(x,y)} p(s)ds. \tag{44}$$

Case 2. $0 \le y < 1$ and $x = 1$. It follows that

$$\begin{aligned}
\int_0^{(1/q)H(Tx,Ty)} p(s)ds &= \int_0^{\frac{10}{9}\times\frac{1}{10}} \frac{1}{2\sqrt{t+1/10}}dt = \sqrt{\frac{19}{90}} - \sqrt{\frac{1}{10}} \\
&< \sqrt{0.212} - 0.316 < 0.6147 < \frac{9}{10}\left(1 - \sqrt{\frac{1}{10}}\right) \\
&= q\int_0^{\rho(x,Tx)} p(s)ds \le q\int_0^{M(x,y)} p(s)ds.
\end{aligned} \tag{45}$$

That is, (19) holds. Thus, Theorem 10 ensures that T has a fixed point $1/10 \in X$. Observe that

$$\begin{aligned}
H\left(T1, T\frac{9}{10}\right) &= H\left(\left\{0, \frac{1}{10}\right\}, \left\{\frac{1}{10}\right\}\right) = \frac{1}{10} \nleq \alpha\frac{1}{10} \\
&= \alpha\rho\left(1, \frac{9}{10}\right), \forall \alpha \in (0, 1),
\end{aligned} \tag{46}$$

which means that Theorem 5 in [15] cannot be used to show the existence of fixed points of T.

4. On Stability of Fixed Point Sets

Now, we discuss the stability of fixed point sets for the integral-type multivalued contractive mappings (19) and (42), respectively. Put $K = \sup_{x\in X} H(T_1x, T_2x)$.

Theorem 12. *Assume that (X, ρ) is a complete metric space and $T_1, T_2 : X \longrightarrow CL(X)$ satisfy*

$$\int_0^{(1/q)(H)(T_ix,T_iy)} p(s)ds \le q\int_0^{M_i(x,y)} p(s)ds, \forall x, y \in X, i \in \{1, 2\}, \tag{47}$$

where q is a constant in $(0,1)$ and $p \in \Phi_2$. Then, $\int_0^{H(F(T_1),F(T_2))} p(s)ds \leq 1/(1-q)\int_0^{(1/q)K} p(s)ds$.

Proof. Without loss of generality, we assume that $K < +\infty$. Note that Theorem 9 yields $F(T_i) \neq \varnothing$ for $i \in \{1,2\}$. Put x_0 in $F(T_1)$. Lemma 5 guarantees

$$\rho(x_0, x_1) \leq \frac{1}{q} H(T_1 x_0, T_2 x_0) \leq \frac{1}{q} K \text{ for some} x_1 \in T_2 x_0, x_2 \in T_2 x_1, \tag{48}$$

$$\rho(x_1, x_2) \leq \frac{1}{q} H(T_2 x_0, T_2 x_1), \tag{49}$$

$$\begin{aligned} \rho(x_0,x_1) &\leq M_2(x_0,x_1) = \max\{\rho(x_0,x_1), \rho(x_0,T_2x_0), \rho(x_1,T_2x_1),\\ &\cdot\frac{1}{2}[\rho(x_0,T_2x_1)+\rho(x_1,T_2x_0)], \frac{\rho(x_0,T_2x_0)\rho(x_1,T_2x_1)}{1+\rho(x_0,x_1)},\\ &\cdot\frac{\rho(x_0,T_2x_1)\rho(x_1,T_2x_0)}{1+\rho(x_0,x_1)}, \frac{\rho(x_0,T_2x_0)\rho(x_1,T_2x_1)}{1+H(T_2x_0,T_2x_1)},\\ &\cdot\frac{\rho(x_0,T_2x_1)\rho(x_1,T_2x_0)}{1+H(T_2x_0,T_2x_1)}, \frac{1+\rho(x_0,T_2x_0)+\rho(x_1,T_2x_1)}{1+\rho(x_0,x_1)+H(T_2x_0,T_2x_1)}\rho(x_0,x_1),\\ &\cdot\frac{1+\rho(x_0,T_2x_1)+\rho(x_1,T_2x_0)}{1+\rho(x_0,x_1)+H(T_2x_0,T_2x_1)}\rho(x_0,x_1)\Big\}\\ &\leq \max\Big\{\rho(x_0,x_1), \rho(x_0,x_1), \rho(x_1,x_2), \frac{1}{2}[\rho(x_0,x_1)+\rho(x_1,T_2x_1)],\\ &\cdot\frac{\rho(x_0,x_1)\rho(x_1,x_2)}{1+\rho(x_0,x_1)}, 0, \frac{\rho(x_0,x_1)\rho(x_1,T_2x_1)}{1+\rho(x_1,T_2x_1)}, 0,\\ &\cdot\frac{1+\rho(x_0,x_1)+\rho(x_1,T_2x_1)}{1+\rho(x_0,x_1)+\rho(x_1,T_2x_1)}\rho(x_0,x_1),\\ &\cdot\frac{1+\rho(x_0,x_1)+\rho(x_1,T_2x_1)}{1+\rho(x_0,x_1)+\rho(x_1,T_2x_1)}\rho(x_0,x_1)\Big\}\\ &\leq \max\{\rho(x_0,x_1), \rho(x_1,x_2),\\ &\cdot\frac{1}{2}[\rho(x_0,x_1)+\rho(x_1,x_2)], \rho(x_1,x_2), \rho(x_0,x_1), \rho(x_0,x_1), \rho(x_0,x_1)\Big\}\\ &= \max\{\rho(x_0,x_1), \rho(x_1,x_2)\}, \end{aligned} \tag{50}$$

which together with (47) and (49) and $p \in \Phi_2$ implies

$$\begin{aligned} \int_0^{\rho(x_2,x_3)} p(s)ds &\leq \int_0^{\left(\frac{1}{q}\right)(H)(T_2x_1,T_2x_2)} p(s)ds \leq q\int_0^{M_2(x_1,x_2)} p(s)ds\\ &\leq q\int_0^{\max\{\rho(x_1,x_2),\rho(x_2,x_3)\}} p(s)ds = q\int_0^{\rho(x_1,x_2)} p(s)ds, \end{aligned} \tag{51}$$

and

$$\rho(x_1,x_2) \leq \rho(x_0,x_1) = M_2(x_0,x_1) \text{and} \int_0^{\rho(x_1,x_2)} p(s)ds \leq q\int_0^{\rho(x_0,x_1)} p(s)ds. \tag{52}$$

Lemma 5 ensures with

$$\rho(x_2,x_3) \leq \frac{1}{q}H(T_2x_1, T_2x_2) \text{for some} x_3 \in T_2x_2. \tag{53}$$

Note that

$$\begin{aligned} \rho(x_1,x_2) &\leq M_2(x_1,x_2)\\ &= \max\{\rho(x_1,x_2), \rho(x_1,T_2x_1), \rho(x_2,T_2x_2),\\ &\cdot\frac{1}{2}[\rho(x_1,T_2x_2)+\rho(x_2,T_2x_1)],\\ &\cdot\frac{\rho(x_1,T_2x_1)\rho(x_2,T_2x_2)}{1+\rho(x_1,x_2)}, \frac{\rho(x_1,T_2x_2)\rho(x_2,T_2x_1)}{1+\rho(x_1,x_2)},\\ &\cdot\frac{\rho(x_1,T_2x_1)\rho(x_2,T_2x_2)}{1+H(T_2x_1,T_2x_2)}, \frac{\rho(x_1,T_2x_2)\rho(x_2,T_2x_1)}{1+H(T_2x_1,T_2x_2)},\\ &\cdot\frac{1+\rho(x_1,T_2x_1)+\rho(x_2,T_2x_2)}{1+\rho(x_1,x_2)+H(T_2x_1,T_2x_2)}\rho(x_1,x_2),\\ &\cdot\frac{1+\rho(x_1,T_2x_2)+\rho(x_2,T_2x_1)}{1+\rho(x_1,x_2)+H(T_2x_1,T_2x_2)}\rho(x_1,x_2)\Big\}\\ &\cdot \leq \max\{\rho(x_1,x_2), \rho(x_1,x_2), \rho(x_2,x_3),\\ &\cdot\frac{1}{2}[\rho(x_1,x_2)+\rho(x_2,T_2x_2)], \frac{\rho(x_1,x_2)\rho(x_2,x_3)}{1+\rho(x_1,x_2)}, 0,\\ &\cdot\frac{\rho(x_1,x_2)\rho(x_2,T_2x_2)}{1+\rho(x_2,Tx_2)}, 0, \frac{1+\rho(x_1,x_2)+\rho(x_2,T_2x_2)}{1+\rho(x_1,x_2)+\rho(x_2,T_2x_2)}\rho(x_1,x_2),\\ &\cdot\frac{1+\rho(x_1,x_2)+\rho(x_2,T_2x_2)}{1+\rho(x_1,x_2)+\rho(x_2,T_2x_2)}\rho(x_1,x_2)\Big\},\\ &\leq \max\{\rho(x_1,x_2), \rho(x_2,x_3),\\ &\cdot\frac{1}{2}[\rho(x_1,x_2)+\rho(x_2,x_3)], \rho(x_2,x_3), \rho(x_1,x_2), \rho(x_1,x_2), \rho(x_1,x_2)\Big\}\\ &= \max\{\rho(x_1,x_2), \rho(x_2,x_3)\}, \end{aligned} \tag{54}$$

which together with (47), (53), and $p \in \Phi_2$ gives

$$\begin{aligned} \int_0^{\rho(x_2,x_3)} p(s)ds &\leq \int_0^{\left(\frac{1}{q}\right)(\mathrm{H})(T_2x_1,T_2x_2)} p(s)ds \leq q\int_0^{M_2(x_1,x_2)} p(s)ds\\ &\leq q\int_0^{\max\{\rho(x_1,x_2),\rho(x_2,x_3)\}} p(s)ds = q\int_0^{\rho(x_1,x_2)} p(s)ds, \end{aligned} \tag{55}$$

and

$$\rho(x_2,x_3) \leq \rho(x_1,x_2) = M_2(x_1,x_2) \text{ and} \int_0^{\rho(x_2,x_3)} p(s)ds \leq q\int_0^{\rho(x_1,x_2)} p(s)ds. \tag{56}$$

From (52) and (56), we deduce that

$$\int_0^{\rho(x_2,x_3)} p(s)ds \leq q\int_0^{\rho(x_1,x_2)} p(s)ds \leq q^2\int_0^{\rho(x_0,x_1)} p(s)ds. \tag{57}$$

Continuing the process, we obtain an order $\{x_n\}_{n\in N_0}$ of T_2 at x_0 satisfying (48) and

$$\begin{aligned} x_n \in T_2x_{n-1}, \rho(x_n,x_{n+1}) &\leq \frac{1}{q}H(T_2x_{n-1}, T_2x_n), \rho(x_n,x_{n+1})\\ &\leq \rho(x_{n-1},x_n) = M_2(x_{n-1},x_n), \int_0^{\rho(x_n,x_{n+1})} p(s)ds\\ &\leq q\int_0^{\rho(x_{n-1},x_n)} p(s)ds, \forall n \in N. \end{aligned} \tag{58}$$

In light of (58), $q \in (0,1)$, and $p \in \Phi_2$, we have

$$0 \le \int_0^{\rho(x_n,x_{n+1})} p(s)ds \le q^n \int_0^{\rho(x_0,x_1)} p(s)ds, \forall n \in N, \tag{59}$$

$$\lim_{n\longrightarrow\infty} \int_0^{\rho(x_n,x_{n+1})} p(s)ds = 0. \tag{60}$$

Combining (60) and Lemma 4, we get

$$\lim_{n\longrightarrow\infty} \rho(x_n, x_{n+1}) = 0. \tag{61}$$

By (59) and $\varphi \in \Phi_2$, we infer

$$\begin{aligned}\int_0^{\rho(x_n,x_m)} p(s)ds &\le \int_0^{\rho(x_n,x_{n+1})+\rho(x_n,x_{n+2})+\cdots+\rho(x_{m-1},x_m)} p(s)ds\\ &\le \int_0^{\rho(x_n,x_{n+1})} p(s)ds + \int_0^{\rho(x_{n+1},x_{n+2})} p(s)ds + \cdots + \int_0^{\rho(x_{m-1},x_m)} p(s)ds\\ &\le q^n\int_0^{\rho(x_0,x_1)} p(s)ds + q^{n+1}\int_0^{\rho(x_0,x_1)} p(s)ds + \cdots + q^{m-1}\int_0^{\rho(x_0,x_1)} p(s)ds\\ &= q^n\left(1+q+\cdots+q^{m-n-1}\right)\int_0^{\rho(x_0,x_1)} p(s)ds\\ &\le \frac{q^n}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds, \forall m, n \in N \text{ with } m > n.\end{aligned} \tag{62}$$

It follows from (62), $q \in (0,1)$, $\varphi \in \Phi_2$, and Lemma 8 that

$$\begin{aligned}0 &\le \int_0^{\sup\{\rho(x_n,x_m):\, m>n\}} p(s)ds = \sup\left\{\int_0^{\rho(x_n,x_m)} p(s)ds : m > n\right\}\\ &\le \frac{q^n}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds \longrightarrow 0 \text{ as } n \longrightarrow \infty,\end{aligned} \tag{63}$$

that is

$$\lim_{n\longrightarrow\infty} \int_0^{\sup\{\rho(x_n,x_m):\, m>n\}} p(s)ds = 0. \tag{64}$$

which together with Lemma 4 means

$$\lim_{n\longrightarrow\infty} \sup\{\rho(x_n, x_m): m > n\} = 0, \tag{65}$$

that is, $\{x_n\}_{n\in N_0}$ is a Cauchy sequence.

Completeness of (X, ρ) guarantees

$$\lim_{n\longrightarrow\infty} x_n = a \text{ for some } a \in X. \tag{66}$$

Letting $m \longrightarrow \infty$ in (62) and making use of (66) and Lemma 3, we deduce

$$\int_0^{\rho(x_n,a)} p(s)ds \le \frac{q^n}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds, \forall n \in N. \tag{67}$$

Observe that (61), (66), and Lemma 7 ensure

$$\begin{aligned}\rho(a, T_2a) &\le M_2(x_{n-1}, a)\\ &= \max\Big\{\rho(x_{n-1}, a), \rho(x_{n-1}, T_2x_{n-1}), \rho(a, T_2a),\\ &\cdot \frac{1}{2}[\rho(x_{n-1}, T_2a) + \rho(a, T_2x_{n-1})],\\ &\cdot \frac{\rho(x_{n-1}, T_2x_{n-1})\rho(a, T_2a)}{1+\rho(x_{n-1}, a)}, \frac{\rho(x_{n-1}, T_2a)\rho(a, T_2x_{n-1})}{1+\rho(x_{n-1}, a)},\\ &\cdot \frac{\rho(x_{n-1}, T_2x_{n-1})\rho(a, T_2a)}{1+H(T_2x_{n-1}, T_2a)}, \frac{\rho(x_{n-1}, T_2a)\rho(a, T_2x_{n-1})}{1+H(T_2x_{n-1}, T_2a)},\\ &\cdot \frac{1+\rho(x_{n-1}, T_2x_{n-1})+\rho(a, T_2a)}{1+\rho(x_{n-1}, a)+H(T_2x_{n-1}, T_2a)}\rho(x_{n-1}, a),\\ &\cdot \frac{1+\rho(x_{n-1}, T_2a)+\rho(a, T_2x_{n-1})}{1+\rho(x_{n-1}, a)+H(T_2x_{n-1}, T_2a)}\rho(x_{n-1}, a)\Big\}\\ &\cdot \le \max\Big\{\rho(x_{n-1}, a), \rho(x_{n-1}, x_n), \rho(a, T_2a),\\ &\cdot \frac{1}{2}[\rho(x_{n-1}, T_2a) + \rho(a, x_n)], \frac{\rho(x_{n-1}, x_n)\rho(a, T_2a)}{1+\rho(x_{n-1}, a)},\\ &\cdot \frac{\rho(x_{n-1}, T_2a)\rho(a, x_n)}{1+\rho(x_{n-1}, a)}, \frac{\rho(x_{n-1}, x_n)\rho(a, T_2a)}{1+\rho(x_n, T_2a)},\\ &\cdot \frac{\rho(x_{n-1}, T_2a)\rho(a, x_n)}{1+\rho(x_n, T_2a)}, \frac{1+\rho(x_{n-1}, x_n)+\rho(a, T_2a)}{1+\rho(x_{n-1}, a)+\rho(x_n, T_2a)}\rho(x_{n-1}, a),\\ &\cdot \frac{1+\rho(x_{n-1}, T_2a)+\rho(a, x_n)}{1+\rho(x_{n-1}, a)+\rho(x_n, T_2a)}\rho(x_{n-1}, a)\Big\} \longrightarrow \max\\ &\cdot \left\{0, 0, \rho(a, T_2a), \frac{1}{2}\rho(a, T_2a), 0, 0, 0, 0, 0, 0\right\} = \rho(a, T_2a) \text{as} n \longrightarrow \infty,\end{aligned} \tag{68}$$

that is

$$\lim_{n\longrightarrow\infty} M_2(x_{n-1}, a) = \rho(a, T_2a). \tag{69}$$

By virtue of (47), (66), (69), $p \in \Phi_2$, and Lemmas 3 and 7, we have

$$\begin{aligned}\int_0^{\rho(a,T_2a)} p(s)ds &= \limsup_{n\longrightarrow\infty}\int_0^{\rho(x_n,T_2a)} p(s)ds \le \limsup_{n\longrightarrow\infty}\int_0^{H(T_2x_{n-1},T_2a)} p(s)ds\\ &\le \limsup_{n\longrightarrow\infty}\int_0^{\frac{1}{q}H(T_2x_{n-1},T_2a)} p(s)ds \le q\limsup_{n\longrightarrow\infty}\int_0^{M_2(x_{n-1},a)} p(s)ds\\ &= q\int_0^{\rho(a,T_2a)} p(s)ds,\end{aligned} \tag{70}$$

which means $\rho(a, T_2a) = 0$ because $q \in (0,1)$, that is, $a \in F(T_2)$.

Taking advantage of (48), (59), (67), $\varphi \in \Phi_2$, and Lemma 4, we get

$$\begin{aligned}\int_0^{\rho(x_0,a)} p(s)ds &\le \int_0^{\rho(x_0,x_1)+\rho(x_1,x_2)+\cdots+\rho(x_n,x_{n+1})+\rho(x_{n+1},a)} p(s)ds\\ &\le \int_0^{\rho(x_0,x_1)} p(s)ds + \int_0^{\rho(x_1,x_2)} p(s)ds + \cdots + \int_0^{\rho(x_n,x_{n+1})} p(s)ds\\ &+ \int_0^{\rho(x_{n+1},a)} p(s)ds \le \int_0^{\rho(x_0,x_1)} p(s)ds\\ &+ q\int_0^{\rho(x_0,x_1)} p(s)ds + \cdots + q^n\int_0^{\rho(x_0,x_1)} p(s)ds + \frac{q^{n+1}}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds\\ &= \frac{1-q^{n+1}+q^{n+1}}{1-q}\int_0^{\rho(x_0,x_1)} p(s)ds \le \frac{1}{1-q}\int_0^{\frac{1}{q}H(T_1x_0,T_2x_0)} p(s)ds\\ &\le \frac{1}{1-q}\int_0^{\frac{1}{q}K} p(s)ds, \forall n \in N.\end{aligned} \tag{71}$$

It follows from Lemma 8 that

$$\int_0^{\rho(x_0,F(T_2))} p(s)ds = \inf_{y\in F(T_2)} \int_0^{\rho(x_0,y)} p(s)ds \le \int_0^{\rho(x_0,a)} p(s)ds \le \frac{1}{1-q}\int_0^{\frac{1}{q}K} p(s)ds, \forall x_0 \in F(0T_1), \tag{72}$$

and

$$\int_0^{\sup\{\rho(x,F(T_2)):\, x\in F(T_1)\}} p(s)ds = \sup_{x\in F(T_1)} \int_0^{\rho(x,F(T_2))} p(s)ds \le \frac{1}{1-q}\int_0^{\frac{1}{q}K} p(s)ds. \tag{73}$$

Reversing the roles of T_1 and T_2, we also conclude

$$\int_0^{\sup\{\rho(y,F(T_1)):\, y\in F(T_2)\}} p(s)ds \le \frac{1}{1-q}\int_0^{\frac{1}{q}K} p(s)ds. \tag{74}$$

Using (73), (74), Lemma 8, and $\varphi \in \Phi_2$, we obtain

$$\begin{aligned}\int_0^{H(F(T_1),F(T_2))} p(s)ds &= \int_0^{\max\{\sup\{\rho(x,F(T_2)):\, x\in F(T_1)\},\sup\{\rho(y,F(T_1)):\, y\in F(T_2)\}\}} p(s)ds \\ &= \max\left\{\int_0^{\sup\{\rho(x,F(T_2)):\, x\in F(T_1)\}} p(s)ds, \int_0^{\sup\{\rho(y,F(T_1)):\, y\in F(T_2)\}} p(s)ds\right\} \\ &\le \frac{1}{1-q}\int_0^{\frac{1}{q}K} p(s)ds.\end{aligned} \tag{75}$$

This completes the proof.□ □

Theorem 13. *Assume that (X,ρ) is a complete metric space and $\{T_i\}_{i\in N_0} : X \longrightarrow CL(X)$ satisfy*

$$\int_0^{\frac{1}{q}H(T_i x,T_i y)} p(s)ds \le q\int_0^{M_i(x,y)} p(s)ds, \forall x,y\in X, i\in N_0, \tag{76}$$

where q is a constant in $(0,1)$ and $p\in\Phi_2$. Assume that

$$\lim_{i\longrightarrow\infty} H(T_i x, T_0 x) = 0 \text{ uniformly for all } x\in X. \tag{77}$$

Then, $\lim_{i\longrightarrow\infty} H(F(T_i), F(T_0)) = 0$.

Proof. Let $\varepsilon > 0$. It follows from Theorem 12.34 in [27] and $p\in\Phi_2$ that there exists $\delta > 0$ with

$$\int_A p(s)ds < (1-q)\varepsilon \text{ for every bounded subset } A\subset R^+ \text{ with } m(A)\le\delta, \tag{78}$$

where $m(A)$ is the Lebesgue measure of A. (77) guarantees that there exists $N\in N$ satisfying

$$\sup_{x\in X} H(T_i x, T_0 x) < q\delta, \forall i\ge N. \tag{79}$$

By means of (78), (79), $p\in\Phi_2$, and Theorem 12, we conclude

$$0 \le \int_0^{H(F(T_i),F(T_0))} p(s)ds \le \frac{1}{1-q}\int_0^{\frac{1}{q}\sup_{x\in X} H(T_i x,T_0 x)} p(s)ds \le \frac{1}{1-q}\int_0^{\delta} p(s)ds < \varepsilon, \forall i\ge N, \tag{80}$$

which implies that

$$\lim_{i\longrightarrow\infty}\int_0^{H(F(T_i),F(T_0))} p(s)ds = 0. \tag{81}$$

Thus, $\lim_{i\longrightarrow\infty} H(F(T_i), F(T_0)) = 0$ follows from (81) and Lemma 4. This completes the proof.□ □

Theorems 12 and 13 infer immediately the following.

Theorem 14. *Assume that (X,ρ) is a complete metric space and $T_1, T_2 : X \longrightarrow CL(X)$ satisfy that*

$$\int_0^{\frac{1}{q}H(T_i x,T_i y)} p(s)ds \le q\int_0^{N_i(x,y)} p(s)ds, \forall x,y\in X, i\in\{1,2\}, \tag{82}$$

where q is a constant in $(0,1)$ and $p\in\Phi_2$. Then, $\int_0^{H(F(T_1),F(T_2))} p(s)ds \le 1/1-q\int_0^{(1/q)K} p(s)ds$.

Theorem 15. *Assume that (X,ρ) is a complete metric space and $\{T_i\}_{i\in N_0} : X \longrightarrow CL(X)$ satisfy (77) and*

$$\int_0^{\frac{1}{q}H(T_i x,T_i y)} p(s)ds \le q\int_0^{N_i(x,y)} p(s)ds, \forall x,y\in X, i\in N_0, \tag{83}$$

where q is a constant in $(0,1)$ and $p\in\Phi_2$. Then, $\lim_{i\longrightarrow\infty} H(F(T_i), F(T_0)) = 0$.

Remark 16. Theorems 14 and 15 extend, respectively, Lemma 1 and Theorem 1 in [11].

5. Conclusion

In this paper, we introduce two classes of integral-type multivalued contractive mappings, which include some known multivalued contractive mappings as special cases, and prove the existence, iterative approximations, and stability of fixed points for these integral-type multivalued contractive mappings under certain conditions. Our results extend several known results in the literature.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Applications of Fixed Point Theory to Investigate a System of Fractional Order Differential Equations

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We investigate a nonlinear system of pantograph-type fractional differential equations (FDEs) via Caputo-Hadamard derivative (CHD). We establish the conditions for existence theory and Ulam-Hyers-type stability for the underlying boundary value system (BVS) of FDE. We use Krasnoselskii's and Banach's fixed point theorems to obtain the desired results for the existence of solution. Stability is an important aspect from a numerical point of view we investigate here. To justify the main work, relevant examples are provided.

1. Introduction

The generalized form of ordinary calculus is called fractional calculus. This newly developed branch of mathematics has numerous applications in many scientific fields including the study of nonlinear oscillations of earthquakes, nanotechnology, and other engineering disciplines. Also, fractional derivatives and integrals have the ability to explore the dynamics of many real-world problems more comprehensively and extensively. To these characteristics of the said area, researchers in the past several decades have taken great interest to investigate FDEs for a different kind of analysis. For applications and usefulness, see [1–5]. The concerned study includes optimization, stability and numerical results, and theoretical analysis. In this regard, existence theory for different kinds of problems of FDEs has been investigated and plenty of research work has been done (see [6–8]).

One of the new emerging classes of FDEs is known as pantograph differential equations (PDEs). The work related to this new research field has been published in large numbers. Initially, pantograph differential equations (PDEs) were studied with delay terms [9, 10], material modeling [11], and modeling lasers, especially quantum dot lasers [12]. Basically, PDEs give change in terms of a dependent variable at a previous time [13]. Some beneficial research has been performed in this area [14–16]. Further, these types of FDEs occur in traffic models, control systems, population dynamics, and many natural phenomena.

In the last few decades, the stability analysis for FDEs has been established very well. Therefore. different kinds of stability notions have been constructed in literature including exponential, Mittag-Leffler, and Lyapunov. The mentioned stability concepts have been very well investigated for FDEs. Among these, UH stability analysis is an important tool that has gained the attention of researchers [17, 18]. The aforesaid UH stability has extended to other forms in large many articles [19, 20]. The UH stability analysis method has been developed for ordinary and FDEs over the last twenty years [21–23].

It is remarkable that great interest has been observed to derive various kinds of results including qualitative and numerical for higher-order problems under BCs [24–26]. Since fractional derivative has various definitions, each and every definition has its own uncharacteristic features. One of the well-known definitions is called the Caputo-Hadamard derivative. The said area has been initiated in the last few years (for detail, see [27–29]). After that, the said definition has been used in large numbers of articles. Motivated from aforesaid work, the qualitative study of a coupled system of FDEs under BCs with fractional CHD has not been investigated properly involving proportional delay term. Therefore, using the results from fixed point theory, we studied the qualitative aspects of the system of FDEs under BCs with CHD given as

$$\begin{cases} {}^{C}D^{\eth}{}_{1+}v(t) + f(t, v(\imath t), \mathscr{Y}(t)) = 0, \\ {}^{C}D^{\eth}1_{+}\mathscr{Y}(t) + g(t, v(t), \mathscr{Y}(\imath t)) = 0, \\ v(1) = v'(1) = 0 = v'(e), v(e) = \varphi(v), \\ \mathscr{Y}(1) = \mathscr{Y}'(1) = 0 = \mathscr{Y}'(e), \mathscr{Y}(e) = \Psi(\mathscr{Y}), \end{cases} \quad (1)$$

with $t \in [1, e] = \mathscr{H}, \eth \in (3, 4], \imath \in (0, 1)$ also the functions $f, g : \mathscr{H} \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ and $\Phi, \Psi : \mathbb{Y} \longrightarrow \mathbf{R}$ are continuous functions. The complete norm space is defined by $\mathbb{Y}, \|,\|$ under the norm $\|y\| = \max_{t\in\mathscr{H}}|y|$.

Consequently, P is a Banach space such that $\mathbf{P} = \mathbb{Y} \times \mathbb{Y}$ with norms $\|(v, \mathscr{Y})\| = \|v\| + \|\mathscr{Y}\|$ or $\|(v, \mathscr{Y})\| = \max\{\|v\|, \|\mathscr{Y}\|\}$. We established sufficient conditions under which the problem under our investigation has at least one solution. Further, some adequate results are studied to check the stability of the UH type for the corresponding solution. These results are derived by using fixed point theory and nonlinear analysis. The analysis is justified by pertinent examples.

2. Preliminaries

Here, we recall some needful preliminary results.

Definition 1. For a function $v : (\mathscr{J}) = (1, e) \longrightarrow \mathbf{R}$, the fractional Hadamard integral is expressed as [30]:

$$I^{\eth}_{1+}v(t) = \frac{1}{\Gamma(\eth)}\int_1^t \left(\ln\frac{t}{\theta}\right)^{\eth-1} v(\theta)\frac{d\theta}{\theta}, \quad (2)$$

if the above integral exists.

Definition 2. For a function $v : (\mathscr{J}) \longrightarrow \boldsymbol{R}$, the fractional Hadamard derivative is denoted as [30]:

$${}^{C}D^{\eth}_{1+}v(t) = \sigma^k I^{k-\eth}_{1+} = \frac{1}{\Gamma(k-\eth)}\left(t\frac{d}{dt}\right)^k \int_1^t \left(\ln\frac{t}{\theta}\right)^{k-\eth-1} v(\theta)\frac{d\theta}{\theta}, \quad (3)$$

where $k = [\eth] + 1$ and $\sigma = t(d/dt)$.

Lemma 3 (see [30]). *Let $v(t) \in AC^k_\sigma[1, e]$, then for fractional differential equation (FDE)*

$${}^{C}D^{\eth}_{1+}v(t) = 0, \eth \in (k-1, k], \quad (4)$$

the solution is given as follows:

$$v(t) = \sum_{j=0}^{k-1} a_j(\ln t)^j, \quad j = 1, 2, 3, \cdots, k-1, \textit{where } a_j \in \mathbf{R}. \quad (5)$$

Lemma 4 (see [30]). *The FDE holds the result in the following:*

$$I^{\eth}_{a+}\left[{}^{C}D_{a}{}^{\eth}{}_{+}v(t)\right] = v(t) + \sum_{j=0}^{k-1} a_j(\ln t)^j, \quad j = 1, 2, 3, \cdots, k-1, \quad (6)$$

where $k = [\eth] + 1$.

Definition 5 (see [31]). Let for operators $V_1, V_2 \ni V_1, V_2 : \mathbf{P} \longrightarrow \mathbb{Y}$, denoted by

$$\begin{cases} v(t) = V_1(v, Y)(t), \\ Y(t) = V_2(v, Y)(t) \end{cases} \quad (7)$$

is called UH stable if for real positive constants $ai(i = 1, 2, 3, 4), \delta i(i = 1, 2)$ and for each solution $(v\wedge, Y\wedge) \in \mathbf{P}$, we have

$$\begin{cases} \|v\wedge - V_1(v\wedge, Y\wedge)\| \le \delta_{1,} \\ \|Y\wedge - V_2(v\wedge, Y\wedge)\| \le \delta_2, \end{cases} \quad (8)$$

there exist a solution $(v, Y) \in \mathbf{P}$ of (7),∍

$$\begin{cases} \|V_1(v, Y) - V_1(v\wedge, Y\wedge)\| \le b_1\|v - v\wedge\| + b_2\|Y - Y\wedge\|, \\ \|V_2(v, Y) - V_2(v\wedge, Y\wedge)\| \le b_3\|v - v\wedge\| + b_4\|Y - Y\wedge\|. \end{cases} \quad (9)$$

Furthermore, if the matrix

$$M = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \quad (10)$$

converges to zero, then the solution of (7) is UH stable.

Theorem 6 (see [32–34]). *Let $E \neq \emptyset$ be closed convex subset of the Banach space P, and there exist two operators $F, \aleph$ such that (a) $Fx + \aleph y \in E$ whenever $x, y \in E$, $(b)F$ is continuous and compact, and $(c)\aleph$ is contraction. So one has $z = (v, Y) \in E$ such that $Fz + \aleph z = z$.*

(M_1) For all $v, Y \in C(H, \boldsymbol{R})$,$\exists \mathfrak{R}^*_\Phi, \mathfrak{R}^*_\psi > 0$∍

$$|\Phi(v) - \Phi(v)| \le \mathfrak{R}^*_\Phi|v - v|, |\psi(Y) - \psi(Y)| \le \mathfrak{R}^*_\psi|Y - Y|. \quad (11)$$

(M_2) *For all* $v, v, Y, Y \in C(H, \mathbf{R}) \forall t \in H \exists L_f^* > 0 \ni$

$$|f(t, v(\jmath t), Y(t)) - f(t, v(\jmath t), Y(t))| \leq L_f^*[|v - v| + |Y - Y|]. \quad (12)$$

(M_3) *For all* $v, v, Y, Y \in C(H, \mathbf{R}) \forall t \in H \exists L^*{}_g > 0 \ni$

$$|\aleph(t, v(t), Y(\jmath t)) - \aleph(t, v(t), Y(\jmath t))| \leq L_g^*[|v - v| + |Y - Y|]. \quad (13)$$

(M_4) *There exist positive real numbers* C_f^*, D_f^*, *and* M_f^*, $M_f^* \ni$

$$\begin{aligned} &|f(t, v(\jmath t), Y(t))| \leq C_f^*|v| + D_f^*|Y| + M_f^*, \\ &|\aleph(t, v(t), Y(\jmath t))| \leq C_g^*|v| + D_g^*|Y| + M_g^*. \end{aligned} \quad (14)$$

(M_5) *There exist positive real numbers* $\kappa_i (i = 1, 2), \beta_\Phi$, $\beta_\Psi \ni$

$$|\Phi(v)| \leq \kappa_1|v| + \beta_\Phi, |\Psi(Y)| \leq \kappa_2|Y| + \beta_\Psi. \quad (15)$$

(M_6) *For simplicity, we introduce the notation as follows:*

$$\hbar(t) = 3(\ln t)^2 - 2(\ln t)^3. \quad (16)$$

3. Main Results

Theorem 7. *Let* $v \in C[1, e]$ *and* $x \in AC_\sigma^k[1, e]$, *the solution for linear problem*

$$^C D_{1+}^{\eth} v(t) = x(t), t \in H, \eth \in (3, 4], \quad (17)$$

$$v(1) = v'(1) = v'(e) = 0, v(e) = \Phi(v) \quad (18)$$

converts to the following form:

$$v(t) = \Phi(v)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} x(\theta) \frac{d\theta}{\theta}. \quad (19)$$

Proof. Thanks to Lemma (4), Equation (18) obtained the form

$$v(t) = I_{1+}^{\eth} x(t) + a_0 + a_1(\ln t) + a_2(\ln t)^2 + a_3(\ln t)^3, \quad (20)$$

by making use of the considered boundary conditions $v(1) = v'(1) = 0$, we get $a_0 = a_1 = 0$ also by

$$\begin{aligned} &v(e) = \Phi(v), v'(e) = 0 \Rightarrow, \\ &\Phi(v) = a_2 + a_3, 0 = 2a_2 + 3a_3, \end{aligned} \quad (21)$$

from this, we can say that $a_2 = 3\Phi(v)$ and $a_3 = -2\Phi(v)$. By making use of a_0, a_1, a_2, and a_3 in (20), we obtain the solution as follows:

$$v(t) = \Phi(v)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} x(\theta) \frac{d\theta}{\theta}. \quad (22)$$

Also, for $Y \in C[1, e]$, and $z \in AC_\sigma^k[1, e]$, the solution of

$$\begin{aligned} &^C D_{1+}^{\eth} Y(t) = z(t), t \in \mathrm{H}, \eth \in (3, 4], \\ &Y(1) = Y'(1) = Y'(e) = 0, \mathrm{Y}(e) = \Psi(Y) \end{aligned} \quad (23)$$

may be expressed as

$$y(t) = \Psi(y)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} z(\theta) \frac{d\theta}{\theta}. \quad (24)$$

□

Corollary 8. *The solution of the concerned problem (1) is expressed as follows:*

$$\begin{cases} v(t) = \Phi(v)\hbar(t) + \dfrac{l}{\Gamma(\eth)} \displaystyle\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} f(\theta, v(\jmath\theta), Y(\theta)), \dfrac{d\theta}{\theta}, \\ y(t) = \Psi(y)\hbar(t) + \dfrac{l}{\Gamma(\eth)} \displaystyle\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} g(\theta, v(\theta), Y(\jmath\theta)), \dfrac{d\theta}{\theta}. \end{cases} \quad (25)$$

Theorem 9. *Consider two functions* f, g *possesses continuity and then the solution of (25) is* $(v, Y) \in \mathbf{P}$, *if* $f(v, Y)$ *is the solution of (1).*

Proof. If (v, Y) is the solution of (25), then by the differentiation of both sides of (25), we have (1). However, if (v, Y) is a solution of (1), then (v, Y) is the solution of (25). □

Let $\mathbf{F}_1, \mathbf{F}_2 : \mathbf{P} \longrightarrow \mathbf{P} \ni$

$$\begin{aligned} &\mathbf{F}_1(v, Y) = \Phi(v)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} f(\theta, v(\jmath\theta), Y(\theta)) \frac{d\theta}{\theta}, \\ &\mathbf{F}_2(v, Y) = \Psi(y)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} g(\theta, v(\theta), Y(\jmath\theta)) \frac{d\theta}{\theta}, \end{aligned} \quad (26)$$

and $\mathbf{F}(v, Y) = \begin{pmatrix} \mathbf{F}_1(v, Y) \\ \mathbf{F}_2(v, Y) \end{pmatrix}$. Hence, solution of (25) is a fixed point of $\mathbf{F}$.

Theorem 10. *If* $\Delta < 1$, *with the help of assumptions* $(M_1) - (M_3)$, *system (1) has at most one solution, where*

$$\Delta = \max \left\{ \mathfrak{R}_\Phi^* + \frac{L_f^*}{\Gamma(\eth + 1)}, \mathfrak{R}_\Psi^* + \frac{L_g^*}{\Gamma(\eth + 1)} \right\}. \quad (27)$$

Proof. Let $(\nu, Y), (\nu, Y) \in \mathbf{P}$ and for all $t \in H$, we have

$$\begin{aligned}
&\|\mathbf{F}_1(\nu, Y) - \mathbf{F}_1(\nu, Y)\| \\
&\quad \le \max_{t\in H}|\Phi(\nu) - \Phi(\nu)|\hbar(t) + \max_{t\in H}\frac{l}{\Gamma(\eth)} \\
&\qquad \cdot \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} |(f(\theta, \nu(\imath\theta), Y(t) - f(\theta, \nu(\imath\theta), Y(t)))| \frac{d\theta}{\theta} \\
&\quad \le \max_{t\in H}\mathfrak{R}^*_\Phi|\nu - \nu|\hbar(t) + \max_{t\in H}(\ln t)^\eth \frac{L^*_f}{\Gamma(\eth + 1)}(|\nu - \nu| + |Y - Y|) \\
&\quad \le \left(\mathfrak{R}^*_\Phi + \frac{L^*_f}{\Gamma(\eth + 1)}\right)\|\nu - \nu\| + \frac{L^*_f}{\Gamma(\eth + 1)}\|Y - Y\| \\
&\quad \le \Delta_1(\|Y - Y\| + \|Y - Y\|),
\end{aligned} \tag{28}$$

where

$$\Delta_1 = \mathfrak{R}^*_\Phi + \frac{L^*_f}{\Gamma(\eth + 1)}. \tag{29}$$

In a similar way, we obtain

$$\|\mathbf{F}_2(\nu, Y) - \mathbf{F}_2(\nu, Y)\| \le \Delta_2(\|\nu - \nu\| + \|Y - Y\|), \tag{30}$$

where

$$\Delta_2 = \mathfrak{R}^*_\Psi + \frac{L^*_g}{\Gamma(\eth + 1)}. \tag{31}$$

Hence, from (28) and (30), one has

$$\begin{aligned}
\|F(\nu, Y) - F(\nu, Y)| &\le \max\,(\Delta_1, \Delta_2)(\|\nu - \nu\| + \|Y - Y\|) \\
&= \Delta(\|\nu - \nu\| + \|Y - Y\|),
\end{aligned} \tag{32}$$

where $\Delta = \max_{t\in H}\{\Delta_1, \Delta_2\}$. Hence, it is obvious that $\mathbf{F}$ is contraction; therefore, (1) has a unique result. □

Theorem 11. *In the light of hypotheses (M_1), (M_4), and (M_5) together with condition* $\max\{Y_1, Y_2\} < 1$, *system (1) has a minimum of one solution.*

Proof. Let

$$\begin{aligned}
J_1 &= \kappa_1 + \kappa_2 + \frac{\left(C^*_f + D^*_f + C^*_g + D^*_g\right)}{\Gamma(\eth + 1)}, \\
J_2 &= \beta_\Phi + \beta_\Psi + \frac{\left(M^*_f + M^*_g\right)}{\Gamma(\eth + 1)}.
\end{aligned} \tag{33}$$

We define a subset $\mathbf{B}$ of $\mathbf{P}$ which is closed. That is,

$$\mathbf{B} = \{(\nu, Y) \in \mathbf{P} : \|(\nu, Y)\| \le \rho\}, \text{for} \rho \ge \max\left\{\frac{J_2}{1 - J_1}\right\}. \tag{34}$$

Let us define the following operators as

$$\begin{aligned}
\aleph_1(\nu, Y) &= \frac{1}{\Gamma(\eth)}\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} f(\theta, \nu(\imath\theta), Y(\theta))\frac{d\theta}{\theta}, \\
\aleph_2(\nu, Y) &= \frac{1}{\Gamma(\eth)}\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} g(\theta, \nu(\theta), Y(\imath\theta))\frac{d\theta}{\theta}, \\
\mathbf{S}_1\nu(t) &= \Phi(\nu)\hbar(t), \\
\mathbf{S}_2Y(t) &= \Psi(Y)\hbar(t).
\end{aligned} \tag{35}$$

It is obvious that $\tilde{T}_1 = \aleph_1 + \mathbf{S}_1$, $\tilde{T}_2 = \aleph_2 + \mathbf{S}_2$. Further, we prove that

$$\tilde{T}(\nu, Y) = \aleph(\nu, Y) + \mathbf{S}(\nu, Y) \in \mathbf{B}, \text{for all } (\nu, Y) \in \mathbf{B}. \tag{36}$$

For any $(\nu, Y) \in \mathbf{B}$, we have

$$\begin{aligned}
|T_1(\nu, Y)| &= \left|\Phi(\nu)\hbar(t) + \frac{l}{\Gamma(\eth)}\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} f(\theta, \nu(\imath\theta), Y(\theta))\frac{d\theta}{\theta}\right| \\
&\le \max_{t\in H} + |\Phi(\nu)\hbar(t)| + \max_{t\in H}\frac{1}{\Gamma(\eth)} \\
&\quad \cdot \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} |f(\theta, \nu(\imath\theta), Y(\theta))|\frac{d\theta}{\theta} \\
&\le \max_{t\in H}\hbar(t)(\kappa_1|\nu| + \beta_\Phi) \\
&\quad + \max_{t\in H}\left(C^*_f|\nu| + D^*_f|Y| + M^*_f\right)\frac{1}{\Gamma(\eth + 1)}\left(\ln \frac{t}{a}\right)^\eth \\
&\le \kappa_1\rho + \beta_\Phi + \frac{\left(C^*_f\rho + D^*_f\rho + M^*_f\right)}{\Gamma(\eth + 1)} \le \frac{\rho}{2}.
\end{aligned} \tag{37}$$

In a similar way, we obtain

$$|\tilde{T}_2(\nu, \mathscr{Y})| \le \kappa_2\rho + \beta_\Psi + \frac{\left(C^*_g\rho + D^*_g\rho + M^*_g\right)}{\Gamma(\eth + 1)} \le \frac{\rho}{2}. \tag{38}$$

The preceding calculations imply that $\|\tilde{T}(\nu, \mathscr{Y})\| \le \rho$, which clarify that $\tilde{T}(\mathbf{B}) \subseteq \mathbf{B}$. For $(\nu, Y), (\nu, Y) \in \mathbf{B}$, we can write it as

$$\begin{aligned}
\|\mathbf{S}_1(\nu) - \mathbf{S}_1(\nu)\| &\le \max_{t\in H}[|\Phi(\nu) - \Phi(\nu)|] \\
&\le \max_{t\in H}\mathfrak{R}^*_\Phi\hbar(t)\|\nu - \nu\| \\
&\le Y_1\|\nu - \nu\|.
\end{aligned} \tag{39}$$

We can also prove that

$$\|\mathbf{S}_2(\mathscr{Y}) - \mathbf{S}_2(\mathscr{Y})\| \le \mathscr{Y}_2\|\mathscr{Y} - \mathscr{Y}\|, \tag{40}$$

where

$$Y_1 = \mathfrak{R}^*_{\Phi},$$
$$Y_2 = \mathfrak{R}^*_{\Psi}. \tag{41}$$

Clearly, (39) and (40) assure the contraction of **S**. Now, we need to show the relative compactness of $\aleph$. Now, as f and g are continuous, hence $\aleph$ is continuous too. For $(\nu, Y) \in \mathbf{B}$, we have

$$\begin{aligned} |\aleph_1(\nu, \mathscr{Y})| &\leq \max_{t\in H} \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} |f(\theta, \nu(\lambda\theta), \mathscr{Y}(\theta))| \frac{d\theta}{\theta} \\ &\leq \frac{\left(C^*_f \rho + D^*_f \rho + M^*_f\right)}{\Gamma(\eth+1)}. \end{aligned} \tag{42}$$

In the same way, one can get

$$|\aleph_2(\nu, \mathscr{Y})| \leq \frac{\left(C^*_g \rho + D^*_g \rho + M^*_g\right)}{\Gamma(\eth+1)}. \tag{43}$$

Therefore, from (42) and (43), it implies

$$\rho \geq \|\aleph(\nu, \mathscr{Y})\|. \tag{44}$$

Hence, from (44), the boundedness of $\aleph$ can also be deduced on **B**. Take any $(\nu, \mathscr{Y}) \in \mathbf{B}$. Subsequently, for t_1, $t_2 \in \mathscr{H}$ with $t_1 \leq t_2 \in [1, e]$, one has

$$\begin{aligned} &|\aleph_1(\nu(t_1), \mathscr{Y}(t_1)) - \aleph_1(\nu(t_2), \mathscr{Y}(t_2)| \\ &\quad\leq \frac{1}{\Gamma(\eth)} \int_1^{t_1} \left(\left(\ln \frac{t_1}{\theta}\right)^{\eth-1} - \left(\ln \frac{t_2}{\theta}\right)^{\eth-1}\right) \\ &\qquad\cdot |f(\theta, \nu(\lambda\theta), \mathscr{Y}(\theta))| \frac{d\theta}{\theta} + \frac{1}{\Gamma(\eth)} \int_{t_1}^{t_2} \left(\ln \frac{t_2}{\theta}\right)^{\eth-1} \\ &\qquad\cdot |f(\theta, \nu(\lambda\theta), \mathscr{Y}(\theta))| \frac{d\theta}{\theta} \\ &\quad\leq \frac{1}{\Gamma(\eth+1)} \left(C^*_f |\nu| + D^*_f |\mathscr{Y}| + M^*_f\right) \\ &\qquad\times \left((\ln t_2)^{\eth} + 2\left(\ln \frac{t_2}{t_1}\right)^{\eth} - (\ln t_1)^{\eth}\right). \end{aligned} \tag{45}$$

From the previous inequality, we can claim that (45) approaches to zero on $t_1 \longrightarrow t_2$. As $\aleph_1$ possesses the properties of continuity and boundedness, it clearly means that $\aleph_1$ possesses uniform boundedness. Therefore, $\|\aleph_1(\nu(t_2), \mathscr{Y}(t_2)) - \aleph_1(\nu(t_1), \mathscr{Y}(t_1))\| \longrightarrow 0$ as t_1 tends to t_2. Similarly, $\|\aleph_2(\nu(t_2), \mathscr{Y}(t_2)) - \aleph_2(\nu(t_1), \mathscr{Y}(t_1))\| \longrightarrow 0$ as t_1 tends to t_2. Hence, all the assumptions of at least one solution for system (1) are achieved. □

4. Stability Results

Theorem 12. *Under the hypothesis $(M_1) - (M_3)$ together with condition $\Delta < 1$, the considered system has UH stable solution.*

Proof. Let for arbitrary solutions $(\nu, \mathscr{Y}), (\nu, \mathscr{Y}) \in \mathbf{P}$, and for all $t \in \mathscr{H}$, we have

$$\begin{aligned} &\|\mathbf{F}_1(\nu, \mathscr{Y}) - \mathbf{F}_1(\nu, \mathscr{Y})\| \\ &\quad\leq \max_{t\in H} |\Phi(\nu) - \Phi(\nu)| \hbar(t) + \max_{t\in H} \frac{1}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} \\ &\qquad\cdot |(f(\theta, \nu(\lambda\theta), \mathscr{Y}(t) - f(\theta, \nu(\lambda\theta), \mathscr{Y}(t)))| \frac{d\theta}{\theta} \\ &\quad\leq \max_{t\in H} \hbar(t) \mathfrak{R}^*_{\Phi} |\nu - \nu| + \max_{t\in H} (\ln t)^{\eth} \frac{L^*_f}{\Gamma(\eth+1)} (|\nu - \nu| + |\mathscr{Y} - \mathscr{Y}|) \\ &\quad\leq \left(\mathfrak{R}^*_{\Phi} + \frac{L^*_f}{\Gamma(\eth+1)}\right) \|\nu - \nu\| + \frac{L^*_f}{\Gamma(\eth+1)} \|\mathscr{Y} - \mathscr{Y}\| \\ &\quad\leq b_1 \|\mathscr{Y} - \mathscr{Y}\| + b_2 \|\mathscr{Y} - \mathscr{Y}\|, \end{aligned} \tag{46}$$

where

$$b_1 = \mathfrak{R}^*_{\Phi} + \frac{L^*_f}{\Gamma(\eth+1)},$$
$$b_2 = \frac{L^*_f}{\Gamma(\eth+1)}. \tag{47}$$

Similarly, one has

$$\|\mathbf{F}_2(\nu, \mathscr{Y}) - \mathbf{F}_2(\nu, \mathscr{Y})\| \leq b_3 \|\mathscr{Y} - \mathscr{Y}\| + b_4 \|\mathscr{Y} - \mathscr{Y}\|, \tag{48}$$

where

$$b_3 = \mathfrak{R}^*_{\Psi} + \frac{L^*_g}{\Gamma(\eth+1)},$$
$$b_4 = \frac{L^*_g}{\Gamma(\eth+1)}. \tag{49}$$

So, from (46) and (48), we get

$$\begin{aligned} &\|\mathbf{F}_1(\nu, \mathscr{Y}) - \mathbf{F}_1(\nu, \mathscr{Y})\| \\ &\quad\leq b_1 \|\mathscr{Y} - \mathscr{Y}\| + b_2 \|\mathscr{Y} - \mathscr{Y}\|, \|\mathbf{F}_2(\nu, \mathscr{Y}) - \mathbf{F}_2(\nu, \mathscr{Y})\| \\ &\quad\leq b_3 \|\mathscr{Y} - \mathscr{Y}\| + b_4 \|\mathscr{Y} - \mathscr{Y}\|. \end{aligned} \tag{50}$$

Using (50), we have

$$\mathscr{M} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}. \tag{51}$$

Since $\mathscr{M}$ converges to zero, hence the result of (1) is UH stable. □

5. Applications

Example 13. Taking a coupled system as.

$$\begin{cases} {}^{C}D_{1^+}^{3.8}\nu(t) + \dfrac{\sin|\nu(0.3t)| + \cos|(\mathscr{Y}t)| + e^t + 4}{(t^2+10)^3} = 0, & t \in \mathscr{H}, \\ {}^{C}D_{1^+}^{3.8}\mathscr{Y}(t) + \dfrac{t^3 + |\nu(t)| - |\mathscr{Y}(0.3t)|}{(e^t + 50)} = 0, & t \in \mathscr{H}, \\ \nu(1) = \nu'(1) = 0 = \nu'(e), \nu(e) = \dfrac{\sin|\nu|}{30}, \\ \mathscr{Y}(1) = \mathscr{Y}'(1) = 0 = \mathscr{Y}'(e), \mathscr{Y}(e) = \dfrac{\cos(\mathscr{Y})}{29}. \end{cases} \tag{52}$$

From above, $\eth = 3.8$, $\jmath = 0.3$ after calculation, we have $L_f^* = 0.0001$, $L_g^* = 0.02$, $\mathfrak{R}_\Phi^* = 0.033$, $\mathfrak{R}_\Psi^* = 0.034$,

$$\begin{aligned} \Delta_1 &= 0.0331, \\ \Delta_2 &= 0.0351. \end{aligned} \tag{53}$$

It is obvious that max $\{\Delta_1, \Delta_2\} = 0.0351 < 1$. So (52) has a unique solution by Theorem 10. Moreover, from the values of b_i, $(i = 1, 2, 3, 4)$, we have

$$\mathscr{M} = \begin{bmatrix} 0.0331 & 0.0001 \\ 0.0351 & 0.0011 \end{bmatrix}, \tag{54}$$

after calculation, the eigenvalues are $\eth_1 = 0.0332$, $\eth_2 = 0.0010$. Therefore $\Lambda(\mathscr{M}) = 0.0332 < 1$. Thus, the given system is HU stable by using Theorem 12.

Example 14. Consider the following problem:

$$\begin{cases} {}^{C}D_{1^+}^{3.7}\nu(t) + \dfrac{\arctan(t)}{10 + |\nu(0.4t)|} = 0, & t \in \mathscr{H}, \\ {}^{C}D_{1^+}^{3.7}\mathscr{Y}(t) + \dfrac{\ln t}{8 + |\mathscr{Y}(0.4t)|} = 0, & t \in \mathscr{H}, \\ \nu(1) = \nu'(1) = 0 = \nu'(e), \nu(e) = \dfrac{|\nu| + t^2}{60}, \\ \mathscr{Y}(1) = \mathscr{Y}'(1) = 0 = \mathscr{Y}'(e), \mathscr{Y}(e) = \dfrac{\sin|\mathscr{Y}|}{25}. \end{cases} \tag{55}$$

From above, $\eth = 3.7$, $\jmath = 0.4$ after calculation, we have $L_f^* = 0.1218$, $L_g^* = 0.125$, $\mathfrak{R}_\Phi^* = 0.016$, $\mathfrak{R}_\Psi^* = 0.04$,

$$\begin{aligned} \Delta_1 &= 0.0239, \\ \Delta_2 &= 0.0481. \end{aligned} \tag{56}$$

It is obvious that max $\{\Delta_1, \Delta_2\} = 0.0481 < 1$. So (55) has a unique solution by Theorem 10. Moreover, from the values of b_i $(i = 1, 2, 3, 4)$, we have

$$\mathscr{M} = \begin{bmatrix} 0.0239 & 0.0079 \\ 0.0481 & 0.0081 \end{bmatrix}, \tag{57}$$

after calculation, the eigenvalues are $\eth_1 = 0.0351$, $\eth_2 = -0.0102$. Therefore, $\Lambda(\mathscr{M}) = 0.0351 < 1$. Thus, the given system is HU stable by using Theorem 12.

6. Conclusion

In this research work, nonlinear BVPs of FDEs containing proportional delay with CHD operator have been successfully investigated. We have utilized the techniques of fixed point theory and nonlinear analysis, to develop the existence and stability results for the proposed system. Through some examples, the main results have been justified. In the future, one can investigate the aforementioned system of FDEs for more complicated boundary conditions.

Data Availability

The data used in this research work is contained in paper.

Conflicts of Interest

There are no conflict of interest that exist.

Authors' Contributions

An equal contribution has been done by all the authors.

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Research Article

Common Fixed-Point Theorems in the Partial *b*-Metric Spaces and an Application to the System of Boundary Value Problems

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In this paper, we investigate the conditions for the existence of the common fixed points of generalized contractions in the partial b-metric spaces endowed with an arbitrary binary relation. We establish some unique common fixed-point theorems. The obtained results may generalize and improve earlier fixed-point results. We provide examples to illustrate our findings. As an application, we discuss the common solution to the system of boundary value problems.

1. Introduction, Preliminaries, and Motivations

The b-metric space was introduced by Czerwik [1]. It is obtained by modifying the triangle property of the metric space. Every metric is a b-metric, but the converse is not true. Almost all the fixed-point theorems in the metric spaces have been proved true in the b-metric spaces; for example, see [2–10] and references therein.

Matthews [11] introduced the notion of the partial metric space as a part of the study of denotational semantics of the dataflow network. In this space, the usual metric is replaced by a partial metric having a property that the self-distance of any point of the space may not be zero. Every metric is a partial metric, but the converse is not true. Matthews [11] also initiated the fixed-point theory in the partial metric space. He proved the Banach contraction principle in this space to be applied in program verification. We can find so many fixed-point theorems in the metric spaces which have been proved in the partial metric spaces by many fixed-point theorists ([12, 26] and references therein).

Shukla [13] introduced the concept of partial b-metric by modifying the triangle property of the partial metric and investigated fixed points of Banach contraction and Kannan contraction in the partial b-metric spaces. Mustafa et al. [14] modified the triangle property of partial b-metric and established a convergence criterion and some working rules in partial b-metric spaces. Moreover, Mustafa et al. [14] investigated common fixed-point results for (ϕ, ψ)-weakly contractive mappings. Dolicanin-Đekic [15] obtained the fixed-point theorems for Ciric-type contractions in the partial b-metric spaces. Singh et al. [16] investigated some conditions to show the existence of the common fixed points of power graphic (F, ψ)-contractions defined on the partial b-metric space endowed with directed graphs. More results on F-contractions can be seen in [8, 17, 18].

Let X be a nonempty set, then the nonempty binary relation $\mathfrak{R}$ is a subset of X^2. The set X^2 itself is known as universal relation, and the empty set is known as an empty relation; both are trivial relations. If any two elements $\alpha, \beta \in X$ are related with respect to $\mathfrak{R}$, then we shall write $(\alpha, \beta) \in \mathfrak{R}$. We shall use the notation $[\alpha, \beta] \in \mathfrak{R}$ if either $(\alpha, \beta) \in \mathfrak{R}$ or $(\beta, \alpha) \in \mathfrak{R}$. $\mathfrak{R}$ is reflexive if $(\alpha, \alpha) \in \mathfrak{R}$, for all $\alpha \in X$. $\mathfrak{R}$ is symmetric if $(\alpha, \beta) \in \mathfrak{R}$ implies $(\beta, \alpha) \in \mathfrak{R}$, for all $\alpha, \beta \in X$. $\mathfrak{R}$ is antisymmetric if $(\alpha, \beta) \in \mathfrak{R}$ and $(\beta, \alpha) \in \mathfrak{R}$ implies $\alpha = \beta$, for all $\alpha, \beta \in X$. $\mathfrak{R}$ is transitive if $(\alpha, \beta) \in \mathfrak{R}$ and $(\beta, \gamma) \in \mathfrak{R}$ implies $(\alpha, \gamma) \in \mathfrak{R}$, for all $\alpha, \beta \in X$. The inverse, transpose, or dual of binary relation $\mathfrak{R}$ is denoted by $\mathfrak{R}^{-1}$ and defined as follows: $\mathfrak{R}^{-1} = \{(\alpha, \beta) \in X \mid (\beta, \alpha) \in \mathfrak{R}\}$. Let $\mathfrak{R}^s = \mathfrak{R} \cup \mathfrak{R}^{-1}$, then it is easy to prove that $(\alpha, \beta) \in \mathfrak{R}^s$ if and only if $[\alpha, \beta] \in \mathfrak{R}$.

Definition 1 (see [19]). Let T be a self-mapping on a nonempty set X. A binary relation $\mathfrak{R}$ on X is said to be T-closed if for all $\alpha, \beta \in X$,

$$(\alpha, \beta) \in \mathfrak{R} \Rightarrow (T(\alpha), T(\beta)) \in \mathfrak{R}. \quad (1)$$

Definition 2 (see [19]). Let $\mathfrak{R}$ be a binary relation on X. A path in $\mathfrak{R}$ from α to β is a sequence $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\} \subseteq X$ such that

(1) $\alpha_0 = \alpha$ and $\alpha_n = \beta$

(2) $(\alpha_j, \alpha_{j+1}) \in \mathfrak{R}$ for all $j \in \{0, 1, 2, \cdots \cdots \cdots, n-1\}$

The set of all paths from α to β in $\mathfrak{R}$ is denoted by $\Gamma(\alpha, \beta, \mathfrak{R})$. The path of length n involves $n+1$ element of X.

Definition 3 (see [19]). A metric space (X, d) equipped with the binary relation $\mathfrak{R}$ is called $\mathfrak{R}$-regular (or d-self-closed) if for each sequence $\{\alpha_n\}$in$X$, whenever $(\alpha_n, \alpha_{n+1}) \in \mathfrak{R}$ and $\alpha_n \xrightarrow{d} \alpha$, we have $(\alpha_n, \alpha) \in \mathfrak{R}$, for all $n \in \mathbb{N} \cup \{0\}$.

Alam and Imdad [19] used nonempty binary relation on the nonempty set X to obtain the following relation-theoretic contraction principle.

Theorem 4 (see [19]). *Let (X, d) be a complete metric space and $\mathfrak{R}$ be a binary relation on X. Let T be a self-mapping defined on (X, d) satisfying the following conditions:*

(a) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is T-closed

(b) Either T is continuous or (X, d) is $\mathfrak{R}$ − regular

(c) There exists $k \in [0, 1)$ such that $d(T(\alpha), T(\beta))) \le kd(\alpha, \beta)$ for $\alpha, \beta \in X$ with $(\alpha, \beta) \in \mathfrak{R}$

Then T admits a fixed point in X. Moreover, if $\Gamma(\alpha, \beta, \mathfrak{R}^s)$ is a nonempty set for all $\alpha, \beta \in X$, then the fixed point is unique.

al-Sulami et al. [20] generalized Theorem 4 by replacing Banach contraction with θ-contraction as follows.

Theorem 5 (see [20]). *Let (X, d) be a complete metric space and $\mathfrak{R}$ be a binary relation on X. Let T be a self-mapping defined on (X, d) satisfying the following conditions:*

(a) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is T-closed

(b) Either T is continuous or (X, d) is $\mathfrak{R}$ − regular

(c) There exists $k \in [0, 1)$ such that $\theta(d(T(\alpha), T(\beta))) \le [\theta(d(\alpha, \beta))]^k$ for $\alpha, \beta \in X$ with $(\alpha, \beta) \in \mathfrak{R}$

Then T admits a fixed point in X. Moreover, if $\Gamma(\alpha, \beta, \mathfrak{R}^s)$ is a nonempty set for all $\alpha, \beta \in X$, then the fixed point is unique.

Definition 6 (see [21]). Let T and S be two self-mappings on a nonempty set X. A binary relation $\mathfrak{R}$ on X is said to be (T, S)-closed if for all $\alpha, \beta \in X$,

$$(\alpha, \beta) \in \mathfrak{R} \Rightarrow (T(\alpha), S(\beta)) \in \mathfrak{R} \text{ or } (S(\alpha), T(\beta)) \in \mathfrak{R}. \quad (2)$$

Zada and Sarwar [21] generalized Theorem 4 by replacing Banach contraction with F-contraction as follows.

Theorem 7 (see [21]). *Let (X, d) be a complete metric space and $\mathfrak{R}$ be a binary relation on X. If the self-mappings T and S defined on (X, d) satisfy the following conditions:*

(a) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed

(b) Either T, S are continuous or (X, d) is $\mathfrak{R}$ − regular

(c) There exists $\tau > 0$, such that for all $(\alpha, \beta) \in \mathfrak{R}$ with $d(T(\alpha), S(\beta)) > 0$,

$$\tau + F(d(T(\alpha), S(\beta))) \le F\left(d(\alpha, \beta) + \frac{d(\alpha, S(\beta))d(\beta, T(\alpha))}{1 + d(\alpha, \beta)}\right) \quad (3)$$

Then T and S have a unique common fixed point in X. Moreover, if $\Gamma(\alpha, \beta, \mathfrak{R}^s)$ is nonempty for all $\alpha, \beta \in X$, then the common fixed point is unique.

Liu et al. [22] introduced the $(D, \mathscr{C})$-contractions where the mapping D maps positive real numbers to positive real numbers and satisfies the conditions $(D_1) - (D_3)$:

(D_1) D is nondecreasing

(D_2) $\lim_{n\longrightarrow\infty} D(t_n) = 0 \Longleftrightarrow \lim_{n\longrightarrow\infty} t_n = 0$, for each positive sequence $\{t_n\}$

(D_3) D is continuous

$\mathscr{C} : (0,\infty) \longrightarrow (0,\infty)$ is a comparison function; that is, it satisfies the following conditions:

(i) $\mathscr{C}$ is monotone increasing, that is,

$$\alpha < \beta \Longrightarrow \mathscr{C}(\alpha) < \mathscr{C}(\beta) \quad (4)$$

(ii) $\lim_{n\longrightarrow\infty} \mathscr{C}^n(t) = 0$ for all $t > 0$, where $\mathscr{C}^n$ stands for the n^{th} iterate of $\mathscr{C}$

Let $\mathscr{D} = \{D : (0,\infty) \longrightarrow (0,\infty) | D \text{ satisfies } (D_1) - (D_3)\}$. If D is defined by $D(t) = t$; $D(t) = \ln t$, then D belongs to $\mathscr{D}$.

Note that if $\mathscr{C}$ is a comparison function, then $\mathscr{C}(t) < t$, for every $t > 0$. The mappings $\mathscr{C}(t) = \alpha t, 0 < \alpha < 1, t > 0$, and $\mathscr{C}(t) = t/(1+t), t > 0$, are comparison functions.

Definition 8 ([22], $(D, \mathscr{C})$-contraction). Let T be a self-mapping defined on the metric space (X, d). Let

$$\mathfrak{J} = \left\{ (\alpha, \beta) \in X^2 : d(T(\alpha), T(\beta)) > 0 \right\}. \quad (5)$$

The mapping T is called $(D, \mathscr{C})$-contraction if it satisfies the following condition:

$$D(d(T(\alpha), T(\beta))) \le \mathscr{C}(D(d(\alpha, \beta))), \text{for all } \alpha, \beta \in \mathfrak{J}. \quad (6)$$

Definition 9 ([22], generalized $(D, \mathscr{C})$-contraction). Let T be a self-mapping defined on the metric space (X, d). If the mapping T satisfies the condition $D(d(T(\alpha), T(\beta))) \le \mathscr{C}(D (M(\alpha, \beta)))$, for all $(\alpha, \beta) \in \mathfrak{J}$, where $M(\alpha, \beta)$ is defined by $M(\alpha, \beta) = \max \{d(\alpha, \beta), d(\alpha, T\alpha), d(\beta, T\beta), (d(\alpha, T\beta) + d(\beta, T\alpha))/2 \}$. Then it is called generalized $(D, \mathscr{C})$-contraction.

Liu et al. established the following theorem for $(D, \mathscr{C})$ -contractions.

Theorem 10 (see [22]). *Every generalized $(D, \mathscr{C})$-contraction has a unique fixed point in a complete metric space (X, d).*

In this paper, in Section 3, we investigate common fixed-point results for generalized contractions in the partial b -metric spaces endowed with binary relation $\mathfrak{R}$. The obtained results generalize Theorems 4, 5, 7, 10. We support the results with a nontrivial example and counter the remarks given in [23].

2. Basic Notions in the Partial b-Metric Spaces

Let X be a nonempty set, and the mapping $P : X \times X \longrightarrow$ $\longrightarrow [0,\infty)$ satisfies the following axioms:

(1) $x = y \Leftrightarrow P(x, x) = P(x, y) = P(y, y), \forall x, y \in X$

(2) $P(x, x) \le P(x, y) \forall x, y \in X$

(3) $P(x, y) = P(y, x) \forall x, y \in X$

(4) $P(x, z) \le P(x, y) + P(y, z) - P(y, y) \forall x, y, z \in X$

(5) There exists a real number $s \ge 1$ such that

$$P(x, z) \le s[P (x, y) + P(y, z)] - P(y, y) \forall x, y, z \in X \quad (7)$$

According to Matthews [11], if the mapping P satisfies axioms (1-4), we say that it is a *partial metric* on the set X and (X, P) is called *partial metric space*. According to Shukla [13], if P satisfies axioms (1, 2, 3, and 5), then it is a *partial b -metric* on the set X and (X, P_b) is called *partial b -metric space*. For convenience, we denote partial b-metric by P_b.

Every partial b-metric P_b induces a b-metric $d_{P_b} : X \times X \longrightarrow [0,\infty)$ defined by

$$d_{P_b}(x, y) = 2P_b(x, y) - P_b(x, x) - P_b(y, y) \forall x, y \in X. \quad (8)$$

It is called *induced b -metric* on X.

Let $B_{P_b}(x, \epsilon) = \{ y \in X : P_b(x, y) < \epsilon + P_b(x, x)\}$, then $\{B_{P_b}(x, \epsilon): x \in X, \epsilon > 0\}$ is a collection of P_b *-balls* which forms a base for *partial b -metric topology*.

The following relation can be observed.

Remark 11.

(1) In (X, P_b), $P_b(x, y) = 0, \Rightarrow x = y, \forall x, y \in X$, but the converse is not true (in this case, (X, P_b) reduces to a b-metric space) Figure 1.

Example 1 (see [13]). Let $X = [0,\infty), l > 1$, be a constant and $P_b : X \times X \longrightarrow [0,\infty)$ be defined by

$$P_b(x, y) = (\max \{x, y\})^l + |x - y|^l \text{ for all } x, y \in X. \quad (9)$$

Then (X, P_b) is a partial b-metric space with coefficient $s = 2^l > 1$, but it is neither a b-metric space nor a partial metric space.

Example 2 (see [13]). Let $P : X \times X \longrightarrow [0,\infty)$ and $d^* : X \times X \longrightarrow [0,\infty)$ be the partial metric and b-metric on X, respectively. Then the mapping $P_b : X \times X \longrightarrow [0,\infty)$ defined by $P_b(x, y) = P(x, y) + d^*(x, y)$ for all $x, y \in X$ defines a partial b-metric on X.

Example 3 (see [13]). Let $P : X \times X \longrightarrow [0,\infty)$ be a partial metric. Then the mapping $P_b : X \times X \longrightarrow [0,\infty)$ defined by $P_b(x, y) = ((P(x, y))^l$ for all $x, y \in X$ and $l \ge 1$ is a partial b-metric on X provided $s = 2^{l-1}$.

Definition 12 (see [13]). A sequence $\{x_n\}_{n\in N}$ in the partial b-metric space (X, P_b, s) is called a convergent sequence if there exists $x \in X$ such that

$$\lim_{n\longrightarrow\infty} P_b (x_n, x) = P_b(x, x). \quad (10)$$

The uniqueness of the limit of a convergent sequence may not be guaranteed in the partial b-metric spaces (see [23]).

Definition 13 (see [13]). A sequence $\{x_n\}_{n\in N}$ in a partial b -metric space (X, P_b, s) is called the Cauchy sequence if

$$\lim P_b(x_{n}, {}_{m,n\longrightarrow\infty} x_m) = P_b(x, x). \quad (11)$$

The partial b-metric space (X, P_b, s) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in N}$ in X converges to a point $x \in X$.

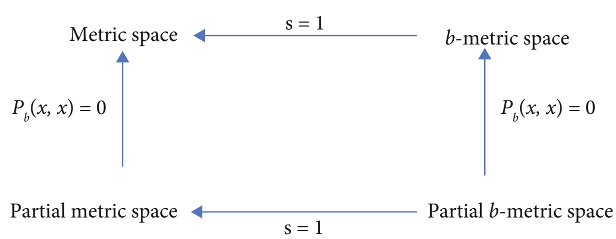

FIGURE 1

Lemma 14 (see [14]).

(1) *Every Cauchy sequence in the b -metric space is also Cauchy in the partial b-metric space and vice versa*

(2) *The partial b-metric space is complete if and only if b -metric space (induced b-metric space) is complete*

(3) *For any sequence* $\{x_n\}_{n\in N}$ *in X,* $\lim_{n\longrightarrow\infty} d_{P_b}(x^*, x_n)$ *= 0 if and only if*

$$\lim_{n\longrightarrow\infty} P_b(x^*, x_n) = P_b(x^*, x^*) = \lim P_{b\,n,m\longrightarrow\infty}(x_n, x_m) \tag{12}$$

3. Common Fixed-Point Theorems in the Partial b-Metric Spaces

This section is the main part of this paper. It contains some new common fixed-point theorems in the partial b-metric spaces. The existence theorems given in [12, 15, 19–22, 24, 27] can be seen as a special case of the results proved in this section.

The results in this paper are based on the following contractive condition.

Definition 15. Let T and S be two self-mappings on the partial b-metric space (X, P_b, s) and $\mathfrak{R}$ be a binary relation on X. Let

$$\mathfrak{I} = \{ (x, y) \in \mathfrak{R} : P_b(T(x), S(y)) > 0 \}. \tag{13}$$

The mappings T and S form a *$D\mathscr{C}$ -contraction* if there exists a continuous comparison function $\mathscr{C}$ and $D \in \mathscr{D}$ such that

$$D(s^2 P_b(T(x), S(y))) \le \mathscr{C}(D(P_b(x, y))), \text{for all } x, y \in \mathfrak{I}. \tag{14}$$

In [23], it was remarked that some contraction conditions on partial b-metric spaces imply contraction conditions on b -metric spaces (see Theorem 2.6 in [23]). In the following example, we show that the contraction condition (14) is independent of these remarks.

Example 4. Let $X = [0,\infty)$ and $\mathfrak{R} = X^2$. Let $P_b : X \times X \longrightarrow [0,\infty)$ be defined by

$$P_b(x, y) = (\max\{x, y\})^2 + |x - y|^2 \text{ for all } x, y \in X. \tag{15}$$

Then (X, P_b) is a partial b-metric space with coefficient $s = 4$. The associated b-metric is given by

$$d_{P_b}(x, y) = 2((\max\{x, y\})^2 + |x - y|^2) - x^2 - y^2. \tag{16}$$

Define $T \equiv S : [0, 1] \longrightarrow [0, 1]$ by $T(x) = x/5$ (if $x \in [0, 1)$) and $T(1) = 0$. Consider

$$D\left(s^2 d_{P_b}\left(T(1), T\left(\frac{5}{6}\right)\right)\right) \le \mathscr{C}\left(D\left(d_{P_b}\left(1, \frac{5}{6}\right)\right)\right). \tag{17}$$

This implies,

$$D\left(\frac{16}{36}\right) \le \mathscr{C}\left(D\left(\frac{13}{36}\right)\right) < D\left(\frac{13}{36}\right), \tag{18}$$

a contradiction to the definition of mapping $D \in \mathscr{D}$. On the other hand, for partial b-metric, we have

$$D\left(s^2 P_b\left(T(1), T\left(\frac{5}{6}\right)\right)\right) = D\left(\frac{32}{36}\right) \le \mathscr{C}\left(D\left(P_b\left(1, \frac{5}{6}\right)\right)\right) < D\left(\frac{37}{36}\right). \tag{19}$$

Note that we have taken $(1, 5/6) \in \mathfrak{R}$. Similarly, it can be shown that the above conclusion holds for all other cases.

Since, in general, b-metric is discontinuous mapping (see [5]), so by Example 2, the partial b-metric is not continuous in general. The following lemma is necessary for the upcoming results.

Lemma 16 (see [14]). *Let* (X, P_b, s) *be a partial b-metric space. If there exists a* $\{x_n\}$ *in* (X, P_b, s) *and* x^*, y^* *such that* $\lim_{n\longrightarrow\infty} x_n = x^*$. *Then*

$$\frac{1}{s} P_b(x^*, y^*) \le \lim_{n\longrightarrow\infty} \inf P_b(x_n, y^*) \le \lim_{n\longrightarrow\infty} \sup P_b(x_n, y^*) \le s P_b(x^*, y^*). \tag{20}$$

3.1. Main Results. We state our main results which describe the conditions for the existence of the common fixed points of $D\mathscr{C}$-contraction in the partial b-metric spaces.

Theorem 17. *Let* (X, P_b) *be a complete partial b-metric space and* $\mathfrak{R}$ *be a transitive binary relation on X. Let T and S form a* $D\mathscr{C}$*-contraction. Then T and S have a common fixed point in X, if the following conditions are satisfied.*

(a) *There exists* $\alpha_0 \in X$ *such that* $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$

(b) $\mathfrak{R}$ *is* (T, S)*-closed*

(c) *T and S are continuous*

Proof. By assumption (a), there exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$. Taking $\alpha_0 \in X$ as the initial point, we define the sequence $\{\alpha_n\}$ in X by

$$\begin{cases} \alpha_1 = T(\alpha_0), \alpha_2 = S(\alpha_1), \text{continuing with the same pattern, we have} \\ \alpha_{2n+1} = T(\alpha_{2n}), \alpha_{2n+2} = S(\alpha_{2n+1}), \text{where } n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{21}$$

□ □

Moreover, by assumptions (a) and (b), we have

$$\begin{aligned} (\alpha_1, \alpha_2) &= (T(\alpha_0), S(\alpha_1)) \in \mathfrak{R}, \\ (\alpha_2, \alpha_3) &= (S(\alpha_1), T(\alpha_2)) \in \mathfrak{R}, \\ (\alpha_3, \alpha_4) &= (T(\alpha_2), S(\alpha_3)) \in \mathfrak{R}, \\ (\alpha_4, \alpha_5) &= (S(\alpha_3), T(\alpha_4)) \in \mathfrak{R}. \end{aligned} \tag{22}$$

In general, we have $(\alpha_{2n}, \alpha_{2n+1}) = (S(\alpha_{2n-1}), T(\alpha_{2n})) \in \mathfrak{R}$ and $(\alpha_{2n+1}, \alpha_{2n+2}) = (T(\alpha_{2n}), S(\alpha_{2n+1})) \in \mathfrak{R}$.

Case 1. If $\alpha_{2n^*} = \alpha_{2n^*+1}$, for some n^*, then

$$\alpha_{2n^*+1} = \alpha_{2n^*+2}. \tag{23}$$

Indeed, on the contrary, if $\alpha_{2n^*+1} \neq \alpha_{2n^*+2}$, then $(\alpha_{2n^*+1}, \alpha_{2n^*+2}) \in \mathfrak{I}$, and by contractive condition (14), we have

$$\begin{aligned} D(P_b(\alpha_{2n^*+1}, \alpha_{2n^*+2}) &\leq D\left(s^2 P_b\left(T\left(\alpha_{2n^*}\right), S\left(\alpha_{2n^*+1}\right)\right)\right) \\ &\leq \mathscr{C}(D(P_b(\alpha_{2n^*}, \alpha_{2n^*+1})). \end{aligned} \tag{24}$$

Since $\mathscr{C}(t) <$ t, for every $t > 0$, we obtain

$$D(P_b(\alpha_{2n^*+1}, \alpha_{2n^*+2}) < D(P_b(\alpha_{2n^*}, \alpha_{2n^*+1}). \tag{25}$$

Since the function D is nondecreasing, so $P_b(\alpha_{2n^*+1}, \alpha_{2n^*+2}) < P_b(\alpha_{2n^*}, \alpha_{2n^*+1})$. This contradicts the second condition of partial b-metric spaces $(P_b(x, x) \leq P_b(x, y) \forall x, y \in X)$. Hence, $\alpha_{2n^*} = \alpha_{2n^*+1}$ implies $\alpha_{2n^*+1} = \alpha_{2n^*+2}$. Consequently, α_{2n^*} is a common fixed point of T, and that is $\alpha_{2n^*} = T(\alpha_{2n^*}) = S(\alpha_{2n^*+1}) = S(\alpha_{2n^*})$.

Case 2. If $\alpha_{2n} \neq \alpha_{2n+1}$ for all $n \in \mathbb{N}$. We have $P_b(T(\alpha_{2n}), S(\alpha_{2n-1})) > 0$ for all $n \in \mathbb{N}$. Since $(\alpha_{2n}, \alpha_{2n-1}) \in \mathfrak{R}$, so $(\alpha_{2n}, \alpha_{2n-1}) \in \mathfrak{I}$. Setting $\alpha = \alpha_{2n}$ and $\beta = \alpha_{2n-1}$ in (14), we get

$$\begin{aligned} D(P_b(\alpha_{2n+1}, \alpha_{2n})) &\leq D\left(s^2 P_b(\alpha_{2n+1}, \alpha_{2n})\right) \\ &= D\left(s^2 P_b(T(\alpha_{2n}), S(\alpha_{2n-1}))\right) \\ &\leq \mathscr{C}(D(P_b(\alpha_{2n}, \alpha_{2n-1}), \end{aligned} \tag{26}$$

for all $n \in \mathbb{N}$.

Similarly, setting $\alpha = \alpha_{2n}$ and $\beta = \alpha_{2n+1}$ in (14), we get

$$\begin{aligned} D(P_b(\alpha_{2n+1}, \alpha_{2n+2}) &\leq D\left(s^2 P_b(T(\alpha_{2n}), S(\alpha_{2n+1}))\right) \\ &\leq \mathscr{C}(D(P_b(\alpha_{2n}, \alpha_{2n+1})). \end{aligned} \tag{27}$$

In general, for all $h(n) \in \mathbb{N}$, either even or odd, we have

$$\begin{aligned} D\left(P_b\left(\alpha_{h(n)}, \alpha_{h(n)+1}\right)\right) &\leq \mathscr{C}\left(D\left(P_b\left(\alpha_{h(n)-1}, \alpha_{h(n)}\right)\right)\right) \\ &\leq \mathscr{C}^2\left(D\left(P_b\left(\alpha_{h(n)-2}, \alpha_{h(n)-1}\right)\right)\right) \vdots \\ &\leq \mathscr{C}^{h(n)}(D(P_b(\alpha_0, \alpha_1))). \end{aligned} \tag{28}$$

Taking limit $n \longrightarrow \infty$ in the above inequality, we get

$$\begin{aligned} 0 &\leq \lim_{n \longrightarrow \infty} D\left(P_b\left(\alpha_{h(n)}, \alpha_{h(n)+1}\right)\right) \\ &\leq \lim_{n \longrightarrow \infty} \mathscr{C}^{h(n)}(D(P_b(\alpha_0, \alpha_1))) = 0. \end{aligned} \tag{29}$$

This implies $\lim_{n \longrightarrow \infty} D(P_b(\alpha_{h(n)}, \alpha_{h(n)+1}) = 0$, and by (D_2), we have
$\lim_{n \longrightarrow \infty} P_b(\alpha_{h(n)}, \alpha_{h(n)+1}) = 0$. This implies (by (8)) that

$$\lim_{n \longrightarrow \infty} d_{P_b}\left(\alpha_{h(n)}, \alpha_{h(n)+1}\right) = 0. \tag{30}$$

By axiom (2), we have $\lim_{n \longrightarrow \infty} P_b(\alpha_{h(n)}, \alpha_{h(n)}) \leq \lim_{n \longrightarrow \infty} P_b(\alpha_{h(n)}, \alpha_{h(n)+1}) = 0$. Thus, for all $n, m \geq 1$, we have

$$\lim_{n,m \longrightarrow \infty} d_{P_b}\left(\alpha_{h(m)}, \alpha_{h(n)}\right) = 2 \lim_{n,m \longrightarrow \infty} P_b\left(\alpha_{h(m)}, \alpha_{h(n)}\right). \tag{31}$$

We claim that $\{\alpha_n\}$ is a Cauchy sequence in (X, d_{P_b}), for this is sufficient to prove that $\{\alpha_{2n}\}$ is Cauchy sequence. On the contrary, if $\{\alpha_{2n}\}$ is not Cauchy, then for some subsequences $\{\alpha_{2n_k}\}_{k=1}^{\infty}$ and $\{\alpha_{2m_k}\}_{k=1}^{\infty}$, there exist $\in$>0, and a positive integer $k(\in)$, such that for all $n_k, m_k > k$, we have $d_{P_b}(\alpha_{2m_k}, \alpha_{2n_k}) \geq \varepsilon$ and $d_{P_b}(\alpha_{2m_k}, \alpha_{2n_k-2}) < \varepsilon$; thus,

$$\varepsilon \leq d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k}\right) \leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2m_k+1}\right) + s d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right). \tag{32}$$

As $k \longrightarrow \infty$ in the above inequality, we have

$$\frac{\varepsilon}{s} \leq \lim_{k \longrightarrow \infty} \sup d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right). \tag{33}$$

By using triangular inequality (axiom (5)), we get

$$d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right) \leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-2}\right) + s d_{P_b}\left(\alpha_{2n_k-2}, \alpha_{2n_k-1}\right). \tag{34}$$

Taking limit $k \longrightarrow \infty$, we have

$$\lim_{k\longrightarrow\infty} \sup d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right) \leq s\varepsilon. \tag{35}$$

Also, we have the following information:

$$\begin{aligned} d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k}\right) &\leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-2}\right) + s d_{P_b}\left(\alpha_{2n_k-2}, \alpha_{2n_k}\right) \\ &\leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-2}\right) + s^2 d_{P_b}\left(\alpha_{2n_k-2}, \alpha_{2n_k-1}\right) \\ &\quad + s^2 d_{P_b}\left(\alpha_{2n_k-1}, \alpha_{2n_k}\right). \end{aligned} \tag{36}$$

Taking limit $k \longrightarrow \infty$, we have

$$\lim_{k\longrightarrow\infty} \sup d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k}\right) \leq s\varepsilon. \tag{37}$$

By axiom (5), we have

$$d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k-1}\right) \leq s d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2m_k}\right) + s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right). \tag{38}$$

Taking limit $k \longrightarrow \infty$ and using (35), we have

$$\lim_{k\longrightarrow\infty} \sup d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k-1}\right) \leq s^2\varepsilon. \tag{39}$$

By using (31), we have the following information from (33), (35), (37), and (39):

$$\frac{\varepsilon}{2s} \leq \lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right), \tag{40}$$

$$\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right) \leq \frac{s\varepsilon}{2}, \tag{41}$$

$$\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k}, \alpha_{2n_k}\right) \leq \frac{s\varepsilon}{2}, \tag{42}$$

$$\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_{k+1}}, \alpha_{2n_k-1}\right) \leq \frac{s^2\varepsilon}{2}, \tag{43}$$

Since $(\alpha_{2m_k}, \alpha_{2n_k-1}) \in \mathfrak{J}$, by (14), we have

$$\begin{aligned} D\left(\frac{s\in}{2}\right) &= D\left(s^2 \cdot \frac{s\in}{2}\right) \leq D\left(s^2 \lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right)\right) \\ &= \lim_{k\longrightarrow\infty} \sup D\left(s^2 P_b\left(T\left(\alpha_{2m_k}\right), S\left(\alpha_{2n_{k-1}}\right)\right)\right) \\ &\leq \lim_{k\longrightarrow\infty} \sup \mathscr{C}\left(D\left(P_b\left(\alpha_{2m_k}, \alpha_{2n_{k-1}}\right)\right)\right) \\ &= \mathscr{C}\left(D\left(\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k}, \alpha_{2n_{k-1}}\right)\right)\right) \\ &\leq \mathscr{C}\left(D\left(\frac{s\in}{2}\right)\right) < D\left(\frac{s\in}{2}\right). \end{aligned} \tag{44}$$

This is a contradiction to the definition of function D. Thus, $\{\alpha_n\}$ is a Cauchy sequence in (X, d_{P_b}). By Lemma 14 (1), $\{\alpha_n\}$ is a Cauchy sequence in (X, P_b). Since (X, P_b) is a complete Partial b-metric space, so by Lemma 14 (2), (X, d_{P_b}) is also a complete metric space. Thus, there exists $\alpha^* \in X$ such that $\alpha_n \longrightarrow \alpha^*$, that is, $\lim_{n\longrightarrow\infty} d_{P_b}(\alpha_n, \alpha^*) = 0$. By Lemma 14 (3), we get

$$\lim_{n\longrightarrow\infty} P_b(\alpha_n, \alpha^*) = P_b(\alpha^*, \alpha^*) = \lim_{n,m\longrightarrow\infty} P_b(\alpha_n, \alpha_m). \tag{45}$$

Since $\lim_{n,m\longrightarrow\infty} P_b(\alpha_n, \alpha_m) = 0$, so that $P_b(\alpha^*, \alpha^*) = 0$. Thus, $\{\alpha_n\}$ converges to α^* in (X, P_b).

Now, we claim that $T(\alpha^*) = S(\alpha^*) = \alpha^*$. By (40), we have

$$\begin{aligned} &\lim_{n\longrightarrow\infty} P_b(\alpha_{2n+1}, \alpha^*) = 0, \\ &\lim_{n\longrightarrow\infty} P_b(\alpha_{2n+2}, \alpha^*) = 0. \end{aligned} \tag{46}$$

Since T and S are continuous, we have

$$\begin{aligned} &\lim_{n\longrightarrow\infty} P_b(T(\alpha_{2n}), T(\alpha^*)) = 0, \\ &\lim_{n\longrightarrow\infty} P_b(S(\alpha_{2n+1}), S(\alpha^*)) = 0. \end{aligned} \tag{47}$$

By Lemma 16, we have

$$\begin{aligned} \frac{1}{s} P_b(\alpha^*, T(\alpha^*)) &\leq \lim_{n\longrightarrow\infty} \inf P_b(\alpha_{2n+1}, T(\alpha^*)) \\ &= \liminf_{n\longrightarrow\infty} P_b(T(\alpha_{2n}), T(\alpha^*)) = 0. \end{aligned} \tag{48}$$

Thus, $P_b(T(\alpha^*), \alpha^*) = P_b(\alpha^*, \alpha^*) = P_b(T(\alpha^*), T(\alpha^*))$.

This implies $T(\alpha^*) = \alpha^*$. Similar arguments lead us to have $S(\alpha^*) = \alpha^*$.Hence,

$T(\alpha^*) = S(\alpha^*) = \alpha^*$; that is, T and S have a common fixed point $\alpha^* \in X$.

If $\Gamma(\alpha, \beta, \mathfrak{R}) \neq \varnothing$, then we have the following theorem.

Theorem 18. *Let (X, P_b) be a complete partial b-metric space and $\mathfrak{R}$ be a transitive binary relation on X. Let T and S form a $D\mathscr{C}$-contraction. Suppose that $\Gamma(\alpha, \beta, \mathfrak{R}) \neq \varnothing$ and statement of Theorem 17 holds, then the mappings T and S admit a unique common fixed point in X.*

Proof. We have proved the existence of a common fixed point in Theorem 17 . On the contrary, suppose that v and v^* are two distinct common fixed points of T and S in X. Then the class of paths of finite length ℓ in $\mathfrak{R}$ from v to v^* is $\Gamma(v, v^*, \mathfrak{R})$. Let one of the paths be $\{A_0, A_1, A_2, \cdots, A_\ell\}$ in X from v to v^* with

$$A_0 = v, A_\ell = v^*, \left(A_{\jmath}, A_{\jmath+1}\right) \in \mathfrak{R}\,; \jmath = 0, 1, 2, 3 \cdots \cdots \cdots \cdots (\ell - 1). \tag{49}$$

By transitivity of $\mathfrak{R}$, we have

$$(\nu, A_1) \in \mathfrak{R}, (A_1, A_2) \in \mathfrak{R}, \cdots, (A_{\ell-1}, \nu^*) \in \mathfrak{R} \Rightarrow (\nu, \nu^*) \in \mathfrak{R}. \tag{50}$$

It is given that T and S form a $D\mathscr{C}$-contraction, that is,

$$D\left(s^2 P_b(T(\nu), S(\nu^*))\right) \leq \mathscr{C}(D(P_b(\nu, \nu^*). \tag{51}$$

This implies $D(s^2 P_b(\nu, \nu^*)) \leq \mathscr{C}(D(P_b(\nu, \nu^*))) < D(P_b(\nu, \nu^*))$. This is a contradiction to the definition of D. Hence, $\nu = \nu^*$. This shows that ν is a unique common fixed point of T and S.☐ ☐

Remark 19. If the mappings T and S are discontinuous, then we have the following theorem.

Theorem 20. *Let (X, P_b) be an $\mathfrak{R}$ – regular complete partial b-metric space. Let T and S form a $D\mathscr{C}$-contraction. Suppose that $\mathfrak{R}$ is an antisymmetric relation, then T and S admit a common fixed point in X if they meet the conditions (a) and (b):*

(a) *There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$*

(b) *$\mathfrak{R}$ is (T, S)-closed*

Proof. By Theorem 17, we know that $(\alpha_n, \alpha_{n+1}) \in \mathfrak{R}$ and $\alpha_n \longrightarrow \alpha^*$ as $n \longrightarrow \infty$. It is given that (X, P_b) is $\mathfrak{R}$ – regular, so $(\alpha_n, \alpha *) \in \mathfrak{R}$, for all $n \in \mathbb{N}$. There are two possible cases.☐ ☐

Case 1. If the sequence $\{\alpha_n\}$ is constant. Let $\alpha_n = \alpha^*$ for each $n \in \mathbb{N}$ so that $\alpha_{2n} = \alpha^*$ and $T(\alpha^*) = T(\alpha_{2n}) = \alpha_{2n+1.}$ Since (X, P_b) is $\mathfrak{R}$ – regular, so $(\alpha_{2n+1}, \alpha^*) = (T(\alpha^*), \alpha^*) \in \mathfrak{R}$. We know that $(\alpha_{2n}, \alpha_{2n+1}) \in \mathfrak{R}$; thus, $(\alpha^*, T(\alpha^*)) \in \mathfrak{R}$. As $\mathfrak{R}$ is an antisymmetric relation, so $\alpha^* = T(\alpha^*)$, by the same arguments we have $\alpha^* = S(\alpha^*)$ as required.

Case 2. If $\{\alpha_n\}$ is not constant and arbitrary, we claim that $P_b(\alpha^*, S(\alpha^*)) = 0$. Let $P_b(\alpha^*, S(\alpha^*)) > 0$. It is proved in Theorem 17 that $\lim_{i \longrightarrow \infty} \alpha_{2i+1} = \alpha^*$, so there must be an integer $n_0 > 0$, such that

$$P_b(\alpha_{2i+1}, S(\alpha^*)) > 0, P_b(\alpha_{2i}, \alpha^*) < \frac{P_b(\alpha^*, S(\alpha^*))}{2}, \text{for all } i \geq n_0. \tag{52}$$

It is assumed that (X, P_b) is $\mathfrak{R}$ – regular, and by Theorem 17, we know that $\alpha_{2i} \longrightarrow \alpha^*$ as $i \longrightarrow \infty$;thus, $(\alpha_{2i}, \alpha^*) \in \mathfrak{R}$. By contractive condition (2.1), monotonicity of D, and Lemma 16, we have

$$\begin{aligned} D(p_b(\alpha^*, S(\alpha^*))) &\leq D\left(s \lim_{i \longrightarrow \infty} \inf P_b(\alpha_{2i+1}, S(\alpha^*))\right) \\ &\leq D\left(s^2 \lim_{i \longrightarrow \infty} \inf P_b(\alpha_{2i+1}, S(\alpha^*))\right) \\ &= \lim_{i \longrightarrow \infty} \inf D\left(s^2 P_b(T(\alpha_{2i}), S(\alpha^*))\right) \\ &\leq \lim_{i \longrightarrow \infty} \inf \mathscr{C}(D(P_b(\alpha_{2i}, \alpha^*))) \\ &< \lim_{i \longrightarrow \infty} \inf \mathscr{C}\left(D\left(\frac{P_b(\alpha^*, S(\alpha^*))}{2}\right)\right) \\ &< D\left(\frac{P_b(\alpha^*, S(\alpha^*))}{2}\right). \end{aligned} \tag{53}$$

This is a contradiction to the definition of mapping D. Thus, $P_b(\alpha^*, S(\alpha^*)) = 0$. Also, we have the following information:

$$P_b(S(\alpha^*), S(\alpha^*)) = 0 = P_b(\alpha^*, \alpha^*). \tag{54}$$

Thus, $\alpha^* = S(\alpha^*)$. By interchanging roles of S and T, we have $\alpha^* = T(\alpha^*)$.

Hence, $T(\alpha^*) = S(\alpha^*) = \alpha^*$; that is, α^* is a common fixed point of T and S in X.

The following is the most general theorem of this section.

Theorem 21. *Let (X, P_b) be an $\mathfrak{R}$ – regular complete partial b-metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. Let T and S form a $D\mathscr{C}$-contraction. Suppose that $\Gamma(\alpha, \beta, \mathfrak{R}) \neq \varnothing$, and assumptions (a) and (b) in Theorem 17 hold. Then the mappings T and S admit a unique common fixed point in X.*

Proof. See the proofs of Theorems 17, 18, and 20, respectively.☐ ☐

Remark 22.

(1) The results in this section are independent of the observation made in [23], and hence, our results are a real generalization of the related results in literature (see [12, 19–22])

(2) Theorem 21 remains true if $P_b(\alpha, \beta)$ is replaced by $M(\alpha, \beta)$

The following example explains the main results.

Example 5. Let $X = \{ \mathscr{a}_n = n(n+1)/2 : n \in \mathbb{N}\}$.Define the partial b-metric function $P_b : X \times X \longrightarrow [0,\infty)$ by

$$P_b(\alpha, \beta) = (\max \{\alpha, \beta\})^2, \text{for all } \alpha, \beta \in X. \tag{55}$$

Then $(X, P_b, 2)$ is a complete partial b-metric space. Define $D:(0,\infty) \longrightarrow (0,\infty)$ by $D(\mathscr{a}) = \mathscr{a} e^{\mathscr{a}}$ for each $\mathscr{a} > 0$, then $D \in \mathscr{D}$. Let the function $\mathscr{C} : (0,\infty) \longrightarrow (0,\infty)$ be defined by $\mathscr{C}(r) = r/2$ for all $r \in (0,\infty)$. Then $\mathscr{C}$ is continuous comparison. Define the binary relation $\mathfrak{R}$ on X by

$$\mathfrak{R} = \{(a_n, a_m): a_n + a_m \geq 2 \text{ for each } m \geq n\}. \quad (56)$$

Define the mappings $T, S: X \longrightarrow X$ by

$$T(a_n) = \begin{cases} a_1, & n = 1, \\ \dfrac{n(n-1)}{2}, & n \geq 2, \end{cases}$$
$$S(a_m) = \begin{cases} a_1, & m \in \{1, 2\}, \\ \dfrac{(m-1)(m-2)}{2}, & m \geq 3, m \in \mathbb{N}. \end{cases} \quad (57)$$

We observe that there exists $a_1 \in X$ such that $(a_1, T(a_1)) \in \mathfrak{R}$ (since $(a_1 + T(a_1) = 2)$ by definition of $\mathfrak{R}$, so assumption (a) is satisfied in Theorem 17. Let $(a_n, a_m) \in \mathfrak{R}$, then we have $T(a_n) + S(a_m) \geq 2$ for each $m \geq n$, so $(T(a_n), S(a_m)) \in \mathfrak{R}$. Thus, $\mathfrak{R}$ is (T, S)-closed (this verifies assumption (b) of Theorem 17. Also, T, S are continuous (assumption (c) is satisfied). Now, we show that T and S form $D\mathscr{C}$-contraction. It is noted that the mappings T, S do not form Banach contraction in the partial b-metric sense. Indeed,

$$\lim_{n \longrightarrow \infty} \frac{P_b(T(a_n), S(a_1))}{P_b a_n, a_1)} = \lim_{n \longrightarrow \infty} \frac{|n^2 - n|^2}{|n^2 + n|^2} = 1. \quad (58)$$

We noticed that $P_b(T(a_n), S(a_m)) > 0$ for each $m \geq n$. Thus, $(a_n, a_m) \in \mathfrak{J}$.Consider

$$4P_b(T(a_n), S(a_m))e^{4P_b(T(a_n),S(a_m))} \leq \frac{1}{2} P_b(a_n, a_m)e^{P_b(a_n,a_m)}. \quad (59)$$

This implies

$$\frac{8P_b(T(a_n), S(a_m))}{P_b(a_n, a_m)} \leq e^{P_b(a_n,a_m) - 4P_b(T(a_n),S(a_m))}. \quad (60)$$

For $n = 1$ and $m = 2$, the inequality (41) reduces to $e^5 \geq 8/9$. Thus, (41) holds for this case. For $n = 2$ and $m = 3$, the inequality (41) gets the form $e^{32} \geq 2/9$. Similarly, for each $m \geq n$, (41) holds true. Thus, we have

$$D(s^2 P_b(T(\alpha), S(\beta))) \leq \mathscr{C}(D(P_b(\alpha, \beta))), \text{ for all } \alpha, \beta \in X. \quad (61)$$

We note that $a_1 = T(a_1) = S(a_1)$.

3.2. Discussions. In this part of the current section, we state some corollaries which are themselves prominent fixed-point theorems in the literature.

The following corollary generalizes the results presented by Jleli and Samet [6] and al-Sulami et al. [20].

Corollary 23. *Let (X, P_b) be a complete partial b-metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. If the self-mappings T and S defined on (X, P_b) satisfy the following conditions:*

(a) $\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$

(b) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed

(c) Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ – regular

(d) There exists a function $\theta \in \Theta$ and $k \in (0, 1)$, such that for all $\alpha, \beta \in \mathfrak{J}$,

$$\theta(s^2 P_b(T(\alpha), S(\beta))) \leq [\theta(P_b(\alpha, \beta))]^k \quad (62)$$

Then the mappings T and S admit a unique common fixed point.

Proof. Setting $\mathscr{C}(t) = (\ln k)t$ and $D(t) = \theta(s^2 t)$ in Theorem 17 and following the proofs of Theorems 17, 18, and 20 respectively, we obtain the required result.□ □

The following corollary generalizes and improves the results presented by Zada and Sarwar [21] and Wardowski [25].

Corollary 24. *Suppose that the self-mappings T and S defined on the complete partial b-metric space (X, P_b) satisfy the following conditions:*

(a) $\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$

(b) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed

(c) Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ – regular

(d) There exists $F \in \mathscr{F}$ and $\tau > 0$,such that for all $\alpha, \beta \in \mathfrak{J}$,

$$\tau + F(s^2 P_b(T(\alpha), S(\beta))) \leq F(P_b(\alpha, \beta)) \quad (63)$$

If $\mathfrak{R}$ is a transitive and antisymmetric binary relation on X, then the mappings T, S admit a unique common fixed point.

Proof. Setting $\mathscr{C}(t) = e^{-\tau} t$ and $D(t) = e^{s^2 F(t)}$ in Theorem 17 and following the proofs of Theorems 17, 18, and 20, respectively, we obtain the required result.□ □

Corollary 25 (see [21]). *Let (X, P_b) be a complete partial b -metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. If the self-mappings T and S defined on (X, P_b) satisfy the following conditions:*

(a) $\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$

(b) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed

(c) *Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ – regular*

(d) *There exists $F \in \mathscr{F}$ and $\tau > 0$, such that for all $\alpha, \beta \in \mathfrak{I}$*
,

$$\tau + F(sP_b(T(\alpha), S(\beta))) \leq F\left(P_b(\alpha, \beta) + \frac{P_b(\alpha, S(\beta))P_b(\beta, T(\alpha))}{1 + P_b(\alpha, \beta)}\right) \tag{64}$$

Then T and S have a unique common fixed point in X.

Proof. This proof follows the proof of Corollary 24.

The following corollary improves the fixed-point results presented by Geraghty [24].□ □

Corollary 26. *Let (X, P_b) be a complete partial b-metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. If the self-mappings T and S defined on (X, P_b) satisfy the following conditions:*

(a) *$\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$*

(b) *There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed*

(c) *Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ – regular*

(d) *For all $\alpha, \beta \in X$ and $(\alpha, \beta) \in \mathfrak{R}$*

$$s^2 P_b(T(\alpha), S(\beta)) \leq \gamma(P_b(\alpha, \beta))P_b(\alpha, \beta), \tag{65}$$

where $\gamma : [0,\infty) \longrightarrow [0,\infty)$ such that $\lim_{r \longrightarrow t^+} \gamma(r) < 1/s$, for each $t \in (0,\infty)$

Proof. By defining $\mathscr{C}(t) = t\gamma(t)$ and $D(t) = s^2 t$ in Theorem 17 and following the proofs of Theorems 17, 18, and 20, respectively, we obtain the required result.□ □

Remark 27.

(1) For $s = 1$, Theorems 17, 18, and 20 establish criteria for the existence of common fixed points of J_c-contractions in the partial metric spaces [12] and correspondingly for Corollaries 23, 24, 25, and 26

(2) For the zero self-distance ($P_b(\alpha, \beta) = 0$ for all α, β) and for the zero self-distance with $s = 1$, the results stated in Remark 27 (1) hold in the b-metric spaces and metric spaces, respectively

4. Application to the System of Boundary Value Problems

We will apply Theorem 17 to achieve the existence of a common solution to the following system of boundary value problems:

$$-\frac{d^2 v}{dt^2} = \mathscr{H}(t, v(t))\,; t \in \mathscr{I}, v(0) = v(1) = 0, \tag{66}$$

$$-\frac{d^2 w}{dt^2} = \mathscr{K}(t, w(t))\,; t \in \mathscr{I}, w(0) = w(1) = 0, \tag{67}$$

where $\mathscr{I} = [0, 1]$, $C(\mathscr{I})$ represents the set of continuous functions defined on $\mathscr{I}$. The functions $\mathscr{H}, \mathscr{K} : [0, 1] \times C(\mathscr{I}) \longrightarrow \mathbb{R}$ are continuous and nondecreasing according to ordinates. We define the binary relation $\mathfrak{R}$ on $C(\mathscr{I})$ as follows:

$$\mathfrak{R} = \{(v, w) \in C(\mathscr{I}) \times C(\mathscr{I}): v(t) \leq w(t) \forall t \in \mathscr{I}\}. \tag{68}$$

The associated Green function $\mathscr{g} : \mathscr{I} \times \mathscr{I} \longrightarrow \mathscr{I}$ to (66) and (67) can be defined as follows:

$$\mathscr{g}(t, \ell) = \begin{cases} t(1 - \ell) \text{ if } 0 \leq t \leq \ell \leq 1, \\ \ell(1 - t) \text{ if } 0 \leq \ell \leq t \leq 1. \end{cases} \tag{69}$$

Let the mapping $d_* : C(\mathscr{I}) \times C(\mathscr{I}) \longrightarrow [0,\infty)$ be defined by

$$\begin{aligned} d_*(v, w) &= \left\|(v - w)^2\right\|_\infty \\ &= \sup |\, v(t) - w(t)|^2, \forall v, w \in C(\mathscr{I}), t \in \mathscr{I}. \end{aligned} \tag{70}$$

It is claimed that $(C(\mathscr{I}), d_*, 2)$ is a complete b-metric space. By integration, we see that (66) and (67) can be written as $v = S(v)$ and $w = T(w)$, where $S, T : C(\mathscr{I}) \longrightarrow C(\mathscr{I})$ are defined by

$$\begin{aligned} S(v)(t) &= \int_0^1 \mathscr{g}(t, \ell)\mathscr{H}(\ell, v(\ell))d\ell, \\ T(w)(t) &= \int_0^1 \mathscr{g}(t, \ell)\mathscr{K}(\ell, w(\ell))d\ell. \end{aligned} \tag{71}$$

It is remarked that the common solution to (66) and (67) is the common fixed point of the operators S, T. Suppose the following conditions:

(a) $\exists \hbar > 0$ such that for $v(t) \neq w(t) (\forall t)$, we have

$$|\mathscr{H}(t, v(t)) - \mathscr{K}(t, w(t))|^2 \leq 16 e^{-\hbar} |v(t) - w(t)|^2 \forall t \in \mathscr{I} \tag{72}$$

(b) $\exists v_0, w_0 \in C(\mathscr{I})$ such that

$$\begin{aligned} v_0(\ell) &\le \int_0^1 g(\ell,\ell)\,\mathscr{H}(\ell, v_0(\ell))d\ell, \\ w_0(\ell) &\le \int_0^1 g(\ell,\ell)\,\mathscr{K}(\ell, w_0(\ell))d\ell \end{aligned} \tag{73}$$

The following theorem states the conditions under which equations (66) and (67) have a common solution.

Theorem 28. *Let the functions $\mathscr{H}, \mathscr{K} : [0,1] \times C(\mathscr{I}) \longrightarrow \mathbb{R}$ satisfy conditions (a) and (b). Then equations (66) and (67) have a common solution.*

Proof. We will apply Theorem 17 to show the existence of the common solution to (66) and (67). By condition (b), there exists v_0 such that $(v_0, S(v_0)) \in \mathfrak{N}$. Since the functions $\mathscr{H}, \mathscr{K}$ are continuous, so $S, T : C(\mathscr{I}) \longrightarrow C(\mathscr{I})$ defined above are continuous. Since it is given that $\mathscr{H}, \mathscr{K}$ are nondecreasing, thus, $\mathfrak{N}$ is (S, T) closed. To show that the mappings S, T form DC-contraction, we proceed as follows:

$$\begin{aligned} |S(v)(\ell) - T(w)(\ell)|^2 &= \left|\int_0^1 g(\ell,\ell)\,(\mathscr{H}(\ell, v(\ell)) - \mathscr{K}(\ell, w(\ell)))d\ell\right|^2 \\ &\le \left(\int_0^1 g(\ell,\ell)|\mathscr{H}(\ell, v(\ell)) - \mathscr{K}(\ell, w(\ell))|d\ell\right)^2 \\ &\le \left(\int_0^1 g(\ell,\ell)\sqrt{16e^{-k}|v(\ell) - w(\ell)|^2}d\ell\right)^2. \end{aligned} \tag{74}$$

□ □

Since $\left(\sup \int_0^1 g(\ell,\ell)d\ell\right)^2 = 1/64$, for all $\ell \in \mathscr{I}$, thus, taking supremum on both sides of the above inequality, we have

$$s^2 d_*(S(v), T(w)) \le e^{-k} d_*(v, w) \forall v(\cdot), w(\cdot) \in C(\mathscr{I}). \tag{75}$$

Define the b-metric d_* on $C(\mathscr{I})$ by

$$d_*(v, w) = \begin{cases} P_b(v, w), & \text{if } v \ne w, \\ 0, & \text{if } v = w. \end{cases} \tag{76}$$

Inequality (75) can be written as

$$s^2 P_b(S(v), T(w)) \le e^{-k} P_b(v, w) \forall v(\cdot), w(\cdot) \in C(\mathscr{I}). \tag{77}$$

Defining the functions $\mathscr{C}$, F, and D by $\mathscr{C}(\ell) = e^{-k}\ell$, $F(\ell) = \ln \ell$, and $D(\ell) = e^{F(\ell)}$, respectively, for all $\ell \in [0,\infty)$, we have

$$\begin{aligned} &k + F\left(s^2 P_b(S(v), T(w))\right) \le F(P_b(v, w)), \\ &e^{k} \cdot e^{F\left(s^2 P_b(S(v),T(w))\right)} \le e^{F(P_b(v,w))}, \\ &e^{F\left(s^2 P_b(S(v),T(w))\right)} \le e^{-\tau} e^{F(P_b(v,w))}, \\ &D\left(s^2 P_b(S(v), T(w))\right) \le \mathscr{C}(D(P_b(v, w))). \end{aligned} \tag{78}$$

Hence, applying Theorem 17, we say that the boundary value problems (66) and (67) have a common solution in $C(\mathscr{I})$.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work.

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Research Article

A Fixed Point Technique for Solving an Integro-Differential Equation Using Mixed-Monotone Mappings

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The objective of this manuscript is to present new tripled fixed point results for mixed-monotone mappings by a pivotal lemma in the setting of partially ordered complete metric spaces. Our outcomes sum up, enrich, and generalize several results in the current writing. Moreover, some examples have been discussed to strengthen and support our theoretical results. Finally, the theoretical results are applied to study the existence and uniqueness of the solution to an integro-differential equation.

1. Brief Introduction

The fixed point (FP) technique is considered a fundamental pillar and a powerful tool in nonlinear analysis because of its many vital applications in many disciplines such as computer science, engineering, economics, biology, chemistry, and physics. In mathematics, this technique plays a prominent role in the study of statistical models, dynamical systems, game-theoretic models, differential equations, and many others. More clearly, for example, this method is mainly applied in finding the analytical solution to some differential and integral equations, fractional equations, integro-differential equations (IDEs), and functional analysis which facilitates the way to find numerical solutions to such problems. These problems were addressed by Fredholm [1], Rus [2], Hammad and De La Sen [3, 4], Ameer et al. [5], Hussain et al. [6, 7] and Younis et al. [8–10]

In [11], the concepts of the coupled FP and a mixed-monotone mapping were initiated, and some exciting work in partially ordered metric spaces (POMSs) have been discussed by the same authors. This idea was investigated by many authors such as Berinde [12], Choudhury and Maity [13], and Aydi et al. [14]. Moreover, in abstract spaces, this concept has many applications in integral and functional equations; see the papers of Cirić et al. [15], Ding et al. [16], Hammad et al. [17, 18], Luong and Thuan [19], Choudhury and Kundu [20], Agarwal et al. [21], Radenović [22], and Hammad et al. [23].

In 2011, coupled FP notions are generalized to tripled fixed points (TFPs) concepts by Berinde and Borcut [24] in the setting of POMSs. Via the mentioned spaces, Borcut and Berinde [25, 26], Karapnar et al. [27] presented pivotal results about TFP theorems and the applications in this direction introduced by Mustafa et al. [28] and Hammad and De la Sen [29, 30].

Definition 1 [25]. We say that a trio $(\omega, \kappa, \nu) \in \aleph^3$ (where $\aleph \times \aleph \times \aleph = \aleph^3$) is a TFP of a self-mapping $\mho : \aleph^3 \longrightarrow \aleph$ if $\omega = \mho(\omega, \kappa, \nu)$, $\kappa = \mho(\kappa, \nu, \omega)$, and $\nu = \mho(\nu, \omega, \kappa)$.

Definition 2 [26]. A trio $(\omega, \kappa, \nu) \in \aleph^3$ on a nonempty set $\aleph$ is called a tripled coincidence point of the two self-mappings $\mho : \aleph^3 \longrightarrow \aleph$ and $\beth : \aleph \longrightarrow \aleph$ if $\beth\omega = \mho(\omega, \kappa, \nu)$, $\beth\kappa = \mho(\kappa, \nu, \omega)$, and $\beth\nu = \mho(\nu, \omega, \kappa)$.

Definition 3 [26]. Assume that $\aleph \neq \varnothing$ is a set. A trio $(\omega, \kappa, \nu) \in \aleph^3$ is called a tripled common FP of $\mho : \aleph^3 \longrightarrow \aleph$ and $\gimel : \aleph \longrightarrow \aleph$, if $\omega = \gimel\omega = \mho(\omega, \kappa, \nu)$, $\kappa = \gimel\kappa = \mho(\kappa, \nu, \omega)$, and $\nu = \gimel\nu = \mho(\nu, \omega, \kappa)$.

Definition 4 [24]. Assume that $(\aleph, \preccurlyeq)$ is a partially ordered set on the product space $\aleph^3$ defined as follows:

$$\begin{gathered}(\omega, \kappa, \nu), (\omega^*, \kappa^*, \nu^*) \in \aleph^3, \\ (\omega, \kappa, \nu) \preccurlyeq (\omega^*, \kappa^*, \nu^*) \Leftrightarrow \omega \preccurlyeq \omega^*, \\ \kappa^* \preccurlyeq \kappa, \nu \preccurlyeq \nu^*.\end{gathered} \tag{1}$$

Under this partial order, we state the following definitions.

Definition 5 [24]. A mapping $\mho : \aleph^3 \longrightarrow \aleph$ on a partially ordered set $(\aleph, \preccurlyeq)$ has a mixed-monotone property, if for any $\omega, \kappa, \nu \in \aleph$, we have

$$\begin{aligned}&\omega_1, \omega_2 \in \aleph, \omega_1 \preccurlyeq \omega_2 \Rightarrow \mho(\omega_1, \kappa, \nu) \preccurlyeq \mho(\omega_2, \kappa, \nu), \\ &\kappa_1, \kappa_2 \in \aleph, \kappa_1 \preccurlyeq \kappa_2 \Rightarrow \mho(\omega, \kappa_1, \nu) \succcurlyeq \mho(\omega, \kappa_2, \nu), \\ &\nu_1, \nu_2 \in \aleph, \nu_1 \preccurlyeq \nu_2 \Rightarrow \mho(\omega, \kappa, \nu_1) \preccurlyeq \mho(\omega, \kappa, \nu_2).\end{aligned} \tag{2}$$

Definition 6 [14]. A mapping $\mho : \aleph^3 \longrightarrow \aleph$ on a partially ordered set $(\aleph, \preccurlyeq)$ has a mixed $\gimel$-monotone property, where $\gimel : \aleph \longrightarrow \aleph$, if for any $\omega, \kappa, \nu \in \aleph$, we have

$$\begin{aligned}&\omega_1, \omega_2 \in \aleph, \gimel\omega_1 \preccurlyeq \gimel\omega_2 \Rightarrow \mho(\omega_1, \kappa, \nu) \preccurlyeq \mho(\omega_2, \kappa, \nu), \\ &\kappa_1, \kappa_2 \in \aleph, \gimel\kappa_1 \preccurlyeq \gimel\kappa_2 \Rightarrow \mho(\omega, \kappa_1, \nu) \succcurlyeq \mho(\omega, \kappa_2, \nu), \\ &\nu_1, \nu_2 \in \aleph, \gimel\nu_1 \preccurlyeq \gimel\nu_2 \Rightarrow \mho(\omega, \kappa, \nu_1) \preccurlyeq \mho(\omega, \kappa, \nu_2).\end{aligned} \tag{3}$$

Definition 7 [27]. Assume that $\aleph$ is a nonempty set. We say that the mappings $\mho : \aleph^3 \longrightarrow \aleph$ and $\gimel : \aleph \longrightarrow \aleph$ are commutative if $\gimel(\mho(\omega, \kappa, \nu)) = \mho(\gimel\omega, \gimel\kappa, \gimel\nu)$, for all $\omega, \kappa, \nu \in \aleph$.

The first contribution of the TFP for a mixed-monotone mapping in a partially ordered set was presented as follows:

Theorem 8 [24]. *Let $(\aleph, \zeta, \preccurlyeq)$ be a complete partially ordered metric space (CPOMS). Assume that $\mho : \aleph^3 \longrightarrow \aleph$, so that*

(i) $\mho$ has a mixed-monotone property

(ii) Either $\mho$ is continuous or $\aleph$ has the following properties:

(a) $\omega_n \preccurlyeq \omega$, if the nondecreasing sequence $\omega_n \longrightarrow \omega$

(b) $\nu_n \succcurlyeq \nu$, if the nonincreasing sequence $\nu_n \longrightarrow \nu$, for all n

(iii) There is $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ such that

$$\zeta(\mho(\omega, \kappa, \nu), \mho(\omega^*, \kappa^*, \nu^*)) \leq \alpha\zeta(\omega, \omega^*) + \beta\zeta(\kappa, \kappa^*) + \gamma\zeta(\nu, \nu^*), \tag{4}$$

for any $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$, for which $\omega \preccurlyeq \omega^*$, $\kappa^* \preccurlyeq \kappa$, and $\nu \preccurlyeq \nu^*$. If there are $\omega_0, \kappa_0, \nu_0 \in \aleph$ so that $\omega_0 \preccurlyeq \mho(\omega_0, \kappa_0, \nu_0)$, $\kappa_0 \succcurlyeq \mho(\kappa_0, \nu_0, \omega_0)$, and $\nu_0 \preccurlyeq \mho(\nu_0, \omega_0, \kappa_0)$. Then $\mho$ has a TFP.

In this manuscript, we utilize a pivotal lemma to obtain new TFP results for mixed-monotone mappings in CPOMSs. Our results unify, extend, and generalize the papers [19, 31, 32]. Also, some examples and a corollary are given. Later on, we apply the theoretical results to obtain the solution of a system of IDEs as an application.

2. Main Results

We begin this part with the pivotal lemma below.

Lemma 9. *Assume that $(\aleph, \preccurlyeq)$ is a partially ordered set and $\mho : \aleph^3 \longrightarrow \aleph$ and $\gimel : \aleph \longrightarrow \aleph$ are two mappings. Suppose that the following assumptions hold:*

(s_1) There is $\beth_0 \in \aleph$, so that for $\rho_1^n\beth_0, \rho_2^n\beth_0, \rho_3^n\beth_0 \in \aleph$, $\rho_1, \rho_2, \rho_3 \in (0, 1)$, and $\rho_1 < \rho_3 < \rho_2$, $n \in \mathbb{N}$

$$\begin{aligned}\mho(\beth_0, \beth_0, \beth_0) &\succcurlyeq \frac{\gimel(\rho_1^n\beth_0)}{\rho_1^n\beth_0}, \\ \mho(\beth_0, \beth_0, \beth_0) &\preccurlyeq \frac{\gimel(\rho_2^n\beth_0)}{\rho_2^n\beth_0}, \\ \mho(\beth_0, \beth_0, \beth_0) &\succcurlyeq \frac{\gimel(\rho_3^n\beth_0)}{\rho_3^n\beth_0}.\end{aligned} \tag{5}$$

(s_2) There is $\Delta : [0,\infty) \longrightarrow [0,\infty)$ with $\Delta(\rho) \in (\rho_1, \rho_2]$, so that for $\rho_1^n\beth_0, \rho_2^n\beth_0, \rho_3^n\beth_0 \in \aleph$, $n \in \mathbb{N}$

$$\mho(\rho_1^n\beth_0, \rho_2^n\beth_0, \rho_3^n\beth_0) \succcurlyeq \Delta(\rho)\mho\left(\rho_1^{n-1}\beth_0, \rho_2^{n-1}\beth_0, \rho_3^{n-1}\beth_0\right), \tag{6}$$

$$\mho(\rho_2^n\beth_0, \rho_3^n\beth_0, \rho_1^n\beth_0) \preccurlyeq \Delta(\rho)\mho\left(\rho_2^{n-1}\beth_0, \rho_3^{n-1}\beth_0, \rho_1^{n-1}\beth_0\right), \tag{7}$$

$$\mho(\rho_3^n\beth_0, \rho_1^n\beth_0, \rho_2^n\beth_0) \succcurlyeq \Delta(\rho)\mho\left(\rho_3^{n-1}\beth_0, \rho_1^{n-1}\beth_0, \rho_2^{n-1}\beth_0\right). \tag{8}$$

Then, there is $\omega_0, \kappa_0, \nu_0 \in \aleph$, so that $\mho(\omega_0, \kappa_0, \nu_0) \succcurlyeq \gimel(\omega_0)$, $\mho(\kappa_0, \nu_0, \omega_0) \preccurlyeq \gimel(\kappa_0)$, and $\mho(\nu_0, \omega_0, \kappa_0) \succcurlyeq \gimel(\nu_0)$.

Proof. We split the proof into three steps:

(Step 1) Since $\Delta(\rho) > \rho_1$, then there is a nonnegative integer $\ell = ((\ln(1/\rho_1))/(\ln(\Delta(\rho)/\rho_1))) + 1$, such that for $n \geq \ell$, we get $(\Delta(\rho)/\rho_1)^n \geq (1/\rho_1)$, i.e., $(\Delta(\rho))^n \geq \rho_1^{n-1}$. Take $\omega_0 = \rho_1^n\beth_0$, $\kappa_0 = \rho_2^n\beth_0$, and $\nu_0 = \rho_3^n\beth_0$. By Stipulation (s_1) and (5), we have

$$
\begin{aligned}
\mho(\omega_0,\kappa_0,\nu_0) &= \mho(\rho_1^n\text{פ}_0,\rho_2^n\text{פ}_0,\rho_3^n\text{פ}_0)\\
&\succcurlyeq \Delta(\rho)\mho\left(\rho_1^{n-1}\text{פ}_0,\rho_2^{n-1}\text{פ}_0,\rho_3^{n-1}\text{פ}_0\right)\\
&\succcurlyeq (\Delta(\rho))^2\mho\left(\rho_1^{n-2}\text{פ}_0,\rho_2^{n-2}\text{פ}_0,\rho_3^{n-2}\text{פ}_0\right)\succcurlyeq\cdots\\
&\succcurlyeq (\Delta(\rho))^n\mho(\text{פ}_0,\text{פ}_0,\text{פ}_0)\succcurlyeq \rho_1^{n-1}\mho(\text{פ}_0,\text{פ}_0,\text{פ}_0)\\
&\succcurlyeq \beth(\rho_1^n\text{פ}_0)=\beth(\omega_0).
\end{aligned}
\tag{9}
$$

(Step 2) Since $\Delta(\rho)\le\rho_2$, then for each $n\in\mathbb{N}$, we obtain $\rho_2\le(\rho_2/\Delta(\rho))^n$, i.e., $(\Delta(\rho))^n\ge\rho_2^{n-1}$. Apply Stipulation (s_1) and (6), we get

$$
\begin{aligned}
\mho(\kappa_0,\nu_0,\omega_0) &= \mho(\rho_2^n\text{פ}_0,\rho_3^n\text{פ}_0,\rho_1^n\text{פ}_0)\\
&\preccurlyeq \Delta(\rho)\mho\left(\rho_2^{n-1}\text{פ}_0,\rho_3^{n-1}\text{פ}_0,\rho_1^{n-1}\text{פ}_0\right)\\
&\preccurlyeq (\Delta(\rho))^2\mho\left(\rho_2^{n-2}\text{פ}_0,\rho_3^{n-2}\text{פ}_0,\rho_1^{n-2}\text{פ}_0\right)\preccurlyeq\cdots\\
&\preccurlyeq (\Delta(\rho))^n\mho(\text{פ}_0,\text{פ}_0,\text{פ}_0)\preccurlyeq \rho_2^{n-1}\mho(\text{פ}_0,\text{פ}_0,\text{פ}_0)\\
&\preccurlyeq \beth(\rho_2^n\text{פ}_0)=\beth(\kappa_0).
\end{aligned}
\tag{10}
$$

(Step 3) Similar to Step 1, since $\Delta(\rho)>\rho_3$, then we can write $(\Delta(\rho))^n\ge\rho_3^{n-1}$. By Stipulation (s_1) and (7), we have

$$
\begin{aligned}
\mho(\nu_0,\omega_0,\kappa_0) &= \mho(\rho_3^n\text{פ}_0,\rho_1^n\text{פ}_0,\rho_2^n\text{פ}_0)\\
&\succcurlyeq \Delta(\rho)\mho\left(\rho_3^{n-1}\text{פ}_0,\rho_1^{n-1}\text{פ}_0,\rho_2^{n-1}\text{פ}_0\right)\\
&\succcurlyeq (\Delta(\rho))\mho\left(\rho_3^{n-1}\text{פ}_0,\rho_1^{n-1}\text{פ}_0,\rho_2^{n-1}\text{פ}_0\right)\succcurlyeq\cdots\\
&\succcurlyeq (\Delta(\rho))^n\mho(\text{פ}_0,\text{פ}_0,\text{פ}_0)\succcurlyeq \rho_3^{n-1}\mho(\text{פ}_0,\text{פ}_0,\text{פ}_0)\\
&\succcurlyeq \beth(\rho_3^n\text{פ}_0)=\beth(\nu_0).
\end{aligned}
\tag{11}
$$

This completes the proof. □

Remark 10. The results (5)-(7) of Lemma 9 still hold if we reserve the symbols "$\succcurlyeq$" and "$\preccurlyeq$," that is, $\omega_0,\kappa_0,\nu_0\in\aleph$ so that $\mho(\omega_0,\kappa_0,\nu_0)\preccurlyeq\beth(\omega_0)$, $\mho(\kappa_0,\nu_0,\omega_0)\succcurlyeq\beth(\kappa_0)$, and $\mho(\nu_0,\omega_0,\kappa_0)\preccurlyeq\beth(\nu_0)$.

Theorem 11. *Let $(\aleph,\zeta,\preccurlyeq)$ be a CPOMS and ϑ be a zero element in $\aleph$. Assume that $\mho:\aleph^3\longrightarrow\aleph$ is mixed $\beth$-monotone mapping, $\beth:\aleph\longrightarrow\aleph$ is self-mapping, and $\Delta:[0,\infty)\longrightarrow[0,\infty)$, so that $\Delta(\rho)\le\rho$ for any $\rho\ge0$. Suppose that the hypotheses below hold:*

(i) $\mho(\aleph^3)\subset\beth(\aleph)$

(ii) $\beth$ *and* $\mho$ *are continuous and commute*

(iii) $\mho(\omega,\kappa,\nu)$ *verifies stipulations* (s_1) *and* (s_2) *of Lemma 9*

(iv) For any $\omega,\kappa,\nu,\omega^*,\kappa^*,\nu^*\in\aleph$ with $\beth(\omega)\succcurlyeq\beth(\omega^*)$, $\beth(\kappa)\preccurlyeq\beth(\kappa^*)$, $\beth(\nu)\succcurlyeq\beth(\nu^*)$, and we have

$$
\begin{aligned}
&\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*))\\
&\quad\le\Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu)),\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right\}\right).
\end{aligned}
\tag{12}
$$

Then, the following conclusions are fulfilled:

(C_1) For a triplet $(\omega_0,\kappa_0,\nu_0)\in\aleph$, construct three sequences $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ in $\aleph$ verifying

$$
\begin{aligned}
\beth(\omega_n)&=\mho(\omega_{n-1},\kappa_{n-1},\nu_{n-1}),\\
\beth(\kappa_n)&=\mho(\kappa_{n-1},\nu_{n-1},\omega_{n-1}),\\
\beth(\nu_n)&=\mho(\nu_{n-1},\omega_{n-1},\kappa_{n-1}),
\end{aligned}
\tag{13}
$$

for all $n\in\mathbb{N}$. Then, $\beth(\omega_n)\longrightarrow\omega''$, $\beth(\kappa_n)\longrightarrow\kappa''$, and $\beth(\nu_n)\longrightarrow\nu''$, as $n\longrightarrow\infty$

(C_2) $\mho$ and $\beth$ have a tripled coincidence point $(\omega'',\kappa'',\nu'')$. Moreover, assume that $\beth(\omega_0)$, $\beth(\kappa_0)$, and $\beth(\nu_0)$ are comparable, and for each $(\omega,\kappa,\nu)\in\aleph$, $(\beth(\omega_0),\beth(\kappa_0),\beth(\nu_0))$ is comparable to $(\mho(\omega,\kappa,\nu),\mho(\kappa,\nu,\omega),\mho(\nu,\omega,\kappa))$

(C_3) $\mho$ and $\beth$ have a unique common fixed point $\beth(\vartheta)$, that is, $\beth(\vartheta)=\beth(\beth(\vartheta))=\mho(\beth(\vartheta),\beth(\vartheta),\beth(\vartheta))$

Proof. We shall prove Conclusion (C_1). By Condition (iii), there are $\omega_0=\rho_1^n\text{פ}_0$, $\kappa_0=\rho_2^n\text{פ}_0$, and $\nu_0=\rho_3^n\text{פ}_0$, so that $\mho(\omega_0,\kappa_0,\nu_0)\succcurlyeq\beth(\omega_0)$, $\mho(\kappa_0,\nu_0,\omega_0)\preccurlyeq\beth(\kappa_0)$, and $\mho(\nu_0,\omega_0,\kappa_0)\succcurlyeq\beth(\nu_0)$, where $\rho_1,\rho_2,\rho_3\in(0,1)$. Since $\mho(\aleph^3)\subset\beth(\aleph)$, by Condition (i), this yields that there exists $\omega_1,\kappa_1,\nu_1\in\aleph$, so that $\beth(\omega_1)=\mho(\omega_0,\kappa_0,\nu_0)$, $\beth(\kappa_1)=\mho(\kappa_0,\nu_0,\omega_0)$, and $\beth(\nu_1)=\mho(\nu_0,\omega_0,\kappa_0)$. Generally, we can build the three sequences $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ in $\aleph$, so that

$$
\begin{aligned}
\beth(\omega_n)&=\mho(\omega_{n-1},\kappa_{n-1},\nu_{n-1}),\\
\beth(\kappa_n)&=\mho(\kappa_{n-1},\nu_{n-1},\omega_{n-1}),\\
\beth(\nu_n)&=\mho(\nu_{n-1},\omega_{n-1},\kappa_{n-1}),\\
&\text{for } n\in\mathbb{N}.
\end{aligned}
\tag{14}
$$

Since $\mho$ is a mixed $\beth$-monotone, then we get

$$
\begin{aligned}
\beth(\omega_2)&=\mho(\omega_1,\kappa_1,\nu_1)\succcurlyeq\mho(\omega_0,\kappa_0,\nu_0)=\beth(\omega_1),\\
\beth(\kappa_2)&=\mho(\kappa_1,\nu_1,\omega_1)\preccurlyeq\mho(\kappa_0,\nu_0,\omega_0)=\beth(\kappa_1),\\
\beth(\nu_2)&=\mho(\nu_1,\omega_1,\kappa_1)\succcurlyeq\mho(\nu_0,\omega_0,\kappa_0)=\beth(\nu_1).
\end{aligned}
\tag{15}
$$

By induction for $n\in\mathbb{N}$, one can write

$$
\begin{aligned}
\beth(\omega_n)&\succcurlyeq\beth(\omega_{n-1}),\\
\beth(\kappa_n)&\preccurlyeq\beth(\kappa_{n-1}),\\
\beth(\nu_n)&\succcurlyeq\beth(\nu_{n-1}).
\end{aligned}
\tag{16}
$$

It follows from (12) and (13) that for $n \in \mathbb{N}$, we have

$$\begin{aligned}\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) &= \zeta(\mho(\omega_n, \kappa_n, \nu_n), \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1}))\\ &\le \Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega_n), \mho(\omega_n, \kappa_n, \nu_n)),\right.\right.\\ &\qquad \left.\left.\zeta(\beth(\omega_{n-1}), \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1}))\right\}\right)\\ &\le \max\left\{\frac{1}{2}\zeta(\beth(\omega_n), \mho(\omega_n, \kappa_n, \nu_n)),\right.\\ &\qquad \left.\zeta(\beth(\omega_{n-1}), \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1}))\right\}\\ &= \max\left\{\frac{1}{2}\zeta(\beth(\omega_n), \beth(\omega_{n+1})), \zeta(\beth(\omega_{n-1}), \beth(\omega_n))\right\}.\end{aligned} \tag{17}$$

If $\zeta(\beth(\omega_{n-1}), \beth(\omega_n)) \le \zeta(\beth(\omega_n), \beth(\omega_{n+1}))$, then by (16), we find directly that

$$\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \le \frac{1}{2}\zeta(\beth(\omega_{n+1}), \beth(\omega_n)), \tag{18}$$

this is a contradiction. So, we should take $\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) = 0$. It is clear that for each $\varepsilon > 0$ and $\sigma \in \mathbb{N}$, we have

$$\zeta(\beth(\omega_n), \beth(\omega_{n+\sigma})) < \varepsilon. \tag{19}$$

Since $\zeta(\beth(\omega_{n-1}), \beth(\omega_n)) > (1/2)\zeta(\beth(\omega_n), \beth(\omega_{n+1}))$, then by (16), we obtain that

$$\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \le \zeta(\beth(\omega_n), \beth(\omega_{n-1})), \quad \text{for } n \in \mathbb{N}. \tag{20}$$

Set $\nabla_n = \zeta(\beth(\omega_{n+1}), \beth(\omega_n))$. Using (12)–(17), one can obtain

$$\nabla_n \le \nabla_{n-1} \le \nabla_{n-2} \le \cdots \le \nabla_0, \quad \text{for } n \in \mathbb{N} \cup \{0\}, \tag{21}$$

thus, we have

$$\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \le \zeta(\beth(\omega_1), \beth(\omega_0)). \tag{22}$$

According to the proof of Lemma 9 and Condition (ii) (the continuity), there is $\text{Ↄ}_0 \in \aleph$, $\rho_1, \rho_2, \rho_3 \in (0,1)$, so that

$$\begin{aligned}\lim_{n\longrightarrow\infty} \beth(\omega_0) &= \lim_{n\longrightarrow\infty} \beth(\rho_1^n \text{Ↄ}_0) \preccurlyeq \lim_{n\longrightarrow\infty} \mho(\rho_1^n \text{Ↄ}_0, \rho_2^n \text{Ↄ}_0, \rho_3^n \text{Ↄ}_0) = \lim_{n\longrightarrow\infty} \mho(\omega_0, \kappa_0, \nu_0),\\ \lim_{n\longrightarrow\infty} \beth(\kappa_0) &= \lim_{n\longrightarrow\infty} \beth(\rho_2^n \text{Ↄ}_0) \succcurlyeq \lim_{n\longrightarrow\infty} \mho(\rho_2^n \text{Ↄ}_0, \rho_3^n \text{Ↄ}_0, \rho_1^n \text{Ↄ}_0) = \lim_{n\longrightarrow\infty} \mho(\kappa_0, \nu_0, \omega_0),\\ \lim_{n\longrightarrow\infty} \beth(\nu_0) &= \lim_{n\longrightarrow\infty} \beth(\rho_3^n \text{Ↄ}_0) \preccurlyeq \lim_{n\longrightarrow\infty} \mho(\rho_3^n \text{Ↄ}_0, \rho_1^n \text{Ↄ}_0, \rho_2^n \text{Ↄ}_0) = \lim_{n\longrightarrow\infty} \mho(\nu_0, \omega_0, \kappa_0),\end{aligned} \tag{23}$$

that is,

$$\begin{aligned}\beth(\vartheta) &\preccurlyeq \mho(\vartheta, \vartheta, \vartheta),\\ \beth(\vartheta) &\preccurlyeq \mho(\vartheta, \vartheta, \vartheta).\end{aligned} \tag{24}$$

Therefore, $\beth(\vartheta) = \mho(\vartheta, \vartheta, \vartheta)$. By the triangle inequality and (21), one can write for any positive integers m, n with $m > n$, so we have

$$\begin{aligned}\zeta(\beth(\omega_m), \beth(\omega_n)) &\le \zeta(\beth(\omega_m), \beth(\omega_{m-1})) + \zeta(\beth(\omega_{m-1}), \beth(\omega_{m-2}))\\ &\quad + \cdots + \zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \le \zeta(\beth(\omega_1), \beth(\omega_0))\\ &\quad + \zeta(\beth(\omega_1), \beth(\omega_0)) + \cdots + \zeta(\beth(\omega_1), \beth(\omega_0))\\ &= \zeta(\mho(\omega_0, \kappa_0, \nu_0), \beth(\omega_0))\\ &\quad + \zeta(\mho(\omega_0, \kappa_0, \nu_0), \beth(\omega_0)) + \cdots\\ &\quad + \zeta(\mho(\omega_0, \kappa_0, \nu_0), \beth(\omega_0))\\ &= \zeta(\mho(\rho_1^n \text{Ↄ}_0, \rho_2^n \text{Ↄ}_0, \rho_3^n \text{Ↄ}_0), \beth(\rho_1^n \text{Ↄ}_0)) + \cdots\\ &\quad + \zeta(\mho(\rho_1^n \text{Ↄ}_0, \rho_2^n \text{Ↄ}_0, \rho_3^n \text{Ↄ}_0), \beth(\rho_1^n \text{Ↄ}_0)).\end{aligned} \tag{25}$$

It follows that $\lim_{m,n\longrightarrow\infty} \zeta(\beth(\omega_m), \beth(\omega_n)) = 0$. Similarly, we can show that $\lim_{m,n\longrightarrow\infty} \zeta(\beth(\kappa_m), \beth(\kappa_n)) = 0$ and $\lim_{m,n\longrightarrow\infty} \zeta(\beth(\nu_m), \beth(\nu_n)) = 0$. This illustrates that $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ are Cauchy sequences. The completeness of $\aleph$ leads to the conclusion that there are $\omega'', \kappa'', \nu'' \in \aleph$ so that

$$\begin{aligned}\lim_{n\longrightarrow\infty} \beth(\omega_n) &= \omega'',\\ \lim_{n\longrightarrow\infty} \beth(\kappa_n) &= \kappa'',\\ \lim_{n\longrightarrow\infty} \beth(\nu_n) &= \nu''.\end{aligned} \tag{26}$$

Next, we shall show Conclusion (C_2). Since $\beth$ is continuous, then by (25), we get

$$\begin{aligned}\lim_{n\longrightarrow\infty} \beth(\beth(\omega_n)) &= \beth\left(\omega''\right),\\ \lim_{n\longrightarrow\infty} \beth(\beth(\kappa_n)) &= \beth\left(\kappa''\right),\\ \lim_{n\longrightarrow\infty} \beth(\beth(\nu_n)) &= \beth\left(\nu''\right).\end{aligned} \tag{27}$$

Also, by the commutativity of $\beth$ and $\mho$, we have

$$\begin{aligned}\beth(\beth(\omega_n)) &= \beth(\mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1})) = \mho(\beth(\omega_{n-1}), \beth(\kappa_{n-1}), \beth(\nu_{n-1})),\\ \beth(\beth(\kappa_n)) &= \beth(\mho(\kappa_{n-1}, \nu_{n-1}, \omega_{n-1})) = \mho(\beth(\kappa_{n-1}), \beth(\nu_{n-1}), \beth(\omega_{n-1})),\\ \beth(\beth(\nu_n)) &= \beth(\mho(\nu_{n-1}, \omega_{n-1}, \kappa_{n-1})) = \mho(\beth(\nu_{n-1}), \beth(\omega_{n-1}), \beth(\kappa_{n-1})).\end{aligned} \tag{28}$$

By (26) and (27), we deduce that

$$\begin{aligned}\beth\left(\omega''\right) &= \lim_{n\longrightarrow\infty} \beth(\beth(\omega_n)) = \lim_{n\longrightarrow\infty} \mho(\beth(\omega_{n-1}), \beth(\kappa_{n-1}), \beth(\nu_{n-1})) = \mho\left(\omega'', \kappa'', \nu''\right),\\ \beth\left(\kappa''\right) &= \lim_{n\longrightarrow\infty} \beth(\beth(\kappa_n)) = \lim_{n\longrightarrow\infty} \mho(\beth(\kappa_{n-1}), \beth(\nu_{n-1}), \beth(\omega_{n-1})) = \mho\left(\kappa'', \nu'', \omega''\right),\\ \beth\left(\nu''\right) &= \lim_{n\longrightarrow\infty} \beth(\beth(\nu_n)) = \lim_{n\longrightarrow\infty} \mho(\beth(\nu_{n-1}), \beth(\omega_{n-1}), \beth(\kappa_{n-1})) = \mho\left(\nu'', \omega'', \kappa''\right).\end{aligned} \tag{29}$$

Therefore, the trio $(\omega'', \kappa'', \nu'')$ is a tripled coincidence point of $\mho$ and $\beth$.

Finally, to prove Conclusion (C_2), assume that $\beth(\omega_0) \preccurlyeq \beth(\kappa_0)$, $\beth(\kappa_0) \preccurlyeq \beth(\nu_0)$, $\beth(\nu_0) \preccurlyeq \beth(\omega_0)$, and $(\beth(\omega_0), \beth(\kappa_0), \beth(\nu_0)) \preccurlyeq (\mho(\omega, \kappa, \nu), \mho(\kappa, \nu, \omega), \mho(\nu, \omega, \kappa))$, for $(\omega, \kappa, \nu) \in \aleph^3$. Since

$(\omega'', \kappa'', \nu''), (\kappa'', \nu'', \omega''), (\nu'', \omega'', \kappa'') \in \aleph^3$ and by Definition 4, we have

$$(\beth(\omega_0), \beth(\kappa_0), \beth(\nu_0)) \preceq \left(\mho\left(\omega'', \kappa'', \nu''\right), \mho\left(\kappa'', \nu'', \omega''\right), \mho\left(\nu'', \omega'', \kappa''\right)\right), \tag{30}$$

which yields

$$\begin{aligned}
\beth(\omega_0) &\preceq \mho\left(\omega'', \kappa'', \nu''\right) = \beth\left(\omega''\right), \\
\beth(\kappa_0) &\succeq \mho\left(\kappa'', \nu'', \omega''\right) = \beth\left(\kappa''\right), \\
\beth(\nu_0) &\preceq \mho\left(\nu'', \omega'', \kappa''\right) = \beth\left(\nu''\right).
\end{aligned} \tag{31}$$

It follows from (13) and (30) and mixed $\beth$-monotonicity of $\mho$ that

$$\begin{aligned}
\mho(\omega_0, \kappa_0, \nu_0) &\preceq \mho\left(\omega'', \kappa'', \nu''\right), \\
\mho(\kappa_0, \nu_0, \omega_0) &\succeq \mho\left(\kappa'', \nu'', \omega''\right), \\
\mho(\omega_0, \kappa_0, \nu_0) &\preceq \mho\left(\nu'', \omega'', \kappa''\right).
\end{aligned} \tag{32}$$

This leads to

$$\begin{aligned}
\beth(\omega_1) &\preceq \beth\left(\omega''\right), \\
\beth(\kappa_1) &\succeq \beth\left(\kappa''\right), \\
\beth(\nu_1) &\preceq \beth\left(\nu''\right).
\end{aligned} \tag{33}$$

By induction, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
\beth(\omega_n) &\preceq \beth\left(\omega''\right), \\
\beth(\kappa_n) &\succeq \beth\left(\kappa''\right), \\
\beth(\nu_n) &\preceq \beth\left(\nu''\right).
\end{aligned} \tag{34}$$

Applying (12), we observe that

$$\begin{aligned}
\zeta\left(\beth\left(\omega''\right), \beth(\omega_{n+1})\right) &= \zeta\left(\mho\left(\omega'', \kappa'', \nu''\right), \mho(\omega_n, \kappa_n, \nu_n)\right) \\
&\leq \Delta\left(\max\left\{\frac{1}{2}\zeta\left(\beth\left(\omega''\right), \mho\left(\omega'', \kappa'', \nu''\right)\right),\right.\right. \\
&\qquad \left.\left.\zeta(\beth(\omega_n), \mho(\omega_n, \kappa_n, \nu_n))\right\}\right) \\
&= \max\left\{\frac{1}{2}\zeta\left(\beth\left(\omega''\right), \beth\left(\omega''\right)\right),\right. \\
&\qquad \left.\zeta(\beth(\omega_n), \beth(\omega_{n+1}))\right\} \\
&\leq \zeta(\beth(\omega_n), \beth(\omega_{n+1})).
\end{aligned} \tag{35}$$

In the same manner, we have

$$\begin{aligned}
\zeta\left(\beth\left(\kappa''\right), \beth(\kappa_{n+1})\right) &\leq \zeta(\beth(\kappa_n), \beth(\kappa_{n+1})), \\
\zeta\left(\beth\left(\nu''\right), \beth(\nu_{n+1})\right) &\leq \zeta(\beth(\nu_n), \beth(\nu_{n+1})).
\end{aligned} \tag{36}$$

From another direction, since $\beth(\omega_0) \preceq \beth(\kappa_0)$, and the mixed $\beth$-monotonicity of $\mho$, then $\mho(\omega_0, \kappa_0, \nu_0) \preceq \mho(\kappa_0, \nu_0, \omega_0)$, i.e., $\beth(\omega_1) \preceq \beth(\kappa_1)$; similarly, one can obtain $\beth(\kappa_1) \preceq \beth(\nu_1)$ and $\beth(\nu_1) \preceq \beth(\omega_1)$. By the proof of Conclusion (C_1), we have

$$\begin{aligned}
&\beth(\omega_0) \preceq \cdots \preceq \beth(\omega_n) \preceq \beth(\kappa_n) \preceq \cdots \preceq \beth(\kappa_0), \\
&\beth(\kappa_0) \preceq \cdots \preceq \beth(\kappa_n) \preceq \beth(\nu_n) \preceq \cdots \preceq \beth(\nu_0), \\
&\beth(\nu_0) \preceq \cdots \preceq \beth(\nu_n) \preceq \beth(\omega_n) \preceq \cdots \preceq \beth(\omega_0).
\end{aligned} \tag{37}$$

Taking the limit as $n \longrightarrow \infty$, we get

$$\lim_{n\longrightarrow\infty} \beth(\omega_n) = \lim_{n\longrightarrow\infty} \beth(\kappa_n) = \lim_{n\longrightarrow\infty} \beth(\nu_n) = \beth(\vartheta). \tag{38}$$

This implies together with (25) that

$$\omega'' = \kappa'' = \nu'' = \beth(\vartheta). \tag{39}$$

Hence, by (34), (35), (37), and (38), we have

$$\lim_{n\longrightarrow\infty} \zeta\left(\beth\left(\omega''\right), \beth(\omega_{n+1})\right) = \lim_{n\longrightarrow\infty} \zeta\left(\beth\left(\kappa''\right), \beth(\kappa_{n+1})\right) = \lim_{n\longrightarrow\infty} \zeta\left(\beth\left(\nu''\right), \beth(\nu_{n+1})\right) = 0. \tag{40}$$

With the help of (39) and the triangle inequality, we can write

$$\begin{aligned}
\zeta\left(\beth\left(\omega''\right), \beth(\vartheta)\right) &\leq \zeta\left(\beth\left(\omega''\right), \beth(\omega_{n+1})\right) + \zeta(\beth(\omega_{n+1}), \beth(\vartheta)) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \\
\zeta\left(\beth\left(\kappa''\right), \beth(\vartheta)\right) &\leq \zeta\left(\beth\left(\kappa''\right), \beth(\kappa_{n+1})\right) + \zeta(\beth(\kappa_{n+1}), \beth(\vartheta)) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \\
\zeta\left(\beth\left(\nu''\right), \beth(\vartheta)\right) &\leq \zeta\left(\beth\left(\nu''\right), \beth(\nu_{n+1})\right) + \zeta(\beth(\nu_{n+1}), \beth(\vartheta)) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{41}$$

Thus, $\beth(\omega'') = \beth(\kappa'') = \beth(\nu'') = \beth(\vartheta)$. According to (39), we obtain $\beth(\vartheta) = \beth(\beth(\vartheta)) = \mho(\beth(\vartheta), \beth(\vartheta), \beth(\vartheta))$, which proves that $\beth(\vartheta)$ is a common FP of $\mho$ and $\beth$.

To discuss the uniqueness, suppose that $\wp$ is another common FP of $\mho$ and $\beth$, thus $\wp = \beth(\wp) = \mho(\wp, \wp, \wp)$. By the above results and Definition 4, we get

$$(\beth(\omega_0), \beth(\kappa_0), \beth(\nu_0)) \preceq (\mho(\wp,\wp,\wp), \mho(\wp,\wp,\wp), \mho(\wp,\wp,\wp)). \tag{42}$$

That is, for $\omega_0 = \rho_1^n \Game_0$, $\kappa_0 = \rho_2^n \Game_0$, and $\nu_0 = \rho_3^n \Game_0$, where $\rho_1, \rho_2, \rho_3 \in (0, 1)$, we have

$$\begin{aligned}
\beth(\omega_0) &\preceq \mho(\wp,\wp,\wp) \preceq \beth(\kappa_0), \\
\beth(\nu_0) &\preceq \mho(\wp,\wp,\wp) \preceq \beth(\kappa_0).
\end{aligned} \tag{43}$$

This leads to

$$\lim_{n\longrightarrow\infty}\beth(\omega_0)=\lim_{n\longrightarrow\infty}\beth(\kappa_0)=\lim_{n\longrightarrow\infty}\beth(\nu_0)=\beth(\vartheta). \tag{44}$$

Hence, $\mho(\wp,\wp,\wp)=\beth(\vartheta)$, which leads to $\wp=\beth(\vartheta)$. Therefore, $\beth(\vartheta)$ is a unique common FP of $\mho$ and $\beth$. This finishes the proof. □

Examples below verify the assumptions of Theorem 11.

Example 1. Let $\aleph=[0,\infty)$ be endowed with

$$\zeta(\omega,\kappa)=|\kappa-\omega|,\quad \kappa,\omega\in\aleph. \tag{45}$$

Define the order relation ≤ by

$$\begin{gathered}\kappa,\omega\in\aleph,\\ \omega\preccurlyeq\kappa\Leftrightarrow\omega\leq\kappa.\end{gathered} \tag{46}$$

It is obvious that $(\aleph,\zeta,\preccurlyeq)$ is a CPOMS. Define the mappings $\mho:\aleph^3\longrightarrow\aleph$ and $\beth:\aleph\longrightarrow\aleph$ by

$$\begin{gathered}\mho(\omega,\kappa,\nu)=\begin{cases}\dfrac{\omega+\kappa-\nu}{4}, & \text{if } \omega\geq\kappa\geq\nu,\\ 0, & \text{otherwise},\end{cases}\\ \beth(\omega)=2\omega,\end{gathered} \tag{47}$$

respectively. It is clear that $\mho(\aleph^3)\subset\beth(\aleph)$, $\beth$ and $\mho$ are continuous, and $\mho$ have a mixed $\beth$-monotone property.

Now, let us verify Condition (12) of Theorem 11 for all $\omega,\kappa,\nu,\omega^*,\kappa^*,\nu^*\in\aleph$ with $\omega\geq\omega^*$, $\kappa\leq\kappa^*$, and $\nu\geq\nu^*$. Consider that the function $\varDelta:[0,+\infty)\longrightarrow[0,+\infty)$ is given by

$$\varDelta(\max\{\omega,\kappa\})=\frac{\omega+\kappa}{2},\quad \omega,\kappa\in[0,\infty). \tag{48}$$

Now, we consider the cases below:

$(\star_1)$ If $\omega\geq\kappa\geq\nu$ and $\omega^*\geq\kappa^*\geq\nu^*$, then $\omega\geq\omega^*\geq\kappa^*\geq\kappa\geq\nu$, and we have

$$\begin{aligned}&\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*))\\ &\quad+\left|\frac{\omega^*+\kappa^*-\nu^*}{4}\right|\leq\frac{(\omega+\omega^*)+(\kappa+\kappa^*)}{4}\leq\frac{2\omega+2\kappa^*}{4}\\ &\leq\frac{\omega+\omega^*}{2}\leq\frac{1}{2}\left(\left(\frac{7\omega}{8}+\frac{|\nu-\kappa|}{8}\right)+\left(\frac{7\omega^*}{4}+\frac{|\nu^*-\kappa^*|}{4}\right)\right)\\ &=\frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu))+\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right)\\ &=\varDelta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu)),\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right\}\right).\end{aligned} \tag{49}$$

$(\star_2)$ If $\omega\geq\kappa\geq\nu$ and $\omega^*<\kappa^*<\nu^*$, then $\omega\geq\nu\geq\nu^*\geq\omega^*\geq\kappa$, and we have

$$\begin{aligned}&\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*))\\ &=\left|\frac{\omega+\kappa-\nu}{4}-0\right|\leq\left(\frac{\omega+\kappa}{4}\right)\leq\left(\frac{\omega+\omega^*}{4}\right)\\ &\leq\frac{1}{2}\left(\left(\frac{7\omega}{8}+\frac{|\nu-\kappa|}{8}\right)+2\omega^*\right)\\ &=\frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu))+\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right)\\ &=\varDelta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu)),\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right\}\right).\end{aligned} \tag{50}$$

$(\star_3)$ If $\omega<\kappa<\nu$ and $\omega^*\geq\kappa^*\geq\nu^*$, then $\nu\geq\omega\geq\omega^*\geq\kappa^*$, and we have

$$\begin{aligned}&\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*))\\ &=\left|0-\frac{\omega^*+\kappa^*-\nu^*}{4}\right|=\left|\frac{\omega^*+\kappa^*-\nu^*}{4}\right|\leq\frac{\omega^*+\kappa^*}{4}\\ &\leq\frac{\omega+\omega^*}{4}\leq\frac{1}{2}\left(2\omega+\left(\frac{7\omega^*}{4}+\frac{|\nu^*-\kappa^*|}{4}\right)\right)\\ &=\frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu))+\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right)\\ &=\varDelta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu)),\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right\}\right).\end{aligned} \tag{51}$$

$(\star_4)$ If $\omega<\kappa<\nu$ and $\omega^*<\kappa^*<\nu^*$, then we have

$$\begin{aligned}&\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*))=0\leq\frac{1}{2}(\omega+2\omega^*)\\ &=\frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu))+\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right)\\ &=\varDelta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega),\mho(\omega,\kappa,\nu)),\zeta(\beth(\omega^*),\mho(\omega^*,\kappa^*,\nu^*))\right\}\right).\end{aligned} \tag{52}$$

The four cases indicate that the requirements of Theorem 11 are fulfilled and $(0,0,0)$ is a unique TFP.

Example 2. Assume that the first requirements of Example 1 hold with the usual order "≤." Define the mappings $\mho:\aleph^3\longrightarrow\aleph$ and $\beth:\aleph\longrightarrow\aleph$ by

$$\begin{gathered}\mho(\omega,\kappa,\nu)=\frac{\omega+\kappa+\nu}{3},\\ \beth(\omega)=\omega,\end{gathered} \tag{53}$$

respectively. It is obvious that $(\aleph,\zeta,\leq)$ is a CPOMS, $\mho(\aleph^3)\subset\beth(\aleph)$, $\beth$ and $\mho$ are continuous, and $\mho$ have a mixed $\beth$-monotone property.

Now, let us verify Condition (12) of Theorem 11 for all $\omega,\kappa,\nu,\omega^*,\kappa^*,\nu^*\in\aleph$ with $\omega\leq\omega^*$, $\kappa\geq\kappa^*$, and $\nu\leq\nu^*$. Let $\varDelta:[0,+\infty)\longrightarrow[0,+\infty)$ be a function defined by

$$\Delta(\max\{\omega,\kappa\}) = \frac{\omega+\kappa}{2}, \quad \text{for all } \omega,\kappa \in [0,\infty). \tag{54}$$

Now, consider

$$\begin{aligned}
\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*)) &= \left|\frac{\omega+\kappa+\nu}{3} - \frac{\omega^*+\kappa^*+\nu^*}{3}\right| \\
&\le \left|\frac{\omega-\omega^*}{3}\right| + \left|\frac{\kappa-\kappa^*}{3}\right| + \left|\frac{\nu-\nu^*}{3}\right| \le \left|\frac{\kappa-\kappa^*}{3}\right| \le \frac{\kappa+\kappa^*}{2} \\
&\le \frac{1}{2}\left(\left(\frac{2\kappa}{6} + \frac{(\omega+\nu)}{6}\right) + \left(\frac{2\kappa^*}{3} + \frac{(\omega^*+\nu^*)}{3}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\kappa),\mho(\kappa,\nu,\omega)) + \zeta(\beth(\kappa^*),\mho(\kappa^*,\nu^*,\omega^*))\right) \\
&= \Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\kappa),\mho(\kappa,\nu,\omega)),\zeta(\beth(\kappa^*),\mho(\kappa^*,\nu^*,\omega^*))\right\}\right).
\end{aligned} \tag{55}$$

Hence, all conditions of Theorem 11 are satisfied and $\mho$ and $\beth$ have a unique common TFP in $\aleph$ for all $\omega=\kappa=\nu$.

If we set $\beth = I_\aleph$ (the identity mapping on $\aleph$) in Theorem 11, we deduce the result below:

Corollary 12. *Let $(\aleph,\zeta,\preccurlyeq)$ be a CPOMS and ϑ be a zero element in $\aleph$. Assume that $\mho : \aleph^3 \longrightarrow \aleph$ is a mixed-monotone mapping and $\Delta : [0,\infty) \longrightarrow [0,\infty)$ is so that $\Delta(\rho) \le \rho$ for any $\rho \ge 0$. Suppose that the assumptions below are satisfied:*

(i) $\mho$ is continuous

(ii) $\mho(\omega,\kappa,\nu)$ verifies stipulations (s_1) and (s_2) of Lemma 9

(iii) For any $\omega,\kappa,\nu,\omega^*,\kappa^*,\nu^* \in \aleph$ with $\omega \succcurlyeq \omega^*$, $\kappa \preccurlyeq \kappa^*$, and $\nu \succcurlyeq \nu^*$, and we have

$$\begin{aligned}
&\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*)) \\
&\quad\le \Delta\left(\max\left\{\frac{1}{2}\zeta(\omega,\mho(\omega,\kappa,\nu)),\zeta(\omega^*,\mho(\omega^*,\kappa^*,\nu^*))\right\}\right).
\end{aligned} \tag{56}$$

Then, the following conclusions are fulfilled:

(C_1) For a triplet $(\omega_0,\kappa_0,\nu_0) \in \aleph$, construct three sequences $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ in $\aleph$ verifying

$$\begin{aligned}
\omega_n &= \mho(\omega_{n-1},\kappa_{n-1},\nu_{n-1}), \\
\kappa_n &= \mho(\kappa_{n-1},\nu_{n-1},\omega_{n-1}), \\
\nu_n &= \mho(\nu_{n-1},\omega_{n-1},\kappa_{n-1}),
\end{aligned} \tag{57}$$

for $n \in \mathbb{N}$. Then, $\omega_n \longrightarrow \omega''$, $\kappa_n \longrightarrow \kappa''$, and $\nu_n \longrightarrow \nu''$, as $n \longrightarrow \infty$

(C_2) $\mho$ has a TFP $(\omega'',\kappa'',\nu'')$. Moreover, assume that ω_0, κ_0, and ν_0 are comparable and for each $(\omega,\kappa,\nu) \in \aleph$, $(\omega_0,\kappa_0,\nu_0)$ is comparable to $(\mho(\omega,\kappa,\nu),\mho(\kappa,\nu,\omega),\mho(\nu,\omega,\kappa))$

(C_3) $\mho$ has a unique FP ϑ, that is $\vartheta = \mho(\vartheta,\vartheta,\vartheta)$

3. An Application to Integro-Differential Equation

In fact, this part is a fundamental pillar of our paper, where the theoretical results presented in the above section are involved in order to obtain the existence of the solution to an IDE of the following form:

$$\begin{aligned}
\omega(\rho) = \frac{1}{2}\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\Big(&\Big[\aleph_1\big(\sigma,\omega(\sigma),\omega'(\sigma)\big) + \mathfrak{R}_1(\sigma,\omega(\sigma))\Big] \\
&+ \Big[\aleph_2\big(\sigma,\kappa(\sigma),\kappa'(\sigma)\big) + \mathfrak{R}_2(\sigma,\kappa(\sigma))\Big] \\
&+ \Big[\aleph_3\big(\sigma,\nu(\sigma),\nu'(\sigma)\big) + \mathfrak{R}_3(\sigma,\nu(\sigma))\Big]\Big)d\sigma,
\end{aligned} \tag{58}$$

for $\rho \in [\ell_1,\ell_2]$, where $\Theta : [\ell_1,\ell_2] \times [\ell_1,\ell_2] \longrightarrow [1,\infty)$, $\aleph_i : [\ell_1,\ell_2] \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, and $\mathfrak{R}_i : [\ell_1,\ell_2] \times [0,\infty) \longrightarrow [0,\infty)$ (for $i = 1,2,3$) are given continuous functions.

Assume that $\omega(\rho)$, $\kappa(\rho)$, $\nu(\rho)$, $\omega'(\rho)$, $\kappa'(\rho)$, and $\nu'(\rho)$ are nonnegative real continuous functions which are differentiable on $[\ell_1,\ell_2]$, where $\omega'(\rho)$, $\kappa'(\rho)$, and $\nu'(\rho)$ are the first derivative of $\omega(\rho)$, $\kappa(\rho)$, and $\nu(\rho)$ with respect to ρ, respectively.

Also, suppose that Θ is continuously differentiable with respect to its first variable, where $(\partial\Theta/\partial\rho) > 0$.

In order to find the existence solution of Problem (57), we shall derive the hypotheses below:

$(\ddagger_i)$ $\aleph_i : [\ell_1,\ell_2] \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ and $\mathfrak{R}_i : [\ell_1,\ell_2] \times [0,\infty) \longrightarrow [0,\infty)$ (for $i = 1,2,3$) are continuous functions so that

$$\begin{aligned}
\mathfrak{R}_1(\rho,\omega(\rho)) &\le \mathfrak{R}_1(\rho,\omega^*(\rho)), \\
\mathfrak{R}_2(\rho,\kappa(\rho)) &\ge \mathfrak{R}_2(\rho,\kappa^*(\rho)), \\
\mathfrak{R}_3(\rho,\nu(\rho)) &\le \mathfrak{R}_3(\rho,\nu^*(\rho)), \\
\aleph_1\big(\rho,\omega(\rho),\kappa'(\rho)\big) &\le \aleph_1\big(\rho,\omega^*(\rho),\kappa^{*'}(\rho)\big), \\
\aleph_2\big(\rho,\kappa(\rho),\nu'(\rho)\big) &\ge \aleph_2\big(\rho,\kappa^*(\rho),\nu^{*'}(\rho)\big), \\
\aleph_3\big(\rho,\nu(\rho),\omega'(\rho)\big) &\le \aleph_3\big(\rho,\nu^*(\rho),\omega^{*'}(\rho)\big),
\end{aligned} \tag{59}$$

for any fixed $\rho \in [\ell_1,\ell_2]$

$(\ddagger_{ii})$ Define the mappings $\mho$ and $\beth$ by

$$\begin{aligned}
\mho(\omega,\kappa,\nu)(\rho) = \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\Big(&\aleph_1\big(\sigma,\omega(\sigma),\kappa'(\sigma)\big) \\
&+ \aleph_2\big(\sigma,\kappa(\sigma),\nu'(\sigma)\big) \\
&+ \aleph_3\big(\sigma,\nu(\sigma),\omega'(\sigma)\big)\Big)d\sigma,
\end{aligned} \tag{60}$$

$$(\beth\omega)(\rho) = \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)(\mathfrak{R}_1(\sigma,\omega(\sigma)) + \mathfrak{R}_2(\sigma,\kappa(\sigma)) + \mathfrak{R}_3(\sigma,\nu(\sigma)))d\sigma, \tag{61}$$

such that

$$\begin{aligned}
&\mathfrak{R}_1(\rho, \mho(\omega,\kappa,\nu)(\rho)) = \aleph_1\left(\rho, \beth\omega(\rho), (\beth\kappa)'(\rho)\right),\\
&\mathfrak{R}_2(\rho, \mho(\omega,\kappa,\nu)(\rho)) = \aleph_2\left(\rho, \beth\kappa(\rho), (\beth\nu)'(\rho)\right),\\
&\mathfrak{R}_3(\rho, \mho(\omega,\kappa,\nu)(\rho)) = \aleph_3\left(\rho, \beth\nu(\rho), (\beth\omega)'(\rho)\right),
\end{aligned} \tag{62}$$

for any fixed $\rho \in [\ell_1, \ell_2]$

($\ddagger_{iii}$) For any fixed $\rho \in [\ell_1, \ell_2]$, there is $\beta \in (0, (1/2))$ so that

$$\begin{aligned}
&\aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right) \ge \mathfrak{R}_1(\rho, \omega(\rho)),\\
&\aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right) \ge \mathfrak{R}_2(\rho, \kappa(\rho)),\\
&\aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right) \ge \mathfrak{R}_3(\rho, \nu(\rho)),
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right)\right|\\
&\quad \le \frac{\beta}{2}\left(\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \mathfrak{R}_1(\rho, \omega(\rho))\right|\right),\\
&\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right)\right|\\
&\quad \le \frac{\beta}{2}\left(\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \mathfrak{R}_2(\rho, \kappa(\rho))\right|\right),\\
&\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right)\right|\\
&\quad \le \frac{\beta}{2}\left(\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \mathfrak{R}_3(\rho, \nu(\rho))\right|\right),
\end{aligned} \tag{64}$$

($\ddagger_{iv}$) there is $\daleth_0(\rho) \in [0,\infty)$, so that for any fixed $\rho \in [\ell_1, \ell_2]$, $\rho_1, \rho_2, \rho_3 \in (0, 1)$,

$$\begin{cases}
\mathfrak{R}\left(\rho, \rho_1^j \daleth_0(\rho)\right) \le \aleph\left(\rho, \rho_1^j \daleth_0(\rho), \rho_2^j \daleth_0'(\rho)\right) \le \mathfrak{R}\left(\rho, \rho_2^j \daleth_0(\rho)\right),\\
\mathfrak{R}\left(\rho, \rho_3^j \daleth_0(\rho)\right) \le \aleph\left(\rho, \rho_3^j \daleth_0(\rho), \rho_2^j \daleth_0'(\rho)\right) \le \mathfrak{R}\left(\rho, \rho_2^j \daleth_0(\rho)\right).
\end{cases} \tag{65}$$

Now, our main theorem of this section is stated as follows:

Theorem 13. *Under hypotheses ($\ddagger_i$)-($\ddagger_{iv}$), System (57) has a unique solution $\beth(\vartheta)$.*

Proof. The proof is splitting into the following steps:

(St_1) Construct a CPOMS. Assume that $\aleph = (C[\ell_1, \ell_2], \mathbb{R}^+)$ is the set of all nonnegative real continuous functions on $[\ell_1, \ell_2]$, $\vartheta \in \aleph$. Define a metric $\zeta : \aleph \times \aleph \longrightarrow [0,\infty)$ on $\aleph$ by

$$\zeta(\omega, \kappa) = |\omega(\rho) - \kappa(\rho)|_1^2, \quad \forall \omega, \kappa \in \aleph, \rho \in [\ell_1, \ell_2], \tag{66}$$

where $|\omega(\rho)|_1^2 = |\omega(\rho)| + |\omega'(\rho)|$. Define the partial ordered $\preccurlyeq$ by

$$\omega \preccurlyeq \kappa \Leftrightarrow \omega(\rho) \le \kappa(\rho), \quad \forall \rho \in [\ell_1, \ell_2]. \tag{67}$$

Then, a trio $(\aleph, \zeta, \preccurlyeq)$ is a CPOMS if $\omega \preccurlyeq \omega^*$, $\kappa^* \preccurlyeq \kappa$, and $\nu \preccurlyeq \nu^*$, whenever $\omega(\rho) \le \omega^*(\rho)$, $\kappa^*(\rho) \le \kappa(\rho)$, and $\nu(\rho) \le \nu^*(\rho)$, for all $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$

(St_2) Construct the mappings $\mho'$ and $\beth'$. For this purpose, we involve a derivative with respect to ρ on both sides of (59) and (60), respectively, for $\omega, \kappa, \nu \in \aleph$, $\rho \in [\ell_1, \ell_2]$, we get

$$\begin{aligned}
&\mho'(\omega,\kappa,\nu)(\rho)\\
&\quad = \Theta(\rho,\rho)\left(\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) + \aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right)\right.\\
&\qquad \left. + \aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right)\right) + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\\
&\qquad \cdot \left(\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) + \aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right)\right.\\
&\qquad \left. + \aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right)\right) d\sigma,
\end{aligned} \tag{68}$$

$$\begin{aligned}
(\beth\omega)'(\rho) = {} & \Theta(\rho,\rho)(\mathfrak{R}_1(\rho, \omega(\rho)) + \mathfrak{R}_2(\rho, \kappa(\rho))\\
& + \mathfrak{R}_3(\rho, \nu(\rho))) + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)(\mathfrak{R}_1(\sigma, \omega(\sigma))\\
& + \mathfrak{R}_2(\sigma, \kappa(\sigma)) + \mathfrak{R}_3(\sigma, \nu(\sigma)))d\sigma.
\end{aligned} \tag{69}$$

(St_3) Show that $\mho$ is a mixed $\beth$-monotone and $\mho$ and $\beth$ are commuting. If $(\beth\kappa)'(\rho) \ge (\beth\kappa^*)'(\rho)$, then by (68), we have

$$\begin{aligned}
&\Theta(\rho,\rho)((\mathfrak{R}_2(\rho, \kappa(\rho)) - \mathfrak{R}_2(\rho, \kappa^*(\rho))) + (\mathfrak{R}_3(\rho, \nu(\rho))\\
&\quad - \mathfrak{R}_3(\rho, \nu^*(\rho))) + (\mathfrak{R}_1(\rho, \omega(\rho)) - \mathfrak{R}_1(\rho, \omega^*(\rho))))\\
&\quad + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)((\mathfrak{R}_2(\rho, \kappa(\rho)) - \mathfrak{R}_2(\rho, \kappa^*(\rho)))\\
&\quad + (\mathfrak{R}_3(\rho, \nu(\rho)) - \mathfrak{R}_3(\rho, \nu^*(\rho))) + (\mathfrak{R}_1(\rho, \omega(\rho))\\
&\quad - \mathfrak{R}_1(\rho, \omega^*(\rho))))d\sigma \ge 0,
\end{aligned} \tag{70}$$

since $\Theta(\rho,\rho) \ge 1$, and $(\partial/\partial\rho)\Theta(\rho,\sigma) > 0$, then, we get $\mathfrak{R}_2(\rho, \kappa(\rho)) \ge \mathfrak{R}_1(\rho, \kappa^*(\rho))$. Moreover by hypothesis ($\ddagger_i$), for any fixed $\rho \in [\ell_1, \ell_2]$, $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$. If

$$\begin{aligned}
(\beth\omega)(\rho) \le (\beth\omega^*)(\rho), (\beth\kappa)'(\rho) \ge (\beth\kappa^*)'(\rho),\\
(\beth\nu)(\rho) \le (\beth\nu^*)(\rho),
\end{aligned} \tag{71}$$

then, we have

$$\mho\left(\omega, \kappa', \nu\right)(\rho) \preccurlyeq \mho\left(\omega^*, \kappa^{*'}, \nu^*\right)(\rho). \tag{72}$$

Similarly, if

$$\begin{gathered}(\gimel\omega)'(\rho) \le (\gimel\omega^*)'(\rho),\\ (\gimel\kappa)(\rho) \ge (\gimel\kappa^*)(\rho),\\ (\gimel\nu)(\rho) \le (\gimel\nu^*)(\rho) \Rightarrow \mho\left(\omega', \kappa, \nu\right)(\rho)\\ \le \mho\left(\omega'_*, \kappa^*, \nu^*\right)(\rho),\end{gathered} \tag{73}$$

and if

$$\begin{gathered}(\gimel\omega)(\rho) \le (\gimel\omega^*)(\rho),\\ (\gimel\kappa)(\rho) \ge (\gimel\kappa^*)(\rho),\\ (\gimel\nu)'(\rho) \le (\gimel\nu^*)'(\rho) \Rightarrow \mho\left(\omega, \kappa, \nu'\right)(\rho)\\ \le \mho\left(\omega^*, \kappa^*, \nu'_*\right)(\rho),\end{gathered} \tag{74}$$

then this implies that $\mho$ is a mixed $\gimel$-monotone. By the definition of $\mho$ and $\gimel$, we can write

$$\begin{aligned}\mho(\gimel\omega, \gimel\kappa, \gimel\nu)(\rho) &= \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\Big(\aleph_1\left(\sigma, \gimel\omega(\sigma), (\gimel\kappa)'(\sigma)\right)\\ &\quad + \aleph_2\left(\sigma, \gimel\kappa(\sigma), (\gimel\nu)'(\sigma)\right) + \aleph_3\left(\sigma, \gimel\nu(\sigma), (\gimel\omega)'(\sigma)\right)\Big)d\sigma\\ &= \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)(\mathfrak{R}_1(\rho, \mho(\omega,\kappa,\nu)(\rho)) + \mathfrak{R}_2(\rho, \mho(\omega,\kappa,\nu)(\rho))\\ &\quad + \mathfrak{R}_3(\rho, \mho(\omega,\kappa,\nu)(\rho)))d\sigma = \gimel(\mho(\omega,\kappa,\nu)(\rho)).\end{aligned} \tag{75}$$

(St_4) Fulfill Condition (12) of Theorem 11. It follows from (60), (63), and (68) that

$$\begin{aligned}&|\mho(\omega,\kappa,\nu)(\rho) - \mho(\omega^*,\kappa^*,\nu^*)(\rho)|\\ &= \Bigg|\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\Big(\left(\aleph_1\left(\sigma, \omega(\sigma), \kappa'(\sigma)\right) - \aleph_1\left(\sigma, \omega^*(\sigma), \kappa^{*'}(\sigma)\right)\right)\\ &\quad + \left(\aleph_2\left(\sigma, \kappa(\sigma), \nu'(\sigma)\right) - \aleph_2\left(\sigma, \kappa^*(\sigma), \nu^{*'}(\sigma)\right)\right)\\ &\quad + \left(\aleph_3\left(\sigma, \nu(\sigma), \omega'(\sigma)\right) - \aleph_3\left(\sigma, \nu^*(\sigma), \omega^{*'}(\sigma)\right)\right)\Big)d\sigma\Bigg|\\ &\le \Bigg|\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\Big(\frac{\beta}{2}\left(\mathfrak{R}_1(\sigma,\omega(\sigma)) - \aleph_1\left(\sigma, \omega(\sigma), \kappa'(\sigma)\right)\right)\\ &\quad + \frac{\beta}{2}\left(\mathfrak{R}_2(\sigma,\kappa(\sigma)) - \aleph_2\left(\sigma, \kappa(\sigma), \nu'(\sigma)\right)\right)\\ &\quad + \frac{\beta}{2}\left(\mathfrak{R}_3(\sigma,\nu(\sigma)) - \aleph_3\left(\sigma, \nu(\sigma), \omega'(\sigma)\right)\right)\Big)d\sigma\Bigg|\\ &= \frac{\beta}{2}\Bigg|\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\Big(\left(\mathfrak{R}_1(\sigma,\omega(\sigma)) - \aleph_1\left(\sigma, \omega(\sigma), \kappa'(\sigma)\right)\right)\\ &\quad + \left(\mathfrak{R}_2(\sigma,\kappa(\sigma)) - \aleph_2\left(\sigma, \kappa(\sigma), \nu'(\sigma)\right)\right)\\ &\quad + \left(\mathfrak{R}_3(\sigma,\nu(\sigma)) - \aleph_3\left(\sigma, \nu(\sigma), \omega'(\sigma)\right)\right)\Big)d\sigma\Bigg|,\end{aligned} \tag{76}$$

$$\begin{aligned}&\left|\mho'(\omega,\kappa,\nu)(\rho) - \mho'(\omega^*,\kappa^*,\nu^*)(\rho)\right|\\ &= \Bigg|\Theta(\rho,\rho)\Big(\left(\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right)\right)\\ &\quad + \left(\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right)\right)\\ &\quad + \left(\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right)\right)\Big)\\ &\quad + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\Big(\Big(\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right)\\ &\quad - \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right)\Big) + \Big(\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right)\\ &\quad - \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right)\Big) + \Big(\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right)\\ &\quad - \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right)\Big)\Big)d\sigma\Bigg|\\ &\le \Bigg|\Theta(\rho,\rho)\Big(\Big(\frac{\beta}{2}\left(\mathfrak{R}_1(\rho,\omega(\rho)) - \aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right)\right)\\ &\quad + \frac{\beta}{2}\left(\mathfrak{R}_2(\rho,\kappa(\rho)) - \aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right)\right)\\ &\quad + \frac{\beta}{2}\left(\mathfrak{R}_3(\rho,\nu(\rho)) - \aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right)\right)\Big)\Big)\\ &\quad + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\Big(\frac{\beta}{2}\left(\mathfrak{R}_1(\sigma,\omega(\sigma)) - \aleph_1\left(\sigma, \omega(\sigma), \kappa'(\sigma)\right)\right)\\ &\quad + \frac{\beta}{2}\left(\mathfrak{R}_2(\sigma,\kappa(\sigma)) - \aleph_2\left(\sigma, \kappa(\sigma), \nu'(\sigma)\right)\right)\\ &\quad + \frac{\beta}{2}\left(\mathfrak{R}_3(\sigma,\nu(\sigma)) - \aleph_3\left(\sigma, \nu(\sigma), \omega'(\sigma)\right)\right)\Big)d\sigma\Bigg|\\ &= \frac{\beta}{2}\Bigg|\Theta(\rho,\rho)\Big(\Big(\left(\mathfrak{R}_1(\rho,\omega(\rho)) - \aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right)\right)\\ &\quad + \left(\mathfrak{R}_2(\rho,\kappa(\rho)) - \aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right)\right)\\ &\quad + \left(\mathfrak{R}_3(\rho,\nu(\rho)) - \aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right)\right)\Big)\Big)\\ &\quad + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\Big(\left(\mathfrak{R}_1(\sigma,\omega(\sigma)) - \aleph_1\left(\sigma, \omega(\sigma), \kappa'(\sigma)\right)\right)\\ &\quad + \left(\mathfrak{R}_2(\sigma,\kappa(\sigma)) - \aleph_2\left(\sigma, \kappa(\sigma), \nu'(\sigma)\right)\right)\\ &\quad + \left(\mathfrak{R}_3(\sigma,\nu(\sigma)) - \aleph_3\left(\sigma, \nu(\sigma), \omega'(\sigma)\right)\right)\Big)d\sigma\Bigg|.\end{aligned} \tag{77}$$

From (59), (60), (67)–(76) and by definition of $|.|_1^2$, we conclude that

$$\begin{aligned}&|\mho(\omega,\kappa,\nu)(\rho) - \mho(\omega^*,\kappa^*,\nu^*)(\rho)|\\ &= \left|\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\begin{pmatrix}\left(\aleph_1\left(\sigma, \omega(\sigma), \kappa'(\sigma)\right) - \aleph_1\left(\sigma, \omega^*(\sigma), \kappa^{*'}(\sigma)\right)\right)\\ +\left(\aleph_2\left(\sigma, \kappa(\sigma), \nu'(\sigma)\right) - \aleph_2\left(\sigma, \kappa^*(\sigma), \nu^{*'}(\sigma)\right)\right)\\ +\left(\aleph_3\left(\sigma, \nu(\sigma), \omega'(\sigma)\right) - \aleph_3\left(\sigma, \nu^*(\sigma), \omega^{*'}(\sigma)\right)\right)\end{pmatrix}d\sigma\right|\\ &\le \left|\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\begin{pmatrix}\frac{\beta}{2}\left(\mathfrak{R}_1(\sigma,\omega(\sigma)) - \aleph_1\left(\sigma, \omega(\sigma), \kappa'(\sigma)\right)\right)\\ +\frac{\beta}{2}\left(\mathfrak{R}_2(\sigma,\kappa(\sigma)) - \aleph_2\left(\sigma, \kappa(\sigma), \nu'(\sigma)\right)\right)\\ +\frac{\beta}{2}\left(\mathfrak{R}_3(\sigma,\nu(\sigma)) - \aleph_3\left(\sigma, \nu(\sigma), \omega'(\sigma)\right)\right)\end{pmatrix}d\sigma\right|\end{aligned}$$

$$= \frac{\beta}{2}\left|\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\begin{pmatrix}\left(\mathfrak{R}_1(\sigma,\omega(\sigma))-\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)\right)\\ +\left(\mathfrak{R}_2(\sigma,\kappa(\sigma))-\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)\right)\\ +\left(\mathfrak{R}_3(\sigma,\nu(\sigma))-\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\right)\end{pmatrix}d\sigma\right|, \tag{78}$$

that is

$$\begin{aligned}\zeta(\mho(\omega,\kappa,\nu)(\rho),\mho(\omega^*,\kappa^*,\nu^*)(\rho)) &\le \beta\times\frac{1}{2}\zeta((\gimel\omega)(\rho),\mho(\omega,\kappa,\nu)(\rho))\\ &\le \beta\max\left\{\frac{1}{2}\zeta((\gimel\omega)(\rho),\mho(\omega,\kappa,\nu)(\rho)),\right.\\ &\left.\quad\zeta((\gimel\omega^*)(\rho),\mho(\omega^*,\kappa^*,\nu^*)(\rho))\right\}.\end{aligned} \tag{79}$$

Set $\Delta(\omega)=\beta\omega$ for $\omega\in[0,\infty),\beta\in(0,(1/2))$. It is clear that $\Delta(\omega)\le\omega$ for $\omega\ge 0$. Through Inequality (78), Hypothesis (iv) of Theorem 11 is verified. Suppose that $\omega_0(\rho)=\rho_1^n\daleth_0$, $\kappa_0(\rho)=\rho_2^n\daleth_0$, and $\nu_0(\rho)=\rho_3^n\daleth_0$, then by ((65)), we have

$$\begin{aligned}&\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\mathfrak{R}(\rho,\omega_0(\rho))d\sigma\\ &\quad\le\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\aleph\left(\rho,\omega_0(\rho),\kappa_0'(\rho)\right)d\sigma\\ &\quad\le\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\mathfrak{R}(\rho,\kappa_0(\rho))d\sigma,\\ &\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\mathfrak{R}(\rho,\nu_0(\rho))d\sigma\\ &\quad\le\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\aleph\left(\rho,\nu_0(\rho),\kappa_0'(\rho)\right)d\sigma\\ &\quad\le\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\mathfrak{R}(\rho,\kappa_0(\rho))d\sigma,\end{aligned} \tag{80}$$

that is

$$\begin{aligned}&(\gimel\omega_0)(\rho)\le\mho(\omega_0,\kappa_0,\nu_0)(\rho)\le(\gimel\kappa_0)(\rho),\\ &(\gimel\nu_0)(\rho)\le\mho(\omega_0,\kappa_0,\nu_0)(\rho)\le(\gimel\kappa_0)(\rho).\end{aligned} \tag{81}$$

This means $\mho(\omega_0,\kappa_0,\nu_0)\succcurlyeq\gimel(\omega_0)$, $\mho(\kappa_0,\nu_0,\omega_0)\preccurlyeq\gimel(\kappa_0)$, and $\mho(\nu_0,\omega_0,\kappa_0)\succcurlyeq\gimel(\nu_0)$. Thus, all requirements of Theorem 11 are fulfilled. So, there is a unique common FP for the mappings $\mho$ and $\gimel$ in the form of $\gimel(\vartheta)\in\aleph$, so that $\gimel(\vartheta)=(1/2)(\mho(\gimel(\vartheta),\gimel(\vartheta),\gimel(\vartheta))+\gimel(\vartheta))$, which is the unique solution to Problem (57). □

The example below supports our application.

Example 3. Consider the following problem:

$$\omega(\rho)=\frac{1}{32}\int_0^{\rho}\rho e^{\rho\sigma}\left(\left(e^{2\omega}+e^{\omega}\right)+\left(e^{2\kappa}+e^{\kappa}\right)+\left(e^{2\nu}+e^{\nu}\right)\right)d\sigma. \tag{82}$$

Problem (80) is a special form of Problem (57), with the following constraints:

$$\begin{aligned}\aleph_1\left(\rho,\omega(\rho),\omega'(\rho)\right)&=\rho\left(\frac{1}{4}e^{\omega}\right)\left(\frac{1}{4}e^{\omega}\right)=\frac{1}{16}\rho e^{2\omega},\\ \mathfrak{R}_1(\rho,\omega(\rho))&=\frac{1}{4}\left(\frac{1}{4}\rho e^{\omega}\right)=\frac{1}{16}\rho e^{\omega},\\ \aleph_2\left(\rho,\kappa(\rho),\kappa'(\rho)\right)&=\frac{1}{16}\rho e^{2\kappa},\\ \mathfrak{R}_2(\rho,\kappa(\rho))&=\frac{1}{16}\rho e^{\kappa},\\ \aleph_3\left(\rho,\nu(\rho),\nu'(\rho)\right)&=\frac{1}{16}\rho e^{2\nu},\\ \mathfrak{R}_3(\rho,\nu(\rho))&=\frac{1}{16}\rho e^{\nu}.\end{aligned} \tag{83}$$

$\Theta(\rho,\sigma)=e^{\rho\sigma}$, for all $\rho\in[0,1]$ and $\ell_1=0$, $\ell_2=1$.

Here, we considered $\omega(\rho)=(1/4)e^{\omega}$, $\kappa(\rho)=(1/4)e^{\kappa}$, and $\nu(\rho)=(1/4)e^{\nu}$. Thus, $\omega'(\rho)=(1/4)e^{\omega}$, $\kappa'(\rho)=(1/4)e^{\kappa}$, and $\nu'(\rho)=(1/4)e^{\nu}$, respectively. It is clear that the function $\Theta:[0,1]\times[0,1]\longrightarrow[1,\infty)$ is continuously differentiable with respect to its first variable, where $(\partial\Theta/\partial\rho)=\sigma e^{\rho\sigma}>0$. Moreover, $(1/4)e^{\omega}$, $(1/4)e^{\kappa}$, and $(1/4)e^{\nu}$ are nonnegative real continuous functions on $[0,1]$. Also, the two mappings take the following form:

$$\mho(\omega,\kappa,\nu)(\rho)=\frac{1}{16}\int_0^1\rho e^{\rho\sigma}\left(e^{2\omega}+e^{2\kappa}+e^{2\nu}\right)d\sigma, \tag{84}$$

$$(\gimel\omega)(\rho)=\frac{1}{16}\int_0^1\rho e^{\rho\sigma}(e^{\omega}+e^{\kappa}+e^{\nu})d\sigma. \tag{85}$$

Now, we are going to satisfy the hypotheses of Theorem 13.

(1) For $\omega\le\omega^*$, $\kappa\ge\kappa^*$, and $\nu\le\nu^*$, the functions $\aleph_i:[0,1]\times[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ and $\mathfrak{R}_i:[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ (for $i=1,2,3$) are continuous functions, so that

$$\begin{aligned}\mathfrak{R}_1(\rho,\omega(\rho))&=\frac{1}{16}\rho e^{\omega}\le\frac{1}{16}\rho e^{\omega^*}=\mathfrak{R}_1(\rho,\omega^*(\rho)),\\ \mathfrak{R}_2(\rho,\kappa(\rho))&=\frac{1}{16}\rho e^{\kappa}\ge\frac{1}{16}\rho e^{\kappa^*}=\mathfrak{R}_2(\rho,\kappa^*(\rho)),\\ \mathfrak{R}_3(\rho,\nu(\rho))&=\frac{1}{16}\rho e^{\nu}\le\frac{1}{16}\rho e^{\nu^*}=\mathfrak{R}_3(\rho,\nu^*(\rho)).\end{aligned} \tag{86}$$

This obviously leads to

$$\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) = \frac{1}{16}\rho e^{\omega+\kappa} \leq \frac{1}{16}\rho e^{\omega^*+\kappa^*} = \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right). \tag{87}$$

Similarly, one can write

$$\begin{aligned} &\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) \geq \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right), \\ &\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) \leq \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right). \end{aligned} \tag{88}$$

(2) From (82) and (83), and assuming that $(\gimel\kappa)' = (\gimel\kappa)$, we have

$$\begin{aligned} &\aleph_1\left(\rho, \gimel\omega(\rho), (\gimel\kappa)'(\rho)\right) \\ &= \frac{1}{16}\rho e^{(\gimel\omega)} e^{(\gimel\kappa)} = \frac{1}{16}\rho e^{\gimel\omega+\gimel\kappa} \\ &= \frac{1}{16}\rho e^{1/16\int_0^1 \rho e^{\rho\sigma}\left(e^{2\omega}+e^{2\kappa}+e^{2\nu}\right)d\sigma} \\ &= \frac{1}{16}\rho e^{\mho(\omega,\kappa,\nu)} = \mathfrak{R}_1(\rho, \mho(\omega, \kappa, \nu)(\rho)). \end{aligned} \tag{89}$$

Similarly, one can write

$$\begin{aligned} &\mathfrak{R}_2(\rho, \mho(\omega, \kappa, \nu)(\rho)) = \aleph_2\left(\rho, \gimel\kappa(\rho), (\gimel\nu)'(\rho)\right), \\ &\mathfrak{R}_3(\rho, \mho(\omega, \kappa, \nu)(\rho)) = \aleph_3\left(\rho, \gimel\nu(\rho), (\gimel\omega)'(\rho)\right). \end{aligned} \tag{90}$$

(3) For any fixed $\rho \in [0, 1]$, there is $\beta = (1/4)$, so that

$$\aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right) = \frac{1}{16}\rho e^{\omega^*+\kappa^*} \geq \frac{1}{4}\left(\frac{1}{4}\rho e^{\omega}\right) = \mathfrak{R}_1(\rho, \omega(\rho)). \tag{91}$$

In the same manner, we have

$$\begin{aligned} &\aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right) \geq \mathfrak{R}_2(\rho, \kappa(\rho)), \\ &\aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right) \geq \beta\mathfrak{R}_3(\rho, \nu(\rho)). \end{aligned} \tag{92}$$

Also, we have

$$\begin{aligned} &\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right)\right| \\ &= \frac{1}{16}\rho\left|e^{\omega+\kappa} - e^{\omega^*+\kappa^*}\right| \leq \frac{1}{8}\rho\left|e^{\omega+\kappa} - e^{\omega}\right|\left(\text{since } e^{\omega^*+\kappa^*} \geq e^{\omega}\right) \\ &= \frac{\beta}{2}\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \mathfrak{R}_1(\rho, \omega(\rho))\right|. \end{aligned} \tag{93}$$

Similarly, with $\beta = (1/4)$, one can write

$$\begin{aligned} &\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right)\right| \\ &\quad \leq \frac{\beta}{2}\left(\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \mathfrak{R}_2(\rho, \kappa(\rho))\right|\right), \\ &\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right)\right| \\ &\quad \leq \frac{\beta}{2}\left(\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \mathfrak{R}_3(\rho, \nu(\rho))\right|\right). \end{aligned} \tag{94}$$

(4) There is $\beth_0(\rho) = e^{\rho} \in [0,\infty)$, so that for any fixed $\rho = 1 \in [0, 1]$, $\rho_1 = (1/16)$, $\rho_2 = (1/2)$, and $\rho_3 = (1/20)$, we have

$$\begin{aligned} &\frac{1}{64}e \leq \frac{1}{32}e^2 \leq \frac{1}{8}e \Rightarrow \frac{1}{4}\left(\frac{1}{16}e^{\rho}\right) \leq \frac{1}{16} \times \frac{1}{2}e^{\rho} \\ &\quad \leq \frac{1}{4}\left(\frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}\left(\rho, \frac{1}{16}e^{\rho}\right) \leq \aleph\left(\rho, \frac{1}{16}e^{\rho}, \frac{1}{2}e^{\rho}\right) \\ &\quad \leq \mathfrak{R}\left(\rho, \frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}\left(\rho, \rho_1^n\beth_0(\rho)\right) \\ &\quad \leq \aleph\left(\rho, \rho_1^n\beth_0(\rho), \rho_2^n\beth_0'(\rho)\right) \leq \mathfrak{R}\left(\rho, \rho_2^n\beth_0(\rho)\right), \\ &\frac{1}{80}e \leq \frac{1}{40}e^2 \leq \frac{1}{8}e \Rightarrow \frac{1}{4}\left(\frac{1}{20}e^{\rho}\right) \leq \frac{1}{20} \times \frac{1}{2}e^{\rho} \\ &\quad \leq \frac{1}{4}\left(\frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}\left(\rho, \frac{1}{20}e^{\rho}\right) \leq \aleph\left(\rho, \frac{1}{20}e^{\rho}, \frac{1}{2}e^{\rho}\right) \\ &\quad \leq \mathfrak{R}\left(\rho, \frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}\left(\rho, \rho_3^n\beth_0(\rho)\right) \\ &\quad \leq \aleph\left(\rho, \rho_3^n\beth_0(\rho), \rho_2^n\beth_0'(\rho)\right) \leq \mathfrak{R}\left(\rho, \rho_2^n\beth_0(\rho)\right). \end{aligned} \tag{95}$$

Hence, we find that all hypotheses of Theorem 13 are fulfilled. So, the Problem (80) has a unique solution.

4. Conclusion

Due to the multiple applications of fixed point theory, it has become widespread in many scientific disciplines, especially in nonlinear analysis. It contributes significantly to the study of the existence and uniqueness of the solution to many differential and integral equations, as well as integro-differential equations. So, the main objectives of this paper have been to present some new tripled fixed point results for mixed monotone mappings in the framework of partially ordered metric spaces, and these new results have extended to a lot of papers in the literature. Furthermore, to support the proposed results, some illustrative examples have been given and the existence and uniqueness of the solution to the integro-differential equation have been obtained.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Research Article

Fixed Points of Generalized α-Meir-Keeler Contraction Mappings in S_b-Metric Spaces

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In this note, we define Meir-Keeler contraction in S_b-metric spaces. Further, by adding the concept of α-admissible mappings, we define generalized α_s-Meir-Keeler contraction and used it for examining the existence and uniqueness of fixed points. Various results are also given as a consequence of our results.

1. Introduction and Preliminaries

The Banach contraction principle has been an important instrument for the study of a fixed point. It has been widely used in different areas like nonlinear analysis, applied mathematics, economics, and physics. Due to its importance, the result has been generalized in different ways. Meir and Keeler [1] introduce a generalization of the Banach contraction principle. According to them, self-mapping A in a metric space (X, d) is called Meir-Keeler contraction if for an $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \leq d(\theta, \phi) < \varepsilon + \delta(\varepsilon)$ implies $d(A\theta, A\phi) < \varepsilon$ for all $\theta, \phi \in X$. They also state and prove that if a self-mapping A in a complete metric space satisfies Meir-Keeler contraction, then there is a unique fixed point for the mapping A. There are a large number of works on Meir-Keeler contraction of which some of the recent works are mentioned here.

Pourhadi et al. [2] introduced the concept of Meir-Keeler expansive mappings and obtained Krasnosel'skii-type fixed point theorem in Banach spaces. A new fixed point theorem was obtained by Du and Rassias [3] for a Meir-Keeler type condition as a generalization of the Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem, etc., simultaneously.

The idea of S_b-metric space [4–6] is defined by combining definitions of S-metric space [7] and b-metric space [8]. Samet et al. [9] introduced the concept of α-admissible mapping. This concept was further extended to G-metric space, S-metric space, S_b-metric space, etc. (for details, see [10–14]). There are various recent results on Meir-Keeler type and related topics which will be helpful to the readers for more information. Some of them can be seen in [15–21].

In this article, we give the concept of α-admissible and Meir-Keeler contraction in S_b-metric space. The new contraction will be known as generalized α_s-Meir-Keeler contraction. By using generalized α_s-Meir-Keeler contraction mappings, we study the existence and uniqueness of the fixed point in S_b-metric space.

The following definitions and properties will be needed.

Definition 1 (see [8]). In a set $X \neq \phi$, suppose $b \geq 1$ is a real number and $d : X \times X \longrightarrow [0,+\infty)$ is a function satisfying

(1) $d(\theta, \phi) = 0$ if and only if $\theta = \phi$ for all $\theta, \phi \in X$

(2) $d(\theta,\phi)=d(\phi,\theta)$ for all $\theta,\phi\in X$

(3) $d(\theta,\phi)\le b[d(\theta,\psi)+d(\psi,\phi)]$ for all $\theta,\phi,\psi\in X$

Then, d is called b-metric on X and the pair (X,d) is called a b-metric space with coefficient b.

Definition 2 (see [7]). In a set $X\neq\phi$, suppose $S:X\times X\times X\longrightarrow[0,+\infty)$ is a function satisfying

(1) $S(\theta,\phi,\psi)=0$ if and only if $\theta=\phi=\psi$ for all $\theta,\phi,\psi\in X$

(2) $S(\theta,\phi,\psi)\le S(\theta,\theta,\omega)+S(\phi,\phi,\omega)+S(\psi,\psi,\omega)$, for all $\theta,\phi,\psi,\omega\in X$

Then, the pair (X,S) is said to be an S-metric space.

Definition 3 (see [5]). In a set $X\neq\phi$, suppose $b\ge1$ is a real number and $S:X\times X\times X\longrightarrow[0,+\infty)$ is a function satisfying

(i) $S(\theta,\phi,\psi)=0$ if and only if $\theta=\phi=\psi$

(ii) $S(\theta,\phi,\psi)\le b[S(\theta,\theta,\omega)+S(\phi,\phi,\omega)+S(\psi,\psi,\omega)]$, for all θ,ϕ,ψ,ω in X

Here, S is said to be a S_b-metric and (X,S) is said to be a S_b-metric space.

Definition 4 (see [4]). A S_b -metric S satisfying $S(\theta,\theta,\phi)=S(\phi,\phi,\theta)$ for all $\theta,\phi\in X$ is called a symmetric S_b-metric.

Definition 5 (see [5]). In a S_b-metric space (X,S), a sequence $\{\theta_n\}$ is called

(i) convergent if and only if $S(\theta_n,\theta_n,\theta)\longrightarrow0$ as $n\longrightarrow\infty$, where $\theta\in X$ and is expressed as $\lim_{n\longrightarrow\infty}\theta_n=\theta$

(ii) Cauchy if and only if $S(\theta_n,\theta_m,\theta)\longrightarrow0$ as $n,m\longrightarrow\infty$, where $\theta\in X$

(iii) complete S_b-metric space if every Cauchy sequence $\{\theta_n\}$ is convergent and converging to θ in X

We recall some types of α-admissible mappings in a metric space (X,d).

Definition 6 (see [9]). Let $A:X\longrightarrow X$ and $\alpha:X\times X\longrightarrow[0,+\infty)$ be functions. Here, A is said to be α-admissible if $\alpha(\theta,\phi)\ge1$ implies $\alpha(A\theta,A\phi)\ge1$ for all $\theta,\phi\in X$.

Definition 7 (see [15]). Let $A,B:X\longrightarrow X$ and $\alpha:X\times X\longrightarrow 0,+\infty)$ are functions. Here, the pair of mappings (A,B) is said to be *an* α-admissible if $\alpha(\theta,\phi)\ge1$ implies $\alpha(A\theta,B\phi)\ge1$ and $\alpha(B\theta,A\phi)\ge1$ for all $\theta,\phi\in X$.

Definition 8 (see [16]). Let $A:X\longrightarrow X$ and $\alpha:X\times X\longrightarrow[0,+\infty)$ be functions. Here, A is known as triangular α-admissible, if

(i) $\alpha(\theta,\phi)\ge1$, which implies $\alpha(A\theta,A\phi)\ge1,\theta,\phi\in X$

(ii) $\alpha(\theta,\phi)\ge1$, $\alpha(\phi,\psi)\ge1$, which implies $\alpha(\theta,\psi)\ge1$, for all $\theta,\phi,\psi\in X$

Definition 9 (see [15]). Let $A,B:X\longrightarrow X$ and $\alpha:X\times X\longrightarrow[0,+\infty)$ be functions. Here, the pair (A,B) is said to be a triangular α-admissible, if

(i) $\alpha(\theta,\phi)\ge1$, which implies $\alpha(A\theta,B\phi)\ge1$ and $\alpha(B\theta,A\phi)\ge1,\theta,\phi\in X$

(ii) $\alpha(\theta,\phi)\ge1$, $\alpha(\phi,\psi)\ge1$, which implies $\alpha(\theta,\psi)\ge1$, for all $\theta,\phi,\psi\in X$

We extend the concept of α-admissible mapping to be suitable for S-metric and S_b-metric spaces. Here, we consider X as S-metric space or S_b-metric space.

Definition 10. Let $A:X\longrightarrow X$ and $\alpha_s:X\times X\times X\longrightarrow[0,+\infty)$ are functions, then A is called α_s-admissible, if $\theta,\phi,\psi\in X$, $\alpha_s(\theta,\phi,\psi)\ge1$ implies $\alpha_s(A\theta,A\phi,A\psi)\ge1$.

Example 11. Consider $X=[0,+\infty)$ and define $A:X\longrightarrow X$ and $\alpha_s:X\times X\times X\longrightarrow[0,+\infty)$ by $A\theta=4\theta$, for all $\theta,\phi,\psi\in X$, and

$$\alpha_s(\theta,\phi,\psi)=\begin{cases}e^{\psi/\theta\phi}, & \text{if } \theta\ge\phi\ge\psi,\theta,\phi\neq0,\\ 0, & \text{if } \theta<\phi<\psi.\end{cases}\tag{1}$$

Then, A is an α_s-admissible mapping.

Definition 12. Let $A,B:X\longrightarrow X$ and $\alpha_s:X\times X\times X\longrightarrow[0,+\infty)$ be three functions. The pair (A,B) is called α_s-admissible if $\theta,\phi,\psi\in X$ such that $\alpha_s(\theta,\phi,\psi)\ge1$, then we have $\alpha_s(A\theta,A\phi,B\psi))\ge1$ and $\alpha_s(B\theta,B\phi,A\psi)\ge1$.

2. Main Result

Here, we give various types of Meir-Keeler contractive mappings in order to extend various results of Gülyaz et al. [17] in S_b-metric space. Throughout this paper, assume (X,S) is a S_b-metric space, $b\ge1$ is a real number, and $A:X\longrightarrow X$ is a mapping.

Definition 13. An α_s-admissible mapping A in (X,S) is known as α_s-Meir-Keeler contraction mapping of type I, if there esists $\delta>0$ for all $\varepsilon>0$ such that

$$\varepsilon\le S(\theta,\phi,\psi)<\varepsilon+\delta\tag{2}$$

implies

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \quad (3)$$

for all $\theta, \phi, \psi \in X$.

Definition 14. An α_s-admissible mapping A in (X, S) is known as α_s-Meir-Keeler contraction mapping of type II, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq S(\theta, \theta, \phi) < \varepsilon + \delta \quad (4)$$

implies

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) < \frac{\varepsilon}{b} \quad (5)$$

for all $\theta, \phi \in X$.

Remark 15.

(i) If A is an α_s-Meir-Keeler contraction of type I, then

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) \leq \frac{S(\theta, \phi, \psi)}{b}, \quad (6)$$

for all $\theta, \phi, \psi \in X$ and equality is true, when $\theta = \phi = \psi$

(ii) If A is an α_s-Meir-Keeler contraction of type II, then

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) \leq \frac{S(\theta, \theta, \phi)}{b}, \quad (7)$$

for all $\theta, \phi \in X$ and equality is true, when $\theta = \phi$

Now, we introduce the following generalization of Meir-Keeler mappings.

Definition 16. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type AI, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \phi, \psi) < \varepsilon + \delta \quad (8)$$

implies

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \quad (9)$$

where

$$\Lambda(\theta, \phi, \psi) = \max\{S(\theta, \phi, \psi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), S(\psi, \psi, A\psi)\} \quad (10)$$

for all $\theta, \phi, \psi \in X$.

Definition 17. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type AII, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \theta, \phi) < \varepsilon + \delta \quad (11)$$

implies

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) < \frac{\varepsilon}{b}, \quad (12)$$

where

$$\Lambda(\theta, \theta, \phi) = \max\{S(\theta, \theta, \phi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi)\} \quad (13)$$

for all $\theta, \phi \in X$.

Definition 18. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type BI, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \phi, \psi) < \varepsilon + \delta \quad (14)$$

implies

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \quad (15)$$

where

$$\Lambda(\theta, \phi, \psi) = \max\{S(\theta, \phi, \psi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), S(\psi, \psi, A\psi), \frac{1}{4}(S(\theta, \theta, A\phi) + S(\phi, \phi, A\psi) + S(\psi, \psi, A\theta))\} \quad (16)$$

for all $\theta, \phi, \psi \in X$.

Definition 19. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type BII, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \theta, \phi) < \varepsilon + \delta \quad (17)$$

implies

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) < \frac{\varepsilon}{b}, \quad (18)$$

where

$$\Lambda(\theta, \theta, \phi) = \max\left\{S(\theta, \theta, \phi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), \frac{1}{4}(S(\theta, \theta, A\theta) + S(\theta, \theta, A\phi) + S(\phi, \phi, A\theta))\right\} \quad (19)$$

for all $\theta, \phi \in X$.

Remark 20.

(i) Let $A : X \longrightarrow X$ be a generalized α_s-Meir-Keeler contraction of type AI or BI. Then

$$\alpha_s(\theta, \phi, \psi) S(A\theta, A\phi, A\psi) \leq \frac{\Lambda(\theta, \phi, \psi)}{b} \tag{20}$$

for all $\theta, \phi, \psi \in X$, where the equality holds only when $\theta = \phi = \psi$

(ii) Let $A : X \longrightarrow X$ be a generalized α_s-Meir-Keeler contraction of type AII or BII. Then

$$\alpha_s(\theta, \theta, \phi) S(A\theta, A\theta, A\phi) \leq \frac{\Lambda(\theta, \theta, \phi)}{b}, \tag{21}$$

for all $\theta, \phi \in X$, where the equality holds only when $\theta = \phi$

Lemma 21. *Let (X, S) be a S_b-metric space and $\{\theta_n\}$ be a sequence satisfying*

(i) $\theta_m \neq \theta_n$ for all $m \neq n$, $m, n \in \mathbb{N}$

(ii) $S(\theta_n, \theta_n, \theta_{n+1}) \leq 1/b S(\theta_{n-1}, \theta_{n-1}, \theta_n)$, for all $n \in \mathbb{N}$

Then, $\{\theta_n\}$ is a Cauchy sequence in (X, S).

Proof. In order to show that sequence $\{\theta_n\}$ is Cauchy, we must prove that $\lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+k}) = 0$ for any $k \in \mathbb{N}$.

From (ii), we have

$$S(\theta_n, \theta_n, \theta_{n+1}) \leq \frac{1}{b^n} S(\theta_0, \theta_0, \theta_1), \text{for all } n \in \mathbb{N}. \tag{22}$$

Applying limit as $n \longrightarrow \infty$, we get

$$\begin{aligned} 0 &\leq \lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+1}) \\ &\leq \frac{1}{b^n} S(\theta_0, \theta_0, \theta_1) \therefore \lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+1}) = 0. \end{aligned} \tag{23}$$

Now,

$$\begin{aligned} S(\theta_n, \theta_n, \theta_{n+k}) &\leq 2bS(\theta_n, \theta_n, \theta_{n+1}) + b^2 S(\theta_{n+1}, \theta_{n+1}, \theta_{n+k}) \\ &\leq 2bS(\theta_n, \theta_n, \theta_{n+1}) + 2b^3 S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) \\ &\quad + b^4 S(\theta_{n+2}, \theta_{n+2}, \theta_{n+k}) \\ &\leq 2\left\{bS(\theta_n, \theta_n, \theta_{n+1}) + b^3 S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) \right. \\ &\quad \left. + \cdots + b^{2(k-1)+1} S(\theta_{n+k-1}, \theta_{n+k-1}, \theta_{n+k})\right\} \\ &\leq 2\left\{b \frac{S(\theta_0, \theta_0, \theta_1)}{b^n} + b^3 \frac{S(\theta_0, \theta_0, \theta_1)}{b^{n+1}} \right. \\ &\quad \left. + \cdots + b^{2(k-1)+1} \frac{S(\theta_0, \theta_0, \theta_1)}{b^{n+k-1}}\right\} \\ &= \frac{2}{b^{n-1}}\left\{1 + b + \cdots + b^k\right\} S(\theta_0, \theta_0, \theta_1) \\ &= \frac{2\left(b^k - 1\right)}{b^{n-1}(b-1)} S(\theta_0, \theta_0, \theta_1) \therefore \lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+k}) \\ &\leq \lim_{n\longrightarrow\infty} \frac{2\left(b^k - 1\right)}{b^{n-1}(b-1)} S(\theta_0, \theta_0, \theta_1) = 0. \end{aligned} \tag{24}$$

Thus, $\{\theta_n\}$ is a Cauchy sequence in S_b-metric space (X, S).□ □

Theorem 22. *Let (X, S) be a complete S_b-metric space and $\alpha_s : X \times X \times X \longrightarrow [0, +\infty)$ be a mapping. Let $A : X \longrightarrow X$ satisfy the following:*

(i) A is a generalized α_s-Meir-Keeler contraction mapping of type AI

(ii) A is α_s-admissible

(iii) There is $\theta_0 \in X$ so that $\alpha_s(\theta_0, \theta_0, A\theta_0) \geq 1$

(iv) A is continuous

Then, there exists a fixed point of A in X.

Proof. Suppose $\theta_0 \in X$ and $\alpha_s(\theta_0, \theta_0, A\theta_0) \geq 1$. Define the sequence $\{\theta_n\}$ in X as

$$\theta_{n+1} = A\theta_n, \text{for all } n \in \mathbb{N}. \tag{25}$$

Suppose $\theta_{n_0} = \theta_{n_0+1}$ for some $n_0 \in \mathbb{N}$ that is $S(\theta_{n_0}, \theta_{n_0}, \theta_{n_0+1}) = 0$ implies that θ_{n_0} is a fixed point of A. Thus, assume that $\theta_n \neq \theta_{n+1}$ for all $n \geq 0$. From (ii), we have

$$\alpha_s(\theta_0, \theta_0, A\theta_0) = \alpha_s(\theta_0, \theta_0, \theta_1) \geq 1 \tag{26}$$

implies that

$$\alpha_s(A\theta_0, A\theta_0, A\theta_1) = \alpha_s(\theta_1, \theta_1, \theta_2) \geq 1\,; \tag{27}$$

continuing on the same lines, we have

$$\alpha_s(\theta_n, \theta_n, \theta_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}. \tag{28}$$

Here, we need to show that sequence $\{\theta_n\}$ satisfies the conditions of Lemma 21. If we put $\theta = \phi = \theta_n$ and $\psi = \theta_{n+1}$ in (9), for all $\varepsilon > 0$, there is $\delta > 0$ satisfying

$$\varepsilon \leq \Lambda(\theta_n, \theta_n, \theta_{n+1}) < \varepsilon + \delta \tag{29}$$

implies

$$\alpha_s(\theta_n, \theta_n, \theta_{n+1}) S(A\theta_n, A\theta_n, A\theta_{n+1}) < \frac{\varepsilon}{b}, \tag{30}$$

where

$$\Lambda(\theta_n, \theta_n, \theta_{n+1}) = \max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_n, \theta_n, A\theta_n), S(\theta_{n+1}, \theta_{n+1}, A\theta_{n+1})\}. \tag{31}$$

From Remark 20(ii), we have

$$\begin{aligned} S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) &= S(A\theta_n, A\theta_n, A\theta_{n+1}) \\ &\le \alpha_s(\theta_n, \theta_n, \theta_{n+1}) S(A\theta_n, A\theta_n, A\theta_{n+1}) \\ &\le \frac{\Lambda(\theta_n, \theta_n, \theta_{n+1})}{b}; \end{aligned} \tag{32}$$

due to the fact that $\theta_n \neq \theta_{n+1}$, we see that equality does not hold, hence,

$$S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) < \frac{\Lambda(\theta_n, \theta_n, \theta_{n+1})}{b}. \tag{33}$$

If $\Lambda(\theta_n, \theta_n, \theta_{n+1}) = S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})$ for some $n \in \mathbb{N}$, then (11) implies

$$S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) < \frac{S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})}{b} \tag{34}$$

which is not possible. Then, $\Lambda(\theta_n, \theta_n, \theta_{n+1}) = S(\theta_n, \theta_n, \theta_{n+1})$ for all $n \in \mathbb{N}$, so that (11) yields

$$S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) < \frac{S(\theta_n, \theta_n, \theta_{n+1})}{b}, \tag{35}$$

which shows that Lemma 21(ii) is true.□ □

Next, we consider the case for $\theta_n \neq \theta_m$ for all $n \neq m$.

If possible, let $\theta_n = \theta_m$ for some $m, n \in \mathbb{N}$. We have $S(\theta_n, \theta_n, \theta_{n+1}) \ge 0$ for some $n \in \mathbb{N}$. In general, let $m > n + 1$.

We have $S(\theta_m, \theta_m, \theta_{m+1}) = S(\theta_n, \theta_n, \theta_{n+1})$; by inequality (12), we have

$$\begin{aligned} S(\theta_n, \theta_n, \theta_{n+1}) = S(\theta_m, \theta_m, \theta_{m+1}) &< \frac{S(\theta_{m-1}, \theta_{m-1}, \theta_m)}{b} \\ &< \frac{S(\theta_{m-2}, \theta_{m-2}, \theta_{m-1})}{b^2} \cdots < \frac{S(\theta_n, \theta_n, \theta_{n+1})}{b^{m-n}} \end{aligned} \tag{36}$$

becomes impossible. Thus, for some $m \neq n, \lambda_n = \lambda_m$ is not true, and hence, it must be $\theta_n \neq \theta_m$ for all $n \neq m$. So, due to Lemma 21, $\{\theta_n\}$ is a Cauchy sequence in (X, S). Thus, $\{\theta_n\}$ converges to $u \in X$, i.e.,

$$\lim_{n \longrightarrow \infty} S(\theta_n, \theta_n, u) = 0. \tag{37}$$

By the continuity of A, we have

$$\lim_{n \longrightarrow \infty} S(A\theta_n, A\theta_n, Au) = \lim_{n \longrightarrow \infty} S(\theta_{n+1}, \theta_{n+1}, Au) = 0, \tag{38}$$

so $\{\theta_n\}$ converges to Au. Since the limit is unique, $Au = u$.

Theorem 23. *Let (X, S) be a complete S_b-metric space and $\alpha_s : X \times X \times X \longrightarrow [0, +\infty)$ be a mapping. Let $A : X \longrightarrow X$ be a mapping such that*

(v) for a pair of fixed points (θ, ϕ) of A, $\alpha_s(\theta, \theta, \phi) \ge 1$

together with the four conditions of Theorem 22, then A has a unique fixed point in X.

Proof. The existence of a fixed point is proved in Theorem 22. Now, for uniqueness, consider θ and ϕ as two different fixed points of A in X.

By (9), we have

$$\varepsilon \le \Lambda(\theta, \theta, \phi) < \varepsilon + \delta \tag{39}$$

implies

$$\alpha_s(\theta, \theta, \phi) S(A\theta, A\theta, A\phi) < \frac{\varepsilon}{b}, \tag{40}$$

where

$$\begin{aligned} \Lambda(\theta, \theta, \phi) &= \max\{S(\theta, \theta, \phi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi)\} \\ &= \max\{S(\theta, \theta, \phi), 0, 0\} = S(\theta, \theta, \phi). \end{aligned} \tag{41}$$

By (v), $\alpha_s(\theta, \theta, \phi) \ge 1$, since $S(\theta, \theta, \phi) > 0$, Remark 20(ii) becomes

$$\begin{aligned} S(\theta, \theta, \phi) = S(A\theta, A\theta, A\phi) &\le \alpha_s(\theta, \theta, \phi) S(A\theta, A\theta, A\phi) \\ &< \frac{\Lambda(\theta, \theta, \phi)}{b} = \frac{S(\theta, \theta, \phi)}{b}, \end{aligned} \tag{42}$$

which is a contradiction, hence, $S(\theta, \theta, \phi) = 0$, i.e., $\theta = \phi$. Thus, the fixed point of A is unique.□ □

Definition 24. In S_b-metric space (X, S), $\alpha_s : X \times X \times X \longrightarrow [0, +\infty)$ is a mapping. Then, S_b-metric space (X, S) is known as an α-regular if for any sequence $\{\theta_n\}$, $\lim_{n \longrightarrow \infty} S(\theta_n, \theta_n, \theta) = 0$ and $\alpha_s(\theta_n, \theta_n, \theta_{n+1}) \ge 1$ for all $n \in \mathbb{N}$; we have $\alpha_s(\theta_n, \theta_n, \theta) \ge 1$ for all $n \in \mathbb{N}$.

Theorem 25. *In a complete S_b-metric space (X, S), $b \ge 1$ is a parameter and $\alpha_s : X \times X \times X \longrightarrow [0, +\infty)$ is an α_s-admissible mapping. Let $A : X \longrightarrow X$ be a generalized α_s-Meir-Keeler contraction of type AI satisfying the following:*

(i) There is $\theta_0 \in X$ so that $\alpha_s(\theta_0, \theta_0, A\theta_0) \ge 1$

(ii) The S_b-metric space (X, S) is an α-regular, then there exists a fixed point of A in X

(iii) For all pairs of fixed points, $\theta, \phi \in X, \alpha_s(\theta, \theta, \phi) \ge 1$

Then, A has unique fixed point.

Proof. Suppose $\theta_0 \in X$ such that $\alpha_s(\theta_0, \theta_0, A\theta_0) \ge 1$. Define a sequence $\{\theta_n\} \in X$ such that $\theta_{n+1} = A\theta_n$ for all $n \in \mathbb{N}$ and converges to $u \in X$ uniquely.

As (X, S) is α_s-regular, $\alpha_s(\theta_n, \theta_n, u) \ge 1$.

By (9), we have

$$\varepsilon \le \Lambda(\theta_n, \theta_n, u) < \varepsilon + \delta \tag{43}$$

implies

$$\alpha_s(\theta_n, \theta_n, u) S(A\theta_n, A\theta_n, Au) < \frac{\varepsilon}{b}, \tag{44}$$

where

$$\Lambda(\theta_n, \theta_n, u) = \max\{S(\theta_n, \theta_n, u), S(\theta_n, \theta_n, A\theta_n), S(u, u, Au)\}. \tag{45}$$

On the other hand, from Remark 20(ii), we have

$$\begin{aligned} S(\theta_{n+1}, \theta_{n+1}, Au) &= S(A\theta_n, A\theta_n, Au) \\ &\le \alpha_s(\theta_n, \theta_n, u) S(A\theta_n, A\theta_n, Au) \\ &< \frac{\Lambda(\theta_n, \theta_n, u)}{b}. \end{aligned} \tag{46}$$

We have

$$\lim_{n\longrightarrow\infty} S(\theta_{n+1}, \theta_{n+1}, Au) = S(u, u, Au). \tag{47}$$

Also,

$$\begin{aligned} \lim_{n\longrightarrow\infty} \Lambda(\theta_n, \theta_n, u) &= \lim_{n\longrightarrow\infty} \max\{S(\theta_n, \theta_n, u), S(\theta_n, \theta_n, A\theta_n), S(u, u, Au)\} \\ &= S(u, u, Au). \end{aligned} \tag{48}$$

Taking the limit as $n \longrightarrow \infty$ in (46), we have

$$S(u, u, Au) \le \frac{S(u, u, Au)}{b}, \tag{49}$$

which conclude that $S(u, u, Au) = 0$.□ □

The uniqueness part is identical to Theorem 23.
Note: Theorems 22, 23, and 25 will be true for generalized α_s-Meir-Keeler contraction mapping of type BI and BII.

Example 26. Let $X = [0,\infty)$ be endowed with S_b-metric

$$S(x, y, z) = |y + z - 2x|, \text{ where } b = 2. \tag{50}$$

Define $A : X \longrightarrow X$ by

$$Ax = \begin{cases} \dfrac{x^2}{8}, & x \in [0, 1], \\ \dfrac{1}{8} + \log x, & x \in (1,\infty), \end{cases} \tag{51}$$

$$\alpha_s(x, y, z) = \begin{cases} 1, & x, y, z \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, mapping A is α_s-admissible and continuous mapping. Let $x, y \in [0, 1]$, without loss of generality, assume that $x \le y$, then

$$S(Ax, Ax, Ay) = S\left(\frac{x^2}{8}, \frac{x^2}{8}, \frac{y^2}{8}\right) = \left|\frac{y^2}{8} - \frac{x^2}{8}\right|. \tag{52}$$

Now, to calculate

$$\Lambda(x, y, z) = \max\{S(x, y, z), S(x, x, Ax), S(y, y, Ay), S(z, z, Az)\}; \tag{53}$$

in our case, if we take $x = y$, then after a simple calculation, we have

$$\begin{aligned} \Lambda(x, x, y) &= \max\{S(x, x, y), S(x, x, Ax), S(y, y, Ay)\} \\ &= \max\left\{|y - x|, \left|x - \frac{x^2}{8}\right|, \left|y - \frac{y^2}{8}\right|\right\}. \end{aligned} \tag{54}$$

Now, suppose that

$$\varepsilon < \Lambda(x, x, y) = \max\left\{|y - x|, \left|x - \frac{x^2}{8}\right|, \left|y - \frac{y^2}{8}\right|\right\} < \varepsilon + \delta \tag{55}$$

for $\delta = 3\varepsilon$. Now, observe that $\max_{x,y\in[0,1]}\{|y - x|\} = 1$ and $\max_{x,y\in[0,1]}\{|y + x|\} = 2$, and assume that $\varepsilon \in (1/2, 1)$, then we have

$$\frac{|y - x||y + x|}{8} < \frac{2}{8} = \frac{1}{4} < \frac{\varepsilon}{2}, \tag{56}$$

which implies that

$$S(Ax, Ax, Ay) = \left|\frac{y^2}{8} - \frac{x^2}{8}\right| < \frac{\varepsilon}{2}. \tag{57}$$

Since $\alpha_s(x, y, z) = 1$ for all $x, y, z \in [0, 1]$; otherwise, $\alpha_s(x, y, z) = 0$, and we have

$$0 = \alpha_s(x, y, z) S(Ax, Ay, Az) < \frac{\varepsilon}{b} = \frac{\varepsilon}{2}. \tag{58}$$

Hence, A satisfies the conditions of generalized α_s-Meir-Keeler contraction mapping of type AI. Also, all the conditions of Theorem 22 are satisfied, and hence, $x = 0$ is the unique fixed point of mapping A.

3. Consequences

Here, we consider some consequences of Theorems 22, 23, and 25.

Corollary 27. *Let (X, S) be complete S_b-metric space and $A : X \longrightarrow X$ be an α_s-admissible mapping satisfying the following:*

(i) *For all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \le N(\theta, \phi, \psi) < \varepsilon + \delta \tag{59}$$

implies

$$\alpha_s(\theta, \phi, \psi) S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \tag{60}$$

where

$$N(\theta, \phi, \psi) = \max \left\{ S(\theta, \phi, \psi), \frac{1}{3}[S(\theta, \theta, A\theta) + S(\phi, \phi, A\phi) + S(\psi, \psi, A\psi)] \right\} \tag{61}$$

for all $\theta, \phi, \psi \in X$

(ii) *There exists $\theta_0 \in X$ such that $\alpha_s(\theta_0, \theta_0, A\theta_0) \ge 1$*

(iii) *A is continuous or S_b-metric space (X, S) is α_s-regular*

Then, A has a fixed point in X.
Also,

(iv) *for every pair of fixed points (θ, ϕ) of A, if $\alpha_s(\theta, \theta, \phi) \ge 1$*

Then, the fixed point of A is unique in X.

Proof. As $N(\theta, \phi, \psi) \le \Lambda(\theta, \phi, \psi)$ for all $\theta, \phi, \psi \in X$, the proof is obvious from Theorems 22, 23, and 25.□ □

Corollary 28. *Let (X, S) be complete S_b-metric space and $A : X \longrightarrow X$ be an α_s-Meir-Keeler contraction of type I; that is, there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \le S(\theta, \phi, \psi) < \varepsilon + \delta \tag{62}$$

implies

$$\alpha_s(\theta, \phi, \psi) S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \tag{63}$$

for all $\theta, \phi, \psi \in X$.

If A is continuous or S_b-metric space (X, S) is α-regular, then A has a fixed point. Further, with condition (v) in Theorem 23, the fixed point of A is unique.

Proof. The proof follows easily from the relation $S(\theta, \phi, \psi) \le \Lambda(\theta, \phi, \psi)$ for all $\theta, \phi, \psi \in X$.□ □

Taking $\alpha(\theta, \phi, \psi) = 1$ in Theorem 25, we get the following.

Corollary 29. *Let (X, S) be a complete S_b-metric space and $A : X \longrightarrow X$ be a continuous mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \le \Lambda(\theta, \phi, \psi) < \varepsilon + \delta \tag{64}$$

implies

$$S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \tag{65}$$

where

$$\Lambda(\theta, \phi, \psi) = \max \{S(\theta, \phi, \psi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), S(\psi, \psi, A\psi)\} \tag{66}$$

for all $\theta, \phi, \psi \in X$. Then, the fixed point of A is unique.

Corollary 30. *Let (X, S) be a complete S_b-metric space and $A : X \longrightarrow X$ be a continuous mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \le N(\theta, \phi, \psi) < \varepsilon + \delta \tag{67}$$

implies

$$S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \tag{68}$$

where

$$N(\theta, \phi, \psi) = \max \left\{ S(\theta, \phi, \psi), \frac{1}{3}[S(\theta, \theta, A\theta) + S(\phi, \phi, A\phi) + S(\psi, \psi, A\psi)] \right\} \tag{69}$$

for all $\theta, \phi, \psi \in X$. Then, A has a unique fixed point.

The Meir-Keeler contraction can be stated on S_b-metric spaces as follows.

Corollary 31. *Let (X, S) be a complete S_b-metric space and $A : X \longrightarrow X$ be a continuous Meir-Keeler mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \le S(\theta, \phi, \psi) < \varepsilon + \delta \tag{70}$$

becomes

$$S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \tag{71}$$

for all $\theta, \phi, \psi \in X$. Then A has a unique fixed point.

4. Conclusion

In this article, we define Meir-Keeler contraction in S_b-metric spaces using the concept of α-admissible mapping. Further, we define generalized α_s-Meir-Keeler contraction. Using these definitions of contractive mappings, we prove theorems for the existence and uniqueness of fixed points. We show that obtained results are potential generalizations of various results in the literature.

Data Availability

No data is used in this research.

Conflicts of Interest

The authors declare not having competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Fixed Points and Continuity for a Pair of Contractive Maps with Application to Nonlinear Volterra Integral Equations

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In this paper, we have established and proved fixed point theorems for the Boyd-Wong-type contraction in metric spaces. In particular, we have generalized the existing results for a pair of mappings that possess a fixed point but not continuous at the fixed point. We can apply this result for both continuous and discontinuous mappings. We have concluded our results by providing an illustrative example for each case and an application to the existence and uniqueness of a solution of nonlinear Volterra integral equations.

1. Introduction and Preliminaries

Continuity is an ideal property which is sometimes difficult to be fulfilled especially in some daily life applications. For instance, most neural network systems like bar code scanning, speech recognition, and handwritten digit recognition lack the continuity property. These neural network systems are some excellent prototypes for learning discontinuity phenomena. Here, we transform different kinds of day to day real-world phenomena into threshold functions which satisfies our desirable continuity of the weaker form and a new type of contraction to provide a solution to some daily life applications. Therefore, it is desirable to relax continuity assumptions because, in some applications, the function may not be continuous. One can see more literature on the topic [1–5].

In 1969, Kannan [6] proved the following fixed point theorem for discontinuous mapping:

Theorem 1 [6]. *If a self mapping T of a complete metric space (X, d) satisfies the condition*

$$d(Tx, Ty) \le a[d(x, Tx) + d(y, Ty)], \quad 0 \le a < \frac{1}{2}, \tag{1}$$

for each $x, y \in X$, then, T has a unique fixed point.

This theorem gave rise to the question of continuity of contractive mappings at their fixed points. In the Kannan contractive condition, continuity of mapping T was not required for the existence of a fixed point.

In 1971, Ciric [7] (see also [8]) introduced the notion of orbital continuity, which is as follows:

Definition 2 (see [7]). Let (X, d) be a metric space and $T : X \longrightarrow X$ be a self mapping. Then, the set $O(x, T) = \{T^n x : n = 0, 1, 2, 3, \cdots,\}$ is called the orbit of T at x and T is called orbitally continuous if for any sequence $\{x_n \subset O(x, T)\}$, $x_n \longrightarrow z$ implies that $Tx_n \longrightarrow Tz$ as $n \longrightarrow \infty$.

In 2017, Pant and Pant [9] introduced the notion of k-continuity which is as follows:

Definition 3 [9]. A mapping $T : X \longrightarrow X$ is called k-continuous for $k = 1, 2, 3, \cdots$, if $T^k x_n \longrightarrow Tt$ whenever a sequence $\{x_n\}$ is in X such that $T^{k-1} x_n \longrightarrow t$.

Continuity of T implies orbital continuity, but the converse is not true (see [7]).

The following are the examples of k-continuity:

Example 1. Let $X=[0,3]$ be equipped with the usual metric and $T:X\longrightarrow X$ be defined by

$$Tx=\begin{cases}1, & \text{if } 0\le x\le 1,\\ 0, & \text{if } x\in(1,3].\end{cases} \tag{2}$$

Then, $Tx_n\longrightarrow t\Rightarrow T^2x_n\longrightarrow t$, since $Tx_n\longrightarrow t$ implies that $t=0$ or $t=1$ and $T^2x_n=1$ for all n, that is, $T^2x_n\longrightarrow 1=Tt$. Hence, T is 2-continuous. However, T is discontinuous at $x=1$.

Example 2. Let $X=[0,5]$ be equipped with the usual metric and $T:X\longrightarrow X$ be defined by

$$Tx=\begin{cases}1, & \text{if } 0\le x\le 1,\\ 0, & \text{if } 1<x\le 4,\\ \dfrac{x}{4}, & \text{if } 4<x\le 5.\end{cases} \tag{3}$$

Then, $T^2x_n\longrightarrow t\Rightarrow T^3x_n\longrightarrow Tt$, since $T^2x_n\longrightarrow t$ implies $t=0$ or $t=1$ and $T^3x_n=1=Tt$ for each n. Hence, T is 3-continuous. However, $Tx_n\longrightarrow t$ does not imply that $T^2x_n\longrightarrow Tt$, that is, T is not 2-continuous.

Example 3 [9]. Let $X=[0,2]$ be equipped with the usual metric and $T:X\longrightarrow X$ be defined by

$$Tx=\begin{cases}\dfrac{(1+x)}{2}, & \text{if } 0\le x\le 1,\\ 0, & \text{if } x>1.\end{cases} \tag{4}$$

Then, it can be verified that T is 2-continuous but not continuous. Moreover, T^k is discontinuous for each positive integer k. Thus 2-continuity of T does not imply continuity of T^2. In general, k-continuity of T does not imply continuity of T^n.

Example 4 [9]. Let $X=[0,3]\cup(4,5)$ be equipped with the usual metric and $T:X\longrightarrow X$ be defined by

$$Tx=\begin{cases}1, & \text{if } 0\le x\le 1,\\ 0, & \text{if } 1<x\le 3,\\ \dfrac{x}{4}, & \text{if } 4<x<5.\end{cases} \tag{5}$$

Then, T^2 is continuous but T is not 2-continuous. If we consider the sequence $\{x_n\}$ given by $x_n=4+1/n$, then, $Tx_n\longrightarrow 1$ but $T^2x_n\longrightarrow 0\neq T1$. Hence, T is not 2-continuous.

From the above examples, one can see that continuity of T^k and k-continuity of T are independent conditions when $k>1$. It is easy to see that 1-continuity is equivalent to continuity and

$$\text{Continuity}\Rightarrow 2-\text{continuity}\Rightarrow 3-\text{continuity}\Rightarrow\cdots. \tag{6}$$

Definition 4 [10]. Let $\{x_n\}$ be a sequence in a metric space (X,d). Then,

(i) A sequence $\{x_n\}$ in X converges to a point $x\in X$ if and only if $\lim_{n\longrightarrow\infty}d(x,x_n)=\lim_{n\longrightarrow\infty}d(x_n,x_n)$

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if there exists $\varepsilon>0$ such that for all $n,m>N$, we have $d(x_n,x_m)<\varepsilon$ for some integers $N\ge 0$, that is $\lim_{n,m\longrightarrow+\infty}d(x_n,x_m)$ exists and it is finite

(iii) A metric space (X,d) is complete if every Cauchy sequence $\{x_n\}$ converges to a point $x\in X$ such that $d(x,x)=\lim_{n,m\longrightarrow+\infty}d(x_n,x_m)$

Pant and Pant [9] proved the following theorem by employing a new type of $(\varepsilon-\delta)$ condition.

Theorem 5 [9]. *Let f be a self mapping of a complete metric space (X,d) such that*

(i) $d(fx,fy)<\max\{d(x,fx),d(y,fy)\}$, $\max\{d(x,fx),d(y,fy)\}>0$

(ii) Given that $\varepsilon>0$, there exists a $\delta>0$ such that $\varepsilon<\max\{d(x,fx),d(y,fy)\}\le\varepsilon+\delta\Rightarrow d(fx,fy)\le\varepsilon$

If f is k-continuous or f^k is continuous for some $k\ge 1$ or f is orbitally continuous, then, f possesses a unique fixed point.

2. Main Results

Pant and Pant [9] used $\varepsilon-\delta$ and k-continuity property to prove the above fixed point theorem for one self map. In this section, we are extending Theorem 5 for a pair of self maps using $\varepsilon-\delta$ conditions as follows:

Theorem 6. *Let X be a nonempty set and let d to be a metric on X. Let T and S be self mappings of a complete metric space (X,d) satisfying*

(i) $d(Tx,Sy)<M(x,y)$, where

$$M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Sy),\frac{d(x,Sy)+d(y,Tx)}{2}\right\}. \tag{7}$$

(ii) Given that $\varepsilon>0$, there exists a $\delta>0$ such that $\varepsilon<M(x,y)\le\varepsilon+\delta\Rightarrow d(Tx,Sy)\le\varepsilon$

If T and S are k-continuous or T^k and S^k are continuous for some $k\ge 1$ or T and S are orbitally continuous, then, T and S have a unique common fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X for $n=0,1,2,\cdots$, as $x_{2n+1}=Tx_{2n}$ and $x_{2n+2}=Sx_{2n+1}$, for all integers $n\ge 0$.

If $x_{2n} = x_{2n+1}$ for some n, then, $x_{2n} = Tx_{2n}$, that is x_{2n} is a fixed point of T. Similarly, if there exists an integer $N \geq 0$ such that $x_{2N+1} = x_{2N+2}$, then, x_{2n+1} is a fixed point of S. This concludes the proof. □

Otherwise, we suppose that $x_{2n} \neq x_{2n+1}$, for all integers $n \geq 0$. Let $d_{2n} = d(x_{2n}, x_{2n+1})$; obviously, $d_{2n+1} = d(x_{2n+1}, x_{2n+2})$.

Then, by using equation (7) with $x = x_{2n}$ and $y = x_{2n+1}$, we have

$$d(x_{2n+1}, x_{2n+2}) < M(x_{2n}, x_{2n+1}), \tag{8}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \frac{d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})}{2}\right\}, \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2}\right\}. \end{aligned} \tag{9}$$

Since,

$$\begin{aligned} &\frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2} \\ &\quad \leq \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2} \\ &\quad = \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}, \end{aligned} \tag{10}$$

then,

$$M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \tag{11}$$

Thus,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, Sx_{2n+1}) \\ &\leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned} \tag{12}$$

Obviously, if $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, we have a contradiction and so $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1})$. Therefore,

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}), \tag{13}$$

which implies that the sequence $\{d_{2n}\}$ is decreasing to a nonnegative real number, say ε, for all integers $n \geq 0$. We claim that $\varepsilon = 0$. In contrary, suppose that $\varepsilon > 0$. Taking the limit as $n \longrightarrow \infty$ in (13), we obtain

$$\varepsilon < d_{2n} \leq \varepsilon + \delta \Longrightarrow d_{2n+1} \leq \varepsilon, \tag{14}$$

which is a contradiction; hence, we conclude that $\varepsilon = 0$ and

$$\lim_{n\longrightarrow\infty} (d_{2n}) = \lim_{n\longrightarrow\infty} d(x_{2n}, x_{2n+1}) = 0. \tag{15}$$

Now, we need to show that a sequence $\{x_{2n}\}$ in X is a Cauchy sequence. In equation (15), it is sufficient to show that a subsequence $\{x_{2n(r)}\}$ is a Cauchy sequence. On the contrary, we claim that $\{x_{2n(r)}\}$ is not a Cauchy sequence. Therefore, there exists $\varepsilon > 0$ and a sequence of integers $m(r), n(r)$ such that

$$d\left(x_{2n(r)}, x_{2m(r)}\right) \geq \varepsilon, \tag{16}$$

for all $n(r) > m(r) \geq r$ for some $r \geq 0$.

Furthermore, suppose that $m(r)$ is the smallest integer which is chosen in such away that (16) holds so that we have

$$dx_{2n(r)} = d\left(x_{2n(r)}, x_{2m(r)-1}\right) < \varepsilon. \tag{17}$$

Now, for all $n(r) > m(r)$, we have

$$d\left(x_{2n(r)}, x_{2m(r)}\right) \leq d\left(x_{2n(r)}, x_{2m(r)-1}\right) + d\left(x_{2m(r)-1}, x_{2m(r)}\right). \tag{18}$$

As $r \longrightarrow \infty$ in (18) and considering (15) and (17), we see that

$$d\left(x_{2n(r)}, x_{2m(r)}\right) \longrightarrow \varepsilon. \tag{19}$$

By similar computations, we see that,

$$dx_{2n(r)-1} = d\left(x_{2n(r)-1}, x_{2m(r)-1}\right) \longrightarrow \varepsilon. \tag{20}$$

To show it, we shall prove that $M(x_{2n(r)-1}, x_{2m(r)-1}) \leq \varepsilon + \delta$. Then, by using equation (7) with $x = x_{2n(r)-1}$ and $y = x_{2m(r)-1}$, we have

$$\begin{aligned} d\left(x_{2n(r)}, x_{2m(r)}\right) &= d\left(Tx_{2n(r)-1}, Sx_{2m(r)-1}\right) \\ &\leq M\left(x_{2n(r)-1}, x_{2m(r)-1}\right), \end{aligned} \tag{21}$$

where

$$\begin{aligned} M\left(x_{2n(r)-1}, x_{2m(r)-1}\right) &= \max\left\{ d\left(x_{2n(r)-1}, x_{2m(r)-1}\right),\right. \\ &\quad d\left(x_{2n(r)-1}, Tx_{2n(r)-1}\right), d\left(x_{2m(r)-1}, Sx_{2m(r)-1}\right), \\ &\quad \left.\frac{d\left(x_{2n(r)-1}, Sx_{2m(r)-1}\right) + d\left(x_{2m(r)-1}, Tx_{2n(r)-1}\right)}{2}\right\} \\ &= \max\left\{ d\left(x_{2n(r)-1}, x_{2m(r)-1}\right),\right. \\ &\quad d\left(x_{2n(r)-1}, x_{2n(r)}\right), d\left(x_{2m(r)-1}, x_{2m(r)}\right), \\ &\quad \left.\frac{d\left(x_{2n(r)-1}, x_{2m(r)}\right) + d\left(x_{2m(r)-1}, x_{2n(r)}\right)}{2}\right\} \\ &= d\left(x_{2n(r)-1}, x_{2m(r)-1}\right). \end{aligned} \tag{22}$$

As $r \longrightarrow \infty$ in (22) and considering (19) and (20), then, (21) becomes

$$\varepsilon < d_{2n(r)-1} \leq \varepsilon + \delta \Longrightarrow d_{2nr} \leq \varepsilon, \tag{23}$$

which is a contradiction. Hence, $\{x_{2n}\}$ in X is a Cauchy sequence and

$$\lim_{n,m\longrightarrow\infty} d(x_{2n}, x_{2m}) = 0. \tag{24}$$

Since X is complete, there exists a point $t \in X$ such that $x_{2n} \longrightarrow t$. Furtheremore, for each $k \geq 1$, we have $T^k x_{2n} \longrightarrow Tt$. Thus, $t = Tt$ as $T^k x_{2n} \longrightarrow t$. Hence, t is a fixed point of T.

Again, for $x_{2n+1} \longrightarrow t$ and for each $k \geq 1$, we have $S^k x_{2n+1} \longrightarrow St$. Hence, $t = St$ as $S^k x_{2n+1} \longrightarrow t$. Therefore, t is a fixed point of S.

In addition, assume that T^k and S^k are k-continuous for some positive integer k. Then, we have $\lim_{n\longrightarrow\infty} d(x_{2n}, t) = 0$. Therefore,

$$\begin{aligned} \lim_{n\longrightarrow\infty} T^k x_{2n} &= \lim_{n\longrightarrow\infty} Tx_{2n} = \lim_{n\longrightarrow\infty} x_{2n+1} = t, \\ \lim_{n\longrightarrow\infty} S^k x_{2n+1} &= \lim_{n\longrightarrow\infty} Sx_{2n+1} = \lim_{n\longrightarrow\infty} x_{2n+2} = t. \end{aligned} \tag{25}$$

Here, we will prove that t is a fixed point of S. Contrarily, suppose that $t \neq St$.

Now,

$$d(x_{2n+1}, St) = d(Tx_{2n}, St) \leq M(x_{2n}, t), \tag{26}$$

where

$$\begin{aligned} M(x_{2n}, t) &= \max\left\{ d(x_{2n}, t), d(x_{2n}, Tx_{2n}), d(t, St), \frac{d(x_{2n}, St) + d(t, Tx_{2n})}{2}\right\} \\ &= \max\left\{ d(x_{2n}, t), d(x_{2n}, x_{2n+1}), d(t, St), \frac{d(x_{2n}, St) + d(t, x_{2n+1})}{2}\right\}. \end{aligned} \tag{27}$$

As $n \longrightarrow \infty$ in (27), we see that

$$M(x_{2n}, t) \longrightarrow d(t, St). \tag{28}$$

Applying the limit as $n \longrightarrow \infty$ in (26), we have

$$d(t, St) \leq d(t, St) < d(t, St), \tag{29}$$

which is a contradiction. Hence, $St = t$.

Now, suppose that T is orbitally continuous. Since $x_{2n} \longrightarrow t$, orbital continuity implies that $Tx_{2n} \longrightarrow Tt$ or $Sx_{2n+1} \longrightarrow St$. This gives $t = Tt$ as $Tx_{2n} \longrightarrow t$ or $t = St$ as $Sx_{2n+1} \longrightarrow t$. Thus, t is a fixed point of T and S.

Next, we will show that a point t is a unique common fixed of T and S. In contrary, suppose that $t \in X$ and $y^* \in X$ are two different common fixed points of T and S, respectively. Thus, $d(t, y^*) > 0$.

Now,

$$d = d \leq M, \tag{30}$$

where

$$\begin{aligned} M(t, y^*) &= \max\left\{ d(t, y^*), d(t, Tt), d(y^*, Sy^*), \frac{d(t, Sy^*) + d(y^*, Tt)}{2}\right\} \\ &= \max\left\{ d(t, y^*), d(t, t), d(y^*, y^*), \frac{d(t, y^*) + d(y^*, t)}{2}\right\} \\ &= d(x^*, y^*). \end{aligned} \tag{31}$$

Hence,

$$d(t, y^*) = d(Tt, Sy^*) < Md(t, y^*) < d(t, y^*), \tag{32}$$

which is a contradiction. Therefore, T and S have a unique common fixed point, that is $t = y^*$.

To prove that any fixed point of T is also a fixed point of S, conversely, we suppose to the contrary that $t = Tt$ and $t \neq St$. Now,

$$d(Tt, St) = d(t, St) \leq M(t, St), \tag{33}$$

where

$$
\begin{aligned}
M(t, St) &= \max \left\{ d(t, St), d(t, Tt), d(St, S^2 t), \frac{d(t, S^2 t) + d(St, Tt)}{2} \right\} \\
&= \max \left\{ d(t, St), d(t, t), d(St, S^2 t), \frac{d(t, S^2 t) + d(St, t)}{2} \right\} \\
&= d(t, St).
\end{aligned}
\tag{34}
$$

Thus,

$$
d(Tt, St) = d(t, St) \le M(t, St) < d(t, St), \tag{35}
$$

which is a contradiction. Therefore, $t = Tt = St$. In a similar way, it is easy to show that any fixed point of S is also a fixed point of T.

Remark 7. If we set $S = T$, we get an improved version of the Bisht and Pant [11] theorem as a corollary for the case $a = 1$ as follows:

Corollary 8. *Let X be a nonempty set and let d to be a metric on X. Let T be self mapping of a complete metric space (X, d) satisfying*

(i) $d(Tx, Ty) < M(x, y)$, where

$$
M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}. \tag{36}
$$

(ii) Given that $\varepsilon > 0$, there exists a $\delta > 0$ such that $\varepsilon < M(x, y) \le \varepsilon + \delta \Rightarrow d(Tx, Ty) \le \varepsilon$

If T is k-continuous or T^k is continuous for some $k \ge 1$ or T is orbitally continuous, then, T has a fixed point.

Corollary 9. *The conclusions of Theorem 6 remain true, if we replace $M(x, y)$ in the contractive equation (7) by any one of the following:*

(i) $M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Sy)\}$.

(ii) $M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2}\}$.

(iii) $M(x, y) = \max \{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Tx) + d(y, Sy)}{2}\}$.

The following example shows the generality of Theorem 6 over Theorem 5.

Example 5. Let $X = [0, 3]$ be equipped with the usual metric. Let S and T be self mappings on X, i.e., $T, S : X \longrightarrow X$ defined by

$$
Tx = \begin{cases} \dfrac{1 + x}{2}, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x \in (1, 3]. \end{cases}
\qquad
Sx = \begin{cases} x^2, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } 1 < x \le 2, \\ \dfrac{x}{2}, & \text{if } 2 < x \le 3. \end{cases}
\tag{37}
$$

Hence, T and S satisfy all the conditions of the above theorem and have a unique fixed point $x = 1$; and S and T are discontinuous at $x = 1$. The mapping T is 2-continuous and S is 3-continuous at $x = 1$. S and T are orbitally continuous. Tt can be easily verified using the following cases:

Case 1. Now, we have

$$
d(Tx, Sy) < M(x, y) \tag{38}
$$

where

$$
M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2} \right\}. \tag{39}
$$

For $x, y \le 1$ and $0 \le x \le 1$, we get

$$
\begin{aligned}
& d(Tx, Sy) = d\left(\frac{1 + x}{2}, y^2\right) = \left| \frac{1 + x}{2} - y^2 \right| = \left| \frac{x - 2y^2 + 1}{2} \right|, \\
& d(x, y) = d(x, y) = |x - y|, \\
& d(x, Tx) = d\left(x, \frac{1 + x}{2}\right) = \left| x - \frac{1 + x}{2} \right| = \left| \frac{x - 1}{2} \right|, \\
& d(y, Sy) = d(y, y^2) = |y - y^2|, \\
& d(x, Sy) = d(x, y^2) = |x - y^2|, \\
& d(y, Tx) = d\left(y, \frac{1 + x}{2}\right) = \left| y - \frac{1 + x}{2} \right| = \left| \frac{2y - x - 1}{2} \right|, \\
& M(x, y) = \max \left\{ |x - y|, \left| \frac{x - 1}{2} \right|, |y - y^2|, \frac{2|x - y^2| + |2y - x - 1|}{4} \right\} \\
& \quad = |x - y|.
\end{aligned}
\tag{40}
$$

By using (i) of Theorem 6, we have

$$
d(Tx, Sy) < M(x, y) \Longrightarrow \left| \frac{x - 2y^2 + 1}{2} \right| < |x - y|, \tag{41}
$$

which is a contradiction. Hence, T and S are discontinuous at $x = 1$.

Remark 10. In case of $x = 1$ and $y = 1$, both functions are discontinuous at this point but have the property of k-continuity. For more details, see Example 1.

Case 2. Next we have,

$$d(Tx, Sy) < M(x, y) \tag{42}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}. \tag{43}$$

For $x, y > 2 \Longrightarrow 2 < x \le 3$, we get

$$\begin{aligned}
d(Tx,Sy) &= d\left(0,\frac{y}{2}\right) = \left|0-\frac{y}{2}\right| = \frac{y}{2},\\
d(x,y) &= d(x,y) = |x-y|,\\
d(x,Tx) &= d(x,0) = |x-0| = |x|,\\
d(y,Sy) &= d\left(y,\frac{y}{2}\right) = \left|y-\frac{y}{2}\right| = \frac{y}{2},\\
d(x,Sy) &= d\left(x,\frac{y}{2}\right) = \left|x-\frac{y}{2}\right| = \frac{2x-y}{2},\\
d(y,Tx) &= d(y,0) = |y-0| = y,
\end{aligned}$$

$$\begin{aligned}
M(x,y) &= \max\left\{d(x,y), d(x,0), d\left(y,\frac{y}{2}\right), \frac{d(x,(y/2))+d(y,0)}{2}\right\},\\
&= \max\left\{|x-y|, x, \frac{y}{2}, \frac{((2x-y)/2)+y}{2}\right\}\\
&= \max\left\{|x-y|, x, \frac{y}{2}, \frac{2x+y}{4}\right\}\\
&= x.
\end{aligned} \tag{44}$$

By using (i) of Theorem 6, we have

$$d(Tx,Sy) < M(x,y) \Longrightarrow \frac{y}{2} < x. \tag{45}$$

Thus, conditions (i) of Theorem 6 satisfy for all $x, y > 2$. This shows that T and S are continuous.

Case 3. Next, we have

$$d(Tx, Sy) < M(x, y) \tag{46}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}. \tag{47}$$

For $x \le 1 \Longrightarrow y \le 2$, we have

$$\begin{aligned}
d(Tx,Sy) &= d\left(\frac{1+x}{2},0\right) = \left|\frac{1+x}{2}-0\right| = \frac{x+1}{2},\\
d(x,y) &= d(x,y) = |x-y|,\\
d(x,Tx) &= d\left(x,\frac{1+x}{2}\right) = \left|x-\frac{1+x}{2}\right| = \frac{x-1}{2},\\
d(y,Sy) &= d(y,0) = |y-0| = y,\\
d(x,Sy) &= d(x,0) = |x-0| = x,\\
d(y,Tx) &= d\left(y,\frac{1+x}{2}\right) = \left|y-\frac{1+x}{2}\right| = \frac{2y-x-1}{2},\\
M(x,y) &= \max\left\{d(x,y), d\left(x,\frac{1+x}{2}\right), d(y,0), \frac{d(x,0)+d(y,((1+x)/2))}{2}\right\},\\
&= \max\left\{|x-y|, \frac{x-1}{2}, y, \frac{x+((2y-x-1)/2)}{2}\right\},\\
&= \max\left\{|x-y|, \frac{x-1}{2}, y, \frac{x+2y-1}{4}\right\}\\
&= y.
\end{aligned} \tag{48}$$

By using (i) of Theorem 6, we have

$$d(Tx,Sy) < M(x,y) \Longrightarrow \frac{x+1}{2} < y. \tag{49}$$

Thus, conditions (i) of Theorem 6 satisfy for all $x \le 1, y \le 2$. This shows that T and S are continuous. Also, $T1 = 1$ and $S2 = 0$. Then, $TSx_{2n} \longrightarrow t \Longrightarrow T^2Sx_{2n} \longrightarrow 1 = TSt$ for each n. Hence, TS is 3-continuous.

Therefore, T and S satisfy condition (ii) of Theorem 6 with $\delta = 1-\varepsilon$ if $\varepsilon < 1$ and $\delta = 1$ for $\varepsilon \ge 1$. To see this, consider Case 1, Case 2, and Case 3 as follows:

By Case 1, using condition (ii) of Theorem 6, $\varepsilon < 1$ and $\delta = 1-\varepsilon$, we get

$$\begin{aligned}
&\varepsilon < M(x,y) < \varepsilon+\delta \Rightarrow d(Tx,Sy) < \varepsilon,\\
&\Rightarrow \varepsilon < \frac{x-1}{2} < \varepsilon+\delta \Rightarrow \left|\frac{x-2y^2+1}{2}\right| < \varepsilon,\\
&\Rightarrow \varepsilon < \frac{x-1}{2} < \varepsilon+1-\varepsilon \Rightarrow \left|\frac{x-2y^2+1}{2}\right| < \varepsilon,\\
&\Rightarrow \varepsilon < 0 < 1 \Rightarrow 0 < \varepsilon.
\end{aligned} \tag{50}$$

which is a contradiction.

By Case 2, using (ii) of Theorem 6, $\delta = 1$ for $\varepsilon \ge 1$, we get

$$\begin{aligned}
&\varepsilon < M(x,y) < \varepsilon+\delta \Rightarrow d(Tx,Sy) < \varepsilon,\\
&\Rightarrow \varepsilon < x < \varepsilon+\delta \Rightarrow \left|\frac{y}{2}\right| < \varepsilon,\\
&\Rightarrow \varepsilon < x < \varepsilon+1 \Rightarrow \left|\frac{y}{2}\right| < \varepsilon,
\end{aligned} \tag{51}$$

satisfying for all $x, y > 2$.

However, this example is not applicable to the conditions imposed in Theorem 5.

Remark 11. It can be seen from the above example that T and S are threshold operation that models firing of a neuron, a function of two diodes, and also a low-pass filter that allows low voltages to pass but not higher voltages (e.g., noise in music systems).

One of the fundamental tools for nonlinear analysis is the Banach fixed point theorem [12]. As a result of its usefulness and applications, this theorem has been massively investigated and generalized by different researchers. One of the important generalization of the Banach fixed point theorem is the Boyd and Wong [13] fixed point theorem. A mapping T satisfying

$$d(Tx, Ty) \leq \phi(d(x, y)), \quad \forall x, y \in M, \tag{52}$$

whereby (M, d) is a complete metric space and a mapping $\phi : [0,\infty) \longrightarrow [0,\infty)$ is upper semicontinuous from the right on $[0, \infty)$ such that $\phi(t) < t$, $\forall t > 0$. Consequently, T has a unique fixed point $z \in M$ and $d(T^n x, z) \longrightarrow 0$ as $n \longrightarrow \infty$, $\forall x \in M$. Pant and Pant [9] proved the following theorem for the Boyd and Wong type fixed point theorem in complete metric spaces:

Theorem 12 [9]. *Let T be a mapping of a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \leq \phi(\max \{d(x, Tx), d(y, Ty)\}), \tag{53}$$

for all $x, y \in X$, where the function $\phi : [0,\infty) \longrightarrow [0,\infty)$ is such that $\phi(t) < t$ for each $t > 0$. If ϕ is upper semicontinuous in the open interval $(0, d(T^k(X)))$, then, T has a unique fixed point.

Now, we will demonstrate an example to explain the above theorem:

Example 6. Let $X = [0, 3]$ with usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let a mapping $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 0, & x \in [0, 1) \\ 1, & x \in [1, 3]. \end{cases} \tag{54}$$

Also, define $\phi : [0,\infty) \longrightarrow [0,\infty)$ as

$$\phi(t) = \begin{cases} \dfrac{1+t}{2}, & t > 1, \\ \dfrac{t}{2}, & t \leq 1. \end{cases} \tag{55}$$

It is clear that the mapping T satisfies the criteria of Theorem 12 with a unique fixed point $T = 1$ but it is discontinuous at this fixed point. Also, we observe that $d(T(X)) = 1$ and ϕ is continuous on $(0, 1)$.

Here, we present an extension of Theorem 12 for a pair of maps to obtain a unique common fixed point.

Theorem 13. *Let X be a nonempty set and let d be a metric on X. Let T and S be self mappings of a complete metric space (X, d) satisfying*

$$d(Tx, Sy) \leq \phi\{M(x, y)\}, \tag{56}$$

for all $x, y \in X$, where the mapping $\phi : [0,\infty) \longrightarrow [0,\infty)$ is such that $\phi(t) < t$ for all $t > 0$ and

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2} \right\}. \tag{57}$$

If ϕ is upper semicontinuous on $(0, d(T^k(X)))$ and $(0, d(S^k(X)))$ for $k = 0, 1, 2, \cdots$, then, T and S have a unique common fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$, for all integers $n \geq 0$. If we assume that there exists a nonnegative integer n_0 such that, $x_{2n_0} = x_{2n_0+1}$, then, $x_{2n} = x_{2n+1} = Tx_{2n}$; this implies that x_{2n} is a fixed point of T. Similarly, if there exists an integer $N \geq 0$ such that $x_{2N+1} = x_{2N+2}$, then, x_{2n+1} is a fixed point of S. This concludes the proof. □

Otherwise, we suppose that $x_{2n} \neq x_{2n+1}$, for all integers $n \geq 0$. Let $\mu_{2n} = d(x_{2n}, x_{2n+1})$, obviously, $\mu_{2n+1} = d(x_{2n+1}, x_{2n+2})$. From (84), we have

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Sx_{2n+1}) \leq \phi(M(x_{2n}, x_{2n+1})), \tag{58}$$

where

$$M(x_{2n}, x_{2n+1}) = \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \frac{d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})}{2} \right\}, \tag{59}$$

Using equations (8) and (12), we have the following:

$$M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \quad (60)$$

Thus,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, Sx_{2n+1}) \\ &\le \phi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}). \end{aligned} \quad (61)$$

If we take $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, then,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, Sx_{2n+1}) \\ &\le \phi\{d(x_{2n+1}, x_{2n+2})\} \\ &< d(x_{2n+1}, x_{2n+2}), \end{aligned} \quad (62)$$

which is a contradiction. Hence, $\max = \{d(x_{2n}, x_{2n+1})\}, d(x_{2n+1}, x_{2n+2}) = d(x_{2n}, x_{2n+1})$.Therefore,

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Sx_{2n+1}) \le \phi\{d(x_{2n}, x_{2n+1})\} < d(x_{2n}, x_{2n+1}), \quad (63)$$

which implies that the sequence $\{\mu_{2n}\}$ is decreasing to a non-negative real number say δ, for all integers $n \ge 0$. We claim that $\delta = 0$. In contrary, suppose that $\delta > 0$. Taking the limit as $n \longrightarrow \infty$ in (63), we obtain

$$0 < \delta \le \phi(\delta) < \delta, \quad (64)$$

which is a contradiction; hence, we conclude that $\delta = 0$ and

$$\lim_{n\longrightarrow\infty} (\mu_{2n}) = \lim_{n\longrightarrow\infty} d(x_{2n}, x_{2n+1}) = 0. \quad (65)$$

Now, we need to show that a sequence $\{x_n\}$ in X is a Cauchy sequence. We claim otherwise. Therefore, there exists $\varepsilon > 0$ and a sequence of integers $m(r), n(r)$ such that

$$d\left(x_{n(r)}, x_{m(r)}\right) \ge \varepsilon, \quad (66)$$

for all $n(r) > m(r) \ge r$ for some $r \ge 0$.

Furthermore, suppose that $m(r)$ is the smallest integer which is chosen in such a way that (66) holds so that we have

$$d\left(x_{(r)}, x_{m(r)-1}\right) < \varepsilon. \quad (67)$$

Now, for all $n(r) > m(r)$, we have

$$\begin{aligned} d\left(x_{n(r)}, x_{m(r)}\right) &\le d\left(x_{n(r)}, x_{m(r)-1}\right) + d\left(x_{m(r)-1}, x_{m(r)}\right) \\ &\le d\left(x_{n(r)}, x_{m(r)-1}\right) + d\left(x_{m(r)-1}, x_{m(r)}\right). \end{aligned} \quad (68)$$

As $r \longrightarrow \infty$ in (68) and considering (65) and (67), we see that

$$d\left(x_{n(r)}, x_{m(r)}\right) \longrightarrow \varepsilon. \quad (69)$$

By similar computations, we see that

$$d\left(x_{n(r)-1}, x_{m(r)-1}\right) \longrightarrow \varepsilon. \quad (70)$$

Thus,

$$d\left(x_{n(r)}, x_{m(r)}\right) = d\left(Tx_{n(r)-1}, Sx_{m(r)-1}\right) \le \phi\left(M\left(x_{n(r)-1}, x_{m(r)-1}\right)\right), \quad (71)$$

where

$$\begin{aligned} M\left(x_{n(r)-1}, x_{m(r)-1}\right) &= \max\left\{d\left(x_{n(r)-1}, x_{m(r)-1}\right), d\left(x_{n(r)-1}, Tx_{n(r)-1}\right), d\left(x_{m(r)-1}, Sx_{m(r)-1}\right), \frac{d\left(x_{n(r)-1}, Sx_{m(r)-1}\right) + d\left(x_{m(r)-1}, Tx_{n(r)-1}\right)}{2}\right\} \\ &= \max\left\{d\left(x_{n(r)-1}, x_{m(r)-1}\right), d\left(x_{n(r)-1}, x_{n(r)-1}\right) d\left(x_{m(r)-1}, x_{m(r)-1}\right), \frac{d\left(x_{n(r)-1}, x_{m(r)-1}\right) + d\left(x_{m(r)-1}, x_{n(r)-1}\right)}{2}\right\}, \end{aligned} \quad (72)$$

As $r \longrightarrow \infty$ in (72) and considering (69) and (70), then, (71) becomes

$$0 < \varepsilon \leq \phi(\varepsilon) < \varepsilon, \tag{73}$$

which is a contradiction. Hence, $\{x_n\}$ in X is a Cauchy sequence and $\lim_{n,m\longrightarrow\infty} d(x_n, x_m) = 0$.

Because X is complete, we can pick a point $z \in X$ such that $\lim_{n\longrightarrow\infty} d(x_{2n}, z) = 0$. Here, we will prove that z is a fixed point of S. In contrary, suppose that $z \neq Sz$.

Now,

$$d(x_{2n+1}, Sz) = \phi(d(Tx_{2n}, Sz)) \leq \phi(M(x_{2n}, z)). \tag{74}$$

where

$$\begin{aligned} M(x_{2n}, z) &= \max\left\{d(x_{2n}, z), d(x_{2n}, Tx_{2n}), d(z, Sz), \frac{d(x_{2n}, Sz) + d(z, Tx_{2n})}{2}\right\} \\ &= \max\left\{d(x_{2n}, z), d(x_{2n}, Tx_{2n}), d(z, Sz), \frac{d(x_{2n}, Sz) + d(z, x_{2n+1})}{2}\right\}. \end{aligned} \tag{75}$$

As $n \longrightarrow \infty$ in (75), we see that

$$M(x_{2n}, z) \longrightarrow d(z, Sz). \tag{76}$$

Applying the limit as $n \longrightarrow \infty$ in (74), we have

$$d(z, Sz) \leq \phi(d(z, Sz)) < d(z, Sz), \tag{77}$$

which is a contradiction. Hence, $Sz = z$.

Now, we will show that a point z is a unique common fix of T and S. In contrary, suppose that $z \in X$ and $w \in X$ are two different common fixed points of T and S, respectively. Thus, $d(z, w) > 0$.

Now,

$$d(z, z) = d(Tz, Sw) \leq \phi(M(z, w)) \tag{78}$$

where

$$\begin{aligned} M(z, w) &= \max\left\{d(z, w), d(z, Tz), d(w, Sw), \frac{d(z, Sw) + d(w, Tz)}{2}\right\} \\ &= \max\left\{d(z, w), d(z, z), d(w, w), \frac{d(z, w) + d(w, z)}{2}\right\} \\ &= d(z, w). \end{aligned} \tag{79}$$

Hence,

$$d(z, w) = d(Tz, Sw) \leq \phi(d(z, w)) < d(z, w), \tag{80}$$

which is a contradiction. Therefore, T and S have a unique common fixed point, that is $z = w$.

To prove that any fixed point of T is also a fixed point of S, conversely, we suppose to the contrary that $z = Tz$ and $z \neq Sz$.

Now,

$$d(Tz, Sz) = d(z, Sz) \leq \phi(M(z, Sw)) \tag{81}$$

where

$$\begin{aligned} M(z, Sw) &= \max\left\{d(z, Sz), d(z, Tz), d(Sz, S^2z), \frac{d(z, S^2z) + d(Sz, Tz)}{2}\right\} \\ &= \max\left\{d(z, Sz), d(z, z), d(Sz, S^2z), \frac{d(z, S^2z) + d(Sz, z)}{2}\right\} \\ &= d(z, Sz). \end{aligned} \tag{82}$$

Thus,

$$d(Tz, Sz) = d(z, Sz) \leq \phi(d(z, Sz)) < d(z, Sz), \tag{83}$$

which is a contradiction. Therefore, $z = Tz = Sz$. In a similar way, it is easy to show that any fixed point of S is also a fixed point of T.

On setting $S = T$, we get the following corollary:

Corollary 14. *Let X be a nonempty set and let d be a metric on X. Let T be a self mapping of a complete metric space (X, d) satisfying*

$$d(Tx, Ty) \leq \phi\{M(x, y)\}, \tag{84}$$

for all $x, y \in X$, where the mapping $\phi : [0,\infty) \longrightarrow [0,\infty)$ is such that $\phi(t) < t$ for all $t > 0$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{85}$$

If ϕ is upper semicontinuous on $(0, d(T^k(X)))$ for $k = 0, 1, 2, \cdots$, then, T has a fixed point.

Example 7. Let $X = [0, 3]$ be equipped with the usual metric. Let S and T be self mappings on X, i.e., $T, S : X \longrightarrow X$ defined by

$$\begin{aligned} Tx &= \begin{cases} \dfrac{1+x}{2}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x \in (1,3], \end{cases} \\ Sx &= \begin{cases} x^2, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } 1 < x \leq 2, \\ \dfrac{x}{2}, & \text{if } 2 < x \leq 3. \end{cases} \end{aligned} \tag{86}$$

Define $\phi = (1 + t)/2$.

Now, we have

$$d(Tx, Sy) < \phi\{M(x, y)\}, \tag{87}$$

where

$$M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Sy),\frac{d(x,Sy)+d(y,Tx)}{2}\right\},\tag{88}$$

For $x, y \le 1$, using $\phi=(1+t)/2$, we have

$$\begin{aligned}
d(Tx,Sy)&=\left|\frac{x-2y^2+1}{2}\right|,\\
M(x,y)&=\left|\frac{x-1}{2}\right|,\\
d(Tx,Sy)&\le\phi(M(x,y)),\\
\left|\frac{x-2y^2+1}{2}\right|&\le\phi\left(\left|\frac{x-1}{2}\right|\right),\\
0&\le\frac{1+x}{2},
\end{aligned}\tag{89}$$

which is true.

For $x, y > 2$, $2 \le x \le 3$ and using $\phi=(1+t)/2$, we have

$$\begin{aligned}
d(Tx,Sy)&=\frac{y}{2},\\
M(x,y)&=x,\\
d(Tx,Sy)&\le\phi(M(x,y)),\\
\frac{y}{2}&\le\phi(x),\quad \frac{y}{2}\le\frac{1+x}{2}.
\end{aligned}\tag{90}$$

For $x \le 1$, $y \le 2$ and using $\phi=(1+t)/2$, we have

$$\begin{aligned}
d(Tx,Sy)&=\frac{x+1}{2},\\
M(x,y)&=y,\\
d(Tx,Sy)&\le\phi(M(x,y)),\\
\frac{x+1}{2}&\le\phi(y),\\
\frac{x+1}{2}&\le\frac{1+y}{2},
\end{aligned}\tag{91}$$

Hence, all conditions imposed in Theorem 13 are satisfied. Thus, T and S satisfy all the conditions of the above theorem and has a unique fixed point $x=1$; S and T are discontinuous at $x=1$. The mapping T is 2-continuous and S is 3-continuous at $x=1$.

3. Fixed Points of Nonexpansive Mappings

Bisht and Pant [11] proved a fixed point theorem (see Theorem 15 [11]) for nonexpansive mapping. In this section, we are extending the result due to Bisht and Pant [11] for a pair of self mappings.

In what follows, we shall denote

$$P((x,y))=\max\left\{d(x,y),d(x,Tx),d(y,Sy),a\left[\frac{d(x,Tx)+d(y,Sy)}{2}\right],b\left[\frac{d(x,Sy)+d(y,Tx)}{2}\right]\right\},\quad 0\le a,b<1.\tag{92}$$

We will use this expression in the following theorem.

Theorem 15. *Let X be a nonempty set and let d be a metric on X. Let T and S be self mappings of a complete metric space (X, d) such that for any $x, y \in X$*

(i) For any given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < P(x,y) < \varepsilon+\delta$ implies $d(Tx,Sy)\le\varepsilon$

(ii) $d(Tx,Sy)\le P(x,y)$

Then, T and S have a unique common fixed point, say z and $T^n x \longrightarrow z$ as well as $S^n x \longrightarrow z$ for each $x \in X$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X as $x_{2n+1}=T^n x_0=Tx_{2n}$ and $x_{2n+2}=Sx_{2n+1}$ for all integers $n\ge 0$. Then, on following the proof of Theorem 6, we can easily prove that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z\in X$ such that $x_n\longrightarrow z$ as $n\longrightarrow\infty$. Also, $Tx_n\longrightarrow z$ and $Sx_n\longrightarrow z$ as $n\longrightarrow\infty$. We claim that $Tz=z$. For if $Tz\ne z$, we get

$$\begin{aligned}
d(Tz,Sx_n)\le\max&\left\{d(z,x_n),d(z,Tx_n),d(x_n,Sx_n),\right.\\
&\left.a\left[\frac{d(z,Tz)+d(x_n,Sx_n)}{2}\right],b\left[\frac{d(z,Sx_n)+d(x_n,Tz)}{2}\right]\right\}.
\end{aligned}\tag{93}$$

On letting $n\longrightarrow\infty$, this yields, $d(Tz,Sz)\le\max\{a[d(Tz,z)+d(z,Sz)]/2,b[d(Sz,z)+d(z,Tz)]/2\}<\max\{d(z,Tz),d(z,Sz)\}$, which is a contradiction since $0\le a,b<1$. Thus, z is a common fixed point of T and S. ☐

Remark 16. By setting $S=T$, one can get Theorem 15 of Bist and Pant [11].

Corollary 17. *Let X be a nonempty set and let d be a metric on X. Let T be a self mapping of a complete metric space (X, d) such that for any $x, y \in X$*

(i) *For any given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < P(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$*

(ii) $d(Tx, Ty) \leq P(x, y)$

Then, T has a unique fixed point, say z and $T^n x \longrightarrow z$ for each $x \in X$.

Example 8. Let $X = [-1, 1]$ be equipped with usual metric and $T, S : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 0, & \text{if } x \in [-1, 0], \\ \dfrac{-1}{2}, & \text{if } x \in (0,1], \end{cases}$$
$$Sx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in [-1, 0], \\ \dfrac{-1}{4}, & \text{if } x \in (0,1]. \end{cases} \quad (94)$$

To verify our contraction condition, let us consider the following two cases:

Case 1. For $x, y \in [-1, 0]$, we have

$$\begin{aligned} d(x, y) &= |x - y|, \\ d(x, Tx) &= d(x, 0) = |x - 0| = x, \\ d(y, Sy) &= \left|y, \frac{y}{2}\right| = \left|y - \frac{y}{2}\right| = \frac{y}{2}, \\ d(x, Sy) &= d\left(x, \frac{y}{2}\right) = \left|x - \frac{y}{2}\right| = \frac{2x - y}{2}, \\ d(y, Tx) &= d(y, 0) = |y - 0| = y, \\ d(Tx, Sy) &= d\left(0, \frac{y}{2}\right) = \frac{y}{2}. \end{aligned} \quad (95)$$

Thus, we have,

$$\begin{aligned} P(x, y) &= \max\left\{d(x, y), d(x, Tx), d(y, Sy), a\left[\frac{d(x, Tx) + d(y, Sy)}{2}\right], b\left[\frac{d(x, Sy) + d(y, Tx)}{2}\right]\right\}, \\ P(x, y) &= \max\left\{|x - y|, x, \frac{y}{2}, a\left[\frac{x + (y/2)}{2}\right], b\left[\frac{(2x - y/2) + y}{2}\right]\right\}, \\ P(x, y) &= \max\left\{|x - y|, x, \frac{y}{2}, a\left[\frac{2x + y}{4}\right], b\left[\frac{2x + y}{4}\right]\right\}, = |x - y|, \quad \text{for } 0 \leq a, b \leq 1. \end{aligned} \quad (96)$$

Using (i) and (ii) of Theorem 15, we have

$$\begin{gathered} \varepsilon < P(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Sy) \leq \varepsilon, \\ \varepsilon < |x - y| < \varepsilon + \delta \Rightarrow \frac{y}{2} \leq \varepsilon, \\ d(Tx, Sy) \leq P(x, y) \Rightarrow \frac{y}{2} \leq |x - y|. \end{gathered} \quad (97)$$

Case 2. For $x, y \in [0, 1]$, we have

$$\begin{aligned} d(x, y) &= |x - y| = |1 - 0| = 1, \\ d(x, Tx) &= d(1, T(0)) = \left|1 - \left(\frac{-1}{2}\right)\right| = \frac{3}{2}, \\ d(y, Sy) &= |0 - S0| = \left|0 - \left(-\frac{1}{4}\right)\right| = \frac{1}{4}, \\ d(x, Sy) &= d(1, S0) = \left|1 - \left(-\frac{1}{4}\right)\right| = \frac{5}{4}, \\ d(y, Tx) &= d(0, T(1)) = \left|0 - \left(-\frac{1}{2}\right)\right| = \frac{1}{2}, \\ d(Tx, Sy) &= d(T(1), S0) = d\left(\frac{-1}{2}, \left(\frac{-1}{4}\right)\right) = \frac{1}{4}. \end{aligned} \quad (98)$$

Thus, we get

$$\begin{aligned} P(x, y) &= \left\{d(x, y), d(x, Tx), d(y, Sy), a\left[\frac{d(x, Tx) + d(y, Sy)}{2}\right], b\left[\frac{d(x, Sy) + d(y, Tx)}{2}\right]\right\}, \\ P(1, 0) &= \max\left\{1, \frac{3}{2}, \frac{1}{4}, a\left[\frac{(3/2) + (1/4)}{2}\right], b\left[\frac{(5/4) + (1/2)}{2}\right]\right\}, \\ P(x, y) &= P(1, 0), = \max\left\{1, \frac{3}{2}, \frac{1}{4}, \frac{7}{8a}, \frac{7}{8b}\right\}, = \frac{3}{2}, 0 \leq a, b \leq 1. \end{aligned} \quad (99)$$

Finally, by (i) and (ii) of Theorem 15, we get

$$\begin{gathered} \varepsilon < P(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Sy) \leq \varepsilon, \\ \varepsilon < \frac{3}{2} < \varepsilon + \delta \Rightarrow \frac{1}{4} \leq \varepsilon, \\ d(Tx, Sy) \leq P(x, y) \Rightarrow \frac{1}{4} \leq \frac{3}{2}. \end{gathered} \quad (100)$$

Thus, we have $d(Tx, Sy) \leq P(x, y)$. Hence, the contraction condition of Theorem 15 is satisfied, and 0 is the common fixed point of T and S. Also, T is not continuous but is 2-continuous. Similarly, S is not continuous at 0 but is 2-continuous.

4. The Existence Solution of Nonlinear Volterra Integral Equation

The integral equation method is very useful for solving many problems in several applied fields like mathematical economics and optimal control theory because problems in these areas are often reduced to integral equations.

Integral equations appear in several forms. However, in this section, we are interested in the integral equation, namely, the Volterra integral differential equation which is of the form

$$u^n(t,x) = f(t,x) + \int_a^x K(x,t,u(t))dt, \tag{101}$$

where $u^n = d^n u/dx^n$.

Now, we present the application of Theorem 6 to study the existence and uniqueness of the solution to nonlinear Volterra integral equations.

The following integral equation is inspired by [14–17].

$$u(x,y) = f(x,y) + \int_0^x g(x,y,\varepsilon,u(\varepsilon,y))d\varepsilon + \int_0^x\int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))d\varepsilon d\sigma, \tag{102}$$

where f, g, h are given functions and u is the unknown function to be found.

Let $C(T,S)$ be the class of continuous functions from set T to set S. We denote $E = \mathbb{R}^+ \times \mathbb{R}^+$, $E_1 = \{f(x,y,s)\colon 0 \leq s \leq x <\infty, y \in \mathbb{R}^+\}$ and $E_2 = \{f(x,y,s,t)\colon 0 \leq s \leq x<\infty, 0 \leq t \leq y<\infty\}$. We denote that $f \in C(E,\mathbb{R})$, $g \in C(E_1 \times \mathbb{R},\mathbb{R})$, and $h \in C(E_2 \times \mathbb{R},\mathbb{R})$.

Let X be the space of functions $z \in C(\mathbb{R}^+ \times \mathbb{R}^+,\mathbb{R})$ satisfying

$$|z(x,t)| = O\left(e^{\lambda(x+y)}\right), \tag{103}$$

where λ is a positive constant, that is,

$$|z(x,y)| \leq M_0\left(e^{\lambda(x+y)}\right), \tag{104}$$

for constant $M_0 > 0$. Let $(X, \|.\|)$ be a Banach space. Define a norm in the space X by

$$|z|_X = \sup_{(x,y)\in X}\left[|z(x,y)|e^{(-\lambda(x+y))}\right]. \tag{105}$$

Define the mapping $T, S : X \times X \longrightarrow [0,\infty)$ by

$$T^k u(x,y) = f(x,y) + \int_0^x g(x,y,\varepsilon,u(\varepsilon,y))d\varepsilon + \int_0^x\int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))d\varepsilon d\sigma, \tag{106}$$

$$S^k u(x,y) = f(x,y) + \int_0^x g(x,y,\varepsilon,v(\varepsilon,y))d\varepsilon + \int_0^x\int_0^y h(x,y,\sigma,\tau,v(\sigma,\tau))d\varepsilon d\sigma, \tag{107}$$

for $u, v \in X$. We assume that T^k and S^k are k-continuous for some positive integer k. For sufficiently large values of k, the mappings T^k and S^k are contraction, where T and S are noncontraction if $(x-a) > 1$, for $x > a$.

Now, we prove our results by establishing the existence of a common fixed point for a pair of self mappings:

Theorem 18. *Suppose that equation (102) satisfies the following conditions:*

(i) For the continuous functions $f, g \in X$, we have

$$\begin{aligned}|g(x,y,\varepsilon,u(\varepsilon,y)) - g(x,y,\varepsilon,v(\varepsilon,y))| &\leq L_1(x,y,\varepsilon)|u-v|,\\ |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,v(\sigma,\tau))| &\leq L_2(x,y,\sigma,\tau)|u-v|,\end{aligned} \tag{108}$$

where $L_1 \in C(E_1,[0,\infty))$ and $L_2 \in C(E_2,[0,\infty))$

(ii) There exist a nonnegative constant γ such that $\gamma < 1$ and

$$\int_0^x L_1(x,y,\varepsilon)e^{\lambda(x+y)} + \int_0^x\int_0^y L_2(x,y,\sigma,\tau)e^{\lambda(\sigma+\tau)} \leq \delta, \tag{109}$$

for all $x, y, \varepsilon, \sigma, \tau \in E_1 \cup E_2$, and

$$\gamma = \frac{[\lambda L_1 + L_2]e^{\lambda(x+y)} - L_1\lambda e^{\lambda y} - 2L_2 e^{\lambda x} + L_2}{\lambda^2}. \tag{110}$$

Then, the nonlinear Volterra integral equation (102) has a unique common solution in $E_1 \cup E_2$.

Proof. Let $T^k, S^k : X \longrightarrow X$ be two operators such that $T^k \in X$ and $S^k \in X$. Now, we verify that T^k and S^k are contractive maps in X. Let $u, v \in X$. On the contrary, we claim that neither T^k nor S^k are contractive maps in X. From (106) and

(107), using condition (i) and (ii) of Theorem 18, we have

$$\begin{aligned}
\left\|T^k u - S^k v\right\| &= f(x,y) + \int_0^x g(x,y,\varepsilon,u(\varepsilon,y))d\varepsilon \\
&\quad + \int_0^x\int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))d\varepsilon d\sigma \\
&\quad - f(x,y) - \int_0^x g(x,y,\varepsilon,v(\varepsilon,y))d\varepsilon \\
&\quad - \int_0^x\int_0^y h(x,y,\sigma,\tau,v(\sigma,\tau))d\varepsilon d\sigma, \\
&\le \int_0^t |g(x,y,\varepsilon,u(\varepsilon,y)) - g(x,y,\varepsilon,v(\varepsilon,y))|d\varepsilon \\
&\quad + \int_0^x\int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,v(\sigma,\tau))|, \\
&\le \left[\int_0^x L_1(x,y,\varepsilon)e^{\lambda(x+y)} + \int_0^x\int_0^y L_2(x,y,\sigma,\tau)e^{\lambda(\sigma+\tau)}\right]\|u-v\|_X, \\
&\le \left[\frac{1}{\lambda}L_1\left[e^{\lambda(x+y)} - e^{\lambda y}\right] + \frac{1}{\lambda^2}L_2\left[e^{\lambda(x+y)} - 2e^{\lambda x} + 1\right]\right]\|u-v\|_X, \\
&\le \left[\frac{[\lambda L_1 + L_2]e^{\lambda(x+y)} - L_1\lambda e^{\lambda y} - 2L_2 e^{\lambda x} + L_2}{\lambda^2}\right]\|u-v\|_X, \\
&\Rightarrow \left\|T^k u - S^k v\right\| \le \gamma\|u-v\|_X \\
&\Rightarrow d(Tu,Sv) \le \gamma M(u,v),
\end{aligned} \tag{111}$$

which is a contradiction. Hence, u is a common fix of T and S and also a solution to integral equation (102).

From (111), let $\gamma = 1$ and using (i) of Theorem 6, where

$$M(u,v) = \max\left\{d(u,v), d(u,Tv), d(v,Sv), \frac{d(u,Sv)+d(v,Tu)}{2}\right\}, \tag{112}$$

we have

$$d(Tu,Sv) < M(u,v). \tag{113}$$

Thus, Theorem 6 is satisfied. □

Data Availability

No data are required for this research article.

Additional Points

Code Availability. There is no coding used for this research article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Fixed Points of Monotone Total Asymptotically Nonexpansive Mapping in Hyperbolic Space via New Algorithm

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In this article, we consider an extensive class of monotone nonexpansive mappings and introduce a new iteration algorithm to approximate the fixed point for monotone total asymptotically nonexpansive mappings in the framework of hyperbolic space. Faster convergence and stability results are proved for that iteration; also, fixed point is approximated numerically in a nontrivial example by using MATLAB.

1. Introduction

The concept of hyperbolic space was given by Reich and Shafrir [1] in 1990, which is defined as a metric space (Σ, σ) that has a family *Fourier* of metric lines; for any two unique end-points $l, m \in \Sigma$, there is a unique metric line in *Fourier*. This metric line works out a unique metric segment symbolize by $[l, m]$, which is an isometric image of $[0, \sigma(l, m)]$. A unique point $z \in [l.m]$ is denoted by $\alpha l \oplus (1-\alpha) m$ which satisfies

$$\begin{aligned} \sigma(l, z) &= (1-\alpha)\sigma(l, m), \\ \sigma(z, m) &= \alpha\sigma(l, m), \end{aligned} \tag{1}$$

where $\alpha \in [0, 1]$. Such a metric space is called a *convex metric space*, such that

$$\sigma(\alpha l \oplus (1-\alpha)m, \alpha p \oplus (1-\alpha)q) \leq \alpha\sigma(l, p) + (1-\alpha)\sigma(m, q), \tag{2}$$

for all $l, m, p, q \in \Sigma$, then Σ is called *hyperbolic metric space* (abbreviated as H.M.S).

The class of hyperbolic spaces contains the normed spaces, CAT(0) spaces, and many others. There are many examples in literature which show that hyperbolic spaces are more general than Banach spaces; for details, see [2] and Example 1.1 of [3].

Recently, a new direction has been discovered dealing with the extension of the Banach Contraction Principle [4] to partially ordered metric spaces. The case of monotone nonexpansive mappings was first considered in [5]. After that, Dehaish and Khamsi [6] gave an analogue to Browder [7] and Göhde [8] fixed point theorems for monotone nonexpansive mappings. In 2018, Alfuraidan and Khamsi [9] extended Goebel and Kirk's fixed point theorem [10] for asymptotically nonexpansive mappings to the case of monotone mappings. Multiple articles [11–14] can be found in the literature on fixed point of asymptotically nonexpansive mapping using multistep iterations and strong convergence analysis.

In 2016, Alber et al. [15] introduced the concept of total asymptotically nonexpansive mappings that generalizes the family of mapping that are the extension of asymptotically nonexpansive mappings. Example 1 of [16] and Example 3.1 of [4] show that total asymptotically nonexpansive mappings properly contain the asymptotically nonexpansive mappings.

In this article, we define monotone total asymptotically nonexpansive mappings (abbreviated as M.T.A.N.M) and also extend Alber's fixed point theorem [10] for the respective class. In addition, this result also generalizes the results of Alfuraidan and Khamsi in hyperbolic space [9].

In Section 4, we introduce a new iteration scheme, prove fast convergence and stability results, and also provide comparison with some famous iterations listed as Banach [17], Mann [18], Ishikawa [19], Agarwal et al. [20], Noor [21], Abbas and Nazir [22], Vatan Two-step [23], an accelerated iteration [24] (by Chen and Wen [24]), and Thakur New [25]. Numerically, we compare the convergence of new iteration with these iterations in a nontrivial example.

2. Preliminaries

Let $(\Sigma, \sigma, \preceq)$ be a partially ordered (abbreviated as P.O) metric space, any two points $a, b \in \Sigma$ are comparable whenever $a \preceq b$ or $b \preceq a$.

Definition 1. Consider $(\Sigma, \sigma, \preceq)$ be a partially ordered space and Y be a self map of Σ, which is said to be

(i) monotone or order preserving [5] if

$$a \preceq b \Rightarrow Ya \preceq Yb \tag{3}$$

(ii) monotone Lipschitzian mapping [5] if Y is order preserving and there exist $L \geq 0$ such that

$$\sigma(Y(a), Y(b)) \leq H\sigma(a, b). \tag{4}$$

If $H = 1$, the mapping Y is said to be *order preserving nonexpansive mapping*

(iii) monotone asymptotically nonexpansive mapping [9] if there exists a sequence $\{H_\upsilon\}$ for $\upsilon \in \mathbb{N}$ such that

$$\begin{gathered}\lim_{\upsilon \longrightarrow \infty} H_\upsilon = 1, \\ \sigma(Y^\upsilon(a), Y^\upsilon(b)) \leq H_\upsilon \sigma(a, b),\end{gathered} \tag{5}$$

for every $a, b \in \Sigma$, such that a and b are comparable

Now, we will define M.T.A.N.M in hyperbolic space.

Definition 2. Let Σ be a hyperbolic metric space having a nonempty subset K. A self map Y is monotone total asymptotically nonexpansive mapping on K if there exist nonnegative sequences $\{\mu_\upsilon\}$ and $\{\xi_\upsilon\}$ with $\mu_\upsilon \longrightarrow 0, \xi_\upsilon \longrightarrow 0$, as $\upsilon \longrightarrow \infty$, a strictly increasing continuous function

$$\vartheta : [0,\infty) \longrightarrow [0,\infty) \text{ with } \vartheta(0) = 0, \tag{6}$$

such that

$$\sigma(Y^\upsilon s, Y^\upsilon t) \leq \sigma(s, t) + \mu_\upsilon \vartheta(\sigma(s, t)) + \xi_\upsilon \text{ for all } \upsilon \geq 1, \tag{7}$$

and there exists a constant $R^* > 0$ such that $\vartheta(\lambda) \leq R^* \lambda$ for $\lambda > 0$, then

$$\sigma(Y^\upsilon s, Y^\upsilon t) \leq (1 + R^* \mu_\upsilon)\sigma(s, t) + \xi_\upsilon, \tag{8}$$

for every comparable elements $s, t \in K$.

Example 1. Consider the real line $\mathbb{R}$ as a hyperbolic metric space and K be the subset of $\mathbb{R}$, $K = [0, \pi/2]$, and $T : K \longrightarrow K$ be a mapping defined by $Tx = \sin x$. Suppose that there exist two nonnegative sequences $\{x_n\}$ and $\{\xi_n\}$ with $x_n \longrightarrow 0$ and $\xi_n \longrightarrow 0$ as $n \longrightarrow 0$ and a strictly increasing continuous function $\theta : [0,\infty) \longrightarrow [0,\infty)$ with $\theta(0) = 0, \theta(\lambda) = \lambda + 1, \lambda \in [0,\infty)$. Then, T is *M.T.A.N.M.*

Example 2. Consider the real plane $\mathbb{R}^2$ as a hyperbolic metric space. Let $\sigma : \Sigma \times \Sigma \longrightarrow \mathbb{R}$ be defined as

$$\sigma(w, h) = |w_1 - h_1| + |w_1 w_2 - h_1 h_2| \text{ where } w = (w_1, w_2), h = (h_1, h_2). \tag{9}$$

Let $K = [0, 1] \times [0, 1] \subset \Sigma$ and $Y : K \longrightarrow K$ be a mapping defined by

$$Y(w, h) = \begin{cases} \left(\dfrac{w}{10}, \dfrac{h}{10}\right) & \text{if } (w, h) \neq \left(\dfrac{1}{2}, \dfrac{1}{2}\right); \\ (0, 0) & \text{if } (w, h) = \left(\dfrac{1}{2}, \dfrac{1}{2}\right). \end{cases} \tag{10}$$

Thus, $(0, 0)$ is the fixed point of Y. Suppose $\exists$ two nonnegative sequences $\{\mu_\upsilon\}$ and $\{\xi_\upsilon\}$ with $\mu_\upsilon \longrightarrow 0, \xi_\upsilon \longrightarrow 0$, as $\upsilon \longrightarrow \infty$, and a strictly increasing continuous function

$$\vartheta : [0,\infty) \longrightarrow [0,\infty) \text{ with } \vartheta(0) = 0. \tag{11}$$

Now, we consider the following cases with assumptions $(w_1, w_2) \preceq (h_1, h_2)$:

Case 1. If $(w_1, w_2) = (1/2, 1/2) = (h_1, h_2)$, then $Y(w_1, w_2) = (0, 0) = Y(h_1, h_2)$. In this case, Y is monotone and satisfies all the conditions of *total asymptotically nonexpansive mapping*.

Case 2. If $(w_1, w_2) = (1/2, 1/2), (h_1, h_2) \neq (1/2, 1/2)$, then $Y(w_1, w_2) = (0, 0)$, $Y(h_1, h_2) = (h_1/10, h_2/10)$, and $Y^\upsilon(w_1, w_2) = (0, 0), Y^\upsilon(h_1, h_2) = (h_1/10^\upsilon, h_2/10^\upsilon)$. Also,

$$\begin{aligned}\sigma(Y^\upsilon(w_1, w_2), Y^\upsilon(h_1, h_2)) &= \frac{h_1}{10^\upsilon} + \frac{h_1 h_2}{10^{2\upsilon}} \leq \left|\frac{1}{2} - h_1\right| \\ &+ \left|\frac{1}{2^2} - h_1 h_2\right| \leq \sigma((w_1, w_2), (h_1, h_2)) \\ &+ \mu_\upsilon \vartheta(\sigma((w_1, w_2), (h_1, h_2))) + \xi_\upsilon\end{aligned} \tag{12}$$

implies that Y is a M.T.A.N.M.

Case 3. If $(w_1, w_2) \neq (1/2, 1/2) \neq (h_1, h_2)$, then $Y(w_1, w_2) = (w_1/10, w_2/10), Y(h_1, h_2) = (h_1/10, h_2/10)$, and $Y^\upsilon(w_1, w_2) = (w_1/10^\upsilon, w_2/10^\upsilon), Y^\upsilon(h_1, h_2) = (h_1/10^\upsilon, h_2/10^\upsilon)$. Now,

$$\begin{aligned}\sigma(Y^\upsilon(w_1, w_2), Y^\upsilon(h_1, h_2)) &= \left|\frac{w_1 - h_1}{10^\upsilon}\right| + \left|\frac{w_1 w_2 - h_1 h_2}{10^{2\upsilon}}\right| \\ &\leq |w_1 - h_1| + |w_1 w_2 - h_1 h_2| \leq \sigma((w_1, w_2), (h_1, h_2)) \\ &+ \mu_\upsilon \vartheta(\sigma((w_1, w_2), (h_1, h_2))) + \xi_\upsilon.\end{aligned} \tag{13}$$

Hence, Y is M.T.A.N.M.

Next, we have some definitions and lemmas that will be useful in the proof of the main result.

Definition 3 (see [26]). A hyperbolic space Σ with metric σ is said to be uniformly convex if for any $w \in \Sigma$, for every $z > 0$, and for each $\varepsilon > 0$

$$\delta(z, \varepsilon) = \inf \left\{1 - \frac{1}{z}\sigma\left(\frac{1}{2}s \oplus \frac{1}{2}t, w\right); \sigma(s, w) \leq z, \sigma(t, w) \leq z, \sigma(s, t) \geq z\varepsilon\right\} > 0. \tag{14}$$

The function δ is called the modulus of uniform convexity of Σ.

A hyperbolic space (Σ, σ) satisfies the *property* (R) [26]. If $\{K_\upsilon\}$ is nonincreasing sequence of nonempty, bounded, closed, and convex subset of Σ, $\cap_{\upsilon=1}^{\infty} K_\upsilon \neq \vartheta$.

Definition 4 (see [27]). A bounded sequence $\{r_\upsilon\} \in \Sigma$ is *Δ-convege to* $r \in \Sigma$, if r is the unique asymptotic centre of every subsequence $\{r_{\upsilon k}\}$ *of* $\{r_\upsilon\}$.

Definition 5 (see [9]). A partially ordered hyperbolic metric space Σ satisfies the monotone weak Opial condition if any sequence in Σ which is monotone and weakly converges to s, then the following

$$\limsup_{\upsilon \longrightarrow \infty} \sigma(s_\upsilon, s) \leq \limsup_{\upsilon \longrightarrow \infty} \sigma(s_\upsilon, t), \tag{15}$$

for every $t \in \Sigma$ such that $s \preceq t$ or $t \preceq s$.

Throughout in article, the order intervals are assumed to be closed and convex and any of the subsets

$$\begin{aligned}[s, \longrightarrow) &= \{y \in \Sigma; s \preceq y\}, \\ (\longleftarrow, t] &= \{y \in \Sigma; y \preceq t\}.\end{aligned} \tag{16}$$

for every $s, t \in \Sigma$.

Lemma 6 [9]. *Suppose (Σ, σ) be uniformly convex H.M.S, and K be a subset of Σ which is closed nonempty and convex. Let $\tau : K \longrightarrow [0,\infty)$ be a type function if $\exists$ a bounded sequence $\{s_\upsilon\} \in \Sigma$ such that*

$$\tau(s) = \limsup_{\upsilon \longrightarrow \infty} \sigma(s_\upsilon, s), \tag{17}$$

for any $s \in K$. Since Σ is hyperbolic space, τ is convex and continuous with distinctive minimum point $u \in K$ such that

$$\tau(u) = \inf \{\tau(s); s \in K\} = \tau_0. \tag{18}$$

Moreover, if $\{Y^\upsilon(u)\}$ in K is the minimizing sequence of τ, i.e.,

$$\lim_{\upsilon \longrightarrow \infty} \tau(Y^\upsilon(u)) = \tau_0, \tag{19}$$

then, $\{Y^\upsilon(u)\}$ strongly converges to u.

3. Main Result

Theorem 7. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preccurlyeq)$ with nonempty convex closed bounded subset K. Let Y be a continuous M.T.A.N.M on K. Assume $\exists s_0 \in K$, such that $s_0 \preccurlyeq Y s_0$. Then, Y has a fixed point.*

Proof. Let $s_0 \in K$ be such that

$$s_0 \preccurlyeq Y s_0. \tag{20}$$

By the monotonicity of Y, we get

$$Y^\upsilon s_0 \preccurlyeq Y^{\upsilon+1} s_0, \tag{21}$$

for each $\upsilon \in \mathbb{N}$, and $\{Y^\upsilon s_0\}$ is a monotone increasing sequence. Also, the order intervals are closed and convex. So, we have

$$K_\infty = \bigcap_{\upsilon \geq 0} [Y^\upsilon s_0, \longrightarrow) \cap K = \cap \{s \in K; Y^\upsilon s_0 \preccurlyeq s\} \neq \phi. \tag{22}$$

Let $s \in K_\infty$, then

$$Y^v s_0 \preccurlyeq s, \tag{23}$$

and the monotonicity of Y implies

$$Y(Y^v s_0) = Y^{v+1} s_0 \preccurlyeq Ys, \tag{24}$$

for every $v \in \mathbb{N}$, i.e., $Y(K_\infty) \subset K_\infty$. Consider the type function $\tau : K_\infty \longrightarrow [0,+\infty)$ produced by $\{Y^v s_0\}$ given as $\tau(s) = \limsup_{v\longrightarrow\infty} \sigma(Y^v s_0, s)$ for any $s \in K_\infty$. Above lemma shows the occurrence of a unique $a \in K_\infty$ such that $\tau(a) = \inf \{\tau(s)\,;\, s \in K_\infty\} = \tau_0$. Since $a \in K_\infty$, we have $Y^d(a) \in K_\infty$, for every $d \in \mathbb{N}$, which implies

$$\begin{aligned}\tau\left(Y^d(a)\right) &= \limsup_{v\longrightarrow\infty} \sigma\left(Y^v s_0, Y^d a\right)\\ &\leq \limsup_{v\longrightarrow\infty} \left[\sigma(Y^v s_0, a) + \mu_p \vartheta(\sigma(Y^v s_0, a)) + \xi_d\right]\\ &= \tau(a) + \mu_d \limsup_{v\longrightarrow\infty} (\vartheta(\sigma(Y^v s_0, a))) + \xi_d.\end{aligned} \tag{25}$$

As Y is total asymptotically nonexpansive mapping, so $\mu_d \longrightarrow 0$, $\xi_d \longrightarrow 0$, when $d \longrightarrow \infty$. Hence,

$$\lim_{d\longrightarrow\infty} \tau\left(Y^d(a)\right) = \tau_0, \tag{26}$$

hence, $\{Y^d(a)\}$ is a minimizing sequence of τ. Using Lemma 6, $Y^d(a)$ converges to a. Since Y is continous, we have

$$\lim_{v\longrightarrow\infty} Y(Y^v a) = Y(a) = a, \tag{27}$$

i.e., a is a fixed point of Y. □

The following corollary is the conclusion of Theorem 3.3 of [9].

Corollary 8. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preccurlyeq)$ with nonempty convex closed bounded subset K. Let Y be a continuous M.T.A.N.M on K. Assume $\exists s_0 \in K$, such that $s_0 \preccurlyeq Ys_0$. Then, Y has a fixed point.*

The corollary given below is the consequence of Theorem 7 by replacing the continuity condition with weak Opial condition.

Corollary 9. *Let (Σ, σ') be a uniformly convex P.O H.M.S, satisfying monotone weak Opial condition with a nonempty convex closed bounded subset K. Let Y be a M.T.A.N.M on K. Assume $\exists s_0 \in K$, such that $s_0 \preccurlyeq Ys_0$. Then, Y has a fixed point.*

4. Convergence Theorem and Stability Results

We introduce the new iteration scheme given below, let K be a nonempty convex subset of a hyperbolic space Σ, for $v \geq 0$, where $\{\alpha_v\}$, $\{\beta_v\}$, and $\{\gamma_v\}$ are sequences in $[0, 1]$, such that

$$\begin{cases} s_0 \in K,\\ \eta_v = (1-\gamma_v)s_v \oplus \gamma_v Y^v s_v,\\ \theta_v = (1-\beta_v)\eta_v \oplus \beta_v Y^v \eta_v,\\ s_{v+1} = Y^v((1-\alpha_v)Y^v s_v \oplus \alpha_v Y^v \theta_v).\end{cases} \tag{28}$$

Fastness and stability play an important role for an iteration process to be preferred on another iteration process, so now, we prove that new iteration is stable and has good speed of convergence than others. For faster convergence and new class of mapping on metric space (Σ, σ) introduced by Berinde [28], satisfying

$$\sigma(Ys, Yt) \leq a\sigma(s, t) + L\sigma(s, Ys), \tag{29}$$

here, we will modify this mapping as

$$\sigma(Y^v s, Y^v t) \leq a\sigma(s, t) + L\sigma(s, Y^v s), \tag{30}$$

for all $s, t \in \Sigma$, where $a \in [0, 1)$ and $L \geq 0$.

The following definitions and lemma will be helpful for faster convergence results given in [29].

Definition 10. Let $\{a_v\}$ and $\{b_v\}$ be two sequences, having convergent points a and b, respectively, then $\{a_v\}$ converges faster than $\{b_v\}$ if

$$\lim_{v\longrightarrow\infty} \frac{\sigma(a_v, a)}{\sigma(b_v, b)} = 0. \tag{31}$$

Definition 11. Let $\{k_v\}$ and $\{l_v\}$ be two fixed point schemes that converge to the same fixed point s. If

$$\sigma(k_v, s) \leq s_v \text{ and } \sigma(l_v, s) \leq t_v \text{ for all } v \geq 0, \tag{32}$$

where s_v and t_v are two sequences that converge to 0. If s_v converges faster than t_v, then $\{k_v\}$ converges faster than $\{l_v\}$ to s.

Lemma 12. *If b is a real number such that $0 \leq b < 1$ and $\{a_v\}_{v=0}^\infty$ be a sequence such that*

$$\lim_{v\longrightarrow\infty} a_v = 0, \tag{33}$$

then for any positive sequence $\{s_v\}_{v=0}^\infty$ satisfying

$$s_{v+1} \leq bs_v + a_v \Rightarrow \lim_{v\longrightarrow\infty} s_v = 0. \tag{34}$$

Lemma 13 (see [4]). *Suppose $\{l_v\}$, $\{m_v\}$, and $\{\delta_v\}$ be* *sequences of nonnegative satisfying*

$$l_{\upsilon+1} \leq (1+\delta_\upsilon) l_\upsilon + m_\upsilon \quad \forall \upsilon \geq 1. \tag{35}$$

If $\Sigma\delta_\upsilon < \infty$ and $\Sigma m_\upsilon < \infty$, then $\lim_{\upsilon\longrightarrow\infty} l_{\upsilon+1}$ exists.

Lemma 14 (see [4]). *Suppose Σ be a uniformly convex H.M.S. Let $P \in [0,\infty)$ be such that*

$$\begin{gathered} \limsup_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, a) \leq P, \\ \limsup_{\upsilon\longrightarrow\infty} \sigma(\theta_\upsilon, a) \leq P, \\ \lim_{\upsilon\longrightarrow\infty} \sigma(a, \alpha_\upsilon s_\upsilon \oplus (1-\alpha_\upsilon)\theta_\upsilon) = P, \end{gathered} \tag{36}$$

where $\alpha_\upsilon \in [a, b]$ with $0 < a \leq b < 1$. Then, we get

$$\lim_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, \theta_\upsilon) = 0. \tag{37}$$

Lemma 15 (see [26]). *Let (Σ, σ') be a P.O hyperbolic space. Let K be a nonempty convex and closed subset of Σ. Let $Y : K \longrightarrow K$ be a monotone mapping. Let $s_1 \in K$, such that $s_1 \preceq Ys_1$ or $(Yt_1 \preceq t_1)$. Then, the sequence $\{s_\upsilon\}$ in (1) then*

(a) $s_\upsilon \preceq Ys_\upsilon \preceq s_{\upsilon+1}$ *or* $(s_{\upsilon+1} \preceq Ys_\upsilon \preceq s_\upsilon)$

(b) $s_\upsilon \preceq s$ *or* $(s \preceq s_\upsilon)$, *provided that* $\{s_\upsilon\}\Delta$*-converge to* $s \in K$, $\forall \upsilon \in \mathbb{N}$

Lemma 16. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preccurlyeq)$ with nonempty convex closed bounded subset K. Let $Y : K \longrightarrow K$ be a M.T.A.N.M with $F(Y) \neq \vartheta$. If the sequence $\{s_\upsilon\}$ is defined by (1) with $s_1 \preceq Ys_1$ or $(Ys_1 \preceq s_1)$. Then, the following holds*

$$\begin{gathered} (a) \lim_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, s) \text{ exist for } s \in F(Y), \\ (b) \lim_{\upsilon\longrightarrow\infty} \sigma(Y^\upsilon s_\upsilon, s_\upsilon) = 0. \end{gathered} \tag{38}$$

Proof. Let $s \in F(Y) \Rightarrow Ys = s$. By the above lemma $s \preceq s_\upsilon$, as Y is monotone

$$Ys \preceq Ys_\upsilon \Rightarrow s \preceq Y^\upsilon s_\upsilon. \tag{39}$$

Now, using Definition 2 and after simplification, we get

$$\sigma(\eta_\upsilon, s) \leq (1 + R^*\mu_\upsilon)\sigma(s_\upsilon, s) + \xi_\upsilon. \tag{40}$$

Again using Definition 2 and (40), we get

$$\sigma(\theta_\upsilon, s) \leq (1 + R^*\mu_\upsilon)^2 \sigma(s_\upsilon, s) + (2 + R^*\mu_\upsilon)\xi_\upsilon. \tag{41}$$

Consider

$$\sigma(s_{\upsilon+1}, s) = \sigma(Y^\upsilon((1-\alpha_\upsilon)Y^\upsilon\eta_\upsilon \oplus \alpha_\upsilon Y^\upsilon\theta_\upsilon), s). \tag{42}$$

Using Definition 2, (40) and (41), we get

$$\sigma(s_{\upsilon+1}, s) \leq (1+\delta_\upsilon)\sigma(s_\upsilon, s) + b_\upsilon, \tag{43}$$

where $\delta_\upsilon = (4 + 6R^*\mu_\upsilon + 4(R^*\mu_\upsilon)^2 + (R^*\mu_\upsilon)^3)R^*\mu_\upsilon$ and $b_\upsilon = (4 + 6R^*\mu_\upsilon + 4(R^*\mu_\upsilon)^2 + (R^*\mu_\upsilon)^3)\xi_\upsilon$. Using Lemma 13, $\lim_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, s)$ exist for $s \in F(Y)$.

For part (b), we have to show that

$$\lim_{\upsilon\longrightarrow\infty} \sigma(Y^\upsilon s_\upsilon, s_\upsilon) = 0, \tag{44}$$

the proof resembles to Theorem 2.1 of [4]. □ □

Theorem 17. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preccurlyeq)$ with convex closed bounded and nonempty subset Y. Let Y be a continuous M.T.A.N.M on Y, with $F(Y) \neq \vartheta$. If $\{s_\upsilon\}$ is defined by (1) with $s_1 \preceq Ys_1$ or $(Ys_1 \preceq s_1)$. If $s \preceq s_1$ or $(s_1 \preceq s)$ for $s \in F(Y)$, then $\{s_\upsilon\}\Delta$ -converges to a fixed point of Y.*

Proof. By Lemma 16,

$$\lim_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, s) \text{ exist} \forall s \in F(Y), \tag{45}$$

sequence is bounded and $\lim_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, Y^\upsilon s_\upsilon) = 0$. Let $\{s_{\upsilon k}\}$ be any subsequence of $\{s_\upsilon\}$ for $k \in \mathbb{N}$, such that $\{s_{\upsilon k}\}\Delta$-converges to $p \in Y$. By Lemma 15, we have

$$s_1 \preceq s_{\upsilon k} \preceq p. \tag{46}$$

Now, we have to show that every Δ-convergent subsequence of $\{s_\upsilon\}$ has a unique Δ-limit in $F(Y)$. Let $\{s_{\upsilon k}\}$ and $\{s_{\upsilon r}\}$ be two subsequences of $\{s_\upsilon\}\Delta$-converging to w and h, respectively. By Lemma 16, $\{s_{\upsilon k}\}$ is bounded and

$$\lim_{\upsilon\longrightarrow\infty} \sigma(s_{\upsilon k}, Y^\upsilon s_{\upsilon k}) = 0. \tag{47}$$

We claim that $w \in F(Y)$, and τ produced by $\{s_{\upsilon k}\}$ is

$$\tau(w) = \limsup_{k\longrightarrow\infty} \sigma(s_{\upsilon k}, w). \tag{48}$$

From Theorem 7, $Y(w) = w$, same for $Y(h) = h$. By the definition of Δ-convergence and Lemma 6, we have

$$\begin{aligned} \limsup_{k\longrightarrow\infty} \sigma(s_\upsilon, w) &= \limsup_{k\longrightarrow\infty} \sigma(s_{\upsilon k}, w) < \limsup_{k\longrightarrow\infty} \sigma(s_{\upsilon k}, h) \\ &= \limsup_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, h) = \limsup_{r\longrightarrow\infty} \sigma(s_{\upsilon r}, h) \\ &< \limsup_{r\longrightarrow\infty} \sigma(s_{\upsilon r}, w) = \limsup_{\upsilon\longrightarrow\infty} \sigma(s_\upsilon, w), \end{aligned} \tag{49}$$

which is contradiction, unless $w = h$. □

Theorem 18. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preccurlyeq)$ with nonempty convex closed bounded subset K. Let Y be a continuous M.T.A.N.M on K with $F(Y) \neq \vartheta$. If $\{s_\upsilon\}$ is defined by (1) with $s_1 \preceq Ys_1$ or $(Ys_1 \preceq s_1)$. If $s \preceq s_1$ or $(s_1 \preceq s)$ for $s \in F(Y)$,* *then* $\{s_v\}$ *converges to a fixed point of* Y *if and only if*

$$\liminf_{v\longrightarrow\infty}\sigma(s_v, F(Y)) = 0. \tag{50}$$

Proof. If $\{s_v\}$ converges to a fixed point of Y, then

$$\liminf_{v\longrightarrow\infty}\sigma(s_v, F(Y)) = 0 \quad \text{for all } v \in \mathbb{N}. \tag{51}$$

Conversely, consider $\liminf_{v\longrightarrow\infty}\sigma(s_v, F(Y)) = 0$. From Lemma 16

$$\lim_{v\longrightarrow\infty}\sigma(s_v, s) \text{ exist for each } s \in F(Y), \tag{52}$$

therefore, $\lim_{v\longrightarrow\infty}\sigma(s_v, F(Y)$ exists. As $\liminf_{v\longrightarrow\infty}\sigma(s_v, F(Y)) = 0$, so we get

$$\lim_{v\longrightarrow\infty}\sigma(s_v, F(Y)) = 0. \tag{53}$$

Now, we prove that $\{s_v\}$ is a Cauchy sequence in K. For $\varepsilon > 0, \exists v_0 \in \mathbb{N}$, such that for all $v \geq v_0$

$$\sigma(s_v, F(Y)) < \frac{\varepsilon}{2}, \tag{54}$$

in particular inf $\{\sigma(s_{v_0}, s)\colon s \in F(Y)\} < \varepsilon/2$, so $\exists$ a fixed point $p \in F(Y)$ such that

$$\sigma\left(s_{v_0}, s\right) < \frac{\varepsilon}{2}. \tag{55}$$

For $m, v \geq v_0$,

$$\sigma(s_{v+m}, s_v) \leq \sigma(s_{v+m}, s) + \sigma(s, s_v) < 2\sigma\left(s_{v_0}, s\right) < \varepsilon. \tag{56}$$

Hence, $\{s_v\}$ is a Cauchy sequence in closed subset K of Σ; therefore, it converges in K such that

$$\lim_{v\longrightarrow\infty} s_v = q \text{ for } q \in K. \tag{57}$$

As we have $\lim_{v\longrightarrow\infty}\sigma(s_v, F(Y)) = 0 \Rightarrow \sigma(q, F(Y)) = 0$, since $F(Y)$ is closed so $q \in F(Y)$.

Now, we prove that newly proposed iteration scheme (1) is faster than Thakur New [25] for a mapping defined in (2) in hyperbolic metric space. □

Theorem 19. *Let* $(\Sigma, \sigma, \preccurlyeq)$ *be a P.O H.M.S. Let* K *be a nonempty convex closed bounded subset of* Σ*, and* Y *be a mapping satisfying (30) with* $F(Y) \neq \vartheta$*. Let* $\{s_v\}$ *be defined by (28), and* $\{u_v\}$ *defined in [25], then* $\{s_v\}$ *converges faster than* $\{u_v\}$.

Proof. Let $s \in F(Y) \Rightarrow Y^v s = s$. Now, using (28) and (30), we have

$$\begin{aligned}\sigma(\eta_v, s) &\leq (1-\gamma_v)\sigma(s_v, s) + \gamma_v[a\sigma(s_v, s) + L\sigma(s, Y^v s)] \\ &= (1-\gamma_v(1-a))\sigma(s_v, s).\end{aligned} \tag{58}$$

Again, using (28), (30), and (58), we have

$$\sigma(\theta_v, s) \leq (1-\beta_v(1-a))(1-\gamma_v(1-a))\sigma(s_v, s). \tag{59}$$

Further, consider

$$\sigma(s_{v+1}, s) = \sigma(Y^v((1-\alpha_v)Y^v\eta_v \oplus \alpha_v Y^v\theta_v), s). \tag{60}$$

Now, using (30) and then (28), we get

$$\sigma(s_{v+1}, s) \leq a[a(1-\gamma_v(1-a))(1-(1-a)\alpha_v\beta_v)]\sigma(s_v, s). \tag{61}$$

Let

$$K_v = a^v[a[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^v]\sigma(s_1, s), \tag{62}$$

and $\sigma_v = a^v[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^v\sigma(u_1, s)$ calculated in Theorem 3.1 of Thakur New [25]. Then,

$$\begin{aligned}\frac{K_v}{\sigma_v} &= \frac{a^v[a[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^v]\sigma(s_1, s)}{a^v[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^v\sigma(u_1, s)} \\ &= \frac{a[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^v\sigma(s_1, s)}{[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^v\sigma(u_1, s)} \longrightarrow 0 \text{ as } v \longrightarrow \infty.\end{aligned} \tag{63}$$

Hence, $\{s_v\}$ converges faster than $\{u_v\}$. □

Now, we will prove the stability result for this we have the following definition by [30].

Definition 20. Let $\{p_v\}_{v=0}^{\infty} \subset K$ be any arbitrary sequence, then iteration sequence s_{v+1} converging to unique fixed point s is said to be Y -stable if for $\varepsilon_v = \sigma(p_v, s_{v+1})v \geq 0$, we have

$$\lim_{v\longrightarrow\infty}\varepsilon_v = 0 \Leftrightarrow \lim_{v\longrightarrow\infty} p_v = s. \tag{64}$$

Theorem 21. *Let* (Σ, σ') *be a P.O H.M.S. Let* K *be a nonempty convex closed bounded subset of* Σ*, and* Y *be a mapping satisfying (2) with* $F(Y) \neq \vartheta$*. Let* $\{s_v\}$ *be defined by (1), satisfying* $\Sigma\alpha_v = \infty$*, then the iteration (1) is* Y *-stable.*

Proof. Let $\{p_v\}_{v=0}^{\infty} \subset K$ be any arbitrary sequence, the sequence defined by (28) converging to unique fixed point s, and

$$\varepsilon_v = \sigma(p_v, s_{v+1}). \tag{65}$$

We have to prove that

$$\lim_{v\longrightarrow\infty}\varepsilon_v = 0 \Leftrightarrow \lim_{v\longrightarrow\infty} p_v = s. \tag{66}$$

Table 1: The convergence behavior of Mann, Ishikawa, Agarwal, and Noor iterations with new iteration for the parameters $\alpha = 0:6$, $\beta = 0:3$, and $\gamma = 0:5$, with the initial values $x_0 = 0$, $y_0 = 3$, and tolerance $= 10^{-6}$.

Steps	Mann	Ishikawa	Agarwal	Noor	New
0	(0, 3)	(0, 3)	(0, 3)	(0, 3)	(0, 3)
1	(0, 2.244506)	(0, 2.013105)	(0, 1.509443)	(0, 1.916515)	(0, 0.311990)
2	(0, 1.519333)	(0, 1.196547)	(0, 0.479098)	(0, 1.097796)	(0, 0.011353)
3	(0, 0.949002)	(0, 0.663545)	(0, 0.116180)	(0, 0.594503)	(0, 0.000384)
4	(0, 0.561764)	(0, 0.355178)	(0, 0.025817)	(0, 0.313100)	(0, 0.000012)
5	(0, 0.321995)	(0, 0.186769)	(0, 0.005616)	(0, 0.162620)	(0, 0.000000)
6	(0, 0.181202)	(0, 0.097333)	(0, 0.001216)	(0, 0.083873)	(0, 0.000000)
7	(0, 0.100930)	(0, 0.050492)	(0, 0.000263)	(0, 0.043105)	(0, 0.000000)
8	(0, 0.055900)	(0, 0.026132)	(0, 0.000056)	(0, 0.022113)	(0, 0.000000)
9	(0, 0.030863)	(0, 0.013508)	(0, 0.000012)	(0, 0.011333)	(0, 0.000000)
10	(0, 0.017010)	(0, 0.006978)	(0, 0.000002)	(0, 0.005806)	(0, 0.000000)
11	(0, 0.009366)	(0, 0.003603)	(0, 0.000000)	(0, 0.002973)	(0, 0.000000)
12	(0, 0.005155)	(0, 0.001860)	(0, 0.000000)	(0, 0.001522)	(0, 0.000000)
13	(0, 0.002836)	(0, 0.000960)	(0, 0.000000)	(0, 0.000779)	(0, 0.000000)
14	(0, 0.001560)	(0, 0.000495)	(0, 0.000000)	(0, 0.000399)	(0, 0.000000)
15	(0, 0.000858)	(0, 0.000256)	(0, 0.000000)	(0, 0.000204)	(0, 0.000000)
16	(0, 0.000472)	(0, 0.000132)	(0, 0.000000)	(0, 0.000104)	(0, 0.000000)
17	(0, 0.000259)	(0, 0.000068)	(0, 0.000000)	(0, 0.000053)	(0, 0.000000)
18	(0, 0.000142)	(0, 0.000035)	(0, 0.000000)	(0, 0.000027)	(0, 0.000000)
19	(0, 0.000078)	(0, 0.000018)	(0, 0.000000)	(0, 0.000014)	(0, 0.000000)
20	(0, 0.000043)	(0, 0.000009)	(0, 0.000000)	(0, 0.000007)	(0, 0.000000)
21	(0, 0.000023)	(0, 0.000004)	(0, 0.000000)	(0, 0.000003)	(0, 0.000000)
22	(0, 0.000013)	(0, 0.000002)	(0, 0.000000)	(0, 0.000001)	(0, 0.000000)
23	(0, 0.000007)	(0, 0.000001)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
24	(0, 0.000003)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
25	(0, 0.000002)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
26	(0, 0.000001)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
27	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)

Let $\lim_{\upsilon\longrightarrow\infty} \varepsilon_\upsilon = 0$, then

$$\begin{aligned}\sigma(p_\upsilon, s) &\le \sigma(p_\upsilon, s_{\upsilon+1}) + \sigma(s_{\upsilon+1}, s)\\ &= \varepsilon_\upsilon + \sigma(Y^\upsilon((1-\alpha_\upsilon)Y^\upsilon s_\upsilon \oplus \alpha_\upsilon Y^\upsilon \theta_\upsilon), s)\\ &\le \varepsilon_\upsilon + a\sigma((1-\alpha_\upsilon)Y^\upsilon \eta_\upsilon \oplus \alpha_\upsilon Y^\upsilon\theta_\upsilon, s)\\ &\quad + L\sigma(s, Y^\upsilon s) = \varepsilon_\upsilon + a\sigma((1-\alpha_\upsilon)\\ &\quad \cdot Y^\upsilon \eta_\upsilon \oplus \alpha_\upsilon Y^\upsilon \theta_\upsilon, s) \le \varepsilon_\upsilon + a[a(1-\gamma_\upsilon\\ &\quad \cdot (1-a))(1-(1-a)\alpha_\upsilon\beta_\upsilon)]\sigma(s_\upsilon, s),\end{aligned} \tag{67}$$

$0 \le a[a(1-\gamma_\upsilon(1-a))(1-(1-a)\alpha_\upsilon\beta_\upsilon)] < 1$, applying $\lim_{\upsilon\longrightarrow\infty}$, we get

$$\lim_{\upsilon\longrightarrow\infty} \sigma(p_\upsilon, s) = 0 \Rightarrow \lim_{\upsilon\longrightarrow\infty} p_\upsilon = s. \tag{68}$$

Conversely, let $\lim_{\upsilon\longrightarrow\infty} p_\upsilon = s$, we have

Table 2: The convergence of Abbas, Thakur, and accelerated iteration with new iteration for the same initial values, parameters, and tolerance 10^{-6}.

Steps	Abbas	An accelerated	Thakur	New
0	(0, 3)	(0, 3)	(0, 3)	(0, 3)
1	(0, 0.917554)	(0, 0.890744)	(0, 0.969760)	(0, 0.311990)
2	(0, 0.133217)	(0, 0.126269)	(0, 0.161770)	(0, 0.011353)
3	(0, 0.015917)	(0, 0.015024)	(0, 0.022578)	(0, 0.000384)
4	(0, 0.001852)	(0, 0.001747)	(0, 0.003065)	(0, 0.000012)
5	(0, 0.000214)	(0, 0.000202)	(0, 0.000414)	(0, 0.000000)
6	(0, 0.000024)	(0, 0.000023)	(0, 0.000056)	(0, 0.000000)
7	(0, 0.000002)	(0, 0.000002)	(0, 0.000007)	(0, 0.000000)
8	(0, 0.000000)	(0, 0.000000)	(0, 0.000001)	(0, 0.000000)
9	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)

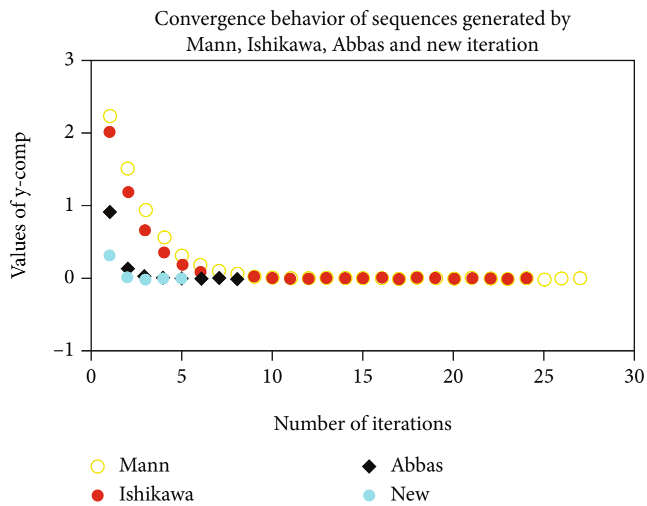

FIGURE 1: Convergence behavior of Mann, Ishikawa, and Abbas with new iteration for the initial values (0,3) with parameters $\alpha = 0.6$, $\beta = 0.3$, $\gamma = 0.5$.

$$\begin{aligned}\varepsilon_{\upsilon} &= \sigma(p_{\upsilon}, s_{\upsilon+1}) \leq \sigma(p_{\upsilon}, s) + \sigma(s_{\upsilon+1}, s) \leq \sigma(p_{\upsilon}, s) \\ &\quad + a[a(1-\gamma_{\upsilon}(1-a))(1-(1-a)\alpha_{\upsilon}\beta_{\upsilon})]\sigma(s_{\upsilon}, s) \\ &\Rightarrow \lim_{\upsilon\longrightarrow\infty} \varepsilon_{\upsilon} = 0.\end{aligned} \tag{69}$$

Hence, it is Y-stable. □

We have a nontrivial example for M.T.A.N.M, and fixed point is numerically approximated by using MATLAB.

Example 3. Let $\Sigma = \mathbb{R}^2$ be a hyperbolic space. Define a relation as

$$(r_1, t_1)'(r_2, t_2) \Leftrightarrow r_1 \leq r_2 \text{and} t_1 \leq t_2. \tag{70}$$

Let $\sigma : \Sigma \times \Sigma \longrightarrow \mathbb{R}$ be defined as

$$\sigma(r, t) = |r_1 - r_2| + |r_1 t_1 - r_2 t_2| \text{ where } r = (r_1, t_1), y = (r_2, t_2). \tag{71}$$

Let $K = [0, 3] \times [0, 3] \subset \Sigma$ and $Y : K \longrightarrow K$ be a mapping defined by

$$Y(r, t) = \left\{ \left(\frac{(1 - \cos r)}{2}, \frac{\exp(t/2) - 1}{2} \right); (r, t) \in K. \right. \tag{72}$$

As $(0, 0)$ is the fixed point of Y, we have to show that Y is monotone for this consider

$$(r_1, t_1)'(r_2, t_2) \Leftrightarrow r_1 \leq r_2 \text{ and } t_1 \leq t_2, \tag{73}$$

then

$$\frac{1 - \cos r_1}{2} \leq \frac{1 - \cos r_2}{2} \text{and} \frac{\exp(t_1/2) - 1}{2} \leq \frac{\exp(t_2/2) - 1}{2}, \tag{74}$$

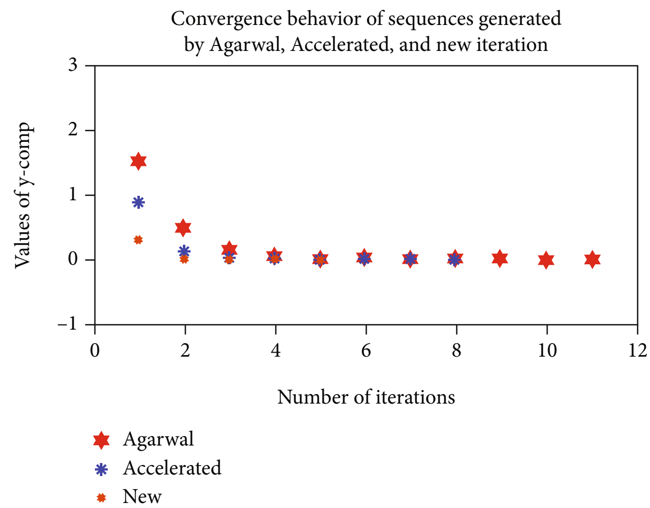

FIGURE 2: Convergence behavior of Agarwal, accelerated with new iteration for the initial values (0,3) with parameters $\alpha = 0.6$, $\beta = 0.3$, $\gamma = 0.5$.

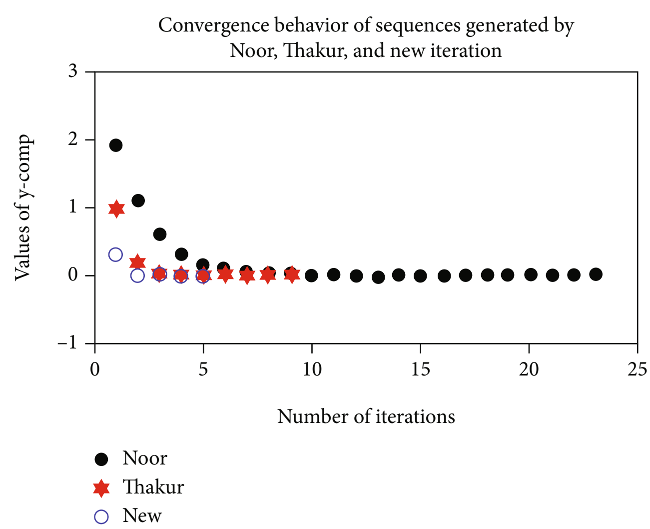

FIGURE 3: Convergence behavior of Noor and Thakur with new iteration for the initial values (0,3) with parameters $\alpha = 0.6$, $\beta = 0.3$, $\gamma = 0.5$.

that is

$$Y(r_1, t_1)' Y(r_2, t_2). \tag{75}$$

Next,

$$\begin{aligned} &Y^{\upsilon}(r_1, t_1) = (0, 0), Y^{\upsilon}(r_2, t_2) = (0, 0), \text{for large } \upsilon, \\ &\quad\Rightarrow \sigma(Y^{\upsilon}(r_1, t_1), Y^{\upsilon}(r_2, t_2)) \\ &\quad= 0 \leq \sigma((r_1, t_1), (r_2, t_2)). \end{aligned} \tag{76}$$

Hence, Y is M.T.A.N.M with $\mu_{\upsilon} = \xi_{\upsilon} = 0$.

The rate of convergence of Mann, Ishikawa, Agarwal, Noor, Abbas, Thakur, and accelerated and new iterations

Table 3: Influence of parameters and initial values by setting stopping parameter at 10^{-15}.

For $\alpha = 0.7$, $\beta = \gamma = 0.1$ with $(x_0, y_0) = (0, 3)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	50	48	26	48	22	22	25	13
For $\alpha = 0.2 = \beta$, $\gamma = 0.3$ with $(x_0, y_0) = (0,2.5)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	221	202	26	203	15	15	22	12
For $\alpha = 0.3 = \beta = 0.7$, $\gamma = 0.2$ with $(x_0, y_0) = (0, 2)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	140	116	24	115	16	16	21	12
For $\alpha = 0.5 = \beta = 0.9$, $\gamma = 0.2$ with $(x_0, y_0) = (0, 1)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	74	57	20	56	18	18	18	11
For $\alpha = 0.3 = \beta = 0.9$, $\gamma = 0.5$ with $(x_0, y_0) = (0,1.5)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	138	109	23	106	14	14	17	11

for the mapping defined in Example 2 is given below. Table 1 shows the convergence behavior of Mann, Ishikawa, Agarwal, and Noor iterations with new iteration for the parameters $\alpha = 0.6, \beta = 0.3$, and $\gamma = 0.5$, with the initial values $x_0 = 0, y_0 = 3$, and tolerance $= 10^{-6}$. New iteration requires less number of iterations for convergence.

Table 2 shows the convergence of Abbas, Thakur, and accelerated iteration with new iteration for the same initial values, parameters, and tolerance.

The following figures show the convergence behavior of different iterations with new iteration in Figures 1–3.

All iterations converges to $(x, y) = (0, 0)$. Comparison shows that new iteration requires the least number of iterations for convergence. Table 3 shows that different parameters have an effect on iterations and by changing the initial values, new iteration not only converges faster but also stable than other iterations

5. Conclusions

In the present article, the concept of monotone asymptotically nonexpansive mapping has been generalized to monotone total asymptotically nonexpansive mapping in the framework of hyperbolic space. New iteration has been introduced to approximate the fixed point for that mapping. We proved the existence of fixed point, faster convergence, and stability results for new iteration. We also constructed a nontrivial example to approximate the fixed point numerically and compare the convergence result of new iteration with some well-known iterations by using MATLAB.

By relaxing the condition of monotonicity, we can also achieve some similar results presented in recent articles [31, 32] by using the proposed iteration.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

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Research Article

Infra Soft Compact Spaces and Application to Fixed Point Theorem

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Infra soft topology is one of the recent generalizations of soft topology which is closed under finite intersection. Herein, we contribute to this structure by presenting two kinds of soft covering properties, namely, infra soft compact and infra soft Lindelöf spaces. We describe them using a family of infra soft closed sets and display their main properties. With the assistance of examples, we mention some classical topological properties that are invalid in the frame of infra soft topology and determine under which condition they are valid. We focus on studying the "transmission" of these concepts between infra soft topology and classical infra topology which helps us to discover the behaviours of these concepts in infra soft topology using their counterparts in classical infra topology and vice versa. Among the obtained results, these concepts are closed under infra soft homeomorphisms and finite product of soft spaces. Finally, we introduce the concept of fixed soft points and reveal main characterizations, especially those induced from infra soft compact spaces.

1. Introduction

In our daily life, we face many types of uncertain phenomena and problems which require looking for adequate approaches to deal with them. The researchers' efforts in this regard lead to proposing various convenient tools to address uncertainty and vagueness. One of the notable tools related to our interest is the soft set which was introduced in 1999 by Molodtsov [1]. He discussed its different applications like smoothness of functions, game theory, theory of measurement, and Riemann integration. Then, soft sets have been applied to several scopes like medical science [2], computer science [3], and decision-making problems [4].

After three years of the emergence of soft sets, Maji with his coauthors [5] put forward the basic concepts of soft set theory. They explored primary operations like the intersection of two soft sets and their union and difference; also, they defined a complement of soft set. Later on, many scholars and researchers interested in soft set theory redefined the concepts given by Maji et al. and displayed new types of soft operations and operators. To keep some properties and results of crisp set theory, Ali et al. [6] proposed new types of these operations and operators. The improvements and contributions to soft set theory have been continued which lead to defining many sorts of soft equality like lower and upper soft equality [7], gf-soft equality [8], and T-soft equality [9].

As it is well known, topologists employed the generalizations of crisp sets to construct novel extensions of topologies. As a continuation of this path, Shabir and Naz [10] and Çağman et al. [11] hybridized classical topology and soft sets to formulate soft topological spaces. In fact, they differently formulated the concept of soft topology. Shabir and Naz stipulated the constant of the universal crisp set and set of parameters which the members of soft topology defined over them, whereas Çağman et al. did not impose any conditions on the universal crisp set and set of parameters. Our approach in this paper goes according to the definition of Shabir and Naz. Special kinds of soft topologies called enriched and extended soft topologies were introduced in [12]. Al-Shami and Kocinac [13] scrutinized the role of extended soft topology to link the concepts in soft topologies with their counterparts in classical topologies.

It was noted that several properties of topological concepts are preserved under some generalizations of soft topology such as supra soft topology [14], soft bitopology [15], and infra soft topology [16]. This means that we can consider that some topology's conditions are superfluous in some cases. In fact, this matter was applied in crisp settings to describe some real-life issues (see [17, 18]). Another merit of generalizations of soft topology is the ease of building counterexamples that show the relationships between the concepts under study. For these reasons, we are interested to study infra soft topology which is one of the recent interesting developments of soft topology.

Compactness and Lindelöfness are one of the interesting concepts in soft topologies. They were studied in some pioneer articles such as [19–21]. Our contribution herein is to analyze the properties of these two concepts in the frame of infra soft topological spaces. We note the validity of some properties of (classical) soft compactness and Lindelöf spaces via infra soft topological structures. This helps us to discuss many topological concepts and reveal the relationships among them in this frame instead of soft topology, so we target in this manuscript to perform an exhaustive analysis of infra soft topological structures.

The fixed point theorem is a hot topic in recent years. It has been investigated in many papers such as [22–25]. In this work, we put forward the basis of fixed soft points in the introduced frame. Some results that associated fixed soft points with infra soft compact spaces and infra parametric infra topological spaces are studied in detail.

The layout followed in the rest of this manuscript is as follows. In Section 2, we recall the definitions and results that we need to comprehend this work. In Section 3, we introduce the concepts of infra soft compact and infra soft Lindelöf spaces and characterize them. We establish their master features and reveal some of their counterparts' properties that are lost. In Section 4, we study fixed soft points in the frame of infra soft topological spaces and explore the role of infra soft compactness to obtain a fixed soft point. Finally, we epitomize the paper's fulfillments and suggest some future works in Section 5.

2. Preliminaries

Through this section, we mention the materials that make this study self-contained. We divide it into two subsections.

2.1. Soft Set Theory

Definition 1 (see [1]). Let $\mathscr{B}$ be a set of parameters, $\mathscr{A}$ a universal set, and $2^{\mathscr{A}}$ the power set of $\mathscr{A}$. A soft set over $\mathscr{A}$ is an ordered pair $(\delta, \mathscr{B})$, where $\delta : \mathscr{B} \longrightarrow 2^{\mathscr{A}}$ is a crisp map. We express the soft set as follows: $(\delta, \mathscr{B}) = \{(\beta, \delta(\beta))\colon \beta \in \mathscr{B}$ and $\delta(\beta) \in 2^{\mathscr{A}}\}$.

A family of all soft sets over $\mathscr{A}$ under a set of parameters $\mathscr{B}$ is symbolized by $S(\mathscr{A}_{\mathscr{B}})$.

Definition 2 (see [6]). A soft set $(\delta^c, \mathscr{B})$ is called a complement of $(\delta, \mathscr{B})$ provided that a map $\delta^c : \mathscr{B} \longrightarrow 2^{\mathscr{A}}$ such that $\delta^c(\beta) = \mathscr{A} \setminus \delta(\beta)$ for each $\beta \in \mathscr{B}$.

Definition 3 (see [5]). If the image of each parameter of $\mathscr{B}$ under a map $\delta : \mathscr{B} \longrightarrow 2^{\mathscr{A}}$ is the universal set $\mathscr{A}$, then $(\delta, \mathscr{B})$ is called the absolute soft set over $\mathscr{A}$. Its complement is called the null soft set. The absolute and null soft sets are symbolized by $\tilde{\mathscr{A}}$ and Φ, respectively.

Definition 4 (see [26, 27]). If all components of a soft set are equal (resp., finite, countable), then we called it a stable (resp., finite, countable) soft set. Otherwise, it is called unstable (resp., infinite, uncountable).

Definition 5 (see [12, 27]). If the image of one parameter, say β, under a map $P : \mathscr{B} \longrightarrow 2^{\mathscr{A}}$ is a singleton set, say $\{\alpha\}$, and the image of each parameter $\beta' \in \mathscr{B} \setminus \{\beta\}$ is the empty set, then a soft set $(P, \mathscr{B})$ is called a soft point over $\mathscr{A}$. It is briefly symbolized by P_β^α.

Definition 6 (see [10, 27]). There are two belong and two nonbelong relations between an element $\alpha \in \mathscr{A}$ and a soft set $(\delta, \mathscr{B})$ defined as follows.

(i) $\alpha \in (\delta, \mathscr{B})$ if $\alpha \in \delta(\beta)$ for all $\beta \in \mathscr{B}$

(ii) $\alpha \notin (\delta, \mathscr{B})$ if $\alpha \notin \delta(\beta)$ for some $\beta \in \mathscr{B}$

(iii) $\alpha \Subset (\delta, \mathscr{B})$ if $\alpha \in \delta(\beta)$ for some $\beta \in \mathscr{B}$

(iv) $\alpha \not\Subset (\delta, \mathscr{B})$ if $\alpha \notin \delta(\beta)$ for all $\beta \in \mathscr{B}$

Definition 7 (see [6]). The intersection of two soft sets $(\delta, \mathscr{B})$ *and* $(\xi, \mathscr{C})$ *over* $\mathscr{A}$, symbolized by $(\delta,\mathscr{B})\tilde{\bigcap}(\xi, \mathscr{C})$, is a soft set $(\lambda, \mathscr{D})$, where $\mathscr{D} = \mathscr{B} \cap \mathscr{C} \neq \varnothing$, and a map $\lambda : \mathscr{D} \longrightarrow 2^{\mathscr{A}}$ is given by $\lambda(\beta) = \delta(\beta) \cap \xi(\beta)$ for each $\beta \in \mathscr{D}$.

Definition 8 (see [5]). The union of two soft sets $(\delta, \mathscr{B})$ and $(\xi, \mathscr{C})$ *over* $\mathscr{A}$, symbolized by $(\delta,\mathscr{B})\tilde{\bigcup}(\xi, \mathscr{C})$, is a soft set $(\lambda, \mathscr{D})$, where $\mathscr{D} = \mathscr{B} \cup \mathscr{C}$ and a map $\lambda : \mathscr{D} \longrightarrow 2^{\mathscr{A}}$ is given as follows:

$$\lambda(\beta) = \begin{cases} \delta(\beta) & : \beta \in \mathscr{B} \setminus \mathscr{C}, \\ \xi(\beta) & : \beta \in \mathscr{C} \setminus \mathscr{B}, \\ \delta(\beta) \cup \xi(\beta) & : \beta \in \mathscr{B} \cap \mathscr{C}. \end{cases} \tag{1}$$

Definition 9 (see [28]). A soft set $(\delta, \mathscr{B})$ is a subset of a soft set $(\xi, \mathscr{C})$, symbolized by $(\delta,\mathscr{B}) \tilde{\subseteq} (\xi, \mathscr{C})$, *if* $\mathscr{B} \subseteq \mathscr{C}$ and $\delta(\beta) \subseteq \xi(\beta)$ for all $\beta \in \mathscr{B}$. The soft sets $(\delta, \mathscr{B})$ and $(\xi, \mathscr{C})$ are called soft equal if each is a subset of the other.

Definition 10 (see [19]). A family of soft sets is said to have the finite (resp., countable) intersection property if the finite (resp., countable) intersection of any members of this family is nonnull.

Definition 11 (see [20]). The Cartesian product of $(\delta, \mathscr{B})$ and $(\xi, \mathscr{C})$, symbolized by $(\delta \times \xi, \mathscr{B} \times \mathscr{C})$, is defined as $(\delta \times \xi)(\beta, \beta') = \delta(\beta) \times \xi(\beta')$ for each $(\beta, \beta') \in \mathscr{B} \times \mathscr{C}$.

Definition 12 (see [29]). A soft map f_{φ} from $S(\mathscr{A}_{\mathscr{B}})$ to $S(\mathscr{Z}_{\mathscr{C}})$ is a pair of crisp maps f and φ, where $f : \mathscr{A} \longrightarrow \mathscr{Z}$, $\varphi : \mathscr{B} \longrightarrow \mathscr{C}$. Let $(\delta, \mathscr{M})$ and $(\xi, \mathscr{N})$ be, respectively, subsets of $S(\mathscr{A}_{\mathscr{B}})$ and $S(\mathscr{Z}_{\mathscr{C}})$. Then, the image of $(\delta, \mathscr{M})$ and preimage of $(\xi, \mathscr{N})$ are given by the following.

(i) $f_{\varphi}(\delta, \mathscr{M}) = (f(\delta), \mathscr{C})$ is a soft set in $S(\mathscr{Z}_{\mathscr{C}})$ such that

$$f(\delta)(\omega) = \begin{cases} \widetilde{\bigcup}_{\beta\in\varphi^{-1}(\omega)\bigcap\mathscr{M}} f(\delta(\beta)) & : \varphi^{-1}(\omega) \neq \varnothing, \\ \varnothing & : \varphi^{-1}(\omega) = \varnothing, \end{cases} \quad (2)$$

for each $\omega \in \mathscr{C}$

(ii) $f_{\varphi}^{-1}(\xi, \mathscr{N}) = (f^{-1}(\xi), \mathscr{B})$ is a soft set in $S(\mathscr{A}_{\mathscr{B}})$ such that

$$f^{-1}(\xi)(\beta) = \begin{cases} f^{-1}(\xi(\varphi(\beta))) & : \varphi(\beta) \in \mathscr{N}, \\ \varnothing & : \varphi(\beta) \in \mathscr{N}, \end{cases} \quad (3)$$

for each $\beta \in \mathscr{B}$

Definition 13 (see [20, 30]). A soft *map* $f_{\varphi} : S(\mathscr{A}_{\mathscr{B}}) \longrightarrow S(\mathscr{Z}_{\mathscr{C}})$ is said to be injective (resp., surjective, bijective) if both f and φ are injective (resp., surjective, bijective).

2.2. Infra Soft Topological Spaces

Definition 14 (see [16]). A family Ω of soft sets over $\mathscr{A}$ with $\mathscr{B}$ as a parameter set is said to be an infra soft topology on $\mathscr{A}$ if it is closed under finite intersection and Φ is a member of Ω.

The triple $(\mathscr{A}, \Omega, \mathscr{B})$ is called an infra soft topological space (briefly, ISTS). We called a member of Ω an infra soft open set and called its complement an infra soft closed set. We called $(\mathscr{A}, \Omega, \mathscr{B})$ stable if all its infra soft open sets are stable and called finite (resp., countable) if $\mathscr{A}$ is finite (resp., countable).

Proposition 15 (see [16]). *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an ISTS. Then, the collection $\Omega_{\beta} = \{\delta(\beta) : (\delta, \mathscr{B}) \in \Omega\}$ forms an infra topology on $\mathscr{A}$ for each $\beta \in \mathscr{B}$.*

We called Ω_{β} a parametric infra topology.

Proposition 16 (see [16]). *Suppose that $\Psi = \{\Omega_{\beta}\}_{\beta\in\mathscr{B}}$ is a class of crisp infra topologies on $\mathscr{A}$. Then,*

$$\Omega(\Psi) = \left\{\{(\beta, F(\beta)) : \beta \in \mathscr{B}\} \in S(\mathscr{A}_{\mathscr{B}}) \text{ such that } F(\beta) \in \Omega_{\beta} \text{ for each } \beta \in \mathscr{B}\right\} \quad (4)$$

defines an infra soft topology on $\mathscr{A}$.

The ISTS given in the above proposition is called an extended infra soft topology on $\mathscr{A}$ or an infra soft topology on $\mathscr{A}$ generated by Ψ.

Definition 17. An ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ is called an infra *pp*-soft T_2 (or an infra *pp*-soft Hausdorff)-space if for every $\alpha \neq \varepsilon \in \mathscr{A}$, there exist disjoint infra soft open sets $(\delta, \mathscr{B})$ and $(\xi, \mathscr{B})$ such that $\alpha \Subset (\delta, \mathscr{B})$, $\varepsilon \Subset (\xi, \mathscr{B})$ and $\varepsilon \in (\delta, \mathscr{B})$, $\alpha \in (\xi, \mathscr{B})$.

In the above definition, if we replace the relations $(\Subset, \in)$ by $(\Subset, \Subset)$ (resp., $(\in, \in)(\in, \Subset)$), then we called $(\mathscr{A}, \Omega, \mathscr{B})$ an infra *pt*-soft T_2 (resp., infra *tp*-soft T_2, infra *tt*-soft T_2)-space.

Definition 18 (see [16]). Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an ISTS and $(\xi, \mathscr{B})$ be a nonnull subset of $\tilde{\mathscr{A}}$. Then, $\Omega_{(\xi,\mathscr{B})} = \{(\xi, \mathscr{B}) \widetilde{\bigcap} (\delta, \mathscr{B}) : (\delta, \mathscr{B}) \in \Omega\}$ is called an infra soft relative topology on $(\xi, \mathscr{B})$ *and* $((\xi, \mathscr{B}), \Omega_{(\xi,\mathscr{B})}, \mathscr{B})$ is called an infra soft subspace of $(\mathscr{A}, \Omega, \mathscr{B})$.

Theorem 19 (see [16]). *Let $((\xi, \mathscr{B}), \Omega_{(\xi,\mathscr{B})}, \mathscr{B})$ be an infra soft subspace of $(\mathscr{A}, \Omega, \mathscr{B})$. Then, $(\lambda, \mathscr{B})$ is an infra soft closed subset of $((\xi, \mathscr{B}), \Omega_{(\xi,\mathscr{B})}, \mathscr{B})$ iff there exists an infra soft closed subset $(\delta, \mathscr{B})$ of $(\mathscr{A}, \Omega, \mathscr{B})$ such that $(\lambda, \mathscr{B}) = (\xi, \mathscr{B}) \widetilde{\bigcap} (\delta, \mathscr{B})$.*

Definition 20 (see [31]). A soft mapping $f_{\varphi} : (\mathscr{A}_1, \Omega_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_2, \Omega_2, \mathscr{B}_2)$ *is said to be*

(i) infra soft continuous provided that the preimage of any infra soft open set is an infra soft open set

(ii) infra soft open (resp., infra soft closed) if the image of any infra soft open (resp., infra soft closed) set is an infra soft open (resp., infra soft closed) set

(iii) an infra soft homeomorphism if it is infra soft continuous, infra soft open, and bijective

A property which is kept by any infra soft homeomorphism is said to be an infra soft topological property.

Proposition 21 (see [31]). *Let $\{(\mathscr{A}_k, \Omega_k, \mathscr{B}_k) : k \in K\}$ be a family of ISTSs. Then, $\Omega = \{\prod_{k\in K}(\delta_k, \mathscr{B}_k) : (\delta_k, \mathscr{B}_k) \in \tau_k\}$ is an infra soft topology on $\mathscr{A} = \prod_{k\in K}\mathscr{A}_k$ under a set of parameters $\mathscr{B} = \prod_{k\in K}\mathscr{B}_k$.*

We called $\mathbf{\Omega}$ given in the proposition above a product of infra soft topologies and $(\mathscr{A}, \mathbf{\Omega}, \mathscr{B})$ a product of infra soft spaces.

3. Infra Soft Compact and Infra Soft Lindelöf Spaces

This section is devoted to investigating compactness and Lindelöfness in infra soft topological spaces. We scrutinize their main properties and bring to light some celebrated results of classical compactness and Lindelöfness that are invalid in the frame of ISTS. For illustration and validation, various examples are offered.

Definition 22.

(i) A family of infra soft open sets $\{(\delta_k, \mathscr{B}): k \in K\}$ is said to be an infra soft open cover of an ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ provided that the union of its elements covers $\tilde{\mathscr{A}}$, i.e., $\tilde{\mathscr{A}} = \tilde{\bigcup}_{k\in K}(\delta_k, \mathscr{B})$

(ii) An ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ is said to be infra soft compact (resp., infra soft Lindelöf) provided that every infra soft open cover of $\tilde{\mathscr{A}}$ has a finite (resp., countable) subcover

(iii) A soft subset $(\delta, \mathscr{B})$ of an ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ is said to be infra soft compact (resp., infra soft Lindelöf) provided that every infra soft open cover of $(\delta, \mathscr{B})$ has a finite (resp., countable) subcover

By the two examples below, we explain the existence and uniqueness of infra soft compact and infra soft Lindelöf spaces.

Example 23. Let $\Omega = \{\tilde{\mathbb{R}}\} \bigcup \{(\delta,\mathscr{B}) \tilde{\subseteq} \tilde{\mathbb{R}} : 3 \tilde{\in} (\delta, \mathscr{B})\}$ be an infra soft topology on the real number set $\mathbb{R}$ with $\mathscr{B}$ as an arbitrary set of parameters. It is easy to check that $(\mathbb{R}, \Omega, \mathscr{B})$ is infra soft compact.

Example 24. Let $\Omega = \{\tilde{\mathbb{R}}\} \bigcup \{(\delta,\mathscr{B}) \tilde{\subseteq} \tilde{\mathbb{R}} : 3 \tilde{\in} (\delta, \mathscr{B})\}$ be an infra soft topology on the real number set $\mathbb{R}$ with $\mathscr{B}$ as an arbitrary set of parameters such that $|\mathscr{B}| \geq 2$. Now, the family $\{(\delta,\mathscr{B}) \tilde{\subseteq} \tilde{\mathbb{R}}$: there is only one parameter $\beta \in \mathscr{B}$ such that $3 \in \delta(\beta)\}$ forms an infra soft open cover of $\tilde{\mathbb{R}}$. It is easy to check that this infra soft open cover does not have a countable subcover; hence, $(\mathbb{R}, \Omega, \mathscr{B})$ is not infra soft Lindelöf.

The proofs of the next two results are easy, so they will be canceled.

Proposition 25. *Every infra soft compact space is infra soft Lindelöf.*

Proposition 26. *A family of infra soft compact (resp., infra soft Lindelöf) sets is closed under a finite (resp., countable) union.*

By the example below, we explain that Proposition 25 is not converse.

Example 27. A family $\Omega = \{\tilde{\mathbb{N}}, (\delta,\mathscr{B}) \tilde{\subseteq} \tilde{\mathbb{N}} : (\delta, \mathscr{B}) \text{ is finite}\}$ represents an infra soft topology on the set of natural numbers $\mathbb{N}$ with $\mathscr{B} = \{\beta_1, \beta_2\}$ as a set of parameters. It is easy to check that $(\mathbb{N}, \Omega, \mathscr{B})$ is an infra soft Lindelöf space, but not infra soft compact.

Proposition 28. *Every infra soft closed subset $(\xi, \mathscr{B})$ of an infra soft compact (resp., infra soft Lindelöf) space $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft compact (resp., infra soft Lindelöf).*

Proof. Consider $\{(\delta_k, \mathscr{B}): k \in K\}$ as an infra soft open cover of $(\xi, \mathscr{B})$ which is a subset of an infra soft compact space $(\mathscr{A}, \Omega, \mathscr{B})$. Then, $\tilde{\bigcup}_{k\in K}(\delta_k,\mathscr{B}) \tilde{\bigcup} (\xi^c, \mathscr{B})$ is an infra soft open cover of $\tilde{\mathscr{A}}$. By hypothesis, $\tilde{\mathscr{A}} = \tilde{\bigcup}_{k=1}^{n}(\delta_k,\mathscr{B}) \tilde{\bigcup} (\xi^c, \mathscr{B})$. Consequently, we obtain $(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n}(\delta_k, \mathscr{B})$. This ends the proof that $(\xi, \mathscr{B})$ is infra soft compact. ☐

Following a similar technique, one can prove the case between parentheses.

The converse of the above proposition fails as illustrated in the next example.

Example 29. Consider an ISTS $(\mathbb{R}, \Omega, \mathscr{B})$ as given in Example 23. Let $\mathscr{B} = \{\beta_1, \beta_2\}$. Then, $(\delta, \mathscr{B}) = \{(\beta_1, \{5\}), (\beta_2, \{5\})\}$ is infra soft compact, but not infra soft closed.

Corollary 30. *The intersection of infra soft compact (resp., infra soft Lindelöf) and infra soft closed sets is infra soft compact (resp., infra soft Lindelöf).*

One of the celebrated results in classical topology reports that the finite (resp., countable) topological space is compact (resp., Lindelöf); this result evaporates in ISTSs as the next example elucidates.

Example 31. Let Ω_1 and Ω_2 be two discrete infra soft topologies on a finite set $\mathscr{A}_1$ and a countable set $\mathscr{A}_2$, respectively. Let the sets of natural numbers $\mathbb{N}$ and real numbers $\mathbb{R}$ be sets of parameters. It is clear that $(\mathscr{A}_1, \Omega_1, \mathbb{N})$ is not infra soft compact in spite of $\mathscr{A}_1$ being finite, and $(\mathscr{A}_2, \Omega_2, \mathbb{R})$ is not infra soft Lindelöf in spite of $\mathscr{A}_2$ being countable.

Note that the intersection of two infra soft compact (resp., infra soft Lindelöf) sets needs not be infra soft compact (resp., infra soft Lindelöf). The example given below confirms this fact.

Example 32. Consider an ISTS $(\mathbb{R}, \Omega, \mathscr{B})$ as given in Example 23. Let $\mathscr{B} = \{\beta_1, \beta_2\}$. Note that the two soft sets $(\delta_1, \mathscr{B}) = \{(\beta_1, \mathbb{R} \setminus \{3\}), (\beta_2, \mathbb{R})\}$ *and* $(\delta_2, \mathscr{B}) = \{(\beta_1, \mathbb{R}), (\beta_2, \mathbb{R} \setminus \{3\})\}$ are infra soft compact. But their intersection is the soft set $\{(\beta_1, \mathbb{R} \setminus \{3\}), (\beta_2, \mathbb{R} \setminus \{3\})\}$ which is not infra soft Lindelöf.

Now, we give a complete description for infra soft compact and infra soft Lindelöf spaces using a family of infra soft closed sets.

Theorem 33. *An ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft compact (resp., infra soft Lindelöf) iff every family of infra soft closed subsets of $(\mathscr{A}, \Omega, \mathscr{B})$, satisfying the finite (resp., countable) intersection property, has, itself, a nonnull intersection.*

Proof. Necessity: consider $\Sigma = \{(\delta_k, \mathscr{B}) \colon k \in K\}$ as a family of infra soft closed subsets of $(\mathscr{A}, \Omega, \mathscr{B})$ which is an infra soft compact space. Suppose that $\widetilde{\bigcap}_{k\in K}(\delta_k, \mathscr{B}) = \Phi$. Then, $\tilde{\mathscr{A}} = \widetilde{\bigcup}_{k\in K}(\delta_k^c, \mathscr{B})$. By the hypothesis of infra soft compactness, we obtain $\tilde{\mathscr{A}} = \widetilde{\bigcup}_{k=1}^{n}(\delta_k^c, \mathscr{B})$. Hence, $\bigcap_{k=1}^{n}(\delta_k, \mathscr{B}) = \Phi$, as required.

Sufficiency: consider $\Sigma = \{(\delta_k, \mathscr{B}) \colon k \in K\}$ as an infra soft open cover of $(\mathscr{A}, \Omega, \mathscr{B})$. Suppose that Σ does not have a finite subcover of $\tilde{\mathscr{A}}$. Then $\tilde{\mathscr{A}} \setminus \bigcup_{k\in I}(\delta_k, \mathscr{B}) \neq \Phi$ for any finite set $I \subseteq K$. Therefore, $\widetilde{\bigcap}_{k\in I}(\delta_k^c, \mathscr{B}) \neq \Phi$. This means that $\{(\delta_k^c, \mathscr{B}) \colon k \in I\}$ is a family of infra soft closed subsets of $\tilde{\mathscr{A}}$ which has the finite intersection property. Thus, $\widetilde{\bigcap}_{k\in K}(\delta_k^c, \mathscr{B}) \neq \Phi$. Consequently, $\tilde{\mathscr{A}} \neq \widetilde{\bigcup}_{k\in I}(\delta_k, \mathscr{B})$. But this contradicts that Σ is an infra soft open cover of $\tilde{\mathscr{A}}$. Hence, $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft compact.

Following a similar technique, one can prove the case between parentheses. □

In the next example, we show that there is no relationship between infra soft closed sets and the different types of infra soft T_2-spaces.

Example 34. Consider an ISTS $(\mathscr{N}, \Omega, \mathscr{B})$ as given in Example 27. Note that a soft set $(\delta_1, \mathscr{B}) = \{(\beta_1, \mathbb{N} \setminus \{1, \sqrt{2}\}), (\beta_2, \mathbb{N} \setminus \{2, \sqrt{3}\})\}$ is infra soft closed. But it is not infra soft compact in spite of $(\mathscr{N}, \Omega, \mathscr{B})$ being an infra tt-soft T_2 (infra *tp -soft* T_2 *, infra pt -soft* T_2 *, infra pp -soft* T_2*)-space.*

Now, we investigate under which conditions the well-known relationship between closed sets and T_2-spaces are satisfied in the frame of ISTSs.

Proposition 35. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an infra tt-soft Hausdorff space such that Ω is closed under a finite union. Then, every stable infra soft compact subset of $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft closed.*

Proof. Let $(\xi, \mathscr{B})$ be a stable infra soft compact subset of an infra tt-soft Hausdorff space $(\mathscr{A}, \Omega, \mathscr{B})$, and let $P_\beta^\alpha \in (\xi, \mathscr{B})^c$. Since $(\xi, \mathscr{B})$ is stable, we get $\alpha \neq \gamma_k$ for each $P_\beta^{\gamma_k} \in (\xi, \mathscr{B})$. Therefore, there are two disjoint infra soft open sets $(\delta_k, \mathscr{B})$ and $(\lambda_k, \mathscr{B})$ such that $\alpha \in (\delta_k, \mathscr{B})$ and $\gamma_k \in (\lambda_k, \mathscr{B})$. Now, $\{(\lambda_k, \mathscr{B}) \colon k \in K\}$ forms an infra soft open cover of $(\xi, \mathscr{B})$. By hypothesis, $(\xi,\mathscr{B}) \tilde{\subseteq} \widetilde{\bigcup}_{k=1}^{n}(\lambda_k, \mathscr{B})$. Since Ω is an infra soft topology closing under a finite union, $\bigcap_{k=1}^{n}(\delta_k, \mathscr{B})$ and $\widetilde{\bigcup}_{k=1}^{n}(\lambda_k, \mathscr{B})$ are disjoint infra soft open sets. Note that $P_\beta^\alpha \in \widetilde{\bigcap}_{k=1}^{n}(\delta_k,\mathscr{B}) \tilde{\subseteq} (\xi, \mathscr{B})^c$. Thus, $(\xi, \mathscr{B})^c$ is infra soft open set which automatically means that $(\xi, \mathscr{B})$ is infra soft closed. □

Corollary 36. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an infra tp-soft Hausdorff space such that Ω is closed under a finite union. Then every stable infra soft compact subset of $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft closed.*

Corollary 37. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be a stable infra pp-soft Hausdorff space such that Ω is closed under a finite union. Then every stable infra soft compact subset of $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft closed.*

Theorem 38. *Let $(\xi, \mathscr{B})$ be an infra soft compact subset of an infra tt -soft Hausdorff space $(\mathscr{A}, \Omega, \mathscr{B})$ such that Ω is closed under a finite union. Then for each $\alpha \Subset (\xi, \mathscr{B})$, there are disjoint infra soft open sets $(\delta, \mathscr{B})$ and $(\lambda, \mathscr{B})$ such that $\alpha \in (\delta, \mathscr{B})$ and $(\xi,\mathscr{B}) \tilde{\subseteq} (\lambda, \mathscr{B})$.*

Proof. Let $\alpha \Subset (\xi, \mathscr{B})$. Then $\alpha \neq \gamma_k$ for each $\gamma_k \Subset (\xi, \mathscr{B})$. By hypothesis, there exist infra soft open sets $(\delta_k, \mathscr{B})$ and $(\lambda_k, \mathscr{B})$ such that $\alpha \in (\delta_k, \mathscr{B})$ and $\gamma_k \in (\lambda_k, \mathscr{B})$. Now, $\{(\lambda_k, \mathscr{B}) \colon k \in K\}$ forms an infra soft open cover of $(\xi, \mathscr{B})$. By the infra soft compactness of $(\xi, \mathscr{B})$ we obtain $(\xi, \mathscr{B}) \subseteq \widetilde{\bigcup}_{k=1}^{n}(\lambda_k, \mathscr{B})$. Since Ω is an infra soft topology closing under a finite union, $\widetilde{\bigcup}_{k=1}^{n}(\lambda_k, \mathscr{B})$ and $\widetilde{\bigcap}_{k=1}^{n}(\delta_k, \mathscr{B})$ are the required disjoint infra soft open sets. Hence, the proof is complete. □

Theorem 39. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an infra tt-soft Hausdorff space which is infra soft compact and closed under a finite union. Then it is infra tt-soft regular.*

Proof. Let $(\xi, \mathscr{B})$ be an infra soft closed set such that $\alpha \Subset (\xi, \mathscr{B})$. Then, $(\xi, \mathscr{B})$ is infra soft compact. According to Theorem 38, there exist disjoint infra soft open sets $(\delta, \mathscr{B})$ and $(\lambda, \mathscr{B})$ such that $\alpha \in (\delta, \mathscr{B})$ and $(\xi,\mathscr{B}) \tilde{\subseteq} (\lambda, \mathscr{B})$. Hence, $(\mathscr{A}, \Omega, \mathscr{B})$ is infra tt-soft regular. □

Corollary 40. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an infra tt-soft Hausdorff space which is infra soft compact and closed under a finite union. Then it is infra tt-soft T_3.*

Definition 41. An ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ is said to be infra soft T_2' -space if for every $P_{\beta_1}^{\alpha_1} \neq P_{\beta_2}^{\alpha_2} (\alpha_1 \neq \alpha_2 \text{ or } \beta_1 \neq \beta_2)$, there are two disjoint infra soft open sets $(\delta, \mathscr{B})$ and $(\lambda, \mathscr{B})$ such that $P_{\beta_1}^{\alpha_1} \in (\delta, \mathscr{B})$ and $P_{\beta_2}^{\alpha_2} \in (\lambda, \mathscr{B})$.

Proposition 42. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an infra soft T_2'-space such that Ω is closed under a finite union. Then every infra soft compact subset of $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft closed.*

Proof. Let $(\xi, \mathscr{B})$ be an infra soft compact subset of $(\mathscr{A}, \Omega, \mathscr{B})$ which is an infra soft T_2'-space. Let $P_\beta^\alpha \in (\xi, \mathscr{B})^c$. Then for each $P_{\rho_k}^{\gamma_k} \in (\xi, \mathscr{B})$, we get $P_{\rho_k}^{\gamma_k} \neq P_\beta^\alpha$. Therefore, there are two disjoint infra soft open sets $(\delta_k, \mathscr{B})$ and $(\lambda_k, \mathscr{B})$ such that $P_{\rho_k}^{\gamma_k} \in (\delta_k, \mathscr{B})$ and $P_\beta^\alpha \in (\lambda_k, \mathscr{B})$. Now, $\{(\delta_k, \mathscr{B}) \colon k \in K\}$ forms an infra soft open cover of $(\xi, \mathscr{B})$. By hypothesis, $(\xi, \mathscr{B}) \tilde{\subseteq} \widetilde{\bigcup}_{k=1}^{n}(\delta_k, \mathscr{B})$. Since Ω is an infra soft topology closing under a finite union, $\widetilde{\bigcap}_{k=1}^{n}(\lambda_k, \mathscr{B})$ and $\widetilde{\bigcup}_{k=1}^{n}(\delta_k, \mathscr{B})$ are disjoint infra soft open sets. Note that $P_\beta^\alpha \in \widetilde{\bigcap}_{k=1}^{n}(\lambda_k,\mathscr{B}) \tilde{\subseteq} (\xi, \mathscr{B})^c$. Thus, $(\xi, \mathscr{B})^c$ is infra soft open set which automatically means that $(\xi, \mathscr{B})$ is infra soft closed. □

Proposition 43. *The infra soft continuous image of an infra soft compact (resp., infra soft Lindelöf) set is infra soft compact (resp., infra soft Lindelöf).*

Proof. Consider $f_\varphi : (\mathscr{A}, \Omega, \mathscr{B}) \longrightarrow (\mathscr{C}, \mho, \mathscr{D})$ as an infra soft continuous map and let $(\xi, \mathscr{B})$ be an infra soft compact subset of $\tilde{\mathscr{A}}$. Suppose that $\{(\delta_k, \mathscr{B}): k \in K\}$ is an infra soft open cover of $f_\varphi(\xi, \mathscr{B})$. Then $(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k\in K} f_\varphi^{-1}(\delta_k, \mathscr{B})$ and $f_\varphi^{-1}(\delta_k, \mathscr{B})$ is an infra soft open set for each $k \in K$. By hypotheses of infra soft compactness of $(\xi, \mathscr{B})$, we obtain $(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n} f_\varphi^{-1}(\delta_k, \mathscr{B})$. Therefore, $f_\varphi(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n} f_\varphi (f_\varphi^{-1}(\delta_k,\mathscr{B})) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n}(\delta_k, \mathscr{B})$. This ends the proof that $f_\varphi(\xi, \mathscr{B})$ is infra soft compact. □

Following similar arguments, one can prove the case between parentheses.

Corollary 44. *The property of being an infra soft compact (infra soft Lindelöf) space is preserved under an infra soft homeomorphism, i.e., it is an infra soft topological property.*

In the next result, we explain that the properties of infra soft compactness and infra soft Lindelöfness transmit from infra soft topology to its parametric infra topologies under an extended condition.

Theorem 45. *If $(\mathscr{A}, \Omega, \mathscr{B})$ is an extended infra soft compact (resp., extended infra soft Lindelöf) space, then $(\mathscr{A}, \Omega_\beta)$ is infra compact (resp., infra Lindelöf) for each $\beta \in \mathscr{B}$.*

Proof. Let $\{H_k : k \in K\}$ be an infra open cover of $(\mathscr{A}, \Omega_\beta)$. Consider $\Theta = \{(\delta_k, \mathscr{B}): k \in K\}$ as a family of soft set which is defined as $\delta_k(\beta) = H_k$ and $\delta_k(\beta') = \mathscr{A}$ for each $\beta' \neq \beta$. Since Ω is extended, Θ is an infra soft open cover of $\tilde{\mathscr{A}}$. By the hypothesis of infra soft compactness, we obtain $\tilde{\mathscr{A}} = \tilde{\bigcup}_{k=1}^{n}(\delta_k, \mathscr{B})$. This implies that $\mathscr{A} = \bigcup_{k=1}^{n} \delta_k(\beta) = \bigcup_{k=1}^{n} H_k$ which proves that $(\mathscr{A}, \Omega_\beta)$ is an infra compact space. □

Following similar arguments, one can prove the case between parentheses.

The converse of the theorem above fails as the next example illustrates.

Example 46. Consider ISTSs $(\mathscr{A}_1, \Omega_1, \mathbb{N})$ and $(\mathscr{A}_2, \Omega_2, \mathbb{R})$ as given in Example 31. It is clear that $(\mathscr{A}_1, \Omega_{1n})$ is an infra compact space for all $n \in \mathbb{N}$ and $(\mathscr{A}_2, \Omega_{2r})$ is an infra Lindelöf space for all $r \in \mathbb{R}$, whereas $(\mathscr{A}_1, \Omega_1, \mathbb{N})$ is not infra soft compact and $(\mathscr{A}_2, \Omega_2, \mathbb{R})$ is not infra soft Lindelöf.

Note that an extended infra soft compactness (resp., extended infra soft Lindelöfness) of $(\mathscr{A}, \Omega, \mathscr{B})$ implies that $\mathscr{B}$ is finite (resp., countable).

In the next result, we determine under which conditions the converse of Theorem 45 hold.

Theorem 47. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be extended such that $\mathscr{B}$ is finite (resp., countable). Then, $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft compact (resp., extended infra soft Lindelöf) space iff $(\mathscr{A}, \Omega_\beta)$ is infra compact (resp., infra Lindelöf) for each $\beta \in \mathscr{B}$.*

Proof. The necessary part was proved in Theorem 45. To prove the sufficient part, consider $\Theta = \{(\delta_k, \mathscr{B}): k \in K\}$ as an infra soft open cover of $(\mathscr{A}, \Omega, \mathscr{B})$. Then, $\mathscr{A} = \bigcup_{k\in K} \delta_k(\beta)$ for each $\beta \in \mathscr{B}$. By hypothesis of infra compactness (resp., resp., infra Lindelöfness), we obtain $\mathscr{A} = \bigcup_{k\in I} \delta_k(\beta)$ for each $\beta \in \mathscr{B}$, where I is finite (resp., countable). Now, we take from Θ all infra soft open sets which (at least) one of their β-components belong to $\{\delta_k(\beta): k \in I\}$. Hence, we obtain the desired result. □

Proposition 48. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be stable. Then, $(\mathscr{A}, \Omega, \mathscr{B})$ is infra soft compact (resp., infra soft Lindelöf) space iff $(\mathscr{A}, \Omega_\beta)$ is infra compact (resp., infra Lindelöf).*

Proof. Straightforward. □

Theorem 49. *$(\xi, \mathscr{B})$ is infra soft compact (resp., infra soft Lindelöf) subset of $(\mathscr{A}, \Omega, \mathscr{B})$ iff a soft subspace $((\xi, \mathscr{B}), \Omega_{(\xi,\mathscr{B})}, \mathscr{B})$ is infra soft compact (resp., infra soft Lindelöf).*

Proof. Necessity: consider $\{(\delta_k, \mathscr{B}): k \in K\}$ as an infra soft open cover of $((\xi, \mathscr{B}), \Omega_{(\xi,\mathscr{B})}, \mathscr{B})$. Then, for each $(\delta_k, \mathscr{B}) \in \Omega_{(\xi,\mathscr{B})}$, there exists $(\pi_k, \mathscr{B}) \in \Omega$ such that $(\delta_k, \mathscr{B}) = (\pi_k,\mathscr{B}) \tilde{\bigcap} (\xi, \mathscr{B})$. Now, $\{(\pi_k, \mathscr{B}): k \in K\}$ is an infra soft open cover of $(\xi, \mathscr{B})$. By hypothesis, $(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n}(\pi_k, \mathscr{B})$. Therefore, $(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n}[(\pi_k,\mathscr{B}) \tilde{\bigcap} (\xi, \mathscr{B})] = \tilde{\bigcup}_{k=1}^{n}(\delta_k, \mathscr{B})$. Thus, $((\xi, \mathscr{B}), \Omega_{(\xi,\mathscr{B})}, \mathscr{B})$ is infra soft compact.

Sufficiency: consider $\{(\delta_k, \mathscr{B}) \in \Omega : k \in K\}$ as an infra soft open cover of $(\xi, \mathscr{B})$. Then, $\{(\delta_k,\mathscr{B}) \tilde{\bigcap} (\xi, \mathscr{B}) : k \in K\}$ is an infra soft open cover of $((\xi, \mathscr{B}), \Omega_{(\xi,\mathscr{B})}, \mathscr{B})$. By hypothesis, $(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n}[(\delta_k,\mathscr{B}) \tilde{\bigcap} (\xi, \mathscr{B})]$. Therefore, $(\xi,\mathscr{B}) \tilde{\subseteq} \tilde{\bigcup}_{k=1}^{n}(\delta_k, \mathscr{B})$. Thus, $(\xi, \mathscr{B})$ is infra soft compact.

Following similar arguments, one can prove the case between parentheses. □

Theorem 50. *The finite product of infra soft compact (resp., infra soft Lindelöf) spaces is infra soft compact (resp., infra soft Lindelöf).*

Proof. Let $(\mathscr{A}_1, \Omega_1, \mathscr{B}_1)$ and $(\mathscr{A}_2, \Omega_2, \mathscr{B}_2)$ be two infra soft Lindelöf spaces. Consider $\{(\xi_k, \mathscr{B}_1 \times \mathscr{B}_2): k \in K\}$ as an infra soft open cover of $\widetilde{\mathscr{A}_1} \times \widetilde{\mathscr{A}_2}$. Now, for each $k \in K$ $(\xi_k, \mathscr{B}_1 \times \mathscr{B}_2) = (\delta_k, \mathscr{B}_1) \times (\lambda_k, \mathscr{B}_2)$, where $(\delta_k, \mathscr{B}_1) \in \Omega_1$ and $(\lambda_k, \mathscr{B}_2) \in \Omega_2$. Then $\widetilde{\mathscr{A}_1} \times \widetilde{\mathscr{A}_2} = \tilde{\bigcup}_{k\in K}(\xi_k, \mathscr{B}_1 \times \mathscr{B}_2) \tilde{\subseteq} [\tilde{\bigcup}_{k\in K}(\delta_k, \mathscr{B}_1)] \times [\tilde{\bigcup}_{k\in K}(\lambda_k, \mathscr{B}_2)]$. That is, $\widetilde{\mathscr{A}_1} = \tilde{\bigcup}_{k\in K}(\delta_k, \mathscr{B}_1)$ and $\widetilde{\mathscr{A}_2} = \tilde{\bigcup}_{k\in K}(\lambda_k, \mathscr{B}_2)$. By hypothesis, there are two countable sets I and J such that $\widetilde{\mathscr{A}_1} = \tilde{\bigcup}_{i\in I}(\delta_i, \mathscr{B}_1)$ and $\widetilde{\mathscr{A}_2} = \tilde{\bigcup}_{j\in J}(\lambda_j, \mathscr{B}_2)$. Thus, $\widetilde{\mathscr{A}_1} \times \widetilde{\mathscr{A}_2} = [\tilde{\bigcup}_{i\in I}(\delta_i, \mathscr{B}_1)] \times [\tilde{\bigcup}_{j\in J}(\lambda_j, \mathscr{B}_2)] = \tilde{\bigcup}_{k\in I \bigcup J}(\xi_k, \mathscr{B}_1 \times \mathscr{B}_2)$. Since $I \bigcup J$ is a countable set, we obtain the desired result. □

Following similar arguments, one can prove the theorem in the case of infra soft compact.

Theorem 51. *Every uncountable (resp., infinite) subset of an infra soft Lindelöf (resp., infra soft compact) space has an infra soft limit point.*

Proof. Let $(\delta, \mathscr{B})$ be an uncountable subset of $(\mathscr{A}, \Omega, \mathscr{B})$ which is an infra soft Lindelöf space. Suppose that no soft point of $\tilde{\mathscr{A}}$ is an infra soft limit point of $(\delta, \mathscr{B})$. Then for each $P^{\alpha}_{\beta} \in \tilde{\mathscr{A}}$, there is an infra soft open set $(\lambda_{\alpha_k}, \mathscr{B})$ containing P^{α}_{β} such that $(\lambda_{\alpha_k}, \mathscr{B}) \tilde{\bigcap} (\delta, \mathscr{B}) \setminus P^{\alpha}_{\beta} = \Phi$. Now, a family $\Lambda = \{(\lambda_{\alpha_k}, \mathscr{B}) : k \in K\}$ forms an infra soft open cover of $\tilde{\mathscr{A}}$. Since $\mathscr{A}$ is infra soft Lindelöf, there is a countable set I such that $\tilde{\mathscr{A}} = \tilde{\bigcup}_{k \in I} (\lambda_{\alpha_k}, \mathscr{B})$. Therefore, $\mathscr{A}$ has at most countable soft points of $(\delta, \mathscr{B})$. This means that $(\delta, \mathscr{B})$ is countable which contradicts the uncountability of $(\delta, \mathscr{B})$. Hence, $(\delta, \mathscr{B})$ has an infra soft limit point. □

Following similar arguments, one can prove the case between parentheses.

4. Fixed Point Theorem in Infra Soft Topological Spaces

Through this portion, we aim to introduce the concept of fixed soft points in the frame of ISTSs and establish its master properties. We present interesting findings that associated fixed soft points with infra soft compact spaces. Finally, the transmission of fixed soft points from infra soft topology to classical infra topology and vice versa is investigated.

Theorem 52. *Let $\{\mathscr{C}_n : n \in \mathbb{N}\}$ be a family of soft sets in an infra soft compact space $(\mathscr{A}, \Omega, \mathscr{B})$. Then $\tilde{\bigcap}_{n \in \mathbb{N}} \mathscr{C}_n \neq \tilde{\Phi}$ provided that the following three conditions are satisfied:*

(i) $\mathscr{C}_n \neq \tilde{\Phi}$ for each $n \in \mathbb{N}$

(ii) $\mathscr{C}_n$ is an infra soft closed set for each $n \in \mathbb{N}$

(iii) $\mathscr{C}_{n+1}$ is a subset of $\mathscr{C}_n$ for each $n \in \mathbb{N}$

Proof. Suppose that $\tilde{\bigcap}_{n \in \mathbb{N}} \mathscr{C}_n = \tilde{\Phi}$. Then, $\tilde{\bigcup}_{n \in \mathbb{N}} \mathscr{C}^c_n = \tilde{\mathscr{A}}$. According to (ii), we obtain $\{\mathscr{C}^c_n : n \in \mathbb{N}\}$ forms an infra soft open cover of $\tilde{\mathscr{A}}$. By the hypothesis of infra soft compactness, there exist $k_1, k_2, \cdots, k_j \in \mathbb{N}$, $k_1 < k_2 < \cdots < k_j$ such that $\tilde{\mathscr{A}} = \mathscr{C}^c_{k_1} \tilde{\bigcup} \mathscr{C}^c_{k_2} \tilde{\bigcup} \cdots \tilde{\bigcup} \mathscr{C}^c_{k_j}$. It comes from (iii) that $\mathscr{C}_{k_j} \tilde{\subseteq} \tilde{\mathscr{A}} = \mathscr{C}^c_{k_1} \tilde{\bigcup} \mathscr{C}^c_{k_2} \tilde{\bigcup} \cdots \tilde{\bigcup} \mathscr{C}^c_{k_j} = [\mathscr{C}_{k_1} \tilde{\bigcap} \mathscr{C}_{k_2} \tilde{\bigcap} \cdots \tilde{\bigcap} \mathscr{C}_{k_j}]^c = \mathscr{C}^c_{k_j}$. This is a contradiction. Hence, $\tilde{\bigcap}_{n \in \mathbb{N}} \mathscr{C}_n \neq \tilde{\Phi}$, as required. □

Proposition 53. *Let $(\mathscr{A}, \Omega, \mathscr{B})$ be an infra soft compact and infra soft T_2 -space such that Ω is closed under finite union. If $f_{\varphi} : (\mathscr{A}, \Omega, \mathscr{B}) \longrightarrow (\mathscr{A}, \Omega, \mathscr{B})$ is infra soft continuous, then f_{φ} has a unique soft point $P^{\alpha}_{\beta} \in \tilde{\mathscr{A}}$.*

Proof. Let $\{\mathscr{C}_n = f_{\varphi}(\mathscr{C}_{n-1}) = f^n_{\varphi}(\tilde{\mathscr{A}})$ for each $n \in \mathbb{N}$, where $\mathscr{C}_1 = f_{\varphi}(\tilde{\mathscr{A}})\}$ be a family of soft sets in $(\mathscr{A}, \Omega, \mathscr{B})$. Obviously, $\mathscr{C}_{n+1} \tilde{\subseteq} \mathscr{C}_n$ for each $n \in \mathbb{N}$. Since f_{φ} is an infra soft continuous map, $\mathscr{C}_n$ is an infra soft compact set for each $n \in \mathbb{N}$, and since $(\mathscr{A}, \Omega, \mathscr{B})$ is an infra soft T_2-space such that Ω is closed under finite union, $\mathscr{C}_n$ is an infra soft closed set for each $n \in \mathbb{N}$. Putting $(\delta, \mathscr{B}) = \tilde{\bigcap}_{n \in \mathbb{N}} \mathscr{C}_n$, according to Theorem 52, we find $(\delta, \mathscr{B})$ is a nonnull soft set. Note that $f_{\varphi}(\delta, \mathscr{B}) = f_{\varphi} (\tilde{\bigcap}_{n \in \mathbb{N}} f^n_{\varphi}(\mathscr{A}^{\sim})) \tilde{\subseteq} \tilde{\bigcap}_{n \in \mathbb{N}} f^{n+1}_{\varphi}(\mathscr{A}^{\sim}) \tilde{\subseteq} \tilde{\bigcap}_{n \in \mathbb{N}} f^n_{\varphi}(\tilde{\mathscr{A}}) = (\delta, \mathscr{B})$. To prove that $(\delta, \mathscr{B}) \tilde{\subseteq} f_{\varphi}(\delta, \mathscr{B})$, suppose that there exists a $P^{\alpha}_{\beta} \in (\delta, \mathscr{B})$ such that $P^{\alpha}_{\beta} \notin f_{\varphi}(\delta, \mathscr{B})$. Let $\mathscr{D}_n = f^{-1}_{\varphi}(P^{\alpha}_{\beta}) \tilde{\bigcap} \mathscr{C}_n$. It is clear that $\mathscr{D}_n \neq \tilde{\Phi}$ and $\mathscr{D}_n \tilde{\subseteq} \mathscr{D}_{n-1}$ for each $n \in \mathbb{N}$. Now, $\mathscr{D}_n$ is an infra soft closed set for each $n \in \mathbb{N}$, and by Theorem 52, there is a soft point P^{γ}_{ρ} such that $P^{\gamma}_{\rho} \in f^{-1}_{\varphi}(P^{\alpha}_{\beta}) \tilde{\bigcap} \mathscr{C}_n$. Therefore, $P^{\alpha}_{\beta} = f_{\varphi}(P^{\gamma}_{\rho}) \in f_{\varphi}(\delta, \mathscr{B})$; this is a contradiction. Hence, $f_{\varphi}(\delta, \mathscr{B}) = (\delta, \mathscr{B})$ which means we obtain the desired result. □

Definition 54. An ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ is said to have a fixed soft point property if every infra soft continuous map $f_{\varphi} : (\mathscr{A}, \Omega, \mathscr{B}) \longrightarrow (\mathscr{A}, \Omega, \mathscr{B})$ has a fixed soft point, i.e., there exists $P^{\alpha}_{\beta} \in \mathscr{A}$ such that $f_{\varphi}(P^{\alpha}_{\beta}) = P^{\alpha}_{\beta}$.

Proposition 55. *The property of being a fixed soft point is an infra soft topological property.*

Proof. Let $(\mathscr{A}_1, \Omega_1, \mathscr{B}_1)$ and $(\mathscr{A}_2, \Omega_2, \mathscr{B}_2)$ be an infra soft homeomorphism. Then there is a bijective soft map $f_{\varphi} : (\mathscr{A}_1, \Omega_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_2, \Omega_2, \mathscr{B}_2)$ such that f_{φ} and f^{-1}_{φ} are infra soft continuous. Consider that $(\mathscr{A}_1, \Omega_1, \mathscr{B}_1)$ has a fixed soft point property, i.e., every infra soft continuous map $f_{\varphi} : (\mathscr{A}_1, \Omega_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_1, \Omega_1, \mathscr{B}_1)$ has a fixed soft point. Now, let $h_{\varphi} : (\mathscr{A}_2, \Omega_2, \mathscr{B}_2) \longrightarrow (\mathscr{A}_2, \Omega_2, \mathscr{B}_2)$ be an infra soft continuous map. Obviously, $h_{\varphi} \circ f_{\varphi} : (\mathscr{A}_1, \Omega_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_2, \Omega_2, \mathscr{B}_2)$ is an infra soft continuous map. Also, $f^{-1}_{\varphi} \circ h_{\varphi} \circ f_{\varphi} : (\mathscr{A}_1, \Omega_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_1, \Omega_1, \mathscr{B}_1)$ is an infra soft continuous. Since $(\mathscr{A}_1, \Omega_1, \mathscr{B}_1)$ has a fixed soft point property, $f^{-1}_{\varphi}(h_{\varphi}(f_{\varphi}(P^{\alpha}_{\beta}))) = P^{\alpha}_{\beta}$ for some $P^{\alpha}_{\beta} \in \tilde{\mathscr{A}}$, so $f_{\varphi}(f^{-1}_{\varphi}(h_{\varphi}(f_{\varphi}(P^{\alpha}_{\beta})))) = f_{\varphi}(P^{\alpha}_{\beta})$. This leads to $h_{\varphi}(f_{\varphi}(P^{\alpha}_{\beta})) = f_{\varphi}(P^{\alpha}_{\beta})$. Thus, $f_{\varphi}(P^{\alpha}_{\beta})$ is a fixed soft point of h_{φ}. This ends the proof that $(\mathscr{A}_2, \Omega_2, \mathscr{B}_2)$ has a fixed soft point property. □

Theorem 56. *Consider Ω as an extended infra soft topology over $\mathscr{A}_1$. A soft map $f_{\varphi} : (\mathscr{A}_1, \Omega, \mathscr{B}_1) \longrightarrow (\mathscr{A}_2, \mho, \mathscr{B}_2)$ is infra soft continuous iff a crisp map $f : (\mathscr{A}_1, \Omega_{\beta}) \longrightarrow (\mathscr{A}_2, \mho_{\varphi(\beta)})$ is infra continuous.*

Proof. [⇒]: let H be an infra open set in $(\mathscr{A}_2, \mho_{\varphi(\beta)})$. Then there is an infra soft open set $(\delta, \mathscr{B}_2)$ in $(\mathscr{A}_2, \mho, \mathscr{B}_2)$ such that $\delta(\varphi(\beta)) = H$. Since f_{φ} is an infra soft continuous map, $f^{-1}_{\varphi}(\delta, \mathscr{B}_2)$ is an infra soft open set. Therefore, a soft set $f^{-1}_{\varphi}(\delta, \mathscr{B}_2) = (f^{-1}(\delta), \mathscr{B}_1)$ in $(\mathscr{A}_1, \Omega, \mathscr{B}_1)$ is given by $f^{-1}_{\varphi}(\delta)(\beta)$

$= f^{-1}(\delta(\varphi(\beta)))$ for each $\beta \in \mathscr{B}_1$. By hypothesis, Ω is extended, a set $f^{-1}(\delta(\varphi(\beta))) = f^{-1}(H)$ in $(\mathscr{A}_1, \Omega_\beta)$ is infra open. Hence, f is an infra continuous map.

[$\Leftarrow$]: let $(\xi, \mathscr{B}_2)$ be an infra soft open set in $(\mathscr{A}_2, \mho, \mathscr{B}_2)$. Then a soft set $f_\phi^{-1}(\xi, \mathscr{B}_2) = (f_\varphi^{-1}(\xi), \mathscr{B}_1)$ in $(\mathscr{A}_1, \Omega, \mathscr{B}_1)$ is given by $f_\varphi^{-1}(\xi)(\beta) = f^{-1}(\xi(\varphi(\beta)))$ for each $\beta \in \mathscr{B}_1$. Since f is an infra continuous map, a subset $f^{-1}(\xi(\varphi(\beta)))$ in $(\mathscr{A}_1, \Omega_\beta)$ is infra open. By hypothesis, Ω is extended, we obtain $f_\varphi^{-1}(\xi, \mathscr{B}_2)$ is an infra soft open set in $(\mathscr{A}_1, \Omega, \mathscr{B}_1)$. Hence, f_φ is an infra soft continuous map. □

Proposition 57. *If an ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ has a fixed soft point, then $(\mathscr{A}, \Omega_\beta)$ has a fixed point property for all $\beta \in \mathscr{B}$.*

Proof. Consider an ISTS $(\mathscr{A}, \Omega, \mathscr{B})$ as a fixed soft point property. This means that any infra soft continuous map $f_\varphi : (\mathscr{A}, \Omega, \mathscr{B}) \longrightarrow (\mathscr{A}, \Omega, \mathscr{B})$ has a fixed soft point. Say, P_β^α. It comes from Theorem 56 that $f : (\mathscr{A}, \Omega_\beta) \longrightarrow (\mathscr{A}, \Omega_{\varphi(\beta)})$ is infra continuous. Since P_β^α is a fixed soft point of f_φ, we obtain $f(\alpha) = \alpha$. Thus, f has a fixed point. □

5. Conclusion and Future Work

This article is aimed at completely presenting and scrutinizing the concepts and notions in the frame of ISTSs. So, we have initiated two sorts of covering properties in the frame of ISTSs which consider two classifications of soft spaces. Also, we have established the concept of fixed soft points and elucidated essential properties, in particular, those induced from infra soft compact spaces.

We sum our accomplishments through this work in the following:

(1) Introduce the concepts of infra soft compact and infra soft Lindelöf spaces

(2) Offer some illustrative examples to point out the relationships between these two spaces and validate the obtained findings and notes

(3) Explain the interchangeable property of these concepts from infra soft topology to classical infra topology and vice versa

(4) Investigate these concepts under finite product of soft spaces and infra soft homeomorphism

(5) Display the concept of fixed soft points and reveal its basic features

Our blueprint in the forthcoming papers is as follows.

(1) Establish new sorts of covering properties like almost infra soft compact (Lindelöf) and nearly infra soft compact (Lindelöf) spaces

(2) Investigate metric spaces in the content of ISTSs which will open a door for more studies on fixed soft points

(3) Carry out further investigations in the areas of rough set theory and infra topologies

(4) Generalize ISTSs to infra soft bitopological and infra soft ordered topological spaces

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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Research Article

Convergence Results for Total Asymptotically Nonexpansive Monotone Mappings in Modular Function Spaces

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In this article, we consider an extensive class of monotone nonexpansive mappings. We use S-iteration to approximate the fixed point for monotone total asymptotically nonexpansive mappings in the settings of modular function space.

1. Introduction

In 1965, the existence results for nonexpansive mapping were initiated by Browder [1], Kirk [2], and Göhde [3] independently. The idea about asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. Fixed point results of nonexpansive mapping were extended for monotone case by Bachar and Khamsi [5] in 2015. Alfuraidan and Khamsi [6] extended the concept of asymptotically nonexpansive for the case of monotone in 2018. Alber et al. [7] introduced the concept of total asymptotically nonexpansive mappings that generalizes family of mapping that are the extension of asymptotically nonexpansive mappings in 2006. Example 2 of [8] and Example 3.1 of [9] show that total asymptotically nonexpansive mappings properly contain the asymptotically nonexpansive mappings.

The notation for modular function (MF) space was initiated in 1950 by Nakano [10], which was further generalized by Musielak and Orlicz [11] in 1959. In 1990, Khamsi et al. [12] were the first who initiated fixed point theory in MF space. Alfuraidan, Bachar, and Khamsi [13] in 2017 extended results of Goebel and Kirk [4] for monotone asymptotically nonexpansive mappings in MF spaces using Mann iteration process.

In this article, we extend the notion of monotone total asymptotically nonexpansive mappings in MF space and generalize the results of Alfuraidan and Khamsi presented in [6, 13]. We use S-iteration process to approximate the fixed point, which is fastly convergent than the classic Picard [14], Mann [15], and Ishikawa [16] iterative processes.

2. Preliminaries

Firstly, we have the definitions of δ-ring and σ-algebra with examples.

Definition 1. Suppose $\Sigma \neq \varnothing$, and R be a nonempty family of subsets of Σ, then R is called ring of sets if $A, B \in R$, satisfies

(i) $A \cup B \in \mathrm{R}$

(ii) $A \setminus B \in \mathrm{R}$

A ring of sets R is called δ -ring of sets if for any sequence of sets $\{A_n\} \in \mathrm{R}$ implies $\cup_{n=1}^{\infty} A_n \in \mathrm{R}$.

Example 2. Let R be the collection of all finite subsets of $\mathbb{N}$, and then R is a ring, but not δ-ring.

Definition 3. Assume that $\Sigma \neq \emptyset$, a collection $\mathscr{A}$ of subsets of Σ is called algebra of sets if $A, B \in \mathscr{A}$, satisfies

(i) $A \cup B \in \mathscr{A}$

(ii) $A' \in \mathscr{A}$, whenever A is in $\mathscr{A}$

An algebra of sets $\mathscr{A}$ is called σ -algebra of sets if for every sequence of sets $\{A_n\} \in \mathscr{A}$ implies $\cup_{n=1}^{\infty} A_n \in \mathscr{A}$.

Example 4. For any set Σ, $P(\Sigma)$ and $\{\phi, \Sigma\}$ are σ-algebras.

In the following, we list some basic concepts of the MF space presented by Kozlowski [12].

Definition 5. Suppose that Σ be a vector space,

(a) A functional $\mu : \Sigma \longrightarrow [0,\infty]$ is known as modular if for $\mathrm{u}, \tau \in \Sigma$, μ, satisfies

(i) $\mu(\mathrm{u}) = 0$ if and only if $\mathrm{u} = 0$

(ii) $\mu(\gamma \mathrm{u}) = \mu(\mathrm{u})$ with $|\gamma| = 1$

(iii) $\mu(\gamma \mathrm{u} + \nu\tau) \leq \mu(\mathrm{u}) + \mu(\tau)$, if $\gamma + \nu = 1$ and $\gamma \geq 0$, $\nu \geq 0$

(b) If condition (iii) is replaced by

(i) $\mu(\gamma \mathrm{u} + \nu\tau) \leq \gamma\mu(\mathrm{u}) + \nu\mu(\tau)$, if $\gamma + \nu = 1$ and $\gamma \geq 0, \nu \geq 0$, then μ is known as convex modular

(c) A modular μ defines a respective MF space, that is, the vector space Σ_μ given by

$$\Sigma_\mu = \{\mathrm{u} \in \Sigma \,; \mu(\lambda \mathrm{u}) \longrightarrow 0 \mathrm{as} \lambda \longrightarrow 0\} \tag{1}$$

Definition 6. A subset $A \in \mathscr{A}$ is said to be μ-null if $\mu(\nu 1_A) = 0$ (the notation 1_A represents the characteristic of A), for any $\nu \in \Sigma$, and a property $\mu(w)$ holds μ-almost everywhere ($\mu - a.e.$) if the set $\{w \in \Sigma \,; \mu(w)$ does not hold$\}$ is μ-null.

A property is considered to hold almost everywhere (a.e) if there is a set of points where this property fails to hold has measure zero.

Definition 7. Let M_∞ stands for the class of all extended functions which are also measurable. A convex and even function $\mu : M_\infty \longrightarrow [0,\infty]$ is said to be regular modular if

(1) $\mu(\mathrm{u}) = 0 \Rightarrow \mathrm{u} = 0\mu$-almost everywhere

(2) $|\mathrm{u}(t)|'|\tau(t)|$ for all $t \in \Omega \Rightarrow \mu(\mathrm{u})'\mu(\tau)$, where $\mathrm{u}, \tau \in M_\infty \mu$ is monotone

(3) $|\mathrm{u}_n(t)| \uparrow |\mathrm{u}(t)|$ for all $t \in \Omega \Rightarrow \mu(\mathrm{u}_n) \uparrow \mu(\mathrm{u})$, where $\mathrm{u}, \in M_\infty \mu$ has Fatou property

Consider

$$M = \{\mathrm{u} \in M_\infty \,; |\mathrm{u}(t)| < \infty \mu - \text{almost everywhere}\}. \tag{2}$$

The *MF space* L_μ is defined as

$$L_\mu = \{\mathrm{u} \in M \,; \mu(\lambda \mathrm{u}) \longrightarrow 0 \mathrm{as} \lambda \longrightarrow 0\}. \tag{3}$$

Following few useful definitions are taken from [17, 18]. From onwards, we assume μ as a convex regular modular.

Definition 8.

(i) $\{\tau_n\}$ is termed as μ-convergent to τ if

$$\lim_{n \longrightarrow \infty} \mu(\tau_n - \tau) = 0. \tag{4}$$

(ii) A sequence $\{\tau_n\}$ is termed as μ -Cauchy if

$$\lim_{m,n \longrightarrow \infty} \mu(\tau_n - \tau_m) = 0. \tag{5}$$

(iii) Let $K \subset L_\mu$ be μ-closed if for any sequence $\{\tau_n\} \in K$, μ-converge to $\tau \Rightarrow \tau \in K$

(iv) Let $K \subset L_\mu$ be μ-bounded if its μ-diameter sup $\{\mu(\tau - h) \,; h \in K\} < \infty$

Definition 9. Suppose that Σ be a vector space, μ is said to satisfy the Δ_2-condition, if $\sup_{n\geq 1} \mu(2u_n, D_k) \longrightarrow 0$ as $k \longrightarrow \infty$ whenever $\{D_k\}$ decreases to ϕ and $\sup_{n\geq 1} \mu(u_n, D_k) \longrightarrow 0$ *as* $k \longrightarrow \infty$.

Remark 10. Consider μ-convergence implies μ-Cauchy if and only if it satisfies the Δ_2-condition.

Definition 11. Let $r > 0$ and $\varepsilon > 0$. Define

$$\delta_\mu(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} \mu\left(\frac{\mathrm{u} + \tau}{2}\right) ; (\mathrm{u}, \tau) \in D(r, \varepsilon) \right\}, \tag{6}$$

where

$$D(r, \varepsilon) = \left\{ (\mathrm{u}, \tau) \,; \mathrm{u}, \tau \in L_\mu, \mu(\mathrm{u}) \leq r, \mu(\tau) \leq r, \mu(\mathrm{u} - \tau) \geq r\varepsilon \right\}. \tag{7}$$

(a) μ is said to satisfy condition (UC) if whenever $R > 0$ and $\varepsilon > 0$, we have $\delta_p(R, \varepsilon) > 0$

(b) μ is considered to satisfy condition (UUC) if whenever $s > 0$ and $\varepsilon > 0$, $\eta(s,\varepsilon) > 0$ exists such that

$$\delta_\mu(R,\varepsilon) > \eta(s,\varepsilon) > 0, for R > s. \tag{8}$$

(c) μ is considered to satisfy condition (SC) if whenever any $\tau, h \in L_\mu$ with $\mu(\tau) = \mu(h)$ and

$$\mu(\gamma\tau + (1-\gamma)h) = \gamma\mu(\tau) + (1-\gamma)\mu(h), \text{for some} \gamma \in (0,1), \tag{9}$$

where $\mathrm{u} = \tau$.

Following definition of μ-type function will be used in the main result taken from [18].

Definition 12. Let $K \subset L_\mu$, and a mapping $\tau : K \longrightarrow [0,\infty]$ is said to be μ-type if a sequence $\{\tau_m\} \in L_\mu$ exists such that

$$\tau(\mathrm{u}) = \limsup_{n\longrightarrow\infty} \mu(\tau_m - \mathrm{u}), \tag{10}$$

for any $\mathrm{u} \in K$. Any sequence $\{\mathrm{u}_n\}$ in K is said to be a minimizing sequence of τ if

$$\lim_{n\longrightarrow\infty} \tau(\mathrm{u}_n) = \inf\{\tau(\mathrm{u})\,;\, \mathrm{u} \in K\}. \tag{11}$$

Following are the definitions of monotone and monotone asymptotically nonexpansive mapping in modular space, and useful remark about property (R), given in [13].

Definition 13. A mapping $\Gamma : K \longrightarrow K$, where K be a nonempty subset of L_μ, is said to be

(i) Monotone if

$$\Gamma(\mathrm{u})'\Gamma(\tau)\mu - \text{a.e.whenever } \mathrm{u}'\tau\mu - \text{a.e.,for } \mathrm{u}, \tau \in K. \tag{12}$$

(ii) Monotone asymptotically nonexpansive if Γ is monotone, and there exists $\{L_n\} \subset [1,+\infty)$ such that $\lim_{n\longrightarrow\infty} L_n = 1$, and

$$\mu(\Gamma^n\tau - \Gamma^n h) \le L_n\mu(\tau - h), \text{for } \mathrm{u}, \tau \in K, \tag{13}$$

such that $\tau' h\mu$-a.e. and $n \ge 1$. Also τ is said to be fixed point if $\Gamma\tau = \tau$.

Remark 14. Let $K \ne \phi$ be a μ-bounded, convex, and μ-closed subset of L_μ where μ is a convex regular modular. Let $\{u_n\}$ be a monotonically increasing sequence in K (due to the convexity and μ-closedness of order intervals in L_μ), then property (R) will imply that

$$\cap_{n\ge1}\left\{\mathrm{u} \in K\,;\, \mathrm{u}_n{}'\mathrm{u}\mu - \text{a.e}\right\} \ne \phi. \tag{14}$$

The following Lemmas taken from [19] will be used in main result.

Lemma 15. *Let $K \ne \phi$ be a μ-bounded, convex, and μ-closed subset of L_μ where μ is a convex regular modular satisfying condition (UUC). Then, every μ-type minimizing sequence defined on K will be μ-convergent, and the limit will not depend upon the minimizing sequence.*

Lemma 16. *Let μ be a convex regular modular satisfying condition (UUC). If there exists $R > 0$ and $\gamma \in (0,1)$ with*

$$\limsup_{n\longrightarrow\infty} \mu(u_n) \le R, \limsup_{n\longrightarrow\infty} \mu(\tau_n) \le R and \lim_{n\longrightarrow\infty} \mu(\gamma u_n + (1-\gamma)\tau_n) = R, \tag{15}$$

then we have

$$\lim_{n\longrightarrow\infty} \mu(u_n - \tau_n) = 0. \tag{16}$$

The μ-distance from $u \in L_\mu$ to $K \subset L_\mu$ is given as

$$dist_\mu(u,K) = \inf\{\mu(u-h) : h \in K\}. \tag{17}$$

Following Lemma taken from [9] will be used in the existence result.

Lemma 17. *Suppose $\{l_n\}$, $\{m_n\}$ and $\{\delta_n\}$ be sequences of nonnegative satisfying*

$$l_{n+1} \le (1+\delta_n)l_n + m_n \forall n \ge 1. \tag{18}$$

If $\Sigma\delta_n < \infty$ and $\Sigma m_n < \infty$, then $\lim_{n\longrightarrow\infty} l_{n+1}$ exists.

Following is the definition of condition (I) taken from [20].

Definition 18. Let $K \ne \phi$ be a subset of L_μ, and a mapping $\Gamma : K \longrightarrow K$ is assumed to fulfill the condition (I) if a nondecreasing function

$$l : [0,\infty) \longrightarrow [0,\infty) \text{ with } l(0) = 0 \text{ and } l(r) > 0, \tag{19}$$

exists for all $r \in (0,\infty)$, such that

$$\mu(\mathrm{u} - \Gamma\mathrm{u}) \ge l\left(\mathrm{dist}_\mu(\mathrm{u}, F_\mu(\Gamma))\right), \tag{20}$$

for all $\mathrm{u} \in K$.

3. Fixed Point Results for Monotone Total Asymptotically Nonexpansive Mapping

Now, we will define monotone total asymptotically nonexpansive mapping in modular space.

Definition 19. Let $K \neq \phi$ be a subset of L_μ where μ is a convex regular modular. A self map Γ of K is said to be monotone total asymptotically nonexpansive mapping if there exists nonnegative sequences $\{\zeta_n\}$ and $\{\xi_n\}$ with $\zeta_n \longrightarrow 0$, $\xi_n \longrightarrow 0$, as $n \longrightarrow \infty$, and a strictly increasing continuous function

$$\phi : [0,\infty) \longrightarrow [0,\infty)\ with\ \phi(0) = 0, \tag{21}$$

such that

$$\mu(\Gamma^n \tau - \Gamma^n h) \leq \mu(\tau - h) + \zeta_n \phi(\mu(\tau - h)) + \xi_n \text{forall} n \geq 1. \tag{22}$$

There exists a constant $M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for $\lambda > 0$, then

$$\mu(\Gamma^n \tau - \Gamma^n h) \leq (1 + M^* \zeta_n)\mu(\tau - h) + \xi_n, \tag{23}$$

for every $\tau, h \in K$ such that τ and h are comparable μ-a.e.

Theorem 20. *Let $K \neq \phi$ be a μ-bounded and μ-closed subset of L_μ where μ is a convex regular modular satisfying condition (UUC). Let a self map Γ of K be a μ-continuous monotone total asymptotically nonexpansive mapping. Assume that there exists $u_0 \in K$, such that $u_0{}'\Gamma(u_0)$ or $(\Gamma(u_0){}'u_0)\mu$ -a.e. Then, Γ has a fixed point u such that $u_0{}'u$ or $(u'u_0)\mu$-a.e.*

Proof. Assume that $\mathrm{u}_0{}'\Gamma(\mathrm{u}_0)\mu$-a.e. Since Γ is monotone, then we have

$$\Gamma^n \mathrm{u}_0{}' \Gamma^{n+1} \mathrm{u}_0, \tag{24}$$

for every $n \in \mathbb{N}$, and the sequence $\{\Gamma^n \mathrm{u}_0\}$ is monotone increasing. From the above Remark,

$$K_\infty = \cap_{n \geq 1} \left\{ \mathrm{u} \in K \,;\, \mathrm{u}_n{}' \mathrm{u}\mu - \mathrm{a.e} \right\} \neq \phi. \tag{25}$$

Consider the μ-type function $\tau : K_\infty \longrightarrow [0,+\infty)$ define by

$$\tau(h) = \lim_{n \longrightarrow \infty} \sup \mu(\Gamma^n \mathrm{u}_0 - h), \text{for any } h \in K_\infty, \tag{26}$$

$$\tau_0 = \inf \{\tau(h) \,;\, h \in K_\infty\}. \tag{27}$$

Let $\{\tau_n\}$ be a minimizing sequence of τ, from the Lemma $\{\tau_n\}\mu$-converges to $\tau \in K_\infty$. We have to show that τ is the fixed point of Γ. Since $h \in K_\infty$, we have $\Gamma^m(h) \in K_\infty$, for every $m \in \mathbb{N}$, which implies

$$\begin{aligned}
\tau(\Gamma^m(h)) &= \lim_{n \longrightarrow \infty} \sup \mu(\Gamma^n \mathrm{u}_0 - \Gamma^m h) \\
&\leq \lim_{n \longrightarrow \infty} \sup \left[\mu(\Gamma^n \mathrm{u}_0 - h) + \mu_m \phi(\mu(\Gamma^n \mathrm{u}_0 - h)) + \xi_m\right] \\
&= \tau(h) + \mu_m \lim_{n \longrightarrow \infty} \sup \left(\phi(\mu(\Gamma^n \mathrm{u}_0 - h))\right) + \xi_m.
\end{aligned} \tag{28}$$

In particular, we have

$$\begin{aligned}
\tau(\Gamma^m(\tau_n)) &= \lim_{n \longrightarrow \infty} \sup \mu(\Gamma^n x_0 - \Gamma^m \tau_n) \\
&\leq \tau(\tau_n) + \mu_m \lim_{n \longrightarrow \infty} \sup \left(\phi(\mu(\Gamma^n x_0 - h))\right) + \xi_m,
\end{aligned} \tag{29}$$

for $n, m \in \mathbb{N}$. As Γ is total asymptotically nonexpansive, so $\mu_m \longrightarrow 0$, $\xi_m \longrightarrow 0$, when $m \longrightarrow \infty$. Hence,

$$\lim_{m \longrightarrow \infty} \tau(\Gamma^m(\tau_n)) = \tau(\tau_n). \tag{30}$$

The sequence $\{\Gamma^{n+p}(\tau_n)\}$ is a minimizing sequence in K_∞, for any $p \in \mathbb{N}$. By Lemma 15, $\{\Gamma^{n+p}(\tau_n)\}$ is μ-converge to τ, for any $p \in \mathbb{N}$. Since Γ is μ-continuous and $\{\Gamma^n(\tau_n)\}$ is μ-convergent to τ, then $\{\Gamma^{n+1}(\tau_n)\}$ is μ-convergent to $\Gamma\tau$ and τ. Since μ-limit of any μ-convergent is unique, we have $\Gamma\tau = \tau$; also, $\tau \in K_\infty$, we have $\mathrm{u}_0{}'\tau$, hence proved. ☐

Example 21. Let f be an extended real valued function defined on a measureable set D, such that $f(x) = c$ for all $x \in D$. The function f is measureable if the set

$$\{x \in D : f(x) > \alpha\} = \begin{cases} D & \text{if } \alpha < c \\ \phi & \text{if } \alpha \geq c \end{cases} \tag{31}$$

is measureable. And the measureability of above set follows directly from the measureability of D and ϕ. So, a constant function is a measureable function. Now, we define a set of extended real valued functions as

$$M_\infty = \left\{ f : f : D \longrightarrow \bar{\mathbb{R}} \text{with} f(x) = c \right\}. \tag{32}$$

Define a function $\mu : M_\infty \longrightarrow [0,\infty)$ by $\mu(f) = f(x)$ for all $f \in M_\infty$, which clearly it is well defined.

Firstly, we need to show that μ is a convex function. For this, we show M_∞ that is a convex set. Consider

$$\begin{aligned}
(\lambda f + (1 - \lambda)g)(x) &= (\lambda f)(x) + ((1 - \lambda)g)(x), \text{ Point wise addition} \\
&= \lambda.f(x) + (1 - \lambda).g(x), \text{ Scaler multiplication} \\
&= \lambda c_1 + (1 - \lambda)c_2,\ As f, g \in M_\infty, \lambda \in (0, 1) = c_3,
\end{aligned} \tag{33}$$

which implies $\lambda f + (1 - \lambda)g \in M_\infty$. Hence, M_∞ is a convex set. Now, for every $f, g \in M_\infty$, it is easy to prove that

$$\mu(\lambda f + (1-\lambda)g) = \lambda.\mu(f) + (1-\lambda).\mu(g), \quad (34)$$

which further implies that μ is a convex function. Now, we check the properties of regular modular.

(1) If $\mu(f) = 0 \Rightarrow f(x) = 0 \Rightarrow c = 0$, which further implies $f = 0$

(2) If

$$\begin{aligned} f(t) &\le g(t), \text{for all } t \in D \\ c_1 &\le c_2, \end{aligned} \quad (35)$$

as $\mu(f) = f(t) = c_1$ and $\mu(g) = g(t) = c_2$. So, $\mu(f) \le \mu(g)$. Thus, μ is monotone.

(3) Clearly, μ is strongly convergent which implies weak convergence

Hence, μ is convex regular modular. Define

$$M = \{f \in M_\infty : |f(x)| < \infty\}, \text{and } L_\mu = \left\{ \begin{array}{c} f \in M : \mu(\lambda f)(x) = (\lambda f)(x) \\ = \lambda.f(x) = \lambda.c \longrightarrow 0, \text{as} \lambda \longrightarrow 0 \end{array} \right\}, \quad (36)$$

and a subset

$$K = \{f \in M : f(x) = c \in [0, 2] \text{ } with \text{ } \mu(\lambda f) \longrightarrow 0 \text{ } as \text{ } \lambda \longrightarrow 0\} \quad (37)$$

of L_μ. Clearly, K is μ-bounded and μ-closed. Let a mapping $\Gamma : K \longrightarrow K$ be defined by $\Gamma(f) = \alpha f$, where $\alpha \in (0, 1)$. Let $(\xi_n)_{n\in\mathbb{N}} = 1/2n$, $(\eta_n)_{n\in\mathbb{N}} = 2/3n^2$ be any positive sequences and $\xi_n, \eta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Define a strictly increasing function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ by $\varphi(x) = x/2$, with $\varphi(0) = 0$. Consider

$$\mu(\Gamma^n(\tau) - \Gamma^n(h)) = \mu(\alpha^n\tau - \alpha^n h) = (\alpha^n\tau - \alpha^n h)(x) = \alpha^n(c_1 - c_2),$$

$$\begin{aligned} \mu(\tau - h) + \eta_n\varphi(\mu(\tau - h)) + \xi_n &= (\tau - h)(x) + \frac{2}{3n^2}\frac{\mu(\tau - h)}{2} + \frac{1}{2n} \\ &= \left(1 + \frac{1}{3n^2}\right)(\tau(x) - h(x)) + \frac{1}{2n} \\ &= \left(1 + \frac{1}{3n^2}\right)(c_1 - c_2) + \frac{1}{2n}. \end{aligned} \quad (38)$$

Clearly,

$$\mu(\Gamma^n(\tau) - \Gamma^n(h)) \le \mu(\tau - h) + \eta_n\varphi(\mu(\tau - h)) + \xi_n. \quad (39)$$

Also, there exists a constant $M^* = 1$, $\varphi(\lambda) = \lambda/2 < 1.\lambda$, and

$$\mu(\Gamma^n(\tau) - \Gamma^n(h)) \le (1 + M^*\eta_n)\mu(\tau - h) + \xi_n. \quad (40)$$

So, Γ is monotone asymptotically nonexpansive mapping. Since all conditions of theorem are satisfied; thus, Γ has a fixed point, since $\Gamma(f) = \alpha f$ implies $(1-\alpha)f = 0$. Thus, $f = 0$. Hence, the 0 function is a fixed point of Γ.

4. Convergence Analysis

Let $K \ne \phi$ be a convex subset of L_μ where μ is a convex regular modular.We modify S-iteration in MF space is defined as

$$\begin{cases} \mathrm{u}_1 \in K \\ y_l = \nu_l\Gamma^l\mathrm{u}_l + (1-\nu_l)\mathrm{u}_l, \\ \mathrm{u}_{l+1} = \gamma_l\Gamma^l y_l + (1-\gamma_l)\Gamma^l\mathrm{u}_l, \end{cases} \quad (41)$$

for $l \in \mathbb{N}$, where $\{\gamma_l\}$ and $\{\nu_l\}$ are sequences in $(0, 1)$.

Theorem 22. *Let $K \ne \phi$ be a μ-bounded subset of L_μ where μ is a convex regular modular satisfying condition (UUC). Let a self map Γ of K be a monotone total asymptotically nonexpansive mapping with $u(\Gamma) \ne \phi$. Assume that there exists $u_0 \in K$, such that $u_0{}'\Gamma(u_0)$ or $(\Gamma(u_0)'u_0)\mu$ -a.e. If the sequence $\{u_l\}$ is defined by (41) where $0 < a'\gamma_l$, $\nu_l{}'b < 1$, then Γ has a fixed point u such that $u_0{}'u$ or $(u'u_0)\mu$ -a.e. Then, the following holds*

(a) $\lim_{l\longrightarrow\infty}\mu(u_l - u)$ *exist for* $u \in u(\Gamma)$.

(b) $\lim_{l\longrightarrow\infty}\mu(\Gamma^l u_l - u_l) = 0$.

Proof. Let $\mathrm{u} \in \mathrm{u}(\Gamma)$, and assume that $\mathrm{u}_0{}'\Gamma(\mathrm{u}_0)\mu$-a.e. Using (41)

$$\begin{aligned} \mu(y_l - \mathrm{u}) &= \mu\left(\left(\nu_l\Gamma^l\mathrm{u}_l + (1-\nu_l)\mathrm{u}_l\right) - \mathrm{u}\right) \\ &\le \nu_l\mu\left(\Gamma^l\mathrm{u}_l - \mathrm{u}\right) + (1-\nu_l)\mu(\mathrm{u}_l - \mathrm{u}), \end{aligned} \quad (42)$$

using (22), we have

$$\begin{aligned} \mu(y_l - \mathrm{u}) &\le \nu_l[\mu(\mathrm{u}_l - \mathrm{u}) + \zeta_l\phi(\mu(\mathrm{u}_l - \mathrm{u})) + \xi_l] + (1-\nu_l)\mu(\mathrm{u}_l - \mathrm{u}) \\ &= \nu_l\mu(\mathrm{u}_l - \mathrm{u}) + \nu_l\zeta_l\phi(\mu(\mathrm{u}_l - \mathrm{u})) + \nu_l\xi_l + \mu(\mathrm{u}_l - \mathrm{u}) - \nu_l\mu(\mathrm{u}_l - \mathrm{u}) \\ &= \nu_l\zeta_l\phi(\mu(\mathrm{u}_l - \mathrm{u})) + \nu_l\xi_l + \mu(\mathrm{u}_l - \mathrm{u}), \end{aligned} \quad (43)$$

upon using (23), and we get

$$\mu(y_l - \mathrm{u}) \le (1 + \nu_l\zeta_l M^*)\mu(\mathrm{u}_l - \mathrm{u}) + \nu_l\xi_l. \quad (44)$$

Now,

$$\begin{aligned} \mu(\mathrm{u}_{l+1} - \mathrm{u}) = \mu\left(\left(\gamma_l\Gamma^l y_l + (1-\gamma_l)\Gamma^l\mathrm{u}_l\right) - \mathrm{u}\right) &\le \gamma_l\mu\left(\Gamma^l y_l - \mathrm{u}\right) \\ &+ (1-\gamma_l)\mu\left(\Gamma^l\mathrm{u}_l - \mathrm{u}\right), \end{aligned} \quad (45)$$

using (22), and we have

$$\mu(\mathrm{u}_{l+1}-\mathrm{u}) \le \gamma_l[\mu(y_l-\mathrm{u})+\zeta_l\phi(\mu(y_l-\mathrm{u}))+\xi_l]+(1-\gamma_l)\cdot[\mu(\mathrm{u}_l-\mathrm{u})+\zeta_l\phi(\mu(\mathrm{u}_l-\mathrm{u}))+\xi_l], \tag{46}$$

upon using (23), and we get

$$\mu(\mathrm{u}_{l+1}-\mathrm{u}) \le (1+\delta_l)\mu(\mathrm{u}_l-\mathrm{u})+b_l\xi_l. \tag{47}$$

where

$$\begin{aligned}\delta_l &= (\gamma_l\nu_l+\gamma_l\nu_l M^*\zeta_l+1)M^*\zeta_l,\\ b_l &= (\gamma_l\nu_l+\gamma_l\nu_l M^*\zeta_l+1).\end{aligned} \tag{48}$$

Using Lemma 17, $\lim_{l\longrightarrow\infty}\mu(\mathrm{u}_l-\mathrm{u})$ exists for $\mathrm{u}\in \mathrm{u}(\Gamma)$. For part (b), we have to show that

$$\lim_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}\right)=0. \tag{49}$$

Assume that

$$\lim_{l\longrightarrow\infty}\mu(\mathrm{u}_l-\mathrm{u})=c\ge 0. \tag{50}$$

Case 1. If $c=0$, then the conclusion is trivial.
Case 2. For $c>0$, we know that

$$\mu(y_l-\mathrm{u})=(1+M^*\nu_l\zeta_l)\mu(\mathrm{u}_l-\mathrm{u})+\nu_l\xi_l. \tag{51}$$

Taking lim sup on both sides of (50),

$$\limsup_{l\longrightarrow\infty}\mu(y_l-\mathrm{u})\le c. \tag{52}$$

Also,

$$\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}\right)=\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l\mathrm{u}\right)\le\mu(\mathrm{u}_l-\mathrm{u})+\zeta_l\phi(\mu(\mathrm{u}_l-\mathrm{u}))+\xi_l \tag{53}$$

applies $\limsup_{l\longrightarrow\infty}$ on both sides:

$$\limsup_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}\right)\le c. \tag{54}$$

Also,

$$\mu\left(\Gamma^l y_l-\mathrm{u}\right)\le\mu(y_l-\mathrm{u})+\zeta_l\phi(\mu(y_l-\mathrm{u}))+\xi_l. \tag{55}$$

Taking lim sup on both sides,

$$\limsup_{l\longrightarrow\infty}\mu\left(\Gamma^l y_l-\mathrm{u}\right)\le c. \tag{56}$$

Now,

$$\lim_{l\longrightarrow\infty}\mu(x_{l+1}-\mathrm{u})=\lim_{l\longrightarrow\infty}\mu\left(W\left(\Gamma^l\mathrm{u}_l,\Gamma^l y_l,\gamma_l\right)-\mathrm{u}\right)=c. \tag{57}$$

By using Lemma 16 and from (54) and (56), we have

$$\lim_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l y_l\right)=0. \tag{58}$$

From (41) and (58),

$$\begin{aligned}\mu\left(x_{l+1},\Gamma^l\mathrm{u}_l\right)&=\mu\left(W\left(\Gamma^l\mathrm{u}_l,\Gamma^l y_l,\gamma_l\right)-\Gamma^l\mathrm{u}_l\right)\\&\le(1-\gamma_l)\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l\mathrm{u}_l\right)+\gamma_l\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l y_l\right),\end{aligned} \tag{59}$$

taking $\lim_{l\longrightarrow\infty}$, and we have

$$\lim_{l\longrightarrow\infty}\mu\left(x_{l+1}-\Gamma^l\mathrm{u}_l\right)=0. \tag{60}$$

Similarly,

$$\lim_{l\longrightarrow\infty}\mu\left(x_{l+1}-\Gamma^l y_l\right)=0. \tag{61}$$

Next,

$$\begin{aligned}\mu(x_{l+1}-\mathrm{u})&\le\mu\left(x_{l+1}-\Gamma^l y_l\right)+\mu\left(\Gamma^l y_l-\mathrm{u}\right)\\&\le\mu\left(x_{l+1}-\Gamma^l y_l\right)+\mu(y_l-\mathrm{u})+\zeta_l\phi(\mu(y_l-\mathrm{u}))+\xi_l,\end{aligned} \tag{62}$$

taking $\liminf_{l\longrightarrow\infty}$, and we get

$$c\le\liminf_{l\longrightarrow\infty}\mu(y_l-\mathrm{u}). \tag{63}$$

From (52) and (63), we get

$$c=\lim_{l\longrightarrow\infty}\mu(y_l-\mathrm{u})=\lim_{l\longrightarrow\infty}\mu\left(W\left(x_l,\Gamma^l x_l,\nu_l\right)-\mathrm{u}\right). \tag{64}$$

By using Lemma 16, we have

$$\lim_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}_l\right)=0, \tag{65}$$

hence proved. □

Data Availability

There is no any data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fixed Point Results for Rational Orbitally (Θ, δ_b)-Contractions with an Application

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The purpose of this paper is to define a rational orbitally (Θ, δ_b)-contraction and prove some new results in the context of b-metric spaces. Our results extend, generalize, and unify some known results in the literature. As application of our main result, we investigate the solution of Fredholm integral inclusion. We also provide an example to substantiate the advantage and usefulness of obtained results.

1. Introduction

The fixed point theory is a very essential tool for nonlinear analysis of solvability of nonlinear integral equations and others. A suitable selection of a generalized and extended metric space allows to get nontrivial conditions guaranteeing the existence of solutions for a considered equation. Therefore, it is necessary to flourish the fixed point theory in various generalization of metric spaces. One of the famous extensions of metric space is the notion of b-metric space which has been given by Bakhtin [1] in 1989. It was properly defined by Czerwik [2] with the aspect of relaxing triangle inequality in metric spaces in 1993 and proved famous Banach Contraction Principle in this generalized metric space. Khamsi and Hussain [3] discussed the topology of b-metric space and established fixed point results for KKM mappings in metric type spaces. Van An et al. [4] proved the Stone-type theorem on b-metric spaces and obtained a sufficient condition for a b-metric space to be metrizable. On the other, Czerwik [5, 6] introduced set-valued mappings in b-metric spaces and generalized Nadler's fixed point theorem. In 2012, Aydi et al. [7, 8] gave fixed point and common fixed point theorems for set-valued quasicontraction mappings and set-valued weak ϕ-contraction mappings in the setting of b-metric spaces, respectively. Many authors followed the concept of b-metric space and established impressive results [9–19].

In 2012, Jleli and Samet [20] introduced a new type of contraction named as Θ-contraction and obtained a fixed point result to generalize the celebrated Banach Contraction Principle in Branciari metric spaces. Ali et al. [21] defined multivalued Suzuki-type θ-contractions and obtained some generalized fixed point results. Afterwards, Jleli et al. [22] established a new fixed point theorem for Θ-contraction in the setting of Branciari metric spaces and extended the main result of Jleli and Samet [20]. Recently, Alamri et al. [23] adapted Jleli's approach to the b-metric space and obtained some generalized fixed point results. For more details in the direction of Θ-contractions, we refer the reader to [21–30].

In this paper, we define the notion of the rational (Θ, δ_b)-contraction in b-metric spaces and explore the existence of solutions for certain integral problems of Fredholm type as applications of our main results. We obtain our results by using fixed point theorems for multivalued mappings, under new contractive conditions, in the setting of complete b-metric spaces. Evidently, the given results generalized some notable results of the literature to b-metric spaces.

2. Preliminaries

In this section, we give some fundamental notations, definitions, and lemmas which will be used throughout the paper. Throughout this paper, we denote $\mathbb{N}$ the set of positive integers and $\mathbb{R}^+$ the set of all nonnegative real numbers.

Czerwik [2] gave the notion of the b-metric space as follows.

Definition 1 (see [2]). Let $\mathscr{M}$ be a nonempty set $\mathscr{M}$ and $s \geq 1$. A function σ_b: $\mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}_0^+$ is called a b-metric if these assertions hold: for all $\rho, \varpi, \omega \in \mathscr{M}$,

$(B1)\sigma_b(\rho, \varpi) = 0 \Leftrightarrow \rho = \varpi$

$(B2)\sigma_b(\rho, \varpi) = \sigma_b(\varpi, \rho)$

$(B3)\sigma_b(\rho, \omega) \leq s(\sigma_b(\rho, \varpi) + \sigma_b(\varpi, \omega))$

The triple $(\mathscr{M}, \sigma_b, s)$ is called a b -metric space.

Now, we give an elementary example of a b-metric space, but it is not a metric space as follows.

Example 2 (see [2]). Let $\mathscr{M} = \mathbb{R}$ and $\sigma_b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}$ be a mapping defined by

$$\sigma_b(\rho, \varpi) = |\rho - \varpi|^2, \quad \forall \rho, \varpi \in \mathscr{M}, s = 2. \tag{1}$$

Then, $(\mathscr{M}, \sigma_b, s)$ is a b-metric space which is not a metric space.

A brief but short history for multivalued mappings defined in $(\mathscr{M}, \sigma_b, s)$ is given in this way.

Let $P_{cb}(\mathscr{M})$ be the family of all bounded and closed subsets of $\mathscr{M}$. For any $E_1, E_2, E_3 \in P_b(\mathscr{M})$, we define

$$\begin{aligned} \sigma_b(E_1, E_2) &= \inf \{\sigma_b(\rho, \varpi): \rho \in E_1, \varpi \in E_2\}, \\ \delta_b(E_1, E_2) &= \sup \{\sigma_b(\rho, \varpi): \rho \in E_1, \varpi \in E_2\}, \end{aligned} \tag{2}$$

with

$$\sigma_b(\rho, E_3) = \sigma_b(\{\rho\}, E_3) = \inf \{\sigma_b(\rho, \omega): \rho \in E_1, \omega \in E_3\}. \tag{3}$$

Here, we provide some useful properties of δ_b and σ_b (see [2, 5, 6]):

(1) If $E_1 = \{\rho\}$ and $E_2 = \{\varpi\}$, then $\sigma_b(E_1, E_2) = \delta_b(E_1, E_2) = \sigma_b(\rho, \varpi)$

(2) $\sigma_b(E_1, E_2) \leq \delta_b(E_1, E_2)$

(3) $\sigma_b(\rho, E_2) \leq \sigma_b(\rho, \varpi)$ for any $\varpi \in E_2$

(4) $\delta_b(E_1, E_3) \leq s[\delta_b(E_1, E_2) + \delta_b(E_2, E_3)]$

(5) $\delta_b(E_1, E_2) = 0 \Leftrightarrow E_1 = E_2 = \{\rho\}$

Moreover, we will always suppose that

(6) the function σ_b is continuous in its variables

Now, we present the concepts of orbit and orbital continuity of a mapping in the setting of $(\mathscr{M}, \sigma_b, s)$ which are generalization and extensions of the same notions for metric spaces given in [31, 32].

Definition 3. Let $\mathscr{J}, \mathscr{J}_{1,} \mathscr{J}_2 : (\mathscr{M}, \sigma_b, s) \longrightarrow P_b(\mathscr{M})$.

(1) An *orbit* $O(\rho_0, \mathscr{J})$ of $\mathscr{J}$ at ρ_0 is any sequence $\{\rho_n\}$ such that $\rho_n \in \mathscr{J}\rho_{n-1}$ for each $n \in \mathbb{N}$

(2) If, for any $\rho_0 \in \mathscr{M}$, there exists $\{\rho_n\}$ in $\mathscr{M}$ such that $\rho_{2n+1} \in \mathscr{J}_2\rho_{2n}$ and $\rho_{2n+2} \in \mathscr{J}_1\rho_{2n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, then $O(\rho_0, \mathscr{J}_1, \mathscr{J}_2)= \{\rho_n\}$ for each $n \in \mathbb{N}$ is said to be an *orbit* of $(\mathscr{J}_1, \mathscr{J}_2)$ at ρ_0

(3) $(\mathscr{M}, \sigma_b, s)$ is said to be $(\mathscr{J}_1, \mathscr{J}_2)$ *-orbitally complete* if any Cauchy subsequence $\{\rho_{n_i}\}$ of $O(\rho_0, \mathscr{J}_1, \mathscr{J}_2)$ converges in $\mathscr{M}$. Specifically, for $\mathscr{J}_1 = \mathscr{J}_2 = \mathscr{J}$, $\mathscr{M}$ is said to be $\mathscr{J}$ *-orbitally complete*

(4) $\mathscr{J}$ is said to be *orbitally continuous* at $\rho_0 \in \mathscr{M}$ if, for any $\{\rho_n\} \subset O(\rho_0, \mathscr{J})$ for each $n \in \mathbb{N} \cup \{0\}$ and $\rho^* \in \mathscr{M}$,$\sigma_b(\rho_n, \rho^*) \longrightarrow 0$ as $n \longrightarrow \infty$ gives that $\delta_b(\mathscr{J}\rho_n, \mathscr{J}\rho^*) \longrightarrow 0$ as $n \longrightarrow \infty$

(5) A graph $G(\mathscr{J})$ of $\mathscr{J}$ is constructed as follows:

$$G(\mathscr{J}) = \{(\rho, \varpi): \rho \in \mathscr{M}, \varpi \in \mathscr{J}\rho\} \tag{4}$$

We need the following property of $G(\mathscr{J})$ in our proof:

(G_p) $G(\mathscr{J})$ is said to be $\mathscr{J}$ *-orbitally closed* if, for any sequence $\{\rho_n\}$ in $\mathscr{M}$, we get $(\rho^*, \rho^*) \in G(\mathscr{J})$ whenever $(\rho_n, \rho_{n+1}) \in G(\mathscr{J})$ and $\lim_{n\longrightarrow\infty}\rho_n = \rho^*$

In 2012, Jleli and Samet [20] gave the notion of Θ-contractions and proved a contemporary fixed point theorem for these contractions in generalized metric spaces. Motivate by Jleli and Samet [20], Alamri et al. [23] present the following definition.

Definition 4 (see [23]). Let $\Omega_s(s \geq 1)$ denote the family of all mappings $\Theta : \mathbb{R}^+ \longrightarrow (1,\infty)$ which satisfy these conditions:

$(\Theta_1)\Theta$ is nondecreasing

(Θ_2)For any $\{\rho_n\} \subseteq \mathbb{R}^+$, $\lim_{n\longrightarrow\infty}\Theta(\rho_n) = 1 \Leftrightarrow \lim_{n\longrightarrow\infty} (\rho_n) = 0$

(Θ_3)There exist $h \in (0, 1)$ and $l \in (0,\infty]$ such that $\lim_{\rho\longrightarrow 0^+}\Theta(\rho) - 1/\rho^h = l$

(Θ_4)For $\{\rho_n\} \subseteq \mathbb{R}^+$ such that $\Theta(s\rho_n) \leq \Theta(\rho_{n-1})]^k$ for each $n \in \mathbb{N}$ and some $0<k<1$, then $\Theta(s_n^n\rho) \leq \Theta(s^{n-1}\rho_{n-1})]^k$ for each $n \in \mathbb{N}$

They provided the following example.

Example 5. Let $\Theta : \mathbb{R}^+ \longrightarrow (1,\infty)$ be a mapping given by $\theta(\varsigma) = e^{\sqrt{\varsigma e^{\varsigma}}}$. Clearly, Θ satisfies the conditions (Θ_1)-(Θ_4). Here, we show only the condition (Θ_4). Suppose that, for each $n \in \mathbb{N}$ and some $0 < k \in <1$, we have $\theta(s\rho_n) \leq \theta(\rho_{n-1})]^k$. This implies that

$$e^{\sqrt{s\rho_n e^{s\rho_n}}} \leq \left[e^{\sqrt{\rho_{n-1}e^{\rho_{n-1}}}}\right]^k, \tag{5}$$

i.e.,

$$\sqrt{s\rho_n e^{s\rho_n}} \le k\sqrt{\rho_{n-1} e^{\rho_{n-1}}}. \tag{6}$$

This implies that

$$\sqrt{s\rho_n e^{s\rho_n - \rho_{n-1}}} \le k\sqrt{\rho_{n-1}}. \tag{7}$$

It follows that $\theta(s\rho_n) \le \theta(\rho_{n-1})]^k \le \theta(\rho_{n-1})$ and θ is non-decreasing, and so, $s\rho_n \le \rho_{n-1}$ and $s\rho_n - \rho_{n-1} \le 0$ implies $e^{s^{n-1}(s\rho_n - \rho_{n-1})} \le e^{s\rho_n - \rho_{n-1}}$. Therefore, (7) implies that

$$\begin{aligned}\sqrt{s\rho_n e^{s^{n-1}(s\rho_n - \rho_{n-1})}} \le k\sqrt{\rho_{n-1}} \Rightarrow \sqrt{\frac{s\rho_n e^{s^n\rho_n}}{e^{s^{n-1}\rho_{n-1}}}} \le k\sqrt{\rho_{n-1}} \Rightarrow \sqrt{s\rho_n e^{s^n\rho_n}} \\ \le k\sqrt{\rho_{n-1} e^{s^{n-1}\rho_{n-1}}} \Rightarrow \sqrt{s^n\rho_n e^{s^n\rho_n}} \\ \le k\sqrt{s^{n-1}\rho_{n-1} e^{s^{n-1}\rho_{n-1}}} \Rightarrow e^{\sqrt{s^n\rho_n e^{s^n\rho_n}}} \\ \le e^{k\sqrt{s^{n-1}\rho_{n-1} e^{s^{n-1}\rho_{n-1}}}} \Rightarrow \theta(s^n\rho_n) \le \left[\theta\left(s^{n-1}\rho_{n-1}\right)\right]^k,\end{aligned} \tag{8}$$

and hence, the condition (Θ_4) holds.

3. Main Results

In this way, we define the notion of rational (Θ, δ_b)-contraction.

Definition 6. Let $(\mathcal{M}, \sigma_b)$ be a b -metric space. A mapping $\mathcal{J} : \mathcal{M} \longrightarrow P_b(\mathcal{M})$ is called a rational (Θ, δ_b)-contraction if $\exists\Theta \in \Omega_s$, $0< k < 1$ and $0 \le L$ such that

$$\Theta(s\delta_b(\mathcal{J}\rho, \mathcal{J}\varpi)) \le [\Theta(m_1(\rho, \varpi) + Lm_2(\rho, \varpi))]^k, \tag{9}$$

$\forall \rho, \varpi \in \mathcal{M}$ with $\min\{\delta_b(\mathcal{J}\rho, \mathcal{J}\varpi), \sigma_b(\rho, \varpi)\} > 0$, where

$$m_1(\rho, \varpi) = \max\left\{\begin{array}{c}\sigma_b(\rho, \varpi), \sigma_b(\rho, \mathcal{J}\rho), \sigma_b(\varpi, \mathcal{J}\varpi), \dfrac{\sigma_b(\rho, \mathcal{J}\varpi) + \sigma_b(\varpi, \mathcal{J}\rho)}{2s}, \\ \dfrac{\sigma_b(\varpi, \mathcal{J}\varpi)[1 + \sigma_b(\rho, \mathcal{J}\rho)]}{s[1 + \sigma_b(\rho, \varpi)]}, \dfrac{\sigma_b(\varpi, \mathcal{J}\rho)[1 + \sigma_b(\rho, \mathcal{J}\rho)]}{s[1 + \sigma_b(\rho, \varpi)]}\end{array}\right\},$$

$$m_2(\rho, \varpi) = \min\{\sigma_b(\rho, \mathcal{J}\rho), \sigma_b(\varpi, \mathcal{J}\varpi), \sigma_b(\rho, \mathcal{J}\varpi), \sigma_b(\varpi, \mathcal{J}\rho)\}. \tag{10}$$

If (9) holds for all $\rho, \varpi \in O(\bar{\rho_0}, \mathcal{J})$ for some $\rho_0 \in \mathcal{M}$, then $\mathcal{J}$ is called a rational orbitally (Θ, δ_b)-contraction.

Theorem 7. *Let $(\mathcal{M}, \sigma_b)$ be a b -metric space such that σ_b is a continuous functional and $\mathcal{J} : \mathcal{M} \longrightarrow P_b(\mathcal{M})$ be a rational orbitally (Θ, δ_b)-contraction. If the following conditions hold:*

(a) $(\mathcal{M}, \sigma_b, s)$ is $\mathcal{J}$-orbitally complete for some $\rho_0 \in \mathcal{M}$

(b) Θ is continuous and $\mathcal{J}\rho$ is closed, $\forall \rho \in O(\bar{\rho_0}, \mathcal{J})$ or the property (Gp) holds, then there exists $\rho^* \in \mathcal{M}$ such that $\rho^* \in \mathcal{J}\rho^*$

Proof. For any $\rho_0 \in \mathcal{M}$, we generate a sequence $\{\rho_n\}$ in $\mathcal{M}$ as $\rho_{n+1} \in \mathcal{J}\rho_n$ for each $n \ge 0$.

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $\rho_{n_0} = \rho_{n_0+1}$, then ρ_{n_0} is a fixed point of $\mathcal{J}$, and so, the proof is completed. Thus, assume that, for each $n \in \mathbb{N} \cup \{0\}$, $\rho_n \ne \rho_{n+1}$. So, we have $\sigma_b(\rho_{n+1}, \rho_{n+2}) > 0$ and $\delta_b(\mathcal{J}\rho_n, \mathcal{J}\rho_{n+1}) > 0$ for each $n \ge 0$. Then, it follows from (9) with $\rho = \rho_n$ and $\varpi = \rho_{n+1}$ that

$$\begin{aligned}\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) &\le \Theta(s\delta_b(\mathcal{J}\rho_n, \mathcal{J}\rho_{n+1})) \\ &\le [\Theta(m_1(\rho_n, \rho_{n+1}) + Lm_2(\rho_n, \rho_{n+1}))]^k,\end{aligned} \tag{11}$$

where

$$\begin{aligned}m_1(\rho_n, \rho_{n+1}) &= \max\left\{\begin{array}{c}\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_n, \mathcal{J}\rho_n), \sigma_b(\rho_{n+1}, \mathcal{J}\rho_{n+1}), \dfrac{\sigma_b(\rho_n, \mathcal{J}\rho_{n+1}) + \sigma_b(\rho_{n+1}, \mathcal{J}\rho_n)}{2s}, \\ \dfrac{\sigma_b(\rho_{n+1}, \mathcal{J}\rho_{n+1})[1 + \sigma_b(\rho_n, \mathcal{J}\rho_n)]}{s[1 + \sigma_b(\rho_n, \rho_{n+1})]}, \dfrac{\sigma_b(\rho_{n+1}, \mathcal{J}\rho_n)[1 + \sigma_b(\rho_n, \mathcal{J}\rho_n)]}{s[1 + \sigma_b(\rho_n, \rho_{n+1})]}\end{array}\right\} \\ &\le \max\left\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2}), \frac{1}{2s}\sigma_b(\rho_n, \rho_{n+2}), \frac{1}{s}\sigma_b(\rho_{n+1}, \rho_{n+2})\right\} \\ &= \max\left\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2}), \frac{1}{2s}\sigma_b(\rho_n, \rho_{n+2})\right\},\end{aligned}$$

$$m_2(\rho_n, \rho_{n+1}) = \min\{\sigma_b(\rho_n, \mathcal{J}\rho_n), \sigma_b(\rho_{n+1}, \mathcal{J}\rho_{n+1}), \sigma_b(\rho_n, \mathcal{J}\rho_{n+1}), \sigma_b(\rho_{n+1}, \mathcal{J}\rho_n)\} = 0. \tag{12}$$

Since

$$\frac{1}{2s}\sigma_b(\rho_n, \rho_{n+2}) \le \max\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2})\}, \quad (13)$$

it follows from (11) that

$$\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) \le [\Theta(\max\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2})\})]^k. \quad (14)$$

Assume that $\sigma_b(\rho_n, \rho_{n+1}) \le \sigma_b(\rho_{n+1}, \rho_{n+2})$ for some $n \in \mathbb{N}$. Then, from (14), we get

$$\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) \le [\Theta(\sigma_b(\rho_{n+1}, \rho_{n+2}))]^k, \quad (15)$$

which is a contradiction with (Θ_1). Hence, we have

$$\max\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2})\} = \sigma_b(\rho_n, \rho_{n+1}), \quad (16)$$

and consequently,

$$\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) \le [\Theta(\sigma_b(\rho_n, \rho_{n+1}))]^k, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (17)$$

It follows from (17) and (Θ_4) that

$$\Theta(s^n\sigma_b(\rho_n, \rho_{n+1})) \le \left[\Theta\left(s^{n-1}\sigma_b(\rho_{n-1}, \rho_n)\right)\right]^k, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (18)$$

Let us represent $\lambda_n = \sigma_b(\rho_n, \rho_{n+1})$ for $n \in \mathbb{N} \cup \{0\}$. Then, $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have

$$\begin{aligned}\Theta(s^n\lambda_n) &\le \left[\Theta\left(s^{n-1}\lambda_{n-1}\right)\right]^k \le \left[\Theta\left(s^{n-2}\lambda_{n-2}\right)\right]^{k^2} \\ &\le \cdots \le [\Theta(\lambda_0)]^{k^n}, \quad \forall n \in \mathbb{N} \cup \{0\}.\end{aligned} \quad (19)$$

Taking $n \longrightarrow \infty$ in (19), we get

$$\lim_{n\longrightarrow\infty} \Theta(s^n\lambda_n) = 1, \quad (20)$$

which implies that

$$\lim_{n\longrightarrow\infty} s^n\lambda_n = 0, \quad (21)$$

by (Θ_2). By (Θ_3), $\exists h \in (0, 1)$ and $\tau \in (0,\infty]$ such that

$$\lim_{n\longrightarrow\infty} \frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h} = \tau. \quad (22)$$

Assume that $\tau < \infty$. For this case, let $q_2 = \tau/2 > 0$. So, $\exists$ $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h} - \tau\right| \le q_2, \quad \forall n > n_1, \quad (23)$$

which implies that

$$\frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h} \ge \tau - q_2 = \frac{\tau}{2} = q_2, \quad \forall n > n_1. \quad (24)$$

Then, we have

$$n(s^n\lambda_n)^h \le q_1 n[\Theta(s^n\lambda_n) - 1], \quad \forall n > n_1, \quad (25)$$

where $q_1 = 1/q_2$.

Now, assume that $\tau = \infty$. Let $q_2 > 0$. So, $\exists n_1 \in \mathbb{N}$ such that

$$q_2 \le \frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h}, \quad \forall n > n_1, \quad (26)$$

which implies that

$$n(s^n\lambda_n)^h \le q_1 n[\Theta(s^n\lambda_n) - 1], \quad \forall n > n_1, \quad (27)$$

where $q_1 = 1/q_2$. Thus, in all cases, there exist $q_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$n(s^n\lambda_n)^h \le q_1 n[\Theta(s^n\lambda_n) - 1], \quad \forall n > n_1. \quad (28)$$

Hence, by (19) and (28), we get

$$n(s^n\lambda_n)^h \le q_1 n\left([\Theta(\lambda_0)]^{k^n} - 1\right). \quad (29)$$

Taking $n \longrightarrow \infty$ in (29), we have

$$\lim_{n\longrightarrow\infty} n(s^n\lambda_n)^h = 0, \quad (30)$$

and hence, $\lim_{n\longrightarrow\infty} n^{1/h}s^n\lambda_n = 0$, which yields that $\sum_{n=1}^{\infty} s^n\lambda_n$ is convergent. Thus, $\{\rho_n\}$ is a Cauchy sequence in $O(\rho_0, \mathcal{J})$. Since $\mathcal{M}$ is $\mathcal{J}$-orbitally complete, there exists $\rho^* \in \mathcal{M}$ such that

$$\rho_n \longrightarrow \rho^* \text{ as } n \longrightarrow \infty. \quad (31)$$

Suppose that $\mathcal{J}\rho^*$ is closed. We notice that if $\exists\{n_k\} \subset \mathbb{N}$ such that $\rho_{n_k} \in \mathcal{J}\rho^*$ for each $k \in \mathbb{N}$. Since $\mathcal{J}\rho^*$ is closed and $\lim_{k\longrightarrow\infty}\rho_{n_k} = \rho^*$, we conclude that $\rho^* \in \mathcal{J}\rho^*$, and so, the proof is finished. Hence, we suppose that there exists $n_0 \in \mathbb{N}$ so that $\rho_n \notin \mathcal{J}\rho^*$ for each $n \in \mathbb{N}$ with $n \ge n_0$. This implies that $\delta_b(\mathcal{J}\rho_n, \mathcal{J}\rho^*) > 0$ for each $n \ge n_0$. Then, it follows from (9) with $\rho = \rho_n$ and $\varpi = \rho^*$ that

$$\begin{aligned}\Theta(s\sigma_b(\rho_{n+1}, \mathcal{J}\rho^*)) &= \Theta(s\delta_b(\mathcal{J}\rho_n, \mathcal{J}\rho^*)) \\ &\le [\Theta(m_1(\rho_n, \rho^*) + Lm_2(\rho_n, \rho^*))]^k,\end{aligned} \quad (32)$$

where

$$
\begin{aligned}
m_1(\rho_n,\rho^*) &= \max\left\{\begin{array}{c}\sigma_b(\rho_n,\rho^*),\sigma_b(\rho_n,\mathscr{J}\rho_n),\sigma_b(\rho^*,\mathscr{J}\rho^*),\dfrac{\sigma_b(\rho_n,\mathscr{J}\rho^*)+\sigma_b(\rho^*,\mathscr{J}\rho_n)}{2s},\\ \dfrac{\sigma_b(\rho^*,\mathscr{J}\rho^*)[1+\sigma_b(\rho_n,\mathscr{J}\rho_n)]}{s[1+\sigma_b(\rho_n,\rho^*)]},\dfrac{\sigma_b(\rho^*,\mathscr{J}\rho_n)[1+\sigma_b(\rho_n,\mathscr{J}\rho_n)]}{s[1+\sigma_b(\rho_n,\rho^*)]}\end{array}\right\}\\
&\le \max\left\{\begin{array}{c}\sigma_b(\rho_n,\rho^*),\sigma_b(\rho_n,\rho_{n+1}),\sigma_b(\rho^*,\mathscr{J}\rho^*),\dfrac{\sigma_b(\rho_n,\mathscr{J}\rho^*)+\sigma_b(\rho^*,\rho_{n+1})}{2s},\\ \dfrac{\sigma_b(\rho^*,\mathscr{J}\rho^*)[1+\sigma_b(\rho_n,\rho_{n+1})]}{s[1+\sigma_b(\rho_n,\rho^*)]},\dfrac{\sigma_b(\rho^*,\rho_{n+1})[1+\sigma_b(\rho_n,\rho_{n+1})]}{s[1+\sigma_b(\rho_n,\rho^*)]}\end{array}\right\}\\
&\longrightarrow \sigma_b(\rho^*,\mathscr{J}\rho^*) \text{ as } n\longrightarrow\infty,\\
m_2(\rho_n,\rho^*) &= \min\{\sigma_b(\rho_n,\mathscr{J}\rho_n),\sigma_b(\rho^*,\mathscr{J}\rho^*),\sigma_b(\rho_n,\mathscr{J}\rho^*),\sigma_b(\rho^*,\mathscr{J}\rho_n)\}\\
&\le \min\{\sigma_b(\rho_n,\rho_{n+1}),\sigma_b(\rho^*,\mathscr{J}\rho^*),\sigma_b(\rho_n,\mathscr{J}\rho^*),\sigma_b(\rho^*,\rho_{n+1})\}\\
&\longrightarrow 0 \text{ as } n\longrightarrow\infty.
\end{aligned}
\tag{33}
$$

Using the continuity of Θ and σ_b, so applying the limit of (32) as $n\longrightarrow\infty$, we get

$$\Theta(s\sigma_b(\rho^*,\mathscr{J}\rho^*)) \le [\Theta(\sigma_b(\rho^*,\mathscr{J}\rho^*)]^k, \tag{34}$$

which is impossible from $0<k\in<1$ and $s\ge 1$. By the condition (Θ_1), we get $\sigma_b(\rho^*,\mathscr{J}\rho^*)=0$. Since $\mathscr{J}\rho^*$ is closed, thus we get $\rho^*\in\mathscr{J}\rho^*$. Assume that $G(\mathscr{J})$ is $\mathscr{J}$-orbitally closed. Since $(\rho_n,\rho_{n+1})\in G(\mathscr{J})$ and $\lim_{n\longrightarrow\infty}\rho_n=\rho^*\forall n\in\mathbb{N}\cup\{0\}$, we get $(\rho^*,\rho^*)\in G(\mathscr{J})$. Hence, $\rho^*\in\mathscr{J}\rho^*$. This completes the proof. □

If $\Theta(\varsigma)=e^{\sqrt{\varsigma}}$ for any $\varsigma>0$ in Theorem 7, we get following result.

Corollary 8. *Let $(\mathscr{M},\sigma_b)$ be a b-metric space such that σ_b is a continuous functional and $\mathscr{J}:(\mathscr{M},\sigma_b)\longrightarrow P_b(\mathscr{M})$ be a mapping satisfying the following condition: for some 0< $k<1$,$\rho_0\in\mathscr{M}$ and $L\ge 0$,*

$$s\delta_b(\mathscr{J}\rho,\mathscr{J}\varpi)\le k(m_1(\rho,\varpi)+Lm_2(\rho,\varpi)) \tag{35}$$

for all $\rho,\varpi\in\mathscr{M}$ with $\min\{\delta_b(\mathscr{J}\rho,\mathscr{J}\varpi),\sigma_b(\rho,\varpi)\}>0$, *where*

$$
\begin{aligned}
m_1(\rho,\varpi) &= \max\left\{\begin{array}{c}\sigma_b(\rho,\varpi),\sigma_b(\rho,\mathscr{J}\rho),\sigma_b(\varpi,\mathscr{J}\varpi),\dfrac{\sigma_b(\rho,\mathscr{J}\varpi)+\sigma_b(\varpi,\mathscr{J}\rho)}{2s},\\ \dfrac{\sigma_b(\varpi,\mathscr{J}\varpi)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]},\dfrac{\sigma_b(\varpi,\mathscr{J}\rho)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}\end{array}\right\},\\
m_2(\rho,\varpi) &= \min\{\sigma_b(\rho,\mathscr{J}\rho),\sigma_b(\varpi,\mathscr{J}\varpi),\sigma_b(\rho,\mathscr{J}\varpi),\sigma_b(\varpi,\mathscr{J}\rho)\}
\end{aligned}
\tag{36}
$$

for all $\rho,\varpi\in O(\bar{\rho_0},\mathscr{J})$. Assume that $(\mathscr{M},\sigma_b,s)$ is $\mathscr{J}$-orbitally complete for some $\rho_0\in\mathscr{M}$. If $\mathscr{J}\rho$ is closed, for all $\rho\in O(\bar{\rho_0},\mathscr{J})$ or the property (Gp) holds, then $\exists\rho^*\in\mathscr{M}$ such that $\rho^*\in\mathscr{J}\rho^*$.

Example 9. Let $\mathscr{M}=[0,1]$. The b-metric is defined by

$$\sigma_b(\rho,\varpi)=(\rho-\varpi)^2, \tag{37}$$

with coefficient $s=2$. Define $\mathscr{J}:\mathscr{M}\longrightarrow P_{cb}(\mathscr{M})$ given by

$$\mathscr{J}\rho=\begin{cases}\left\{\dfrac{1}{3}\right\}, & 0\le\rho<1,\\ \left[0,\dfrac{1}{4}\right], & \rho=1.\end{cases} \tag{38}$$

If $\rho,\varpi\in[0,1)$, then $\delta_b(\mathscr{J}\rho,\mathscr{J}\varpi)=0$. Let $\rho\in[0,1)$ and $\varpi=1$. Then, $\mathscr{J}\rho=\{1/3\}$,$\mathscr{J}\varpi=[0,1/4]$, and $\delta_b(\mathscr{J}\rho,\mathscr{J}\varpi)=1/12$,

$$
\begin{aligned}
m_1(\rho,\varpi) &= \max\left\{\begin{array}{c}(1-\rho)^2,\left(\dfrac{1}{3}-\rho\right)^2,\left(\dfrac{3}{4}\right)^2,\dfrac{1}{4}\left[\sigma_b(\rho,\mathscr{J}\varpi)+\left(\dfrac{2}{3}\right)^2\right],\\ \dfrac{(3/4)^2\left[1+(1/3-\rho)^2\right]}{2\left[1+(1-\rho)^2\right]},\dfrac{(2/3)^2\left[1+(1/3-\rho)^2\right]}{2\left[1+(1-\rho)^2\right]}\end{array}\right\}\ge\frac{9}{16},\\
m_2(\rho,\varpi) &= \min\left\{\left(\frac{1}{3}-\rho\right)^2,\left(\frac{3}{4}\right)^2,\sigma_b(\rho,\mathscr{J}\varpi),\left(\frac{2}{3}\right)^2\right\}\ge 0.
\end{aligned}
\tag{39}
$$

Take $k=2/9$, $\Theta(\varsigma)=e^{\sqrt{\varsigma}}$ and $0\le L$. Then,

$$
\begin{aligned}
\Theta(s\delta_b(\mathscr{J}\rho,\mathscr{J}\varpi)) &= \Theta\left(2\cdot\frac{1}{12}\right)=e^{\sqrt{1/6}}<e^{2/3}=e^{8/9\cdot 3/4}\\
&= [\Theta(m_1(\rho,\varpi)+Lm_2(\rho,\varpi))]^k.
\end{aligned}
\tag{40}
$$

Thus, all conditions of Theorem 7 are satisfied and $\rho^*=1/3$ is the required point.

The family Ω_s contains a wide set of functions; that is, if we take

$$\Theta(\rho) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{\rho^{\beta}}\right), \tag{41}$$

where $\rho > 0$ and $0 < \beta < 1$, then we can obtain the following corollary from Theorem 7.

Corollary 10. *Let $(\mathscr{M}, \sigma_b)$ be a b-metric space such that σ_b is a continuous functional. Assume that $\mathscr{J} : \mathscr{M} \longrightarrow P_b(\mathscr{M})$ be a mapping such that for some $0 < k, \beta < 1, \rho_0 \in \mathscr{M}$ and $0 \le L$,*

$$\begin{aligned} &2 - \frac{2}{\pi} \arctan\left(\frac{1}{(s\delta_b(\mathscr{J}\rho, \mathscr{J}\varpi))^{\beta}}\right) \\ &\quad \le \left[2 - \frac{2}{\pi} \arctan\left(\frac{1}{(m_1(\rho,\varpi) + Lm_2(\rho,\varpi))^{\beta}}\right)\right]^k, \end{aligned} \tag{42}$$

$\forall \rho, \varpi \in \mathscr{M}$ with $\min\{\delta_b(\mathscr{J}\rho, \mathscr{J}\varpi), \sigma_b(\rho, \varpi)\} > 0$, where

$$m_1(\rho,\varpi) = \max\left\{\begin{array}{c} \sigma_b(\rho,\varpi), \sigma_b(\rho,\mathscr{J}\rho), \sigma_b(\varpi,\mathscr{J}\varpi), \dfrac{\sigma_b(\rho,\mathscr{J}\varpi) + \sigma_b(\varpi,\mathscr{J}\rho)}{2s}, \\ \dfrac{\sigma_b(\varpi,\mathscr{J}\varpi)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}, \dfrac{\sigma_b(\varpi,\mathscr{J}\rho)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]} \end{array}\right\},$$

$$m_2(\rho,\varpi) = \min\{\sigma_b(\rho,\mathscr{J}\rho), \sigma_b(\varpi,\mathscr{J}\varpi), \sigma_b(\rho,\mathscr{J}\varpi), \sigma_b(\varpi,\mathscr{J}\rho)\} \tag{43}$$

for all $\rho, \varpi \in O(\bar{\rho_0}, \mathscr{J})$. Suppose that $(\mathscr{M}, \sigma_b, s)$ is $\mathscr{J}$-orbitally complete for some $\rho_0 \in \mathscr{M}$. If $\mathscr{J}\rho$ is closed for all $\rho \in O(\bar{\rho_0}, \mathscr{J})$ or the property (Gp) holds, then there exists $\rho^* \in \mathscr{M}$ such that $\rho^* \in \mathscr{J}\rho^*$.

If we replace self-mapping $\mathscr{J} : \mathscr{M} \longrightarrow \mathscr{M}$ on the place of multivalued mapping in Theorem 7, we can get following results as consequences.

Corollary 11. *Let $(\mathscr{M}, \sigma_b)$ be a b-metric space such that σ_b is a continuous functional and $\mathscr{J} : \mathscr{M} \longrightarrow \mathscr{M}$ be a mapping such that $\mathscr{M}$ is $\mathscr{J}$-orbitally complete at some ρ_0. Assume that there exist $\Theta \in \Omega_s$, $k \in (0, 1)$ and $L \ge 0$ such that*

$$\Theta(s\sigma_b(\mathscr{J}\rho, \mathscr{J}\varpi)) \le \left[\Theta(m_1^{\prime}(\rho,\varpi) + Lm_2^{\prime}(\rho,\varpi))\right]^k \tag{44}$$

for all $\rho, \varpi \in \mathscr{M}$ with $\min\{\sigma_b(\mathscr{J}\rho, \mathscr{J}\varpi), \sigma_b(\rho, \varpi)\} > 0$, where

$$m_1^{\prime}(\rho,\varpi) = \max\left\{\begin{array}{c} \sigma_b(\rho,\varpi), \sigma_b(\rho,\mathscr{J}\rho), \sigma_b(\varpi,\mathscr{J}\varpi), \dfrac{\sigma_b(\rho,\mathscr{J}\varpi) + \sigma_b(\varpi,\mathscr{J}\rho)}{2s}, \\ \dfrac{\sigma_b(\varpi,\mathscr{J}\varpi)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}, \dfrac{\sigma_b(\varpi,\mathscr{J}\rho)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]} \end{array}\right\},$$

$$m_2^{\prime}(\rho,\varpi) = \min\{\sigma_b(\rho,\mathscr{J}\rho), \sigma_b(\varpi,\mathscr{J}\varpi), \sigma_b(\rho,\mathscr{J}\varpi), \sigma_b(\varpi,\mathscr{J}\rho)\}. \tag{45}$$

If Θ is continuous, then there exists $\rho^* \in \mathscr{M}$ such that $\rho^* = \mathscr{J}\rho^*$.

4. Applications

In this section, we solve the Fredholm integral inclusion:

$$\rho(\varsigma) \in g(\varsigma) + \int_a^b K(\varsigma, \zeta, \rho(\zeta))\sigma\zeta, \quad \forall \varsigma \in [a, b], \tag{46}$$

given in the start of this paper.

Let σ_b on $C[a, b]$ be given by

$$\sigma_b(\rho,\varpi) = \left(\max_{\varsigma\in[a,b]} |\rho(\varsigma) - \varpi(\varsigma)|\right)^p = \max_{\varsigma\in[a,b]} |\rho(\varsigma) - \varpi(\varsigma)|^p, \quad \text{with } p \ge 1. \tag{47}$$

Then, $(C[a, b], \sigma_b, 2^{p-1})$ is a complete b-metric space. Assume that for $\varsigma, \zeta \in [a, b]$, these conditions hold:

(a) $\forall \rho \in C[a, b], K_{\rho}(\varsigma, \zeta) = K(\varsigma, \zeta, \rho(\zeta))$ is continuous

(b) $\exists Y : [a, b] \times [a, b] \longrightarrow [0,+\infty)$ such that

$$\begin{aligned} &\left|k_{\varphi}(\varsigma,\zeta) - k_{\psi}(\varsigma,\zeta)\right|^p \le Y(\varsigma,\zeta) \\ &\cdot \left[\begin{array}{c} \max\left\{\begin{array}{c} \sigma_b(\varphi(\zeta),\psi(\zeta)), \sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta))), \\ \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\psi(\zeta))), \\ \dfrac{\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\psi(\zeta))) + \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))}{2^p}, \\ \dfrac{\sigma_b(\psi(\zeta), K(\varsigma,\zeta,\psi(\zeta)))[1+\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))]}{2^{p-1}[1+\sigma_b(\varphi(\zeta),\psi(\zeta))]}, \\ \dfrac{\sigma_b(\psi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))[1+\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))]}{2^{p-1}[1+\sigma_b(\varphi(\zeta),\psi(\zeta))]} \end{array}\right\} + \\ \lambda \min\{\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta))), \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\psi(\zeta))), \\ \sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\psi(\zeta))), \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))\} \end{array}\right] \end{aligned} \tag{48}$$

$\forall \varsigma, \zeta \in [a, b], \varphi, \psi \in C[a, b]$ and $k_{\varphi}(\varsigma, \zeta) \in K_{\varphi}(\varsigma, \zeta), k_{\psi}(\varsigma, \zeta) \in K_{\psi}(\varsigma, \zeta)$, where $p > 1$ and $\lambda \ge 0$

(c) $\exists 0 < k < 1$ such that

$$\sup_{\varsigma\in[a,b]} \int_a^b Y(\varsigma,\zeta)\sigma\zeta \le \frac{k}{2^{p-1}} \tag{49}$$

Theorem 12. *Under the conditions (a)–(c), the integral inclusion given in (46) has a solution in $C[a, b]$.*

Proof. Let $\mathscr{M} = C[a, b]$. Define b-metric σ_b as in (47) and a multivalued mapping $\mathscr{J} : \mathscr{M} \longrightarrow P_{cb}(\mathscr{M})$ by

$$\mathscr{J}\rho = \left\{\varpi \in \mathscr{M} : \varpi(\varsigma) \in g(\varsigma) + \int_a^b K(\varsigma, \zeta, \rho(\zeta))\sigma\zeta, \varsigma \in [a, b]\right\}. \tag{50}$$

Let $\rho \in \mathscr{M}$. For a mapping $K_{\rho}(\varsigma, \zeta)$: $[a, b] \times [a, b] \longrightarrow P_{cb}(\mathbb{R})$, there exists $k_{\rho}(\varsigma, \zeta)$: $[a, b] \times [a, b] \longrightarrow \mathbb{R}$ such that

$$k_{\rho}(\varsigma, \zeta) \in K_{\rho}(\varsigma, \zeta), \tag{51}$$

$\forall \varsigma, \zeta \in [a, b]$. Then,

$$g(\varsigma) + \int_a^b k_\rho(\varsigma, \zeta)\sigma\zeta \in \mathcal{J}\rho. \tag{52}$$

Hence, $\mathcal{J}\rho \neq \emptyset$. Since $K_\rho(\varsigma, \zeta)$ is continuous and $g \in C([a, b])$, the ranges of both functions are bounded. This shows that $\mathcal{J}\rho$ is also bounded.

Now, we show that (9) holds for any $\mathcal{J}$ on $\mathscr{M}$ with some $0 < k < 1$, $0 \leq L$ and $\Theta \in \Omega_\zeta$, i.e.,

$$\begin{aligned}
&\Theta(\zeta\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2)) \\
&\leq \left[\Theta\left[\begin{array}{c} \max\left\{\begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]} \end{array}\right\} \\ +L \min\{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \sigma_b(\rho_1, \mathcal{J}\rho_2), \sigma_b(\rho_2, \mathcal{J}\rho_1)\} \end{array}\right]\right]^k
\end{aligned} \tag{53}$$

for all $\rho_1, \rho_2 \in \mathscr{M}$. Let $\varpi_1 \in \mathcal{J}\rho_1$ be such that

$$\varpi_1(\varsigma) \in g(\varsigma) + \int_a^b K(\varsigma, \zeta, \rho_1(\zeta))\sigma\zeta, \quad \forall \varsigma \in [a, b]. \tag{54}$$

It follows that $\exists k_{\rho_1}(\varsigma, \zeta) \in K_{\rho_1}(\varsigma, \zeta) = K(\varsigma, \zeta, \rho_1(\zeta))$ such that

$$\varpi_1(\varsigma) = g(\varsigma) + \int_a^b k_{\rho_1}(\varsigma, \zeta)\sigma\zeta, \quad \forall \varsigma, \zeta \in [a, b]. \tag{55}$$

Now, for all $\rho_1, \rho_2 \in \mathscr{M}$, it follows from (b) that

$$\begin{aligned}
&\left|k_{\rho_1}(\varsigma, \zeta) - k_{\rho_2}(\varsigma, \zeta)\right|^p \leq Y(\varsigma, \zeta) \\
&\cdot\left[\begin{array}{c} \max\left\{\begin{array}{c} \sigma_b(\rho_1(\zeta), \rho_2(\zeta)), \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta))), \\ \sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))), \\ \dfrac{\sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))) + \sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))}{2^p} \\ \dfrac{\sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_2(\zeta)))[1 + \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))]}{2^{p-1}[1 + \sigma_b(\rho_1(\zeta), \rho_2(\zeta))]}, \\ \dfrac{\sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))[1 + \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))]}{2^{p-1}[1 + \sigma_b(\rho_1(\zeta), \rho_2(\zeta))]} \end{array}\right\} \\ +L \min\{\sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta))), \\ \sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))), \\ \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))), \\ \sigma_b(\rho_2(\rho), K(\varsigma, \zeta, \rho_1(\zeta)))\} \end{array}\right]
\end{aligned} \tag{56}$$

It follows that $\exists \omega(\varsigma, \zeta) \in K_{\rho_2}(\varsigma, \zeta)$ such that

$$\begin{aligned}
&\left|k_{\rho_1}(\varsigma, \zeta) - \omega(\varsigma, \zeta)\right|^p \leq Y(\varsigma, \zeta) \\
&\cdot\left[\begin{array}{c} \max\left\{\begin{array}{c} \sigma_b(\rho_1(\zeta), \rho_2(\zeta)), \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta))), \\ \sigma_b(\rho_2(\zeta), \omega(\varsigma, \zeta)), \\ \dfrac{\sigma_b(\rho_1(\zeta), \omega(\varsigma, \zeta)) + \sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))}{2^p}, \\ \dfrac{\sigma_b(\rho_2(\zeta), \omega(\varsigma, \zeta))[1 + \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))]}{2^{p-1}[1 + \sigma_b(\rho_1(\zeta), \rho_2(\zeta))]}, \\ \dfrac{\sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))[1 + \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))]}{2^{p-1}[1 + \sigma_b(\rho_1(\zeta), \rho_2(\zeta))]} \end{array}\right\} \\ +L \min\{\sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta))), \sigma_b(\rho_2(\zeta), \omega(\varsigma, \zeta)), \\ \sigma_b(\rho_1(\zeta), \omega(\varsigma, \zeta)), \sigma_b(\rho_2(\rho), K(\varsigma, \zeta, \rho_1(\zeta)))\} \end{array}\right] \\
&= R(\varsigma, \zeta), \forall \varsigma, \zeta \in [a, b].
\end{aligned} \tag{57}$$

Now, we show the function $U(\varsigma, \zeta)$: $[a, b] \times [a, b] \longrightarrow P_{cb}(\mathbb{R})$ by

$$U(\varsigma, \zeta) = K_{\rho_2}(\varsigma, \zeta) \bigcap \left\{\varphi \in \mathbb{R} : \sigma_b\left(k_{\rho_1}(\varsigma, \zeta), \varphi\right) \leq R(\varsigma, \zeta)\right\}. \tag{58}$$

Thus, by (a), U is lower semicontinuous; this implies that there exists $k_{\rho_2}(\varsigma, \zeta)$: $[a, b] \times [a, b] \longrightarrow \mathbb{R}$ such that $k_{\rho_2}(\varsigma, \zeta) \in U(\varsigma, \zeta), \forall \varsigma, \zeta \in [a, b]$. Then, $\varpi_2(\varsigma) = g(\varsigma) + \int_a^b k_{\rho_1}(\varsigma, \zeta)\sigma\zeta$ satisfies

$$\varpi_1(\varsigma) \in g(\varsigma) + \int_a^b K(\varsigma, \zeta, \rho_1(\zeta))\sigma\zeta, \quad \forall \varsigma \in [a, b], \tag{59}$$

which implies that $\varpi_2 \in \mathcal{J}\rho_2$ and

$$\begin{aligned}
&\sigma_b(\varpi_1, \varpi_2) \leq \max_{\varsigma \in [a,b]} \int_a^b \left|k_{\rho_1}(\varsigma, \zeta) - k_{\rho_2}(\varsigma, \zeta)\right|^p \sigma\zeta \leq \max_{\varsigma \in [a,b]} \int_a^b Y(\varsigma, \zeta) \\
&\cdot\left[\begin{array}{c} \max\left\{\begin{array}{c} \sigma_b(\rho_1(\zeta), \rho_2(\zeta)), \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta))), \\ \sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))), \\ \dfrac{\sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))) + \sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))}{2^p}, \\ \dfrac{\sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_2(\zeta)))[1 + \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))]}{2^{p-1}[1 + \sigma_b(\rho_1(\zeta), \rho_2(\zeta))]}, \\ \dfrac{\sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))[1 + \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta)))]}{2^{p-1}[1 + \sigma_b(\rho_1(\zeta), \rho_2(\zeta))]} \end{array}\right\} \\ +L \min\{\sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_1(\zeta))), \\ \sigma_b(\rho_2(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))), \\ \sigma_b(\rho_1(\zeta), K(\varsigma, \zeta, \rho_2(\zeta))), \\ \sigma_b(\rho_2(\rho), K(\varsigma, \zeta, \rho_1(\zeta)))\} \end{array}\right] \sigma\zeta \leq \frac{k}{2^{p-1}} \\
&\cdot\left[\begin{array}{c} \max\left\{\begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{\zeta[1 + \sigma_b(\rho_1, \rho_2)]}, \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{\zeta[1 + \sigma_b(\rho_1, \rho_2)]} \end{array}\right\} \\ +L \min\{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_2), \sigma_b(\rho_2 \mathcal{J}\rho_1)\} \end{array}\right],
\end{aligned} \tag{60}$$

$\forall \varsigma, \zeta \in [a, b]$. Thus, we get

$$\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2) \le \frac{k}{2^{p-1}} \left[\begin{array}{c} \max \left\{ \begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]} \end{array} \right\} \\ +L \min \{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_2), \\ \sigma_b(\rho_2, \mathcal{J}\rho_1)\} \end{array} \right],$$

$$\zeta\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2) \le k \left[\begin{array}{c} \max \left\{ \begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]} \end{array} \right\} \\ +L \min \{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \sigma_b(\rho_1, \mathcal{J}\rho_2), \sigma_b(\rho_2, \mathcal{J}\rho_1)\} \end{array} \right]. \tag{61}$$

Taking the exponential on both sides, we have

$$e^{\zeta\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2)}\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2) \le ek \left[\begin{array}{c} \max \left\{ \begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]} \end{array} \right\} \\ +L \min \{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_2), \sigma_b(\rho_2, \mathcal{J}\rho_1)\} \end{array} \right]. \tag{62}$$

Taking the function $\Theta \in \Omega_s$, it follows that (18) is fulfilled. Therefore, by Theorem 7, we show that the integral inclusion (46) has a solution. This completes the proof. □

5. Conclusion

In this paper, we have defined a rational (Θ, δ_b)-contraction and established some fixed point results in b-metric spaces. In this way, we generalized some known results of literature. We also discussed the solution of Fredholm integral inclusion as application of our obtained result. We expect that the results given in this article will make new directions for those who are working in the theory of fixed points.

In this direction, our future work will pivot on studying the fixed points of fuzzy mappings and L-fuzzy mappings for Θ-contractions in b-metric spaces, with fractional differential inclusion problems as applications.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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Research Article

An Analysis on the Positive Solutions for a Fractional Configuration of the Caputo Multiterm Semilinear Differential Equation

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In this paper, we consider a multiterm semilinear fractional boundary value problem involving Caputo fractional derivatives and investigate the existence of positive solutions by terms of different given conditions. To do this, we first study the properties of Green's function, and then by defining two lower and upper control functions and using the wellknown Schauder's fixed-point theorem, we obtain the desired existence criteria. At the end of the paper, we provide a numerical example based on the given boundary value problem and obtain its upper and lower solutions, and finally, we compare these positive solutions with exact solution graphically.

1. Introduction

At a vast level, it is understood that the hereditary properties and the memory of most processes, phenomena, and materials are predictable with the help of different modelings under the fractional operators. In this regard, differential equations involving fractional derivatives have recently been confirmed to be a useful tool in modeling of a considerable variety of structures in miscellaneous branches of sciences. For the sake of the increasing acceleration and advancements of studies and researches in the field of fractional calculus, several works have been done; see [1, 2]. Since theoretical findings are used to achieve a deep understanding for the fractional models, a large number of mathematicians have also assigned their focus on studying the existence aspects of solutions for several structures of fractional equations by means of different techniques and methods. For instance, see [3–10].

In the next periods, a large number of researchers studied the notion of positive solutions for nonlinear fractional differential equations, and accordingly, many papers have been published in this direction. In 2003, Zhang [11] investigated the multiple and infinitely solvability of positive solutions for a nonlinear generalized fractional differential equation by relying on fixed point methods on cones. In 2007, El-Shahed [12] investigated the existence and nonexistence of positive solutions for a nonlinear fractional boundary value problem in the Riemann-Liouville settings. The author used Krasnoselskii's fixed point theorem on cone preserving operators for deriving some required criteria. In [13], Guezane-Lakoud et al. presented a fourth-order mathematical model of elastic beam in three separate points of domain and studied the existence of positive solutions with the help of fixed point techniques.

In [14], Tian et al. turned to investigate positive solutions for a new class of fourpoint boundary value problem of fractional differential equations with p-laplacian operator and used the Leggett-Williams fixed point theorem on a cone to prove the multiplicity results of such solutions. More recently, Seemab et al. [15] established the existence results of positive solutions for a boundary value problem defined within generalized Riemann-Liouville and Caputo fractional operators by studying the properties of Green functions in three different types. Along with these, some other researchers investigated numerical methods and nonsingular fractional operators for obtaining numerical solutions of different fractional differential equations such as [16, 17].

More specifically, in [18], Zhang studied the multiplicity and existence of positive solutions for the fractional nonlinear boundary value problem given by

$$\begin{cases} \mathfrak{D}^{\nu} w(z) = \Xi(z, w(z)), (1 < \nu < 2, 0 \leq z \leq 1) \\ w(0) + w'(0) = 0, w(1) + w'(1) = 0, \end{cases} \tag{1}$$

where $\mathfrak{D}^{\nu}$ stands for the Caputo fractional derivative. To obtain the existence conditions, Zhang applied a method based on cones. Bai and Lu [19] also employed some nonlinear methods to establish the multiplicity and existence of positive solutions of the given problem as

$$\begin{cases} {}^{R}\mathfrak{D}^{\nu} w(z) = \Xi(z, w(z)), (0 \leq z \leq, 1 < \nu \leq 2), \\ w(0) = 0, w(1) = 0, \end{cases} \tag{2}$$

where $\mathfrak{D}^{\nu}$ denotes the Riemann-Liouville fractional derivative and $\Xi : [0, 1] \times \mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$ is a continuous function. Their method is based upon the reduction of the given boundary value problem to the equivalent Fredholm integral equation of the second kind.

Inspired by the above works, in this paper, we derive some sufficient conditions to establish our main results on the existence of positive solutions to multiterm semilinear fractional boundary value problem given by

$$\begin{cases} \mathfrak{D}^{r} w(z) = \Xi(z, w(z), \mathfrak{D}^{p} w(z)), \left(z \in \bar{J} = [0, 1]\right), \\ w(0) = 0, \mathfrak{D}^{r-1} w(1) = 0, \end{cases} \tag{3}$$

where $1 < r < 2$, $0 < p < 1$ and Ξ is a continuous positive function on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ and $\mathfrak{D}^{(\cdot)}$ denotes the Caputo fractional derivative. Note that the main method of this paper is to convert our multiterm semilinear boundary value problem (3) to an equivalent integral equation which allows us to convert it to a fixed point problem. In the following path, the criteria establishing the existence of positive solutions are guaranteed by imposing several sufficient conditions. In fact, we introduce two upper and lower control functions to achieve such aims, and after surveying some properties of the Green's function, the fundamental theorems in relation to the existence results are derived. To prove the existence results, by applying lower and upper control functions, we use the standard Schauder's fixed point theorem to obtain lower and upper solutions. In contrast to other works, we generalize and consider two terminal boundary conditions in the context of the Caputo fractional derivative of the unknown function and show our findings graphically. In other words, by plotting the graphs of lower and upper solutions, one can compare our results with the exact positive solution. It is notable that in the suggested problem (3), we have considered a multiterm semilinear boundary value problem, and in the future researches, one can implement this technique for the complicated boundary conditions and one can also cover other generalized nonlinear fractional boundary problems arising in real-world phenomena.

The structure of the article is presented as follows: Sect. 2 recall the auxiliary and preliminary notions in relation to the fractional calculus and also some basic lemmas are provided. Sect. 3 is assigned to derive sufficient conditions to obtain positive solutions of the multiterm semilinear boundary value problem (3). Finally, a special example is provided to validate our theoretical findings according to the method implemented in theorems.

2. Basic and Auxiliary Concepts

Before proving some preliminary lemmas, we need to present some definitions and properties on fractional calculus which are useful throughout our research work.

Definition 1 (see [20]). Let $\nu > 0$ and $\phi : (0,+\infty) \longrightarrow \mathbb{R}$ be continuous. The integral

$$\mathcal{I}^{\nu} \varphi(z) = \frac{1}{\Gamma(\nu)} \int_0^z (z - s)^{\nu - 1} \varphi(s) ds, \tag{4}$$

is called the fractional integral in the Riemann-Liouville settings of order ν such that it has finite values.

Definition 2 (see [20]). Let $k - 1 < \nu < k$ with $k = [\nu] + 1$ and $\phi : (0,+\infty) \longrightarrow \mathbb{R}$ belongs to $AC(k)((0,\infty), \mathbb{R})$. Then

$$\mathfrak{D}^{\gamma} \varphi(z) = \frac{1}{\Gamma(k - \nu)} \int_0^z (z - s)^{k - \nu - 1} \varphi^{(k)}(s) ds, \tag{5}$$

is called the Caputo fractional derivative of order ν such that it exists.

Remark 3. We have the following:

(RE1) For $0 \leq \nu < \gamma$, the equality $\mathfrak{D}^{\nu} I^{\gamma} \varphi(z) = I^{\gamma - \nu} \varphi(z)$ holds

(RE2) For $\gamma > -1$ with $\gamma \neq \nu - j (j = 1, 2, \cdots, n)$ and for each $z \geq 0$, we have

$$\mathfrak{D}^{\nu} z^{\gamma} = \frac{\Gamma(1 + \gamma)}{\Gamma(\gamma - \nu + 1)} z^{\gamma - \nu} \text{and} \mathfrak{D}^{\nu} z^{\nu - 1} = 0, (j = 1, 2, \cdots, n). \tag{6}$$

Proposition 4 (see [21]). *Suppose that ϕ is contained in the space $\mathcal{L}(0, 1) \cap \mathcal{C}(0, 1)$ and $k = [\nu] + 1$. Then*

$$\mathcal{I}^{\gamma} \mathfrak{D}^{\gamma} w(z) = w(z) + c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-1} z^{k-1}, \tag{7}$$

such that $c_0, c_2, \cdots, c_{k-1} \in \mathbb{R}$.

The following proposition is important and specifies the structure of the equivalent solution of the integral equation arising in the multiterm semilinear boundary value problem (3).

Proposition 5. *Consider $\varrho \in \mathscr{C}([0,1],\mathbb{R})$ and $1<r<2$. Then, the solution of the linear problem*

$$\begin{cases} \mathfrak{D}^r w(z)=\varrho(z),\ z\in \bar{J} \\ w(0)=0,\ \mathfrak{D}^{r-1}w(1)=0, \end{cases} \tag{8}$$

is given by the following integral equation

$$w(z)=\int_0^1 H(z,s)\varrho(s)ds, \tag{9}$$

where

$$H(z,s)=\begin{cases} \dfrac{(z-s)^{r-1}}{\Gamma(r)}-\Gamma(3-r)z, & 0\le s\le z\le 1 \\ -\Gamma(3-r)z, & 0\le z\le s\le 1. \end{cases} \tag{10}$$

Proof. If w is a solution of the linear boundary value problem (8), then from Proposition (7), it is followed that

$$w(z)=c_0+c_1z+I^r\varrho(z)=c_0+c_1z+\frac{1}{\Gamma(r)}\int_0^z (z-s)^{r-1}\varrho(s)ds. \tag{11}$$

Then, the first boundary condition gives $c_0=0$. By applying the operator $\mathfrak{D}^{r-1}$ to both sides of (11) and using (6), we find that

$$\mathfrak{D}^{r-1}w(z)=\frac{c_1}{\Gamma(3-r)}z^{2-r}+I^1_{0+}\varrho(z), \tag{12}$$

which in view of the second boundary condition, gives

$$c_1=-\Gamma(3-r)\int_0^1 \varrho(s)ds. \tag{13}$$

By substituting c_0 and c_1 in (11), we get

$$\begin{aligned} \mathrm{w}(\mathrm{z})&=\frac{1}{\Gamma(r)}\int_0^z (z-s)^{r-1}\varrho(s)ds-\Gamma(3-1)z\int_0^1\varrho(s)ds \\ &=\int_0^1 H(z,s)\varrho(s)ds, \end{aligned} \tag{14}$$

where $H(z,s)$ is given by (10). In this case, we follow that w will be a solution of (9). This completes the proof. □

Remark 6. It is easy to show by a simple computation that the function H satisfies

$$\int_0^1 |H(z,s)|ds \le \frac{1}{\Gamma(r-1)}+\Gamma(3-\mathrm{r}). \tag{15}$$

Lemma 7. *The function $|\partial H(z,s)/\partial z|$ is integrable for each $z\in[0,1]$.*

Proof. We have

$$\frac{\partial H(z,s)}{\partial z}=\begin{cases} \dfrac{(z-s)^{r-2}}{\Gamma(r-1)}-\Gamma(3-\mathrm{r}), & 0\le s\le z\le 1, \\ -\Gamma(3-\mathrm{r}), & 0\le z\le s\le 1. \end{cases} \tag{16}$$

Then,

$$\begin{aligned} \int_0^1\left|\frac{\partial H(z,s)}{\partial z}\right|ds &\le \int_0^z \frac{(z-s)^{r-2}}{\Gamma(r-1)}ds+\int_0^z\Gamma(3-\mathrm{r})ds+\int_z^1\Gamma(3-\mathrm{r})ds \\ &=\frac{z^{r-1}}{\Gamma(\mathrm{r})}+\Gamma(3-\mathrm{r})z+\Gamma(3-\mathrm{r})(1-z) \\ &\le \frac{1}{\Gamma(\mathrm{r})}+\Gamma(3-\mathrm{r})<\infty. \end{aligned} \tag{17}$$

This completes the proof. □

Remark 8. Consider the space $\mathbb{X}=\mathscr{C}^1([0,1],\mathbb{R})$. For $0<p<1$ and $w\in\mathbb{X}$, define the norm of w by

$$\|w\|_{\mathbb{X}}=\max_{z\in[0,1]}|w(z)|+\max_{z\in[0,1]}\left|w'(z)\right|+\max_{z\in[0,1]}\left|\mathfrak{D}^p w(z)\right|. \tag{18}$$

Then, clearly, $(\mathbb{X},\|.\|_{\mathbb{X}})$ is a Banach space.

3. Existence Criteria for Positive Solutions

In this section, several conditions are derived for which the existence of positive solutions to the multiterm semilinear boundary value problem (3) is guaranteed. Let $\alpha_1,\alpha_3\in\mathbb{R}^+$ and $\alpha^2,\alpha^4\in\mathbb{R}$ with $\alpha_1<\alpha_3$ and $\alpha_2<\alpha_4$. The upper control function

$$\widehat{\Delta}:[0,1]\times[\alpha_1,+\infty)\times[\alpha_2,+\infty)\longrightarrow\mathbb{R}^+, \tag{19}$$

and the lower control function $\widehat{\delta}:[0,1]\times[-\infty,\alpha_3)\times[-\infty,\alpha_4)\longrightarrow\mathbb{R}^+$ is defined by

$$\begin{aligned} \widehat{\Delta}(z,u,v)&=\sup_{\substack{\alpha_1\le\theta\le u\\ \alpha_2\le\mu\le v}}\Xi(z,\theta,\mu)\text{ and} \\ \widehat{\delta}(z,u,v)&=\inf_{\substack{u\le\theta\le\alpha_3\\ v\le\mu\le\alpha_4}}\Xi(z,\theta,\mu), \end{aligned} \tag{20}$$

respectively. We clearly have

$$\begin{aligned} &\widehat{\delta}(z,u,v)\le\Xi(z,u,v)\le\widehat{\Delta}(z,u,v),\text{ for }0 \\ &\le z\le 1,\alpha_1\le u\le\alpha_3,\alpha_2\le v\le\alpha_4. \end{aligned} \tag{21}$$

In addition to these, define the set

$$\tilde{\Lambda}=\{w\in\mathbb{X}:w(z)\ge 0,0\le z\le 1\}, \tag{22}$$

which is used in the sequel. Here, we mean by a positive solution, each function $\boldsymbol{w}$ satisfies $\boldsymbol{w} \in X, \boldsymbol{w}(0) = 0$ and $\boldsymbol{w}(z) > 0$ for each $0 < z \le 1$; in other words, $w \in \tilde{\Lambda}$.

3.1. Required Assumptions. Now, for our main results, we need some assumptions given as follows:

(A1) There are $w^\star, w_\star \in \tilde{\Lambda}$ which satisfy $\alpha_1 \le \boldsymbol{w}_\star(z) \le \boldsymbol{w}^\star(z) \le \alpha_3$ and $\alpha_2 \le \mathfrak{D}^p \boldsymbol{w}_\star(z) \le \mathfrak{D}^p \boldsymbol{w}^\star(z) \le \alpha_4$ along with

$$\begin{aligned} w^\star(z) &\ge \int_0^1 |H(z,s)| \widehat{\Delta}(s, w^\star(s), \mathfrak{D}^p w^\star(s)) ds, \\ w_\star(z) &\le \int_0^1 |H(z,s)| \widehat{\delta}(s, w_\star(s), \mathfrak{D}^p w_\star(s)) ds, \\ \mathfrak{D}^p w^\star(z) &\ge -\frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \widehat{\Delta}(s, w^\star(s), \mathfrak{D}^p w^\star(s)) ds \\ &\quad + \frac{\Gamma(3-r)}{\Gamma(2-p)} z^{1-p} \int_0^1 \widehat{\delta}(s, w^\star(s), \mathfrak{D}^p w^\star(s)) ds, \\ \mathfrak{D}^p w_\star(z) &\le -\frac{1}{\Gamma(r-p)} \int_0^z (z-s)^{r-p-1} \widehat{\delta}(s, w_\star(s), \mathfrak{D}^p w_\star(s)) ds \\ &\quad + \frac{\Gamma(3-r)}{\Gamma(2-p)} z^{1-p} \int_0^1 \widehat{\Delta}(s, w_\star(s), \mathfrak{D}^p w_\star(s)) ds. \end{aligned} \tag{23}$$

(A2) There exist $\xi > 0$ and nonnegative function $\theta \in \mathscr{L}^1(0,1)$ such that

$$\Xi(z, w, v) \le \theta(z) + \xi(|w| + |v|), 0 \le z \le 1, w, v \in \mathbb{R}. \tag{24}$$

(A3) There exists $\zeta > 0$ such that

$$\begin{aligned} A + B + \frac{B}{\Gamma(2-p)} + \xi\zeta \bigg(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(r+1)} \\ + 2\Gamma(3-r) + \frac{1}{\Gamma(2-p)\Gamma(r)} + \frac{\Gamma(3-r)}{\Gamma(2-p)} \bigg) \le \zeta. \end{aligned} \tag{25}$$

with

$$A = \max_{z \in [0,1]} \int_0^1 |H(z,s)\theta(s)| ds \text{and} B = \max_{z \in [0,1]} \int_0^1 \left| \frac{\partial H(z,s)}{\partial z} \theta(s) \right| ds. \tag{26}$$

At this moment, we are ready to present the first existence theorem.

Theorem 9. *Suppose that the assumptions (A1)–(A3) hold. Then, the multiterm semilinear boundary value problem (3) has at least a positive solution w in $\mathbb{X}$ such that all inequalities $w_\star(z) \le w(z) \le w^\star(z)$ and $\mathfrak{D}^p w_\star(z) \le \mathfrak{D}^p w(z) \le \mathfrak{D}^p w^\star(z)$ hold for each $0 \le z \le 1$.*

Proof. For each $\zeta > 0$, define the set Γ_ζ as

$$\begin{aligned} \Gamma_\zeta = \Big\{ w \in \tilde{\Lambda} : \|w\|_{\mathbb{X}} \le \zeta, w_\star(z) \le w(z) \le w^\star(z), \mathfrak{D}^p w_\star(z) \\ \le \mathfrak{D}^p w(z) \le \mathfrak{D}^p w^\star(z), 0, \le z \le 1 \Big\}. \end{aligned} \tag{27}$$

Obviously, Γ_ζ is a convex, closed, and bounded set in $\mathbb{X}$. Consider the operator $\mathfrak{P} : \Gamma_\zeta \longrightarrow \mathbb{X}$ under the following rule

$$\begin{aligned} (\mathfrak{P}w)(z) &= -\frac{1}{\Gamma(r)} \int_0^z (z-s)^{r-1} \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\ &\quad + \Gamma(3-r) z \int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s)) ds \\ &= \int_0^1 H(z,s) \Xi(s, w(s), \mathfrak{D}^p w(s)) ds. \end{aligned} \tag{28}$$

To prove Theorem 9, we will show that the hypotheses of Schauder's fixed point theorem hold. So, the process of proof will be done in several steps.

Step 1. P is continuous in X. To prove such a claim, we consider a sequence $\{w_n\}$ which converges to w in X. We have

$$\begin{aligned} &|\mathfrak{P}w_n(z) - \mathfrak{P}w(z)| \\ &= \bigg| \int_0^1 H(z,s) (\Xi(s, w_n(s), \mathfrak{D}^p w_n(s)) \\ &\quad - \Xi(s, w(s), \mathfrak{D}^p_{0_+} w(s))) ds \bigg| \\ &\le \max_{z \in [0,1]} |\Xi(\mathrm{z}, \mathrm{w_n}(\mathrm{z}), \mathfrak{D}^p w_n(z)) \\ &\quad - \Xi(\mathrm{z}, w(\mathrm{t}), \mathfrak{D}^p w_n(z))| \int_0^1 |H(z,s)| ds \\ &\le \left(\frac{1}{\Gamma(r+1)} + \Gamma(3-r) \right) \max_{z \in [0,1]} |\Xi(\mathrm{z}, \mathrm{w_n}(\mathrm{z}), \mathfrak{D}^p w_n(z)) \\ &\quad - \Xi(\mathrm{z}, w(\mathrm{z}), \mathfrak{D}^p w(z))|, \end{aligned} \tag{29}$$

$$\begin{aligned} &\left| \mathfrak{D}^p \mathfrak{P} w_n(z) - \mathfrak{D}^p \mathfrak{P} w(z) \right| \\ &= \left| \frac{1}{\Gamma(1-p)} \int_0^z (z-s)^{-p} \Big((\mathfrak{P}w_n)'(s) - (\mathfrak{P}w)'(s) \Big) ds \right| \\ &\le \Gamma(1-\mathrm{p}) \int_0^z (z-s)^{-p} \bigg(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} [\Xi(\lambda, w_n(\lambda), \mathfrak{D}^p w_n(\lambda)) \right. \\ &\quad \left. - \Xi(\lambda, w(\lambda), \mathfrak{D}^p w(\lambda))] \right| d\lambda \bigg) ds \\ &\le \max_{z \in [0,1]} \left| \Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) - \Xi(z, w(z), \mathfrak{D}^p w(z)) \right| \\ &\quad \cdot \frac{1}{\Gamma(1-p)} \int_0^z (z-s)^{-p} \bigg(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \right| d\lambda \bigg) ds \\ &\le \max_{z \in [0,1]} \left| \Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) - \Xi(z, w(z), \mathfrak{D}^p w(z)) \right| \\ &\quad \cdot \frac{1}{\Gamma(1-p)} \left(\frac{1}{\Gamma(r)} + \frac{\Gamma(3-\mathrm{r})}{\Gamma(2)} \right) \int_0^z (z-s)^{-p} ds \\ &\le \frac{1}{\Gamma(2-p)} \left(\frac{1}{\Gamma(r)} + \Gamma(3-r) \right) \max_{z \in [0,1]} |\Xi(z, w_n(z), \mathfrak{D}^p w_n(z)) \\ &\quad - \Xi(z, w(z), \mathfrak{D}^p w(z))|, \end{aligned} \tag{30}$$

$$\begin{aligned}
&\left|(\mathfrak{P}w_n)'(z)-(\mathfrak{P}w)'(z)\right| \\
&\quad=\left|\int_0^1 \frac{\partial H(z,s)}{\partial z}\left(\Xi(s,w_n(s),\mathfrak{D}^p w_n(s))\right.\right. \\
&\qquad\left.\left.-\Xi(s,w(s),\mathfrak{D}^p w(s))\right)ds\right| \\
&\quad\le \max_{z\in[0,1]}\left|\Xi(z,w_n(z),\mathfrak{D}^p w_n(z))\right. \\
&\qquad\left.-\Xi(z,w(z),\mathfrak{D}^p w(z))\right|\left|\int_0^1\left|\frac{\partial H(z,s)}{\partial z}\right|ds\right. \\
&\quad\le\left(\frac{1}{\Gamma(r)}+\Gamma(3-\mathrm{r})\right)w_{z\in[0,1]}\left|\Xi(z,w_n(z),\mathfrak{D}^p w_n(z))\right. \\
&\qquad\left.-\Xi(z,w(z),\mathfrak{D}^p w(z))\right|.
\end{aligned} \tag{31}$$

By tending $n\longrightarrow\infty$ and from the inequalities (29), (30), and (31), we follow that $\mathfrak{P}$ is continuous in $\mathbb{X}$.

Step 2. Now, we show that $\mathfrak{P}:\Gamma_\zeta\longrightarrow\Gamma_\zeta$ is a selfmap on Γ_ζ. Let $w\in\Gamma_\zeta$. By inequalities (15) and (17) along with the assumptions (A2) and (A3), we get

$$\begin{aligned}
|\mathfrak{P}w(z)|&=\left|\int_0^1 H(z,s)\Xi(s,w(s),\mathfrak{D}^p w(s))ds\right| \\
&\le\int_0^1\left|H(z,s)\Xi(\mathrm{s},\mathrm{w}(\mathrm{s}),\mathfrak{D}^p w(s))\right|ds \\
&\le\int_0^1\left|H(z,s)\left[\theta(s)+\xi\left(|w(s)|+\left|\mathfrak{D}^p w(s)\right)\right|\right)\right]\right|ds \\
&\le\int_0^1|H(z,s)\theta(s)|ds+\xi\zeta\int_0^1|H(z,s)|ds \\
&\le B+\xi\zeta\left(\frac{1}{\Gamma(r)}+\Gamma(3-\mathrm{r})\right),
\end{aligned} \tag{32}$$

$$\begin{aligned}
\left|(\mathfrak{P}w)'(z)\right|&=\left|\int_0^1\frac{\partial H(z,s)}{\partial z}\Xi(s,w(s),\mathfrak{D}^p w(s))ds\right| \\
&\le\int_0^1\left|\frac{\partial H(z,s)}{\partial z}\left[\theta(s)+\xi\left(|w(s)|+\left|\mathfrak{D}^p w(s)\right)\right|\right)\right]\right|ds \\
&\le\int_0^1\left|\frac{\partial H(z,s)}{\partial z}\theta(s)\right|ds+\xi\zeta\int_0^1\left|\frac{\partial H(z,s)}{\partial z}\right|ds \\
&\le B+\xi\zeta\left(\frac{1}{\Gamma(\mathrm{r})}+\Gamma(3-\mathrm{r})\right),
\end{aligned} \tag{33}$$

$$\begin{aligned}
\left|\mathfrak{D}^p\mathfrak{P}w(z)\right|&=\left|\frac{1}{\Gamma(1-p)}\int_0^z(z-s)^{-p}(\mathfrak{P}w)'(s)ds\right| \\
&\le\frac{1}{\Gamma(1-p)}\int_0^z(z-s)^{-p} \\
&\quad\cdot\left(\int_0^1\left|\frac{\partial H(s,\lambda)}{\partial s}\Xi(\lambda,w(\lambda),\mathfrak{D}^p w(\lambda))\right|d\lambda\right)ds \\
&\le\frac{1}{\Gamma(1-p)}\int_0^1(z-s)^{-p}\left(\int_0^1\left|\frac{\partial H(s,\lambda)}{\partial s}\theta(\lambda)\right|d\lambda\right. \\
&\quad\left.+\int_0^1\frac{\partial H(s,\lambda)}{\partial s}\xi\left(|x(\lambda)|+\left|\mathfrak{D}^p w(\lambda)\right|\right)d\lambda\right)ds \\
&\le\frac{B}{\Gamma(1-\mathrm{p})}\int_0^z(z-s)^{-p}ds \\
&\quad+\frac{\xi\zeta}{\Gamma(1-\mathrm{p})}\int_0^z(z-s)^{-p}\left(\int_0^1\left|\frac{\partial H(s,\lambda)}{\partial s}\right|d\lambda\right)ds \\
&\le\frac{B}{\Gamma(2-\mathrm{p})}z^{1-p}+\frac{\xi\zeta}{\Gamma(2-\mathrm{p})}\left(\frac{1}{\Gamma(\mathrm{r})}+\Gamma(3-\mathrm{r})\right)z^{1-p} \\
&\le\frac{B}{\Gamma(2-\mathrm{p})}+\frac{\xi\zeta}{\Gamma(2-\mathrm{p})}\left(\frac{1}{\Gamma(\mathrm{r})}+\Gamma(3-\mathrm{r})\right).
\end{aligned} \tag{34}$$

By virtue of inequalities (32), (33), (34), and the assumption (A3), we get $\|\mathfrak{P}x\|_{\mathbb{X}}\le\zeta$.

In the sequel, we investigate the inequalities $w_\star(z)\le\mathfrak{P}w(z)\le w^\star(z)$ and also $\mathfrak{D}^p w_\star(z)\le\mathfrak{D}^p\mathfrak{P}w(z)\le\mathfrak{D}^p w^\star(z)$ for each $0\le z\le 1$. Since w belongs to Γ_ζ, we obviously have $\boldsymbol{w}_\star(z)\le w(z)\le\boldsymbol{w}^\star(z)$. By using definitions of upper and lower control functions together with the assumption (A1), we get

$$\begin{aligned}
\mathfrak{P}w(z)&\le\int_0^1|H(z,s)|\widehat{\Delta}(s,w(s),\mathfrak{D}^p w(s))ds \\
&\le\int_0^1|H(z,s)|\widehat{\Delta}(s,w^\star(s),\mathfrak{D}^p w^\star(s))ds\le w^\star(z), \\
\mathfrak{P}w(z)&\ge\int_0^1|H(z,s)|\widehat{\delta}(s,w(s),\mathfrak{D}^p w(s))ds \\
&\ge\int_0^1|H(z,s)|\widehat{\delta}(s,w_\star(s),\mathfrak{D}^p w_\star(s))ds\ge w_\star(z),
\end{aligned} \tag{35}$$

Hence, we obtain $w_\star(z)\le\mathfrak{P}w(z)\le w^\star(z)$. Now, we need to show that $\mathfrak{D}^p w_\star(z)\le\mathfrak{D}^p\mathfrak{P}w(z)\le\mathfrak{D}^p w^\star(z)$. We have

$$\begin{aligned}
D_{0^+}^p\mathfrak{P}w(z)&=-\frac{1}{\Gamma(r-p)}\int_0^z(z-s)^{r-p-1}\Xi(s,w(s),\mathfrak{D}^p w(s))ds \\
&\quad+\frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)}\int_0^1\Xi(s,w(s),\mathfrak{D}^p w(s))ds \\
&\le-\frac{1}{\Gamma(r-\mathrm{p})}\int_0^z(z-s)^{r-p-1}\widehat{\Delta}(s,w^\star(s),\mathfrak{D}^p w^\star(s))ds \\
&\quad+\frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)}\int_0^1\widehat{\delta}(s,w^\star(s),\mathfrak{D}^p w^\star(s))ds \\
&\le\mathfrak{D}^p w^\star(z).
\end{aligned} \tag{36}$$

Similarly, we showed that $\mathfrak{D}^p\mathfrak{P}w(z)\ge\mathfrak{D}^p w_\star(z)$. Therefore, $\mathfrak{P}(\Gamma_\zeta)\subseteq\Gamma_\zeta$.

Step 3. At the final step, we aim to prove that P has the complete continuity property.

To see this, let $w \in \Gamma_\zeta$ and take $M = \max_{z\in[0,1]}\Xi(z, w(z), \mathfrak{D}^p w(z))$. We have

$$\begin{aligned}
|\mathfrak{P}w(z)| &= \left|\frac{1}{\Gamma(r)}\int_0^z (z-s)^{r-1}\Xi(s, w(s), \mathfrak{D}^p w(s))ds - \Gamma(3-r)\mathrm{z}\int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s))ds\right| \\
&\le \frac{1}{\Gamma(r)}\int_0^z \left|(z-s)^{r-1}\Xi(s, w(s), \mathfrak{D}^p w(s))\right|ds + \Gamma(3-r)\int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s))ds \\
&\le \left(\frac{z^r}{\Gamma(\mathrm{r}+1)} + \Gamma(3-r)\mathrm{z}\right)M \\
&\le \left(\frac{1}{\Gamma(\mathrm{r}+1)} + \Gamma(3-r)\right)M, \\
\left|(\mathfrak{P}w)'(z)\right| &= \left|\int_0^1 \frac{\partial H(z,s)}{\partial z}\Xi(s, w(s), \mathfrak{D}^p w(s))ds\right| \\
&\le M\int_0^1 \left|\frac{\partial H(z,s)}{\partial z}\right|ds \le \left(\frac{1}{\Gamma(r)} + \Gamma(3-r)\right)M, \\
\left|\mathfrak{D}^p\mathfrak{P}w(z)\right| &= \left|\frac{1}{\Gamma(r-p)}\int_0^z (z-s)^{r-p-1}\Xi(s, w(s), \mathfrak{D}^p w(s))ds - \frac{\Gamma(3-r)\mathrm{z}^{1-p}}{\Gamma(2-p)}\int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s))ds\right| \\
&\le \frac{1}{\Gamma(r-p)}\int_0^1 (z-s)^{r-p-1}\Xi(s, w(s), \mathfrak{D}^p w(s))ds + \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)}\int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s))ds \\
&\le \left(\frac{z^{r-p}}{\Gamma(r-p+1)} + \frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)}\right)M \\
&\le \left(\frac{z^{r-p}}{\Gamma(r-p+1)} + \frac{\Gamma(3-r)}{\Gamma(2-p)}\right)M.
\end{aligned} \tag{37}$$

Thus,

$$\|\mathfrak{P}w\|_{\mathbb{m}} \le \left(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(r+1)} + \frac{1}{\Gamma(r-p+1)} + 2F(3-r) + \frac{\Gamma(3-r)}{\Gamma(2-p)}\right)M. \tag{38}$$

Hence, $\mathfrak{P}(\Gamma_\zeta)$ has the property of the uniform boundedness. Next, we show that $\mathfrak{P}$w is equicontinuous. To do this, for each $w \in \Gamma_\zeta$ and $z_1, z_2 \in [0,1]$ with $z_1 < z_2$, we have

$$\begin{aligned}
&|\mathfrak{P}\mathbf{w}(z_2) - \mathfrak{P}\mathrm{w}(z_1)| \\
&= |\frac{1}{\Gamma(r)}\int_0^{z_1} (z_2-s)^{r-1}\Xi(s, w(s), \mathfrak{D}^p w(s))ds + \frac{1}{\Gamma(r)}\int_{z_1}^{z_2} (z_2-s)^{r-1}\Xi(s, w(s), \mathfrak{D}^p w(s))ds \\
&\quad - \frac{1}{\Gamma(r)}\int_0^{z_1} (z_1-s)^{r-1}\Xi(s, w(s), \mathfrak{D}^p w(s))ds - \Gamma(3-r)(z_2-z_1)\int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s))ds \\
&\le \frac{1}{\Gamma(r)}\int_0^{z_1} \left[(z_2-s)^{r-1} - (z_1-s)^{r-1}\right]\Xi(s, w(s), \mathfrak{D}^p w(s))ds + \frac{1}{\Gamma(r)}\int_0^{z_2} (z_2-s)^{r-1}\Xi(s, w(s), \mathfrak{D}^p w(s))ds \\
&\quad - \Gamma(3-r)(z_2-z_1)\int_0^1 \Xi(s, w(s), \mathfrak{D}^p w(s))ds \\
&= \frac{M}{\Gamma(r)}\int_0^{z_1} \left[(z_2-s)^{r-1} - (z_1-s)^{r-1}\right]ds + \frac{M}{\Gamma(r)}\int_{z_1}^{z_2} (z_2-s)^{r-1}ds + M\Gamma(3-r)(z_2-z_1) \\
&= \frac{M(z_2^r - z_1^r)}{\Gamma(r+1)} + M\Gamma(3-r)(z_2-z_1).
\end{aligned} \tag{39}$$

It is seen that the right-hand side of (39) does not depend on w and tends to zero whenever $z_1 \longrightarrow z_2$ which leads to $|Bw(z_2) - Bw(z_1)| \longrightarrow 0$. Further, we have

$$\begin{aligned}
&\left|(P\mathrm{w})'(z_2) - (P\mathrm{w})'(z_1)\right| \\
&= \left|\int_0^1 \frac{\partial H(z_2,s)}{\partial z_2}\Xi(\mathrm{s}, \mathrm{w}(\mathrm{s}), D^p\mathrm{w}(\mathrm{s}))ds - \int_0^1 \frac{\partial H(z_1,s)}{\partial z_1}\Xi(\mathrm{s}, \mathrm{w}(\mathrm{s}), D^p\mathrm{w}(\mathrm{s}))ds\right| \\
&\le \int_0^1 \frac{\partial H(z_2,s)}{\partial z_2} - \frac{\partial H(z_1,s)}{\partial z_1}\Xi(\mathrm{s}, \mathrm{w}(\mathrm{s}), D^p\mathrm{w}(\mathrm{s}))ds \\
&\le \frac{M}{\Gamma(r-1)}\left[\int_0^{z_1}\left[(z_2\text{-}s)^{r-2} - (z_1\text{-}s)^{r-2}\right]ds + \int_{z_z}^{z_2}(z_2\text{-}s)^{r-2}ds\right] \\
&= \frac{M}{\Gamma(r)}\left[z_2^{r-1} - z_1^{r-1}\right],
\end{aligned} \tag{40}$$

which tends to zero whenever $z_1 \longrightarrow z_2$. In addition,

$$\begin{aligned}
&|D^pBw(z_2) - D^pBw(z_1)| \\
&= \frac{1}{\Gamma(1-p)}\left|\int_0^{z_2}(z_2-s)^{-p}(Bw)'(s)ds - \int_0^{z_1}(z_1-s)^{-p}(Bw)'(s)ds\right| \\
&= \frac{1}{\Gamma(1-p)}\left|\int_0^{z_1}\left[(z_1-s)^{-p} - (z_2-s)^{-p}\right]\right. \\
&\quad \cdot\left(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s}\Xi(\lambda, w(\lambda), D^p w(\lambda))d\lambda\right)ds
\end{aligned}$$

$$+\int_{z_1}^{t_2}(z_2-s)^{-p}\left(\int_0^1 \frac{\partial H(s,\lambda)}{\partial_s}\Xi(\lambda,w(\lambda),D^p w(\lambda))d\lambda\right)ds\Bigg|$$
$$\le \frac{1}{\Gamma(1-p)}\int_0^{z_1}\left[(z_1-s)^{-p}-(z_2-s)^{-p}\right]$$
$$\cdot\left(\int_0^1\left|\frac{\partial H(s,\lambda)}{\partial s}\right|\Xi(\lambda,w(\lambda),D^p w(\lambda))d\lambda\right)ds$$
$$+\frac{1}{\Gamma(1-p)}\int_{z_1}^{z_2}(z_2-s)^{-p}$$
$$\cdot\left(\int_0^1\left|\frac{\partial H(s,\lambda)}{\partial s}\right|\Xi(\lambda,w(\lambda),D^p w(\lambda))d\lambda\right)ds$$
$$\le \frac{M}{\Gamma(1-p)}\int_0^{z_1}\left[(z_1-s)^{-p}-(z_2-s)^{-p}\right]$$
$$\cdot\left(\int_0^1\left|\frac{\partial H(s,\lambda)}{\partial s}\right|d\lambda\right)ds$$
$$+\frac{M}{\Gamma(1-p)}\int_{z_1}^{z_2}(z_2-s)^{-p}\left(\int_0^1\left|\frac{\partial H(s,\lambda)}{\partial s}\right|d\lambda\right)ds$$
$$\le \frac{M}{\Gamma(2-p)}\left(\frac{1}{\Gamma(r)}+(3-r)\right)\left(2(z_2-z_1)^{1-p}+z_1^{1-p}-z_2^{1-p}\right), \tag{41}$$

which tends to zero as $z_1 \longrightarrow z_2$. Therefore, inequalities (39), (40), and (41) imply that Pw is equicontinuous. Knowing that it is uniformly bounded, we find that P is completely continuous. Schauder's fixed point theorem implies that P has a fixed point $w \in \Gamma_\zeta$ which is a solution for the multiterm semilinear boundary value problem (3) and the proof is completed.□

Corollary 10. *Let Ξ be continuous on $[0,1]\times\mathbb{R}\times\mathbb{R}$ and there exists $\xi>0$ such that*

$$\xi \le \frac{1}{((1/\Gamma(r))+(1/\Gamma(r+1))+2\Gamma(3-r)+(1/\Gamma(2-p)\Gamma(r))+(\Gamma(3-r)/\Gamma(2-p)))}. \tag{42}$$

Then, a solution exists for the multiterm semilinear boundary value problem (3).

Proof. We choose $\theta(z)=0$ and $\max_{z\in[0,1]}\Xi(z,w,v)=\xi(|w|+|v|)$. So, the assumption (42) allows us to apply Theorem 9 which affirms the existence of a solution for the mentioned multiterm semilinear problem (3). □

Corollary 11. *Assume that there exist two real numbers η, $v>0$ such that*

$$\eta \ge \sup_{\substack{0<z\le 1\\ w\in\mathbb{R}_+,v\in\mathbb{R}}}\Xi(z,\mathbf{w},\mathbf{v}) \text{ and } v \le \inf_{\substack{0<z\le 1\\ w\in\mathbb{R}_+,v\in\mathbb{R}}}\Xi(z,\mathbf{w},\mathbf{v}). \tag{43}$$

Then, the multiterm semilinear boundary value problem (3) has at least a positive solution on $[0,Z^*]$, where

$$Z^*=\left(\frac{v\Gamma(r+1)\Gamma(3-r)}{\eta}\right)^{1/r-1}. \tag{44}$$

□

Proof. From definitions of the functions $\widehat{\delta}(z,u,v)$ and $\widehat{\Delta}(z,u,v)$, it is followed that

$$v\le\widehat{\delta}(z,u,v)\le\widehat{\Delta}(z,u,v)\le\eta,\ (0\le z\le Z^*, w\in\mathbb{R}_+, v\in\mathbb{R}). \tag{45}$$

Define

$$w^*(z)=\Gamma(3-r)\eta z-\frac{vz^r}{\Gamma(r+1)},\quad 0\le z\le Z^*,$$
$$w_*(z)=\Gamma(3-r)v-\frac{\eta z^r}{\Gamma(r+1)},\quad 0\le z\le Z^*, \tag{46}$$

So, we get $0\le \mathbf{w}_*(z)\le \mathbf{w}^*(z)$for $0\le z\le Z^*$ and also

$$w^*(z)\ge\eta\int_0^1|H(z,s)|ds$$
$$\ge\int_0^1|H(z,s)|\widehat{\Delta}(s,w^*(s),D^p w^*(s))ds,\quad 0\le z\le Z^*,$$
$$w_*(z)\le v\int_0^1|H(z,s)|ds$$
$$\le\int_0^1|H(z,s)|\widehat{\delta}(s,w_*(s),D^p w_*(s))ds,\quad 0\le z\le Z^*. \tag{47}$$

Moreover, by some direct computations, we get

$$D^p w^*(z)=\frac{\Gamma(3-r)\eta z^{1-p}}{\Gamma(2-p)}-\frac{vz^{r-p}}{\Gamma(r-p+1)},\quad 0\le z\le Z^*,$$
$$D^p w_*(z)=\frac{\Gamma(3-r)vz^{1-p}}{\Gamma(2-p)}-\frac{\eta z^{r-p}}{\Gamma(r-p+1)},\quad 0\le z\le Z^*. \tag{48}$$

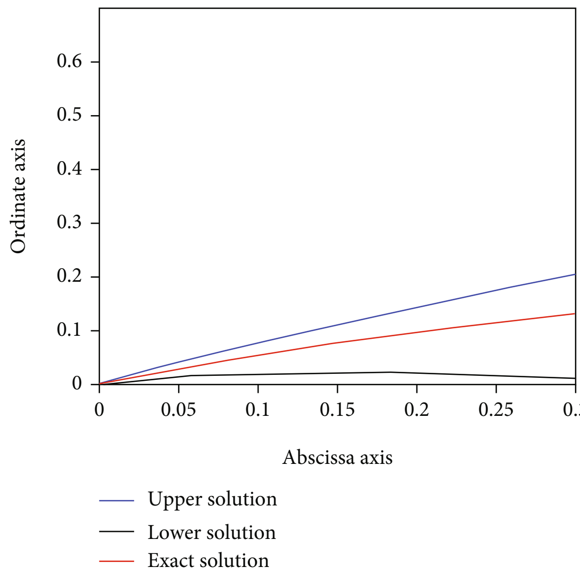

Figure 1: Graphs of w, w_*, and w^*.

Thus,

$$\begin{aligned}
&\frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)}\int_0^1 \widehat{\delta}(s, w^*(s), D^p w^*(s))ds \\
&\quad - \frac{1}{\Gamma(r-p)}\int_0^z (z-s)^{r-p-1}\widehat{\Delta}(s, w^*(s), D^p w^*(s))ds \\
&\le \frac{\Gamma(3-r)\eta z^{1-p}}{\Gamma(2-p)} - \frac{\nu z^{r-p}}{\Gamma(r-p+1)} = D^p_{0+} w^*(z), \\
&\frac{\Gamma(3-r)z^{1-p}}{\Gamma(2-p)}\int_0^1 \widehat{\Delta}(s, w_*(s), D^p w_*(s))ds \\
&\quad - \frac{1}{\Gamma(r-p)}\int_0^z (z-s)^{r-p-1}\widehat{\delta}(s, w_*(s), D^p w_*(s))ds \\
&\ge \frac{\Gamma(3-r)\nu z^{1-p}}{\Gamma(2-p)} - \frac{\eta z^{r-p}}{\Gamma(r-p+1)} = D^p_{0+} w_*(z),
\end{aligned} \tag{49}$$

This means that the assumption (A1) is satisfied. Finally, if (A2) holds, then we can choose ζ such that

$$\begin{aligned}
\zeta \ge A + B + \frac{B}{\Gamma(2-p)} + \eta\bigg(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(r+1)} + 2\Gamma(3-r) \\
+ \frac{1}{\Gamma(2-p)\Gamma(r)} + \frac{\Gamma(3-r)}{\Gamma(2-p)}\bigg).
\end{aligned} \tag{50}$$

Now, all hypotheses of Theorem 9 hold. Consequently, the multiterm semilinear boundary value problem (3) has at least a positive solution $\mathbf{w} \in \Gamma_\zeta$, where $0 \le w_*(z) \le w(z) \le w^*(z)$ and $D^p w_*(z) \le D^p w(z) \le D^p w^*(z)$ for each $z \in [0, Z^*]$ and the corollary is proved. □

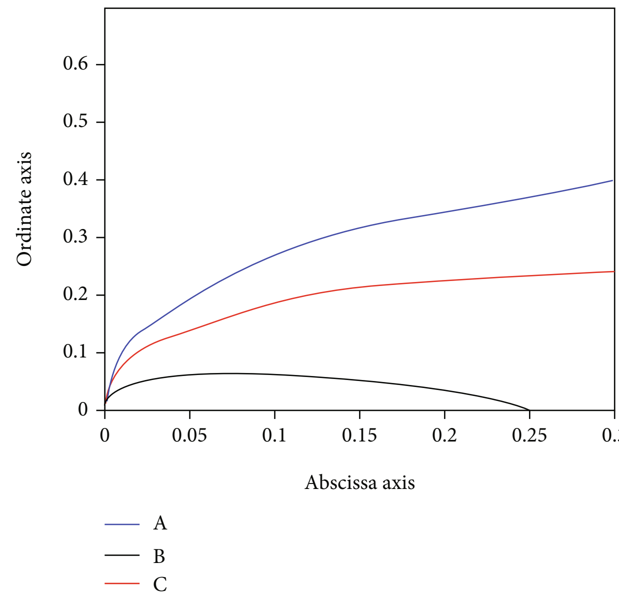

Figure 2: Graphs of $A = D^p w_*$, $B = D^p w^*$, and $C = D^p w$.

To validate the theoretical findings, we provide a special example corresponding to the suggested multiterm semilinear boundary value problem (3).

Example 12. According to the multiterm semilinear boundary value problem (3), in the present example, we take $r = 1.5, p = 0.5, \eta = 1, \nu = 0.5, Z^* = 0.3$ and

$$\Xi(z, w, y) = \nu + (\eta - \nu)z = 0.5 + 0.5z. \tag{51}$$

By taking into account the definition of the function Ξ, we clearly have $\nu \le \Xi(z, \mathbf{w}, \boldsymbol{\nu}) \le \eta$. Now, we choose upper and lower control functions $\widehat{\Delta}(z, u, v) = \eta$ and $\widehat{\delta}(z, u, v) = \nu$, respectively, and then, we get

$$\begin{aligned}
\mathbf{W}^*(z) &= \Gamma(1.5)z - \frac{1}{2\Gamma(2.5)}z^{1.5} = 0.8862z - 0.3761z^{1.5}, \\
\mathbf{W}_*(z) &= 0.5 \times \Gamma(1.5)z - \frac{1}{\Gamma(2.5)}z^{1.5} = 0.4431z - 0.7522z^{1.5}, \\
\mathbf{w}(z) &= \frac{3\Gamma(1.5)}{4}z - \frac{1}{3\Gamma(1.5)}z^{1.5} - \frac{2}{15\Gamma(1.5)}z^{2.5} \\
&= 0.6647z - 0.3761z^{1.5} - 0.1505z^{2.5}.
\end{aligned} \tag{52}$$

Therefore, by some simple calculations, we obtain

$$\begin{aligned}
D^p w^*(z) &= z^{0.5} - 0.5z, \\
D^p w_*(z) &= 0.5z - z, \\
D^p w(z) &= 0.75z^{0.5} - 0.5z - 0.25z^2.
\end{aligned} \tag{53}$$

Note that in this example, we have taken the interval $[0, Z^*] \subset [0, 1]$ to hold the essential condition $0 \le w_*(z) \le w(z)$

$\leq w^*(z)$. The graphs of positive solutions and their derivatives are illustrated in Figures 1 and 2.

4. Conclusion

In this paper, we considered a new fractional class of the multiterm semilinear differential equation in the context of the standard Caputo differentiation operator. The main purpose here is to derive several criteria of the existence of positive solutions for mentioned multiterm boundary value problem. To achieve such an aim, we first obtained the relevant Green function of the equivalent integral equation and showed that the absolute value of the first-order partial derivative of this function is integrable. After introducing two lower and upper control functions, we defined an operator on the given Banach space and used Schauder's fixed point theorem for establishing the existence of positive solutions. With the help of a special numerical example at the end of the study, we validated our theoretical findings according to the method implemented here. By plotting upper and lower positive solutions and their derivatives, we compared our results with the exact solution graphically. The results and methods presented in this research can be widely applied in different complicated classes of boundary value problems involving modern integral conditions in the future works. Also, one can design such models by using new generalized fractional operators involving singular or nonsingular kernels for describing their dynamical behaviors in better settings.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

On Suzuki and Wardowski-Type Contraction Multivalued Mappings in Partial Symmetric Spaces

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The purpose of this paper is to provide some fixed-point results for Suzuki and Wardowski-type contraction multivalued mappings in partial symmetric spaces. We give some examples to support and substantiate the developed notions and obtained results. Also, we use one of our main results to establish the existence and uniqueness of the solution for a system of integral inclusions.

1. Introduction

In 1922, Banach [1] formulated a very famous, fruitful, useful, and core fixed-point result known as the Banach contraction principle on a complete metric space. This celebrated result has been generalized and extended in several directions by various authors on some generalized metric spaces such as partial metric spaces [2], partial JS-metric spaces [3], metric-like spaces [4], b-metric spaces [5], rectangular metric spaces [6], controlled and doubled metric spaces [7, 8], generalized b-metric spaces [9], and extended rectangular b-metric spaces [10]. Sometimes, one may come across a situation wherein the full force of metric conditions is not required in the enunciation of fixed-point results. Motivated by this observation, several researchers established fixed and common fixed-point results in symmetric spaces which did not require triangular inequality, i.e., a symmetric χ on a nonempty set X is a mapping $\chi : X \times X \longrightarrow [0,\infty)$ which satisfies $d(\rho, \rho) = 0$ iff $\rho = \varsigma$, and $d(\rho, \varsigma) = d(\varsigma, \rho)$ for all $\rho, \varsigma \in X$.

Very recently, Asim et al. [11] introduced the class of partial symmetric spaces and proved some related fixed-point results for single-valued and multivalued mappings. The theory of multivalued mappings plays an important role in many areas of mathematics due to its diverse applications, namely in differential equations, integral equations, optimization problems, game theory, control theory, economics, etc. Aydi et al. [12, 13] proved fixed and common fixed-point results for multivalued maps in partial Hausdorff metric spaces to generalize fixed-point results due to Nadler [14].

In 2008, Suzuki [15] introduced a class of new types of contraction mappings and established fixed-point theorems for such mappings, which are genuine improvements of the Banach contraction principle. Thereafter, many authors attempted to extend such results for multivalued maps (employing the Pompieu-Hausdorff metric). In 2012, Wardowski [16] initiated the notion of a new kind of nonlinear contractions, namely F-contractions and proved some related fixed-point results. In recent years, the idea of F-contractions has been generalized and improved in several ways and directions. For more details, see [17–21].

In our paper, based on Suzuki and Wardowski-type contractions, we consider multivalued mappings, and we establish some related fixed-point results in the setting of partial symmetric spaces. Some examples are also furnished to demonstrate the utility of our results. As an application of Theorem 16, we establish the existence and uniqueness of the solution for a system of integral inclusions.

2. Preliminaries

In what follows, we collect some relevant definitions and auxiliary results needed in the sequel.

Definition 1 [11]. Let X be a nonempty set. The function $\chi : X \times X \longrightarrow [0,\infty)$ satisfying

$\chi^1 \chi(\rho, \varsigma) = \chi(\rho, \rho) = \chi(\varsigma, \varsigma)$ iff $\rho = \varsigma$,

$\chi^2 \chi(\rho, \rho) \leq \chi(\rho, \varsigma)$,

$\chi^3 \chi(\rho, \varsigma) = \chi(\varsigma, \rho)$is said to be a partial symmetric, and the pair (X, χ) is called a partial symmetric space.

A partial symmetric space (X, χ) reduces to a symmetric space if $\chi(x, x) = 0$, for all $x \in X$. Observe that every symmetric space is a partial symmetric space. The converse is not true as it is shown in the following examples.

Example 2. Let $X = \mathbb{R}$ and define $\chi : X \times X \longrightarrow [0,\infty)$ by

$$\chi(\rho, \varsigma) = |\rho - \varsigma|^p + |\rho - \varsigma|^q + \alpha, \tag{1}$$

where $p, q > 1$ and $\alpha \geq 0$.

Example 3. Let $X = [0,\infty)$ and define $\chi : X \times X \longrightarrow [0,\infty)$ by

$$\chi(\rho, \varsigma) = (\max \{\rho, \varsigma\})^p + |\rho - \varsigma|^q, \tag{2}$$

where $p, q > 1$.

Example 4. Let $X = [0, \pi)$ and define $\chi : X \times X \longrightarrow [0,\infty)$ by

$$\chi(\rho, \varsigma) = \sin |\rho - \varsigma| + \alpha, \tag{3}$$

where $\alpha \geq 0$.

Example 5. Let $X = [0, \pi)$ and define $\chi : X \times X \longrightarrow [0,\infty)$ by

$$\chi(\rho, \varsigma) = (\max \{\rho, \varsigma\})^p + e^{|\rho-\varsigma|}, \tag{4}$$

where $p > 1$.

Let (X, χ) be a partial symmetric space. The χ-open ball with a center $\rho \in X$ and a radius $\varepsilon > 0$ is defined by

$$B_\chi(\rho, \varepsilon) = \{\varsigma \in \rho : \chi(\rho, \varsigma) < \chi(\rho, \rho) + \varepsilon\}. \tag{5}$$

Similarly, the χ-closed ball with a center $\rho \in X$ and a radius $\varepsilon > 0$ is defined by

$$B_\chi[\rho, \varepsilon] = \{\varsigma \in X : \chi(\rho, \varsigma) \leq \chi(\rho, \rho) + \varepsilon\}. \tag{6}$$

The family of χ-open balls denoted by

$$\mathcal{U}_\chi = \left\{B_\chi(\rho, \varepsilon): \rho \in X, \varepsilon > 0\right\} \tag{7}$$

forms on X a basis of some topology τ_χ.

Lemma 6. *Let (X, τ_χ) be a topological space and $f : X \longrightarrow X$ be given mapping. If f is continuous; then, for every convergent sequence $\{\rho_n\}$ to a point ρ in X, the sequence $\{f\rho_n\}$ converges to $f\rho$. The converse holds if X is metrizable.*

We require some more definitions, namely, a convergent sequence, a Cauchy sequence, and a complete partial symmetric space, in our forthcoming discussions.

Definition 7 [11]. *A sequence $\{\rho_n\}$ in $(X, \chi)\chi$ -converges to $\rho \in X$, with respect to τ_χ, if*

$$\chi(\rho, \rho) = \lim_{n\longrightarrow\infty} \chi(\rho_n, \rho). \tag{8}$$

Definition 8 [11]. *A sequence $\{\rho_n\}$ in (X, χ) is χ -Cauchy iff $\lim_{n,m\longrightarrow\infty} \chi(\rho_n, \rho_m)$ exists and is finite.*

Definition 9 [11]. *A partial symmetric space (X, χ) is χ -complete if each χ -Cauchy sequence $\{\rho_n\}$ in X is χ -convergent with respect to τ_χ to a point $\rho \in X$ such that*

$$\chi(\rho, \rho) = \lim_{n\longrightarrow\infty} \chi(\rho_n, \rho) = \lim_{n,m\longrightarrow\infty} \chi(\rho_n, \rho_m). \tag{9}$$

Let (X, χ) be a partial symmetric space and $\mathscr{CB}^\chi(X)$ be the set of all nonempty, χ-closed, and bounded subsets of (X, χ). Moreover, for $\mathscr{L}, \mathscr{M} \in \mathscr{CB}^\chi(X)$ and $\rho \in X$, we define

$$\begin{aligned} &dist_\chi(\rho, \mathscr{L}) = \inf \{\chi(\rho, l): l \in \mathscr{L}\},\\ &\delta_\chi(\mathscr{L}, \mathscr{M}) = \sup \left\{dist_\chi(l, \mathscr{M}): l \in \mathscr{L}\right\},\\ &\delta_\chi(\mathscr{M}, \mathscr{L}) = \sup \left\{dist_\chi(m, \mathscr{L}): m \in \mathscr{M}\right\}. \end{aligned} \tag{10}$$

Let $H_\chi(.,.)$ be the partial Pompieu-Hausdorff symmetric, that is

$$H_\chi(\mathscr{L}, \mathscr{M}) = \max \left\{\delta_\chi(\mathscr{L}, \mathscr{M}), \delta_\chi(\mathscr{M}, \mathscr{L})\right\}. \tag{11}$$

3. Main Results

Given that $f : X \longrightarrow \mathscr{CB}^\chi(X)$, then for every $\rho \in X$, we define

$$\mathfrak{S}(\chi, f, \rho) = \sup \left\{\chi\left(\rho_i, \rho_j\right): i, j \in \mathbb{N}\right\}, \tag{12}$$

where $\rho_i \in f\rho_{i-1}$ and $\rho_j \in f\rho_{j-1}$.

Now, we present the following lemma proved in [11], which is needed in the sequel.

Lemma 10. *Let (X, χ) be a partial symmetric space and $\mathscr{L}$, $\mathscr{M} \in \mathscr{CB}^\chi(X)$. Then, for all $h > 1$ and $l \in \mathscr{L}$, there is $m \in \mathscr{M}$ so that*

$$\chi(l, m) \leq hH_\chi(\mathscr{L}, \mathscr{M}). \tag{13}$$

Next, let $\psi : [0, 1) \longrightarrow (0, 1]$ be the nonincreasing function defined by

$$\psi(r) = \begin{pmatrix} 1 & \text{if } 0 \leq r < \frac{1}{2} \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1. \end{pmatrix} \tag{14}$$

Theorem 11. *Let (X, χ) be a complete partial symmetric space and $f : X \longrightarrow \mathscr{CB}^{\chi}(X)$.*

Assume the following:

(Si) There exists $0 \leq r < 1$ such that f satisfies the condition

$$\psi(r) dist_{\chi}(\rho, f\rho) \leq \chi(\rho, \varsigma) \text{ implies } H_{\chi}(f\rho, f\varsigma) \leq rM(\rho, \varsigma), \tag{15}$$

for all $\rho, \varsigma \in X$, where ψ is defined by (14) and

$$M(\rho, \varsigma) = \max\left\{\chi(\rho, \varsigma), dist_{\chi}(\rho, f\rho), dist_{\chi}(\varsigma, f\varsigma), dist_{\chi}(\rho, f\varsigma), dist_{\chi}(\varsigma, f\rho)\right\} \tag{16}$$

(Sii) There is $\rho_0 \in X$ so that $\mathfrak{S}(\chi, f, \rho_0) < \infty$

($Siii$) The function $\rho \longrightarrow dist_{\chi}(\rho, f\rho)$ is lower semicontinuous.

Then, f has a unique fixed point.

Proof. By (Si), there exists $0 \leq r < 1$ such that (15) holds. Let $r_1 \in (0, 1)$ be such that $0 \leq r \leq r_1 < 1$. For such $\rho_0 \in X$, define a sequence $\{\rho_n\} \in X$ by $\rho_{n+1} \in f\rho_n$. If we assume that $\rho_{n_0} \in f\rho_{n_0}$ for some $n_0 \in \mathbb{N}$, then ρ_{n_0} is a fixed point of f, and we are done. Now, suppose that $\rho_n \notin f\rho_n$ for each $n \in \mathbb{N}$, and so $\chi(\rho_n, \rho_{n+1}) > 0$. From Lemma 10 (with $h = 1/\sqrt{r_1}$) and condition (15) (for any arbitrary $n \in \mathbb{N}$ and for all $i, j \in \mathbb{N}$), we have

$$\chi\left(\rho_{n+i}, \rho_{n+j}\right) \leq \frac{1}{\sqrt{r_1}} H_{\chi}\left(f\rho_{n-1+i}, f\rho_{n-1+j}\right). \tag{17}$$

Since $\psi(r) \leq 1$, we get

$$\psi(r)\text{dist}_{\chi}(\rho_{n+i}, f\rho_{n+i}) \leq \text{dist}_{\chi}(\rho_{n+i}, f\rho_{n+i}) \leq \chi(\rho_{n+i}, \rho_{n+1+i}). \tag{18}$$

From (15) and (17), we have

$$\begin{aligned}\chi\left(\rho_{n+i}, \rho_{n+j}\right) &\leq \frac{1}{\sqrt{r_1}} H_p\left(f\rho_{n-1+i}, f\rho_{n-1+j}\right) \\ &\leq \sqrt{r_1} M\left(\rho_{n-1+i}, \rho_{n-1+j}\right) \\ &\leq \sqrt{r_1} \max\left\{\chi\left(\rho_{n-1+i}, \rho_{n-1+j}\right),\right. \\ &\quad \text{dist}_{\chi}(\rho_{n-1+i}, f\rho_{n-1+i}), \text{dist}_{\chi}\left(\rho_{n-1+j}, f\rho_{n-1+j}\right), \\ &\quad \left.\text{dist}_{\chi}\left(\rho_{n-1+i}, f\rho_{n-1+j}\right), \text{dist}_{\chi}\left(\rho_{n-1+j}, f\rho_{n-1+i}\right)\right\}.. \end{aligned} \tag{19}$$

The above inequality is true for all $i, j \in N$; therefore, by conditions (Sii) and (12), we have

$$\mathfrak{S}(\chi, f, f^n\rho_0) \leq \sqrt{r_1}\mathfrak{S}\left(\chi, f, f^{n-1}\rho_0\right). \tag{20}$$

By repeating this process, we have (for all $n \geq 1$)

$$\mathfrak{S}(\chi, f, f^n\rho_0) \leq (\sqrt{r_1})^n \mathfrak{S}(\chi, f, \rho_0). \tag{21}$$

Now

$$\chi(\rho_n, \rho_{n+m}) \leq \mathfrak{S}(\chi, f, f^n\rho_0) \leq (\sqrt{r_1})^n \mathfrak{S}(\chi, f, \rho_0), \forall n, m \in \mathbb{N}. \tag{22}$$

Since $\mathfrak{S}(\chi, f, \rho_0) < \infty$ and $r_1 \in (0, 1)$, then

$$\lim_{n,m \longrightarrow \infty} \chi(\rho_n, \rho_{n+m}) = 0, \tag{23}$$

so that $\{\rho_n\}$ is a χ-Cauchy sequence in X. In view of the χ-completeness of X, there exists $\rho \in X$ such that $\{\rho_n\}$ χ-converges to ρ. Thus

$$\chi(\rho, \rho) = \lim_{n \longrightarrow \infty} \chi(\rho_n, \rho) = \lim_{n,m \longrightarrow \infty} \chi(\rho_n, \rho_{n+m}) = 0. \tag{24}$$

Assume that $\rho \longrightarrow \text{dist}_{\chi}(\rho, f\rho)$ is lower semicontinuous. Then

$$\text{dist}_{\chi}(\rho, f\rho) \leq \liminf_{n \longrightarrow \infty} \text{dist}_{\chi}(\rho_n, f\rho_n) \leq \lim_{n \longrightarrow \infty} \chi(\rho_n, \rho_{n+1}) = 0. \tag{25}$$

Therefore, $\text{dist}_{\chi}(\rho, f\rho) = 0$, that is, $\rho \in f\rho$. Hence, ρ is a fixed point of f.

Next, let us show that f has a unique fixed point. Suppose there are $\rho, \varsigma \in X$ such that $\rho \in f\rho$ and $\varsigma \in f\varsigma$. Then, by conditions (Si) and ($\chi 2$), we have

$$\psi(r)\text{dist}_{\chi}(\rho, f\rho) \leq \text{dist}_{\chi}(\rho, f\rho) \leq \chi(\rho, \rho) \leq \chi(\rho, \varsigma), \tag{26}$$

which implies (by Lemma 10 with $h = 1/\sqrt{r_1}$ and $0 \leq r \leq r_1 < 1$)

$$\begin{aligned}\chi(\rho, \varsigma) &\leq \frac{1}{\sqrt{r_1}} H_{\chi}(f\rho, f\varsigma) \leq \sqrt{r_1} M((\rho, \varsigma)) \\ &= \sqrt{r_1} \max\left\{\chi(\rho, \varsigma), \text{dist}_{\chi}(\rho, f\rho), \text{dist}_{\chi}(\varsigma, f\varsigma),\right. \\ &\quad \left.\text{dist}_{\chi}(\rho, f\varsigma), \text{dist}_{\chi}(\rho, f\varsigma)\right\} \\ &\leq \sqrt{r_1} \max\left\{\chi(\rho, \varsigma), \chi(\rho, \rho), \chi(\varsigma, \varsigma), \chi(\rho, \varsigma), \chi(\varsigma, \rho)\right\} \\ &= \sqrt{r_1}\chi(\rho, \varsigma) < \chi(\rho, \varsigma), \end{aligned} \tag{27}$$

a contradiction, so that $\chi(\rho, \varsigma) = 0$, which implies that $\rho = \varsigma$. Hence, f has a unique fixed point. This completes the proof. □

Example 12. *Take* $X = \{0, 1/10, 1/5\}$. *The partial symmetric* $\chi : X \times X \longrightarrow [0,\infty)$ *is defined by*

$$\chi(\rho, \varsigma) = \frac{1}{2}|\rho - \varsigma|^2 + \frac{1}{4}(\max\{\rho, \varsigma\})^2, \forall \rho, \varsigma \in X. \quad (28)$$

Then, (X, χ) *is a* χ *-complete symmetric space. Note that* $\{0\}$ *and* $\{1/10\}$ *are bounded sets in* (X, χ). *In fact, if* $\rho \in \{0, 1/10, 1/5\}$, *then*

$$\rho \in \overline{\{0\}} \Longleftrightarrow dist_\chi(\rho, \{0\}) = \chi(\rho, \rho) \Longleftrightarrow \frac{3\rho^2}{4} = \frac{\rho^2}{4} \Longleftrightarrow \rho = 0 \Longleftrightarrow \rho \in \{0\}. \quad (29)$$

Hence, $\{0\}$ *is closed. Next*

$$\rho \in \overline{\left\{\frac{1}{10}\right\}} \Longleftrightarrow \mathrm{dist}_\chi\left(\rho, \left\{\frac{1}{10}\right\}\right) = \chi(\rho, \rho) \Longleftrightarrow \min\left\{\frac{3\rho^2}{4}, \frac{1}{2}\left|\rho - \frac{1}{10}\right|^2 + \frac{1}{4}\left(\max\left\{\rho, \frac{1}{10}\right\}\right)^2\right\} = \frac{\rho^2}{4} \Longleftrightarrow \rho \in \left\{\frac{1}{10}\right\}. \quad (30)$$

Hence, $\{1/10\}$ *is also closed. Now, define* $f : X \longrightarrow \mathscr{C}\mathscr{B}^\chi(X)$ *by*

$$f0 = f\frac{1}{10} = \{0\} \text{ and } f\frac{1}{5} = \left\{\frac{1}{10}\right\}. \quad (31)$$

Clearly, by a routine calculation, one can easily show that $x \longrightarrow \mathrm{dist}_\chi(\rho, f\rho)$ *is lower semicontinuous. To prove the contractive condition* (Si) *of* Theorem 11, *we need the following.*

Case 13. *Let* $\rho = 0$. *Then*

$$\psi(r)\mathrm{dist}_\chi(0, f0) = 0 \le \chi(0, \varsigma), \text{for all } \varsigma \in X. \quad (32)$$

For $\varsigma = 0$ or $\varsigma = 1/10$, *we have.*

$$H_\chi(f0, f\varsigma) = H_\chi(0, 0) = 0 \le r\chi(0, \varsigma). \quad (33)$$

For $\varsigma = 1/5$, *we have*

$$\begin{aligned} H_\chi\left(f0, f\frac{1}{5}\right) &= H_\chi\left(0, \frac{1}{10}\right) = \frac{1}{2}\left|0 - \frac{1}{10}\right|^2 + \frac{1}{4}\left(\max\left\{0, \frac{1}{10}\right\}\right)^2 \\ &= \frac{1}{4}\left(\frac{1}{2}\left|0 - \frac{1}{5}\right|^2 + \frac{1}{4}\left(\max\left\{0, \frac{1}{5}\right\}\right)^2\right) \\ &= \frac{1}{4}\chi\left(0, \frac{1}{5}\right). \end{aligned} \quad (34)$$

Case 14. *Let* $\rho = 1/10$. *Then,* $\psi(r)\mathrm{dist}_\chi(1/10, f1/10) = 3/400 \le \chi(0, \varsigma)$, for all $\varsigma \in X$.

For $\varsigma = 0$ or $\varsigma = 1/10$, *we have*

$$H_\chi\left(f\frac{1}{10}, f\varsigma\right) = 0 < r\frac{3\varsigma^2}{4} \le r\chi\left(\frac{1}{10}, \varsigma\right). \quad (35)$$

For $\varsigma = 1/5$, *we have*

$$\begin{aligned} H_\chi\left(f\frac{1}{10}, f\frac{1}{5}\right) &= H_\chi\left(\{0\}, \left\{\frac{1}{10}\right\}\right) \\ &= \frac{1}{2}\left|0 - \frac{1}{10}\right|^2 + \frac{1}{4}\left(\max\left\{0, \frac{1}{10}\right\}\right)^2 \\ &= \frac{1}{2}\left(\frac{1}{2}\left|\frac{1}{10} - \frac{1}{5}\right|^2 + \frac{1}{4}\left(\max\left\{\frac{1}{10}, \frac{1}{5}\right\}\right)^2\right) \\ &= \frac{1}{2}\chi\left(\frac{1}{10}, \frac{1}{5}\right). \end{aligned} \quad (36)$$

Case 15. *Let* $\rho = 1/5$. *Then*

$$\psi(r)\mathrm{dist}_\chi\left(\frac{1}{5}, f\frac{1}{5}\right) = \frac{3}{200} \le \chi\left(\frac{1}{5}, \varsigma\right), \text{for } \varsigma \in X. \quad (37)$$

For $\varsigma = 0$, *we have*

$$H_\chi\left(f\frac{1}{5}, f\varsigma\right) = H_\chi\left(\left\{\frac{1}{10}\right\}, \{0\}\right) \le \frac{1}{4}\chi\left(\frac{1}{5}, 0\right). \quad (38)$$

For $\varsigma = 1/10$, *we get*

$$H_\chi\left(f\frac{1}{5}, f\frac{1}{10}\right) = H_\chi\left(\left\{\frac{1}{10}\right\}, \{0\}\right) \le \frac{1}{4}\chi\left(\frac{1}{5}, 0\right). \quad (39)$$

For $\varsigma = 1/5$, *we have*

$$H_\chi\left(f\frac{1}{5}, f\frac{1}{5}\right) = H_\chi\left(\left\{\frac{1}{10}\right\}, \left\{\frac{1}{10}\right\}\right) \le \frac{1}{4}\chi\left(\frac{1}{5}, \frac{1}{5}\right). \quad (40)$$

Hence, the contractive condition (Si) *of Theorem 11 is satisfied. Observe that* f *has a unique fixed point (namely,* $\rho = 0$ *).*

Denote by $\mathscr{F}$ the class of functions $F : (0,\infty) \longrightarrow \mathbb{R}$ such that

(F_1) F is increasing

(F_2) For all $\alpha_n > 0, \lim_{n\longrightarrow\infty}\alpha_n = 0$ iff $\lim_{n\longrightarrow\infty}F(\alpha_n) = -\infty$.

The related fixed-point Wardowski-type result for multi-valued mappings in the setting of partial symmetric spaces is as follows.

Theorem 16. *Let* (X, χ) *be a complete partial symmetric space and* $f : X \longrightarrow \mathscr{C}\mathscr{B}^\chi(X)$. *Assume that*

(i) There are $F \in \mathscr{F}$ *and* $\tau > 0$ *such that*

$$\tau + F\left(H_\chi(f\rho, f\varsigma)\right) \le F(M(\rho,\varsigma)) \quad (41)$$

(ii) There is $\rho_0 \in X$ so that $\mathfrak{S}(\chi, f, \rho_0) < \infty$

(iii) The function $\rho \longrightarrow dist_\chi(\rho, f\rho)$ is lower semicontinuous.

Then, f has a unique fixed point.

Proof. For such $\rho_0 \in X$, take $\rho_1 \in f\rho_0$. Define the sequence $\{\rho_n\} \in X$ by $\rho_{n+1} \in f\rho_n$. If we assume that $\rho_{n_0} \in f\rho_{n_0}$ for some $n_0 \in \mathbb{N}$, then ρ_{n_0} is a fixed point of f. Suppose $\rho_n \notin f\rho_n$ for all $n \in \mathbb{N}$, so $\chi(\rho_n, \rho_{n+1}) > 0$. Since f is a generalized multivalued F-contraction, we have (for all $i, j \in \mathbb{N}$)

$$\begin{aligned} F\left(\chi\left(\rho_{n+i}, \rho_{n+j}\right)\right) &\le F\left(H\left(f\rho_{n-1+i}, f\rho_{n-1+j}\right)\right) \\ &\le F\left(M\left(\rho_{n-1+i}, \rho_{n-1+j}\right)\right) - \tau \\ &= F\Big(\max\Big\{\chi\left(\rho_{n-1+i}, \rho_{n-1+j}\right), \\ &\quad \mathrm{dist}_\chi(\rho_{n-1+i}, f\rho_{n-1+i}), \mathrm{dist}_\chi\left(\rho_{n-1+j}, f\rho_{n-1+j}\right), \\ &\quad \mathrm{dist}_\chi\left(\rho_{n-1+i}, f\rho_{n-1+j}\right), \mathrm{dist}_\chi\left(\rho_{n-1+j}, f\rho_{n-1+i}\right)\Big\}\Big) - \tau \\ &\le F\Big(\max\Big\{\chi\left(\rho_{n-1+i}, \rho_{n-1+j}\right), \chi(\rho_{n-1+i}, \rho_{n+i}), \\ &\quad \chi\left(\rho_{n-1+j}, \rho_{n+j}\right), \chi\left(\rho_{n-1+i}, \rho_{n+j}\right), \chi\left(\rho_{n-1+j}, \rho_{n+i}\right)\Big\}\Big) - \tau. \end{aligned} \quad (42)$$

The above inequality is true for all $i, j \in \mathbb{N}$. Therefore, by conditions (ii) and (12), we have

$$F(\mathfrak{S}(\chi, f, \rho_n)) \le F(\mathfrak{S}(\chi, f, \rho_{n-1})) - \tau. \quad (43)$$

By continuing this process, we find that

$$F(\mathfrak{S}(\chi, f, \rho_n)) \le F(\mathfrak{S}(\chi, f, \rho_0)) - n\tau. \quad (44)$$

Now, for all $n, m \in \mathbb{N}$, we have

$$\chi(\rho_n, \rho_{n+m}) \le \mathfrak{S}(\chi, f, \rho_n) F(\mathfrak{S}(\chi, f, \rho_n)) \le F(\mathfrak{S}(\chi, f, \rho_0)) - n\tau. \quad (45)$$

On making the limit as $n \longrightarrow \infty$ in (45), we have

$$\lim_{n\longrightarrow\infty} F(\mathfrak{S}(\chi, f, \rho_n)) = -\infty. \quad (46)$$

Owing to conditions (ii) and (F_2), we have

$$\lim_{n\longrightarrow\infty} \mathfrak{S}(\chi, f, \rho_n) = 0 \lim_{n,m\longrightarrow\infty} \chi(\rho_n, \rho_{n+m}) = 0, \quad (47)$$

so that $\{\rho_n\}$ is a χ-Cauchy sequence in X. In view of the χ-completeness of X, there is $\rho \in X$ in order that $\{\rho_n\}\chi$-converges to ρ. Thus, we have

$$\chi(\rho, \rho) = \lim_{n\longrightarrow\infty} \chi(\rho_n, \rho) = \lim_{n,m\longrightarrow\infty} \chi(\rho_n, \rho_{n+m}) = 0. \quad (48)$$

Assume that $\rho \longrightarrow \mathrm{dist}_\chi(\rho, f\rho)$ is lower semicontinuous. Then, we have

$$\mathrm{dist}_\chi(\rho, f\rho) \le \liminf_{n\longrightarrow\infty} \mathrm{dist}_\chi(\rho_n, f\rho_n) \le \lim_{n\longrightarrow\infty} \chi(\rho_n, \rho_{n+1}) = 0. \quad (49)$$

Therefore, $\mathrm{dist}_\chi(\rho, f\rho) = 0$ implies that $\rho \in f\rho$. Hence, ρ is a fixed point of f.

For the uniqueness part, suppose there exist $\rho \ne \varsigma \in X$ such that $\rho \in f\rho$ and $\varsigma \in f\varsigma$. Thus, by the condition (i), we have

$$\begin{aligned} F\left(H_\chi(\rho, \varsigma)\right) &= F\left(H_\chi(f\rho, f\varsigma)\right) \le F(M(\rho, \varsigma)) - \tau \\ &= F\Big(\max\Big\{\chi(\rho, \varsigma), \mathrm{dist}_\chi(\rho, f\rho), \mathrm{dist}_\chi(\varsigma, f\varsigma), \\ &\quad \mathrm{dist}_\chi(\rho, f\varsigma), \mathrm{dist}_\chi(\varsigma, f\rho)\Big\}\Big) - \tau \\ &= F(\max\{\chi(\rho, \varsigma), \chi(\rho, \rho), \chi(\varsigma, \varsigma), \chi(\rho, \varsigma), \chi(\varsigma, \rho)\}) - \tau \\ &= F(\chi(\rho, \varsigma)) - \tau, \end{aligned} \quad (50)$$

which is a contradiction. Therefore, $\chi(\rho, \varsigma) = 0$, that is, $\rho = \varsigma$. It completes the proof. ☐

Example 17. Let $X = \{0\} \cup \{1/2^n : n \in \mathbb{N}\}$ be equipped with the partial symmetric $\chi : X \times X \longrightarrow [0,\infty)$ defined by

$$\chi(\rho, \varsigma) = \frac{1}{2}|\rho - \varsigma|^2 + \frac{1}{4}(\max\{\rho, \varsigma\})^2 \text{ for all} \rho, \varsigma \in X. \quad (51)$$

Then, (X, χ) is a χ -complete symmetric space. Note that $\{1/2^{n+1}\}$ and $\{0\}$ are bounded sets in (X, χ) . In fact

$$\begin{aligned} \rho \in \overline{\left\{\frac{1}{2^{n+1}}\right\}} &\Longleftrightarrow \mathrm{dist}_\chi\left(\rho, \left\{\frac{1}{2^{n+1}}\right\}\right) \\ &= \chi(\rho, \rho) \Longleftrightarrow \min\left\{\frac{1}{2}\left|\rho - \frac{1}{2^{n+1}}\right|^2 + \frac{1}{4}\left(\max\left\{\rho, \frac{1}{2^{n+1}}\right\}\right)^2\right\} \\ &= \frac{\rho^2}{4} \Longleftrightarrow \rho = \frac{1}{2^{n+1}} \Longleftrightarrow \rho \in \left\{\frac{1}{2^{n+1}}\right\}. \end{aligned} \quad (52)$$

Hence, $\{1/2^{n+1}\}$ is closed. Next

$$\rho \in \overline{\{0\}} \Longleftrightarrow dist_\chi(\rho, \{0\}) = \chi(\rho, \rho) \Longleftrightarrow \frac{3\rho^2}{4} = \frac{\rho^2}{4} \Longleftrightarrow \rho \in \{0\}. \quad (53)$$

Hence, $\{0\}$ is also closed with respect to the partial symmetric χ.

Define $f : X \longrightarrow \mathscr{CB}^\chi(X)$ and $F : (0,\infty) \longrightarrow \mathbb{R}$ *by*

$$f\rho = \begin{cases} \left\{\frac{1}{2^{n+1}}\right\}, & \text{if } \rho = \frac{1}{2^n} \\ \{0\}, & \text{if } \rho = 0, \end{cases} \quad (54)$$

and $F(\alpha) = \ln(\alpha)$ for $\alpha > 0$, *respectively. Then*

$$\mathrm{dist}_\chi(\rho, f\rho) = \begin{cases} \dfrac{3}{2}\left(\dfrac{1}{2^{n+1}}\right)^2, & \text{if } \rho = \dfrac{1}{2^n} \\ 0, & \text{if } \rho = 0. \end{cases} \tag{55}$$

Hence, $\rho \longrightarrow \mathrm{dist}_\chi(\rho, f\rho)$ is lower semicontinuous. Now, we will show that the contractive condition (i) of Theorem 16 is satisfied.

Case 18. *Let $\rho, \varsigma \in \{1/2^n\}$* (say $\rho = 1/2^{n_1}$ and $\rho = 1/2^{n_2}$ for all $n_1, n_2 \in \mathbb{N}$). *Here*

$$\begin{aligned} H_\chi(f\rho, f\varsigma) &= H_\chi\left(\left\{\frac{1}{2^{n_1+1}}\right\}, \left\{\frac{1}{2^{n_2+1}}\right\}\right) \\ &= \frac{1}{2}\left|\frac{1}{2^{n_1+1}} - \frac{1}{2^{n_2+1}}\right|^2 + \frac{1}{4}\left(\max\left\{\frac{1}{2^{n_1+1}}, \frac{1}{2^{n_2+1}}\right\}\right)^2 \\ &= \frac{1}{4}\left(\frac{1}{2}\left|\frac{1}{2^{n_1}} - \frac{1}{2^{n_2}}\right|^2 + \frac{1}{4}\left(\max\left\{\frac{1}{2^{n_1}}, \frac{1}{2^{n_2}}\right\}\right)^2\right) \\ &= \frac{1}{4}\chi(\rho, \varsigma) \le \frac{1}{4}M(\rho, \varsigma). \end{aligned} \tag{56}$$

That is (for $\tau = \ln(4)$)

$$\tau + F\left(H_\chi(f\rho, f\varsigma)\right) \le F(M(\rho, \varsigma)). \tag{57}$$

Case 19. *Let $\rho \in \{1/2^n\}$ and $\varsigma \in \{0\}$. Then*

$$\begin{aligned} H_\chi(f\rho, f\varsigma) &= H_\chi\left(\left\{\frac{1}{2^{n+1}}\right\}, \{0\}\right) \\ &= \max\left\{\delta_\chi\left(\left\{\frac{1}{2^{n+1}}\right\}, \{0\}\right), \delta_\chi\left(\{0\}, \left\{\frac{1}{2^{n+1}}\right\}\right)\right\} \\ &= \frac{3}{4}\left(\frac{1}{2^{n+1}}\right)^2. \end{aligned} \tag{58}$$

Also

$$\begin{aligned} M\left(\frac{1}{2^n}, 0\right) &= \max\left\{\chi\left(\frac{1}{2^n}, 0\right), \mathrm{dist}_\chi\left(\frac{1}{2^n}, f\frac{1}{2^n}\right), \mathrm{dist}_\chi(0, \{0\}),\right. \\ &\qquad \left.\mathrm{dist}_\chi\left(\frac{1}{2^n}, \{0\}\right)\mathrm{dist}_\chi\left(0, \frac{1}{2^{n+1}}\right)\right\} \\ &= \max\left\{\frac{3}{4}\left(\frac{1}{2^n}\right)^2, \frac{3}{8}\left(\frac{1}{2^n}\right)^2, 0, \frac{3}{4}\left(\frac{1}{2^n}\right)^2, \frac{3}{16}\left(\frac{1}{2^n}\right)^2\right\} \\ &= \frac{3}{4}\left(\frac{1}{2^n}\right)^2. \end{aligned} \tag{59}$$

For $\tau = \ln(4)$, one writes

$$\tau + F\left(H_\chi(f\rho, f\varsigma)\right) \le F(M(\rho, \varsigma)). \tag{60}$$

Therefore, all the conditions of Theorem16 are satisfied, and f has a unique fixed point (namely, $\varsigma = 0$).

4. An Application to an Integral Inclusion

Let $X = C([a, b], \mathbb{R})$ be the set of all continuous real valued functions defined on $[a, b]$. Now, we consider the following integral inclusion of Volterra type:

$$\rho(t) \in \int_a^t G(t, s, \rho(s))ds + h(t), \forall t \in [a, b] \text{and} h \in X, \tag{61}$$

where $G : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow K_{cv}(\mathbb{R})$ is a multivalued operator and $K_{cv}(\mathbb{R})$ is the set of nonempty compact and convex subsets of $\mathbb{R}$.

Define $\chi : X \times X \longrightarrow \mathbb{R}^+$by

$$\chi(\rho(t), \varsigma(t)) = \sup_{t\in[a,b]} |\rho(t) - \varsigma(t)|^p + \sup_{t\in[a,b]} |\rho(t) - \varsigma(t)|^q, \quad q > p > 1. \tag{62}$$

Then, (X, χ) is a χ-complete partial symmetric space.

Now, we are ready to present our result as follows:

Theorem 20. *Suppose that for all $\rho, \varsigma \in C([a, b], \mathbb{R})$ there exist a continuous function $j : X \longrightarrow \mathbb{R}$ and $\int_a^t j(s)ds \le (e^{-\tau})^{1/p}$ with $\tau > 0$ such that*

$$H_\chi(G(t, s, \rho(s)), G(t, s, \varsigma(s))) \le j(s)|\rho(s) - \varsigma(s)|, \forall t, s \in [a, b]. \tag{63}$$

Then, the integral inclusion (61) has a unique solution.

Proof. Let us define the multivalued operator $f : X \longrightarrow \mathscr{C}\mathscr{B}^\chi(X)$ by $f\rho(t) = \in X : \{v \in \int_0^1 G(t, s, \rho(s))ds + h(t)$, for all $t \in [0, 1]$ and $h \in X\}$.

Let $\rho \in X$ and consider $G_\rho = G(t, s, \rho(s)), t, s \in [a, b]$. For a multivalued operator $G_\rho : [a, b] \times [a, b] \longrightarrow K_{cv}(\mathbb{R})$, there exists a continuous operator $g_\rho : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ (by Michael's selection theorem) such that $g_\rho(t, s) \in G_\rho(t, s)$, for all $t, s \in [a, b]$. It implies that $\int_a^t g_\rho(t, s)ds + h(t) \in f\rho(t)$. Thus, the operator $f\rho$ is nonempty. Note that $f\rho$ is closed. For more details, see [22].

Firstly, we check the condition (i) of Theorem 16. Let $\rho, \varsigma \in X$ be such that $u \in f\rho$. Then, there exists $g_\rho(t, s) \in G_\rho(t, s)$, for all $t, s \in [a, b]$ such that $u(t) = \int_a^t g_\rho(t, s)ds + h(t), t \in [a, b]$.

Besides, for all $\rho, \varsigma \in X$, we have

$$H_\chi(G(t, s, \rho(s)), G(t, s, \varsigma(s))) \le j(s)|\rho(s) - \varsigma(s)|, \forall t, s \in [a, b]. \tag{64}$$

Consequently, there exists $v(t, s) \in G_\varsigma(t, s)$ such that

$$|g_\rho(t, s) - v(t, s)| \le j(s)|\rho(s) - \varsigma(s)|, \forall t, s \in [a, b]. \tag{65}$$

Let us consider the multivalued operator S defined by

$$S(t,s)=G_{\varsigma}(t,s)\cap\Big\{w\in\mathbb{R}:|g_{\rho}(t,s)-w|\leq j(s)\mid\rho(s)-\varsigma(s)|,\forall t,s\in[a,b]\Big\}. \tag{66}$$

Since f is a lower semicontinuous, it follows that there exists $g_{\varsigma}:[a,b]\times[a,b]\longrightarrow\mathbb{R}$ such that $g_{\varsigma}(t,s)\in S(t,s)$, for all $t,s\in[a,b]$. Then, we have

$$\begin{aligned}
z(t)&=\int_a^t g_{\varsigma}(t,s)ds+h(t)\in\int_a^t G(t,s,\varsigma(s))ds+h(t),t\in[a,b],\\
|v(t)-z(t)|^p+|v(t)-z(t)|^q&=\left(\int_a^t\mid g_{\rho}(t,s)-g_{\varsigma}(t,s)\mid ds\right)^p+\left(\int_a^t\mid g_{\rho}(t,s)-g_{\varsigma}(t,s)\mid ds\right)^q\\
&\leq\left(\int_a^t j(s)\mid\rho(s)-\varsigma(s)\mid ds\right)^p+\left(\int_a^t j(s)\mid\rho(s)-\varsigma(s)\mid ds\right)^q\\
&\leq\left(\sqrt[p]{\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^p}\int_a^t j(s)ds\right)^p+\left(\sqrt[q]{\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^q}\int_a^t j(s)ds\right)^q\\
&=\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^p\left(\int_a^t j(s)ds\right)^p+\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^q\left(\int_a^t j(s)ds\right)^q\\
&\leq\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^p e^{-\tau}+\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^q\left(e^{-\tau}\right)^{q/p}\\
&\leq\left(\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^p+\sup_{t\in[a,b]}|\rho(t)-\varsigma(t)|^q\right)e^{-\tau}(\text{since}, q>p)\\
&=\chi(\rho,\varsigma)e^{-\tau}.
\end{aligned} \tag{67}$$

Hence, we have

$$\chi(v,z)\leq e^{-\tau}\chi(\rho,\varsigma). \tag{68}$$

On interchanging the job of ρ and ς, we have

$$H_{\chi}(f\rho,f\varsigma)\leq e^{-\tau}\chi(\rho,\varsigma). \tag{69}$$

By using $F(\alpha)=\ln(\alpha)$, we have

$$\tau+F\left(H_{\chi}(f\rho,f\varsigma)\right)\leq\chi(\rho,\varsigma)\leq F(M(\rho,\varsigma)). \tag{70}$$

Thus, all the hypotheses of Theorem 16 are satisfied. Hence, the operator f has exactly one fixed point, that is, the Volterra integral inclusion (61) has a unique solution. □

Remark 21. Consider the following differential inclusion (for $\rho\in C([a,b],\mathbb{R})$):

$$\rho'(t)\in G(t,s,\rho(s)),t,s\in[a,b], \tag{71}$$

where $G:[a,b]\times[a,b]\times\mathbb{R}\longrightarrow K_{cv}(\mathbb{R})$ is a multivalued operator. Observe that, for $h(t)=0$, (71) *is equivalent to* (61). *Also, G satisfies the condition* (63) *of Theorem20. Then, applying Theorem20, the differential inclusion* (71) *has a unique solution.*

Example 22. For all $\rho,\varsigma\in C([a,b],\mathbb{R})$, let

$$G(t,s,\rho(s))=\{v\in\mathbb{R}:g_1(t,s,\rho(s))\leq v\leq g_2(t,s,\rho(s))\}, \tag{72}$$

where $g_1(t,s,\rho(s))$ (resp., $g_2(t,s,\rho(s))$) is upper semicontinuous (resp., lower semicontinuous) on $[a,b]\times[a,b]$. For $\rho\in C([a,b],\mathbb{R})$, *assume the following:*

$$\rho'(t)\in G(t,s,\rho(s)),t,s\in[a,b]. \tag{73}$$

For all $\rho,\varsigma\in C([a,b],\mathbb{R})$, there exist a continuous function $j:X\longrightarrow\mathbb{R}$ and $\tau>0$ such that $\int_a^t j(s)ds\leq(e^{-\tau})^{1/p}$ and

$$\max|g_1(t,s,\rho(s))-g_2(t,s,\varsigma(s))|\leq j(s)|\rho(s)-\varsigma(s)|,\text{ for }all\ t,s\in[a,b]. \tag{74}$$

Hence, G is a compact and convex valued operator. Thus, G satisfies the condition (63) *of Theorem 20 (for $h(t)=0$), so* (74) *has a unique solution.*

5. Conclusion

We are concerned with Suzuki and Wardowski-type contraction multivalued mappings in the setting of partial symmetric spaces. We also studied a system of integral inclusions. It would be interesting to work on more generalized contraction mappings involving simulation or control functions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

A Generalization of Caristi's Fixed Point Theorem in the Variable Exponent Weighted Formal Power Series Space

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Suppose (p_n) be sequence of positive reals. By $\mathscr{H}_w((p_n))$, we represent the space of all formal power series $\sum_{n=0}^{\infty} a_n z^n$ equipped with $\sum_{n=0}^{\infty} |\lambda a_n/(n+1)|^{p_n} < \infty$, for some $\lambda > 0$. Various topological and geometric behavior of $\mathscr{H}_w((p_n))$ and the prequasi ideal constructs by s-numbers and $\mathscr{H}_w((p_n))$ have been considered. The upper bounds for s-numbers of infinite series of the weighted n-th power forward shift operator on $\mathscr{H}_w((p_n))$ with applications to some entire functions are granted. Moreover, we investigate an extrapolation of Caristi's fixed point theorem in $\mathscr{H}_w((p_n))$.

1. Introduction

While a statement of fixed Lebesgue spaces, variable exponent Lebesgue spaces go back many years, and in successive centuries, variable Lebesgue and Sobolev spaces have been systematically examined. As then, many variable exponent real function spaces and complex function spaces have presented, for example, Hardy spaces, Besov spaces, Bessel potential spaces, Trieble-Lizorkin spaces, Morrey spaces, Herz-Morrey spaces, Fock spaces, Bergman spaces, and Herz spaces with variable exponents. For three centuries, variable exponent function spaces have been widely applied in approximation theory, image processing, and differential equations. The learn about of the variable exponent Lebesgue spaces obtained in addition impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids [1, 2]. Applications of non-Newtonian fluids moreover regarded as electrorheological, vary from their use in army science to civil engineering and orthopedics. We will use the next conventions during the article; if others are used, we will state them.

Conventions 1. $\mathbb{N} = \{0, 1, 2, \cdots\}$.$\mathbb{C}$:The space of all complex numbers

$\mathbb{R}^{\mathbb{N}}$:The space of all sequences of real numbers

ℓ_∞:The space of bounded sequences of real numbers

ℓ^r:The space of r-absolutely summable sequences of real numbers

c_0:The space of null sequences of real numbers

$e_l = (0, 0, \cdots, 1, 0, 0, \cdots)$,:As 1 lies at the l^{th} coordinate, for all $l \in N$

$\mathscr{F}$:The space of each sequences with finite nonzero coordinates

card $(\mathfrak{G})$:The number of elements of G

mi_{Z}:The space of all monotonic increasing sequences of positive reals

L:The ideal of all bounded linear mappings between any arbitrary Banach spaces

F:The ideal of finite rank mappings between any arbitrary Banach spaces

Λ:The ideal of approximable mappings between any arbitrary Banach spaces

L_c:The ideal of compact mappings between any arbitrary Banach spaces

$L(X, Y)$:The space of all bounded linear mappings from a Banach space X into a Banach space Y

$L(X)$:The space of all bounded linear mappings from a Banach space X into itself

$F(X, Y)$:The space of finite rank mappings from a Banach space X into a Banach space Y

$F(X)$:The space of finite rank mappings from a Banach space X into itself

$\Lambda(X, Y)$:The space of approximable mappings from a Banach space X into a Banach space Y

$\Lambda(X)$:The space of approximable mappings from a Banach space X into itself

$L_c(X, Y)$:The space of compact mappings from a Banach space X into a Banach space Y

$L_c(X)$:The space of compact mappings from a Banach space X into itself

$(s_a(G))_{a\in\mathbb{N}}$:The sequence of s-numbers of the bounded linear operator G

$(\alpha_a(G))_{a\in\mathbb{N}}$:The sequence of approximation numbers of the bounded linear operator G

$(s_a(G))_{a\in\mathbb{N}}$:The sequence of Kolmogorov numbers of the bounded linear operator G

$S_{\mathscr{V}}$:The operator ideals constructed by the sequence of s-numbers in any sequence space V

$S_{\mathscr{V}}^{\mathrm{app}}$:The operator ideals constructed by the sequence of approximation numbers in any sequence space V

$S_{\mathscr{V}}^{\mathrm{Kol}}$:The operator ideals constructed by the sequence of Kolmogorov numbers in any sequence space V.

The theory of operator ideals has many activities in fixed point theorems, eigenvalue distributions, geometry of Banach spaces, spectral theorems, etc. Some of operator ideals in the class of Banach spaces or Hilbert spaces are generated by sequence of real numbers. For example, L_c is constructed by $(d_a(G))_{a\in\mathbb{N}}$ and c_0. Pietsch [3] examined the quasi ideals $S_{\ell^t}^{\mathrm{app}}$ for $0 < t < \infty$. He proved that the ideals of Hilbert Schmidt operators and of nuclear operators between Hilbert spaces are generated by ℓ^2 and ℓ^1, respectively. Also, he examined that $\bar{F} = S_{\ell^t}$, for $1 \le t < \infty$, and S_{ℓ^t} is simple Banach space. Pietsch [4] showed that S_{ℓ^t}, where $0 < t < \infty$, is small. Makarov and Faried [5] investigated that for any Banach spaces X and Y with dim $(X) = $ dim $(Y) = \infty$. They have for all $r > t > 0$ that $S_{\ell^t}^{\mathrm{app}}(X, Y)$Þ$S_{\ell^r}^{\mathrm{app}}(X, Y)UL(X, Y)$. Faried and Bakery [6] generalized the concept of quasi ideal by introducing the concept of prequasi ideal. They introduced some geometric and topological structure of the spaces $S_{ces(t)}$ and S_{ℓ_M}. Yaying et al. [7] suggested the sequence space, χ_r^t, with its r-Cesàro matrix in ℓ^t, with $r \in (0, 1]$ and $1 \le t \le \infty$. They examined the quasi Banach ideal of type χ_r^t, for $r \in (0, 1]$ and $1 < t < \infty$. They found its Schauder basis, $\alpha -$, $\beta -$, and $\gamma -$ duals and committed to certain matrix classes linked to this sequence space. Mursaleen and Noman [8, 9] investigated the compact operators on some difference sequence spaces. The multiplication maps on Cesàro sequence spaces with the Luxemburg norm explored by Komal et al. [10]. İlkhan et al. [11] affected the multiplication maps on Cesàro second order function spaces. Recently, many authors in the literature have investigated some nonabsolute kind sequence spaces and brought recent splendid papers; for example, Mursaleen and Noman [12] defined the sequence spaces ℓ_p^λ and ℓ_∞^λ of nonabsolute type and proved that the spaces ℓ_p^λ and ℓ_p^λ are linearly isomorphic for $0 < p \le \infty$, ℓ_p^λ is a p-normed space, and a BK -space in the cases for $0 < p < 1$ and $1 \le p \le \infty$ and modeled the basis for the space ℓ_p^λ for $1 \le p < \infty$. In [13], they considered the $\alpha -$, $\beta -$, and $\gamma -$ duals of ℓ_p^λ and ℓ_∞^λ of nonabsolute type, for $1 \le p < \infty$. They were given a picture of some related matrix classes and shown the characterizations of some other classes by means of a given basic lemma. On Cesàro summable sequences, Mursaleen and Basar [14] defined some spaces of double sequences whose Cesàro transforms are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent, and absolutely q-summable, respectively, and investigated some topological properties of those sequence spaces. The Banach fixed point theorem [15] gave many mathematicians the way to examine many generalizations for the contraction operators defined on the space or on the space itself. Kannan [16] examined an example of a class of operators with the identical fixed point actions as contractions but fails to be continuous. Ghoncheh [17] was the only one who investigated Kannan operators in modular vector spaces. He showed that the existence of a fixed point of Kannan mapping in complete modular spaces that have Fatou property. Bakery and Mohamed [18] explored the concept of the prequasi norm on Nakano sequence space so as to its variable exponent in $(0, 1]$. They investigated the sufficient conditions on it equipped with the definite prequasi norm to generate prequasi Banach and closed space and approved the Fatou property of distinct prequasi norms on it. Moreover, they showed that the existence of a fixed point of Kannan prequasi norm contraction maps on it and on the prequasi Banach operator ideal generated by s-numbers which belong to this sequence space. According to the spectral decomposition theorem [3], for $A \in L_c(H)$, where H is a Hilbert space, then $A(y) = \sum_{a=0}^{\infty} \alpha_a \prec y, r_a \succ w_a$, where $\{r_a\}$ and $\{w_a\}$ are orthonormal families in H. Assume $(t_a)_{a\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be decreasing and $D : (\eta_a) \longrightarrow (t_a \eta_a)$ be the diagonal operator on ℓ^p with $p \ge 1$. Hence, $s_a(D) = t_a$. Shields [19] gave an illustration to the weighted shift operators as formal power series in unilateral shifts and formal Laurent series in bilateral shifts. Hedayatian [20] explored the space of formal power series with power r, $\mathscr{H}^r((b_a))$, where $((b_a))$ is a sequence of positive numbers with $b_0 = 1$ and $r > 0$. By the space $\mathscr{H}^p((b_a))$, he thought that the set of all formal power series $\sum_{a=0}^{\infty} \widehat{f}_a z^a$ with $\sum_{a=0}^{\infty} |b_a \widehat{f}_a|^p < \infty$. He investigated cyclic vectors for the forward shift operator and super cyclic vectors for the backward shift operator on the space $\mathscr{H}^p((b_a))$. However, Emamirad and Heshmati [21], studied the idea of functions evident on the Bargmann space by $f(z) = \sum_{a=0}^{\infty} c_a (z^a/\sqrt{a!})$ with $\|f\| = \sum_{a=0}^{\infty} |c_a|^2 < \infty$, where $\{(z^a/\sqrt{a!}) : a \in \mathbb{N}\}$ is an orthonormal basis. Faried et al. [22] examined the upper bounds for s-numbers of infinite series of the weighted n-th power forward shift operator on $\mathscr{H}^r((b_a))$, for $1 \le r < \infty$, with some applications to some entire functions.

The aim of this paper is organized as follows: In Section 3, we introduce the definition of the space $\mathscr{H}_w((p_n))$ under the function ρ. We offer the enough setup on $\mathscr{H}_w((p_n))$ to become premodular special space of formal power series which implies that $\mathscr{H}_w((p_n))$ is a prequasi normed space. In Section 4, firstly, the operator ideals generated by s -numbers and $\mathscr{H}_w((p_n))$ so as to $S_{\mathscr{H}_w((p_n))}$ constructs an operator ideal are presented. Secondly, we offer the sufficient conditions (not necessary) on $(\mathscr{H}_w((p_n)))_\rho$, such that F is dense in $S_{(\mathscr{H}_w((p_n)))_\rho}$. This presents the nonlinearity of s -type $(\mathscr{H}_w((p_n)))_\rho$ spaces which offers an answer of Rhoades [23] open problem. Thirdly, we examine the setup on $(\mathscr{H}_w((p_n)))_\rho$ so that the elements of prequasi ideal $S_{(\mathscr{H}_w((p_n)))_\rho}$ are complete and closed. Fourthly, we study the enough setup on $(\mathscr{H}_w((p_n)))_\rho$ such that $S_{(\mathscr{H}_w((p_n)))_\rho}$ is strictly contained for distinct powers. We investigate the smallness of $S_{(\mathscr{H}_w((p_n)))_\rho}$. Fifthly, we explain the simpleness of $S_{(\mathscr{H}_w((p_n)))_\rho}$. Sixthly, we offer the enough conditions on $(\mathscr{H}_w((p_n)))_\rho$ so as to the class L with its sequence of eigenvalues in $(\mathscr{H}_w((p_n)))_\rho$ is strictly contained in $S_{(\mathscr{H}_w((p_n)))_\rho}$. In Section 5, we evaluate the upper bounds for s-numbers of infinite series of the weighted n-th power forward shift operator on $\mathscr{H}_w((p_n))$ with applications to some entire functions. Finally, in Section 6, we appraise a generalization of Caristi's fixed point theorem in $(\mathscr{H}_w((p_n)))_\rho$.

2. Definitions and Preliminaries

Definition 2 [24]. A map $s : L(X, Y) \longrightarrow [0,\infty)^{\mathbb{N}}$ is named an s-number, if the sequence $(s_b(B))_{a=0}^{\infty}$, for every $B \in L(X, Y)$, verifies the next conditions:

(a) ssume $B \in L(X, Y)$, then $\|B\| = s_0(B) \geq s_1(B) \geq s_2(B) \geq \cdots \geq 0$

(b) $s_{b+a}(B_1 + B_2) \leq s_b(B_1) + s_a(B_2)$, for all $B_1, B_2 \in L(X, Y)$, $b, a \in \mathbb{N}$

(c) The inequality $s_a(ABD) \leq \|A\| s_a(B) \|D\|$ satisfies, if $D \in L(X_0, X)$, $B \in L(X, Y)$, and $A \in L(Y, Y_0)$, where X_0 and Y_0 are any two Banach spaces

(d) If $A \in L(X, Y)$ and $\lambda \in \mathbb{R}$, then $s_a(\lambda A) = |\lambda| s_a(A)$

(e) Assume rank $(A) \leq b$ then $s_b(A) = 0$, whenever $A \in L(X, Y)$

(f) Suppose I_a denotes the unit map on the a-dimensional Hilbert space ℓ_2^a, then $s_{r\geq a}(I_a) = 0$ or $s_{r<a}(I_a) = 1$

We introduce a few examples of s-numbers as indicated:

(i) The n-th approximation number, $\alpha_b(A)$, where

$$\alpha_b(A) = \inf \{\|A - B\|: B \in L(X, Y) \text{ and rank } (B) \leq b\}. \quad (1)$$

(ii) The n-th Kolmogorov number, $d_b(A)$, where

$$d_b(A) = \inf_{\dim Y \leq b} \sup_{\|u\| \leq 1} \inf_{v \in Y} \|Au - v\|. \quad (2)$$

Remark 3 [24]. Assume $A \in L_c(H)$, where H be a Hilbert space, then all the s -numbers equal the eigenvalues of $|A|$, where $|A| = \sqrt{A^* A}$.

Lemma 4 [3]. *Suppose $A \in L(X, Y)$ and $A \notin \Lambda(X, Y)$, then there are maps $D \in L(X)$ and $M \in L(Y)$ with $MADe_b = e_b$, for all $b \in \mathbb{N}$.*

Definition 5 [3]. A Banach space Y is called simple if $L(Y)$ has a unique nontrivial closed ideal.

Theorem 6 [3]. *Assume D be a Banach space with* dim $(D) = \infty$, *one has*

$$F(D) \subsetneqq \Lambda(D) \subsetneqq L_c(D) \subsetneqq L(D). \quad (3)$$

Definition 7 [3]. A class $U \subseteq L$ is called an operator ideal if each element $U(X, Y) = U \cap L(X, Y)$ verifies the next conditions:

$$F \subseteq U. \quad (4)$$

(i) $U(X, Y)$ is linear space on $\mathbb{R}$

(ii) Suppose $D \in L(X_0, X)$, $B \in U(X, Y)$, and $A \in L(Y, Y_0)$ then, $ABD \in U(X_0, Y_0)$

Definition 8 [6]. A map $g : U \longrightarrow [0,\infty)$ is named a prequasi norm on the ideal U if it verifies the next conditions:

(1) Assume $A \in L(X, Y)$, $g(A) \geq 0$, and $g(A) = 0 \Leftrightarrow A = 0$

(2) We have $M \geq 1$ with $g(\mu A) \leq M|\mu| g(A)$, for each $\mu \in \mathbb{R}$ and $A \in U(X, Y)$

(3) We find $K \geq 1$ with $g(A_1 + A_2) \leq K[g(A_1) + g(A_2)]$, for every $A_1, A_2 \in U(X, Y)$

(4) We obtain $C \geq 1$ so that if $A \in L(X_0, X)$, $B \in U(X, Y)$, and $D \in L(Y, Y_0)$ then $g(DBA) \leq C \|D\| g(B) \|A\|$, where X_0 and Y_0 are normed spaces

Theorem 9 [6]. *If g is a quasi norm on the ideal U, then g is a prequasi norm on the ideal U.*

Theorem 10 [25]. *If s-type $\mathscr{V}_v := \{f = (s_r(T)) \in \mathbb{R}^{\mathbb{N}} : T \in L(X, Y)$ and $v(f) < \infty\}$. Assume $S_{\mathscr{V}_v}$ be an operator ideal, one has*

(1) $\mathscr{F} \subset s-$ type $\mathscr{V}_v$

(2) *Suppose* $(s_r(T_1))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$ *and* $(s_r(T_2))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$, *then* $(s_r(T_1 + T_2))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$

(3) *Assume* $\lambda \in \mathbb{R}$ *and* $(s_r(T))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$, *then* $|\lambda| (s_r(T))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$

(4) *The sequence space* $\mathscr{V}_v$ *is solid; i.e., if* $(s_r(G))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$ *and* $s_r(T) \le s_r(G)$, *for all* $r \in \mathbb{N}$ *and* $T, G \in L(X, Y)$, *then* $(s_r(T))_{r=0}^{\infty} \in$ *s-type* $\mathscr{V}_v$

Lemma 11 [26]. *Assume* $\{\xi_i\}_{i\in\Psi}$ *be a bounded family of* $\mathbb{R}$. *Then*

$$\sup_{\operatorname{card}(\mathfrak{G})=a+1} \inf_{i\in\mathfrak{G}} \xi_i = \inf_{\operatorname{card}(\mathfrak{G})=a} \sup_{i\notin\mathfrak{G}} \xi_i. \tag{5}$$

The next inequality in [27] will be used in the sequel. For $(r_a), (t_a) \in \mathbb{R}^{\mathbb{N}}$, *and* $(q_a) \in (0,\infty)^{\mathbb{N}}$, *one has*

$$|r_a + t_a|^{q_a} \le K(|r_a|^{q_a} + |t_a|^{q_a}), \tag{6}$$

where $K = \max\{1, 2^{\varpi_q - 1}\}$ *and* $\varpi_q = \max\{1, \sup_a q_a\}$.

3. Main Results

3.1. The Space of Functions $(\mathscr{H}_w((p_n)))_\rho$. In this section, we current the definition of the space $(\mathscr{H}_w((p_n)))_\rho$ equipped with the function ρ. We introduce the sufficient setup on $(\mathscr{H}_w((p_n)))_\rho$ to form premodular special space of formal power series (ssfps). This gives that $(\mathscr{H}_w((p_n)))_\rho$ is a pre-quasi normed (ssfps).

Assume $p = (p_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$, we define the following space:

$$(\mathscr{H}_w((p_n)))_\rho = \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n \text{ converges for any } z \in \mathbb{C} \text{ and } \rho(\lambda f) < \infty, \text{for some } \lambda > 0 \right\}, \tag{7}$$

where

$$\rho(f) = \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n}. \tag{8}$$

When $(p_n) \in \ell_\infty$, we have

$$\begin{aligned}
(\mathscr{H}_w((p_n)))_\rho &= \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n \text{ converges for any } z \in \mathbb{C} \text{ and } \rho(\lambda f) < \infty, \text{for some } \lambda > 0 \right\} \\
&= \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n \text{ converges for any } z \in \mathbb{C} \text{ and } \sum_{n=0}^{\infty} \left| \frac{\lambda \widehat{f_n}}{n+1} \right|^{p_n} < \infty, \text{for some } \lambda > 0 \right\} \\
&= \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n \text{ converges for any } z \in \mathbb{C} \text{ and } \inf_n |\lambda|^{p_n} \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n} < \infty, \text{for some } \lambda > 0 \right\} \\
&= \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n \text{ converges for any } z \in \mathbb{C} \text{ and } \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n} < \infty \right\} \\
&= \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n \text{ converges for any } z \in \mathbb{C} \text{ and } \rho(\lambda f) < \infty, \text{for any } \lambda > 0 \right\}.
\end{aligned} \tag{9}$$

Definition 12. The linear space of formal power series,

$$\mathscr{H} = \left\{ f : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n \text{ converges for any } z \in \mathbb{C} \right\}, \quad (10)$$

is named a (ssfps) when:

(1) $e^{(m)} \in \mathscr{H}$, for every $m \in \mathbb{N}$, where $e^{(m)}(z) = \sum_{n=0}^{\infty} \widehat{e_n^{(m)}} z^n = z^m$

(2) Suppose $g \in \mathscr{H}$ and $|\widehat{f_n}| \le |\widehat{g_n}|$, for all $n \in \mathbb{N}$, then $f \in \mathscr{H}$

(3) Let $f \in \mathscr{H}$, then $f_{[.]} \in \mathscr{H}$, where $f_{[.]}(z) = \sum_{b=0}^{\infty} \widehat{f_{[b/2]}} z^b$ and $[b/2]$ indicates the integral part of $b/2$

By $\mathfrak{F}$, we indicate the space of finite formal power series, i.e, for $f \in \mathfrak{F}$; then, there is $l \in \mathbb{N}$ so that $f(z) = \sum_{n=0}^{l} \widehat{f_n} z^n$.

Definition 13. A subspace $\mathscr{H}_\rho$ of the (ssfps) is called a premodular (ssfps), if there is a map $\rho : \mathscr{H} \longrightarrow [0,\infty)$ satisfies the following setup:

(i) Assume $f \in \mathscr{H}$, one has $\rho(f) \ge 0$ and $f = \theta \Leftrightarrow \rho(f) = 0$, where θ is the zero function of $\mathscr{H}$

(ii) If $f \in \mathscr{H}$ and $\lambda \in \mathbb{R}$, one has $l \ge 1$ with $\rho(\lambda f) \le |\lambda| l \rho(f)$

(iii) If $f, g \in \mathscr{H}$, one has $K \ge 1$ so as to $\rho(f + g) \le K(\rho(f) + \rho(g))$

(iv) Suppose $|\widehat{f_b}| \le |\widehat{g_b}|$, for every $b \in \mathbb{N}$, then $\rho(f) \le \rho(g)$

(v) One has $K_0 \ge 1$ such that $\rho(f) \le \rho(f_{[.]}) \le K_0 \rho(f)$

$$\bar{\mathfrak{F}} = \mathscr{H}_\rho. \quad (11)$$

(vi) We have $\xi > 0$ with $\rho(\lambda e^{(0)}) \ge \xi |\lambda| \rho(e^{(0)})$, where $\lambda \in \mathbb{R}$

Remark that the continuity of $\rho(f)$ at θ follows from part (ii). The part (1) in Definition 12 and part (vi) in Definition 13 explain that $(e^{(m)})_{m \in \mathbb{N}}$ is a Schauder basis of $\mathscr{H}_\rho$.

The (ssfps) $\mathscr{H}_\rho$ is named a prequasi normed (ssfps) if ρ satisfies the parts (i)–(iii) of Definition 13, and if the space $\mathscr{H}$ is complete equipped with ρ, then $\mathscr{H}_\rho$ is named a prequasi Banach (ssfps).

Theorem 14. *Every premodular (ssfps) $\mathscr{H}_\rho$ is a prequasi normed (ssfps).*

Theorem 15. *If $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$, then $(\mathscr{H}_w((p_n)))_\rho$ is a premodular Banach (ssfps).*

Proof. (1-i) Assume $f, g \in \mathscr{H}_w((p_n))$. Hence, $f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n$ and $g(z) = \sum_{n=0}^{\infty} \widehat{g_n} z^n$ converge for any $z \in \mathbb{C}$. One has $(f + g)(z) = \sum_{n=0}^{\infty} (\widehat{f_n} + \widehat{g_n}) z^n$ converges for any $z \in \mathbb{C}$. As $(p_n) \in \ell_\infty$, one obtains

$$\sum_{n=0}^{\infty} \left| \frac{\widehat{f_n} + \widehat{g_n}}{n+1} \right|^{p_n} \le K \left(\sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n} + \sum_{n=0}^{\infty} \left| \frac{\widehat{g_n}}{n+1} \right|^{p_n} \right) < \infty, \quad (12)$$

then $f + g \in \mathscr{H}_w((p_n))$.

(1-ii) Suppose $\lambda \in \mathbb{R}$ and $f \in \mathscr{H}_w((p_n))$. Hence, $f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n$ converges for any $z \in \mathbb{C}$. One has $(\lambda f)(z) = \sum_{n=0}^{\infty} \lambda \widehat{f_n} z^n$ converges for any $z \in \mathbb{C}$. As $(p_n) \in \ell_\infty$, one can see

$$\sum_{n=0}^{\infty} \left| \frac{\lambda \widehat{f_n}}{n+1} \right|^{p_n} \le \sup_n |\lambda|^{p_n} \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n} < \infty. \quad (13)$$

So $\lambda f \in \mathscr{H}_w((p_n))$. Hence, from parts (1-i) and (1-ii), one gets that the space $\mathscr{H}_w((p_n))$ is linear. Also $e^{(m)} \in \mathscr{H}_w((p_n))$, for every $m \in \mathbb{N}$, where $e^{(m)}(z) = \sum_{n=0}^{\infty} \widehat{e_n^{(m)}} z^n = z^m$ and $\sum_{n=0}^{\infty} |\widehat{e_n^{(m)}}/(n+1)|^{p_n} = 1/(m+1)^{p_m}$.

(2) Suppose $|\widehat{f_n}| \le |\widehat{g_n}|$, for every $n \in \mathbb{N}$ and $g \in \mathscr{H}_w((p_n))$. Hence, $g(z) = \sum_{n=0}^{\infty} \widehat{g_n} z^n$ converges for any $z \in \mathbb{C}$. We have

$$\sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n} \le \sum_{n=0}^{\infty} \left| \frac{\widehat{g_n}}{n+1} \right|^{p_n} < \infty. \quad (14)$$

Therefore, $f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n$ converges for any $z \in \mathbb{C}$ and $\rho(f) < \infty$. So, $f \in \mathscr{H}_w((p_n))$.

(3) If $f \in \mathscr{H}_w((p_n))$ and $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$. Hence, $f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n$ converges for any $z \in \mathbb{C}$ and $\rho(f) < \infty$. We obtain

$$\begin{aligned} \rho(f_{[.]}) &= \sum_{n=0}^{\infty} \left| \frac{\widehat{f_{[n/2]}}}{n+1} \right|^{p_n} = \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{2n+1} \right|^{p_{2n}} + \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{2n+2} \right|^{p_{2n+1}} \\ &\le 2 \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n} = 2\rho(f). \end{aligned} \quad (15)$$

Therefore, $f_{[.]}(z) = \sum_{n=0}^{\infty} \widehat{f_{[n/2]}} z^n$ converges for any $z \in \mathbb{C}$ and $\rho(f_{[.]}) < \infty$. Then, $f_{[.]} \in \mathscr{H}_w((p_n))$.

(i) Definitely, for all $f \in \mathscr{H}_w((p_n))$, then $\rho(f) \geq 0$ and $\rho(f) = 0 \Leftrightarrow f = \theta$

(ii) One has $l = \max\{1, \sup_n |\eta|^{p_n - 1}\} \geq 1$, for every $\eta \in \mathbb{R} \setminus \{0\}$ and $l \geq 1$, for $\eta = 0$ so that

$$\rho(\eta f) = \sum_{n=0}^{\infty} \left|\eta b_n \widehat{f_n}\right|^{p_n} \leq \sup_n |\eta|^{p_n} \sum_{n=0}^{\infty} \left|b_n \widehat{f_n}\right|^{p_n} \leq l|\eta|\rho(f), \tag{16}$$

for all $f \in \mathscr{H}_w((p_n))$.

(iii) One gets some $K = \max\{1, 2^{\sup_n p_n - 1}\} \geq 1$ satisfy the inequality

$$\rho(f+g) = \sum_{n=0}^{\infty} \left|\frac{\widehat{f_n} + \widehat{g_n}}{n+1}\right|^{p_n} \leq K(\rho(f) + \rho(g)), \tag{17}$$

for every $f, g \in \mathscr{H}_w((p_n))$.

(iv) Definitely from the proof part (2)

(v) From the proof part (3), one has $K_0 = 2 \geq 1$

(vi) Definitely $\bar{\mathfrak{F}} = \mathscr{H}_w((p_n))$

(vii) We have ζ with $0 < \zeta \leq \eta^{p_0 - 1}$ so that $\rho(\eta e^{(0)}) \geq \zeta|\eta| \rho(e^{(0)})$, for all $\eta \neq 0$ and $\zeta > 0$, when $\eta = 0$

So, the space $(\mathscr{H}_w((p_n)))_\rho$ is premodular (ssfps). To prove that $(\mathscr{H}_w((p_n)))_\rho$ is a premodular Banach (ssfps), let $f^{(i)}$ is a Cauchy sequence in $(\mathscr{H}_w((p_n)))_\rho$; hence, for all $\varepsilon \in (0,1)$, one has $i_0 \in \mathbb{N}$ so that for every $i, j \geq i_0$, we have

$$\rho\left(f^{(i)} - f^{(j)}\right) = \sum_{n=0}^{\infty} \left|\frac{\widehat{f_n^{(i)}} - \widehat{f_n^{(j)}}}{n+1}\right|^{p_n} < \varepsilon^{\varpi_p}. \tag{18}$$

For $i, j \geq i_0$, and $k \in \mathbb{N}$, one gets

$$\left|\widehat{f_k^{(i)}} - \widehat{f_k^{(j)}}\right| < \varepsilon. \tag{19}$$

Therefore, $(\widehat{f_k^{(j)}})$ is a Cauchy sequence in $\mathbb{R}$, for constant $k \in \mathbb{N}$; hence, $\lim_{j\longrightarrow\infty} \widehat{f_k^{(j)}} = \widehat{f_k^{(0)}}$, for constant $k \in \mathbb{N}$. So, $\rho(f^{(i)} - f^{(0)}) < \varepsilon^{\varpi_p}$, for all $i \geq i_0$. Finally, to investigate that $f^{(0)} \in \mathscr{H}_w((p_n))$, one has

$$\rho\left(f^{(0)}\right) = \rho\left(f^{(0)} - f^{(n)} + f^{(n)}\right) \leq K\left(\rho\left(f^{(n)} - f^{(0)}\right) + \rho\left(f^{(n)}\right)\right) < \infty. \tag{20}$$

Therefore, $f^{(0)} \in \mathscr{H}_w((p_n))$. So, the space $(\mathscr{H}_w((p_n)))_\rho$ is a premodular Banach (ssfps).

By using Theorem 14, we offer the next theorem.

Theorem 16. *Assume $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$, then the space $(\mathscr{H}_w((p_n)))_\rho$ be a prequasi Banach (ssfps), where*

$$\rho(f) = \sum_{n=0}^{\infty} \left|\frac{\widehat{f_n}}{n+1}\right|^{p_n}, \text{for all } f \in \mathscr{H}_w((p_n)). \tag{21}$$

Theorem 17. *If $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$, then the space $(\mathscr{H}_w((p_n)))_\rho$ is a prequasi closed (ssfps), where*

$$\rho(f) = \sum_{n=0}^{\infty} \left|\frac{\widehat{f_n}}{n+1}\right|^{p_n}, \text{for all } f \in \mathscr{H}_w((p_n)). \tag{22}$$

Proof. From Theorem 16, the space $(\mathscr{H}_w((p_n)))_\rho$ is a prequasi normed (ssfps). To prove that $(\mathscr{H}_w((p_n)))_\rho$ is a prequasi closed (ssfps), suppose $\{f^{(i)}\}_{i=0}^{\infty} \in (\mathscr{H}_w((p_n)))_\rho$ and $\lim_{i\longrightarrow\infty} \rho(f^{(i)} - f^{(0)}) = 0$; then, for each $\varepsilon \in (0,1)$, one has $i_0 \in \mathbb{N}$ so that for every $i \geq i_0$, we have

$$\varepsilon > \rho\left(f^{(i)} - f^{(0)}\right) = \left[\sum_{a=0}^{\infty} \left|\frac{\widehat{f_a^{(i)}} - \widehat{f_a^{(0)}}}{n+1}\right|^{p_a}\right]^{1/\varpi_p}. \tag{23}$$

Therefore, for $i \geq i_0$ and $a \in \mathbb{N}$, one obtains $|\widehat{f_a^{(i)}} - \widehat{f_a^{(0)}}| < \varepsilon$. Hence, $(\widehat{f_a^{(i)}})$ is a convergent sequence in $\mathbb{R}$, for constant $a \in \mathbb{N}$. Hence, $\lim_{i\longrightarrow\infty} \widehat{f_a^{(i)}} = \widehat{f_a^{(0)}}$, for constant $a \in \mathbb{N}$. Finally, to show that $f^{(0)} \in (\mathscr{H}_w((p_n)))_\rho$, one has

$$\rho\left(f^{(0)}\right) = \rho\left(f^{(0)} - f^{(i)} + f^{(i)}\right) \leq \rho\left(f^{(i)} - f^{(0)}\right) + \rho\left(f^{(i)}\right) < \infty, \tag{24}$$

so $f^{(0)} \in (\mathscr{H}_w((p_n)))_\rho$. This explains that $(\mathscr{H}_w((p_n)))_\rho$ is a prequasi closed (ssfps).

4. Properties of Operator Ideal

In this section, we offer some geometric and topological structure of the operator ideals generated by s-numbers and $(\mathscr{H}_w((p_n)))_\rho$.

4.1. Prequasi Ideal

Notations 18.

$$
\begin{aligned}
S_{\mathscr{H}} &:= \{S_{\mathscr{H}}(X,Y)\,;X \text{ and } Y \text{ are Banach Spaces}\}, \text{ where}\\
S_{\mathscr{H}}(X,Y) &:= \left\{P \in L(X,Y): f_s \in \mathscr{H}, \text{ where, } f_s(z) = \sum_{n=0}^{\infty} s_n(P) z^n \text{ converges for any } z \in \mathbb{C}\right\}.\\
S_{\mathscr{H}}^{\mathrm{app}} &:= \{S_{\mathscr{H}}^{\mathrm{app}}(X,Y)\,;X \text{ and } Y \text{ are Banach Spaces}\}, \text{ where}\\
S_{\mathscr{H}}^{\mathrm{app}}(X,Y) &:= \left\{P \in L(X,Y): f_{\mathrm{app}} \in \mathscr{H}, \text{ where, } f_{\mathrm{app}}(z) = \sum_{n=0}^{\infty} \alpha_n(P) z^n \text{ converges for any } z \in \mathbb{C}\right\}.\\
S_{\mathscr{H}}^{\mathrm{Kol}} &:= \left\{S_{\mathscr{H}}^{\mathrm{Kol}}(X,Y)\,;X \text{ and } Y \text{ are Banach Spaces}\right\}, \text{ where}\\
S_{\mathscr{H}}^{\mathrm{Kol}}(X,Y) &:= \left\{P \in L(X,Y): f_{\mathrm{Kol}} \in \mathscr{H}, \text{ where, } f_{\mathrm{Kol}}(z) = \sum_{n=0}^{\infty} d_n(P) z^n \text{ converges for any } z \in \mathbb{C}\right\}
\end{aligned}
\tag{25}
$$

Theorem 19. *$S_{\mathscr{H}}$ is an operator ideal, whenever $\mathscr{H}$ is a (ssfps).*

Proof. To prove $S_{\mathscr{H}}$ is an operator ideal:

(i) Assume $B \in F(X,Y)$ and rank $(B) = n$, for all $n \in \mathbb{N}$, as $e^{(k)} \in \mathscr{H}$, for every $k \in \mathbb{N}$ and $\mathscr{H}$ is a linear space, then $(s_i(B))_{i=0}^{\infty} = (s_0(B), s_1(B), \cdots, s_{n-1}(B), 0, 0, 0, \cdots)$. Hence

$$
f_s(z) = \sum_{k=0}^{\infty} s_k(B) e^{(k)}(z) = \sum_{k=0}^{n-1} s_k(B) e^{(k)}(z) = \sum_{k=0}^{n-1} s_k(B) z^k,
\tag{26}
$$

converges for any $z \in \mathbb{C}$. So $f_s \in \mathscr{H}$ and hence $B \in S_{\mathscr{H}}(X,Y)$ which gives $F(X,Y) \subseteq S_{\mathscr{H}}(X,Y)$.

(ii) Suppose $P_1, P_2 \in S_{\mathscr{H}}(X,Y)$ and $b_1, b_2 \in \mathbb{R}$. Hence f_s, $g_s \in \mathscr{H}$ so that $f_s(z) = \sum_{n=0}^{\infty} s_n(P_1) z^n$ and $g_s(z) = \sum_{n=0}^{\infty} s_n(P_2) z^n$ converge for any $z \in \mathbb{C}$. From Definition 12 part (3), we have $f_{[.]}, g_{[.]} \in \mathscr{H}$, where $f_{[.]}(z) = \sum_{n=0}^{\infty} s_{[n/2]}(P_1) z^n$ and $g_{[.]}(z) = \sum_{n=0}^{\infty} s_{[n/2]}(P_2) z^n$ converge for any $z \in \mathbb{C}$. Since $n \geq 2[n/2]$, by using the definition of s-numbers and $s_n(P)$ is a decreasing sequence, one has

$s_n(b_1P_1 + b_2P_2) \leq s_{2[n/2]}(b_1P_1 + b_2P_2) \leq s_{[n/2]}(b_1P_1) + s_{[n/2]}(b_2P_2) = |b_1| s_{[n/2]}(P_1) + |b_2| s_{[n/2]}(P_2)$, for every $n \in \mathbb{N}$. As from Definition 12 part (2) and $\mathscr{H}$ is a linear space, one gets $h_s \in \mathscr{H}$, where $h_s(z) =: \sum_{n=0}^{\infty} s_n(b_1P_1 + b_2P_2) z^n$ converges for any $z \in \mathbb{C}$. Hence, $b_1P_1 + b_2P_2 \in S_{\mathscr{H}}(X,Y)$.

(iii) Assume $P \in L(X_0, X)$, $T \in S_{\mathscr{H}}(X,Y)$, and $R \in L(Y, Y_0)$. Hence, $f_s \in \mathscr{H}$, where $f_s(z) = \sum_{n=0}^{\infty} s_n(T) z^n$ converges for any $z \in \mathbb{C}$. As $s_n(RTP) \leq \|R\| s_n(T) \|P\|$, from Definition 12 parts (1) and (2), one obtains $h_s \in \mathscr{H}$, where $h_s(z) =: \sum_{n=0}^{\infty} s_n(RTP) z^n$ converges for any $z \in \mathbb{C}$. This gives $RTP \in S_{\mathscr{H}}(X_0, Y_0)$.

Corollary 20. *Assume $(p_n) \in mi_{\nearrow} \cap \ell_\infty$ with $p_0 > 0$, then $S_{\mathscr{H}_w((p_n))}$ be an operator ideal.*

4.2. Ideal of Finite Rank Operators. In this section, the sufficient conditions (not necessary) on $(\mathscr{H}_w((p_n)))_\rho$ such that F is dense in $S_{(\mathscr{H}_w((p_n)))_\rho}$ are presented. This investigates the nonlinearity of the s-type $(\mathscr{H}_w((p_n)))_\rho$ spaces (Rhoades open problem [23]).

Theorem 21. *$\bar{F(X, Y)} = S_{(\mathscr{H}_w((p_n)))_\rho}(X,Y)$, where*

$$
\rho(f) = \sum_{n=0}^{\infty} \left| \frac{\widehat{f_n}}{n+1} \right|^{p_n}, \textit{for all } f \in \mathscr{H}_w((p_n)),
\tag{27}
$$

whenever $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$.

Proof. Evidently, $\bar{F(X, Y)} \subset S_{(\mathscr{H}_w((p_n)))_\rho}(X,Y)$, as the space $S_{(\mathscr{H}_w((p_n)))_\rho}$ is an operator ideal. Hence, we have to show that $S_{(\mathscr{H}_w((p_n)))_\rho}(X,Y) \subseteq \bar{F(X, Y)}$. By choosing $T \in S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y)$, hence, $f_s \in (\mathscr{H}_w((p_n)))_\rho$, with $f_s(z) = \sum_{n=0}^{\infty} s_n(T) z^n$ converges for any $z \in \mathbb{C}$. Therefore, $\rho(f_s) < \infty$; let $\varepsilon \in (0,1)$, one has $m \in \mathbb{N} - \{0\}$ so that $\rho(f_s - \sum_{n=0}^{m-1} e^{(n)}) < \varepsilon/4$. Since $(s_n(T))_{n \in \mathbb{N}}$ is decreasing, one obtains

$$
\sum_{n=m+1}^{2m} \left(\frac{s_{2m}(T)}{n+1}\right)^{p_n} \leq \sum_{n=m+1}^{2m} \left(\frac{s_n(T)}{n+1}\right)^{p_n} \leq \sum_{n=m}^{\infty} \left(\frac{s_n(T)}{n+1}\right)^{p_n} < \frac{\varepsilon}{4}.
\tag{28}
$$

So, one has $A \in F_{2m}(X, Y)$, rank$A \le 2m$, and

$$\sum_{n=2m+1}^{3m} \left(\frac{\|T-A\|}{n+1}\right)^{p_n} \le \sum_{n=m+1}^{2m} \left(\frac{\|T-A\|}{n+1}\right)^{p_n} < \frac{\varepsilon}{4}. \quad (29)$$

Because $(p_n) \in \ell_\infty$, so we have

$$\sum_{n=0}^{m} \left(\frac{\|T-A\|}{n+1}\right)^{p_n} < \frac{\varepsilon}{4}. \quad (30)$$

As $T - A \in S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y)$, hence $h_s \in (\mathscr{H}_w((p_n)))_\rho$, where $h_s(z) := \sum_{n=0}^{\infty} s_n(T-A) z^n$ converges for any $z \in \mathbb{C}$. In view of (p_n) is increasing also the inequalities (2)–(4) explain that

$$\begin{aligned} d(T, A) = \rho(h_s) &= \sum_{n=0}^{3m-1} \left(\frac{s_n(T-A)}{n+1}\right)^{p_n} + \sum_{n=3m}^{\infty} \left(\frac{s_n(T-A)}{n+1}\right)^{p_n} \\ &\le \sum_{n=0}^{3m} \left(\frac{\|T-A\|}{n+1}\right)^{p_n} + \sum_{n=m}^{\infty} \left(\frac{s_{n+2m}(T-A)}{n+2m+1}\right)^{p_{n+2m}} \\ &\le 3\sum_{n=0}^{m} \left(\frac{\|T-A\|}{n+1}\right)^{p_n} + \sum_{n=m}^{\infty} \left(\frac{s_n(T)}{n+1}\right)^{p_n} < \varepsilon. \end{aligned} \quad (31)$$

As $I_7 \in S_{(\mathscr{H}((0)))_\rho}(X, Y)$ but the condition $p_0 > 0$ is not satisfied which implies a negative example of the converse statement. This completes the proof.

From Theorem 21, we can say that if $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$, then each compact operators can be approximated by finite rank operators, and the converse is not always true.

4.3. Banach and Closed Prequasi Ideal. For which space $\mathscr{H}_\rho$ are the elements of prequasi operator ideal $S_{\mathscr{H}_\rho}$ complete?

Theorem 22. *If $\mathscr{H}_\rho$ is a premodular (ssfps), then the function $g(P) = \rho(f_s)$ is a prequasi norm on $S_{\mathscr{H}_\rho}$, where $f_s(z) = \sum_{n=0}^{\infty} s_n(P) z^n$ converges for any $z \in \mathbb{C}$.*

Proof. Let $\mathscr{H}_\rho$ be a premodular (ssfps), so g satisfies the following conditions:

(1) Suppose $P \in S_{\mathscr{H}_\rho}(X, Y)$, $g(P) = \rho(f_s) \ge 0$, and $g(P) = \rho(f_s) = 0$, if and only if, $s_n(P) = 0$, for all $n \in \mathrm{N}$, if and only if, $P = 0$

(2) One has $l \ge 1$ with $g(\lambda P) = \rho(\lambda f_s) \le l|\lambda| \rho(f_s) = l|\lambda| g(P)$, for all $P \in S_{\mathscr{H}_\rho}(X, Y)$ and $\lambda \in \mathbb{R}$

(3) We have $KK_0 \ge 1$ so that for $P_1, P_2 \in S_{\mathscr{H}_\rho}(X, Y)$. Hence, $f1_s(z) = \sum_{n=0}^{\infty} s_n(P_1) z^n$ and $f2_s(z) = \sum_{n=0}^{\infty} s_n(P_2) z^n$ converge for any $z \in \mathbb{C}$. Hence, for $h_s(z) =: \sum_{n=0}^{\infty} s_n(P_1 + P_2) z^n$, we have

$$\begin{aligned} g(P_1 + P_2) = \rho(h_s) &\le \rho\left((f1_s)_{[.]} + (f2_s)_{[.]}\right) \le K\left(\rho\left((f1_s)_{[.]}\right) \right. \\ &\left. + \rho\left((f2_s)_{[.]}\right)\right) \le KK_0(g(P_1) + g(P_2)). \end{aligned} \quad (32)$$

(4) There are $C \ge 1$, let $A \in L(X_0, X)$, $B \in S_{\mathscr{H}_\rho}(X, Y)$, and $D \in L(Y, Y_0)$. Hence, $f_s(z) = \sum_{n=0}^{\infty} s_n(B) z^n$ converges for all $z \in \mathbb{C}$. Then, for $h_s(z) =: \sum_{n=0}^{\infty} s_n(DBA) z^n$, one has

$$g(DBA) = \rho(h_s) \le \rho(\|A\| \|D\| f_s) \le C \|A\| g(B) \|D\|. \quad (33)$$

Theorem 23. *Suppose X and Y be Banach spaces, and $\mathscr{H}_\rho$ be a premodular (ssfps), then $(S_{\mathscr{H}_\rho}, g)$ be a prequasi Banach operator ideal, where $g(P) = \rho(f_s)$ and $f_s(z) = \sum_{n=0}^{\infty} s_n(P) z^n$ converge for any $z \in \mathbb{C}$.*

Proof. Since $\mathscr{H}_\rho$ is a premodular (ssfps), then the function $g(P) = \rho(f_s)$ is a prequasi norm on $S_{\mathscr{H}_\rho}$. Assume (P_m) be a Cauchy sequence in $S_{\mathscr{H}_\rho}(X, Y)$. Hence, $f_s^{(m)} \in \mathscr{H}_\rho$ and $f_s^{(m)}(z) = \sum_{n=0}^{\infty} s_n(P_m) z^n$ converge for any $z \in \mathbb{C}$. Let $h_s(z) =: \sum_{n=0}^{\infty} s_n(P_i - P_j) z^n$, then by using conditions (iv) and (vii) of Definition 13 and as $L(X, Y) \supseteq S_{\mathscr{H}_\rho}(X, Y)$, one has

$$\begin{aligned} g(P_i - P_j) = \rho(h_s) &\ge \rho\left(s_0(P_i - P_j) e^{(0)}\right) = \rho\left(\|P_i - P_j\| e^{(0)}\right) \\ &\ge \xi \|P_i - P_j\| \rho\left(e^{(0)}\right), \end{aligned} \quad (34)$$

hence, $(P_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $L(X, Y)$. But the space $L(X, Y)$ is a Banach space, so one has $P \in L(X, Y)$ with $\lim_{m \longrightarrow \infty} \|P_m - P\| = 0$ and since $f_s^{(m)} \in \mathscr{H}_\rho$, for all $m \in \mathbb{N}$. Therefore, from Theorem 22 and the continuously of ρ at θ, one gets

$$g(P) = g(P - P_m + P_m) \le KK_0(g(P_m - P) + g(P_m)) = KK_0\rho\left(\|P_m - P\| \sum_{m=0}^{\infty} e^{(m)}\right) + KK_0\rho\left(f_s^{(m)}\right) < \varepsilon, \quad (35)$$

then $f_s \in \mathscr{H}_\rho$, this gives $P \in S_{\mathscr{H}_\rho}(X, Y)$.

Corollary 24. *Suppose X and Y be Banach spaces, and $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$, then $S_{\mathscr{H}_w((p_n))}$ be a prequasi Banach operator ideal.*

Theorem 25. *Assume X and Y be Banach spaces and $\mathscr{H}_\rho$ be a premodular (ssfps), then $(S_{\mathscr{H}_\rho}, g)$ is a prequasi closed operator ideal, where $g(P) = \rho(f_s)$, where $f_s(z) = \sum_{n=0}^{\infty} s_n(P) z^n$ converges for any $z \in \mathbb{C}$.*

Proof. Since $\mathscr{H}_\rho$ is a premodular (ssfps), then the function $g(P) = \rho(f_s)$ is a prequasi norm on $S_{\mathscr{H}_\rho}$. Assume $P_m \in S_{\mathscr{H}_\rho}(X, Y)$, with $m \in \mathbb{N}$ and $\lim_{m\longrightarrow\infty} g(P_m - P) = 0$. Hence, $f_s^{(m)} \in \mathscr{H}_\rho$ and $f_s^{(m)}(z) = \sum_{n=0}^{\infty} s_n(P_m) z^n$ converge for any $z \in \mathbb{C}$. Let $h_s(z) =: \sum_{n=0}^{\infty} s_n(P_i - P_j) z^n$, then by using conditions (iv) and (vii) of Definition 13 and as $L(X, Y) \supseteq S_{\mathscr{H}_\rho}(X, Y)$, one obtains

$$g(P - P_j) = \rho(h_s) \ge \rho\left(s_0(P - P_j) e^{(0)}\right) = \rho\left(\|P - P_j\| e^{(0)}\right) \ge \xi \|P - P_j\| \rho\left(e^{(0)}\right), \quad (36)$$

hence, $(P_m)_{m\in\mathbb{N}}$ is a convergent sequence in $L(X, Y)$. But the space $L(X, Y)$ is a Banach space, so one gets $P \in L(X, Y)$ with $\lim_{m\longrightarrow\infty} \|P_m - P\| = 0$ and since $f_s^{(m)} \in \mathscr{H}_\rho$, for all $m \in \mathbb{N}$, so from Theorem 22 and the continuously of ρ at θ, we have

$$g(P) = g(P - P_m + P_m) \le KK_0(g(P_m - P) + g(P_m)) = KK_0\rho\left(\|P_m - P\| \sum_{m=0}^{\infty} e^{(m)}\right) + KK_0\rho\left(f_s^{(m)}\right) < \varepsilon, \quad (37)$$

then $f_s \in \mathscr{H}_\rho$, this implies $P \in S_{\mathscr{H}_\rho}(X, \mathrm{Y})$.

Corollary 26. *Suppose X and Y be Banach spaces and $(p_n) \in mi_\nearrow \cap \ell_\infty$ with $p_0 > 0$, then $S_{\mathscr{H}_w((p_n))}$ be a prequasi closed operator ideal.*

In view of Theorem 10, we explain the next properties of the s-type $(\mathscr{H}_w((p_n)))_\rho$.

Theorem 27. *For s-type $(\mathscr{H}_w((p_n)))_\rho := \{(s_n(T)) \in \mathbb{R}^{\mathbb{N}} : T \in S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y)\}$. The following conditions are satisfied:*

(1) One has s-type $(\mathscr{H}_w((p_n)))_\rho \supset \mathscr{F}$

(2) Assume $(s_r(T_1))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_w((p_n)))_\rho$ and $(s_r(T_2))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_w((p_n)))_\rho$, then $(s_r(T_1 + T_2))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_w((p_n)))_\rho$

(3) We have $\lambda \in \mathbb{R}$ and $(s_r(T))_{r=0}^{\infty} \in$ s-type $(\mathscr{H}_w((p_n)))_\rho$, then $|\lambda|(s_r(T))_{r=0}^{\infty} \in$ s- ype $(\mathscr{H}_w((p_n)))_\rho$

(4) The s-type $(\mathscr{H}_w((p_n)))_\rho$ is solid

4.4. Small Prequasi Banach Ideal. We offer here a few inclusion relations for the space $S_{(\mathscr{H}_w((p_n)))_\rho}$ for distinct (p_n).

Theorem 28. *Suppose X and Y be Banach spaces with $\dim(X) = \dim(Y) = \infty$ and $(p_n), (q_n) \in mi_\nearrow \cap \ell_\infty$ with $p_0 > 0$ and $p_n < q_n$, for every $n \in \mathbb{N}$, one has*

$$S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y) \subsetneqq S_{(\mathscr{H}_w((q_n)))_\rho}(X, Y) UL(X, Y). \quad (38)$$

Proof. Let $T \in S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y)$. Hence, $f_s \in (\mathscr{H}_w((p_n)))_\rho$ and $f_s(z) = \sum_{n=0}^{\infty} s_n(T) z^n$ converge for any $z \in \mathbb{C}$. We have

$$\sum_{n=0}^{\infty} \left(\frac{s_n(T)}{n+1}\right)^{q_n} < \sum_{n=0}^{\infty} \left(\frac{s_n(T)}{n+1}\right)^{p_n} < \infty, \quad (39)$$

so $T \in S_{(\mathscr{H}_w((q_n)))_\rho}(X, Y)$. After, by choosing T with $s_n(T) = (n+1)^{1-1/p_n}$, hence $T \notin S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y)$ and $T \in S_{(\mathscr{H}_w((q_n)))_\rho}(X, Y)$. Obviously, $S_{(\mathscr{H}_w((q_n)))_\rho}(X, Y) \subset L(X, Y)$. Next, by taking T with $s_n(T) = (n+1)^{1-1/q_n}$, then $T \notin S_{(\mathscr{H}_w((q_n)))_\rho}(X, Y)$ and $T \in L(X, Y)$. This completes the proof.

In this part, we explore the conditions for which $S^{\mathrm{app}}_{(\mathscr{H}_w((p_n)))_\rho}$ is small.

Theorem 29. *Assume X and Y be Banach spaces with $\dim(X) = \dim(Y) = \infty$. Suppose $(p_n) \in mi_\nearrow \cap (0, 1]^{\mathbb{N}}$, then $S^{app}_{(\mathscr{H}_w((p_n)))_\rho}$ be small.*

Proof. Evidently, the space $(S^{\mathrm{app}}_{(\mathscr{H}_w((p_n)))_\rho}, g)$ constructs a prequasi Banach operator ideal, with $g(T) = \sum_{j=0}^{\infty} (\alpha_j(T)/(j+1))^{p_j}$. Assume $S^{\mathrm{app}}_{(\mathscr{H}_w((p_n)))_\rho}(X, Y) = L(X, Y)$. Therefore, one has $C > 0$ with $g(T) \le C\|T\|$, for all $T \in L(X, Y)$. From Dvoretzky's Theorem [28] with $r \in \mathbb{N}$, we have quotient spaces X/λ_r and subspaces η_r of Y that transformed onto ℓ_2^r by isomorphisms D_r and B_r with $\|D_r\|\|D_r^{-1}\| \leq 2$ and $\|B_r\|\|B_r^{-1}\| \leq 2$. Assume I_r be the identity operator on ℓ_2^r, ζ_r be the quotient operator from X onto X/λ_r, and J_r be the natural embedding operator from η_r into Y. Let h_a be the Bernstein numbers [29], one obtains

$$\begin{aligned} 1 = h_a(I_r) &= h_a\left(B_r B_r^{-1} I_r D_r D_r^{-1}\right) \leq \|B_r\| h_a\left(B_r^{-1} I_r D_r\right)\left\|D_r^{-1}\right\| \\ &= \|B_r\| h_a\left(J_r B_r^{-1} I_r D_r\right)\left\|D_r^{-1}\right\| \leq \|B_r\| d_a\left(J_r B_r^{-1} I_r D_r\right)\left\|D_r^{-1}\right\| \\ &= \|B_r\| d_a\left(J_r B_r^{-1} I_r D_r \zeta_r\right)\left\|D_r^{-1}\right\| \leq \|B_r\| \alpha_a\left(J_r B_r^{-1} I_r D_r \zeta_r\right)\left\|D_r^{-1}\right\|, \end{aligned} \tag{40}$$

for $0 \leq j \leq r$. Hence, for $l \geq 1$, we have

$$\begin{aligned} \left(\frac{1}{j+1}\right)^{p_j} &\leq \left(\|B_r\|\left\|D_r^{-1}\right\|\right)^{p_j}\left(\frac{\alpha_j\left(J_r B_r^{-1} I_r D_r \zeta_r\right)}{j+1}\right)^{p_j} \Rightarrow \left(\frac{1}{j+1}\right)^{p_j} \\ &\leq l\|B_r\|\left(\frac{\alpha_j\left(J_r B_r^{-1} I_r D_r \zeta_r\right)}{j+1}\right)^{p_j}\left\|D_r^{-1}\right\| \Rightarrow \sum_{j=0}^{r}\left(\frac{1}{j+1}\right)^{p_j} \\ &\leq l\|B_r\|\left\|D_r^{-1}\right\| \sum_{j=0}^{r}\left(\frac{\alpha_j\left(J_r B_r^{-1} I_r D_r \zeta_r\right)}{j+1}\right)^{p_j} \Rightarrow \sum_{j=0}^{r}\left(\frac{1}{j+1}\right)^{p_j} \\ &\leq l\|B_r\|\left\|D_r^{-1}\right\| g\left(J_r B_r^{-1} I_r D_r \zeta_r\right) \Rightarrow \sum_{j=0}^{r}\left(\frac{1}{j+1}\right)^{p_j} \\ &\leq lC\|B_r\|\left\|D_r^{-1}\right\|\left\|J_r B_r^{-1} I_r D_r \zeta_r\right\| \Rightarrow \sum_{j=0}^{r}\left(\frac{1}{j+1}\right)^{p_j} \\ &\leq lC\|B_r\|\left\|D_r^{-1}\right\|\left\|J_r B_r^{-1}\right\|\|I_r\|\|D_r \zeta_r\| \\ &= lC\|B_r\|\left\|D_r^{-1}\right\|\left\|B_r^{-1}\right\|\|I_r\|\|D_r\| \Rightarrow \sum_{j=0}^{r}\left(\frac{1}{j+1}\right)^{p_j} \leq 4lC. \end{aligned} \tag{41}$$

Since $r \longrightarrow \infty$, then $\sum_{j=0}^{\infty}(1/(j+1))^{p_j} < \infty$. As $\sum_{j=0}^{\infty}(1/(j+1))^{p_j} \geq \sum_{j=0}^{\infty} 1/(j+1) = \infty$. Therefore, the space $S^{app}_{(\mathscr{H}_w((p_n)))_\rho}$ is small.

Theorem 30. *Suppose X and Y be Banach spaces with* dim $(X) =$ dim $(Y) = \infty$. *Let $(p_n) \in mi_{\nearrow} \cap (0,1]^{\mathbb{N}}$, then $S^{Kol}_{(\mathscr{H}_w((p_n)))_\rho}$ be small.*

4.5. Simple Prequasi Ideal. We declare an answer of the following investigation; for which $(\mathscr{H}_w((p_n)))_\rho$ is the space $S_{(\mathscr{H}_w((p_n)))_\rho}$ simple?

Theorem 31. *Suppose $(p_n), (q_n) \in mi_{\nearrow} \cap \ell_\infty$ with $1 \leq p_n < q_n$, for all $n \in \mathbb{N}$, hence*

$$L\left(S_{(\mathscr{H}_w((q_n)))_\rho}, S_{(\mathscr{H}_w((p_n)))_\rho}\right) = \Lambda\left(S_{(\mathscr{H}_w((q_n)))_\rho}, S_{(\mathscr{H}_w((p_n)))_\rho}\right). \tag{42}$$

Proof. Assume we have $T \in L(S_{(\mathscr{H}_w((q_n)))_\rho}, S_{(\mathscr{H}_w((p_n)))_\rho})$ and $T \notin \Lambda(S_{(\mathscr{H}_w((q_n)))_\rho}, S_{(\mathscr{H}_w((p_n)))_\rho})$. From Lemma 4, there are $G \in L(S_{(\mathscr{H}_w((q_n)))_\rho})$ and $B \in L(S_{(\mathscr{H}_w((p_n)))_\rho})$ with $BTGI_m = I_m$. For all $m \in \mathbb{N}$, we have

$$\begin{aligned} \|I_m\|_{S_{(\mathscr{H}_w((p_n)))_\rho}} &= \left(\sum_{n=0}^{\infty}\left(\frac{s_n(I_m)}{n+1}\right)^{p_n}\right)^{1/\sup p_n} = \left(\sum_{n=0}^{m-1}\left[\frac{1}{n+1}\right]^{p_n}\right)^{1/\sup p_n} \\ &\leq \|BTG\|\|I_m\|_{S_{(\mathscr{H}_w((q_n)))_\rho}} \leq \left(\sum_{n=0}^{\infty}\left(\frac{s_n(I_m)}{n+1}\right)^{q_n}\right)^{1/\sup q_n} \\ &= \left(\sum_{n=0}^{m-1}\left[\frac{1}{n+1}\right]^{q_n}\right)^{1/\sup q_n}. \end{aligned} \tag{43}$$

This defies Theorem 28.

Corollary 32. *Assume $(p_n), (q_n) \in mi_{\nearrow} \cap \ell_\infty$ with $1 \leq p_n < q_n$, for all $n \in \mathbb{N}$, hence*

$$L\left(S_{(\mathscr{H}_w((q_n)))_\rho}, S_{(\mathscr{H}_w((p_n)))_\rho}\right) = L_C\left(S_{(\mathscr{H}_w((q_n)))_\rho}, S_{(\mathscr{H}_w((p_n)))_\rho}\right). \tag{44}$$

Proof. Definitely from $\Lambda \subseteq L_c$.

Theorem 33. *Suppose $(p_n) \in mi_{\nearrow} \cap \ell_\infty$ with $p_0 \geq 1$, hence $S_{(\mathscr{H}_w((p_n)))_\rho}$ is simple.*

Proof. Let $T \in L_C(S_{(\mathscr{H}_w((p_n)))_\rho})$ and $T \notin \Lambda(S_{(\mathscr{H}_w((p_n)))_\rho})$. By using Lemma 4, there are $G, B \in L(S_{(\mathscr{H}_w((p_n)))_\rho})$ such that $BTGI_k = I_k$. One obtains $I_{S_{(\mathscr{H}_w((p_n)))_\rho}} \in L_C(S_{(\mathscr{H}_w((p_n)))_\rho})$. Hence, $L(S_{(\mathscr{H}_w((p_n)))_\rho}) = L_C(S_{(\mathscr{H}_w((p_n)))_\rho})$. This gives a unique nontrivial closed ideal $\Lambda(S_{(\mathscr{H}_w((p_n)))_\rho})$ in $L(S_{(\mathscr{H}_w((p_n)))_\rho})$.

4.6. Spectrum of Prequasi Ideal. In this part, we give the sufficient conditions on $(\mathscr{H}_w((p_n)))_\rho$ such that the class L with sequence of eigenvalues in $(\mathscr{H}_w((p_n)))_\rho$ is strictly contained in $S_{(\mathscr{H}_w((p_n)))_\rho}$.

Notations 34.

$$\begin{aligned} \left(S_{\mathscr{H}_\rho}\right)^\lambda &:= \left\{\left(S_{\mathscr{H}_\rho}\right)^\lambda(X,Y)\,; X \text{ and } Y \text{ are Banach Spaces}\right\}, \text{where } \left(S_{\mathscr{H}_\rho}\right)^\lambda(X,Y) \\ &:= \{T \in L(X,Y) \colon f_\lambda \in \mathscr{H}_\rho, \text{where}, f_\lambda(z) \\ &= \sum_{n=0}^{\infty}\lambda_n(T) z^n \text{ converges for any } z \in \mathbb{C} \text{ and } \|T - \lambda_l(T) I\| \\ &= 0, \text{for every } l \in \mathbb{N}\}. \end{aligned} \tag{45}$$

Theorem 35. *Pick up any Banach spaces X and Y with* dim $(X) =$ dim $(Y) = \infty$. *Assume $(p_n) \in mi_{\nearrow} \cap \ell_\infty$ with $p_0 > 0$, one has*

$$\left(S_{(\mathscr{H}_w((p_n)))_\rho}\right)^\lambda (X, Y) US_{(\mathscr{H}_w((p_n)))_\rho}(X, Y). \tag{46}$$

Proof. Suppose $T \in (S_{(\mathscr{H}_w((p_n)))_\rho})^\lambda(X, Y)$, therefore $f_\lambda \in (\mathscr{H}_w((p_n)))_\rho$, where $f_\lambda(z) = \sum_{n=0}^\infty \lambda_n(T) z^n$ converges for every $z \in \mathbb{C}$ with $\rho(f_\lambda) = \sum_{n=0}^\infty (|\lambda_n(T)|/(n+1))^{p_n} < \infty$, and $\|T - \lambda_l(T)I\| = 0$, for every $l \in \mathbb{N}$. One can see $T = \lambda_l(T)I$, with $l \in \mathbb{N}$, hence $s_l(T) = s_l(\lambda_l(T)I) = |\lambda_l(T)|$, with $l \in \mathbb{N}$. As a result, $f_s \in (\mathscr{H}_w((p_n)))_\rho$, hence $T \in S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y)$. Finally, by taking $s_n(T) = (n+1)^{1-(2/p_n)}$ and $\lambda_n(T) = (n+1)^{1-(1/p_n)}$. Therefore, $T \notin (S_{(\mathscr{H}_w((p_n)))_\rho})^\lambda(X, Y)$ and $T \in S_{(\mathscr{H}_w((p_n)))_\rho}(X, Y)$. This finishes the proof.

5. Weighted Shift Operators on $(\mathscr{H}_w((p_n)))_\rho$

In this section, we offer the upper bounds of s-numbers for infinite series of the weighted n-th power forward shift operator on $\mathscr{H}_w((p_n))$ with applications to some entire functions.

Definition 36. If $\mathscr{H}_\rho$ is a prequasi normed (ssfps). An operator $V_z : \mathscr{H}_\rho \longrightarrow \mathscr{H}_\rho$ is named forward shift operator if $V_z f = zf$, for every $f \in \mathscr{H}_\rho$, where $V_z f(z) = \sum_{n=0}^\infty \widehat{f}_n z^{n+1}$ converges for any $z \in \mathbb{C}$ and $\rho(V_z f) < \infty$.

Definition 37. If $\mathscr{H}_\rho$ is a prequasi normed (ssfps). An operator $B_z : \mathscr{H}_\rho \longrightarrow \mathscr{H}_\rho$ is named backward shift operator if $B_z f(z) = (f(z) - f(0))/z$, for every $f \in \mathscr{H}_\rho$, where $B_z f(z) = \sum_{n=0}^\infty \widehat{f}_{n+1} z^n$ converges for all $z \in \mathbb{C}$ and $\rho(B_z f) < \infty$.

Theorem 38. *If $(p_n) \in mi_\nearrow \cap \ell_\infty$ with $p_0 > 0$, hence $V_z \in L((\mathscr{H}_w((p_n)))_\rho)$ with*

$$\sup_r \left(\frac{(r+1)^{p_r}}{(r+2)^{p_{r+1}}}\right)^{1/\varpi_p} \leq \|V_z\| \leq \sup_r \left(\frac{r+1}{r+2}\right)^{p_r/\varpi_p}, \tag{47}$$

where $\rho(f) = [\sum_{r=0}^\infty |\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, for each $f \in (\mathscr{H}_w((p_n)))_\rho$.

Proof. Assume the setup be verified. For $f \in (\mathscr{H}_w((p_n)))_\rho$. As $(p_n) \in mi_\nearrow \cap \ell_\infty$ with $p_0 > 0$, hence

$$\begin{aligned} \rho(V_z f) = \rho(zf) &= \left[\sum_{r=0}^\infty \left|\frac{\widehat{f}_r}{r+2}\right|^{p_{r+1}}\right]^{1/\varpi_p} \leq \left[\sum_{r=0}^\infty \left|\frac{\widehat{f}_r}{r+2}\right|^{p_r}\right]^{1/\varpi_p} \\ &\leq \sup_r \left(\frac{r+1}{r+2}\right)^{p_r/\varpi_p} \left[\sum_{r=0}^\infty \left|\frac{\widehat{f}_r}{r+1}\right|^{p_r}\right]^{1/\varpi_p} \\ &= \sup_r \left(\frac{r+1}{r+2}\right)^{p_r/\varpi_p} \rho(f). \end{aligned} \tag{48}$$

Then, $V_z \in L((\mathscr{H}_w((p_n)))_\rho)$ with $\|V_z\| \leq \sup_r((r+1)/(r+2))^{p_r/\varpi_p}$. As $V_z \in L((\mathscr{H}_w((p_n)))_\rho)$. Hence, one has $A > 0$ so that $\rho(V_z f) \leq A\rho(f)$, for every $f \in (\mathscr{H}_w((p_n)))_\rho$. Therefore, $\rho(V_z e^{(r)}) \leq A\rho(e^{(r)})$, then $\sup_r((r+1)^{p_r}/(r+2)^{p_{r+1}})^{1/\varpi_p} \leq \|V_z\|$. This finishes the proof.

Theorem 39. *If $(p_n) \in mi_\nearrow \cap \ell_\infty$ with $p_0 > 0$, hence $B_z \in L((\mathscr{H}_w((p_n)))_\rho)$ with*

$$\sup_r \left(\frac{(r+2)^{p_{r+1}}}{(r+1)^{p_r}}\right)^{1/\varpi_p} \leq \|B_z\| \leq \sup_r \left(\frac{r+2}{r+1}\right)^{p_r/\varpi_p}, \tag{49}$$

where $\rho(f) = [\sum_{r=0}^\infty |\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, for any $f \in (\mathscr{H}_w((p_n)))_\rho$.

Proof. Assume the setup be verified. For $f \in (\mathscr{H}_w((p_n)))_\rho$. As $(p_n) \in mi_Z \cap \ell_\infty$ with $p_0 > 0$, hence

$$\begin{aligned} \rho(B_z f) &= \left[\sum_{r=0}^\infty \left|\frac{\widehat{f_{r+1}}}{r+1}\right|^{p_r}\right]^{1/\varpi_p} \leq \sup_r \left(\frac{r+2}{r+1}\right)^{p_r/\varpi_p} \left[\sum_{r=0}^\infty \left|\frac{\widehat{f_{r+1}}}{r+2}\right|^{p_r}\right]^{1/\varpi_p} \\ &\leq \sup_r \left(\frac{r+2}{r+1}\right)^{p_r/\varpi_p} \left[\sum_{r=0}^\infty \left|\frac{\widehat{f}_r}{r+1}\right|^{p_r}\right]^{1/\varpi_p} = \sup_r \left(\frac{r+2}{r+1}\right)^{p_r/\varpi_p} \rho(f). \end{aligned} \tag{50}$$

Then, $B_z \in L((\mathscr{H}_w((p_n)))_\rho)$ with $\|B_z\| \leq \sup_r((r+2)/(r+1))^{p_r/\varpi_p}$. As $B_z \in L((\mathscr{H}_w((p_n)))_\rho)$. Hence, one has $A > 0$ so that $\rho(B_z f) \leq A\rho(f)$, for every $f \in (\mathscr{H}_w((p_n)))_\rho$. Therefore, $\rho(B_z e^{(r+1)}) \leq A\rho(e^{(r+1)})$, so $\sup_r((r+2)^{p_{r+1}}/(r+1)^{p_r})^{1/\varpi_p} \leq \|B_z\|$. This finishes the proof.

Theorem 40. *If $(p_n) \in mi_\nearrow \cap \ell_\infty$ with $p_0 \geq 1$. Assume $\limsup_{n\longrightarrow\infty}(1/(n+1))^{p_n/n} = 1$, hence all function in $(\mathscr{H}_w((p_n)))_\rho$ is analytic on the open unit disc $\mathbb{D}$. More, the convergence in $(\mathscr{H}_w((p_n)))_\rho$ gives the uniform convergence on compact subsets of $\mathbb{D}$, where $\rho(f) = [\sum_{r=0}^\infty |\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, for any $f \in (\mathscr{H}_w((p_n)))_\rho$.*

Proof. Assume $\limsup_{n\longrightarrow\infty}(1/(n+1))^{p_n/n} = 1$, and $f \in (\mathscr{H}_w((p_n)))_\rho$. Hence, $f(z) = \sum_{n=0}^\infty \widehat{f}_n z^n$ converges for any $z \in \mathbb{C}$ and $\rho(f) = [\sum_{n=0}^\infty |\widehat{f}_n/(n+1)|^{p_n}]^{1/\varpi_p} < \infty$. So, $\limsup_{n\longrightarrow\infty} \sqrt[n]{|\widehat{f}_n/(n+1)|^{p_n}} < 1$. One has

$$\limsup_{n\longrightarrow\infty} \sqrt[n]{\left|\widehat{f}_n\right|^{p_n}} < \frac{1}{\limsup_{n\longrightarrow\infty}(1/(n+1))^{p_n/n}} = 1. \tag{51}$$

As $(p_n) \in mi_{\mathrm{Z}} \cap \ell_\infty$ with $p_0 \geq 1$, one gets $\limsup_{n\longrightarrow\infty} \sqrt[n]{|\widehat{f_n}|}|z| < |z| < 1$, for every $z \in \mathbb{D}$. Then, $f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n$ converges for any complex value of $z \in \mathbb{D}$. Suppose A be a compact subset of $\mathbb{D}$, and $f^k(z) \in A$, for every $k \in \mathbb{N}$. Assume f^k converges to $f \in (\mathscr{H}_w((p_n)))_\rho$, one obtains

$$\begin{aligned} \left|f^k(z) - f(z)\right| &= \left|\sum_{n=0}^{\infty}\left(\widehat{f_n^k} - \widehat{f_n}\right)z^n\right| \leq \sum_{n=0}^{\infty}\left|\widehat{f_n^k} - \widehat{f_n}\right||z^n| \\ &\leq \left[\sum_{n=0}^{\infty}\left|\frac{\widehat{f_n^k} - \widehat{f_n}}{n+1}\right|^{p_n}\right]^{1/\varpi_p}\left[\sum_{n=0}^{\infty}(n+1)^{q_n}|z|^{nq_n}\right]^{1/\varpi_p} \\ &= \left[\sum_{n=0}^{\infty}(n+1)^{q_n}|z|^{nq_n}\right]^{1/\varpi_p}\rho\left(f^k - f\right), \end{aligned} \tag{52}$$

where $(q_n) \in mi_{\mathrm{Z}} \cap \ell_\infty$ with $q_0 \geq 1$ and $1/p_n + 1/q_n = 1$, for every $n \in \mathbb{N}$. Obviously, $\limsup_{n\longrightarrow\infty}(n+1)^{q_n/n}|z|^{q_n} < 1$, hence $[\sum_{n=0}^{\infty}(n+1)^{q_n}|z|^{nq_n}]^{1/\varpi_q} < \infty$. Then, $\lim_{k\longrightarrow\infty} f^k(z) = f(z) \in A$.

Theorem 41. *Suppose V_z be the forward shift operator on $(\mathscr{H}_w((p_n)))_\rho$, with $\rho(f) = [\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, for every $f \in (\mathscr{H}_w((p_n)))_\rho$. One has*

$$\begin{aligned} &\sup_{\text{card } \xi = r+1}\inf_{k\in\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p}\frac{1}{A_n} \\ &\quad\leq s_r(V_z^n) \leq \sup_{\text{card } \xi = r+1}\inf_{k\in\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p}, \end{aligned} \tag{53}$$

where $A_n = [\sum_{k=0}^{\infty}|\widehat{f_k}/(k+1)|^{p_k}]^{1/\varpi_p}/[\sum_{k\in\xi}|\widehat{f_k}/(k+1)|^{p_{k+n}}]^{1/\varpi_p}$.

Proof. Assume card $\xi = r+1$ and since $V_z^n f \in (\mathscr{H}_w((p_n)))_\rho$, for every $f \in (\mathscr{H}_w((p_n)))_\rho$, where $f(z) = \sum_{k=0}^{\infty}\widehat{f_k}z^k$ converges for any $z \in \mathbb{C}$ and $\rho(f) = [\sum_{k=0}^{\infty}|\widehat{f_k}/(k+1)|^{p_k}]^{1/\varpi_p} < \infty$. Therefore, $V_z^n f(z) = \sum_{k=0}^{\infty}\widehat{f_k}z^{k+n}$ and $\rho(V_z^n f) = [\sum_{k=0}^{\infty}|\widehat{f_k}/(k+n+1)|^{p_{k+n}}]^{1/\varpi_p} < \infty$.

Suppose P_ξ be an operator on $(\mathscr{H}_w((p_n)))_\rho$ with rank $P_\xi = r+1$ evident by

$$\left(P_\xi g\right)(z) = P_\xi\left(\sum_{k=0}^{\infty}\widehat{f_k}z^{k+n}\right) = \sum_{k\in\xi}\widehat{f_k}z^{k+n}. \tag{54}$$

As $\rho(P_\xi g) = [\sum_{k\in\xi}|\widehat{f_k}/(k+n+1)|^{p_{k+n}}]^{1/\varpi_p} \leq [\sum_{k=0}^{\infty}|\widehat{f_k}/(k+n+1)|^{p_{k+n}}]^{1/\varpi_p} = \rho(g)$. This implies $\|P_\xi\| \leq 1$. Define an operator S_z^n by $(S_z^n h)(z) = S_z^n(\sum_{k\in\xi}\widehat{f_k}z^{k+n}) = \sum_{k=0}^{\infty}\widehat{f_k}z^k$, one gets

$$\rho(S_z^n h) = \left[\sum_{k=0}^{\infty}\left|\frac{\widehat{f_k}}{k+1}\right|^{p_k}\right]^{1/\varpi_p} \leq U_n\left[\sum_{k\in\xi}\left|\frac{\widehat{f_k}}{k+n+1}\right|^{p_{k+n}}\right]^{1/\varpi_p} = U_n\rho(g). \tag{55}$$

Therefore, $\|S_z^n\| \leq U_n$, where $1 \leq U_n = [\sum_{k=0}^{\infty}|\widehat{f_k}/(k+1)|^{p_k}]^{1/\varpi_p}/[\sum_{k\in\xi}|\widehat{f_k}/(k+n+1)|^{p_{k+n}}]^{1/\varpi_p} < \infty$. Hence, the identity operator will be $I_{r+1} = P_\xi V_z^n S_z^n$, in view of the definition of s-numbers, one has

$$\begin{aligned} s_r(I_{r+1}) = 1 &\leq \left\|P_\xi\right\|s_r(V_z^n)\|S_z^n\| \leq s_r(V_z^n)\|S_z^n\| \Rightarrow s_r(V_z^n) \\ &\geq \frac{1}{\|S_z^n\|} \geq \frac{1}{U_n} = \frac{\left[\sum_{k\in\xi}\left|\widehat{f_k}/(k+n+1)\right|^{p_{k+n}}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty}\left|\widehat{f_k}/(k+1)\right|^{p_k}\right]^{1/\varpi_p}} \\ &\geq \inf_{k\in\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p}\frac{1}{A_n}. \end{aligned} \tag{56}$$

This inequality is confirmed for all card $\xi = r+1$, we have

$$s_r(V_z^n) \geq \sup_{\text{card } \xi = r+1}\inf_{k\in\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p}\frac{1}{A_n}. \tag{57}$$

On the other hand, suppose ξ be a subset of $\mathbb{N}$ with card $\xi = r$. Define the finite rank operator R_z^n by $(R_z^n v)(z) = R_z^n(\sum_{k=0}^{\infty}\widehat{f_k}z^k) = \sum_{k\in\xi}\widehat{f_k}z^{k+n}$. From the definition of approximation numbers, one gets

$$\begin{aligned} s_r(V_z^n) &\leq \alpha_r(V_z^n) \leq \|V_z^n - R_z^n\| \leq \sup_{|f(z)|\neq 0}\frac{|(V_z^n - R_z^n)f(z)|}{|f(z)|} \\ &= \sup_{|f(z)|\neq 0}\frac{\left|\sum_{k\notin\xi}\widehat{f_k}z^{k+n}\right|}{|f(z)|} \leq \sup_{|f(z)|\neq 0}\frac{\left[\sum_{k\notin\xi}\left|\widehat{f_k}/(k+n+1)\right|^{p_{k+n}}\right]^{1/\varpi_p}}{|f(z)|} \\ &\leq \sup_{k\notin\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p}. \end{aligned} \tag{58}$$

This inequality is satisfied for any card $\xi = r$ and from Lemma 11, we have

$$\begin{aligned} &\sup_{\text{card } \xi = r+1}\inf_{k\in\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p}\frac{1}{A_n} \leq s_r(V_z^n) \\ &\quad\leq \inf_{\text{card } \xi = r}\sup_{k\notin\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p} \\ &\quad= \sup_{\text{card } \xi = r+1}\inf_{k\in\xi}\left(\frac{k+1}{k+n+1}\right)^{p_{k+n}/\varpi_p}. \end{aligned} \tag{59}$$

This finishes the proof.

Here, the upper and lower bounds of norm $\sum_{m=0}^{\infty}c_m V_z^m$ on the space $(\mathscr{H}_w((p_n)))_\rho$ have been introduced.

Theorem 42. *Acting* $\sum_{m=0}^{\infty} c_m V_z^m$ *on the space* $(\mathscr{H}_w((p_n)))_\rho$, *where* $\rho(f)=[\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, *for every* $f\in(\mathscr{H}_w((p_n)))_\rho$, $(c_m)_{m=0}^{\infty}\in\ell^{(p_m)/\varpi_p}$, *and* $(p_n)\in mi_\nearrow\cap\ell_\infty$ *with* $p_0\geq 1$, *one has*

$$\begin{aligned}\sup_k\left[\sum_{m=0}^{\infty}|c_m|^{p_{m+k}}\frac{(k+1)^{p_k}}{(m+k+1)^{p_{m+k}}}\right]^{1/\varpi_p}&\leq\left\|\sum_{m=0}^{\infty}c_mV_z^m\right\|\\ &\leq\sup_{m,k}\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}.\end{aligned}\tag{60}$$

Proof. Let $f\in(\mathscr{H}_w((p_n)))_\rho$, one gets $\sum_{m=0}^{\infty}c_mV_z^mf(z)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}c_m\widehat{f}_kz^{k+m}$. We have

$$\begin{aligned}\left\|\sum_{m=0}^{\infty}c_mV_z^m\right\|&\geq\frac{\rho(\sum_{m=0}^{\infty}c_mV_z^me^{(k)})}{\rho(e^{(k)})}=\left[\frac{\sum_{m=0}^{\infty}|c_m/(m+k+1)|^{p_{m+k}}}{(1/(k+1))^{p_k}}\right]^{1/\varpi_p}\\ &\geq\sup_k\left[\sum_{m=0}^{\infty}|c_m|^{p_{m+k}}\frac{(k+1)^{p_k}}{(m+k+1)^{p_{m+k}}}\right]^{1/\varpi_p}.\end{aligned}\tag{61}$$

As ρ verifies the triangle inequality, one can see

$$\begin{aligned}\left\|\sum_{m=0}^{\infty}c_mV_z^m\right\|&=\sup_{\rho(f)\neq0}\frac{\rho(\sum_{m=0}^{\infty}c_mV_z^mf)}{\rho(f)}\\ &\leq\sup_{\rho(f)\neq0}\frac{\left[\sum_{m=0}^{\infty}\left[\sum_{k=0}^{\infty}\left(|c_m|\left|\widehat{f}_k/(m+k+1)\right|\right)^{p_{m+k}}\right]\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty}\left|\widehat{f}_k/(k+1)\right|^{p_k}\right]^{1/\varpi_p}}\\ &\leq\sup_{m,k}\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\frac{\sum_{m=0}^{\infty}\left[\sum_{k=0}^{\infty}\left(|c_m|\left|\widehat{f}_k/(k+1)\right|\right)^{p_{m+k}}\right]^{1/\varpi_p}}{\left[\sum_{k=0}^{\infty}\left|\widehat{f}_k/(k+1)\right|^{p_k}\right]^{1/\varpi_p}}\\ &\leq\sup_{m,k}\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}.\end{aligned}\tag{62}$$

The next theorem investigates an upper estimation to the s-numbers of $\sum_{m=0}^{\infty}c_mV_z^m$ on the space $(\mathscr{H}_w((p_n)))_\rho$.

Theorem 43. *Acting* $\sum_{m=0}^{\infty}c_mV_z^m$ *on the space* $(\mathscr{H}_w((p_n)))_\rho$, *where* $\rho(f)=[\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, *for every* $f\in(\mathscr{H}_w((p_n)))_\rho$, *the s – numbers of this operator are supposed by*

$$s_r\left(\sum_{m=0}^{\infty}c_mV_z^m\right)\leq\sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\sup_m\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p},\tag{63}$$

for all $(c_m)_{m=0}^{\infty}\in\ell^{(p_m)/\varpi_p}$ *and* $(p_n)\in mi_Z\cap\ell_\infty$ *with* $p_0\geq1$.

Proof. Assume ξ be a subset of $\mathbb{N}$ and card $\xi=r$. From the definition of s-numbers. Define the finite rank operator R by $Rf(z)=R(\sum_{k=0}^{\infty}\widehat{f}_kz^k)=\sum_{k\in\xi}\sum_{m=0}^{k}c_m\widehat{f_{k-m}}z^k$. From the definition of approximation numbers and as ρ verifies the triangle inequality, one has

$$\begin{aligned}s_r\left(\sum_{m=0}^{\infty}c_mV_z^m\right)&\leq\alpha_r\left(\sum_{m=0}^{\infty}c_mV_z^m\right)\leq\left\|\sum_{m=0}^{\infty}c_mV_z^m-R\right\|\\ &\leq\sup_{\rho(f)\neq0}\frac{\rho(\sum_{m=0}^{\infty}c_mV_z^mf-Rf)}{\rho(f)}\\ &\leq\sup_{\rho(f)\neq0}\frac{\sum_{m=0}^{\infty}\left[\sum_{k\notin\xi}\left|c_m\left(\widehat{f}_k/(k+m+1)\right)\right|^{p_{k+m}}\right]}{\rho(f)}\\ &\leq\sup_{k\notin\xi,m}\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}.\end{aligned}\tag{64}$$

This inequality is satisfied for all card $\xi=r$, we have

$$\begin{aligned}s_r\left(\sum_{m=0}^{\infty}c_mV_z^m\right)&\leq\inf_{\text{card }\xi=r}\sup_{k\notin\xi,m}\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}\\ &=\sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\sup_m\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\sum_{m=0}^{\infty}|c_m|^{p_m/\varpi_p}.\end{aligned}\tag{65}$$

This performs the proof.

Definition 44. By using the power series of an entire function $g(z)=\sum_{m=0}^{\infty}a_mz^m$, the shift operator $V_{g(z)}$ is defined as:

$$V_{g(z)}(f(z))=\left(\sum_{m=0}^{\infty}a_mV_z^m\right)(f(z)).\tag{66}$$

The next theorem implies a direct actions of the last theorem for some entire functions, for example, the exponential and the sine functions.

Corollary 45. *Assume* $(p_n)\in mi_\nearrow\cap\ell_\infty$ *with* $p_0\geq1$. *Suppose* V_{e^z} *be shift operator on* $(\mathscr{H}_w((p_n)))_\rho$, *where* $\rho(f)=[\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, *for every* $f\in(\mathscr{H}_w((p_n)))_\rho$ *and* $e^z=\sum_{m=0}^{\infty}z^m/m!$. *The upper estimation of the s-numbers of* V_{e^z} *is pretended by*

$$s_r(V_{e^z})\leq\sup_{\text{card }\xi=r+1}\inf_{k\in\xi}\sup_m\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p}\sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)^{p_m/\varpi_p}.\tag{67}$$

Proof. Clearly, from Theorem 43.

Corollary 46. *Suppose* $(p_n)\in mi_\nearrow\cap\ell_\infty$ *with* $p_0\geq1$. *Assume* $V_{\sin(z)}$ *be shift operator on* $(\mathscr{H}_w((p_n)))_\rho$, *where* $\rho(f)=[\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, *for every* $f\in(\mathscr{H}_w((p_n)))_\rho$ *and* sin

$(z) = \sum_{m=0}^{\infty}(-1)^m(z^{2m+1}/(2m+1)!)$. *The upper estimation of the s-numbers of* $V_{\sin(z)}$ *is presented by*

$$s_r\left(V_{\sin(z)}\right) \le \sup_{\operatorname{card}\xi=r+1} \inf_{k\in\xi} \sup_{m}\left(\frac{k+1}{m+k+1}\right)^{p_{m+k}/\varpi_p} \sum_{m=0}^{\infty}\left(\frac{1}{(2m+1)!}\right)^{p_m/\varpi_p}. \tag{68}$$

Proof. Obviously, from Theorem 43.

6. Caristi's Fixed Point Theorem in $(\mathscr{H}_w((p_n)))_\rho$

The Ekeland variational principle (EVP) can not examine in modular spaces as the modular does not satisfy the triangle inequality. In this section in view of Farkas [30], we offer a development of Caristi's fixed point theorem in $(\mathscr{H}_w((p_n)))_\rho$.

Definition 47.

(a) The prequasi normed (ssfps) ρ on $(\mathscr{H}_w((p_n)))_\rho$ is named ρ-convex, if $\rho(\omega v + (1-\omega)t) \le \omega\rho(v) + (1-\omega)\rho(t)$, for each $\omega \in [0,1]$ and $v, t \in (\mathscr{H}_w((p_n)))_\rho$

(b) $\{v^{(a)}\}_{a\in\mathbb{N}} \subseteq (\mathscr{H}_w((p_n)))_\rho$ is ρ-convergent to $v \in (\mathscr{H}_w((p_n)))_\rho$, if and only if, $\lim_{a\longrightarrow\infty}\rho(v^{(a)} - v) = 0$. If the ρ-limit exists, then it is unique

(c) $\{v^{(a)}\}_{a\in\mathbb{N}} \subseteq (\mathscr{H}_w((p_n)))_\rho$ is ρ-Cauchy, if $\lim_{a,b\longrightarrow\infty}\rho(v^{(a)} - v^{(b)}) = 0$

(d) $Y \subset (\mathscr{H}_w((p_n)))_\rho$ is ρ-closed, if for every ρ-converging $\{u^{(a)}\}_{a\in\mathbb{N}} \subset Y$ to u, then $u \in Y$

(e) $Y \subset (\mathscr{H}_w((p_n)))_\rho$ is ρ-bounded, if $\delta_\rho(Y) = \sup\{\rho(v-t): v, t \in Y\} < \infty$

(f) The ρ-ball of radius $d \ge 0$ and center v, for all $v \in (\mathscr{H}_w((p_n)))_\rho$, is defined as:

$$\mathscr{B}_\rho(v,d) = \left\{t \in (\mathscr{H}_w((p_n)))_\rho : \rho(v-t) \le d\right\}. \tag{69}$$

(g) A prequasi normed (ssfps) ρ on $\mathscr{H}_w((p_n))$ verifies the Fatou property, if for every sequence $\{t^{(a)}\} \subseteq (\mathscr{H}_w((p_n)))_\rho$ with $\lim_{a\longrightarrow\infty}\rho(t^{(a)} - t) = 0$ and every $v \in (\mathscr{H}_w((p_n)))_\rho$ hence

$$\rho(v-t) \le \sup_j \inf_{a\ge j}\rho\left(v - t^{(a)}\right). \tag{70}$$

Contemplate that the Fatou property gives the ρ-closedness of the ρ-balls.

Theorem 48. *The function* $\rho(f) = [\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, *for all* $f \in (\mathscr{H}_w((p_n)))_\rho$, *verifies the Fatou property when* $(p_n) \in mi_{\nearrow} \cap \ell_\infty$ *with* $p_0 > 0$.

Proof. Let $\{f^{(i)}\} \subseteq (\mathscr{H}_w((p_n)))_\rho$ with $\lim_{i\longrightarrow\infty}\rho(f^{(i)} - f) = 0$. As the space $(\mathscr{H}_w((p_n)))_\rho$ is a prequasi closed space, hence $f \in (\mathscr{H}_w((p_n)))_\rho$. So for all $g \in (\mathscr{H}_w((p_n)))_\rho$, we have

$$\begin{aligned}\rho(g-f) &= \left[\sum_{a=0}^{\infty}\left|\frac{\widehat{g_a} - \widehat{f_a}}{a+1}\right|^{p_a}\right]^{1/\varpi_p} \\ &\le \left[\sum_{a=0}^{\infty}\left|\frac{\widehat{g_a} - \widehat{f_a^{(i)}}}{a+1}\right|^{p_a}\right]^{1/\varpi_p} + \left[\sum_{a=0}^{\infty}\left|\frac{\widehat{f_a^{(i)}} - \widehat{f_a}}{a+1}\right|^{p_a}\right]^{1/\varpi_p} \\ &\le \sup_j \inf_{i\ge j}\rho\left(g - f^{(i)}\right).\end{aligned} \tag{71}$$

Theorem 49. *The function* $\rho(f) = \sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}$, *for all* $f \in (\mathscr{H}_w((p_n)))_\rho$, *does not verify the Fatou property, when* $(p_n) \in mi_{\nearrow} \cap \ell_\infty$ *with* $p_0 > 1$.

Proof. Suppose $\{f^{(i)}\} \subseteq (\mathscr{H}_w((p_n)))_\rho$ with $\lim_{i\longrightarrow\infty}\rho(f^{(i)} - f) = 0$. As the space $(\mathscr{H}_w((p_n)))_\rho$ is a prequasi closed space, hence $f \in (\mathscr{H}_w((p_n)))_\rho$. So for every $g \in (\mathscr{H}_w((p_n)))_\rho$, one gets

$$\begin{aligned}\rho(g-f) &= \sum_{a=0}^{\infty}\left|\frac{\widehat{g_a} - \widehat{f_a}}{a+1}\right|^{p_a} \\ &\le 2^{\sup_a p_a - 1}\left[\sum_{a=0}^{\infty}\left|\frac{\widehat{g_a} - \widehat{f_a^{(i)}}}{a+1}\right|^{p_a} + \sum_{a=0}^{\infty}\left|\frac{\widehat{f_a^{(i)}} - \widehat{f_a}}{a+1}\right|^{p_a}\right] \\ &\le 2^{\sup_a p_a - 1}\sup_j \inf_{i\ge j}\rho\left(g - f^{(i)}\right).\end{aligned} \tag{72}$$

Therefore, ρ does not verify the Fatou property.

Example 50. The function $\rho(f) = [\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p_r}]^{1/\varpi_p}$, for every $f \in (\mathscr{H}_w((p_n)))_\rho$, is a prequasi normed (ssfps), not quasi normed (ssfps), and not a normed (ssfps), for all $f \in (\mathscr{H}_w((p_n)))_\rho$.

Example 51. The function $\rho(f) = [\sum_{r=0}^{\infty}|\widehat{f}_r/(r+1)|^{p}]^{1/p}$, for every $f \in \mathscr{H}((b_n),(p))_\rho$, is a prequasi normed (ssfps), quasi normed (ssfps), and not a normed (ssfps) on $(\mathscr{H}_w((p)))_\rho$, for $0 < p < 1$.

Example 52. The function $\rho(f)=\inf\{\kappa>0:\sum_{r=0}^{\infty}|\widehat{f}_r/(\kappa(r+1))|^{p_r}\leq 1\}$, is a prequasi normed (ssfps), a quasi normed (ssfps), and a normed (ssfps) on $\mathscr{H}_w((p_n))$.

Definition 53. Suppose $g:(\mathscr{H}_w((p_n)))_\rho \longrightarrow (-\infty,\infty]$ be a function. The function g is named lower semicontinuous at $f^{(0)}\in(\mathscr{H}_w((p_n)))_\rho$ if $\liminf_{f\longrightarrow f^{(0)}} g(f)=g(f^{(0)})$, where $\liminf_{f\longrightarrow f^{(0)}} g(f)=\sup_{V\in\mathscr{V}(f^{(0)})}\inf_{f\in V} g(f)$, where $\mathscr{V}(f^{(0)})$ is a neighborhood system of $f^{(0)}$.

Definition 54. Assume $g:(\mathscr{H}_w((p_n)))_\rho \longrightarrow (-\infty,\infty]$ be a function. The function g is named proper, if

$$\mathscr{D}(g)=\left\{f\in(\mathscr{H}_w((p_n)))_\rho : g(f)<\infty\right\}\neq\varnothing. \tag{73}$$

Theorem 55. *Pick up $Y\neq\varnothing$ and Y be a ρ-closed subset of $(\mathscr{H}_w((p_a)))_\rho$, with $\rho(f)=[\sum_{a=0}^{\infty}|\widehat{f}_a/(a+1)|^{p_a}]^{1/\varpi_p}$, for every $f\in(\mathscr{H}_w((p_a)))_\rho$, and $g:Y\longrightarrow(-\infty,\infty]$ be a proper, ρ-lower semicontinuous function with $\inf_{f\in Y} g(f)>-\infty$. Suppose $\gamma>0$, $\{\beta_a\}\subset(0,\infty)$, and $f^{(0)}\in Y$ with $g(f^{(0)})\leq\inf_{f\in Y} g(f)+\gamma$. Therefore, one has $\{f^{(a)}\}\in Y$ which ρ-converges to some $f^{(\gamma)}$, with*

(i) *$\rho(f^{(\gamma)}-f^{(a)})\leq\gamma/2^a\beta_0$, for all $a\in\mathbb{N}$*

$$g\left(f^{(\gamma)}\right)+\sum_{a=0}^{\infty}\beta_a\rho\left(f^{(\gamma)}-f^{(a)}\right)\leq g\left(f^{(0)}\right), \tag{74}$$

(ii) *if $f\neq f^{(\gamma)}$, one has*

$$\begin{aligned} g\left(f^{(\gamma)}\right)&+\sum_{a=0}^{\infty}\beta_a\rho\left(f^{(\gamma)}-f^{(a)}\right)\\ &<g(f)+\sum_{a=0}^{\infty}\beta_a\rho\left(f-f^{(a)}\right). \end{aligned} \tag{75}$$

Proof. Place $S(f^{(0)})=\{f\in Y: g(f)+\beta_0\rho(f-f^{(0)})\leq g(f^{(0)})\}$. As $f^{(0)}\in S(f^{(0)})$, hence $S(f^{(0)})\neq\varnothing$. Since g is ρ-lower semicontinuous, ρ verifies the Fatou property, and Y is ρ-closed, one has $S(f^{(0)})$ is ρ-closed. Choose $f^{(1)}\in S(f^{(0)})$ with

$$\begin{aligned} g\left(f^{(1)}\right)&+\beta_0\rho\left(f^{(1)}-f^{(0)}\right)\\ &\leq\inf_{f\in S\left(f^{(0)}\right)}\left\{g(f)+\beta_0\rho\left(f-f^{(0)}\right)\right\}+\frac{\gamma\beta_1}{2\beta_0}. \end{aligned} \tag{76}$$

After place

$$\begin{aligned} S\left(f^{(1)}\right)=\Bigg\{f\in S\left(f^{(0)}\right)&: g(f)+\sum_{j=0}^{1}\beta_j\rho\left(f-f^{(j)}\right)\\ &\leq g\left(f^{(1)}\right)+\beta_0\rho\left(f^{(1)}-f^{(0)}\right)\Bigg\}. \end{aligned} \tag{77}$$

Similar to $S(f^{(0)})$, one has $S(f^{(1)})\neq\varnothing$ and ρ-closed. Assume that one has built $\{f^{(0)},f^{(1)},f^{(2)},,f^{(a)}\}$ and $\{S(f^{(0)}),S(f^{(1)}),S(f^{(2)}),,S(f^{(a)})\}$. Next that, choose $f^{(a+1)}\in S(f^{(a)})$ with

$$\begin{aligned} g\left(f^{(a+1)}\right)&+\sum_{j=0}^{a}\beta_j\rho\left(f^{(a+1)}-f^{(j)}\right)\\ &\leq\inf_{f\in S\left(f^{(a)}\right)}\left\{g(f)+\sum_{j=0}^{a}\beta_j\rho\left(f-f^{(j)}\right)\right\}+\frac{\gamma\beta_a}{2^a\beta_0}. \end{aligned} \tag{78}$$

Let

$$\begin{aligned} S\left(f^{(a+1)}\right)\coloneqq\Bigg\{f\in S\left(f^{(a)}\right)&: g(f)+\sum_{j=0}^{a+1}\beta_j\rho\left(f-f^{(j)}\right)\\ &\leq g\left(f^{(a+1)}\right)+\sum_{j=0}^{a}\beta_j\rho\left(f^{(a+1)}-f^{(j)}\right)\Bigg\}. \end{aligned} \tag{79}$$

Hence, we form the sequences $\{f^{(a)}\}$ and $\{S(f^{(a)})\}$ by induction. For fixed $a\in\mathbb{N}$. Suppose $z\in S(f^{(a)})$. We have

$$g(z)+\sum_{j=0}^{a}\beta_j\rho\left(z-f^{(j)}\right)\leq g\left(f^{(a)}\right)+\sum_{j=0}^{a-1}\beta_j\rho\left(f^{(a)}-f^{(j)}\right), \tag{80}$$

hence

$$\begin{aligned} \beta_a\rho\left(z-f^{(a)}\right)&\leq g\left(f^{(a)}\right)+\sum_{j=0}^{a-1}\beta_j\rho\left(f^{(a)}-f^{(j)}\right)\\ &\quad-\left[g(z)+\sum_{j=0}^{a-1}\beta_j\rho\left(z-f^{(j)}\right)\right]\\ &\leq g\left(f^{(a)}\right)+\sum_{j=0}^{a-1}\beta_j\rho\left(f^{(a)}-f^{(j)}\right)\\ &\quad-\inf_{f\in S\left(f^{(a-1)}\right)}\left[g(f)+\sum_{j=0}^{a-1}\beta_j\rho\left(f-f^{(j)}\right)\right]\\ &\leq\frac{\gamma\beta_a}{2^a\beta_0}. \end{aligned} \tag{81}$$

As $\{S(f^{(a)})\}$ is decreasing with $f^{(a)} \in S(f^{(a)})$, for every $n \in \mathbb{N}$, we obtain

$$\rho\left(f^{(a+h)} - f^{(a)}\right) \leq \frac{\gamma}{2^a \beta_0}, \tag{82}$$

for all $a, h \in \mathbb{N}$. This implies $\{f^{(a)}\}$ is ρ-Cauchy. As $(\mathscr{H}_w((p_a)))_\rho$ is ρ-Banach space. Hence, $\{f^{(a)}\}$ has ρ-limits $f^{(\gamma)}$, and $\bigcap_{a\in\mathbb{N}} S(f^{(a)}) = \{f^{(\gamma)}\}$ verifies. Since $f^{(a+1)} \in S(f^{(a)})$, we can see

$$g\left(f^{(a+1)}\right) + \sum_{j=0}^{a} \beta_j \rho\left(f^{(a+1)} - f^{(j)}\right) \leq g\left(f^{(a)}\right) + \sum_{j=0}^{a-1} \beta_j \rho\left(f^{(a)} - f^{(j)}\right), \tag{83}$$

this gives $\{g(f^{(a)}) + \sum_{j=0}^{a-1} \beta_j \rho(f^{(a)} - f^{(j)})\}$ is decreasing. Next, suppose $f \neq f^{(\gamma)}$. Hence, we have $m \in \mathbb{N}$ with $f \notin S(f^{(a)})$, for all $a \geq m$, i.e.,

$$g\left(f^{(a)}\right) + \sum_{j=0}^{a-1} \beta_j \rho\left(f^{(a)} - f^{(j)}\right) < g(f) + \sum_{j=0}^{a} \beta_j \rho\left(f - f^{(j)}\right). \tag{84}$$

Since $f^{(\gamma)} \in S(f^{(a)})$, with $a \geq m$, we get

$$\begin{aligned} g\left(f^{(\gamma)}\right) &+ \sum_{j=0}^{a} \beta_j \rho\left(f^{(\gamma)} - f^{(j)}\right) \\ &\leq g\left(f^{(a)}\right) + \sum_{j=0}^{a-1} \beta_j \rho\left(f^{(a)} - f^{(j)}\right) \\ &\leq g\left(f^{(m)}\right) + \sum_{j=0}^{m-1} \beta_j \rho\left(f^{(m)} - f^{(j)}\right). \end{aligned} \tag{85}$$

Put $a \longrightarrow \infty$ in the previous inequality, we have

$$\begin{aligned} g\left(f^{(\gamma)}\right) &+ \sum_{j=0}^{\infty} \beta_j \rho\left(f^{(\gamma)} - f^{(j)}\right) \\ &\leq g(x_m) + \sum_{j=0}^{m-1} \beta_j \rho\left(f^{(m)} - f^{(j)}\right) \\ &< g(f) + \sum_{j=0}^{m} \beta_j \rho\left(f - f^{(j)}\right) \\ &\leq g(f) + \sum_{j=0}^{\infty} \beta_j \rho\left(f - f^{(j)}\right). \end{aligned} \tag{86}$$

This gives

$$\begin{aligned} g\left(f^{(\gamma)}\right) &+ \sum_{a=0}^{\infty} \beta_a \rho\left(f^{(\gamma)} - f^{(a)}\right) \\ &< g(f) + \sum_{a=0}^{\infty} \beta_a \rho\left(f - f^{(a)}\right). \end{aligned} \tag{87}$$

This completes the proof.

In view of Theorem 55, we present a generalization of Caristi's fixed point theorem in $(\mathscr{H}_w((p_a)))_\rho$.

Theorem 56. *Suppose $Y \neq \varnothing$ and Y be a ρ-closed subset of $(\mathscr{H}_w((p_a)))_\rho$, with $\rho(f) = [\sum_{a=0}^{\infty} |\widehat{f}_a/(a+1)|^{p_a}]^{1/\varpi_p}$, for every $f \in (\mathscr{H}_w((p_a)))_\rho$. Let $\gamma > 0$ and $\{\beta_n\}$ with $0 < \chi = \sum_{n=0}^{\infty} \beta_n < \infty$. Assume $T : Y \longrightarrow Y$ be a mapping and one has a function $g : Y \longrightarrow (-\infty, \infty]$ which is a proper and ρ-lower semicontinuous with $\inf_{f\in Y} g(f) > -\infty$ and*

(1) $\rho(T(f) - h) - \rho(f - h) \leq \rho(T(f) - f)$, for any $f, h \in Y$

(2) $\rho(T(f) - f) \leq g(f) - g(T(f))$, for any $f \in Y$

Therefore, T has a fixed point in Y.

Proof. As $0 < \chi = \sum_{n=0}^{\infty} \beta_n < \infty$, one has $g^* \coloneqq \chi g$ is also proper, ρ-lower semicontinuous, and bounded from below. Suppose $f \in Y$, we have

$$\chi \rho(T(f) - f) \leq g^*(f) - g^*(T(f)). \tag{88}$$

Since $\inf_{f\in Y} g^*(f) > -\infty$, one obtains $f^{(0)} \in Y$ with $g^*(f^{(0)}) < \inf_{f\in Y} g^*(f) + \gamma$. From Theorem 55, there is $\{f^{(a)}\}$ which ρ-converges to some $f^{(\gamma)} \in Y$, with

$$g^*\left(f^{(\gamma)}\right) + \sum_{a=0}^{\infty} \beta_a \rho\left(f^{(\gamma)} - f^{(a)}\right) < g^*(f) + \sum_{a=0}^{\infty} \beta_a \rho\left(f - f^{(a)}\right), \tag{89}$$

for every $f \neq f^{(\gamma)}$. Assume that $T(f^{(\gamma)}) \neq f^{(\gamma)}$, we have

$$\begin{aligned} g^*\left(f^{(\gamma)}\right) &+ \sum_{a=0}^{\infty} \beta_a \rho\left(f^{(\gamma)} - f^{(a)}\right) \\ &< g^*\left(T\left(f^{(\gamma)}\right)\right) + \sum_{a=0}^{\infty} \beta_a \rho\left(T\left(f^{(\gamma)}\right) - f^{(a)}\right), \end{aligned} \tag{90}$$

then

$$\begin{aligned} &g^*\left(f^{(\gamma)}\right) - g^*\left(T\left(f^{(\gamma)}\right)\right) \\ &\quad < \sum_{a=0}^{\infty} \beta_a \rho\left(T\left(f^{(\gamma)}\right) - f^{(a)}\right) - \sum_{a=0}^{\infty} \beta_a \rho\left(f^{(\gamma)} - f^{(a)}\right) \\ &\quad = \sum_{a=0}^{\infty} \beta_a \left(\rho\left(T\left(f^{(\gamma)}\right) - f^{(a)}\right) - \rho\left(f^{(\gamma)} - f^{(a)}\right)\right). \end{aligned} \tag{91}$$

By using setting (6), we get

$$\begin{aligned} g^*\left(f^{(\gamma)}\right) - g^*\left(T\left(f^{(\gamma)}\right)\right) &< \sum_{a=0}^{\infty} \beta_a \rho\left(T\left(f^{(\gamma)}\right) - f^{(\gamma)}\right) \\ &= \chi\rho\left(T\left(f^{(\gamma)}\right) - f^{(\gamma)}\right). \end{aligned} \tag{92}$$

The inequality 5 implies that

$$\begin{aligned} \chi\rho\left(T\left(f^{(\gamma)}\right) - f^{(\gamma)}\right) &\le g^*\left(f^{(\gamma)}\right) - g^*\left(T\left(f^{(\gamma)}\right)\right) \\ &< \chi\rho\left(T\left(f^{(\gamma)}\right) - f^{(\gamma)}\right). \end{aligned} \tag{93}$$

This is a disagreement. Therefore, $T(f^{(\gamma)}) = f^{(\gamma)}$. This finishes the proof.

Data Availability

No data were used.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Hyers-Ulam Stability and Existence Criteria for the Solution of Second-Order Fuzzy Differential Equations

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In this paper, existence, uniqueness, and Hyers-Ulam stability for the solution of second-order fuzzy differential equations (FDEs) are studied. To deal a physical model, it is required to insure whether unique solution of the model exists. The natural transform has the speciality to converge to both Laplace and Sumudu transforms only by changing the variables. Therefore, this method plays the rule of checker on the Laplace and Sumudu transforms. We use natural transform to obtain the solution of the proposed FDEs. As applications of the established results, some nontrivial examples are provided to show the authenticity of the presented work.

1. Introduction

Zadeh [1] introduced the concept of fuzziness in the set theory. The complexity of uncertainty as ambiguity in a real-life scenario is dealt properly with fuzzy set theory. The mathematical tool of fuzzy set theory deals with uncertainty in the real-life problem in a better way. The fuzzy set theory has suggestions for nonclassical and higher order fuzzy sets for different specialized purposes. In this direction, Chang and Zadeh used fuzzy sets and initiate the concept of fuzzy mapping and control [2]. The work of many researchers on fuzzy mappings and control puts the foundation of elementary fuzzy calculus. For detail, see [3–7]. For the last two decades, the valuable interest of fuzzy integral and differential equations has extended classical calculus to modern fuzzy calculus. Fuzzy differential and integral equations are the well-equipped mathematical tools to deal properly with physical models in the fuzzy environments. The solutions of every fuzzy differential and integral equation do not exist. Therefore, some strategy is required to insure whether the solution of FDEs exists or not. The existence theory is one of the best research areas in the field of fuzzy differential equations (FDEs). Before dealing a physical model, it is important to know whether its solution exists. Now, if unique solution of a physical model exists, then a physical model is dealt properly. The existence of a unique solution of fuzzy differential equation properties of differentiable fuzzy mappings was studied by Kaleva [5]. Liu and Liu [8] introduced self-duality credibility measure for the measurement of fuzzy event. Moreover, they studied the existence of unique solutions of FDEs. The existence and uniqueness result for FDEs with linear growth and Lipschitz conditions was discussed by Fei et al. [9]. The uniqueness result for the FDEs with non-Lipschitz coefficients was investigated by Chen and Qin [10] for more detail (see [6, 11, 12]).

Stability analysis of differential equations (DEs) is another most important and remarkable area in the qualitative theory. The stability of DEs has been studied by various researchers with different concepts like Lyapunov stability, asymptotic stability, and Ulam stability. The first effort of Ulam stability was initiated by Ulam [13], and just after one year, Hyers [14] studied the stability of the linear functional equation known as Hyers-Ulam stability. Oblaza [15] proved the Hyers-Ulam stability of linear DEs; for further detail of Hyers-Ulam, see [16–19]. Shen [20] investigated the Ulam stability of first-order linear FDEs.

The Laplace and Sumudu transforms are commonly used for the solutions of differential equations. The natural transform introduced by Khan and Khan [21] has speciality to converge to both Laplace and Sumudu transforms only by changing the variables. Therefore, the natural transform method plays the rule of checker on the Laplace and Sumudu transforms. The applications of the natural transform method turn out to be well for solutions of differential equations, see [21–23].

The aim of this work is to study the existence, uniqueness, and Hyers-Ulam stability of second-order FDEs. For this purpose, the corresponding second-order FDEs are reduced to equivalent systems of fuzzy integral equations. Using the concept of Hukuhara generalized differentiability, existence, uniqueness, and Hyers-Ulam stability of the equivalent system of integral equations are discussed. We use the natural transform method to solve second-order FDEs. The last Hyers-Ulam stability of the numerical problem is discussed. Two nontrivial examples are given to show the authenticity of the presented work.

2. Preliminaries

Here, some basic results are provided from the existing literature.

Definition 1 (see [24]). Let $s : R \to [0, 1]$ satisfy the conditions, where $y_0, y_1, y_2, z \in R$ (set of real numbers)

(i) s is upper semicontinuous

(ii) $s(\sigma y_1 + (1-\sigma)y_2) \geq \min\{s(y_1), s(y_2)\}$

(iii) $\exists y_0 \in R$ such that $\sigma(y_0) = 1$

(iv) $cl\{z \in R, \sigma(z) > 0\}$ is compact

Then, s is a fuzzy number.

Throughout this paper, $\widehat{F}_R$ represent the set that contains all fuzzy numbers.

Definition 2 (see [25]). The fuzzy number can be written as $(\underline{k}(q), \overline{k}(q))$ in order pair form, with $0 \leq q \leq 1$, and holds the following conditions:

(i) Nondecreasing bounded function, $\underline{s}(q)$ is left-continuous in $[0, 1]$

(ii) Nonincreasing bounded function, $\overline{s}(q)$ is left-continuous in $[0, 1]$

(iii) $\underline{s}(q) \leq \overline{s}(q)$

q is a crisp nunmber when $\underline{s}(q) = \overline{s}(q) = q$.

Theorem 3. *Let the mapping $F : Y \to Y$ is contractive with $k > 1$, where (Y, d) is a generalized complete metric space. If for $z \in Y$, $d(F^{n+1}(z), F^n(z)) < \infty$, with $n \geq 0$ and $F(z_0) = z_0 \in Y$, then*

(i) $F^n(z)$ converge to z_0

(ii) The point z_0 is in $X_0 = \{p \in Y \mid d(F_n(z), p) < \infty\}$

(iii) If $p \in Y$, then

$$d(p, z_0) \leq \frac{1}{1-k} d(F(p), p) \tag{1}$$

Definition 4. The mapping $d : Y_1 \times Y_1 \to R^+$ is a generalized metric on Y_1 if and only if, $\forall x, y$ and $z \in Y_1$.

(F_1) $d(x, y) = 0$ if and only if $x = y$

(F_2) $d(x, y) = d(x, y)$

(F_3) $d(x, z) \leq d(x, y) + d(y, z)$. Then, (Y_1, d) is a metric space

Definition 5 (see [26]). The mapping $d_H : \widehat{F}_R \times \widehat{F}_R \to R^+$, defined by

$$\begin{aligned} d_H(x, y) &= \sup_{a_1 \in [0,1]} d(x(a_1), y(a_1)) \\ &= \sup_{a_1 \in [0,1]} \max\left\{\left|\underline{x}(a_1) - \underline{y}(a_1)\right|, \left|\overline{x}(a_1) - \overline{y}(a_1)\right|\right\}. \end{aligned} \tag{2}$$

The pair $(d_H, \widehat{F}_R)$ is a generalized complete metric space. Moreover, $\forall x_1, x_2, x_3, x_4 \in \widehat{F}_R$ and $\alpha \in R+$ hold

(D_1) $d_H(x_1 + x_3, x_2 + x_3) = d_H(x_1, x_2)$

(D_2) $d_H(\alpha x_1, \alpha x_2) = \alpha d_H(x_1, x_2)$

(D_3) $d_H(x_1 + x_4, x_2 + x_3) \leq d_H(x_1, x_2) + d_H(x_4, x_3)$

Definition 6 (see [27]). The fuzzy function, $\chi : (a, b) \to \widehat{F}_R$, at $y \in (a, b)$, is generalized Hukuhara differentiable if there exist $\chi'(y) \in \widehat{F}_R$ such that

(i) For $r > 0$, sufficiently small, the Hukuhara difference, $\chi(y+r) \ominus \chi(y)$, $\chi(y) \ominus \chi(y-r)$, and limits exist in the complete metric space $X \in \widehat{F}_R$

$$\lim_{r \to 0} \frac{\chi(y) \ominus \chi(y+r)}{r} = \lim_{r \to 0} \frac{\chi(y-r) \ominus \chi(y)}{r} = \chi'(y) \tag{3}$$

(ii) For $r > 0$, sufficiently small, the Hukuhara difference, $\chi(y) \ominus \chi(y+r)$, $\chi(y-r) \ominus \chi(y)$, and limits exist in the complete metric space $X \in \widehat{F}_R$

$$\lim_{r \to 0} \frac{\chi(y+r) \ominus \chi(y)}{(-r)} = \lim_{r \to 0} \frac{\chi(y) \ominus \chi(y-r)}{(-r)} = \chi'(y) \tag{4}$$

The first one is referred to (i)-differentiable and second one to (ii)-differentiable.

Definition 7 (see [27]). The fuzzy function, $\chi : (a, b) \to \widehat{F}_R$, at $y \in (a, b)$, is generalized Hukuhara second-order differentiable if there exist, $\chi''(y) \in \widehat{F}_R$ such that

(i) For $r > 0$, sufficiently small, the Hukuhara difference, $\chi'(y+r) \ominus \chi'(y)$, $\chi'(y) \ominus \chi'(y-r)$, and limits exist in the complete metric space $X \in \widehat{F}_R$

$$\lim_{r\to0}\frac{\chi'(y)\ominus\chi'(y+r)}{r}=\lim_{r\to0}\frac{\chi'(y-r)\ominus\chi'(y)}{r}=\chi''(y) \quad (5)$$

(ii) For $r>0$, sufficiently small, the Hukuhara difference, $\chi'(y)\ominus\chi'(y+r)$, $\chi'(y-r)\ominus\chi'(y)$, and limits exist in the complete metric space $X\in\widehat{F}_R$

$$\lim_{r\to0}\frac{\chi'(y+r)\ominus\chi'(y)}{(-r)}=\lim_{r\to0}\frac{\chi'(y)\ominus\chi'(y-r)}{(-r)}=\chi''(y) \quad (6)$$

The first one is referred to (i)-differentiable and second one to (ii)-differentiable.

Definition 8 (see [28]). The crisp set $\{e\in R|\mu(e)\geq\gamma\}$ is the γ -level set of fuzzy number $\mu\in\widehat{F}_R$. Moreover, the γ-level set is bounded and closed with upper and lower bond, $\bar{\mu}(q)$ and $\underline{\mu}(q)$, respectively, denoted by $(\underline{\mu}(q),\bar{\mu}(q))$.

Theorem 9 (see [29]). *Let a continuous fuzzy function $\mu:R\rightarrow F$, such that, $\mu(p)=(\underline{\mu}(p),\bar{\mu}(p))$, and $0\leq p\leq 1$:*

(i) $\mu(p)$ is (i)-differentiable, then $\mu'(p)=(\underline{\mu}'(p),\bar{\mu}'(p))$

(ii) $\mu(p)$ is (ii)-differentiable, then $\mu'(p)=(\bar{\mu}'(p),\underline{\mu}'(p))$

Corollary 10 (see [26]). *The continuous mapping $V:I\rightarrow\widehat{F}_R$ is integrable, where I is an interval.*

Theorem 11 (see [27]). *The generalized differentiable mapping $V:I\rightarrow\widehat{F}_R$ has integrable derivative V' on I.*

(i) When V is (i)-differentiable, $\forall x\in I$, we have

$$V(x)=V(t_0)+\int_{t0}^{x}(-1)V'(t)dt \quad (7)$$

(ii) When V is (ii)-differentiable, $\forall x\in I$, we have

$$V(x)=V(t_0)+\int_{t0}^{x}(-1)V'(t)dt \quad (8)$$

Definition 12 (see [21]). The natural transform of $g(t)$ is $R(s,p)$ given by the following formula:

$$R(s,p)=N\{g(t)\}=\int_0^\infty e^{-st}g(pt)dt, \quad (9)$$

where s and p are transform parameter which is real and positive.

Lemma 13 (see [22]). *The duality relation of natural and Laplace transform is given by*

$$R(s,p)=N\{g(t)\}=\frac{1}{p}\int_0^\infty e^{-t/p}g(t)dt=\frac{1}{p}L\{g(t)\}, \quad (10)$$

where L is the Laplace transform. The natural transform converts to Laplace transform by taking parameter $p=1$ and Sumudu transform by taking parameter $s=1$.

Definition 14 (see [23]). The natural transform of nth order derivative of $g(t)$ is

$$N\{g^n(t)\}=\frac{s^n}{p^n}R(s,p)-\sum_{k=0}^{n-1}\frac{s^{n-(k-1)}}{p^{n-(k-1)}}\frac{1}{u}g^k(0). \quad (11)$$

The natural transform of $g'(t)$and $g''(t)$ first- and second-order derivative of $g(t)$ is given by

$$\begin{aligned}N\left\{g'(t)\right\}&=\frac{s}{p}R(s,p)-\frac{1}{u}g(0),\\ N\left\{g''(t)\right\}&=\frac{s^2}{p^2}R(s,p)-\frac{s}{p^2}g(0)-\frac{1}{p}g'(0).\end{aligned} \quad (12)$$

3. Existence, Uniqueness, and Stability Analysis

Before dealing a physical model, it is important to know whether its solution exists. Now, if unique solution of a physical model exists, then the model is dealt properly. But if the solution is not unique then who is to say that the solution we found is what will actually happen in real life. Therefore, we need additional conditions to get unique solution. In this section, we carry out results for the existence of unique solutions of second-order fuzzy differential equations by using the contraction principle.

Let us consider FDEs

$$\begin{cases}u''(t)=F\left(t,u(t),u'(t)\right),\\ u(t_0)=u_0=(\underline{u}_0,\bar{u}_0),\\ u'(t_0)=u_0'=\left(\underline{u}_0',\bar{u}_0'\right),\end{cases} \quad (13)$$

where $Y=[t_0,t_0+T]$ and $F:Y\times\widehat{F_R}\times\widehat{F_R}\longrightarrow\widehat{F_R}$, and u_0, $u_0'\in\widehat{F}_R$.

Definition 15. The fuzzy-valued function, $u\in C(Y,\widehat{F}_R)$, is the solution of differential Equation (13) if $u(t)$ satisfies Equation (13).

Let $u\in C(Y,\widehat{F}R)$, which is the solution of differential Equation (13), we reduce Equation (13) to an equivalent system of fuzzy-valued integral equations. Let $v:Y\rightarrow F\hat{}R$, which is continuous fuzzy-valued function, such that $u^{\cdot}(t)=v(t)$; therefore, we deduce

$$\begin{cases} u'(t) = v(t), \text{with } u(t_0) = u_0 = (\underline{u}_0, \bar{u}_0), \\ v'(t) = F(t, u(t), v(t)), v(t_0) = u'_0 = \left(\underline{u}'_0, \bar{u}'_0\right). \end{cases} \tag{14}$$

Using Corollary 10, Theorem 11, and initial condition, then the following four systems of equations are obtained:

(1) If $u(t)$ and $u'(t) = v(t)$ are (i)-differentiable.

$$\begin{cases} u(t) = u_0 + \int_{t_0}^{t} v(s)ds, \\ v(t) = u'_0 + \int_{t_0}^{t} F(s, u(s), v(s))ds \end{cases} \tag{15}$$

(2) If $u(t)$ is (i)-differentiable and $u'(t) = v(t)$ is (ii)-differentiable.

$$\begin{cases} u(t) = u_0 + \int_{t_0}^{t} v(s)ds, \\ v(t) = u'_0 \ominus \int_{t_0}^{t} (-1)F(s, u(s), v(s))ds \end{cases} \tag{16}$$

(3) If $u(t)$ is (ii)-differentiable and $u'(t) = v(t)$ is (i)-differentiable.

$$\begin{cases} u(t) = u_0 \ominus \int_{t_0}^{t} (-1)v(s)ds, \\ v(t) = u'_0 + \int_{t_0}^{t} F(s, u(s), v(s))ds \end{cases} \tag{17}$$

(4) If $u(t)$ and $u'(t) = v(t)$ are (ii)-differentiable.

$$\begin{cases} u(t) = u_0 \ominus \int_{t_0}^{t} (-1)v(s)ds, \\ v(t) = u'_0 \ominus \int_{t_0}^{t} (-1)F(s, u(s), v(s))ds \end{cases} \tag{18}$$

Theorem 16. *Let the continuous fuzzy-valued function $u \in C(Y, \widehat{F}_R)$ and $\chi = C(Y, F\wedge_R)^2$ such that $H : \chi \times \chi \to R^+ \cup \{+\infty\}$, with*

$$H((u_1, v_1), (u_2, v_2)) = \inf \{N \in R^+ \cup \{+\infty\}: (d_H(u_1, u_2) + d_H(v_1, v_2))e^{-C(t-t_0)} \leq N\widehat{\phi}(t), \forall t \in Y\}, \tag{19}$$

where $\widehat{\phi} : \widehat{Y} \to (0, \infty)$ be continuous mapping; then, (χ, H) is a generalized complete metric space.

Proof. The first two conditions F_1 and F_2 are easy to show; therefore, we only show F_3. Assume that for every (u_1, v_1), (u_2, v_2), $(u_3, v_3) \in \chi$, there exist $t \in Y$, such that

$$\begin{aligned} H((u_1, v_1), (u_3, v_3)) &= (d_H(u_1, u_3) + d_H(v_1, v_3))e^{-C(t-t_0)} \\ &\leq (d_H(u_1, u_2) + d_H(u_2, u_3))e^{-C(t-t_0)} \\ &\quad + (d_H(v_1, v_2) + d_H(v_2, v_3))e^{-C(t-t_0)} \\ &= (d_H(u_1, u_2) + d_H(v_1, v_2))e^{-C(t-t_0)} \\ &\quad + (d_H(u_2, u_3) + d_H(v_2, v_3))e^{-C(t-t_0)}, \end{aligned}$$

$$H((u_1, v_1), (u_3, v_3)) \leq H((u_1, v_1), (u_2, v_2)) + H((u_2, v_2), (u_3, v_3)). \tag{20}$$

Hence, (χ, H) is a generalized metric space.

Now, we need to show that (χ, H) is complete. Let us consider the Cauchy sequence, (u_n, v_n) on (χ, H), then $H((u_n, v_n), (u_m, v_m) \leq \varepsilon$, for all $n, m \geq N(\varepsilon)$, using definition, (19), one can get

$$(d_H(u_n, u_m) + d_H(v_n, v_m))e^{-C(t-t_0)} \leq \epsilon\widehat{\phi}(t), \quad \forall n, m \geq N(\varepsilon), t \in Y. \tag{21}$$

Since $(\widehat{F}_R, d_H)$ is a complete metric space, then there is $\exists u(t), v(t)$: $Y \to \widehat{F}_R$, such that Cauchy sequence, $u_n(t)$ and $v_n(t)$ converges to $u(t)$ and $v(t)$, respectively.

When $m \to \infty$, then (21) produced

$$\begin{aligned} (d_H(u_n, u(t)) + d_H(v_n, v(t)))e^{-C(t-t_0)} &\leq \epsilon\widehat{\phi}(t), \quad \forall n, \geq N(\varepsilon), t \in Y, \\ H((u_n(t), v_n(t)), (u(t), v(t))) &\leq \varepsilon\widehat{\phi}(t), \quad \forall n, \geq N(\varepsilon), t \in Y. \end{aligned} \tag{22}$$

Hence, $(u_n(t), v_n(t))$ converge to $(u(t), v(t))$, in (χ, H). Therefore, (χ, H) is a complete generalized metric space.

Theorem 17. *The fuzzy problem (13) has a unique and Hyers-Ulam stable solution $\widehat{u}, \widehat{v} : Y \to \widehat{F}_R$, defined by*

$$\begin{cases} \widehat{u} = u_0 + \int_{t_0}^{t} v(s)ds, \\ \widehat{v} = u'_0 + \int_{t_0}^{t} F(s, u(s), v(s))ds. \end{cases} \tag{23}$$

If the following conditions hold for some nonnegative L, ε and $T = t - t_0$.

(i) $d_H(F(t,u_1,v_1), F(t,u_2,v_2) \le L d_H(u_1,u_2) + (L-1) d_H(v_1,v_2)$,

where $F: Y \times \widehat{F}_R \times \widehat{F}_R \to \widehat{F}_R$ *be fuzzy-valued continuous function* $\forall u_1, v_1, u_2, v_2 \in \widehat{F}_R$ *and* $t \in Y$

(ii) $d_H(Du(t), v(t)) + d_H(Dv(t), F(t, u(t), v(t))) \le \varepsilon$,

where $u, v : Y \to \widehat{F}_R$ *are (i)-differentiable*

(iii) $d_H(u, \widehat{u}) \le (1+L)\,\varepsilon\, T$ *and* $d_H(v, \widehat{v}) \le (1+L)\,\varepsilon\, T$

Proof. Let $C(Y, \widehat{F}_R) = \{u : Y \to \widehat{F}_R,$ be continuous fuzzy − valued function $\}$ and $\chi = C(Y, F\wedge_R)^2$ such that $H : \chi \times \chi \to R^+ \cup \{+\infty\}$, with

$$H((u_1,v_1),(u_2,v_2)) = \inf\{N \in R^+ \cup \{+\infty\}: (d_H(u_1,u_2) + d_H(v_1,v_2))e^{-(L+1)(t-t_0)} \le N\widehat{\phi}(t) \forall t \in Y\}, \tag{24}$$

where $\widehat{\phi} : \widehat{Y} \to (0,\infty)$ be continuous mapping, then (χ, H) is a generalized complete metric space, in view of Theorem 16. Let us define a self operator $G : \chi \to \chi$, $\forall (u,v) \in \chi$ and $t \in Y$, by

$$\begin{aligned} G(u(t),v(t)) &= (G(u),G(v)) \\ &= \left(u_0 + \int_{t_0}^t v(s)ds, u_0' + \int_{t_0}^t F(s,u(s),v(s)ds)ds\right). \end{aligned} \tag{25}$$

First, we show $G(u,v) \in \chi$. Let $(p_0, q_0) \in \chi$ and $t \in Y$,

$$\begin{aligned} &\{d_H(G(p_0),p_0) + d_H(G(q_0),q_0)\}e^{-(L+1)(t-t_0)} < \infty, \\ &H((G(p_0,q_0),(p_0,q_0)) < \infty. \end{aligned} \tag{26}$$

Hence, $G(u(t), v(t)) = (G(u), G(v))$ is well defined. Now, we can show for all $(u,v) \in \chi$, the operator G is strictly contractive on (χ, H).

$$\begin{aligned} &d_H(G(u_1),G(u_2)) + d_H(G(v_1),G(v_2)) \\ &= d_H\left(u_0 + \int_{t_0}^t v_1(s)ds, u_0 + \int_{t_0}^t v_2(s)ds\right) \\ &\quad + d_H\left(u_0' + \int_{t_0}^t F(s,u_1(s),v_1)(s)ds, u_0' + \int_{t_0}^t F(s,u_2(s),v_2)(s)ds\right) \\ &= d_H\left(\int_{t_0}^t v_1(s)ds, \int_{t_0}^t v_2(s)ds\right) \\ &\quad + d_H\left(\int_{t_0}^t F(s,u_1(s),v_1(s))ds, \int_{t_0}^t F(s,u_2(s),v_2(s))ds\right) \\ &\le L\int_{t_0}^t (d_H(u_1,u_2) + d_H(v_1,v_2))e^{-(L+1)(s-t_0)}e^{(L+1)(s-t_0)}ds \\ &= \frac{L}{L+1}H((u_1,v_1),(u_2,v_2))e^{(L+1)(t-t_0)}. \end{aligned} \tag{27}$$

To multiply $e^{-(L+1)(t-t_0)}$, on both sides of the above inequality, we have

$$\begin{aligned} &(d_H(G(u_1),G(u_2)) + d_H(G(v_1),G(v_2)))e^{-(L+1)(t-t_0)} \\ &\quad \le \frac{L}{L+1}H((u_1,v_1),(u_2,v_2)). \end{aligned} \tag{28}$$

Now, from the definition of H, for all $t \in Y$, one can get

$$H(G(u_1,v_1),G(u_2,v_2)) \le \frac{L}{L+1}H((u_1,v_1),(u_2,v_2)). \tag{29}$$

This indicates that G is strictly contractive on (χ, H). Therefore, there exist unique solution of problem (13).

The $u(t) \ominus u(t_0)$ and $v(t) \ominus u'(t_0)$, Hukuhara differences exist for $t \in Y$, where u and v are (i)-differentiable, then using condition (ii), we have

$$\begin{aligned} d_H(u,G(u)) + d_H(v,G(v)) &= d_H\left(u(t), u_0 + \int_{t_0}^t v(s)ds\right) + d_H \\ &\quad \cdot \left(v(t), u_0' + \int_{t_0}^t F(s,u(s),v(s))ds\right) \\ &= d_H\left(u(t) - u_0, \int_{t_0}^t v(s)ds\right) + d_H \\ &\quad \cdot \left(v(t) - u_0', \int_{t_0}^t F(s,u(s),v(s))ds\right) \\ &= d_H\left(\int_{t_0}^t Du(t), \int_{t_0}^t v(s)ds\right) + d_H \\ &\quad \cdot \left(\int_{t_0}^t Dv(t), \int_{t_0}^t F(s,u(s),v(s))ds\right) \\ &= \int_{t_0}^t \{d_H(Du(t),v(t)) + d_H \\ &\quad \cdot (D^2u(t), F(t,u(t),v(t)))\}dt \\ &\le \varepsilon(t - t_0). \end{aligned} \tag{30}$$

To multiply $e^{-(L+1)(t-t_0)}$, on both sides of the above inequality, we have

$$(d_H(u,G(u)) + d_H(v,G(v))e^{-(L+1)(t-t_0)} \le \varepsilon(t-t_0)e^{-(L+1)(t-t_0)}. \tag{31}$$

Now, from the definition of H and substitute $T = t - t_0$, for all $t \in Y$, we can obtain

$$H((u,v),G(u,v)) \le \varepsilon\, T\, e^{-(L+1)(t-t_0)}. \tag{32}$$

From, Theorem 3 condition (iii), there exist unique solution $(\widehat{u}, \widehat{v})$, of problem (13), such that

$$H((u,v),(\widehat{u},\widehat{v})) = \frac{1}{1-} H((u,v), G(u,v)) \le (1+L)\varepsilon\, T\, e^{-(L+1)(t-t_0)}. \tag{33}$$

Using definition of H, one can get

$$d_H(u,\widehat{u}) + d_H(v,\widehat{v})) \le (1+L)\varepsilon\, T, \quad \forall t \in Y. \tag{34}$$

Hence, for all $t \in Y$, we can get

$$\begin{aligned} d_H(u,\widehat{u}) &\le (1+L)\varepsilon\, T, \\ d_H(v,\widehat{v}) &\le (1+L)\varepsilon\, T. \end{aligned} \tag{35}$$

Hence, the fuzzy solutions of the problem (13) are Hyers-Ulam stable.

Theorem 18. *Let $F : Y \times \widehat{F}_R \times \widehat{F}_R \to \widehat{F}_R$ be the fuzzy-valued continuous function such that*

$$d_H(F(t,u_1,v_1), F(t,u_2,v_2)) \le d_H(u_1,u_2), \quad \forall u_1, v_1, u_2, v_2 \in \widehat{F}_R, t \in Y. \tag{36}$$

If the functions, $u : Y \to \widehat{F}_R$, are (i)-differentiable and $v : Y \to \widehat{F}_R$ is (ii)-differentiable, the following condition holds:

$$d_H(Du(t), v(t)) + d_H(Dv(t), F(t,u(t),v(t))) \le \varepsilon, \textit{some nonnegative}, \varepsilon. \tag{37}$$

Then, there exist unique solution of the problem (13), $\widehat{u}$, $\widehat{v} : Y \to \widehat{F}_R$, defined by

$$\begin{cases} \widehat{u} = u_0 + \displaystyle\int_{t_0}^{t} v(s)ds, \\ \widehat{v} = u_0'(-1) \ominus \displaystyle\int_{t_0}^{t} F(s,u(s),v(s))ds. \end{cases} \tag{38}$$

The fuzzy solution is Hyers-Ulam stable, if the following conditions are satisfied:

$$\begin{aligned} d_H(u,\widehat{u}) &\le (L+1)\varepsilon\, T, \\ d_H(v,\widehat{v}) &\le (L+1)\varepsilon\, T, \end{aligned} \tag{39}$$

where L is nonnegative constant and $T = t - t_0$.

Proof. The proof can be easily obtained on the similar procedure of Theorem 17.

Theorem 19. *Let $F : Y \times \widehat{F}_R \times \widehat{F}_R \to \widehat{F}_R$, be the fuzzy-valued continuous function such that*

$$d_H(F(t,u_1,v_1), F(t,u_2,v_2) \le d_H(u_1,u_2), \quad \forall u_1, v_1, u_2, v_2 \in \widehat{F}_R, t \in Y. \tag{40}$$

If the function $u : Y \to \widehat{F}_R$ is (ii)-differentiable and $v : Y \to \widehat{F}_R$ is (i)-differentiable, the following condition holds:

$$d_H(Du(t), v(t)) + d_H(Dv(t), F(t,u(t),v(t))) \le \varepsilon, \textit{some nonnegative}, \varepsilon. \tag{41}$$

Then, there exist unique solutions of the problem (13), $\widehat{u}$, $\widehat{v} : Y \to \widehat{F}_R$, defined by

$$\begin{cases} \widehat{u} = u_0(-1) \ominus \displaystyle\int_{t_0}^{t} v(s)ds, \\ \widehat{v} = u_0' + \displaystyle\int_{t_0}^{t} F(s,u(s),v(s))ds. \end{cases} \tag{42}$$

The fuzzy solution is Hyers-Ulam stable, if the following conditions are satisfied:

$$\begin{aligned} d_H(u,\widehat{u}) &\le (L+1)\varepsilon\, T, \\ d_H(v,\widehat{v}) &\le (L+1)\varepsilon\, T, \end{aligned} \tag{43}$$

where L is nonnegative constant and $T = t - t_0$.

Proof. The proof can be easily obtained on the similar procedure of Theorem 17.

Theorem 20. *Let $F : Y \times \widehat{F}_R \times \widehat{F}_R \to \widehat{F}_R$ be the fuzzy-valued continuous function such that*

$$d_H(F(t,u_1,v_1), F(t,u_2,v_2) \le d_H(u_1,u_2), \quad \forall, u_1, v_1, u_2, v_2 \in \widehat{F}_R, t \in Y. \tag{44}$$

If the functions $u, v : Y \to \widehat{F}_R$ are (ii)-differentiable, the following conditions hold:

$$d_H(Du(t), v(t)) + d_H(Dv(t), F(t,u(t),v(t))) \le \varepsilon, \textit{some nonnegative}, \varepsilon. \tag{45}$$

Then, there exist unique solution of the problem (13), $\widehat{u}$, $\widehat{v} : Y \to \widehat{F}_R$, defined by

$$\begin{cases} \widehat{u} = u_0 + \displaystyle\int_{t_0}^{t} v(s)ds, \\ \widehat{v} = u_0' + \displaystyle\int_{t_0}^{t} F(s,u(s),v(s))ds. \end{cases} \tag{46}$$

The fuzzy solution is Hyers-Ulam stable, if the following conditions are satisfied:

$$\begin{aligned} d_H(u,\widehat{u}) &\le (L+1)\varepsilon\, T, \\ d_H(v,\widehat{v}) &\le (L+1)\varepsilon\, T, \end{aligned} \tag{47}$$

where L is nonnegative constant and $T = t - t_0$.

Proof. Using a procedure similar to Theorem 17, one can easily prove.

4. Natural Transforms and Solutions of the Proposed Fuzzy Differential Equations

Theorem 21. *Let fuzzy-valued function $\chi(w)$ be continuous such that $e^{-sw}\chi(uw)$, $e^{-sw}\chi'(uw)$, and $e^{-sw}\chi''(uw)$ exist and are continuous. Moreover, they are Riemann integrable on $[0,\infty)$; then,*

(A) If $\chi(w)$ and $\chi'(w)$ are (i)-differentiable, then

$$\mathcal{N}\left\{\chi''(w)\right\} = \left[\frac{s^2}{u^2}\widehat{R}(s,u) \ominus \frac{s}{u^2}\chi(0)\right] \ominus \frac{1}{u}\chi'(0) \quad (48)$$

(B) If $\chi(w)$ is (i)-differentiable and $\chi'(w)$ is (ii)-differentiable, then

$$\mathcal{N}\left\{\chi''(w)\right\} = \left(-\frac{1}{u}\chi'(0)\right) \ominus \left[\left(-\frac{s^2}{u^2}\widehat{R}(s,u)\right) \ominus \left(-\frac{s}{u^2}\chi(0)\right)\right] \quad (49)$$

(C) If $\chi(w)$ is (ii)-differentiable and $\chi'(w)$ is (i)-differentiable, then

$$\mathcal{N}\left\{\chi''(w)\right\} = \left[\left(-\frac{s}{u^2}\chi(0)\right) \ominus \left(-\frac{s^2}{u^2}\widehat{R}(s,u)\right)\right] \ominus \frac{1}{u}\chi'(0). \quad (50)$$

(D) If $\chi(w)$ and $\chi'(w)$ are (ii)-differentiable, then

$$\mathcal{N}\left\{\chi''(w)\right\} = \left(-\frac{1}{u}\chi'(0)\right) \ominus \left[\frac{s}{u^2}\chi(0) \ominus \frac{s^2}{u^2}\widehat{R}(s,u)\right] \quad (51)$$

Proof.

(A) Let $\chi(w)$ and $\chi'(w)$ are (i)-differentiable, then

$$\begin{aligned}\mathcal{N}\left\{\chi'(w)\right\} &= \frac{s}{u}\left[\widehat{R}(s,u)\right] \ominus \frac{1}{u}\chi(0),\\ \text{and } \mathcal{N}\left\{\chi''(w)\right\} &= \frac{s}{u}\left[\mathcal{N}\left\{\chi'(w)\right\}\right] \ominus \frac{1}{u}\chi'(0).\end{aligned} \quad (52)$$

Using (52), this identity yields

$$\begin{aligned}\mathcal{N}\left\{\chi''(w)\right\} &= \frac{s}{u}\left[\frac{s}{u}\left[\widehat{R}(s,u)\right] \ominus \frac{1}{u}\chi(0)\right] \ominus \frac{1}{u}\chi'(0),\\ \mathcal{N}\left\{\chi''(w)\right\} &= \left[\frac{s^2}{u^2}\widehat{R}(s,u) \ominus \frac{s}{u^2}\chi(0)\right] \ominus \frac{1}{u}\chi'(0)\end{aligned} \quad (53)$$

(B) If, $\chi(w)$ and $\chi'(w)$ are (i)-differentiable and (ii)-differentiable, respectively, then

$$\begin{aligned}\mathcal{N}\left\{\chi'(w)\right\} &= \frac{s}{u}\left[\widehat{R}(s,u)\right] \ominus \frac{1}{u}\chi(0),\\ \mathcal{N}\left\{\chi''(w)\right\} &= \left(-\frac{1}{u}\chi'(0)\right) \ominus \left(-\frac{s}{u}\left[\mathcal{N}\left\{\chi'(w)\right\}\right]\right).\end{aligned} \quad (54)$$

Using (54), this identity yields

$$\begin{aligned}\mathcal{N}\left\{\chi''(w)\right\} &= \left(-\frac{1}{u}\chi'(0)\right) \ominus \left[-\frac{s}{u}\left(\frac{s}{u}\left[\widehat{R}(s,u)\right] \ominus \frac{1}{u}\chi(0)\right)\right],\\ \mathcal{N}\left\{\chi''(w)\right\} &= \left(-\frac{1}{u}\chi'(0)\right) \ominus \left[\left(-\frac{s^2}{u^2}\widehat{R}(s,u)\right) \ominus \left(-\frac{s}{u^2}\chi(0)\right)\right]\end{aligned} \quad (55)$$

(C) If $\chi'(w)$ and $\chi(w)$ are (i)-differentiable and (ii)-differentiable, respectively, then

$$\begin{aligned}\mathcal{N}\left\{\chi'(w)\right\} &= \left(-\frac{1}{u}\chi(0)\right) \ominus \left(-\frac{s}{u}\left[\widehat{R}(s,u)\right]\right),\\ \mathcal{N}\left\{\chi''(w)\right\} &= \frac{s}{u}\left[\mathcal{N}\left\{\chi'(w)\right\}\right] \ominus \frac{1}{u}\chi(0).\end{aligned} \quad (56)$$

Using (56), this identity yields

$$\begin{aligned}\mathcal{N}\left\{\chi''(w)\right\} &= \left(-\frac{s}{u^2}\chi'(0)\right) \ominus \left(-\frac{s^2}{u^2}\left[\widehat{R}(s,u)\right]\right) \ominus \frac{1}{u}\chi(0)),\\ \mathcal{N}\left\{\chi''(w)\right\} &= \left[\left(-\frac{1}{u}\chi(0)\right) \ominus \left(-\frac{s^2}{u^2}\widehat{R}(s,u)\right)\right] \ominus \frac{1}{u}\chi'(0)\end{aligned} \quad (57)$$

(D) If $\chi(w)$ and $\chi'(w)$ are (ii)-differentiable, then

$$\mathcal{N}\left\{\chi'(w)\right\} = \left(-\frac{1}{u}\chi(0)\right)!\left(-\frac{s}{u}\left[\widehat{R}(s,u)\right]\right),$$
$$\mathcal{N}\left\{\chi''(w)\right\} = \left(-\frac{1}{u}\chi'(0)\right) \ominus \left(-\frac{s}{u}\left[\mathcal{N}\left\{\chi'(w)\right\}\right]\right). \tag{58}$$

Using (58), this identity yields

$$\mathcal{N}\left\{\chi''(w)\right\} = \left(-\frac{1}{u}\chi'(0)\right) \ominus \left[-\frac{s}{u}\left(\left(-\frac{1}{u}\chi'(0)\right) \ominus \left(-\frac{s}{u}\left[\widehat{R}(s,u)\right]\right)\right)\right],$$
$$\mathcal{N}\left\{\chi''(w)\right\} = \left(-\frac{1}{u}\chi'(0)\right) \ominus \left[\frac{s}{u^2}\chi(0) \ominus \frac{s^2}{u^2}\widehat{R}(s,u))\right] \tag{59}$$

Next, we have to solve problem (13), with natural transform. The fuzzy solution of the concerned problem can be discussed in four cases. Taking natural transform of (13), we can get

$$\mathcal{N}\left[F\left(w,\chi(w),\chi'(w)\right)\right] = \mathcal{N}\left\{\chi''(w)\right\}. \tag{60}$$

Case I. Let $\chi(w)$ and $\chi'(w)$ are (i)-differentiable; then, using Theorem 21, in (60), one can get

$$\begin{cases} \mathcal{N}\left[\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\underline{\widehat{R}}(s,u) \ominus \frac{s}{u^2}\underline{\chi}(0,\gamma)\right] \ominus \frac{1}{u}\underline{\chi}'(0,\gamma), \\ \mathcal{N}\left[\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\bar{\widehat{R}}(s,u) \ominus \frac{s}{u^2}\bar{\chi}(0,\gamma)\right] \ominus \frac{1}{u}\bar{\chi}'(0,\gamma), \end{cases} \tag{61}$$

where

$$\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right) = \min\left\{F\left(w,\chi(w),\chi'(w)\right) \mid \chi(w) \in \cdot\left(\underline{\chi}(w,\gamma),\bar{\chi}(w,\gamma)\right); \chi'(w) \in \cdot\left(\underline{\chi}'(w,\gamma),\bar{\chi}'(w,\gamma)\right)\right\},$$
$$\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right) = \max\left\{F\left(w,\chi(w),\chi'(w)\right) \mid \chi(w) \in \cdot\left(\underline{\chi}(w,\gamma),\bar{\chi}(w,\gamma)\right); \chi'(w) \in \cdot\left(\underline{\chi}'(w,\gamma),\bar{\chi}'(w,\gamma)\right)\right\}. \tag{62}$$

Assume that solving Equation (61), satisfying the conditions $\chi(w_0) = \chi_0 = (\underline{\chi}_0, \bar{\chi}_0)$ and $\chi'(w_0) = \chi_0' = (\underline{\chi}_0', \bar{\chi}_0')$, one can obtain the solutions as follows:

$$\begin{cases} \underline{\widehat{R}}(s,u,\gamma) = R_1(s,u,\gamma), \\ \bar{\widehat{R}}(s,u,\gamma) = G_1(s,u,\gamma). \end{cases} \tag{63}$$

Taking inverse fuzzy natural transform of Equation (63), one can find $\underline{\chi}(w,\gamma)$ and $\bar{\chi}(w,\gamma)$, as follows:

$$\begin{cases} \underline{\chi}(w,\gamma) = \mathcal{N}^{-1}\{R_1(s,u,\gamma)\}, \\ \bar{\chi}(w,\gamma) = \mathcal{N}^{-1}\{G_1(s,u,\gamma)\}. \end{cases} \tag{64}$$

Case II. Let $\chi(w)$ is (i)-differentiable and $\chi'(w)$ is (ii)-differentiable, then using Theorem 21, in (60), one can get

$$\begin{cases} \mathcal{N}\left[\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left(-\frac{1}{u}\bar{\chi}'(0,\gamma)\right) \ominus \left[\left(-\frac{s^2}{u^2}\underline{\widehat{R}}(s,u,\gamma)\right) \ominus \left(-\frac{s}{u^2}\underline{\chi}(0,\gamma)\right)\right], \\ \mathcal{N}\left[\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left(-\frac{1}{u}\underline{\chi}'(0,\gamma)\right) \ominus \left[\left(-\frac{s^2}{u^2}\bar{\widehat{R}}(s,u,\gamma)\right) \ominus \left(-\frac{s}{u^2}\bar{\chi}(0,\gamma)\right)\right]. \end{cases} \tag{65}$$

Using initial condition in (65), $\chi(w_0) = \chi_0 = (\underline{\chi}_0, \bar{\chi}_0)$ and $\chi'(w_0) = \chi_0' = (\bar{\chi}_0', \underline{\chi}_0')$, one can get

$$\begin{cases} \mathcal{N}\left[\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\underline{\widehat{R}}(s,u,\gamma) \ominus \frac{s}{u^2}\underline{\chi}_0(\gamma)\right] \ominus \frac{1}{u}\bar{\chi}_0'(\gamma), \\ \mathcal{N}\left[\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\bar{\widehat{R}}(s,u,\gamma) \ominus \frac{s}{u^2}\bar{\chi}_0(\gamma)\right] \ominus \frac{1}{u}\underline{\chi}_0'(\gamma). \end{cases} \tag{66}$$

Assume that by solving Equation (66), we can get the solutions as follows:

$$\begin{cases} \underline{\widehat{R}}(s,u,\gamma) = R_2(s,u,\gamma), \\ \bar{\widehat{R}}(s,u,\gamma) = G_2(s,u,\gamma). \end{cases} \tag{67}$$

Taking inverse fuzzy natural transform of Equation (67), one can find $\underline{\chi}(w,\gamma)$ and $\bar{\chi}(w,\gamma)$,

$$\begin{cases} \underline{\chi}(w,\gamma) = \mathcal{N}^{-1}\{R_2(s,u,\gamma)\}, \\ \bar{\chi}(w,\gamma) = \mathcal{N}^{-1}\{G_2(s,u,\gamma)\}. \end{cases} \tag{68}$$

Case III. Let $\chi(w)$ is (ii)-differentiable and $\chi'(w)$ is (i)-differentiable, then using Theorem 21, in (60), one can get

$$\begin{cases} \mathcal{N}\left[\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\left(-\frac{s}{u^2}\bar{\chi}(0,\gamma)\right) \ominus \left(-\frac{s^2}{u^2}\bar{\widehat{R}}(s,u,\gamma)\right)\right] \ominus \left(-\frac{1}{u}\underline{\chi}'(0,\gamma)\right), \\ \mathcal{N}\left[\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\left(-\frac{s}{u^2}\underline{\chi}(0,\gamma)\right) \ominus \left(-\frac{s^2}{u^2}\underline{\widehat{R}}(s,u,\gamma)\right)\right] \ominus \left(-\frac{1}{u}\bar{\chi}'(0,\gamma)\right). \end{cases} \tag{69}$$

Using initial condition in (69), $\chi(w_0) = \chi_0 = (\bar{\chi}_0, \underline{\chi}_0)$ and $\chi'(w_0) = \chi_0' = (\underline{\chi}_0', \bar{\chi}_0')$, one can get

$$\begin{cases} \mathcal{N}\left[\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\bar{\widehat{R}}(s,u,\gamma) \ominus \frac{s}{u^2}\bar{\chi}_0(\gamma)\right] \ominus \frac{1}{u}\underline{\chi}_0'(\gamma), \\ \mathcal{N}\left[\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\underline{\widehat{R}}(s,u,\gamma) \ominus \frac{s}{u^2}\underline{\chi}_0(\gamma)\right] \ominus \frac{1}{u}\bar{\chi}_0'(\gamma). \end{cases} \tag{70}$$

Assume that by solving Equation (70), one can obtain the solution as follows:

$$\begin{cases} \widehat{\underline{R}}(s,u,\gamma) = R_3(s,u,\gamma), \\ \widehat{\overline{R}}(s,u,\gamma) = G_3(s,u,\gamma). \end{cases} \tag{71}$$

Taking inverse fuzzy natural transform of Equation (71), one can find

$$\begin{cases} \underline{\chi}(w,\gamma) = \mathcal{N}^{-1}\{R_3(s,u,\gamma)\}, \\ \bar{\chi}(w,\gamma) = \mathcal{N}^{-1}\{G_3(s,u,\gamma)\}. \end{cases} \tag{72}$$

Case IV. Let $\chi(w)$ and $\chi'(w)$ are (ii)-differentiable, then using Theorem 21, in (60), one can get

$$\begin{cases} \mathcal{N}\left[\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\left(-\frac{s}{u^2}\bar{\chi}(0,\gamma)\right)\ominus\left(-\frac{s^2}{u^2}\widehat{\bar{R}}(s,u,\gamma)\right)\right]\ominus\left(-\frac{1}{u}\bar{\chi}'(0,\gamma)\right), \\ \mathcal{N}\left[\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\left(-\frac{s}{u^2}\underline{\chi}(0,\gamma)\right)\ominus\left(-\frac{s^2}{u^2}\widehat{\underline{R}}(s,u,\gamma)\right)\right]\ominus\left(-\frac{1}{u}\underline{\chi}'(0,\gamma)\right). \end{cases} \tag{73}$$

Using initial condition in (73), $\chi(w_0) = \chi_0 = (\bar{\chi}_0, \underline{\chi}_0)$ and $\chi'(w_0) = \chi'_0 = (\bar{\chi}'_0, \underline{\chi}'_0)$, one can get

$$\begin{cases} \mathcal{N}\left[\bar{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\widehat{\bar{R}}(s,u,\gamma)\ominus\frac{s}{u^2}\bar{\chi}_0(\gamma)\right]\ominus\frac{1}{u}\bar{\chi}'_0(\gamma), \\ \mathcal{N}\left[\underline{F}\left(w,\chi(w),\chi'(w),\gamma\right)\right] = \left[\frac{s^2}{u^2}\widehat{\underline{R}}(s,u,\gamma)\ominus\frac{s}{u^2}\underline{\chi}_0(\gamma)\right]\ominus\frac{1}{u}\underline{\chi}'_0(\gamma). \end{cases} \tag{74}$$

Assume that by solving Equation (74), one can obtain the solution as follows:

$$\begin{cases} \widehat{\underline{R}}(s,u,\gamma) = R_4(s,u,\gamma), \\ \widehat{\overline{R}}(s,u,\gamma) = G_4(s,u,\gamma). \end{cases} \tag{75}$$

Taking inverse fuzzy natural transform of Equation (75), one can find

$$\begin{cases} \underline{\chi}(w,\gamma) = \mathcal{N}^{-1}\{R_4(s,u,\gamma)\}, \\ \bar{\chi}(w,\gamma) = \mathcal{N}^{-1}\{G_4(s,u,\gamma)\}. \end{cases} \tag{76}$$

5. Examples

Here, we discuss the numerical problem.

$$\begin{cases} \chi''(w) = p(w)\chi(w) + q(w)\chi'(w) + c(w), \\ \chi(w_0) = \chi_0 = \left(\underline{\chi}_0, \bar{\chi}_0\right), \\ \chi'(w_0)\chi'_0 = \left(\underline{\chi}'_0, \bar{\chi}'_0\right), \end{cases} \tag{77}$$

where $p, q : Y \to R^+$ be continuous functions such that for nonnegative $L, |p(w)| \le L, |q(w)| \le L-1$, and $c : Y \to \widehat{F}_R$ be continuous function for all $w \in Y$, while $Y = [0, 3/L]$. If the functions $\chi : Y \to \widehat{F}_R$, the following conditions for some nonnegative ε hold:

$$d_H\left(D\chi(w), \chi'(w)\right) + d_H \cdot \left(D\chi'(w), p(w)\chi(w) + q(w)\chi'(w) + c(w)\right) \le \varepsilon. \tag{78}$$

Assume that, on setting $F(w,\chi(w),\chi'(w)) = p(w)\chi(w) + q(w)\chi'(w) + w(w)$, it is easy to obtain condition (i), as follows

$$\begin{aligned} d_H(F&\left(w,\chi_1,\chi'_1\right), F\left(w,\chi_2,\chi'_2\right) \\ &= d_H\left(p(w)\chi_1(w) + q(w)\chi'_1(w) + c(w), p(w)\chi_2(w)\right. \\ &\quad \left. + q(w)\chi'_2(w) + c(w)\right) \le d_H(p(w)\chi_1(w), p(w)\chi_2(w)) \\ &\quad + d_H\left(q(w)\chi'_1(w), q(w)\chi'_2(w)\right) = |p(w)| d_H(\chi_1(w),\chi_2(w)) \\ &\quad + |q(w)| d_H\left(\chi'_1(w),\chi'_2(w)\right) \le L d_H(\chi_1(w),\chi_2(w)) \\ &\quad + (L-1) d_H\left(\chi'_1(w),\chi'_2(w)\right), \quad \text{for all } w \in Y \end{aligned} \tag{79}$$

Then, there exist unique solution of the problem (77), $\hat{u}, \hat{v} : Y \to \widehat{F}_R$, defined by

$$\begin{cases} \widehat{\chi} = \chi_0 + \int_{3/2L}^{w} \chi'(s)ds, \\ \widehat{\chi}' = \chi'_0 + \int_{3/2L}^{w} F\left(s,\chi(s),\chi'(s)\right)ds, \end{cases} \tag{80}$$

$$\begin{aligned} d_H(\chi,\widehat{\chi}) &\le (1+L)\frac{3}{2L}\varepsilon, \\ d_H\left(\chi',\widehat{\chi}'\right) &\le (1+L)\frac{3}{2L}\varepsilon, \quad \text{for all, } w \in Y. \end{aligned} \tag{81}$$

This shows that the fuzzy solution of differential equations, (77), is Hyers-Ulam stable on Y.

When $p(w) = p$ and $q(w) = q$, then $R_1(s,u,\gamma)$ and $G_1(s,u,\gamma)$ of system (60) can be calculated as

Case 1. When $p, q \ge 0$, then system (61) is equivalent to

$$\begin{cases} \frac{s^2}{u^2}\widehat{\underline{R}}(s,u,\gamma)\ominus\frac{s}{u^2}\underline{\chi}(0,\gamma)\ominus\frac{1}{u}\underline{\chi}'(0,\gamma) = \left(p+q\frac{s}{u}\right)\widehat{\underline{R}}(s,u,\gamma)\ominus q\frac{1}{u}\underline{\chi}'(0,\gamma) + N(\underline{c}(w)), \\ \frac{s^2}{u^2}\widehat{\bar{R}}(s,u,\gamma)\ominus\frac{s}{u^2}\bar{\chi}(0,\gamma)\ominus\frac{1}{u}\bar{\chi}'(0,\gamma) = \left(p+q\frac{s}{u}\right)\widehat{\bar{R}}(s,u,\gamma)\ominus q\frac{1}{u}\bar{\chi}'(0,\gamma) + N(\bar{c}(w)). \end{cases} \tag{82}$$

After some process, one can get

$$\begin{cases} R_1(s,u,\gamma) = \widehat{\underline{R}}(s,u,\gamma) = \dfrac{(s/u^2 - q/u)\underline{\chi}(0,\gamma) + (1/u)\underline{\chi}'(0,\gamma) + N(\underline{c}(w))}{s^2/u^2 - qs/u - p}, \\ G_1(s,u,\gamma) = \widehat{\bar{R}}(s,u,\gamma) = \dfrac{(s/u^2 - q/u)\bar{\chi}(0,\gamma) + (1/u)\bar{\chi}'(0,\gamma) + N(\bar{c}(w))}{s^2/u^2 - qs/u - p}. \end{cases} \tag{83}$$

Case 2. When $p \geq 0$ and $q < 0$, then system (65) is equivalent to the following:

$$\begin{cases} \dfrac{s^2}{u^2}\widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}\underline{\chi}(0,\gamma) \ominus \dfrac{1}{u}\underline{\chi}'(0,\gamma) = p\widehat{\underline{R}}(s,u,\gamma) + q\dfrac{s}{u}\widehat{\overline{R}}(s,u,\gamma) \ominus \dfrac{q}{u}\overline{\chi}'(0,\gamma) + N(\underline{c}(w)), \\ \dfrac{s^2}{u^2}\widehat{\overline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}\overline{\chi}(0,\gamma) \ominus \dfrac{1}{u}\overline{\chi}'(0,\gamma) = p\widehat{\overline{R}}(s,u,\gamma) + q\dfrac{s}{u}\widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{q}{u}\underline{\chi}'(0,\gamma) + N(\overline{c}(w)). \end{cases} \tag{84}$$

Let us denote

$$\begin{cases} f(s,u,\gamma) = \dfrac{s}{u^2}\underline{\chi}(0,\gamma) + \dfrac{1}{u}\underline{\chi}'(0,\gamma) - \dfrac{q}{u}\overline{\chi}'(0,\gamma) + N(\underline{c}(w)), \\ g(s,u,\gamma) = \dfrac{s}{u^2}\overline{\chi}(0,\gamma) + \dfrac{1}{u}\overline{\chi}'(0,\gamma) - \dfrac{q}{u}\underline{\chi}'(0,\gamma) + N(\overline{c}(w)). \end{cases} \tag{85}$$

Solving the above system, one can get

$$\begin{cases} R_1(s,u,\gamma) = \widehat{\underline{R}}(s,u,\gamma) = \dfrac{\left(s^2/u^2 - p\right)f(s,u,\gamma) + (qs/u)g(s,u,\gamma)}{\left(s^2/u^2 - p\right)^2 + (qs/u)^2}, \\ G_1(s,u,\gamma) = \widehat{\overline{R}}(s,u,\gamma) = \dfrac{\left(s^2/u^2 - p\right)g(s,u,\gamma) - (qs/u)f(s,u,\gamma)}{\left(s^2/u^2 - p\right)^2 + (qs/u)^2}. \end{cases} \tag{86}$$

Case 3. When $p < 0$ and $q \geq 0$, then system (69) is equivalent to the following system:

$$\begin{cases} \dfrac{s^2}{u^2}\widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}\underline{\chi}(0,\gamma) \ominus \dfrac{1}{u}\underline{\chi}'(0,\gamma) = p\widehat{\overline{R}}(s,u,\gamma) + q\dfrac{s}{u}\widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{q}{u}\underline{\chi}'(0,\gamma) + N(\underline{c}(w)), \\ \dfrac{s^2}{u^2}\widehat{\overline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}\overline{\chi}(0,\gamma) \ominus \dfrac{1}{u}\overline{\chi}'(0,\gamma) = p\widehat{\underline{R}}(s,u,\gamma) + q\dfrac{s}{u}\widehat{\overline{R}}(s,u,\gamma) \ominus \dfrac{q}{u}\overline{\chi}'(0,\gamma) + N(\overline{c}(w)). \end{cases} \tag{87}$$

Let us denote

$$\begin{cases} B(s,u,\gamma) = \dfrac{s}{u^2}\underline{\chi}(0,\gamma) + \dfrac{1}{u}\underline{\chi}'(0,\gamma) - \dfrac{q}{u}\underline{\chi}'(0,\gamma) + N(\underline{c}(w)), \\ C(s,u,\gamma) = \dfrac{s}{u^2}\overline{\chi}(0,\gamma) + \dfrac{1}{u}\overline{\chi}'(0,\gamma) - \dfrac{q}{u}\overline{\chi}'(0,\gamma) + N(\overline{c}(w)). \end{cases} \tag{88}$$

Solving the above system, one can get

$$\begin{cases} R_1(s,u,\gamma) = \widehat{\underline{R}}(s,u,\gamma) = \dfrac{\left(s^2/u^2 - qs/u\right)B(s,u,\gamma) + pC(s,u,\gamma)}{\left(s^2/u^2 - qs/u\right)^2 + (p)^2}, \\ G_1(s,u,\gamma) = \widehat{\overline{R}}(s,u,\gamma) = \dfrac{\left(s^2/u^2 - qs/u\right)C(s,\mathrm{u},\gamma) + pB(s,u,\gamma)}{\left(s^2/u^2 - qs/u\right)^2 + (p)^2}. \end{cases} \tag{89}$$

Case 4. When $p < 0$ and $q < 0$, then system (73) is equivalent to the following:

$$\begin{cases} \dfrac{s^2}{u^2}\widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}\underline{\chi}(0,\gamma) \ominus \dfrac{1}{u}\underline{\chi}'(0,\gamma) = p\widehat{\overline{R}}(s,u,\gamma) + q\dfrac{s}{u}\widehat{\overline{R}}(s,u,\gamma) \ominus \dfrac{q}{u}\overline{\chi}'(0,\gamma) + N(\underline{c}(w)), \\ \dfrac{s^2}{u^2}\widehat{\overline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}\overline{\chi}(0,\gamma) \ominus \dfrac{1}{u}\overline{\chi}'(0,\gamma) = p\widehat{\underline{R}}(s,u,\gamma) + q\dfrac{s}{u}\widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{q}{u}\underline{\chi}'(0,\gamma) + N(\overline{c}(w)). \end{cases} \tag{90}$$

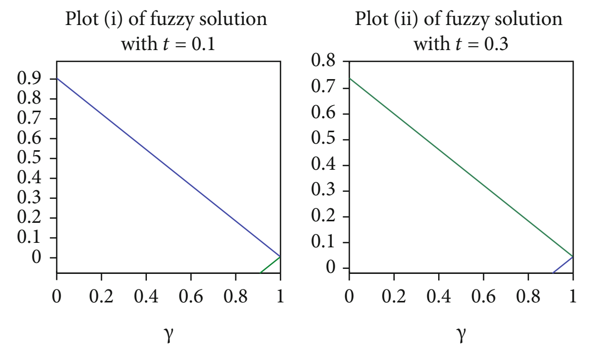

FIGURE 1: Present fuzzy solutions of Example 1 for different values of γ. The subplot (i) is 2D plot of lower and upper solutions with $t = 0.1$, and subplot (ii) is 2D plot of lower and upper solutions with $t = 0.3$.

For the remaining system, we can find the same processes. Solving the above system, one can get

$$\begin{cases} R_1(s,u,\gamma) = \widehat{\underline{R}}(s,u,\gamma) = \dfrac{\left(s^2/u^2\right)\mathrm{f}(s,u,\gamma) + (p+qs/u)g(s,u,\gamma)}{\left(s^2/u^2\right)^2 + (p+qs/u)^2}, \\ G_1(s,u,\gamma) = \widehat{\bar{R}}(s,u,\gamma) = \dfrac{\left(s^2/u^2\right)g(s,u,\gamma) + (p+(qs/u))f(s,u,\gamma)}{\left(s^2/u^2\right)^2 + (p+qs/u)^2}. \end{cases} \tag{91}$$

Example 1. Now, we discuss the following system:

$$\begin{cases} \chi''(t) = -\chi(t) + w_0, \quad w_0 = (\gamma, 2-\gamma), \\ \chi(0,\gamma) = (\gamma-1, 1-\gamma), \\ \chi'(0,\gamma) = (\gamma-1, 1-\gamma). \end{cases} \tag{92}$$

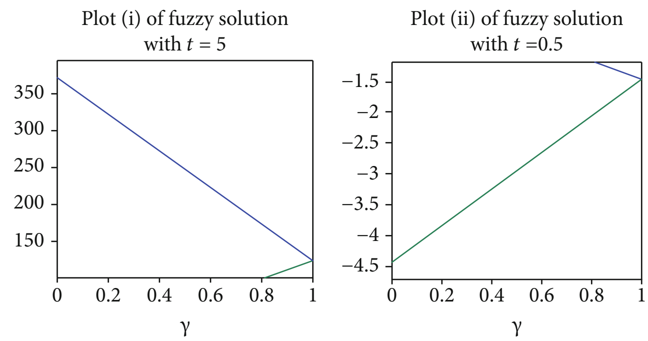

FIGURE 2: Present fuzzy solutions of Example 2 for different values of γ. The subplot (i) is 2D plot of lower and upper solutions with $t = 5$, and subplot (ii) is 2D plot of lower and upper solutions with $t = 0.5$.

This differential equation is equivalent to Case 3 of problem (77); therefore, the solution exists.

$$\begin{cases} \dfrac{s^2}{u^2}\widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}(\gamma-1) \ominus \dfrac{1}{u}(\gamma-1) = -\widehat{\bar{R}}(s,u,\gamma) + N(\underline{w}(t)), \\ \dfrac{s^2}{u^2}\widehat{\bar{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}(1-\gamma) \ominus \dfrac{1}{u}(1-\gamma) = -\widehat{\underline{R}}(s,u,\gamma) + N(\bar{w}(t)). \end{cases} \tag{93}$$

Solving the above system, one can get

$$\begin{cases} R_1(s,u,\gamma) = \widehat{\underline{R}}(s,u,\gamma) = \gamma\left(\dfrac{1}{2(s+u)} - \dfrac{1}{2(s-u)} + \dfrac{1}{s}\right) + \dfrac{1}{2(s-u)} - \dfrac{1}{2(s+u)} - \dfrac{s}{s^2+u^2}, \\ G_1(s,u,\gamma) = \widehat{\bar{R}}(s,u,\gamma) = \gamma\left(\dfrac{1}{2(s-u)} - \dfrac{1}{2(s+u)} - \dfrac{1}{s}\right) + \dfrac{2}{s} + \dfrac{1}{2(s+u)} - \dfrac{1}{2(s-u)} - \dfrac{s}{s^2+u^2}. \end{cases} \tag{94}$$

Now, applying inverse natural transform, we deduce

$$\begin{cases} \underline{\chi}(t,\gamma) = \gamma(1-\sinh(t)) + \sinh(t) - \cos(t), \\ \bar{\chi}(t,\gamma) = (2-\gamma)(1-\sinh(t)) + \sinh(t) - \cos(t). \end{cases} \tag{95}$$

This solution is acceptable for all $t \in (0, \ln(1+\sqrt{2}))$, see [30] (Figure 1).

Example 2. Here, we consider the following system:

$$\begin{cases} \chi''(t) = \chi(t) + t + 1, \\ \chi(0,\gamma) = (\gamma-2, 1-2\gamma), \\ \chi'(0,\gamma) = (\gamma-2, 1-2\gamma). \end{cases} \tag{96}$$

This differential equation is equivalent to Case 1 of problem (77); therefore, the solution exists.

$$\begin{cases} \dfrac{s^2}{u^2}\, \widehat{\underline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}(\gamma-2) \ominus \dfrac{1}{u}(\gamma-2) = -\widehat{\overline{R}}(s,u,\gamma) + \dfrac{u}{s^2} + \dfrac{1}{s}, \\ \dfrac{s^2}{u^2}\, \widehat{\overline{R}}(s,u,\gamma) \ominus \dfrac{s}{u^2}(1-\gamma) \ominus \dfrac{1}{u}(1-\gamma) = -\widehat{\underline{R}}(s,u,\gamma) + \dfrac{u}{s^2} + \dfrac{1}{s}. \end{cases} \tag{97}$$

Solving Equation (97), one can write

$$\begin{cases} R_1(s,u,\gamma) = \widehat{\underline{R}}(s,u,\gamma) = \dfrac{\gamma}{s-u} - \dfrac{2}{s-u} + \dfrac{1}{s^2(s-u)} + \dfrac{1}{s^3(s-u)}, \\ G_1(s,u,\gamma) = \widehat{\overline{R}}(s,u,\gamma) = \dfrac{1}{s-u} - \dfrac{2\gamma}{s-u} + \dfrac{1}{s^2(s-u)} + \dfrac{1}{s^3(s-u)}. \end{cases} \tag{98}$$

Now, applying inverse natural transform, we deduce

$$\begin{cases} \underline{\chi}(t,\gamma) = \gamma \exp(t) - \dfrac{t^2}{2} - 2t - 2, \\ \overline{\chi}(t,\gamma) = (3-2\gamma) \exp(t) - \dfrac{t^2}{2} - 2t - 2. \end{cases} \tag{99}$$

This solution is acceptable for all $t \in R$ (real numbers) (Figure 2).

6. Conclusion and Future Direction

In this work, we study the existence, uniqueness, and Hyers-Ulam stability of second-order FDEs. We discuss the natural transform of second-order fuzzy differential equations for different cases of fuzzy differentiability. The natural transform converts to Laplace transform by taking parameter $p = 1$ and Sumudu transform by taking parameter $s = 1$. Therefore, we use natural transform to solve the proposed FDEs. In the future, the natural transform method will be used to solve fuzzy problems due to significant duality with Laplace and Sumudu transforms.

Data Availability

No date were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interest regarding this manuscript.

Authors' Contributions

All authors contributed equally to the writing of this manuscript. All authors read and approve the final version.

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Research Article

Rational Type Compatible Single-Valued Mappings via Unique Common Fixed Point Findings in Complex-Valued b-Metric Spaces with an Application

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In this paper, we establish some new generalized rational type common fixed point results for compatible three self-mappings in complex-valued b-metric space, in which a one self-map is continuous. In support of our results, we present some illustrative examples to verify the validity of our main work. Moreover, we present the application of two Urysohn integral type equations (UITEs) for the existence of a common solution to support our work. The UITEs are $\nu_1(p) = \int_{k_1}^{k_2} Q_1(p, r, \nu_1(r))dr + \hbar_1(p)$ and $\nu_2(p) = \int_{k_1}^{k_2} Q_2(p, r, \nu_2(r))dr + \hbar_2(p)$, where $p \in [k_1, k_2], \nu_1, \nu_2, \hbar_1, \hbar_2 \in V$, where $V = C([k_1, k_2], \mathbb{R}^n)$ is the set of all real-valued continuous functions defined on $[k_1, k_2]$ and $Q_1, Q_2 : [k_1, k_2] \times [k_1, k_2] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

1. Introduction

The theory of fixed point is one of the most interesting area of research in Mathematics. Initially, the concept of this theory was given by Banach [1] and he proved "a Banach contraction theorem for fixed point" which is stated as "a single-valued contractive type mapping in a complete metric space has a unique fixed point." After the publication of this research article, many authors have contributed their ideas to the theory of fixed point in the context of metric spaces and proved different contractive type fixed point results for single-valued and multivalued mappings with different types of applications. Some fixed soft points and a-fixed soft points results can be found in [2, 3].

The concept of b-metric space was first introduced by Bakhtin [4], while Czerwik [5] proved some fixed point results for nonlinear set-valued contractive type mappings in b-metric spaces. Later on, Akkouchi [6] established some common fixed point theorems (CFP-theorems) for single-valued mappings under an implicit relation in b-metric spaces. In [7], Aghajani et al. proved CFP-results under the generalized weak contraction in partially ordered b-metric spaces. Further, Aydi et al. [8, 9] presented some FP-theorems and CFP-theorems for set-valued quasi-contraction and weak ϕ-contraction, respectively, in b-metric spaces. In 2013, Roshan et al. [10] established some generalized contractive type CFP-theorems in b-metric spaces and they proved that the b-metric function used in the theorems and results are not necessarily continuous. Some more FP-results in b-metric space can be found in [11–19]; the references are therein.

In 2011, Azam et al. [20] introduced the notion of complex-valued metric space and proved some CFP-theorems for a pair of self-mappings. Though complex-valued metric space forms a special class of cone metric space, so far this concept is proposed to define rational type expressions that are not significant in cone metric spaces, and therefore, some results of the analysis cannot be generalized to cone metric spaces. Properly the notion of complex-valued metric space was introduced by Rouzkard and Imdad [21] which generalized the expression of Azam et al. [20] and proved some CFP-theorems. Some more FP-results in the context of complex-valued metric spaces can be found in [22–24].

In 2013, Rao et al. [25] introduced the notion of complex-valued b-metric space which generalized the notion of complex-valued metric spaces given by Azam et al. [20] in 2011. They presented some CFP-results for generalized contraction conditions in complex-valued b-metric space. Later on, Mukheimer [26] extended and improved the results of [20, 25] and established some unique CFP-theorems in complex-valued metric spaces with illustrative examples.

In this paper, we establish some new generalized rational type CFP-theorems for compatible three self-mappings on complex-valued b-metric spaces in which one is a continuous self-map. Our results extend and modify many results given in the literature. This paper is organized as follows: Section 2 consists of preliminary concepts. In Section 3, we present some generalized unique CFP-theorems for compatible three self-mappings in complex-valued b-metric spaces with some illustrative examples to verify the validity of our work. In Section 4, we present an application of the two UITEs for the existence of a common solution to support our main result, while in Section 5, we discuss the conclusion.

2. Preliminaries

Consider $\mathbb{C}$ represents a set of complex numbers and $z_i, z_{ii} \in \mathbb{C}$. Define $\leq$ as $z_i \leq z_{ii}$, iff $R_e(z_i) \leq R_e(z_{ii})$ and $I_m(z_i) \leq I_m(z_{ii})$, where R_e denotes the real part and I_m denotes the imaginary part of a complex number. Accordingly, $z_i \leq z_{ii}$, if any one of the following conditions holds:

(C_1) $R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$,

(C_2) $R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$,

(C_3) $R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$,

(C_4) $R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$,

In particular, we can write $z_i \lneq z_{ii}$ if $z_i \neq z_{ii}$ and one of (C_2), (C_3), and (C_4) is satisfied.

Remark 1 (see [26]). The properties given below hold and can be verified:

(R_1) if $a_1, a_2 \in \mathbb{R}$ and $a_1 \leq a_2 \Rightarrow a_1 y \leq a_2 y \forall y \in \mathbb{C}$,

(R_2) $0 \leq z_i \lneq z_{ii} \Rightarrow |z_i| < |z_{ii}|$,

(R_3) $z_i \leq z_{ii}$ and $z_{ii} < z_{iii} \Rightarrow z_i < z_{iii}$.

Definition 2 (see [5]). Let V be a nonempty set and let $b \geq 1$ be a given real number. A function $\delta : V \times V \longrightarrow [0,\infty)$ is said to be a b-metric on V if it holds the following conditions:

(b_m1) $\delta(v_1, v_2) = 0 \Leftrightarrow v_1 = v_2$,

(b_m2) $\delta(v_1, v_2) = \delta(v_2, v_1)$,

(b_m3) $\delta(v_1, v_2) \leq b[\delta(v_1, v_3) + \delta(v_3, v_2)]$,

for all $v_1, v_2, v_3 \in V$. The pair (V, δ) is called a b-metric space, where b is a coefficient of (V, δ).

Definition 3 (see [25]). Let V be a nonempty set and let $b \geq 1$ be a given real number. A function $\delta : V \times V \longrightarrow \mathbb{C}$ is said to be a complex-valued b-metric on V if it holds the following conditions:

(Cb_m1) $\delta(v_1, v_2) \geq 0$ and $\delta(v_1, v_2) = 0$ if and only if $v_1 = v_2$,

(Cb_m2) $\delta(v_1, v_2) = \delta(v_2, v_1)$,

(Cb_m3) $\delta(v_1, v_2) \leq b[\delta(v_1, v_3) + \delta(v_3, v_2)]$,

for all $v_1, v_2, v_3 \in V$. The pair (V, δ) is called a complex-valued b-metric space, where b is a coefficient of (V, δ).

Example 1. Let $V = [0,\infty)$. The mapping $\delta : V \times V \longrightarrow \mathbb{C}$ is defined by

$$\delta(v_1, v_2) = \frac{7}{17}|v_1 - v_2|^2 + i\frac{7}{17}|v_1 - v_2|^2, \forall v_1, v_2 \in \mathbb{V}. \quad (1)$$

Then (V, δ) is a complex-valued b-metric space with $b = 2$.

Definition 4 (see [25, 26]). Let (V, δ) is a complex-valued b-metric space and $\{v_n\}$ be a sequence in V and $v \in V$. Consider the following:

(1) If there is $N_1 \in \mathbb{N}$ for every $c_1 \in \mathbb{C}$ and $0 < c_1$ such that for all $n > N_1$, $\delta(v_n, v) < c_1$, then $\{v_n\}$ is called convergent, $\{v_n\}$ converges to v, and v is a limit point of $\{v_n\}$. Mathematically, it can be written as $\lim_{n\longrightarrow\infty} v_n = v$ or $\{v_n\} \longrightarrow v$ as $n \longrightarrow \infty$.

(2) If there is $N_1 \in \mathbb{N}$ for every $c_1 \in \mathbb{C}$ and $0 < c_1$ such that for all $n > N_1$, $\delta(v_n, v_{n+m}) < c_1$, where $m \in \mathbb{N}$, then $\{v_n\}$ is said to be Cauchy sequence.

(3) If every Cauchy sequence is convergent, then (V, δ) is said to be complete complex-valued b-metric space.

Lemma 5 (see [25, 26]). *Let (V, δ) be a complex-valued b-metric space and let $\{v_n\}$ be a sequence in V. Then, $\{v_n\}$ converges to v iff $|\delta(v_n, v)| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Lemma 6 (see [25, 26]). *Let (V, δ) be a complex-valued b-metric space and let $\{v_n\}$ be a sequence in V. Then, $\{v_n\}$ is a cauchy sequence iff $|\delta(v_n, v_{n+m})| \longrightarrow 0$ as $n \longrightarrow \infty$.*

To prove the main result, we will use following lemma.

Lemma 7 (see [10]). *Let (V, δ) be a complex-valued b-metric space. Consider $\{v_n\}$ and $\{w_n\}$ be two sequences such that $\lim_{n\longrightarrow\infty} |\delta(v_n, w_n)| = 0$, whenever $\{v_n\}$ is a sequence in V such that $\lim_{n\longrightarrow\infty} v_n = z$ for some $z \in V$; then, $\lim_{n\longrightarrow\infty} w_n = z$.*

Proof. Given that

$$\lim_{n\longrightarrow\infty} |\delta(v_n, w_n)| = 0, \lim_{n\longrightarrow\infty} v_n = z. \quad (2)$$

By triangular property of (V, δ),

$$\delta(w_n, z) \leq b[\delta(w_n, v_n) + \delta(v_n, z)] \Rightarrow |\delta(w_n, z)| \leq b[|\delta(w_n, v_n)| + |\delta(v_n, z)|]. \quad (3)$$

Now by applying $\lim_{n\longrightarrow\infty}$ and using (2), we have

$$\lim_{n\longrightarrow\infty} |\delta(w_n, z)| \leq \lim_{n\longrightarrow\infty} b|\delta(w_n, v_n)| + \lim_{n\longrightarrow\infty} b|\delta(v_n, z)| = 0 + 0. \quad (4)$$

Hence, we proved that $\lim_{n\longrightarrow\infty} w_n = z$.

Definition 8 (see [27]). Let (V, δ) be a complex-valued b-metric space. A pair (J, K) is said to be compatible *iff* $\lim_{n\longrightarrow\infty} |\delta(JKv_n, KJv_n)| = 0$, whenever $\{v_n\}$ is a sequence in V such that

$$\lim_{n\longrightarrow\infty} Jv_n = \lim_{n\longrightarrow\infty} Kv_n = z \text{ for some } z \in V. \quad (5)$$

3. Main Result

Theorem 9. *Let (V, δ) be a complete complex-valued b-metric space and let $J, K, f : V \longrightarrow V$ be three self-mappings satisfying the following:*

$$\begin{aligned}\delta(Jv_1, Kv_2) \leq{}& \eta_1\delta(fv_1, fv_2) + \eta_2 \frac{\delta(fv_1, Jv_1)\delta(fv_2, Kv_2)}{1 + \delta(fv_1, fv_2)} \\ &+ \eta_3 \frac{[\delta(fv_1, Jv_1)\delta(fv_1, Kv_2) + \delta(fv_2, Kv_2)\delta(fv_2, Jv_1)]}{\delta(fv_1, Kv_2) + \delta(fv_2, Jv_1)} \\ &+ \eta_4 \max\{\delta(fv_1, Jv_1), \delta(fv_2, Kv_2), \delta \\ &\cdot (fv_1, Kv_2), \delta(fv_2, Jv_1)\},\end{aligned} \quad (6)$$

for all $v_1, v_2 \in V$, and $\eta_1, \eta_2, \eta_3, \eta_4 \in [0, 1)$ with $(\eta_1 + \eta_2 + \eta_3 + 2b\eta_4) < 1$, where $b \geq 1$. If f is continuous and (f, J), (f, K) are compatible, then f, J and K have a unique common fixed point in V.

Proof. Fix $v_0 \in V$, and we define some sequences in V such that

$$\hbar_{2n} = fv_{2n+1} = Jv_{2n}, \hbar_{2n+1} = fv_{2n+2} = Kv_{2n+1}, \forall n \geq 0. \quad (7)$$

Now by using (6),

$$\begin{aligned}\delta(\hbar_{2n}, \hbar_{2n+1}) ={}& \delta(Jv_{2n}, Kv_{2n+1}) \leq \eta_1\delta(fv_{2n}, fv_{2n+1}) \\ &+ \eta_2 \frac{\delta(fv_{2n}, Jv_{2n})\delta(fv_{2n+1}, Kv_{2n+1})}{1 + \delta(fv_{2n}, fv_{2n+1})} \\ &+ \eta_3 \frac{[\delta(fv_{2n}, Jv_{2n})\delta(fv_{2n}, Kv_{2n+1}) + \delta(fv_{2n+1}, Kv_{2n+1})\delta(fv_{2n+1}, Jv_{2n})]}{\delta(fv_{2n}, Kv_{2n+1}) + \delta(fv_{2n+1}, Jv_{2n})} \\ &+ \eta_4 \max\{\delta(fv_{2n}, Jv_{2n}), \delta(fv_{2n+1}, Kv_{2n+1}), \delta \\ &\cdot (fv_{2n}, Kv_{2n+1}), \delta(fv_{2n+1}, Jv_{2n})\} \\ ={}& \eta_1\delta(\hbar_{2n-1}, \hbar_{2n}) + \eta_2 \frac{\delta(\hbar_{2n-1}, \hbar_{2n})\delta(\hbar_{2n}, \hbar_{2n+1})}{1 + \delta(\hbar_{2n-1}, \hbar_{2n})} \\ &+ \eta_3 \frac{[\delta(\hbar_{2n-1}, \hbar_{2n})\delta(\hbar_{2n-1}, \hbar_{2n+1}) + \delta(\hbar_{2n}, \hbar_{2n+1})\delta(\hbar_{2n}, \hbar_{2n})]}{\delta(\hbar_{2n-1}, \hbar_{2n+1}) + \delta(\hbar_{2n}, \hbar_{2n})} \\ &+ \eta_4 \max\{\delta(\hbar_{2n-1}, \hbar_{2n}), \delta(\hbar_{2n}, \hbar_{2n+1}), \delta(\hbar_{2n-1}, \hbar_{2n+1}), \delta(\hbar_{2n}, \hbar_{2n})\}.\end{aligned} \quad (8)$$

This implies that

$$\begin{aligned}|\delta(\hbar_{2n}, \hbar_{2n+1})| \leq{}& \eta_1|\delta(\hbar_{2n-1}, \hbar_{2n})| + \eta_2 \frac{|\delta(\hbar_{2n-1}, \hbar_{2n})||\delta(\hbar_{2n}, \hbar_{2n+1})|}{|1 + \delta(\hbar_{2n-1}, \hbar_{2n})|} \\ &+ \eta_3 \frac{[|\delta(\hbar_{2n-1}, \hbar_{2n})||\delta(\hbar_{2n-1}, \hbar_{2n+1})| + |\delta(\hbar_{2n}, \hbar_{2n+1})||\delta(\hbar_{2n}, \hbar_{2n})|]}{|\delta(\hbar_{2n-1}, \hbar_{2n+1})| + |\delta(\hbar_{2n}, \hbar_{2n})|} \\ &+ \eta_4 \max\{|\delta(\hbar_{2n-1}, \hbar_{2n})|, |\delta(\hbar_{2n}, \hbar_{2n+1})|, |\delta(\hbar_{2n-1}, \hbar_{2n+1})|, \\ &\cdot |\delta(\hbar_{2n}, \hbar_{2n})|\}.\end{aligned} \quad (9)$$

After simplification, we get that

$$\begin{aligned}|\delta(\hbar_{2n}, \hbar_{2n+1})| \leq{}& (\eta_1 + \eta_3)|\delta(\hbar_{2n-1}, \hbar_{2n})| + \eta_2|\delta(\hbar_{2n}, \hbar_{2n+1})| \\ &+ \eta_4 \max\{|\delta(\hbar_{2n-1}, \hbar_{2n})|, |\delta(\hbar_{2n}, \hbar_{2n+1})|, \\ &\cdot |\delta(\hbar_{2n-1}, \hbar_{2n+1})|\}.\end{aligned} \quad (10)$$

Now there are three possibilities:

(i) If $\delta(\hbar_{2n-1}, \hbar_{2n})$ is a maximum term in $\{|\delta(\hbar_{2n-1}, \hbar_{2n})|, |\delta(\hbar_{2n}, \hbar_{2n+1})|, |\delta(\hbar_{2n-1}, \hbar_{2n+1})|\}$, then after simplification, (10) can be written as follows:

$$|\delta(\hbar_{2n}, \hbar_{2n+1})| \leq a_1|\delta(\hbar_{2n-1}, \hbar_{2n})|, \text{ where } a_1 = \frac{\eta_1 + \eta_3 + \eta_4}{1 - \eta_2} < 1. \quad (11)$$

(ii) If $\delta(\hbar_{2n}, \hbar_{2n+1})$ is a maximum term in $\{|\delta(\hbar_{2n-1}, \hbar_{2n})|, |\delta(\hbar_{2n}, \hbar_{2n+1})|, |\delta(\hbar_{2n-1}, \hbar_{2n+1})|\}$, then after simplification, (10) can be written as follows:

$$|\delta(\hbar_{2n}, \hbar_{2n+1})| \leq a_2|\delta(\hbar_{2n-1}, \hbar_{2n})|, \text{ where } a_2 = \frac{\eta_1 + \eta_3}{1 - \eta_2 - \eta_4} < 1. \quad (12)$$

(iii) If $\delta(\hbar_{2n-1}, \hbar_{2n+1})$ is a maximum term in $\{|\delta(\hbar_{2n-1}, \hbar_{2n})|, |\delta(\hbar_{2n}, \hbar_{2n+1})|, |\delta(\hbar_{2n-1}, \hbar_{2n+1})|\}$, and by using the triangular property of complex-valued b-metric space, then after simplification, (10) can be written as follows:

$$|\delta(\hbar_{2n}, \hbar_{2n+1})| \leq a_3 |\delta(\hbar_{2n-1}, \hbar_{2n})|, \text{where } a_3 = \frac{\eta_1 + \eta_3 + \eta_4 b}{1 - \eta_2 - \eta_4 b} < 1. \tag{13}$$

Let $a := \max\{a_1, a_2, a_3\} < 1$; then, from (11), (12), and (13), for all $n \geq 0$, we have

$$|\delta(\hbar_{2n}, \hbar_{2n+1})| \leq a |\delta(\hbar_{2n-1}, \hbar_{2n})|. \tag{14}$$

Similarly,

$$|\delta(\hbar_{2n-1}, \hbar_{2n})| \leq a |\delta(\hbar_{2n-2}, \hbar_{2n-1})|. \tag{15}$$

Now from (15) and (14), and by induction, we have that,

$$\begin{aligned} |\delta(\hbar_{2n}, \hbar_{2n+1})| &\leq a |\delta(\hbar_{2n-1}, \hbar_{2n})| \leq a^2 |\delta(\hbar_{2n-2}, \hbar_{2n-1})| \\ &\leq \cdots \leq a^{2n} |\delta(\hbar_0, \hbar_1)|. \end{aligned} \tag{16}$$

Next, we show that $\{\hbar_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ and $m > n$, then we have

$$\begin{aligned} |\delta(\hbar_n, \hbar_m)| &\leq b|\delta(\hbar_n, \hbar_{n+1})| + b|\delta(\hbar_{n+1}, \hbar_m)| \\ &\leq b|\delta(\hbar_n, \hbar_{n+1})| + b^2|\delta(\hbar_{n+1}, \hbar_{n+2})| + \cdots + b^{m-n}|\delta(\hbar_{m-1}, \hbar_m)| \\ &\leq ba^n|\delta(\hbar_0, \hbar_1)| + b^2 a^{n+1}|\delta(\hbar_0, \hbar_1)| + \cdots + b^{m-n} a^{m-1}|\delta(\hbar_0, \hbar_1)| \\ &\leq \left[ba^n + b^2 a^{n+1} + \cdots + b^{m-n} a^{m-1}\right] |\delta(\hbar_0, \hbar_1)| \\ &= \left[ba^n + b^2 a^{n+1} + \cdots + b^{m-n} a^{m-1}\right] |\delta(\hbar_0, \hbar_1)| \\ &= ba^n \left[1 + ba + b^2 a^2 \cdots + b^{m-(n+1)} a^{m-(n+1)}\right] |\delta(\hbar_0, \hbar_1)| \\ &= ba^n \sum_{t=0}^{m-(n+1)} b^t a^t |\delta(\hbar_0, \hbar_1)| \leq ba^n \sum_{t=0}^{\infty} b^t a^t |\delta(\hbar_0, \hbar_1)| \\ &= \frac{ba^n}{1 - ba} |\delta(\hbar_0, \hbar_1)| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned} \tag{17}$$

Hence, $\{\hbar_n\}$ is a Cauchy sequence. Since V is a complete complex-valued b-metric space, there exists $p \in V$ such that $\hbar_n \longrightarrow p$, as $n \longrightarrow \infty$ or $\lim_{n \longrightarrow \infty} \hbar_n = p$, and from (7), we have

$$\lim_{n \longrightarrow \infty} f v_{2n+1} = p, \lim_{n \longrightarrow \infty} J v_{2n} = p, \lim_{n \longrightarrow \infty} K v_{2n+1} = p. \tag{18}$$

Since f is a continuous self-map on V, therefore

$$\lim_{n \longrightarrow \infty} f(f v_{2n+1}) = fp, \lim_{n \longrightarrow \infty} f(J v_{2n}) = fp, \lim_{n \longrightarrow \infty} f(K v_{2n+1}) = fp. \tag{19}$$

As a pair (f, J) is compatible, so for some sequence $\{v_{2n}\}$ in V and by the definition of compatibility, we have that

$$\lim_{n \longrightarrow \infty} |\delta(J(f v_{2n}), f(J v_{2n}))| = 0. \tag{20}$$

Now from (19), (20), and by using Lemma 7, we have

$$\lim_{n \longrightarrow \infty} J(f v_{2n}) = fp. \tag{21}$$

Next, we have to show that $fp = p$, so by putting $v_1 = f v_{2n}$ and $v_2 = v_{2n+1}$, in (6),

$$\begin{aligned} \delta(J(f v_{2n}), K v_{2n+1}) \leq{}& \eta_1 \delta(f(f v_{2n}), f v_{2n+1}) + \eta_2 \frac{\delta(f(f v_{2n}), J(f v_{2n}))\delta(f v_{2n+1}, K v_{2n+1})}{1 + \delta(f(f v_{2n}), f v_{2n+1})} \\ &+ \eta_3 \frac{[\delta(f(f v_{2n}), J(f v_{2n}))\delta(f(f v_{2n}), K v_{2n+1}) + \delta(f v_{2n+1}, K v_{2n+1})\delta(f v_{2n+1}, J(f v_{2n}))]}{\delta(f(f v_{2n}), K v_{2n+1}) + \delta(f v_{2n+1}, J(f v_{2n}))} \\ &+ \eta_4 \max\{\delta(f(f v_{2n}), J(f v_{2n})), \delta(f v_{2n+1}, K v_{2n+1}), \delta \\ &\cdot (f(f v_{2n}), K v_{2n+1})), \delta(f v_{2n+1}, J(f v_{2n}))\}. \end{aligned} \tag{22}$$

This implies that

$$\begin{aligned} |\delta(J(f v_{2n}), K v_{2n+1})| \leq{}& \eta_1 |\delta(f(f v_{2n}), f v_{2n+1})| + \eta_2 \frac{|\delta(f(f v_{2n}), J(f v_{2n}))||\delta(f v_{2n+1}, K v_{2n+1})|}{|1 + \delta(f(f v_{2n}), f v_{2n+1})|} \\ &+ \eta_3 \frac{[|\delta(f(f v_{2n}), J(f v_{2n}))||\delta(f(f v_{2n}), K v_{2n+1})| + |\delta(f v_{2n+1}, K v_{2n+1})||\delta(f v_{2n+1}, J(f v_{2n}))|]}{|\delta(f(f v_{2n}), K v_{2n+1})| + |\delta(f v_{2n+1}, J(f v_{2n}))|} \\ &+ \eta_4 \max\{|\delta(f(f v_{2n}), J(f v_{2n}))|, |\delta(f v_{2n+1}, K v_{2n+1})|, \\ &\cdot |\delta(f(f v_{2n}), K v_{2n+1})|, |\delta(f v_{2n+1}, J(f v_{2n}))|\}. \end{aligned} \tag{23}$$

Now applying $\lim_{n \longrightarrow \infty}$ on both sides and from (18), (19), and (21), we get that

$$\begin{aligned} |\delta(fp, p)| \leq{}& \eta_1 |\delta(fp, p)| + \eta_2 \frac{|\delta(fp, fp)||\delta(p, p)|}{|1 + \delta(fp, p)|} \\ &+ \eta_3 \frac{[|\delta(fp, fp)||\delta(fp, p)| + |\delta(p, p)||\delta(p, fp)|]}{|\delta(fp, p)| + |\delta(p, fp)|} \\ &+ \eta_4 \max\{|\delta(fp, fp)|, |\delta(p, p)|, |\delta(fp, p)|, |\delta(p, fp)|\}. \end{aligned} \tag{24}$$

After simplification, we get that

$$|\delta(fp, p)| \leq (\eta_1 + \eta_4)|\delta(fp, p)| \Rightarrow (1 - \eta_1 - \eta_4)|\delta(fp, p)| \leq 0. \tag{25}$$

Since $(1 - \eta_1 - \eta_4) \neq 0 \Rightarrow |\delta(fp, p)| = 0$, hence we get that

$$fp = p. \tag{26}$$

Next, we have to show that $Jp = p$, and by the view of (6),

$$\begin{aligned} \delta(Jp, f v_{2n+2}) = \delta(Jp, K v_{2n+1}) \leq{}& \eta_1 \delta(fp, f v_{2n+1}) + \eta_2 \frac{\delta(fp, Jp)\delta(f v_{2n+1}, K v_{2n+1})}{1 + \delta(fp, f v_{2n+1})} \\ &+ \eta_3 \frac{[\delta(fp, Jp)\delta(fp, K v_{2n+1}) + \delta(f v_{2n+1}, K v_{2n+1})\delta(f v_{2n+1}, Jp)]}{\delta(fp, K v_{2n+1}) + \delta(f v_{2n+1}, Jp)} \\ &+ \eta_4 \max\{\delta(fp, Jp), \delta(f v_{2n+1}, K v_{2n+1}), \delta(fp, K v_{2n+1}), \delta \\ &\cdot (f v_{2n+1}, Jp)\}. \end{aligned} \tag{27}$$

This implies that

$$\begin{aligned}|\delta(Jp, fv_{2n+2})| \le{}& \eta_1|\delta(fp, fv_{2n+1})| + \eta_2 \frac{|\delta(fp, Jp)||\delta(fv_{2n+1}, Kv_{2n+1})|}{|1+\delta(fp, fv_{2n+1})|}\\ &+ \eta_3 \frac{[|\delta(fp, Jp)||\delta(fp, Kv_{2n+1})| + |\delta(fv_{2n+1}, Kv_{2n+1})||\delta(fv_{2n+1}, Jp)|]}{|\delta(fp, Kv_{2n+1})| + |\delta(fv_{2n+1}, Jp)|}\\ &+ \eta_4 \max\{|\delta(fp, Jp)|, |\delta(fv_{2n+1}, Kv_{2n+1})|, |\delta(fp, Kv_{2n+1})|,\\ &\cdot |\delta(fv_{2n+1}, Jp)|\}.\end{aligned} \tag{28}$$

Now again applying $\lim_{n\longrightarrow\infty}$ on both sides and by using (18) and (26), we have that

$$\begin{aligned}|\delta(Jp, p)| \le{}& \eta_1|\delta(fp, p)| + \eta_2 \frac{|\delta(fp, Jp)||\delta(p, p)|}{|1+\delta(fp, p)|}\\ &+ \eta_3 \frac{[|\delta(fp, Jp)||\delta(fp, p)| + |\delta(p, p)||\delta(p, Jp)|]}{|\delta(fp, p)| + |\delta(p, Jp)|} + \eta_4 \max\\ &\cdot \{|\delta(fp, Jp)|, |\delta(p, p)|, |\delta(fp, p)|, |\delta(p, Jp)|\} = \eta_4|\delta(p, Jp)|.\end{aligned} \tag{29}$$

This implies that $(1-\eta_4)|\delta(Jp, p)| \le 0$. Since $(1-\eta_4) \ne 0 \Rightarrow |\delta(Jp, p)| = 0$, hence

$$Jp = p. \tag{30}$$

Now, we have to show that $Kp = p$, and by using (6),

$$\begin{aligned}\delta(fv_{2n+1}, Kp) = \delta(Jv_{2n}, Kp) \le{}& \eta_1\delta(fv_{2n}, fp) + \eta_2 \frac{\delta(fv_{2n}, Jv_{2n})\delta(fp, Kp)}{1+\delta(fv_{2n}, fp)}\\ &+ \eta_3 \frac{[\delta(fv_{2n}, Jv_{2n})\delta(fv_{2n}, Kp) + \delta(fp, Kp)\delta(fp, Jv_{2n})]}{\delta(fv_{2n}, Kp) + \delta(fp, Jv_{2n})}\\ &+ \eta_4 \max\{\delta(fv_{2n}, Jv_{2n}), \delta(fp, Kp), \delta(fv_{2n}, Kp), \delta\\ &\cdot (fp, Jv_{2n})\}.\end{aligned} \tag{31}$$

This implies that

$$\begin{aligned}|\delta(fv_{2n+1}, Kp)| \le{}& \eta_1|\delta(fv_{2n}, fp)| + \eta_2 \frac{|\delta(fv_{2n}, Jv_{2n})||\delta(fp, Kp)|}{|1+\delta(fv_{2n}, fp)|}\\ &+ \eta_3 \frac{[|\delta(fv_{2n}, Jv_{2n})||\delta(fv_{2n}, Kp)| + |\delta(fp, Kp)||\delta(fp, Jv_{2n})|]}{|\delta(fv_{2n}, Kp)| + |\delta(fp, Jv_{2n})|}\\ &+ \eta_4 \max\{|\delta(fv_{2n}, Jv_{2n})|, |\delta(fp, Kp)|, |\delta(fv_{2n}, Kp)|,\\ &\cdot |\delta(fp, Jv_{2n})|\}.\end{aligned} \tag{32}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides and by using (18) and (26), we have that

$$\begin{aligned}|\delta(p, Kp)| \le{}& \eta_1|\delta(p, fp)| + \eta_2 \frac{|\delta(p, p)||\delta(fp, Kp)|}{|1+\delta(p, fp)|}\\ &+ \eta_3 \frac{[|\delta(p, p)||\delta(p, Kp)| + |\delta(fp, Kp)||\delta(fp, p)|]}{|\delta(p, Kp)| + |\delta(fp, p)|} + \eta_4 \max\\ &\cdot \{|\delta(p, p)|, |\delta(fp, Kp)|, |\delta(p, Kp)|, |\delta(fp, p)|\} = \eta_4|\delta(p, Kp)|.\end{aligned} \tag{33}$$

This implies that $(1-\eta_4)|\delta(p, Kp)| \le 0$. Since $(1-\eta_4) \ne 0 \Rightarrow |\delta(p, Kp)| = 0$, hence

$$Kp = p. \tag{34}$$

Now from (26), (30), and (34), we get that p is a common fixed point of f, J and K, i.e.,

$$fp = Jp = Kp = p. \tag{35}$$

Uniqueness: assume that $p^* \in V$ is an other common fixed point of f, J and K along with p, i.e.,

$$\begin{aligned}fp &= Jp = Kp = p,\\ fp^* &= Jp^* = Kp^* = p^*.\end{aligned} \tag{36}$$

Then, from (6), we have that

$$\begin{aligned}\delta(p, p^*) = \delta(Jp, Kp^*) \le{}& \eta_1\delta(fp, fp^*) + \eta_2 \frac{\delta(fp, Jp)\delta(fp^*, Kp^*)}{1+\delta(fp, fp^*)}\\ &+ \eta_3 \frac{[\delta(fp, Jp)\delta(fp, Kp^*) + \delta(fp^*, Kp^*)\delta(fp^*, Jp)]}{\delta(fp, Kp^*) + \delta(fp^*, Jp)} + \eta_4 \max\\ &\cdot \{\delta(fp, Jp), \delta(fp^*, Kp^*), \delta(fp, Kp^*), \delta(fp^*, Jp)\} = (\eta_1+\eta_4)\delta(p, p^*).\end{aligned} \tag{37}$$

This implies that $|\delta(p, p^*)| \le (\eta_1+\eta_4)|\delta(p, p^*)| \Rightarrow (1-\eta_1-\eta_4)|\delta(p, p^*)| \le 0$, since $(1-\eta_1-\eta_4) \ne 0 \Rightarrow |\delta(p, p^*)| = 0 \Rightarrow p = p^*$. Hence, we proved that f, J and K have a unique common fixed point in V.

Remark 10.

(i) If we put $\eta_1 = \lambda$, $\eta_2 = \mu$, and $\eta_3 = \eta_4 = 0$ in Theorem 9, we get the results of [26] Theorem 15.

(ii) If we put $\eta_2 = a$ and $\eta_1 = \eta_3 = \eta_4 = 0$ in Theorem 9, we get the results of [26] Theorem 19.

Example 2. Let (V, δ) be a complex-valued b-metric space, where $V = [0, 1)$ and $\delta : V \times V \longrightarrow \mathbb{C}$ with $\delta(v_1, v_2) = 4|v_1 - v_2|^2/9 + i(4|v_1 - v_2|^2/9)$, for all $v_1, v_2 \in V$. Now to find the value of b, we have that

$$\begin{aligned}\delta(v_1, v_2) &= \frac{4|v_1-v_2|^2}{9} + i\frac{4|v_1-v_2|^2}{9}\\ &= \frac{4|(v_1-v_3)+(v_3-v_2)|^2}{9} + i\frac{4|(v_1-v_3)+(v_3-v_2)|^2}{9}\\ &\le \left[\frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9} + \frac{4}{9}(2|v_1-v_3||v_3-v_2|)\right]\\ &\quad + i\left[\frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9} + \frac{4}{9}(2|v_1-v_3||v_3-v_2|)\right]\\ &\le \left[\frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9} + \frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9}\right]\\ &\quad + i\left[\frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9} + \frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9}\right]\\ &= 2\left[\frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9}\right] + i2\left[\frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9}\right]\\ &= 2\left[\frac{4|v_1-v_3|^2}{9} + i\frac{4|v_1-v_3|^2}{9} + \frac{4|v_3-v_2|^2}{9} + i\frac{4|v_3-v_2|^2}{9}\right]\\ &= 2[\delta(v_1, v_3) + \delta(v_3, v_2)].\end{aligned} \tag{38}$$

That is, $\delta(v_1, v_2) \leq b[\delta(v_1, v_3) + \delta(v_3, v_2)]$ with $b = 2$. Now we define $J, K, f : V \longrightarrow V$ as

$$Jv_1 = Kv_1 = \frac{2v_1}{35} \text{ and } fv_1 = \frac{v_1}{5}. \tag{39}$$

Notice that

$$\left\{ \begin{array}{c} |\delta(fv_1, fv_2)|, \dfrac{|\delta(fv_1, Jv_1)||\delta(fv_2, Kv_2)|}{|1 + \delta(fv_1, fv_2)|}, \dfrac{[|\delta(fv_1, Jv_1)||\delta(fv_1, Kv_2)| + |\delta(fv_2, Kv_2)||\delta(fv_2, Jv_1)|]}{|\delta(fv_1, Kv_2)| + |\delta(fv_2, Jv_1)|}, \\ \max\{|\delta(fv_1, Jv_1)|, |\delta(fv_2, Kv_2)|, |\delta(fv_1, Kv_2)|, |\delta(fv_2, Jv_1)|\} \end{array} \right\} \geq 0. \tag{40}$$

In all regards, it is enough to show that $\delta(Jv_1, Kv_2) \leq \eta_1 \delta(fv_1, fv_2)$, for all $v_1, v_2 \in [0, 1]$ and $\eta_1, \eta_2, \eta_3, \eta_4 \in [0, 1)$, with $(\eta_1 + \eta_2 + \eta_3 + 2\eta_4 b) < 1$.

$$\begin{aligned} \delta(Jv_1, Kv_2) &= \left[\frac{4|Jv_1 - Kv_2|^2}{9} + i\frac{4|Jv_1 - Kv_2|^2}{9}\right] \\ &= \left[\frac{4|2v_1/35 - 2v_2/35|^2}{9} + i\frac{4|2v_1/35 - 2v_2/35|^2}{9}\right] \\ &= \left(\frac{2}{7}\right)^2 \left[\frac{4|v_1/5 - v_2/5|^2}{9} + i\frac{4|v_1/5 - v_2/5|^2}{9}\right] \\ &= \frac{4}{49}\left[\frac{4|v_1/5 - v_2/5|^2}{9} + i\frac{4|v_1/5 - v_2/5|^2}{9}\right]. \end{aligned} \tag{41}$$

And

$$\begin{aligned} \delta(fv_1, fv_2) &= \left[\frac{4|fv_1 - fv_2|^2}{9} + i\frac{4|fv_1 - fv_2|^2}{9}\right] \\ &= \left[\frac{4|v_1/5 - v_2/5|^2}{9} + i\frac{4|v_1/5 - v_2/5|^2}{9}\right]. \end{aligned} \tag{42}$$

For $v_1, v_2 \in [0, 1]$, we discuss different cases with $\eta_1 = 2/5, \eta_2 = 1/5, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$. Hence,

$$\eta_1 + \eta_2 + \eta_3 + 2\eta_4 b = \frac{2}{5} + \frac{1}{5} + \frac{1}{10} + 2\left(\frac{1}{20}\right)2 < 1. \tag{43}$$

Case 1. Let $v_1 = 0, v_2 = 0$; then, from (41) and (42), directly we get that $\delta(Jv_1, Kv_2) \leq \eta_1 \delta(fv_1, fv_2)$. Hence, (6) is satisfied with $\eta_1 = 2/5, \eta_2 = 1/5, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Case 2. Let $v_1 = 1, v_2 = 0$; then, from (41) and (42), we find $\delta(Jv_1, Kv_2) \leq \eta_1 \delta(fv_1, fv_2)$ is satisfied with $\eta_1 = 2/5$, i.e,

$$\begin{aligned} &\frac{4}{49}\left[\frac{4|1/5 - 0/5|^2}{9} + i\frac{4|1/5 - 0/5|^2}{9}\right] \\ &\leq \eta_1 \left[\frac{4|1/5 - 0/5|^2}{9} + i\frac{4|1/5 - 0/5|^2}{9}\right] 0.0014[1 + i] \\ &\leq 0.0071[1 + i]. \end{aligned} \tag{44}$$

Hence, (6) is satisfied with $\eta_1 = 2/5, \eta_2 = 1/5, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Case 3. Let $v_1 = 1/2, v_2 = 1/4$; then from (41) and (42), we find $\delta(Jv_1, Kv_2) \leq \eta_1 \delta(fv_1, fv_2)$ is satisfied with $\eta_1 = 2/5$, i.e,

$$\begin{aligned} \frac{4}{49}\left[\frac{4}{3600} + i\frac{4}{3600}\right] &\leq \eta_1 \left[\frac{4}{3600} + i\frac{4}{3600}\right] 0.000090[1 + i] \\ &\leq 0.00044[1 + i]. \end{aligned} \tag{45}$$

Hence, (6) is satisfied with $\eta_1 = 2/5, \eta_2 = 1/5, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Case 4. Let $v_1 = 1, v_2 = 1$; then, from (41) and (42), directly we get that $\delta(Jv_1, Kv_2) \leq \eta_1 \delta(fv_1, fv_2)$. Hence, (6) is satisfied with $\eta_1 = 2/5, \eta_2 = 1/5, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$. Thus, all the conditions of Theorem 9 are satisfied with noticing that the point $0 \in V$, which remains fixed under mappings f, J and K, is indeed unique.

Corollary 11. *Let (V, δ) be a complete complex-valued b-metric space and let $J, K, f : V \longrightarrow V$ be three self-mappings satisfying the following:*

$$\begin{aligned} \delta(Jv_1, Kv_2) \leq{}& \eta_1 \delta(fv_1, fv_2) + \eta_2 \frac{\delta(fv_1, Jv_1)\delta(fv_2, Kv_2)}{1 + \delta(fv_1, fv_2)} \\ &+ \eta_3 \frac{[\delta(fv_1, Jv_1)\delta(fv_1, Kv_2) + \delta(fv_2, Kv_2)\delta(fv_2, Jv_1)]}{\delta(fv_1, Kv_2) + \delta(fv_2, Jv_1)} \\ &+ \eta_4[\delta(fv_1, Jv_1) + \delta(fv_2, Kv_2)], \end{aligned} \tag{46}$$

for all $v_1, v_2 \in V$ and $\eta_1, \eta_2, \eta_3, \eta_4 \in [0, 1)$ such that $(\eta_1 + \eta_2 + \eta_3 + 2\eta_4) < 1$. If f is a continuous self-mapping and (f, J), (f, K) are compatible, then f, J and K have a unique common fixed point in V.

Corollary 12. *Let (V, δ) be a complete complex-valued b-metric space and let $J, K, f : V \longrightarrow V$ be three self-mappings satisfying the following:*

$$\begin{aligned}\delta(Jv_1,Kv_2)\leq{}&\eta_1\delta(fv_1,fv_2)+\eta_2\frac{\delta(fv_1,Jv_1)\delta(fv_2,Kv_2)}{1+\delta(fv_1,fv_2)}\\&+\eta_3\frac{[\delta(fv_1,Jv_1)\delta(fv_1,Kv_2)+\delta(fv_2,Kv_2)\delta(fv_2,Jv_1)]}{\delta(fv_1,Kv_2)+\delta(fv_2,Jv_1)}\\&+\eta_4[\delta(fv_1,Kv_2)+\delta(fv_2,Jv_1)],\end{aligned}\tag{47}$$

for all $v_1,v_2\in V$ and $\eta_1,\eta_2,\eta_3,\eta_4\in[0,1)$ such that $(\eta_1+\eta_2+\eta_3+2\eta_4 b)<1$, where $b\geq 1$. If f is continuous and (f,J), (f,K) are compatible, then f, J and K have a unique common fixed point in V.

Theorem 13. *Let (V,δ) be a complete complex-valued b-metric space and let $J,K,f:V\longrightarrow V$ be three self-mappings satisfying:*

$$\begin{aligned}\delta(Jv_1,Kv_2)\leq{}&\eta_1\delta(fv_1,fv_2)+\eta_2\frac{\delta(fv_1,Jv_1)\delta(fv_2,Kv_2)}{1+\delta(fv_1,fv_2)}\\&+\eta_3\frac{[\delta(fv_1,Jv_1)\delta(fv_1,Kv_2)+\delta(fv_2,Kv_2)\delta(fv_2,Jv_1)]}{\delta(fv_1,Kv_2)+\delta(fv_2,Jv_1)}\\&+\eta_4[\delta(fv_1,Jv_1)+\delta(fv_2,Kv_2)+\delta(fv_1,Kv_2)+\delta(fv_2,Jv_1)],\end{aligned}\tag{48}$$

for all $v_1,v_2\in V$, $\eta_1,\eta_2,\eta_3\in[0,1)$, and $\eta_4\in[0,1/4)$, such that $(\eta_1+\eta_2+\eta_3+2\eta_4+2\eta_4 b)<1$, where $b\geq 1$. If f is a continuous self-mapping and (f,J), (f,K) are compatible, then f, J and K have a unique common fixed point in V.

Proof. Fix $v_0\in V$, and we define some sequences in V such that

$$\hbar_{2n}=fv_{2n+1}=Jv_{2n},\hbar_{2n+1}=fv_{2n+2}=Kv_{2n+1},\text{ for all }n\geq 0.\tag{49}$$

Now by view of (48) and (49),

$$\begin{aligned}\delta(\hbar_{2n},\hbar_{2n+1})={}&\delta(Jv_{2n},Kv_{2n+1})\leq\eta_1\delta(fv_{2n},fv_{2n+1})\\&+\eta_2\frac{\delta(fv_{2n},Jv_{2n})\delta(fv_{2n+1},Kv_{2n+1})}{1+\delta(fv_{2n},fv_{2n+1})}\\&+\eta_3\frac{[\delta(fv_{2n},Jv_{2n})\delta(fv_{2n},Kv_{2n+1})+\delta(fv_{2n+1},Kv_{2n+1})\delta(fv_{2n+1},Jv_{2n})]}{\delta(fv_{2n},Kv_{2n+1})+\delta(fv_{2n+1},Jv_{2n})}\\&+\eta_4[\delta(fv_{2n},Jv_{2n})+\delta(fv_{2n+1},Kv_{2n+1})+\delta(fv_{2n},Kv_{2n+1})\\&+\delta(fv_{2n+1},Jv_{2n})]=\eta_1\delta(\hbar_{2n-1},\hbar_{2n})+\eta_2\frac{\delta(\hbar_{2n-1},\hbar_{2n})\delta(\hbar_{2n},\hbar_{2n+1})}{1+\delta(\hbar_{2n-1},\hbar_{2n})}\\&+\eta_3\frac{[\delta(\hbar_{2n-1},\hbar_{2n})\delta(\hbar_{2n-1},\hbar_{2n+1})+\delta(\hbar_{2n},\hbar_{2n+1})\delta(\hbar_{2n},\hbar_{2n})]}{\delta(\hbar_{2n-1},\hbar_{2n+1})+\delta(\hbar_{2n},\hbar_{2n})}\\&+\eta_4[\delta(\hbar_{2n-1},\hbar_{2n})+\delta(\hbar_{2n},\hbar_{2n+1})+\delta(\hbar_{2n-1},\hbar_{2n+1})+\delta(\hbar_{2n},\hbar_{2n})].\end{aligned}\tag{50}$$

This implies that

$$\begin{aligned}|\delta(\hbar_{2n},\hbar_{2n+1})|\leq{}&\eta_1|\delta(\hbar_{2n-1},\hbar_{2n})|+\eta_2\frac{|\delta(\hbar_{2n-1},\hbar_{2n})||\delta(\hbar_{2n},\hbar_{2n+1})|}{|1+\delta(\hbar_{2n-1},\hbar_{2n})|}\\&+\eta_3\frac{[|\delta(\hbar_{2n-1},\hbar_{2n})||\delta(\hbar_{2n-1},\hbar_{2n+1})|+|\delta(\hbar_{2n},\hbar_{2n+1})||\delta(\hbar_{2n},\hbar_{2n})|]}{|\delta(\hbar_{2n-1},\hbar_{2n+1})|+|\delta(\hbar_{2n},\hbar_{2n})|}\\&+\eta_4[|\delta(\hbar_{2n-1},\hbar_{2n})|+|\delta(\hbar_{2n},\hbar_{2n+1})|+|\delta(\hbar_{2n-1},\hbar_{2n+1})|\\&+|\delta(\hbar_{2n},\hbar_{2n})|].\end{aligned}\tag{51}$$

Now by using triangular inequality of (V,δ) and after simplification, we get that

$$|\delta(\hbar_{2n},\hbar_{2n+1})|\leq\left(\frac{\eta_1+\eta_3+\eta_4+\eta_4 b}{1-\eta_2-\eta_4-\eta_4 b}\right)|\delta(\hbar_{2n-1},\hbar_{2n})|.\tag{52}$$

Again by view of (48) and (49),

$$\begin{aligned}\delta(\hbar_{2n-1},\hbar_{2n})={}&\delta(Kv_{2n-1},Jv_{2n})=\delta(Jv_{2n},Kv_{2n-1})\\\leq{}&\eta_1\delta(fv_{2n},fv_{2n-1})+\eta_2\frac{\delta(fv_{2n},Jv_{2n})\delta(fv_{2n-1},Kv_{2n-1})}{1+\delta(fv_{2n},fv_{2n-1})}\\&+\eta_3\frac{[\delta(fv_{2n},Jv_{2n})\delta(fv_{2n},Kv_{2n-1})+\delta(fv_{2n-1},Kv_{2n-1})\delta(fv_{2n-1},Jv_{2n})]}{\delta(fv_{2n},Kv_{2n-1})+\delta(fv_{2n-1},Jv_{2n})}\\&+\eta_4[\delta(fv_{2n},Jv_{2n})+\delta(fv_{2n-1},Kv_{2n-1})+\delta(fv_{2n},Kv_{2n-1})\\&+\delta(fv_{2n-1},Jv_{2n})]=\eta_1\delta(\hbar_{2n-1},\hbar_{2n-2})\\&+\eta_2\frac{\delta(\hbar_{2n-1},\hbar_{2n})\delta(\hbar_{2n-2},\hbar_{2n-1})}{1+\delta(\hbar_{2n-1},\hbar_{2n-2})}\\&+\eta_3\frac{[\delta(\hbar_{2n-1},\hbar_{2n})\delta(\hbar_{2n-1},\hbar_{2n-1})+\delta(\hbar_{2n-2},\hbar_{2n-1})\delta(\hbar_{2n-2},\hbar_{2n})]}{\delta(\hbar_{2n-1},\hbar_{2n-1})+\delta(\hbar_{2n-2},\hbar_{2n})}\\&+\eta_4[\delta(\hbar_{2n-1},\hbar_{2n})+\delta(\hbar_{2n-2},\hbar_{2n-1})+\delta(\hbar_{2n-1},\hbar_{2n-1})+\delta(\hbar_{2n-2},\hbar_{2n})].\end{aligned}\tag{53}$$

This implies that

$$\begin{aligned}|\delta(\hbar_{2n-1},\hbar_{2n})|\leq{}&\eta_1|\delta(\hbar_{2n-1},\hbar_{2n-2})|+\eta_2\frac{|\delta(\hbar_{2n-1},\hbar_{2n})||\delta(\hbar_{2n-2},\hbar_{2n-1})|}{|1+\delta(\hbar_{2n-1},\hbar_{2n-2})|}\\&+\eta_3\frac{[|\delta(\hbar_{2n-1},\hbar_{2n})||\delta(\hbar_{2n-1},\hbar_{2n-1})|+|\delta(\hbar_{2n-2},\hbar_{2n-1})||\delta(\hbar_{2n-2},\hbar_{2n})|]}{|\delta(\hbar_{2n-1},\hbar_{2n-1})|+|\delta(\hbar_{2n-2},\hbar_{2n})|}\\&+\eta_4[|\delta(\hbar_{2n-1},\hbar_{2n})|+|\delta(\hbar_{2n-2},\hbar_{2n-1})|+|\delta(\hbar_{2n-1},\hbar_{2n-1})|\\&+|\delta(\hbar_{2n-2},\hbar_{2n})|].\end{aligned}\tag{54}$$

By using triangular inequality of (V,δ) and after simplification we get that

$$|\delta(\hbar_{2n-1},\hbar_{2n})|\leq\left(\frac{\eta_1+\eta_3+\eta_4+\eta_4 b}{1-\eta_2-\eta_4-\eta_4 b}\right)|\delta(\hbar_{2n-2},\hbar_{2n-1})|.\tag{55}$$

Now from (55) and (52) and by induction, we have

$$\begin{aligned}|\delta(\hbar_{2n},\hbar_{2n+1})|\leq q|\delta(\hbar_{2n-1},\hbar_{2n})|&\leq q^2|\delta(\hbar_{2n-2},\hbar_{2n-1})|\\&\leq\cdots\leq q^{2n}|\delta(\hbar_0,\hbar_1)|,\end{aligned}\tag{56}$$

where $q=(\eta_1+\eta_3+\eta_4+\eta_4 b)/(1-\eta_2-\eta_4-\eta_4 b)<1$. Next, we have to show that $\{\hbar_n\}$ is a Cauchy sequence. Let $m,n\in\mathbb{N}$ and $m>n$. Then, we have

$$\begin{aligned}|\delta(\hbar_n,\hbar_m)|&\leq b|\delta(\hbar_n,\hbar_{n+1})|+b|\delta(\hbar_{n+1},\hbar_m)|\\&\leq b|\delta(\hbar_n,\hbar_{n+1})|+b^2|\delta(\hbar_{n+1},\hbar_{n+2})|+\cdots+b^{m-n}|\delta(\hbar_{m-1},\hbar_m)|\\&\leq bq^n|\delta(\hbar_0,\hbar_1)|+b^2q^{n+1}|\delta(\hbar_0,\hbar_1)|+\cdots+b^{m-n}q^{m-1}|\delta(\hbar_0,\hbar_1)|\\&\leq\left[bq^n+b^2q^{n+1}+\cdots+b^{m-n}q^{m-1}\right]|\delta(\hbar_0,\hbar_1)|\\&=bq^n\left[1+bq+b^2q^2\cdots+b^{m-(n+1)}q^{m-(n+1)}\right]|\delta(\hbar_0,\hbar_1)|\\&=bq^n\sum_{t=0}^{m-(n+1)}b^tq^t|\delta(\hbar_0,\hbar_1)|\leq bq^n\sum_{t=0}^{\infty}b^tq^t|\delta(\hbar_0,\hbar_1)|\\&=\frac{bq^n}{1-bq}|\delta(\hbar_0,\hbar_1)|\longrightarrow 0,\text{ as }n\longrightarrow\infty.\end{aligned}\tag{57}$$

Hence, $\{\hbar_n\}$ is a Cauchy sequence. Since V is complete, there exists some $p \in V$, such that $\hbar_n \longrightarrow p$, as $n \longrightarrow \infty$, and from (49), we have that

$$\lim_{n\longrightarrow\infty} f v_{2n+1} = p, \lim_{n\longrightarrow\infty} J v_{2n} = p, \lim_{n\longrightarrow\infty} K v_{2n+1} = p. \tag{58}$$

f is a continuous self-mapping on V, so that

$$\lim_{n\longrightarrow\infty} f(f v_{2n+1}) = fp, \lim_{n\longrightarrow\infty} f(J v_{2n}) = fp, \lim_{n\longrightarrow\infty} f(K v_{2n+1}) = fp. \tag{59}$$

Since (f, J) is compatible and for some sequence $\{v_{2n}\}$ in V, we have that

$$\lim_{n\longrightarrow\infty} |\delta(J(f v_{2n}), f(J v_{2n}))| = 0. \tag{60}$$

From (59), (60), and by using Lemma 7, we get that

$$\lim_{n\longrightarrow\infty} J(f v_{2n}) = fp. \tag{61}$$

Now, we have to show that $fp = p$. So, by putting $v_1 = f v_{2n}$ and $v_2 = v_{2n+1}$ in (48),

$$\begin{aligned}\delta(J(fv_{2n}), Kv_{2n+1}) \le{}& \eta_1 \delta(f(fv_{2n}), fv_{2n+1}) + \eta_2 \frac{\delta(f(fv_{2n}), J(fv_{2n}))\delta(fv_{2n+1}, Kv_{2n+1})}{1 + \delta(f(fv_{2n}), fv_{2n+1})} \\ &+ \eta_3 \frac{[\delta(f(fv_{2n}), J(fv_{2n}))\delta(f(fv_{2n}), Kv_{2n+1}) + \delta(fv_{2n+1}, Kv_{2n+1})\delta(fv_{2n+1}, J(fv_{2n}))]}{\delta(f(fv_{2n}), Kv_{2n+1}) + \delta(fv_{2n+1}, J(fv_{2n}))} \\ &+ \eta_4[\delta(f(fv_{2n}), J(fv_{2n})) + \delta(fv_{2n+1}, Kv_{2n+1}) + \delta(f(fv_{2n}), Kv_{2n+1}) \\ &+ \delta(fv_{2n+1}, J(fv_{2n}))].\end{aligned} \tag{62}$$

This implies that

$$\begin{aligned}|\delta(J(fv_{2n}), Kv_{2n+1})| \le{}& \eta_1 |\delta(f(fv_{2n}), fv_{2n+1})| + \eta_2 \frac{|\delta(f(fv_{2n}), J(fv_{2n}))||\delta(fv_{2n+1}, Kv_{2n+1})|}{|1 + \delta(f(fv_{2n}), fv_{2n+1})|} \\ &+ \eta_3 \frac{[|\delta(f(fv_{2n}), J(fv_{2n}))||\delta(f(fv_{2n}), Kv_{2n+1})| + |\delta(fv_{2n+1}, Kv_{2n+1})||\delta(fv_{2n+1}, J(fv_{2n}))|]}{|\delta(f(fv_{2n}), Kv_{2n+1})| + |\delta(fv_{2n+1}, J(fv_{2n}))|} \\ &+ \eta_4[|\delta(f(fv_{2n}), J(fv_{2n}))| + |\delta(fv_{2n+1}, Kv_{2n+1})| + |\delta(f(fv_{2n}), Kv_{2n+1})| \\ &+ |\delta(fv_{2n+1}, J(fv_{2n}))|].\end{aligned} \tag{63}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides and using (58), (59), and (61), we get that

$$\begin{aligned}|\delta(fp, p)| \le{}& \eta_1 |\delta(fp, p)| + \eta_2 \frac{|\delta(fp, fp)||\delta(p, p)|}{|1 + \delta(fp, p)|} \\ &+ \eta_3 \frac{[|\delta(fp, fp)||\delta(fp, p)| + |\delta(p, p)||\delta(p, fp)|]}{|\delta(fp, p)| + |\delta(p, fp)|} \\ &+ \eta_4[|\delta(fp, fp)| + |\delta(p, p)| + |\delta(fp, p)| + |\delta(p, fp)|] \\ ={}& (\eta_1 + 2\eta_4)|\delta(fp, p)|.\end{aligned} \tag{64}$$

This implies that $(1 - \eta_1 - 2\eta_4)|\delta(fp, p)| \le 0$. Since $(1 - \eta_1 - 2\eta_4) \ne 0 \Rightarrow |\delta(fp, p)| = 0$, hence,

$$fp = p. \tag{65}$$

Next, we have to show that $Jp = p$, and by using (48),

$$\begin{aligned}\delta(Jp, fv_{2n+2}) = \delta(Jp, Kv_{2n+1}) \le{}& \eta_1 \delta(fp, fv_{2n+1}) \\ &+ \eta_2 \frac{\delta(fp, Jp)\delta(fv_{2n+1}, Kv_{2n+1})}{1 + \delta(fp, fv_{2n+1})} \\ &+ \eta_3 \frac{[\delta(fp, Jp)\delta(fp, Kv_{2n+1}) + \delta(fv_{2n+1}, Kv_{2n+1})\delta(fv_{2n+1}, Jp)]}{\delta(fp, Kv_{2n+1}) + \delta(fv_{2n+1}, Jp)} \\ &+ \eta_4[\delta(fp, Jp) + \delta(fv_{2n+1}, Kv_{2n+1}) + \delta(fp, Kv_{2n+1}) \\ &+ \delta(fv_{2n+1}, Jp)].\end{aligned} \tag{66}$$

This implies that

$$\begin{aligned}|\delta(Jp, fv_{2n+2})| \le{}& \eta_1 |\delta(fp, fv_{2n+1})| + \eta_2 \frac{|\delta(fp, Jp)||\delta(fv_{2n+1}, Kv_{2n+1})|}{|1 + \delta(fp, fv_{2n+1})|} \\ &+ \eta_3 \frac{[|\delta(fp, Jp)||\delta(fp, Kv_{2n+1})| + |\delta(fv_{2n+1}, Kv_{2n+1})||\delta(fv_{2n+1}, Jp)|]}{|\delta(fp, Kv_{2n+1})| + |\delta(fv_{2n+1}, Jp)|} \\ &+ \eta_4[|\delta(fp, Jp)| + |\delta(fv_{2n+1}, Kv_{2n+1})| + |\delta(fp, Kv_{2n+1})| \\ &+ |\delta(fv_{2n+1}, Jp)|].\end{aligned} \tag{67}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides and using (58) and (65), we have

$$\begin{aligned}|\delta(Jp, p)| \le{}& \eta_1 |\delta(fp, p)| + \eta_2 \frac{|\delta(fp, Jp)||\delta(p, p)|}{|1 + \delta(fp, p)|} \\ &+ \eta_3 \frac{[|\delta(fp, Jp)||\delta(fp, p)| + |\delta(p, p)||\delta(p, Jp)|]}{|\delta(fp, p)| + |\delta(p, Jp)|} \\ &+ \eta_4[|\delta(fp, Jp)| + |\delta(p, p)| + |\delta(fp, p)| + |\delta(p, Jp)|] \\ ={}& \eta_1 |\delta(p, p)| + \eta_2 \frac{|\delta(p, Jp)||\delta(p, p)|}{|1 + \delta(p, p)|} \\ &+ \eta_3 \frac{[|\delta(p, Jp)||\delta(p, p)| + |\delta(p, p)||\delta(p, Jp)|]}{|\delta(p, p)| + |\delta(p, Jp)|} \\ &+ \eta_4[|\delta(p, Jp)| + |\delta(p, p)| + |\delta(p, p)| + |\delta(p, Jp)|].\end{aligned} \tag{68}$$

Thus, we get that $|\delta(Jp, p)| \le 2\eta_4 |\delta(p, Jp)| \Rightarrow (1 - 2\eta_4)|\delta(Jp, p)| \le 0$. Since $(1 - 2\eta_4) \ne 0$, as $(\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + 2\eta_4 b) < 1$, therefore $|\delta(Jp, p)| = 0$. Hence,

$$Jp = p. \tag{69}$$

Now, we have to show that $Kp = p$, and by using (48),

$$\begin{aligned}\delta(fv_{2n+1}, Kp) = \delta(Jv_{2n}, Kp) \le{}& \eta_1 \delta(fv_{2n}, fp) \\ &+ \eta_2 \frac{\delta(fv_{2n}, Jv_{2n})\delta(fp, Kp)}{1 + \delta(fv_{2n}, fp)} \\ &+ \eta_3 \frac{[\delta(fv_{2n}, Jv_{2n})\delta(fv_{2n}, Kp) + \delta(fp, Kp)\delta(fp, Jv_{2n})]}{\delta(fv_{2n}, Kp) + \delta(fp, Jv_{2n})} \\ &+ \eta_4[\delta(fv_{2n}, Jv_{2n}) + \delta(fp, Kp) + \delta(fv_{2n}, Kp) \\ &+ \delta(fp, Jv_{2n})].\end{aligned} \tag{70}$$

This implies that

$$|\delta(fv_{2n+1}, Kp)| \le \eta_1 |\delta(fv_{2n}, fp)| + \eta_2 \frac{|\delta(fv_{2n}, Jv_{2n})||\delta(fp, Kp)|}{|1+\delta(fv_{2n}, fp)|} + \eta_3 \frac{[|\delta(fv_{2n}, Jv_{2n})||\delta(fv_{2n}, Kp)| + |\delta(fp, Kp)||\delta(fp, Jv_{2n})|]}{|\delta(fv_{2n}, Kp)| + |\delta(fp, Jv_{2n})|} + \eta_4[|\delta(fv_{2n}, Jv_{2n})| + |\delta(fp, Kp)| + |\delta(fv_{2n}, Kp)| + |\delta(fp, Jv_{2n})|]. \tag{71}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides and by view of (58) and (65), we have that

$$|\delta(p, Kp)| \le \eta_1 |\delta(p, fp)| + \eta_2 \frac{|\delta(p,p)||\delta(fp, Kp)|}{|1+\delta(p, fp)|} + \eta_3 \frac{[|\delta(p,p)||\delta(p, Kp)| + |\delta(fp, Kp)||\delta(fp, p)|]}{|\delta(p, Kp)| + |\delta(fp, p)|} + \eta_4[|\delta(p,p)| + |\delta(fp, Kp)| + |\delta(p, Kp)| + |\delta(fp, p)|] = 2\eta_4|\delta(p, Kp)|. \tag{72}$$

This implies that $(1-2\eta_4)|\delta(p, Kp)| \le 0$. Since $(1-2\eta_4) \ne 0$, therefore $|\delta(p, Kp)| = 0$. Hence,

$$Kp = p. \tag{73}$$

Thus, from (65), (69), and (73), we get that p is a common fixed point of f, J and K, i.e.,

$$fp = Jp = Kp = p. \tag{74}$$

Uniqueness: we contrary suppose that p^* is another common fixed point of f, J and K such that $fp^* = Jp^* = Kp^* = p^*$. Now, by (48),

$$\delta(p, p^*) = \delta(Jp, Kp^*) \le \eta_1 \delta(fp, fp^*) + \eta_2 \frac{\delta(fp, Jp)\delta(fp^*, Kp^*)}{1+\delta(fp, fp^*)} + \eta_3 \frac{[\delta(fp, Jp)\delta(fp, Kp^*) + \delta(fp^*, Kp^*)\delta(fp^*, Jp)]}{\delta(fp, Kp^*) + \delta(fp^*, Jp)} + \eta_4[\delta(fp, Jp) + \delta(fp^*, Kp^*) + \delta(fp, Kp^*) + \delta(fp^*, Jp)] = (\eta_1 + 2\eta_4)\delta(p, p^*). \tag{75}$$

This implies that $|\delta(p, p^*)| \le (\eta_1 + 2\eta_4)|\delta(p, p^*)| \Rightarrow (1 - \eta_1 - 2\eta_4)|\delta(p, p^*)| \le 0$.Since $(1 - \eta_1 - \eta_4) \ne 0$, therefore $|\delta(p, p^*)| \le 0 \longrightarrow p = p^*$. Hence, we proved that f, J and K have a unique common fixed point in V.

Example 3. Let $V = [0,\infty)$ and $\delta : V \times V \longrightarrow \mathbb{C}$ defined by $\delta(v_1, v_2) = 3|v_1 - v_2|^2/13 + i(3|v_1 - v_2|^2/13)$ for all $v_1, v_2 \in V$ is a b-metric on V and (V, δ) is a complex-valued b-metric space. Now, first, we show that V is a b-metric with $b = 2$, so that

$$\begin{aligned}
\delta(v_1, v_2) &= \frac{3|v_1 - v_2|^2}{13} + i\frac{3|v_1 - v_2|^2}{13} \le \frac{3|(v_1 - v_3) + (v_3 - v_2)|^2}{13} + i\frac{3|(v_1 - v_3) + (v_3 - v_2)|^2}{13} \\
&\le \left[\frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13} + \frac{3}{13}(2|v_1 - v_3||v_3 - v_2|)\right] + i\left[\frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13} + \frac{3}{13}(2|v_1 - v_3||v_3 - v_2|)\right] \\
&\le \left[\frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13} + \frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13}\right] + i\left[\frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13} + \frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13}\right] \\
&= 2\left[\frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13}\right] + i2\left[\frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13}\right] \\
&= 2\left[\frac{3|v_1 - v_3|^2}{13} + i\frac{3|v_1 - v_3|^2}{13} + \frac{3|v_3 - v_2|^2}{13} + i\frac{3|v_3 - v_2|^2}{13}\right] \\
&= 2[\delta(v_1, v_3) + \delta(v_3, v_2)].
\end{aligned} \tag{76}$$

That is, $\delta(v_1, v_2) \le b[\delta(v_1, v_3) + \delta(v_3, v_2)]$, with $b = 2$.

Define $J, K, f : V \longrightarrow V$ by $Jv_1 = Kv_1 = \ln(1 + v_1/3 + v_1)$, and $fv_1 = e^{3v_1} - 1$.

Notice that

$$\left\{ \begin{array}{c} |\delta(fv_1, fv_2)|, \dfrac{|\delta(fv_1, Jv_1)||\delta(fv_2, Kv_2)|}{|1+\delta(fv_1, fv_2)|}, \dfrac{[|\delta(fv_1, Jv_1)||\delta(fv_1, Kv_2)| + |\delta(fv_2, Kv_2)||\delta(fv_2, Jv_1)|]}{|\delta(fv_1, Kv_2)| + |\delta(fv_2, Jv_1)|} \\ [|\delta(fv_1, Jv_1)| + |\delta(fv_2, Kv_2)| + |\delta(fv_1, Kv_2)| + |\delta(fv_2, Jv_1)|] \end{array} \right\} \ge 0, \tag{77}$$

in all regards. It is enough to show that $\delta(Jv_1, Kv_2) \le \eta_1 \delta(fv_1, fv_2)$, for all $v_1, v_2 \in [0,\infty)$ and $\eta_1, \eta_2, \eta_3 \in [0,1)$, $\eta_4 \in [0, 1/4)$ such that $(\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + 2\eta_4 b) < 1$, where $b \ge 1$, and we have

$$\begin{aligned}\delta(Jv_1, Kv_2) &= \left[\frac{3|Jv_1 - Kv_2|^2}{13} + i\frac{3|Jv_1 - Kv_2|^2}{13}\right] \\ &= \left[\frac{3|\ln(1+v_1/3+v_1) - \ln(1+v_2/3+v_2)|^2}{13}\right. \\ &\quad \left. + i\frac{3|\ln(1+v_1/3+v_1) - \ln(1+v_2/3+v_2)|^2}{13}\right] \\ &\le \left[\frac{3|v_1/3+v_1 - v_2/3+v_2|^2}{13} + i\frac{3|v_1/3+v_1 - v_2/3+v_2|^2}{13}\right] \\ &\le \left[\frac{3|3v_1 - 3v_2/9|^2}{13} + i\frac{3|3v_1 - 3v_2/9|^2}{13}\right] \\ &= \frac{1}{9^2}\left[\frac{3|3v_1 - 3v_2|^2}{13} + i\frac{3|3v_1 - 3v_2|^2}{13}\right] \\ &\le \frac{1}{81}\left[\frac{3\left|e^{3v_1} - e^{3v_2}\right|^2}{13} + i\frac{3\left|e^{3v_1} - e^{3v_2}\right|^2}{13}\right].\end{aligned} \tag{78}$$

And

$$\begin{aligned}\delta(fv_1, fv_2) &= \left[\frac{3|fv_1 - fv_2|^2}{13} + i\frac{3|fv_1 - fv_2|^2}{13}\right] \\ &= \left[\frac{3\left|\left(e^{3v_1}-1\right) - \left(e^{3v_2}-1\right)\right|^2}{13} + i\frac{3\left|\left(e^{3v_1}-1\right) - \left(e^{3v_2}-1\right)\right|^2}{13}\right] \\ &= \left[\frac{3\left|e^{3v_1} - e^{3v_2}\right|^2}{13} + i\frac{3\left|e^{3v_1} - e^{3v_2}\right|^2}{13}\right].\end{aligned} \tag{79}$$

For $v_1, v_2 \in [0,\infty)$, we discuss different cases with $\eta_1 = 1/5, \eta_2 = 1/4, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$. Notice that $\eta_1 + \eta_2 + \eta_3 + 2\eta_4 b = 1/5 + 1/4 + 1/10 + 2(1/20)2 < 1$.

Case 1. Let $v_1 = 0, v_2 = 0$. Then, from (78) and (79), directly we get that $\delta(Jv_1, Kv_2) \le \eta_1 \delta(fv_1, fv_2)$. Hence, (48) is satisfied with $\eta_1 = 1/5, \eta_2 = 1/4, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Case 2. Let $v_1 = 0, v_2 = 1$; then, from (78) and (79), we find $\delta(Jv_1, Kv_2) \le \eta_1 \delta(fv_1, fv_2)$ is satisfied with $\eta_1 = 1/5$, as

$$\frac{1}{81}\left[\frac{3\left|1 - e^3\right|^2}{13} + i\frac{3\left|1 - e^3\right|^2}{13}\right] \le \eta_1\left[\frac{3\left|1 - e^3\right|^2}{13} + i\frac{3\left|1 - e^3\right|^2}{13}\right]. \tag{80}$$

By using $\eta_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}&\frac{1}{81}\left[\frac{3|-19.0855|^2}{13} + i\frac{3|-19.0855|^2}{13}\right] \\ &\quad \le \frac{1}{5}\left[\frac{3|-19.0855|^2}{13} + i\frac{3|-19.0855|^2}{13}\right] 1.04[1+i] \\ &\quad \le 16.81[1+i].\end{aligned} \tag{81}$$

Hence, (48) is satisfied with $\eta_1 = 1/5, \eta_2 = 1/4, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Case 3. Let $v_1 = 1/2, v_2 = 1/4$; then, from (78) and (79), we find $\delta(Jv_1, Kv_2) \le \eta_1 \delta(fv_1, fv_2)$, is true for $\eta_1 = 1/5$, as

$$\begin{aligned}&\frac{1}{81}\left[\frac{3\left|e^{3/2} - e^{3/4}\right|^2}{13} + i\frac{3\left|e^{3/2} - e^{3/4}\right|^2}{13}\right] \\ &\quad \le \eta_1\left[\frac{3\left|e^{3/2} - e^{3/4}\right|^2}{13} + i\frac{3\left|e^{3/2} - e^{3/4}\right|^2}{13}\right].\end{aligned} \tag{82}$$

By using $\eta_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}&\frac{1}{81}\left[\frac{3|2.3646|^2}{13} + i\frac{3|2.3646|^2}{13}\right] \\ &\quad \le \frac{1}{5}\left[\frac{3|2.3646|^2}{13} + i\frac{3|2.3646|^2}{13}\right] 0.02[1+i] \le 0.26[1+i].\end{aligned} \tag{83}$$

Hence, (48) is satisfied with $\eta_1 = 1/5, \eta_2 = 1/4, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Case 4. Let $v_1 = 1/2, v_2 = 1$; then, from (78) and (79), we get that $\delta(Jv_1, Kv_2) \le \eta_1 \delta(fv_1, fv_2)$ is true for $\eta_1 = 1/5$, as

$$\begin{aligned}&\frac{1}{81}\left[\frac{3\left|e^{3/2} - e^{3}\right|^2}{13} + i\frac{3\left|e^{3/2} - e^{3}\right|^2}{13}\right] \\ &\quad \le \eta_1\left[\frac{3\left|e^{3/2} - e^{3}\right|^2}{13} + i\frac{3\left|e^{3/2} - e^{3}\right|^2}{13}\right].\end{aligned} \tag{84}$$

By using $\eta_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}&\frac{1}{81}\left[\frac{3|-15.6038|^2}{13} + i\frac{3|-15.6038|^2}{13}\right] \\ &\quad \le \frac{1}{5}\left[\frac{3|-15.6038|^2}{13} + i\frac{3|-15.6038|^2}{13}\right] 0.69[1+i] \\ &\quad \le 11.24[1+i].\end{aligned} \tag{85}$$

Hence, (48) is satisfied with $\eta_1 = 1/5, \eta_2 = 1/4, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Case 5. Let $v_1 = 1, v_2 = 5$; then, from (78) and (79), we find $\delta(Jv_1, Kv_2) \le \eta_1 \delta(fv_1, fv_2)$ is true for $\eta_1 = 1/5$, as

$$\frac{1}{81}\left[\frac{3|e^3 - e^{15}|^2}{13} + i\frac{3|e^3 - e^{15}|^2}{13}\right] \le \eta_1 \left[\frac{3|e^3 - e^{15}|^2}{13} + i\frac{3|e^3 - e^{15}|^2}{13}\right]. \quad (86)$$

By using $\eta_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}&\frac{1}{81}\left[\frac{3|-3268997.28|^2}{13} + i\frac{3|-3268997.28|^2}{13}\right]\\ &\le \frac{1}{5}\left[\frac{3|-3268997.28|^2}{13} + i\frac{3|-3268997.28|^2}{13}\right] 0.304 \times 10^{11}[1+i]\\ &\le 4.932 \times 10^{11}[1+i].\end{aligned} \quad (87)$$

Hence, (48) is satisfied with $\eta_1 = 1/5, \eta_2 = 1/4, \eta_3 = 1/10, \eta_4 = 1/20$, and $b = 2$.

Thus, all conditions of Theorem 13 are satisfied with noticing that the point $0 \in V$ remains fixed under mappings f, J and K and is indeed unique.

Corollary 14. *Let (V, δ) be a complete complex valued b-metric space and let $J, K, f : V \longrightarrow V$ be three self-mappings satisfying the following:*

$$\begin{aligned}\delta(Jv_1, Kv_2) \le{}& \eta_1 \delta(fv_1, fv_2) + \eta_2 \\ &\cdot \frac{[\delta(fv_1, Jv_1)\delta(fv_1, Kv_2) + \delta(fv_2, Kv_2)\delta(fv_2, Jv_1)]}{1 + \delta(fv_1, fv_2)} \\ &+ \eta_3[\delta(fv_1, Jv_1) + \delta(fv_2, Kv_2) + \delta(fv_1, Kv_2) \\ &+ \delta(fv_2, Jv_1)],\end{aligned} \quad (88)$$

for all $v_1, v_2 \in V$ and $\eta_1, \eta_2 \in [0, 1)$, $\eta_3 \in [0, 1/4)$, such that $(\eta_1 + 2\eta_2 b + 2\eta_3 + 2\eta_3 b) < 1$, where $b \ge 1$. If f is continuous and (f, J), (f, K) are compatible, then f, J and K have a unique common fixed point in V.

4. Applications

In this section, we present an integral type application to support our main work. For this purpose, we use the two UITEs to get the existing result of a common solution to verify the validity of our work. Let $V = C([k_1, k_2], \mathbb{R}^n)$ be the set of all real-valued continuous functions defined on $[k_1, k_2]$. In the following, we apply Theorem 9 to get the existing result of a common solution by using the two UITEs. Now we are in the position to present a theorem based on the two UITEs to get the existing result of a common solution to support our work.

Theorem 15 (see [23]). *Let $V = C([k_1, k_2], \mathbb{R}^n)$, where $[k_1, k_2] \subseteq \mathbb{R}$ and $\delta : V \times V \longrightarrow \mathbb{C}$ is defined as*

$$\delta(v_1, v_2) = \|v_1(p) - v_2(p)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (89)$$

for all $v_1, v_2 \in V$ and $p \in [k_1, k_2]$. Consider the UITEs are

$$\begin{aligned} v_1(p) &= \int_{k_1}^{k_2} Q_1(p, r, v_1(r))dr + \hbar_1(p), \\ v_2(p) &= \int_{k_1}^{k_2} Q_2(p, r, v_2(r))dr + \hbar_2(p), \end{aligned} \quad (90)$$

where $r \in [k_1, k_2]$. Let $Q_1, Q_2 : [k_1, k_2] \times [k_1, k_2] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are such that $D_{v_1}, E_{v_2} \in V$ for every $v_1, v_2 \in V$, and we have that

$$\begin{aligned} D_{v_1}(p) &= \int_{k_1}^{k_2} Q_1(p, r, v_1(r))dr, \\ E_{v_2}(p) &= \int_{k_1}^{k_2} Q_2(p, r, v_2(r))dr. \end{aligned} \quad (91)$$

If there exists $\mu \in (0, 1)$ such that for all $v_1, v_2 \in V$,

$$\|D_{v_1}(p) - E_{v_2}(p) + \hbar_1(p) - \hbar_2(p)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1} \le \mu M(v_1, v_2), \quad (92)$$

where

$$M(v_1, v_2) = \max\{A_1(v_1, v_2)(p), A_2(v_1, v_2)(p), A_3(v_1, v_2)(p), A_4(v_1, v_2)(p)\} \quad (93)$$

with

$$A_1(v_1, v_2)(p) = \|v_1(p) - v_2(p)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (94)$$

$$A_2(v_1, v_2)(p) = \frac{\|D_{v_1}(p) + \hbar_1(p) - v_1(p)\|^2 \|E_{v_2}(p) + \hbar_2(p) - v_2(p)\|^2 \left(\sqrt{1 + k_1^2} e^{i \cot k_1}\right)^2}{1 + \|v_1(p) - v_2(p)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}, \quad (95)$$

$$\begin{aligned} A_3(v_1, v_2)(p) ={}& \frac{\left(\|D_{v_1}(p) + \hbar_1(p) - v_1(p)\|^2 \|E_{v_2}(p) + \hbar_2(p) - v_1(p)\|^2 + \|E_{v_2}(p) + \hbar_2(p) - v_2(p)\|^2 \|D_{v_1}(p) + \hbar_2(p) - v_2(p)\|^2\right)}{\|E_{v_2}(p) + \hbar_2(p) - v_1(p)\|^2 + \|D_{v_1}(p) + \hbar_1(p) - v_2(p)\|^2} \\ &\times \sqrt{1 + k_1^2} e^{i \cot k_1}, \end{aligned} \quad (96)$$

$$A_4(v_1, v_2)(p) = \max\{a_1(v_1, v_2)(p), a_2(v_1, v_2)(p), a_3(v_1, v_2)(p), a_4(v_1, v_2)(p)\}, \quad (97)$$

where

$$\begin{aligned}
a_1(v_1,v_2)(p) &= \left\|D_{v_1}(p)+\hbar_1(p)-v_1(p)\right\|^2\sqrt{1+k_1^2}e^{i\cot k_1},\\
a_2(v_1,v_2)(p) &= \left\|E_{v_2}(p)+\hbar_2(p)-v_2(p)\right\|^2\sqrt{1+k_1^2}e^{i\cot k_1},\\
a_3(v_1,v_2)(p) &= \left\|E_{v_2}(p)+\hbar_2(p)-v_1(p)\right\|^2\sqrt{1+k_1^2}e^{i\cot k_1},\\
a_4(v_1,v_2)(p) &= \left\|D_{v_1}(p)+\hbar_1(p)-v_2(p)\right\|^2\sqrt{1+k_1^2}e^{i\cot k_1}.
\end{aligned} \tag{98}$$

Then, the two UITEs, i.e., (90), have a unique common solution.

Proof. Define $J, K, f : V \longrightarrow V$ as

$$\begin{aligned}
Jv_1 &= Jv_1(p) = D_{v_1}(p)+\hbar_1(p) = D_{v_1}+\hbar_1, fv_1 = fv_1(p) = v_1(p) = v_1,\\
Kv_2 &= Kv_2(p) = E_{v_2}(p)+\hbar_2(p) = E_{v_2}+\hbar_2, fv_2 = fv_2(p) = v_2(p) = v_2.
\end{aligned} \tag{99}$$

Then, we have the following four cases:

(1) If $A_1(v_1,v_2)(p)$ is the maximum term in $\{A_1(v_1,v_2)(p), A_2(v_1,v_2)(p), A_3(v_1,v_2)(p), A_4(v_1,v_2)(p)\}$, then from (92), (93), and (99), we have that

$$\delta(Jv_1,Kv_2) \le \mu\|v_1-v_2\|^2\sqrt{1+k_1^2}e^{i\cot k_1}, \tag{100}$$

for all $v_1, v_2 \in V$. Hence, the mappings J, K and f satisfy all the conditions of Theorem 9 with $\mu = \eta_1$ and $\eta_2 = \eta_3 = \eta_4 = 0$ in (6). Then, the given two UITEs, i.e., (90), have a unique common solution in V.

(2) If $A_2(v_1,v_2)(p)$ is the maximum term in $\{A_1(v_1,v_2)(p), A_2(v_1,v_2)(p), A_3(v_1,v_2)(p), A_4(v_1,v_2)(p)\}$, then from (92), (93), and (99), we have that

$$\delta(Jv_1,Kv_2) \le \mu\frac{\|D_{v_1}+\hbar_1-v_1\|^2\|E_{v_2}+\hbar_2-v_2\|^2\left(\sqrt{1+k_1^2}e^{i\cot k_1}\right)^2}{1+\|v_1-v_2\|^2\sqrt{1+k_1^2}e^{i\cot k_1}}, \tag{101}$$

for all $v_1, v_2 \in V$. Hence, the mappings J, K and f satisfy all the conditions of Theorem 9 with $\mu = \eta_2$ and $\eta_1 = \eta_3 = \eta_4 = 0$ in (6). Then, the given two UITEs, i.e., (90), have a unique common solution in V.

(3) If $A_3(v_1,v_2)(p)$ is the maximum term in $\{A_1(v_1,v_2)(p), A_2(v_1,v_2)(p), A_3(v_1,v_2)(p), A_4(v_1,v_2)(p)\}$, then from (92), (93), and (99), we have that

$$\delta(Jv_1,Kv_2) \le \mu\frac{\left(\|D_{v_1}+\hbar_1-v_1\|^2\|E_{v_2}+\hbar_2-v_1\|^2+\|E_{v_2}+\hbar_2-v_2\|^2\|D_{v_1}+\hbar_1-v_2\|^2\right)\sqrt{1+k_1^2}e^{i\cot k_1}}{\|E_{v_2}+\hbar_2-v_1\|^2+\|D_{v_1}+\hbar_1-v_2\|^2}, \tag{102}$$

for all $v_1, v_2 \in V$. Hence, the mappings J, K and f satisfy all the conditions of Theorem 9 with $\mu = \eta_3$ and $\eta_1 = \eta_2 = \eta_4 = 0$ in (6). Then, the given two UITEs, i.e., (90), have a unique common solution in V.

(4) If $A_4(v_1,v_2)(p)$ is the maximum term in $\{A_1(v_1,v_2)(p), A_2(v_1,v_2)(p), A_3(v_1,v_2)(p), A_4(v_1,v_2)(p)\}$, then from (93), we have that

$$M(v_1,v_2) = A_4(v_1,v_2)(p), \tag{103}$$

Then, there are furthermore four subcases arise:

(i) If $a_1(v_1,v_2)(p)$ is the maximum term in $\{a_1(v_1,v_2)(p), a_2(v_1,v_2)(p), a_3(v_1,v_2)(p), a_4(v_1,v_2)(p)\}$. Then, from (92), (97), (99), and (103), we have that

$$\delta(Jv_1,Kv_2) \le \mu\left\|D_{v_1}+\hbar_1-v_1\right\|^2\sqrt{1+k_1^2}e^{i\cot k_1}, \tag{104}$$

for all $v_1, v_2 \in V$. Hence, the mappings J, K and f satisfy all the conditions of Theorem 9 with $\mu = \eta_4$ and $\eta_1 = \eta_2 = \eta_3 = 0$ in (6). Then, the given two UITEs, i.e., (90), have a unique common solution in V.

(ii) If $a_2(v_1,v_2)(p)$ is the maximum term in $\{a_1(v_1,v_2)(p), a_2(v_1,v_2)(p), a_3(v_1,v_2)(p), a_4(v_1,v_2)(p)\}$. Then, from (92), (97), (99), and (103), we have that

$$\delta(Jv_1,Kv_2) \le \mu\left\|E_{v_2}+\hbar_2-v_2\right\|^2\sqrt{1+k_1^2}e^{i\cot k_1}, \tag{105}$$

for all $v_1, v_2 \in V$. Hence, the mappings J, K and f satisfy all the conditions of Theorem 9 with $\mu = \eta_4$ and $\eta_1 = \eta_2 = \eta_3 = 0$ in (6). Then, the given two UITEs, i.e., (90), have a unique common solution in V.

(iii) If $a_3(v_1,v_2)(p)$ is the maximum term in $\{a_1(v_1,v_2)(p), a_2(v_1,v_2)(p), a_3(v_1,v_2)(p), a_4(v_1,v_2)(p)\}$. Then, from (92), (97), (99), and (103), we have that

$$\delta(Jv_1, Kv_2) \le \mu \left\| E_{v_2} + \hbar_2 - v_1 \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (106)$$

for all $v_1, v_2 \in V$. Hence, the mappings J, K and f satisfy all the conditions of Theorem 9 with $\mu = \eta_4$ and $\eta_1 = \eta_2 = \eta_3 = 0$ in (6). Then, the given two UITEs, i.e., (90), have a unique common solution in V.

(iv) If $a_4(v_1, v_2)(p)$ is the maximum term in $\{a_1(v_1, v_2)(p), a_2(v_1, v_2)(p), a_3(v_1, v_2)(p), a_4(v_1, v_2)(p)\}$. Then, from (92), (97), (99), and (103), we have that

$$\delta(Jv_1, Kv_2) \le \mu \left\| D_{v_1} + \hbar_1 - v_2 \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (107)$$

for all $v_1, v_2 \in V$. Hence, the mappings J, K and f satisfy all the conditions of Theorem 9 with $\mu = \eta_4$ and $\eta_1 = \eta_2 = \eta_3 = 0$ in (6). Then, the given two UITEs, i.e., (90), have a unique common solution in V.

5. Conclusions

We proved some unique CFP-theorems in complex-valued b-metric spaces under the more generalized rational type contraction conditions for compatible three self-mappings in which a one self-map is continuous. Our results extend and improved many results given in the literature (e.g., see [26, 23]). In the support of our work, we presented some illustrative examples for three self-mappings in complex-valued b-metric spaces. Moreover, we presented an application of the two UITEs to get the existing result of a common solution to support our main work. In this direction, many results can be contributed to the complex-valued b-metric spaces by using different contractive type single-valued mappings with different types of applications.

Data Availability

Data sharing is not applicable to this article as no data set were generated or analysed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Research Article

Some Novel Generalized Strong Coupled Fixed Point Findings in Cone Metric Spaces with Application to Integral Equations

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Fixed point (FP) has been the heart of several areas of mathematics and other sciences. FP is a beautiful mixture of analysis (pure and applied), topology, and geometry. To construct the link between FP and applied mathematics, this paper will present some generalized strong coupled FP theorems in cone metric spaces. Our consequences give the generalization of “cyclic coupled Kannan-type contraction” given by Choudhury and Maity. We present illustrative examples in support of our results. This new concept will play an important role in the theory of fixed point results and can be generalized for different contractive-type mappings in the context of metric spaces. In addition, we also establish an application in integral equations for the existence of a common solution to support our work.

1. Introduction

In a wide range of engineering and mathematical problems, the occurrence of a solution is identical to the existence of a FP for a suitable mapping. The occurrence of a FP is therefore of greatest importance in many diverse fields of engineering and applied mathematics. FP consequences make provision to provide many conditions under which maps have solutions. In the last five decades or so, the theory of FP has been disclosed as an important tool in the field of nonlinear phenomena. Particularly, FP methods have been used in several areas such as engineering, physics, chemistry, economics, biology, and game theory. The objective of this paper is to find new FP consequences and their use in which the indispensability of the FP results is mentioned.

In 2003, Kirk et al. [1] proved the result for FP on an exceptional variety of maps which are familiar as cyclic contractive maps. Some of these works are in [2–5] (and the references therein). Later, Lakshmikantham and Ćirić [6] established a new idea of coupled FP, which is a beautiful mixture of analysis (pure and applied), topology, and geometry. It has also a lot of applications in a wide range of mathematical problems. In 2014, Choudhury and Maity [7] proved a strong coupled FP result for cyclic coupled Kannan-type mapping.

Hang and Zhang [8] in 2007 discovered the idea of cone metric space (CMS). Moreover, they have presented some preliminary concepts and proved a cone Banach contraction theorem. After the publication of this article, many researchers, i.e., Abbas and Jungck [9], Ilić and Rakočević [10], and P. Vetro [11] generalize their results for the FP, common FP, and coincidence points in CMS by using the contraction conditions. Some other related results can be found in (see [12–23] and the references are therein).

In this manuscript, we study the more general cyclic coupled cone contraction results in complete CMS (U, d_c) and prove that a cyclic mapping $\Gamma : U \times U \longrightarrow U$ has a strong coupled FP theorem in (U, d_c). Moreover, we present an integral-type application by using the concept of Chen et al. [24] and Jabeen et al. [25] to support our work. We shall present the illustrative examples and the two Urysohn integral equations (UIEs) for finding the solution of problems to hold up our consequences. This paper is organized as follows: Section 2 consists of preliminary concepts. In Section 3, we present some strong coupled FP results by using cyclic-type mappings in complete cone metric spaces with illustrative examples. In Section 4, we present an application of Voltera integral equations for the existence of a solution to support our main work. In the last section, which is Section 5, we discuss the conclusion.

2. Basic Definitions

Definition 1 (see [18]). A subset P of a real Banach space $\mathbf{E}$ is called a cone if

(i) $P \neq \varnothing$, closed, and $P \neq \{\theta\}$, where θ is the zero elements of $\mathbf{E}$

(ii) If $a, b \in [0,\infty)$ and any $\mu, \nu \in P$, then $a\mu + b\nu \in P$

(iii) If both $-u, \mu \in P$, then $\mu = \theta$

A partial ordering $\preccurlyeq$ on a given cone $P \subseteq \mathbf{E}$ is defined as $\mu \preccurlyeq \nu$ iff $\mu - \nu \in P$ and $\mu \prec \nu$ and if $\mu \preccurlyeq \nu$ and $\mu \neq \nu$ while $\mu \ll \nu$ iff $\mu - \nu \in \operatorname{int}(P)$. A cone P is known as a normal cone if $\exists \kappa > 0$ such that $\forall \mu, \nu \in E$:

$$\theta \preccurlyeq \mu \preccurlyeq \nu \Longrightarrow \|\mu\| \leq \kappa \|\nu\|, \quad \forall \mu, \nu \in \mathbf{E}. \tag{1}$$

Then, κ is known as a normal constant of P.

Definition 2 (see [8]). Let $U \neq \varnothing$ be the set and a mapping $d_c : U \times U \longrightarrow E$ is known as a cone metric, if d_c satisfies the following:

(i) $\theta \preccurlyeq d_c(\mu_1, \mu_2)$ and $d_c(\mu_1, \mu_2) = \theta \Longleftrightarrow \mu_1 = \mu_2$

$$\begin{aligned} d_c(\mu_1, \mu_2) &= d_c(\mu_2, \mu_1), \\ d_c(\mu_1, \mu_3) &\preccurlyeq d_c(\mu_1, \mu_2) + d_c(\mu_2, \mu_3), \end{aligned} \tag{2}$$

$\forall \mu_1, \mu_2, \mu_3 \in U$. Then, a pair (U, d_c) is called a CMS.

Definition 3 (see [8]). Let (U, d_c) be a CMS, and let $\mu \in U$ and $\{\mu_\ell\}$ be a sequence in U. Then,

(i) $\{\mu_\ell\}$ is said to be convergent and converges to μ if for any $c \gg \theta$ in E and N is a positive integer such that $d_c(\mu_\ell, \mu) \ll c$, for $\ell \geq N$. We denote this by $\lim_{\ell \longrightarrow \infty} \mu_\ell = \mu$ or $\mu_\ell \longrightarrow \mu$, as $\ell \longrightarrow \infty$

(ii) $\{\mu_\ell\}$ is said to be a Cauchy sequence if for any $c \gg \theta$ in E and N is a positive integer such that $d_c(\mu_\ell, \mu_k) \ll c$, for $\ell, k \geq N$

(iii) (U, d_c) is known as complete if every Cauchy sequence is convergent in U

Throughout this paper, $\mathbf{E}$ is a real Banach space, P may be a normal cone in E with normal constant κ, and $\operatorname{int}(P) \neq \varnothing$, and "$\preccurlyeq$" is a partial ordering w.r.t. P.

Lemma 4 (see [8]). *Let $\{\mu_\ell\}$ and $\{\nu_\ell\}$ be two sequences in (U, d_c). Then, the following statements hold:*

(i) $\lim_{\ell \longrightarrow \infty} \mu_\ell \longrightarrow \mu$ iff $\lim_{\ell \longrightarrow \infty} d_c(\mu_\ell, \mu) \longrightarrow \theta$, and the limit of a convergent sequence is unique

(ii) $\{\mu_\ell\}$ is a Cauchy sequence iff $\lim_{\ell,k \longrightarrow \infty} d_c(\mu_\ell, \mu_k) \longrightarrow \theta$

(iii) $\lim_{\ell \longrightarrow \infty} \mu_\ell \longrightarrow \mu$ and $\lim_{\ell \longrightarrow \infty} \nu_\ell \longrightarrow \nu$ imply that $\lim_{\ell \longrightarrow \infty} d_c(\mu_\ell, \nu_\ell) \longrightarrow d_c(\mu, \nu)$

For more details, we shall refer the readers to study [8].

Definition 5 (see [7]). Let G and H be two nonempty subsets of a given set U. We call that a function $\Gamma : U \times U \longrightarrow U$ such that $\Gamma(\mu, \nu) \in G$ if $\mu \in H$ and $\nu \in G$ and $\Gamma(\mu, \nu) \in H$ if $\mu \in G$ and $\nu \in H$ is a cyclic mapping w.r.t. G and H.

Definition 6 (see [7]). Let $U \neq \varnothing$ be the set, and an element $(\mu, \nu) \in U \times U$ is called a coupled FP of a mapping $\Gamma : U \times U \longrightarrow U$ if $\Gamma(\mu, \nu) = \mu$ and $\Gamma(\nu, \mu) = \nu$. Moreover, μ is called a strong coupled FP of Γ, if $\mu = \nu$, that is, $\Gamma(\mu, \mu) = \mu$.

Definition 7 (see [7]). Let G and H be two nonempty subsets of a metric space U. Then, a mapping $\Gamma : U \times U \longrightarrow U$ is known as a cyclic coupled Kannan-type contraction w.r.t. G and H if Γ is cyclic w.r.t. G and H and satisfies the inequality for some $a \in (0, 1/2)$, such that

$$d_c(\Gamma(x, y), \Gamma(\mu, \nu)) \leq a(d_c(x, \Gamma(x, y)) + d_c(\mu, \Gamma(\mu, \nu))), \tag{3}$$

where $x, \nu \in G$ and $y, \mu \in H$.

The following "cyclic coupled Kannan-type contraction theorem" was obtained in [7].

Theorem 8 (see [7]). *Assume that G and H are two nonempty closed subsets of a complete metric space (U, d_c), and let $\Gamma : U \times U \longrightarrow U$ be a cyclic coupled Kannan-type contraction w.r.t. G and H and $G \cap H \neq \varnothing$. Then, Γ has a strong coupled FP in $G \cap H$.*

3. Main Result

Now, we are in the position to present our main results.

Let $\Gamma : U \times U \longrightarrow U$ be a cyclic mapping w.r.t. G and H, where G and H are subsets of a CMS (U, d_c), under the generalized coupled cone contractive-type condition:

$$d_c(\Gamma(x, y), \Gamma(\mu, \nu)) \leq \alpha(d_c(x, \Gamma(x, y)) + d_c(\mu, \Gamma(\mu, \nu))) + \beta(d_c(x, \Gamma(\mu, \nu)) + d_c(\mu, \Gamma(x, y))), \tag{4}$$

where $x, \nu \in G$, $y, \mu \in H$, and $\alpha, \beta \in [0,\infty)$. Our results generalize and improve Theorem 8 (that is, Theorem 5 in [7]) (see Remark 12). Moreover, some illustrate examples, and the integral-type application is given in the paper to support our work.

Theorem 9. *Let G and H be two nonempty closed subsets of a complete CMS (U, d_c), and let $\Gamma : U \times U \longrightarrow U$ be a cyclic mapping w.r.t. G and H. Assume that Γ satisfies (4) with $2(\alpha + \beta) < 1$. Then, $G \cap H \neq \varnothing$ and Γ has a strong coupled FP in $G \cap H$.*

Proof. Fix $\nu_0 \in G$ and $\mu_0 \in H$, and let (ν_ℓ) and (μ_ℓ) be two sequences defined as

$$\nu_{\ell+1} = \Gamma(\mu_\ell, \nu_\ell), \quad \mu_{\ell+1} = \Gamma(\nu_\ell, \mu_\ell), \quad \forall \ell \geq 0. \tag{5}$$

Then, $\{\nu_\ell\} \subset G$ and $\{\mu_\ell\} \subset H$, since Γ is cyclic mapping w.r.t. G and H. Denote

$$\lambda = \frac{\alpha + \beta}{1 - (\alpha + \beta)}. \tag{6}$$

Then, $\lambda \in (0, 1)$ for $\alpha + \beta < 1/2$. We claim that

$$d_c(\nu_\ell, \mu_{\ell+1}) + d_c(\mu_\ell, \nu_{\ell+1}) \preccurlyeq \lambda^\ell (d_c(\nu_0, \mu_1) + d_c(\mu_0, \nu_1)), \quad \ell \geq 0. \tag{7}$$

It is clear that (7) holds for $\ell = 0$. Assume that (7) holds for $\ell = k$; then, by (4),

$$\begin{aligned} d_c(\nu_{k+1}, \mu_{k+2}) &= d_c(\Gamma(\mu_k, \nu_k), \Gamma(\nu_{k+1}, \mu_{k+1})) \\ &\preccurlyeq \alpha(d_c(\mu_k, \Gamma(\mu_k, \nu_k)) + d_c(\nu_{k+1}, \Gamma(\nu_{k+1}, \mu_{k+1}))) \\ &\quad + \beta(d_c(\mu_k, \Gamma(\nu_{k+1}, \mu_{k+1})) + d_c(\nu_{k+1}, \Gamma(\mu_k, \nu_k)) \\ &= \alpha(d_c(\mu_k, \nu_{k+1}) + d_c(\nu_{k+1}, \mu_{k+2})) \\ &\quad + \beta(d_c(\mu_k, \mu_{k+2}) + d_c(\nu_{k+1}, \nu_{k+1})) \\ &\preccurlyeq \alpha(d_c(\mu_k, \nu_{k+1}) + d_c(\nu_{k+1}, \mu_{k+2})) \\ &\quad + \beta(d_c(\mu_k, \nu_{k+1}) + d_c(\nu_{k+1}, \mu_{k+2})), \end{aligned} \tag{8}$$

which implies that

$$d_c(\nu_{k+1}, \mu_{k+2}) \preccurlyeq \lambda d_c(\mu_k, \nu_{k+1}). \tag{9}$$

Similarly, we can get

$$d_c(\mu_{k+1}, \nu_{k+2}) \preccurlyeq \lambda d_c(\nu_k, \mu_{k+1}). \tag{10}$$

Thus, by the induction hypothesis, i.e., (7) with $\ell = k$, we have

$$\begin{aligned} d_c(\nu_{k+1}, \mu_{k+2}) + d_c(\mu_{k+1}, \nu_{k+2}) &\preccurlyeq \lambda(d_c(\nu_k, \mu_{k+1}) + d_c(\mu_k, \nu_{k+1})) \\ &\preccurlyeq \lambda^2(d_c(\nu_{k-1}, \mu_k) + d_c(\mu_{k-1}, \nu_k)) \\ &\preccurlyeq \cdots \preccurlyeq \lambda^{k+1}(d_c(\nu_0, \mu_1) + d_c(\mu_0, \nu_1)). \end{aligned} \tag{11}$$

That is, (7) holds for $\ell = k + 1$. Therefore, we have proven that (7) holds for all $\ell \geq 0$ by induction. Meanwhile, by (4), for $\ell \geq 0$,

$$\begin{aligned} d_c(\nu_\ell, \nu_{\ell+1}) + d_c(\mu_\ell, \mu_{\ell+1}) &\preccurlyeq d_c(\nu_\ell, \mu_{\ell+1}) + d_c(\mu_{\ell+1}, \nu_{\ell+1}) \\ &\quad + d_c(\mu_\ell, \nu_{\ell+1}) + d_c(\nu_{\ell+1}, \mu_{\ell+1}) \\ &= d_c(\nu_\ell, \mu_{\ell+1}) + d_c(\mu_\ell, \nu_{\ell+1}) \\ &\quad + 2d_c(\Gamma(\mu_\ell, \nu_\ell), \Gamma(\nu_\ell, \mu_\ell)) \\ &\preccurlyeq d_c(\nu_\ell, \mu_{\ell+1}) + d_c(\mu_\ell, \nu_{\ell+1}) \\ &\quad + 2(\alpha(d_c(\mu_\ell, \nu_{\ell+1}) + d_c(\nu_\ell, \mu_{\ell+1})) \\ &\quad + \beta(d_c(\mu_\ell, \mu_{\ell+1}) + d_c(\nu_\ell, \nu_{\ell+1}))) \\ &= (1 + 2\alpha)(d_c(\nu_\ell, \mu_{\ell+1}) + d_c(\mu_\ell, \nu_{\ell+1})) \\ &\quad + 2\beta(d_c(\mu_\ell, \mu_{\ell+1}) + d_c(\nu_\ell, \nu_{\ell+1})). \end{aligned} \tag{12}$$

This together with (7) implies that

$$d_c(\nu_\ell, \nu_{\ell+1}) + d_c(\mu_\ell, \mu_{\ell+1}) \preccurlyeq \frac{1 + 2\alpha}{1 - 2\beta} \lambda^\ell (d_c(\nu_0, \mu_1) + d_c(\mu_0, \nu_1)), \quad \ell \geq 0. \tag{13}$$

Then, for $\ell, j \geq 0$, without loss of generality, we assume that $j > \ell$:

$$\begin{aligned} d_c(\nu_\ell, \nu_j) &\preccurlyeq \sum_{k=\ell}^{j-1} d_c(\nu_k, \nu_{k+1}) \preccurlyeq \sum_{k=\ell}^{j-1} \frac{1 + 2\alpha}{1 - 2\beta} \lambda^k (d_c(\nu_0, \mu_1) + d_c(\mu_0, \nu_1)) \\ &= \frac{1 + 2\alpha}{(1 - 2\beta)(1 - \lambda)} \lambda^\ell (d_c(\nu_0, \mu_1) + d_c(\mu_0, \nu_1)) \\ &\longrightarrow \theta, \quad \text{as } \ell \longrightarrow \infty. \end{aligned} \tag{14}$$

Hence, $\{\nu_\ell\}$ is a Cauchy sequence by Lemma 4 (ii), and the completeness of U yields that there exists $\nu \in U$ such that

$$\lim_{\ell \longrightarrow \infty} \nu_\ell = \nu \in G. \tag{15}$$

Similarly, we can get that

$$\lim_{\ell \longrightarrow \infty} \mu_\ell = \mu \in H. \tag{16}$$

Then, from Lemma 4 (iii), we have

$$\lim_{\ell \longrightarrow \infty} d_c(\nu_\ell, \mu_\ell) = d_c(\nu, \mu). \tag{17}$$

On the other hand, by (7) and (13),

$$\begin{aligned} d_c(\nu_\ell, \mu_\ell) &\preccurlyeq d_c(\nu_\ell, \nu_{\ell+1}) + d_c(\nu_{\ell+1}, \mu_{\ell+1}) \\ &\preccurlyeq \left(\frac{1+2\alpha}{1-2\beta} + 1\right)\lambda^\ell (d_c(\nu_0, \mu_1) + d_c(\mu_0, \nu_1) \longrightarrow \theta, \quad \text{as } \ell \longrightarrow \infty. \end{aligned} \tag{18}$$

Therefore, $d_c(\nu, \mu) = \theta$ by Lemma 4 (i), and $\nu = \mu \in G \cap H$.

Now, we have to prove that ν is a strong coupled FP of Γ. Indeed, by (4), (15), and (16), we have

$$\begin{aligned} d_c(\nu, \Gamma(\nu, \mu)) &\le d_c(\nu, \nu_{\ell+1}) + d_c(\nu_{\ell+1}, \Gamma(\nu, \mu)) \\ &= d_c(\nu, \nu_{\ell+1}) + d_c(\Gamma(\mu_\ell, \nu_\ell), \Gamma(\nu, \mu)) \\ &\preccurlyeq d_c(\nu, \nu_{\ell+1}) + \alpha(d_c(\mu_\ell, \Gamma(\mu_\ell, \nu_\ell)) \\ &\quad + d_c(\nu, \Gamma(\nu, \mu))) + \beta(d_c(\mu_\ell, \Gamma(\nu, \mu)) \\ &\quad + d_c(x, \Gamma(\mu_\ell, \nu_\ell))) \preccurlyeq (1+\beta) d_c(\nu, \nu_{\ell+1}) \\ &\quad + \alpha(d_c(\mu_\ell, \nu_{\ell+1}) + d_c(\nu, \Gamma(\nu, \mu))) \\ &\quad + \beta d_c(\mu_\ell, \Gamma(\nu, \mu)) \\ &\longrightarrow (\alpha+\beta) d_c(\nu, \Gamma(\nu, \mu)), \quad \text{as } \ell \longrightarrow \infty. \end{aligned} \tag{19}$$

This implies that $d_c(\nu, \Gamma(\nu, \mu)) = \theta$ since $\alpha + \beta < 1/2$. Then, $\Gamma(\nu, \mu) = \nu = \mu$, and ν is a strong coupled FP of Γ.

Corollary 10. *Let G and H be two nonempty closed subsets of a complete CMS (U, d_c), and let $\Gamma : U \times U \longrightarrow U$ be a cyclic coupled Kannan-type cone contraction w.r.t. G and H. That is,*

$$d_c(\Gamma(x, y), \Gamma(\mu, \nu)) \le \alpha(d_c(x, \Gamma(x, y)) + d_c(\mu, \Gamma(\mu, \nu))), \tag{20}$$

where $\alpha \in ((0, 1)/2)$, $x, \nu \in G$, and $y, \mu \in H$. Then, Γ has a strong coupled FP in $G \cap H$.

Corollary 11. *Let G and H be two nonempty closed subsets of a complete CMS (U, d_c), and let $\Gamma : U \times U \longrightarrow U$ be a cyclic coupled Chatterjea-type cone contraction w.r.t. G and H. That is,*

$$d_c(\Gamma(x, y), \Gamma(\mu, \nu)) \le \alpha(d_c(\mu, \Gamma(x, y)) + d_c(x, \Gamma(\mu, \nu))), \tag{21}$$

where $\alpha \in ((0, 1)/2)$, $x, \nu \in G$, and $y, \mu \in H$. Then, Γ has a strong coupled FP in $G \cap H$.

Remark 12. In the special case, when (U, d_c) is a complete metric space, Corollary 10 is the same as Theorem 8 (Theorem 5 in [7]), if $\beta = 0$ in (4). Therefore, our results generalize the result given in [7]. Moreover, the following example shows that Theorem 8 does not apply while Theorem 9 does.

Example 13. Let $U = \mathbb{R}$ in CMS which is defined as $d_c(\mu, \nu) = |\mu - \nu|$, $\forall \mu, \nu \in U$, and let $G = [-1, 0]$ and $H = [0, 1]$. Then, G and H are two nonempty closed subsets of a set U and $d(G, H) = 0$. A mapping $\Gamma : U \times U \longrightarrow U$ can be defined as

$$\Gamma(x, y) = \frac{-2x}{3}. \tag{22}$$

Then, easily one can prove that Γ is a cyclic mapping w.r.t. G and H, for any $x, \nu \in G$ and $y, \mu \in H$. A mapping Γ is not a cyclic coupled Kannan-type contraction, since

$$d_c(\Gamma(x, y), \Gamma(\mu, \nu)) = |\Gamma(x, y) - \Gamma(\mu, \nu)| = \frac{2|x-\mu|}{3}, \tag{23}$$

where $a = 2/3 \notin ((0, 1)/2)$. Now, from (4), we have

$$\begin{aligned} d_c(\Gamma(x, y), \Gamma(\mu, \nu)) &= \frac{2|x-\mu|}{3} \le \frac{2|x+\mu|}{3} \le \frac{17|x+\mu|}{21} \\ &= \frac{10|x+\mu|}{21} + \frac{|x+\mu|}{3} = \frac{2}{7}\left|\frac{5x+5\mu}{3}\right| \\ &\quad + \frac{1}{5}\left|\frac{5x+5\mu}{3}\right| = \frac{2}{7}\left(\left|x + \frac{2x}{3} + \mu + \frac{2\mu}{3}\right|\right) \\ &\quad + \frac{1}{5}\left(\left|x + \frac{2\mu}{3} + \mu + \frac{2x}{3}\right|\right) \\ &\le \frac{2}{7}\left(\left|x + \frac{2x}{3}\right| + \left|\mu + \frac{2\mu}{3}\right|\right) \\ &\quad + \frac{1}{5}\left(\left|x + \frac{2\mu}{3}\right| + \left|\mu + \frac{2x}{3}\right|\right) \\ &= \frac{2}{7}(d_c(x, \Gamma(x, y)) + d_c(\mu, \Gamma(\mu, \nu))) \\ &\quad + \frac{1}{5}(d_c(x, \Gamma(\mu, \nu)) + d_c(\mu, \Gamma(x, y))). \end{aligned} \tag{24}$$

Hence, all the axioms of Theorem 9 are fulfilled at $\alpha = 2/7$ and $\beta = 1/5$. A mapping Γ has a strong coupled FP which is 0, that is, $\Gamma(0, 0) = 0$.

Theorem 14. *Assume that G and H are two nonempty closed subsets of a complete CMS (U, d_c) and $\Gamma : U \times U \longrightarrow U$ is a cyclic coupled contractive-type mapping w.r.t. G and H which satisfies*

$$\begin{aligned} d_c(\Gamma(x, y), \Gamma(\mu, \nu)) \le a(\max\{&d_c(x, \Gamma(x, y)), d_c(\mu, \Gamma(\mu, \nu)), d_c \\ &\cdot (\mu, \Gamma(x, y)), d_c(x, \Gamma(\mu, \nu))\}), \end{aligned} \tag{25}$$

where $x, \nu \in G$, $y, \mu \in H$, and $a \in [0, 1)$. Then, $G \cap H \ne \varnothing$ and Γ has a strong coupled FP in $G \cap H$.

Proof. Fix $\nu_0 \in G$ and $\mu_0 \in H$, and let (ν_ℓ) and (μ_ℓ) be two sequences defined as

$$\begin{aligned} \nu_{\ell+1} &= \Gamma(\mu_\ell, \nu_\ell), \\ \mu_{\ell+1} &= \Gamma(\nu_\ell, \mu_\ell), \quad \text{for all } \ell \ge 0. \end{aligned} \tag{26}$$

Then, $\{\nu_\ell\} \subset G$ and $\{\mu_\ell\} \subset H$, since Γ is a cyclic mapping w.r.t. G and H. Now, we shall show that (ν_ℓ) is a Cauchy sequence. We claim that, for all $\ell \ge 0$,

$$d_c(\nu_{\ell+1}, \mu_{\ell+2}) + d_c(\mu_{\ell+1}, \nu_{\ell+2}) \le \lambda^{\ell+1}(d_c(\nu_0, \mu_1) + d_c(\mu_0, \nu_1)), \quad \text{where } \lambda = \frac{a}{1-a} < 1. \tag{27}$$

First, we have to prove that

$$d_c(\nu_{\ell+1}, \mu_{\ell+2}) \le \lambda d_c(\mu_\ell, \nu_{\ell+1}), \quad \text{where } \lambda = \frac{a}{1-a} < 1. \tag{28}$$

Then, by (25), we have

$$\begin{aligned} d_c(\nu_{\ell+1}, \mu_{\ell+2}) &= d_c(\Gamma(\mu_\ell, \nu_\ell), \Gamma(\nu_{\ell+1}, \mu_{\ell+1})) \\ &\le a(\max\{d_c(\mu_\ell, \Gamma(\mu_\ell, \nu_\ell)), d_c(\nu_{\ell+1}, \Gamma(\nu_{\ell+1}, \mu_{\ell+1})), d_c \\ &\quad \cdot (\nu_{\ell+1}, \Gamma(\mu_\ell, \nu_\ell)), d_c(\mu_\ell, \Gamma(\nu_{\ell+1}, \mu_{\ell+1}))\}) \\ &= a(\max\{d_c(\mu_\ell, \nu_{\ell+1}), d_c(\nu_{\ell+1}, \mu_{\ell+2}), d_c \\ &\quad \cdot (\nu_{\ell+1}, \nu_{\ell+1}), d_c(\mu_\ell, \mu_{\ell+2})\}) \\ &= a(\max\{d_c(\mu_\ell, \nu_{\ell+1}), d_c(\nu_{\ell+1}, \mu_{\ell+2}), d_c(\mu_\ell, \mu_{\ell+2})\}). \end{aligned} \tag{29}$$

Now, we have three cases:

(i) If $d_c(\mu_\ell, \nu_{\ell+1})$ is maximum, then from (29), we have

$$d_c(\nu_{\ell+1}, \mu_{\ell+2}) \le a d_c(\mu_\ell, \nu_{\ell+1}) \le \frac{a}{1-a} d_c(\mu_\ell, \nu_{\ell+1}). \tag{30}$$

It holds (28), as $a < a/(1-a)$, since $a \in [0, 1)$.

(ii) If $d_c(\nu_{\ell+1}, \mu_{\ell+2})$ is maximum, then from (29), we have

$$d_c(\nu_{\ell+1}, \mu_{\ell+2}) \le a d_c(\nu_{\ell+1}, \mu_{\ell+2}), \tag{31}$$

which is not possible.

(iii) If $d_c(\mu_\ell, \mu_{\ell+2})$ is maximum, then from (29), we have

$$\begin{aligned} d_c(\nu_{\ell+1}, \mu_{\ell+2}) &\le a d_c(\mu_\ell, \mu_{\ell+2}) \le a(d_c(\mu_\ell, \nu_{\ell+1}) + d_c(\nu_{\ell+1}, \mu_{\ell+2})) \\ &\le \frac{a}{1-a}(d_c(\mu_\ell, \nu_{\ell+1})). \end{aligned} \tag{32}$$

It follows that (28) holds.

Hence, from all cases, we get that

$$d_c(\nu_{\ell+1}, \mu_{\ell+2}) \le \lambda d_c(\mu_\ell, \nu_{\ell+1}), \quad \text{where } \lambda = \frac{a}{1-a} < 1. \tag{33}$$

Similarly, we can prove

$$d_c(\mu_{\ell+1}, \nu_{\ell+2}) \le \lambda d_c(\nu_\ell, \mu_{\ell+1}), \quad \text{where } \lambda = \frac{\mathrm{a}}{1-a} < 1. \tag{34}$$

Then, again from (25), we have

$$\begin{aligned} d_c(\mu_{\ell+1}, \nu_{\ell+2}) &= d_c(\Gamma(\nu_\ell, \mu_\ell), \Gamma(\mu_{\ell+1}, \nu_{\ell+1})) \\ &\le a(\max\{d_c(\nu_\ell, \Gamma(\nu_\ell, \mu_\ell)), d_c(\mu_{\ell+1}, \Gamma(\mu_{\ell+1}, \nu_{\ell+1})), d_c \\ &\quad \cdot (\mu_{\ell+1}, \Gamma(\nu_\ell, \mu_\ell)), d_c(x_n, \Gamma(\mu_{\ell+1}, \nu_{\ell+1}))\}) \\ &= a(\max\{d_c(\nu_\ell, \mu_{\ell+1}), d_c(\mu_{\ell+1}, \nu_{\ell+2}), d_c(\mu_{\ell+1}, \mu_{\ell+1}), d_c \\ &\quad \cdot (\nu_\ell, \nu_{\ell+2}, t)\}) = a(\max\{d_c(\nu_\ell, \mu_{\ell+1}), d_c(\mu_{\ell+1}, \nu_{\ell+2}), d_c \\ &\quad \cdot (\nu_\ell, \nu_{\ell+2})\}). \end{aligned} \tag{35}$$

Then again, we have the further three cases:

(i) If $d_c(\nu_\ell, \mu_{\ell+1})$ is maximum, then from (35), we have

$$d_c(\mu_{\ell+1}, \nu_{\ell+2}) \le a d_c(\nu_\ell, \mu_{\ell+1}) \le \frac{a}{1-a} d_c(\nu_\ell, \mu_{\ell+1}). \tag{36}$$

It holds (34), as $a < a/(1-a)$, since $a \in [0, 1)$.

(ii) If $d_c(\mu_{\ell+1}, \nu_{\ell+2})$ is maximum, then from (35), we have

$$d_c(\mu_{\ell+1}, \nu_{\ell+2}) \le a d_c(\mu_{\ell+1}, \nu_{\ell+2}), \tag{37}$$

which is not possible.

(iii) If $d_c(\nu_\ell, \nu_{\ell+2})$ is maximum, then from (35), we have

$$\begin{aligned} d_c(\mu_{\ell+1}, \nu_{\ell+2}, t) &\le a d_c(\nu_\ell, \nu_{\ell+2}) \le a(d_c(\nu_\ell, \mu_{\ell+1}) + d_c(\mu_{\ell+1}, \nu_{\ell+2})) \\ &\le \frac{a}{1-a} d_c(\nu_\ell, \mu_{\ell+1}). \end{aligned} \tag{38}$$

It follows that (34) holds.

Hence, from all cases, we get that

$$d_c(\mu_{\ell+1}, \nu_{\ell+2}) \le \lambda d_c(\nu_\ell, \mu_{\ell+1}), \quad \text{where } \lambda = \frac{a}{1-a} < 1. \tag{39}$$

Now, by adding (33) and (39), we have that

$$d_c(\nu_{\ell+1}, \mu_{\ell+2}) + d_c(\mu_{\ell+1}, \nu_{\ell+2}) \le \lambda(d_c(\mu_\ell, \nu_{\ell+1}) + d_c(\nu_\ell, \mu_{\ell+1})). \tag{40}$$

Now, again by (25) and similar as above, we can get

$$d_c(\mu_\ell, \nu_{\ell+1}) \le \lambda d_c(\nu_{\ell-1}, \mu_\ell), \quad \text{where } \lambda = \frac{a}{1-a} < 1, \tag{41}$$

$$d_c(\nu_\ell, \mu_{\ell+1}) \le \lambda d_c(\mu_{\ell-1}, \nu_\ell), \quad \text{where } \lambda = \frac{a}{1-a} < 1. \tag{42}$$

Now, again by adding (41) and (42) and then putting in (40), we have that

$$d_c(\nu_{\ell+1},\mu_{\ell+2})+d_c(\mu_{\ell+1},\nu_{\ell+2})\le\lambda^2(d_c(\nu_{\ell-1},\mu_\ell)+d_c(\mu_{\ell-1},\nu_\ell)). \tag{43}$$

Continuing this process, we have that

$$d_c(\nu_{\ell+1},\mu_{\ell+2})+d_c(\mu_{\ell+1},\nu_{\ell+2})\le\lambda^{\ell+1}(d_c(\nu_0,\mu_1)+d_c(\mu_0,\nu_1)). \tag{44}$$

Hence, it is proven that (27) is exact $\forall\ell\ge0$. Now, for integer k, we have

$$\begin{aligned}d_c(\nu_{k+1},\mu_{k+1})&=d_c(\Gamma(\mu_k,\nu_k),\Gamma(\nu_k,\mu_k))\\&\le a(\max\{d_c(\mu_k,\Gamma(\mu_k,\nu_k)),d_c(\nu_k,\Gamma(\nu_k,\mu_k)),d_c\\&\quad\cdot(\nu_k,\Gamma(\mu_k,\nu_k)),d_c(\mu_k,\Gamma(\nu_k,\mu_k))\})\\&\le a(\max\{d_c(\mu_k,\nu_{k+1}),d_c(\nu_k,\mu_{k+1}),d_c\\&\quad\cdot(\nu_k,\nu_{k+1}),d_c(\mu_k,\mu_{k+1})\}).\end{aligned} \tag{45}$$

Then, we have the following four cases:

(a) If $d_c(\mu_k,\nu_{k+1})$ is maximum, then from (45), we have

$$d_c(\nu_{k+1},\mu_{k+1})\le ad_c(\mu_k,\nu_{k+1})\le\lambda d_c(\mu_k,\nu_{k+1}), \tag{46}$$

where $a<\lambda=a/(1-a)<1$, since $a\in[0,1)$.

(b) If $d_c(\nu_k,\mu_{k+1})$ is maximum, then from (45), we have

$$d_c(\nu_{k+1},\mu_{k+1})\le ad_c(\nu_k,\mu_{k+1})\le\lambda d_c(\nu_k,\mu_{k+1}), \tag{47}$$

where $a<\lambda=a/(1-a)<1$, since $a\in[0,1)$.

(c) If $d_c(\nu_k,\nu_{k+1})$ is maximum, then from (45), we have

$$\begin{aligned}d_c(\nu_{k+1},\mu_{k+1})&\le ad_c(\nu_k,\nu_{k+1})\le a(d_c(\nu_k,\mu_{k+1})+d_c(\mu_{k+1},\nu_{k+1}))\\&\le\lambda d_c(\nu_k,\mu_{k+1}),\end{aligned} \tag{48}$$

where $a<\lambda=a/(1-a)<1$.

(d) If $d_c(\mu_k,\mu_{k+1},t)$ is maximum, then from (45), we have

$$\begin{aligned}d_c(\nu_{k+1},\mu_{k+1})&\le ad_c(\mu_k,\mu_{k+1})\le a(d_c(\mu_k,\nu_{k+1})+d_c(\nu_{k+1},\mu_{k+1}))\\&\le\lambda d_c(\mu_k,\nu_{k+1}),\end{aligned} \tag{49}$$

where $a<\lambda=a/(1-a)<1$.

Hence, from (a) and (d), we have

$$d_c(\nu_{k+1},\mu_{k+1})\le\lambda d_c(\mu_k,\nu_{k+1}),\quad\text{where }\lambda=\frac{a}{1-a}<1. \tag{50}$$

And from (b) and (c), we have

$$d_c(\nu_{k+1},\mu_{k+1})\le\lambda d_c(\nu_k,\mu_{k+1}),\quad\text{where }\lambda=\frac{a}{1-a}<1. \tag{51}$$

By adding (50) and (51), we have

$$d_c(\nu_{k+1},\mu_{k+1})\le\lambda(d_c(\nu_k,\mu_{k+1})+d_c(\mu_k,\nu_{k+1}), \tag{52}$$

where $\eta=\lambda/2$, and in view of (44), we have

$$d_c(\nu_{k+1},\mu_{k+1})\le\eta\lambda^k(d_c(\nu_0,\mu_1)+d_c(\mu_0,\nu_1)),\quad\text{for}k\ge0. \tag{53}$$

Since, by triangular inequality (44) and (53), we have

$$\begin{aligned}d_c(\nu_\ell,\nu_{\ell+1})+d_c(\mu_\ell,\mu_{\ell+1})&\le(d_c(\nu_\ell,\mu_\ell)+d_c(\mu_\ell,\nu_{\ell+1}))\\&\quad+(\mathrm{d}_c(\mu_\ell,\nu_\ell)+d_c(\nu_\ell,\mu_{\ell+1}))\\&=(d_c(\nu_\ell,\mu_\ell)+d_c(\mu_\ell,\nu_\ell))\\&\quad+(d_c(\mu_\ell,\nu_{\ell+1})+d_c(\nu_\ell,\mu_{\ell+1}))\\&\le2\eta\lambda^{\ell-1}(d_c(\nu_0,\mu_1)+d_c(\mu_0,\nu_1))\\&\quad+\lambda^\ell(d_c(\nu_0,\mu_1,t)+d_c(\mu_0,\nu_1))=\left(1+\frac{2\eta}{\lambda}\right)\lambda^\ell\\&\quad\cdot(d_c(\nu_0,\mu_1)+d_c(\mu_0,\nu_1)),\quad\text{for }\ell\ge0.\end{aligned} \tag{54}$$

Now, for $\ell,j\ge0$ and $j>\ell$, we have

$$\begin{aligned}d_c(\nu_\ell,\nu_j)&\le\sum_{k=\ell}^{j-1}d_c(\nu_k,\nu_{k+1})\le\sum_{k=\ell}^{j-1}\left(1+\frac{2\eta}{\lambda}\right)\lambda^k(d_c(\nu_0,\mu_1)+d_c(\mu_0,\nu_1))\\&\le\left(1+\frac{2\eta}{\lambda}\right)\frac{\lambda^\ell}{1-\lambda}(d_c(\nu_0,\mu_1)+d_c(\mu_0,\nu_1))\\&\longrightarrow\theta,\quad\text{as }\ell\longrightarrow\infty.\end{aligned} \tag{55}$$

Hence, it is proven that (ν_ℓ) is a Cauchy sequence and it is convergent in U. Since G is a closed subset of U, therefore

$$\nu_\ell\longrightarrow\nu\in G,\quad\text{as }\ell\longrightarrow\infty. \tag{56}$$

Similarly,

$$\mu_\ell\longrightarrow\mu\in H,\quad\text{as }\ell\longrightarrow\infty. \tag{57}$$

Hence, from (56) and (57), we have

$$\lim_{\ell\longrightarrow\infty}d_c(\nu_\ell,\mu_\ell)=d_c(\nu,\mu). \tag{58}$$

By triangular inequality (44) and (54), we have

$$\begin{aligned}d_c(\nu_\ell,\mu_\ell)&\le d_c(\nu_\ell,\nu_{\ell+1})+d_c(\nu_{\ell+1},\mu_\ell)\\&\le\left(\frac{\lambda+2\eta}{\lambda}+1\right)\lambda^\ell(d_c(\nu_0,\mu_1)+d_c(\mu_0,\nu_1))\\&\longrightarrow\theta,\quad\text{as }\ell\longrightarrow\infty.\end{aligned} \tag{59}$$

Therefore, $d_c(\nu,\mu)=0$, which implies that $\nu=\mu\in G\cap H$.

Now, we prove a strong coupled FP of Γ in (U, d_c); therefore,

$$d_c(\nu, \Gamma(\nu, \mu)) \leq d_c(\nu, \nu_{\ell+1}) + d_c(\nu_{\ell+1}, \Gamma(\nu, \mu)). \tag{60}$$

Then, by the view of (25), (56), and (57), we have

$$\begin{aligned} &\leq a\left(\max\left\{d_c(\mu_\ell, \Gamma(\mu_\ell, \nu_\ell)), d_c(\nu, \Gamma(\nu, \mu)), d_c(\nu, \Gamma(\mu_\ell, \nu_\ell)), d_c\left(\mu_j, \Gamma(\nu, \mu)\right)\right\}\right) \\ &\leq a(\max\{d_c(\mu_\ell, \nu_{\ell+1}), d_c(\nu, \Gamma(\nu, \mu)), d_c(\nu, \nu_{\ell+1}), d_c(\mu_\ell, \Gamma(\nu, \mu))\}) \\ &\longrightarrow a d_c(\nu, \Gamma(\nu, \mu)), \quad \text{as } \ell \longrightarrow \infty. \end{aligned} \tag{61}$$

Hence, from (60),

$$\begin{aligned} d_c(\nu, \Gamma(\nu, \mu)) &\leq d_c(\nu, \nu_{\ell+1}) + d_c(\nu_{\ell+1}, \Gamma(\nu, \mu)) \\ &\longrightarrow a d_c(\nu, \Gamma(\nu, \mu)), \quad \text{as } \ell \longrightarrow \infty, \end{aligned} \tag{62}$$

which implies that $d_c(\nu, \Gamma(\nu, \mu)) = \theta$, since $1 - a \neq 0$. Hence, $\Gamma(\nu, \mu) = \nu = \mu$, which shows that ν is a strong coupled FP of Γ.

Corollary 15. *Assume that G and H are two nonempty closed subsets of a complete CMS (U, d_c) and $\Gamma : U \times U \longrightarrow U$ is a cyclic coupled contractive-type mapping w.r.t. G and H which satisfies:*

$$d_c(\Gamma(x, y), \Gamma(\mu, \nu)) \leq a(\max\{d_c(x, \Gamma(x, y)), d_c(\mu, \Gamma(\mu, \nu))\}), \tag{63}$$

where $x, \nu \in G$, $y, \mu \in H$, and $a \in [0, 1)$. Then, $G \cap H \neq \emptyset$ and Γ has a strong coupled FP in $G \cap H$.

Example 16. From Example 13, a mapping $\Gamma : U \times U \longrightarrow U$ can be defined as

$$\Gamma(x, y) = \frac{-3x}{7}. \tag{64}$$

Then, easily one can verify that Γ is a cyclic mapping w.r.t. G and H, for $x, \nu \in G$ and $y, \mu \in H$. A mapping Γ is a cyclic coupled Kannan-type contraction, since

$$d_c(\Gamma(x, y), \Gamma(\mu, \nu)) = |\Gamma(x, y) - \Gamma(\mu, \nu)| = \frac{3}{7}|x - \mu| \tag{65}$$

where $a = 3/7 \in ((0, 1)/2)$. Now, from (25), we have

$$\begin{aligned} d_c(\Gamma(x, y), \Gamma(\mu, \nu)) &= \frac{3|x - \mu|}{7} \leq \frac{30|x - \mu|}{49} \leq \frac{3}{7} \cdot \frac{10}{7} \\ &\cdot \left(\max\left\{x, \mu, \frac{7\mu + 3x}{10}, \frac{7x + 3\mu}{10}\right\}\right) \\ &= \frac{3}{7}\left(\max\left\{\frac{10x}{7}, \frac{10\mu}{7}, \frac{7\mu + 3x}{7}, \frac{7x + 3\mu}{7}\right\}\right) \\ &\leq \frac{3}{7}\left(\max\left\{\left|x + \frac{3x}{7}\right|, \left|\mu + \frac{3\mu}{7}\right|, \left|\frac{7\mu + 3x}{7}\right|, \left|\frac{7x + 3\mu}{7}\right|\right\}\right) \\ &= \frac{3}{7}(\max\{d_c(x, \Gamma(x, y)), d_c(\mu, \Gamma(\mu, \nu)), d_c \\ &\cdot (\mu, \Gamma(x, y)), d_c(x, \Gamma(\mu, \nu))\}). \end{aligned} \tag{66}$$

Hence, Example 16 is satisfying all conditions of Theorem 14 with $a = 3/5$, and $\Gamma(0, 0) = 0$, which is a strong coupled FP in U.

4. Integral-Type Application

This section is intended to present two UIEs for the existence of a common solution to support our main result. Let $U = C([a_0, b_0], \mathbb{R})$ be the Banach space of all continuous functions defined on $[a_0, b_0]$ with the supremum norm

$$\|\mu\| = \sup_{r \in [a_0, b_0]} |\mu(r)|, \quad \text{where } \mu \in C([a_0, b_0], \mathbb{R}), \tag{67}$$

and the induced metric d_c can be defined as

$$d_c(\mu, \nu) = \sup_{r \in [a_0, b_0]} |\mu(r) - \nu(r)|, \quad \text{where } \mu, \nu \in C([a_0, b_0], \mathbb{R}). \tag{68}$$

Now, we are in the position to introduce two UIEs for finding the solution of problems to hold up our consequences.

Theorem 17. *The two UIEs are*

$$\mu(l) = \int_{a_0}^{b_0} K_1(l, s, \mu(s)) ds + f_1(l), \tag{69}$$

$$\mu(l) = \int_{a_0}^{b_0} K_2(l, s, \mu(s)) ds + f_2(l), \tag{70}$$

where $l \in [a_0, b_0] \subset \mathbb{R}$ and $\mu, f_1, f_2 \in U$.

Let $K_1, K_2 : [a_0, b_0]^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $G_{(x,y)}, H_{(\mu,\nu)} \in U$, for $x, \nu \in G$, $y, \mu \in H$, and $G, H \subseteq U$; therefore,

$$G_{(x,y)}(l) = \int_{a_0}^{b_0} K_1(l, s, (x, y)(s)) ds, H_{(\mu,\nu)}(l) = \int_{a_0}^{b} K_2(l, s, (\mu, \nu)(s)) ds, \tag{71}$$

for all $l \in [a_0, b_0]$. If there exists $a \in (0, 1)$ such that

$$\left\|\left(G_{(x,y)} + f_1\right) - \left(H_{(\mu,\nu)} + f_2\right)\right\| \leq a N((x, y), (\mu, \nu)), \tag{72}$$

where

$$N((x,y),(\mu,\nu)) = \max\left\{\begin{array}{l}\left\|G_{(x,y)}+f_1-x\right\|, \left\|H_{(\mu,\nu)}+f_2-\mu\right\|, \\ \left\|G_{(x,y)}+f_1-\mu\right\|, \left\|H_{(\mu,\nu)}+f_2-x\right\|\end{array}\right\}, \tag{73}$$

then the two UIEs (69) and (70) have a unique common solution.

Proof. Define a mapping $\Gamma : U \times U \longrightarrow U$:

$$\begin{aligned}\Gamma(x,y) &= G_{(x,y)} + f_1, \\ \Gamma(\mu,\nu) &= H_{(\mu,\nu)} + f_2.\end{aligned} \tag{74}$$

Then, we may have the following four cases:

(a) If $\|G_{(x,y)} + f_1 - x\|$ is the maximum in (73), then we have

$$N((x,y),(\mu,\nu)) = \left\|G_{(x,y)} + f_1 - x\right\|. \tag{75}$$

Hence, we get that

$$\begin{aligned}d_c(\Gamma(x,y),\Gamma(\mu,\nu)) &= \|\Gamma(x,y)-\Gamma(\mu,\nu)\| \le a(\|\Gamma(x,y)-x\|) \\ &= ad_c(x,\Gamma(x,y)).\end{aligned} \tag{76}$$

(b) If $\|H_{(\mu,\nu)} + f_2 - \mu\|$ is the maximum in (73), then we have

$$N((x,y),(\mu,\nu)) = \left\|H_{(\mu,\nu)} + f_2 - \mu\right\|. \tag{77}$$

Hence, we get that

$$\begin{aligned}d_c(\Gamma(x,y),\Gamma(\mu,\nu)) &= \|\Gamma(x,y)-\Gamma(\mu,\nu)\| \le a(\|\Gamma(\mu,\nu)-\mu\|) \\ &= ad_c(\mu,\Gamma(\mu,\nu)).\end{aligned} \tag{78}$$

(c) If $\|G_{(x,y)} + f_1 - \mu\|$ is the maximum in (73), then we have

$$N((x,y),(\mu,\nu)) = \left\|G_{(x,y)} + f_1 - \mu\right\|. \tag{79}$$

Hence, we get that

$$\begin{aligned}d_c(\Gamma(x,y),\Gamma(\mu,\nu)) &= \|\Gamma(x,y)-\Gamma(\mu,\nu)\| \le a(\|\Gamma(x,y)-\mu\|) \\ &= ad_c(\mu,\Gamma(x,y)).\end{aligned} \tag{80}$$

(d) If $\|H_{(\mu,\nu)} + f_2 - x\|$ is the maximum in (73), then we have

$$N((x,y),(\mu,\nu)) = \left\|H_{(\mu,\nu)} + f_2 - x\right\|. \tag{81}$$

Hence, we get that

$$\begin{aligned}d_c(\Gamma(x,y),\Gamma(\mu,\nu)) &= \|\Gamma(x,y)-\Gamma(\mu,\nu)\| \le a(\|\Gamma(\mu,\nu)-x\|) \\ &= ad_c(x,\Gamma(\mu,\nu)).\end{aligned} \tag{82}$$

Hence, from all cases, for every $x, \nu \in G$, $y, \mu \in H$, and $G, H \subseteq U$ with $a \in (0,1)$ in Theorem 14, the two UIEs (69) and (70) have a common solution in U.

5. Conclusions

It is well known that FP has been the heart of several areas of mathematics and other sciences. In this work, we have presented some generalized strong coupled FP theorems in cone metric spaces by using the cyclic-type mappings with illustrative examples. The result of our article gives the generalization of "cyclic coupled Kannan-type contraction" given by Choudhury and Maity in [7]. Finally, to support our main result, we have established an application of the two UIEs for the existence of a common solution to support our work. This concept will be very useful for finding different contractive-type strong coupled FP results in the context of metric spaces with different types of applications.

Data Availability

There is no any data availability.

Conflicts of Interest

This work does not have any conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this article. All authors read and approved the final manuscript.

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Research Article

S^{*p}-b-Partial Metric Spaces with some Results in Common Fixed Point Theorems

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In this paper, we introduce the notion of S^{*P}-b-partial metric spaces which is a generalization each of S-b-metric spaces and partial-metric space. Also, we study and prove some topological properties, to know the convergence of the sequences and Cauchy sequence. Finally, we study a new common fixed point theorem in these spaces.

1. Introduction

There are a large number of generalizations of the Banach contraction principle with different use forms of contractual terms in a variety of generalized metric spaces. For some of these, the circulares are obtained through contractual terms expressed in reasonable terms. Latif et al. [1] introduced the notion of G-rational Geraghty contractive mappings in the setup of ordered generalized b-metric spaces and investigated the existence of fixed points for such mappings. They also provided an example to illustrate the presented results and show that they are more general than some existing ones.

One such generalization is a partial-metric space introduced by Matthews [2] in 1994. Since in this space, a self-distance of an arbitrary point need not to be equal zero. Shahkoohi and Razani [3] introduced new classes of rational Geraghty contractive mappings in the setup of b-metric spaces and the existence of some fixed point for such mappings in ordered b-metric spaces. Bakhtin [4] and Bourbaki [5] introduced a notion of b-metric spaces. Later on, Czerwik [6] generalized the Banach contraction mapping theorem by formally defining a b-metric space and giving a postulate which was weaker than the condition of triangular inequality. After that, Fagin and Stockmeyer [7] defined some kind of relaxation in triangular inequality and called this new distance measure as the nonlinear elastic mathing. All these notions and applications pushed us to introduce the concept of extended b-metric space by S-metric space and partial metric space.

In 2007, Sedghi et al. [8] introduced D^*-metric spaces which are the generalization of the notion of D-metric spaces introduced by Dhage [9]. Also, Sedghi et al. proved some of the basic properties in D^*-metric spaces. In 2012, Sedghi et al. [10] introduced S-metric spaces and give some properties and fixed point theorem for a complete S-metric space. Soliman and Zidan [11] introduced a new coupled fixed point theorem in a generalized metric space, and they utilized the same to study the stability for a system of set-valued functional equations. In 2015, Gupta and Deep [12] proved fixed point theorems for nonlinear contractive mappings in S-metric spaces. On the other way, Chauhan and Gupta [13] introduced the notion of fuzzy cone b-metric space and defined fuzzy cone b-contractive mapping and proved Banach contraction theorem for a single mapping in the setting of fuzzy cone b-metric space.

Recently, in 2019, Mustafa et al. [14] introduced the structure of S_b-metric spaces which is a generalization of S-metric space and gave some properties and fixed point theorem for this spaces.

2. Preliminaries and Definitions

We begin the section with some basic definitions and concepts.

Definition 1. Let Y be a nonempty set and $\zeta \geq 1$ be a given real number. A function $d_b : Y \times Y \longrightarrow [0,\infty)$ is called *b -metric*, if it satisfies the following properties for each $w, t, r \in Y$:

(b_1) $d_b(w,t) = 0 \Leftrightarrow w = t$

(b_2) $d_b(w,t) = d_b(t,w)$

(b_3) $d_b(w,t) = \leq \zeta[d_b(w,r) + d_b(r,t)]$

The pair (Y, d_b) is called a b-metric space.

Example 2. Let $Y = l_{\mho}\mathbb{R}$ with $0 < \mho < 1$ where $l_{\mho}\mathbb{R} : \{\{w_n\} \subset \mathbb{R} : \Sigma_{n=1}^{\infty}|w_n|^{\mho} < \infty\}$. Define $d_b : Y \times Y \longrightarrow [0,\infty)$ as

$$d_b(w,t) = \left(\sum_{n=1}^{\infty} |w_n - t_n|^{\mho}\right)^{1/\mho}, \tag{1}$$

where $w = \{w_n\}$, $t = \{t_n\}$. Then, d_b is a b-metric space.

Definition 3 (see [2]). Let Y be a set. A function $P : Y \times Y \longrightarrow [0,\infty)$ is said to be a partial metric spaces on a nonempty set Y, if for all $w, t, r \in Y$, the following conditions hold:

(P_1) $0 \leq P(w,w) \geq P(w,t)$ (nonnegativity and small self-distances)

(P_2) If $P(w,w) = P(w,t) = P(t,t)$ then $w = t$ (indistancy implies equality)

(P_3) If $P(w,t) = P(t,w)$ (symmetry)

(P_4) $P(w,t) \leq P(w,r) + P(r,t) - P(r,r)$ (triangularity)

Hence, the function P is called a P-metric on Y and the pair (Y, P) is called a partial-metric space.

Mustafa and Sims [15] extended the Banach principle by introducing the notation of generalized metric spaces the so-called G-metric spaces as follows.

Definition 4. Let Y be a set. A function $G : Y \times Y \times Y \longrightarrow [0,\infty)$ is said to be generalized metric spaces on a *nonempty* set Y, if for all $w, t, r, \eta \in Y$, the following conditions hold:

(G_1) $G(w,t,r) = 0 \Leftrightarrow w = t = r$

(G_2) $G(w,t,r) > 0$ and $w \neq t$

(G_3) $G(w,w,t) \leq G(w,t,r)$ and $t \neq r$

(G_4) $G(w,t,r) = G(w,r,t) = G(t,r,w) = \cdots$, (symmetry in all three variables)

(G_5) $G(w,t,r) \leq G(w,\eta,\eta) + G(\eta,r,w)$

Hence, the function G is called a G-metric on Y and the pair (Y, G) is called a generalized metric space.

Mustafa and Sims [15] found some basic properties and examples of G-metric spaces.

After that, Zand and Nezhad [16] introduced the definition of G^p-metric space by generalization and unification of both the partial-metric space and G-metric space as follows:

Definition 5. Let Y be a set. A function $G^p : Y \times Y \times Y \longrightarrow [0,\infty)$ is said to be a generalized partial metric spaces on a nonempty set Y, if for all $w, t, r, \eta \in Y$, the following conditions hold:

(G_1^p) $w = t = r$ if $G^p(w,w,w) = G^p(t,t,t) = G^p(r,r,r)$

(G_2^p) $0 \leq G^p(w,w,w) \leq G^p(w,w,t) \leq G^p(w,t,y)$

(G_3^p) $G^p(w,t,r) = G^p(w,r,t) = G^p(t,r,w) = \cdots$, (symmetry in all three variables)

(G_4^p) $G^p(w,t,r) \leq G^p(w,\eta,\eta) + G^p(\eta,r,w) - G^p(\eta,\eta,\eta)$

Hence, the function G^p is called a G^p-metric on Y and the pair (Y, G^p) is called a generalized partial metric space.

Example 6 (see [16]). Let $Y = [0,\infty$ and $G^p(w,t,r) = \max\{w,t,r\}$ for all $w, t, r \in Y$, then (Y, G^p) is a G^p *-metric* space. Also, (Y, G^p) is not a G-metric space.

Proposition 7. *Let (Y, G^p) is a G^p -metric space, then for any $w, t, r \in Y$ and $\eta \in Y$, it follows that*

(i) $G^p(w,t,r) \leq G^p(w,w,t) + G^p(w,w,r) - G^p(w,w,w)$

(ii) $G^p(w,t,t) \leq 2G^p(w,w,t) - G^p(w,w,w)$

(iii) $G^p(w,t,r) \leq G^p(w,\eta,\eta) + G^p(t,\eta,\eta) + G^p(r,\eta,\eta) - 2G^p(\eta,\eta,\eta)$

Sedghi et al. [8] introduced the notion of D^*-metric which is a modification of the definition of D-metric introduced by Dhage [9, 17], and they proved some basic properties in D^*-metric spaces.

Definition 8 (see [8]). Let Y be a set. A function $D^* : Y \times Y \times Y \longrightarrow [0,\infty)$ is said to be a D^* *-metric* spaces on a non-empty set Y, if for all $w, t, r \in Y$, the following conditions hold:

(D_1^*) $D^*(w,t,r) \geq 0$

(D_2^*) $D^*(w,t,r) \Leftrightarrow w = t = w$

(D_3^*) $D^*(w,t,r) = D^*(\aleph\{w,t,r\})$, (where $\aleph$ is a permutation function)

Hence, the function D^* is called a D^*-metric on Y and the pair (Y, D^*) is called a generalized partial metric space.

Example 9 (see [8]). Let $Y = \mathbb{R}$. Denote $D^*(w,t,r) = |w - t| + |t - r| + |r - w|$, for all $w, t, r \in \mathbb{R}$. Since

$$B_{D^*}(w,\lambda) = \{t \in Y : D^*(w,t,t) < \lambda\}, \tag{2}$$

Hence,

$$\begin{aligned} &B_{D^*}(1,2) \\ &\quad = \{t \in \mathbb{R} : D^*(1,t,t) < 2\} \\ &\quad = \{t \in \mathbb{R} : |t-1| + |t-1| < 2\} \\ &\quad = \{t \in \mathbb{R} : |t-1| < 1\} = (0,2). \end{aligned} \tag{3}$$

On the other way, Sedghi et al. introduced S-metric spaces as follows.

Definition 10 (see [10]). Let Y be a set. A function $S : Y^3 \longrightarrow [0,\infty)$ is said to be S *-metric* spaces on a nonempty set Y, if for all $w, t, r, \eta \in Y$ the following conditions hold:

(S_1) $S(w,t,r) > 0$

(S_2) $S(w,t,r) \Leftrightarrow w = t = r$

(S_3) $S(w, t, r) \leq S(w, w, \eta) + S(t, t, \eta) + S(r, r, \eta)$ for all $w, t, r, \eta \in Y$ (rectangle inequality)

Hence, the function S is called a S-metric on Y and the pair (Y, S) is called an S-metric space.

Example 11 (see [10]). Let $Y = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Y, then

$$S(w, t, r) = \|t + r - 2w\| + \|t - r\|, \tag{4}$$

is a S-metric on Y.

Remark 12. Every *D^* -metric is S -metric*, but in general, the converse is not true, see the following example.

Example 1. Let $Y = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Y, then $S(w, t, r) = \|t + r - 2w\| + \|t - r\|$ *is S -metric on* Y, but it is not *D^* -metric* because it is not symmetric.

Example 2. Let $Y = \mathbb{R}^2$ and d is an ordinary metric on Y. Therefore, $S(w, t, r) = d(w, t) + d(w, r) + d(t, r)$ is an *S -metric* on Y. Then, if the points w, t, r connected by a line. Hence, the triangle when choose a point a mediating this triangle then the inequality

$$S(w, t, r) \leq s(w, w, \eta) + s(t, t, \eta) + (r, r, \eta), \tag{5}$$

for all $w, t, r, \eta \in Y$ holds.

Definition 13 (see [10]). Let (Y, S) be a *S-metric* space. Then, for $w_0 \in Y$ and $\lambda > 0$, the *S -open ball* and *S -closed ball* of radius λ with centered at w_0 is

$$\begin{aligned} B_S(w_0, \lambda) &= \{t \in Y : S(w_0, w_0, t) < \lambda\}, \\ \bar{B}_S(w_0, \lambda) &= \{t \in Y : S(w_0, w_0, t) \leq \lambda\}. \end{aligned} \tag{6}$$

Proposition 14 (see [10]). *Let (Y, S) be a S -metric space and $A \subset Y$:*

(1) If for every $w_0 \in A$ there exists $\lambda > 0$ such that $B_S(w_0, \lambda)$, then the subset A is called an open subset of Y

(2) A subset A of Y is said to be S-bounded if there exists $\lambda > 0$ such that $S(w, w, t) < \lambda$ for all $w, t \in A$

(3) A sequence $\{w_n\}$ in Y converges to w if and only if S $(w_n, w_n, w) \longrightarrow 0$ as $n \longrightarrow \infty$. That is, for each $\lambda > 0$, there exists $n_0 \in N$ such that for all $n > n_0$, $S(w_n, w_n, w) < \lambda$ and we denote this by $\lim_{n \longrightarrow \infty} w_n = w$

Lemma 15 (see [10]). *Let (Y, S) be an S -metric space, then the following is satisfied:*

(1) If $\lambda > 0$ and $w \in Y$, then the ball $B_S(w_0, \lambda)$ is an open subset of Y

(2) If the sequence $\{w_n\}$ in Y converges to w, then w is unique

(3) If the sequence $\{w_n\}$ in Y converges to w, then $\{w_n\}$ is a Cauchy sequence

Definition 16 (see [15]). Two classes of the following mappings are

(1) $\Pi = \{\pi : \pi : [0,\infty) \rightarrow [0,\infty)\}$ is nondecreasing, continuous and $\pi^{-1}(0) = 0$

(2) $\Gamma = \{\gamma : \gamma : [0,\infty) \rightarrow [0,\infty)\}$ is nondecreasing, lower semi-continuous and $\pi^{-1}(0) = 0$

Definition 17. Let $(Y, \preccurlyeq)$ be a partially ordered set. Two maps $\mathfrak{f}, \mathfrak{g} : Y \times Y$ are said to be weak increasing if $\mathfrak{f}w \preccurlyeq \mathfrak{g}\mathfrak{f}w$ and $\mathfrak{g}w^{\circ}\mathfrak{f}\mathfrak{g}w$ for all $w \in Y$.

Barakat and Zidan [18] proved a common fixed point theorem for weak contractive maps by using the concept of G^p-metric spaces.

Theorem 18. *Let $(Y, \preccurlyeq)$ be a partially ordered set wit $\mathfrak{f}$ and $\mathfrak{g}$ be weakly increasing self mapping on a complete G^p -partial metric space. Suppose that there exist $\pi \in \Pi$ and $\gamma \in \Gamma$ such that*

$$\pi(G^p(\mathfrak{f}(w), \mathfrak{f}(w), \mathfrak{g}(t))) \leq \pi(M(w, w, t)) - \gamma(M(t, t, w)), \tag{7}$$

for all $w, t \in Y$, where

$$\begin{aligned} M(w, w, t) = {} & \zeta_1 G^p(w, w, t) + \zeta_2 G^p(w, w, \mathfrak{f}(w)) + \zeta_3 G^p(t, t, \mathfrak{g}(t)) \\ & + \zeta_4[G^p(w, w, \mathfrak{g}(t)) + G^p(t, t, \mathfrak{f}(w))], \end{aligned} \tag{8}$$

where $\zeta_1 > 0$ for $i = \{1, 2, 3, 4\}$ with $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \geq 1$.

Then, of the following two cases, assume that one of the following cases is satisfied:

(a) If a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\}^{\circ} r$ for all $n \in \mathbb{N}$

(b) $\mathfrak{f}$ or $\mathfrak{g}$ is continuous

Therefore, the maps $\mathfrak{f}$ or $\mathfrak{g}$ have a common fixed point.

The present paper is aimed at introducing the notion of S_b^*-partial metric spaces which is a generalization each of S -metric spaces and partial-metric space. Also, we give some of the topological properties that are important in knowing the convergence of the sequences and Cauchy sequence. Finally, we study a new fixed point theory in this spaces.

3. S_b^*-Partial Metric Spaces and some Properties

We first introduce the concept of a S_b^{*P}-partial metric space or (S_b^{*P}).

Definition 19. Let Y be a set and $\zeta \geq 1$. A function $S_b^{*P}: Y^3 \longrightarrow [0,\infty)$ is said to be a S_b^{*P} *-partial metric spaces* on a nonempty set Y, if for all $w, t, r, \eta \in Y$, the following conditions hold:

$(S_b^{*P}1)$

$S_b^{*P}(w,t,r) = S_b^{*P}(w,w,w) = S_b^{*P}(t,t,t) = S_b^{*P}(r,r,r)$ then, $w = t = r$

(S_2^{*P})

$S_b^{*P}(w,t,r) \leq \zeta[S_b^{*P}(w,w,\eta) + S_b^{*P}(t,t,\eta) + S_b^{*P}(r,r,\eta) - S_b^{*P}(\eta,\eta,\eta)]$ for all $w,t,r,\eta \in Y$ (rectangle inequality)

Hence, the function S_b^{*P} is called an S_b^*-partial metric on Y and the pair (Y, S_b^{*P}) is called an S_b^*-partial metric space.

Example 20. Let $Y = \mathbb{R}$ and $\|\cdot\|$ a norm on Y, then we have

$$S_b^{*P}(w,t,r) = \max\{w,t,r\} + \|w-r\| + \|t-w\|, \tag{9}$$

is a S_b^*-partial metric on Y.

Remark 21. From Example 20, we get every S *-metric* is S_b^{*P} *-metric*, but the converse is not true at all.

Definition 22. Let (Y, S_b^{*P}) be a S_b^*-partial metric space. Then, for $w_0 \in Y$ and $\lambda > 0$, the S_b^{*P} *-open ball* and S_b^{*P} *-open closed of radius* λ with centered at w_0 *is*

$$\begin{aligned} B_{S_b^{*P}}(w_0,\lambda) &= \{t \in Y : S_b^{*P}(w_0,w_0,t) < \lambda + S_b^{*P}(w_0,w_0,w_0) \\ &\quad - S_b^{*P}(t,t,w_0)\}, \\ \bar{B}_{S_b^{*P}}(w_0,\lambda) &= \{t \in Y : S_b^{*P}(w_0,w_0,t) \leq \lambda + S_b^{*P}(w_0,w_0,w_0) \\ &\quad - S_b^{*P}(t,t,w_0)\}. \end{aligned} \tag{10}$$

Proposition 23. Let (Y, S_b^{*P}) be a S_b^{*P}-partial metric space. *Then for $w_0 \in Y$ and $\lambda > 0$, the following statements are satisfying:*

(1) *If $S_b^{*P}(w_0,w_0,w) \leq \lambda + S_b^{*P}(w_0,w_0,w_0) - S_b^{*P}(w,w,,w_0)$ and $S_b^{*P}(w_0,w_0,t) \leq \lambda + S_b^{*P}(w_0,w_0,w_0) - S_b^{*P}(t,t,w_0)$, then $w,t \in B_{S_b^{*P}}(w_0,\lambda)$*

(2) *If $t \in B_{S_b^{*P}}(w_0,\lambda)$, then their exist $\rho > 0$, such that $B_{S_b^{*P}}(t,\rho) \subseteq B_{S_b^{*P}}(w_0,\lambda)$*

Proof.

(1) The proof is straightforward.

(2) Let $t \in B_{S_b^{*P}}(w_0,\lambda)$, then we have

$$S_b^{*P}(w_0,w_0,t) \leq \lambda + S_b^{*P}(w_0,w_0,w_0) - S_b^{*P}(t,t,w_0) \tag{11}$$

Also, we suppose

$$\rho = \lambda + S_b^{*P}(w_0,w_0,w_0) - S_b^{*P}(r,r,w_0) - 2S_b^{*P}(w_0,w_0,t), \tag{12}$$

and $r \in B_{S_b^{*P}}(t,\rho)$, then we get

$$S_b^{*P}(t,t,r) \leq \rho + S_b^{*P}(t,t,t) - S_b^{*P}(r,r,t). \tag{13}$$

Therefore, $t \in B_{S_b^{*P}}(w_0,\lambda)$, and so (2) holds.

$$\begin{aligned} S_b^{*P}(w_0,w_0,r) &\leq \zeta\left[S_b^{*P}(w_0,w_0,t) + S_b^{*P}(w_0,w_0,t)\right. \\ &\quad \left.+ S_b^{*P}(r,r,t) - S^{*P}(t,t,t)\right] \\ &\leq \zeta\left[2S_b^{*P}(w_0,w_0,t) + S_b^{*P}(r,r,t)\right. \\ &\quad + \lambda + S_b^{*P}(w_0,w_0,w_0) - S_b^{*P}(r,r,w_0) \\ &\quad \left.- 2S_b^{*P}(w_0,w_0,t) - S_b^{*P}(t,t,t)\right] \\ &\leq \zeta\left[\lambda + S_b^{*P}(w_0,w_0,w_0) - S_b^{*P}(r,r,t)\right] \\ &\leq \lambda + S_b^{*P}(w_0,w_0,w_0) - S_b^{*P}(r,r,t). \end{aligned} \tag{14}$$

Hence, $r \in B_{S_b^{*P}}(w_0,\lambda)$.

Definition 24. Let (Y, S_b^{*P}) be a S_b^{*P} *-partial metric space* and $\{w_n\}$ a sequence in Y. A point $w \in Y$ is said to be the limit of the sequence if

$$\lim_{n,m \longrightarrow \infty} S_b^{*P}(w,w_n,w_m) = S_b^{*P}(w,w,w). \tag{15}$$

Hence, the sequence $\{w_n\}$ is S_b^{*P}-convergent to w.

Therefore, if $\{w_n\} \longrightarrow w$ in a S_b^{*P}-partial metric space (Y, S_b^{*P}). Then, for any $\hbar > 0$, there exists $\mathfrak{J} \in N$ such that, for all $n, m > \mathfrak{J}$, we have

$$|S_b^{*P}(w,w_n,w_m) - S_b^{*P}(w,w,w)| < \hbar. \tag{16}$$

Definition 25. A S_b^{*P} *-partial metric space* (Y, S_b^{*P}) is called a S_b^{*P} *-partial asymmetric space if*

$$S_b^{*P}(w,w,t) = S_b^{*P}(t,t,w). \tag{17}$$

Lemma 26. (Y, S_b^{*P}) be a S_b^*-partial metric space. *If the sequence $\{w_n\}$ in Y converges to w. Therefore, we get w is unique.*

Proof. Let $\{w_n\}$ converges to w and t. Therefore, for each $\lambda > 0$, there exist $n_1, n_2 \in N$, then we have

$$\begin{aligned} n \geq n_1 &\Rightarrow S_b^{*P}(w_n,w_n,w) < \frac{\lambda}{3}, \\ n \geq n_2 &\Rightarrow S_b^{*P}(w_n,w_n,t) < \frac{\lambda}{3}. \end{aligned} \tag{18}$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$. Also, we have a third condition of S_b^{*P}-partial metric

$$\begin{aligned} S_b^{*P}(w,w,t) &\le \zeta\left[S_b^{*P}(w,w,w_n)+S_b^{*P}(w,w,w_n)\right.\\ &\quad \left.+S_b^{*P}(t,t,w_n)-S_b^{*P}(w_n,w_n,w_n)\right]\\ &\le \zeta\left[2S_b^{*P}(w,w,w_n)+S_b^{*P}(t,t,w_n)-S_b^{*P}(w_n,w_n,w_n)\right]\\ &\le \zeta\left[2S_b^{*P}(w,w,w_n)+S_b^{*P}(t,t,w_n)\right]\\ &\le 2S_b^{*P}(w,w,w_n)+S_b^{*P}(t,t,w_n)<2\frac{\lambda}{3}+\frac{\lambda}{3}=\lambda. \end{aligned} \tag{19}$$

Hence, $S_b^{*P}(w,w,t)=0 \Rightarrow w=t$, but the converse is not necessarily true.

Definition 27. Let (Y,S_b^{*P}) be a S_b^{*P}-partial metric space. Then, for a s' *sequence* $\{w_n\}\subseteq Y$ and a point $w\in Y$, the following are equivalent:

(1) $\{w_n\}$ is a S_b^{*P} convergent to w

(2) $\lim_{n,m\longrightarrow\infty} S_b^{*P}(w_n,w_n,w)=S_b^{*P}(w,w,w)$

Proposition 28. Let (Y,S_b^{*P}) be a S_b^{*P}-partial metric space. *If then, we get*

(1) A sequence $\{w_n\}\in Y$ is called a Cauchy sequence if for each $\lambda>0$, there exists $n_0\in N$ such that

$$S_b^{*P}(w_n,w_n,w_m)<\lambda+S_b^{*P}(w,w,w), \tag{20}$$

for each $n,m\ge 0$.

Definition 29. Let (Y,S_b^{*P}) be a S_b^{*P} -partial metric space. Then, (Y,S_b^{*P}) is said to be complete if every Cauchy sequence is convergent.

Definition 30. Let (Y,S_b^{*P}) and let (Y_2,S_{b2}^{*P}) be two S_b^{*P}-partial metric space. Also, we suppose a function

$$\mathfrak{F}:\left(Y_1,S_{b1}^{*P}\right)\longrightarrow\left(Y_2,S_{b2}^{*P}\right), \tag{21}$$

then $\mathfrak{F}$ is said to be S_b^{*P}- continuous at a point a $x\in Y_1$ if and only if, for given $\hbar>0$, there exists $\lambda>0$ such that $w,t\in Y_1$ and the inequality

$$S_{b1}^{*P}(x,w,t)<\hbar+S_{b1}^{*P}(w,w,w). \tag{22}$$

This is an indication

$$S_{b2}^{*P}(\mathfrak{F}(x),\mathfrak{F}(w),\mathfrak{F}(t))<\lambda+S_{b2}^{*P}(\mathfrak{F}(w),\mathfrak{F}(w),\mathfrak{F}(w)). \tag{23}$$

Hence, a function $\mathfrak{F}$ is S_b^{*P}- continuous on Y_1 if and only if it S_b^{*P}-continuous at all $x\in Y_1$.

Definition 31. The two classes of following mappings are defined $\Pi=\{\pi\,;\pi:[0,\infty)\to[0,\infty)\}$ is continuous, nondecreasing, and $\pi^{-1}(0)=0$. $\Gamma=\{\gamma\,;\gamma:[0,\infty)\to[0,\infty)\}$ is lower semicontinuous, nondecreasing, and $\gamma^{-1}(0)=0$.

4. A Generalization of Common Point Theorems in S_b^*-Partial Metric Spaces

Theorem 32. *Let $(Y,\preccurlyeq)$ be a partially ordered set, $\mathfrak{f}$ and $\mathfrak{g}$ be weakly increasing self-mapping on a complete S_b^{*P} -metric space with $\zeta\ge 1$. Suppose that there exist $\pi\in\Pi$ and $\gamma\in\Gamma$ such that*

$$\pi\left(S_b^{*P}(\mathfrak{f}(w),\mathfrak{f}(w),\mathfrak{g}(t)\right)\le\pi(M(w,w,t))-\gamma(M(t,t,w)), \tag{24}$$

for all $w,t\in Y$, where

$$\begin{aligned} M(w,w,t)=\max\Big\{&S_b^{*P}(w,w,t),S_b^{*P}(w,w,\mathfrak{f}(w)),S_b^{*P}(t,t,\mathfrak{g}(t)),\\ &\cdot\left[S_b^{*P}(w,w,\mathfrak{g}(t))+S_b^{*P}(t,t,\mathfrak{f}(w))\right]\Big\}. \end{aligned} \tag{25}$$

Assume that one of the following cases is satisfied:

(a) if a nondecreasing sequence $\{w_n\}$ converges to $r\in Y$ implies $\{w_n\}^{\circ}r$ for all $n\in\mathbb{N}$

(b) $\mathfrak{f}$ or $\mathfrak{g}$ is continuous

Therefore, the maps $\mathfrak{f}$ or $\mathfrak{g}$ have a common fixed point.

Proof. Suppose that $\hbar$ is a fixed point of $\mathfrak{f}$ and $S^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar))>0$. From 4.1 with $w=t=\hbar$, we have

$$\begin{aligned} \pi\left(S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar))\right)&=\pi\left(S_b^{*P}(\mathfrak{f}(\hbar),\mathfrak{f}(\hbar),\mathfrak{g}(\hbar))\right)\\ &\le\pi(M(\hbar,\hbar,\hbar))-\gamma(M(\hbar,\hbar,\hbar)), \end{aligned} \tag{26}$$

where

$$\begin{aligned} M(\hbar,\hbar,\hbar)&=\max\Big\{S_b^{*P}(\hbar,\hbar,\hbar),S_b^{*P}(\hbar,\hbar,\mathfrak{f}(\hbar)),S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar)),\\ &\qquad\cdot\left[S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar))+S_b^{*P}(\hbar,\hbar,\mathfrak{f}(\hbar))\right]\Big\}\\ &=\max\Big\{S_b^{*P}(\hbar,\hbar,\hbar),S_b^{*P}(\hbar,\hbar,\hbar),S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar)),\\ &\qquad\cdot\left[S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar))+S_b^{*P}(\hbar,\hbar,\hbar)\right]\Big\}\\ &=\max\left\{S_b^{*P}(\hbar,\hbar,\hbar),S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar))\right\}\\ &=S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar)). \end{aligned} \tag{27}$$

Hence, we have

$$\begin{aligned} \pi\left(S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar))\right)&=\pi\left(S_b^{*P}(\mathfrak{f}(\hbar),\mathfrak{f}(\hbar),\mathfrak{g}(\hbar))\right)\\ &\le\pi\left(S_b^{*P}(\mathfrak{f}(\hbar),\mathfrak{f}(\hbar),\mathfrak{g}(\hbar))\right)\\ &\quad-\gamma\left(S_b^{*P}(\mathfrak{f}(\hbar),\mathfrak{f}(\hbar),\mathfrak{g}(\hbar))\right)\\ &\Rightarrow\gamma\left(S_b^{*P}(\hbar,\hbar,\mathfrak{g}(\hbar))\right)\le 0. \end{aligned} \tag{28}$$

A contradiction. Therefore, $S_b^{*P}(\mathfrak{f}(\hbar), \mathfrak{f}(\hbar), \mathfrak{g}(\hbar)) = 0$. So, $\hbar$ is common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$. Similarly, if $\hbar$ is a fixed point of $\mathfrak{g}$, then one can deduce that $\hbar$ is also fixed point of $\mathfrak{f}$.

If, we let w_0 be an arbitrary point of Y with $\mathfrak{f}w_0 = w_0$, then the proof is finished, so we assume that $\mathfrak{f}w_0 \neq w_0$.

Now, one can construct a sequence $\{w_n\} \in Y$ as follows:

$$\begin{aligned} w_1 &= \mathfrak{f}w_0 \preccurlyeq \mathfrak{g}\mathfrak{f}w_0 = \mathfrak{g}w_1 = w_2, \\ w_2 &= \mathfrak{f}w_1 \preccurlyeq \mathfrak{g}\mathfrak{f}w_1 = \mathfrak{g}w_1 = w_3, \\ &\vdots \\ w_n &\preccurlyeq w_{n+1}. \end{aligned} \tag{29}$$

Since w_{2n} and w_{2n+1} are comparable, then we may assume that $S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}) > 0$, for every $n \in N$. If not, then $w_{2n} = w_{2n+1}$ for some n. For all those n, using (24), we obtain

$$\begin{aligned} \pi\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right) &= \pi\left(S_b^{*P}(\mathfrak{f}w_{2n}, \mathfrak{f}w_{2n}, \mathfrak{g}w_{2n+1})\right) \\ &\leq \pi(M(w_{2n}, w_{2n}, w_{2n+1})) \\ &\quad - \gamma(M(w_{2n}, w_{2n}, gw_{2n+1})), \end{aligned}$$

$$\begin{aligned} M(w_{2n}, w_{2n}, w_{2n+1}) &= \max \\ &\cdot\Big\{S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}), S_b^{*P}(w_{2n}, w_{2n}, \mathfrak{f}w_{2n}), S_b^{*P}(w_{2n+1}, w_{2n+1}, \mathfrak{g}w_{2n+1}), \\ &\cdot\frac{\left[S_b^{*P}(w_{2n}, w_{2n}, \mathfrak{g}w_{2n+1}), S_b^{*P}(w_{2n+1}, w_{2n+1}, \mathfrak{f}w_{2n})\right]}{2}\Big\} \\ &= \max\Big\{S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}), S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}), S_b^{*P} \\ &\cdot(w_{2n+1}, w_{2n+1}, w_{2n+2}), \left[\frac{S_b^{*P}(w_{2n}, w_{2n}, w_{2n+2}), S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1})}{2}\right]\Big\} \\ &\leq \max\Big\{S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}), \frac{1}{2}\Big[\zeta\Big[S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}) \\ &\cdot + S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}) + S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1}) \\ &\cdot - S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1})\Big] + S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1})\Big]\Big\} \\ &= S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}). \end{aligned} \tag{30}$$

Therefore

$$\begin{aligned} \pi\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right) &\leq \pi\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right) \\ &\quad - \gamma\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right). \end{aligned} \tag{31}$$

It means that $\gamma(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) = 0$ and $w_{2n+1} = w_{2n+2}$. Following the similar arguments, we obtain $w_{2n+2} = w_{2n+3}$ and hence w_{2n} becomes a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$.

By taking $S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}) > 0$ for $n = 1, 2, 3, \cdots$, now, we consider

$$\begin{aligned} \pi\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right) &= \pi\left(S_b^{*P}(\mathfrak{f}w_{2n}, \mathfrak{f}w_{2n}, \mathfrak{g}w_{2n+1})\right) \\ &\leq \pi(M(w_{2n}, w_{2n}, w_{2n+1})) \\ &\quad - \gamma(M(w_{2n}, w_{2n}, w_{2n+1})). \end{aligned}$$

$$\begin{aligned} &M(w_{2n}, w_{2n}, w_{2n+1}) \\ &\quad = \max\left\{S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}), S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right\}. \end{aligned} \tag{32}$$

Now if $S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}) \geq S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1})$ for some $n = 0, 1, 2, \cdots$, then $M(w_{2n}, w_{2n}, w_{2n+1}) = S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})$ and from (32), we have

$$\begin{aligned} \pi\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right) &\leq \pi\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right), \\ &\quad -\gamma\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\right), \end{aligned} \tag{33}$$

implying that $\gamma(S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) = 0$ is a contradiction. Therefore, for all $n \geq 0$,

$$S_b^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}) \leq S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}). \tag{34}$$

Similarly, we have

$$S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}) \leq S_b^{*P}(w_{2n-1}, w_{2n-1}, w_{2n}), \tag{35}$$

for all $n \geq 0$. Hence, we get

$$S_b^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) \leq S_b^{*P}(w_n, w_n, w_{n+1}). \tag{36}$$

Also, $\{S_b^{*P}(w_{n+1}, w_{n+1}, w_{n+2})\}$ is a nonincreasing sequence, then there exists $\mathfrak{L}$, such that

$$\lim_{n \longrightarrow \infty} S_b^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) = \mathfrak{L}. \tag{37}$$

Hence, by the lower semicontinuity of γ,

$$\gamma(\mathfrak{L}) \leq \liminf_{n \longrightarrow \infty} \gamma\left(S_b^{*P}(w_n, w_n, w_{n+1})\right) \tag{38}$$

Now, we claim that $\mathfrak{L} = 0$. By lower semicontinuity of γ, taking the upper limit as $n \longrightarrow \infty$ on either side of

$$\begin{aligned} \pi\left(S_b^{*P}(w_{n+1}, w_{n+1}, w_{n+2})\right) &\leq \pi(M(w_n, w_n, w_{n+1})) \\ &\quad - \gamma(M(w_n, w_n, w_{n+1})), \end{aligned} \tag{39}$$

then, we get

$$\pi(\mathfrak{L}) \leq \pi(\mathfrak{L}) - \liminf_{n \longrightarrow \infty} \gamma(M(w_n, w_n, w_{2n+1})) \leq \pi(\mathfrak{L}) - \gamma(\mathfrak{L}), \tag{40}$$

this implies that $\gamma(\mathfrak{L}) \leq 0 \Rightarrow \gamma(\mathfrak{L}) = 0$, then we have

$$\lim_{n \longrightarrow \infty} S_b^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) = 0. \tag{41}$$

To show that $\{w_{n+1}\}$ is a S^{*P}-Cauchy sequence for each n, m, and $n, m > N$, we have

$$S_b^{*P}(w_n, w_n, w_m) \leq \zeta\left[2S_b^{*P}(w_n, w_n, w_{n+1}) + 2\zeta S_b^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) + 2\zeta^2 S_b^{*P}(w_{n+2}, w_{n+2}, w_{n+3}) + \cdots + 2\zeta^m S_b^{*P}(w_m, w_m, w_{m+1})\right] - \zeta\left[\left\{2S_b^{*P}(w_{n+1}, w_{n+1}, w_{n+1}) + \cdots + + 2\zeta^m S_b^{*P} \cdot (w_{m+1}, w_{m+1}, w_{m+1})\right\}\right]. \tag{42}$$

By taking the limit as $n, m \longrightarrow \infty$ to both sides of the above inequality and from (41), we have

$$\lim_{n,m\longrightarrow\infty} S_b^{*P}(w_n, w_n, w_m) = 0. \tag{43}$$

It follows that $\{w_n\}$ is a S_b^{*P}-Cauchy sequence and by S_b^{*P} -completeness of Y, so there exist $r \in Y$ such that $\{w_n\}$ converges to r as $n \longrightarrow \infty$. Now, we will distinguish the cases (a) and (b) of this theorem.

(a) Suppose $\mathfrak{g}$ is continuous, since $w_{2n+1} \longrightarrow r$, we obtain that

$$w_{2n+1} = \mathfrak{g}(w_{2n+1}) = \mathfrak{g}(r). \tag{44}$$

But $w_{2n+1} \longrightarrow r$, as a subsequence of $\{w_n\}$. It follows that $\mathfrak{g}(r) = r$ and from the beginning of the prove we get $\mathfrak{g}(r) = r = \mathfrak{f}(r)$.The proof, assuming that $\mathfrak{f}$ is continuous, is similar to the above.

(b) Suppose that $S_b^{*P}(r, r, \mathfrak{g}(r))$ and for $\{w_n\}$ and a nondecreasing sequence with $w_n \longrightarrow r$ in Y indicate that $w_{2n+1} \preccurlyeq r$, $\forall n \in N$. Therefore, from (24), we have

$$\pi\left(S_b^{*P}(w_{2n+1}, w_{2n+1}, \mathfrak{g}(r))\right) = \pi\left(S_b^{*P}(\mathfrak{f}w_{2n}, \mathfrak{f}w_{2n}, \mathfrak{g}(r))\right), \leq \pi(M(w_{2n}, w_{2n}, r)) - \gamma(M(w_{2n}, w_{2n}, r)), \tag{45}$$

where

$$M(w_{2n}, w_{2n}, r) = \max \cdot \left\{S_b^{*P}(w_{2n}, w_{2n}, r), S_b^{*P}(w_{2n}, w_{2n}, \mathfrak{f}w_{2n}), S_b^{*P}(r, r, \mathfrak{g}r), \frac{1}{2} \cdot \left[S_b^{*P}(w_{2n}, w_{2n}, \mathfrak{g}r) + S_b^{*P}(r, r, \mathfrak{f}w_{2n})\right]\right\} = \max\left\{S_b^{*P}(w_{2n}, w_{2n}, r), S_b^{*P}(w_{2n}, w_{2n}, w_{2n+1}), S_b^{*P}(r, r, \mathfrak{g}r), \frac{1}{2} \cdot \left[S_b^{*P}(w_{2n}, w_{2n}, \mathfrak{g}r) + S_b^{*P}(r, r, w_{2n+1})\right]\right\}. \tag{46}$$

By taking limit as $n \longrightarrow \infty$, this implies

$$\lim_{n\longrightarrow\infty} M(w_{2n}, w_{2n}, r) = S_b^{*P}(r, r, \mathfrak{g}r). \tag{47}$$

Hence,

$$\pi\left(S_b^{*P}(r, r, \mathfrak{g}r)\right) = \lim_{n\longrightarrow\infty} \sup \pi\left(S_b^{*P}(\mathfrak{f}w_{2n}, \mathfrak{f}w_{2n}, \mathfrak{g}r)\right) \leq \lim_{n\longrightarrow\infty} \sup \left[\pi(M(w_{2n}, w_{2n}, r)) - \gamma(M(w_{2n}, w_{2n}, r))\right]. \tag{48}$$

This is a contradiction. Thus, we have

$$S_b^{*P}(r, r, \mathfrak{g}r) = 0 \Rightarrow r = \mathfrak{f}r = \mathfrak{g}r. \tag{49}$$

Corollary 33. *Let $(Y, \preccurlyeq)$ be a partially ordered set, $\mathfrak{f}$ and $\mathfrak{g}$ be weakly increasing self-mapping on a complete S_b^{*P} -metric space with $\zeta \geq 1$. Suppose that there exist $\gamma \in \Gamma$ such that*

$$S_b^{*P}(\mathfrak{f}(w), \mathfrak{f}(w), \mathfrak{g}(t)) \leq M(w, w, t) - \gamma(M(t, t, w)), \tag{50}$$

for all $w, t \in Y$, where

$$M(w, w, t) = \max\left\{S_b^{*P}(w, w, t), S_b^{*P}(w, w, \mathfrak{f}(w)), S_b^{*P}(t, t, \mathfrak{g}(t)) + \frac{1}{2}\left[S_b^{*P}(w, w, \mathfrak{g}(t)) + S_b^{*P}(t, t, \mathfrak{f}(w))\right]\right\}. \tag{51}$$

Assume that one of the following cases is satisfied:

(a) if a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \preccurlyeq r$ for all $n \in \mathbb{N}$

(b) $\mathfrak{f}$ or $\mathfrak{g}$ is continuous

Therefore, the maps $\mathfrak{f}$ or $\mathfrak{g}$ have a common fixed point.

Proof. Put $\pi(\mathfrak{t}) = \mathfrak{t}$ in Theorem 32.

Corollary 34. *Let $(Y, \preccurlyeq)$ be a partially ordered set, $\mathfrak{f}$ and $\mathfrak{g}$ be weakly increasing self-mapping on a complete S_b^{*P} -metric space. Suppose that there exist $\pi \in \Pi$ and $\gamma \in \Gamma$ such that*

$$\pi\left(S_b^{*P}(\mathfrak{f}(w), \mathfrak{f}(w), \mathfrak{g}(t))\right) \leq \pi(M(w, w, t)) - \gamma(M(t, t, w)), \tag{52}$$

for all $w, t \in Y$, where

$$M(w, w, t) = \mu_1 S_b^{*P}(w, w, t) + \mu_2 S_b^{*P}(w, w, \mathfrak{f}(w)) + \mu_3 S_b^{*P}(t, t, \mathfrak{g}(t)) + \mu_4 \cdot \left[S_b^{*P}(w, w, \mathfrak{g}(t)) + S_b^{*P}(t, t, \mathfrak{f}(w))\right], \tag{53}$$

where $\mu_1 > 0$ for $i = \{1, 2, 3, 4\}$ with $\mu_1 + \mu_2 + \mu_3 + \mu_4 \geq 1$.

Then, of the following two cases, assume that one of the following cases is satisfied:

(a) if a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \preccurlyeq r$ for all $n \in \mathbb{N}$

(b) $\mathfrak{f}$ or $\mathfrak{g}$ is continuous

Therefore, the maps $\mathfrak{f}$ or $\mathfrak{g}$ have a common fixed point.

Proof. A corollary is S^{*P}-partial metric spaces version of Theorem 18.

Corollary 35. *Let $(Y, \preccurlyeq)$ be a partially ordered set, $\mathfrak{f}$ and $\mathfrak{g}$ be weakly increasing self-mapping on a complete S_b^{*P} -metric space. Suppose that there exist $\gamma \in \Gamma$ such that*

$$S_b^{*P}(\mathfrak{f}(w), \mathfrak{f}(w), \mathfrak{g}(t)) \leq M(w, w, t) - \gamma(M(t, t, w)), \quad (54)$$

for all $w, t \in Y$, where

$$\begin{aligned} M(w, w, t) = {} & \mu_1 S_b^{*P}(w, w, t) + \mu_2 S_b^{*P}(w, w, \mathfrak{f}(w)) \\ & + \mu_3 S_b^{*P}(t, t, \mathfrak{g}(t)) + \mu_4 \\ & \cdot \left[S_b^{*P}(w, w, \mathfrak{g}(t)) + S_b^{*P}(t, t, \mathfrak{f}(w))\right], \end{aligned} \quad (55)$$

where $\zeta_1 > 0$ for $i = \{1, 2, 3, 4\}$ with $\mu_1 + \mu_2 + \mu_3 + \mu_4 \geq 1$.

Then, of the following two cases, assume that one of the following cases is satisfied:

(a) if a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \preccurlyeq r$ for all $n \in \mathbb{N}$

(b) $\mathfrak{f}$ or $\mathfrak{g}$ is continuous

Therefore, the maps $\mathfrak{f}$ or $\mathfrak{g}$ have a common fixed point.

Proof. Put $\pi(\mathfrak{t}) = \mathfrak{t}$ in Corollary 34.

Remark 36. It will be interesting to find more applications to our current paper in other fields, see [11, 19–29].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Blow-Up of Solutions for a Class Quasilinear Wave Equation with Nonlinearity Variable Exponents

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This work deals with the blow-up of solutions for a new class of quasilinear wave equation with variable exponent nonlinearities. To clarify more, we prove in the presence of dispersion term $-\Delta u_{tt}$ a finite-time blow-up result for the solutions with negative initial energy and also for certain solutions with positive energy. Our results are extension of the recent work (Appl Anal. 2017; 96(9): 1509-1515).

1. Introduction

We study in this paper the following nonlinear wave equation:

$$\begin{cases} u_{tt} - \operatorname{div}\left(\left|\nabla u^{s(.)-2}\right|\nabla u\right) - \Delta u_{tt} + \eta u_t |u_t|^{q(.)-2} = \mu u |u|^{p(.)-2}, \text{in}\Omega \times (0, T), \\ u(x, t) = 0, \text{on}\Omega \times (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \text{in}\Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^n (n \geq 1)$, be a bounded domain with a smooth boundary $\partial\Omega$, and$\eta, \mu > 0$ are constants, and the exponents $q(.), p(.)$, and $s(.)$ are given measurable functions on Ω satisfying

$$2 \leq \max \{q_2, s_2\} < p_1 \leq p(x) \leq p_2 \leq s^*(x), \tag{2}$$

with

$$\begin{gathered} p_1 \coloneqq \operatorname{ess}\inf_{x\in\Omega} p(x), p_2 \coloneqq \operatorname{ess}\sup_{x\in\Omega} p(x), \\ s_1 \coloneqq \operatorname{ess}\inf_{x\in\Omega} s(x), s_2 \coloneqq \operatorname{ess}\sup_{x\in\Omega} s(x), \\ q_1 \coloneqq \operatorname{ess}\inf_{x\in\Omega} q(x), q_2 \coloneqq \operatorname{essp}\sup_{x\in\Omega} q(x), \\ s^*(x) = \begin{cases} \dfrac{Ns(x)}{\operatorname{ess}\sup_{x\in\Omega}(N - q(x))} & \text{if } s_2 < n, \\ +\infty & \text{if } s_2 \geq n. \end{cases} \end{gathered} \tag{3}$$

Also, we suppose that $q(.)$, $p(.)$, and $s(.)$ satisfy the log-Holder continuity condition:

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|} \text{ for a.e.} x, y \in \Omega, \text{with} |x-y| < \delta, \tag{4}$$

$A > 0, 0 < \delta < 1$. In (4), if $x = y$, the inequality is undefined because log 0 is undefined. The inequality is defined for x not equal to y, but the condition that δ is completely greater than zero always makes x not equal to y because $|x - y| < \delta$. The term $\Delta_{s()} u = \text{div}\ (|\nabla u^{s(.)-2}|\nabla u)$ is called $s(.)$-Laplacian.

There are many studies that have studied the problem (P) in the case of constant and variable-exponent nonlinearities.

In the case of constant exponent nonlinearity when $\eta = 0$, $s = 2$, and $\Delta u_{tt} = 0$, the explosion term $\mu u|u|^{p-2}$ forces the negative-energy solutions to explode in finite time ([1, 2]), whereas when $\mu = 0$, $s = 2$, and $\Delta u_{tt} = 0$, the dissipation term $\eta|u_t|^{q-2}u_t$ guarantees the existence of global solutions for any initial data [3].

The problem was first treated by (Levine [2] and Vitillaro [4]) in the case when the both terms are present (the dissipation and source). He debated the case when $q = 2, s = 2$, and determined the result of blow-up for solutions with negative initial energy. To extend Levine's results in [5] considered a different method when $q > 2$ and discussed the cases when $q \geq p$ and $p > q$.

Chen et al. in [6] looked into the nonlinear p − Laplacian wave equation:

$$\partial_{tt}u - \nabla.\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + q(x, u) = f(x), \tag{5}$$

when $2 \leq p < n$ and f, q are given functions. Under suitable conditions on the initial data and the functions f, q, they realized global existence and uniqueness and also discussed the long-time behavior of the solution.

In [7], Benaissa and Mokeddem considered

$$u_{tt} - \text{div}\ \left(|\nabla u|^{p-2}\nabla u\right) - \sigma(t)\ \text{div}\ \left(|\nabla u_t|^{m-2}\nabla u_t\right) = 0, \tag{6}$$

and they achieved an energy decay estimate for the solutions where $p, m \geq 2$, σ, is a positive function and expanded Yang [8] and Messaoudi [9] results. Recently, Mokeddem and Mansour [10] added some modification in the problem of Benaissa and Mokeddem [10] and established the same decay result.

Messaoudi and Houari [11] studied the nonlinear wave equation:

$$\begin{aligned} &u_{tt} - \Delta u_t - c\ \text{div}\ \left(|\nabla u|^{\alpha-2}\nabla u\right) - \text{div}\ \left(|\nabla u_t|^{\beta-2}\nabla u_t\right) \\ &\quad + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \end{aligned} \tag{7}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$,$a, b, c > 0$, and $\alpha, \beta, m, p > 2$. They investigated with appropriate conditions imposed on $\alpha, \beta, m, p > 2$, a global nonexistence result for solutions associated with negative initial energy. In [12], Kafini and Messaoudi treated a nonlinear wave equation with delay term and proved, under appropriate hypotheses on the initial data, that the energy of solutions explodes in a finite time. For more results, see the previous studies ([13–35]). In the case of variable-exponent nonlinearity, Antontsev [36] looked into the problem:

$$\partial_{tt}u - \text{div}\ \left(a(x, t)|\nabla u|^{p(x,t)-2}\nabla u\right) - \alpha\Delta u_t = b(x, t)u|u|^{\sigma(x,t)-2}, \tag{8}$$

when α is a nonnegative constant, and a, b, p, σ are given functions. He discussed the case when $\alpha = 0$ and $\alpha > 0$ and demonstrated a blow-up result under particular hypothesis on a, b, p, σ.

Thereafter, Antontsev in [36] considered the same equation and established a local, global existence of weak solutions for specific conditions on a, b, p, σ and realized a blow-up results for solutions with nonpositive initial energy. In [5], Guo and Gao considered the same problem of Antontsev [13], they picked the constant $\sigma(x, t) = r > 2$ and realized a blow-up result in finite time, and also, they alleged without any proof the same blow-up result for $\sigma(x, t) = r(x)$.

Sun et al. in [37] studied the blow-up result for solutions with positive initial energy for the following equation:

$$u_{tt} - \text{div}\ (a(x, t)\nabla u) + c(x, t)u_t|u_t|^{q(x,t)-1} = b(x, t)u|u|^{p(x,t)-1}. \tag{9}$$

They also gave lower and upper bounds for the blow-up time and provided a numerical illustrations for their result.

Lately, Messaoudi and Talahmeh [38] looked into

$$u_{tt} - \text{div}\ \left(|\nabla u|^{m(x)-2}\nabla u\right) + \mu u_t = |u|^{p(x)-2}u, \tag{10}$$

where $\mu \geq 0$. They proved a blow-up result for certain solutions with arbitrary positive initial energy. Korpusov [39] generalized this result and established (10), with m and p are constants.

In this paper, we care to find sufficient conditions on q, p, s and the initial data for which the blowup happens.

In addition to the introduction, our paper is divided into three sections. The second section deals with variable-exponent Lebesgue and Sobolev spaces and some of their characteristics. We also mention the result of existence, but without demonstration, and the second one deals with the result of blow-up for solutions with negative initial energy.

In the fourth one, we present and demonstrate the theorem of blow-up for certain solutions with positive initial energy.

2. Background and Preliminaries

This section contains some essential concepts and definitions about the Lebesgue and Sobolev spaces with variable exponents which will be useful to us later (see Fan and Zhao [40] and Lars et al. [41], Mezouar and Boulaaras [42]).

Let Ω that is a domain of $\mathbb{R}^n$, $p : \Omega \longrightarrow 1, \infty]$ be a measurable function. We introduce the Lebesgue space with a variable exponent $p(.)$ by

$$\begin{aligned} L^{p(.)}(\Omega) := \Big\{ &w : \Omega \longrightarrow \mathbb{R}\ ; \text{measurable in}\ \Omega : \rho_{p(.)}(\lambda w) \\ &< +\infty, \text{for } \textit{some}\ \lambda > 0 \Big\}, \end{aligned} \tag{11}$$

where

$$\rho_{p(.)}(w) = \int_{\Omega} |w(x)|^{p(x)} dx. \quad (12)$$

Equipped with the Luxembourg-type norm,

$$\|w\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{w(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \quad (13)$$

$L^{p(.)}(\Omega)$ is a Banach space (see Lars et al. [41]).

Now, we introduce the following variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$:

$$W^{1,p(.)}(\Omega) = \left\{ w \in L^{p(.)}(\Omega) \text{ such that} \nabla w \text{ exists and } |\nabla w| \in L^{p(.)}(\Omega) \right\}. \quad (14)$$

This space is a Banach space with respect to the norm $\|w\|_{W^{1,p(.)}(\Omega)} = \|w\|_{p(.)} + \|\nabla w\|_{p(.)}$. Otherwise, we put $W_0^{1,p(.)}(\Omega)$ to be the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(.)}(\Omega)$.

Remark 1. The space $W^{1,p(.)}(\Omega)$ is generally defined in a different way for the variable exponent case.

However, the two definitions are equivalent under condition (4) (see Lars et al. [4]). We define $W^{-1,p'(.)}(\Omega)$ as the dual of $W^{1,p(.)}(\Omega)$, in the same way as the classical Sobolev spaces, where $(1/p(.)) + (1/p'(.)) = 1$.

Lemma 2. *(Poincare inequality [41]).*

Let Ω be a bounded domain and assume that $p(.)$ satisfies (4), then

$$\|w\|_{p(.)} \le C \|\nabla w\|_{p(.)}, \text{for all } w \in W^{1,p(.)}(\Omega), \quad (15)$$

for $C > 0$ that is a constant depends only on Ω and p_1, p_2. Particularly, $\|\nabla w\|_{p()}$ determines an equivalent norm on $W_0^{1,p(.)}(\Omega)$.

Lemma 3. *(Lars et al. [41]).*

If $m : \Omega \longrightarrow [1,\infty)$ is a measurable function and $s(.) \in C(\bar{\Omega})$ such that

$$\begin{aligned} &\operatorname*{ess\,inf}_{x\in\Omega} (s^*(x) - m(x)) > 0 \text{ with } s^*(x) \\ &= \begin{cases} \dfrac{ns(x)}{\operatorname*{ess\,sup}_{x\in\Omega}(n - s(x))} & \text{if } s_2 < n, \\ \infty & \text{if } s_2 \ge n. \end{cases} \end{aligned} \quad (16)$$

Then, the embedding $W_0^{1,s(\bullet)}(\Omega)^{\circ} L^{m()}(\Omega)$ is continuous and compact.

Lemma 4. *(Hölder's inequality [41]).*

Assume that $p, m, r \ge 1$ are measurable functions defined on Ω such that

$$\frac{1}{r(y)} = \frac{1}{p(y)} + \frac{1}{m(y)}, \text{for a.e.} y \in \Omega. \quad (17)$$

If $u \in L^{p(.)}(\Omega)$ and $w \in L^{m(.)}(\Omega)$, then $uw \in L^{r(.)}(\Omega)$, with $\|uw\|_{r(.)} \le 2\|u\|_{p(.)}\|w\|_{m(.)}$.

Lemma 5. *(Unit ball property [41]).*

Assume that p is a measurable function on Ω. Then,

$$\|f\|_{p(.)} \le 1 \text{ if and only if } \rho_{p(.)}(f) \le 1. \quad (18)$$

Lemma 6. *(Lars etal [41]).*

If p is a measurable function on Ω satisfying (4), then for a.e. $x \in \Omega$, we get

$$\min \left\{ \|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2} \right\} \le \rho_{p(.)}(u) \le \max \left\{ \|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2} \right\}, \quad (19)$$

for any $u \in L^{p(.)}(\Omega)$.

Proposition 7. *Let $(u_0, u_1) \in (W_0^{1,s()}(\Omega) \times L^2(\Omega))$ and suppose that the exponents p, q, s satisfy (1) and (2). Then, problem (P) admits a unique weak solution such that*

$$\begin{aligned} &u \in L^{\infty}\left((0,T), W_0^{1,s(\bullet)}(\Omega)\right), \\ &u_t \in L^{\infty}\left((0,T), H_0^1(\Omega)\right), \\ &u_{tt} \in L^{\infty}\left((0,T), W_0^{-1,s'(\bullet)}(\Omega)\right), \end{aligned} \quad (20)$$

where $(1/s()) + (1/s'()) = 1$.

Remark 8. We can achieve the proof of the previous proposition by using the Galerkin method as in [13].

3. Blowing Up for Negative Initial Energy

To introduce and demonstrate our results, we first define our energy as follows:

$$\begin{aligned} E(t) := &\frac{1}{2}\int_{\Omega} u_t^2 dx + \int_{\Omega} \frac{1}{s(x)} |\nabla u|^{s(x)} dx + \frac{1}{2}\int_{\Omega} |\nabla u_t|^2 dx \\ &- \mu \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx. \end{aligned} \quad (21)$$

Theorem 9. *Assume that the assumptions of Proposition1 hold true and suppose that*

$$E(0) < 0. \quad (22)$$

Then, the solution of problem (P) blows up in finite time.

To state and demonstrate our result, In order to prove our result, we give the following Lemma

Lemma 10. *Suppose the conditions of Lemma 3 hold. Then, we have*

$$\rho_{p(.)}^{r/p_1}(u) \le C\left(\|\nabla u\|_{s_{(.)}}^{s_1} + \rho_{p(.)}(u)\right), s_1 \le r \le p_1, \tag{23}$$

for any $u \in W_0^{1,s()}(\Omega)$, where $C > 1$ is a positive constant which depends on Ω only.

Corollary 11. *Let the assumptions of Lemma 10 hold. Then, for any $u \in W_0^{1,s()}$, we get*

$$\|u\|_{p_1}^r \le C\left(\|\nabla u\|_{s_{(.)}}^{s_1} + \|u\|_{p_1}^{p_1}\right), \tag{24}$$

where $s_1 \le r \le p_1$ and C are a positive constant.

Now, we set

$$H(t) := -E(t) \tag{25}$$

and use, throughout this paper, C to denote a generic positive constant depending on Ω only. From the result of (21) and (23), we give the following Corollary

Corollary 12. *Let the assumptions of Lemma 10 hold. Then, we have*

$$\rho_{p(.)}^{r/p_1}(u) \le C\left(|H(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \rho_{p(.)}(u)\right), \tag{26}$$

for any $u \in u \in W_0^{1,s()}$ and $s_1 \le r \le p_1$.

Corollary 13. *Let the assumptions of Lemma 10 hold. Then, we have*

$$\|u\|_{p_1}^r \le C\left(|H(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right), \tag{27}$$

for any $u \in W_0^{1,s()}$ and $s_1 \le r \le p_1$.

Lemma 14. *Assume that (2) and (4) hold and $E(0) < 0$. Then, the solution of (P) satisfies, for some $c > 0$,*

$$\rho_{p(.)}(u) \ge c\|u\|_{p_1}^{p_1}. \tag{28}$$

Lemma 15. *Let u be the solution of problem (P) and assume that (2) holds. Then,*

$$\int_\Omega |u|^{q(x)} dx \le C\left(\left(\rho_{p(.)}(u)\right)^{q_1/p_1} + \left(\rho_{p(.)}(u)\right)^{q_2/p_1}\right). \tag{29}$$

Remark 16. We can achieve the proof of the previous Lemmas and Corollaries as in the paper of Messaoudi and Talahmeh [39].

Lemma 17. *Let u be the solution of (P). Then, there exists a constant $c_1 > 0$ such that*

$$\|\nabla u(.,t)\|_{s(.)} \ge c_1, \forall t \ge 0. \tag{30}$$

Proof. Assume, by contradiction, there exists a sequence t_j such that

$$\left\|\nabla u\left(.,t_j\right)\right\|_{s(.)} \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{31}$$

Then, Lemmas 3 and 6 give us

$$\rho_{p(.)}\left(u\left(.,t_j\right)\right) \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{32}$$

This yields

$$\lim_{j\longrightarrow\infty} E\left(t_j\right) \ge 0, \tag{33}$$

that contrasts with the fact that $E(t) < 0, \forall t \ge 0$.

Proof. Proof of Theorem 9.

As usual, multiplying by u_t and integrating over Ω in (P) to get

$$E'(t) = -\eta \int_\Omega |u_t(x,t)|^{q(x)} dx \le 0, \tag{34}$$

for almost every t in $[0, T)$ since $E(t)$ is absolutely continuous; hence, $H'(t) \ge 0$ and

$$0 < H(0) \le H(t) \le \frac{\mu}{p_1}\rho_{p(.)}(u), \tag{35}$$

for every t in $[0, T)$, by remembering the condition that $E(0) < 0$. We then introduce

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t) dx, \tag{36}$$

for ε small to be chosen later and

$$0 < \alpha \le \min\left\{\frac{p_1 - 2}{2p_1}, \frac{p_1 - q_2}{p_1(q_2 - 1)}\right\}. \tag{37}$$

By taking the derivative of (35) and using (1), we obtain

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2(x,t) dx + \varepsilon \int_\Omega u u_{tt}^2(x,t) dx, \tag{38}$$

so

$$\left\{L'(t) + \frac{d}{dt}\left(\varepsilon\int_\Omega \{\nabla u_t \nabla u\}\right)\right.$$
$$= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega \left[u_t^2 - |\nabla u|^{s(x)} + |\nabla u_t|^2\right]$$
$$+ \varepsilon\mu\int_\Omega |u|^{p(x)} - \eta\varepsilon\int_\Omega u u_t |u_t|^{q(x)-2}. \tag{39}$$

Adding and subtracting the term $\varepsilon(1-\xi)p_1H(t)$, for $0<\xi<1$, from the right side of (37), we get

$$L'(t) + \frac{d}{dt}\left(\varepsilon\int_\Omega \{\nabla u_t \nabla u\}\right)$$
$$\geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(1-\xi)p_1H(t)$$
$$+ \varepsilon\mu\xi\int_\Omega |u|^{p(x)} + \varepsilon\left(\frac{(1-\xi)p_1}{2}+1\right)\|u_t\|_2^2$$
$$+ \varepsilon\left(\frac{(1-\xi)p_1}{s_2}-1\right)\int_\Omega |\nabla u|^{s(x)}\varepsilon\left(\frac{(1-\xi)p_1}{2}+1\right)$$
$$\cdot \int_\Omega |\nabla u_t|^2 - \eta\varepsilon\int_\Omega u u_t |u_t|^{q(x)-2}dx. \tag{40}$$

So, for ξ small enough, we obtain

$$L'(t) + \frac{d}{dt}\left(\varepsilon\int_\Omega \{\nabla u_t \nabla u\}\right)$$
$$\geq \varepsilon\beta\left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \rho_{s(.)}(\nabla u) + \rho_{p(.)}(u)\right]$$
$$+ (1-\alpha)H^{-\alpha}(t)H'(t) - \eta\varepsilon\int_\Omega u u_t |u_t|^{q(x)-2}dx. \tag{41}$$

where

$$\beta = \min\left\{(1-\xi)p_1, \mu\xi, \frac{(1-\xi)p_1}{2}+1, \frac{(1-\xi)p_1}{s_2}-1\right\} > 0. \tag{42}$$

By using Young's inequality, the last term in (40) yields

$$\int_\Omega |u_t|^{q(x)-1}|u|dx \leq \frac{1}{q_1}\int_\Omega \delta^{q(x)}|u|^{q(x)} + \frac{q_2-1}{q_2} \cdot \int_\Omega \delta^{-q(x)/q(x)-1}|u_t|^{q(x)}dx \forall\delta>0. \tag{43}$$

Thus, by picking δ such that

$$\delta^{-q(x)/q(x)-1} = kH^{-\alpha}(t), \tag{44}$$

for a large constant k to be given later, and replacing in (41), we reach to

$$\int_\Omega |u_t|^{q(x)-1}|u|dx \leq \frac{1}{q_1}\int_\Omega k^{1-q(x)}|u|^{q(x)}H^{\alpha(q(x)-1)}(t) + \frac{q_2-1}{q_2}kH^{-\alpha}(t)H'(t)\forall\delta>0. \tag{45}$$

Combining (40) and (43) yields

$$L'(t) + \frac{d}{dt}\left(\varepsilon\int_\Omega \{\nabla u_t \nabla u\}\right)$$
$$\geq \varepsilon\beta\left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \rho_{s(.)}(\nabla u) + \rho_{p(.)}(u)\right]$$
$$+ \left[(1-\alpha) - \varepsilon\frac{q_2-1}{q_2}k\right]H^{-\alpha}(t)H'(t)$$
$$- \eta\varepsilon\frac{k^{1-q_1}}{q_1}C_1H^{\alpha(q_2-1)}(t)\int_\Omega |u|^{q(x)}dx. \tag{46}$$

Exploiting Lemma 15 and (34) to get

$$H^{\alpha(q_2-1)}(t)\int_\Omega |u|^{q(x)}dx \leq C\left[(\rho(u))^{q_1/p_1+\alpha(q_2-1)} + (\rho(u))^{q_2/p_1+\alpha(q_2-1)}\right]. \tag{47}$$

Now, we employ Lemma 10 and (36), and we get

$$r = q_2 + \alpha p_1(q_2-1) \leq p_1 \text{ and } r = q_1 + \alpha p_1(q_2-1) \leq p_1, \tag{48}$$

And it is easy to see from (46) that

$$H^{\alpha(q_2-1)}(t)\int_\Omega |u|^{q(x)}dx \leq C\left(\|\nabla u\|_{s(.)}^{s_1} + \rho_{p(.)}(u)\right). \tag{49}$$

So, by using Lemmas 6 and 17, we obtain

$$\rho_{s(.)}(\nabla u) \geq c_2\|\nabla u\|_{s(.)}^{s_1}. \tag{50}$$

Collecting of (45), (47), and (49), we get

$$L'(t) + \frac{d}{dt}\left(\varepsilon\int_\Omega \{\nabla u_t \nabla u\}\right)$$
$$\geq \left[(1-\alpha) - \frac{q_2-1}{q_2}\varepsilon k\right]H^{-\alpha}(t)H'(t)$$
$$+ \varepsilon\left(\beta - \eta\frac{k^{1-q_1}}{q_1}C\right)\left[H(t) + \|u_t\|_2^2\right.$$
$$\left. + \|\nabla u_t\|_2^2 + \|\nabla u\|_{s(.)}^{s_1} + \rho_{p(.)}(u)\right]. \tag{51}$$

In this step, we choose k so large that the coefficient

$$\gamma = \beta - \eta\frac{k^{1-q_1}}{q_1}C > 0. \tag{52}$$

Once k is fixed (thus γ), we put sufficiently small ε so that

$$(1-\alpha)-\frac{q_2-1}{q_2}\varepsilon k\geq 0 \text{ and } L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0u_1(x)dx>0. \tag{53}$$

Subsequently, (50) becomes

$$\begin{aligned}\left\{L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\right.&\geq\varepsilon\gamma\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\|\nabla u\|_{s(.)}^{s_1}+\rho_{p(.)}(u)\right]\\&\geq\varepsilon\gamma\left[H(t)+\|u_t\|_2^2+\|u\|_{p_1}^{p_1}\right],\end{aligned} \tag{54}$$

by virtue of (28). Therefore, we get

$$L(t)\geq L(0)>0, \text{ for } all\ t\geq 0. \tag{55}$$

Next, we are in the position to obtain an inequality of the form

$$L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\geq\Gamma L^{1/1-\alpha}(t), \text{ for all } t\geq 0. \tag{56}$$

Here, Γ is a positive constant that depends on $\varepsilon\gamma$, C (the constant of Corollary 1).

To achieve (54), we estimate the term$|\int_\Omega uu_t(x,t)dx|\leq\|u\|_2\|u_t\|_2\leq C(\|u\|_{p_1}\|u_t\|_2)$. (32)Hence,

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)}\leq C\|u\|_{p_1}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}. \tag{57}$$

From Young's inequality that yields the following estimate

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)}\leq C\left[\|u\|_{p_1}^{\omega/(1-\alpha)}+\|u_t\|_2^{\chi/(1-\alpha)}\right], \tag{58}$$

where $1/\omega+1/\chi=1$. Putting $\chi=2/(1-\alpha)$, we find $\omega/(1-\alpha)=2/(1-2\alpha)\leq p_1$ by (37). Thus, (56) becomes

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)}\leq C\left[\|u\|_{p_1}^{r}+\|u_t\|_2^2\right], \tag{59}$$

with $r=2/(1-2\alpha)\leq p_1$. We obtain after using Corollary 3

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)}\leq C\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\|u\|_{p_1}^{p_1}\right], \text{ for all } t\geq 0. \tag{60}$$

In the end, by noting that

$$\begin{aligned}L^{1/(1-\alpha)}(t)&=\left[H^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t(x,t)dx\right]^{1/(1-\alpha)}\\&\leq 2^{1/(1-\alpha)}\left[H(t)+\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)}\right],\end{aligned} \tag{61}$$

and combining it with (51) and (58)), the inequality (54) is achieved.

The proof is completed.

4. Blowing Up for Positive Initial Energy

Now, we are in the position to present and prove one of the main results of this section which is the blow-up for certain solutions with positive energy. For this goal, let A be the best constant of the Sobolev embedding $W_0^{1,s()}(\Omega)^\circ L^{p()}(\Omega)$ and set

$$A_1=\max\left\{1,A,\left(\frac{1}{\mu}\right)^{1/s_2}\right\},\alpha_1=\left(\left(\frac{1}{\mu A_1^{p_1}}\right)^{s_2/(p_1-s_2)}\right), \tag{62}$$

$$\alpha_0=\|\nabla u_0\|_{s(.)}^{s_2},E_1=\left(\frac{1}{s_2}-\frac{1}{p_1}\right)\alpha_1, \tag{63}$$

$$H(t)=E_1-E(t), \tag{64}$$

$$K(t)=H^{1-\lambda}(t)+\varepsilon\int_\Omega uu_t(x,t)dx, \tag{65}$$

for $0<\lambda<1$, $\varepsilon>0$ that are to be specified later.

We state here the following theorem which will be our main result.

Theorem 18. *Assume that the conditions of Proposition 7 hold true and suppose that*

$$E(0)<E_1,\alpha_1<\alpha_0\leq A_1^{-s_2}. \tag{66}$$

Then, the solution of (P) blows up in a finite time.

To demonstrate our theorem, we refer the following two lemmas.

Lemma 19. *Let the assumptions in Theorem 18 be fulfilled, and then there exists a constant $\alpha_2>\alpha_1$ such that*

$$\|\nabla u(.,t)\|_{s(.)}^{s_2}\geq\alpha_2,\forall t\geq 0. \tag{67}$$

Proof. . Exploiting (21), we get

$$\begin{aligned} E(t) &\geq \frac{1}{s_2}\rho_{s(.)}(\nabla u) - \frac{\mu}{p_1}\rho_{p(.)}(u) \\ &\geq \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} \\ &\quad - \frac{\mu}{p_1}\max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \\ &\geq \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} - \frac{\mu}{p_1}\max \\ &\quad \cdot\left\{\left(A_1\|\nabla u\|_{s(.)}\right)^{p_1}, \left(A_1\|\nabla u\|_{s(.)}\right)^{p_2}\right\} \\ &= \frac{1}{s_2}\min\left\{\alpha^{s_1/s_2}, \alpha\right\} - \frac{\mu}{p_1}\max\left\{\left(A_1^{s_2}\alpha\right)^{p_1/s_2}, \left(A_1^{s_2}\alpha\right)^{p_2/s_2}\right\} \\ &\coloneqq h(\alpha), \forall \alpha \in [0,\infty), \end{aligned} \tag{68}$$

where $\alpha = \|\nabla u\|_{s(.)}^{s_2}$. Let

$$g(\alpha) = \frac{1}{s_2}\alpha - \frac{\mu}{p_1}\left(A_1^{s_2}\alpha\right)^{p_1/r_2}. \tag{69}$$

By noting that $g(\alpha) = h(\alpha)$, for $0 < \alpha \leq A_1^{s_2}$, we can easily verify that the function $g(\alpha)$ is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha_1 < \alpha \leq +\infty$.

Because $E(0) < E_1 = g(\alpha_1)$, there exists a positive constant $\alpha_2 \in (\alpha_1, \infty)$ such that $g(\alpha_2) = E(0)$. So, we get $g(\alpha_0) = h(\alpha_0) \leq E(0) = g(\alpha_2)$. This means that $\alpha_0 \geq \alpha_2$.

To demonstrate (67), we suppose that $\|\nabla u(t_0)\|_{s(.)}^{s_2} < \alpha_2$, for some $t_0 > 0$. Then, there exists $t_1 > 0$ such that $\alpha_1 < \|\nabla u(t_1)\|_{s(.)}^{s_2} < \alpha_2$. Exploiting the monotonicity of $g(\alpha)$ to find

$$E(t_1) \geq g\left(\|\nabla u(t_1)\|_{s(.)}^{s_2}\right) > g(\alpha_2) = E(0), \tag{70}$$

which contradicts $E(t) < E(0)$, for all $t \in (0, T)$. Consequently, (67) is determined.

Lemma 20. *Let the assumptions in Theorem 18 be fulfilled, and so we have*

$$0 < H(0) \leq H(t) \leq \frac{\mu}{p_1}\rho_{p(.)}(u). \tag{71}$$

Proof. Exploiting (21), (30), and (64) to get

$$\begin{aligned} 0 < H(0) \leq H(t) \leq E_1 &- \left[\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{s(x)}|\nabla u|^{s(x)} dx \right. \\ &\left. + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx\right] + \mu\int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx, \end{aligned} \tag{72}$$

then from (67), we find

$$\begin{aligned} & E_1 - \left[\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{s(x)}|\nabla u|^{s(x)} dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx\right] \\ &\leq E_1 - \int_\Omega \frac{1}{s_2}|\nabla u|^{s(x)} dx \leq E_1 - \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} \\ &\leq E_1 - \frac{1}{s_2}\min\left\{\alpha_2^{s_1/s_2}, \alpha_2\right\} \leq E_1 - \frac{1}{s_2}\min\left\{\alpha_1^{s_1/s_2}, \alpha_1\right\} \\ &= E_1 - \frac{1}{s_2}\alpha_1 = -\frac{\alpha_1}{p_1} < 0, \forall t \geq 0. \end{aligned} \tag{73}$$

Therefore,

$$0 < H(0) \leq H(t) \leq \frac{\mu}{p_1}\rho_{p(.)}(u). \forall t \geq 0. \tag{74}$$

Proof of Theorem 2. It is not hard to determine the proof precisely by repeating the same steps (35) to (58) of the proof of Theorem 1, with the use of Lemma 20.

Data Availability

No data were used to support the study.

Conflicts of Interest

The author(s) declare(s) that they have no conflicts of interest.

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Research Article

On Extended Branciari b-Distance Spaces and Applications to Fractional Differential Equations

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In this work, we define new $\alpha - \lambda$-rational contractive conditions and establish fixed-points results based on aforesaid contractive conditions for a mapping in extended Branciari b-distance spaces. We furnish two examples to justify the work. Further, we discuss results on weak well-posed property, weak limit shadowing property, and generalized w-Ulam-Hyers stability in the underlying space. Finally, as an application of our main result, we obtain sufficient conditions for the existence of solutions of a nonlinear fractional differential equation with integral boundary conditions.

1. Introduction and Preliminaries

The distance notion in the metric fixed-point theory is introduced and generalized in different ways by many authors [1–5]. Bakhtin [6] defined the notion of b-metric space which is further used by Czerwik in [7, 8]. In [9], Branciari extended the metric space and introduced the notion of the Branciari distance by changing the property of triangle inequality with quadrilateral one.

Definition 1 [9]. Let $\Xi \neq \varnothing$ be a set and let $b : \Xi^2 \longrightarrow \mathbb{R}_+$ such that, for all $\vartheta, \nu \in \Xi$ and all $u, v \in \Xi \setminus \{\vartheta, \nu\}$

(bd1) $b(\vartheta, \nu) = 0$ if and only if $\vartheta = \nu$ (self-distance/indistancy)

(bd2) $b(\vartheta, \nu) = b(\nu, \vartheta)$ (symmetry)

(bd3) $b(\vartheta, \nu) \leq b(\vartheta, u) + b(u, v) + b(v, \nu)$ (quadrilateral inequality).

The symbol (Ξ, b) the denotes Branciari distance space and abbreviated as "BDS."

In [10], Kamran et al. introduced the notion of extended b-metric space as a generalization of b-metric space and proved the following result.

Definition 2 [10]. Let $\Xi \neq \varnothing$ be a set and $w : \Xi^2 \longrightarrow \mathbb{R}_+ \setminus (0, 1)$. We say that a function $\rho_e : \Xi^2 \longrightarrow \mathbb{R}_+$ is an extended b-metric (ρ_e-metric, in short) if it satisfies

(eb1) $\rho_e(\vartheta, \nu) = 0$ if and only if $\vartheta = \nu$

(eb2) $\rho_e(\vartheta, \nu) = \rho_e(\nu, \vartheta)$ (symmetry) (eb3) $\rho_e(\vartheta, \nu) \le w(\vartheta, \nu)[\rho_e(\vartheta, \upsilon) + \rho_e(\upsilon, \nu)]$,

for all $\vartheta, \nu, \upsilon \in \Xi$. The symbol (Ξ, ρ_e) denotes a ρ_e-metric space.

Theorem 3 [10]. *Let (Ξ, ρ_e) be a complete extended b-metric space such that ρ_e is a continuous functional. Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ satisfy $\rho_e(\mathfrak{T}\vartheta, \mathfrak{T}\nu) \le k\,\rho_e(\vartheta, \nu)$ for all $\vartheta, \nu \in \Xi$ where $k \in [0, 1)$ such that for each $\vartheta_0 \in \Xi$, $\lim_{n,m\longrightarrow\infty} w(\vartheta_n, \vartheta_m) < 1/k$, here $\vartheta_n = \mathfrak{T}^n \vartheta_0$, $n = 1, 2, \cdots$. Then $\mathfrak{T}$ has precisely one fixed-point ϑ. Moreover, for each $\nu \in \Xi$, $\mathfrak{T}^n \nu \longrightarrow \vartheta$.*

In [3], Mitrović et al. extended Theorem 3 and proved the following:

Theorem 4 [3]. *Let (Ξ, ρ_e) be a complete extended b-metric space such that ρ_e is a continuous functional. Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ satisfy*

$$\rho_e(\mathfrak{T}\vartheta, \mathfrak{T}\nu) \le a\,\rho_e(\vartheta, \nu) + b\,\rho_e(\vartheta, \mathfrak{T}\vartheta) + c\,\rho_e(\nu, \mathfrak{T}\nu), \quad (1)$$

for all $\vartheta, \nu \in \Xi$ where a, b, c are nonnegative real numbers with $a + b + c < 1$. Then, $\mathfrak{T}$ has a unique fixed-point ϑ. Moreover, there exists a sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ in Ξ which converges to ϑ such that $\vartheta_{n+1} = \mathfrak{T}\vartheta_n$ for every $n \in \mathbb{N}$.

In [11], Abdeljawad et al. defined the notion of extended Branciari b-distance (EBbDS, in short) by combining the extended b-metric and Branciari distance.

Definition 5 [11]. Let $\Xi \ne \emptyset$ be a set and $w : \Xi^2 \longrightarrow \mathbb{R}_+ \setminus (0, 1)$. We say that a function $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ is an extended Branciari b-metric (e_b-metric, in short) if it satisfies

(ebb1) $e_b(\vartheta, \upsilon) = 0$ if and only if $\vartheta = \upsilon$

(ebb2) $e_b(\vartheta, \upsilon) = e_b(\upsilon, \vartheta)$

(ebb3) $e_b(\vartheta, \upsilon) \le w(\vartheta, \upsilon)[e_b(\vartheta, \nu) + e_b(\nu, \rho) + e_b(\rho, \upsilon)]$,

for all $\vartheta, \upsilon \in \Xi$, all distinct $\nu, \rho \in \Xi \setminus \{\vartheta, \upsilon\}$. The symbol (Ξ, e_b) denotes the extended Branciari b-distance space. For $w(\vartheta, \upsilon) = 1$, (Ξ, e_b) will be called a Branciari b-distance space (BbDS, in short).

Example 1. Let $\Xi = C([0, 1], \mathbb{R})$ and define $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ by $e_b(P, Q) = \int_0^1 (P(t) - Q(t))^2 dt$ with $w(P, Q) = |P(t)| + |Q(t)| + 2$. Note that $e_b(P, Q) \ge 0$ for all $P, Q \in \Xi$, and $e_b(P, Q) = 0$ if and only if $P = Q$. Also, $e_b(P, Q) = e_b(Q, P)$. Hence, it is clear that (Ξ, e_b) is an EBbDS, but it is neither an BDS nor metric space.

Definition 6 [11]. Let $\Xi \ne \emptyset$ be a set endowed with extended Branciari b-distance e_b.

(a) A sequence $\{\vartheta_n\}$ in Ξ converges to ϑ if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $e_b(\vartheta_n, \vartheta) < \varepsilon$ for all $n \ge N$. For this particular case, we write $\lim_{n\longrightarrow\infty} \vartheta_n = \vartheta$

(b) A sequence $\{\vartheta_n\}$ in Ξ is called Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $e_b(\vartheta_m, \vartheta_n) < \varepsilon$ for all $m, n \ge N$

(c) An e_b-metric space (Ξ, e_b) is complete if every Cauchy sequence in Ξ is convergent.

On the other hand, in [12], Samet et al. define the notion of α-admissible mappings which is further extended by Sintunavarat [13] and named as weakly α-admissible mapping.

Definition 7. For a $\Xi \ne \emptyset$ set, let $\alpha : \Xi \times \Xi \longrightarrow [0,\infty)$ and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be two mappings. Then $\mathfrak{T}$ is called

(1) [12] α-admissible if

$$\vartheta, \nu \in \Xi \text{ with } \alpha(x, \nu) \ge 1 \Rightarrow \alpha(\mathfrak{T}\vartheta, \mathfrak{T}\nu) \ge 1 \quad (2)$$

(2) [13] weakly α-admissible if

$$\vartheta \in \Xi \text{ with } \alpha(\vartheta, \mathfrak{T}\vartheta) \ge 1 \Rightarrow \alpha(\mathfrak{T}\vartheta, \mathfrak{T}\mathfrak{T}\vartheta) \ge 1. \quad (3)$$

For a $\Xi \ne \emptyset$ set and a mapping $\alpha : \Xi \times \Xi \longrightarrow [0,\infty)$, we use

$$\begin{aligned} \mathscr{A}(\Xi, \alpha) &:= \text{The set of all } \alpha-\text{admissible mappings on } \Xi, \\ \mathscr{W}\mathscr{A}(\Xi, \alpha) &:= \text{The set of all weakly } \alpha-\text{admissible mappings on } \Xi. \end{aligned} \quad (4)$$

It is noted that

$$\mathscr{A}(\Xi, \alpha) \subset \mathscr{W}\mathscr{A}(\Xi, \alpha). \quad (5)$$

The notion of well-posedness of a fixed-point problem (fpp) has evoked much interest of several mathematicians, for example, Popa [14, 15] and others. In the paper [16], authors defined a weak well-posed (wwp) property in BbDS and in the papers [17, 18]; the authors have discussed limit shadowing property of fixed-point problems.

The aim of this work is to introduce $\alpha-\lambda$-rational contraction in an EBbDS and prove the existence of fixed points of such rational contraction in an EBbDS. We also discuss the weak well-posedness, limit shadowing property, and generalized weak-Ulam-Hyers stability of fixed-point problems in a EBbDS. As an application of our main result, we obtain sufficient conditions for the existence of solutions of a nonlinear fractional differential equation with integral boundary conditions. By doing these work, we generalize Theorems 3 and 4 in the sense that we use a more general contractive condition which depends on the variable (Lipschitz constants), function $w(x, y)$ on the left-side of contractive condition, and proved results on the weakly α-admissible mapping on more general space structures. It is justifies the usefulness of these terms through illustrations, and the results are real generalization as the considered distances are neither metric space not Branciari distance space.

2. Main Results

2.1. $\alpha-\lambda$-Rational Contractive Mapping and Fixed Points. We start with introducing the notion of $\alpha-\lambda$-rational contraction in a EBbDS as follows.

Definition 8. Let (Ξ, e_b) be an EBbDS and $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$. A mapping $\mathfrak{I} : \Xi \longrightarrow \Xi$ is said to be an $\alpha-\lambda$ -rational contraction, if there exist

$$\vartheta, \nu \in \Xi, \tag{6}$$

with

$$\begin{gathered}\alpha(\vartheta, \nu) \geq 1,\\ e_b(\vartheta, \nu) > 0, e_b(\mathfrak{I}\vartheta, \mathfrak{I}\nu) > 0,\end{gathered} \tag{7}$$

which implies

$$\begin{aligned}&w(\vartheta, \nu)e_b(\mathfrak{I}\vartheta, \mathfrak{I}\nu)\\ &\quad\leq \lambda(u) \max\left\{\begin{array}{c} e_b(\vartheta, \nu), e_b(\vartheta, \mathfrak{I}\vartheta), e_b(\nu, \mathfrak{I}\nu),\\ \dfrac{e_b(\nu, \mathfrak{I}\nu)[1+e_b(\vartheta, \mathfrak{I}\vartheta)]}{w(\vartheta, \nu)[1+e_b(\vartheta, \nu)]}, \dfrac{e_b(\vartheta, \mathfrak{I}\vartheta).e_b(\nu, \mathfrak{I}\nu)}{w(\vartheta, \nu).e_b(\vartheta, \nu)}\end{array}\right\}.\end{aligned} \tag{8}$$

We denote by $\Lambda(\Xi, \alpha)$ the collection of all $\alpha-\lambda$-rational contractive mappings on (Ξ, e_b).

The set of all fixed points of a self-mapping $\mathfrak{I}$ on a set $\Xi \neq \varnothing$ will be denoted by $Fix(\mathfrak{I})$.

We are now in a position to state and prove the result.

Theorem 9. *Let (Ξ, e_b) be a complete EBbDS and $\alpha : \Xi \times \Xi \longrightarrow [0,\infty)$. Let $\mathfrak{I} : \Xi \longrightarrow \Xi$ be a mapping satisfying the following:*

(A1) $\mathfrak{I} \in \Lambda(\Xi, \alpha) \cap \mathscr{W}\mathscr{A}(\Xi, \alpha)$

(A2) There exists $u_0 \in \Xi$ such that $\alpha(u_0, \mathfrak{I}u_0) \geq 1$

(A3) $\mathfrak{I}$ is continuous.

Then, $Fix(\mathfrak{I}) \neq \varnothing$. Furthermore, for any $u_0 \in \Xi$, the sequence u_n satisfying $u_n = \mathfrak{I}u_{n-1}$ is convergent.

Proof. By virtue of condition (A2), there exists $u_0 \in \Xi$ such that $\alpha(u_0, \mathfrak{I}u_0) \geq 1$. Define the sequence $\{u_n\} \in \Xi$ by $u_{n+1} = \mathfrak{I}u_n$. If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} \in Fix(\mathfrak{I})$, and we are complete. Therefore, we assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$.

It follows that

$$e_b(u_n, u_{n+1}) > 0, \forall n \in \mathbb{N}. \tag{9}$$

It follows from $\mathfrak{I} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$ and $\alpha(u_0, \mathfrak{I}u_0) \geq 1$ that

$$\alpha(u_1, u_2) = \alpha(\mathfrak{I}u_0, \mathfrak{I}\mathfrak{I}u_0) \geq 1. \tag{10}$$

Continuing this process, we obtain

$$\alpha(u_n, u_{n+1}) \geq 1 \forall n \in \mathbb{N}. \tag{11}$$

Step 1. First, we prove that

$$\lim_{n\longrightarrow\infty} e_b(u_n, u_{n+1}) = 0. \tag{12}$$

It follows from $\mathfrak{I} \in \Lambda(\Xi, \alpha)$ that

$$\begin{aligned}&w(u_{n-1}, u_n)e_b(\mathfrak{I}u_{n-1}, \mathfrak{I}u_n))\\ &\quad\leq \lambda(u_{n-1}) \max\left\{\begin{array}{c} e_b(u_{n-1}, u_n), e_b(u_{n-1}, \mathfrak{I}u_{n-1}), e_b(u_n, \mathfrak{I}u_n),\\ \dfrac{e_b(u_n, \mathfrak{I}u_n)[1+e_b(u_{n-1}, \mathfrak{I}u_{n-1})]}{w(u_{n-1}, u_n)[1+e_b(u_{n-1}, u_n)]}, \dfrac{e_b(u_{n-1}, \mathfrak{I}u_{n-1}).e_b(u_n, \mathfrak{I}u_n)}{w(u_{n-1}, u_n).e_b(u_{n-1}, u_n)}\end{array}\right\}\\ &\quad= \lambda(u_{n-1}) \max\left\{\begin{array}{c} e_b(u_{n-1}, u_n), e_b(u_{n-1}, u_n), e_b(u_n, u_{n+1}),\\ \dfrac{e_b(u_n, u_{n+1})[1+e_b(u_{n-1}, u_n)]}{w(u_{n-1}, u_n)[1+e_b(u_{n-1}, u_n)]}, \dfrac{e_b(u_{n-1}, u_n).e_b(u_n, u_{n+1})}{w(u_{n-1}, u_n).e_b(u_{n-1}, u_n)}\end{array}\right\},\end{aligned} \tag{13}$$

$$\begin{aligned}&\leq\lambda(u_{n-1}) \max\left\{\begin{array}{c} e_b(u_{n-1}, u_n), e_b(u_{n-1}, u_n), e_b(u_n, u_{n+1}),\\ e_b(u_n, u_{n+1}), e_b(u_n, u_{n+1})\end{array}\right\}\\ &\leq \lambda(u_{n-1}) \max\{e_b(u_{n-1}, u_n), e_b(u_n, u_{n+1})\}.\end{aligned} \tag{14}$$

If $e_b(u_{n-1}, u_n) \leq e_b(u_n, u_{n+1})$ for some $n \in \mathbb{N}$, then from (13), we have $w(u_n, u_{n+1})e_b(u_{n-1}, u_n) \leq \lambda(u_{n-1})e_b(u_n, u_{n+1})$, which is a contradiction since $w \geq 1$ and $\lambda < 1$. Thus, $e_b(u_n, u_{n+1}) \leq e_b(u_{n-1}, u_n)$ for all $n \in \mathbb{N}$, and the sequence $\{e_b(u_n, u_{n+1})\}$ is a decreasing sequence of real numbers. Therefore, there exists ζ such that

$$\lim_{n\longrightarrow\infty} e_b(u_n, u_{n+1}) = \zeta. \tag{15}$$

Again applying the limit in (13), we get

$$\lim_{n\longrightarrow\infty} w(u_{n-1}, u_n)\zeta \leq \lim_{n\longrightarrow\infty} \lambda(u_{n-1})\zeta, \tag{16}$$

which leads to $\zeta = 0$ as $w \geq 1$. Thus, we get

$$\lim_{n\longrightarrow\infty} e_b(u_n, u_{n+1}) = 0. \tag{17}$$

Step 2. At this step, we will prove that $\{u_n\}$ is a Cauchy sequence, that is, for $m > n$, we prove

$$\lim_{n,m\longrightarrow\infty} e_b(u_n, u_m) = 0. \tag{18}$$

Using (ebb3), we have

$$
\begin{aligned}
e_b(u_n,u_m) \le\ & w(u_n,u_m)[e_b(u_n,u_{n+1}) \\
& + e_b(u_{n+1},u_{n+2}) + e_b(u_{n+2},u_{n+m})] \\
\le\ & w(u_n,u_m)e_b(u_n,u_{n+1}) + w(u_n,u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)e_b(u_{n+2},u_m) \\
\le\ & w(u_n,u_m)e_b(u_n,u_{n+1}) + w(u_n,u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)w(u_{n+2},u_m)[e_b(u_{n+2},u_{n+3}) \\
& + e_b(u_{n+3},u_{n+4}) + e_b(u_{n+4},u_m)] \\
\le\ & w(u_n,u_m)e_b(u_n,u_{n+1}) + w(u_n,u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)w(u_{n+2},u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)w(u_{n+2},u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)w(u_{n+2},u_m)e_b(u_{n+4},u_m) \vdots \\
\le\ & w(u_n,u_m)e_b(u_n,u_{n+1}) + w(u_n,u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)w(u_{n+2},u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)w(u_{n+2},u_m)e_b(u_n,u_{n+1}) + \cdots \\
& + w(u_n,u_m)w(u_{n+2},u_m)\cdots w(u_{m-2},u_m)e_b \\
& \cdot (u_n,u_{n+1}) + w(u_n,u_m)w(u_{n+2},u_m)\cdots w \\
& \cdot (u_{m-2},u_m)e_b(u_n,u_{n+1}) \\
\le\ & w(u_n,u_m)e_b(u_n,u_{n+1}) + w(u_n,u_m)w \\
& \cdot (u_{n+1},u_m)e_b(u_n,u_{n+1}) + w(u_n,u_m)w \\
& \cdot (u_{n+1},u_m)w(u_{n+2},u_m)e_b(u_n,u_{n+1}) \\
& + w(u_n,u_m)w(u_{n+1},u_m)w(u_{n+2},u_m)w \\
& \cdot (u_{n+3},u_m)e_b(u_n,u_{n+1}) + \cdots + w(u_n,u_m)w \\
& \cdot (u_{n+1},u_m)w(u_{n+2},u_m)\cdots w(u_{m-2},u_m)e_b \\
& \cdot (u_n,u_{n+1}) + w(u_n,u_m)w(u_{n+1},u_m)w \\
& \cdot (u_{n+2},u_m)\cdots w(u_{m-2},u_m)w(u_{m-1},u_m)e_b(u_n,u_{n+1}).
\end{aligned} \tag{19}
$$

Applying $n, m \longrightarrow \infty$ and using (12), we get

$$\lim_{n,m\longrightarrow\infty} e_b(u_n,u_m) = 0. \tag{20}$$

Hence, $\{u_n\}$ is a Cauchy sequence. Since (Ξ, e_b) is a complete EBbDS, then there exists a point $u^* \in \Xi$ such that $u_n \longrightarrow u^*$ as $n \longrightarrow +\infty$, that is

$$\lim_{n\longrightarrow+\infty} e_b(u_n,u^*) = 0. \tag{21}$$

Next, we prove that $u^* \in Fix(\mathfrak{T})$. Indeed, we write

$$
\begin{aligned}
e_b(u^*,\mathfrak{T}u^*) \le\ & w(u^*,\mathfrak{T}u^*)[e_b(u^*,u_n) \\
& + e_b(u_n,u_{n+1}) + e_b(u_{n+1},\mathfrak{T}u^*)].
\end{aligned} \tag{22}
$$

Since $\mathfrak{T}$ is continuous, on letting $n \longrightarrow +\infty$, we obtain $e_b(u^*,\mathfrak{T}u^*) = 0$, that is, $\mathfrak{T}u^* = u^*$, and hence, u^* is a fixed point of $\mathfrak{T}$.

To prove the uniqueness of fixed-point u^*, we impose an additional requirement.

(A4) For every pair u^* and v^* of fixed points of $\mathfrak{T}$, $\alpha(u^*, v^*) \ge 1$.

Theorem 10. *In addition of condition (A4) in Theorem 9, Fi x($\mathfrak{T}$) is a singleton set.*

Proof. Following Theorem 9, $u^* \in Fix(\mathfrak{T})$. To prove $Fix(\mathfrak{T})$ is a singleton set, assume that there exist $u^*, v^* \in Fix(\mathfrak{T})$ with $u^* \neq v^*$, and by (A4), we have $\alpha(u^*, v^*) \ge 1$. It follows from $\mathfrak{T} \in \Lambda(\Xi, \alpha)$ that

$$
\begin{aligned}
& w(u^*,v^*)e_b(\mathfrak{T}u^*,\mathfrak{T}v^*) \\
& \le \lambda(u^*) \max\left\{\begin{array}{c} e_b(u^*,v^*), e_b(u^*,\mathfrak{T}u^*), e_b(v^*,\mathfrak{T}v^*), \\ \dfrac{e_b(v^*,\mathfrak{T}v^*)[1+e_b(u^*,\mathfrak{T}v^*)}{w(u^*,v^*)[1+e_b(u^*,v^*)]}, \dfrac{e_b(u^*,\mathfrak{T}u^*).e_b(v^*,\mathfrak{T}v^*)}{w(u^*,v^*)).e_b(u^*,v^*)} \end{array}\right\} \\
& \le \lambda(u^*) \max\{e_b(u^*,v^*), 0, 0, 0, 0\},
\end{aligned} \tag{23}
$$

which implies that

$$w(u^*,v^*)e_b(u^*,v^*) \le \lambda(u^*)e_b(u^*,v^*), \tag{24}$$

a contradiction, and hence, $u^* = v^*$.

2.2. Illustrations

Example 2. Let $\Xi = \{0.2, 0.25, 0.3, 0.5, 1\}$. Define $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ so that $e_b(\zeta,\xi) = e_b(\xi,\zeta)$ for all $\zeta, \xi \in \Xi$, and

$$
\begin{gathered}
e_b(0.5,0.3) = 0.07, e_b(0.5,0.25) = 0.015, e_b(0.25,0.2) = 0.02, \\
e_b(0.3,0.25) = 0.02, e_b(0.3,0.2) = 0.02,
\end{gathered} \tag{25}
$$

$e_b(\zeta,\xi) = (\zeta - \xi)^2$, otherwise. Then (Ξ, e_b) is a EBbDS with $w(\zeta,\xi) = \zeta + \xi + 2$ but neither a BDS (Ξ, b) nor a metric space (Ξ, d). For instance

$$
\begin{aligned}
& e_b(0.5,0.3) = 0.07 \leq 0.035 = e_b(0.5,0.25) + e_b(0.25,0.3), \\
& e_b(0.5,0.3) = 0.07 \leq 0.055 = e_b(0.5,0.25) + e_b(0.25,0.2) + e_b(0.2,0.3),
\end{aligned} \tag{26}
$$

but

$$
\begin{aligned}
e_b(0.5,0.3) = 0.07 \le 0.154 = w(\zeta,\nu)[& e_b(0.5,0.25) \\
& + e_b(0.25,0.2) + e_b(0.2,0.3)].
\end{aligned} \tag{27}
$$

Consider the self-mapping $\mathfrak{T}$ on Ξ, $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0,1)$

$$
\mathfrak{T} : \begin{pmatrix} 0.2 & 0.25 & 0.3 & 0.5 & 1 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.25 \end{pmatrix},
$$

$$
\alpha(\zeta,\nu) = \begin{cases} 1, & (\zeta,\nu) \in (0.2,1) \cup (0.5,1) \cup (1,0.2) \cup (1,0.5) \\ 0, & \text{otherwise}, \end{cases} \tag{28}
$$

and $\lambda(\zeta) = 2\zeta/3$ for all $\zeta \in \Xi$.

It is easy to see that $\mathfrak{T} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$. We will check that $\mathfrak{T}$ satisfies (8) for $\zeta \neq \xi$ with $\mathfrak{T}(\zeta) \neq \mathfrak{T}(\xi)$ and $\alpha(\zeta, \xi) > 1$. We demonstrate by three nontrivial possible cases. Here, $w(\zeta, \xi) \in [2.4,6]$.

Case 1. $\zeta = 0.5$, $\xi = 1$ (or vice versa if ζ, ξ change places). Then, $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) = 0.015$, $w(\zeta, \xi) = 3.5$, $\lambda(\zeta) = 0.333$ and

$$\mathscr{M}_w(\zeta, \xi) = \max\left\{0.25,0,0.5625\frac{(0.5625)[1+0]}{(3.5)[1+0.25]}, \frac{(0)(0.5625)}{(3.5)(0.25)}\right\} = 0.5625. \tag{29}$$

Therefore, (8) implies that $0.0525 < 0.1873$, and (8) holds true.

Case 2. $\zeta = 0.2$, $\xi = 1$ (or vice versa if ζ, ξ change places). Then, $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) = 0.02$, $w(\zeta, \xi) = 3.2$, $\lambda(\zeta) = 0.1333$ and

$$\mathscr{M}_w(\zeta, \xi) = \max\left\{0.64,0.02,0.5625, \frac{0.5625[1+0.02]}{3.2[1+0.64]}, \frac{(0.2)(0.5625)}{(3.2)(0.64)}\right\} = 0.64, \tag{30}$$

and it is easily seen that (8) is fulfilled.

Thus, all the conditions are fulfilled, and $\mathfrak{T}$ has a unique fixed point (which is $\zeta^* = 0.5$).

Note that in this example the use of weakly α-admissibility and $\lambda(\zeta)$ was crucial because, e.g., if we take $\zeta = 0.2$, $\xi = 0.5$, we get $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) = 0.07$, $w(\zeta, \xi) = 2.7$ and

$$\mathscr{M}_w(\zeta, \xi) = \max\left\{0.09,0.02,0, \frac{(0)[1+0.02]}{2.7[1+0.09]}, \frac{(0.02)(0)}{(2.7)(0.09)}\right\} = 0.09, \tag{31}$$

and no contractive condition for any $\lambda(\zeta) < 1$ can be chosen which would holds for these points.

Example 3. Consider $\Xi = [0, 1]$ and define $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ by $e_b(\zeta, \xi) = |\zeta - \xi|^2$. Then, (Ξ, e_b) is a EBbDS with $w(\zeta, \xi) = \zeta + \xi + 2.5$ but neither a BDS (Ξ, b) not a metric space (Ξ, d). For instance

$$\begin{aligned} e_b(0, 1) &= 1 \leq 0.5 = e_b(0,0.5) + e_b(0.5,1), \\ e_b(0, 1) &= 1 \leq 0.4902 = e_b(0,0.5) + e_b(0.5,0.99) + e_b(0.99,1), \end{aligned} \tag{32}$$

but

$$\begin{aligned} e_b(\zeta, \xi) &= |\zeta - \xi|^2 = |\zeta - \mu + \mu - \upsilon + \upsilon - \xi|^2 \leq |\zeta - \mu|^2 \\ &\quad + |\mu - \upsilon|^2 + |\upsilon - \xi|^2 + 2|\zeta - \mu||\mu - \upsilon| \\ &\quad + 2|\mu - \upsilon||\upsilon - \xi| + 2|\upsilon - \xi||\zeta - \mu| \\ &\leq \left(\zeta + \xi + \frac{5}{2}\right)\left[|\zeta - \mu|^2 + |\mu - \upsilon|^2 + |\upsilon - \xi|^2\right] \\ &= w(\zeta, \xi)[e_b(\zeta, \mu) + e_b(\mu, \upsilon) + e_b(\upsilon, \xi)], \end{aligned} \tag{33}$$

for all $\zeta, \xi, \mu, \upsilon \in \Xi$.

Consider the self-mapping $\mathfrak{T}$ on Ξ given by $\mathfrak{T}(\zeta) = \zeta^2/2$. Taking $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$ such that $\lambda(\zeta) = 8.95 + \zeta/10$ for all $\zeta \in \Xi$, and $\alpha(\zeta, \xi) = 1$ for $\zeta, \xi \in \Xi$, it is obvious to see $\mathfrak{T} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$. Here, $w(\zeta, \xi) \in (2, 4)$.

Then equation (8) for $\zeta \neq \xi$ would be of the form

$$\begin{aligned} &(\zeta + \xi + 2.5)\left|\frac{\zeta^2}{2} - \frac{\xi^2}{2}\right|^2 \\ &\leq \left(\frac{8.95 + \zeta}{10}\right)\max\left\{\begin{array}{c} |\zeta - \xi|^2, \left|\zeta - \dfrac{\zeta^2}{2}\right|^2, \left|\xi - \dfrac{\xi^2}{2}\right|^2, \\ \dfrac{\left|\xi - \xi^2/2\right|^2\left[1 + \left|\zeta - \zeta^2/2\right|^2\right]}{(\zeta + \xi + 2.5)\left[1 + |\zeta - \xi|^2\right]}, \dfrac{\left|\zeta - \zeta^2/2\right|^2 \cdot \left|\xi - \xi^2/2\right|^2}{(\zeta + \xi + 2.5) \cdot |\zeta - \xi|^2} \end{array}\right\} \end{aligned} \tag{34}$$

holds whenever $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) > 0$ and $\alpha(\zeta, \xi) \geq 1$.

For example, we demonstrate (34) is true for two cases:

Case 1. $\zeta = 0$, $\xi = 1$ (or vice versa if ζ, ξ change places). Then, (34) will be

$$\begin{aligned} (3.5)(0.25) &= 0.875 \\ &\leq \left(\frac{8.95}{10}\right)\max\left\{1, 0, \frac{1}{4}, \frac{1/4(1+0)}{(3.5)(1+1)}, 0\right\} \\ &= 0.895, \end{aligned} \tag{35}$$

which is true.

Case 2. $\zeta = 1$, $\xi = 0.9$ (or vice versa if ζ, ξ change places). Then, (34) will be

$$\begin{aligned} &(4.4)(0.19)^2 = 0.15884 \\ &\leq \left(\frac{9.95}{10}\right)\max\left\{\begin{array}{c} 0.01,0.444,0.3969, \\ \dfrac{(0.3969)[1+0.444]}{(4.4)(1+0.01)}, \dfrac{(0.444)(0.3969)}{(4.4)(0.01)} \end{array}\right\} \\ &= 0.44178, \end{aligned} \tag{36}$$

which holds true.

Similarly, it can be verified for any $\zeta \neq \xi \in \Xi$ with $\alpha(\zeta, \xi) \geq 1$. Thus, all the conditions are fulfilled, and the $Fix(\mathfrak{T}) = \{0\}$ is a singleton set.

2.3. Weak Well-Posedness, Weak Limit Shadowing, and Generalized w-Ulam-Hyers Stability. The notion of well-posedness of an fpp has evoked much interest of several mathematicians, for example, Popa [14, 15] and others. In the paper [16], the authors defined a weak well-posed (wwp) property in BbDS. In what follows, we extend this notion to EBbDS.

Definition 11. Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a mapping. The fpp of $\mathfrak{T}$ is said to be weak well-posed if it satisfies the following:

(1) $u^* \in Fix(\mathfrak{T})$ is a singleton set in Ξ

(2) For any sequence $\{u_p\}$ in Ξ with $\lim_{p\longrightarrow\infty} e_b(u_p, \mathfrak{T}(u_p)) = 0$ and

$$\lim_{p,q\longrightarrow\infty} e_b\left(\mathfrak{T}\left(u_p\right), \mathfrak{T}\left(u_q\right)\right) = 0, \text{ one has } \lim_{p\longrightarrow\infty} e_b\left(u_p, u^*\right) = 0. \tag{37}$$

Theorem 12. *Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a mapping satisfying all the conditions of Theorem 9 and a sequence $\{u_n\}$ in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{T}u_n) = 0$, $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{T}u_n, \mathfrak{T}u_m) = 0$, and $u^* \in Fix(\mathfrak{T})$. Then, the fpp of $\mathfrak{T}$ is wwp.*

Proof. Let $\{u_n\}$ be a sequence in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{T}(u_n)) = 0$ and $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{T}u_n, \mathfrak{T}u_m) = 0$, for $m > n$; we obtain from (ebb3) that

$$\begin{aligned} e_b(u_n, u^*) \le w(u_n, u^*)\{&e_b(u_n, \mathfrak{T}u_m) \\ &+ e_b(\mathfrak{T}u_m, \mathfrak{T}u_n) + e_b(\mathfrak{T}u_n, u^*)\}. \end{aligned} \tag{38}$$

Taking limit $n \longrightarrow \infty$

$$\lim_{n\longrightarrow\infty} e_b(u_n, u^*) \le \lim_{n\longrightarrow\infty} w(u_n, u^*)\{e_b(u_n, \mathfrak{T}u_m) + e_b(\mathfrak{T}u_n, u^*)\}. \tag{39}$$

WLOG, we can assume that there exists a distinct subsequence $\{\mathfrak{T}u_{n_k}\}$ of $\{\mathfrak{T}u_n\}$. Otherwise, there exists $u_0 \in \Xi$ and $n_1 \in \mathbb{N}$ such that $\mathfrak{T}u_n = u_0$ for $n \ge n_1$. Since $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{T}u_n) = 0$, we get $\lim_{n\longrightarrow\infty} e_b(u_n, u_0) = 0$. If $u_0 \ne u^*$, then $u_0 \ne \mathfrak{T}u_0$ due to uniqueness of the fixed point of $\mathfrak{T}$. For $n \ge n_1$, we obtain $u_0 = \mathfrak{T}u_n \ne \mathfrak{T}u_0$. So, we have

$$e_b(u_0, \mathfrak{T}u_0) = e_b(\mathfrak{T}u_n, \mathfrak{T}u_0) \le w(u_n, u_0)e_b(\mathfrak{T}u_n, \mathfrak{T}u_0). \tag{40}$$

For $\alpha(u_0, \mathfrak{T}u_0) \ge 1$ and $\mathfrak{T} \in \Lambda(\Xi, \alpha)$, we have

$$\begin{aligned} w(u_n, u_0)e_b(\mathfrak{T}u_n, \mathfrak{T}u_0) &\le \lambda(u_n) \max \left\{ \begin{array}{c} e_b(u_n, u_0), e_b(u_n, \mathfrak{T}u_n), e_b(u_0, \mathfrak{T}u_0), \\ \dfrac{e_b(u_0, \mathfrak{T}u_0)[1 + e_b(u_n, \mathfrak{T}u_n)]}{w(u_n, u_0))[1 + e_b(u_n, u_0)]}, \dfrac{e_b(u_n, \mathfrak{T}u_n).e_b(u_0, \mathfrak{T}u_0)}{w(u_n, u_0).e_b(u_n, u_0)} \end{array} \right\} \\ &\le \lambda(u_n) \max \{ e_b(u_n, u_0), e_b(u_n, u_0), e_b(u_0, \mathfrak{T}u_0), 0, e_b(u_0, \mathfrak{T}u_0) \} = \lambda(u_n) \max \{ e_b(u_n, u_0), e_b(u_0, \mathfrak{T}u_0) \}. \end{aligned} \tag{41}$$

Therefore, since $\lim_{n\longrightarrow\infty} e_b(u_n, u_0)) = 0$, we get

$$\lim_{n\longrightarrow\infty} w(u_n, u_0)e_b(u_0, \mathfrak{T}u_0) \le \lim_{n\longrightarrow\infty} \lambda(u_n)e_b(u_0, \mathfrak{T}u_0). \tag{42}$$

So $e_b(u_0, \mathfrak{T}u_0) = 0$, i.e., $u_0 = \mathfrak{T}u_0$, a contradiction. Hence, there exist $m, q, n > n_0 (m > q > n)$ such that $\mathfrak{T}u_m \ne \mathfrak{T}u_q \ne \mathfrak{T}u_n \ne u_n$. Then

$$\begin{aligned} e_b(u_n, \mathfrak{T}u_m) \le w(u_n, \mathfrak{T}u_m)\{&e_b(u_n, \mathfrak{T}u_n) \\ &+ e_b\left(\mathfrak{T}u_n, \mathfrak{T}u_q\right) + e_b\left(\mathfrak{T}u_q, \mathfrak{T}u_m\right)\}, \end{aligned} \tag{43}$$

which $\longrightarrow 0$ as $n \longrightarrow \infty$. On replacing the value in (39), we get

$$\lim_{n\longrightarrow\infty} e_b(u_n, u^*) \le \lim_{n\longrightarrow\infty} w(u_n, u^*)e_b(\mathfrak{T}u_n, u^*). \tag{44}$$

Again, since $\alpha(u_n, u^*) \ge 1$ and $\mathfrak{T} \in \Lambda(\Xi, \alpha)$, we have

$$\begin{aligned} w(u_n, u^*)e_b(\mathfrak{T}u_n, \mathfrak{T}u^*) &\le \lambda(u_n) \max \left\{ \begin{array}{c} e_b(u_n, u^*), e_b(u_n, \mathfrak{T}u_n), e_b(u^*, \mathfrak{T}u^*), \\ \dfrac{e_b(u_n, \mathfrak{T}u_n)[1 + e_b(u^*, \mathfrak{T}u^*)]}{w(u_n, u^*)[1 + e_b(u_n, u^*)]}, \dfrac{e_b(u^*, \mathfrak{T}u^*).e_b(u_n, \mathfrak{T}u_n)}{w(u_n, u^*)).e_b(u_n, u^*)} \end{array} \right\} \\ &\le \lambda(u_n) \max \left\{ e_b(u_n, u^*), e_b(u_n, \mathfrak{T}u_n), 0, \frac{e_b(u_n, \mathfrak{T}u_n)}{w(u_n, u^*)[1 + e_b(u_n, u^*)]}, 0 \right\}, \end{aligned} \tag{45}$$

which implies

$$\lim_{n\longrightarrow\infty} w(u_n, u^*)e_b(\mathfrak{I}u_n, \mathfrak{I}u^*) \le \lim_{n\longrightarrow\infty} \lambda(u_n)e_b(u_n, u^*). \quad (46)$$

On placing in (39), we get

$$\begin{aligned}\lim_{n\longrightarrow\infty} e_b(u_n, u^*) &\le \lim_{n\longrightarrow\infty} w(u_n, u^*)e_b(\mathfrak{I}u_n, \mathfrak{I}u^*)\\ &\le \lim_{n\longrightarrow\infty} \lambda(u_n)e_b(u_n, u^*).\end{aligned} \quad (47)$$

Therefore, $\lim_{n\longrightarrow\infty} e_b(u_n, u^*) = 0$.

The limit shadowing property of fpps has been discussed in the papers [17, 18]. We define weak limit shadowing property (wlsp) in EBbDS.

Definition 13. Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be a mapping. The fpp of $\mathfrak{I}$ is said to have wlsp in Ξ if assuming that $\{u_n\}$ in Ξ satisfies $e_b(u_n, \mathfrak{I}u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $e_b(\mathfrak{I}u_n, \mathfrak{I}u_m) \longrightarrow 0$, it follows that there exists $u \in \Xi$ such that $e_b(u_n, \mathfrak{I}^n u) \longrightarrow 0$ as $n \longrightarrow \infty$.

Theorem 14. *Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be an $\alpha - \lambda$-contractive mapping for $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$ with $\{u_n\}$ in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{I}u_n) = 0$, $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{I}u_n, \mathfrak{I}u_m) = 0$ and $u^* \in Fix(\mathfrak{I})$. Then, $\mathfrak{I}$ has the wlsp.*

Proof. Since u^* is a fixed point of $\mathfrak{I}$, we have $e_b(u^*, \mathfrak{I}u^*) = 0$, and let $\{u_n\}$ in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{I}u_n) = 0$, $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{I}u_n, \mathfrak{I}u_m) = 0$; then, by virtue of Theorem 12, we have $\lim_{n\longrightarrow\infty} e_b(u_n, u^*) = 0$, and therefore, we can write $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{I}^n u^*) = 0$.

In the following, we define the generalized w-Ulam-Hyers stability (Gw-UHS) of fixed-point problem (fpp) in EBbDS as an extension of b-metric space case discussed in [19, 20] (see also [21]).

Definition 15. Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be a mapping. The fixed-point equation (FPE)

$$u = \mathfrak{I}u, u \in \Xi \quad (48)$$

is called the generalized weak-Ulam-Hyers stable (Gw-UHS in short) in the setting of EBbDS if there exists an increasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, continuous at 0, with $\phi(0) = 0$, such that for each $\varepsilon > 0$ and an ε-solution $v \in \Xi$, that is

$$e_b(v, \mathfrak{I}v) \le \varepsilon, \quad (49)$$

there exists a solution $u^* \in \Xi$ of (48) such that

$$e_b(v, u^*) \le \phi(w(u^*, v)\varepsilon). \quad (50)$$

If $\phi(\xi) = \alpha\xi$ for all $\xi \in \mathbb{R}_+$, where $\alpha > 0$, then FPE (48) is said to be w-UHS in the setting of EBbDS.

Theorem 16. *Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be an $\alpha - \lambda$-contractive mapping for $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$ and also that the function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing and onto. Then the FPE (48) is Gw-UHS.*

Proof. Following Theorem 14, we have $\mathfrak{I}u^* = u^*$, that is, $u^* \in \Xi$ is a solution of the FPE (48) with $e_b(u^*, u^*) = 0$. Let $\varepsilon > 0$ and $v^* \in \Xi$ be an ε-solution of FPE (48), that is

$$e_{4b}(v^*, \mathfrak{I}v^*) \le \varepsilon. \quad (51)$$

Since $e_b(u^*, \mathfrak{I}u^*) = e_b(u^*, u^*) = 0 \le \varepsilon$, u^* and v^* are ε-solutions. Since we have $\alpha(u^*, v^*) \ge 1$, so

$$\begin{aligned}e_b(u^*, v^*) &\le w(u^*, v^*)[e_b(u^*, \mathfrak{I}u^*) + e_b(\mathfrak{I}u^*, \mathfrak{I}v^*) + e_b(\mathfrak{I}v^*, v^*)]\\ &\le w(u^*, v^*)e_b(\mathfrak{I}u^*, \mathfrak{I}v^*) + \varepsilon w(u^*, v^*)\\ &\le \lambda(u^*) \max\left\{\begin{array}{c} e_b(u^*, v^*), e_b(u^*, \mathfrak{I}u^*), e_b(v^*, \mathfrak{I}v^*),\\ \dfrac{e_b(v^*, \mathfrak{I}v^*)[1 + e_b(u^*, \mathfrak{I}u^*)]}{w(u^*, v^*)[1 + e_b(u^*, v^*)]}, \dfrac{e_b(u^*, \mathfrak{I}u^*).e_b(v^*, \mathfrak{I}v^*)}{w(u^*, v^*)).e_b(u^*, v^*)} \end{array}\right\} + \varepsilon w(u^*, v^*)\\ &\le \lambda(u^*) \max\{e_b(u^*, v^*), 0, \varepsilon, 0, 0\} + \varepsilon w(u^*, v^*).\end{aligned} \quad (52)$$

Let us discuss the two possible cases.

Case 1. If $e_b(u^*, v^*) > \varepsilon$, then we get

$$e_b(u^*, v^*) \le \lambda(u^*)e_b(u^*, v^*) + w(u^*, v^*)\varepsilon, \quad (53)$$

that is

$$e_b(u^*, v^*)[1 - \lambda(u^*)] \le w(u^*, v^*)\varepsilon, \quad (54)$$

which implies that

$$e_b(u^*, v^*) \leq \frac{1}{1-\lambda(u^*)} w(u^*, v^*)\varepsilon = \phi(w(u^*, v^*)\varepsilon). \tag{55}$$

Case 2. If $e_b(u^*, v^*) < \varepsilon$, then (12) gives

$$\begin{aligned} e_b(u^*, v^*) &\leq \lambda(u^*)\varepsilon + w(u^*, v^*)\varepsilon \\ &\leq \lambda(u^*)w(u^*, v^*)\varepsilon + w(u^*, v^*)\varepsilon \\ &= (\lambda(u^*)+1)w(u^*, v^*)\varepsilon = \phi(w(u^*, v^*)\varepsilon). \end{aligned} \tag{56}$$

It shows that the inequality (50) is true for all cases, and thus the FPE (48) is Gw-UHS.

3. Application

In this section, we discuss the existence of solutions of a nonlinear fractional differential equation (FDE) [22] as an application of Theorem 9. Some other FDE-related work can be seen in [23–25].

The Caputo fractional derivative of order β is defined as

$$\begin{aligned} {}^c\mathscr{D}^\beta(p(\rho)) = {} & \frac{1}{\Gamma(n-\beta)} \int_0^\rho (\rho-\sigma)^{n-\beta-1} p^{(n)}(\sigma) ds \\ & \cdot (n-1<\beta<n, n=[\beta]+1), \end{aligned} \tag{57}$$

where $p : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function, $[\beta]$ denotes the integer part of the positive real number β, and Γ is the gamma function.

Consider the nonlinear FDE

$${}^c\mathscr{D}^\beta(\vartheta(\rho)) = \hbar(\rho, \vartheta(\rho))(0<\rho<1, 1<\beta\leq 2), \tag{58}$$

with the integral boundary conditions

$$\vartheta(0) = 0, \vartheta(1) = \int_0^\eta \vartheta(\sigma)\, ds \, (0<\eta<1), \tag{59}$$

where $J = [0, 1]$, $\vartheta \in C(J, \mathbb{R})$, and $\hbar : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are a continuous function.

Let $\Xi = C(J, \mathbb{R})$ be endowed with the EBbDS function

$$e_b(\vartheta, \nu) = \max_{\rho\in J} |\vartheta(\rho) - \nu(\rho)|^2, \tag{60}$$

and $w(\vartheta, \nu) = |\vartheta(\rho)| + |\nu(\rho)| + 2$.

Theorem 17. *Let $\mathfrak{I} : \Xi \longrightarrow \Xi$ be the operator defined by*

$$\begin{aligned} \mathfrak{I}\vartheta(\rho) = {} & \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho-\sigma)^{\beta-1} \hbar(\sigma, \vartheta(\sigma)) d\sigma \\ & - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \hbar(\sigma, \vartheta(\sigma)) d\sigma \\ & + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^\sigma (\sigma-\varsigma)^{\beta-1} \hbar(z, \vartheta(z))\, d\varsigma \right) d\sigma, \end{aligned} \tag{61}$$

for $\vartheta \in \Xi$, $\rho \in J$. Also, let $\zeta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a given function. Assume the following:

(F1) $\hbar : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, nondecreasing in the second variable

(F2) There exists $\vartheta_0 \in \Xi$ such that $\zeta(\vartheta_0(\rho), \mathfrak{I}\vartheta_0(\rho)) \geq 0$ for all $\rho \in J$

(F3) $(\vartheta, \nu) \in \Xi^2$ and $\zeta(\vartheta(\rho), \mathfrak{I}\vartheta(\rho)) \geq 0$ for all $\rho \in J$ imply that $\zeta(\mathfrak{I}\vartheta(\rho), \mathfrak{I}\mathfrak{I}\vartheta(\rho)) \geq 0$ for all $\rho \in J$

(F4) There exists $\lambda : \Xi \longrightarrow [0, 1)$ such that for $\vartheta, \nu \in \Xi$ with $\zeta(\vartheta, \nu) \geq 0$, and $\rho \in J$, we have

$$|\hbar(\rho, \vartheta(\rho)) - \hbar(\rho, \nu(\rho))|^2 \leq \frac{\lambda(\vartheta(\rho))\Theta(\vartheta, \nu)(\rho)}{\theta \times \max_{\rho\in J}(|\vartheta(\rho)| + |\nu(\rho)| + 2)}, \tag{62}$$

where

$$\Theta(\vartheta, \nu)(\rho) = \max \left\{ \begin{array}{c} |\vartheta(\rho)-\nu(\rho)|^2, |\vartheta(\rho)-\mathfrak{I}\vartheta(\rho)|^2, |\nu(\rho)-\mathfrak{I}\nu(\rho)|^2, \\ \dfrac{\left(|\nu(\rho)-\mathfrak{I}\nu(\rho)|^2\right)\left(1+|\vartheta(\rho)-\mathfrak{I}\vartheta(\rho)|^2\right)}{\max_{\rho\in J}(|\vartheta(\rho)|+|\nu(\rho)|+2)\left(1+\max_{\rho\in J}|\vartheta(\rho)-\nu(\rho)|^2\right)}, \\ \dfrac{\left(|\vartheta(\rho)-\mathfrak{I}\vartheta(\rho)|^2\right)\left(|\nu(\rho)-\mathfrak{I}\nu(\rho)|^2\right)}{\max_{\rho\in J}(|\vartheta(\rho)|+|\nu(\rho)|+2)\left(\max_{\rho\in J}|\vartheta(\rho)-\nu(\rho)|^2\right)}, \end{array} \right\} \tag{63}$$

and $\theta = (2\beta-1)\Gamma(\beta)\Gamma(\beta+1)/2(5\beta+2)$. Then, the problems (58) and (59) have at least one solution $\vartheta^* \in \Xi$.

Proof. Define a function $\alpha : \Xi^2 \longrightarrow [0,\infty)$ by

$$\alpha(\vartheta, \nu) = \begin{cases} 1, & \text{if for} \zeta(\vartheta(\rho), \nu(\rho)) \geq 0, \text{for all} \rho \in J \\ \gamma, & \text{otherwise,} \end{cases} \tag{64}$$

where $\gamma \in (0, 1)$. It is obvious to check that the assumption (F2) implies the condition (A2) of Theorem 9. Assumption (F3) clearly implies that $\mathfrak{I} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$.

Let $\vartheta, \nu \in \Xi$ be $\alpha(\vartheta, \nu) \geq 1$, i.e., $\zeta(\vartheta(\rho), \nu(\rho)) \geq 0$ for all $\rho \in J$. For each $\rho \in J$, by the definition (61) of operator $\mathfrak{I}$, we have (using Cauchy-Schwartz inequality)

$$\begin{aligned}
&|\mathfrak{T}\vartheta(\rho) - \mathfrak{T}\nu(\rho)|^2 \\
&= \left| \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho-\sigma)^{\beta-1} \hbar(\sigma, \vartheta(\sigma))\, d\sigma \right. \\
&\quad - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \hbar(\sigma, \vartheta(\sigma)) d\sigma \\
&\quad + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^\sigma (\sigma-\varsigma)^{\beta-1} \hbar(\varsigma, \vartheta(z))\, d\varsigma \right) d\sigma \\
&\quad - \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho-\sigma)^{\beta-1} \hbar(\sigma, \nu(\sigma))\, d\sigma \\
&\quad - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \hbar(\sigma, \nu(\sigma))\, d\sigma \\
&\quad \left. + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^\sigma (\sigma-\varsigma)^{\beta-1} \hbar(\varsigma, \nu(\varsigma))\, d\varsigma \right) d\sigma \right|^2 \\
&\le \frac{2}{\Gamma^2(\beta)} \left\{ \int_0^\rho (\rho-\sigma)^{\beta-1} |\hbar(\sigma, \vartheta(\sigma)) - \hbar(\sigma, \nu(\sigma))|^2\, d\sigma \right\} \\
&\quad + \frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)} \left\{ \int_0^1 (1-\sigma)^{\beta-1} |\hbar(s, \vartheta(\sigma)) \right. \\
&\quad \left. - \hbar(\sigma, \nu(\sigma))|^2\, d\sigma \right\} + \frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)} \\
&\quad \cdot \left\{ \int_0^\eta \left(\int_0^\sigma (\sigma-\varsigma)^{\beta-1} |\hbar(\varsigma, \vartheta(\varsigma)) - \hbar(\varsigma, \nu(\varsigma))|^2\, d\varsigma \right) d\sigma \right\},
\end{aligned} \tag{65}$$

that is

$$\begin{aligned}
|\mathfrak{T}\vartheta(\rho) - \mathfrak{T}\nu(\rho)|^2 &\le \frac{2}{\Gamma^2(\beta)} \int_0^\rho (\rho-\sigma)^{2\beta-2} d\sigma \\
&\quad \cdot \int_0^\rho |\hbar(\sigma, \vartheta(\sigma)) - \hbar(\sigma, \nu(\sigma))|^2\, d\sigma \\
&\quad + \frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)} \int_0^1 (1-\sigma)^{2\beta-2} d\sigma \\
&\quad \cdot \int_0^1 |\hbar(\sigma, \vartheta(\sigma)) - \hbar(\sigma, \nu(\sigma))|^2\, d\sigma \\
&\quad + \frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)} \int_0^\eta \int_0^\sigma (\sigma-\varsigma)^{2\beta-2} d\varsigma\, d\sigma \\
&\quad \times \int_0^\eta \int_0^\sigma |\hbar(\varsigma, \vartheta(\varsigma)) - \hbar(\varsigma, \nu(\varsigma))|^2\, d\varsigma\, d\sigma.
\end{aligned} \tag{66}$$

Applying (F4) and small calculations, we get

$$|\mathfrak{T}\vartheta(\rho) - \mathfrak{T}\nu(\rho)|^2 \le \frac{\lambda(\vartheta(\rho))\Theta_{\mathfrak{T}}(\vartheta, \nu)(\rho)}{(|\vartheta(\rho)| + |\nu(\rho)| + 2)}. \tag{67}$$

This implies that

$$\begin{aligned}
w(\vartheta, \nu) e_b(\mathfrak{T}\vartheta, \mathfrak{T}\nu) &= w(\vartheta, \nu) \max_{t \in I} \left(|(\mathfrak{T}\vartheta)(\rho) - (\mathfrak{T}\nu)(\rho)|^2 \right) \\
&\le \lambda(\vartheta)\Theta_{\mathfrak{T}}(\vartheta, \nu),
\end{aligned} \tag{68}$$

for all $\vartheta, \nu \in \Xi$ with $e_b(\mathfrak{T}\vartheta, \mathfrak{T}\nu) > 0$ where

$$\Theta_{\mathfrak{T}}(\vartheta, \nu) = \max \left\{ \begin{array}{c} e_b(\vartheta, \nu), e_b(\vartheta, \mathfrak{T}\vartheta), e_b(\nu, \mathfrak{T}\nu), \\ \dfrac{e_b(\nu, \mathfrak{T}\nu)[1 + e_b(\vartheta, \mathfrak{T}\vartheta)]}{w(\vartheta, \nu)[1 + e_b(\vartheta, \nu)]}, \dfrac{e_b(\vartheta, \mathfrak{T}\vartheta).e_b(\nu, \mathfrak{T}\nu)}{w(\vartheta, \nu).e_b(\vartheta, \nu)} \end{array} \right\}. \tag{69}$$

Thus, $\mathfrak{T} \in \Lambda(\Xi, \alpha)$. Therefore, all the requirements of Theorem 9 are fulfilled, and we conclude that there is a fixed-point $\vartheta^* \in \Xi$ of the operator $\mathfrak{T}$. It is well known (see, e.g., [22], Theorem 17) that in this case ϑ^* is also a solution of the integral equation (61) and the FDE (58) with the condition (59).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Common Best Proximity Point Theorems in JS-Metric Spaces Endowed with Graphs

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In this paper, we introduce a notion of G-proximal edge preserving and dominating G-proximal Geraghty for a pair of mappings, which will be used to present some existence and uniqueness results for common best proximity points. Here, the mappings are defined on subsets of a JS-metric space endowed with a directed graph. An example is also provided to support the results. Moreover, we apply our result to a similar setting, where the JS-metric space is endowed with a binary relation.

1. Introduction

Problems concerning objects that remain unchanged have been of great interest in sciences. Geneticists, for instance, have discovered that gene mutations may be delayed by raising the number of DNAs kept unaltered under exposure and hence cancer prevention. Mathematicians may interpret such objects in their own environment as points kept fixed under self-mapping, known as fixed points.

Theory of fixed points and their related notions have been expansively explored, not only in pure mathematics itself but also in real-world problems, where optimal solutions are sought. There have been a large number of publications that contribute to the subject in various approaches (see [1, 2] for some fixed point theorems and see [3–9] for results regarding best proximity points and pairs, to mention but a few). The reader may also be referred to [10] for some applications of the theory in economics.

Best proximity and common best proximity points, in particular, have become one of the most studied topics in the field of fixed point theory. These notions generalize fixed points and allow us to deal with nonself-mappings. Several settings and techniques have been used in order to determine which circumstances a best (common best) proximity point can be guaranteed. Hussain and his coauthors are amongst those who have actively contributed results to this research area (see [11, 12], where different kinds of contractions are employed; see [13, 14], where generalized notions of metric spaces are considered; and see [15] for best proximity results in nonlinear dynamical systems). There are many more results on common best proximity points in the literature (see [16–20] for some of the key works).

One of the most popular research approaches in the theory of fixed points is to appropriately adjust mappings that control the distance between two points. A well-known result by S. Banach, known as Banach contraction principle, gives rise to a variety of modifications. Instead of contractions, in [21], Ayari considered a new class of mappings containing all $\theta : [0,+\infty) \rightarrow [0, 1]$ with property that

$$\lim_{n\rightarrow+\infty} \theta(t_n) = 1 \text{ implies } \lim_{n\rightarrow+\infty} t_n = 0, \quad (1)$$

which generalizes contractions and also Geraghty's work [22]. With this generalization, existence and uniqueness results for best proximity points in a closed subset of a complete metric space can be established. A recent work by Khemphet et al. [20] also benefits from this class of mappings—a notion of dominating proximal generalized Geraghty property of a pair of mappings is presented for some existence and uniqueness results of common best proximity coincidence points in complete metric spaces, improving Chen's work [19].

One may try to control the distance between two points in a metric space using directed graphs. This idea was first introduced by Jachymski [23]. Given a directed graph $G = (V, E)$, the set of edges E is contained in the Cartesian product $V \times V$. If V is also a metric space, one could impose some conditions on the distance between points x and y, provided that $(x, y) \in E$. There have been a number of articles employing this graph-like theoretic approach (see [24, 25] for some fixed point theorems, see [26–28] for some results on Hilbert spaces, and see [29–31] for some assertions regarding common fixed points, coincidence points, and best proximity points).

This paper is aimed at establishing some common best proximity point theorems (Theorems 12 and 13) on a more general setting, the so-called JS-metric spaces, introduced in [32]. Our space will also be endowed with a directed graph G. More specifically, the main theorems rely on two key assumptions that a pair of mappings is G-proximal edge preserving and dominating G-proximal Geraghty. To the best of our knowledge, this study approach has not yet been investigated.

The paper is organized as follows. Section 2 collects basic definitions and facts regarding JS-metric spaces, common proximity points, and spaces endowed with directed graphs. Section 3 presents our main results and some example. Section 4 provides some consequences of our main theorems by concretely selecting Geraghty-like functions. Last but not least, Section 5 is devoted for an application of our results for a pair of mappings between subsets of a JS-metric space endowed with a binary relation.

2. Preliminaries

2.1. JS-Metric Spaces. In [32], a weaker notion of metrics was introduced by Jleli and Samet, known as JS-metrics. These generalized metrics lack the triangle inequality, which somewhat ruins the intuition of distance. It turns out, however, that various types of topological spaces are JS-metric spaces.

Let X be a nonempty set, and let $D : X \times X \to [0,+\infty]$ be a function. For each $x \in X$, let us define

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to+\infty} D(x_n, x) = 0 \right\} \quad (2)$$

to be the set of sequences in X that converges to x, with respect to D.

Definition 1 (see [32]). Let X be a nonempty set. A function $D : X \times X \to [0,+\infty]$ is called *JS-metric* on a set X if it satisfies the following conditions:

(JS_1) For any $x, y \in X$, $D(x, y) = 0$ implies $x = y$

(JS_2) For any $x, y \in X$, $D(x, y) = D(y, x)$

(JS_3) There is a constant $C_X > 0$ such that

$$D(x, y) \le C_X \limsup_{n\to+\infty} D(x_n, y), \quad (3)$$

whenever $x, y \in X$ and $\{x_n\} \in C(D, X, x)$.

The pair (X, D) is called a *JS-metric space*.

Conditions (JS_1) and (JS_3) imply the following fact.

Proposition 2 (see [33]). *Let (X, D) be a JS-metric space and $x \in X$. If $C(D, X, x) \neq \varnothing$, then $D(x, x) = 0$.*

As in a metric space, convergence and completeness for a JS-metric space can be defined in a similar way.

Definition 3 (see [32]). *Let (X, D) be a JS-metric space and $\{x_n\}$ a sequence in X.*

(i) $\{x_n\}$ *D-converges* to x if $\{x_n\} \in C(D, X, x)$

(ii) $\{x_n\}$ is called a *D-Cauchy* sequence if $\lim_{m,n\to+\infty} D(x_m, x_n) = 0$

(iii) The space (X, D) is said to be *complete* if every D-Cauchy sequence is D-convergent

In a metric space, triangle inequality forces any convergent sequence to have a unique limit. Here, the condition (JS_3) plays that role.

Proposition 4 (see [32]). *Let (X, D) be a JS-metric space, and let $\{x_n\}$ be a sequence in X. For any $x, y \in X$, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, then $x = y$.*

Pointwise continuity can then be defined, using convergence of sequences.

Definition 5. Let (X, D) be a JS-metric space. A mapping $P : X \to X$ is said to be continuous at a point $x_0 \in X$ if $\{x_n\} \in C(D, X, x_0)$ implies $\{Px_n\} \in C(D, X, Px_0)$. In addition, P is said to be continuous if it is continuous at each x in X.

2.2. Common Proximity. Throughout the paper, for nonempty subsets A and B of a JS-metric space (X, D), the following notations will be used:

$$\begin{aligned} D(A, B) &:= \inf \{D(a, b) : a \in A, b \in B\}, \\ A_0 &:= \{a \in A : \text{there exists } b \in B \text{ such that } D(a, b) = D(A, B)\}, \\ B_0 &:= \{b \in B : \text{there exists } a \in A \text{ such that } D(a, b) = D(A, B)\}. \end{aligned} \quad (4)$$

Notice that A_0 and B_0 can be empty or even undefined. If $D(A, B) = +\infty$ and $D(a, b) = +\infty$, then it is unclear whether $a \in A_0$. Throughout the paper, the distance $D(A, B)$ will be always assumed to be finite.

For nonempty subsets A and B of a JS-metric space (X, D), let us recall that a *best proximity point* of a mapping $T : A \to B$ is a point $x^* \in A$ with $D(x^*, Tx^*) = D(A, B)$. If $D(A, B) = 0$, then x^* becomes a fixed point. The term "common" comes into play when we deal with a pair of mappings.

Definition 6 (see [18]). *Let $P, Q : A \to B$ be mappings. An element $x^* \in A$ is said to be a common best proximity point of the pair (P, Q) if*

$$D(x^*, Qx^*) = D(A, B) = D(x^*, Px^*). \tag{5}$$

The set of common best proximity points of P and Q is denoted by $\mathrm{CP}(P, Q)$.

Observe also that if $D(A, B) = 0$, then x^* becomes a common fixed point of the pair (P, Q).

Definition 7 (see [17]). Let $P, Q : A \to B$ be mappings. A pair (P, Q) is said to *commute proximally* if for each $x, u, v \in A$,

$$D(u, Qx) = D(A, B) = D(v, Px) \text{ implies } Pu = Qv. \tag{6}$$

2.3. Graph Endowment. Given a nonempty set X, a directed graph $G = (V_G, E_G)$ will be constructed as follows.

(i) The set of vertices V_G is the set X itself

(ii) The set of edges $E_G \subseteq X \times X$ contains all the loops; that is, $\{(x, x) \mid x \in X\} \subseteq E_G$

(iii) E_G contains no parallel edges

We say that X is said to be *endowed with a directed graph* $G = (V_G, E_G)$.

Let (X, D) be a JS-metric space endowed with a directed graph $G = (V_G, E_G)$, denoted by (X, D, G). We next introduce a notion of continuity with respect to the graph G.

Definition 8 (see [23]). *A mapping $P : X \to X$ is called G-continuous at $x_0 \in X$ if for any sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E_G$ and $\{x_n\} \in C(D, X, x_0)$, it follows that $\{Px_n\} \in C(D, X, Px_0)$.*

Notice that G-continuity is a weaker notion than usual continuity, with respect to the JS-metric. In other words, any continuous mapping on a space endowed with a directed graph G is G-continuous. A counterexample of the converse is easily found, when G has an isolated vertex, for example.

Let us now introduce some more terminology used in our main results.

Definition 9. Let A, B be nonempty subsets of (X, D, G) and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be G-proximal edge preserving if the following hold:

(i) If $(Px, Py) \in E_G$, $(Qx, Qy) \in E_G$

(ii) For any $x, u, v \in A$ with $(Px, Qx) \in E_G$ and

$$D(u, Qx) = D(A, B) = D(v, Px), \tag{7}$$

it follows that $(v, u) \in E_G$

Define the class Γ of functions to be

$$\Gamma = \left\{\gamma : [0,+\infty] \to [0, 1] \mid \lim_{n\to+\infty} \gamma(t_n) = 1 \text{ implies } \lim_{n\to+\infty} t_n = 0\right\}. \tag{8}$$

This extends the class of functions in [19, 21]. Notice that any function in $\gamma \in \Gamma$ has a property that

$$\gamma(t) = 1 \text{ implies } t = 0. \tag{9}$$

Definition 10. Let A, B be nonempty subsets of (X, D, G) and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be dominating G-proximal Geraghty if there exists a function $\gamma \in \Gamma$ such that for each $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ satisfying

$$\begin{gathered}(Px_1, Qx_1) \in E_G,\\ (Px_2, Qx_2) \in E_G,\\ D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2),\end{gathered} \tag{10}$$

it follows that

$$D(u_1, u_2) \le \gamma(M(v_1, v_2, u_1, u_2))M(v_1, v_2, u_1, u_2), \tag{11}$$

where $M(v_1, v_2, u_1, u_2) = \max\{D(v_1, v_2), D(v_1, u_1), D(v_2, u_2)\}$.

3. Existence Theorems

Throughout this section, let (X, D, G) be a complete JS-metric space endowed with a directed graph G, A, B be nonempty subsets of X, and $P, Q : A \to B$ be mappings. The following assumptions will be imposed.

(A0) $A_0, B_0 \neq \varnothing$

(A1) A_0 is bounded with respect to D and closed, in the sense that any convergent sequence in A_0 has its limit in A_0

(A2) $Q(A_0) \subseteq B_0$

(A3) For any $u, v \in A$, if there exist $x, y \in A$ such that

$$D(u, Qx) = D(A, B) = D(v, Py), \tag{12}$$

then $D(u, v) < +\infty$

(A4) The pair (P, Q) commutes proximally

(A5) The pair (P, Q) is dominating G-proximal Geraghty

Lemma 11. *If $Pu = Qu$ for some $u \in A_0$, then $CP(P, Q) \neq \varnothing$.*

Proof. Let $u \in A_0$ such that $Pu = Qu$. By (A2), there exists $x^* \in A_0$ such that

$$D(x^*, Qu) = D(A, B) = D(x^*, Pu). \tag{13}$$

Since the pair (P, Q) commutes proximally, we have $Px^* = Qx^*$. Again, the assumption (A2) implies that

$$D(z^*, Qx^*) = D(A, B) = D(z^*, Px^*), \tag{14}$$

for some $z^* \in A_0$. From (13) and (14), the assumption (A3) yields $D(x^*, x^*) < +\infty$, $D(z^*, z^*) < +\infty$, and $D(x^*, z^*) < +\infty$. Since $(Pu, Qu) = (Pu, Pu) \in E_G$, it follows from (13) and (A5) that

$$\begin{aligned} D(x^*, x^*) &\leq \gamma(M(x^*, x^*, x^*, x^*))M(x^*, x^*, x^*, x^*) \\ &= \gamma(D(x^*, x^*))D(x^*, x^*) \leq D(x^*, x^*). \end{aligned} \tag{15}$$

That is, $\gamma(D(x^*, x^*)) = 1$, and hence, $D(x^*, x^*) = 0$. Similarly, by (14) and $(Px^*, Qx^*) = (Px^*, Px^*) \in E_G$, we also get $D(z^*, z^*) = 0$.

Now, observe that (14) will prove the lemma if $x^* = z^*$. In fact, the equality can be achieved by a similar argument above. Since

$$\begin{aligned} M(x^*, z^*, x^*, z^*) &= \max \{D(x^*, z^*), D(x^*, x^*), D(z^*, z^*)\} \\ &= D(x^*, z^*), \\ D(x^*, z^*) &\leq \gamma(M(x^*, z^*, x^*, z^*))M(x^*, z^*, x^*, z^*) \\ &= \gamma(D(x^*, z^*))D(x^*, z^*) \leq D(x^*, z^*), \end{aligned} \tag{16}$$

it follows that $\gamma(D(x^*, z^*)) = 1$ yielding $D(x^*, z^*) = 0$. Thus, $x^* \in \mathrm{CP}(P, Q)$.

Theorem 12. *Assume (A0)–(A5) as before. In addition, if the following are satisfied*

(i) $Q(A_0) \subseteq P(A_0)$

(ii) there exists $x_0 \in A_0$ *such that* $(Px_0, Qx_0) \in E_G$

(iii) (P, Q) *is G-proximal edge preserving*

(iv) P and Q are G-continuous,

then $CP(P, Q) \neq \varnothing$. *Moreover, if* $(Px, Qx) \in E_G$, *for all* $x \in CP(P, Q)$, *then the pair* (P, Q) *has a unique common best proximity point.*

Proof. Let $x_0 \in A_0$ be such that $(Px_0, Qx_0) \in E_G$. From the assumption $Q(A_0) \subseteq P(A_0)$ and the G-proximal edge preserving property of (P, Q), we can construct a sequence $\{x_n\}$ in A_0 satisfying

$$\begin{aligned} Qx_n &= Px_{n+1}, \\ (Px_n, Qx_n) &\in E_G, \end{aligned} \tag{17}$$

for all integers $n \geq 0$. For each n, since $Q(A_0) \subseteq B_0$, there exists an element $u_n \in A_0$ such that

$$D(u_n, Qx_n) = D(A, B) = D(u_n, Px_{n+1}). \tag{18}$$

If $u_{n_0} = u_{n_0+1}$ for some n_0, we then have

$$D(u_{n_0+1}, Qx_{n_0+1}) = D(A, B) = D(u_{n_0}, Px_{n_0+1}). \tag{19}$$

It follows from (A4) that $Q(u_{n_0}) = P(u_{n_0+1}) = P(u_{n_0})$. Applying Lemma 11 yields $\mathrm{CP}(P, Q) \neq \varnothing$.

Let us assume $u_n \neq u_{n+1}$ for all integers $n \geq 0$. From (18), we get

$$\begin{aligned} D(u_n, Qx_n) = D(u_{n+1}, Qx_{n+1}) &= D(A, B) = D(u_{n-1}, Px_n) \\ &= D(u_n, Px_{n+1}), \end{aligned} \tag{20}$$

and (A3) gives $D(u_n, u_{n+1}) < +\infty$ for all n. Since (P, Q) is dominating G-proximal Geraghty, we have that

$$\begin{aligned} D(u_n, u_{n+1}) &\leq \gamma(M(u_{n-1}, u_n, u_n, u_{n+1}))M(u_{n-1}, u_n, u_n, u_{n+1}) \\ &\leq M(u_{n-1}, u_n, u_n, u_{n+1}) \\ &= \max \{D(u_{n-1}, u_n), D(u_n, u_{n+1})\}, \end{aligned} \tag{21}$$

implying that $D(u_n, u_{n+1}) \leq D(u_{n-1}, u_n)$ for all integers $n \geq 1$. Let $\{(D(u_n, u_{n+1})\}$ converge to a real number $r \geq 0$. So does $\{M(u_{n-1}, u_n, u_n, u_{n+1})\}$.

If r were positive, we would obtain

$$1 = \lim_{n\to+\infty} \frac{D(u_n, u_{n+1})}{M(u_{n-1}, u_n, u_n, u_{n+1})} \leq \lim_{n\to+\infty} \gamma(M(u_{n-1}, u_n, u_n, u_{n+1})) \leq 1, \tag{22}$$

which leads to a contradiction as follows:

$$\lim_{n\to+\infty} \gamma(M(u_{n-1}, u_n, u_n, u_{n+1})) = 1 \text{ implies} \lim_{n\to+\infty} M(u_{n-1}, u_n, u_n, u_{n+1}) = 0. \tag{23}$$

Therefore, $\lim_{n\to+\infty} D(u_n, u_{n+1}) = 0$.

Let us next show that $\{u_n\}$ is a D-Cauchy sequence. Suppose that this is not the case. Then, we can construct subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ satisfying

$$\begin{aligned} D(u_{n_k}, u_{m_k}) &\geq \varepsilon, \\ m_k > n_k &\geq k, \end{aligned} \tag{24}$$

for some $\varepsilon > 0$. Notice that $D(u_{n_k}, u_{m_k}) < +\infty$ for all positive integers k, by (18) and (A3), and $(Px_{n_k}, Qx_{n_k}) \in E_G$ and (Px_{m_k}, Qx_{m_k}) belong to E_G, by (17). Thus,

$$\begin{aligned} D(u_{n_k}, Qx_{n_k}) = D(u_{m_k}, Qx_{m_k}) &= D(A, B) = D(u_{n_k-1}, Px_{n_k}) \\ &= D(u_{m_k-1}, Px_{m_k}) \end{aligned} \tag{25}$$

for all k. The dominating G-proximal Geraghty property of (P, Q) implies

$$D(u_{n_k}, u_{m_k}) \leq \gamma(M(u_{n_k-1}, u_{m_k-1}, u_{n_k}, u_{m_k}))M(u_{n_k-1}, u_{m_k-1}, u_{n_k}, u_{m_k}), \tag{26}$$

where

$$M\left(u_{n_k-1}, u_{m_k-1}, u_{n_k}, u_{m_k}\right) \\ = \max\left\{D\left(u_{n_k-1}, u_{m_k-1}\right), D\left(u_{n_k-1}, u_{n_k}\right), D\left(u_{m_k-1}, u_{m_k}\right)\right\}. \tag{27}$$

Observe that $M(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k})$ is clearly neither $D(u_{n_k-1}, u_{n_k})$ nor $D(u_{m_k-1}, u_{m_k})$ for sufficiently large k, as $\lim_{n\to+\infty} D(u_n, u_{n+1}) = 0$. Without loss of generality, we may assume

$$M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right) = D\left(u_{n_k-1}, u_{m_k-1}\right), \tag{28}$$

implying

$$D\left(u_{n_k}, u_{m_k}\right) \le \gamma\left(D\left(u_{n_k-1}, u_{m_k-1}\right)\right) D\left(u_{n_k-1}, u_{m_k-1}\right) \tag{29}$$

for all k. Moreover, by induction, we obtain

$$D\left(u_{n_k-i}, u_{m_k-i}\right) \le \gamma\left(D\left(u_{n_k-i-1}, u_{m_k-i-1}\right)\right) D\left(u_{n_k-i-1}, u_{m_k-i-1}\right), \tag{30}$$

where $i = 0, 1, 2, \cdots, n_k - 1$. Therefore,

$$D\left(u_{n_k}, u_{m_k}\right) \le \prod_{i=1}^{n_k} \gamma\left(D\left(u_{n_k-i}, u_{m_k-i}\right)\right) D\left(u_0, u_{m_k-n_k}\right). \tag{31}$$

Define

$$\gamma\left(D\left(u_{n_k-i_k}, u_{m_k-i_k}\right)\right) = \max\left\{\gamma\left(D\left(u_{n_k-i}, u_{m_k-i}\right)\right) : 1 \le i \le n_k\right\}, \\ \eta = \limsup_{k\to+\infty}\left\{\gamma\left(D\left(u_{n_k-i_k}, u_{m_k-i_k}\right)\right)\right\}. \tag{32}$$

Note that $0 \le \eta \le 1$. If $\eta < 1$, then $\lim_{k\to+\infty}\prod_{i=1}^{n_k}\gamma(D(u_{n_k-i}, u_{m_k-i})) = 0$. If $\eta = 1$, then γ forces $\{D(u_{n_k-i_k}, u_{n_k+m_k-i_k})\}$ to possess a subsequence converging to 0 as $k \to +\infty$. Both cases above contradict the fact that $D(u_{n_k}, u_{m_k}) \ge \varepsilon$ for all k.

We have shown that $\{u_n\}$ is a D-Cauchy sequence. By the assumption (A1), we have $\lim_{n\to+\infty} u_n = u$ for some $u \in A_0$. Notice that, from (18), we may write

$$D(u_{n+1}, Qx_{n+1}) = D(A, B) = D(u_n, Px_{n+1}) \tag{33}$$

for all integers $n \ge 0$. Since the pair (P, Q) commutes proximally and is G-proximal edge preserving, we have

$$Pu_{n+1} = Qu_n, \\ (u_n, u_{n+1}) \in E_G, \tag{34}$$

for all n. The G-continuity of P and Q implies

$$Pu = \lim_{n\to+\infty} Pu_n = \lim_{n\to+\infty} Qu_{n-1} = Qu. \tag{35}$$

Lemma 11 then guarantees a common best proximity point.

For uniqueness, we assume that any common proximity point of (P, Q) satisfies the property $(Px, Qx) \in E_G$. Let x^*, $y^* \in CP(P, Q)$. Then,

$$D(x^*, Px^*) = D(y^*, Py^*) = D(A, B) = D(x^*, Qx^*) = D(y^*, Qy^*). \tag{36}$$

As seen in the proof of Lemma 11, we have $D(x^*, x^*) = D(y^*, y^*) = 0$ and

$$M(x^*, y^*, x^*, y^*) = \max\{D(x^*, y^*), D(x^*, x^*), D(y^*, y^*)\} \\ = D(x^*, y^*) < +\infty. \tag{37}$$

Since (P, Q) is dominating G-proximal Geraghty, we obtain

$$D(x^*, y^*) \le \gamma(D(x^*, y^*)) D(x^*, y^*) \le D(x^*, y^*). \tag{38}$$

The inequalities above together with the property of γ yield $D(x^*, y^*) = 0$, as required.

The following is a modification of Theorem 12. Note that the G-continuity of P and Q is dropped and replaced by (iv), which helps facilitate the existence of a common best proximity point.

Theorem 13. *Assume (A0)–(A5) and (i), (ii), and (iii) as in Theorem12. In addition, if the following holds*

(iv) For any sequence $\{x_n\}$ in A that D-converges to $x \in A$ and satisfies $(Qx_n, Px_{n+1}) \in E_G$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D\left(x, Qx_{n_k}\right) = D(A, B) = D\left(x, Px_{n_k}\right), \tag{39}$$

then $CP(P, Q) \ne \varnothing$. Moreover, if $(Px, Qx) \in E_G$, for all $x \in CP(P, Q)$, then (P, Q) has a unique common best proximity point.

Proof. The conditions (i) and (ii) are used to construct $\{x_n\}$ and $\{u_n\}$, as in the proof of Theorem 12. Let us assume that $\lim_{n\to+\infty} u_n = u$ for some $u \in A_0$. By (18), we have

$$D(u_n, Qx_n) = D(A, B) = D(u_{n-1}, Px_n), \tag{40}$$

implying, since (P, Q) commutes proximally, that

$$Pu_n = Qu_{n-1} \tag{41}$$

for all integers $n \ge 1$. Since $(Qu_n, Pu_{n+1}) = (Qu_n, Qu_n) \in E_G$ and the condition (iv), there exists subsequence $\{u_{n_k}\}$ of

$\{u_n\}$ such that

$$D\left(u, Qu_{n_k}\right) = D(A,B) = D\left(u, Pu_{n_k}\right). \quad (42)$$

This, again, yields $Pu = Qu$. Lemma 11 shows $\mathrm{CP}(P, Q) \neq \varnothing$.

The uniqueness part is shown in the same fashion as in Theorem 12.

Example 14. Let $X = \mathbb{R}^3$ be equipped with the JS-metric D given by

$$D((x_1, y_1, z_1), (x_2, y_2, z_2)) = \begin{cases} \dfrac{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|}{3}, & (y_1, y_2) = (0,0) \text{ or } (z_1, z_2) = (0,0), \\ |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|, & \text{otherwise}. \end{cases} \quad (43)$$

Let $A = \{(x, 1, 1) : 0 \le x \le 3\}$ and $B = \{(x, -1, 1) : 0 \le x \le 3\}$. It is easy to see that $D(A, B) = 2$.

Define the mappings $P, Q : A \to B$ by

$$\begin{aligned} P(x, 1, 1) &= (x, -1, 1), \\ Q(x, 1, 1) &= (\ln(2 + x), -1, 1) \end{aligned} \quad (44)$$

for all $(x, 1, 1) \in A$. Notice that P and Q are continuous.

Let

$$E_G = \left\{((x, y, z), (u, v, w)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x \ge u \text{ and } y \le v \text{ and } z \ge w\right\}. \quad (45)$$

We will show that (P, Q) is G-proximal edge preserving.

(1) Let $(x^*, 1, 1), (y^*, 1, 1) \in A$ and

$$(P(x^*, 1, 1), P(y^*, 1, 1)) = ((x^*, -1, 1), (y^*, -1, 1)) \in E_G, \quad (46)$$

we have $x^* \ge y^*$ and $\ln(2 + x^*) \ge \ln(2 + y^*)$. Thus,

$$(Q(x^*, 1, 1), Q(y^*, 1, 1) = ((\ln(2 + x^*), -1, 1), (\ln(2 + y^*), -1, 1))) \in E_G. \quad (47)$$

(2) Let $x, u, v \in A$. Observe that they must have the following forms:

$$\begin{aligned} x &= (\hat{x}, 1, 1), \\ u &= (\hat{u}, 1, 1), \\ v &= (\hat{v}, 1, 1), \end{aligned} \quad (48)$$

such that $(Px, Qx) \in E_G$ and

$$D(u, Qx) = D(A, B) = D(v, Px). \quad (49)$$

Thus, $\hat{x} \ge \ln(2 + \hat{x})$ and

$$D((\hat{u}, 1, 1), (\ln(2 + \hat{x}), -1, 1)) = 2 = D((\hat{v}, 1, 1), (\hat{x}, -1, 1)). \quad (50)$$

We have $\hat{u} = \ln(2 + \hat{x})$, $\hat{v} = \hat{x}$ implying that $\hat{v} \ge \hat{u}$; that is, $(v, u) \in E_G$.

To show that the pair (P, Q) is dominating G-proximal Geraghty, define the mapping $\gamma : [0, +\infty] \to [0, 1]$ by

$$\gamma(t) = \begin{cases} 1, & t = 0, \\ \dfrac{\ln(1 + t)}{t}, & 0 < t, \\ 0, & t = +\infty. \end{cases} \quad (51)$$

Then, $\gamma \in \Gamma$.

Let $x_1, x_2, u_1, u_2, v_1, v_2 \in A$. Notice that they must have the following forms:

$$\begin{aligned} x_1 &= (\hat{x}_1, 1, 1), \\ x_2 &= (\hat{x}_2, 1, 1), \\ u_1 &= (\hat{u}_1, 1, 1), \\ u_2 &= (\hat{u}_2, 1, 1), \\ v_1 &= (\hat{v}_1, 1, 1), \\ v_2 &= (\hat{v}_2, 1, 1), \end{aligned} \quad (52)$$

such that $(Px_1, Qx_1), (Px_2, Qx_2) \in E_G$ and

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2). \quad (53)$$

Thus, $\hat{x}_1 \ge \ln(2 + \hat{x}_1), \hat{x}_2 \ge \ln(2 + \hat{x}_2)$ and

$$\begin{aligned} \hat{u}_1 &= \ln(2 + \hat{x}_1), \\ \hat{u}_2 &= \ln(2 + \hat{x}_2), \\ \hat{v}_1 &= \hat{x}_1, \\ \hat{v}_2 &= \hat{x}_2. \end{aligned} \quad (54)$$

To obtain the inequality (11), if $u_1 = u_2$ or $v_1 = v_2$, then we are done. Assume that $u_1 \neq u_2$. Then, $\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2$ are all distinct. As a consequence, $M(v_1, v_2, u_1, u_2) > 0$. Thus, we have that

$$\begin{aligned} D(u_1, u_2) &= |\hat{u}_1 - \hat{u}_2| = |\ln(2+\hat{v}_1) - \ln(2+\hat{v}_2)| \\ &= \left|\ln\left(\frac{2+\hat{v}_2+\hat{v}_1-\hat{v}_2}{2+\hat{v}_2}\right)\right| \le \ln(1+|\hat{v}_1-\hat{v}_2|) \\ &\le \ln(1+M(v_1, v_2, u_1, u_2)) \\ &= \left[\frac{\ln(1+M(v_1, v_2, u_1, u_2))}{M(v_1, v_2, u_1, u_2)}\right] M(v_1, v_2, u_1, u_2) \\ &= \gamma(M(v_1, v_2, u_1, u_2)) M(v_1, v_2, u_1, u_2). \end{aligned} \tag{55}$$

Therefore, the pair (P, Q) is dominating G-proximal Geraghty.

Next, consider, by the definition of A_0 and B_0, that $A_0 = A$ and $B_0 = B$. Additionally,

$$\begin{aligned} Q(A_0) &= \{(x, -1, 1): \ln 2 \le x \le \ln 5\} \subseteq \{(x, -1, 1): 0 \le x \le 3\} \\ &= B_0 = P(A_0). \end{aligned} \tag{56}$$

Now, it remains to show that (P, Q) commutes proximally. Let $x, u, v \in A$ be such that

$$D(u, Qx) = D(A, B) = D(v, Px). \tag{57}$$

Consequently, $x = (\hat{x}, 1, 1), u = (\hat{u}, 1, 1), v = (\hat{v}, 1, 1)$, where $\hat{u} = \ln(2+\hat{x})$ and $\hat{v} = \hat{x}$. Thus,

$$Qv = (\ln(2+\hat{v}), -1, 1) = (\ln(2+\hat{x}), -1, 1) = (\hat{u}, -1, 1) = Pu. \tag{58}$$

Thus, (P, Q) commutes proximally.

Finally, by Theorem 12, we can conclude that there is a unique common best proximity point of the pair (P, Q). In fact, the point $(0, 1, 1)$ is the unique common best proximity point of (P, Q).

4. Some Special Cases

Recall that

$$\Gamma = \left\{\gamma : [0, +\infty] \to [0, 1] \middle| \lim_{n\to+\infty} \gamma(t_n) = 1 \text{ implies } \lim_{n\to+\infty} t_n = 0\right\}. \tag{59}$$

In this section, we present some existence results where functions in Γ are concretely chosen. These results are direct consequences of Theorems 12 and 13.

First of all, let (X, D, G) be a complete JS-metric space endowed with a directed graph, A, B be nonempty subsets of X, and $P, Q : A \to B$ be mappings.

Corollary 15. *Assume (A0)–(A4) and (i), (ii), and (iii) as in Theorem12. In addition, if the following are satisfied*

(i) *either of the following holds*

(a) *P and Q are G-continuous*

(b) *For any sequence $\{x_n\}$ in A that D-converges to $x \in A$ and satisfies $(Qx_n, Px_{n+1}) \in E_G$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$D(x, Qx_{n_k}) = D(A, B) = D(x, Px_{n_k}) \tag{60}$$

(ii) *there exists $k \in [0, 1)$ such that for any $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ with $(Px_i, Qx_i) \in E_G$ and*

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2), \tag{61}$$

it follows

$$D(u_1, u_2) \le kM(v_1, v_2, u_1, u_2); \tag{62}$$

then, $CP(P, Q) \neq \emptyset$. Moreover, if $(Px, Qx) \in E_G$, for all $x \in CP(P, Q)$, then (P, Q) has a unique common best proximity point.

Proof. Define $\gamma : [0, +\infty] \to [0, 1]$ by $\gamma(t) = k$ for some $k \in [0, 1)$. Clearly, $\gamma \in \Gamma$, and hence, (A5) is satisfied.

Corollary 16. *Assume (A0)–(A4) and (i), (ii), (iii), and (iv) as in Corollary15. In addition, if the following holds*

(i) *for any $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ with $(Px_i, Qx_i) \in E_G$ and*

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2), \tag{63}$$

it follows

$$D(u_1, u_2) \le \frac{M(v_1, v_2, u_1, u_2)}{1 + M(v_1, v_2, u_1, u_2)}; \tag{64}$$

then $CP(P, Q) \neq \emptyset$. Moreover, if $(Px, Qx) \in E_G$, for all $x \in CP(P, Q)$, then (P, Q) has a unique common best proximity point.

Proof. Define $\psi : [0, +\infty] \to [0, 1]$ by $\psi(t) = 1/(1+t)$ for all $t \in [0, +\infty)$ and $\psi(+\infty) = 0$. For any sequence $\{t_n\}$ with $\lim_{n\to\infty} \psi(t_n) = 1$, we easily have $\lim_{n\to\infty} t_n = 0$. Thus, $\psi \in \Gamma$, and hence, (A5) is satisfied.

5. Application on a JS-Metric Space Endowed with an Arbitrary Relation

In this section, it is shown that our result gives rise to a common best proximity point theorem for a mapping on a JS-metric space endowed with a binary relation $\mathscr{R}$ on X denoted by $(X, D, \mathscr{R})$. To begin with, let us introduce some terminology.

Definition 17. *Let A, B be nonempty subsets of $(X, D, \mathscr{R})$ and $x_0 \in X$. A mapping $P : A \to B$ is called $\mathscr{R}$ -continuous at x_0 if for any sequence $\{x_n\}$ in A that D -converges to x_0 and $x_n \mathscr{R} x_{n+1}$ for all n , the sequence $Px_n D$ -converges to Px_0.*

Definition 18. *Let A, B be nonempty subsets of $(X, D, \mathscr{R})$ and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be $\mathscr{R}$ -proximally comparative preserving if the following assertions hold:*

(i) If $Px \mathscr{R} Py$, then $Qx \mathscr{R} Qy$

(ii) For any $x, u, v \in A$ such that $Px \mathscr{R} Qx$ and

$$D(u, Qx) = D(A, B) = D(v, Px), \tag{65}$$

it follows $v \mathscr{R} u$.

Definition 19. *Let A, B be nonempty subsets of $(X, D, \mathscr{R})$ and $P, Q : A \to B$ be mappings. A pair (P, Q) is said to be dominating $\mathscr{R}$ -proximally comparative Geraghty if there exists a function $\gamma \in \Gamma$ such that for any $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ satisfying $Px_1 \mathscr{R} Qx_1$, $Px_2 \mathscr{R} Qx_2$, and*

$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2), \tag{66}$$

it follows that

$$D(u_1, u_2) \le \gamma(M(v_1, v_2, u_1, u_2))M(v_1, v_2, u_1, u_2), \tag{67}$$

where $M(v_1, v_2, u_1, u_2) = \max\{D(v_1, v_2), D(v_1, u_1), D(v_2, u_2)\}$.

Corollary 20. *Let A, B be nonempty subsets of a complete JS-metric space $(X, D, \mathscr{R})$ and let $P, Q : A \to B$ be mappings. Suppose that the pair (P, Q) is dominating $\mathscr{R}$ -proximally comparative Geraghty. Assume that A_0 and B_0 are nonempty such that A_0 is closed and bounded. If the following assertions hold:*

(i) $Q(A_0) \subseteq B_0$ and $Q(A_0) \subseteq P(A_0)$

(ii) P and Q commute proximally

(iii) For each $u, v \in A$, if there exist $x, y \in A$ such that

$$D(u, Qx) = D(A, B) = D(v, Py), \tag{68}$$

then $D(u, v) < +\infty$

(iv) There exists $x_0 \in A_0$ such that $Px_0 \mathscr{R} Qx_0$

(v) (P, Q) is $\mathscr{R}$-proximally comparative preserving

(vi) Suppose that one of the following holds

(a) P and Q are $\mathscr{R}$-continuous

(b) For sequence $\{x_n\}$ in A that D-converges to $x \in A$ and satisfies $Qx_n \mathscr{R} Px_{n+1}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D(x, Qx_{n_k}) = D(A, B) = D(x, Px_{n_k}), \tag{69}$$

then $CP(P, Q) \neq \emptyset$. Moreover, if $Px \mathscr{R} Qx$ for all $x \in CP(P, Q)$, then the pair (P, Q) has a unique common best proximity point.

Proof. We define a directed graph $G = (V_G, E_G)$ with $V_G = X$ and

$$E_G = \{(x, y) \in X \times X : x \mathscr{R} y\}. \tag{70}$$

In order to apply Theorem 12 or 13, all the hypotheses must hold.

(1) We will show that (P, Q) is dominating G-proximal Geraghty. To this end, let $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ be such that

$$(Px_1, Qx_1) \in E_G,$$
$$(Px_2, Qx_2) \in E_G,$$
$$D(u_1, Qx_1) = D(u_2, Qx_2) = D(A, B) = D(v_1, Px_1) = D(v_2, Px_2). \tag{71}$$

Then, we have $Px_1 \mathscr{R} Qx_1$ and $Px_2 \mathscr{R} Qx_2$. Since (P, Q) is dominating $\mathscr{R}$-proximally comparative Geraghty, the pair (P, Q) is dominating G-proximally Geraghty.

(2) The condition (iv) implies that there exists $x_0 \in A_0$ such that

$$(Px_0, Qx_0) \in E_G \tag{72}$$

(3) Let (P, Q) be $\mathscr{R}$-proximally comparative preserving. We will show that (P, Q) is G-proximal edge preserving. Notice that $(Px, Py) \in E_G$ implies $Px \mathscr{R} Py$. Since (P, Q) is $\mathscr{R}$-proximally comparative preserving, we get $Qx \mathscr{R} Qy$ implying that $(Qx, Qy) \in E_G$

Let $x, u, v \in A$ such that $(Px, Qx) \in E_G$ and

$$D(u, Qx) = D(A, B) = D(v, Px). \tag{73}$$

By the definition of E_G, we have $Px \mathscr{R} Qx$. Since (P, Q) is

$\mathscr{R}$-proximally comparative preserving, we obtain $v\mathscr{R}u$. This yields $(v, u) \in E_G$.

(4) The condition (a) implies that P and Q are G-continuous on A. Applying Theorem 12, we achieve CP $(P, Q) \neq \varnothing$

Suppose the condition (b) holds. Let $\{x_n\}$ be a sequence in A that D-converges to $x \in A$ and satisfy $(Qx_n, Px_{n+1}) \in E_G$. Then, $Qx_n\mathscr{R}Px_{n+1}$. By (b), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D\left(x, Qx_{n_k}\right) = D(A, B) = D\left(x, Px_{n_k}\right). \tag{74}$$

Theorem 13 now applies and gives $\mathrm{CP}(P, Q) \neq \varnothing$.

Finally, if $Px\mathscr{R}Qx$ for all $x \in \mathrm{CP}(P, Q)$, then $(Px, Qx) \in E_G$ for all $x \in \mathrm{CP}(P, Q)$. Therefore, (P, Q) has a unique common best proximity point.

6. Conclusion

This work has proposed some existence and uniqueness theorems for common best proximity points of any two mappings in a JS-metric space endowed with a directed graph G. The results obtained were mainly due to the assumptions that the pair of mappings is G-proximal edge preserving and dominated G-proximal Geraghty. The results have been further applied to a situation where the JS-metric space enjoys a binary relation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

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Research Article

Kannan Contraction Operator on the Domain of $r(.)$-Cesàro Matrix in $\ell_{t(.)}$ and Related Prequasi Ideal with an Application of Nonlinear Difference Equations

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In this article, the sequence space $(\Xi(r,t))_v$ has been built by the domain of (r_l)-Cesàro matrix in Nakano sequence space $\ell_{(t_l)}$, where $t=(t_l)$ and $r=(r_l)$ are sequences of positive reals with $1 \le t_l < \infty$, and $v(f)=\sum_{l=0}^{\infty}(|\sum_{z=0}^{l} r_z f_z| / \sum_{z=0}^{l} r_z)^{t_l}$, with $f=(f_z) \in \Xi(r,t)$. Some topological and geometric behavior of $(\Xi(r,t))_v$, the multiplication maps acting on $(\Xi(r,t))_v$, and the eigenvalues distribution of operator ideal constructed by $(\Xi(r,t))_v$ and s-numbers have been examined. The existence of a fixed point of Kannan prequasi norm contraction mapping on this sequence space and on its prequasi operator ideal are investigated. Moreover, we indicate our results by some explanative examples and actions to the existence of solutions of nonlinear difference equations.

1. Introduction

As a remark of constant Lebesgue spaces, variable exponent Lebesgue spaces go again many years, and in successive centuries, variable Lebesgue and Sobolev spaces have been regularly studied. Next, many variable exponent real function spaces and complex function spaces have presented, for instance, Morrey spaces, Herz-Morrey spaces, Herz spaces, Hardy spaces, Besov spaces, Trieble-Lizorkin spaces, Fock spaces, Bessel potential spaces, and Bergman spaces with variable exponents. For three centuries, variable exponent function spaces have been extensively applied in approximation theory, image processing, and differential equations, and many variable exponent real function spaces and complex function spaces have shown. Thus far, the theory of variable exponent function spaces has pensively built upon on the boundedness of the Hardy-Littlewood maximal operator. This confines its technique in differential equations, optimization, and approximation. The spaces of all, bounded, r-absolutely summable and convergent to zero sequences of complex numbers will be denoted by $\mathscr{C}^{\mathrm{N}}$, ℓ_∞, ℓ_r, and c_0. $\mathrm{N}=\{0,1,2,\cdots\}$. We denote the space of all, finite rank, approximable, and compact bounded linear maps from a Banach space $\mathscr{P}$ into a Banach space $\mathscr{Q}$ by $\mathbb{B}(\mathscr{P},\mathscr{Q})$, $\mathbb{F}(\mathscr{P},\mathscr{Q})$, $\mathscr{A}(\mathscr{P},\mathscr{Q})$, and $\mathscr{K}(\mathscr{P},\mathscr{Q})$, and if $\mathscr{P}=\mathscr{Q}$, we indicate $\mathbb{B}(\mathscr{P})$, $\mathbb{F}(\mathscr{P})$, $\mathscr{A}(\mathscr{P})$, and $\mathscr{K}(\mathscr{P})$, respectively. The ideal of all, finite rank, approximable, and compact maps are indicated by $\mathbb{B}$, $\mathbb{F}$, $\mathscr{A}$, and $\mathscr{K}$. We label $e_l=(0,0,\cdots,1,0,0,\cdots)$, as 1 lies at the lth coordinate, with $l \in \mathrm{N}$.

Definition 1 [1]. A function $s:\mathbb{B}(\mathscr{P},\mathscr{Q}) \longrightarrow [0,\infty)^N$ is called an s-number, if the sequence $(s_b(X))_{b=0}^{\infty}$, for any $X \in \mathbb{B}(\mathscr{P},\mathscr{Q})$, satisfies the following setup:

(a) $\|X\|=s_0(X) \ge s_1(X) \ge s_2(X) \ge \cdots \ge 0$, with $X \in \mathbb{B}(\mathscr{P},\mathscr{Q})$

(b) $s_{l+b-1}(X_1+X_2) \leq s_l(X_1)+s_b(X_2)$, with $X_1, X_2 \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ and $l, b \in \mathrm{N}$

(c) $s_b(ZYX) \leq \|Z\| s_b(Y)\|X\|$, for all $X \in \mathbb{B}(\mathscr{P}_0, \mathscr{P})$, $Y \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q}, \mathscr{Q}_0)$, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are any two Banach spaces

(d) If $G \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ and $\gamma \in \mathscr{C}$, then $s_a(\gamma G) = |\gamma| s_a(G)$

(e) Suppose rank $(X) \leq b$, then $s_b(X) = 0$, for all $X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$

(f) $s_{l \geq b}(I_b) = 0$ or $s_{l<b}(I_b) = 1$, where I_b indicates the unit operator on the b-dimensional Hilbert space ℓ_2^b

We give some examples of s-numbers as follows:

(1) The l-th Kolmogorov number, $d_l(X)$, where

$$d_l(X) = \inf_{\dim J \leq l} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|. \quad (1)$$

(2) The l-th approximation number, $\alpha_l(X)$, where

$$\alpha_l(X) = \inf \{\|X - Y\| : Y \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) \text{ and rank } (Y) \leq l\}. \quad (2)$$

Notations 2 [2].

$$\begin{aligned}
\mathbb{B}_{\mathscr{V}}^{s} &:= \{\mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q})\,; \mathscr{P} \text{ and } \mathscr{Q} \text{ are Banach Spaces}\},\\
&\quad \text{where } \mathbb{B}_{\mathscr{V}}^{s}(\mathscr{P}, \mathscr{Q}) := \left\{X \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) : \left((s_a(X))_{a=0}^{\infty} \in \mathscr{V}\right\}.\right.\\
\mathbb{B}_{\mathscr{V}}^{\alpha} &:= \{\mathbb{B}_{\mathscr{V}}^{\alpha}(\mathscr{P}, \mathscr{Q})\,; \mathscr{P} \text{ and } \mathscr{Q} \text{ are Banach Spaces}\},\\
&\quad \text{where } \mathbb{B}_{\mathscr{V}}^{\alpha}(\mathscr{P}, \mathscr{Q}) := \left\{X \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) : \left((\alpha_a(X))_{a=0}^{\infty} \in \mathscr{V}\right\}.\right.\\
\mathbb{B}_{\mathscr{V}}^{d} &:= \left\{\mathbb{B}_{\mathscr{V}}^{d}(\mathscr{P}, \mathscr{Q}) \mathscr{P} \text{ and } \mathscr{Q} \text{ areBanach Spaces}\right\},\\
&\quad \text{where } \mathbb{B}_{\mathscr{V}}^{d}(\mathscr{P}, \mathscr{Q}) := \left\{X \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) : \left((d_a(X))_{a=0}^{\infty} \in \mathscr{V}\right\}.\right.
\end{aligned} \quad (3)$$

Some of ideals in the class of Banach spaces or Hilbert spaces are generated by scalar sequence spaces. For example, the ideal of compact maps is constructed by the space c_0 and $d_a(X)$, for $X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$. Pietsch [3] investigated the quasi-ideals $\mathbb{B}_{\ell_b}^{\alpha}$, for $0 < b < \infty$. He examined that the ideals of nuclear maps and of Hilbert Schmidt maps between Hilbert spaces are formed by ℓ_1 and ℓ_2, respectively. He discussed that $\mathbb{F}(\ell_b)$ are dense in $\mathbb{B}(\ell_b)$, and the algebra $\mathbb{B}(\ell_b)$, where $(1 \leq b < \infty)$, generated simple Banach space. Pietsch [4] established that $\mathbb{B}_{\ell_b}^{\alpha}$, with $0 < b < \infty$, is small. Makarov and Faried [5] proved that for every infinite dimensional Banach spaces $\mathscr{P}$, $\mathscr{Q}$, and $r > b > 0$, hence $\mathbb{B}_{\ell_b}^{\alpha}(\mathscr{P}, \mathscr{Q}) Þ \mathbb{B}_{\ell_r}^{\alpha}(\mathscr{P}, \mathscr{Q}) \mathscr{U} \mathbb{B}(\mathscr{P}, \mathscr{Q})$. Yaying et al. [6] constructed the sequence space, χ_r^t, whose its r-Cesàro matrix in ℓ_t, with $r \in (0, 1]$ and $1 \leq t \leq \infty$. They studied the quasi Banach ideal of type χ_r^t, with $r \in (0, 1]$ and $1 < t < \infty$. They introduced its Schauder basis, $\alpha-$, $\beta-$, and $\gamma-$ duals and determined certain matrix classes related to this sequence space. On sequence spaces, Basarir and Kara examined the compact maps on some Euler $B(m)$-difference sequence spaces [7], some difference sequence spaces of weighted means [8], the Riesz $B(m)$-difference sequence space [9], the B-difference sequence space derived by weighted mean [10], and the mth order difference sequence space of generalized weighted mean [11]. Mursaleen and Noman [12, 13] established the compact operators on some difference sequence spaces. Alotaibi et al. [14, 15] examined the compact operators on some Fibonacci difference sequence spaces and on a new sequence space related to ℓ_p spaces. The multiplication maps on Cesàro sequence spaces equipped with the Luxemburg norm investigated by Komal et al. [16]. İlkhan et al. [17], investigated the multiplication maps on Cesàro second order function spaces. Recently, many authors in the literature have investigated some nonabsolute type sequence spaces and brought current exquisite papers, for examples, Mursaleen and Noman [18] introduced the sequence space ℓ_p^{λ} and ℓ_{∞}^{λ} of nonabsolute type and proved that the spaces ℓ_p^{λ} and ℓ_p^{λ} are linearly isomorphic for $0 < p \leq \infty$, ℓ_p^{λ} is a p-normed space and a BK-space in the cases for $0 < p < 1$ and $1 \leq p \leq \infty$ and constructed the basis of the space ℓ_p^{λ} for $1 \leq p < \infty$. In [19], they examined the $\alpha-$, $\beta-$, and $\gamma-$ duals of ℓ_p^{λ} and ℓ_{∞}^{λ} of nonabsolute type, with $1 \leq p < \infty$. They detailed some related matrix classes and developed the properties of some other classes by means of a given basic lemma. On Cesàro summable sequences, Mursaleen and Basar [20] introduced some spaces of double sequences whose Cesàro transforms are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent, and absolutely q-summable, respectively, and investigated some topological properties of those sequence spaces. The Banach fixed point theorem [21] gave many mathematicians the way to examine many generalizations for the contraction maps defined on the space or on the space itself. Kannan [22] investigated an example of a class of operators with the identical fixed point actions as contractions though that fails to be continuous. Ghoncheh [23] was the only one who examined Kannan operators in modular vector spaces. He showed that the existence of a fixed point of Kannan mapping in complete modular spaces that have Fatou property. Bakery and Mohamed [24] explored the concept of the prequasi norm on Nakano sequence space such that its variable exponent in (0, 1]. They explained the sufficient conditions on its equipped with the definite prequasi norm to generate prequasi Banach and closed space and examined the Fatou property of different prequasi norms on it. More, they showed the existence of a fixed point of Kannan prequasi norm contraction maps on it and on the prequasi Banach operator ideal constructed by s-numbers which belong to this sequence space. The next inequality will used in the sequel [25]: Assume $t_a \geq 1$ and $x_a, z_a \in \mathscr{C}$, for all $a \in \mathrm{N}$, and $\hbar = \sup_a t_a$, then

$$|x_a + z_a|^{t_a} \leq 2^{\hbar - 1}\left(|x_a|^{t_a} + |z_a|^{t_a}\right). \quad (4)$$

The aim of this paper is arranged as follows: In Section 3, we introduce the definition and some inclusion relations of the sequence space $(\Xi(r,t))_{\upsilon}$ under the function υ. In Section 4, we investigate the enough setup on $\Xi(r,t)$ with definite function υ to form premodular private sequence space ($\mathfrak{pss}$), which gives that $(\Xi(r,t))_{\upsilon}$ is a prequasi normed $\mathfrak{pss}$. In Section 5, we examine a multiplication map on $(\Xi(r,t))_{\upsilon}$ and introduce the necessity and sufficient conditions on this sequence space such that the multiplication map is bounded, approximable, invertible, Fredholm, and closed range. In Section 6, firstly, we introduce the sufficient settings (not necessary) on$(\Xi(r,t))_{\upsilon}$, such that$\mathbb{F}$is dense in$\mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}$. This investigates a negative answer of Rhoades [26] open problem about the linearity of s-type $(\Xi(r,t))_{\upsilon}$ spaces. Secondly, we introduce the conditions on $(\Xi(r,t))_{\upsilon}$ so that the components of prequasi ideal $\mathbb{B}^{s}_{\Xi(r,t)}$ are complete and closed. Thirdly, we investigate the sufficient settings on $(\Xi(r,t))_{\upsilon}$ such that $\mathbb{B}^{\alpha}_{(\Xi(r,t))_{\upsilon}}$ is strictly included for different weights and powers. We investigate the conditions for which the prequasi ideal $\mathbb{B}^{\alpha}_{(\Xi(r,t))_{\upsilon}}$ is minimum. Fourthly, we explore the setting for which the Banach prequasi ideal $\mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}$ is simple. Fifthly, we examine the sufficient setting on $(\Xi(r,t))_{\upsilon}$ such that the class $\mathbb{B}$ which sequence of eigenvalues in $(\Xi(r,t))_{\upsilon}$ equals $\mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}$. In Section 7, the existence of a fixed point of Kannan prequasi norm contraction mapping on this sequence space and on its prequasi operator ideal generated by $(\Xi(r,t))_{\upsilon}$, and s-numbers are presented. Additionally, in Section 8, we illustrate our results by some examples and applications to the existence of solutions of nonlinear difference equations. Finally, we introduce our conclusion in Section 9.

2. Definitions and Preliminaries

Lemma 3 [3]. *Assume $U \in \mathbb{B}(\mathscr{P},\mathscr{Q})$ and $U \notin \mathscr{A}(\mathscr{P},\mathscr{Q})$, then there exist operators $X \in \mathbb{B}(\mathscr{P})$ and $Y \in \mathbb{B}(\mathscr{Q})$ such that $YUXe_b = e_b$, for all $b \in N$.*

Definition 4 [3]. A Banach space $\mathscr{V}$ is called simple if the algebra $\mathbb{B}(\mathscr{V})$ contains one and only one nontrivial closed ideal.

Theorem 5 [3]. *If $\mathscr{V}$ is a Banach space with* $\dim(\mathscr{V}) = \infty$, *hence*

$$\mathbb{F}(\mathscr{V}) \text{Þ} \mathscr{A}(\mathscr{V}) \text{Þ} \mathscr{K}(\mathscr{V}) \text{Þ} \mathbb{B}(\mathscr{V}). \quad (5)$$

Definition 6 [27]. An operator $U \in \mathbb{B}(\mathscr{V})$ is called Fredholm if $\dim(\mathrm{Range}(U))^{c} < \infty$, $\dim(\ker(U)) < \infty$, and $\mathrm{Range}(U)$ is closed, where $(\mathrm{Range}(U))^{c}$ indicates the complement of $\mathrm{Range}(U)$.

Definition 7 [28]. A class $\mathbb{W} \subseteq \mathbb{B}$ is called an operator ideal if each element $\mathbb{W}(\mathscr{P},\mathscr{Q}) = \mathbb{W} \cap \mathbb{B}(\mathscr{P},\mathscr{Q})$ verifies the following conditions:

(i) $I_{\Omega} \in \mathbb{W}$, if Ω indicates Banach space of one dimension

(ii) $\mathbb{W}(\mathscr{P},\mathscr{Q})$ is a linear space on $\mathscr{C}$

(iii) Assume $X \in \mathbb{B}(\mathscr{P}_0,\mathscr{P})$, $Y \in \mathbb{W}(\mathscr{P},\mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q},\mathscr{Q}_0)$, then $ZYX \in \mathbb{W}(\mathscr{P}_0,\mathscr{Q}_0)$, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are normed spaces

Definition 8 [2]. A map $\Psi : \mathbb{W} \longrightarrow [0,\infty)$ is called a prequasi norm on the operator ideal $\mathbb{W}$, if it satisfies the following conditions:

(1) For every $X \in \mathbb{W}(\mathscr{P},\mathscr{Q})$, $\Psi(X) \geq 0$ and $\Psi(X) = 0 \Leftrightarrow X = 0$

(2) One has $E_0 \geq 1$ such that $\Psi(\kappa X) \leq E_0|\kappa|\Psi(X)$, with $X \in \mathbb{W}(\mathscr{P},\mathscr{Q})$ and $\kappa \in \mathscr{C}$

(3) One has $G_0 \geq 1$ such that $\Psi(Z_1 + Z_2) \leq G_0[\Psi(Z_1) + \Psi(Z_2)]$, for all $Z_1, Z_2 \in \mathbb{W}(\mathscr{P},\mathscr{Q})$

(4) One has $D_0 \geq 1$ such that if $X \in \mathbb{B}(\mathscr{P}_0,\mathscr{P})$, $Y \in \mathbb{W}(\mathscr{P},\mathscr{Q})$, and $Z \in \mathbb{B}(\mathscr{Q},\mathscr{Q}_0)$ hence $\Psi(ZYX) \leq D_0\|Z\|\Psi(Y)\|X\|$

Theorem 9 [2]. *Every quasi norm on the ideal $\mathbb{W}$ is a prequasi norm on the same ideal.*

Definition 10 [29]. The linear space of sequences $\mathscr{V}$ is called a private sequence space ($\mathfrak{pss}$), if it verifies the next setup:

(1) $e_b \in \mathscr{V}$, with $b \in \mathrm{N}$

(2) $\mathscr{V}$ is solid, i.e., for $f = (f_b) \in \mathscr{C}^{\mathrm{N}}$, $|g| = (|g_b|) \in \mathscr{V}$, and $|f_b| \leq |g_b|$, with $b \in \mathrm{N}$, then $|f| \in \mathscr{V}$

(3) $(|f_{[b/2]}|)_{b=0}^{\infty} \in \mathscr{V}$, where $[b/2]$ indicates the integral part of $b/2$, if $(|f_b|)_{b=0}^{\infty} \in \mathscr{V}$

Theorem 11 [29]. *Suppose the linear sequence space $\mathscr{V}$ be a $\mathfrak{pss}$, then $\mathbb{B}^{s}_{\mathscr{V}}$ be an operator ideal.*

Definition 12 [29]. A subspace of the $\mathfrak{pss}$ is called a premodular $\mathfrak{pss}$, if there exists a map $\upsilon : \mathscr{V} \longrightarrow [0,\infty)$ verifies the next setup:

(i) For all $f \in \mathscr{V}$, $f = \theta \Leftrightarrow \upsilon(|f|) = 0$, and $\upsilon(f) \geq 0$, with θ is the zero vector of $\mathscr{V}$

(ii) Assume $f \in \mathscr{V}$ and $\rho \in \mathscr{C}$, one has $E_0 \geq 1$ with $\upsilon(\rho f) \leq |\rho| E_0 \upsilon(f)$

(iii) $\upsilon(f + g) \leq G_0(\upsilon(f) + \upsilon(g))$ includes for some $G_0 \geq 1$, with $f, g \in \mathscr{V}$

(iv) If $b \in \mathrm{N}$, $|f_b| \leq |g_b|$, one has $\upsilon((|f_b|)) \leq \upsilon((|g_b|))$

(v) The inequality, $\upsilon((|f_b|)) \leq \upsilon((|f_{[b/2]}|)) \leq D_0 \upsilon((|f_b|))$ holds, for $D_0 \geq 1$

(vi) $\bar{\mathscr{F}} = \mathscr{V}_\upsilon$, where $\mathscr{F}$ indicates the space of all sequences with finite none zero coordinates

(vii) There are $\varpi > 0$ such that $\upsilon(\rho, 0, 0, 0, \cdots) \geq \varpi |\rho| \upsilon(1, 0, 0, 0, \cdots)$, with $\rho \in \mathscr{C}$

Definition 13 [29]. The $\mathfrak{pss}\mathscr{V}_\upsilon$ is called a prequasi normed $\mathfrak{pss}$, if υ verifies the conditions (i)–(iii) of Definition 12. When $\mathscr{V}$ is complete equipped with υ, then $\mathscr{V}_\upsilon$ is called a prequasi Banach $\mathfrak{pss}$.

Theorem 14 [29]. *Every premodular $\mathfrak{pss}\mathscr{V}_\upsilon$ is a prequasi normed $\mathfrak{pss}$.*

Theorem 15 [29]. *The function Ψ is a prequasi norm on $\mathbb{B}^s_{(\mathscr{V})_\upsilon}$, where $\Psi(Y) = \upsilon(s_b(Y))_{b=0}^\infty$, for all $Y \in \mathbb{B}^s_{(\mathscr{V})_\upsilon}(\mathscr{P}, \mathscr{Q})$, whenever $(\mathscr{V})_\upsilon$ is a premodular $\mathfrak{pss}$.*

Definition 16 [24]. A prequasi norm υ on $\mathscr{V}$ satisfies the Fatou property, if for each sequence $\{t^a\} \subseteq \mathscr{V}_\upsilon$ with $\lim_{a\longrightarrow\infty} \upsilon(t^a - t) = 0$ and all $z \in \mathscr{V}_\upsilon$ then $\upsilon(z - t) \leq \sup_j \inf_{a\geq j} \upsilon(z - t^a)$.

Definition 17 [24]. A prequasi norm Ψ on the ideal $\mathbb{B}^s_{\mathscr{V}}$, where $\Psi(W) = \upsilon((s_a(W))_{a=0}^\infty)$, satisfies the Fatou property if for each sequence $\{W_a\}_{a\in N} \subseteq \mathbb{B}^s_{\mathscr{V}}(Z, M)$ with $\lim_{a\longrightarrow\infty} \Psi(W_a - W) = 0$ and all $V \in \mathbb{B}^s_{\mathscr{V}}(Z, M)$, then

$$\Psi(V - W) \leq \sup_a \inf_{i\geq a} \Psi(V - W_i). \tag{6}$$

Definition 18. $(\upsilon(Wz - z) + \upsilon(Wt - t))$, for all $z, t \in \mathscr{V}_\upsilon$.

An element $z \in \mathscr{V}_\upsilon$ is called a fixed point of W, if $W(z) = z$.

Definition 19 [24]. An operator $W : \mathbb{B}^s_{\mathscr{V}}(Z, M) \longrightarrow \mathbb{B}^s_{\mathscr{V}}(Z, M)$ is named a Kannan Ψ-contraction, if there exists $\lambda \in [0, 1/2)$, so as to $\Psi(WV - WT) \leq \lambda(\Psi(WV - V) + \Psi(WT - T))$, for all $V, T \in \mathbb{B}^s_{\mathscr{V}}(Z, M)$.

Definition 20 [24]. If $\mathscr{V}_\upsilon$ is a prequasi normed (sss), $W : \mathscr{V}_\upsilon \longrightarrow \mathscr{V}_\upsilon$ and $b \in \mathscr{V}_\upsilon$. The operator W is called υ-sequentially continuous at b, if and only if, when $\lim_{a\longrightarrow\infty} \upsilon(t_a - b) = 0$, then $\lim_{a\longrightarrow\infty} \upsilon(Wt_a - Wb) = 0$.

Definition 21 [24]. For the prequasi norm Ψ on the ideal $\mathbb{B}^s_{\mathscr{V}}$, where $\Psi(W) = \upsilon((s_a(W))_{a=0}^\infty)$, $G : \mathbb{B}^s_{\mathscr{V}}(Z, M) \longrightarrow \mathbb{B}^s_{\mathscr{V}}(Z, M)$, and $B \in \mathbb{B}^s_{\mathscr{V}}(Z, M)$. The operator G is called Ψ-sequentially continuous at B, if and only if, when $\lim_{p\longrightarrow\infty} \Psi(W_p - B) = 0$, then $\lim_{p\longrightarrow\infty} \Psi(GW_p - GB) = 0$.

Definition 22 [29]. If $\omega = (\omega_k) \in \mathscr{C}^N$ and $\mathscr{V}_\upsilon$ is a prequasi normed $\mathfrak{pss}$. The operator $H_\omega : \mathscr{V}_\upsilon \longrightarrow \mathscr{V}_\upsilon$ is called a multiplication operator on $\mathscr{V}_\upsilon$, when $H_\omega f = (\omega_b f_b) \in \mathscr{V}_\upsilon$, with $f \in \mathscr{V}_\upsilon$. The multiplication operator is called created by ω, if $H_\omega \in \mathbb{B}(\mathscr{V}_\upsilon)$.

Theorem 23 [30]. *Assume s-type $\mathscr{V}_\upsilon := \{f = (s_r(X)) \in R^N : X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ and $\upsilon(f) < \infty\}$. If $\mathbb{B}^s_{\mathscr{V}_\upsilon}$ is an operator ideal, then the next setups are confirmed:*

(1) $\mathscr{F} \subset$ s-type $\mathscr{V}_\upsilon$

(2) Suppose $(s_r(X_1))_{r=0}^\infty \in$ s-type $\mathscr{V}_\upsilon$ and $(s_r(X_2))_{r=0}^\infty \in$ s-type $\mathscr{V}_\upsilon$, then $(s_r(X_1 + X_2))_{r=0}^\infty \in$ s-type $\mathscr{V}_\upsilon$

(3) Assume $\lambda \in \mathscr{C}$ and $(s_r(X))_{r=0}^\infty \in$ s-type $\mathscr{V}_\upsilon$, then $|\lambda| (s_r(X))_{r=0}^\infty \in$ s-type $\mathscr{V}_\upsilon$

(4) The sequence space $\mathscr{V}_\upsilon$ is solid, i.e., if $(s_r(Y))_{r=0}^\infty \in$ s-type $\mathscr{V}_\upsilon$ and $s_r(X) \leq s_r(Y)$, for all $r \in N$ and $X, Y \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$, then $(s_r(X))_{r=0}^\infty \in$ s-type $\mathscr{V}_\upsilon$

3. Main Results

3.1. The Sequence Space $(\Xi(r, t))_\upsilon$. The definition and some inclusion relations of the sequence space $(\Xi(r, t))_\upsilon$ under the function υ in this section are presented.

Definition 24. Suppose $(r_l), (t_l) \in R^{+N}$, where R^{+N} be the space of all sequences of positive reals. The sequence space $(\Xi(r, t))_\upsilon$ with the function υ is evident by:

$$(\Xi(r, t))_\upsilon = \{f = (f_k) \in \mathscr{C}^N : \upsilon(\rho f) < \infty, \text{for some } \rho > 0\},$$
$$\text{where } \upsilon(f) = \sum_{l=0}^\infty \left(\frac{\left|\sum_{z=0}^l r_z f_z\right|}{\sum_{z=0}^l r_z} \right)^{t_l}. \tag{7}$$

Theorem 25. *Assume $(t_l) \in R^{+N} \cap \ell_\infty$, one has*

$$(\Xi(r, t))_\upsilon = \{f = (f_k) \in \mathscr{C}^N : \upsilon(\rho f) < \infty, \textit{for any } \rho > 0\}. \tag{8}$$

Proof. Assume $(t_l) \in \mathrm{R}^{+\mathrm{N}} \cap \ell_\infty$, one has

$$\begin{aligned}
(\Xi(\Delta, r))_\upsilon &= \{f = (f_k) \in \mathscr{C}^{\mathrm{N}} : \upsilon(\rho f) < \infty, \text{for some } \rho > 0\} \\
&= \left\{ f = (f_k) \in \mathscr{C}^{\mathrm{N}} : \sum_{l=0}^\infty \left(\frac{\left|\sum_{z=0}^l \rho r_z f_z\right|}{\sum_{z=0}^l r_z} \right)^{t_l} < \infty, \text{for some } \rho > 0 \right\} \\
&= \left\{ f = (f_k) \in \mathscr{C}^{\mathrm{N}} : \inf_l \rho^{t_l} \sum_{l=0}^\infty \left(\frac{\left|\sum_{z=0}^l r_z f_z\right|}{\sum_{z=0}^l r_z} \right)^{t_l} < \infty, \text{for some } \rho > 0 \right\} \\
&= \left\{ f = (f_k) \in \mathscr{C}^{\mathrm{N}} : \sum_{l=0}^\infty \left(\frac{\left|\sum_{z=0}^l r_z f_z\right|}{\sum_{z=0}^l r_z} \right)^{t_l} < \infty \right\} \\
&= \{f = (f_k) \in \mathscr{C}^{\mathrm{N}} : \upsilon(\rho f) < \infty, \text{for any } \rho > 0\}.
\end{aligned} \tag{9}$$

Remark 26.

(1) Assume $t_z = t$, $r_z = r^z$, for all $z \in \mathrm{N}$, $0 < r \leq 1$, and $t \geq 1$, the sequence space $\Xi(r, t) = \chi_r^t$ examined by Yaying et al. [6]

(2) If $t_z = t$, $r_z = 1$, for all $z \in \mathrm{N}$ and $t \geq 1$, hence $\Xi(r,t) = ces^t$, defined and studied by Ng and Lee [31]

Theorem 27. *Pick up $(r_l), (t_l) \in R^{+N}$ and $1 \leq t_l < \infty$, then $(\Xi(r,t))_\upsilon$ be nonabsolute type.*

Proof. Let $f = (1,-1,0,0,0,)$, then $|f| = (1,1,0,0,0,)$. One has

$$\begin{aligned} \upsilon(f) = 1 + \left(\frac{|r_0 - r_1|}{r_0 + r_1}\right)^{t_1} + \left(\frac{|r_0 - r_1|}{r_0 + r_1 + r_2}\right)^{t_2} + \neq 2 \\ + \left(\frac{|r_0 + r_1|}{r_0 + r_1 + r_2}\right)^{t_2} + = \upsilon(|f|). \end{aligned} \tag{10}$$

Hence, the sequence space $(\Xi(r,t))_\upsilon$ is nonabsolute type.

Note that, we call the sequence space $(\Xi(r,t))_\upsilon$ as (r_l)-generalized Cesàro sequence space of nonabsolute type since it is generated by the domain of (r_l)-Cesàro matrix in $\ell_{(t_l)}$, where the (r_l)-Cesàro matrix, $\Lambda(r) = (\lambda_{lz}(r))$, is defined as:

$$\lambda_{lz}(r) = \begin{cases} \dfrac{r_z}{\sum_{z=0}^{l} r_z}, & 0 \leq z \leq l, \\ 0, & z > l. \end{cases} \tag{11}$$

The (r_l)-Cesàro matrix may be shown clearly as:

$$\Lambda(r) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \dfrac{r_0}{r_0 + r_1} & \dfrac{r_1}{r_0 + r_1} & 0 & 0 & \cdots \\ \dfrac{r_0}{r_0 + r_1 + r_2} & \dfrac{r_1}{r_0 + r_1 + r_2} & \dfrac{r_2}{r_0 + r_1 + r_2} & 0 & \cdots \\ \dfrac{r_0}{r_0 + r_1 + r_2 + r_3} & \dfrac{r_1}{r_0 + r_1 + r_2 + r_3} & \dfrac{r_2}{r_0 + r_1 + r_2 + r_3} & \dfrac{r_3}{r_0 + r_1 + r_2 + r_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{12}$$

Definition 28. Pick up $(r_l), (t_l) \in R^{+N}$. The (r_l)-generalized Cesàro sequence space of absolute type $(ces(r,t))_\varphi$ is defined as:

$$\begin{aligned} (ces(r,t))_\varphi = \{f = (f_k) \in \mathscr{C}^{\mathrm{N}} : \varphi(\rho f) < \infty, \text{for some } \rho > 0\}, \\ \text{where } \varphi(f) = \sum_{l=0}^{\infty} \left(\frac{\sum_{z=0}^{l} r_z |f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l}. \end{aligned} \tag{13}$$

Theorem 29. *Assume $(r_l), (t_l) \in R^{+N} \cap \ell_\infty$ with $\inf_l r_l > 0$, then $(ces(r,t))_\varphi U(\Xi(r,t))_\upsilon$.*

Proof. Suppose $f \in (ces(r,t))_\varphi$, as

$$\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z f_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \leq \sum_{l=0}^{\infty} \left(\frac{\sum_{z=0}^{l} r_z |f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \infty. \tag{14}$$

Therefore, $f \in (\Xi(r,t))_\upsilon$. For $(t_l) \in (1,\infty)^{\mathscr{N}} \cap \ell_\infty$, we take $g = ((-1)^z / r_z)_{z \in \mathscr{N}}$; we have $g \in (\Xi(r,t))_\upsilon$ and $g \notin (ces(r,t))_\varphi$. For $(t_l) \in (0,1]^{\mathscr{N}}$, we take $h = (1/r_0, -1/r_1, 0, 0, 0, \cdots)$; we have $h \in (\Xi(r,t))_\upsilon$ and $h \notin (ces(r,t))_\varphi = \{(0,0,\cdots)\}$.

4. Premodular Private Sequence Space

In this section, we investigate the sufficient conditions on $\Xi(r,t)$ with definite function υ to form premodular $\mathfrak{pss}$, which gives that $\Xi(r,t)$ is a prequasi normed $\mathfrak{pss}$.

Here and after, the space of all monotonic decreasing and monotonic increasing sequences of positive reals will be indicated by $\mathfrak{I}_\searrow$ and $\mathfrak{I}_\nearrow$, respectively.

Theorem 30. *$\Xi(r,t)$ is a $\mathfrak{pss}$, if the following conditions are satisfied:*

(f1) $(t_l) \in \mathfrak{I}_\nearrow \cap \ell_\infty$ with $t_0 > 1$

(f2) $(r_z)_{z=0}^{\infty} \in \mathfrak{I}_\searrow$ with $\inf_z r_z > 0$ or $(r_z)_{z=0}^{\infty} \in \mathfrak{I}_\nearrow \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2z+1} \leq C r_z$

Proof.

(1-i) Let $f, g \in \Xi(r,t)$. We have

$$\begin{aligned} &\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z f_z + r_z g_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &\leq 2^{\hbar - 1} \left(\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z f_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} + \sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z g_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right) < \infty, \end{aligned} \tag{15}$$

therefore, $f + g \in \Xi(r,t)$.

(1-ii) Assume $\rho \in \mathscr{C}$, $f \in \Xi(r,t)$ and since $(t_l) \in \mathfrak{I}_Z \cap \ell_\infty$, one has

$$\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z \rho f_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \leq \sup_l |\rho|^{t_l} \sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z f_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \infty. \tag{16}$$

Therefore, $\rho f \in \Xi(r,t)$. According to conditions (1-i) and (1-ii), one has $\Xi(r,t)$ is a linear space.

Since $(t_l) \in \mathfrak{I}_Z \cap \ell_\infty$ and $t_0 > 1$, we have

$$\begin{aligned} \sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z (e_b)_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} &= \sum_{l=b}^{\infty} \left(\frac{r_b}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &\leq \sup_l r_b^{t_l} \sum_{l=b}^{\infty} \left(\frac{1}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \infty. \end{aligned} \tag{17}$$

Hence, $e_b \in \Xi(r,t)$, for all $b \in \mathrm{N}$.

(1) Assume $|f_b| \le |g_b|$, with $b \in \mathrm{N}$ and $|g| \in \Xi(r,t)$. We get

$$\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r_z |f_z|\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \le \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r_z |g_z|\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \infty, \tag{18}$$

then $|f| \in \Xi(r,t)$.

(2) Suppose $(|f_z|) \in \Xi(r,t)$, with $(t_l), (r_z) \in \mathfrak{I}_Z \cap \ell_\infty$ and there exists $C \ge 1$ with $r_{2z+1} \le C r_z$, one has

$$\begin{aligned}
\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z \left|f_{[z/2]}\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{2l} r_z \left|f_{[z/2]}\right|}{\sum_{z=0}^{2l} r_z}\right)^{t_{2l}} + \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{2l+1} r_z \left|f_{[z/2]}\right|}{\sum_{z=0}^{2l+1} r_z}\right)^{t_{2l+1}} \\
&\le \sum_{l=0}^{\infty}\left(\frac{r_{2l}|f_l| + \sum_{z=0}^{l}(r_{2z}+r_{2z+1})|f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}(r_{2z}+r_{2z+1})|f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\
&\le 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_{2z}|f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l} + \sum_{l=0}^{\infty}\left(\frac{2C\sum_{z=0}^{l} r_z|f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right) \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{2C\sum_{z=0}^{l} r_z\,|f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\
&\le \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)C^{\hbar}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z|f_z|}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \infty,
\end{aligned} \tag{19}$$

then $(|f_{[z/2]}|) \in \Xi(r,t)$.

From Theorem 11, one has the following theorem.

Theorem 31. *If the conditions (f1) and (f2) are confirmed, then $\mathbb{B}^s_{\Xi(r,t)}$ is an operator ideal.*

Theorem 32. *$(\Xi(r,t))_v$ is a premodular $\mathfrak{pss}$, if the conditions (f1) and (f2) are verified.*

Proof.

(i) Clearly, $v(f) \ge 0$ and $v(|f|) = 0 \Leftrightarrow f = \theta$

(ii) One has $E_0 = \max\{1, \sup_l |\rho|^{t_l-1}\} \ge 1$ with $v(\rho f) \le E_0|\rho| v(f)$, for each $f \in \Xi(r,t)$ and $\rho \in \mathscr{C}$

(iii) The inequality $v(f+g) \le 2^{\hbar-1}(v(f)+v(g))$ holds, with $f, g \in \Xi(r,t)$

(iv) Obviously, from the proof part (31) of Theorem 30

(v) Clearly, the proof part (32) of Theorem 30 that $D_0 \ge (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar})C^{\hbar} \ge 1$

(vi) Definitely, $\bar{\mathscr{F}} = \Xi(r,t)$

(vii) There is $0 < \varpi \le \sup_l |\rho|^{t_l-1}$ with $v(\rho,0,0,0,\cdots) \ge \varpi |\rho| v(1,0,0,0,\cdots)$, for each $\rho \ne 0$ and $\varpi > 0$, if $\rho = 0$

Theorem 33. *If the conditions (f1) and (f2) are verified, then $(\Xi(r,t))_v$ is a prequasi Banach $\mathfrak{pss}$.*

Proof. From Theorem 32, the space $(\Xi(r,t))_v$ is a premodular $\mathfrak{pss}$. From Theorem 14, the space $(\Xi(r,t))_v$ is a prequasi normed $\mathfrak{pss}$. To prove that $(\Xi(r,t))_v$ is a prequasi Banach $\mathfrak{pss}$, suppose $f^a = (f^a_z)_{z=0}^{\infty}$ be a Cauchy sequence in $(\Xi(r,t))_v$, then for every $\varepsilon \in (0,1)$, there exists $a_0 \in \mathrm{N}$ such that for every $a, b \ge a_0$, we have

$$v\left(f^a - f^b\right) = \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r_z\left(f^a_z - f^b_z\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \varepsilon^{\hbar}. \tag{20}$$

Therefore, for $a, b \ge a_0$ and $z \in \mathrm{N}$, one has $|f^a_z - f^b_z| < \varepsilon$. Therefore, (f^b_z) is a Cauchy sequence in $\mathscr{C}$; for constant $z \in \mathrm{N}$, this implies $\lim_{b\longrightarrow\infty} f^b_z = f^0_z$, for constant $z \in \mathrm{N}$. Then, $v(f^a - f^0) < \varepsilon^{\hbar}$, for every $a \ge a_0$. Finally, to prove that $f^0 \in (\Xi(r,t))_v$, we obtain $v(f^0) \le 2^{\hbar-1}(v(f^a - f^0) + v(f^a)) < \infty$, hence $f^0 \in (\Xi(r,t))_v$. This gives that $(\Xi(r,t))_v$ is a prequasi Banach $\mathfrak{pss}$.

According to Theorem 23, we explain the following properties of the s-type $(\Xi(r,t))_v$.

Theorem 34. *Assume s -type $(\Xi(r,t))_v := \{f = (s_n(X)) \in \mathscr{C}^N : X \in \mathbb{B}(\mathscr{P}, \mathscr{Q})$ and $v(f) < \infty\}$. The following settings are satisfied:*

(1) We have s-type $(\Xi(r,t))_v \supset \mathscr{F}$

(2) If $(s_r(X_1))_{r=0}^{\infty} \in s$-type $(\Xi(r,t))_v$ and $(s_r(X_2))_{r=0}^{\infty} \in s$-type $(\Xi(r,t))_v$, then $(s_r(X_1 + X_2))_{r=0}^{\infty} \in s$-type $(\Xi(r,t))_v$

(3) For all $\lambda \in \mathscr{C}$ and $(s_r(X))_{r=0}^{\infty} \in s$-type $(\Xi(r,t))_v$, then $|\lambda|(s_r(X))_{r=0}^{\infty} \in s$-type $(\Xi(r,t))_v$

(4) The s-type $(\Xi(r,t))_v$ is solid

5. Multiplication Operators on $(\Xi(r,t))_v$

In this section, we examine the multiplication map on the prequasi normed $\mathfrak{pss}$, $(\Xi(r,t))_v$, and introduce the necessity and sufficient conditions on $(\Xi(r,t))_v$ such that the multiplication operator is bounded, invertible, approximable, Fredholm, and closed range.

Theorem 35. *If $\omega \in \mathscr{C}^N$, the conditions (f1) and (f2) are confirmed, then $\omega \in \ell_\infty$, if and only if, $H_\omega \in \mathbb{B}(\Xi(r,t))_v)$.*

Proof. Assume $\omega \in \ell_\infty$. Therefore, there exists $\nu > 0$ such that $|\omega_b| \le \nu$, for all $b \in \mathrm{N}$. Suppose $f \in (\Xi(r,t))_v$, we have

$$v(H_\omega f) = v(\omega f) = \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r_z \omega_z f_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \le \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} \nu r_z f_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \le \sup_l \nu^{t_l} \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r_z f_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} = \sup_l \nu^{t_l} v(f). \tag{21}$$

Hence, $H_\omega \in \mathbb{B}((\Xi(r,t))_v)$.

On the other hand, suppose $H_\omega \in \mathbb{B}((\Xi(r,t))_v)$ and $\omega \notin \ell_\infty$. Then, for every $b \in N$, there exist $x_b \in N$ such that $\omega_{x_b} > b$. One has

$$v(H_\omega e_{x_b}) = v(\omega e_{x_b}) = \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l} r_z \omega_z (e_{x_b})_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} = \sum_{l=x_b}^{\infty}\left(\frac{r_{x_b}|\omega_{x_b}|}{\sum_{z=0}^{l} r_z}\right)^{t_l} > \sum_{l=x_b}^{\infty}\left(\frac{b r_{x_b}}{\sum_{z=0}^{l} r_z}\right)^{t_l} > b^{t_0} v(e_{x_b}). \tag{22}$$

Therefore, $H_\omega \notin \mathbb{B}((\Xi(r,t))_v)$. Hence, $\omega \in \ell_\infty$.

Theorem 36. *If $\omega \in \mathscr{C}^N$ and $(\Xi(r,t))_v$ is a prequasi normed $\mathfrak{pss}$. Then, $\omega_b = g$, for all $b \in N$ and $g \in \mathscr{C}$ with $|g| = 1$, if and only if, H_ω is an isometry.*

Proof. Assume the sufficient condition is satisfied. We have

$$v(H_\omega f) = v(\omega f) = \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k \omega_k f_k\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} = \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} |g| r_k f_k\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} = v(f), \tag{23}$$

for all $f \in (\Xi(r,t))_v$. Hence, H_ω is an isometry.

Assume the necessity condition is verified and $|\omega_b| < 1$, for some $b = b_0$. One has

$$v(H_\omega e_{b_0}) = v(\omega e_{b_0}) = \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k \omega_k (e_{b_0})_k\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} = \sum_{l=b_0}^{\infty}\left(\frac{r_{b_0}|\omega_{b_0}|}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \sum_{l=b_0}^{\infty}\left(\frac{r_{b_0}}{\sum_{z=0}^{l} r_z}\right)^{t_l} = v(e_{b_0}). \tag{24}$$

Also, if $|\omega_{b_0}| > 1$, clearly $v(H_\omega e_{b_0}) > v(e_{b_0})$, which is a contradiction for the two cases. Hence, $|\omega_b| = 1$, for each $b \in N$.

The space of all sets with finite number of elements will be indicated by $\mathfrak{F}$.

Theorem 37. *If $\omega \in \mathscr{C}^N$, the conditions (f1) and (f2) are verified. Then, $H_\omega \in \mathscr{A}((\Xi(r,t))_v)$, if and only if, $(\omega_b)_{b=0}^{\infty} \in c_0$.*

Proof. Assume $H_\omega \in \mathscr{A}((\Xi(r,t))_v)$, hence $H_\omega \in \mathscr{K}((\Xi(r,t))_v)$. Let $\lim_{b\longrightarrow\infty}\omega_b \neq 0$. Hence, one has $\rho > 0$ so that the set $K_\rho = \{b \in N : |\omega_b| \ge \rho\} \mathfrak{a}\mathfrak{F}$. Suppose $\{\alpha_b\}_{b\in N} \subset K_\rho$. Therefore, $\{e_{\alpha_b} : \alpha_b \in K_\rho\} \in \ell_\infty$ is an infinite set in $(\Xi(r,t))_v$. As

$$v(H_\omega e_{\alpha_a} - H_\omega e_{\alpha_b}) = v(\omega e_{\alpha_a} - \omega e_{\alpha_b}) = \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k \omega_k \left((e_{\alpha_a})_k - (e_{\alpha_b})_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \ge \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k \rho \left((e_{\alpha_a})_k - (e_{\alpha_b})_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \ge \inf_l \rho^{t_l} v(e_{\alpha_a} - e_{\alpha_b}), \tag{25}$$

for all $\alpha_a, \alpha_b \in K_\rho$. Hence, $\{e_{\alpha_b} : \alpha_b \in K_\rho\} \in \ell_\infty$, which cannot have a convergent subsequence under H_ω. Then, $H_\omega \notin \mathscr{K}((\Xi(r,t))_v)$. Which gives $H_\omega \notin \mathscr{A}((\Xi(r,t))_v)$, this implies a contradiction. Therefore, $\lim_{b\longrightarrow\infty}\omega_b = 0$. On the contrary, suppose $\lim_{b\longrightarrow\infty}\omega_b = 0$. Hence, for each $\rho > 0$, we have $K_\rho = \{b \in N : |\omega_b| \ge \rho\} \subset \mathfrak{F}$. Therefore, for all $\rho > 0$, one has dim $(((\Xi(r,t))_v)_{K_\rho}) = $ dim $(\mathscr{C}^{K_\rho}) < \infty$. Then, $H_\omega \in \mathbb{F}(((\Xi(r,t))_v)_{K_\rho})$. Let $\omega_a \in \mathscr{C}^N$, for every $a \in N$, where

$$(\omega_a)_b = \begin{cases} \omega_b, & b \in K_{1/a+1}, \\ 0, & \text{otherwise}. \end{cases} \tag{26}$$

Clearly, $H_{\omega_a} \in \mathbb{F}(((\Xi(r,t))_v)_{B_{1/(a+1)}})$ as dim $(((\Xi(r,t))_v)_{B_{1/(a+1)}}) < \infty$, for every $a \in N$. From $(t_l) \in \mathfrak{I}_Z \cap \ell_\infty$ with $t_0 > 1$, one has

$$\begin{aligned} v((H_\omega - H_{\omega_a})f) &= v\left(\left((\omega_b - (\omega_a)_b)f_b\right)_{b=0}^{\infty}\right) \\ &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b(\omega_b - (\omega_a)_b)f_b\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &= \sum_{l=0, l\in K_{1/a+1}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b(\omega_b - (\omega_a)_b)f_b\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} + \sum_{l=0, l\notin K_{1/a+1}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b(\omega_b - (\omega_a)_b)f_b\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &= \sum_{l=0, l\notin K_{1/a+1}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b \omega_b f_b\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &\le \frac{1}{(a+1)^{t_0}} \sum_{l=0, l\notin K_{1/a+1}}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b f_b\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &< \frac{1}{(a+1)^{t_0}} \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b f_b\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} = \frac{1}{(a+1)^{t_0}} v(f). \end{aligned} \tag{27}$$

Therefore, $\|H_\omega - H_{\omega_a}\| \le (1/(a+1)^{t_0})$. This implies H_ω is a limit of finite rank maps. Hence, $H_\omega \in \mathscr{A}((\Xi(r,t))_v)$.

Theorem 38. *If $\omega \in \mathscr{C}^N$, the conditions (f1) and (f2) are confirmed. Then, $H_\omega \in \mathscr{K}((\Xi(r,t))_v)$, if and only if, $(\omega_b)_{b=0}^{\infty} \in c_0$.*

Proof. Clearly, as $\mathscr{A}((\Xi(r,t))_v) Þ \mathscr{K}((\Xi(r,t))_v)$.

Corollary 39. *If the conditions (f1) and (f2) are verified, then $\mathscr{K}((\Xi(r,t))_v) Þ \mathbb{B}((\Xi(r,t))_v)$.*

Proof. Since $\omega = (1, 1,)$ is created the multiplication map I on $(\Xi(r,t))_v$. This implies $I \notin \mathscr{K}((\Xi(r,t))_v)$ and $I \in \mathbb{B}((\Xi(r,t))_v)$.

Theorem 40. *Pick up $(\Xi(r,t))_v$ be a prequasi Banach 𝔭𝔰𝔰 and $H_\omega \in \mathbb{B}((\Xi(r,t))_v)$. Then, there exist $\alpha > 0$ and $\eta > 0$ so that $\alpha < |\omega_b| < \eta$, with $b \in (\ker(\omega))^c$, if and only if, Range(H_ω) is closed.*

Proof. Let the sufficient setup be verified. Therefore, there exists $\rho > 0$ such that $|\omega_b| \ge \rho$, for each $b \in (\ker(\omega))^c$. To prove that Range(H_ω) is closed. Suppose g is a limit point of Range(H_ω). One has $H_\omega f_b \in (\Xi(r,t))_v$, for all $b \in \mathrm{N}$ such that $\lim_{b\longrightarrow\infty} H_\omega f_b = g$. Clearly, the sequence $H_\omega f_b$ is a Cauchy sequence. Since $(t_l) \in \mathfrak{I}_Z \cap \ell_\infty$ with $t_0 > 1$, we have

$$\begin{aligned} v(H_\omega f_a - H_\omega f_b) &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k\left(\omega_k(f_a)_k - \omega_k(f_b)_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &= \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k\left(\omega_k(f_a)_k - \omega_k(f_b)_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &\quad + \sum_{l=0, l\notin(\ker(\omega))^c}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k\left(\omega_k(f_a)_k - \omega_k(f_b)_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &\ge \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k\left(\omega_k(f_a)_k - \omega_k(f_b)_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k\left(\omega_k(u_a)_k - \omega_k(u_b)_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &> \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{k=0}^{l} r_k \rho\left((u_a)_k - (u_b)_k\right)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &\ge \inf_l \rho^{t_l} v(u_a - u_b), \end{aligned} \tag{28}$$

where

$$(u_a)_k = \begin{cases} (f_a)_k, & k \in (\ker(\omega))^c, \\ 0, & k \notin (\ker(\omega))^c. \end{cases} \tag{29}$$

Therefore, $\{u_a\}$ is a Cauchy sequence in $(\Xi(r,t))_v$. Since $(\Xi(r,t))_v$ is complete. Then, there exists $f \in (\Xi(r,t))_v$ such that $\lim_{b\longrightarrow\infty} u_b = f$. As $H_\omega \in \mathbb{B}((\Xi(r,t))_v)$, one gets $\lim_{b\longrightarrow\infty} H_\omega u_b = H_\omega f$. But $\lim_{b\longrightarrow\infty} H_\omega u_b = \lim_{b\longrightarrow\infty} H_\omega f_b = g$. Hence, $H_\omega f = g$. So $g \in$ Range(H_ω). Then, Range(H_ω) is closed. Next, let the necessity condition be verified. Therefore, there exists $\rho > 0$ such that $v(H_\omega f) \ge \rho v(f)$, with $f \in ((\Xi(r,t))_v)_{(\ker(\omega))^c}$. When $K = \{b \in (\ker(\omega))^c : |\omega_b| < \rho\} \ne \varnothing$, then for $a_0 \in K$, we obtain

$$\begin{aligned} v(H_\omega e_{a_0}) = v\left(\left(\omega_b\left(e_{a_0}\right)_b\right)_{b=0}^{\infty}\right) &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b \omega_b\left(e_{a_0}\right)_b\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &< \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{b=0}^{l} r_b\left(e_{a_0}\right)_b \rho\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \le \sup_l \rho^{t_l} v\left(e_{a_0}\right), \end{aligned} \tag{30}$$

this implies a contradiction. Then, $K = \phi$, one has $|\omega_b| \ge \rho$, with $b \in (\ker(\omega))^c$. This shows the theorem.

Theorem 41. *If $\omega \in \mathscr{C}^N$ and $(\Xi(r,t))_v$ is a prequasi Banach 𝔭𝔰𝔰. Then, there exist $\alpha > 0$ and $\eta > 0$ such that $\alpha < |\omega_b| < \eta$, for all $b \in N$, if and only if, $H_\omega \in \mathbb{B}((\Xi(r,t))_v)$ is invertible.*

Proof. Let the sufficient condition be confirmed. If $\kappa \in \mathscr{C}^{\mathrm{N}}$ with $\kappa_b = 1/\omega_b$. From Theorem 35, the operators H_ω and H_κ are bounded linear. One has $H_\omega . H_\kappa = H_\kappa . H_\omega = I$. Then, $H_\kappa = H_\omega^{-1}$. After, assume H_ω be invertible. Hence, Range(H_ω) $= ((\Xi(r,t))_v)_{\mathrm{N}}$. Therefore, Range($H_\omega$) is closed. Hence, from Theorem 40, there exists $\alpha > 0$ such that $|\omega_b| \ge \alpha$, for all $b \in (\ker(\omega))^c$. One gets $\ker(\omega) = \varnothing$, if $\omega_{b_0} = 0$, with $b_0 \in \mathrm{N}$; this implies $e_{b_0} \in \ker(H_\omega)$ which is an inconsistency, since $\ker(H_\omega)$ is trivial. Hence, $|\omega_b| \ge \alpha$, for all $b \in \mathrm{N}$. As $H_\omega \in \ell_\infty$. By using Theorem 35, there exists $\eta > 0$ such that $|\omega_b| \le \eta$, for all $b \in \mathrm{N}$. Then, one obtains $\alpha \le |\omega_b| \le \eta$, with $b \in \mathrm{N}$.

Theorem 42. *If $(\Xi(r,t))_v$ is a prequasi Banach 𝔭𝔰𝔰 and $H_\omega \in \mathbb{B}((\Xi(r,t))_v)$. Then, H_ω is Fredholm operator, if and only if, (i) $\ker(\omega) \subsetneq N$ is a finite and (ii) $|\omega_b| \ge \rho$, with $b \in (\ker(\omega))^c$.*

Proof. Assume the sufficient setup be confirmed. Suppose $\ker(\omega)$UN be an infinite, then $e_b \in \ker(H_\omega)$, for all $b \in \ker(\omega)$. As e_b's are linearly independent, we have that $\dim(\ker(H_\omega)) = \infty$; this gives an inconsistency. Therefore, $\ker(\omega)$U N must be finite. The setup (ii) follows from Theorem 40. After, assume the setups (i) and (ii) be satisfied. By using Theorem 40, the setup (ii) gives that Range(H_ω) is closed. The condition (i) implies that $\dim(\ker(H_\omega)) < \infty$ and $\dim((\mathrm{Range}(H_\omega))^c) < \infty$. Then, H_ω is Fredholm.

6. Prequasi Ideal Properties

In this section, firstly, we give the sufficient conditions (not necessary) on$(\Xi(r,t))_v$so that$\mathbb{F}$is dense in$\mathbb{B}^s_{(\Xi(r,t))_v}$. This explains a negative answer of Rhoades [26] open problem about the linearity of s-type $(\Xi(r,t))_v$ spaces. Secondly, for which setup on $(\Xi(r,t))_v$ are $\mathbb{B}^s_{(\Xi(r,t))_v}$ complete and closed? Thirdly, we investigate the sufficient conditions on $(\Xi(r,t))_v$ so that $\mathbb{B}^\alpha_{(\Xi(r,t))_v}$ is strictly contained for different weights and powers. We introduce the conditions such that $\mathbb{B}^\alpha_{(\Xi(r,t))_v}$ is minimum. Fourthly, we give the setup such that the Banach prequasi ideal $\mathbb{B}^s_{(\Xi(r,t))_v}$ is simple. Fifthly, we explore the sufficient setup on $(\Xi(r,t))_v$ so that the space of all bounded linear operators which sequence of eigenvalues in $(\Xi(r,t))_v$ equals $\mathbb{B}^s_{(\Xi(r,t))_v}$.

6.1. Finite Rank Prequasi Ideal

Theorem 43. *$\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q}) = \bar{\mathbb{F}(\mathscr{P},\mathscr{Q})}$, if the conditions $(f1)$ and $(f2)$ are confirmed. But the converse is not necessarily true.*

Proof. To prove that $\bar{\mathbb{F}(\mathscr{P},\mathscr{Q})} \subseteq \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q})$. Since $e_l \in (\Xi(r,t))_v$, for all $l \in \mathrm{N}$ and $(\Xi(r,t))_v$ is a linear space. Assume $Z \in \mathbb{F}(\mathscr{P},\mathscr{Q})$, we have $(s_l(Z))_{l=0}^\infty \in \mathscr{F}$. To prove that $\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q}) \subseteq \bar{\mathbb{F}(\mathscr{P},\mathscr{Q})}$. We have $\sum_{l=0}^\infty (1/\sum_{z=0}^l r_z)^{t_l} < \infty$. Suppose $Z \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q})$, one has $(s_l(Z))_{l=0}^\infty \in (\Xi(r,t))_v$. Since $v(s_l(Z))_{l=0}^\infty < \infty$, assume $\rho \in (0,1)$, then there exists $l_0 \in \mathrm{N} - \{0\}$ with $v((s_l(Z))_{l=l_0}^\infty) < \rho/2^{\hbar+3}\eta d$, for some $d \geq 1$, where $\eta = \max\{1, \sum_{l=l_0}^\infty (1/\sum_{z=0}^l r_z)^{t_l}\}$. As $s_l(Z)$ is decreasing, one has

$$\sum_{l=l_0+1}^{2l_0} \left(\frac{\sum_{j=0}^l r_j s_{2l_0}(Z)}{\sum_{z=0}^l r_z}\right)^{t_l} \leq \sum_{l=l_0+1}^{2l_0} \left(\frac{\sum_{j=0}^l r_j s_j(Z)}{\sum_{z=0}^l r_z}\right)^{t_l} \leq \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^l r_j s_j(Z)}{\sum_{z=0}^l r_z}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{31}$$

Then, there exists $Y \in \mathbb{F}_{2l_0}(\mathscr{P},\mathscr{Q})$ such that rank $(Y) \leq 2l_0$ and

$$\sum_{l=2l_0+1}^{3l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} \leq \sum_{l=l_0+1}^{2l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}, \tag{32}$$

as $(t_l) \in \mathfrak{I}_Z \cap \ell_\infty$, one gets

$$\sup_{l=l_0}^\infty \left(\sum_{j=0}^{l_0} r_j \|Z-Y\|\right)^{t_l} < \frac{\rho}{2^{2\hbar+2}\eta}. \tag{33}$$

Hence, we have

$$\sum_{l=0}^{l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{34}$$

From inequalities (4)–(34), we have

$$\begin{aligned}
d(Z,Y) = v(s_l(Z-Y))_{l=0}^\infty &= \sum_{l=0}^{3l_0-1} \left(\frac{\sum_{j=0}^l r_j s_j(Z-Y)}{\sum_{z=0}^l r_z}\right)^{t_l} \\
&+ \sum_{l=3l_0}^{\infty} \left(\frac{\sum_{j=0}^l r_j s_j(Z-Y)}{\sum_{z=0}^l r_z}\right)^{t_l} \leq \sum_{l=0}^{3l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} \\
&+ \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^{l+2l_0} r_j s_j(Z-Y)}{\sum_{j=0}^{l+2l_0} r_j}\right)^{t_{l+2l_0}} \leq \sum_{l=0}^{3l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} \\
&+ \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^{l+2l_0} r_j s_j(Z-Y)}{\sum_{z=0}^l r_z}\right)^{t_l} \leq 3\sum_{l=0}^{l_0} \left(\frac{\sum_{j=0}^i r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} \\
&+ \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^{2l_0-1} r_j s_j(Z-Y) + \sum_{j=2l_0}^{l+2l_0} r_j s_j(Z-Y)}{\sum_{z=0}^l r_z}\right)^{t_l} \\
&\leq 3\sum_{l=0}^{l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} + 2^{\hbar-1}\left[\sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^{2l_0-1} r_j s_j(Z-Y)}{\sum_{z=0}^l r_z}\right)^{t_l}\right. \\
&\left.+ \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=2l_0}^{l+2l_0} r_j s_j(Z-Y)}{\sum_{z=0}^l r_z}\right)^{t_l}\right] \leq 3\sum_{l=0}^{l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} \\
&+ 2^{\hbar-1}\left[\sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^{2l_0-1} r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l}\right. \\
&\left.+ \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^l r_{j+2l_0} s_{j+2l_0}(Z-Y)}{\sum_{z=0}^l r_z}\right)^{t_l}\right] \leq 3\sum_{l=0}^{l_0} \left(\frac{\sum_{j=0}^l r_j \|Z-Y\|}{\sum_{z=0}^l r_z}\right)^{t_l} \\
&+ 2^{\hbar-1} \sup_{l=l_0}^{\infty} \left(\sum_{j=0}^{2l_0-1} r_j \|Z-Y\|\right)^{t_l} \sum_{l=l_0}^{\infty} \left(\sum_{z=0}^l r_z\right)^{-t_l} \\
&+ 2^{\hbar-1} \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^l r_j s_j(Z)}{\sum_{z=0}^l r_z}\right)^{t_l} < \rho.
\end{aligned} \tag{35}$$

Conversely, we have a counter example as $I_4 \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q})$, where $r = (0,0,0,0,1,1,)$ and $t = (1,1,1,)$, but $t_0 > 1$ is not satisfied. This gives the proof.

6.2. Banach and Closed Prequasi Ideal

Theorem 44. *Suppose the conditions $(f1)$ and $(f2)$ be verified, then $(\mathbb{B}^s_{(\Xi(r,t))_v}, \Psi)$ be a prequasi Banach ideal, where $\psi(X) = v((s_l(X))_{l=0}^\infty)$.*

Proof. Since $(\Xi(r,t))_v$ is a premodular $\mathfrak{pss}$, then from Theorem 15, Ψ is a prequasi norm on $\mathbb{B}^s_{(\Xi(r,t))_v}$. Let $(X_b)_{b\in\mathrm{N}}$ be a Cauchy sequence in $\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q})$. Since $\mathbb{B}(\mathscr{P},\mathscr{Q}) \supseteq \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q})$, we have

$$\Psi(X_a - X_b) = \sum_{l=0}^\infty \left(\frac{\sum_{z=0}^l r_z s_z(X_a - X_b)}{\sum_{z=0}^l r_z}\right)^{t_l} \geq \|X_a - X_b\|^{t_0}, \tag{36}$$

then $(X_b)_{b\in\mathrm{N}}$ is a Cauchy sequence in $\mathbb{B}(\mathscr{P},\mathscr{Q})$. As $\mathbb{B}(\mathscr{P},\mathscr{Q})$ is a Banach space, hence there exists $X \in \mathbb{B}(\mathscr{P},\mathscr{Q})$ with $\lim_{b\longrightarrow\infty}\|X_b - X\| = 0$. As $(s_l(X_b))_{l=0}^\infty \in (\Xi(r,t))_v$, for all $b \in$

N. Then, from Definition 12 conditions (ii), (iii), and (v), we obtain

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_z(X)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_{[z/2]}(X-X_b)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &+ 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_{[z/2]}(X_b)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \leq 2^{\hbar-1}\sum_{l=0}^{\infty}\|X-X_b\|^{t_l} \\ &+ 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_z(X_b)}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \infty.\end{aligned} \tag{37}$$

Then, $(s_l(X))_{l=0}^{\infty} \in (\Xi(r,t))_{\upsilon}$, hence $X \in \mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}(\mathscr{P},\mathscr{Q})$.

Theorem 45. *If $\mathscr{P}$ and $\mathscr{Q}$ are normed spaces, the conditions $(f1)$ and $(f2)$ are confirmed; then, $(\mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}},\Psi)$ is a prequasi closed ideal, where $\Psi(X)=\upsilon((s_l(X))_{l=0}^{\infty})$.*

Proof. Since $(\Xi(r,t))_{\upsilon}$ is a premodular 𝔭𝔰𝔰, from Theorem 15, Ψ is a prequasi norm on $\mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}$. Let $X_b \in \mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}(\mathscr{P},\mathscr{Q})$, for all $b\in\mathbb{N}$ and $\lim_{b\longrightarrow\infty}\Psi(X_b-X)=0$. Since $\mathbb{B}(\mathscr{P},\mathscr{Q}) \supseteq \mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}(\mathscr{P},\mathscr{Q})$, one gets

$$\Psi(X-X_b)=\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_z(X-X_b)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \geq \|X-X_b\|^{t_0}, \tag{38}$$

then $(X_b)_{b\in\mathbb{N}}$ is a convergent sequence in $\mathbb{B}(\mathscr{P},\mathscr{Q})$. As $(s_l(X_b))_{l=0}^{\infty}\in(\Xi(r,t))_{\upsilon}$, for all $b\in\mathbb{N}$. From Definition 12 conditions (ii), (iii), and (v), we have

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_z(X)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_{[z/2]}(X-X_b)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &+ 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_{[z/2]}(X_b)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \leq 2^{\hbar-1}\sum_{l=0}^{\infty}\|X-X_b\|^{t_l} \\ &+ 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_z(X_b)}{\sum_{z=0}^{l} r_z}\right)^{t_l} < \infty.\end{aligned} \tag{39}$$

We get $(s_l(X))_{l=0}^{\infty}\in(\Xi(r,t))_{\upsilon}$, so $X\in\mathbb{B}^{s}_{(\Xi(r,t))_{\upsilon}}(\mathscr{P},\mathscr{Q})$.

6.3. Minimum Prequasi Ideal

Theorem 46. *If $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces with* dim $(\mathscr{P})$ = dim $(\mathscr{Q})=\infty$ *and the conditions $(f1)$ and $(f2)$ are verified with $1<t_l^{(1)}<t_l^{(2)}$ and $0<(r_l^{(2)}/\sum_{z=0}^{l} r_z^{(2)})\leq(r_l^{(1)}/\sum_{z=0}^{l} r_z^{(1)})$, for every $l\in N$, then*

$$\mathbb{B}^{s}_{\left(\Xi\left(\left(r_l^{(1)}\right),\left(t_l^{(1)}\right)\right)\right)_{\upsilon}}(\mathscr{P},\mathscr{Q})\text{Þ}\mathbb{B}^{s}_{\left(\Xi\left(\left(r_l^{(2)}\right),\left(t_l^{(2)}\right)\right)\right)_{\upsilon}}(\mathscr{P},\mathscr{Q})\mathscr{U}\mathbb{B}(\mathscr{P},\mathscr{Q}). \tag{40}$$

Proof. Assume $Z\in\mathbb{B}^{s}_{(\Xi((r_l^{(1)}),(t_l^{(1)})))_{\upsilon}}(\mathscr{P},\mathscr{Q})$, then $(s_l(Z))\in(\Xi((r_l^{(1)}),(t_l^{(1)})))_{\upsilon}$. We have

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z^{(2)} s_z(Z)}{\sum_{z=0}^{l} r_z^{(2)}}\right)^{t_l^{(2)}} < \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z^{(1)} s_z(Z)}{\sum_{z=0}^{l} r_z^{(1)}}\right)^{t_l^{(1)}} < \infty, \tag{41}$$

then $Z\in\mathbb{B}^{s}_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_{\upsilon}}(\mathscr{P},\mathscr{Q})$. Next, if we take $(s_l(Z))_{l=0}^{\infty}$ with $\sum_{z=0}^{l} r_z^{(1)} s_z(Z)=\sum_{z=0}^{l} r_z^{(1)}/\sqrt[tl1]{l+1}$, one obtains $Z\in\mathbb{B}(\mathscr{P},\mathscr{Q})$ so that

$$\begin{aligned}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z^{(1)} s_z(Z)}{\sum_{z=0}^{l} r_z^{(1)}}\right)^{t_l^{(1)}} &= \sum_{l=0}^{\infty}\frac{1}{l+1}=\infty, \\ \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z^{(2)} s_z(Z)}{\sum_{z=0}^{l} r_z^{(2)}}\right)^{t_l^{(2)}} &\leq \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z^{(1)} s_z(Z)}{\sum_{z=0}^{l} r_z^{(1)}}\right)^{t_l^{(2)}} \\ &= \sum_{l=0}^{\infty}\left(\frac{1}{l+1}\right)^{t_l^{(2)}/t_l^{(1)}} < \infty.\end{aligned} \tag{42}$$

Hence, $Z\notin\mathbb{B}^{s}_{(\Xi((r_l^{(1)}),(t_l^{(1)})))_{\upsilon}}(\mathscr{P},\mathscr{Q})$ and $Z\in\mathbb{B}^{s}_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_{\upsilon}}(\mathscr{P},\mathscr{Q})$. Obviously, $\mathbb{B}^{s}_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_{\upsilon}}(\mathscr{P},\mathscr{Q})\subset\mathbb{B}(\mathscr{P},\mathscr{Q})$. After, if we choose $(s_l(Z))_{l=0}^{\infty}$ so that $\sum_{z=0}^{l} r_z^{(2)} s_z(Z)=\sum_{z=0}^{l} r_z^{(2)}/\sqrt[tl2]{l+1}$. One gets $Z\in\mathbb{B}(\mathscr{P},\mathscr{Q})$ so that $Z\notin\mathbb{B}^{s}_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_{\upsilon}}(\mathscr{P},\mathscr{Q})$. This completes the proof.

Theorem 47. *If $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces with* dim $(\mathscr{P})$ = dim $(\mathscr{Q})=\infty$ *and the conditions $(f1)$ and $(f2)$ are verified, then $\mathbb{B}^{\alpha}_{(\Xi(r,t))_{\upsilon}}$ is minimum.*

Proof. Let the sufficient conditions be verified. Hence, $(\mathbb{B}^{\alpha}_{\Xi(r,t)},\Psi)$, where $\Psi(Z)=\sum_{l=0}^{\infty}((1/\sum_{z=0}^{l} r_z)\sum_{z=0}^{l} r_z\alpha_z(Z))^{t_l}$, is a prequasi Banach ideal. Assume $\mathbb{B}^{\alpha}_{\Xi(r,t)}(\mathscr{P},\mathscr{Q})=\mathbb{B}(\mathscr{P},\mathscr{Q})$, then there exists $\eta>0$ with $\Psi(Z)\leq\eta\|Z\|$, for all $Z\in\mathbb{B}(\mathscr{P},\mathscr{Q})$. From Dvoretzky's theorem [32], for all $b\in\mathbb{N}$, we have quotient spaces $\mathscr{P}/Y_b$ and subspaces M_b of $\mathscr{Q}$ which can be operated onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\|\leq 2$ and $\|X_b\|\|X_b^{-1}\|\leq 2$. If I_b is the identity map on ℓ_2^b, T_b is the quotient map from $\mathscr{P}$ to $\mathscr{P}/Y_b$, and J_b is the natural embedding map from M_b to $\mathscr{Q}$. Assume m_z be the Bernstein numbers [33] hence

$$\begin{aligned}1=m_z(I_b)&=m_z\left(X_bX_b^{-1}I_bV_bV_b^{-1}\right)\leq\|X_b\|m_z\left(X_b^{-1}I_bV_b\right)\left\|V_b^{-1}\right\| \\ &=\|X_b\|m_z\left(J_bX_b^{-1}I_bV_b\right)\left\|V_b^{-1}\right\|\leq\|X_b\|d_z\left(J_bX_b^{-1}I_bV_b\right)\left\|V_b^{-1}\right\| \\ &=\|X_b\|d_z\left(J_bX_b^{-1}I_bV_bT_b\right)\left\|V_b^{-1}\right\|\leq\|X_b\|\alpha_z\left(J_bX_b^{-1}I_bV_bT_b\right)\left\|V_b^{-1}\right\|,\end{aligned} \tag{43}$$

for $0 \le l \le b$. One gets

$$\sum_{z=0}^{l} r_z \le \sum_{z=0}^{l} \|X_b\| r_z \alpha_z \left(J_b X_b^{-1} I_b V_b T_b\right) \left\|V_b^{-1}\right\| \Rightarrow 1 \le \left(\|X_b\| \left\|V_b^{-1}\right\|\right)^{t_l} \left(\frac{\sum_{z=0}^{l} r_z \alpha_z \left(J_b X_b^{-1} I_b V_b T_b\right)}{\sum_{z=0}^{l} r_z}\right)^{t_l}. \quad (44)$$

Therefore, for some $\rho \ge 1$, we have

$$\begin{aligned} b+1 &\le \rho \|X_b\| \left\|V_b^{-1}\right\| \sum_{l=0}^{b} \left(\frac{\sum_{z=0}^{l} r_z \alpha_z \left(J_b X_b^{-1} I_b V_b T_b\right)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \Rightarrow b \\ &+1 \le \rho \|X_b\| \left\|V_b^{-1}\right\| \Psi\left(J_b X_b^{-1} I_b V_b T_b\right) \Rightarrow b+1 \\ &\le \rho\eta \|X_b\| \left\|V_b^{-1}\right\| \left\|J_b X_b^{-1} I_b V_b T_b\right\| \Rightarrow b+1 \\ &\le \rho\eta \|X_b\| \left\|V_b^{-1}\right\| \left\|J_b X_b^{-1} I_b V_b T_b\right\| \Rightarrow b+1 \\ &\le \rho\eta \|X_b\| \left\|V_b^{-1}\right\| \left\|J_b X_b^{-1}\right\| \|I_b\| \|V_b T_b\| \\ &= \rho\eta \|X_b\| \left\|V_b^{-1}\right\| \left\|X_b^{-1}\right\| \|I_b\| \|V_b\| \le 4\rho\eta. \end{aligned} \quad (45)$$

Hence, there is an inconsistency, when $b \longrightarrow \infty$. Hence, $\mathscr{P}$ and $\mathscr{Q}$ both cannot be infinite dimensional when $\mathbb{B}^{\alpha}_{\Xi(r,t)}(\mathscr{P}, \mathscr{Q}) = \mathbb{B}(\mathscr{P}, \mathscr{Q})$. This finishes the proof.

Theorem 48. *If $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces with* $\dim(\mathscr{P}) = \dim(\mathscr{Q}) = \infty$ *and the conditions $(f1)$ and $(f2)$ are verified, then $\mathbb{B}^{d}_{\Xi(r,t)}$ is minimum.*

6.4. Simple Banach Prequasi Ideal

Theorem 49. *If $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces with* $\dim(\mathscr{P}) = \dim(\mathscr{Q}) = \infty$ *and the conditions $(f1)$ and $(f2)$ are verified with $1 < t_l^{(1)} < t_l^{(2)}$ and $0 < (r_l^{(2)}/\sum_{z=0}^{l} r_z^{(2)}) \le (r_l^{(1)}/\sum_{z=0}^{l} r_z^{(1)})$, for all $l \in \mathrm{N}$, then*

$$\mathbb{B}\left(\mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(2)}\right),\left(t_l^{(2)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q}), \mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(1)}\right),\left(t_l^{(1)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q})\right) = \mathscr{A}\left(\mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(2)}\right),\left(t_l^{(2)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q}), \mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(1)}\right),\left(t_l^{(1)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q})\right). \quad (46)$$

Proof. Suppose $X \in \mathbb{B}(\mathbb{B}^s_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_v}(\mathscr{P}, \mathscr{Q}), \mathbb{B}^s_{(\Xi((r_l^{(1)}),(t_l^{(1)})))_v}(\mathscr{P}, \mathscr{Q}))$ and $X \notin \mathscr{A}(\mathbb{B}^s_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_v}(\mathscr{P}, \mathscr{Q}), \mathbb{B}^s_{(\Xi((r_l^{(1)}),(t_l^{(1)})))_v}(\mathscr{P}, \mathscr{Q}))$. From Lemma 3, one has $Y \in \mathbb{B}(\mathbb{B}^s_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_v}(\mathscr{P}, \mathscr{Q}))$ and $Z \in \mathbb{B}(\mathbb{B}^s_{(\Xi((r_l^{(1)}),(t_l^{(1)})))_v}(\mathscr{P}, \mathscr{Q}))$ with $ZXYI_b = I_b$. Then, for all $b \in \mathrm{N}$, one has

$$\begin{aligned} \|I_b\|_{\mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(1)}\right),\left(t_l^{(1)}\right)\right)\right)_v}(\mathscr{P},\mathscr{Q})} &= \sum_{l=0}^{\infty} \left(\frac{\sum_{z=0}^{l} r_z^{(1)} s_z(I_b)}{\sum_{z=0}^{l} r_z^{(1)}}\right)^{t_l^{(1)}} \\ &\le \|ZXY\| \|I_b\|_{\mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(2)}\right),\left(t_l^{(2)}\right)\right)\right)_v}(\mathscr{P},\mathscr{Q})} \le \sum_{l=0}^{\infty} \left(\frac{\sum_{z=0}^{i} r_z^{(2)} s_z(I_j)}{\sum_{z=0}^{l} r_z^{(2)}}\right)^{t_l^{(2)}}. \end{aligned} \quad (47)$$

This contradicts Theorem 46. Hence, $X \in \mathscr{A}(\mathbb{B}^s_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_v}(\mathscr{P}, \mathscr{Q}), \mathbb{B}^s_{(\Xi((r_l^{(1)}),(t_l^{(1)})))_v}(\mathscr{P}, \mathscr{Q}))$, which completes the proof.

Corollary 50. *If $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces with* $\dim(\mathscr{P}) = \dim(\mathscr{Q}) = \infty$ *and the conditions $(f1)$ and $(f2)$ are verified with $1 < t_l^{(1)} < t_l^{(2)}$ and $0 < (r_l^{(2)}/\sum_{z=0}^{l} r_z^{(2)}) \le (r_l^{(1)}/\sum_{z=0}^{l} r_z^{(1)})$, for all $l \in \mathrm{N}$, then*

$$\mathbb{B}\left(\mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(2)}\right),\left(t_l^{(2)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q}), \mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(1)}\right),\left(t_l^{(1)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q})\right) = \mathscr{K}\left(\mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(2)}\right),\left(t_l^{(2)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q}), \mathbb{B}^s_{\left(\Xi\left(\left(r_l^{(1)}\right),\left(t_l^{(1)}\right)\right)\right)_v}(\mathscr{P}, \mathscr{Q})\right). \quad (48)$$

Proof. Obviously, since $\mathscr{A} \subset \mathscr{K}$.

Theorem 51. *If $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces with* $\dim(\mathscr{P}) = \dim(\mathscr{Q}) = \infty$ *and the conditions $(f1)$ and $(f2)$ are confirmed, then $\mathbb{B}^s_{(\Xi(r,t))_v}$ is simple.*

Proof. Suppose the closed ideal $\mathscr{K}(\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q}))$ contains an operator $X \notin \mathscr{A}(\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q}))$. By using Lemma 3, there are $Y, Z \in \mathbb{B}(\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q}))$ with $ZXYI_b = I_b$. This implies that $I_{\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P},\mathscr{Q})} \in \mathscr{K}(\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q}))$. Hence, $\mathbb{B}(\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})) = \mathscr{K}(\mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q}))$. Therefore, $\mathbb{B}^s_{(\Xi(r,t))_v}$ is simple Banach space.

6.5. Eigenvalues of s-Type Operators

Notation 52.

$$(\mathbb{B}^s_{\mathscr{V}})^{\rho} := \{(\mathbb{B}^s_{\mathscr{V}})^{\rho}(\mathscr{P}, \mathscr{Q})\,;\, \mathscr{P} \text{ and } \mathscr{Q} \text{ are Banach Spaces}\}, \text{ where}$$

$$(\mathbb{B}^s_{\mathscr{V}})^{\rho}(\mathscr{P}, \mathscr{Q}) := \left\{X \in \mathbb{B}(\mathscr{P}, \mathscr{Q}) \colon \left((\rho_l(X))_{n=0}^{\infty} \in \mathscr{V} \text{ and } \|X - \rho_l(X) I\| \text{ is notinvertible, for all } l \in \mathrm{N}\right\}. \quad (49)$$

Theorem 53. *If $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces with* $\dim(\mathscr{P}) = \dim(\mathscr{Q}) = \infty$ *and the conditions $(f1)$ and $(f2)$ are verified, then*

$$\left(\mathbb{B}^s_{(\Xi(r,t))_v}\right)^{\rho}(\mathscr{P}, \mathscr{Q}) = \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q}). \quad (50)$$

Proof. Assume $X \in (\mathbb{B}^s_{(\Xi(r,t))_v})^{\rho}(\mathscr{P}, \mathscr{Q})$, then $(\rho_l(X))_{l=0}^{\infty} \in (\Xi(r,t))_v$ and $\|X - \rho_l(X)I\| = 0$, for every $l \in \mathrm{N}$. One gets $X = \rho_l(X)I$, for each $l \in \mathrm{N}$, then $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$, for all $l \in \mathrm{N}$. Hence, $(s_l(X))_{l=0}^{\infty} \in (\Xi(r,t))_{\upsilon}$, so $X \in \mathbb{B}^s_{(\Xi(r,t))_\upsilon}(\mathscr{P},\mathscr{Q})$.

Secondly, assume $X \in \mathbb{B}^s_{(\Xi(r,t))_\upsilon}(\mathscr{P},\mathscr{Q})$. Hence, $(s_l(X))_{l=0}^{\infty} \in (\Xi(r,t))_{\upsilon}$. Therefore, one can see

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_z s_z(X)}{\sum_{z=0}^{l} r_z}\right)^{t_l} \geq \sum_{l=0}^{\infty}[s_l(X)]^{t_l}. \tag{51}$$

Hence, $\lim_{l\longrightarrow\infty} s_l(X)=0$. Suppose $\|X-s_l(X)I\|^{-1}$ exists, for all $l \in \mathrm{N}$. Then, $\|X-s_l(X)I\|^{-1}$ exists and bounded, for all $l \in \mathrm{N}$. Hence, $\lim_{l\longrightarrow\infty}\|X-s_l(X)I\|^{-1}=\|X\|^{-1}$ exists and bounded. Since $(\mathbb{B}^s_{(\Xi(r,t))_\upsilon},\Psi)$ is a prequasi operator ideal, one obtains

$$I = XX^{-1} \in \mathbb{B}^s_{(\Xi(r,t))_\upsilon}(\mathscr{P},\mathscr{Q}) \Rightarrow (s_l(I))_{l=0}^{\infty} \in \Xi(r,t) \Rightarrow \lim_{l\longrightarrow\infty} s_l(I)=0. \tag{52}$$

Therefore, there is a contradiction, as $\lim_{l\longrightarrow\infty} s_l(I)=1$. Then, $\|X-s_l(X)I\|=0$, for all $l \in \mathrm{N}$. This implies $X \in (\mathbb{B}^s_{(\Xi(r,t))_\upsilon})^{\rho}(\mathscr{P},\mathscr{Q})$. This confirms the proof.

7. Kannan Contraction Operator

Theorem 54. *The function $\upsilon(f)=[\sum_{l=0}^{\infty}(|\sum_{z=0}^{l} r_z f_z|/\sum_{z=0}^{l} r_z)^{t_l}]^{1/\hbar}$ confirms the Fatou property, for every $f \in \Xi(r,t)$, if the conditions $(f1)$ and $(f2)$ are verified.*

Proof. Let $\{g^b\} \subseteq (\Xi(r,t))_\upsilon$ with $\lim_{b\longrightarrow\infty}\upsilon(g^b-g)=0$. Since the space $(\Xi(r,t))_\upsilon$ is a prequasi closed space, hence $g \in (\Xi(r,t))_\upsilon$. Therefore, for each $f \in (\Xi(r,t))_\upsilon$, one has

$$\begin{aligned}\upsilon(f-g) &= \left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(f_z-g_z)r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\leq \left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(f_z-g_z^b)r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\quad + \left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(g_z^b-g_z)r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\leq \sup_j \inf_{b\geq j}\upsilon\left(f-g^b\right).\end{aligned} \tag{53}$$

Theorem 55. *The function $\upsilon(f)=\sum_{l=0}^{\infty}(|\sum_{z=0}^{l} r_z f_z|/\sum_{z=0}^{l} r_z)^{t_l}$ does not satisfy the Fatou property, for every $f \in \Xi(r,t)$, if the conditions $(f1)$ and $(f2)$ are verified.*

Proof. Let $\{g^b\} \subseteq (\Xi(r,t))_\upsilon$ with $\lim_{b\longrightarrow\infty}\upsilon(g^b-g)=0$. Since the space $(\Xi(r,t))_\upsilon$ is a prequasi closed space, hence $g \in (\Xi(r,t))_\upsilon$. Therefore, for each $f \in (\Xi(r,t))_\upsilon$, one gets

$$\begin{aligned}\upsilon(f-g) &= \sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(f_z-g_z)r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l} \\ &\leq 2^{\hbar-1}\left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(f_z-g_z^b)r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right. \\ &\quad \left.+\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(g_z^b-g_z)r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right] \\ &\leq 2^{\hbar-1}\sup_j \inf_{b\geq j}\upsilon\left(f-g^b\right).\end{aligned} \tag{54}$$

Hence, υ does not satisfy the Fatou property.

Now, we give the sufficient conditions on $(\Xi(r,t))_\upsilon$ equipped with definite prequasi norm such that there exists a unique fixed point of Kannan contraction operator.

Theorem 56. *If the conditions $(f1)$ and $(f2)$ are verified, and $W:(\Xi(r,t))_\upsilon \longrightarrow (\Xi(r,t))_\upsilon$ is Kannan υ-contraction operator, where $\upsilon(f)=[\sum_{l=0}^{\infty}(|\sum_{z=0}^{l} r_z f_z|/\sum_{z=0}^{l} r_z)^{t_l}]^{1/\hbar}$, for all $f \in \Xi(r,t)$, hence W has a unique fixed point.*

Proof. Assume $f \in \Xi(r,t)$, hence $W^p f \in \Xi(r,t)$. As W is a Kannan υ-contraction operator, one has

$$\begin{aligned}\upsilon(W^{p+1}f-W^p f) &\leq \lambda\left(\upsilon\left(W^{p+1}f-W^p f\right)+\upsilon\left(W^p f-W^{p-1}f\right)\right) \\ &\Rightarrow \upsilon\left(W^{p+1}f-W^p f\right) \leq \frac{\lambda}{1-\lambda}\upsilon\left(W^p f-W^{p-1}f\right) \\ &\leq \left(\frac{\lambda}{1-\lambda}\right)^2\upsilon\left(W^{p-1}f-W^{p-2}f\right) \\ &\leq \leq \left(\frac{\lambda}{1-\lambda}\right)^p\upsilon(Wf-f).\end{aligned} \tag{55}$$

Hence, for all $p,q \in \mathrm{N}$ with $q>p$, one can see

$$\begin{aligned}\upsilon(W^p f-W^q f) &\leq \lambda\left(\upsilon\left(W^p f-W^{p-1}f\right)+\upsilon\left(W^q f-W^{q-1}f\right)\right) \\ &\leq \lambda\left(\left(\frac{\lambda}{1-\lambda}\right)^{p-1}+\left(\frac{\lambda}{1-\lambda}\right)^{q-1}\right)\upsilon(Wf-f).\end{aligned} \tag{56}$$

Therefore, $\{W^p f\}$ is a Cauchy sequence in $(\Xi(r,t))_\upsilon$. As the space $(\Xi(r,t))_\upsilon$ is prequasi Banach space. Hence, there is $g \in (\Xi(r,t))_\upsilon$ such that $\lim_{p\longrightarrow\infty} W^p f = g$. To prove that $W g = g$. Since υ has the Fatou property, one has

$$\begin{aligned}\upsilon(Wg-g) &\leq \sup_i \inf_{p\geq i}\upsilon\left(W^{p+1}f-W^p f\right) \\ &\leq \sup_i \inf_{p\geq i}\left(\frac{\lambda}{1-\lambda}\right)^p\upsilon(W\mathrm{f}-f)=0,\end{aligned} \tag{57}$$

so $Wg=g$. Therefore, g is a fixed point of W. To show that the fixed point is unique. Let we have two different fixed points $b, g \in (\Xi(r,t))_\upsilon$ of W. Hence, we have

$$\upsilon(b-g) \le \upsilon(Wb - Wg) \le \xi(\upsilon(Wb-b) + \upsilon(Wg-g)) = 0. \tag{58}$$

Therefore, $b = g$.

Corollary 57. *If the conditions $(f1)$ and $(f2)$ are verified, and $W : (\Xi(r,t))_\upsilon \longrightarrow (\Xi(r,t))_\upsilon$ is Kannan υ-contraction operator, where $\upsilon(f) = [\sum_{l=0}^{\infty}(|\sum_{z=0}^{l} r_z f_z| / \sum_{z=0}^{l} r_z)^{t_l}]^{1/h}$, for every $f \in \Xi(r,t)$, then W has one and only one fixed point b with $\upsilon(W^p f - b) \le \lambda(\lambda/(1-\lambda))^{p-1}\upsilon(Wf - f)$.*

Proof. From Theorem 56, there exists a unique fixed point b of W. Hence, we have

$$\begin{aligned} \upsilon(W^p f - b) &= \upsilon(W^p f - Wb) \\ &\le \lambda\left(\upsilon\left(W^p f - W^{p-1} f\right) + \upsilon(Wb - b)\right) \\ &= \lambda\left(\frac{\lambda}{1-\lambda}\right)^{p-1}\upsilon(Wf - f). \end{aligned} \tag{59}$$

Theorem 58. *If the conditions $(f1)$ and $(f2)$ are verified, and $W : (\Xi(r,t))_\upsilon \longrightarrow (\Xi(r,t))_\upsilon$, where $\upsilon(f) = \sum_{l=0}^{\infty}(|\sum_{z=0}^{l} r_z f_z| / \sum_{z=0}^{l} r_z)^{t_l}$, for every $f \in \Xi(r,t)$. The point $g \in (\Xi(r,t))_\upsilon$ is the only fixed point of W, if the following conditions are confirmed:*

(a) W is Kannan υ-contraction operator

(b) W is υ-sequentially continuous at $g \in (\Xi(r,t))_\upsilon$

(c) One has $v \in (\Xi(r,t))_\upsilon$ so that the sequence of iterates $\{W^p v\}$ has a subsequence $\{W^{p_i} v\}$ converging to g

Proof. Let the sufficient conditions be verified. Assume g be not a fixed point of W, hence $Wg \ne g$. From the conditions (b) and (c), we have

$$\lim_{p_i \longrightarrow \infty} \upsilon(W^{p_i} f - g) = 0 \text{ and } \lim_{p_i \longrightarrow \infty} \upsilon\left(W^{p_i+1} f - Wg\right) = 0. \tag{60}$$

As the operator W is Kannan υ-contraction, one gets

$$\begin{aligned} 0 < \upsilon(Wg - g) &= \upsilon\left(\left(Wg - W^{p_i+1} f\right) + \left(W^{p_i} f - g\right) \right.\\ &\quad \left.+ \left(W^{p_i+1} f - W^{p_i} f\right)\right) \le 2^{2h-2}\upsilon\left(W^{p_i+1} v - Wg\right) \\ &\quad + 2^{2h-2}\upsilon(W^{p_i} v - g) + 2^{h-1}\lambda\left(\frac{\lambda}{1-\lambda}\right)^{p_i-1}\upsilon(Wf - f). \end{aligned} \tag{61}$$

As $p_i \longrightarrow \infty$, we have a contradiction. Therefore, g is a fixed point of W. To prove that the fixed point g is unique. Let we have two different fixed points $g, b \in (\Xi(r,t))_\upsilon$ of W. Then, one obtains

$$\upsilon(g-b) \le \upsilon(Wg - Wb) \le \lambda(\upsilon(Wg - g) + \upsilon(Wb - b)) = 0. \tag{62}$$

Hence, $g = b$.

Example 59. Assume $T : (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon \longrightarrow (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$, where $\upsilon(p) = \sqrt{\sum_{t=0}^{\infty}}\ (|\sum_{x=0}^{t}((x+2)/(x+1)) p_x| / \sum_{x=0}^{t}((x+2)/(x+1)))^{(2t+3)/(t+2)}$, with $p \in \Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty})$ and

$$T(p) = \begin{cases} \dfrac{p}{4}, & \upsilon(p) \in [0,1), \\ \dfrac{p}{5}, & \upsilon(p) \in [1,\infty). \end{cases} \tag{63}$$

As for each $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\upsilon(p), \upsilon(q) \in [0,1)$, one gets

$$\begin{aligned} \upsilon(Tp - Tq) &= \upsilon\left(\frac{p}{4} - \frac{q}{4}\right) \le \frac{1}{\sqrt[4]{27}}\left(\upsilon\left(\frac{3p}{4}\right) + \upsilon\left(\frac{3q}{4}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(\upsilon(Tp - p) + \upsilon(Tq - q)). \end{aligned} \tag{64}$$

For every $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\upsilon(p), \upsilon(q) \in [1,\infty)$, one can see

$$\begin{aligned} \upsilon(Tp - Tq) &= \upsilon\left(\frac{p}{5} - \frac{q}{5}\right) \le \frac{1}{\sqrt[4]{64}}\left(\upsilon\left(\frac{4p}{5}\right) + \upsilon\left(\frac{4q}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{64}}(\upsilon(Tp - p) + \upsilon(Tq - q)). \end{aligned} \tag{65}$$

For all $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\upsilon(p) \in [0,1)$ and $\upsilon(q) \in [1,\infty)$, one has

$$\begin{aligned} \upsilon(Tp - Tq) &= \upsilon\left(\frac{p}{4} - \frac{q}{5}\right) \le \frac{1}{\sqrt[4]{27}}\upsilon\left(\frac{3p}{4}\right) + \frac{1}{\sqrt[4]{64}}\upsilon\left(\frac{4q}{5}\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\upsilon\left(\frac{3p}{4}\right) + \upsilon\left(\frac{4q}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(\upsilon(Tp - p) + \upsilon(Tq - q)). \end{aligned} \tag{66}$$

Hence, the operator T is Kannan υ-contraction. As υ verifies the Fatou property. From Theorem 56, the operator T has a unique fixed point $\theta \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$.

Assume $\{p^{(a)}\} \subseteq (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\lim_{a\longrightarrow\infty}\upsilon(p^{(a)} - p^{(0)}) = 0$, where $p^{(0)} \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\upsilon(p^{(0)}) = 1$. As the prequasi norm υ is continuous, one has

$$\lim_{a\longrightarrow\infty} \upsilon\left(Tp^{(a)} - Tp^{(0)}\right) = \lim_{a\longrightarrow\infty} \upsilon\left(\frac{p^{(a)}}{4} - \frac{p^{(0)}}{5}\right) = \upsilon\left(\frac{p^{(0)}}{20}\right) > 0. \tag{67}$$

Therefore, T is not υ-sequentially continuous at $p^{(0)}$. Hence, the operator T is not continuous at $p^{(0)}$.

Assume $\upsilon(p) = \sum_{t=0}^{\infty}(|\sum_{x=0}^{t}((x+2)/(x+1))p_x|/\sum_{x=0}^{t}((x+2)/(x+1)))^{(2t+3)/(t+2)}$, for all $p \in \Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty})$.

As for each $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\upsilon(p), \upsilon(q) \in [0, 1)$, one obtains

$$\begin{aligned}\upsilon(Tp - Tq) &= \upsilon\left(\frac{p}{4} - \frac{q}{4}\right) \le \frac{2}{\sqrt{27}}\left(\upsilon\left(\frac{3p}{4}\right) + \upsilon\left(\frac{3q}{4}\right)\right) \\ &= \frac{2}{\sqrt{27}}(\upsilon(Tp - p) + \upsilon(Tq - q)).\end{aligned} \tag{68}$$

If $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\upsilon(p), \upsilon(q) \in [1,\infty)$, one has

$$\begin{aligned}\upsilon(Tp - Tq) &= \upsilon\left(\frac{p}{5} - \frac{q}{5}\right) \le \frac{1}{4}\left(\upsilon\left(\frac{4p}{5}\right) + \upsilon\left(\frac{4q}{5}\right)\right) \\ &= \frac{1}{4}(\upsilon(Tp - p) + \upsilon(Tq - q)).\end{aligned} \tag{69}$$

For all $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $\upsilon(p) \in [0, 1)$ and $\upsilon(q) \in [1,\infty)$, one gets

$$\begin{aligned}\upsilon(Tp - Tq) &= \upsilon\left(\frac{p}{4} - \frac{q}{5}\right) \le \frac{2}{\sqrt{27}}\upsilon\left(\frac{3p}{4}\right) + \frac{1}{4}\upsilon\left(\frac{4q}{5}\right) \\ &\le \frac{2}{\sqrt{27}}\left(\upsilon\left(\frac{3p}{4}\right) + \upsilon\left(\frac{4q}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}(\upsilon(Tp - p) + \upsilon(Tq - q)).\end{aligned} \tag{70}$$

Hence, the operator T is Kannan υ-contraction, and $T^r(p) = \begin{cases} p/4^r, & \upsilon(p) \in [0, 1), \\ p/5^r, & \upsilon(p) \in [1,\infty). \end{cases}$

Obviously, T is υ-sequentially continuous at $\theta \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$, and $\{T^r p\}$ has a subsequence $\{T^{r_j} p\}$ converging to θ. From Theorem 58, the vector $\theta \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ is the only fixed point of T.

Example 60. Suppose $T : (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon \longrightarrow (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$, with $\upsilon(p) = \sum_{t=0}^{\infty}(|\sum_{x=0}^{t}((x+2)/(x+1))p_x|/\sum_{x=0}^{t}((x+2)/(x+1)))^{(2t+3)/(t+2)}$, for every $p \in \Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty})$ and

$$T(p) = \begin{cases} \frac{1}{4}(e_1 + p), & p_0 \in \left(-\infty, \frac{1}{3}\right), \\ \frac{1}{3}e_1, & p_0 = \frac{1}{3}, \\ \frac{1}{4}e_1, & p_0 \in \left(\frac{1}{3},\infty\right). \end{cases} \tag{71}$$

As for every $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $p_0, q_0 \in (-\infty,1/3)$, one has

$$\begin{aligned}\upsilon(Tp - Tq) &= \upsilon\left(\frac{1}{4}(p_0 - q_0, p_1 - q_1, p_2 - q_2, \cdots)\right) \\ &\le \frac{2}{\sqrt{27}}\left(\upsilon\left(\frac{3p}{4}\right) + \upsilon\left(\frac{3q}{4}\right)\right) \\ &\le \frac{2}{\sqrt{27}}(\upsilon(Tp - p) + \upsilon(Tq - q)).\end{aligned} \tag{72}$$

For every $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $p_0, q_0 \in (1/3,\infty)$, then for every $\varepsilon > 0$ one gets

$$\upsilon(Tp - Tq) = 0 \le \varepsilon(\upsilon(Tp - p) + \upsilon(Tq - q)). \tag{73}$$

For each $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $p_0 \in (-\infty,1/3)$ and $q_0 \in (1/3,\infty)$, one can see

$$\begin{aligned}\upsilon(Tp - Tq) &= \upsilon\left(\frac{p}{4}\right) \le \frac{1}{\sqrt{27}}\upsilon\left(\frac{3p}{4}\right) = \frac{1}{\sqrt{27}}\upsilon(Tp - p) \\ &\le \frac{1}{\sqrt{27}}(\upsilon(Tp - p) + \upsilon(Tq - q)).\end{aligned} \tag{74}$$

Hence, the operator T is Kannan υ-contraction. Clearly, T is υ-sequentially continuous at $(1/3)e_1 \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$, and there exists $p \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $p_0 \in (-\infty,1/3)$ so that the sequence of iterates $\{T^r p\} = \{\sum_{a=1}^{r}(1/4^a)e_1 + (1/4^r)p\}$ has a subsequence $\{T^{r_j} p\} = \{\sum_{a=1}^{r_j}(1/4^a)e_1 + (1/4^{r_j})p\}$ converging to $(1/3)e_1$. From Theorem 58, the operator T has one fixed point $(1/3)e_1 \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$. Recall that T is not continuous at $(1/3)e_1 \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$.

Suppose $\upsilon(p) = \sqrt{\sum_{t=0}^{\infty}(|\sum_{x=0}^{t}((x+2)/(x+1))p_x|/\sum_{x=0}^{t}((x+2)/(x+1)))^{(2t+3)/(t+2)}}$, for every $p \in \Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty})$. As for every $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\upsilon$ with $p_0, q_0 \in (-\infty,1/3)$, one has

$$\begin{aligned}\upsilon(Tp - Tq) &= \upsilon\left(\frac{1}{4}(p_0 - q_0, p_1 - q_1, p_2 - q_2, \cdots)\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\upsilon\left(\frac{3p}{4}\right) + \upsilon\left(\frac{3q}{4}\right)\right) \\ &\le \frac{1}{\sqrt[4]{27}}(\upsilon(Tp - p) + \upsilon(Tq - q)).\end{aligned} \tag{75}$$

For every $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))

$)_{t=0}^{\infty}))_v$ with $p_0, q_0 \in (1/3,\infty)$, then for every $\varepsilon > 0$, one can see

$$v(Tp - Tq) = 0 \le \varepsilon(v(Tp - p) + v(Tq - q)). \tag{76}$$

For every $p, q \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_v$ with $p_0 \in (-\infty, 1/3)$ and $q_0 \in (1/3,\infty)$, one has

$$\begin{aligned} v(Tp - Tq) &= v\left(\frac{p}{4}\right) \le \frac{1}{\sqrt[4]{27}} v\left(\frac{3p}{4}\right) = \frac{1}{\sqrt[4]{27}} v(Tp - p) \\ &\le \frac{1}{\sqrt[4]{27}} (v(Tp - p) + v(Tq - q)). \end{aligned} \tag{77}$$

Hence, the operator T is Kannan v-contraction. As v verifies the Fatou property. From Theorem 56, the operator T has a unique fixed point $(1/3)e_1 \in (\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_v$.

We examine the existence of a fixed point of Kannan contraction operator in the prequasi Banach operator ideal constructed by $(\Xi(r,t))_v$ and s-numbers.

Theorem 61. *The prequasi norm* $\Psi(W) = [\sum_{l=0}^{\infty} (|\sum_{z=0}^{l} r_z s_z(W)|/\sum_{z=0}^{l} r_z)^{t_l}]^{1/\hbar}$ *does not satisfy the Fatou property, for all* $W \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$, *if the conditions* $(f1)$ *and* $(f2)$ *are confirmed.*

Proof. Assume the setup be satisfied and $\{W_p\}_{p\in\mathbb{N}} \subseteq \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$ with $\lim_{p\longrightarrow\infty} \Psi(W_p - W) = 0$. Since the space $\mathbb{B}^s_{(\Xi(r,t))_v}$ is a prequasi closed ideal. Therefore, $W \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$. Hence, for every $V \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$, we have

$$\begin{aligned} \Psi(V - W) &= \left[\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z s_z(V - W)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\le \left[\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z s_{[z/2]}(V - W_i)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\quad + \left[\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z s_{[z/2]}(W - W_i)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\le 2^{1/\hbar} \sup_p \inf_{i\ge p} \left[\sum_{l=0}^{\infty} \left(\frac{\left|\sum_{z=0}^{l} r_z s_z(V - W_i)\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar}. \end{aligned} \tag{78}$$

Then, Ψ does not satisfy the Fatou property.

Theorem 62. *Suppose the conditions* $(f1)$ *and* $(f2)$ *be confirmed and* $G : \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q}) \longrightarrow \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$, *where* $\Psi(W) = [\sum_{l=0}^{\infty} (|\sum_{z=0}^{l} r_z s_z(W)|/\sum_{z=0}^{l} r_z)^{t_l}]^{1/\hbar}$, *for all* $W \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$. *The point* $A \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$ *is the unique fixed point of* G, *if the following conditions are satisfied:*

(a) G is Kannan Ψ-contraction mapping

(b) G is Ψ-sequentially continuous at a point $A \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$

(c) We have $B \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$ such that the sequence of iterates $\{G^p B\}$ has a subsequence $\{G^{p_i} B\}$ converging to A

Proof. Assume the sufficient conditions be confirmed. Let A be not a fixed point of G, hence $GA \neq A$. From the setups (b) and (c), we get

$$\lim_{p_i\longrightarrow\infty} \Psi(G^{p_i} B - A) = 0 \text{ and } \lim_{p_i\longrightarrow\infty} \Psi\left(G^{p_i+1} B - GA\right) = 0. \tag{79}$$

Since G is Kannan Ψ-contraction operator, one has

$$\begin{aligned} 0 < \Psi(GA - A) &= \Psi\left(\left(GA - G^{p_i+1} B\right) + \left(G^{p_i} B - A\right)\right. \\ &\quad \left. + \left(G^{p_i+1} B - G^{p_i} B\right)\right) \le 2^{1/\hbar} \Psi\left(G^{p_i+1} B - GA\right) \\ &\quad + 2^{2/\hbar} \Psi(G^{p_i} B - A) + 2^{2/\hbar} \lambda \left(\frac{\lambda}{1-\lambda}\right)^{p_i-1} \Psi(GB - B). \end{aligned} \tag{80}$$

As $p_i \longrightarrow \infty$, we have a contradiction. Therefore, A is a fixed point of G. To show that the fixed point A is unique. Let we have two different fixed points $A, D \in \mathbb{B}^s_{(\Xi(r,t))_v}(\mathscr{P}, \mathscr{Q})$ of G. Hence, one can see

$$\Psi(A - D) \le \Psi(GA - GD) \le \lambda(\Psi(GA - A) + \Psi(GD - D)) = 0. \tag{81}$$

Hence, $A = D$.

Example 63. Let $M : S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_v}(\mathscr{P}, \mathscr{Q}) \longrightarrow S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_v}(\mathscr{P}, \mathscr{Q})$, where $\Psi(H) = \sqrt{\sum_{t=0}^{\infty} (|\sum_{x=0}^{t} ((x+1)/(x+2)) s_x| / \sum_{x=0}^{t} ((x+1)/(x+2)))^{(2t+3)/(t+2)}}$, for all $H \in S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_v}(\mathscr{P}, \mathscr{Q})$ and

$$M(H) = \begin{cases} \dfrac{H}{6}, & \Psi(H) \in [0, 1), \\ \dfrac{H}{7}, & \Psi(H) \in [1, \infty). \end{cases} \tag{82}$$

As for every $H_1, H_2 \in S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_v}$ with $\Psi(H_1), \Psi(H_2) \in [0, 1)$, one has

$$\begin{aligned}\Psi(MH_1 - MH_2) &= \Psi\left(\frac{H_1}{6} - \frac{H_2}{6}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Psi\left(\frac{5H_1}{6}\right) + \Psi\left(\frac{5H_2}{6}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{125}}(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)).\end{aligned} \tag{83}$$

For every $H_1, H_2 \in S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty},((2t+3)/(t+2))_{t=0}^{\infty}))_v}$ with $\Psi(H_1), \Psi(H_2) \in [1,\infty)$, one gets

$$\begin{aligned}\Psi(MH_1 - MH_2) &= \Psi\left(\frac{H_1}{7} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Psi\left(\frac{6H_1}{7}\right) + \Psi\left(\frac{6H_2}{7}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)).\end{aligned} \tag{84}$$

For every $H_1, H_2 \in S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty},((2t+3)/(t+2))_{t=0}^{\infty}))_v}$ with $\Psi(H_1) \in [0, 1)$ and $\Psi(H_2) \in [1,\infty)$, one has

$$\begin{aligned}\Psi(MH_1 - MH_2) &= \Psi\left(\frac{H_1}{6} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}}\Psi\left(\frac{5H_1}{6}\right) \\ &\quad + \frac{\sqrt{2}}{\sqrt[4]{216}}\Psi\left(\frac{6H_2}{7}\right) \\ &\leq \frac{\sqrt{2}}{\sqrt[4]{125}}(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)).\end{aligned} \tag{85}$$

Hence, the operator M is Kannan Ψ-contraction and $M^r(H) = \begin{cases} H/6^r, & \Psi(H) \in [0, 1), \\ H/7^r, & \Psi(H) \in [1,\infty). \end{cases}$

Obviously, M is Ψ-sequentially continuous at the zero operator $\Theta \in S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty},((2t+3)/(t+2))_{t=0}^{\infty}))_v}$, and $\{M^r H\}$ has a subsequence $\{M^{r_j}H\}$ converging to Θ. From Theorem 62, the zero operator $\Theta \in S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty},((2t+3)/(t+2))_{t=0}^{\infty}))_v}$ is the only fixed point of M. Let $\{H^{(a)}\} \subseteq S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty},((2t+3)/(t+2))_{t=0}^{\infty}))_v}$ be so that $\lim_{a\longrightarrow\infty}\Psi(H^{(a)} - H^{(0)}) = 0$, where $H^{(0)} \in S_{(\Xi(((t+1)/(t+2))_{t=0}^{\infty},((2t+3)/(t+2))_{t=0}^{\infty}))_v}$ with $\Psi(H^{(0)}) = 1$. As the prequasi norm Ψ is continuous, one obtains

$$\begin{aligned}\lim_{a\longrightarrow\infty}\Psi\left(MH^{(a)} - MH^{(0)}\right) &= \lim_{a\longrightarrow\infty}\Psi\left(\frac{H^{(0)}}{6} - \frac{H^{(0)}}{7}\right) \\ &= \Psi\left(\frac{H^{(0)}}{42}\right) > 0.\end{aligned} \tag{86}$$

Therefore, M is not Ψ-sequentially continuous at $H^{(0)}$. Hence, the operator M is not continuous at $H^{(0)}$.

8. Application to the Existence of Solutions of Nonlinear Difference Equations

Summable equations as (87) are examined by Salimi et al. [34], Agarwal et al. [35], and Hussain et al. [36]. In this section, we search for a solution to (87) in $(\Xi(r,t))_v$, where the conditions $(f1)$ and $(f2)$ are verified and $v(f) = [\sum_{l=0}^{\infty}(|\sum_{z=0}^{l} r_z f_z|/\sum_{z=0}^{l} r_z)^{t_l}]^{1/\hbar}$, for every $f \in \Xi(r,t)$. Consider the summable equations

$$f_z = p_z + \sum_{m=0}^{\infty} A(z,m)g(m,f_m), \tag{87}$$

and suppose $W : (\Xi(r,t))_v \longrightarrow (\Xi(r,t))_v$ be defined by

$$W(f_z)_{z\in N} = \left(p_z + \sum_{m=0}^{\infty} A(z,m)g(m,f_m)\right)_{z\in N}. \tag{88}$$

Theorem 64. *The summable equation (87) has a solution in $(\Xi(r,t))_v$, if $A : N^2 \longrightarrow R$, $g : N\times R \longrightarrow R$, $d : N \longrightarrow R$, and $p : N \longrightarrow R$, suppose there is a number λ such that $\sup_l \lambda^{t_l/\hbar} \in [0, 1/2)$ and for every $l \in N$, one has*

$$\begin{aligned}&\left|\sum_{z=0}^{l}\left(\sum_{m\in N} A(z,m)[g(m,f_m) - g(m,d_m)]\right)r_z\right| \\ &\leq \lambda\left[\left|\sum_{z=0}^{l}\left(p_z - f_z + \sum_{m=0}^{\infty} A(z,m)g(m,f_m)\right)r_z\right|\right. \\ &\quad \left.+ \left|\sum_{z=0}^{l}\left(p_z - d_z + \sum_{m=0}^{\infty} A(z,m)g(m,d_m)\right)r_z\right|\right].\end{aligned} \tag{89}$$

Proof. Assume the setup be verified. Let the mapping $W : (\Xi(r,t))_v \longrightarrow (\Xi(r,t))_v$ defined by equation (88). One has

$$\begin{aligned}v(Wf - Wd) &= \left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(Wf_z - Wd_z)r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &= \left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(\sum_{m\in N} A(z,m)[g(m,f_m) - g(m,d_m)])r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\leq \sup_l \lambda^{t_l/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(p_z - f_z + \sum_{m=0}^{\infty} A(z,m)g(m,f_m))r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &\quad + \sup_l \lambda^{t_l/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\left|\sum_{z=0}^{l}(p_z - d_z + \sum_{m=0}^{\infty} A(z,m)g(m,d_m))r_z\right|}{\sum_{z=0}^{l} r_z}\right)^{t_l}\right]^{1/\hbar} \\ &= \sup_l \lambda^{t_l/\hbar}(v(Wf - f) + v(Wd - d)).\end{aligned} \tag{90}$$

From Theorem 56, one gets a solution of equation (87) in $(\Xi(r,t))_v$.

Example 65. Pick up the sequence space $(\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}))_\phi$, where $v(f) = \sqrt{\sum_{t=0}^{\infty}(|\sum_{x=0}^{t}((x+2)/(x+1))f_x|/\sum_{x=0}^{t}((x+2)/(x+1)))^{(2t+3)/(t+2)}}$, for all $f \in \Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty})$. Consider the nonlinear difference equations:

$$f_z = e^{-(3z+6)} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{f_{z-2}^{p}}{f_{z-1}^{q} + m^2 + 1}, \tag{91}$$

with $p, q, f_{-2}, f_{-1} > 0$ and assume $W : \Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty}) \longrightarrow \Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty})$, defined by

$$W(f_z)_{z=0}^{\infty} = \left(e^{-(3z+6)} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{f_{z-2}^{p}}{f_{z-1}^{q} + m^2 + 1}\right)_{z=0}^{\infty}. \tag{92}$$

Obviously, there exists a number λ so that $\sup_l \lambda^{(2l+3)/(2l+4)} \in [0, 1/2)$ and for every $l \in \mathrm{N}$, one obtains

$$\begin{aligned} &\left|\sum_{z=0}^{l}\left(\sum_{m=0}^{\infty} (-1)^{z} \frac{f_{z-2}^{p}}{f_{z-1}^{q} + m^2 + 1}\left((-1)^m - (-1)^m\right)\right)\frac{z+2}{z+1}\right| \\ &\le \lambda\left|\sum_{z=0}^{l}\left(e^{-(3z+6)} - f_z + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{f_{z-2}^{p}}{f_{z-1}^{q} + m^2 + 1}\right)\frac{z+2}{z+1}\right| \\ &\quad + \lambda\left|\sum_{z=0}^{l}\left(e^{-(3z+6)} - d_z + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{d_{z-2}^{p}}{d_{z-1}^{q} + m^2 + 1}\right)\frac{z+2}{z+1}\right|. \end{aligned} \tag{93}$$

From Theorem 64, the nonlinear difference equations (91) has a solution in $\Xi(((t+2)/(t+1))_{t=0}^{\infty}, ((2t+3)/(t+2))_{t=0}^{\infty})$.

9. Conclusion

In this article, we present some topological and geometric properties of $(\Xi(r,t))_v$, of the multiplication maps acting on $(\Xi(r,t))_v$, of the class $\mathbb{B}^{s}_{(\Xi(r,t))_v}$, and of the class $(\mathbb{B}^{s}_{(\Xi(r,t))_v})^{\rho}$. We explain the existence of a fixed point of Kannan contraction map acting on these spaces. Some several numerical experiments are introduced to illustrate our results. More, some successful applications to the existence of solutions of nonlinear difference equations are discussed. This article has a number of advantages for researchers such as studying the fixed points of any contraction maps on this prequasi normed sequence space which is a generalization of the quasi normed sequence spaces, a new general space of solutions for many difference equations, examining the eigenvalue problem in this new settings and note that the closed operator ideals are certain to play an important function in the principle of Banach lattices.

Data Availability

No data were used.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

On Best Approximations in Hyperconvex Spaces

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In this manuscript, we present further extensions of the best approximation theorem in hyperconvex spaces obtained by Khamsi.

1. Introduction

The importance of fixed point theory emerges from the fact that it gives a unified approach and constitutes an essential tool in resolving problems which are not necessarily linear. A variant number of problems can be expressed as nonlinear equations of the form $f(u) = u$, where f is a self-mapping, see [1–6]. Nevertheless, an equation of the type $f(u) = u$ does not necessarily have a solution if f is a non-self-mapping. Let (X, d) be a metric space. Here, we search an optimal solution in the sense that $d(u, f(u))$ is minimum. That is, we resolve a problem of searching an element $u \in X$ so that u is in best proximity to f in some sense. A best proximity point result presents the condition under which the optimisation problem, i.e., $\inf_{u\in A} d(u, f(u))$, possesses a solution. The element u is called the best proximity point of $f : A \longrightarrow B$ if $d(u, f(u)) = d(A, B) = \inf \{d(a, b), a \in A, b \in B\}$. Observe that the best proximity point is reduced to a fixed point if f is a self-mapping. For more related works, see [7–11].

The concept of a hyperconvex space was initiated in [12] by Aronszajn and Panitchpakdi. In hyperconvex spaces, many results on coincidence points, fixed points, best approximations, and coupled best approximations are obtained. See, for example, [13–23]. For more details on the best approximation and KKM principle, we refer readers to the classic book [24]. Due to Aronszajn and Panitchpakdi [12], the definition of a hyperconvex metric space is as follows.

A metric space (Λ, ω) is named to be a hyperconvex space if for any set of points $\{\ell_\alpha\}$ of Λ and for any family of nonnegative real numbers $\{r_\alpha\}$ with $\omega(\ell_\alpha, \ell_\beta) \le r_\alpha + r_\beta$, we have $\cap_\alpha B(\ell_\alpha, r_\alpha) \ne \varnothing$, where $B(\ell, r) = \{j \in \Lambda : \omega(\ell, j) \le r\}$ represents the closed ball with center $\ell \in \Lambda$ and radius r.

Suppose that a subset A of Λ is bounded. Consider,

$$\begin{aligned} co(A) &= \cap \{B \subseteq \Lambda : B \text{ is a closed ball so that } A \subseteq B\}, \\ \mathscr{A}(\ell) &= \{A \subseteq \Lambda : A = co(A)\}, \end{aligned} \tag{1}$$

i.e., $A \in \mathscr{A}(\ell)$ iff A is an intersection of closed balls. Here, A is named to be an admissible subset of Λ. In the linear case, the notation $\operatorname{conv}(A)$ describes the convex hull of A. Note that $co(A)$ is always defined and is in $\mathscr{A}(\ell)$. If (Λ, ω) is a hyperconvex space, then it is complete [17].

Let Λ be a nonempty set. We denote by $\langle \Lambda \rangle$ and 2^Λ the set of all nonempty finite subsets of Λ and the set of all nonempty subsets of Λ, respectively. Let Λ and Ω be topological spaces with $A \subseteq \Lambda$ and $B \subseteq \Omega$. Given a set-valued map $F : \Lambda \longrightarrow 2^\Omega$, the image of A under F is the set $F(A) = \cup_{a\in A} F(a)$ and the inverse image of B under F is $F^-(B) = \{\ell \in \Lambda : F(\ell) \cap B \ne \varnothing\}$. The map F is lower (upper) semicontinuous if, for each open (closed) set $B \subseteq \Omega$,

$F^{-}(B)$ is open (closed) set in Λ. The map F is continuous if F is both upper semicontinuous and lower semicontinuous.

Let A be an admissible subset of Λ. The set-valued map $F : A \longrightarrow 2^{\Lambda}$ is named to be quasiconvex if for any admissible set A of Λ, $F^{-}(A)$ is also admissible (see [15]). Observe that if F is a quasiconvex map, then the set $F^{-}(B(\ell, r))$ is admissible for each closed ball $B(\ell, r)$. Note that, if $A \in \mathscr{A}(\ell)$, then $A + r \in \mathscr{A}(\ell)$ (see [25]), where $A + r = \cup_{a\in A} B(a, r)$.

Khamsi [17] presented a hyperconvex version of the KKM principle in hyperconvex spaces. As an application, he gave a hyperconvex version of the best approximation result of Fan for continuous single-valued maps. In this manuscript, we ensure the existence of a solution of a best approximation problem for set-valued maps F and G: for a set K, find $\ell_0 \in K$ so that

$$\omega(G(\ell_0), F(\ell_0)) \leq \omega(G(\ell), F(\ell_0)) \text{ for all } \ell \in K. \tag{2}$$

Let Λ be a metric space and $K \subset \Lambda$. A multivalued map $H : K \longrightarrow 2^{\Lambda}$ is said to be a KKM map if

$$co(A) \subset H(A) \text{ for any } A \in \langle K \rangle. \tag{3}$$

Theorem 1 (see [17], KKM principle). *Let Λ be a hyperconvex space, K be an arbitrary subset of Λ, and $H : K \longrightarrow 2^{\Lambda}$ be a KKM map so that $H(\ell_0)$ is compact for some $\ell_0 \in K$ and $H(\ell)$ is closed for any $\ell \in K$. Then, $\bigcap_{\ell\in K} H(\ell) \neq \varnothing$.*

Theorem 2 (see [17], best approximation). *Let Λ be a hyperconvex space and $K \in \mathscr{A}(\ell)$ be compact. Given a continuous map $f : K \longrightarrow \Lambda$, there is $\ell_0 \in K$ so that*

$$\omega(\ell_0, f(\ell_0)) = \inf_{\ell\in K} \omega(\ell, f(\ell_0)). \tag{4}$$

This result has been generalized to other forms of maps. For more details, see [13–16, 18, 22].

Now, we give the definition of a measure of noncompactness of Pasicki [26].

Definition 3 (see [26]). Let Λ be a metric space. An arbitrary function $\theta : 2^{\Lambda} \longrightarrow [0,\infty]$ is named to be a measure of noncompactness on Λ if

(1) $\theta(A) = 0$ iff A is a totally bounded set

(2) for $A, B \in 2^{\Lambda} A \subseteq B$, implies $\theta(A) \leq \theta(B)$

(3) for all $A \subseteq \Lambda$ and $\ell \in \Lambda$, $\theta(A \cup \{\ell\}) = \theta(A)$

Definition 4 (see [19]). Let Λ be a metric space, θ be a measure of noncompactness on Λ, and $K \subset \Lambda$. The map $H : K \longrightarrow 2^{\Lambda}$ is condensing if for any $\varepsilon > 0$, there is $A \in \langle K \rangle$ so that $\theta(\bigcap_{a\in A} H(a)) < \varepsilon$. A condensing map $H : K \longrightarrow 2^{\Lambda}$ is a condensing KKM map if it is a KKM map.

In this paper, we present further extensions of the best approximation result (Theorem 2) obtained by Khamsi. Finally, we present a problem related to the Schauder conjecture.

2. Results

The following result generalizes Theorem 1. The proof is essentially the same as Theorem 3.1 in [19].

Theorem 5. *Let θ be a measure of noncompactness on Λ a hyperconvex space, K be an arbitrary subset of Λ, and $H : K \longrightarrow 2^{\Lambda}$ be a condensing KKM map such that each $H(\ell)$ is closed, then $\bigcap_{\ell\in K} H(\ell)$ is nonempty and compact set.*

We introduce the concept of a φ-quasiconvex map in hyperconvex spaces.

Definition 6. Let Λ be a hyperconvex space and $K \in \mathscr{A}(\ell)$. A set-valued map $G : K \longrightarrow 2^{\Lambda}$ is said to be a φ-quasiconvex if for any $\ell \in K$ and $r > 0$,

$$coG^{-}(B(\ell, r)) \subseteq G^{-}(B(\ell, \varphi(r))), \tag{5}$$

where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous monotone increasing function so that $\varphi(u) \geq u$ for any $u \geq 0$ and $\varphi(0) = 0$.

Let $i_\omega : [0,\infty) \longrightarrow [0,\infty)$ be the identity map and $I_\omega : K \longrightarrow 2^{K}$ be the identity set-valued map so that $I_\omega(\ell) = \{\ell\}$ for any $\ell \in K$. Note that a quasiconvex map is i_ω-quasiconvex and I_ω is i_ω-quasiconvex in hyperconvex spaces.

If (Λ, ω) is a linear metric space, then I_ω may not be i_ω-quasiconvex.

Example 1. Denote by S the linear space of real sequences. The Fréchet metric ω_F for S is given as follows (see [27]):

Let $\ell = (\ell_1, \ell_2, \cdots, \ell_n, \cdots)$, $j = (j_1, j_2, \cdots, j_n, \cdots)$, and $0 = (0, 0, \cdots, 0, \cdots)$,

$$d_F(\ell, j) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{|\ell_n - j_n|}{1 + |\ell_n - j_n|}, \tag{6}$$

then, we obtain

$$\text{conv}B\left(0, \frac{1}{3}\right) \subseteq B\left(0, \frac{1}{3}\right). \tag{7}$$

Namely, for $\ell = (1, 0, 1, 0, \cdots)$ and $j = (1/2, 1/2, \cdots)$, we have that $\ell \in B(0, 1/3)$, $j \in B(0, 1/3)$, and $(\ell + j)/2 \notin B(0, 1/3)$.

Note that for any $\ell \in S$ and $r > 0$, one writes

$$\text{conv}B(\ell, r) \subseteq B(\ell, 2r). \tag{8}$$

So, in the linear metric space (S, ω_F), the map I_ω is not i_ω-quasi-convex and I_ω is φ-quasiconvex, where $\varphi(u) = 2u$, $u \in [0,\infty)$.

Theorem 7. *Let (Λ, ω) be a hyperconvex space, $K \in \mathscr{A}(\ell)$ be closed, θ be a measure of noncompactness on Λ, $F, G : K$*

$\longrightarrow \mathscr{A}(\ell)$ *be continuous maps with compact values and G be* φ *-quasi-convex. If for any* $u > 0$ *there is* $j \in K$ *so that*

$$\theta(\{\ell \in K : \omega(G(\ell), F(\ell)) \leq \varphi(\omega(G(j), F(\ell)))\}) \leq u, \quad (9)$$

then, there is $\ell_0 \in K$ *so that*

$$\omega(G(\ell_0), F(\ell_0)) \leq \inf_{\ell \in K} \varphi(\omega(G(\ell), F(\ell_0))). \quad (10)$$

Proof. Define $H : K \longrightarrow 2^{\Lambda}$ by

$$H(j) = \{\ell \in K : \omega(G(\ell), F(\ell)) \leq \varphi(\omega(G(j), F(\ell)))\}. \quad (11)$$

From condition $\varphi(t) \geq t$, we obtain

$$\omega(G(j), F(j)) \leq \varphi(\omega(G(j), F(j))) \text{for all } j \in K, \quad (12)$$

so $H(j)$ is a nonempty set for all $j \in K$, because $j \in H(j)$.

From condition (9), one asserts that H is a condensing map.

Since F and G are continuous maps and φ is a continuous function, we get that $H(j)$ is a closed set for all $j \in K$.

The map $j \longmapsto H(j)$ is KKM. Indeed, suppose that for some $A \in \langle K \rangle$,

$$co(A) \nsubseteq H(A). \quad (13)$$

Then, there is $j \in co(A)$ so that $j \notin H(a)$ for any $a \in A$. Thus,

$$\omega(G(j), F(j)) > \varphi(\omega(G(a), F(j))) \text{for all } a \in A. \quad (14)$$

The function φ is increasing, so

$$\varphi^{-1}(\omega(G(j), F(j))) > \omega(G(a), F(j)) \text{ for all } a \in A. \quad (15)$$

Let $\varepsilon > 0$ be so that

$$\varphi^{-1}(\omega(G(j), F(j))) - \varepsilon \geq \omega(G(a), F(j)) \text{ for all } a \in A. \quad (16)$$

Then,

$$A \subseteq G^{-}\left(F(j) + \varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right), \quad (17)$$

and hence,

$$co(A) \subseteq co\left(G^{-}\left(F(j) + \varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right)\right). \quad (18)$$

Since G is φ-quasiconvex, one gets from (18),

$$co(A) \subseteq G^{-}\left(F(j) + \varphi\left(\varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right)\right). \quad (19)$$

Since $j \in co(A)$, we deduce that

$$\omega(G(j), F(j)) \leq \varphi\left(\varphi^{-1}(\omega(G(j), F(j))) - \varepsilon\right). \quad (20)$$

Consequently,

$$\varphi^{-1}(\omega(G(j), F(j)))) \leq \varphi^{-1}(\omega(G(j), F(j))) - \varepsilon, \quad (21)$$

which is not possible. Therefore, H must be a KKM map. Now, form Theorem 5, there is $\ell_0 \in K$ so that

$$\ell_0 \in \bigcap_{\ell \in K} H(\ell). \quad (22)$$

Therefore,

$$\omega(G(\ell_0), F(\ell_0)) \leq \inf_{\ell \in K} \varphi(\omega(G(\ell), F(\ell_0))). \quad (23)$$

Taking the set K to be compact, we state from Theorem 7 the next results.

Theorem 8. *Let* (Λ, ω) *be a hyperconvex space,* $K \in \mathscr{A}(\ell)$ *be compact,* $F, G : K \longrightarrow \mathscr{A}(\ell)$ *be continuous maps with compact values, and G be* φ*-quasiconvex. Then, there is* $\ell_0 \in K$ *so that*

$$\omega(G(\ell_0), F(\ell_0)) \leq \inf_{\ell \in K} \varphi(\omega(G(\ell), F(\ell_0))). \quad (24)$$

Theorem 9. *Let* (Λ, ω) *be a hyperconvex space,* $K \in \mathscr{A}(\ell)$ *be compact,* $F, G : K \longrightarrow \mathscr{A}(K)$ *be continuous maps with compact values, and G be* φ*-quasiconvex onto a map. Then, there is* $\ell_0 \in K$ *so that* $G(\ell_0) \cap F(\ell_0) \neq \varnothing$.

Theorem 10. *Let* (Λ, ω) *be a hyperconvex space,* $K \in \mathscr{A}(\ell)$ *be compact, and* $F, G : K \longrightarrow \mathscr{A}(\ell)$ *be continuous maps with compact values. If there is* $\lambda \geq 1$ *such that for any* $\ell \in X$ *and* $r > 0$,

$$coG^{-}(B(\ell, r)) \subseteq G^{-}(B(\ell, \lambda r)), \quad (25)$$

then there is $\ell_0 \in K$ *so that*

$$\omega(G(\ell_0), F(\ell_0)) \leq \inf_{\ell \in K} \lambda \omega(G(\ell), F(\ell_0)). \quad (26)$$

Theorem 11. *Let* (Λ, ω) *be a hyperconvex space,* $K \in \mathscr{A}(\ell)$ *be compact, and* $F : K \longrightarrow \mathscr{A}(\ell)$ *be a continuous map with compact values so that* $F(\ell) \cap K \neq \varnothing$ *for any* $\ell \in K$*. Then, there is* $\ell_0 \in K$ *so that* $\ell_0 \in F(\ell_0)$.

Remark 12. If F is a single-valued map, we deduce from Theorem 11 the main theorem of Park [22] (Theorem 5. (iv)).

Theorem 13 (see [18], Theorem 3.2). *Let* (Λ, ω) *be a hyperconvex space,* $K \in \mathscr{A}(\ell)$ *be compact,* $F, G : K \longrightarrow \mathscr{A}(\ell)$ *be continuous maps with compact values, and G be quasiconvex. Then, there is* $\ell_0 \in K$ *so that*

$$\omega(G(\ell_0), F(\ell_0)) \leq \inf_{\ell \in K} \omega(G(\ell), F(\ell_0)). \quad (27)$$

Theorem 14 (see [16], Theorem 2.9). *Let* (Λ, ω) *be a hyperconvex space,* $K \in \mathscr{A}(\ell)$ *be compact, and* $F : K \longrightarrow \mathscr{A}(\ell)$ *be* *a continuous map with compact values. Then, there is $\ell_0 \in K$ so that*

$$\omega(\ell_0, F(\ell_0)) = \inf_{\ell \in K} \omega(\ell, F(\ell_0)). \quad (28)$$

Finally, we give the following problems.

Problem 15. Does for every linear metric space (Λ, ω) and for a compact subset K of Λ, there is a map φ so that the identity map $I_\omega : K \longrightarrow 2^K$ (i.e., $I_\omega(\ell) = \{\ell\}, \ell \in K$) is φ-quasiconvex?

In other words, does for a linear metric space, there is a continuous monotone increasing function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ so that $\varphi(u) \geq u$ for any $u \geq 0$, $\varphi(0) = 0$ and

$$\mathrm{conv} B(\ell, r) \subseteq B(\ell, \varphi(r)), \quad (29)$$

for any $\ell \in K$ and $r > 0$?

In 2001, Cauty [28] obtained the affirmative solution to the Schauder conjecture as follows:

Problem 16. Let K be a compact convex subset of a (metrizable) topological vector space. Does any continuous map $f : K \longrightarrow K$ have a fixed point?

Remark 17. Note that if Problem 15 is affirmative, then Problem 16 is affirmative.

Data Availability

Data sharing is not applicable to this article as no data set was generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors have read and agreed to the published version of the manuscript.

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Research Article

Ulam-Hyers-Rassias Stability of Stochastic Functional Differential Equations via Fixed Point Methods

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The Ulam-Hyers-Rassias stability for stochastic systems has been studied by many researchers using the Gronwall-type inequalities, but there is no research paper on the Ulam-Hyers-Rassias stability of stochastic functional differential equations via fixed point methods. The main goal of this paper is to investigate the Ulam-Hyers Stability (HUS) and Ulam-Hyers-Rassias Stability (HURS) of stochastic functional differential equations (SFDEs). Under the fixed point methods and the stochastic analysis techniques, the stability results for SFDE are investigated. We analyze two illustrative examples to show the validity of the results.

1. Introduction

In recent years, SFDEs play an important role in different areas such as physics, mechanics, population dynamics, ecology, medicine biology, and other areas of sciences. SFDEs have great applications and have been developed very fast, see for example [1–5]. Stability investigation is conducted for stochastic nonlinear differential equations with constant delay. The Lyapunov method is used for stability investigation of different mathematical models such as predator-prey relationships and inverted controlled pendulum.

The HUS problem of functional systems began from a question of S. Ulam, queried in 1940, about the stability of functional differential equations for homomorphism as follows. The question regarding the stability problem of homomorphisms is as follows:

Denote by H_1 the group, and H_2 the metric group with a metric $\tilde{\delta}$ and a constant $\vartheta > 0$. The question is to study if there exists $\lambda > 0$ satisfies for every $h : H_1 \longrightarrow H_2$ such that

$$\tilde{\delta}(h(\sigma\upsilon), h(\sigma)h(\upsilon)) \le \lambda, \forall \sigma, \upsilon \in H_1, \tag{1}$$

there exists a homomorphism $f : H_1 \longrightarrow H_2$ satisfies

$$\tilde{\delta}(h(\sigma), f(\sigma)) \le \vartheta, \forall \sigma \in H_1. \tag{2}$$

In 1941, Hyers [6] presented a partial solution to the question of S. Ulam assuming that D_1, D_2 be two Banach spaces in the case of λ-linear transformations, that is

Let D_1, D_2 be two Banach spaces and set $h : D_1 \longrightarrow D_2$ be a linear transformation satisfying

$$\|h(\sigma + \upsilon) - h(\sigma) - h(\upsilon)\| \le \lambda, \forall \sigma, \upsilon \in D_1 \lambda > 0. \tag{3}$$

There exists a unique linear transformation $\Delta : D_1 \longrightarrow D_2$ such that the limit $\Delta(\sigma) = \lim_{n \longrightarrow +\infty} h(2^n\sigma)/2^n$ exists for each $\sigma \in D_1$ and $\|h(\sigma) - \Delta(\sigma)\| \le \lambda$ for all $\sigma \in D_1$, which was the first step towards more answers in this area. Many researchers have analyzed the HUS of various classes of differential systems (see, for instance, [1, 6–21]). In 1978, Rassias [22] provided a generalized answer to the Ulam question for approximate λ-linear transformations. In [23], Rassias obtained an extension of the Hyers's answer.

In 1994, Gavruta [24] gave a generalization form of Rassias's Theorem for the unbounded Cauchy difference $h(\sigma + \upsilon) - h(\sigma) - h(\upsilon)$ and introduced the notion of generalized HURS in the sense of Rassias approach.

In the last decades, there is an increasing interest and work on the Ulam stability and the Ulam-Hyers stability of some deterministic systems using the Banach contraction principle and Schaeferâ€™s fixed point theorem (see [25, 26]).

In the literature, there are a few papers about the HUS and the HURS of stochastic systems (see [13, 27–30]). The stability of SFDEs has attracted much more attention (see [2, 19] etc.). Consequently, it is interesting to extend the research results on the deterministic functional systems to the stochastic case.

Let us outline the framework of this paper. After some basic notions and assumptions (see Section 2), in Section 3, the HUS and HURS of the solution of the system are proved by using the fixed point methodology. In the last section, two numerical examples are presented to illustrate the main results.

2. Preliminary

Denote by $\{\Omega, \mathscr{F}, (\mathscr{F}_\sigma)_{\sigma\geq 0}, P\}$ the complete probability space where $\{\mathscr{F}_\sigma\}_{\sigma\geq 0}$ is a filtration satisfying the usual conditions. $W(\sigma)$ is an m-dimensional Brownian motion defined on the probability space. Denote by $\mathscr{L}^2([\nu, w], \mathbb{R}^b)$ the set of $\mathbb{R}^b$-valued $\mathscr{F}_\sigma$-adapted processes $\{\psi(\sigma)\}_{\nu\leq\sigma\leq w}$ such that $\int_\nu^w |\psi(\sigma)|^2 d\sigma < \infty$ a.s. and $\mathscr{M}^2([\nu, w], \mathbb{R}^b)$ the set of processes $\{\psi(\sigma)\}_{\nu\leq\sigma\leq w}$ in $\mathscr{L}^2([\nu, w], \mathbb{R}^b)$ satisfies $\mathbb{E}\int_\nu^w |\psi(\sigma)|^2 d\sigma < \infty$. Let $C([-\varrho, 0]; \mathbb{R}^b)$ denote the set of functions φ from $[-\varrho, 0]$ to $\mathbb{R}^b$ that are right-continuous and have limits on the left. $C([-\varrho, 0]; \mathbb{R}^b)$ is equipped with the norm $\|\varphi\| = \sup_{-\varrho\leq s\leq 0}|\varphi(s)|$ and $|x| = \sqrt{x^T x}$ for any $x \in \mathbb{R}^b$. Denote by $C^b_{\mathscr{F}_0}([-\varrho, 0]; \mathbb{R}^b)$ the set of all $\mathscr{F}_0$-measurable bounded $C([-\varrho, 0]; \mathbb{R}^b)$-valued random variables $\chi = \{\chi(\kappa): -\varrho \leq \kappa \leq 0\}$. Let $L^2_{\mathscr{F}_t}([-\varrho, 0]; \mathbb{R}^b)$, $t \geq 0$, denote the set of all $\mathscr{F}_t$-measurable, $C([-\varrho, 0]; \mathbb{R}^b)$-valued random variables $\varsigma = \{\varsigma(\kappa): -\varrho \leq \kappa \leq 0\}$ satisfies $\sup_{-\varrho\leq\kappa\leq 0} E|\varsigma(\kappa)|^2 < \infty$.

Consider the following SFDE for $0 \leq \omega_0 < T$ fixed:

$$d\zeta(\omega) = f(\omega, \zeta_\omega)d\omega + g(\omega, \zeta_\omega)dW(\omega), \omega_0 \leq \omega \leq T, \quad (4)$$

with the initial condition

$$\zeta_{\omega_0} = \chi \in L^2_{\mathscr{F}_{\omega_0}}\left([-\varrho, 0]; \mathbb{R}^b\right), \quad (5)$$

and recall that, given $\zeta \in C([\omega_0, T]; \mathbb{R}^b)$, for each $\omega \in [\omega_0, T]$, we denote by $\zeta_\omega(\cdot)$ the function in $C([\omega_0 - \varrho, 0]; \mathbb{R}^b)$ defined as $\zeta_\omega(p) = \zeta(\omega + p)$, $-\varrho \leq p \leq 0$. We assume that

$$f: [\omega_0, T] \times C\left([-\varrho, 0]; \mathbb{R}^b\right) \longrightarrow \mathbb{R}^b, g : [\omega_0, T] \times C\left([-\varrho, 0]; \mathbb{R}^b\right) \longrightarrow \mathbb{R}^{b\times q}. \quad (6)$$

Using the definition of Itô's stochastic differential and integrating the two sides of equation (4) from ω_0 to ω, we have

$$\zeta(\omega) = \zeta(\omega_0) + \int_{\omega_0}^{\omega} f(\nu, \zeta_\nu)d\nu + \int_{\omega_0}^{\omega} g(\nu, \zeta_\nu)dW(\nu), \omega_0 \leq \omega \leq T. \quad (7)$$

We consider the following assumption:

$\mathscr{A}_1$: (Uniform Lipschitz condition): Suppose that there is a constant $K > 0$ satisfies

$$|f(\omega, \upsilon_1) - f(\omega, \upsilon_2)|^2 \vee |g(\omega, \upsilon_1) - g(\omega, \upsilon_2)|^2 \leq K^2\|\upsilon_1 - \upsilon_2\|^2, \quad (8)$$

$\forall\omega \in [\omega_0, T]$ and $\upsilon_1, \upsilon_2 \in C([-\varrho, 0]; \mathbb{R}^b)$, where $z_1 \vee z_2$ define the maximum of z_1 and z_2.

3. Stability Results

In this part, we discuss the HUS and the HURS of equation (4) under the assumption $\mathscr{A}_1$.

Definition 1. Equation (4) is called HUS with respect to (w.r.t) ε if there exists a constant $M > 0$ such that for each $\varepsilon > 0$ and for each solution $\tilde{\zeta} \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, of the following inequation:

$$E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f\left(\nu, \tilde{\zeta}_\nu\right)d\nu - \int_{\omega_0}^{\omega} g\left(\nu, \tilde{\zeta}_\nu\right)dW(\nu)\right|^2 \leq \varepsilon, \forall\omega \in [\omega_0, T], \quad (9)$$

there exists a solution $\zeta \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $E\left|\tilde{\zeta}(\omega) - \zeta(\omega)\right|^2 \leq M\varepsilon$,

$$\forall\omega \in [\omega_0, T]. \quad (10)$$

Definition 2. Equation (4) is called HURS w.r.t $(\varepsilon, \theta(\omega))$, with $\theta(\cdot) \in C([\omega_0, T]; \mathbb{R}_+)$, if there exists a constant $M > 0$ such that for each $\varepsilon > 0$ and for each solution

$\tilde{\zeta} \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, satisfying

$$E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f\left(\nu, \tilde{\zeta}_\nu\right)d\nu - \int_{\omega_0}^{\omega} g\left(\nu, \tilde{\zeta}_\nu\right)dW(\nu)\right|^2 \leq \varepsilon\theta(\omega), \forall\omega \in [\omega_0, T], \quad (11)$$

there exists a solution $\zeta(\omega) \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$, $E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq \varepsilon M \theta(\omega)$.

Definition 3. Equation (4) is generalized HURS w.r.t $\theta(\omega)$, with $\theta(\cdot) \in C([\omega_0 - \varrho, T]; \mathbb{R}_+)$, if there exists a constant $M > 0$ such that for each solution.

$\tilde{\zeta} \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, satisfying

$$E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f\left(\nu, \tilde{\zeta}_\nu\right) d\nu - \int_{\omega_0}^{\omega} g\left(\nu, \tilde{\zeta}_\nu\right) dW(\nu)\right|^2 \leq \varepsilon\theta(\omega), \forall \omega \in [\omega_0, T], \tag{12}$$

there exists a solution $\zeta(\omega) \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$, $E|\tilde{\zeta}(\omega) - \zeta(\omega)|^2 \leq M\theta(\omega)$.

Lemma 4 (see [2]). *Set $\mathscr{M} = \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$. Let $d : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}_+$ be the function such that*

$$d^2(\zeta_1, \zeta_2) = \inf\left\{\Lambda \in [0, +\infty), \frac{E|\zeta_1(\omega) - \zeta_2(\omega)|^2}{h_1(\omega)} \leq \Lambda h_2(\omega), \forall \omega \in [\omega_0 - \varrho, T]\right\}, \tag{13}$$

where $h_1, h_2 \in C([\omega_0 - \varrho, T], \mathbb{R}_+^*)$. Then, $(\mathscr{M}, d)$ is a complete metric space.

Theorem 5 (see [29]). *Suppose (F, d) is a complete metric space and $L : F \longrightarrow F$ is a contraction (with $\tau \in [0, 1)$). Suppose that $\nu \in F$, $\lambda > 0$ and $d(\nu, L(\nu)) \leq \lambda$. So, there exists a unique $\beta \in F$ satisfies $\beta = L(\beta)$. Moreover,*

$$d(\nu, \beta) \leq \frac{\lambda}{1 - \tau}. \tag{14}$$

Theorem 6. *Suppose that $\mathscr{A}_1$ hold. Let $\tilde{\zeta} \in \mathscr{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, be a stochastic process satisfies*

$$E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f\left(\nu, \tilde{\zeta}_\nu\right) d\nu - \int_{\omega_0}^{\omega} g\left(\nu, \tilde{\zeta}_\nu\right) dW(\nu)\right|^2 \leq \varepsilon\theta(\omega), \forall \omega \in [\omega_0, T], \tag{15}$$

where $\varepsilon > 0$ and $\theta(\cdot) \in C([\omega_0, T]; \mathbb{R}_+^*)$ is a nondecreasing function. Then, there is a solution $\zeta \in \mathscr{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E\left|\tilde{\zeta}(\omega) - \zeta(\omega)\right|^2 \leq \frac{1}{\left(1 - \sqrt{\alpha/\alpha + \delta}\right)^2} \exp\left((\alpha + \delta)(T - \omega_0)\right)\varepsilon\theta(\omega), \tag{16}$$

where $\alpha = 2K^2[(T - \omega_0) + 1]$ and δ is any positive constant.

Proof. Consider $d : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}_+$ such that

$$d^2(\zeta_1, \zeta_2) = \inf\left\{\Lambda \in [0, +\infty), \frac{E|\zeta_1(\omega) - \zeta_2(\omega)|^2}{h(\omega)} \leq \Lambda\widetilde{\theta}(\omega), \forall \omega \in [\omega_0 - \rho, T]\right\}, \tag{17}$$

with $h(\omega) = e^{\gamma(\omega - \omega_0)}$ for $\omega \in [\omega_0, T]$ and $h(\omega) = 1$ for $\omega \in [\omega_0 - \varrho, \omega_0]$, where $\gamma = \alpha + \delta$, and $\widetilde{\theta}(\omega) = \theta(\omega)$ for $\omega \in [\omega_0, T]$ and $\widetilde{\theta}(\omega) = \theta(\omega_0)$ for $\omega \in [\omega_0 - \rho, \omega_0]$.

Let the operator $R : \mathscr{M} \longrightarrow \mathscr{M}$ such that $(R\zeta)(\omega) = \tilde{\zeta}(\omega)$, for $\omega \in [\omega_0 - \varrho, \omega_0]$, and

$$(R\zeta)(\omega) = \tilde{\zeta}(\omega_0) + \int_{\omega_0}^{\omega} f(\nu, \zeta_\nu) d\nu + \int_{\omega_0}^{\omega} g(\nu, \zeta_\nu) dW(\nu), \tag{18}$$

for $\omega \in [\omega_0, T]$.

It is easy to prove that R is well defined.

Let $\zeta_1, \zeta_2 \in \mathscr{M}$, for $\omega \in [\omega_0 - \varrho, \omega_0]$, we get $(R\zeta_1)(\omega) - (R\zeta_2)(\omega) = 0$.

For $\omega \in [\omega_0, T]$, we have

$$|(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 \leq 2\left|\int_{\omega_0}^{\omega} [f(\nu, \zeta_{1\nu}) - f(s, \zeta_{2\nu})] d\nu\right|^2 + 2\left|\int_{\omega_0}^{\omega} [g(\nu, \zeta_{1\nu}) - g(\nu, \zeta_{2\nu})] dW(\nu)\right|^2. \tag{19}$$

Taking the expectation on both sides and using assumption $\mathscr{A}_1$, we have

$$E|(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 \leq 2K^2\left[(T - \omega_0)\int_{\omega_0}^{\omega} E\|\zeta_{1\nu} - \zeta_{2\nu}\|^2 d\nu + \int_{\omega_0}^{\omega} E\|\zeta_{1\nu} - \zeta_{2\nu}\|^2 d\nu\right]. \tag{20}$$

Then,

$$E|(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 \leq \alpha\int_{\omega_0}^{\omega} E\|\zeta_{1\nu} - \zeta_{2\nu}\|^2 d\nu. \tag{21}$$

For $\nu \in [\omega_0, \omega]$, we have $E\|\zeta_{1\nu} - \zeta_{2\nu}\|^2 = E|\zeta_1(\nu + \sigma) - \zeta_2(\nu + \sigma)|^2$ where $\sigma \in [-\varrho, 0]$. Then,

$$E\|\zeta_{1\nu} - \zeta_{2\nu}\|^2 = \frac{E|\zeta_1(\nu+\sigma) - \zeta_2(\nu+\sigma)|^2}{h(\nu+\sigma)\widetilde{\theta}(\nu+\sigma)} h(\nu+\sigma)\widetilde{\theta}(\nu+\sigma) \leq d^2(\zeta_1,\zeta_2)h(\nu)\widetilde{\theta}(\nu). \tag{22}$$

Therefore,

$$E|(R\zeta_1)(\omega) - (R\zeta_2)(\omega)|^2 \leq \alpha d^2(\zeta_1,\zeta_2)\theta(\omega)\int_{\omega_0}^{\omega} h(\nu)d\nu \leq \frac{\alpha}{\gamma} d^2(\zeta_1,\zeta_2)\theta(\omega)h(\omega). \tag{23}$$

Then, $d^2(R\zeta_1, R\zeta_2) \leq (\alpha/\gamma)d^2(\zeta_1,\zeta_2)$. Thus, $d(R\zeta_1, R\zeta_2) \leq \sqrt{\alpha/\gamma}d(\zeta_1,\zeta_2)$. Therefore, R is strictly contractive for $c = \sqrt{\alpha/\alpha+\delta}$.

For $\omega \in [\omega_0 - \varrho, \omega_0]$, we get $(R\tilde{\zeta})(\omega) - \tilde{\zeta}(\omega) = 0$.

From (15), we get

$$\frac{E\left|\tilde{\zeta}(\omega) - R\tilde{\zeta}(\omega)\right|^2}{\theta(\omega)h(\omega)} = \frac{E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f(\nu,\zeta_\nu)d\nu - \int_{\omega_0}^{\omega} g(\nu,\zeta_\nu)dW(\nu)\right|^2}{\theta(\omega)h(\omega)} \leq \varepsilon, \tag{24}$$

for all $\omega \in [\omega_0, T]$.

Then, $d^2(R\tilde{\zeta}, \tilde{\zeta}) \leq \varepsilon$. Therefore, $d(R\tilde{\zeta}, \tilde{\zeta}) \leq \sqrt{\varepsilon}$. Using the fixed point theorem, there is a solution ζ^* of (4) such that $d(\tilde{\zeta}, \zeta^*) \leq (1/1-c)\sqrt{\varepsilon}$. Then,

$$\frac{E\left|\tilde{\zeta}(\omega) - \zeta^*(\omega)\right|^2}{\theta(\omega)h(\omega)} \leq \frac{1}{(1-c)^2}\varepsilon, \forall \omega \in [\omega_0, T]. \tag{25}$$

Therefore,

$$E\left|\tilde{\zeta}(\omega) - \zeta^*(\omega)\right|^2 \leq \frac{e^{(\alpha+\delta)(T-\omega_0)}}{(1-c)^2}\varepsilon\theta(\omega), \forall \omega \in [\omega_0, T], \tag{26}$$

as desired.

Remark 7. In our analysis of the HURS, we do not suppose any condition on K unlike the case of the Theorem 6 in [29].

Theorem 8. *Assume that $\mathscr{A}_1$ hold. Let $\tilde{\zeta} \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, be a stochastic process satisfies*

$$E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f\left(\nu, \tilde{\zeta}_\nu\right)d\nu - \int_{\omega_0}^{\omega} g\left(\nu, \tilde{\zeta}_\nu\right)dW(\nu)\right|^2 \leq \varepsilon, \forall \omega \in [\omega_0, T], \tag{27}$$

where $\varepsilon > 0$. Then, there is a solution $\zeta \in \mathscr{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$ of (4), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E\left|\tilde{\zeta}(\omega) - \zeta(\omega)\right|^2 \leq \frac{1}{\left(1 - \sqrt{\alpha/\alpha+\delta}\right)^2} \exp\left((\alpha+\delta)(T-\omega_0)\right)\varepsilon, \tag{28}$$

where $\alpha = 2K^2[(T-\omega_0) + 1]$ and δ is a positive constant.

Proof. The proof of this theorem is similar to Theorem 6.

Theorem 9. *Assume that $\mathscr{A}_1$ hold. Let $\tilde{\zeta} \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}^b)$, with $\tilde{\zeta}_{\omega_0} = \chi$, be a stochastic process satisfies*

$$E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f\left(\nu, \tilde{\zeta}_\nu\right)d\nu - \int_{\omega_0}^{\omega} g\left(\nu, \tilde{\zeta}_\nu\right)dW(\nu)\right|^2 \leq \theta(\omega), \forall \omega \in [\omega_0, T], \tag{29}$$

where $\theta(\cdot) \in C([\omega_0, T]; \mathbb{R}_+^*)$ is a nondecreasing function. Then, there is a solution $\zeta \in \mathscr{M}^2([\omega_0 - \rho, T], \mathbb{R}^b)$ of (4), with $\zeta_{t_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E\left|\tilde{\zeta}(\omega) - \zeta(\omega)\right|^2 \leq \frac{1}{\left(1 - \sqrt{\alpha/\alpha+\delta}\right)^2} \exp\left((\alpha+\delta)(T-\omega_0)\right)\theta(\omega), \tag{30}$$

where $\alpha = 2K^2[(T-\omega_0) + 1]$ and $\delta > 0$.

Proof. The proof of this theorem is similar to Theorem 6.

4. Examples

Two examples are studied to show the interest of the main results.

Example 10. Consider the following SFDE for each $\varepsilon > 0$

$$\begin{cases} d\zeta(\omega) = f(\omega,\zeta_\omega)d\omega + g(\omega,\zeta_\omega)dW(\omega), \\ E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(\omega_0) - \int_{\omega_0}^{\omega} f\left(\nu, \tilde{\zeta}_\nu\right)d\nu - \int_{\omega_0}^{\omega} g\left(\nu, \tilde{\zeta}_\nu\right)dW(\nu)\right|^2 \leq \varepsilon(\omega+1), \end{cases} \tag{31}$$

where

$$\begin{aligned} &\chi \in L^2_{\mathscr{F}_{\omega_0}}([-\rho, 0]; \mathbb{R}), \zeta(\omega) \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R}), \\ &f(\omega, \xi_1) = \sin(\omega)\xi_1(0) + \cos(\omega)\xi_1(-\varrho), \xi_1 \in C([-\varrho, 0]; \mathbb{R}), \\ &g(\omega, \xi_1) = \frac{1}{\sqrt{1+\omega^2}}\xi_1(0) + \frac{e^{-\omega}}{\sqrt{1+\omega^2}}\xi_1(-\varrho), \xi_1 \in C([-\varrho, 0]; \mathbb{R}), \end{aligned} \tag{32}$$

with $\varrho > 0$. Then, replacing now ξ_1 by the segment of a solution ζ_ω, we get

$$g(\omega, \zeta_\omega) = \frac{1}{\sqrt{1+\omega^2}}\zeta(\omega) + \frac{e^{-\omega}}{\sqrt{1+\omega^2}}\zeta(\omega - \varrho),$$
$$f(\omega, \zeta_\omega) = \sin(\omega)\zeta(\omega) + \cos(\omega)\zeta(\omega - \varrho). \tag{33}$$

Let $\xi_1, \xi_2 \in C([-\varrho, 0]; \mathbb{R})$, then

$$\begin{aligned}|f(\omega,\xi_1) - f(\omega,\xi_2)|^2 &= |\sin(\omega)(\xi_1(0) - \xi_2(0)) + \cos(\omega)(\xi_1(-\varrho) - \xi_2(-\varrho))|^2 \\ &\le 2\sin(\omega)^2|\xi_1(0) - \xi_2(0)|^2 + 2\cos(\omega)^2|\xi_1(-\varrho) - \xi_2(-\varrho)|^2 \\ &\le 4\|\xi_1 - \xi_2\|^2,\end{aligned}$$

$$\begin{aligned}|g(\omega,\xi_1) - g(\omega,\xi_2)|^2 &= \left|\frac{1}{\sqrt{1+\omega^2}}(\xi_1(0) - \xi_2(0)) + \frac{e^{-\omega}}{\sqrt{1+\omega^2}}(\xi_1(-\varrho) - \xi_2(-\varrho))\right|^2 \\ &\le \frac{2}{1+\omega^2}|\xi_1(0) - \xi_2(0)|^2 + \frac{2e^{-2\omega}}{1+\omega^2}|\xi_1(-\varrho) - \xi_2(-\varrho)|^2 \\ &\le 4\|\xi_1 - \xi_2\|^2.\end{aligned} \tag{34}$$

Hence, the uniform Lipschitz condition is satisfied.

Therefore, by Theorem 6, there is a solution $\zeta \in \mathscr{M}^2([\omega_0 - \varrho, T], \mathbb{R})$ of (31), with $\zeta_{\omega_0} = \chi$, such that $\forall \omega \in [\omega_0, T]$,

$$E\left|\tilde{\zeta}(\omega) - \zeta(\omega)\right|^2 \le \frac{1}{\left(1 - \sqrt{\alpha/\alpha+\delta}\right)^2}\exp((\alpha+\delta)(T-\omega_0))\varepsilon(\omega+1), \tag{35}$$

where $\alpha = 32[(T - \omega_0) + 1]$ and $\delta > 0$.

For System (31), we conduct a simulation based on the Euler-Maruyama scheme with step size 10^{-3}, for which we set $\omega_0 = 0$, $\varrho = 0.5$ and the initial, data χ as a map, namely, $\chi = \omega^2$ for all $-0.5 \le \omega \le 0$. In Figure 1, we give a sequence of computer simulations of the exact solution path $\zeta(\omega)$ and the rough solution path $\tilde{\zeta}(\omega)$ for System (31) on the interval $[-0.5,8]$. Choosing $\varepsilon = 10^{-4}$, $\delta = 15$, $\alpha = 35.2$, and $T = 0.1$ one obtain $M = (1/(1 - \sqrt{\alpha/\alpha+\delta})^2)\exp((\alpha+\delta)(T - \omega_0)) = 5.72 \times 10^3$. We use the time step 10^{-4} of the interval $[0,0.1]$ and 10000 realizations for this discretisation; we give in Figure 2 the trajectory of $\varepsilon \times M \times (\omega+1)$ and simulation of the mean square of $|\tilde{\zeta}(\omega) - \zeta(\omega)|$ on the interval $[0,0.1]$. It is clear that the convergence plot verifies the theoretical findings.

Example 11. Consider the following SFDE for each $\varepsilon > 0$

$$\begin{cases} d\zeta(\omega) = f(\omega,\zeta_\omega)d\omega + g(\omega,\zeta_\omega)dW(\omega), \\ E\left|\tilde{\zeta}(\omega) - \tilde{\zeta}(0) - \int_0^\omega f\left(\nu,\tilde{\zeta}_\nu\right)d\nu - \int_0^\omega g\left(\nu,\tilde{\zeta}_\nu\right)dW(\nu)\right|^2 \le \varepsilon, \end{cases} \tag{36}$$

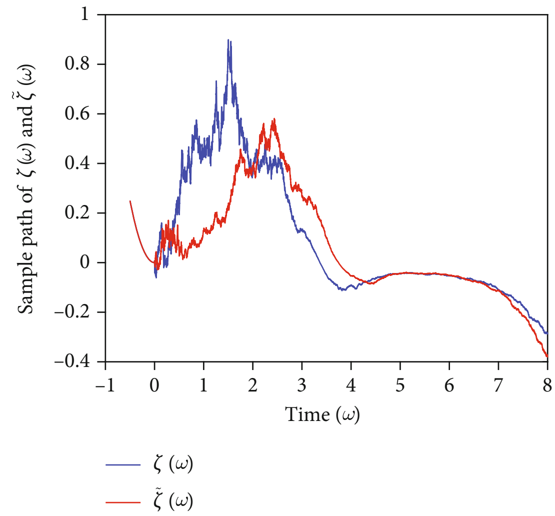

FIGURE 1: Simulation of $\zeta(\omega)$ and $\tilde{\zeta}(\omega)$ are trajectory in System (31) with $\varrho = 0.5$ and $\chi(\omega) = \omega^2$ for $\omega \in [-0.5,8]$.

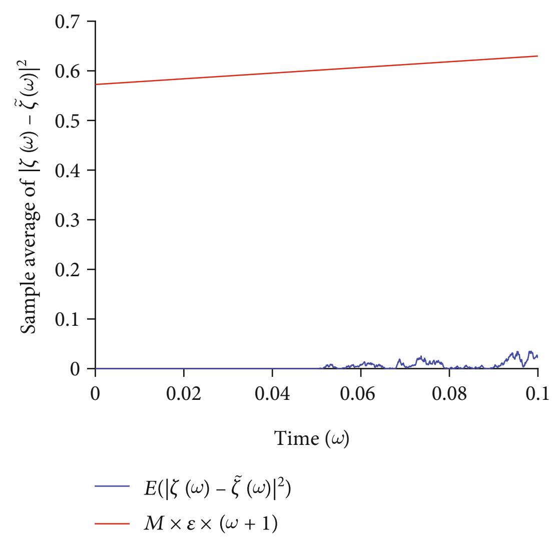

FIGURE 2: HURS with respect to $(\varepsilon, \theta(\omega))$ of $\zeta(\omega)$ on the interval $[0,0.1]$ with $\chi(\omega) = \omega^2$.

where

$$\chi \in L^2_{\mathscr{F}_0}([-\rho, 0]; \mathbb{R}), \zeta(\omega) \in \mathscr{M}^2([-\varrho, 3], \mathbb{R}),$$
$$f(\omega,\xi_1) = \omega\xi_1(0) + \omega^2\xi_1(-\varrho), \xi_1 \in C([-\varrho, 0]; \mathbb{R}), \tag{37}$$
$$g(\omega,\xi_1) = \omega^2\xi_1(0) + \omega\xi_1(-\varrho), \xi_1 \in C([-\varrho, 0]; \mathbb{R}),$$

with $\rho > 0$. Then, replacing now ξ_1 by the segment of a solution ζ_ω, we have

$$f(\omega,\zeta_\omega) = \omega\zeta(\omega) + \omega^2\zeta(\omega - \varrho),$$
$$g(\omega,\zeta_\omega) = \omega^2\zeta(\omega) + \omega\zeta(\omega - \varrho). \tag{38}$$

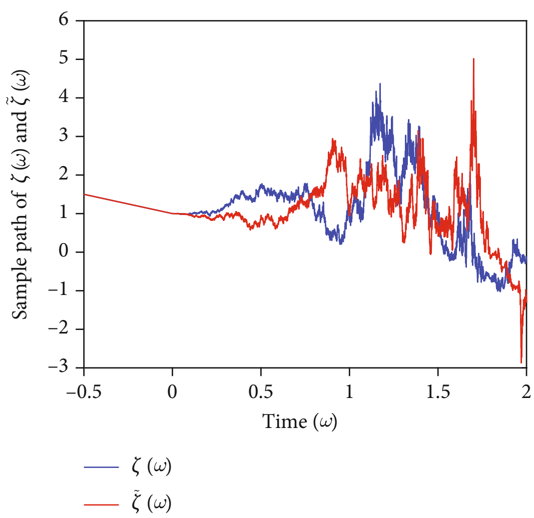

Figure 3: Simulation of $\zeta(\omega)$ and $\tilde{\zeta}(\omega)$ are trajectory in System (36) with $\varrho = 0.5$ and $\chi(\omega) = -\omega + 1$ for $\omega \in [-0.5,2]$.

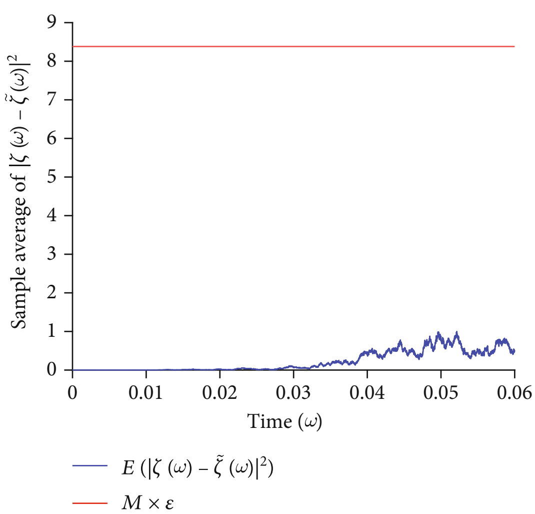

Figure 4: HUS with respect to ε of $\zeta(\omega)$ on the interval $[0,0.06]$ with $\chi(\omega) = -\omega + 1$.

Let $\xi_1, \xi_2 \in C([-\varrho, 0]; \mathbb{R})$, then

$$\begin{aligned} |f(\omega,\xi_1) - f(\omega,\xi_2)|^2 &= \left|\omega(\xi_1(0) - \xi_2(0)) + \omega^2(\xi_1(-\varrho) - \xi_2(-\varrho))\right|^2 \\ &\le 2\omega^2|\xi_1(0) - \xi_2(0)|^2 + 2\omega^4|\xi_1(-\varrho) - \xi_2(-\varrho)|^2 \\ &\le 180\|\xi_1 - \xi_2\|^2, \end{aligned}$$

$$\begin{aligned} |g(\omega,\xi_1) - g(\omega,\xi_2)|^2 &= \left|\omega^2(\xi_1(0) - \xi_2(0)) + \omega(\xi_1(-\varrho) - \xi_2(-\varrho))\right|^2 \\ &\le 2\omega^4|\xi_1(0) - \xi_2(0)|^2 + 2\omega^2|\xi_1(-\varrho) - \xi_2(-\varrho)|^2 \\ &\le 180\|\xi_1 - \xi_2\|^2. \end{aligned} \tag{39}$$

Hence, the uniform Lipschitz condition is satisfied.

Therefore, by Theorem 8, there is a solution $\zeta \in \mathcal{M}^2([-\varrho, 3], \mathbb{R})$ of (36), with $\zeta_0 = \chi$, such that $\forall \omega \in [0, 3]$,

$$E\left|\tilde{\zeta}(\omega) - \zeta(\omega)\right|^2 \le \frac{1}{\left(1 - \sqrt{\alpha/\alpha + \delta}\right)^2} \exp\left(3(\alpha + \delta)\right)\varepsilon, \tag{40}$$

where $\alpha = 1440$ and $\delta > 0$.

We use again Euler-Maruyama scheme with step size 1 0^{-5} to conduct a simulation for System (36). We fix $\rho = 0.5$ and the initial data χ as a linear mapping, namely, $\chi = -\omega + 1$ for all $-0.5 \le \omega \le 0$. In Figure 3, we plot the path of the exact solution $\zeta(\omega)$ and the rough solution path $\tilde{\zeta}(\omega)$ for System (36) on the interval $[-0.5,2]$. Choosing $\varepsilon = 10^{-9}$, $\delta = 10$, $\alpha = 1440$, and $T = 0.06$ one obtain a large value of $M = (1/(1 - \sqrt{\alpha/\alpha + \delta})^2) \exp(3(\alpha + \delta))$. We use the time step 10^{-4} of the interval $[0,0.06]$ and 10000 realizations for this discretisation; we give in Figure 4 the trajectory of the constant function $\varepsilon \times M$ and simulation of the mean square of $|\tilde{\zeta}(\omega) - \zeta(\omega)|$ on the interval $[0,0.06]$. It is clear that the convergence plot verifies the theoretical findings.

5. Conclusion

In this paper, we investigate the Ulam-Hyers-Rassias stability of stochastic functional differential equations. To obtain the main results, we used the fixed point theorem and the classical stochastic calculus techniques. Moreover, we extend the Ulam-Hyers-Rassias stability for a generalization version. An example is presented to show the applicability of our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Common Fixed Points of Two *G*-Nonexpansive Mappings via a Faster Iteration Procedure

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In this work, we study the convergence of a new faster iteration in which two G-nonexpansive mappings are involved in the setting of uniformly convex Banach spaces with a directed graph. Moreover, by constructing a numerical example, we show the fastness of our iteration procedure over other existing iteration procedures in the literature.

1. Introduction

In 1922, Banach originated a great tool for solving the existence problems of nonlinear mappings, which is familiar as Banach's contraction principle [1]. After it, this principle has been generalized in so many different directions.

Over the last 50 years, many authors introduced and studied various iteration schemes for different classes of contractive and nonexpansive mappings. In 2008, Jachymski [2] introduced the concept of joining fixed point theory and graph theory and established the Banach contraction principle in a complete metric space endowed with a directed graph. In 2012, Aleomraninejad et al. [3] presented some iterative scheme for G-contraction and G-nonexpansive mappings in a Banach space with a graph. Tiammee et al. [4] familiarized Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space with a directed graph. In 2016, Tripak [5] showed the convergence of a sequence developed by the Ishikawa iteration to some common fixed points of two G-nonexpansive mappings in a Banach space combined with a graph. In 2017, Suparatulatorn et al. [6] introduced and studied the modified S-iteration for two G-nonexpansive mappings in a uniformly convex Banach space associated with a graph. Recently, Thianwan and Yambangwai [7] introduced a new two-step iteration process involving two G-nonexpansive mappings and studied its convergence analysis in a uniformly convex Banach space endowed with a graph. There is vast literature in this direction for more details (see [8–19] and references therein).

Recently, Ullah et al. [20] introduced a new three-step iteration process known as the K-iteration process and proved its convergence for Suzuki's generalized nonexpansive mapping.

Inspired by the above work, we proposed a modified iteration process containing two G-nonexpansive mappings, by generating the sequence $\{r_n\}$ as follows:

Let $\mathscr{C}$ be a nonempty convex subset of a Banach space $\mathscr{X}$, for any random $r_0 \in \mathscr{C}$,

$$t_n = (1-\varsigma_n)r_n + \varsigma_n P_2 r_n, \tag{1}$$

$$s_n = P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n), \tag{2}$$

$$r_{n+1} = P_1 s_n, \tag{3}$$

where $\{\sigma_n\}$ and $\{\varsigma_n\}$ are appropriate real sequences in $(0, 1)$ and $P_1, P_2 : \mathscr{C} \to \mathscr{C}$ are G-nonexpansive mappings. Under some certain conditions, we demonstrate the convergence analysis of (1) for approximating common fixed points of two G-nonexpansive mappings in a uniformly convex

Banach space $\mathscr{X}$ with graph. We also construct a numerical example, and by using MATLAB R2018a, we clarify that the proposed iteration procedure converges faster than modified Ishikawa iteration, modified S-iteration, and Thianwan's new iteration (see [5–7]).

2. Preliminaries

In this part, we gather some familiar concepts and applicable conclusions which will be used often.

Let $\mathscr{C}$ be a nonempty subset of a Banach space $\mathscr{X}$. Let Δ denote the diagonal of the cartesian product $\mathscr{C}\times\mathscr{C}$, i.e., $\Delta=\{(r,r): r\in\mathscr{C}\}$.

$V(G)$ denotes the set of vertices that coincides with $\mathscr{C}$ in a directed graph G, and the set $E(G)$ of its edges contains all loops, i.e., $E(G)\supseteq\Delta$. By assuming G has no parallel edge to identify the graph G with the pair $(V(G),E(G)$. G^{-1} denotes the conversion of a graph G. So we have

$$E\left(G^{-1}\right)=\{(r,s)\in\mathscr{C}\times\mathscr{C} : (s,r)\in E(G)\}. \quad (4)$$

A set B is said to be dominated by r_0 if for each $r\in B$, $(r_0,r)\in E(G)$ and dominates r_0 if for each $r\in B$, $(r,r_0)\in E(G)$.

Let $P:\mathscr{C}\to\mathscr{C}$ be a self map. An edge preserving mapping, i.e., $(r,s)\in E(G)\Rightarrow(Pr,Ps)\in E(G)$, is said to be G-nonexpansive if

$$\|Pr-Ps\|\le\|r-s\| \quad \forall(r,s)\in E(G). \quad (5)$$

A mapping $P:\mathscr{C}\to\mathscr{C}$ is said to be G-demiclosed at 0, if for any sequence $\{r_n\}$ in $\mathscr{C}$ such that $r_n\rightharpoonup r$ for all $(r_n,r_{n+1})\in E(G)$ and $Pr_n\to 0$ then $Pr=0$.

Let us recall that a Banach space $\mathscr{X}$ is said to satisfy Opial's property if $r_n\rightharpoonup r$ and then

$$\limsup_{n\to\infty}\|r_n-r\|<\limsup_{n\to\infty}\|r_n-s\|, \quad \forall s\neq r. \quad (6)$$

Lemma 1 [21]. *Let $\mathscr{C}$ be a subset of a metric space $(\mathscr{X},d)$. A mapping $P:\mathscr{C}\to\mathscr{C}$ is semicompact if for a sequence $\{r_n\}$ in $\mathscr{C}$ with $\lim_{n\to\infty}d(r_n,Pr_n)=0$ there exists a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ such that $r_{n_j}\to p\in\mathscr{C}$.*

Let $\mathscr{C}$ be a subset of a normed space $\mathscr{X}$ and let $G=(V(G),E(G))$ be a directed graph such that $V(G)=\mathscr{C}$. Then, $\mathscr{C}$ is said to have property WG (SG) if for each sequence $\{r_n\}$ in $\mathscr{C}$ converging weakly (strongly) to $r\in\mathscr{C}$ and $(r_n,r_{n+1})\in E(G)$ there is a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ such that $(r_{n_j},r)\in E(G)$ for all $n\in\mathbb{N}$.

Lemma 2 [6]. *Suppose that $\mathscr{X}$ is the Banach space having Opial's condition, $\mathscr{C}$ has property WG, and let $P:\mathscr{C}\to\mathscr{C}$ be a G-nonexpansive mapping. Then, $I-P$ is G-demiclosed at 0, i.e., if $r_n\rightharpoonup r$ and $(r_n-Pr_n)\to 0$, then $r\in F(P)$, where $F(P)$ is the set of fixed points of P.*

Lemma 3 [22]. *Let $\mathscr{X}$ be uniformly convex Banach space and $\{\sigma_n\}$ a sequence in $[\delta,1-\delta]$ for some $\delta\in(0,1)$. Suppose that the sequences $\{r_n\}$ and $\{s_n\}$ in $\mathscr{X}$ are such that $\limsup_{n\to\infty}\|r_n\|\le c$, $\limsup_{n\to\infty}\|s_n\|\le c$, and $\limsup_{n\to\infty}\|\sigma_n r_n+(1-\sigma_n)s_n\|=c$ for some $c\ge 0$. Then, $\lim_{n\to\infty}\|r_n-s_n\|=0$.*

Lemma 4 [23]. *Let $\mathscr{X}$ be the Banach space that satisfies Opial's condition and let $\{r_n\}$ be a sequence in $\mathscr{X}$. Let $u,v\in X$ be such that $\lim_{n\to\infty}\|r_n-u\|$ and $\lim_{n\to\infty}\|r_n-v\|$ exist. If $\{r_{n_j}\}$ and $\{r_{n_k}\}$ are subsequences of $\{r_n\}$ that converges weakly to u and v, respectively, then $u=v$.*

Lemma 5 [24]. *Let $\{r_n\}$ be a bounded sequence in a reflexive Banach space $\mathscr{X}$. If for any weakly convergent subsequences $\{r_{n_j}\}$ of $\{r_n\}$, both $\{r_{n_j}\}$ and $\{r_{n_{j+1}}\}$ converge weakly to the same point in $\mathscr{X}$, then the sequence $\{r_n\}$ is weakly convergent.*

Lemma 6 [7]. *Let $\mathscr{C}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$ and suppose that $\mathscr{C}$ has property WG. Let P be a G-nonexpansive mapping on $\mathscr{C}$. Then, $I-P$ is G-demiclosed at 0.*

3. Main Results

We initiate this section by proving the following proposition.

Proposition 7. *Let P_1 and P_2 be two self G-nonexpansive mappings on $\mathscr{C}$ with $F=F(P_1)\cap F(P_2)$ nonempty, where $\mathscr{C}$ is a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$ endowed with a directed graph. Let $V(G)=\mathscr{C}$, $E(G)$ is convex and the graph G is transitive. For random $r_0\in C$, define the sequence $\{r_n\}$ by (1). Let $p_0\in F$ be such that (r_0,p_0), (p_0,r_0) are in $E(G)$. Then, (r_n,p_0), (s_n,p_0), (t_n,p_0), (p_0,r_n), (p_0,s_n), (p_0,t_n), (r_n,s_n), (r_n,t_n), and (r_n,r_{n+1}) are in $E(G)$.*

Proof. We go ahead by induction. By using the edge preserving property of P_2 and assumption $(r_0,p_0)\in E(G)$, we get $(P_2r_0,p_0)\in E(G)$. By convexity of $E(G)$, we obtain $(t_0,p_0)\in E(G)$. Again, by edge preservingness of P_2 and $(t_0,p_0)\in E(G)$, we have $(P_2t_0,p_0)\in E(G)$. By convexity of $E(G)$ and $(P_1r_0,p_0),(P_2t_0,p_0)\in E(G)$ and applying the edge preservingness of P_2 again, we have $(s_0,p_0)\in E(G)$. Again, by using the property of edge preserving of P_1 and $(s_0,p_0)\in E(G)$, we get $(r_1,p_0)\in E(G)$. Again, by applying edge preserving of P_2, $(P_2r_1,p_0)\in E(G)$, we get $(t_1,p_0)\in E(G)$ as $E(G)$ is convex. Thus, by edge preserving of P_2, $(P_2t_1,p_0)\in E(G)$. Again, by using convexity of $E(G)(P_1r_1,p_0),(P_2t_1,p_0)\in E(G)$ and edge preservingness of P_2, we get $(s_1,p_0)\in E(G)$. By edge preserving of P_1, we get $(P_1s_1,p_0)\in E(G)$, and we get $(r_2,p_0)\in E(G)$. Next, we assume that $(r_k,p_0)\in E(G)$. By edge preserving of P_2 and convexity of $E(G)$, we get $(P_2r_k,p_0)\in E(G)$ and $(t_k,p_0)\in E(G)$. By applying edge preserving of P_2 on $(t_k,p_0)\in E(G)$, we get $(P_2t_k,p_0)\in E(G)$. By using convexity of $E(G)$ and $(P_1r_k,p_0),(P_2t_k,p_0)\in E(G)$ and edge preserving property of P_2, we have $(s_k,p_0)\in E(G)$. As P_1 is edge preserving, we get $(r_{k+1}, p_0) \in E(G)$. Owing to edge preserving of P_2, we obtain $(P_2 r_{k+1}, p_0) \in E(G)$ and so $(t_{k+1}, p_0) \in E(G)$, since $E(G)$ is convex. By convexity of $E(G)$ and $(P_1 r_{k+1}, p_0), (P_2 t_{k+1}, p_0) \in E(G)$ and applying the edge preservingness of P_2 again, we have $(s_{k+1}, p_0) \in E(G)$. Therefore, $(r_n, p_0), (s_n, p_0), (t_n, p_0) \in E(G)$ for all $n \geq 1$. Using a similar argument, we can show that $(p_0, r_n), (p_0, s_n), (p_0, t_n) \in E(G)$ under the assumption that $(p_0, r_0) \in E(G)$. By using transitivity of G, we get $(r_n, s_n), (r_n, t_n), (s_n, t_n), (r_n, r_{n+1}) \in E(G)$. This completes the proof.

Lemma 8. *Let $\mathscr{X}$, $\mathscr{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7. Suppose that $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$ and $(r_0, p_0), (p_0, r_0) \in E(G)$ for arbitrary $r_0 \in \mathscr{C}$ and $p_0 \in F$. Then,*

(i) $\lim_{n\to\infty} \|r_n - p_0\|$ exists

(ii) $\lim_{n\to\infty} \|P_1 r_n - r_n\| = 0 = \lim_{n\to\infty} \|P_2 r_n - r_n\|$

Proof.

(i) Let $p_0 \in F$. By Proposition 7, we have $(r_n, p_0), (s_n, p_0), (t_n, p_0) \in E(G)$. As P_1 and P_2 are G-nonexpansive mappings and using the iterative sequence $\{r_n\}$, we have

$$\begin{aligned}\|t_n - p_0\| &= \|(1-\varsigma_n)r_n + \varsigma_n P_2 r_n - p_0\| \leq (1-\varsigma_n)\|r_n \\ &\quad - p_0\| + \varsigma_n \|P_2 r_n - p_0\| \leq (1-\varsigma_n)\|r_n - p_0\| \\ &\quad + \varsigma_n \|r_n - p_0\| = \|r_n - p_0\|,\end{aligned} \tag{7}$$

$$\begin{aligned}\|s_n - p_0\| &= \|P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n) - p_0\| \\ &\leq \|(1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n - p_0\| \\ &\leq (1-\sigma_n)\|P_1 r_n - p_0\| + \sigma_n \|P_2 t_n - p_0\| \\ &\leq (1-\sigma_n)\|r_n - p_0\| + \sigma_n \|t_n - p_0\| \\ &\leq (1-\sigma_n)\|r_n - p_0\| + \sigma_n \|r_n - p_0\| \leq \|r_n - p_0\|,\end{aligned} \tag{8}$$

$$\|r_{n+1} - p_0\| = \|P_1 s_n - p_0\| \leq \|s_n - p_0\| \leq \|r_n - p_0\|. \tag{9}$$

This implies that sequence $\{\|r_n - p_0\|\}$ is decreasing and bounded below for all $p_0 \in F(P)$. Hence, $\lim_{n\to\infty} \|r_n - p_0\|$ exists.

(ii) Assume that $\lim_{n\to\infty} \|r_n - p_0\| = c$. If $c = 0$; then, by G-nonexpansiveness of P_1 and P_2, we get

$$\|r_n - P_i r_n\| \leq \|r_n - p_0\| + \|p_0 - P_i r_n\| \leq \|r_n - p_0\| + \|p_0 - r_n\|. \tag{10}$$

Therefore, the result follows.
Suppose that $c > 0$.

Taking the lim sup on both sides in the inequality (7) and (8), we obtain

$$\limsup_{n\to\infty} \|t_n - p_0\| \leq \limsup_{n\to\infty} \|r_n - p_0\| = c, \tag{11}$$

$$\limsup_{n\to\infty} \|s_n - p_0\| \leq \limsup_{n\to\infty} \|r_n - p_0\| = c. \tag{12}$$

On the other hand, using (1), we have

$$\begin{aligned}\|r_{n+1} - p_0\| &= \|P_1 s_n - p_0\| \leq \|s_n - p_0\| \\ &\leq \|P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n) - p_0\| \\ &\leq \|(1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n - p_0\| \\ &= \|(1-\sigma_n)(P_1 r_n - p_0) + \sigma_n(P_2 t_n - p_0)\| \\ &\leq (1-\sigma_n)\|r_n - p_0\| + \sigma_n \|t_n - p_0\| \\ &\leq \|r_n - p_0\| - \sigma_n \|r_n - p_0\| + \sigma_n \|t_n - p_0\|.\end{aligned} \tag{13}$$

This implies that

$$\frac{\|r_{n+1} - p_0\| - \|r_n - p_0\|}{\sigma_n} \leq \|t_n - p_0\| - \|r_n - p_0\|. \tag{14}$$

So

$$\begin{aligned}\|r_{n+1} - p_0\| - \|r_n - p_0\| &\leq \frac{\|r_{n+1} - p_0\| - \|r_n - p_0\|}{\sigma_n} \\ &\leq \|t_n - p_0\| - \|r_n - p_0\|.\end{aligned} \tag{15}$$

This implies

$$\|r_{n+1} - p_0\| \leq \|t_n - p_0\|. \tag{16}$$

By taking lim inf both sides, we have

$$c \leq \liminf_{n\to\infty} \|t_n - p_0\|. \tag{17}$$

From (11) and (17), we get

$$c = \lim_{n\to\infty} \|t_n - p_0\|, \tag{18}$$

$$c = \lim_{n\to\infty} \|(1-\varsigma_n)r_n + \varsigma_n P_2 r_n - p_0. \tag{19}$$

By using (11) and (19) and Lemma 3, we get

$$\lim_{n\to\infty} \|P_2 r_n - r_n\| = 0, \tag{20}$$

$$\|r_{n+1} - p_0\| \leq \|P_1 s_n - p_0\| \leq \|s_n - p_0\|. \tag{21}$$

By taking lim inf both sides, we have

$$c \leq \liminf_{n\to\infty} \|s_n - p_0\|. \tag{22}$$

By using (12) and (22), we have

$$\lim_{n\to\infty}\|s_n - p_0\| = c,$$
$$\lim_{n\to\infty}\|P_2((1-\sigma_n)P_1r_n + \sigma_nP_2t_n) - p_0\| = c. \tag{23}$$

We also have

$$\|P_2((1-\sigma_n)P_1r_n + \sigma_nP_2t_n) - p_0\| \le \|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\|. \tag{24}$$

By taking limit infimum on both sides, we get

$$c \le \liminf_{n\to\infty}\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\|. \tag{25}$$

By using edge preserving property of P_1 and P_2, we have

$$\|P_1r_n - p_0\| \le \|r_n - p_0\|, \tag{26}$$

$$\|P_2t_n - p_0\| \le \|t_n - p_0\| \le \|r_n - p_0\|. \tag{27}$$

By taking limit sup on both sides in (26) and (27), we get

$$\limsup_{n\to\infty}\|P_1r_n - p_0\| \le c, \tag{28}$$

$$\limsup_{n\to\infty}\|P_2t_n - p_0\| \le c. \tag{29}$$

By using (28) and (29), we obtain

$$\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\| \le (1-\sigma_n)\|P_1r_n - p_0\| + \sigma_n\|P_2t_n - p_0\| \le (1-\sigma_n)\|r_n - p_0\| + \sigma_n\|t_n - p_0\| \le \|r_n - p_0\|. \tag{30}$$

By taking limit sup on both sides in (30), we get

$$\limsup_{n\to\infty}\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\| \le c. \tag{31}$$

By using (25) and (31), we have

$$\lim_{n\to\infty}\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\| = c. \tag{32}$$

By (28), (29), and (32) and Lemma 3, we have

$$\lim_{n\to\infty}\|P_1r_n - P_2t_n\| = 0. \tag{33}$$

In addition,

$$\|t_n - r_n\| = \|(1-\varsigma_n)r_n + \varsigma_nP_2r_n - r_n\| = \|\varsigma_n(P_2r_n - r_n)\| \to 0 (\text{as} n \to \infty). \tag{34}$$

By using (20), (33), and (34), we have

$$\|P_1r_n - r_n\| = \|P_1r_n - P_2t_n + P_2t_n - r_n\| \le \|P_1r_n - P_2t_n\| + \|P_2t_n - r_n\| \le \|P_1r_n - P_2t_n\| + \|P_2t_n - P_2r_n\| + \|P_2r_n - r_n\| \le \|P_1r_n - P_2t_n\| + \|t_n - r_n\| + \|P_2r_n - r_n\| \to 0 (\text{as } n \to \infty). \tag{35}$$

Therefore, we conclude $\lim_{n\to\infty}\|P_1r_n - r_n\| = 0 = \lim_{n\to\infty}\|P_2r_n - r_n\|$.

The next proof is for the weak convergence of the sequence generated by (1) in a uniformly convex Banach space with directed graph satisfying Opial's condition.

Theorem 9. *Let $\mathcal{X}$, $\mathcal{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7 with $\mathcal{X}$ satisfying Opial's condition and $\mathcal{C}$ has property WG. Suppose that $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0,1)$ and $(r_0,p_0),(p_0,r_0) \in E(G)$ for arbitrary $r_0 \in \mathcal{C}$ and $p_0 \in F$ and then $\{r_n\}$ weakly converges to a common fixed point of P_1 and P_2.*

Proof. Let $p_0 \in F$ be such that $(r_0,p_0),(p_0,r_0) \in E(G)$. From Lemm 8(i), $\lim_{n\to\infty}\|r_n - p_0\|$ exists, so $\{r_n\}$ is bounded. It follows from Lemma 8(ii) that $\lim_{n\to\infty}\|P_1r_n - r_n\| = 0 = \lim_{n\to\infty}\|P_2r_n - r_n\|$. Since $\mathcal{X}$ is uniformly convex and $\{r_n\}$ is bounded, we may assume that $r_n \rightharpoonup u$ as $n \to \infty$, without loss of generality. By Lemma 2, we have $u \in F$. Suppose that subsequences $\{r_{n_k}\}$ and $\{r_{n_j}\}$ of $\{r_n\}$ converges weakly to u and v, respectively. By Lemma 8(ii), we obtain that $\|P_ir_{n_k} - r_{n_k}\| \to 0$ and $\|P_ir_{n_j} - r_{n_j}\| \to 0$ as $k,j \to \infty$. Using Lemma 2, we have $u,v \in F$. By Lemma 8(i), $\lim_{n\to\infty}\|r_n - u\|$ and $\lim_{n\to\infty}\|r_n - v\|$ exist. It follows from Lemma 4 that $u = v$. Therefore, $\{r_n\}$ converges weakly to a common fixed point of P_1 and P_2.

Next, we prove weak convergence of the sequence $\{r_n\}$ generated by (1) without assuming Opial's condition in a uniformly convex Banach space with a directed graph.

Theorem 10. *Let $\mathcal{X}$, $\mathcal{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7 with C having property WG, $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0,1)$, F is dominated by r_0, and F dominates r_0. Then, $\{r_n\}$ weakly converges to a common fixed point of P_1 and P_2.*

Proof. Let $p_0 \in F$ be such that $(r_0,p_0),(p_0,r_0) \in E(G)$. From Lemma 8(i), $\lim_{n\to\infty}\|r_n - p_0\|$ exists, so $\{r_n\}$ is bounded in $\mathcal{C}$. Since $\mathcal{C}$ is nonempty closed convex subset of a uniformly convex Banach space $\mathcal{X}$, it is weakly compact and hence there exists a subsequence $\{r_{n_j}\}$ of the sequence $\{r_n\}$ which converges weakly to some point $p \in \mathcal{C}$. By Lemma 8(ii), we obtain that

$$\|r_{n_j} - P_1r_{n_j}\| = 0 = \|r_{n_j} - P_2r_{n_j}\|. \tag{36}$$

By using (20), (34), and (35), we have

$$\begin{aligned}\|s_n - r_n\| &= \|P_2((1-\sigma_n)P_1r_n + \sigma_nP_2t_n) - r_n\| \\ &\le \|P_2((1-\sigma_n)P_1r_n + \sigma_nP_2t_n) - P_2r_n\| + \|P_2r_n - r_n\| \\ &\le \|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - r_n\| + \|P_2r_n - r_n\| \\ &\le \|(1-\sigma_n)(P_1r_n - r_n) + \sigma_n(P_2t_n - r_n)\| + \|P_2r_n - r_n\| \\ &\le (1-\sigma_n)\|P_1r_n - r_n\| + \sigma_n\|P_2t_n - r_n\| + \|P_2r_n - r_n\| \\ &\le (1-\sigma_n)\|P_1r_n - r_n\| + \sigma_n\|P_2t_n - P_2r_n\| + \|P_2r_n - r_n\| \\ &\le (1-\sigma_n)\|P_1r_n - r_n\| + \sigma_n\|t_n - r_n\| \\ &\quad + \|P_2r_n - r_n\| \to 0 \ (\text{as } n\to\infty).\end{aligned} \tag{37}$$

In addition,

$$\begin{aligned}\|P_2t_n - t_n\| &\le \|P_2t_n - P_1r_n\| + \|P_1r_n - t_n\| \le \|P_2t_n - P_1r_n\| \\ &\quad + \|P_1r_n - r_n\| + \|r_n - t_n\| \to 0 \ (\text{as } n\to\infty).\end{aligned} \tag{38}$$

And by using (35) and (37),

$$\begin{aligned}\|P_1s_n - s_n\| &\le \|P_1s_n - P_1r_n\| + \|P_1r_n - s_n\| \le \|s_n - r_n\| \\ &\quad + \|P_1r_n - r_n\| + \|r_n - s_n\| \le 2\|s_n - r_n\| \\ &\quad + \|P_1r_n - r_n\| \to 0 \ (\text{as } n\to\infty).\end{aligned} \tag{39}$$

Using Lemma 6, $I - P_1$ and $I - P_2$ are G-demiclosed at 0 so that $p \in F$. To complete the proof, it suffices to show that $\{r_n\}$ converges weakly to p. To this end, we need to show that $\{r_n\}$ satisfies the hypothesis of Lemma 5. Let $\{r_{n_j}\}$ be a subsequence of $\{r_n\}$ which converges weakly to some $q \in \mathscr{C}$. By similar argument as above $q \in F$. Now, for each $j \ge 1$, using (1), we have

$$r_{n_{j+1}} = P_1 s_{n_j}. \tag{40}$$

By using (34), we get

$$P_1 r_{n_j} = \left(P_1 r_{n_j} - r_{n_j}\right) + r_{n_j} \rightharpoonup q. \tag{41}$$

By using (20), we have

$$P_2 r_{n_j} = \left(P_2 r_{n_j} - r_{n_j}\right) + r_{n_j} \rightharpoonup q. \tag{42}$$

We have

$$t_{n_j} = \left(1 - \varsigma_{n_j}\right) r_{n_j} + \varsigma_{n_j} P_2 r_{n_j} \rightharpoonup q. \tag{43}$$

It follows from (38)

$$P_2 t_{n_j} = \left(P_2 t_{n_j} - t_{n_j}\right) + t_{n_j} \rightharpoonup q. \tag{44}$$

By using (41) and (44), we get

$$s_{n_j} = P_2\left(\left(1 - \sigma_{n_j}\right) P_1 r_{n_j} + \sigma_{n_j} P_2 t_{n_j}\right) \rightharpoonup q. \tag{45}$$

It follows from (39) and (45), we get

$$\begin{aligned}P_1 s_{n_j} &= \left(P_1 s_{n_j} - s_{n_j}\right) + s_{n_j} \rightharpoonup q, \\ r_{n_{j+1}} &= P_1 s_{n_j} \rightharpoonup q.\end{aligned} \tag{46}$$

Therefore, the sequence $\{r_n\}$ satisfies the hypothesis of Lemma 5 which in turn implies that $\{r_n\}$ weakly converges to q so that $p = q$.

Next, we recall condition (B) for strong convergence.

Let $\mathscr{C}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$. The mappings P_1 and P_2 on $\mathscr{C}$ are said to satisfy condition (B) [21] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(z) > 0$ for all $z > 0$ such that for all $r \in \mathscr{C}$,

$$\max\{\|r - P_1 r\|, \|r - P_2 r\|\} \ge f(d(r, F)), \tag{47}$$

where $d(r, F) = \inf\{\|r - q\| : q \in F\}$.

Theorem 11. *Let $\mathscr{X}$, $\mathscr{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7. Suppose that $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$, $P_i (i = 1, 2)$ satisfy condition (B), F is dominated by r_0 , and F dominates r_0. Then, the sequence $\{r_n\}$ converges strongly to a common fixed point of P_1 and P_2.*

Proof. From Lemma 8(i), $\lim_{n\to\infty}\|r_n - q\|$ exists and so $\lim_{n\to\infty} d(r_n, F)$ exists for any $q \in F$. Also, from Lemma 8(ii), $\lim_{n\to\infty}\|r_n - P_1 r_n\| = 0 = \lim_{n\to\infty}\|r_n - P_2 r_n\|$. Owing to condition (B),

$$f(d(r, F)) \le \max\{\|r - P_1 r\|, \|r - P_2 r\|\}. \tag{48}$$

We have $\lim_{n\to\infty} f(d(r_n, F)) = 0$. As $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(z) > 0$ for all $z \in [0,\infty)$, we obtain that $\lim_{n\to\infty} d(r_n, F) = 0$.

Hence, we can find a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ and a sequence $\{u_j\} \subset F$ such that $\|r_{n_j} - u_j\| \le (1/2^j)$. Put $n_{j+1} = n_j + h$ for some $h \ge 1$. Then,

$$\begin{aligned}\|r_{n_{j+1}} - u_j\| &\le \|r_{n_j+h-1} - u_j\| \le \|r_{n_j} - u_j\| \le \frac{1}{2^j}, \\ \|u_{j+1} - u_j\| &\le \|u_{j+1} - r_{n_{j+1}}\| + \|r_{n_{j+1}} - u_j\| \le \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}} \to 0 \ (\text{as } j\to\infty).\end{aligned} \tag{49}$$

So $\{u_j\}$ is a Cauchy sequence. We assume that $u_j \to q_0 \in C$ as $j \to \infty$. Since F is closed, we get $q_0 \in F$. So we have

Table 1: Convergence of iterative schemes.

Iteration no.	Modified Ishikawa iteration	Modified S-iteration	Thianwan's new iteration	Proposed iteration
1.0	0.5000000000000	0.5000000000000	0.5000000000000	0.5000000000000
3.0	0.5839729465079	0.9045303242056	0.9160290275994	0.9953507671826
5.0	0.7228781259509	0.9831124491843	0.9875389515921	0.9999589583885
7.0	0.8483210253860	0.9970590402716	0.9982054854975	0.9999996090513
9.0	0.9299336356828	0.9995059080444	0.9997519971246	0.9999999961503
11.0	0.9720289950766	0.9999208072730	0.9999673170979	0.9999999999616
13.0	0.9901611017681	0.9999879343875	0.9999958969892	0.9999999999996
15.0	0.9969024367323	0.9999982522932	0.9999995085433	1.00000000000000

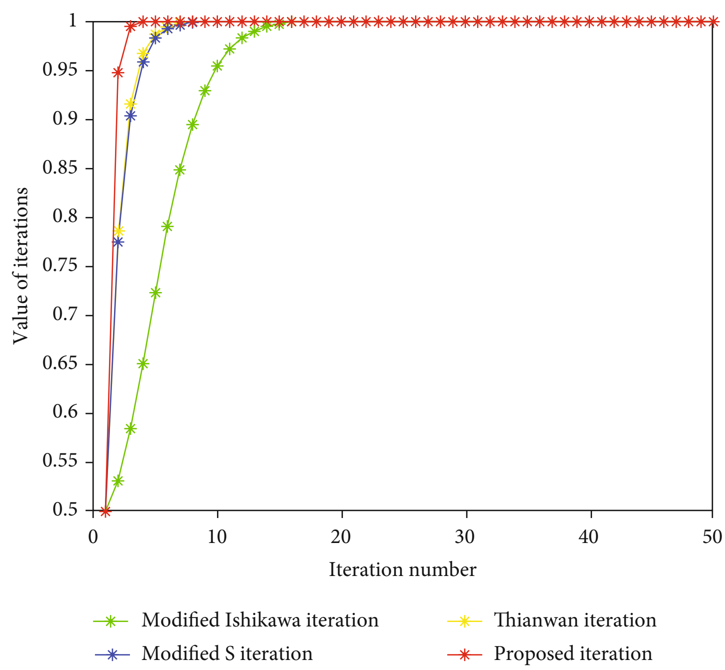

Figure 1: Numerical experiment of Example 1 using the modified Ishikawa iteration, modified S-iteration, Thianwan new iteration, and proposed iteration.

$r_{n_j} \to q_0$ as $j \to \infty$. Since $\lim_{n\to\infty} \|r_n - q_0\|$ exists, we get $r_n \to q_0$.

We prove another strong convergence theorem as follows.

Theorem 12. *Let $\mathscr{X}$, $\mathscr{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7 with C having property SG, $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$, F is dominated by r_0, and F dominates r_0. If one of $P_i (i = 1, 2)$ is semicompact, then $\{r_n\}$ converges strongly to a common fixed point of P_1 and P_2.*

Proof. It follows from Lemma 8 that $\{r_n\}$ is bounded and $\lim_{n\to\infty} \|P_1 r_n - r_n\| = 0 = \lim_{n\to\infty} \|P_2 r_n - r_n\|$. Since one of P_1 and P_2 is semicompact, then there exist subsequences $\{r_{n_j}\}$ of $\{r_n\}$ such that $r_{n_j} \to q \in \mathscr{C}$ as $j \to \infty$. Since $\mathscr{C}$ has property SG and transitivity of graph G, we obtain $(r_{n_j}, q) \in E(G)$. Notice that, for each $i \in 0, 1$, $\lim_{j\to\infty} \|r_{n_j} - P_i r_{n_j}\| = 0$.

Then,

$$\begin{aligned}\|q - P_i q\| &\le \|q - r_{n_j}\| + \|r_{n_j} - P_i r_{n_j}\| + \|P_i r_{n_j} - P_i q\| \\ &\le \|q - r_{n_j}\| + \|r_{n_j} - \mathrm{P}_i r_{n_j}\| + \|r_{n_j} - q\| \to 0 \; (\text{as } j \to \infty).\end{aligned} \tag{50}$$

Hence, $q \in F$. Thus, $\lim_{n\to\infty} d(r_n, F)$ exists by Theorem 11. We note that $d(r_{n_j}, F) \le d(r_{n_j}, q) \to 0$ as $j \to \infty$. Hence, $\lim_{n\to\infty} d(r_n, F) = 0$. It follows, as in the proof of Theorem 11, that $\{r_n\}$ converges strongly to a common fixed point of P_1 and P_2.

4. Numerical Examples

This section contains a numerical example which supports our main theorem. It is worth mentioning here that this example is motivated by [25].

Example 1. Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{C} = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = \mathcal{C}$ and $(r, s) \in E(G)$ if and only if $0.50 \leq r \neq s \leq 1.70$ or $r = s \in \mathcal{C}$. Define mappings $P_1, P_2 : \mathcal{C} \rightarrow \mathcal{C}$

$$\begin{aligned} P_1 r &= r^{1/3}, \\ P_2 r &= \frac{40}{62} \arcsin (r-1) + 1, \end{aligned} \tag{51}$$

for any $r \in \mathcal{C}$.

It is easy to show that P_1, P_2 are G-nonexpansive mappings but P_1, P_2 are not nonexpansive mappings because

$$\begin{aligned} |P_1 r - P_1 s| &> |r - s|, \\ |P_2 u - P_2 v| &> |u - v|, \end{aligned} \tag{52}$$

when $r = 0.5$, $s = .03$, $u = 1.9$, and $v = 1.5$. Choose $\sigma_n = n/(n+5)$, $\varsigma_n = n/(n+5)$, for all $n \in N$ and initial points $r_0 = s_0 = t_0 = 0.5$. In Table 1 and Figure 1, we have shown the convergence rate of the modified Ishikawa iteration, modified S-iteration, Thianwan iteration, and proposed iteration (1).

Figure 1 shows the convergence of the modified Ishikawa iteration, modified S-iteration, Thianwan new iteration, and proposed iteration (1) to the common fixed point of P_1 and P_2 which is 1 in this numerical experiment, and it is clear that the proposed iteration process converges faster than others.

5. Conclusion

The purpose of this paper was to study the convergence of a new faster iteration in which two G-nonexpansive mappings were involved in the setting of uniformly convex Banach spaces with a directed graph. Also, we constructed a numerical example to show the fastness of our iteration procedure over other existing iteration procedures in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors have read and agreed to the published version of the manuscript.

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Research Article

On Some Common Fixed Point Results for Weakly Contraction Mappings with Application

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In this paper, we introduce a new class of generalized weakly contractive mappings and prove common fixed point results by using different algorithms involving this new class of mappings in the framework of b-metric spaces, which generalize the results of Cho. We also provide two examples to show the applicability and validity of our results. As an application of our result, we obtain a solution to an integral equation. Our results extend and improve several comparable results in the existing literature.

1. Introduction

The Banach fixed point theorem [1] popularly known as the Banach contraction mapping principle is a rewarding result in fixed point theory. It has widespread applications in both pure and applied mathematics and has been extended in many different directions. One of the most popular and interesting topics among them is the study of new classes of spaces and their fundamental properties.

In 1993, Czerwik [2] introduced firstly the concept of b-metric space and proved some fixed point theorems of contractive mappings in b-metric space. After that, some authors have researched on the fixed point theorems of various new types of contractive conditions in b-metric space. Aydi et al. in [3] proved common fixed point results for single-valued and multivalued mappings satisfying a weak ϕ-contraction in b-metric spaces. Starting from the results of Berinde [4], Pacurar [5] proved the existence and uniqueness of the fixed point of ϕ-contractions and Zada et al. [6] established fixed point results satisfying contractive conditions of rational type. In 2019, Hussain et al. studied the existence and uniqueness of periodic common fixed point for pairs of mappings via rational type contraction in [7]. After that, in [8], the authors obtained fixed point theorems for cyclic $(\alpha, \beta)-(\psi, \phi)_s$-rational type contractions and discussed the existence of a unique solution to nonlinear fractional differential equations. Also using rational type contractive conditions, Hussain et al. [9] got the existence and uniqueness of a common n-tupled fixed point for a pair of mappings. Using a contraction condition defined by means of a comparison function, [10] established results regarding the common fixed points of two mappings. In 2014, Abbas et al. obtained the results on common fixed point of four mappings in b-metric space in [11]. Iqbal et al. [12] introduced a generalized multivalued (α, L)-almost contraction and proved the existence and uniqueness of the fixed point for a specific mapping in the b-metric space.

Inspired by Czerwik's results, Hussain and Shah in [13] introduced the notion of a cone b-metric space, which means that it is a generalization of b-metric spaces and cone metric spaces; they considered topological properties of cone b-metric spaces and obtained some results on KKM mappings in the setting of cone b-metric spaces. Younis et al. [14] studied the existence of fixed points of a new class of generalized F-contraction in partial b-metric space. In [15], some fixed point results for weakly contractive mappings in ordered partial metric space were obtained. Recently, Samet et al. [16] introduced the concept of α-admissible and $\alpha-\psi$-contractive mappings and presented fixed point theorems for them. In [17, 18], Zoto et al. studied generalized α_{sp} contractive mappings and $(\alpha-\psi, \phi)$-contractions in b-metric-like space. In 2020, Isik et al. [19] firstly introduced the structure of extended quasi b-metric-like spaces as a generalization of both quasi metric-like spaces and quasi b-metric-like spaces. Also, they presented the notion of JSR-contractive mappings in the setup of extended quasi b-metric-like spaces and investigated the existence of fixed point for such mappings. Abu-Donia et al. [20] proved the uniqueness and existence of the fixed points for five mappings from a complete intuitionistic fuzzy 3-metric space into itself under weak compatible of type (α) and asymptotically regular. In 2015, Ege [21] introduced complex valued rectangular b-metric space and proved an analogue of the Banach contraction principle in this space. Recently, Younis et al. [22] provided much simpler and shorter proofs of some new results in rectangular metric spaces, and Mitrovic et al. [23] gave a proof of the results of Miculescu and Mihail [24] and Suzuki [25] in extended b-metric spaces. In graphical b-metric spaces, Younis et al. presented fixed point results for Kannan-type and Reich-type mappings in [26, 27]. Lately, Gholidahneh et al. [28] introduced the notion of a modular p-metric space (an extended modular b-metric space) and established some fixed point results for α-$\widehat{\nu}$-Meir-Keeler contractions in this new space.

In 1997, Alber and Guerre-Delabriere [29] generalized the Banach fixed point theorem by introducing the concept of weak contraction mappings in Hilbert spaces. Weak contraction principle states that every weak contraction mapping on a complete Hilbert space has a unique fixed point. Rhoades [30] extended weak contraction principle in Hilbert spaces to metric spaces. Since then, many authors (for example, [31–42]) obtained generalizations and extensions of the weak contraction principle. Recently, in [43], Jamal et al. used (ψ, ϕ)-weak contraction to generalize coincidence point results which are established in the context of partially ordered b-metric spaces.

In particular, Choudhury et al. [36] obtained a generalization of the weak contraction principle in metric spaces by using altering distance functions as follows:

Theorem 1 (see [36]). *Suppose that a mapping $g : X \longrightarrow X$, where X is a metric space with metric d, satisfies the following condition:*

$$\psi(d(gx, gy)) \le \psi\left(\max\left\{d(x, y), d(x, gx), d(y, gy), \frac{1}{2}\{d(x, gy) + d(y, gx)\}\right\}\right) - \varphi(\max\{d(x, y), d(y, gy)\}), \tag{1}$$

for all $x, y \in X$, where $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous function and $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ is an altering function, that is, ψ is a nondecreasing and continuous function, and $\psi(t) = 0$ if and only if $t = 0$. Then, g has a unique fixed point.

Let X be a metric space with metric d, let $T : X \longrightarrow X$, and let $\varphi : X \longrightarrow [0,+\infty)$ be a lower semicontinuous function. Then, T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \le \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi)), \tag{2}$$

where $\psi \in \Psi$, $\phi \in \Phi$, and

$$m(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}\{d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)\}\}, \tag{3}$$

$$l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}. \tag{4}$$

Cho [44] extended the results of Choudhury et al. [36] to generalized weakly contractive mappings in the setting of metric spaces and obtained the following result:

Theorem 2 (see [44]). *Let X be complete. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

Motivated and inspired by Theorem 2.1 in [44], in this paper, our purpose is to introduce a new class of generalized weakly contractive mappings and obtain a few of common fixed point results by using different algorithms involving generalized weakly contractive conditions in the framework of b-metric space, which generalize the results of Cho. Furthermore, we provide examples that elaborated the useability of our results. Meanwhile, we present an application to the existence of solutions to an integral equation by means of one of our results.

2. Preliminaries

In this section, in order to get our main results, we will introduce some definitions and lemmas first.

Definition 3 (see [2]). Let X be a nonempty set and $s \ge 1$ be a given real number. A mapping $d : X \times X \longrightarrow [0,+\infty)$ is said to be a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \le s(d(x, z) + d(y, z))$.

In general, (X, d) is called a b-metric space with parameter $s \ge 1$.

Remark 4. We should note that a b-metric space with $s = 1$ is a metric space. We can find several examples of b-metric spaces which are not metric spaces (see [45]).

Example 5 (see [46]). Let (X, ρ) be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then, $d(x, y)$ is a b -metric space with $s = 2^{p-1}$.

Definition 6 (see [11]). Let (X, d) be a b -metric space with parameter $s \geq 1$. Then, a sequence $\{x_n\}$ in X is said to be

(i) b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow +\infty$,

(ii) a Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ when $n, m \longrightarrow +\infty$.

In addition, a b-metric space is called complete if and only if each Cauchy sequence in this space is b-convergent.

Definition 7 (see [47]). Let f and g be two self-mappings on a nonempty set X. If $w = fx = gx$, for some $x \in X$, then x is said to be the coincidence point of f and g, where w is called the point of coincidence of f and g. Let $C(f, g)$ denote the set of all coincidence points of f and g.

Definition 8 (see [47]). Let f and g be two self-mappings defined on a nonempty set X. Then, f and g is said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$ for every $x \in C(f, g)$.

The following lemma plays an important role to obtain our main results:

Lemma 9 (see [46]). *Let (X, d) be a b -metric space with parameter $s \geq 1$. Assume that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \longrightarrow +\infty} d(x_n, y_n) \leq \limsup_{n \longrightarrow +\infty} d(x_n, y_n) \leq s^2 d(x, y). \tag{5}$$

In particular, if $x = y$, then we have $\lim_{n \longrightarrow +\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n \longrightarrow +\infty} d(x_n, z) \leq \limsup_{n \longrightarrow +\infty} d(x_n, z) \leq s d(x, z). \tag{6}$$

3. Main Results

In this section, we will establish common fixed point theorems for generalized weakly contractive mappings in complete b-metric space. Furthermore, we also provide two examples to support our results.

A function $f : X \longrightarrow [0, +\infty)$, where (X, d) is a b-metric space, is called lower semicontinuous if, for all $x \in X$ and $\{x_n\}$ are b-convergent to x, we have

$$f(x) \leq \liminf_{n \longrightarrow +\infty} f(x_n). \tag{7}$$

We shall consider that the contractive conditions in this section are constructed via auxiliary functions defined with the families Ψ, Φ, respectively:

$$\Psi = \{\psi : [0, +\infty) \rightarrow [0, +\infty) \text{ is a nondecreasing and continuous function}\}, \tag{8}$$

$$\Phi = \{\phi : [0, +\infty) \longrightarrow [0, +\infty) \text{ is a nondecreasing and lower semicontinuous function and } \phi(t) = 0 \text{ if and only if } t = 0\}. \tag{9}$$

Theorem 10. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ be given self-mappings satisfying g as injective and $f(X) \subset g(X)$ where $g(X)$ is closed. Suppose $\varphi : X \longrightarrow [0, +\infty)$ is a lower semicontinuous function and $p \geq 2$ is a constant. If there are functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned} &\psi(s^p[d(fx, fy) + \varphi(fx) + \varphi(fy)]) \\ &\quad \leq \psi(m(x, y, d, f, g, \varphi)) - \phi(l(x, y, d, f, g, \varphi)), \end{aligned} \tag{10}$$

where

$$\begin{aligned} m(x, y, d, f, g, \varphi) = \max \Big\{ &d(gx, gy) + \varphi(gx) + \varphi(gy), \frac{1}{2}\{d(fx, gx) + \varphi(fx) \\ &+ \varphi(gx) + d(fy, gy) + \varphi(fy) + \varphi(gy)\}, \frac{1}{2s}\{d(fx, gy) \\ &+ \varphi(fx) + \varphi(gy) + d(fy, gx) + \varphi(fy) + \varphi(gx)\}\Big\}, \end{aligned}$$

$$l(x, y, d, f, g, \varphi) = \max\{d(gx, gy) + \varphi(gx) + \varphi(gy), d(fy, gy) + \varphi(fy) + \varphi(gy)\}, \tag{11}$$

then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Proof. Let $x_0 \in X$. As $f(X) \subset g(X)$, there exists $x_1 \in X$ with $f x_0 = gx_1$. Now we define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$; then, we have $y_n = y_{n+1} = fx_{n+1} = gx_{n+1}$ and f and g have a coincidence point. Without loss of generality, we assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Applying (10) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$\begin{aligned} &\psi(d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})) \\ &\quad \leq \psi(s^p[d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})]) \\ &\quad = \psi(s^p[d(fx_n, fx_{n+1}) + \varphi(fx_n) + \varphi(fx_{n+1})]) \\ &\quad \leq \psi(m(x_n, x_{n+1}, d, f, g, \varphi)) - \phi(l(x_n, x_{n+1}, d, f, g, \varphi)), \end{aligned} \tag{12}$$

where

$$\begin{aligned} m(x_n, x_{n+1}, d, f, g, \varphi) &= \max\{d(gx_n, gx_{n+1}) + \varphi(gx_n) \\ &\quad + \varphi(gx_{n+1}), \frac{1}{2}\{d(fx_n, gx_n) + \varphi(fx_n) + \varphi(gx_n) \\ &\quad + d(fx_{n+1}, gx_{n+1}) + \varphi(fx_{n+1}) \\ &\quad + \varphi(gx_{n+1})\}, \frac{1}{2s}\{d(fx_n, gx_{n+1}) + \varphi(fx_n) + \varphi(gx_{n+1}) \\ &\quad + d(fx_{n+1}, gx_n) + \varphi(fx_{n+1}) + \varphi(gx_n)\}\} \\ &\le \max\left\{d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n), \frac{1}{2}\{d(y_n, y_{n-1}) \right. \\ &\quad + \varphi(y_n) + \varphi(y_{n-1}) + d(y_{n+1}, y_n) + \varphi(y_{n+1}) \\ &\quad + \varphi(y_n)\}, \frac{1}{2s}\{d(y_n, y_n) + \varphi(y_n) + \varphi(y_n) \\ &\quad + d(y_{n+1}, y_{n-1}) + \varphi(y_{n+1}) + \varphi(y_{n-1})\}\} \\ &\le \max\{d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n), d(y_{n+1}, y_n) \\ &\quad + \varphi(y_{n+1}) + \varphi(y_n)\}, \end{aligned} \tag{13}$$

$$\begin{aligned} l(x_n, x_{n+1}, d, f, g, \varphi) &= \max\{d(gx_n, gx_{n+1}) + \varphi(gx_n) + \varphi(gx_{n+1}), d(fx_{n+1}, gx_{n+1}) \\ &\quad + \varphi(fx_{n+1}) + \varphi(gx_{n+1})\} = \max\{d(y_{n-1}, y_n) + \varphi(y_{n-1}) \\ &\quad + \varphi(y_n), d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n)\}. \end{aligned} \tag{14}$$

If $d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1}) > d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1})$, for some $n \in \mathbb{N}$, in view of (12), (13), and (14), we have

$$\begin{aligned} &\psi(d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})) \\ &\quad \le \psi(m(x_n, x_{n+1}, d, f, g, \varphi)) - \phi(l(x_n, x_{n+1}, d, f, g, \varphi)) \\ &\quad \le \psi(d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n)) \\ &\qquad - \phi(d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n)), \end{aligned} \tag{15}$$

which implies $\phi(d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})) = 0$. Hence, $y_n = y_{n+1}$, a contradiction.

Thus, we have

$$d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1}) \le d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}), \tag{16}$$

$$m(x_n, x_{n+1}, d, f, g, \varphi) \le d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}), \tag{17}$$

$$l(x_n, x_{n+1}, d, f, g, \varphi) = d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}). \tag{18}$$

It follows from (16) that $\{d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})\}$ is a nonincreasing sequence, and so there exists $r \ge 0$ such that

$$\lim_{n \longrightarrow +\infty} (d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})) = r. \tag{19}$$

By virtue of (12), (17), and (18), one can obtain

$$\begin{aligned} &\psi(d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})) \\ &\quad \le \psi(m(x_n, x_{n+1}, d, f, g, \varphi)) - \phi(l(x_n, x_{n+1}, d, f, g, \varphi)) \\ &\quad \le \psi(d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1})) \\ &\qquad - \phi(d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1})). \end{aligned} \tag{20}$$

Now assume that $r > 0$. Taking the upper limit as $n \longrightarrow \infty$ in (20), we have

$$\begin{aligned} &\limsup_{n \longrightarrow +\infty} \psi(d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})) \\ &\quad \le \limsup_{n \longrightarrow +\infty} \psi(m(x_n, x_{n+1}, d, f, g, \varphi)) \\ &\qquad - \limsup_{n \longrightarrow +\infty} \phi(l(x_n, x_{n+1}, d, f, g, \varphi)) \\ &\quad \le \limsup_{n \longrightarrow +\infty} \psi(d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1})) \\ &\qquad - \liminf_{n \longrightarrow +\infty} \phi(d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1})), \end{aligned} \tag{21}$$

which implies that $\psi(r) \le \psi(r) - \phi(r)$, a contradiction. This yields that

$$\lim_{n \longrightarrow +\infty} (d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})) = r = 0. \tag{22}$$

It follows that $\lim_{n \longrightarrow +\infty} d(y_n, y_{n+1}) = 0$ and $\lim_{n \longrightarrow +\infty} \varphi(y_n) = 0$.

Now we shall prove that $\{y_n\}$ is a Cauchy sequence in X. Suppose on the contrary that $\{y_n\}$ is not Cauchy. It follows that there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ satisfying n_k is the smallest index for which $n_k > m_k > k$,

$$\varepsilon \le d\left(y_{m_k}, y_{n_k}\right), \tag{23}$$

$$d\left(y_{m_k}, y_{n_k - 1}\right) < \varepsilon. \tag{24}$$

By the triangle inequality in b-metric space and (23) and (24), we have

$$\begin{aligned} \varepsilon \le d\left(y_{m_k}, y_{n_k}\right) &\le sd\left(y_{m_k}, y_{n_k - 1}\right) + sd\left(y_{n_k - 1}, y_{n_k}\right) \\ &< s\varepsilon + sd\left(y_{n_k - 1}, y_{n_k}\right). \end{aligned} \tag{25}$$

Taking the upper limit as $k \longrightarrow +\infty$ in the above inequality, we have

$$\varepsilon \le \limsup_{k \longrightarrow +\infty} d\left(y_{m_k}, y_{n_k}\right) \le s\varepsilon. \tag{26}$$

Also,

$$d\left(y_{m_k}, y_{n_k}\right) \le sd\left(y_{m_k}, y_{n_k - 1}\right) + sd\left(y_{n_k - 1}, y_{n_k}\right), \tag{27}$$

$$d\left(y_{m_k}, y_{n_k}\right) \le sd\left(y_{m_k}, y_{m_k-1}\right) + sd\left(y_{m_k-1}, y_{n_k}\right), \quad (28)$$

$$d\left(y_{m_k-1}, y_{n_k}\right) \le sd\left(y_{m_k-1}, y_{m_k}\right) + sd\left(y_{m_k}, y_{n_k}\right). \quad (29)$$

From (23), (24), and (27), we obtain

$$\frac{\varepsilon}{s} \le \limsup_{k\longrightarrow+\infty} d\left(y_{m_k}, y_{n_k-1}\right) \le \varepsilon. \quad (30)$$

Using (23), (28), and (29), we get

$$\frac{\varepsilon}{s} \le \limsup_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k}\right) \le s^2\varepsilon. \quad (31)$$

Similarly,

$$\begin{aligned} d\left(y_{m_k-1}, y_{n_k-1}\right) &\le sd\left(y_{m_k-1}, y_{m_k}\right) + sd\left(y_{m_k}, y_{n_k-1}\right),\\ d\left(y_{m_k}, y_{n_k}\right) &\le sd\left(y_{m_k}, y_{m_k-1}\right) + s^2 d\left(y_{m_k-1}, y_{n_k-1}\right)\\ &\quad + s^2 d\left(y_{n_k-1}, y_{n_k}\right), \end{aligned} \quad (32)$$

so there is

$$\frac{\varepsilon}{s^2} \le \limsup_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k-1}\right) \le s\varepsilon. \quad (33)$$

Using the same method, one can obtain that

$$\begin{aligned} &\varepsilon \le \liminf_{k\longrightarrow+\infty} d\left(y_{m_k}, y_{n_k}\right) \le s\varepsilon,\ \frac{\varepsilon}{s} \le \liminf_{k\longrightarrow+\infty} d\left(y_{m_k}, y_{n_k-1}\right) \le \varepsilon,\\ &\frac{\varepsilon}{s} \le \liminf_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k}\right) \le s^2\varepsilon,\ \frac{\varepsilon}{s^2} \le \liminf_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k-1}\right) \le s\varepsilon. \end{aligned} \quad (34)$$

In view of the definition of $m(x, y, d, f, g, \varphi)$, we deduce

$$\begin{aligned} m(x_{m_k}, x_{n_k}, d, f, g, \varphi) &= \max\left\{d(gx_{m_k}, gx_{n_k}) + \varphi(gx_{m_k}) + \varphi(gx_{n_k}), \frac{1}{2}\{d(fx_{m_k}, gx_{m_k})\right.\\ &\quad + \varphi(fx_{m_k}) + \varphi(gx_{m_k}) + d(fx_{n_k}, gx_{n_k}) + \varphi(fx_{n_k})\\ &\quad + \varphi(gx_{n_k})\}, \frac{1}{2s}\{d(fx_{m_k}, gx_{n_k}) + \varphi(fx_{m_k}) + \varphi(gx_{n_k})\\ &\quad \left.+ d(fx_{n_k}, gx_{m_k}) + \varphi(fx_{n_k}) + \varphi(gx_{m_k})\}\right\}\\ &= \max\left\{d\left(y_{m_k-1}, y_{n_k-1}\right) + \varphi\left(y_{m_k-1}\right) + \varphi\left(y_{n_k-1}\right), \frac{1}{2}\right.\\ &\quad \cdot\left\{d\left(y_{m_k}, y_{m_k-1}\right) + \varphi\left(y_{m_k}\right) + \varphi\left(y_{m_k-1}\right) + d\left(y_{n_k}, y_{n_k-1}\right)\right.\\ &\quad \left.+ \varphi\left(y_{n_k}\right) + \varphi\left(y_{n_k-1}\right)\right\}, \frac{1}{2s}\left\{d\left(y_{m_k}, y_{n_k-1}\right) + \varphi\left(y_{m_k}\right)\right.\\ &\quad \left.\left.+ \varphi\left(y_{n_k-1}\right) + d\left(y_{n_k}, y_{m_k-1}\right) + \varphi\left(y_{n_k}\right) + \varphi\left(y_{m_k-1}\right)\right\}\right\}. \end{aligned} \quad (35)$$

Taking the upper limit as $k \longrightarrow +\infty$ in (35), we obtain

$$\limsup_{k\longrightarrow+\infty} m(x_{m_k}, x_{n_k}, d, f, g, \varphi) \le \max\left\{s\varepsilon, 0, \frac{\varepsilon + s^2\varepsilon}{2s}\right\} = s\varepsilon. \quad (36)$$

Also, we have

$$\begin{aligned} l(x_{m_k}, x_{n_k}, d, f, g, \varphi) &= \max\left\{d(gx_{m_k}, gx_{n_k}) + \varphi(gx_{m_k}) + \varphi(gx_{n_k}), d(fx_{n_k}, gx_{n_k})\right.\\ &\quad \left.+ \varphi(fx_{n_k}) + \varphi(gx_{n_k})\right\}\\ &= \max\left\{d\left(y_{m_k-1}, y_{n_k-1}\right) + \varphi\left(y_{m_k-1}\right) + \varphi\left(y_{n_k-1}\right), d\left(y_{n_k}, y_{n_k-1}\right)\right.\\ &\quad \left.+ \varphi\left(y_{n_k}\right) + \varphi\left(y_{n_k-1}\right)\right\}. \end{aligned} \quad (37)$$

It follows that

$$s\varepsilon \ge \liminf_{k\longrightarrow+\infty} l(x_{m_k}, x_{n_k}, d, f, g, \varphi) \ge \frac{\varepsilon}{s^2}. \quad (38)$$

Applying (10) with $x = x_{m_k}$ and $y = x_{n_k}$, one can get

$$\begin{aligned} \psi(s\varepsilon) \le \psi(s^p\varepsilon) &\le \psi\left(s^p \limsup_{n\longrightarrow+\infty}\left[d\left(y_{m_k}, y_{n_k}\right) + \varphi\left(y_{m_k}\right) + \varphi\left(y_{n_k}\right)\right]\right)\\ &\le \psi\left(\limsup_{n\longrightarrow+\infty} m(x_{m_k}, x_{n_k}, d, f, g, \varphi)\right)\\ &\quad - \liminf_{n\longrightarrow+\infty} \phi\left(l(x_{m_k}, x_{n_k}, d, f, g, \varphi)\right)\\ &\le \psi(s\varepsilon) - \phi\left(\liminf_{n\longrightarrow+\infty} l(x_{m_k}, x_{n_k}, d, f, g, \varphi)\right), \end{aligned} \quad (39)$$

which implies that

$$\liminf_{n\longrightarrow+\infty} l(x_{m_k}, x_{n_k}, d, f, g, \varphi) = 0, \quad (40)$$

a contradiction to (38). It follows that $\{y_n\}$ is a Cauchy sequence in X. The completeness of X ensures that there exists a $u \in X$ such that

$$\begin{aligned} \lim_{n\longrightarrow+\infty} d(y_n, u) &= \lim_{n\longrightarrow+\infty} d(fx_n, u) = \lim_{n\longrightarrow+\infty} d(gx_{n+1}, u)\\ &= \lim_{n,m\longrightarrow+\infty} d(y_n, y_m) = 0. \end{aligned} \quad (41)$$

Furthermore, we have $u \in g(X)$ since $g(X)$ is closed. It follows that one can choose a $z \in X$ such that $u = gz$, and one can write (41) as

$$\lim_{n\longrightarrow+\infty} d(y_n, gz) = \lim_{n\longrightarrow+\infty} d(fx_n, gz) = \lim_{n\longrightarrow+\infty} d(gx_{n+1}, gz) = 0. \quad (42)$$

Following from the definition of φ, we get

$$\varphi(gz) = \varphi(u) \le \liminf_{n\longrightarrow+\infty} \varphi(y_n) = 0. \quad (43)$$

That is, $\varphi(gz) = \varphi(u) = 0$.

If $fz \neq gz$, taking $x = x_{n_k}$ and $y = z$ in contractive condition (10), we deduce that

$$\begin{aligned}\psi\left(d\left(fx_{n_k}, fz\right) + \varphi\left(fx_{n_k}\right) + \varphi(fz)\right) &\leq \psi\left(s^p\left[d\left(fx_{n_k}, fz\right) + \varphi\left(fx_{n_k}\right) + \varphi(fz)\right]\right) \\ &\leq \psi\left(m\left(x_{n_k}, z, d, f, g, \varphi\right)\right) - \phi\left(l\left(x_{n_k}, z, d, f, g, \varphi\right)\right),\end{aligned} \tag{44}$$

where

$$\begin{aligned}m\left(x_{n_k}, z, d, f, g, \varphi\right) = \max &\left\{d\left(gx_{n_k}, gz\right) + \varphi\left(gx_{n_k}\right) + \varphi(gz), \frac{1}{2}\left\{d\left(fx_{n_k}, gx_{n_k}\right)\right.\right. \\ &+ \varphi\left(fx_{n_k}\right) + \varphi\left(gx_{n_k}\right) + d(fz, gz) + \varphi(fz) + \varphi(gz)\}, \frac{1}{2s} \\ &\cdot \left\{d\left(fx_{n_k}, gz\right) + \varphi\left(fx_{n_k}\right) + \varphi(gz) + d\left(fz, gx_{n_k}\right) + \varphi(fz)\right. \\ &\left.\left. + \varphi\left(gx_{n_k}\right)\right\}\right\} = \max\left\{d\left(y_{n_k-1}, gz\right) + \varphi\left(y_{n_k-1}\right) + \varphi(gz), \frac{1}{2}\right. \\ &\cdot \left\{d\left(y_{n_k}, y_{n_k-1}\right) + \varphi\left(y_{n_k}\right) + \varphi\left(y_{n_k-1}\right) + d(fz, gz) + \varphi(fz)\right. \\ &+ \varphi(gz)\}, \frac{1}{2s}\left\{d\left(y_{n_k}, gz\right) + \varphi\left(y_{n_k}\right) + \varphi(gz)\right. \\ &\left.\left. + d\left(fz, y_{n_k-1}\right) + \varphi(fz) + \varphi\left(y_{n_k-1}\right)\right\}\right\},\end{aligned}$$

$$\begin{aligned}l\left(x_{n_k}, z, d, f, g, \varphi\right) = \max &\left\{d\left(gx_{n_k}, gz\right) + \varphi\left(gx_{n_k}\right) + \varphi(gz), d(fz, gz)\right. \\ &\left. + \varphi(fz) + \varphi(gz)\right\} = \max\left\{d\left(y_{n_k-1}, gz\right)\right. \\ &\left. + \varphi\left(y_{n_k-1}\right) + \varphi(gz), d(fz, gz) + \varphi(fz) + \varphi(gz)\right\}.\end{aligned} \tag{45}$$

By simple calculation, we obtain

$$\limsup_{k \longrightarrow +\infty} m\left(x_{n_k}, z, d, f, g, \varphi\right) \leq d(fz, gz) + \varphi(fz), \tag{46}$$

$$\liminf_{k \longrightarrow +\infty} l\left(x_{n_k}, z, d, f, g, \varphi\right) \geq d(fz, gz) + \varphi(fz). \tag{47}$$

By taking the upper limit as $k \longrightarrow +\infty$ in (44) and using (46) and (47), one can get

$$\begin{aligned}\psi(d(fz, gz) + \varphi(fz)) \leq \psi(d(fz, gz) + \varphi(fz)) \\ - \phi(d(fz, gz) + \varphi(fz)).\end{aligned} \tag{48}$$

Hence, $d(fz, gz) + \varphi(fz) = 0$, which implies that $fz = gz$ and $\varphi(fz) = 0$.

Now we claim that z is the unique coincidence point of f and g. If not, there exist $z, z' \in C(f, g)$ and $z \neq z'$; applying (10) with $x = z$ and $y = z'$, we obtain that

$$\begin{aligned}\psi\left(d\left(fz, fz'\right) + \varphi(fz) + \varphi\left(fz'\right)\right) &\leq \psi\left(s^p\left[d\left(fz, fz'\right) + \varphi(fz) + \varphi\left(fz'\right)\right]\right) \\ &\leq \psi\left(m\left(z, z', d, f, g, \varphi\right)\right) - \phi\left(l\left(z, z', d, f, g, \varphi\right)\right).\end{aligned} \tag{49}$$

Here,

$$\begin{aligned}m\left(z, z', d, f, g, \varphi\right) = \max &\left\{d\left(gz, gz'\right) + \varphi(gz) + \varphi\left(gz'\right), \frac{1}{2}\{d(fz, gz)\right. \\ &+ \varphi(fz) + \varphi(gz) + d\left(fz', gz'\right) + \varphi\left(fz'\right) + \varphi\left(gz'\right)\}, \frac{1}{2s} \\ &\cdot \left\{d\left(fz, gz'\right) + \varphi(fz) + \varphi\left(gz'\right) + d\left(fz', gz\right) + \varphi\left(fz'\right)\right. \\ &\left.\left. + \varphi(gz)\right\}\right\} \leq d\left(gz, gz'\right) + \varphi\left(gz'\right),\end{aligned}$$

$$\begin{aligned}l\left(z, z', d, f, g, \varphi\right) = \max &\left\{d\left(gz, gz'\right) + \varphi(gz) + \varphi\left(gz'\right), d\left(fz', gz'\right)\right. \\ &\left. + \varphi\left(fz'\right) + \varphi\left(gz'\right)\right\} \geq d\left(gz, gz'\right) + \varphi\left(gz'\right).\end{aligned} \tag{50}$$

It follows from (49) that

$$\psi\left(d\left(gz, gz'\right) + \varphi\left(gz'\right)\right) \leq \psi\left(d\left(gz, gz'\right) + \varphi\left(gz'\right)\right) - \phi\left(d\left(gz, gz'\right) + \varphi\left(gz'\right)\right). \tag{51}$$

Hence, we get that $d(gz, gz') + \varphi(gz') = 0$, which implies that $gz = gz'$ and $\varphi(gz') = 0$. Since g is an injective mapping, then $z = z'$; that is, z is a unique coincidence point of f and g. Further, if f and g are weakly compatible, then it is easy to show that z is a unique common fixed point of f and g. This completes the proof.

Example 11. Let $X = [0, +\infty)$ and $d(x, y) = (x - y)^2$. Define mappings $f, g, \varphi : X \longrightarrow X$ by

$$fx = \frac{\left(x + x^2\right)}{8}, \quad gx = \frac{7\left(x + x^2\right)}{8}, \varphi x = x^2, \quad x \in [0, +\infty). \tag{52}$$

Define mappings $\psi, \phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\psi(t) = t$, $\phi(t) = 35t/98$.

It is clear that $f(X) \subset g(X)$. For all $x, y \in X$, we have

$$\begin{aligned}&\psi\left(s^p[d(fx, fy) + \varphi(fx) + \varphi(fy)]\right) \\ &= 4 \cdot \left[\left(\frac{x + x^2}{8} - \frac{y + y^2}{8}\right)^2 + \left(\frac{x + x^2}{8}\right)^2 + \left(\frac{y + y^2}{8}\right)^2\right] \\ &\leq 4 \cdot \frac{1}{64} \cdot 2\left(\left(x + x^2\right)^2 + \left(y + y^2\right)^2\right) = \frac{1}{8}\left(\left(x + x^2\right)^2 + \left(y + y^2\right)^2\right),\end{aligned}$$

$$\begin{aligned}&\psi(m(x, y, d, f, g, \varphi)) \\ &\geq \psi\left(\frac{1}{2}[d(fx, gx) + \varphi(fx) + \varphi(gx) + d(fy, gy) + \varphi(fy) + \varphi(gy)]\right) \\ &= \frac{1}{2}\left[\left(\frac{x + x^2}{8} - \frac{7\left(x + x^2\right)}{8}\right)^2 + \left(\frac{x + x^2}{8}\right)^2 + \left(\frac{7\left(x + x^2\right)}{8}\right)^2\right. \\ &\quad \left. + \left(\frac{y + y^2}{8} - \frac{7\left(y + y^2\right)}{8}\right)^2 + \left(\frac{y + y^2}{8}\right)^2 + \left(\frac{7\left(y + y^2\right)}{8}\right)^2\right] \\ &= \frac{43}{64}\left(\left(x + x^2\right)^2 + \left(y + y^2\right)^2\right),\end{aligned}$$

$$\begin{aligned}\phi(l(x,y,d,f,g,\varphi)) &= \frac{35}{98}\max\{d(gx,gy)+\varphi(gx)+\varphi(gy), d(fy,gy)+\varphi(fy)+\varphi(gy)\}\\ &= \frac{35}{98}\max\left\{\left(\frac{7(x+x^2)}{8}-\frac{7(y+y^2)}{8}\right)^2+\left(\frac{7(x+x^2)}{8}\right)^2\right.\\ &\quad+\left(\frac{7(y+y^2)}{8}\right)^2, \left(\frac{y+y^2}{8}-\frac{7(y+y^2)}{8}\right)^2+\left(\frac{y+y^2}{8}\right)^2\\ &\quad\left.+\left(\frac{7(y+y^2)}{8}\right)^2\right\}\\ &\le \frac{35}{98}\max\left\{\frac{49}{32}\left(\left(x+x^2\right)^2+\left(y+y^2\right)^2\right), \frac{43}{32}\left(y+y^2\right)^2\right\}\\ &\le \frac{35}{64}\left(\left(x+x^2\right)^2+\left(y+y^2\right)^2\right).\end{aligned} \tag{53}$$

According to above inequalities, we get that

$$\begin{aligned}&\psi(s^p[d(fx,fy)+\varphi(fx)+\varphi(fy)])\\ &\quad\le \frac{1}{8}\left(\left(x+x^2\right)^2+\left(y+y^2\right)^2\right)=\frac{43}{64}\left(\left(x+x^2\right)^2+\left(y+y^2\right)^2\right)\\ &\quad\quad-\frac{35}{64}\left(\left(x+x^2\right)^2+\left(y+y^2\right)^2\right)\\ &\quad\le \psi(m(x,y,d,f,g,\varphi))-\phi(l(x,y,d,f,g,\varphi)).\end{aligned} \tag{54}$$

It follows that all conditions of Theorem 10 are satisfied with $s=2, p=2$. It is easy to obtain that 0 is the unique common fixed point of f and g.

Note that, for $x=0, y\in(0,+\infty)$, one can calculate that

$$\begin{aligned}sd(fx,gy) &= \frac{49\left(y+y^2\right)^2}{32}=\max\left\{0,\frac{49\left(y+y^2\right)^2}{32}, L\frac{113\left(y+y^2\right)^2}{64}\right\}\\ &> \max\{sd(x,fx), sd(y,gy), L(d(x,gy)+d(fx,y))\}\\ &> \varphi(\max\{sd(x,fx), sd(y,gy), L(d(x,gy)+d(fx,y))\}),\end{aligned} \tag{55}$$

which implies that Theorem 1 of [10] cannot be applied to testify the existence of common fixed points of the mappings f and g in X.

If $\varphi=0$ in Theorem 10, we can get the following result:

Corollary 12. *Let (X,d) be a complete b-metric space with parameter $s\ge 1$, and let $f,g:X\longrightarrow X$ be given self-mappings satisfying g as injective and $f(X)\subset g(X)$ where $g(X)$ is closed. Suppose $p\ge 2$ is a constant. If there are functions $\psi\in\Psi$ and $\phi\in\Phi$ such that*

$$\psi(s^p d(fx,fy))\le\psi(m_1(x,y,d,f,g))-\phi(l_1(x,y,d,f,g)), \tag{56}$$

where

$$\begin{aligned}m_1(x,y,d,f,g) &= \max\left\{d(gx,gy),\frac{1}{2}\{d(fx,gx)+d(fy,gy)\},\frac{1}{2s}\right.\\ &\quad\left.\cdot\{d(fx,gy)+d(fy,gx)\}\right\},\\ l_1(x,y,d,f,g) &= \max\{d(gx,gy), d(fy,gy)\},\end{aligned} \tag{57}$$

then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

If we consider the corresponding problem in the setting of metric space, that is, $s=1$ in Theorem 10, one can obtain the following:

Corollary 13. *Let (X,d) be a complete metric space, and let $f,g:X\longrightarrow X$ be given self-mappings satisfying g as injective and $f(X)\subset g(X)$ where $g(X)$ is closed. Suppose $\varphi:X\longrightarrow[0,+\infty)$ is a lower semicontinuous function. If there are functions $\psi\in\Psi$ and $\phi\in\Phi$ such that*

$$\begin{aligned}&\psi(d(fx,fy)+\varphi(fx)+\varphi(fy))\\ &\quad\le\psi(m_2(x,y,d,f,g,\varphi))-\phi(l(x,y,d,f,g,\varphi)),\end{aligned} \tag{58}$$

where

$$\begin{aligned}m_2(x,y,d,f,g,\varphi) &= \max\left\{d(gx,gy)+\varphi(gx)+\varphi(gy),\frac{1}{2}\{d(fx,gx)\right.\\ &\quad+\varphi(fx)+\varphi(gx)+d(fy,gy)+\varphi(fy)+\varphi(gy)\},\frac{1}{2}\\ &\quad\cdot\{d(fx,gy)+\varphi(fx)+\varphi(gy)+d(fy,gx)+\varphi(fy)\\ &\quad\left.+\varphi(gx)\}\right\},\end{aligned} \tag{59}$$

and $l(x,y,d,f,g,\varphi)$ is the same as Theorem 10, then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Theorem 14. *Let (X,d) be a complete b-metric space with parameter $s\ge 1$ and let $f,g:X\longrightarrow X$ be given self-mappings, and one of f and g is continuous. Suppose $\varphi:X\longrightarrow[0,+\infty)$ is a lower semicontinuous function and $p\ge 3, 0<\lambda\le 1/4$ are two constants. If there are functions $\psi\in\Psi$ and $\phi\in\Phi$ such that*

$$\begin{aligned}&\psi(s^p[d(fx,gy)+\varphi(fx)+\varphi(gy)])\\ &\quad\le\psi(n(x,y,d,f,g,\varphi))-\phi(r(x,y,d,g,\varphi)),\end{aligned} \tag{60}$$

where

$$\begin{aligned}n(x,y,d,f,g,\varphi) &= \lambda\max\left\{d(x,y)+\varphi(x)+\varphi(y), d(fx,x)+\varphi(fx)\right.\\ &\quad+\varphi(x)+d(y,gy)+\varphi(y)+\varphi(gy),\frac{1}{s}\{d(fx,y)\\ &\quad\left.+\varphi(fx)+\varphi(y)+d(x,gy)+\varphi(x)+\varphi(gy)\}\right\},\\ r(x,y,d,g,\varphi) &= \max\{d(x,y)+\varphi(x)+\varphi(y), d(y,gy)+\varphi(y)+\varphi(gy)\},\end{aligned} \tag{61}$$

then f and g have a unique common fixed point in X.

Proof. Let $x_0\in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_{2i+1}=fx_{2i}, x_{2i+2}=gx_{2i+1}$ for $i=0,1,2,\cdots$. Firstly, we prove that f and g have at most one common fixed point.

Suppose that v and w are two different common fixed points; then, $f(v) = v \neq w = g(w)$. It follows that $d(f(v), g(w)) = d(v, w) > 0$. Applying (60) with $x = v$ and $y = w$, we obtain

$$\begin{aligned}\psi(d(v,w)+\varphi(v)+\varphi(w)) &\leq \psi(s^p[d(fv,gw)+\varphi(fv)+\varphi(gw)]) \\ &\leq \psi(n(v,w,d,f,g,\varphi)) - \phi(r(v,w,d,g,\varphi)),\end{aligned} \tag{62}$$

where

$$\begin{aligned}n(v,w,d,f,g,\varphi) &= \lambda \max\{d(v,w)+\varphi(v)+\varphi(w), d(fv,v)+\varphi(fv) \\ &\quad +\varphi(v)+d(w,gw)+\varphi(w)+\varphi(gw), \frac{1}{s}\{d(fv,w) \\ &\quad +\varphi(fv)+\varphi(w)+d(v,gw)+\varphi(v)+\varphi(gw)\}\} \\ &= \lambda \max\{d(v,w)+\varphi(v)+\varphi(w), d(v,v)+\varphi(v) \\ &\quad +\varphi(v)+d(w,w)+\varphi(w)+\varphi(w), \frac{1}{s}\{d(v,w) \\ &\quad +\varphi(v)+\varphi(w)+d(v,w)+\varphi(v)+\varphi(w)\}\} \\ &\leq d(v,w)+\varphi(v)+\varphi(w), \\ r(v,w,d,g,\varphi) &= \max\{d(v,w)+\varphi(v)+\varphi(w), d(w,gw)+\varphi(w)+\varphi(gw)\} \\ &= \max\{d(v,w)+\varphi(v)+\varphi(w), d(w,w)+\varphi(w)+\varphi(w)\} \\ &\geq d(v,w)+\varphi(v)+\varphi(w).\end{aligned} \tag{63}$$

It follows from (62) that

$$\begin{aligned}\psi(d(v,w)+\varphi(v)+\varphi(w)) &\leq \psi(d(v,w)+\varphi(v)+\varphi(w)) - \phi(d(v,w)+\varphi(v)+\varphi(w)),\end{aligned} \tag{64}$$

which implies that $d(v,w)+\varphi(v)+\varphi(w) = 0$. That is, $v = w$ and $\varphi(v) = 0$. Hence, the pair (f, g) has at most one common fixed point.

We suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. If not, for some k, $x_{2k} = x_{2k+1}$ and from (60), we obtain

$$\begin{aligned}\psi(d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2})) &\leq \psi(s^p[d(fx_{2k},gx_{2k+1})+\varphi(fx_{2k})+\varphi(gx_{2k+1})]) \\ &\leq \psi(n(x_{2k},x_{2k+1},d,f,g,\varphi)) - \phi(r(x_{2k},x_{2k+1},d,g,\varphi)),\end{aligned} \tag{65}$$

where

$$\begin{aligned}n(x_{2k},x_{2k+1},d,f,g,\varphi) &= \lambda \max\{d(x_{2k},x_{2k+1})+\varphi(x_{2k})+\varphi(x_{2k+1}), d(fx_{2k},x_{2k}) \\ &\quad +\varphi(fx_{2k})+\varphi(x_{2k})+d(x_{2k+1},gx_{2k+1})+\varphi(x_{2k+1}) \\ &\quad +\varphi(gx_{2k+1}), \frac{1}{s}\{d(fx_{2k},x_{2k+1})+\varphi(fx_{2k})+\varphi(x_{2k+1}) \\ &\quad +d(x_{2k},gx_{2k+1})+\varphi(x_{2k})+\varphi(gx_{2k+1})\}\} \\ &= \lambda \max\{d(x_{2k},x_{2k+1})+\varphi(x_{2k})+\varphi(x_{2k+1}), d(x_{2k+1},x_{2k}) \\ &\quad +\varphi(x_{2k+1})+\varphi(x_{2k})+d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1}) \\ &\quad +\varphi(x_{2k+2}), \frac{1}{s}\{d(x_{2k+1},x_{2k+1})+\varphi(x_{2k+1})+\varphi(x_{2k+1}) \\ &\quad +d(x_{2k},x_{2k+2})+\varphi(x_{2k})+\varphi(x_{2k+2})\}\} \\ &\leq \frac{1}{2}\max\{d(x_{2k+1},x_{2k})+\varphi(x_{2k+1})+\varphi(x_{2k}), d(x_{2k+1},x_{2k+2}) \\ &\quad +\varphi(x_{2k+1})+\varphi(x_{2k+2})\} \leq d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2}), \\ r(x_{2k},x_{2k+1},d,g,\varphi) &= \max\{d(x_{2k},x_{2k+1})+\varphi(x_{2k})+\varphi(x_{2k+1}), d(x_{2k+1},gx_{2k+1}) \\ &\quad +\varphi(x_{2k+1})+\varphi(gx_{2k+1})\} = \max\{d(x_{2k},x_{2k+1})+\varphi(x_{2k}) \\ &\quad +\varphi(x_{2k+1}), d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2})\} \\ &\geq d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2}).\end{aligned} \tag{66}$$

By virtue of (65) and the above inequalities, we have

$$\begin{aligned}\psi(d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2})) &\leq \psi(n(x_{2k},x_{2k+1},d,f,g,\varphi)) - \phi(r(x_{2k},x_{2k+1},d,g,\varphi)) \\ &\leq \psi(d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2})) \\ &\quad - \phi(d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2})),\end{aligned} \tag{67}$$

which implies that $d(x_{2k+1},x_{2k+2})+\varphi(x_{2k+1})+\varphi(x_{2k+2}) = 0$. That is, $x_{2k+1} = x_{2k+2}$. Thus, x_{2k} is a common fixed point of f and g. If $x_{2k+1} = x_{2k+2}$, then using the same arguments as in the case $x_{2k} = x_{2k+1}$, it can be shown that x_{2k+1} is a common fixed point of f and g.

Now take $d(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}$. Letting $x = x_{2n}$, $y = x_{2n+1}$ in (60), as the same arguments, we obtain

$$\begin{aligned}\psi(d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1})+\varphi(x_{2n+2})) &\leq \psi(n(x_{2n},x_{2n+1},d,f,g,\varphi)) - \phi(r(x_{2n},x_{2n+1},d,g,\varphi)),\end{aligned} \tag{68}$$

where

$$\begin{aligned}n(x_{2n},x_{2n+1},d,f,g,\varphi) &= \lambda \max\{d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1}), d(x_{2n+1},x_{2n}) \\ &\quad +\varphi(x_{2n+1})+\varphi(x_{2n})+d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1}) \\ &\quad +\varphi(x_{2n+2}), \frac{1}{s}\{d(x_{2n+1},x_{2n+1})+\varphi(x_{2n+1})+\varphi(x_{2n+1}) \\ &\quad +d(x_{2n},x_{2n+2})+\varphi(x_{2n})+\varphi(x_{2n+2})\}\} \\ &\leq \frac{1}{2}\max\{d(x_{2n+1},x_{2n})+\varphi(x_{2n+1})+\varphi(x_{2n}), d(x_{2n+1},x_{2n+2}) \\ &\quad +\varphi(x_{2n+1})+\varphi(x_{2n+2})\},\end{aligned} \tag{69}$$

$$\begin{aligned}r(x_{2n},x_{2n+1},d,g,\varphi) &= \max\{d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1}), d(x_{2n+1},x_{2n+2}) \\ &\quad +\varphi(x_{2n+1})+\varphi(x_{2n+2})\}.\end{aligned} \tag{70}$$

If for some n, $d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1})+\varphi(x_{2n+2}) > d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})$, then it follows from (68), (69), and (70) that

$$\begin{aligned}\psi(d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1})+\varphi(x_{2n+2})) &\leq \psi(d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1})+\varphi(x_{2n+2})) \\ &\quad - \phi(d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1})+\varphi(x_{2n+2})),\end{aligned} \tag{71}$$

which yields that $\phi(d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1}) = 0$ or equivalently

$$d(x_{2n+1},x_{2n+2})+\varphi(x_{2n+1})+\varphi(x_{2n+2}) = 0. \tag{72}$$

That is, $d(x_{2n+1}, x_{2n+2}) = 0$, a contradiction. Hence,

$$\begin{aligned} &d(x_{2n+1}, x_{2n+2}) + \varphi(x_{2n+1}) + \varphi(x_{2n+2}) \\ &\quad \le d(x_{2n}, x_{2n+1}) + \varphi(x_{2n}) + \varphi(x_{2n+1}), \end{aligned} \tag{73}$$

for all $n \in N$. By similar arguments, we get

$$d(x_{2n+2}, x_{2n+3}) + \varphi(x_{2n+2}) + \varphi(x_{2n+3}) \le d(x_{2n+1}, x_{2n+2}) + \varphi(x_{2n+1}) + \varphi(x_{2n+2}). \tag{74}$$

Therefore, $\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})\}$ is a nonincreasing sequence, and there exists a $r \ge 0$ such that

$$\lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) = r. \tag{75}$$

If $r > 0$, by virtue of (68), (69), (70), and (73), one can obtain that

$$\begin{aligned} &\psi(d(x_{2n+1}, x_{2n+2}) + \varphi(x_{2n+1}) + \varphi(x_{2n+2})) \\ &\quad \le \psi(d(x_{2n}, x_{2n+1}) + \varphi(x_{2n}) + \varphi(x_{2n+1})) \\ &\qquad - \phi(d(x_{2n}, x_{2n+1}) + \varphi(x_{2n}) + \varphi(x_{2n+1}). \end{aligned} \tag{76}$$

Taking the upper limit as $n \longrightarrow +\infty$ in (76), we have

$$\begin{aligned} &\limsup_{n \longrightarrow +\infty} \psi(d(x_{2n+1}, x_{2n+2}) + \varphi(x_{2n+1}) + \varphi(x_{2n+2})) \\ &\quad \le \limsup_{n \longrightarrow +\infty} \psi(d(x_{2n}, x_{2n+1}) + \varphi(x_{2n}) + \varphi(x_{2n+1})) \\ &\qquad - \limsup_{n \longrightarrow +\infty} \phi(d(x_{2n}, x_{2n+1}) + \varphi(x_{2n}) + \varphi(x_{2n+1})) \\ &\quad \le \limsup_{n \longrightarrow +\infty} \psi(d(x_{2n}, x_{2n+1}) + \varphi(x_{2n}) + \varphi(x_{2n+1})) \\ &\qquad - \liminf_{n \longrightarrow +\infty} \phi(d(x_{2n}, x_{2n+1}) + \varphi(x_{2n}) + \varphi(x_{2n+1})), \end{aligned} \tag{77}$$

which implies that $\psi(r) \le \psi(r) - \phi(r)$, a contradiction. It follows that

$$\lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) = 0, \tag{78}$$

which yields that

$$\lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) = 0, \lim_{n \longrightarrow +\infty} \varphi(x_n) = 0. \tag{79}$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X. To do this, it is sufficient to prove that $\{x_{2n}\}$ is Cauchy. Suppose on the contrary that $\{x_{2n}\}$ is not Cauchy. It follows that there exists $\varepsilon > 0$ for which one can find sequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_{2n}\}$ satisfying n_k as the smallest index for which $2m_k > 2n_k > k$,

$$\begin{aligned} &\varepsilon \le d(x_{2m_k}, x_{2n_k}), \\ &d(x_{2m_k-2}, x_{2n_k}) < \varepsilon. \end{aligned} \tag{80}$$

Using the same technique in the proof of Theorem 10, we can deduce that

$$\varepsilon \le \liminf_{k \longrightarrow +\infty} d(x_{2m_k}, x_{2n_k}) \le \limsup_{k \longrightarrow +\infty} d(x_{2m_k}, x_{2n_k}) \le s\varepsilon, \tag{81}$$

$$\frac{\varepsilon}{s} \le \liminf_{k \longrightarrow +\infty} d(x_{2m_k-1}, x_{2n_k}) \le \limsup_{k \longrightarrow +\infty} d(x_{2m_k-1}, x_{2n_k}) \le s^2\varepsilon, \tag{82}$$

$$\frac{\varepsilon}{s^2} \le \liminf_{k \longrightarrow +\infty} d(x_{2m_k-1}, x_{2n_k+1}) \le \limsup_{k \longrightarrow +\infty} d(x_{2m_k-1}, x_{2n_k+1}) \le s^3\varepsilon, \tag{83}$$

$$\frac{\varepsilon}{s} \le \liminf_{k \longrightarrow +\infty} d(x_{2m_k}, x_{2n_k+1}) \le \limsup_{k \longrightarrow +\infty} d(x_{2m_k}, x_{2n_k+1}). \tag{84}$$

Letting $x = x_{2n_k}$ and $y = x_{2m_k-1}$ in (60), we obtain

$$\begin{aligned} &\psi(d(x_{2n_k+1}, x_{2m_k}) + \varphi(x_{2n_k+1}) + \varphi(x_{2m_k})) \\ &\quad \le \psi(s^p[d(fx_{2n_k}, gx_{2m_k-1}) + \varphi(fx_{2n_k}) + \varphi(gx_{2m_k-1})]) \\ &\quad \le \psi(n(x_{2n_k}, x_{2m_k-1}, d, f, g, \varphi)) - \phi(r(x_{2n_k}, x_{2m_k-1}, d, g, \varphi)). \end{aligned} \tag{85}$$

Here,

$$\begin{aligned} n(x_{2n_k}, x_{2m_k-1}, d, f, g, \varphi) &= \lambda \max \{d(x_{2n_k}, x_{2m_k-1}) + \varphi(x_{2n_k}) \\ &\quad + \varphi(x_{2m_k-1}), d(fx_{2n_k}, x_{2n_k}) + \varphi(fx_{2n_k}) \\ &\quad + \varphi(x_{2n_k}) + d(x_{2m_k-1}, gx_{2m_k-1}) \\ &\quad + \varphi(x_{2m_k-1}) + \varphi(gx_{2m_k-1}), \frac{1}{s}\{d(fx_{2n_k}, x_{2m_k-1}) \\ &\quad + \varphi(fx_{2n_k}) + \varphi(x_{2m_k-1}) + d(x_{2n_k}, gx_{2m_k-1}) \\ &\quad + \varphi(x_{2n_k}) + \varphi(gx_{2m_k-1})\}\} \\ &= \lambda \max \{d(x_{2n_k}, x_{2m_k-1}) + \varphi(x_{2n_k}) \\ &\quad + \varphi(x_{2m_k-1}), d(x_{2n_k+1}, x_{2n_k}) + \varphi(x_{2n_k+1}) \\ &\quad + \varphi(x_{2n_k}) + d(x_{2m_k-1}, x_{2m_k}) + \varphi(x_{2m_k-1}) \\ &\quad + \varphi(x_{2m_k}), \frac{1}{s}\{d(x_{2n_k+1}, x_{2m_k-1}) + \varphi(x_{2n_k+1}) \\ &\quad + \varphi(x_{2m_k-1}) + d(x_{2n_k}, x_{2m_k}) + \varphi(x_{2n_k}) + \varphi(x_{2m_k})\}\}, \end{aligned}$$

$$\begin{aligned} r(x_{2n_k}, x_{2m_k-1}, d, g, \varphi) &= \max \{d(x_{2n_k}, x_{2m_k-1}) + \varphi(x_{2n_k}) + \varphi(x_{2m_k-1}), d(x_{2m_k-1}, gx_{2m_k-1}) \\ &\quad + \varphi(x_{2m_k-1}) + \varphi(gx_{2m_k-1})\} = \max \{d(x_{2n_k}, x_{2m_k-1}) + \varphi(x_{2n_k}) \\ &\quad + \varphi(x_{2m_k-1}), d(x_{2m_k-1}, x_{2m_k}) + \varphi(x_{2m_k-1}) + \varphi(x_{2m_k})\}. \end{aligned} \tag{86}$$

It follows from (79) to (84) that

$$\begin{aligned} &\limsup_{k \longrightarrow +\infty} n(x_{2n_k}, x_{2m_k-1}, d, f, g, \varphi) \\ &\quad \le \lambda \max \left\{s^2\varepsilon, 0, \frac{s^3\varepsilon + s\varepsilon}{s}\right\} \le s^2\varepsilon, \end{aligned} \tag{87}$$

$$s^2\varepsilon \ge \liminf_{k \longrightarrow +\infty} r(x_{2n_k}, x_{2m_k-1}, d, g, \varphi) \ge \frac{\varepsilon}{s}. \tag{88}$$

By virtue of (85), (87), and (88), we have

$$\begin{aligned}\psi(s^2\varepsilon) &= \psi\left(s^3\frac{\varepsilon}{s}\right) \le \psi\left(s^p \limsup_{k\longrightarrow+\infty}\left[d\left(x_{2n_k+1}, x_{2m_k}\right) + \varphi\left(x_{2n_k+1}\right) + \varphi\left(x_{2m_k}\right)\right]\right) \\ &\le \psi(s^2\varepsilon) - \liminf_{k\longrightarrow+\infty}\phi\left(r\left(x_{2n_k}, x_{2m_k-1}, d, g, \varphi\right)\right),\end{aligned} \tag{89}$$

which implies that

$$\liminf_{k\longrightarrow+\infty} r\left(x_{2n_k}, x_{2m_k-1}, d, g, \varphi\right) = 0, \tag{90}$$

a contradiction to (88). Hence, $\{x_n\}$ is a Cauchy sequence. The completeness of X ensures that there exists a x^* in X such that

$$\lim_{n\longrightarrow+\infty} fx_{2n} = \lim_{n\longrightarrow+\infty} gx_{2n+1} = x^*. \tag{91}$$

By the definition of φ, we deduce that

$$\varphi(x^*) \le \liminf_{n\longrightarrow+\infty}\varphi(x_n) = 0. \tag{92}$$

Now we will show that if one of the mappings f and g is continuous, then $fx^* = gx^* = x^*$. Without loss of generality, we can suppose that f is continuous. It follows from (91) that

$$x^* = \lim_{n\longrightarrow+\infty} fx_{2n} = f\left(\lim_{n\longrightarrow+\infty} x_{2n}\right) = f(x^*). \tag{93}$$

That is, x^* is a fixed point of f.

From the contractive conditions (60), we get

$$\begin{aligned}&\psi(d(x^*, gx^*) + \varphi(x^*) + \varphi(gx^*)) \\ &\quad\le \psi(s^p[d(fx^*, gx^*) + \varphi(fx^*) + \varphi(gx^*)]) \\ &\quad\le \psi(n(x^*, x^*, d, f, g, \varphi)) - \phi(r(x^*, x^*, d, g, \varphi)),\end{aligned} \tag{94}$$

where

$$\begin{aligned}n(x^*, x^*, d, f, g, \varphi) &= \lambda \max\{d(x^*, x^*) + \varphi(x^*) + \varphi(x^*), d(fx^*, x^*) \\ &\quad + \varphi(fx^*) + \varphi(x^*) + d(x^*, gx^*) + \varphi(x^*) \\ &\quad + \varphi(gx^*), \frac{1}{s}\{d(fx^*, x^*) + \varphi(fx^*) + \varphi(x^*) \\ &\quad + d(x^*, gx^*) + \varphi(x^*) + \varphi(gx^*)\}\} \\ &= \lambda \max\left\{0, d(x^*, gx^*) + \varphi(gx^*), \frac{1}{s}\{d(x^*, gx^*) + \varphi(gx^*)\}\right\} \\ &\le d(x^*, gx^*) + \varphi(gx^*), \\ r(x^*, x^*, d, g, \varphi) &= \max\{d(x^*, x^*) + \varphi(x^*) + \varphi(x^*), d(x^*, gx^*) + \varphi(x^*) + \varphi(gx^*)\} \\ &\ge d(x^*, gx^*) + \varphi(gx^*).\end{aligned} \tag{95}$$

It follows from (94) that

$$\begin{aligned}&\psi(d(x^*, gx^*) + \varphi(x^*) + \varphi(gx^*)) \\ &\quad\le \psi(d(x^*, gx^*) + \varphi(gx^*)) - \phi(d(x^*, gx^*) + \varphi(gx^*)),\end{aligned} \tag{96}$$

Hence, $d(x^*, gx^*) + \varphi(gx^*) = 0$, that is, $x^* = gx^*$ and $\varphi(gx^*) = 0$. This implies that x^* is the unique common fixed point of f and g. This completes the proof.

Example 15. Let $X = [0, 1]$ and $d(x, y) = (x - y)^2$. Define mappings $f, g, \varphi : X \longrightarrow X$ by

$$fx = \begin{cases} 0, & x \in \left[0, \dfrac{1}{2}\right] \\ \dfrac{x}{16}, & x \in \left(\dfrac{1}{2}, 1\right] \end{cases}, gx = \frac{x}{16}, \varphi x = \frac{x^2}{4}, \quad x \in [0, 1]. \tag{97}$$

Define mappings $\psi, \phi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\psi(t) = 2t$, $\phi(t) = 3 \cdot 256t/8 \cdot 1413$.

Now we consider two cases:

Case 1. $x \in [0, 1/2]$. For $y \in X$, we have

$$\begin{aligned}\psi(s^p[d(fx, gy) + \varphi(fx) + \varphi(gy)]) &= 2 \cdot 8 \cdot \left[\left(\frac{y}{16}\right)^2 + \frac{1}{4}\left(\frac{y}{16}\right)^2\right] \\ &= \frac{5}{64}y^2 \le \frac{1}{4}\left(\frac{5x^2}{8} + \frac{1157y^2}{2048}\right),\end{aligned}$$

$$\begin{aligned}\psi(n(x, y, d, f, g, \varphi)) &\ge 2 \cdot \frac{1}{4}[d(fx, x) + \varphi(fx) + \varphi(x) + d(y, gy) \\ &\quad + \varphi(y) + \varphi(gy)]) = \frac{1}{2}\left[x^2 + \frac{x^2}{4} + \left(y - \frac{y}{16}\right)^2 \right.\\ &\quad \left. + \frac{y^2}{4} + \frac{1}{4}\left(\frac{y}{16}\right)^2\right] = \frac{5}{8}x^2 + \frac{1157}{2048}y^2,\end{aligned}$$

$$\begin{aligned}\phi(r(x, y, d, g, \varphi)) &= \frac{3 \cdot 256}{8 \cdot 1413} \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, gy) \\ &\quad + \varphi(y) + \varphi(gy)\} \\ &\le \frac{3 \cdot 1157}{4 \cdot 6922} \max\left\{(x - y)^2 + \frac{x^2}{4} + \frac{y^2}{4}, \frac{1157y^2}{1024}\right\} \\ &\le \frac{3 \cdot 1157}{4 \cdot 6922}\left(2x^2 + 2y^2 + \frac{x^2}{4} + \frac{y^2}{4} + \frac{1157y^2}{1024}\right) \\ &\le \frac{3 \cdot 1157}{4 \cdot 6922}\left(\frac{9}{4}x^2 + \frac{3461y^2}{1024}\right) \le \frac{3}{4}\left(\frac{5}{8}x^2 + \frac{1157}{2048}y^2\right).\end{aligned} \tag{98}$$

It follows that

$$\begin{aligned}&\psi(s^p[d(fx, gy) + \varphi(fx) + \varphi(gy)]) \\ &\quad\le \frac{1}{4}\left(\frac{5x^2}{8} + \frac{1157y^2}{2048}\right) = \left(\frac{5x^2}{8} + \frac{1157y^2}{2048}\right) \\ &\quad - \frac{3}{4}\left(\frac{5x^2}{8} + \frac{1157y^2}{2048}\right) \le \psi(n(x, y, d, f, g, \varphi)) \\ &\quad - \phi(r(x, y, d, g, \varphi)).\end{aligned} \tag{99}$$

Case 2. $x \in (1/2, 1]$. For $y \in X$, one can obtain

$$
\begin{aligned}
\psi(s^p[d(fx, gy) + \varphi(fx) + \varphi(gy)]) &= 2 \cdot 8 \cdot \left[\left(\frac{x}{16} - \frac{y}{16}\right)^2 + \frac{1}{4}\left(\frac{x}{16}\right)^2 + \frac{1}{4}\left(\frac{y}{16}\right)^2\right] \\
&= \frac{1}{16}\left((x-y)^2 + \frac{x^2+y^2}{128}\right) \le \frac{1}{8}\left((x-y)^2 + \frac{x^2+y^2}{4}\right),
\end{aligned}
$$

$$
\begin{aligned}
\psi(n(x, y, d, f, g, \varphi)) &\ge 2 \cdot \frac{1}{4}[d(x, y) + \varphi(x) + \varphi(y)] \\
&= \frac{1}{2}\left((x-y)^2 + \frac{x^2+y^2}{4}\right),
\end{aligned}
$$

$$
\begin{aligned}
\phi(r(x, y, d, g, \varphi)) &= \frac{3 \cdot 256}{8 \cdot 1413} \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, gy) + \varphi(y) + \varphi(gy)\} \\
&= \frac{3 \cdot 256}{8 \cdot 1413} \max\left\{(x-y)^2 + \frac{x^2}{4} + \frac{y^2}{4}, \frac{1157y^2}{1024}\right\} \\
&\le \frac{3}{8}\left((x-y)^2 + \frac{x^2+y^2}{4}\right).
\end{aligned}
\tag{100}
$$

It follows that

$$
\begin{aligned}
&\psi(s^p[d(fx, gy) + \varphi(fx) + \varphi(gy)]) \\
&\quad \le \frac{1}{8}\left((x-y)^2 + \frac{x^2+y^2}{4}\right) = \frac{1}{2}\left((x-y)^2 + \frac{x^2+y^2}{4}\right) \\
&\quad - \frac{3}{8}\left((x-y)^2 + \frac{x^2+y^2}{4}\right) \le \psi(n(x, y, d, f, g, \varphi)) \\
&\quad - \phi(r(x, y, d, g, \varphi)).
\end{aligned}
\tag{101}
$$

Therefore, all conditions of Theorem 10 are satisfied with $\lambda = 1/4$, $s = 2, p = 3$. Theorem 10 ensures that f and g has a unique common fixed point. It is easy to get that 0 is the unique common fixed point of f and g.

Note that, taking $S = T = I_x$ in Theorem 2.1 of [11], Roshan et al. give the existence of common fixed point for mappings f, g such that

$$
d(fx, gy) \le \frac{q}{s^4} \max\left\{d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, fy) + d(fx, y))\right\},
\tag{102}
$$

where $q \in (0, 1)$ is a constant. For $x = 0, y \in (1/2, 1]$, it is easy to calculate that for $q < 1/16$,

$$
\begin{aligned}
d(fx, gy) &= \frac{y^2}{256} > \frac{q}{16}y^2 = \frac{q}{16} \max\left\{y^2, 0, \frac{225y^2}{256}, \frac{y^2}{2 \cdot 256} + \frac{y^2}{2}\right\} \\
&= \frac{q}{s^4} \max\left\{d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, fy) + d(fx, y))\right\},
\end{aligned}
\tag{103}
$$

which implies that Theorem 2.1 of [11] cannot be applied to testify the existence of common fixed points of the mappings f and g in X.

If $\varphi = 0$ in Theorem 10, we can get the following result:

Corollary 16. *Let (X, d) be a complete b-metric space with parameter $s \ge 1$ and let $f, g : X \longrightarrow X$ be given self-mappings, and one of f and g is continuous. Suppose $p \ge 3$, $0 < \lambda \le 1/4$ are two constants. If there are functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$
\psi(s^p d(fx, gy)) \le \psi(n_1(x, y, d, f, g, \varphi)) - \phi(r_1(x, y, d, g, \varphi)),
\tag{104}
$$

where

$$
\begin{aligned}
n_1(x, y, d, f, g, \varphi) &= \lambda \max\{d(x, y), d(fx, x) \\
&\quad + d(y, gy), \frac{1}{s}\{d(fx, y) + d(x, gy)\}, \\
r_1(x, y, d, g, \varphi) &= \max\{d(x, y), d(y, gy)\},
\end{aligned}
\tag{105}
$$

then f and g have a unique common fixed point in X.

If we consider the corresponding problem in the setting of metric space, that is, $s = 1$ in Theorem 10, we get the following:

Corollary 17. *Let (X, d) be a complete metric space and let $f, g : X \longrightarrow X$ be given self-mappings, and one of f and g is continuous. Suppose $\varphi : X \longrightarrow [0, \infty)$ is a lower semicontinuous function and $0 < \lambda \le 1/4$ is a constant. If there are functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$
\begin{aligned}
&\psi(d(fx, gy) + \varphi(fx) + \varphi(gy)) \\
&\quad \le \psi(n_2(x, y, d, f, g, \varphi)) - \phi(r(x, y, d, g, \varphi)),
\end{aligned}
\tag{106}
$$

where

$$
\begin{aligned}
n_2(x, y, d, f, g, \varphi) &= \lambda \max\{d(x, y) + \varphi(x) + \varphi(y), d(fx, x) \\
&\quad + \varphi(fx) + \varphi(x) + d(y, gy) + \varphi(y) \\
&\quad + \varphi(gy), \frac{1}{s}\{d(fx, y) + \varphi(fx) + \varphi(y) \\
&\quad + d(x, gy) + \varphi(x) + \varphi(gy)\}\},
\end{aligned}
\tag{107}
$$

and $r(x, y, d, g, \varphi)$ is the same as Theorem 10, then f and g have a unique common fixed point in X.

Theorem 18. *Let (X, d) be a complete b-metric space with parameter $s \ge 1$, and let $f : X \longrightarrow X$ be a given self-mapping and $\varphi : X \longrightarrow [0, +\infty)$ be a lower semicontinuous function with $\varphi(t) = 0$ for $t \in Fix(f)$. Suppose $p \ge 2$ is a constant. If there are functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$
\begin{aligned}
&\psi(s^p[d(fx, fy) + \varphi(fx) + \varphi(fy)]) \\
&\quad \le \psi(h(x, y, d, f, \varphi)) - \phi(q(x, y, d, f, \varphi)),
\end{aligned}
\tag{108}
$$

where

$$h(x,y,d,f,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y),d(x,fx)+\varphi(x)+\varphi(fx),d(y,fy)+\varphi(y)+\varphi(fy),\frac{1}{2s}\{d(x,fy)+\varphi(x)+\varphi(fy)+d(y,fx)+\varphi(y)+\varphi(fx)\}\},$$

$$q(x,y,d,f,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y),d(y,fy)+\varphi(y)+\varphi(fy)\}, \tag{109}$$

then f has a unique fixed point $z\in X$.

Proof. The proof is similar to that of Theorem 10, so we omit it.

Remark 19.

(i) If $s=1$ in Theorem 18, then we get Theorem 2.

(ii) If $s=1$ and $\varphi=0$ in Theorem 18, then we get Theorem 1.

According to Theorem 18, we can obtain the following result:

Corollary 20. *Let (X,d) be a complete b-metric space with parameter $s\geq 1$, and let $f:X\longrightarrow X$ be a given self-mapping and $\varphi:X\longrightarrow[0,+\infty)$ be a lower semicontinuous function with $\varphi(t)=0$ for $t\in Fix(f)$. Suppose $p\geq 2$ is a constant. If there are functions $\psi\in\Psi$ and $\phi\in\Phi$ such that*

$$\psi(s^p[d(fx,fy)+\varphi(fx)+\varphi(fy)])\leq\psi(h(x,y,d,f,\varphi))-\phi(h(x,y,d,f,\varphi)), \tag{110}$$

where $h(x,y,d,f,\varphi)$ is the same as Theorem 18, then f has a unique fixed point $z\in X$.

4. Application

It is well known that an automobile suspension system is the realistic application for the spring mass system in engineering problems. Consider the motion of a spring of a car when it moves along a rough and pitted road, where the forcing term is the rough road and shock absorbers provide the damping. The external forces under which the system operates may be gravity, ground vibrations, earthquake, tension force, etc. Let m be the mass of the spring and F be the external force acting on it; then, the critical damped motion of this system subjected to the external force F is governed by the following initial value problem:

$$\begin{cases} m\dfrac{d^2x}{dt^2}+l\dfrac{dx}{dt}-mF(t,x(t))=0,\\ x(0)=0,\\ x'(0)=0,\end{cases} \tag{111}$$

where $l>0$ is the damping constant and $F:[0,T]\times\mathbb{R}^+\longrightarrow\mathbb{R}$ is a continuous function.

It is easy to show that the problem (111) is equivalent to the integral equation:

$$x(t)=\int_0^T\gamma(t,r)F(r,x(r))dr,\quad t\in[0,T], \tag{112}$$

where $\gamma(t,r)$ is Green's function given by

$$\gamma(t,r)=\begin{cases}\dfrac{1-e^{\mu(t-r)}}{\mu}, & 0\leq r\leq t\leq T\\ 0, & 0\leq t\leq r\leq T\end{cases}, \tag{113}$$

where $\mu=l/m$ is a constant.

In this section, by using Corollary 20, we will show the existence of a solution to the integral equation:

$$x(t)=\int_0^T G(t,r,x(r))dr. \tag{114}$$

Let $X=C([0,T])$ be the set of real continuous functions defined on $[0,T]$. For $p\geq 1$, we define

$$d(x,y)=(\rho(x,y))^p=\sup_{t\in[0,T]}|x(t)-y(t)|^p\quad\text{for all } x,y\in X. \tag{115}$$

It is easy to prove that (X,d) is a complete b-metric space with $s=2^{p-1}$.

Consider the mapping $f:X\longrightarrow X$ defined by

$$fx(t)=\int_0^T G(t,r,x(r))dr. \tag{116}$$

Theorem 21. *Consider equation (114) and suppose that*

(i) $G:[0,T]\times[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}^+$ is continuous

(ii) there exists a continuous function $\gamma:[0,T]\times[0,T]\longrightarrow\mathbb{R}^+$ such that

$$\sup_{t\in[0,T]}\int_0^T\gamma(t,r)dr\leq 1, \tag{117}$$

(iii) there exists a constant $L\in(0,1)$ such that for $(t,r)\in[0,T]\times[0,T]$,

$$|G(t,r,x(r))-G(t,r,y(r))|\leq\sqrt[p]{\frac{1-L}{s^p}}\gamma(t,r)|x(r)-y(r)|. \tag{118}$$

Then, the integral equation (114) has a unique solution $x\in X$.

Proof. For $x, y \in X$, by virtue of assumptions (i)–(iii), we have

$$\begin{aligned} s^p d(fx(t), fy(t)) &= s^p \sup_{t\in[0,T]} |fx(t) - fy(t)|^p \\ &= s^p \sup_{t\in[0,T]} \left| \int_0^T G(t, r, x(r))dr - \int_0^T G(t, r, y(r))dr \right|^p \\ &\le s^p \sup_{t\in[0,T]} \left(\int_0^T |G(t, r, x(r)) - G(t, r, y(r))|dr \right)^p \\ &\le s^p \sup_{t\in[0,T]} \left(\int_0^T \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r) \mid x(r) - y(r) \mid dr \right)^p \\ &\le s^p \sup_{t\in[0,T]} \left(\int_0^T \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r)dr \right)^p \sup_{t\in[0,T]} |x(t) - y(t)|^p \\ &\le (1 - L)h(x(t), y(t), d, f, \varphi), \end{aligned} \tag{119}$$

which implies that

$$s^p d(fx(t), fy(t)) \le (1 - L)h(x(t), y(t), d, f, \varphi). \tag{120}$$

Therefore, letting $\psi(t) = t$, $\phi(t) = Lt$, and $\varphi(t) = 0$, all the conditions of Corollary 20 are satisfied. As a result, the mapping f has a unique fixed point $x \in X$, which is a solution of the integral equation (114).

Remark 22. If we let $G(t, r, x(r)) = \gamma(t, r)F(r, x(r))$, $|F(r, x(r)) - F(r, y(r))| \le \sqrt[p]{(1 - L)/s^p} \mid x(r) - y(r) \mid$ and μ, T satisfy $\mu \ge T$, then all the conditions of Theorem 21 are satisfied, which implies that the problem (111) has a unique solution.

5. Conclusions

In this manuscript, we introduced a new class of generalized weakly contractive mappings and established common fixed point results involving this new class of mappings in the framework of b-metric spaces. Further, we provided examples that elaborated the useability of our results. Meanwhile, we presented an application to the existence of solutions to an integral equation by means of one of our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Solving a Split Feasibility Problem by the Strong Convergence of Two Projection Algorithms in Hilbert Spaces

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The goal of this manuscript is to establish strong convergence theorems for inertial shrinking projection and CQ algorithms to solve a split convex feasibility problem in real Hilbert spaces. Finally, numerical examples were obtained to discuss the performance and effectiveness of our algorithms and compare the proposed algorithms with the previous shrinking projection, hybrid projection, and inertial forward-backward methods.

1. Introduction

Assume that Y is a real HS defined on the induced norm $\|.\|$ and the inner product $\langle ., . \rangle$. Let C be a NCC subset of Y.

The mapping $T : C \longrightarrow C$ is called NE, if for all $\kappa, \omega \in C$, the following inequality holds:

$$\|T\kappa - T\omega\| \leq \|\kappa - \omega\|. \qquad (1)$$

For the mapping $T, F(T) = \{\kappa \in C : T\kappa = \kappa\}$ refers to the set of all FPs of it.

Here, we study the following inclusion problem:

$$\text{Find } \tilde{\kappa} \in Y \text{ so that } 0 \in A\tilde{\kappa} + B\tilde{\kappa}, \qquad (2)$$

where $A : Y \longrightarrow Y$ and $B : Y \longrightarrow 2^Y$ are single-valued and set-valued operators, respectively.

There are many important applications enjoyed by approximating FP problems for NEMs, such as monotone variational inequalities, image restoration problems, convex optimization problems, and SCFPs, for example, see [1–3]. For more accuracy, these problems can be expressed as mathematical models such a machine learning and the linear inverse problem.

In the past, the solution of problem (2) was described by $(A + B)^{-1}(0)$ and it relied on the forward-backward splitting method [4–10]. This technique is described as follows: $\kappa_1 \in Y$ and

$$\kappa_{n+1} = (I + \tau B)^{-1}(\kappa_n - \tau A\kappa_n), \quad n \geq 1, \tau > 0. \qquad (3)$$

In this scene, we do not mean the sum of A and B in the iterates, but each step of iterates includes only A as the forward term and B as the backward term. As special cases, this technique gets involved heavily in a study of the proximal point algorithm [11–13] and the gradient method [14–17].

In 1979, a good splitting iterative scheme in a real HS was introduced by Lions and Mercier [18]. It is described as follows:

$$\kappa_{n+1} = \left(2J_\tau^A - I\right)\left(2J_\tau^B - I\right)\kappa_n, \quad n \geq 1, \qquad (4)$$

$$\kappa_{n+1} = J_\tau^A \left(2J_\tau^B - I\right)\kappa_n + \left(I - J_\tau^B\right)\kappa_n, \quad n \geq 1, \tag{5}$$

where $J_\tau^S = (I + \tau S)^{-1}$. In the previous literature, the algorithm (4) is called Peaceman-Rachford [7] and the scheme (5) is called Douglas-Rachford [19]. Generally, the convergence of both procedures is weak [7].

In 2001, a heavy ball method involved for studying maximal monotone operators is introduced by Alvarez and Attouch [20]; this idea was developed in [21, 22], where an inertial term was added. This procedure is called the inertial proximal point algorithm and it takes the shape

$$\begin{cases} \omega_n = \kappa_n + \theta_n(\kappa_n - \kappa_{n-1}), \\ \kappa_{n+1} = (I + \tau_n B)_n^{-1}\omega_n, \quad n \geq 1. \end{cases} \tag{6}$$

They got the weak convergence for the mapping B, if $\{\tau_n\}$ is nondecreasing and $\{\theta_n\} \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|\kappa_n - \kappa_{n-1}\|^2 < \infty. \tag{7}$$

In particular, condition (7) is true for $\theta_n < 1/3$. Here, θ_n is an extrapolation factor and the inertia is represented by the term $\theta_n(\kappa_n - \kappa_{n-1})$.

It should be noted that the inertial term improves and increases the convergence speed of the algorithm [23–25].

An inertial proximal point algorithm improved by Moudafi and Oliny [26], where the single-valued, cocoercive, and Lipschitz continuous operator was added, is as follows:

$$\begin{cases} \omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \kappa_{n+1} = (I + \tau_n B)_n^{-1}(\omega_n - \tau_n A\omega_n), \quad n \geq 1. \end{cases} \tag{8}$$

The problem of weak convergence still persists for the algorithm (8) via stipulation (7) and $\tau_n < 2/L$ where L is the Lipschitz constant of A.

Besides that, the strong convergence is of interest to many researchers, but the study of convergence via norm convergence in infinite-dimensional spaces is often much more desirable than weak convergence [27].

The first contribution of researchers to the strong convergence is the algorithm presented by Nakajo and Takahashi [28]. They added the CQ terms to the Mann algorithm as follows: for an arbitrary point $\kappa_0 \in C$, define the sequence $\{\kappa_n\}$ iteratively by

$$\begin{cases} \omega_n = \alpha_n\kappa_n + (1 - \alpha_n)T\kappa_n, \\ C_n = \{p \in C : \|\omega_n - p\| \leq \|\kappa_n - p\|\}, \\ Q_n = \{p \in C : \langle \kappa_0 - \kappa_n, \kappa_n - p\rangle \geq 0\}, \\ \kappa_{n+1} = P_{Q_n \cap C_n}\kappa_0, \quad n \geq 0. \end{cases} \tag{9}$$

They showed that the sequence $\{\kappa_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}\kappa_0$, whenever the sequence $\{\alpha_n\}$ is bounded above by 1. We highly recommend seeing [24, 29], for more details on the CQ algorithms for NEMs.

Based on the algorithm (9), Dong et al. [30] introduced a strong convergence result by implicating an inertial forward-backward algorithm for monotone inclusions as the following: assume that $A : Y \longrightarrow Y$ is an α-ISM operator and $B : Y \longrightarrow 2^Y$ is a MM operator so that $(A + B)^{-1}(0) \neq \varnothing$. Suppose $\{\alpha_n\} \in \mathbb{R}$ and $\{\kappa_n\} \in Y$ is a sequence maked iteratively by $\kappa_\circ, \kappa_1 \in Y$,

$$\begin{cases} \omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \upsilon_n = (I + \tau_n B)^{-1}(\omega_n - \tau_n A\omega_n), \\ C_n = \left\{p \in H : \|\upsilon_n - p\|^2 \leq \|\kappa_n - p\|^2 - 2\alpha_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \alpha_n^2\|\kappa_{n-1} - \kappa_n\|^2\right\}, \\ Q_n = \{p \in H : \langle \kappa_0 - \kappa_n, \kappa_n - p\rangle \leq 0\}, \\ \kappa_{n+1} = P_{Q_n \cap C_n}\kappa_0, \quad n \geq 1. \end{cases} \tag{10}$$

Recently, nice convergence analysis for NEMs via suitable stipulations has been discussed by Dong et al. [31]. They extended the inertial Mann algorithm as follows:

$$\begin{cases} \omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}), \\ \kappa_{n+1} = (1 - \lambda_n)\omega_n + \lambda_n T\upsilon_n, \quad n \geq 1, \end{cases} \tag{11}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are real sequences that justify the stipulations D_1 and D_2 [31].

Believing in the idea of strong convergence of algorithms in this manuscript, the two-step inertial shrinking projection algorithm is introduced which analyzes the strong convergence. As an application to our main results, the SCFP is solved. Finally, to see the behavior and performance of our algorithms in terms of convergence, numerical results are presented and discussed.

2. Preliminaries

This section is devoted to collect some important preliminaries, which we need in the sequel. Let C be a NCC subset of a real HS Y and $\{\kappa_n\}$ be a sequence in Y. Here, the strong convergence of $\{\kappa_n\}$ to a point κ is written as $\kappa_n \longrightarrow \kappa$. The metric projection of Y onto C is described by P_C, that is, $\|\kappa - P_C\kappa\| \le \|\kappa - \omega\|$ for all $\kappa \in Y$ and $\omega \in C$.

Lemma 1 (see [32]). *Let C be a NCC subset of a real HS Y, the metric projection P_C is firmly NE, i.e.,*

$$\|P_C\kappa - P_C\omega\|^2 \le \langle P_C\kappa - P_C\omega, \kappa - \omega\rangle, \tag{12}$$

for all $\kappa, \omega \in Y$. Furthermore, for all $\kappa \in Y$ and $\omega \in C$, $\langle \kappa - P_C\kappa, \omega - P_C\omega\rangle \le 0$ is satisfied.

Lemma 2 (see [32]). *Assume that Y is a real HS. Then, we get*

(i) $\|\kappa + \omega\|^2 \le \|\kappa\|^2 + 2\langle \omega, \kappa + \omega\rangle$

(ii) $\|\rho\kappa + (1-\rho)\omega\|^2 = \rho\|\kappa\|^2 + (1-\rho)\|\omega\|^2 - \rho(1-\rho)\|\kappa - \omega\|^2$

for each $\kappa, \omega \in Y$ and for a real number ρ.

Lemma 3 (see [33]). *Suppose that Y is a real HS and $\{\kappa_n\}$ is a sequence in Y. Then, the following hypotheses hold:*

(i) If $\kappa_n \rightharpoonup \kappa$ and $\|\kappa_n\| \longrightarrow \|\kappa\|$ as $n \longrightarrow \infty$, then $\kappa_n \longrightarrow \kappa$ as $n \longrightarrow \infty$; that is, the HS Y has the Kadec-Klee property

(ii) If $\kappa_n \rightharpoonup \kappa$ as $n \longrightarrow \infty$, then $\|\kappa\| \le \liminf_{n\to\infty}\|\kappa_n\|$

Lemma 4 (see [34]). *Let C be a NCC subset of a real HS Y. For each $\kappa, \omega, \upsilon \in Y$ and $b \in \mathbb{R}$, the following set is closed and convex:*

$$\left\{\delta \in C : \|\omega - \delta\|^2 \le \|\kappa - \delta\|^2 + \langle \upsilon, \delta\rangle + b\right\}. \tag{13}$$

Lemma 5 (see [28]). *Let C be a NCC subset of a real HS Y and $P_C : Y \longrightarrow C$ be the metric projection. Then, for all $\kappa \in Y$ and $\omega \in C$, the following inequality holds:*

$$\|\omega - P_C\kappa\|^2 + \|\kappa - P_C\kappa\|^2 \le \|\kappa - \omega\|^2. \tag{14}$$

Lemma 6 (see [35]). *Let T be a NE self-mapping of a NCC subset C of a real HS Y. The mapping $I - T$ is demiclosed, i.e., the sequence $\{\kappa_n\}$ in C weakly converges to some $\kappa \in C$ and the sequence $\{(I-T)(\kappa_n)\}$ strongly converges to some ω ; it follows that $(I-T)(\kappa) = \omega$.*

Definition 7. Assume that $D(A) \subset Y$ is the domain of the mapping A, then for all $\kappa, \omega \in D(A)$, the mapping A is called

(i) monotone if $\langle \kappa - \omega, A\kappa - A\omega\rangle \ge 0$

(ii) σ-strongly monotone if there is $\beta > 0$ so that $\langle \kappa - \omega, A\kappa - A\omega\rangle \ge \sigma\|\kappa - \omega\|^2$

(iii) α-ISM if there is $\alpha > 0$ so that $\langle \kappa - \omega, A\kappa - A\omega\rangle \ge \alpha \|A\kappa - A\omega\|^2$

Lemma 8 (see [5]). *Let Y be a real HS, $A : Y \longrightarrow Y$ be an α-ISM operator, and $B : Y \longrightarrow 2^Y$ be a MM operator. For each $\ell > 0$, we consider*

$$T_\ell = J_\ell^B(I - \ell A) = (I + \ell B)^{-1}(I - \ell A), \tag{15}$$

then the following statements hold:

(i) for $\ell > 0, F(T_\ell) = (A+B)^{-1}(0)$

(ii) for $0 < s \le \ell$ and $\kappa \in Y$, $\|\kappa - T_s\kappa\| \le 2\|\kappa - T_\ell\kappa\|$

Lemma 9 (see [36]). *Let H be a real HS, $A : Y \longrightarrow Y$ be an α-ISM operator and $B : Y \longrightarrow 2^Y$ be a MM operator, then for all $\ell > 0$ and all $\kappa, \omega \in Y$, we have*

$$\|T_\ell\kappa - T_\ell\omega\|^2 \le \|\kappa - \omega\|^2 - \ell(2\alpha - \ell)\|A\kappa - A\omega\|^2. \tag{16}$$

3. Strong Convergence Results

From now on, we assume that C be a NCC subset of a real HS Y,$A : Y \longrightarrow Y$ is α-ISM operator, $B : Y \longrightarrow 2^Y$ is MM operator, $T : Y \longrightarrow Y$ is quasi-NEM so that $I - T$ is demiclosed at zero and $\Omega = (A+B)^{-1}(0)$.

Now, we build our algorithms to finding an element in Ω as follows:

Now, we shall discuss the strong convergence of Algorithm 1: by introducing the following theorem.

Theorem 10. *Let the sequence $\{\alpha_n\}$, $\{\beta_n\}$ be bounded and $\{\gamma_n\}$ be a sequence in $(0,1]$ and $\{\tau_n\}$ be a sequence of positive real numbers so that the following two stipulations hold:*

(i) $\inf_n\{\gamma_n\} \ge \gamma > 0$

(ii) $0 < \inf_n\{\tau_n\} \le \sup_n\{\tau_n\} < 2\alpha$

If $\Omega \ne \emptyset$, then the sequence $\{\kappa_n\}$ created by Algorithm 1: converges strongly to $\Theta = P_\Omega(\kappa_1)$.

Proof. The proof will be divided into the following steps:

Step (i). For each $\kappa_1 \in H$, $\Omega \subset C_{n+1}$, and for $n \ge 0$. Prove that $P_{C_{n+1}}\kappa_1$ is well-defined.

From the stipulation (ii) and Lemma 9, we get $T_{\tau_n} = (I + \tau_n B)^{-1}(I - \tau_n A)$ is NEM. Thus, it follows from Lemma 8 that the set Ω is closed and convex. Moreover, Lemma 4 leads that for all $n \ge 1$,C_{n+1} is closed and convex. Considering

Let $\{\alpha_n\}$,$\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of real numbers. Select initial $\kappa_0, \kappa_1 \in C_1 = Y$.
Step (1). Compute
$$\omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}),$$
$$\upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}).$$
Step (2). Compute
$$\mu_n = (1-\gamma_n)\omega_n + \gamma_n J^B_{\tau_n}(\upsilon_n - \tau_n A\upsilon_n).$$
Step (3). Compute
$$C_{n+1} = \{p \in C_n : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + (\alpha_n^2(1-\gamma_n) + \gamma_n\beta_n^2)\|\kappa_{n-1} - \kappa_n\|^2\},$$
$$\kappa_{n+1} = P_{C_{n+1}}(\kappa_1), n \ge 1.$$

Algorithm 1: Shrinking projection algorithm.

$p \in \Omega$, we get

$$\|\omega_n - p\|^2 = \|(\kappa_n - p) - \alpha_n(\kappa_{n-1} - \kappa_n)\|^2 = \|\kappa_n - p\|^2 - 2\alpha_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \alpha_n^2\|\kappa_{n-1} - \kappa_n\|^2. \tag{17}$$

By the same manner, one can write

$$\|\upsilon_n - p\|^2 = \|\kappa_n - p\|^2 - 2\beta_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \beta_n^2\|\kappa_{n-1} - \kappa_n\|^2. \tag{18}$$

Furthermore, by Lemma 2 (ii) and Lemma 9, we obtain that

$$\begin{aligned}\|\mu_n - p\|^2 &= \left\|(1-\gamma_n)\omega_n + \gamma_n J^B_{\tau_n}(\upsilon_n - \tau_n A\upsilon_n) - p\right\|^2 \\ &= \left\|(1-\gamma_n)(\omega_n - p) + \gamma_n\left(T_{\tau_n}\upsilon_n - p\right)\right\|^2 \\ &= \gamma_n\left\|T_{\tau_n}\upsilon_n - p\right\|^2 + (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\left\|T_{\tau_n}\upsilon_n - \omega_n\right\|^2 \le (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad + \gamma_n\left\|T_{\tau_n}\upsilon_n - p\right\|^2 = (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad + \gamma_n\left\|T_{\tau_n}\upsilon_n - T_{\tau_n}p\right\|^2 \le (1-\gamma_n)\|\omega_n - p\|^2 \\ &\quad + \gamma_n\left(\|\upsilon_n - p\|^2 - \tau_n(2\alpha - \tau_n)\|A\upsilon_n - Ap\|^2\right) \\ &\le (1-\gamma_n)\|\omega_n - p\|^2 + \gamma_n\|\upsilon_n - p\|^2.\end{aligned} \tag{19}$$

Applying (17) and (18) in (19) and by stipulation (ii) of Theorem 10, we can write

$$\begin{aligned}\|\mu_n - p\|^2 &\le (1-\gamma_n)\left(\|\kappa_n - p\|^2 - 2\alpha_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \alpha_n^2\|\kappa_{n-1} - \kappa_n\|^2\right) \\ &\quad + \gamma_n\left(\|\kappa_n - p\|^2 - 2\beta_n\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \beta_n^2\|\kappa_{n-1} - \kappa_n\|^2\right) \\ &= \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle \\ &\quad + \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2.\end{aligned} \tag{20}$$

It can be easily obtained $\Omega \subset C_1 = Y$. For some $n \ge 1$, assume that $\Omega \subset C_n$, then $p \in C_n$, and by (20), we conclude that $p \in C_{n+1}$. Therefore, $\Omega \subset C_{n+1}$ for all $n \ge 1$, and this finishes the requirement of Claim 1.

Step (ii). Prove that the boundedness of $\{\kappa_n\}$. Because Ω is a NCC subset of Y, there is a unique $u \in \Omega$ so that $u = P_\Omega\kappa_1$.

From $\kappa_n = P_{C_n}\kappa_1$,$C_{n+1} \subset C_n$, and $\kappa_{n+1} \in C_n$ for all $n \ge 1$, we get

$$\|\kappa_n - \kappa_1\| \le \|\kappa_{n+1} - \kappa_1\|, \quad \text{for all } n \ge 1. \tag{21}$$

Also, since $\Omega \subset C_n$, we get

$$\|\kappa_n - \kappa_1\| \le \|u - \kappa_1\|, \quad \text{for all } n \ge 1. \tag{22}$$

By (21) and (22), we obtain that $\lim_{n\to\infty}\|\kappa_n - \kappa_1\|$ exists; this leads to $\{\kappa_n\}$ being bounded.

Step (iii). Prove that $\kappa_n \longrightarrow \Theta$ as $n \longrightarrow \infty$, for some $\Theta \in Y$. By the structure of C_n, for $m > n$, one sees that $\kappa_m = P_{C_m}\kappa_1 \in C_m \subset C_n$. From Lemma 5, we have

$$\|\kappa_m - \kappa_n\|^2 \le \|\kappa_m - \kappa_1\|^2 - \|\kappa_n - \kappa_1\|^2. \tag{23}$$

From Step (ii), we obtain that $\|\kappa_m - \kappa_n\|^2 \longrightarrow 0$ as $m, n \longrightarrow \infty$. This proves that the sequence $\{\kappa_n\}$ is a Cauchy. Therefore, $\kappa_n \longrightarrow \Theta$ as $n \longrightarrow \infty$. In particular, we can obtain

$$\lim_{n\to\infty}\|\kappa_{n+1} - \kappa_n\| = 0. \tag{24}$$

Step (iv). Prove that $\Theta \in \Omega$. Because $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded, then by (24), we have

$$\|\omega_n - \kappa_n\| = |\alpha_n|\|\kappa_n - \kappa_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{25}$$

$$\|\upsilon_n - \kappa_n\| = |\beta_n|\|\kappa_n - \kappa_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{26}$$

From (24), (25), and (26), we have

$$\|\kappa_{n+1} - \omega_n\| \le \|\kappa_{n+1} - \kappa_n\| + \|\omega_n - \kappa_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{27}$$

$$\|\kappa_{n+1} - \upsilon_n\| \le \|\kappa_{n+1} - \kappa_n\| + \|\upsilon_n - \kappa_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{28}$$

Let $\{\alpha_n\}$,$\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of real numbers. Select initial $\kappa_0, \kappa_1 \in Y$.
Step (1). Compute

$$\omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}),$$
$$\upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}).$$

Step (2). Compute

$$\mu_n = (1-\gamma_n)\omega_n + \gamma_n J^B_{\tau_n}(\upsilon_n - \tau_n A\upsilon_n).$$

Step (3). Compute

$$C_n = \{p \in Y : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + (\alpha_n^2(1-\gamma_n) + \gamma_n\beta_n^2)\|\kappa_{n-1} - \kappa_n\|^2\},$$
$$Q_n = \{p \in Y : \langle p - \kappa_n, \kappa_1 - \kappa_n\rangle \le 0\},$$
$$\kappa_{n+1} = P_{C_n\cap Q_n}(\kappa_1), n \ge 1.$$

Algorithm 2: CQ algorithm.

Since $\kappa_{n+1} \in C_{n+1}$, we obtain that

$$\begin{aligned}\|\mu_n - \kappa_{n+1}\| &\le \|\kappa_n - \kappa_{n+1}\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - \kappa_{n+1}, \kappa_{n-1} - \kappa_n\rangle \\ &+ \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2 \le \|\kappa_n - \kappa_{n+1}\|^2 \\ &+ 2|\alpha_n(1-\gamma_n) + \gamma_n\beta_n|\|\kappa_n - \kappa_{n+1}\|\|\kappa_{n-1} - \kappa_n\| \\ &+ \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2.\end{aligned} \tag{29}$$

Thus, from the boundedness of $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ and (24), (29), we obtain that

$$\|\mu_n - \kappa_{n+1}\| \longrightarrow 0. \tag{30}$$

By (24), (30), (27), and (28) and the following inequalities

$$\begin{aligned}\|\mu_n - \kappa_n\| &\le \|\mu_n - \kappa_{n+1}\| + \|\kappa_n - \kappa_{n+1}\|, \\ \|\mu_n - \omega_n\| &\le \|\mu_n - \kappa_{n+1}\| + \|\omega_n - \kappa_{n+1}\|, \\ \|\mu_n - \upsilon_n\| &\le \|\mu_n - \kappa_{n+1}\| + \|\upsilon_n - \kappa_{n+1}\|,\end{aligned} \tag{31}$$

we get that

$$\|\mu_n - \kappa_n\| \longrightarrow 0, \|\mu_n - \omega_n\| \longrightarrow 0, \|\mu_n - \upsilon_n\| \longrightarrow 0. \tag{32}$$

We now have

$$\begin{aligned}\left\|T_{\tau_n}\upsilon_n - \upsilon_n\right\| &= \left\|\frac{1}{\gamma_n}[\mu_n - (1-\gamma_n)\omega_n] - \upsilon_n\right\| \\ &= \frac{1}{\gamma_n}\|\gamma_n\upsilon_n + (1-\gamma_n)\omega_n - \mu_n\| \\ &= \frac{1}{\gamma_n}\|\gamma_n(\upsilon_n - \mu_n) + (1-\gamma_n)(\omega_n - \mu_n)\| \\ &\le \frac{1}{\gamma_n}[\gamma_n\|\upsilon_n - \mu_n\| + (1-\gamma_n)\|\omega_n - \mu_n\|].\end{aligned} \tag{33}$$

Again using the stipulation (i) and (32), we get

$$\lim_{n\to\infty}\left\|T_{\tau_n}\upsilon_n - \upsilon_n\right\| = 0. \tag{34}$$

As $\liminf_{n\to\infty}\tau_n > 0$, there is $\varepsilon > 0$ so that $\tau_n \ge \varepsilon$ and $\varepsilon \in (0, 2\alpha)$ for all $n \ge 1$. Hence, from Lemma 8 (ii) and (34), one can write

$$\|T_\varepsilon\upsilon_n - \upsilon_n\| \le 2\left\|T_{\tau_n}\upsilon_n - \upsilon_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{35}$$

Based on (32), as $\kappa_n \longrightarrow \Theta$, we get $\upsilon_n \longrightarrow \Theta$. Because T_ε is NE, T_ε is continuous mapping. Hence, using (35), we have $\Theta \in \Omega$.

Step (v). Prove that $\Theta = P_\Omega(\kappa_1)$. Since $\kappa_n = P_{C_n}\kappa_1$, and $\Omega \subset C_n$, we obtain that

$$\langle \kappa_1 - \kappa_n, \kappa_n - p\rangle \ge 0, \quad \forall p \in \Omega. \tag{36}$$

By taking the limit in (36), we have

$$\langle \kappa_1 - \Theta, \Theta - p\rangle \ge 0, \quad \forall p \in \Omega. \tag{37}$$

This shows that $\Theta = P_\Omega\kappa_1$ and this finishes the requirement.

Next, we shall discuss the strong convergence of Algorithm 2: by presenting the following theorem.

Theorem 11. *Let the sequence $\{\alpha_n\}$, $\{\beta_n\}$ be bounded and $\{\gamma_n\}$ be a sequence in $(0, 1]$ and $\{\tau_n\}$ be a sequence of positive real numbers so that the following two stipulations hold:*

(i) $\inf_n\{\gamma_n\} \ge \gamma > 0$

(ii) $0 < \inf_n\{\tau_n\} \le \sup_n\{\tau_n\} < 2\alpha$

If $\Omega \ne \varnothing$, then the sequence $\{\kappa_n\}$ marked by Algorithm 2: converges strongly to $\Theta = P_\Omega(\kappa_1)$.

Proof. In the same way as proving Theorem 10, we will discuss the following steps:

Step (i). Prove that for all $\kappa_1 \in Y$, $\Omega \subset Q_n \cap C_n$ and for each $n \ge 0$.$\{\kappa_n\}_{n=0}^\infty$ is well-defined.

It is clear that C_n is closed and a convex subset of Y (by Lemma 4). So, we can rewrite the set Q_n in the shape

$$Q_n = \{p \in Y : \langle \kappa_1 - \kappa_n, p\rangle \le \langle \kappa_1 - \kappa_n, \kappa_n\rangle\}. \tag{38}$$

It follows that Q_n is closed and convex subset of Y too. Thus, $Q_n \cap C_n$ is also closed and convex, for each $n \geq 0$.

Let $p \in \Omega$. One can obtain by similar way of Theorem 10 that

$$\|\mu_n - p\|^2 \leq \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n)\langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + \left[(1-\gamma_n)\alpha_n^2 + \gamma_n\beta_n^2\right]\|\kappa_{n-1} - \kappa_n\|^2. \tag{39}$$

Hence, $p \in C_n$ for all $n \geq 1$; this implies that $\Omega \subset C_n$.

When $n = 1$, we get $Q_1 = Y$, and hence, $\Omega \subset C_1 \cap Q_1$. Suppose that $\Omega \subset C_l \cap Q_l$ for some $l \geq 1$. It follows from $\kappa_{l+1} = P_{C_l \cap Q_l}(\kappa_1)$ that

$$\langle \kappa_{l+1} - a, \kappa_1 - \kappa_{l+1}\rangle \geq 0, \tag{40}$$

for each $a \in C_l \cap Q_l$. Since $\Omega \subseteq C_l \cap Q_l$, and $p \in \Omega$, we have

$$\langle \kappa_{l+1} - p, \kappa_1 - \kappa_{l+1}\rangle \geq 0. \tag{41}$$

This yields $p \in Q_{l+1}$, and hence, $\Omega \subseteq Q_{l+1}$. This implies that $\Omega \subseteq C_{l+1} \cap Q_{l+1}$, and hence, $\{\kappa_n\}$ is well-defined as well as $\Omega \subset C_n \cap Q_n$.

Step (ii). Prove that the boundedness of $\{\kappa_n\}$. Based on Algorithm 2:, one gets

$$\langle \xi - \kappa_n, \kappa_1 - \kappa_n\rangle \leq 0, \quad \forall \xi \in Q_n, n \geq 1. \tag{42}$$

This implies that $\kappa_n = P_{Q_n}(\kappa_1)$. Since $\Omega \subset Q_n$, we have

$$\|\kappa_n - \kappa_1\| \leq \|\kappa_1 - \xi\|, \quad \forall \xi \in \Omega. \tag{43}$$

Also, since $\kappa_{n+1} \in Q_n$, we can write

$$\|\kappa_n - \kappa_1\| \leq \|\kappa_{n+1} - \kappa_1\|. \tag{44}$$

By (43) and (44), we obtain $\lim_{n\to\infty}\|\kappa_n - \kappa_1\|$ exists, and hence, $\{\kappa_n\}$ is bounded.

Step (iii). Prove that $\|\kappa_{n+1} - \kappa_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Because $\kappa_{n+1} \in Q_n$ and $\kappa_n = P_{Q_n}(\kappa_1)$, it follows from Lemma 5 that

$$\|\kappa_{n+1} - \kappa_n\|^2 \leq \|\kappa_{n+1} - \kappa_1\|^2 - \|\kappa_n - \kappa_1\|^2 \longrightarrow 0. \tag{45}$$

This implies that $\|\kappa_{n+1} - \kappa_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Step (iv). Prove that $\Theta = P_\Omega(\kappa_1)$. In the same manner as the proof of Step (iv) of Theorem 10, we can write

$$\|T_\varepsilon \nu_n - \nu_n\| \longrightarrow 0, \|\nu_n - \kappa_n\| \longrightarrow 0, \tag{46}$$

where $\varepsilon \in (0, 2\alpha)$. It follows from the nonexpansivity of T_ε that

$$\begin{aligned}\|T_\varepsilon\kappa_n - \kappa_n\| &= \|T_\varepsilon\kappa_n - T_\varepsilon\nu_n\| + \|T_\varepsilon\nu_n - \nu_n\| + \|\nu_n - \kappa_n\| \\ &\leq 2\|\nu_n - \kappa_n\| + \|T_\varepsilon\nu_n - \nu_n\|.\end{aligned} \tag{47}$$

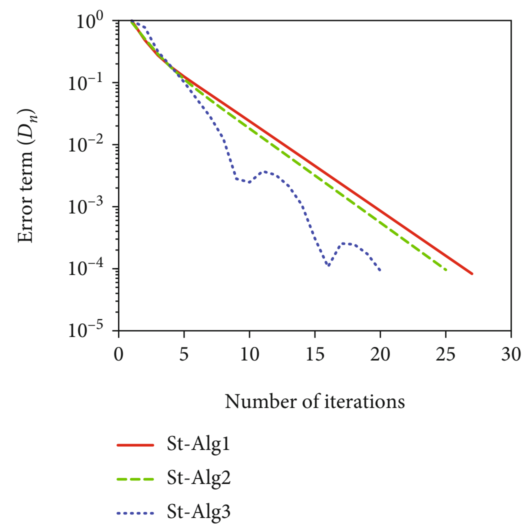

FIGURE 1: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = 2t + 3$.

From (46) and (47), we can get

$$\|T_\varepsilon\kappa_n - \kappa_n\| \longrightarrow 0. \tag{48}$$

By the boundedness of $\{\kappa_n\}$, there exists a subsequence $\{\kappa_{n_k}\}$ of $\{\kappa_n\}$ so that $\kappa_{n_k} \rightharpoonup \kappa^*$. This combines with (48), and from Lemma 6, we have $\kappa^* \in F(T_\varepsilon)$; this means that $\kappa^* \in \Omega$.

Since $\Theta = P_\Omega(\kappa_1)$ and $\kappa^* \in \Omega$, (43) and Lemma 3 (ii) imply that

$$\begin{aligned}\|\kappa_1 - \Theta\| &\leq \|\kappa_1 - \kappa^*\| \leq \liminf_{k\to\infty}\|\kappa_{n_k} - \kappa_1\| \\ &\leq \limsup_{k\to\infty}\|\kappa_{n_k} - \kappa_1\| \leq \|\kappa_1 - \Theta\|.\end{aligned} \tag{49}$$

Since the nearest point Θ is unique, then we have $\Theta = \kappa^*$. Also, we get $\|\kappa_{n_k} - \kappa_1\| \longrightarrow \|\kappa_1 - \Theta\|$. Applying Lemma 3 (i), we have $\kappa_{n_k} \longrightarrow \Theta$ as $k \longrightarrow \infty$. Again, the uniqueness of Θ leads to $\kappa_n \longrightarrow \Theta$ as $n \longrightarrow \infty$.

This finishes the requirement.

4. Application to Solve Split Convex Feasibility Problem

This part is devoted to applying our methods to find a solution to the SCFP. Assume that $T : Y_1 \longrightarrow Y_2$ is a bounded linear operator and T^* its adjoint defined on real HSs Y_1 and Y_2. Let $C \subset Y_1$ and $Q \subset Y_2$ be a NCC sets. Censor and Elfving [37] formulated the SCFP as follows:

$$\text{Find } \bar{\kappa} \in C \text{ so that } T(\bar{\kappa}) \in Q. \tag{50}$$

Censor and Elfving in [37] have introduced the SCFP in HSs while using a multidistance approach to find an adaptive approach for resolving it. Many of the problems that emerge from state retrieval and restoration of medical image can be formulated as split variational feasibility problems [38, 39].

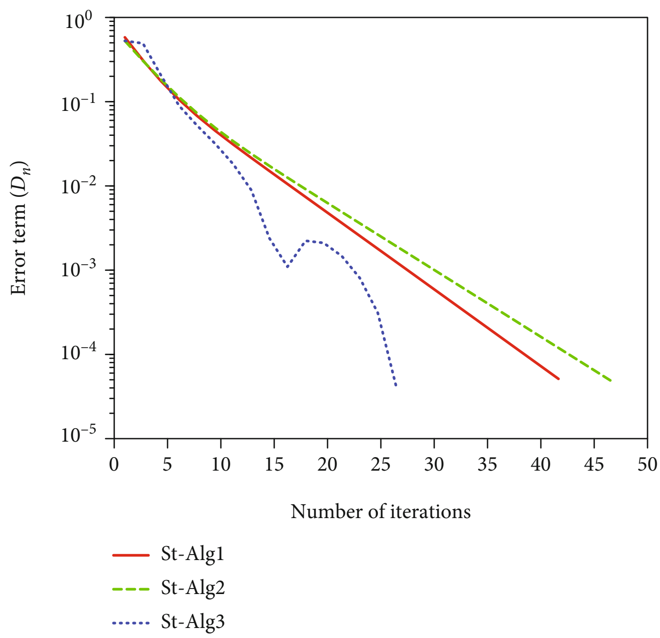

Figure 2: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = (5 + t^2)/5$.

This problem is also used in a variety of disciplines like image restoration, dynamic emission tomographic image reconstruction, and radiation therapy treatment planning [40–42]. Let consider

$$A(\kappa) = \nabla\left(\frac{1}{2}\left\|T\kappa - P_Q(T\kappa)\right\|^2\right) = T^*\left(I - P_Q\right)T\kappa, \tag{51}$$

where P_Q is the metric projection on to Q, ∇ is the gradient, and $B = \partial i_C$. Due to the above construction, the problem (50) has an inclusion format as described in (2). It can be seen that$A$ is Lipschitz which continues with constant $L = \|T\|^2$ and B is MM, see, for example, [43].

For any NCC subset C of a real HS Y, the indicator function i_C of C is defined by

$$i_C(\kappa) = \begin{cases} 0, & \text{if } \kappa \in C, \\ \infty, & \text{otherwise.} \end{cases} \tag{52}$$

Now, on the basis of the main results, we can deduce the following results for a SCFP.

Theorem 12. *Let $\{\kappa_n\}$ be a sequence iterated as follows: choose initial points $\kappa_0, \kappa_1 \in C_1 = Y$ and let $\{\alpha_n\}$, $\{\beta_n\}$ be bounded sequences and $\{\gamma_n\}$ be a sequence in $(0, 1]$. Consider that $\{\tau_n\}$ is a sequence of positive real numbers so that the following assumptions are fulfilled:*

(i) $\inf_n\{\gamma_n\} \geq \gamma > 0$

(ii) $0 < \inf_n\{\tau_n\} \leq \sup_n\{\tau_n\} < 2\alpha$

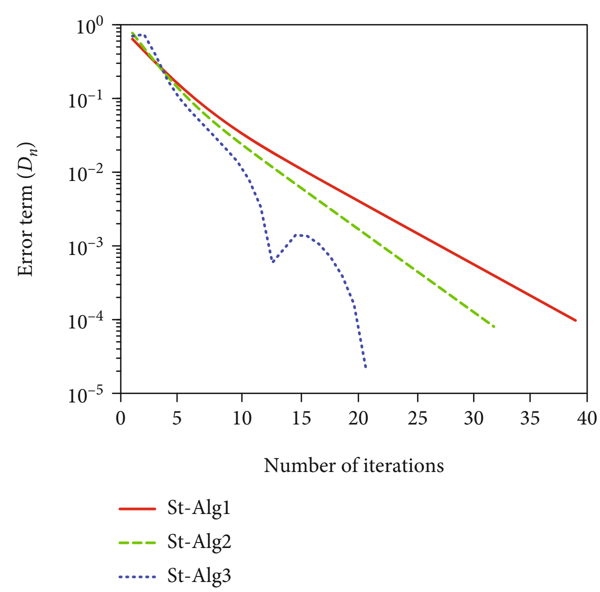

Figure 3: Numerical computational behavior of Algorithm 1:, Algorithm 2:, Algorithm (4.2) while $\kappa_0 = \kappa_1 = 3e^{2t}t^4$.

Step (1). Compute

$$\begin{aligned} \omega_n &= \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}), \\ \upsilon_n &= \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}). \end{aligned} \tag{53}$$

Step (2). Compute

$$\mu_n = (1 - \gamma_n)\omega_n + \gamma_n P_C\left[\upsilon_n - \tau_n T^*\left(I - P_Q\right)T\upsilon_n\right]. \tag{54}$$

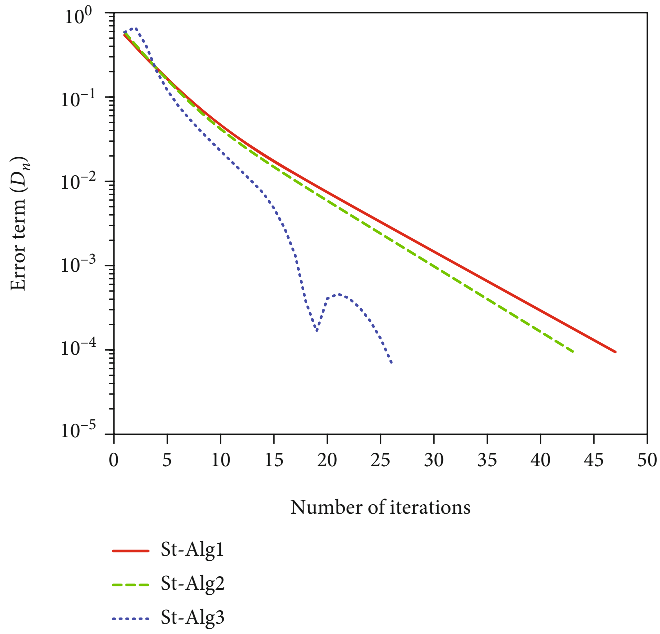

FIGURE 4: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = 3e^t \sin(t)$.

Step (3). Compute

$$C_{n+1} = \{p \in C_n : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n) \cdot \langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + (\alpha_n^2(1-\gamma_n) + \gamma_n\beta_n^2)\|\kappa_{n-1} - \kappa_n\|^2\},$$
$$\kappa_{n+1} = P_{C_{n+1}}(\kappa_1), \quad n \ge 1. \tag{55}$$

If the solution set Γ_{SCFP} is nonempty, then the sequence $\{\kappa_n\}$ converges weakly to an element of solution set Γ_{SCFP}.

Theorem 13. *Let $\{\kappa_n\}$ be a sequence iterated as follows: choose initial points $\kappa_0, \kappa_1 \in C_1 = Y$ and let $\{\alpha_n\}$, $\{\beta_n\}$ be bounded sequences and $\{\gamma_n\}$ be a sequence in $(0,1]$. Consider that $\{\tau_n\}$ is a sequence of positive real numbers so that the following assumptions are fulfilled:*

(i) $\inf_n\{\gamma_n\} \ge \gamma > 0$

(ii) $0 < \inf_n\{\tau_n\} \le \sup_n\{\tau_n\} < 2\alpha$

Step (1). Compute

$$\omega_n = \kappa_n + \alpha_n(\kappa_n - \kappa_{n-1}),$$
$$\upsilon_n = \kappa_n + \beta_n(\kappa_n - \kappa_{n-1}). \tag{56}$$

Step (2). Compute

$$\mu_n = (1-\gamma_n)\omega_n + \gamma_n P_C[\upsilon_n - \tau_n T^*(I - P_Q)T\upsilon_n]. \tag{57}$$

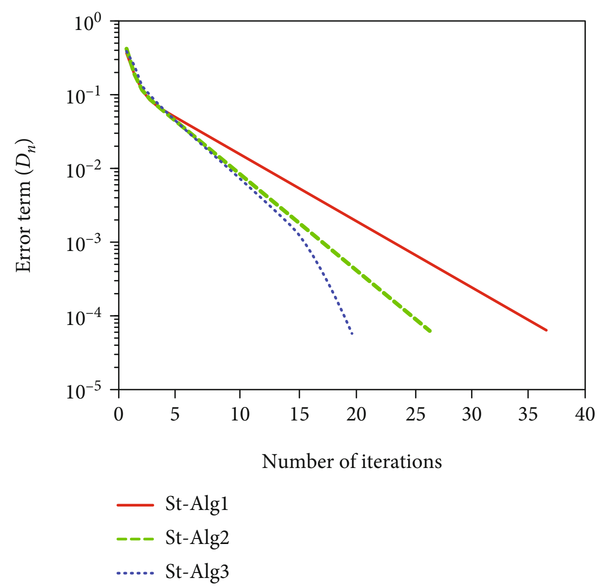

FIGURE 5: Numerical computational behavior of Algorithm 1:, Algorithm 2:, and Algorithm (4.2) while $\kappa_0 = \kappa_1 = (2t^3 - 3e^t)\cos(t)$.

Step (3). Compute

$$C_n = \{p \in Y : \|\mu_n - p\|^2 \le \|\kappa_n - p\|^2 - 2(\alpha_n(1-\gamma_n) + \gamma_n\beta_n) \cdot \langle \kappa_n - p, \kappa_{n-1} - \kappa_n\rangle + (\alpha_n^2(1-\gamma_n) + \gamma_n\beta_n^2)\|\kappa_{n-1} - \kappa_n\|^2\},$$
$$Q_n = \{p \in Y : \langle p - \kappa_n, \kappa_1 - \kappa_n\rangle \le 0\}, \kappa_{n+1} = P_{C_n \cap Q_n}(\kappa_1), \quad n \ge 1. \tag{58}$$

If the solution set Γ_{SCFP} is nonempty, then the sequence $\{\kappa_n\}$ converges weakly to an element of solution set Γ_{SCFP}.

Table 1: Numerical computational results of Algorithm 1:, Algorithm 2:, and Algorithm (4.2).

	Number of iterations			Execution time in seconds		
Initial point ($\kappa_0 = \kappa_1$)	St-Alg1	St-Alg2	St-Alg3	St-Alg1	St-Alg2	St-Alg3
$2t + 3$	27	25	20	0.0497	0.0184	0.0184
$5 + t^2/5$	25	28	16	0.0497	0.0184	0.0184
$3e^{2t}t^4$	39	32	21	0.0497	0.0184	0.0184
$3e^t \sin(t)$	47	43	26	0.0497	0.0184	0.0184
$(2t^3 - 3e^t)\cos(t)$	55	40	30	0.0497	0.0184	0.0184

5. Supportive Numerical Examples

This section is the mainstay of the paper as it studies the behavior and performance of our algorithms numerically and graphically. The program used here is MATLAB R2014a running on an HP Compaq 510, Core™ 2 Duo CPU T5870 with 2.0 GHz and 2 GB RAM.

Example 14. Let $Y_1 = Y_2 = L_2([0, 2\pi])$ be two HSs with an inner product

$$\langle x, y\rangle := \int_0^{2\pi} x(t)y(t)dt, \quad \forall x, y \in L_2([0, 2\pi]), \tag{59}$$

and the induced norm defined by

$$\|x\| := \sqrt{\int_0^{2\pi} |x(t)|^2 dt}, \quad \forall x \in L_2([0, 2\pi]). \tag{60}$$

Next, consider the feasible set $C \subset Y_1$ as

$$C = \left\{x \in Y_1 : \int_0^{2\pi} x(t)dt \le 1\right\}, \tag{61}$$

and $Q \subset Y_2$ is

$$Q = \left\{x \in Y_2 : \int_0^{2\pi} |x(t) - \sin(t)|^2 dt \le 16\right\}. \tag{62}$$

Consider the mapping $T : Y_1 \longrightarrow Y_2$ so that $(Tx)(s) = x(s), x \in Y_1$. Then, $(T^* x)(s) = x(s)$ and $\|T\| = 1$. So, we wish to solve the following problem:

$$\text{Find } \bar{\kappa} \in C \text{ so that } T(\bar{\kappa}) \in Q. \tag{63}$$

We can also observe that since $(Tx)(s) = x(s), x \in Y_1$, the above problem is actually a SCFP in the form of

$$\text{Find } \bar{\kappa} \in C \cap Q. \tag{64}$$

Figures 1–5 and Table 1 show the numerical computational results of Algorithm (4.2) by Dong et al. in [30] (St-Alg1), Algorithm 1: (St-Alg2), and Algorithm 2: (St-Alg3) by assuming $D_n = \|\kappa_n - \kappa_{n-1}\| \le 10^{-4}$.

Remark 15. It is important to note that the different choices of initial points have substantial effect on the CPU (time) and a number of iterations on the proposed algorithms and also for those existing algorithms that are used for comparison. These facts can be seen from Figures 1–5 and Table 1. We conclude from these numerical results that our algorithms are faster in convergence than their counterpart presented by Dong et al. in [30].

6. Conclusion

The quality of the algorithm is measured by two main factors: velocity in convergence and time. When the convergence is faster in a short time, the results are faster and more accurate. Given the importance of algorithms in many applications in real society, many researchers have studied this logic and try to obtain a strong convergence, which has a prominent role in studying the efficiency and effectiveness of these algorithms. On the basis of this principle, in this paper was studied the effect of shrinking projection and CQ term on two inertial terms to get the strong convergence of new algorithms called two inertial shrinking projection and CQ algorithms. These results have been implicated to obtain a solution to SCFP in HSs. Finally, some numerical results were formulated to illustrate the efficiency and effectiveness of algorithms.

Abbreviations

HSs: Hilbert spaces
NCC: Nonempty closed convex
NEMs: Nonexpansive mappings
FPs: Fixed points
SCFPs: Split convex feasibility problems
ISM: Inverse strongly monotone
MM: Maximal monotone.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

On Fixed Point Results in Partial b-Metric Spaces

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Partial b-metric spaces are characterised by a modified triangular inequality and that the self-distance of any point of space may not be zero and the symmetry is preserved. The spaces with a symmetric property have interesting topological properties. This manuscript paper deals with the existence and uniqueness of fixed point points for contraction mappings using triangular weak α-admissibility with regard to η and C-class functions in the class of partial b-metric spaces. We also introduce an example to demonstrate the obtained results.

1. Introduction and Preliminaries

The contribution of fixed point theorems has been appeared as a really effective and successful strategy for understanding many mathematical issues, which rise from actual applications. Fixed point theory is an essential tool to solve several equations in diverse fields, such as integral and differential (fractional) equations appeared in economics, game theory, engineering, physics, and chemistry. For instance, see [1–8]. As the significance and accessibility of Banach contraction mapping (BCM), various authors have amplified and generalized this contraction and set up various interesting supplements and alterations. In this continuation, Czerwik [9] and Bakhtin [10] originated the idea of a b-metric space as a suitable generalization of metric spaces. They demonstrated the truth of new weak contraction mapping standards in b-metric spaces that generalize the celebrated Banach contraction. Subsequently, several authors have dealt with related fixed point results in this setting (see [11–17]). Furthermore, Matthews [18] presented the idea of a partial metric space by considering the distance between any point to itself that requires not be equal to zero. For more related works in this setting, see [19–21]. Recently, Shukla [22] studied the connection, as topological properties, between a partial metric and a b-metric space to introduce an unification of the concept of a partial b-metric space.

Definition 1 ([22]). Given a nonempty set W. A function $\rho_{pb} : W \times W \longrightarrow [0,\infty)$ is denominated as a b-partial metric if for all $\upsilon, \kappa, z \in W$

$(\rho_{pb1})\, \upsilon = \kappa$ if and only if $\rho_{pb}(\upsilon, \upsilon) = \rho_{pb}(\upsilon, \kappa) = \rho_{pb} b(\kappa, \kappa)$,

$$\begin{aligned} &(\rho_{pb2})\, \rho_{pb}(\upsilon, \upsilon) \leq \rho_{pb}(\upsilon, \kappa), \\ &(\rho_{pb3})\, \rho_{pb}(\upsilon, \kappa) = \rho_{pb}(\kappa, \upsilon), \end{aligned} \tag{1}$$

(ρ_{pb4}) there is a real number $s \geq 1$ such that $\rho_{pb}(\upsilon, \kappa)$
$\leq s\left[\rho_{pb}(\upsilon, z) + \rho_{pb}(z, \kappa)\right] - \rho_{pb}(z, z)$.

If $\rho_{pb}(\upsilon, \kappa) = 0$, then $\upsilon = \kappa$. The converse does not hold in general. Now, consider $W = [0,\infty)$ and $p > 1$. Take $\rho_{pb} : W \times W \longrightarrow \mathbb{R}^+$ as

$$\rho_{pb}(\upsilon, \kappa) = [\max \{\upsilon, \kappa\}]^p - |\upsilon - \kappa|^p, \tag{2}$$

for all $\upsilon, \kappa \in W$. Then, (X, ρ_{pb}) is a partial b-metric space (Here, $s = 2p > 1$). It is neither a partial metric, nor a b-metric space. Let ρ_p be a partial metric and ρ_b be a $b-$ metric (with $s > 1$) on a nonempty set W. Then, $\rho_{pb} : W \times W \longrightarrow [0,\infty)$ defined by $\rho_{pb}(\upsilon, \kappa) = \rho_p(\upsilon, \kappa) + \rho_b(\upsilon, \kappa)$ for all $\upsilon, \kappa \in W$, is a partial b-metric on W.

Definition 2 ([22, 23]). Let (W, ρ_{pb}) be a partial b-metric space with $s \geq 1$. Let $\{\upsilon_n\}$ be a sequence in W and $\upsilon \in W$. Then,

(i) $\{\upsilon_n\} \subseteq W$ converges to $\upsilon \in W$ if $\lim_{n\longrightarrow\infty}\rho_{pb}(\upsilon_n, \upsilon) = \rho_{pb}(\upsilon, \upsilon)$

(ii) $\{\upsilon_n\} \subseteq W$ is Cauchy in (W, ρ_{pb}) if $\lim_{n,m\longrightarrow\infty}\rho_{pb}(\upsilon_n, \upsilon_m)$ exists and is finite

(iii) (W, ρ_{pb}) is complete if for every Cauchy sequence $\{\upsilon_n\} \subseteq W$, there is $\upsilon \in W$ so that

$$\lim_{n,m\longrightarrow\infty}\rho_{pb}(\upsilon_n, \upsilon_m) = \lim_{n\longrightarrow\infty}\rho_{pb}(\upsilon_n, \upsilon) = \rho_{pb}(\upsilon, \upsilon). \tag{3}$$

Given a nonempty set W. Given $f : W \longrightarrow W$ and $\alpha : W \times W \longrightarrow [0,\infty)$. Then, f is called α-admissible [24] if for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \geq 1$, we have $\alpha(f\upsilon, f\kappa) \geq 1$. The conception of triangular α-admissibility was initiated in [25]. This concept has been weakened as follows.

Definition 3 ([26]). Given a nonempty set W. Given α, $\eta : W \times W \rightarrow [0,\infty)$. $S : W \longrightarrow W$ is triangular weak α-admissible with regard to η if

(1) $\alpha(\upsilon, S^n\upsilon) \geq \eta(\upsilon, S^n\upsilon)$, implies that $\alpha(S^n\upsilon, S^{n+1}\upsilon) \geq \eta(S^n\upsilon, S^{n+1}\upsilon)$, for all $n \in \mathbb{N}$

(2) $\alpha(\upsilon, z) \geq \eta(\upsilon, z)$ and $\alpha(z, \kappa) \geq \eta(z, \kappa)$, imply that $\alpha(\upsilon, \kappa) \geq \eta(\upsilon, \kappa)$.

Let $W = [0,\infty)$. Given $S : W \longrightarrow W$ as $S\upsilon = \upsilon^2$. Take α, $\eta : W \times W \rightarrow [0,\infty)$ as $\alpha(\upsilon, \kappa) = e^{\upsilon+\kappa}$ and $\eta(\upsilon, \kappa) = e^{\kappa-\upsilon}$. Here, S is triangular weak α-admissible with regard to η.

Lemma 4 ([26]). *Given $\alpha, \eta : W \times W \longrightarrow \mathbb{R}$. Let $S : W \longrightarrow W$ be triangular weak α-admissible with regard to η. Assume that there is $\upsilon_0 \in W$ such that $\alpha(\upsilon_0, S\upsilon_0) \geq \eta(\upsilon_0, S\upsilon_0)$. Define $\{\upsilon_n\}$ in W by $S\upsilon_n = \upsilon_{n+1}$. Then, for all integers $n < m$*

$$\alpha(\upsilon_n, \upsilon_m) \geq \eta(\upsilon_n, \upsilon_m). \tag{4}$$

Let (W, σ) be a metric space. Given $\alpha, \eta : W \times W \longrightarrow [0,\infty)$. Following [27], the mapping $T : W \longrightarrow W$ is said to be α-η-continuous if every sequence $\{\upsilon_n\}$ in W with $\alpha(\upsilon_n, \upsilon_{n+1}) \geq \eta(\upsilon_n, \upsilon_{n+1})$ for all $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty}\sigma(\upsilon_n, \upsilon) = 0$, we have $\lim_{n\longrightarrow\infty}\sigma(T\upsilon_n, T\upsilon) = 0$. This concept could be extended to b-partial metric spaces as follows.

Definition 5. Let (W, ρ_{pb}) be a b-partial metric space. Given $\alpha, \eta : W \times W \longrightarrow [0,\infty)$. The mapping $T : W \longrightarrow W$ is said to be α-η-continuous if every sequence $\{\upsilon_n\}$ in W with $\alpha(\upsilon_n, \upsilon_{n+1}) \geq \eta(\upsilon_n, \upsilon_{n+1})$ for all $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty}\rho_{pb}(\upsilon_n, \upsilon) = \rho_{pb}(\upsilon, \upsilon)$, we have $\lim_{n\longrightarrow\infty}d(T\upsilon_n, T\upsilon) = \rho_{pb}(T\upsilon, T\upsilon)$.

The conception of C-class functions is posed as follows.

Definition 6 ([28]). We say that $\mathcal{F} : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is a C-class function if it is continuous and for $\upsilon, \kappa \in [0,\infty)$, we have

(i) $\mathcal{F}(\upsilon, \kappa) \leq \upsilon$

(ii) $\mathcal{F}(\upsilon, \kappa) = \upsilon$ yields that either $\upsilon = 0$, or $\kappa = 0$

Denote by $\mathcal{C}$ the set of C-class functions. The following functions are elements in $\mathcal{C}$ (for all $\upsilon, \kappa \in [0,\infty)$):

(1) $\mathcal{F}(\upsilon, \kappa) = \upsilon - \kappa$

(2) $\mathcal{F}(\upsilon, \kappa) = \mu\upsilon$ (with $0 < \mu < 1$)

(3) $\mathcal{F}(\upsilon, \kappa) = \Theta(\upsilon)$ (with $\Theta : [0,\infty) \longrightarrow [0, 1)$ is continuous)

Denote by Ψ the set of functions [29] $\psi : [0,\infty) \longrightarrow [0,\infty)$ verifying that:

(i) ψ is monotone and nondecreasing

(ii) $\psi(\upsilon) = 0 \Longleftrightarrow \upsilon = 0$

Let Φ be the set of functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ so that

(i) φ is continuous

(ii) $\varphi(\upsilon) = 0 \Longleftrightarrow \upsilon = 0$

In this paper, we initiate the concept of generalized $\alpha - \eta - \psi - \varphi - F$-contraction self-mappings via C-class functions and the connotation of triangular weak α-admissibility with regard to η. The goal is to prove some related fixed point results in the context of partial b-metric spaces.

2. Main Results

We start with the following.

Definition 7. Let (W, ρ_{pb}) be a complete b-partial metric space with coefficient $s \geq 1$ and $S : W \longrightarrow W$. If there are $\alpha, \eta : W \times W \longrightarrow [0,\infty)$, $F \in \mathscr{C}$, $\psi \in \Psi$, and $\varphi \in \Phi$, so that for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \geq \eta(\upsilon, \kappa)$,

$$\psi\left(s\rho_{pb}(S\upsilon, S\kappa)\right) \leq \begin{cases} F(\psi(\mathscr{P}(\upsilon, \kappa)), \varphi(\mathscr{P}(\upsilon, \kappa))) & \text{if } \rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) \neq 0, \\ 0 & \text{otherwise } \rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) = 0, \end{cases} \tag{5}$$

where either

$$\mathscr{P}(\upsilon, \kappa) = \max\left\{\rho_{pb}(\upsilon, \kappa), \rho_{pb}(\upsilon, S\upsilon), \rho_{pb}(\kappa, S\kappa), \frac{\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(\kappa, S\upsilon)}{2s}\right\}, \tag{6}$$

or

$$\mathscr{P}(\upsilon, \kappa) = \left[\frac{\rho_{pb}(\upsilon, S\upsilon) + \rho_{pb}(\kappa, S\kappa)}{2}\right], \tag{7}$$

then S is called a generalized $\alpha - \eta - \psi - \varphi - F$-contraction.

Theorem 8. *Let (W, ρ_{pb}) be a complete b-partial metric space (with $s \geq 1$) and $S : W \longrightarrow W$. Assume that*

(a) S is a generalized $\alpha - \eta - \psi - \varphi - F$-contraction

(b) S is triangular weak α-admissible

(c) There is $\upsilon_0 \in W$ such that $\alpha(\upsilon_0, S\upsilon_0) \geq \eta(\upsilon_0, S\upsilon_0)$

(d) S is $\alpha - \eta -$ continuous

Then, S possesses a unique fixed point.

Proof. Let $\upsilon_0 \in W$ be in order that $\alpha(\upsilon_0, S\upsilon_0) \geq \eta(\upsilon_0, S\upsilon_0)$. Realize a sequence $\{\upsilon_n\}$ in W in order that $\upsilon_{n+1} = S\upsilon_n$ for all $n \in \mathbb{N}$. If for some n, $\upsilon_n = \upsilon_{n+1}$, the υ_n is a fixed point. Now, suppose that $\upsilon_n \neq \upsilon_{n+1}$ for all $n \in \mathbb{N}$. By using assumption (b), the triangular weak α-admissibility and Lemma 4, we get that for all integers $m > n$,

$$\alpha(\upsilon_n, \upsilon_m) \geq \eta(\upsilon_n, \upsilon_m). \tag{8}$$

In particular, $\alpha(\upsilon_n, \upsilon_{n+1}) \geq \eta(\upsilon_n, \upsilon_{n+1})$ for all $n \in \mathbb{N}$.

Case 1. Here, we consider $\mathscr{P}(\upsilon, \kappa) = \max\{\rho_{pb}(\upsilon, \kappa), \rho_{pb}(\upsilon, S\upsilon), \rho_{pb}(\kappa, S\kappa), ((\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(\kappa, S\upsilon))/2s)\}$. We need three steps.

Step 1. In view of assumption (a), one writes

$$\begin{aligned}\psi\left(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)\right) &= \psi\left(\rho_{pb}(S\upsilon_n, S\upsilon_{n-1})\right) \leq \psi\left(s\rho_{pb}(S\upsilon_n, S\upsilon_{n-1})\right) \\ &\leq F(\psi(\mathscr{P}(\upsilon_n, \upsilon_{n-1})), \varphi(\mathscr{P}(\upsilon_n, \upsilon_{n-1}))) \\ &\leq \psi(\mathscr{P}(\upsilon_n, \upsilon_{n-1})),\end{aligned} \tag{9}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}\mathscr{P}(\upsilon_n, \upsilon_{n-1}) &= \max\left\{\rho_{pb}(\upsilon_n, \upsilon_{n-1}), \rho_{pb}(\upsilon_n, S\upsilon_n), \rho_{pb}(\upsilon_{n-1}, S\upsilon_{n-1}), \frac{\rho_{pb}(\upsilon_n, S\upsilon_{n-1}) + \rho_{pb}(\upsilon_{n-1}, S\upsilon_n)}{2s}\right\} \\ &= \max\left\{\rho_{pb}(\upsilon_n, \upsilon_{n-1}), \rho_{pb}(\upsilon_n, \upsilon_{n+1}), \rho_{pb}(\upsilon_{n-1}, \upsilon_n), \frac{\rho_{pb}(\upsilon_n, \upsilon_n) + \rho_{pb}(\upsilon_{n-1}, \upsilon_{n+1})}{2s}\right\} \\ &= \max\left\{\rho_{pb}(\upsilon_n, \upsilon_{n-1}), \rho_{pb}(\upsilon_n, \upsilon_{n+1})\right\}.\end{aligned} \tag{10}$$

If $\mathscr{P}(\upsilon_n, \upsilon_{n-1}) = \rho_{pb}(\upsilon_n, \upsilon_{n+1})$ for some n, then we get

$$\begin{aligned}\psi\left(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)\right) &\leq F(\psi(\mathscr{P}(\upsilon_n, \upsilon_{n-1})), \varphi(\mathscr{P}(\upsilon_n, \upsilon_{n-1}))) \\ &= F\left(\psi\left(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)\right), \varphi\left(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)\right)\right) \\ &\leq \psi\left(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)\right).\end{aligned} \tag{11}$$

This implicates that either $\psi(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)) = 0$ or $\varphi(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)) = 0$, so $\rho_{pb}(\upsilon_{n+1}, \upsilon_n) = 0$; hence, $\upsilon_{n+1} = \upsilon_n$. It is a contradiction. Thus. $\mathscr{P}(\upsilon_n, \upsilon_{n-1}) = \rho_{pb}(\upsilon_n, \upsilon_{n-1})$ for all $n \geq 1$. By (5), we have

$$\psi\left(\rho_{pb}(\upsilon_{n+1}, \upsilon_n)\right) \leq \psi\left(\rho_{pb}(\upsilon_n, \upsilon_{n-1})\right). \tag{12}$$

Since ψ is nondecreasing, we get $\rho_{pb}(\upsilon_{n+1}, \upsilon_n) \leq \rho_{pb}(\upsilon_n, \upsilon_{n-1})$ for all $n \geq 1$. Thus, $\{\rho_{pb}(\upsilon_n, \upsilon_{n+1})\}$ is nonincreasing. Thus, there is $\omega \geq 0$ in order that $\lim_{n\longrightarrow\infty}\rho_{pb}(\upsilon_n, \upsilon_{n+1}) = \omega$.

We claim that $\omega = 0$. We have

$$\begin{aligned}\psi\left(\rho_{pb}(\upsilon_{n+1}, \upsilon_{n+2})\right) &\leq \psi\left(s\rho_{pb}(S\upsilon_n, S\upsilon_{n+1})\right)\\ &\leq F\left(\psi\left(\rho_{pb}(\upsilon_n, \upsilon_{n+1})\right), \varphi\left(\rho_{pb}(\upsilon_n, \upsilon_{n+1})\right)\right)\\ &\leq \psi\left(\rho_{pb}(\upsilon_n, \upsilon_{n+1})\right).\end{aligned} \tag{13}$$

At the limit, we acquire that

$$\psi(\omega) \leq F(\psi(\omega), \varphi(\omega)) \leq \psi(\omega). \tag{14}$$

This infers that either $\psi(\omega) = 0$, or $\varphi(\omega) = 0$. That is, $\omega = 0$, i.e.,

$$\lim_{n\longrightarrow\infty} \rho_{pb}(\upsilon_n, \upsilon_{n+1}) = 0. \tag{15}$$

To prove that $\{\upsilon_n\}$ is a Cauchy sequence in the b-partial metric space (W, ρ_{pb}), we argue by contradiction. For this, suppose that there exist $\varepsilon > 0$ and subsequences $\{m(l)\}_{l=1}^{\infty}$ and $\{n(l)\}_{l=1}^{\infty}$ of positive integers with $n(l) > m(l) > l$ such that

$$\rho_{pb}\left(\upsilon_{n(l)}, \upsilon_{m(l)}\right) \geq \varepsilon \text{ and } \rho_{pb}\left(\upsilon_{m(l)}, \upsilon_{n(l)-1}\right) < \varepsilon. \tag{16}$$

Using (ρ_{pb4}) and applying (15) and (16), one easily gets

$$\frac{\varepsilon}{s^2} \leq \limsup_{l\longrightarrow\infty} \rho_{pb}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right) \leq \varepsilon s, \tag{17}$$

$$\frac{\varepsilon}{s} \leq \limsup_{l\longrightarrow\infty} \rho_{pb}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)}\right) \leq \varepsilon, \tag{18}$$

$$\frac{\varepsilon}{s} \leq \limsup_{l\longrightarrow\infty} \rho_{pb}\left(\upsilon_{n(l)}, \upsilon_{m(l)-1}\right) \leq \varepsilon s^2. \tag{19}$$

In view of assumption (a), we have

$$\begin{aligned}\psi\left(s\rho_{pb}\left(\upsilon_{n(l)}, \upsilon_{m(l)}\right)\right) &= \psi\left(s\rho_{pb}\left(S\upsilon_{n(l)-1}, S\upsilon_{m(l)-1}\right)\right)\\ &\leq F\left(\psi\left(\mathscr{P}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right)\right), \varphi\left(\mathscr{P}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right)\right)\right)\\ &\leq \psi\left(\mathscr{P}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right)\right),\end{aligned} \tag{20}$$

where

$$\begin{aligned}\mathscr{P}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right) &= \max\left\{\rho_{pb}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right), \rho_{pb}\left(\upsilon_{n(l)-1}, S\upsilon_{n(l)-1}\right),\right.\\ &\quad\left.\rho_{pb}\left(\upsilon_{m(l)-1}, S\upsilon_{m(l)-1}\right), \frac{\rho_{pb}\left(\upsilon_{n(l)-1}, S\upsilon_{m(l)-1}\right) + \rho_{pb}\left(\upsilon_{m(l)-1}, S\upsilon_{n(l)-1}\right)}{2s}\right\}\\ &= \max\left\{\rho_{pb}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right), \rho_{pb}\left(\upsilon_{n(l)-1}, \upsilon_{n(l)}\right), \rho_{pb}\left(\upsilon_{m(l)-1}, \upsilon_{m(l)}\right),\right.\\ &\quad\left.\frac{\rho_{pb}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)}\right) + \rho_{pb}\left(\upsilon_{m(l)-1}, \upsilon_{n(l)}\right)}{2s}\right\}.\end{aligned} \tag{21}$$

By (15)–(19), one gets

$$\limsup_{l\longrightarrow\infty} \mathscr{P}\left(\upsilon_{n(l)-1}, \upsilon_{m(l)-1}\right) \leq \varepsilon s. \tag{22}$$

Letting $l \longrightarrow \infty$ and having in mind properties of ψ, φ and F, we deduce that

$$\psi(s\varepsilon) \leq F(\psi(s\varepsilon), \varphi(s\varepsilon)) \leq \psi(s\varepsilon). \tag{23}$$

That is,

$$F(\psi(s\varepsilon), \varphi(s\varepsilon)) = \psi(s\varepsilon), \tag{24}$$

so either $\psi(s\varepsilon) = 0$ or $\varphi(s\varepsilon) = 0$. Thus, $\varepsilon = 0$, which is, a contradiction. Hence, $\{\upsilon_n\}$ is a Cauchy sequence.

Step 2. We claim that S has a fixed point. The sequence $\{\upsilon_n\}$ is Cauchy in the complete b-partial metric space, so there is $\upsilon^* \in W$ in order that

$$\lim_{n,m\longrightarrow\infty} \rho_{pb}(\upsilon_n, \upsilon^*) = \lim_{n,m\longrightarrow\infty} \rho_{pb}(\upsilon_n, \upsilon_m) = \rho_{pb}(\upsilon^*, \upsilon^*) = 0. \tag{25}$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\rho_{pb}(\upsilon^*, S\upsilon^*) &\leq s\left[\rho_{pb}(\upsilon^*, \upsilon_{n+1}) + \rho_{pb}(\upsilon_{n+1}, S\upsilon^*)\right] - \rho_{pb}(\upsilon_{n+1}, \upsilon_{n+1})]\\ &\leq s\left[\rho_{pb}(\upsilon^*, \upsilon_{n+1}) + \rho_{pb}(S\upsilon_n, S\upsilon^*)\right] \leq s\rho_{pb}(\upsilon^*, \upsilon_{n+1})\\ &\quad + s\rho_{pb}(S\upsilon_n, S\upsilon^*).\end{aligned} \tag{26}$$

By assumption (d) and Definition 5, $\rho_{pb}(S\upsilon_n, S\upsilon^*) \longrightarrow \rho_{pb}(S\upsilon^*, S\upsilon^*)$ as $n \longrightarrow \infty$. Using (25), we deduce that $\rho_{pb}(\upsilon^*, S\upsilon^*) \leq s\rho_{pb}(S\upsilon^*, S\upsilon^*)$. We will show that $\upsilon^* = S\upsilon^*$. Suppose that $\upsilon^* \neq S\upsilon^*$, then we obtain that $\rho_{pb}(\upsilon^*, S\upsilon^*) \neq 0$. This implies that

$$\begin{aligned}\psi\left(\rho_{pb}(S\upsilon^*, S\upsilon^*)\right) &\leq \psi\left(s\rho_{pb}(S\upsilon^*, S\upsilon^*)\right)\\ &\leq F(\psi(\mathscr{P}(\upsilon^*, \upsilon^*)), \varphi(\mathscr{P}(\upsilon^*, \upsilon^*)))\\ &\leq \psi(\mathscr{P}(\upsilon^*, \upsilon^*)),\end{aligned} \tag{27}$$

where

$$\begin{aligned}\mathscr{P}(\upsilon^*, \upsilon^*) &= \max\left\{\rho_{pb}(\upsilon^*, \upsilon^*), \rho_{pb}(\upsilon^*, S\upsilon^*), \rho_{pb}(\upsilon^*, S\upsilon^*),\right.\\ &\quad\left.\frac{\rho_{pb}(\upsilon^*, S\upsilon^*) + \rho_{pb}(\upsilon^*, S\upsilon^*)}{2s}\right\} = \rho_{pb}(\upsilon^*, S\upsilon^*).\end{aligned} \tag{28}$$

So,

$$\psi\left(\rho_{pb}(Sv^*, Sv^*)\right) \le \psi\left(\rho_{pb}(v^*, Sv^*)\right) \le \psi\left(\rho_{pb}(Sv^*, Sv^*)\right). \tag{29}$$

Hence, $\psi(\rho_{pb}(Sv^*, Sv^*)) = \psi(\rho_{pb}(Sv^*, v^*))$. We can write

$$\begin{aligned}\psi\left(\rho_{pb}(Sv^*, v^*)\right) &\le F(\psi\left(\rho_{pb}(Sv^*, v^*)\right), \varphi\left(\rho_{pb}(Sv^*, v^*)\right) \\ &\le \psi\left(\rho_{pb}(Sv^*, v^*)\right).\end{aligned} \tag{30}$$

Then,

$$F(\psi\left(\rho_{pb}(Sv^*, v^*)\right), \varphi\left(\rho_{pb}(Sv^*, v^*)\right) = \psi\left(\rho_{pb}(Sv^*, v^*)\right). \tag{31}$$

Thus, $\psi(\rho_{pb}(Sv^*, v^*)) = 0$ or $\varphi(\rho_{pb}(Sv^*, v^*)) = 0$. Hence, $v^* = Sv^*$. It is a contradiction. Consequently, S has a fixed point.

Step 3. We demonstrate that the fixed point of S is unique.

Given $u \neq \hat{u} \in W$ so that $Su = u$ and $S\hat{u} = \hat{u}$. From (5),

$$\begin{aligned}&\psi\left(\rho_{pb}(u, \hat{u})\right) \le \psi\left(s\rho_{pb}(u, \hat{u})\right) = \psi\left(s\rho_{pb}(Su, S\hat{u})\right) \\ &\le F(\psi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, Su), \rho_{pb}(\hat{u}, S\hat{u}), \frac{\rho_{pb}(u, S\hat{u}) + \rho_{pb}(\hat{u}, Su)}{2s}\right\}\right), \\ &\varphi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, Su), \rho_{pb}(\hat{u}, S\hat{u}), \frac{\rho_{pb}(u, S\hat{u}) + \rho_{pb}(\hat{u}, Su)}{2s}\right\}\right)\Big) \\ &= F(\psi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, u), \rho_{pb}(\hat{u}, \hat{u}), \frac{\rho_{pb}(u, \hat{u}) + \rho_{pb}(\hat{u}, u)}{2s}\right\}\right), \\ &\varphi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, u), \rho_{pb}(\hat{u}, \hat{u}), \frac{\rho_{pb}(u, \hat{u}) + \rho_{pb}(\hat{u}, u)}{2s}\right\}\right)\Big) \\ &= F\left(\psi\left(\rho_{pb}(u, \hat{u})\right), \varphi\left(\rho_{pb}(u, \hat{u})\right)\right) \le \psi\left(\rho_{pb}(u, \hat{u})\right).\end{aligned} \tag{32}$$

This infers that either $\psi(\rho_{pb}(u, \hat{u})) = 0$ or $\varphi(\rho_{pb}(u, \hat{u})) = 0$. This implies that $\rho_{pb}(u, \hat{u}) = 0$, so $u = \hat{u}$.

Case 1. We consider $\mathscr{P}(v, \kappa) = [\rho_{pb}(v, Sv) + \rho_{pb}(\kappa, S\kappa)/2]$.

Again, in view of assumption (a), one writes

$$\begin{aligned}\psi\left(\rho_{pb}(v_{n+1}, v_n)\right) &\le \psi\left(s\rho_{pb}(v_{n+1}, v_n)\right) = \psi\left(s\rho_{pb}(Sv_n, Sv_{n-1})\right) \\ &\le F(\psi(\mathscr{P}(v_n, v_{n-1})), \varphi(\mathscr{P}(v_n, v_{n-1}))) \le \psi(\mathscr{P}(v_n, v_{n-1})),\end{aligned} \tag{33}$$

for all $n \geq 1$, where

$$\begin{aligned}\mathscr{P}(v_n, v_{n-1}) &= \left[\frac{\rho_{pb}(v_n, Sv_n) + \rho_{pb}(v_{n-1}, Sv_{n-1})}{2}\right] \\ &= \left[\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}\right].\end{aligned} \tag{34}$$

Then,

$$\begin{aligned}\psi\left(\rho_{pb}(v_{n+1}, v_n)\right) &\le F(\psi(\mathscr{P}(v_n, v_{n-1})), \varphi(\mathscr{P}(v_n, v_{n-1}))) \\ &= F\left((\psi\left(\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}\right),\right. \\ &\quad \left.\varphi\left(\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}\right)\right) \\ &\le \psi\left(\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}\right).\end{aligned} \tag{35}$$

We deduce that

$$\rho_{pb}(v_{n+1}, v_n) \le \frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}. \tag{36}$$

That is,

$$\rho_{pb}(v_{n+1}, v_n) \le \rho_{pb}(v_{n-1}, v_n), \quad n \geq 1. \tag{37}$$

Let $\{\rho_{pb}(v_{n+1}, v_n)\} = \theta_n$. Then, $\theta_n < \theta_{n-1}$.

This emphasizes that $\{\theta_n\}$ is a bounded decreasing sequence and bounded below, so it must be a convergent sequence.

For each $n \in \mathbb{N}$, we must have,

$$\theta_n < \theta_{n-1} < \cdots < \theta_1 = M \text{ (bounded value)}. \tag{38}$$

Again, for all $n, m \in \mathbb{N}$, we get

$$\rho_{pb}(v_n, v_m) \le \frac{1}{2}(\theta_{n-1} + \theta_{m-1}) < M. \tag{39}$$

Therefore, $\{v_n\}$ is a bounded sequence in W. By Bolzano Weierstrass theorem that every bounded sequence of real numbers has a convergent subsequence, $\{v_n\}$ must have a convergent subsequence, like $\{v_{n_k}\}$ which converges to some $z \in W$. So,

$$\lim_{k\longrightarrow\infty} \rho_{pb}\left(v_{n_k}, v_{n_{k+1}}\right) = \rho_{pb}\left(\lim_{k\longrightarrow\infty} v_{n_k}, \lim_{k\longrightarrow\infty} v_{n_{k+1}}\right) = \rho_{pb}(z, z) = 0. \tag{40}$$

This confirms that the subsequence $\{\theta_{n_k}\}$ itself converges to 0. So, the sequence $\{\theta_n\}$ must converge to 0. Hence, for all $n, m \in \mathbb{N}$,

$$\rho_{pb}(\upsilon_n, \upsilon_m) \le \frac{1}{2}\left\{\rho_{pb}(\upsilon_{n-1}, \upsilon_n) + \rho_{pb}(\upsilon_{m-1}, \upsilon_m)\right\} \longrightarrow 0, \tag{41}$$

as $n, m \longrightarrow \infty$.

This concludes that $\{\upsilon_n\}$ is a Cauchy sequence. As the subsequence $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ converges to z, the limit of $\{\upsilon_n\}$ must be z. Also, we have

$$\begin{aligned}\rho_{pb}(z, Sz) &\le \rho_{pb}(z, S\upsilon_{n+1}) + \rho_{pb}(S\upsilon_{n+1}, Sz) < \rho_{pb}(z, S\upsilon_{n+1}) \\ &\quad + \frac{1}{2}\left\{\rho_{pb}\left(\upsilon_n, S\upsilon_{n+1} + \rho_{pb}(z, Sz)\right) \Rightarrow \frac{1}{2}\rho_{pb}(z, Sz)\right. \\ &\le \rho_{pb}(z, S\upsilon_{n+1}) + \frac{1}{2}\rho_{pb}(\upsilon_n, S\upsilon_{n+1} \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned} \tag{42}$$

This infers that $z = Sz$. Hence, z is a fixed point of S.

Next, we assay the uniqueness of z. Debating by contradiction, let $\dot{z} \ne z$ be another fixed point of S, then $\rho_{pb}(z, \dot{z}) > 0$. We have

$$\begin{aligned}\psi\left(\rho_{pb}(z, \dot{z})\right) &\le \psi\left(s\rho_{pb}(Sz, S\dot{z})\right) \le F(\psi(\mathscr{P}(z, \dot{z}), \varphi(\mathscr{P}(z, \dot{z})))) \\ &= F\left(\psi\left(\frac{\rho_{pb}(z, Sz) + \rho_{pb}(\dot{z}, S\dot{z})}{2}\right), \varphi\left(\frac{\rho_{pb}(z, Sz) + \rho_{pb}(\dot{z}, S\dot{z})}{2}\right)\right) \\ &\le \psi\left(\frac{\rho_{pb}(z, z) + \rho_{pb}(\dot{z}, \dot{z})}{2}\right) \le \psi\left(\rho_{pb}(z, \dot{z})\right).\end{aligned} \tag{43}$$

Thus, $\rho_{pb}(z, \dot{z}) = 0$, which is a contradiction. Then, $z = \dot{z}$.

Now, the continuity of S in Theorem 8 has been dropped.

Theorem 9. *Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \ge 1$) and $S : W \longrightarrow W$. Assume that:*

(1) S is a generalized $\alpha - \eta - \psi - \varphi - F$-contraction

(2) S is triangular weak α-admissible

(3) There is $\upsilon_0 \in W$ in order that $\alpha(\upsilon_0, S\upsilon_0) \ge \eta(\upsilon_0, S\upsilon_0)$

(4) If $\{\upsilon_n\}$ is a sequence in W in order that $\alpha(\upsilon_n, \upsilon_{n+1}) \ge \eta(\upsilon_n, \upsilon_{n+1})$ and $\upsilon_n \longrightarrow \upsilon^* \in W$ as $n \longrightarrow \infty$, then there is $\{\upsilon_{n(k)}\}$ in order that $\alpha(\upsilon_{n(k)}, \upsilon^*) \ge \eta(\upsilon_{n(k)}, \upsilon^*)$

Then, S possesses a unique fixed point.

Proof. As in the proof of Theorem 8, we build the sequence $\{\upsilon_n\}$ by $\upsilon_{n+1} = S\upsilon_n$ for all $n \in \mathbb{N}$, which converges to $\upsilon^* \in W$ in order that $\alpha(\upsilon_n, \upsilon_{n+1}) \ge \eta(\upsilon_n, \upsilon_{n+1})$ for all $n \in \mathbb{N}$. By condition (4), there is a subsequence $\{\upsilon_{n(k)}\}$ of $\{\upsilon_n\}$ in order that $\alpha(\upsilon_{n(k)}, \upsilon^*) \ge \eta(\upsilon_{n(k)}, \upsilon^*)$ for all $k \in \mathbb{N}$. Consequently,

$$\begin{aligned}\psi\left(\rho_{pb}\left(\upsilon_{n(k)+1}, S\upsilon^*\right)\right) &\le \psi\left(s\rho_{pb}\left(S\upsilon_{n(k)}, S\upsilon^*\right)\right) \\ &= \psi\left(s\rho_{pb}\left(S\upsilon_{n(k)}, S\upsilon^*\right)\right), \le F(\psi\left(\mathscr{P}\left(\upsilon_{n(k)}, \upsilon^*\right), \varphi\left(\mathscr{P}\left(\upsilon_{n(k)}, \upsilon^*\right)\right)\right), \\ &\le F\left(\psi\left(\mathscr{P}\left(\upsilon_{n(k)}, \upsilon^*\right)\right)\right),\end{aligned} \tag{44}$$

for all $n \in \mathbb{N}$, where

Case 1.

$$\begin{aligned}\mathscr{P}\left(\upsilon_{n(k)}, \upsilon^*\right) &= \max\left\{\rho_{pb}(\upsilon_n, \upsilon^*), \rho_{pb}\left(\upsilon_{n(k)}, S\upsilon_{n(k)}\right), \rho_{pb}(\upsilon^*, S\upsilon^*), \frac{\rho_{pb}\left(\upsilon_{n(k)}, S\upsilon^*\right) + \rho_{pb}\left(\upsilon^*, S\upsilon_{n(k)}\right)}{2s}\right\}, \\ &= \max\left\{\rho_{pb}\left(\upsilon_{n(k)}, \upsilon^*\right), \rho_{pb}\left(\upsilon_{n(k)}, \upsilon_{n(k)+1}\right), \rho_{pb}(\upsilon^*, \upsilon^*), \frac{\rho_{pb}\left(\upsilon_{n(k)}, \upsilon^*\right) + \rho_{pb}\left(\upsilon^*, S\upsilon_{n(k)}\right)}{2s}\right\}, \\ &\le \max\left\{\rho_{pb}\left(\upsilon_{n(k)}, \upsilon^*\right), \rho_{pb}\left(\upsilon_{n(k)}, \upsilon_{n(k)+1}\right), \rho_{pb}\left(\upsilon^*, S\upsilon_{n(k)}\right)\right\}.\end{aligned} \tag{45}$$

By letting $n \longrightarrow \infty$, then

$$\mathscr{P}\left(\upsilon_{n(k)}, \upsilon^*\right) \le \rho_{pb}(\upsilon^*, S\upsilon^*). \tag{46}$$

Case 2.

$$\begin{aligned}\mathscr{P}\left(\upsilon_{n(k)}, \upsilon^*\right) &= \frac{\rho_{pb}\left(\upsilon_{n(k)}, S\upsilon_{n(k)} + \rho_{pb}(\upsilon^*, S\upsilon^*)\right)}{2}, \\ &= \frac{\rho_{pb}\left(\upsilon_{n(k)}, \upsilon_{n(k)+1}\right) + \rho_{pb}(\upsilon^*, S\upsilon^*)}{2}.\end{aligned} \tag{47}$$

By letting $n \longrightarrow \infty$, then

$$\mathscr{P}\left(\upsilon_{n(k)}, \upsilon^*\right) \le \frac{\rho_{pb}(\upsilon^*, S\upsilon^*)}{2} \le \rho_{pb}(\upsilon^*, S\upsilon^*). \tag{48}$$

By using (44), we get

$$\psi\left(\rho_{pb}(\upsilon^*, S\upsilon^*)\right) \le F\left(\psi\left(\rho_{pb}(\upsilon^*, S\upsilon^*)\right), \phi\left(\rho_{pb}(\upsilon^*, S\upsilon^*)\right)\right), \tag{49}$$

which infers that $\rho_{pb}(\upsilon^*, S\upsilon^*) = 0$. Hence, $S\upsilon^* = \upsilon^*$.

Following Theorem 8 and Theorem 9, we have

Corollary 10. *Let (W, ρ_{pb}) be a complete b-partial metric space (with $s \ge 1$). Let S be a self-mapping on W. Given $\alpha : W \times W \longrightarrow [0,\infty)$. Assume that*

(1) for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \ge 1$ and $\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) \ne 0$, we have $\psi(s\rho_{pb}(S\upsilon, S\kappa)) \le F(\psi(\rho_{pb}(\upsilon, \kappa)), \varphi(\rho_{pb}(\upsilon, \kappa)))$

(2) *S is triangular weak α-admissible*

(3) *There is $\upsilon_0 \in W$ in order that $\alpha(\upsilon_0, S\upsilon_0) \geq 1$*

(4) *S is $\alpha - \eta -$ continuous*

Then, S admits a unique fixed point.

Proof. We consider $\eta : W \times W \longrightarrow \mathbb{R}$ as $\eta(\upsilon, \kappa) = 1$.

Corollary 11. *Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$). Let S be a self-mapping on W. Given α : $W \times W \longrightarrow [0,\infty)$. Assume that*

(1) *for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \geq 1$ and $\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) \neq 0$, we have $\psi(s\rho_{pb}(S\upsilon, S\kappa)) \leq \lambda F(\psi(\rho_{pb}(\upsilon, \kappa)), \varphi(\rho_{pb}(\upsilon, \kappa)))$*

(2) *S is triangular α-admissible*

(3) *There is $\upsilon_0 \in W$ in order that $\alpha(\upsilon_0, S\upsilon_0) \geq 1$*

(4) *If $\{\upsilon_n\}$ is a sequence in W in order that $\alpha(\upsilon_n, \upsilon_{n+1}) \geq 1$ and $\upsilon_n \longrightarrow \upsilon^* \in W$ as $n \longrightarrow \infty$, then there is $\{\upsilon_{n(k)}\}$ in order that $\alpha(\upsilon_{n(k)}, \upsilon^*) \geq 1$*

Then, S possesses a unique fixed point.

Proof. Take $\eta(\upsilon, \kappa) = 1$ in Theorem 9.

Now, by taking $\mathcal{F}(\upsilon, \kappa) = \upsilon\beta(\upsilon)$ where $\Theta : [0,\infty) \longrightarrow [0, 1)$ is continuous, we have

Corollary 12. *Let (W, ρ_{pb}) be a complete b-partial metric space (with $s \geq 1$). Assume S is a self-mapping on W. Given $\alpha, \eta : W \times W \longrightarrow 0, \infty)$. Assume that for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \geq \eta(\upsilon, \kappa)$ and $\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) \neq 0$*

$$\psi\left(s\rho_{pb}(S\upsilon, S\kappa)\right) \leq \Theta\left(\psi\left(\rho_{pb}(\upsilon, \kappa)\right)\right)\psi\left(\rho_{pb}(\upsilon, \kappa)\right), \quad (50)$$

where $\psi \in \Psi$ and $\Theta : [0,\infty) \longrightarrow 0, 1)$ is continuous. If

(1) *S is triangular weak α-admissible*

(2) *There is $\upsilon_0 \in W$ in order that $\alpha(\upsilon_0, S\upsilon_0) \geq \eta(\upsilon_0, S\upsilon_0)$*

(3) *S is $\alpha - \eta -$ continuous*

then S possesses a unique fixed point.

Corollary 13. *Let (W, ρ_{pb}) be a complete b-partial metric space (with $s \geq 1$) and S be a self-mapping on W. Given α, η : $W \times W \longrightarrow 0, \infty)$. Assume that*

$$\psi\left(s\rho_{pb}(S\upsilon, S\kappa)\right) \leq \Theta\left(\psi\left(\rho_{pb}(\upsilon, \kappa)\right)\right), \psi\left(\rho_{pb}(\upsilon, \kappa)\right), \quad (51)$$

where $\psi \in \Psi$ and $\Theta : [0,\infty) \longrightarrow 0, 1)$ is continuous, for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \geq \eta(\upsilon, \kappa)$ and $\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) \neq 0$. If

(1) *S is triangular weak α-admissible*

(2) *There is $\upsilon_0 \in W$ in order that $\alpha(\upsilon_0, S\upsilon_0) \geq \eta(\upsilon_0, S\upsilon_0)$*

(3) *$\{\upsilon_n\}$ is a sequence in W so that $\alpha(\upsilon_n, \upsilon_{n+1}) \geq \eta(\upsilon_n, \upsilon_{n+1})$ and $\upsilon_n \longrightarrow \upsilon^* \in W$ as $n \longrightarrow \infty$, then there is $\{\upsilon_{n(k)}\}$ of $\{\upsilon_n\}$ in order that $\alpha(\upsilon_{n(k)}, \upsilon^*) \geq \eta(\upsilon_{n(k)}, \upsilon^*)$*

then S possesses a unique fixed point.

3. Consequences

Let Δ be the family of nonnegative functions δ defined on $[0, \infty)$ so that:

(1) δ is Lebesgue-integrable on every compact of $[0, \infty)$

(2) for every $\varpi > 0$

$$\int_0^{\varpi} \delta(\iota)d\iota > 0. \quad (52)$$

Theorem 14. *Let (W, ρ_{pb}) be a complete b-partial metric space (with $s \geq 1$) and S be a self-mapping on W. Suppose there are $F \in \mathcal{C}$, $\delta_1, \delta_2 \in \Delta$, and $\alpha, \eta : W \times W \longrightarrow [0,\infty)$ such for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \geq \eta(\upsilon, \kappa)$ and $\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) \neq 0$, we get*

$$\begin{aligned}
&\int_0^{s\rho_{pb}(S\upsilon, T\kappa)} \delta_1(z))dz \\
&\leq F\left(\int_0^{\max\left\{\rho_{pb}(\upsilon,\kappa), \rho_{pb}(\upsilon,S\upsilon), \rho_{pb}(S\upsilon,S\upsilon), \left(\left(\rho_{pb}(\upsilon,S\kappa)+\rho_{pb}(\kappa,S\upsilon)\right)/2s\right)\right\}} \delta_1(z)dz,\right. \\
&\left.\int_0^{\max\left\{\rho_{pb}(\upsilon,\kappa), \rho_{pb}(\upsilon,S\upsilon), \rho_{pb}(S\upsilon,S\upsilon), \left(\left(\rho_{pb}(\upsilon,S\kappa)+\rho_{pb}(\kappa,S\upsilon)\right)/2s\right)\right\}} \delta_2(z)dz\right).
\end{aligned} \quad (53)$$

Also, suppose that

(1) S is triangular weak α-admissible

(2) There is $\upsilon_0 \in W$ in order that $\alpha(\upsilon_0, S\upsilon_0) \geq \eta(\upsilon_0, S\upsilon_0)$

(3) S is $\alpha - \eta -$ continuous

Then, S possesses a unique fixed point.

Proof. Take $\psi(t) = \int_0^t \delta_1(\iota))d\iota$ and $\varphi(t) = \int_0^t \delta_2(\iota))d\iota$. By using Theorem 8, S possesses a fixed point.

Theorem 15. *Let (W, ρ_{pb}) be a complete b-partial metric space (with $s \geq 1$) and $S : W \longrightarrow W$. Suppose there are $F \in \mathcal{C}$, $\delta_1, \delta_2 \in \Delta$, and $\alpha, \eta : W \times W \longrightarrow [0,\infty)$ such for all $\upsilon, \kappa \in W$ with $\alpha(\upsilon, \kappa) \geq \eta(\upsilon, \kappa)$ and $\rho_{pb}(\upsilon, S\kappa) + \rho_{pb}(S\upsilon, \kappa) \neq 0$, we have*

$$\int_0^{s\rho_{pb}(Sv,S\kappa)} \delta_1(z))dz \le F\left(\int_0^{\max\left\{\rho_{pb}(v,\kappa),\rho_{pb}(v,Sv),\rho_{pb}(Sv,S\kappa),\left(\left(\rho_{pb}(v,S\kappa)+\rho_{pb}(\kappa,Sv)\right)/2s\right)\right\}} \delta_1(z)dz, \int_0^{\max\left\{\rho_{pb}(v,\kappa),\rho_{pb}(v,Sv),\rho_{pb}(Sv,S\kappa),\left(\left(\rho_{pb}(v,S\kappa)+\rho_{pb}(\kappa,Sv)\right)/2s\right)\right\}} \delta_2(z)dz\right). \tag{54}$$

Assume that:

(1) *S is triangular weak α-admissible*

(2) *There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \ge \eta(v_0, Sv_0)$*

(3) *If $\{v_n\}$ is a sequence in W in order that $\alpha(v_n, v_{n+1}) \ge \eta(v_n, v_{n+1})$ and $v_n \longrightarrow v^* \in W$ as $n \longrightarrow \infty$, then there is $\{v_{n(k)}\}$ in order that $\alpha(v_{n(k)}, v^*) \ge \eta(v_{n(k)}, v^*)$*

Then, S possesses a unique fixed point.

Proof. Consider $\psi(t) = \int_o^t \delta_1(\iota))d\iota$ and $\varphi(t) = \int_o^t \delta_2(\iota))d\iota$ in Theorem 9.

Example 1. Given $W = [0,\infty)$ and $\rho_{pb} : W \times W \longrightarrow \mathbb{R}$ as

$$\rho_{pb}(v,\kappa) = \begin{cases} [\max\{v,\kappa\}]^2 & \text{if } v \ne \kappa, \\ 0 & \text{if } v = \kappa, \end{cases} \tag{55}$$

for all $v, \kappa \in W$. Choose $\psi(t) = t$ and $\phi(t) = (13/16)t$. Consider $Sv = e^v/4$. Given $\alpha, \eta : W \times W \longrightarrow 0, \infty)$ as

$$\alpha(v,\kappa) = \begin{pmatrix} 2 & \text{if } v,\kappa \in [0,1], \\ 0 & \text{if not,} \end{pmatrix} \qquad \eta(v,\kappa) = \begin{pmatrix} 1 & \text{if } v,\kappa \in [0,1], \\ 0 & \text{if not.} \end{pmatrix} \tag{56}$$

Put $F(\tau, \omega) = \tau - \omega$ for all $\tau, \omega \in W$. We are going to demonstrate the following:

(1) (W, ρ_{pb}) is a complete partial b-metric space

(2) S is triangular weak α-admissible

(3) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \ge \eta(v_0, Sv_0)$

(4) If $\{v_n\}$ is a sequence in W in order that $\alpha(v_n, v_{n+1}) \ge \eta(v_n, v_{n+1})$ and $v_n \longrightarrow v^* \in W$ as $n \longrightarrow \infty$

Proof. Note that (W, ρ_{pb}) is a complete partial b-metric space (Here, $s = 3$).

If $\alpha(v, Sv) \ge \eta(v, Sv)$, then $\alpha(Sv, S^2v) \ge \eta(Sv, S^2v)$, So $\alpha(v, (e^v/4)) = 2 > 1 = \eta(x, (e^v/4))$, then $\alpha((e^v/4), ((e^{(e^v/4)})/4)) = 2 \ge 1 = \eta((e^v/4), ((e^{(e^v/4)})/4))$. So $v \ge 0$ and so $Sv \le 0$. Hence, $\alpha(v, Sv) \ge \eta(v, Sv)$.

Recall that $\alpha(v, Sv) \ge \eta(v, Sv)$ for all $v, \kappa \in 0, 1]$. In this case,

$$\begin{aligned} \psi\left(s\rho_{pb}(Sv, S\kappa)\right) &= 3\left[\max \frac{e^v}{4}, \frac{e^\kappa}{4}\right]^2 = \frac{3}{16}\left[\max e^v, e^\kappa\right]^2 \\ &= \frac{3}{16}\rho_{pb}(v,\kappa) \le \frac{3}{16}\mathscr{P}(v,\kappa) = \mathscr{P}(v,\kappa) \\ &\quad - \frac{13}{16}\mathscr{P}(v,\kappa) = \psi(\mathscr{P}(v,\kappa)) - \phi(\mathscr{P}(v,\kappa)) \\ &= F(\psi(\mathscr{P}(v,\kappa)), \phi(\mathscr{P}(v,\kappa))). \end{aligned} \tag{57}$$

All hypotheses of Corollary 11 hold, and so, S admits a unique fixed point.

4. Conclusion

We ensured the existence of a unique fixed point for generalized contraction type mappings involving triangular weak α-admissibility with regard to a function η and C-class functions in the class of partial b-metric spaces. Some illustrated examples have been also provided.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

On Fuzzy b-Metric-Like Spaces

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The aim of this manuscript is to introduce the concept of fuzzy b-metric-like spaces and discuss some related fixed point results. Some examples are imparted to illustrate the feasibility of the proposed methods. Finally, to validate the superiority of the obtained results, an application is provided to solve a first kind of Fredholm type integral equations.

1. Introduction and Preliminaries

The concept of fuzzy sets was initiated by Zadeh [1], which gave a new aspect to research activity leading with the improvement of fuzzy systems. Afterwards, several researchers contributed towards some basic significant results in fuzzy sets.

Kramosil and Michlek [2] introduced the concept of fuzzy metric spaces by generalizing the concepts of probabilistic metric spaces to fuzzy metric spaces. George and Veeramani [3] derived a Hausdorff topology which was initiated by a fuzzy metric to modify the concept of fuzzy metric spaces. Later on, the fixed point theory dealing with a fuzzy metric has been enriched with a number of different generalizations. Garbiec displayed the fuzzy version of Banach contraction principle in fuzzy metric spaces. For some necessary definitions, examples, and basic results, we refer to [4–6] and the references herein.

As we know, fixed point theory plays a crucial role in proving the existence of solutions for different mathematical models and has a wide range of applications in different fields related to mathematics. This theory has intrigued many researchers. Recently, Harandi [7] initiated the concept of metric-like spaces, which generalizes the notion of metric spaces in a nice way. Alghamdi et al. [8] used the concept metric-like spaces to introduce the notion of b-metric-like spaces. Since then, a number of authors contributed in same. For more details, we refer to [7, 9–16]. In this sequel, Shukla and Abbas [8] generalized the concept of metric-like spaces and introduced fuzzy metric-like spaces. For more details on this topic, please see [17–29].

In this article, our aim is to generalize the concept of b-metric-like spaces by introducing the concept of fuzzy b-metric-like spaces and prove some related fixed point results. We also support this work by some examples and an application to solve an integral equation.

First, we write some notations used throughout this paper, as C-t-norm for a continuous triangular norm, b-metric-l for b-metric-like, F-metric-l for fuzzy metric-like, F-b-metric for fuzzy b-metric, F-b-metric-l for fuzzy b-metric-like, and s.t. for such that.

Definition 1 [17]. A binary operation $^*: [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a C-t-norm if it satisfies the following assertions:

(1) $a * b = b * a, (\forall)\, a, b \in [0, 1]$

(2) $a * 1 = a, (\forall)\, a \in [0, 1]$

(3) $(a * b) * c = a * (b * c), (\forall)\, a, b, c \in [0, 1]$

(4) If $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$, then $a * b \leq c * d$

Some fundamental examples of a t-norm are $a * b = a \bullet b$, $a * b = \min\{a, b\}$ and $a * b = \max\{a + b - 1, 0\}$.

Definition 2 [8]. A b-metric-l on a set $\mho \neq \Phi$ is a function $\mu : \mho \times \mho \longrightarrow [0,+\infty)$ such that for all $w, k, z \in \mho$ and $b \geq 1$, it satisfies the following conditions:

(1) If $\mu(w, k) = 0 \Rightarrow w = k$

(2) $\mu(w, k) = w(k, w)$

(3) $\mu(w, k) \leq b[\mu(w, z) + \mu(z, k)]$

The pair $(\mho, \mu)$ is called a b-metric-l space.

Example 1 [8]. Let $\mho = [0,\infty)$. Define the function $\mu : \mho \times \mho \longrightarrow [0,+\infty)$ by

$$\mu(w, k) = (w + k)^2. \tag{1}$$

Then, $(\mho, \mu)$ is called a b-metric-l space with $b = 2$.

Example 2 [8]. Let $\mho = [0,\infty)$. Define the function $\mu : \mho \times \mho \longrightarrow [0,+\infty)$ by.

$$\mu(w, k) = (\max\{w, k\})^2. \tag{2}$$

Then, $(\mho, \mu)$ is named as a b-metric-l space with $b = 2$.

Definition 3 [18]. A 3-tuple $(\mho, \Omega, *)$ is said an F-metric-l space if $\mho \neq \Phi$ is a random set, $*$ is a C-t-norm, and Ω is a fuzzy set on $\mho \times \mho \times (0,\infty)$ meeting the conditions below for all $w, k, z \in \mho, \tau, s > 0$:
FL1) $\Omega(w, k, \tau) > 0$
FL2) If $\Omega(w, k, \tau) = 1$, then $w = k$
FL3) $\Omega(w, k, \tau) = \Omega(k, w, \tau)$
FL4) $\Omega(w, z, \tau + s) \geq \Omega(w, k, \tau) * \Omega(k, z, s)$
FL5) $\Omega(w, k, \bullet)$: $(0,\infty) \longrightarrow [0, 1]$ is continuous.

Example 3 [18]. Let $\mho = \mathbb{R}^+$, $q \in \mathbb{R}^+$, and $m > 0$. Define a t-norm by $a * b = ab$ and the fuzzy set Ω on $\mho \times \mho \times (0,\infty)$ by

$$\Omega(w, k, \tau) = \frac{q\tau}{q\tau + m(\max\{w, k\})}, \quad \forall w,\ k \in \mho,\ \tau > 0. \tag{3}$$

Then, $(\mho, \Omega, *)$ is an F-metric-l space.

Definition 4 [5]. A 3-tuple $(\mho, \Omega, *)$ is said an F-b-metric space if $\mho$ is a random (nonempty) set, $*$ is a C-t-norm, and Ω is a fuzzy set on $\mho \times \mho \times (0,\infty)$ meeting the conditions below for all $w, k, z \in \mho, \tau, s > 0$ and a provided real number $b \geq 1$:
FB1) $\Omega(w, k, \tau) > 0$
FB2) $\Omega(w, k, \tau) = 1$ iff $w = k$
FB3) $\Omega(w, k, \tau) = \Omega(k, w, \tau)$
FB4) $\Omega(w, z, \tau + s) \geq \Omega(w, k, (\tau/b)) * \Omega(k, z, (s/b))$
FB5) $\Omega(w, k, \bullet)$: $(0,\infty) \longrightarrow [0, 1]$ is continuous.

Example 4 [9]. Let $\Omega(w, k, \tau) = e^{-|w-k|^p/\tau}$, where $p > 1$ is a real number. It is then simple to show that Ω is an F-b-metric with $b = 2^{p-1}$.

2. Main Results

We start this section with the introduction of F-b-metric-l spaces and prove some related fixed point results.

Definition 5. A 4-tuple $(\mho, \Omega, *, b)$ is named an F-b-metric-l space if $\mho \neq \Phi$ is a random set, $*$ is a C-t-norm, and Ω is a fuzzy set on $\mho \times \mho \times (0,\infty)$ meeting the following conditions below for all $w, k, z \in \mho, \tau, s > 0$:
B1) $\Omega(w, k, \tau) > 0$
B2) If $\Omega(w, k, \tau) = 1$, then $w = k$
B3) $\Omega(w, k, \tau) = \Omega(k, w, \tau)$
B4) $\Omega(w, z, b(\tau + s)) \geq \Omega(w, k, \tau) * \Omega(k, z, s)$, for $b \in \mathbb{N}$
B5) $\Omega(w, k, \bullet)$: $(0,\infty) \longrightarrow [0, 1]$ is continuous.

Remark 6. In the above definition, a set $\mho$ is endowed by an F-b-metric-l with a t-norm $(*)$. An F-b-metric-l space does not satisfy the (FB2) condition of F-b-metric spaces, that is, the self-distance may not be equal to 1, i.e., $\Omega(w, w, \tau) \neq 1$ for all $\tau > 0$,for some or may be for all $w \in \mho$. But all other conditions are the same. Hence, the F-b-metric-l may not be a F-b-metric, but the converse is true.

Definition 7. Let $(\mho, \Omega, *, b)$ be an F-b-metric-l space. For $x \in X, \Theta \in (0, 1), \tau > 0$, we define the open ball as

$$B(\mathrm{x}, \Theta, \tau) = \{k \in \mho : \Omega(w, k, \tau) > 1 - \Theta\}. \tag{4}$$

Then,

$$T_M = \{T \subset \mho : w \in T \Longleftrightarrow \text{there exist}\ \tau > 0\ \text{and}\ \Theta \in (0, 1)\,;\, B(x, \Theta, \tau) \subseteq T\}, \tag{5}$$

is a topology on $\mho$.

The following simplest example shows that an F-b-metric-l need not to be an F-b-metric.

Example 5. Take $\mho = (0,\infty)$. Consider the t-norm defined by $a * b = ab$, then

$$\Omega(w, k, \tau) = \left[e^{(w+k)^2/\tau}\right]^{-1}, \quad \forall w,\ k \in \mho,\ \tau > 0. \tag{6}$$

is an F-b-metric-l. But it is not an F-b-metric.

Proof. (B1), (B2), (B3), and (B5) are obvious. Here, we prove (B4). We have

$$|w+z| \leq \left(\frac{b(\tau+s)}{\tau}\right)|w+k| + \left(\frac{b(\tau+s)}{s}\right)|k+z|. \quad (7)$$

Here, b is an arbitrary integer. We have

$$\frac{|w+z|}{b(\tau+s)} \leq \frac{|w+k|}{\tau} + \frac{|k+z|}{s}. \quad (8)$$

Hence,

$$e^{|w+z|/(b(\tau+s))} \leq e^{|w+k|/\tau} + e^{|k+z|/s}. \quad (9)$$

Since e^w is an increasing function for $w>0$, one writes

$$\left[e^{(w+z)^2/(b(\tau+s))}\right]^{-1} \geq \left[e^{(w+k)^2/\tau}\right]^{-1} + \left[e^{(k+z)^2/s}\right]^{-1}. \quad (10)$$

That is,

$$\Omega(w,z,b(\tau+s)) \geq \Omega(w,k,\tau) * \Omega(k,z,s), \quad \forall w,k,z \in \mho, \forall \tau, s>0 \quad (11)$$

Hence, (B4) satisfied.

Now, we have to prove that $(\mho,\Omega,*,b)$ is not an F-b-metric space. For this purpose, we investigate the self-distance. Indeed,

$$\Omega(w,w,\tau) = \left[e^{(w+w)^2/\tau}\right]^{-1} = \frac{1}{e^{(w+w)^2/\tau}} \neq 1, \quad \forall \tau>0,\ \forall w \in \mho. \quad (12)$$

Hence, $(\mho,\Omega,*,b)$ is not an F-b-metric space.

The following example shows that an F-b-metric-l space need not be continuous.

Example 6. Let $\mho=[0,\infty)$, $\Omega(w,k,\tau)=(e^{-(d(w,k)/\tau)})$, $\forall w,k \in \mho$, $\tau>0$ and

$$d(w,k) = \begin{cases} 0, & \text{if } w=k, \\ 2(k+w)^2, & \text{if } k,w \in [0,1], \\ \frac{1}{2}(k+w)^2, & \text{otherwise}. \end{cases} \quad (13)$$

If we consider the t-norm defined by $a*b=ab$, then $(\mho,\Omega,*,b)$ is an F-b-metric-l space with a coefficient $b=4$. To illustrate the discontinuity, we have

$$\lim_{n\longrightarrow\infty} \Omega\left(0, 1-\frac{1}{n}, \tau\right) = \lim_{n\longrightarrow\infty} e^{-2(1-(1/n))^2} = e^{-2} = \Omega(0,1,\tau). \quad (14)$$

However, since

$$\lim_{n\longrightarrow\infty} \Omega\left(1, 1-\frac{1}{n}, \tau\right) = \lim_{n\longrightarrow\infty} e^{-2(2-(1/n))^2} = e^{-8} \neq 1 = \Omega(1,1,\tau). \quad (15)$$

One can assert that $\Omega(w,k,\tau)$ is not continuous.

Proposition 8. *Let $(\mho,\mu)$ be a b-metric-l space. Then, $(\mho,\Omega,*,b)$ is an F-b-metric-l space defined as*

$$\Omega(w,k,\tau) = e^{-\mu(w,k)/\tau^n}, \quad \forall \tau>0, \forall w,k \in \mho, n \in \mathbb{N}. \quad (16)$$

Proof. (B1), (B2), (B3), and (B5) are obvious. Here, we prove (B4). We have

$$\mu(w,z) \leq b[\mu(w,k)+\mu(k,z)]. \quad (17)$$

Therefore,

$$\frac{\mu(w,z)}{(\tau+s)^n} \leq \frac{b[\mu(w,k)+\mu(k,z)]}{(\tau+s)^n}. \quad (18)$$

That is,

$$\frac{\mu(w,z)}{b(\tau+s)^n} \leq \frac{\mu(w,k)}{\tau^n} + \frac{\mu(k,z)}{s^n}. \quad (19)$$

Hence,

$$e^{\mu(w,z)/(b(\tau+s)^n)} \geq e^{\mu(w,k)/\tau^n} + e^{\mu(k,z)/s^n}. \quad (20)$$

That is, $\Omega(w,z,b(\tau+s)) \geq \Omega(w,k,\tau) * \Omega(k,z,s)$. Hence, (B4) is satisfied and $(\mho,\Omega,*,b)$ is an F-b-metric-l space.

Definition 9. A sequence $\{w_n\}$ in a F-b-metric-l space $(\mho,\Omega,*,b)$ is said to be convergent to $w \in \mho$, if

$$\lim_{n\longrightarrow\infty} \Omega(w_n,w,\tau) = \Omega(w,w,\tau), \quad \forall \tau>0. \quad (21)$$

Definition 10. A sequence $\{w_n\}$ in an F-b-metric-l space $(\mho,\Omega,*,b)$ is said to be Cauchy if

$$\lim_{n\longrightarrow\infty} \Omega(w_n,w_{n+p},\tau), \quad \forall \tau>0,\ p\geq 1, \quad (22)$$

exists and is finite.

Definition 11. An F-b-metric-l space $(\mho,\Omega,*,b)$ is said to be complete if every Cauchy sequence $\{w_n\}$ in $\mho$ converges to some $w \in \mho$ such that

$$\lim_{n\longrightarrow\infty} \Omega(w_n,w,\tau) = \Omega(w,w,\tau) = \lim_{n\longrightarrow\infty} \Omega(w_n,w_{n+p},\tau), \quad \forall \tau>0,\ p\geq 1. \quad (23)$$

Definition 12. Let $(\mho,\Omega,*,b)$ be an F-b-metric-l space. A mapping $\mathscr{F}:\mho\longrightarrow\mho$ is said to be fuzzy contractive if there exists $q\in(0,1)$ such that

$$\frac{1}{\Omega(\mathcal{F}w, \mathcal{F}k, \tau)} - 1 \le q\left[\frac{1}{\Omega(w, k, \tau)} - 1\right], \quad (24)$$

for all $w, k \in \mho$ and $\tau > 0$. Here, q is called the fuzzy contractive constant of $\mathcal{F}$.

Theorem 13. *Let $(\mho, \Omega, *, b)$ be a complete F-b-metric-l space and $\mathcal{F} : \mho \longrightarrow \mho$ be a fuzzy contractive mapping with a fuzzy contractive constant q, then $\mathcal{F}$ has a unique fixed point $v \in \mho$ so that $\Omega(v, v, \tau) = 1$ for all $\tau > 0$.*

Proof. For an arbitrary $w_0 \in \mho$, define a sequence $\{w_n\}$ in $\mho$ by

$$w_1 = \mathcal{F}w_0, w_2 = \mathcal{F}^2 w_0 = \mathcal{F}w_1, \cdots, w_n = \mathcal{F}^n w_0 = \mathcal{F}w_{n-1} \text{ for all } n \in \mathbb{N}. \quad (25)$$

If $w_n = w_{n-1}$ for some $n \in \mathbb{N}$, then w_n is a fixed point of $\mathcal{F}$. We assume that $w_n \neq w_{n-1}$ for all $n \in \mathbb{N}$. For $\tau > 0$ and $n \in \mathbb{N}$, we get from (24)

$$\begin{aligned}\frac{1}{\Omega(w_n, w_{n+1}, \tau)} - 1 &= \frac{1}{\Omega(\mathcal{F}w_{n-1}, \mathcal{F}w_n, \tau)} - 1 \\ &\le q\left[\frac{1}{\Omega(w_{n-1}, w_n, \tau)} - 1\right].\end{aligned} \quad (26)$$

We have

$$\frac{1}{\Omega(w_n, w_{n+1}, \tau)} \le \frac{q}{\Omega(w_{n-1}, w_n, \tau)} + (1-q), \quad \forall \tau > 0. \quad (27)$$

Therefore,

$$\begin{aligned}\frac{q}{\Omega(\mathcal{F}w_{n-2}, \mathcal{F}w_{n-1}, \tau)} + (1-q) &\le \frac{q^2}{\Omega(w_{n-2}, w_{n-1}, \tau)} \\ &\quad + q(1-q) + (1-q).\end{aligned} \quad (28)$$

Continuing in this way, we get

$$\begin{aligned}\frac{1}{\Omega(w_n, w_{n+1}, \tau)} &\le \frac{q^n}{\Omega(w_0, w_1, \tau)} + q^{n-1}(1-q) + q^{n-2}(1-q) \\ &\quad + \cdots + q(1-q) + (1-q) \le \frac{q^n}{\Omega(w_0, w_1, \tau)} \\ &\quad + \left(q^{n-1} + q^{n-2} + \cdots + 1\right)(1-q) \\ &\le \frac{q^n}{\Omega(w_0, w_1, \tau)} + (1-q^n).\end{aligned} \quad (29)$$

We have

$$\frac{1}{(q^n/(\Omega(w_0, w_1, \tau))) + (1-q^n)} \le \Omega(w_n, w_{n+1}, \tau), \quad \forall \tau > 0, \ n \in \mathbb{N}. \quad (30)$$

Now, for $p \ge 1$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\Omega(w_n, w_{n+p}, \tau) &\ge \Omega\left(w_n, w_{n+1}, \frac{\tau}{b}\right) * \Omega\left(w_{n+1}, w_{n+p}, \frac{\tau}{b}\right) \\ &\ge \Omega\left(w_n, w_{n+1}, \frac{\tau}{b}\right) * \Omega\left(w_{n+1}, w_{n+2}, \frac{\tau}{b^2}\right) * \Omega \\ &\quad \cdot \left(w_{n+2}, w_{n+p}, \frac{\tau}{b^2}\right).\end{aligned} \quad (31)$$

Continuing in this way, we get

$$\begin{aligned}\Omega(w_n, w_{n+p}, \tau) &\ge \Omega\left(w_n, w_{n+1}, \frac{\tau}{b}\right) * \Omega\left(w_{n+1}, w_{n+2}, \frac{\tau}{b^2}\right) * \cdots * \Omega \\ &\quad \cdot \left(w_{n+p-1}, w_{n+p}, \frac{\tau}{b^{p-1}}\right).\end{aligned} \quad (32)$$

By using (30) in the above inequality, we have

$$\begin{aligned}\Omega(w_n, w_{n+p}, \tau) &\ge \frac{1}{(q^n/\Omega(w_0, w_1, (\tau/b))) + (1-q^n)} * \\ &\quad \cdot \frac{1}{\left(q^{n+1}/\left(\Omega\left(w_0, w_1, \left(\tau/b^2\right)\right)\right)\right) + (1-q^{n+1})} * \cdots * \\ &\quad \cdot \frac{1}{\left(q^{n+p-1}/\left(\Omega\left(w_0, w_1, \left(\tau/b^{p-1}\right)\right)\right)\right) + (1-q^{n+p-1})} \\ &\ge \frac{1}{(q^n/(\Omega(w_0, w_1, (\tau/b)))) + 1} * \\ &\quad \cdot \frac{1}{\left(q^{n+1}/\left(\Omega\left(w_0, w_1, \left(\tau/b^2\right)\right)\right)\right) + 1} * \cdots * \\ &\quad \cdot \frac{1}{\left(q^{n+p-1}/\left(\Omega\left(w_0, w_1, \left(\tau/b^{p-1}\right)\right)\right)\right) + 1}.\end{aligned} \quad (33)$$

Here, b is an arbitrary positive integer, and as $q \in (0, 1)$, we deduce from the above expression that

$$\lim_{n \longrightarrow \infty} \Omega\left(w_n, w_{n+p}, \tau\right) = 1 \quad \text{for all } \tau > 0, \ p \ge 1. \quad (34)$$

Therefore, $\{w_n\}$ is a Cauchy sequence in $(\mho, \Omega, *, b)$. By the completeness of $(\mho, \Omega, *, b)$,
there is $v \in \mho$ such that

$$\begin{aligned}\lim_{n \longrightarrow \infty} \Omega(w_n, v, \tau) &= \lim_{n \longrightarrow \infty} \Omega\left(w_n, w_{n+p}, \tau\right) = \lim_{n \longrightarrow \infty} \Omega(v, v, \tau) \\ &= 1, \quad \forall \tau > 0, \ p \ge 1.\end{aligned} \quad (35)$$

Now, we prove that v is a fixed point for $\mathcal{F}$. For this, we obtain from (24) that

$$\frac{1}{\Omega(\mathscr{F}w_n,\mathscr{F}v,\tau)}-1\le q\left[\frac{1}{\Omega(w_n,v,\tau)}-1\right]=\frac{q}{\Omega(w_n,v,\tau)}-q,$$
$$\frac{1}{(q/(\Omega(w_n,v,\tau)))+1-q}\le\Omega(\mathscr{F}w_n,\mathscr{F}v,\tau) \tag{36}$$

Using the above inequality, we obtain

$$\begin{aligned}\Omega(v,\mathscr{F}v,\tau)&\ge\Omega\left(v,w_{n+1},\frac{\tau}{2b}\right)*\Omega\left(w_{n+1},\mathscr{F}v,\frac{\tau}{2b}\right)\\&=\Omega\left(v,w_{n+1},\frac{\tau}{2b}\right)*\Omega\left(\mathscr{F}w_n,\mathscr{F}v,\frac{\tau}{2b}\right)\\&\ge\Omega\left(v,w_{n+1},\frac{\tau}{2b}\right)*\\&\quad\cdot\frac{1}{(q/(\Omega(w_n,v,(\tau/2b))))+1-q}.\end{aligned} \tag{37}$$

Taking limit as $n\longrightarrow\infty$ and using (35) in the above expression, we get that $\Omega(v,\mathscr{F}v,\tau)=1$, that is, $\mathscr{F}v=v$. Therefore, v is a fixed point of $\mathscr{F}$ and $\Omega(v,v,\tau)=1$ for all $\tau>0$.

Now, we investigate the uniqueness of the fixed point v of $\mathscr{F}$. Let ν be another fixed point of $\mathscr{F}$, s.t. $\Omega(v,\nu,t)<1$ for some $\tau>0$. It follows from (24) that

$$\begin{aligned}\frac{1}{\Omega(v,\nu,\tau)}-1&=\frac{1}{\Omega(\mathscr{F}v,\mathscr{F}\nu,\tau)}-1\le q\left[\frac{1}{\Omega(v,\nu,\tau)}-1\right]\\&<\frac{1}{\Omega(v,\nu,\tau)}-1,\end{aligned} \tag{38}$$

a contradiction. Therefore, we must have $\Omega(v,\nu,\tau)=1$, for all $\tau>0$, and hence $v=\nu$.

Corollary 14. *Let $(\mho,\Omega,*,b)$ be a complete F-b-metric-l space and $\mathscr{F}:\mho\longrightarrow\mho$ be a mapping satisfying*

$$\frac{1}{\Omega(\mathscr{F}^nw,\mathscr{F}^nk,\tau)}-1\le q\left[\frac{1}{\Omega(w,k,\tau)}-1\right], \tag{39}$$

for some $n\in\mathbb{N},\forall w,k\in\mho,\tau>0$, where $0<q<1$. Then, $\mathscr{F}$ has a unique fixed point $v\in\mho$ and $\Omega(v,v,\tau)=1,\forall\tau>0$.

Proof. $v\in\mho$ is the unique fixed point of $\mathscr{F}^n$ by using Theorem 13, and $\Omega(v,v,\tau)=1,\forall\tau>0$.$\mathscr{F}v$ is also a fixed point of $\mathscr{F}^n$ as $\mathscr{F}^n(\mathscr{F}v)=\mathscr{F}v$ and from Theorem 13, $\mathscr{F}v=v$, v is the unique fixed point, since the unique fixed point of $\mathscr{F}$ is also the unique fixed point of $\mathscr{F}^n$.

Example 7. Let $\mho=[0,2]$ and the t-norm be defined as $a*b=ab$. Given Ω as

$$\Omega(w,k,\tau)=e^{-(\max\{w,k\})^2/\tau},\ \text{for all } w,k\in\mho \text{ and } \tau>0. \tag{40}$$

(From Example 2 and Proposition 8, Ω is an F-b-metric-l).

Then, $(\mho,\Omega,*,b)$ is a complete F-b-metric-l space. Define $\mathscr{F}:\mho\longrightarrow\mho$ as

$$\mathscr{F}w=\begin{cases}0, & w=1,\\ \dfrac{w}{2}, & w\in[0,1),\\ \dfrac{w}{4}, & w\in(1,2].\end{cases} \tag{41}$$

Then, we have 8 cases:

Case 1. If $w=k=1$, then $\mathscr{F}w=\mathscr{F}k=0$.

Case 2. If $w=1$ and $k\in[0,1)$, then $\mathscr{F}w=0$ and $\mathscr{F}k=k/2$.

Case 3. If $w=1$ and $k\in(1,2]$, then $\mathscr{F}w=0$ and $\mathscr{F}k=k/4$.

Case 4. If $w\in[0,1)$ and $k\in(1,2]$, then $\mathscr{F}w=w/2$ and $\mathscr{F}k=k/4$.

Case 5. If $w\in[0,1)$ and $k\in[0,1)$, then $\mathscr{F}w=w/2$ and $\mathscr{F}k=k/2$.

Case 6. If $w\in[0,1)$ and $k=1$, then $\mathscr{F}w=w/2$ and $\mathscr{F}k=0$.

Case 7. If $w\in(1,2]$ and $k=1$, then $\mathscr{F}w=w/4$ and $\mathscr{F}k=0$.

Case 8. If $w\in(1,2]$ and $k\in(1,2]$, then $\mathscr{F}w=w/4$ and $\mathscr{F}k=k/4$.

All above cases satisfy the fuzzy contraction:

$$\frac{1}{\Omega(\mathscr{F}w,\mathscr{F}k,\tau)}-1\le q\left[\frac{1}{\Omega(w,k,\tau)}-1\right], \tag{42}$$

with $q\in[1/2,1)$ the fuzzy contractive constant . Hence $\mathscr{F}$ is a fuzzy contractive mapping with $q\in[1/2,1)$. All conditions of Theorem 13 are satisfied. Also, 0 is the unique fixed point of $\mathscr{F}$ and $\Omega(0,0,\tau)=1,\forall\tau>0$.

Theorem 15. *Let $(\mho,\Omega,*,b)$ be a complete F-b-metric-l space such that*

$$\lim_{\tau\longrightarrow\infty}\Omega(w,k,\tau)=1, \tag{43}$$

for all $w,k\in\mho,\tau>0$, and $\mathscr{F}:\mho\longrightarrow\mho$ be a mapping satisfying the condition

$$\Omega(\mathscr{F}w,\mathscr{F}k,\alpha\tau)\ge\Omega(w,k,\tau), \tag{44}$$

for all $w,k\in\mho,\tau>0$, where $\alpha\in(0,1)$. Then $\mathscr{F}$ has a unique fixed point $v\in\mho$ and

$$\Omega(v,v,\tau)=1,\quad\forall\tau>0. \tag{45}$$

Proof. For an arbitrary $w_0\in\mho$, define a sequence $\{w_n\}$ in $\mho$

by

$$w_1 = \mathcal{F}w_0, w_2 = \mathcal{F}^2 w_0 = \mathcal{F}w_1, \cdots, w_n = \mathcal{F}^n w_0 = \mathcal{F}w_{n-1} \quad \text{for all } n \in \mathbb{N}. \tag{46}$$

If $w_n = w_{n-1}$ for some $n \in \mathbb{N}$, then w_n is a fixed point of $\mathcal{F}$. We assume that $w_n \neq w_{n-1}$ for all $n \in \mathbb{N}$. For $\tau > 0$ and $n \in \mathbb{N}$, we get from (44) that

$$\begin{aligned}\Omega(w_n, w_{n+1}, \tau) &\geq \Omega(w_{n+1}, w_n, \alpha\tau) = \Omega(\mathcal{F}w_n, \mathcal{F}w_{n-1}, \alpha\tau)\\ &\geq \Omega(w_n, w_{n-1}, \tau),\end{aligned} \tag{47}$$

for all $n \in \mathbb{N}$ and $\tau > 0$. Therefore, by applying the above expression, we can deduce that

$$\begin{aligned}\Omega(w_{n+1}, w_n, \tau) &\geq \Omega(w_{n+1}, w_n, \alpha\tau) = \Omega(\mathcal{F}w_n, \mathcal{F}w_{n-1}, \alpha\tau)\\ &\geq \Omega(w_n, w_{n-1}, \tau) = \Omega(\mathcal{F}w_{n-1}, \mathcal{F}w_{n-2}, \tau)\\ &\geq \Omega\left(w_{n-1}, w_{n-2}, \frac{\tau}{\alpha}\right) \geq \cdots \geq \Omega\left(w_1, w_0, \frac{\tau}{\alpha^n}\right),\end{aligned} \tag{48}$$

for all $n \in \mathbb{N}$, $p \geq 1$, and $\tau > 0$. Thus, we have

$$\Omega(w_n, w_{n+p}, \tau) \geq \Omega\left(w_n, w_{n+1}, \frac{\tau}{b}\right) * \Omega\left(w_{n+1}, w_{n+p}, \frac{\tau}{b}\right). \tag{49}$$

Continuing in this way, we get

$$\begin{aligned}\Omega(w_n, w_{n+p}, \tau) \geq \Omega\left(w_n, w_{n+1}, \frac{\tau}{b}\right) * \Omega\left(w_{n+1}, w_{n+2}, \frac{\tau}{b^2}\right) * \cdots * \Omega\\ \cdot \left(w_{n+p-1}, w_{n+p}, \frac{\tau}{b^{p-1}}\right).\end{aligned} \tag{50}$$

Using (48) in the above inequality, we deduce

$$\begin{aligned}\Omega(w_n, w_{n+p}, \tau) \geq \Omega\left(w_0, w_1, \frac{\tau}{b\alpha^n}\right) * \Omega\left(w_0, w_1, \frac{\tau}{b^2\alpha^{n+1}}\right) * \cdots * \Omega\\ \cdot \left(w_0, w_1, \frac{\tau}{b^{p-1}\alpha^{n+p-1}}\right).\end{aligned} \tag{51}$$

Here, b is an arbitrary positive integer. We know that $\lim_{n \longrightarrow \infty} \Omega(w, k, \tau) = 1, \forall w, k \in \mathcal{G}$ and $\tau > 0$, $\alpha \in (0, 1)$. So, from (51), we deduce that

$$\lim_{n \longrightarrow \infty} \Omega(w_n, w_{n+p}, \tau) = 1 * 1 * \cdots * 1 = 1, \quad \forall \tau > 0,\ p \geq 1. \tag{52}$$

Hence, $\{w_n\}$ is a Cauchy sequence. The hypothesis of completeness of the F-b-metric-l space $(\mathcal{G}, \Omega, *, b)$ ensures that there exists $v \in \mathcal{G}$ such that

$$\begin{aligned}\lim_{n \longrightarrow \infty} \Omega(w_n, v, \tau) &= \lim_{n \longrightarrow \infty} \Omega(w_n, w_{n+p}, \tau) = \Omega(v, v, \tau)\\ &= 1, \quad \forall \tau > 0,\ p \geq 1.\end{aligned} \tag{53}$$

Now, we derive that $v \in \mathcal{G}$ is a fixed point of $\mathcal{F}$. We have

$$\begin{aligned}\Omega(v, \mathcal{F}v, \tau) &\geq \Omega\left(v, w_{n+1}, \frac{\tau}{2b}\right) * \Omega\left(w_{n+1}, \mathcal{F}v, \frac{\tau}{2b}\right), \forall \tau > 0\\ &= \Omega\left(v, w_{n+1}, \frac{\tau}{2b}\right) * \Omega\left(\mathcal{F}w_n, \mathcal{F}v, \frac{\tau}{2b}\right)\\ &\geq \Omega\left(v, w_{n+1}, \frac{\tau}{2b}\right) * \Omega\left(w_n, v, \frac{\tau}{2b\alpha}\right).\end{aligned} \tag{54}$$

Taking limit as $n \longrightarrow +\infty$, and by (53), we get

$$\Omega(v, \mathcal{F}v, \tau) = 1 * 1 = 1. \tag{55}$$

Therefore, v is a fixed point of $\mathcal{F}$ and $F(v, v, \tau) = 1, \forall \tau > 0$.

Now, we investigate the uniqueness of fixed point. For this, assume that ν and v are two fixed points of $\mathcal{F}$. Then, by (44), we have

$$\begin{aligned}&\Omega(v, \nu, \tau) = \Omega(\mathcal{F}v, \mathcal{F}\nu, \tau) \geq \Omega\left(v, \nu, \frac{\tau}{\alpha}\right),\\ &\Omega(v, \nu, \tau) \geq \Omega\left(v, \nu, \frac{\tau}{\alpha}\right), \quad \forall \tau > 0\end{aligned} \tag{56}$$

We obtain

$$\Omega(v, \nu, \tau) \geq \Omega\left(v, \nu, \frac{\tau}{\alpha^n}\right), \quad \forall n \in \mathbb{N}. \tag{57}$$

Taking limit as $n \longrightarrow +\infty$ and using the fact $\lim_{\tau \longrightarrow \infty} \Omega(w, k, \tau) = 1$, so $v = \nu$; hence the fixed point is unique.

Example 8. Let $\mathcal{G} = [0, 1]$ and the t-norm be defined as $a * b = ab$. Also, Ω is defined as

$$\Omega(w, k, \tau) = e^{-(w+k)^2/\tau}, \quad \forall w, k \in \mathcal{G},\ \tau > 0. \tag{58}$$

Then, $(\mathcal{G}, \Omega, *, b)$ is a complete F-b-metric-l space. Define $\mathcal{F} : \mathcal{G} \longrightarrow \mathcal{G}$ by

$$\mathcal{F}w = \begin{cases} 0, & w \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{w}{6}, & w \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{59}$$

Now, $\lim_{\tau \longrightarrow \infty} \Omega(w, k, \tau) = \lim_{\tau \longrightarrow \infty} e^{-(w+k)^2/\tau} = 1$. For $\alpha \in [1/2, 1)$, we have four cases:

Case 1. If $w, k \in [0, 1/2]$, then $\mathscr{F}w = \mathscr{F}k = 0$. Here,

$$\Omega(\mathscr{F}w, \mathscr{F}k, \alpha\tau) = \Omega(0, 0, \alpha\tau) = e^0 \geq e^{-(w+k)^2/\tau} = \Omega(w, k, \tau). \tag{60}$$

Case 2. If $w \in [0, 1/2]$ and $k \in (1/2, 1]$, then $\mathscr{F}w = 0$ and $\mathscr{F}k = k/6$. We have

$$\begin{aligned}\Omega(\mathscr{F}w, \mathscr{F}k, \alpha\tau) &= \Omega\left(0, \frac{w}{6}, \alpha\tau\right) = e^{-(0+(k/6))^2/\alpha\tau} = e^{-\left(k^2/36\alpha\tau\right)} \\ &\geq e^{-(w+k)^2/\tau} = \Omega(w, k, \tau).\end{aligned} \tag{61}$$

Case 3. If $w, k \in (1/2, 1]$, then $\mathscr{F}w = w/6$ and $\mathscr{F}k = k/6$. Here,

$$\begin{aligned}\Omega(\mathscr{F}w, \mathscr{F}k, \alpha\tau) &= \Omega\left(\frac{w}{6}, \frac{k}{6}, \alpha\tau\right) = e^{-(w+k)^2/36\alpha\tau} \\ &\geq e^{-(w+k)^2/\tau} = \Omega(w, k, \tau).\end{aligned} \tag{62}$$

Case 4. If $w \in (1/2, 1]$ and $k \in [0, 1/2]$, then $\mathscr{F}w = w/6$ and $\mathscr{F}k = 0$. Then,

$$\Omega(\mathscr{F}w, \mathscr{F}k, \alpha\tau) = \Omega\left(\frac{w}{6}, 0, \alpha\tau\right) = e^{-w^2/36\alpha\tau} \geq e^{-(w+k)^2/\tau} = \Omega(w, k, \tau). \tag{63}$$

From all 4 cases, we obtain that

$$\Omega(\mathscr{F}w, \mathscr{F}k, \alpha\tau) \geq \Omega(w, k, \tau). \tag{64}$$

Hence, all conditions of Theorem 13 are satisfied, and 0 is the unique fixed point of $\mathscr{F}$. Also,

$$\Omega(v, v, \tau) = \Omega(0, 0, \tau) = e^0 = 1, \quad \forall\tau > 0. \tag{65}$$

3. An Application to an Integral Equation

Consider the following integral equation:

$$w(l) = \int_0^J \partial(l, r, w(r))dr, \tag{66}$$

where $J > 0$ and $\partial : [0, J] \times [0, J] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$.

Let $\mathcal{G} = C[0, J]$ be the set of all continuous real valued functions defined on $[0, J]$. Consider the b-metric-l given as

$$\mu(w, k) = (|w|+|k\,|\,)^2. \tag{67}$$

Then, by Proposition 8,

$$\Omega(w(l), k(l), \tau) = \max_{l\in[0,J]} e^{-(|w(l)|+|k(l)|)^2/\tau}, \quad \forall w, k \in \mathcal{G}. \tag{68}$$

Clearly, $(\mathcal{G}, \Omega, *, b)$ is a complete F-b-metric-l space. Let

$$\mathscr{F}w(l) = \int_0^J \partial(l, r, w(r))dr, \quad \forall w \in \mathcal{G}, \tag{69}$$

for $l \in [0, J]$. Observe that the existence of a solution of (66) is equivalent to the existence of a fixed point of $\mathscr{F}$.

Theorem 16. *Assume that the following hypotheses hold:*

(1) $\partial : [0, J] \times [0, J] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous

(2) $\forall l, r \in [0, J]$, there is a continuous function $\beta : [0, J] \times [0, J] \longrightarrow \mathbb{R}^+$ such that

$$|\partial(l, r, w(r)| + |\partial(l, r, k(r)| \leq \alpha^{1/2}\beta(l, r)(|w(l)| + |k(l)|, \tag{70}$$

where $\alpha \in (0, 1)$ and $b > 1$.

(3) $\lim_{\tau\longrightarrow\infty} \Omega(w(l), k(l), \tau) = 1$ and $\mathrm{Sup}_{l\in[0,J]} \int_0^J \beta(l, r)dr \leq 1$

Then, the integral Equation (66) has a unique solution.

Proof. For all $l \in [0, J]$, we have

$$\begin{aligned}\Omega(\mathscr{F}w(l), \mathscr{F}k(l), \alpha\tau) &= e^{-\left((|\mathscr{F}w(l)|+|\mathscr{F}k(l)|)^2/\alpha\tau\right)} \\ &= e^{-\left(\left(|\int_0^J \partial(l,r,w(r))dr|+|\int_0^J \partial(l,r,k(r))dr|\right)^2/\alpha\tau\right)} \\ &\geq e^{-\left(\left(\int_0^J |\partial(l,r,w(r))|dr+\int_0^J |\partial(l,r,k(r))|dr\right)^2/\alpha\tau\right)} \\ &= e^{-\left(\left(\int_0^J |\partial(l,r,w(r))|+|\partial(l,r,k(r))|dr\right)^2/\alpha\tau\right)} \\ &\geq e^{-\left(\left(\int_0^J \left(\alpha^{1/2}\beta(l,r)(|w(l)|+|k(l)|)dr\right)\right)^2/\alpha\tau\right)} \\ &= e^{-\left(\left(\int_0^J \left(\alpha^{1/2}\beta(l,r)(|w(l)|)+|k(l)|)^2\right)^{1/2}dr\right)^2/\alpha\tau\right)} \\ &= \mathrm{Sup}_{l\in[0,J]}\, e^{-\left(\alpha(|w(l)|+|k(l)|)^2\left(\int_0^J \beta(l,r)dr\right)^2/\alpha\tau\right)} \\ &\geq e^{-\left((|w(l)|+|k(l)|)^2/\tau\right)} = \Omega(w(l), k(l), \tau).\end{aligned} \tag{71}$$

Therefore, $\Omega(\mathscr{F}w(l), \mathscr{F}k(l), \alpha\tau) \geq \Omega(w(l), k(l), \tau)$. Also, observe that all conditions of Theorem 15 are satisfied. Hence, the operator $\mathscr{F}$ has a unique fixed point. This means that the integral Equation (66) has a unique solution.

Example 9. Consider the nonlinear integral equation below

$$w(l) = \frac{1}{8}\int_0^1 rw(r)dr. \tag{72}$$

Then, it has a solution in $\mathcal{G}$.

Proof. Let $\mathscr{F} : \mathcal{G} \longrightarrow \mathcal{G}$ be defined by $\mathscr{F}w(l) = (1/8)\int_0^1 rw(r)dr$. Set $\partial(l, r, w(r)) = (1/8)rw(r)$ in Theorem 16, we get

(i) $\partial : [0, J] \times [0, J] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous

(ii) $\forall l, r \in [0, J]$, there is a continuous function $\beta(l, r) = r$ such that

$$
\begin{aligned}
|\partial(l, r, w(r)| + |\partial(l, r, k(r)| &= \frac{1}{4} \times \frac{r}{2}[|w(r)| + k(r)] \\
&\leq \frac{1}{4} r[|w(r)| + k(r)] \\
&= \left(\frac{1}{16}\right)^{1/2} r[|w(r)| + k(r)] \\
&= \alpha^{1/2}\beta(l, r)(|w(l)| + |k(l)|,
\end{aligned} \tag{73}
$$

where $\alpha = (1/16) \in (0, (1/b))$

(iii) $\lim_{\tau \longrightarrow \infty} \Omega(w(l), k(l), \tau) = 1$ and $\operatorname{Sup}_{l \in [0, J]} \int_0^1 \beta(l, r) = \operatorname{Sup}_{l \in [0, J]} \int_0^1 r dr = (1/2) < 1$

Hence, all hypotheses of Theorem 16 are fulfilled. Therefore, the problem (72) has a solution on $\mho$.

4. Conclusion

Fixed point techniques are used to solve many mathematical problems, as differential and integral equations, integro-differential equations, game theory, and economics. The intent of this manuscript is to present a new space, so-called a fuzzy b-metric-like space. Topological properties and related examples are addressed, so our fixed point results are new. Ultimately, to illustrate the practical side of the theoretical results, a solution of a nonlinear integral equation is given.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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Research Article

Common Fixed Points of Two Mappings regarding a Generalized c-Distance over a Banach Algebra

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In this article, applying the concept of a generalized c-distance in cone b-metric spaces over Banach algebra with a nonnormal solid cone therein, we establish several common fixed point theorems for two noncontinuous mappings satisfying the Han-Xu-type contraction. Our results are interesting, since they are not equivalent to former well-known results regarding a wt-distance in b-metric spaces while they contain recent results corresponding to a generalized c-distance in cone b-metric spaces.

1. Introduction and Preliminaries

In 2015, Bao et al. [1] suggested a generalized c-distance, which extended many former definitions in [2–10]) and references therein. Moreover, with regard to a survey on fixed point theory corresponding to this distance, see [11–13]).

In 2013, Liu and Xu [14] offered a cone metric space over Banach algebra by replacing a Banach space E with a Banach algebra $\mathscr{E}$. After this definition, some other researchers suggested new several various spaces over a Banach algebra and extended available results in [15–17] and their references. In 2015, Huang et al. [18] proposed a c-distance in cone metric spaces over a Banach algebra $\mathscr{E}$. In 2018, Han and Xu [19] proved some common fixed point results by removing the assumption of continuity of the mappings and by deleting the hypothesis of the normality of the cone. Recently, Arabnia et al. [20] suggested a generalized c-distance in cone b-metric spaces over a Banach algebra.

Here, we review some basic definitions and preliminary lemmas which are needed to continue.

Let $\mathscr{E}$ be a Banach algebra with a unit element e, a zero element θ, a norm $\|x\|$, and a cone P therein. Define a partial order $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ iff $y - x \in P$. Also, $x \precnsim y$ if $x \preccurlyeq y$ and $x \neq y$, and $x \ll y$ iff $y - x \in \operatorname{int} P$ (int P is the same as the interior of P). If int $P \neq \emptyset$, then the cone P is named solid. Further, if there is $M > 0$ so that $\theta \preccurlyeq x \preccurlyeq y$ deduced that $\|x\| \leq M\|y\|$ for every $x, y \in \mathscr{E}$, then the cone P is named a normal cone.

Definition 1 (see [17]). Let $\mathfrak{X}$ be a nonempty set, $s \geq 1$ be a constant, and $\mathscr{E}$ be a Banach algebra. For all $x, y, z \in \mathfrak{X}$, assume that $d : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathscr{E}$ satisfies the following items: 2

$(\mathrm{d}_1)\theta \preccurlyeq \mathrm{d}(x, y)$ and $\mathrm{d}(x, y) = \theta$ iff $x = y$

$(\mathrm{d}_2)\mathrm{d}(x, y) = \mathrm{d}(y, x)$

$(\mathrm{d}_3)\mathrm{d}(x, z) \preccurlyeq s[\mathrm{d}(x, y) + \mathrm{d}(y, z)]$

Then, d is named a cone b-metric on $\mathfrak{X}$, and $(\mathfrak{X}, \mathrm{d})$ is named a cone b-metric space over the Banach algebra $\mathscr{E}$.

For definitions such as convergent and Cauchy sequences, c-sequence, completeness, continuity, and examples, see [16, 17]. In the sequel, assume that $(\mathfrak{X}, \mathrm{d})$ is a cone b-metric space with the coefficient $s \geq 1$ over a Banach algebra $\mathscr{E}$ and P is a solid cone therein.

Lemma 2 (see [17, 21]). *Let $\mathscr{E}$ be a Banach algebra and $u, v \in \mathscr{E}$. Then, the following items hold:*

(l_1) if $\rho(u) < |w|$ where $\rho(u)$ is the spectral radius and w is a complex constant, then $we - u$ is invertible in $\mathscr{E}$ and

$$(we - u)^{-1} = \sum_{i=0}^{\infty} \frac{u^i}{w^{i+1}}, \ \rho\left((we - u)^{-1}\right) \leq \frac{1}{|w| - \rho(u)} \tag{1}$$

(l_2)if u commutes with v, then $\rho(u+v)\leq\rho(u)+\rho(v)$ and $\rho(uv)\leq\rho(u)\rho(v)$

(l_3)if $u,k_1,k_2\in P$ with $k_1\preccurlyeq k_2$, $u\preccurlyeq k_1 u$, and $\rho(k_2)<1$, then $u=\theta$

(l_4)if $\rho(u)<1$, then $\{u^n\}$ is a c-sequence. Furthermore, $\{ku^n\}$ is a c-sequence for each arbitrarily vector $k\in P$

Definition 3 (see [20]). Let $(\mathfrak{X},d)$ be a cone b-metric space over a Banach algebra $\mathscr{E}$ with a constant $s\geq 1$. A function $\nu : \mathfrak{X}\times\mathfrak{X}\to\mathscr{E}$ is named a generalized c-distance on $\mathfrak{X}$ if, for all $x,y,z\in\mathfrak{X}$, it satisfies in the following properties:

$(\nu_1)\theta\preccurlyeq\nu(x,y)$

$(\nu_2)\nu(x,y)\preccurlyeq s[\nu(x,z)+\nu(z,y)]$

(ν_3) for $x\in\mathfrak{X}$ and a sequence $\{y_n\}$ converges to y in $\mathfrak{X}$, if $\nu(x,y_n)\preccurlyeq u$ for some $u=u_x\in P$ and all $n\geq 1$, then $\nu(x,y)\preccurlyeq su$

$(\nu_4)$for all $c\in\mathscr{E}$ with $\theta\ll c$, there exists $e\in\mathscr{E}$ with $\theta\ll e$ so that $\nu(z,x)\ll e$ and $\nu(z,y)\ll e$ imply $\mathrm{d}(x,y)\ll c$

Notice that a generalized c-distance contains both wt -distance and c-distance. Further, $\nu(x,y)=\nu(y,x)$ is not necessarily true and the $\nu(x,y)=\theta$ dose not imply that $x=y$ for every $x,y\in\mathfrak{X}$.

Example 4. Take $\mathfrak{X}=[0,1]$, $\mathscr{E}=C^1_{\mathbb{R}}[0,1]$ with $\|f\|=\|f\|_\infty+\|f'\|_\infty$. Let multiplication in $\mathscr{E}$ be just pointwise multiplication. Then, $\mathscr{E}$ is a Banach algebra with a unit $e(t)=1$ for all $t\in[0,1]$. Also, let $P=\{f\in\mathscr{A}\mid f(t)\geq 0,\ \forall t\in[0,1]\}$ be a solid cone. Now, define $d:X\times X\to P\subset\mathscr{A}$ by $d(a,b)(t)=|a-b|^2 2^t$ for all $a,b\in\mathfrak{X}$, where $2^t\in P$. Then, (X,d) is a cone b -metric space over a Banach algebra $\mathscr{E}$. Consider a mapping $\nu:\mathfrak{X}\times\mathfrak{X}\to\mathscr{E}$ by $\nu(a,b)(t)=(a^2+b^2)2^t$ for all $a,b\in\mathfrak{X}$. Then, ν is a generalized c-distance on $\mathfrak{X}$.

Lemma 5 (see [20]). *Consider a generalized c-distance ν on $\mathfrak{X}$ with two sequences $\{t_n\}$ and $\{s_n\}$ in $\mathfrak{X}$, $\alpha,\beta,\gamma\in\mathfrak{X}$, and $\{p_n\}$ and $\{q_n\}$ be two c-sequences. Then, the following cases hold:*

(1) if $\nu(t_n,\beta)\preccurlyeq p_n$ and $\nu(t_n,\gamma)\preccurlyeq q_n$ for $n\in\mathbb{N}$, then $\beta=\gamma$. In particular, if $\nu(\alpha,\beta)=\theta$ and $\nu(\alpha,\gamma)=\theta$, then $\beta=\gamma$

(2) if $\nu(t_n,s_n)\preccurlyeq p_n$ and $\nu(t_n,\gamma)\preccurlyeq q_n$ for $n\in\mathbb{N}$, then $\{s_n\}$ converges to γ

(3) if $\nu(t_n,t_m)\preccurlyeq p_n$ for $m>n$, then $\{t_n\}$ is a Cauchy sequence in $\mathfrak{X}$

(4) if $\nu(\beta,t_n)\preccurlyeq p_n$ for $n\in\mathbb{N}$, then $\{t_n\}$ is a Cauchy sequence in $\mathfrak{X}$

Lemma 6 (see [20]). *Let ν be a generalized c-distance on $\mathfrak{X}$. If $\nu(\alpha,\beta)=\nu(\beta,\alpha)=\theta$ for $\alpha,\beta\in\mathfrak{X}$, then $\alpha=\beta$.*

In this work, we establish several common fixed point theorems regarding a generalized c-distance over a Banach algebra by removing the normality of the cone and the continuity of the mappings.

2. Main Results

The following theorem is the principal result of this paper using Han-Xu-type contraction [19].

Theorem 7. *Consider a generalized c-distance ν on a complete cone b-metric space $(\mathfrak{X},d)$ over a Banach algebra $\mathscr{E}$. Assume that two mappings $F,G:\mathfrak{X}\to\mathfrak{X}$ for every $a,b\in\mathfrak{X}$ satisfy the following relations:*

$$\nu(Fa,Gb)\preccurlyeq h_1\nu(a,b)+h_2\nu(a,Fa)+h_3\nu(a,Gb), \tag{2}$$

$$\nu(Ga,Fb)\preccurlyeq h_1\nu(a,b)+h_2\nu(a,Ga)+h_3\nu(a,Fb), \tag{3}$$

where $h_1,h_2,h_3\in P$ so that sh_3 commutes with $(h_1+h_2+sh_3)$ and

$$\rho(sh_3)+\rho\left(sh_1+sh_2+s^2h_3\right)<1. \tag{4}$$

Then, F and G have a unique common fixed point.

Proof. Assume that $a_0\in\mathfrak{X}$ is an arbitrary point with $Fa_0\neq a_0$. Consider the sequence $\{a_n\}$ by putting $a_{2n+1}=Fa_{2n}$ and $a_{2n+2}=Ga_{2n+1}$ for all $n\in\mathbb{N}$. Applying relation (2) by $a=a_{2n}$ and $b=a_{2n+1}$, we get

$$\begin{aligned}\nu(a_{2n+1},a_{2n+2})&=\nu(Fa_{2n},Ga_{2n+1})\preccurlyeq h_1\nu(a_{2n},a_{2n+1})\\&\quad+h_2\nu(a_{2n},Fa_{2n})+h_3\nu(a_{2n},Ga_{2n+1})\\&\preccurlyeq h_1\nu(a_{2n},a_{2n+1})+h_2\nu(a_{2n},a_{2n+1})\\&\quad+sh_3[\nu(a_{2n},a_{2n+1})+\nu(a_{2n+1},a_{2n+2})],\end{aligned} \tag{5}$$

for all $n\in\mathbb{N}$, which induces that

$$(e-sh_3)\nu(a_{2n+1},a_{2n+2})\preccurlyeq(h_1+h_2+sh_3)\nu(a_{2n},a_{2n+1}). \tag{6}$$

Similarly, applying relation (3) by $a=a_{2n+1}$ and $b=a_{2n+2}$, we get

$$\begin{aligned}\nu(a_{2n+2},a_{2n+3})&=\nu(Ga_{2n+1},Fa_{2n+2})\\&\preccurlyeq h_1\nu(a_{2n+1},a_{2n+2})+h_2\nu(a_{2n+1},a_{2n+2})\\&\quad+sh_3[\nu(a_{2n+1},a_{2n+2})+\nu(a_{2n+2},a_{2n+3})],\end{aligned} \tag{7}$$

for all $n\in\mathbb{N}$, which induces that

$$(e-sh_3)\nu(a_{2n+2},a_{2n+3})\preccurlyeq(h_1+h_2+sh_3)\nu(a_{2n+1},a_{2n+2}). \tag{8}$$

Now, the inequalities (6) and (8) show that

$$(e-sh_3)\nu(a_n,a_{n+1})\preccurlyeq(h_1+h_2+sh_3)\nu(a_{n-1},a_n). \tag{9}$$

Since $\rho(sh_3)<1$ (by relation (4)), it follows from Lemma 2, (l_1), that $(e-sh_3)$ is invertible and $(e-sh_3)^{-1}=\sum_{i=0}^{\infty}(sh_3)^i$. Let $h=(e-sh_3)^{-1}(h_1+h_2+sh_3)$. Since sh_3 commutes with $h_1+h_2+sh_3$, we obtain

$$(e - sh_3)^{-1}(h_1 + h_2 + sh_3) = (h_1 + h_2 + sh_3)(e - sh_3)^{-1}. \quad (10)$$

Now, set $h = (e - sh_3)^{-1}(h_1 + h_2 + sh_3)$. Then, by Lemma 2, (l_1) and (l_2), we obtain

$$\begin{aligned}\rho(h) &= \rho\left((e - sh_3)^{-1}(h_1 + h_2 + sh_3)\right) \\ &\leq \frac{1}{1 - \rho(sh_3)}\rho(h_1 + h_2 + sh_3) < \frac{1}{s},\end{aligned} \quad (11)$$

which implies that $(e - sh)^{-1} = \sum_{i=0}^{\infty}(sh)^i$. Moreover, by multiplying $(e - sh_3)^{-1}$ in relation (9), we get

$$\begin{aligned}\nu(a_n, a_{n+1}) &\preccurlyeq (e - sh_3)^{-1}(h_1 + h_2 + sh_3)\nu(a_{n-1}, a_n) \\ &= h\nu(a_{n-1}, a_n) \preccurlyeq \cdots \preccurlyeq h^n \nu(a_0, a_1).\end{aligned} \quad (12)$$

Consider $m, n \in \mathbb{N}$ with $m > n \geq 1$. Using relation (12) and (ν_2), we deduce by a simple computation that

$$\nu(a_n, a_m) \preccurlyeq (e - sh)^{-1} sh^n \nu(a_0, a_1). \quad (13)$$

Since $\rho(h) < 1/s$ and $s \geq 1$, we have $\rho(h) < 1$ which means that $\{h^n\}$ is a c-sequence by Lemma 2, (l_4). Using Lemma 5, (3), $\{a_n\}$ is a Cauchy sequence. Due to the completeness of the space $\mathfrak{X}$, there is a $u \in \mathfrak{X}$ so that $a_n \to u$ as $n \to \infty$. Using relation (13) and (ν_3), we have

$$\nu(a_n, u) \preccurlyeq (e - sh)^{-1} s^2 h^n \nu(a_0, a_1), \quad (14)$$

which shows that

$$\nu(a_{2n+1}, u) \preccurlyeq (e - sh)^{-1} s^2 h^{2n+1} \nu(a_0, a_1), \quad (15)$$

$$\nu(a_{2n}, u) \preccurlyeq (e - sh)^{-1} s^2 h^{2n} \nu(a_0, a_1). \quad (16)$$

Now, we establish that $Fu = Gu = u$. In relation (2), set $a = a_{2n}$ and $b = u$. Then, we get

$$\begin{aligned}\nu(a_{2n+1}, Gu) = \nu(Fa_{2n}, Gu) &\preccurlyeq h_1\nu(a_{2n}, u) + h_2\nu(a_{2n}, a_{2n+1}) \\ &\quad + sh_3[\nu(a_{2n}, a_{2n+1}) + \nu(a_{2n+1}, Gu)],\end{aligned} \quad (17)$$

which induces that $(e - sh_3)\nu(a_{2n+1}, Gu) \preccurlyeq h_1\nu(a_{2n}, u) + (h_2 + sh_3)\nu(a_{2n}, a_{2n+1})$. Note that $e - sh_3$ is invertible. Thus, by the inequalities (12) and (16), we get

$$\begin{aligned}\nu(a_{2n+1}, Gu) &\preccurlyeq (e - sh_3)^{-1}[h_1\nu(a_{2n}, u) + (h_2 + sh_3)\nu(a_{2n}, a_{2n+1})] \\ &\preccurlyeq (e - sh_3)^{-1}\left[h_1(e - sh)^{-1}s^2h^{2n}\nu(a_0, a_1) + (h_2 + sh_3)sh^{2n}\nu(a_0, a_1)\right] \\ &= (e - sh_3)^{-1}\left[h_1(e - sh)^{-1}s^2 + s(h_2 + sh_3)\right]h^{2n}\nu(a_0, a_1).\end{aligned} \quad (18)$$

By considering the inequalities (15) and (18), Lemma 2, (l_4), and Lemma 5, (1), we conclude that $Gu = u$. Now, in relation (3), set $a = a_{2n+1}$ and $b = u$. Then, we get

$$\begin{aligned}\nu(a_{2n+2}, Fu) = \nu(Ga_{2n+1}, Fu) &\preccurlyeq h_1\nu(a_{2n+1}, u) \\ &\quad + h_2\nu(a_{2n+1}, a_{2n+2}) \\ &\quad + sh_3[\nu(a_{2n+1}, a_{2n+2}) + \nu(a_{2n+2}, Fu)],\end{aligned} \quad (19)$$

which induces that $(e - sh_3)\nu(a_{2n+2}, Fu) \preccurlyeq h_1\nu(a_{2n+1}, u) + (h_2 + sh_3)\nu(a_{2n+1}, a_{2n+2})$. Note that $e - sh_3$ is invertible. Thus, by the inequalities (12) and (15), we get

$$\begin{aligned}\nu(a_{2n+2}, Fu) &\preccurlyeq (e - sh_3)^{-1}[h_1\nu(a_{2n+1}, u) \\ &\quad + (h_2 + sh_3)\nu(a_{2n+1}, a_{2n+2})] \\ &\preccurlyeq (e - sh_3)^{-1}\left[h_1(e - sh)^{-1}s^2h^{2n+1}\nu(a_0, a_1)\right. \\ &\quad \left.+ (h_2 + sh_3)sh^{2n+1}\nu(a_0, a_1)\right] \\ &= (e - sh_3)^{-1}\left[h_1(e - sh)^{-1}s^2\right. \\ &\quad \left.+ s(h_2 + sh_3)\right]h^{2n+1}\nu(a_0, a_1).\end{aligned} \quad (20)$$

By considering the inequalities (16) and (20), Lemma 2, (l_4), and Lemma 5, (1), we conclude that $Fu = u$. Consequently, $Fu = Gu = u$; that is, u is a common fixed point of F and G. Also, by using the relation (2), we have

$$\begin{aligned}\nu(u, u) = \nu(Fu, Gu) &\preccurlyeq h_1\nu(u, u) \\ &\quad + h_2\nu(u, Fu) + h_3\nu(u, Gu) \\ &= (h_1 + h_2 + h_3)\nu(u, u),\end{aligned} \quad (21)$$

which induces that $(e - h_1 - h_2 - h_3)\nu(u, u) \preccurlyeq \theta$. Now, notice that $h_1 + h_2 + h_3 \preccurlyeq sh_1 + sh_2 + s^2h_3$. Thus, by relation (4), $(e - h_1 - h_2 - h_3)$ is invertible. Hence, by Lemma 2, (l_3), we have $\nu(u, u) = \theta$. Next, we prove that the common fixed point of F and G is unique. Assume that v is another common fixed point F and G. It follows from relation (2) that

$$\begin{aligned}\nu(u, v) = \nu(Fu, Gv) &\preccurlyeq h_1\nu(u, v) \\ &\quad + h_2\nu(u, Fu) + h_3(u, Gv) \\ &= (h_1 + h_3)\nu(u, v).\end{aligned} \quad (22)$$

Since $h_1 + h_3 \preccurlyeq sh_1 + sh_2 + s^2h_3$ and by using relation (4), we have $\nu(u, v) = \theta$ by Lemma 2, (l_3). Also, it follows from relation (3) that

$$\begin{aligned}\nu(v, u) = \nu(Gv, Fu) &\preccurlyeq h_1\nu(v, u) \\ &\quad + h_2\nu(v, Gv) + h_3(v, Fu) \\ &= (h_1 + h_3)\nu(v, u),\end{aligned} \quad (23)$$

which implies by the above procedure that $\nu(v, u) = \theta$. Now, by Lemma 6, we obtain $u = v$. Consequently, the common fixed point of F and G is unique. Here, the proof ends.

Corollary 8. *Consider a generalized c-distance ν on a complete cone b-metric space $(\mathfrak{X}, d)$ over a Banach algebra $\mathscr{E}$. Assume that two mappings $F, G : \mathfrak{X} \to \mathfrak{X}$ for every $a, b \in \mathfrak{X}$ satisfy the following relations:*

$$\nu(Fa, Gb) \preccurlyeq h_1 \nu(a, b) + h_2 \nu(a, Fa),$$
$$\nu(Ga, Fb) \preccurlyeq h_1 \nu(a, b) + h_2 \nu(a, Ga), \tag{24}$$

where $h_1, h_2 \in P$ with $\rho(sh_1 + sh_2) < 1$. Then, F and G have a unique common fixed point.

Proof. It is sufficient to set $h_3 = \theta$ in Theorem 7.

Corollary 9. *Consider a generalized c-distance ν on a complete cone b-metric space $(\mathfrak{X}, d)$ over a Banach algebra $\mathscr{E}$. Assume that a mapping $F : \mathfrak{X} \to \mathfrak{X}$ for every $a, b \in \mathfrak{X}$ satisfies the following relation:*

$$\nu(Fa, Fb) \preccurlyeq h_1 \nu(a, b) + h_2 \nu(a, Fa), \tag{25}$$

where $h_1, h_2 \in P$ with $\rho(sh_1 + sh_2) < 1$. Then, F has a unique fixed point.

Proof. It follows by taking $F = G$ in Corollary 8.

Corollary 10. *Consider a generalized c-distance ν on a complete cone b-metric space $(\mathfrak{X}, d)$ over a Banach algebra $\mathscr{E}$. Assume that a mapping $F : \mathfrak{X} \to \mathfrak{X}$ for every $a, b \in \mathfrak{X}$ satisfies the following relation:*

$$\nu(Fa, Fb) \preccurlyeq h_1 \nu(a, b), \tag{26}$$

where $h_1 \in P$ with $\rho(h_1) < 1/s$. Then, F has a unique fixed point.

Proof. It is sufficient to set $h_2 = \theta$ in Corollary 9.

Example 11. Let $\mathfrak{X} = [0, 1]$, $\mathscr{E} = C^1_{\mathbb{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and multiplication in $\mathscr{E}$ be just pointwise multiplication. Then, $\mathscr{E}$ is a real Banach algebra with a unit $e(t) = 1$ for all $t \in [0, 1]$. Take a solid cone $P = \{f \in \mathscr{E} \mid f(t) \geq 0 \text{ for all } t \in [0, 1]\}$ and define the cone b-metric $d : \mathfrak{X} \times \mathfrak{X} \to P \subseteq \mathscr{E}$ by $d(a, b) = |a - b|^s 2^t$, where $2^t \in P \subset \mathscr{E}$ and $s = 2$. Consider a mapping $\nu : \mathfrak{X} \times \mathfrak{X} \to \mathscr{E}$ by $\nu(a, b)(t) = b^2 2^t$ for all $a, b, t \in \mathfrak{X}$. Then, ν is a generalized c-distance in cone b-metric space d over Banach algebra $\mathscr{E}$. Take $h_1 = 2/121 + (3/121)t$ and define the mapping $F : \mathfrak{X} \to \mathfrak{X}$ by

$$F(a) = \begin{cases} \dfrac{\sqrt{2}}{11} a, & a \in \mathbb{Q} \cap \mathfrak{X}, \\ \dfrac{\sqrt{2}}{12} a, & \text{otherwise}. \end{cases} \tag{27}$$

Clearly, F is not continuous. Also,

$$\rho(sh_1) = \frac{10}{121} < 1. \tag{28}$$

On the other hand, we have the following two cases:

(i) for all $a \in \mathfrak{X}$ and $b \in \mathbb{Q} \cap \mathfrak{X}$, we get

$$\nu(Fa, Fb)(t) = (Fb)^2 2^t = \frac{2}{121} b^2 2^t \preccurlyeq h_1 \nu(a, b)(t) \tag{29}$$

(ii) for all $a \in \mathfrak{X}$ and $b \in \mathbb{Q} \cap \mathfrak{X}$, we get

$$\nu(Fa, Fb)(t) = (Fb)^2 2^t = \frac{2}{144} b^2 2^t \preccurlyeq h_1 \nu(a, b)(t) \tag{30}$$

That is, all hypotheses of Corollary 10 are held. Thus, F has a unique fixed point at $a = 0$.

Corollary 12. *Consider a generalized c-distance ν on a complete cone b-metric space $(\mathfrak{X}, d)$. Assume that two mappings $F, G : \mathfrak{X} \to \mathfrak{X}$ for every $a, b \in \mathfrak{X}$ satisfy the following relations:*

$$\nu(Fa, Gb) \preccurlyeq h_1 \nu(a, b) + h_2 \nu(a, Fa) + h_3 \nu(a, Gb),$$
$$\nu(Ga, Fb) \preccurlyeq h_1 \nu(a, b) + h_2 \nu(a, Ga) + h_3 \nu(a, Fb), \tag{31}$$

where $h_1, h_2, h_3 \in P$ so that $s(h_1 + h_2) + (s^2 + s)h_3 < 1$. Then, F and G have a unique common fixed point.

Proof. In Theorem 7, put $\rho(h_1) = h_1$, $\rho(h_2) = h_2$, and $\rho(h_3) = h_3$. The proof is evident.

Remark 13. In Theorem 7 and its corollaries, we take $s = 1$. Then, we obtain the same Theorem 16 and its next corollaries from Han and Xu [19] regarding a c-distance ν over a Banach algebra $\mathscr{E}$. Also, these results generalize some main theorems and its next corollaries in [1, 3, 12, 13, 15, 18, 20].

3. Conclusions

In this paper, we established several fixed point results for two mappings F and G regarding a generalized c-distance ν over a Banach algebra $\mathscr{E}$. Notice that the class of these distances is bigger than the class of usual c-distances over the same Banach algebra. Also, this class is not equivalent to the class of wt-distances in b-metric spaces. Further, we removed the continuity condition of the mappings F and G in expressing our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Estimation of Newly Established Iterative Scheme for Generalized Nonexpansive Mappings

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We introduce a new iterative method in this article, called the D iterative approach for fixed point approximation. Analytically, and also numerically, we demonstrate that our established D I.P is faster than the well-known I.P of the prior art. Finally, in a uniformly convex Banach space environment, we present weak as well as strong convergence theorems for Suzuki's generalized nonexpansive maps. Our findings are an extension, refinement, and induction of several existing iterative literatures.

1. Introduction and Preliminaries

In many branches of mathematics and various sciences, the existence of a fixed point is crucial. A completely fine synthesis of analysis, geometry, and topology is the fixed point theory. In particular, fixed point techniques are used in economics, biology, engineering, biochemistry, game theory, and physics. So, once the presence of a fixed point is found, it is difficult to find its value; that is why we use an I.P to calculate them. A variety of I.Ps have been introduced and it is not possible to discuss all I.Ps. The famous Banach contraction theorem uses the Picard I.P to approximate fixed points. Other well-known I.Ps can be found in references [1–8, 15–18]. The approximation speed of the I.P has a major role, and an I.P tends to be chosen in another iterative process. In [2], the writer believes that the approximation rate of the Agarwal I.P is like that of the Picard I.P and is faster than the contraction mapping of the Mann I.P. In [9], the writers help by numerical examples to prove for nonexpansive mapping, that the approximation rate of the Picard-S I.P is better than existing literature. They proved that the convergence rate was good. In [6], the creators illustrate that the approximation rate of the M^* I.P is better than existing literature. Recently, in [7], another I.P, namely, M I.P, was developed, and it is shown that its approximation rate is better than existing literature. In [10], one more I.P called K I.P was developed, in which they have shown that the approximation rate is better than previous literature. By hypothesis, we introduce a new I.P, namely, the D iterative process.

Numerically, the approximation rate for the latest I.P is contrasted with the Agarwal I.P, Picard-S I.P, M I.P, M^* I.P, and K I.P. We present the weak as well as strong convergence theorems of Suzuki generalized nonexpansive maps and contraction map for our newly developed I.P. We first remember those definitions, ideas, and lemmas which we have to use in the upcoming two sections. A Banach space X is referred to as uniformly convex [11] if $\forall \varepsilon \in (0, 2] \exists \delta > 0$ s.t for $\xi, \eta \in X$,

$$\left.\begin{array}{l} \|\xi\| \le 1 \\ \|\eta\| \le 1 \le 1 \\ \|\xi - \eta\| > \varepsilon > \varepsilon \end{array}\right\} \Rightarrow \left\| \frac{\xi + \eta}{2} \right\| \le \delta. \quad (1)$$

X is referred to obey the Opial property [4] if $\forall \{\xi_n\} \in X$, approaching weakly to $\xi \in X$, we have

$$\lim_{n\to\infty} \sup\|\xi_n - \xi\| < \lim_{n\to\infty} \sup\|\xi_n - \eta\|, \quad (2)$$

$$\forall \eta \in X \text{ s.t } \eta \neq \xi. \quad (3)$$

If $F(p) = p$, $T(F)$ denotes the set of all fixed points of F; a point p is referred to as the fixed point of mapping F.

Let $C \neq \phi \subseteq X$, and $F : C \to C$ is referred as contraction if $\exists \beta \in (0, 1)$ s.t $\|F\xi - F\eta\| \leq \beta\|\xi - \eta\|$, $\forall \xi, \eta \in C$.

$F : C \to C$ is referred to as nonexpansive if $\|F\xi - F\eta\| \leq \|\xi - \eta\| \forall \xi, \eta \in C$, and quasinonexpansive if $\forall \xi \in C$ and $p \in T(F)$, so $\|F\xi - p\| \leq \|\xi - p\|$. In [12], $F : C \to C$ is referred to satisfy condition (M) if $\forall \xi, \eta \in C$, we have

$$\frac{1}{2}\|\xi - F\xi\| \leq \|\xi - \eta\| \Rightarrow \|F\xi - F\eta\| \leq \|\xi - \eta\|. \quad (4)$$

In [12], the author approved that the mapping holding condition (M) is weaker than the nonexpansive mapping and stronger than the quasinonexpansive mapping. The mapping which holds condition (M) is called the Suzuki generalized nonexpansive mapping. Reference [12] includes the fixed point theorem and convergence theorem for Suzuki's generalized nonexpansive mapping. Recently, the fixed point theorem of the Suzuki generalized nonexpansive mapping has been studied by many authors [5–7]. Now, we define properties of Suzuki generalized nonexpansive mappings.

Proposition 1. *Let $C \neq \phi \subseteq X$ and $F : C \to C$. Then,*

(i) ([12], Proposition 1) If F is nonexpansive, so F must be a Suzuki generalized nonexpansive

(ii) ([12], Proposition 2) If F is a Suzuki generalized nonexpansive having a fixed point, then F must be a quasinonexpansive

(iii) ([12], Lemma 7) If F is a Suzuki generalized nonexpansive, then $\|\xi - F_\eta\| \leq 3\|F\xi - \xi\| + \|\xi - \eta\| \forall \xi, \eta \in C$.

Lemma 2 ([12], Proposition 3). *Let $F : C \to C$ which satisfies the Opial property. Suppose that F is a Suzuki generalized nonexpansive. If $\{\xi_n\}$ approaches weakly to t and $\lim_{n\to\infty}\|F\xi_n - \xi_n\| = 0$, then $Ft = t$. So, I-F is demiclosed at zero.*

Lemma 3 ([12], Theorem 5). *Let $F : C \to C$ and F is a Suzuki generalized nonexpansive. Then F must have a fixed point.*

Lemma 4 ([13], Lemma 1.3). *Let $\{t_n\}$ be any real sequence in X s.t $0 < p \leq t_n \leq q < 1 \forall n \geq 1$. Let $\{\xi_n\}$ and $\{\eta_n\}$ be any two sequences of X s.t $\limsup_{n\to\infty}\|\xi_n\| \leq r$, $\limsup_{n\to\infty}\|\eta_n\| \leq r$, and $\limsup_{n\to\infty}\|t_n\xi_n + (1 - t_n)\eta_n\| = r$ hold for some $r \geq 0$. Then, $\lim_{n\to\infty}\|\xi_n - \eta_n\| = 0$.*

Let $C \neq \phi \subseteq X$, and let $\{\xi_n\}$ be a bounded sequence $\in X$. For $\xi \in X$, we set

$$r(\xi, \{\xi_n\}) = \limsup_{n\to\infty}\|\xi_n - \xi\|. \quad (5)$$

The asymptotic radius of $\{\xi_n\}$ relative to C is given by

$$r(C, \{\xi_n\}) = \inf \{r(\xi, \{\xi_n\}) : \xi \in C\}, \quad (6)$$

and the asymptotic center of $\{\xi_n\}$ relative to C is defined as

$$A(C, \{\xi_n\}) = \{\xi \in C : r(\xi, \{\xi_n\}) = r(C, \{\xi_n\}). \quad (7)$$

Definition 5 (see [2]). Let $\{\xi_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ be two fixed point I.P sequences that approach the single fixed point p and $\|\xi_n - p\| \leq u_n$ and $\|\eta_n - p\| \leq v_n$, $\forall n \geq 0$. If the sequence $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ approaches to u and v,respectively, and $\lim_{n\to\infty}\|u_n - u\|/\|v_n - v\| = 0$, then $\{\xi_n\}_{n=0}^{\infty}$ approaches faster as $\{\eta_n\}_{n=0}^{\infty}$ to p.

2. The D Iteration Process

In this section, let $n \geq 0$ and $\{\theta_n\}$ and $\{\vartheta_n\}$ are real sequences $\in [0, 1]$ and $C \neq \phi \subseteq X$. For detail on I.P, please see [14].

Following is two step Agrawal I.P or S I.P as

$$\begin{cases} \xi_0 \in C, \\ \eta_n = (1 - \vartheta_n)\xi_n + \vartheta_n F\xi_n, \\ \xi_{n+1} = (1 - \theta_n)F\xi_n + \theta_n F\eta_n. \end{cases} \quad (8)$$

In [9], the authors develop a new I.P called Picard-S I.P as given below:

$$\begin{cases} \xi_0 \in C, \\ \omega_n = (1 - \vartheta_n)\xi_n + \vartheta_n F\xi_n, \\ \eta_n = (1 - \theta_n)F\xi_n + \theta_n T\omega_n, \\ \xi_{n+1} = F\eta_n. \end{cases} \quad (9)$$

They have demonstrated that with Picard-S I.P, we approximate the fixed point of contraction mapping. Also, by a numerical example, they proved that the Picard-S I.P have a better approximation rate than all previously developed I.P.

In [7], the authors introduced a new I.P, namely, M I.P, which is given below:

$$\begin{cases} \xi_0 \in C, \\ \omega_n = (1 - \theta_n)\xi_n + \theta_n F\xi_n, \\ \eta_n = F\omega_n, \\ \xi_{n+1} = F\eta_n. \end{cases} \quad (10)$$

In [6], the authors introduced one new I.P, namely, M^* I.P, which is defined as

$$\begin{cases} \xi_0 \in C, \\ \omega_n = (1-\vartheta_n)\xi_n + \vartheta_n F\xi_n, \\ y_n = F((1-\theta_n)\xi_n + \theta_n F\omega_n), \\ \xi_{n+1} = F\eta_n. \end{cases} \quad (11)$$

They proved that each of (10) and (11) is moving faster as compared to (8) and (9).

In [10], the authors introduced a three-step I.P, called K I.P, which is given below:

$$\begin{cases} \xi_0 \in C, \\ \omega_n = (1-\vartheta_n)\xi_n + \vartheta_n F\xi_n, \\ \eta_n = F((1-\theta_n)F\xi_n + \theta_n F\omega_n), \\ \xi_{n+1} = F\eta_n. \end{cases} \quad (12)$$

They insisted that the new I.P converged quickly. By providing an example, they showed that the K I.P has a better approximating rate than the S I.P., Picard-S I.P, M I.P, and M^* I.P.

In this competition, we developed following (new) three-step I.P, namely, D I.P, defined by

$$\begin{cases} \xi_0 \in C, \\ \omega_n = F((1-\vartheta_n)\xi_n + \vartheta_n F\xi_n), \\ \eta_n = F((1-\theta_n)F\xi_n + \theta_n F\omega_n), \\ \xi_{n+1} = F\eta_n. \end{cases} \quad (13)$$

In order to prove that our new I.P (13) have a better approximation rate as compared to (8),(9), (10), (11), and (12), first, we generally prove that our I.P strongly converges to unique fixed point, and then it is supported with a numerical example.

Theorem 6. *Let the contraction map $F : C \to C$, where $C \neq \phi \subseteq X$. Assume $\{\xi_n\}_{n=0}^{\infty}$ to be an iterative sequence generated by D I.P, where $\{\theta_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty} \in [0\ 1]$ are real sequences satisfying $\Sigma_{n=0}^{\infty}\theta_n = \infty$ or $\Sigma_{n=0}^{\infty}\vartheta_n = \infty$. Then, $\{\xi_n\}_{n=0}^{\infty}$ approaches strongly to a unique fixed point of F.*

Proof. As F is a contraction in X, so F must have a unique fixed point in C. Assume p is a particular fixed point of F. From D I.P, we get

$$\begin{aligned} \|\omega_n - p\| &= \|F((1-\vartheta_n)\xi_n + \vartheta_n F\xi_n) - Fp\| \\ &\leq k\|(1-\vartheta_n)\xi_n + \vartheta_n F\xi_n - p\| \\ &\leq k\|(1-\vartheta_n)(\xi_n - p) + \beta_n(F\xi_n - p)\| \\ &\leq k(1-\vartheta_n)\|\xi_n - p\| + \vartheta_n\|F\xi_n - p\| \\ &\leq k\{(1-\vartheta_n)\|\xi_n - p\| + k\vartheta_n\|\xi_n - p\| \\ &\leq k\{1-\vartheta_n(1-k)\}\|\xi_n - p\|. \end{aligned} \quad (14)$$

Now,

$$\begin{aligned} \|\eta_n - p\| &= \|F((1-\theta_n)F\xi_n + \theta_n F\omega_n) - Fp\| \\ &\leq k[(1-\theta_n)\|F\xi_n - p\| + \theta_n\|F\omega_n - p\|] \\ &\leq k[(1-\theta_n)k\|\xi_n - p\| + \theta_n k\|\omega_n - p\|] \\ &\leq k^2[(1-\theta_n)\|(\xi_n - p)\| + \theta_n\|\omega_n - p\|] \\ &\leq k^2[(1-\theta_n)\|(\xi_n - p)\| + \theta_n(k\{1-\vartheta_n(1-k)\}\|\xi_n - p\|) \\ &\leq k^2[1-(\theta_n + k\theta_n\vartheta_n)(1-k)]\|(\xi_n - p)\|. \end{aligned} \quad (15)$$

Then,

$$\begin{aligned} \|\xi_{n+1} - p\| &= \|F\eta_n - Fp\| \leq k\|\eta_n - p\| \\ &\leq k^3[1-(\theta_n + k\theta_n\vartheta_n)(1-k)]\|(\xi_n - p)\|. \end{aligned} \quad (16)$$

After repetition, we get

$$\begin{gathered} \|\xi_n - p\| \leq k^3[1-(\theta_{n-1} + k\theta_{n-1}\vartheta_{n-1})(1-k)]\|(\xi_{n-1} - p)\| \\ \|\xi_{n-1} - p\| \leq k^3[1-(\theta_{n-2} + k\theta_{n-2}\vartheta_{n-2})(1-k)]\|(\xi_{n-2} - p)\| \\ \vdots \\ \|\xi_{n-2} - p\| \leq k^3[1-(\theta_{n-3} + k\theta_{n-3}\vartheta_{n-3})(1-k)]\|(\xi_{n-3} - p)\| \\ \|\xi_1 - p\| \leq k^3[1-(\theta_0 + k\theta_0\vartheta_0)(1-k)]\|(\xi_0 - p). \end{gathered} \quad (17)$$

Therefore, we obtain $\|\xi_{n+1} - p\| \leq k^{3(n+1)}\|(\xi_0 - p)\|\prod_{i=0}^{n}[1 - (\theta_i + k\theta_i\vartheta_i)(1-k)]$. Now, $k < 1$ so $(1-k) > 0$ and $\theta_n, \vartheta_n \leq 1 \forall n \in N$. Thus, we get $[1-(\theta_i + k\theta_i\vartheta_i)(1-k)] < 1$ and $\forall n \in N$. As $1 - x \leq e^{-x}$, $\forall \xi \in [0\ 1]$. So we have

$$\|\xi_{n+1} - p\| \leq k^{3(n+1)}\|(\xi_0 - p)\|e^{-(1-k)}\sum_{i=0}^{n}\{\theta_i + k\theta_i\vartheta_i\}. \quad (18)$$

Taking the limits $n \to \infty$ both sides, we get $\lim_{n\to\infty}\|\xi_n - p\| = 0$.

Theorem 7. *Let $C \neq \phi \subseteq X$, and let contraction map $F : C \to C$ holding condition (M), having a fixed point p. For a given $\xi_0 = \eta_0$, let $\{\xi_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ be iterative sequences developed by D I.P and Picard-S I.P as in [11], respectively, where $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$, and $\{\eta_n\}_{n=0}^{\infty} \in [0\,;1]$ are real sequences satisfying $\theta \leq \theta_n < 1$ and $\vartheta \leq \vartheta_n < 1$, for some $\theta, \vartheta > 0$ and $\forall n \in N$. Then, $\{\xi_n\}_{n=0}^{\infty}$ approaches to p firstly rather than $\{\eta_n\}_{n=0}^{\infty}$.*

Proof. By Theorem 2.5 in [11], we have

$$\|\eta_{n+1} - p\| \leq k^{2(n+1)}\|(\eta_0 - p)\|\prod_{i=0}^{n}\{1-\theta_i\vartheta_i(1-k)\}. \quad (19)$$

Since $\theta \leq \theta_n$ and for all $n \in \mathrm{N}$, we obtain $\|\eta_{n+1} - p\| \leq k^{2(n+1)}\|(\eta_0 - p)\|\{1 - \theta_i\vartheta_i(1-k)\}^{n+1}$. Let $a_n = k^{2(n+1)}\|(\eta_0 - p)\|\{1 - \theta_i\vartheta_i(1-k)\}^{n+1}$.

Now, from Theorem 6, we get

$$\|\xi_{n+1} - p\| \leq k^{3(n+1)}\|(\xi_0 - p)\| \prod_{i=0}^{n} \{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}. \tag{20}$$

Again $\theta \leq \theta_n$ for all $n \in N$ gives

$$\|\xi_{n+1} - p\| \leq k^{3(n+1)}\|(\xi_0 - p)\|\{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}^{n+1}. \tag{21}$$

Let $b_n = k^{3(n+1)}\|(\xi_0 - p)\|\{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}^{n+1}$. Then,

$$\frac{b_n}{a_n} = \frac{k^{3(n+1)}\|(\xi_0 - p)\|\{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}^{n+1}}{k^{2(n+1)}\|(\eta_0 - p)\|\{1 - \theta_i\vartheta_i(1-k)\}}. \tag{22}$$

Thus, taking the limit as $n \to \infty \lim_{n\to\infty}(b_n/a_n) = 0$. Hence, the result follows.

Next, we present a result which defines the better approximation rate between the D and M I.P [7].

Theorem 8. *Let $C \neq \phi \subseteq X$, also let contraction map $F : C \to C$ holding condition (M), having a fixed point p. For a given $\xi_0 = \eta_0$, let $\{\xi_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be iterative sequences developed by D I.P and M I.P as in [7], respectively, where $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$, and $\{\eta_n\}_{n=0}^{\infty} \in [0;1]$ are real sequences satisfying $\theta \leq \theta_n < 1$ and $\vartheta \leq \vartheta_n < 1$, for some $\theta, \vartheta > 0$ and $\forall n \in N$. Then, $\{\xi_n\}_{n=0}^{\infty}$ approaches to p firstly rather than $\{u_n\}_{n=0}^{\infty}$.*

Proof. By Theorem 6, we have

$$\|\xi_{n+1} - p\| \leq k^{3(n+1)}\|(\xi_0 - p)\| \prod_{i=0}^{n} \{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}. \tag{23}$$

Since $\theta \leq \theta_n$ and for all $n \in N$, we obtain

$$\|\xi_{n+1} - p\| \leq k^{3(n+1)}\|(\xi_0 - p)\|\{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}^{n+1}. \tag{24}$$

Let $a_n = k^{3(n+1)}\|(\xi_0 - p)\|\{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}^{n+1}$. Then, for the M iteration, we have

$$\begin{cases} u_0 \in C, \\ w_n = (1-\vartheta_n)u_n + \vartheta_n F u_n, \\ v_n = F w_n, \\ u_{n+1} = F v_n, \end{cases}$$

$$\begin{aligned} \|w_n - p\| &= \|((1-\vartheta_n)u_n + \vartheta_n F u_n) - Fp\| \\ &\leq \|(1-\vartheta_n)u_n + \vartheta_n F u_n - p\| \\ &\leq \|(1-\vartheta_n)(u_n - p) + \vartheta_n(F u_n - p)\| \\ &\leq (1-\vartheta_n)\|(u_n - p)\| + \vartheta_n\|(F u_n - p)\| \\ &\leq \{(1-\vartheta_n)\|(u_n - p)\| + k\vartheta_n\|(u_n - p)\|\} \\ &\leq k\{1 - \vartheta_n(1-k)\}\|(u_n - p)\|. \end{aligned} \tag{25}$$

Now,

$$\begin{aligned} \|v_n - p\| &\leq \|F w_n - p\| \leq k\|w_n - p\| \\ &\leq k\{1 - \vartheta_n(1-k)\}\|(u_n - p)\|. \end{aligned} \tag{26}$$

Therefore, we get $\|u_{n+1} - p\| \leq \|F v_n - p\| \leq k\|v_n - p\| \leq \mathrm{k}^2\{1 - \vartheta_n(1-k)\}\|(u_n - p)\|$.

After repetition,

$$\begin{aligned} \|u_n - p\| &\leq k^2\{1 - \vartheta_{n-1}(1-k)\}\|(u_{n-1} - p)\|, \\ \|u_{n-1} - p\| &\leq k^2\{1 - \vartheta_{n-2}(1-k)\}\|(u_{n-2} - p)\|, \\ \|u_{n-2} - p\| &\leq k^2\{1 - \vartheta_{n-3}(1-k)\}\|(u_{n-3} - p)\|, \\ &\vdots \\ \|u_1 - p\| &\leq k^2\{1 - \vartheta_0(1-k)\}\|(u_0 - p)\|. \end{aligned} \tag{27}$$

Thus, we have $\|u_{n+1} - p\| \leq k^{2(n+1)}\|(u_0 - p)\|\prod_{i=0}^{n} \{1 - \vartheta_i(1-k)\}$. Now, since $\vartheta \leq \vartheta_n$ and for all $n \in N$, we obtain $\|u_{n+1} - p\| \leq k^{2(n+1)}\|(u_0 - p)\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}$. Let $b_n = k^{2(n+1)}\|(u_0 - p)\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}$.

Then,

$$\frac{a_n}{b_n} = \frac{k^{3(n+1)}\|(\xi_0 - p)\|\{1 - (\theta_i + k\theta_i\vartheta_i)(1-k)\}^{n+1}}{k^{2(n+1)}\|(u_0 - p)\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}}. \tag{28}$$

Thus, taking the limit as $n \to \infty \lim_{n\to\infty}(b_n/a_n) = 0$. Hence, the result follows.

Next, we prove that D I.P is faster than that of the M^* I.P [6].

Theorem 9. *Let $C \neq \phi \subseteq X$; also, let a contraction mapping $F : C \to C$ holding condition (M), having a particular fixed point p. For a given $\xi_0 = \eta_0$, let $\{\xi_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be iterative sequences developed by D I.P and M I.P as in [6], respectively, where $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$, and $\{\eta_n\}_{n=0}^{\infty} \in [0;1]$ are real sequences satisfying $\theta \leq \theta_n < 1$ and $\vartheta \leq \vartheta_n < 1$, for some $\theta, \vartheta > 0$ and $\forall n \in N$. Then, $\{\xi_n\}_{n=0}^{\infty}$ approaches to p firstly rather than $\{u_n\}_{n=0}^{\infty}$.*

Proof. By result 2.1, we have

$$\|\xi_{n+1}-p\| \leq k^{3(n+1)}\|(\xi_0-p)\| \prod_{i=0}^{n} \{1-(\theta_i+k\theta_i\vartheta_i)(1-k)\}. \tag{29}$$

Since $\theta \leq \theta_n$ and for all $n \in N$, we obtain

$$\|\xi_{n+1}-p\| \leq k^{3(n+1)}\|(\xi_0-p)\|\{1-(\theta_i+k\theta_i\vartheta_i)(1-k)\}^{n+1}. \tag{30}$$

Let $a_n = k^{3(n+1)}\|(\xi_0-p)\|\{1-(\theta_i+k\theta_i\vartheta_i)(1-k)\}^{n+1}$. Then, for the M^*iteration, we have

$$\begin{cases} u_0 \in C, \\ w_n = (1-\vartheta_n)u_n + \vartheta_n F u_n, \\ v_n = F((1-\theta_n)u_n + \theta_n F w_n), \\ u_{n+1} = F v_n, \end{cases}$$

$$\begin{aligned} \|w_n-p\| &= \|((1-\vartheta_n)u_n+\vartheta_n F u_n) - Fp\| \\ &\leq \|(1-\vartheta_n)u_n+\vartheta_n F u_n - p\| \\ &\leq \|(1-\vartheta_n)(u_n-p)+\vartheta_n(Fu_n-p)\| \\ &\leq (1-\vartheta_n)\|(u_n-p)\|+\vartheta_n\|(Fu_n-p)\| \\ &\leq \{(1-\vartheta_n)\|(u_n-p)\|+k\vartheta_n\|(u_n-p)\|\} \\ &\leq \{1-\vartheta_n(1-k)\}\|(u_n-p)\|. \end{aligned} \tag{31}$$

Now,

$$\begin{aligned} \|v_n-p\| &= \|F((1-\theta_n)u_n+\theta_n F w_n) - Fp\| \\ &\leq k\|(1-\theta_n)u_n+\theta_n F w_n - p\| \\ &\leq k\|(1-\theta_n)(u_n-p)+\theta_n(Fw_n-p)\| \\ &\leq k(1-\theta_n)\|(u_n-p)\|+\theta_n\|(Fw_n-p)\| \\ &\leq k\{(1-\theta_n)\|(u_n-p)\|+k\theta_n\|(w_n-p)\|\} \\ &\leq k\{(1-\theta_n)\|(u_n-p)\|+k\theta_n\{1-\vartheta_n(1-k)\}\|(u_n-p)\|\} \\ &\leq k\{(1-\theta_n)+k\theta_n-k\theta_n\vartheta_n(1-k)\}\|(u_n-p)\| \\ &\leq k\{(1-(1-k)\theta_n-k\theta_n\vartheta_n(1-k)\}\|(u_n-p)\| \\ &\leq k\{(1-\theta_n(1-k)(1-k\vartheta_n)\}\|(u_n-p)\|\|u_{n+1}-p\| \\ &\leq \|Fv_n-p\| \leq k^2\{(1-\theta_n(1-k)(1-k\vartheta_n)\}\|(u_n-p)\|. \end{aligned} \tag{32}$$

After repetition,

$$\begin{aligned} \|u_n-p\| &\leq k^2\{1-\vartheta_{n-1}(1-k)\}\|(u_{n-1}-p)\|, \\ \|u_{n-1}-p\| &\leq k^2\{1-\vartheta_{n-2}(1-k)\}\|(u_{n-2}-p)\|, \\ \|u_{n-1}-p\| &\leq k^2\{1-\vartheta_{n-2}(1-k)\}\|(u_{n-2}-p)\|, \\ &\vdots \\ \|u_1-p\| &\leq k^2\{1-\vartheta_0(1-k)\}\|(u_0-p)\|. \end{aligned} \tag{33}$$

Table 1: Sequence generated by D, K, M^*, M, Picard-S, and S I.P for mapping F of Example 10.

	S	Picard-S	M	M^*	K	D
ξ_0	3.5	3.5	3.5	3.5	3.5	3.5
ξ_1	3.2	2.96	2.96	2.96	2.768	2.768
ξ_2	2.9024	2.57754	2.55296	2.52347	2.36962	2.33502
ξ_3	2.66692	2.34146	2.31345	2.27631	2.17483	2.14147
ξ_4	2.48921	2.20038	2.17653	2.14387	2.08208	2.05893
ξ_5	2.35737	2.1171	2.09908	2.07434	2.03837	2.02436
ξ_6	2.26037	2.06825	2.05548	2.03823	2.01789	2.01002
ξ_7	2.18935	2.03971	2.03102	2.0196	2.00833	2.00028
ξ_8	2.13752	2.02307	2.01733	2.01002	2.00387	2.00168
ξ_9	2.09977	2.01339	2.00967	2.00512	2.0018	2.00005
ξ_{10}	2.07233	2.00777	2.00539	2.00261	2.00083	2.00028
ξ_{11}	2.0524	2.0045	2.00301	2.00133	2.00039	2.00011
ξ_{12}	2.03794	2.00261	2.00168	2.00068	2.00018	2.00005
ξ_{13}	2.02746	2.00151	2.00093	2.00034	2.00008	2.00002
ξ_{14}	2.01987	2.00087	2.00052	2.00017	2.00004	2.00001
ξ_{15}	2.01437	2.00051	2.00029	2.00009	2.00002	2
ξ_{16}	2.01039	2.00029	2.00016	2.00004	2.00001	2
ξ_{17}	2.00751	2.00017	2.00009	2.00002	2	2
ξ_{18}	2.00543	2.0001	2.00005	2.00001	2	2
ξ_{19}	2.00392	2.00006	2.00003	2.00001	2	2
ξ_{20}	2.00283	2.00003	2.00002	2	2	2

Thus, $\|u_{n+1}-p\| \leq k^{2(n+1)}\|(u_0-p)\|\prod_{i=0}^{n}\{(1-\vartheta_i)(1-k)\}$. Now, since $\theta \leq \theta_n \vartheta \leq \vartheta_n$ and for all $n \in N$, we obtain $\|u_{n+1}-p\| \leq k^{2(n+1)}\|(u_0-p)\|\{1-\vartheta\theta_i(1-k)\}^{n+1}$. Let $b_n = k^{2(n+1)}\|(u_0-p)\|\{1-\vartheta\theta_i(1-k)\}^{n+1}$.

Then,

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{k^{3(n+1)}\|(\xi_0-p)\|\{1-(\theta_i+k\theta_i\vartheta_i)(1-k)\}^{n+1}}{k^{2(n+1)}\|(u_0-p)\|\{1-\theta_i(1-k)(1-k\vartheta_i)\}^{n+1}} \\ &= \frac{k^{n+1}\{1-\theta_i(1-k)\}^{n+1}}{\{1-\theta_i(1-k)(1-k\vartheta_i)\}^{n+1}}. \end{aligned} \tag{34}$$

Thus, taking the limit as $n \to \infty \lim_{n\to\infty}(b_n/a_n) = 0$. Hence, the result follows.

Now, we prove by numerical example that our D I.P has a better approximation rate than existing I.Ps in literature.

Example 10. Let a contraction mapping $F : R \to R$ be defined by $(\xi) = (4\xi + 2)/5$. Let $\theta_n = 2n/(3n+1)$ and $\vartheta_n = 3n/(4n+1)$. The iterative values for $\xi_0 = 3.5$ are defined in Table 1. Figures 1(a) and 1(b) present the convergence graph.

By Figures 1(a) and 1(b) and Table 1, it is clear that the new D I.P has a better approximation rate then the K, M^*, M, Picard-S, and S I.P.

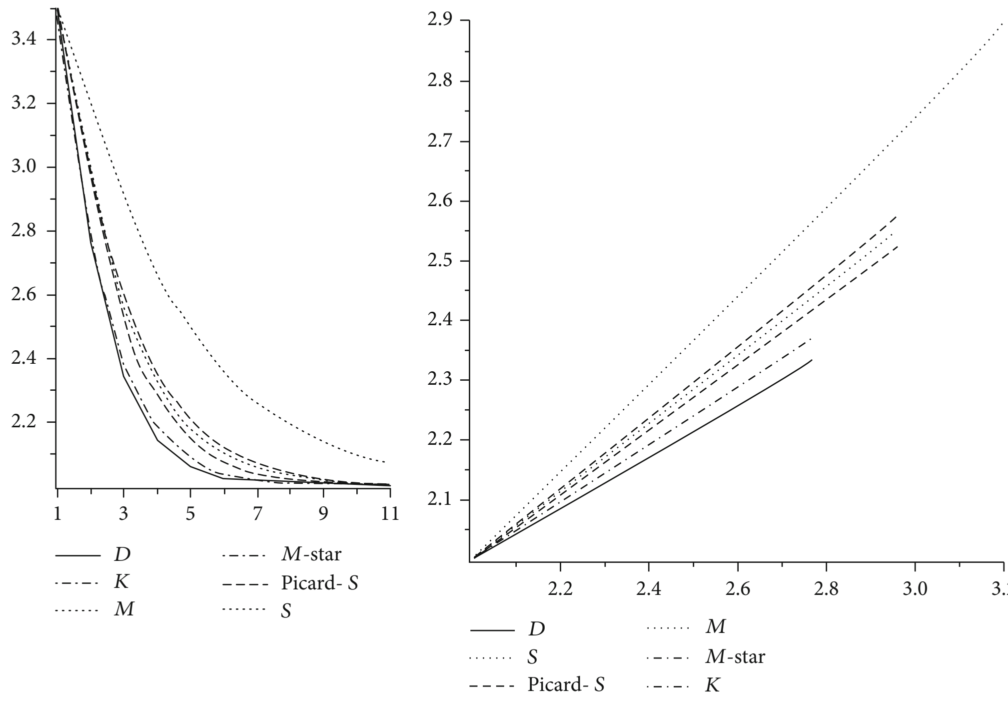

Figure 1: (a, b) Convergence of D, K, M^*, M, Picard-S, and S I.P to the fixed point 2 for F of Example 10.

Table 2: Sequence generated by D, K, M^*, M, Picard-S, and S I.P for F of Example 11.

	S	Picard-S	M	M^*	K	D
ξ_0	1.5	1.5	1.5	1.5	1.5	1.5
ξ_1	1.732050808	1.981969534	1.981969534	1.98197	2.219882	2.219882
ξ_2	2.022743362	2.453392689	2.482109844	2.506389	2.727322	2.75395
ξ_3	2.302462597	2.747657664	2.776630296	2.801381	2.920085	2.936761
ξ_4	2.530373744	2.892112825	2.910917465	2.926554	2.977931	2.984781
ξ_5	2.69661837	2.955490157	2.965667023	2.973809	2.994026	2.99642
ξ_6	2.809384444	2.98193592	2.986964491	2.990812	2.998385	2.999166
ξ_7	2.882389407	2.992726282	2.995085112	2.996804	2.999571	2.999807
ξ_8	2.928277354	2.997083159	2.998153608	2.998893	2.999885	2.999955
ξ_9	2.956589022	2.998833116	2.9999307853	2.999618	2.999969	2.99999
ξ_{10}	2.973852778	2.99953392	2.999740915	2.999869	2.999992	2.999998
ξ_{11}	2.984301526	2.999814051	2.999903123	2.999955	2.999998	2.999999
ξ_{12}	2.990595194	2.999925877	2.999963806	2.999985	2.999999	3
ξ_{13}	2.994374126	2.999970477	2.999986488	2.999995	3	3
ξ_{14}	2.996638281	2.999988247	2.999994958	2.999998	3	3
ξ_{15}	2.997992835	2.999995323	2.99999812	2.999999	3	3
ξ_{16}	2.998802339	2.99999814	2.999999298	3	3	3
ξ_{17}	2.999285722	2.99999926	2.999999738	3	3	3
ξ_{18}	2.999574185	2.999999705	2.999999902	3	3	3
ξ_{19}	2.999746241	2.999999883	2.999999963	3	3	3
ξ_{20}	2.999848822	2.999999953	2.999999985	3	3	3

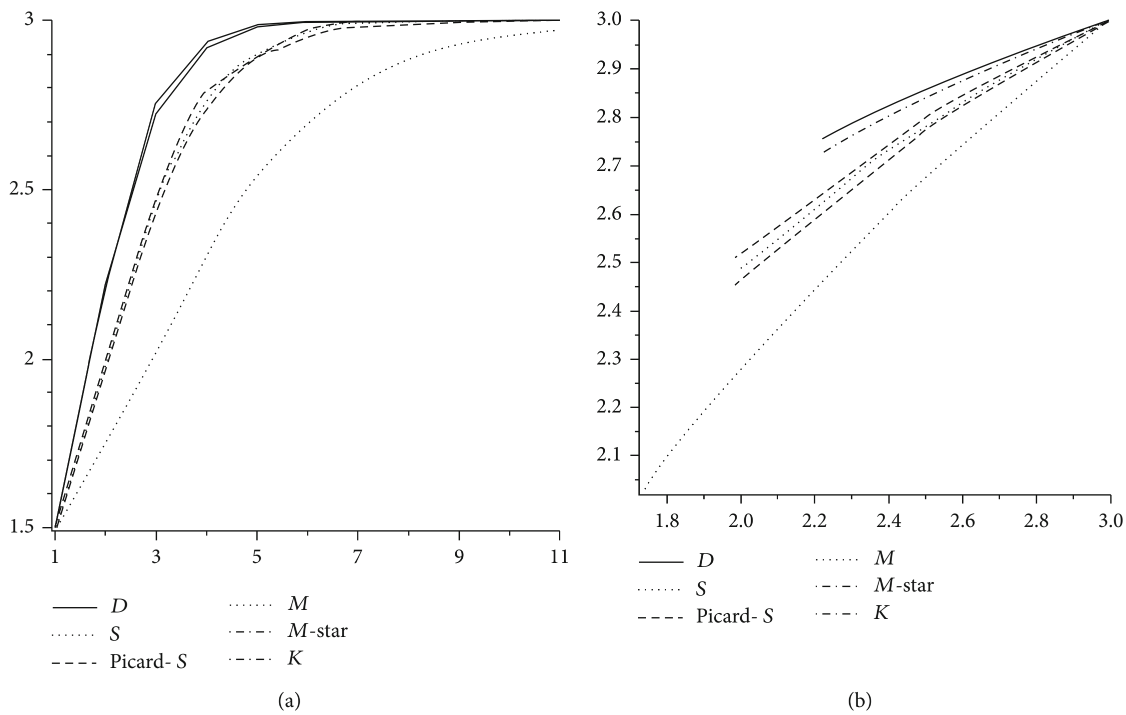

FIGURE 2: (a, b) Convergence of D, K, M^*, M, Picard-S, S I.P to the fixed point 3 for F of Example 11.

Example 11. Let a contraction mapping $F : [0, 50) \rightarrow [0, 50]$ by $F(\xi) = \sqrt{4\xi - 3}$. Let $\theta_n = 2n/(3n + 1)$ and $\vartheta_n = 3n/(4n + 1)$; the iterative values for the initial guess $\xi_0 = 1.5$ are given in Table 2. Figures 2(a) and 2(b) represent the convergence graph. We can see that our new I.P (13) has a better approximation rate as compared to *S*, Picard-*S*, *M*, M^*, and *K* I.P.

By Figures 2(a) and 2(b) and Table 2, it is proven that the new *D* iteration has a better approximation rate than the *K*, M^*, *M*, Picard-*S*, and *S* I.P.

3. Convergence Results of a Sequence Generated by *D* I.P

Here, we prove some strong and weak convergence result of sequence generated by *D* I.P for the Suzuki generalized nonexpansive mapping in uniformly convex Banach spaces.

Lemma 12. *Let $C \neq \phi \subseteq X$ and let contraction mapping $F : C \rightarrow C$ be a holding condition (M) and also $T(F) \neq \varnothing$. For randomly chosen $\xi_0 \in C$, let the sequence $\{\xi_n\}$ be generated by (13), then $lim_{n\rightarrow\infty}\|\xi_n - p\|$ exists for any $p \in T(F)$.*

Proof. Let $p \in T(F)$ and $\omega \in C$. As F holds the (M) condition, so

$$\frac{1}{2}\|p - Fp\| = 0 \leq \|p - \omega\| \Rightarrow \|Fp - F\omega\| \leq \|p - \omega\|. \qquad (35)$$

So from 1.1(ii), we have

(a) $\|\omega_n - p\| = \|(1 - \vartheta_n)\xi_n + \vartheta_n F\xi_n - p\| \leq (1 - \vartheta_n)\|\xi_n - p\| + (\vartheta_n\|F\xi_n - p\| \leq (1 - \vartheta_n)\|\xi_n - p\| + (\vartheta_n\|\xi_n - p\| = \|\xi_n - p\|$.

So by using (a), we get

(b) $\|\eta_n - p\| = \|F((1 - \theta_n)F\xi_n + \theta_n F\omega_n) - p\| \leq \|(1 - \theta_n)F\xi_n + \theta_n F\omega_n - p \leq (1 - \theta_n)\|F\xi_n - p\| + \theta_n\|F\omega_n - p\| \leq (1 - \theta_n\|\xi_n - p\| + \theta_n\|\omega_n - p\| \leq (1 - \theta_n)\|\xi_n - p\| + \theta_n\|\xi_n - p\| = \|\xi_n - p\|$.

So using (b), we have

(c) $\|\xi_{n+1} - p\| = \|F\eta_n - p\| \leq \|\eta_n - p\| \leq \|\xi_n - p\|$,

which means that $\|\xi_n - p\|$ is nonincreasing and bounded, $\forall p \in T(F)$. So $\lim_{n\rightarrow\infty}\|\xi_n - p\|$ exists, hence proven.

Theorem 13. *Let $C \neq \phi \subseteq X$, and let contraction mapping $F : C \rightarrow C$ holding condition (M). For randomly chosen $\xi_0 \in C$, let $\{\xi_n\}$ be the sequence generated by (13) $\forall n \geq 1$ where θ_n and ϑ_n are sequences of real numbers $\in [0\ 1]$. Then, $T(F) \neq \varnothing$ iff ξ_n is bounded and $\lim_{n\rightarrow\infty}\|F\xi_n - \xi_n\| = 0$.*

Proof. Assume $F(T) \neq \phi$ and consider $p \in F(T)$. So by result 3.1, $\lim_{n\rightarrow\infty}\|\xi_n - p\|$ exists and ξ_n is bounded put

(d) $\lim_{n\rightarrow\infty}\|\xi_n - p\| = r$.

From (a) and (d), we have

(e) $\lim_{n\rightarrow\infty} \sup\|\omega_n - p\| \leq \lim_{n\rightarrow\infty} \sup\|\xi_n - p\| = r$.

By result 1.1(ii), we have

(f) $\lim_{n\to\infty}\sup\|T\xi_n - p\|\le\lim_{n\to\infty}\sup\|\xi_n - p\| = r$.

On the other hand,

$$\begin{aligned}\|\xi_{n+1} - p\| &= \|F\eta_n - p\| \le \|\eta_n - p\| \\ &= \|F((1-\theta_n)F\xi_n + \theta_n F\omega_n) - p\| \\ &\le \|(1-\theta_n)F\xi_n + \theta_n F\omega_n - p\| \\ &\le (1-\theta_n)\|F\xi_n - p\| + \theta_n\|F\omega_n - p\| \\ &\le (1-\theta_n\|\xi_n - p\| + \theta_n\|\omega_n - p\| \\ &\le \|\xi_n - p\| - \theta_n\|\xi_n - p\| + \theta_n\|\omega_n - p\|.\end{aligned} \tag{36}$$

This implies that

$$\frac{\|\xi_{n+1} - p\| - \|\xi_n - p\|}{\theta_n} \le \|\omega_n - p\| - \|\xi_n - p\|. \tag{37}$$

So

$$\begin{aligned}\|\xi_{n+1} - p\| - \|\xi_n - p\| &\le \frac{\|\xi_{n+1} - p\| - \|\xi_n - p\|}{\theta_n} \\ &\le \|\omega_n - p\| - \|\xi_n - p\|,\end{aligned} \tag{38}$$

implies that

$$\|\xi_{n+1} - p\| \le \|\omega_n - p\|. \tag{39}$$

Therefore,

(g) $r \le \lim_{n\to\infty}\inf\|\omega_n - p\|$.

From (e) and (g), we get

(h) $r = \lim_{n\to\infty}\|\omega_n - p\| = \lim_{n\to\infty}\|(1-\vartheta_n)\xi_n + \vartheta_n F\xi_n - p\| = \lim_{n\to\infty}\|(\vartheta_n)(F\xi_n - p) + (1-\vartheta_n)(\xi_n - p)$.

From (d), (f), and (h) together with Lemma 4, we have

$$\lim_{n\to\infty}\|F\xi_n - \xi_n\| = 0. \tag{40}$$

On the other hand, assume that ξ_n is a bounded sequence and $\lim_{n\to\infty}\|F\xi_n - \xi_n\| = 0$. Also, $p \in A(C, \xi_n)$. So by Proposition 1,

$$\begin{aligned}r(Fp, \xi_n) &= \lim_{n\to\infty}\sup\|\xi_n - Fp\| \\ &\le \lim_{n\to\infty}\sup\,(3\|F\xi_n - \xi_n\| + \|\xi_n - p\|) \\ &\le \lim_{n\to\infty}\sup\|\xi_n - p\| = r(p, \xi_n).\end{aligned} \tag{41}$$

Hence, $Fp \in A(C, \xi_n)$. As X is uniformly convex, and $A(C, \xi_n)$ is a singleton set, therefore $Fp = p$. Hence, $(F) \ne \varnothing$.

3.1. Weak and Strong Convergence Theorem

Theorem 14. *Let $C \ne \phi \subseteq X$, where X satisfies Opial condition, and let $F : C \to C$ holding condition (M), where θ_n and ϑ_n are real sequences $\in[0\,1]$, such that $T(F) \ne \varnothing$. For randomly selected $\xi_0 \in C$, consider x_n to be the sequence generated by (13) $\forall n \ge 1$. Then, the sequence generated by the D iterative process converges weakly to $p \in T(F)$.*

Proof. As $T(F) \ne \phi$, and from Theorem 13, ξ_n is a bounded sequence and $\lim_{n\to\infty}\|F\xi_n - \xi_n\| = 0$. Being uniformly convex X must be reflexive, so by Eberlin's theorem $\exists$, a subsequence ξ_{n_j} of ξ_n which approaches weakly to $p_1 \in X$. As C is closed and convex, by Mazur's theorem, $p_1 \in C$. By Lemma 2, $p_1 \in T(F)$. Next, we prove that ξ_n approaches weakly to p_1. Actually, if it is not true, so $\exists$ a subsequence ξ_{n_k} for ξ_n s.t ξ_{n_k} converges weakly to $p_2 \in C$ and $p_2 \ne p_1$. By Lemma 2, $p_2 \in T(F)$. Since $\lim_{n\to\infty}\|\xi_n - p\|$ defined $\forall p \in T(F)$. Thus, from Theorem 13 and Opial's property,

$$\begin{aligned}\lim_{n\to\infty}\|\xi_n - p_1\| &= \lim_{j\to\infty}\|\xi_{n_j} - p_1\| < \lim_{j\to\infty}\|\xi_{n_j} - p_2\| \\ &= \lim_{n\to\infty}\|\xi_n - p_2\| = \lim_{k\to\infty}\|\xi_{n_k} - p_2\| \\ &< \lim_{k\to\infty}\|\xi_{n_k} - p_1\| = \lim_{n\to\infty}\|\xi_n - p_1\|,\end{aligned} \tag{42}$$

which is a contradiction to fact. So $p_1 = p_2 \Rightarrow \xi_n$ converges weakly to a single fixed point of F.

Next, we prove the strong convergence theorem.

Theorem 15. *Let $C \ne \phi \subseteq X$, and let contraction mapping $F : C \to C$ holding condition (M), where θ_n and $\vartheta_n \in [0\,1]$ are real sequences, s.t $T(F) \ne \varnothing$. Then, ξ_n approaches strongly to $p \in T(F)$.*

Proof. By Lemma 3, $T(F) \ne \phi$, and by Theorem 13, we have $\lim_n \to \infty\|T\xi_n - \xi_n\| = 0$. As C is compact, so $\exists$ a subsequence ξ_{n_j} of ξ_n which approaches strongly to p where $p \in C$. By Proposition 1 (iii), we have $\|\xi_{n_k} - Tp\|\le 3\|T\xi_{n_k} - \xi_{n_k}\| + \|\xi_{n_k} - p\|$, $\forall n \ge 1$. Letting $k \to \infty$, we attain $p \in T(F)$. As, by Lemma 12, $\lim_n \to \|\xi_n - p\|$ has a defined value, for every $p \in F(T)$, so ξ_n converge strongly to p.

4. Conclusion

In this article, we present the new fastest iteration method to approximating fixed point of contraction mapping. First, we present the D iteration process and prove weak and strong convergence results. Also, we analytically and numerically proved that the D iteration process has a better approximation rate than existing iteration processes as defined in [1–4, 6, 7, 9].

Data Availability

No data use for this study.

Conflicts of Interest

The writers announce that they do not have any competing interests.

Authors' Contributions

All authors contributed fairly and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

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Research Article

On Unique and Nonunique Fixed Points and Fixed Circles in $\mathscr{M}_v^b$-Metric Space and Application to Cantilever Beam Problem

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We introduce $\mathscr{M}_v^b$-metric to generalize and improve $\mathscr{M}_v$-metric and unify numerous existing distance notions. Further, we define topological notions like open ball, closed ball, convergence of a sequence, Cauchy sequence, and completeness of the space to discuss topology on $\mathscr{M}_v^b$-metric space and to create an environment for the survival of a unique fixed point. Also, we introduce a notion of a fixed circle and a fixed disc to study the geometry of the set of nonunique fixed points of a discontinuous self-map and establish fixed circle and fixed disc theorems. Further, we verify all these results by illustrative examples to demonstrate the authenticity of the postulates. Towards the end, we solve a fourth order differential equation arising in the bending of an elastic beam.

1. Introduction

A Greek mathematician Euclid of Alexandria (323-283 BC) was the first to communicate the notion of distance. Indeed, Euclidean distance is a noteworthy measure of closeness among two quantities, which is one of the initial conceptions appreciated by mankind. Fréchet [1] deliberated the general and more axiomatic form of distance as "L-space." Hausdorff [2] reexamined it as a metric space in the setting of points which has been refined, discussed, and generalized in numerous ways. Bakhtin [3], Branciari [4], Asadi et al. [5], George et al. [6], Mitrović and Radenović [7], Özgür et al. [8], Karahan and Isik [9], and Asim et al. [10] introduced the notions of a b-metric, a rectangular metric and a v-generalized metric, an M-metric, a rectangular b-metric, a $b_v(s)$-metric, a rectangular M-metric, a generalized p_v^b-partial metric, and an M_v-metric, respectively. Further, one may also refer to Fernandez et al. [11] and Kanwal et al. [12, 13] for work in N_b-cone metric spaces over Banach algebra, orthogonal F-metric spaces, and weak partial b-metric spaces, respectively. One may allude to Kirk and Shahzed [14], to study in detail about the generalizations of the metric notion.

The aim of this work is to introduce a novel distance structure called an M_v^b-metric space which is an improvement and generalization of an M_v-metric space. Also, we introduce topological notions like open ball, closed ball, convergence of a sequence, Cauchy sequence, and completeness of the space to discuss topology and to create an environment for the survival of a unique fixed point in an M_v^b-metric space. Further, we demonstrate that the collection of open balls, which forms a basis on M_v^b-metric space, generates a $\mathscr{T}_0$ topology on it. In the sequel, with the help of examples and remarks, we demonstrate that an M_v^b-metric space unifies and combines numerous distance conceptions and marks supremacy over all those spaces wherein the continuity of a map is required for the survival of a fixed point. We also introduce a notion of a fixed circle and a fixed disc to study the geometry of the set of nonunique fixed points of a discontinuous self-map in an M_v^b-metric space and establish fixed circle and greatest fixed disc theorems. We verify these results by illustrative examples with geometric interpretations to establish the authenticity of the postulates. Towards the end, we solve fourth order differential equation arising in the Cantilever beam problem.

2. Preliminaries

In this section, we use notations $m_{v_{u,w}} = \min\{m_v(u,u), m_v(w,w)\}$ and $M_{v_{u,w}} = \max\{m_v(u,u), m_v(w,w)\}$.

In 2017, Mitrović and Radenović [7] introduced b_v^s-metric space.

Definition 1. A $b_v(s)$-metric on a nonempty set $\mathscr{M}$ with $s \ge 1$ is a map $b_v(s)$: $\mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ satisfying

$$(b_v(s)(\mathrm{i}))\ b_v(s)(\mathfrak{u},\mathfrak{v}) = 0 \text{ iff } \mathfrak{u} = \mathfrak{v},$$

$$(b_v(s)(\mathrm{ii}))\ b_v(s)(\mathfrak{u},\mathfrak{v}) \ge 0,$$

$$(b_v(s)(\mathrm{iii}))\ b_v(s)(\mathfrak{u},\mathfrak{w}) = b_v(s)(\mathfrak{w},\mathfrak{u}),$$

$$(b_v(s)(\mathrm{iv}))\ b_v(s)(\mathfrak{u},\mathfrak{w}) \le s[b_v(s)(\mathfrak{u},\mathfrak{z}_1) + b_v(s)(\mathfrak{z}_1,\mathfrak{z}_2) + \cdots + b_v(s)(\mathfrak{z}_v,\mathfrak{w})], \quad (1)$$

$u, z_1, z_2, \cdots, z_v, w \in \mathscr{M}$ and are distinct. A pair $(\mathscr{M}, b_v(s))$ is a $b_v(s)$-metric space.

Remark 2. A $b_v(s)$-metric may be reduced to a v-generalized metric [4] for $s = 1$, a rectangular metric [4] for $v = 2$ and $s = 1$, a rectangular b-metric [6] for $v = 2$, a b-metric [3] for $v = 1$, and a usual metric [1] for $v = s = 1$.

In 2018, Karahan and Isik [9] introduced the p_v^b-partial metric.

Definition 3. A partial $b_v(s)$-metric on a nonempty set $\mathscr{M}$ with $s \ge 1$ is a map $pb_v(s)$: $\mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ satisfying

$$(pb_v(s)(\mathrm{i}))\ pb_v(s)(\mathfrak{u},\mathfrak{u}) = pb_v(s)(\mathfrak{w},\mathfrak{w}) = pb_v(s)(\mathfrak{u},\mathfrak{w}) \text{ iff } \mathfrak{u} = \mathfrak{w},$$

$$(pb_v(s)(\mathrm{ii}))\ pb_v(s)(\mathfrak{u},\mathfrak{u}) \le pb_v(s)(\mathfrak{u},\mathfrak{w}),$$

$$(pb_v(s)(\mathrm{iii}))\ pb_v(s)(\mathfrak{u},\mathfrak{w}) = pb_v(s)(\mathfrak{w},\mathfrak{u}),$$

$$(pb_v(s)(\mathrm{iv}))\ pb_v(s)(\mathfrak{u},\mathfrak{w}) \le s[pb_v(s)(\mathfrak{u},\mathfrak{z}_1) + pb_v(s)(\mathfrak{z}_1,\mathfrak{z}_2) + \cdots + pb_v(s)(\mathfrak{z}_v,\mathfrak{w})] - \Sigma_{i=1}^{v} pb_v(s)(\mathfrak{z}_i,\mathfrak{z}_i), \quad (2)$$

$\mathfrak{u}, \mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_v, \mathfrak{w} \in \mathscr{M}$ and are distinct. A pair $(\mathscr{M}, pb_v(s))$ is a partial $b_v(s)$-metric space.

Remark 4. A partial $b_v(s)$-metric may be reduced to a rectangular partial metric [15] for $v = 2$ and $s = 1$, a rectangular partial b-metric for $v = 2$, a partial b-metric [16] for $v = 1$, and a partial metric [17] for $v = s = 1$.

In 2019, Asim et al. [10] have familiarized the notion of an M_v-metric space.

Definition 5. An M_v-metric on a nonempty set $\mathscr{M}$ is a map $m_v : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ *satisfying*

$$(m_v\mathrm{i})\ m_v(\mathfrak{u},\mathfrak{u}) = m_v(\mathfrak{w},\mathfrak{w}) = m_v(\mathfrak{u},\mathfrak{w}) \text{ iff } \mathfrak{u} = \mathfrak{w},$$

$$(m_v\mathrm{ii})\ m_{v_{\mathfrak{u},\mathfrak{w}}} \le m_v(\mathfrak{u},\mathfrak{w}),$$

$$(m_v\mathrm{iii}) m_v(\mathfrak{u},\mathfrak{w}) = m_v(\mathfrak{w},\mathfrak{u}),$$

$$(m_v\mathrm{iv}) \left(m_v(\mathfrak{u},\mathfrak{w}) - m_{v_{\mathfrak{u},\mathfrak{w}}}\right) \le \left(m_v(\mathfrak{u},\mathfrak{z}_1) - m_{v_{\mathfrak{u},\mathfrak{z}_1}}\right) + \left(m_v(\mathfrak{z}_1,\mathfrak{z}_2) - m_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\right) + \cdots + \left(m_v(\mathfrak{z}_v,\mathfrak{w}) - m_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right), \quad (3)$$

$\mathfrak{u}, \mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_v, \mathfrak{w} \in \mathscr{M}$ and are distinct. A pair $(\mathscr{M}, m_v)$ is an M_v-metric space.

Example 6 (see [10]). Let $\mathscr{M} = \mathbb{R}$. Define $m_v : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ *by* $m_v(\mathfrak{u},\mathfrak{w}) = |\mathfrak{u}| + |\mathfrak{w}|/2$, $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$ *, then* $(\mathscr{M}, m_v)$ *is an* M_v *-metric space.*

Remark 7. In the above example, if $v = 1$, m_v will be an M-metric [5]. In this case, for $1, 2, 3 \in \mathscr{M}$,

$$\left(m_v(1,3) - m_{v_{(1,3)}}\right) > \left(m_v(1,2) - m_{v_{(1,2)}}\right) + \left(m_v(2,3) - m_{v_{(2,3)}}\right) - m_v(2,2). \quad (4)$$

If $v = 2$, it will be a rectangular M-metric [18]. Again, in this case, for $1, 2, 3, 4 \in \mathscr{M}$,

$$\left(m_v(1,4) - m_{v_{(1,4)}}\right) > \left(m_v(1,2) - m_{v_{(1,2)}}\right) + \left(m_v(2,3) - m_{v_{(2,3)}}\right) + \left(m_v(3,4) - m_{v_{(3,4)}}\right) - m_v(2,2) - m_v(3,3). \quad (5)$$

This suggests that if in an M_v-metric, we subtract the term $\Sigma_{i=1}^{v} m_v(\mathfrak{z}_i,\mathfrak{z}_i)$, then the function m_v will no longer be a metric. However, the terms like $\Sigma_{i=1}^{v} m_v(\mathfrak{z}_i,\mathfrak{z}_i)$ are indispensable in the analogous generalizations of a partial metric as a self-distance of a point in this space is not essentially zero.

For instance, if $m_v(\mathfrak{u},\mathfrak{w}) = 3^{-n}$, then for a finite n and $\mathfrak{u} = \mathfrak{w}$, $m_v(\mathfrak{u},\mathfrak{u})$ is not equal to zero.

Definition 8 (see [19]). A function $\psi : [0,\infty) \longrightarrow [0,\infty)$ is a subadditive altering distance function if

(i) ψ is an altering distance function (i.e., ψ is continuous, strictly increasing, and $\psi(t) = 0$ iff $t = 0$)

(ii) $\psi(\mathfrak{u}+\mathfrak{w}) \leq \psi(\mathfrak{u}) + \psi(\mathfrak{w})$, $\mathfrak{u}, \mathfrak{w} \in [0,\infty)$

3. Main Results

We use notations $m^b_{v_{\mathfrak{u},\mathfrak{w}}} = \min\{m^b_v(\mathfrak{u},\mathfrak{u}), m^b_v(\mathfrak{w},\mathfrak{w})\}$ and $M^b_{v_{\mathfrak{u},\mathfrak{w}}} = \max\{m^b_v(\mathfrak{u},\mathfrak{u}), m^b_v(\mathfrak{w},\mathfrak{w})\}$. First, we introduce an M^b_v-metric space.

Definition 9. An M^b_v -metric on a nonempty set $\mathscr{M}$ with $s \geq 1$ is a map $m^b_v : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ satisfying

$$\left(m^b_v\mathrm{i}\right)\; m^b_v(\mathfrak{u},\mathfrak{u}) = m^b_v(\mathfrak{w},\mathfrak{w}) = m^b_v(\mathfrak{u},\mathfrak{w}) \text{ iff } \mathfrak{u} = \mathfrak{w},$$

$$\left(m^b_v\mathrm{ii}\right)\; m^b_{v_{u,w}} \leq m^b_v(u,w),$$

$$\left(m^b_v\mathrm{iii}\right)\; m^b_v(\mathfrak{u},\mathfrak{w}) = m^b_v(\mathfrak{w},\mathfrak{u}),$$

$$\begin{aligned}\left(m^b_v\mathrm{iv}\right)\; &\left(m^b_v(\mathfrak{u},\mathfrak{w}) - m^b_{v_{u,w}}\right)\\ &\leq s\left[\left(m^b_v(\mathfrak{u},\mathfrak{z}_1) - m^b_{v_{\mathfrak{u},\mathfrak{z}_1}}\right) + \left(m^b_v(\mathfrak{z}_1,\mathfrak{z}_2) - m^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\right)\right.\\ &\quad\left. + \cdots + \left(m^b_v(\mathfrak{z}_v,\mathfrak{w}) - m^b_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right)\right] - \Sigma^v_{i=1} m^b_v(\mathfrak{z}_i,\mathfrak{z}_i),\end{aligned} \tag{6}$$

$\mathfrak{u}, \mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_v, \mathfrak{w} \in \mathscr{M}$ and are distinct. A pair $(\mathscr{M}, m^b_v)$ is called an M^b_v-metric space.

Example 10. Let $\mathscr{M} = \mathbb{R}^+$ and $m^b_v : \mathscr{M} \times \mathscr{M} \longrightarrow [0,\infty)$ be defined by $m^b_v(\mathfrak{u},\mathfrak{w}) = |\mathfrak{u}-\mathfrak{w}|^\alpha + \max\{\mathfrak{u},\mathfrak{w}\}^\alpha$, $\alpha > 1$. By routine calculation, we verify that $(\mathscr{M}, m^b_v)$ is an M^b_v -metric space with $s \geq 3^{\alpha-1}$. But $(\mathscr{M}, m^b_v)$ is not an M_v -metric space. Since, for $\mathfrak{u} = 1, \mathfrak{w} = n$ and $\mathfrak{z}_1 = 2, \mathfrak{z}_2 = 3, \cdots, \mathfrak{z}_v = n-1$, we attain

$$\begin{aligned} m^b_v(1,n) - m^b_{v_{1,n}} &= |1-n|^\alpha + \max\{1,n\}^\alpha - 1^\alpha = 1 + n^\alpha - 1 = n^\alpha,\\ m^b_v(1,2) - m^b_{v_{1,2}} &= |1-2|^\alpha + \max\{1,2\}^\alpha - 1^\alpha = 1 + 2^\alpha - 1 = 2^\alpha,\\ m^b_v(2,3) - m^b_{v_{2,3}} &= |2-3|^\alpha + \max\{2,3\}^\alpha - 2^\alpha = 1 + 3^\alpha - 2^\alpha, \vdots\\ m^b_v(n-2,n-1) - m^b_{v_{n-2,n-1}} &= |n-2-n+1|^\alpha + \max\{n-2,n-1\}^\alpha\\ &\quad - (n-2)^\alpha = 1 + (n-1)^\alpha - (n-2)^\alpha.\end{aligned} \tag{7}$$

Therefore, $m^b_v(1,n) - m^b_{v_{1,n}} > m^b_v(1,2) - m^b_{v_{1,2}} + m^b_v(2,3) - m^b_{v_{2,3}} + \cdots + m^b_v(n-2,n-1) - m^b_{v_{n-2,n-1}}$.

Also, if $s = 1$, $(\mathscr{M}, m^b_v)$ is an improvement and extension of an M_v-metric space [10]. Inclusion of the terms containing nonzero self-distances demonstrate that it is a proper generalization of a notion of an M_v-metric space and consequently partial metric space as well. In particular, $(\mathscr{M}, m^b_v)$ is an M_v -metric [18] for $v = 1$ and rectangular M_b-metric space [20] for $v = 2$.

Theorem 11. *Let m^b_v be an M^b_v -metric on a set $\mathscr{M}$. Let $(pb_v(s))^*(\mathfrak{u},\mathfrak{w}) = m^b_v(\mathfrak{u},\mathfrak{w}) + M^b_{v_{\mathfrak{u},\mathfrak{w}}} - m^b_{v_{\mathfrak{u},\mathfrak{w}}}$, then $(pb_v(s))^*$ is a generalized partial $b_v(s)$ -metric on $\mathscr{M}$.*

Proof. Let $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$:

$(pb_v(s)(i))$

Let$(pb_v(s))^*(\mathfrak{u},\mathfrak{u}) = (pb_v(s))^*(\mathfrak{w},\mathfrak{w}) = (pb_v\;\;(s))^*(\mathfrak{u},\mathfrak{w}) \Leftrightarrow m^b_v(\mathfrak{u},\mathfrak{u}) = m^b_v(\mathfrak{w},\mathfrak{w}) = (pb_v(s))^*(\mathfrak{u},\mathfrak{w})$.

So, $(pb_v(s))^*(\mathfrak{u},\mathfrak{w}) = m^b_v(\mathfrak{u},\mathfrak{w})$, i.e., $m^b_v(\mathfrak{u},\mathfrak{u}) = m^b_v(\mathfrak{w},\mathfrak{w}) = m^b_v(\mathfrak{u},\mathfrak{w}) \Longrightarrow \mathfrak{u} = \mathfrak{w}$.

Hence, $(pb_v(s))^*(\mathfrak{u},\mathfrak{u}) = (pb_v(s))^*(\mathfrak{v},\mathfrak{v}) = (pb_v(s))^*(\mathfrak{u},\mathfrak{v}) \Leftrightarrow \mathfrak{u} = \mathfrak{v}$.

$(pb_v(s)(\mathrm{ii}))$ For $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$, $(pb_v(s))^*(\mathfrak{u},\mathfrak{u}) = m^b_v(\mathfrak{u},\mathfrak{u}) + M^b_{v_{\mathfrak{u},\mathfrak{u}}} - m^b_{v_{\mathfrak{u},\mathfrak{u}}} = m^b_v(\mathfrak{u},\mathfrak{u})$ and $m^b_v(\mathfrak{u},\mathfrak{w}) - m^b_{v_{\mathfrak{u},\mathfrak{w}}} \geq 0$.

Now, $(pb_v(s))^*(\mathfrak{u},\mathfrak{u}) = m^b_v(\mathfrak{u},\mathfrak{u}) \leq M^b_{v_{\mathfrak{u},\mathfrak{w}}} \leq M^b_{v_{\mathfrak{u},\mathfrak{w}}} + m^b_v(\mathfrak{u},\mathfrak{w}) - m^b_{v_{\mathfrak{u},\mathfrak{w}}} = (pb_v(s))^*(\mathfrak{u},\mathfrak{w})$.

$(pb_v(s)(\mathrm{iii}))$ Symmetric condition is trivially satisfied.

$(pb_v(s)(\mathrm{iv}))$ For $\mathfrak{u}, \mathfrak{z}_1, \mathfrak{z}_2, \cdots \mathfrak{z}_v, \in \mathscr{M}$,

$$\begin{aligned}(pb_v(s))^*(\mathfrak{u},\mathfrak{w}) &= m^b_v(\mathfrak{u},\mathfrak{w}) + M^b_{v_{\mathfrak{u},\mathfrak{w}}} - m^b_{v_{\mathfrak{u},\mathfrak{w}}}\\ &\leq s\left[\left(m^b_v(\mathfrak{u},\mathfrak{z}_1) - m^b_{v_{u,\mathfrak{z}_1}}\right)\right.\\ &\quad + \left(m^b_v(\mathfrak{z}_1,\mathfrak{z}_2) - m^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\right)\\ &\quad\left. + \cdots + \left(m^b_v(\mathfrak{z}_v,\mathfrak{w}) - m^b_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right)\right] - \Sigma^v_{i=1} m^b_v(\mathfrak{z}_i,\mathfrak{z}_i)\\ &= s\left[\left(m^b_v(\mathfrak{u},\mathfrak{z}_1) - m^b_{v_{\mathfrak{u},\mathfrak{z}_1}}\right) + \left(m^b_v(\mathfrak{z}_1,\mathfrak{z}_2) - m^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\right)\right.\\ &\quad\left. + \cdots + \left(m^b_v(\mathfrak{z}_v,\mathfrak{w}) - m^b_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right)\right] - \Sigma^v_{i=1} m^b_v(\mathfrak{z}_i,\mathfrak{z}_i) + M^b_{v_{\mathfrak{u},\mathfrak{w}}}\\ &\leq s\left[\left(m^b_v(\mathfrak{u},\mathfrak{z}_1) - m^b_{v_{\mathfrak{u},\mathfrak{z}_1}}\right) + \left(m^b_v(\mathfrak{z}_1,\mathfrak{z}_2) - m^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\right)\right.\\ &\quad\left. + \cdots + \left(m^b_v(\mathfrak{z}_v,\mathfrak{w}) - m^b_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right)\right] + M^b_{v_{\mathfrak{u},\mathfrak{z}_1}} + M^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\\ &\quad + \cdots + M^b_{v_{\mathfrak{z}_v,\mathfrak{w}}} - \Sigma^v_{i=1} m^b_v(\mathfrak{z}_i,\mathfrak{z}_i)\\ &\leq s\left[\left(m^b_v(\mathfrak{u},\mathfrak{z}_1) - m^b_{v_{\mathfrak{u},\mathfrak{z}_1}}\right) + \left(m^b_v(\mathfrak{z}_1,\mathfrak{z}_2) - m^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\right)\right.\\ &\quad\left. + \cdots + \left(m^b_v(\mathfrak{z}_v,\mathfrak{w}) - m^b_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right)\right]\\ &\quad + s\left[M^b_{v_{\mathfrak{u},\mathfrak{z}_1}} + M^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}} + \cdots + M^b_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right] - \Sigma^v_{i=1}(pb_v(s))^*(\mathfrak{z}_i,\mathfrak{z}_i))\\ &= s\left[\left(m^b_v(\mathfrak{u},\mathfrak{z}_1) + M^b_{v_{\mathfrak{u},\mathfrak{z}_1}} - m^b_{v_{\mathfrak{u},\mathfrak{z}_1}}\right)\right.\\ &\quad + \left(m^b_v(\mathfrak{z}_1,\mathfrak{z}_2) + M^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}} - m^b_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}\right)\\ &\quad\left. + \cdots + \left(m^b_v(\mathfrak{z}_v,\mathfrak{w}) + M^b_{v_{\mathfrak{z}_v,\mathfrak{w}}} - m^b_{v_{\mathfrak{z}_v,\mathfrak{w}}}\right)\right] - \Sigma^v_{i=1}(pb_v(s))^*(\mathfrak{z}_i,\mathfrak{z}_i))\\ &= s[(pb_v(s))^*(\mathfrak{u},\mathfrak{z}_1) + (pb_v(s))^*(\mathfrak{z}_1,\mathfrak{z}_2) + \cdots + (pb_v(s))^*(\mathfrak{z}_v,\mathfrak{w})]\\ &\quad - \Sigma^v_{i=1}(pb_v(s))^*(\mathfrak{z}_i,\mathfrak{z}_i)).\end{aligned} \tag{8}$$

Thus, $(\mathscr{M}, (pb_\nu(s))^*)$ is a generalized partial $b_\nu(s)$-metric space.

Theorem 12. *Let m_ν^b be an M_ν^b -metric on a set $\mathscr{M}$. Let $(b_\nu(s))^*(\mathfrak{u},\mathfrak{w}) = m_\nu^b(\mathfrak{u},\mathfrak{w}) - m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b$, then $(b_\nu(s))^*$ is a $b_\nu(s)$ -metric on $\mathscr{M}$.*

Proof. $(b_\nu(s)(\text{i}))$ For $\mathfrak{u},\mathfrak{w} \in \mathscr{M}$,

$$\begin{aligned}\text{let } (b_\nu(s))^*(\mathfrak{u},\mathfrak{w}) = 0 &\Leftrightarrow m_\nu^b(\mathfrak{u},\mathfrak{w}) = m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b \Leftrightarrow m_\nu^b(\mathfrak{u},\mathfrak{w}) \\ &= m_\nu^b(\mathfrak{u},\mathfrak{u}) = m_\nu^b(\mathfrak{w},\mathfrak{w}) \Leftrightarrow \mathfrak{u} = \mathfrak{w}.\end{aligned} \tag{9}$$

$(b_\nu(s)(\text{ii}))$ Since, $m_\nu^b(\mathfrak{u},\mathfrak{w}) \geq m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b, m_\nu^b(\mathfrak{u},\mathfrak{w}) - m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b \geq 0$, i.e., $(b_\nu(s))^*(\mathfrak{u},\mathfrak{w}) \geq 0$, $\mathfrak{u},\mathfrak{w} \in \mathscr{M}$.

$(b_\nu(s)(\text{iii}))$ Symmetric condition is trivially satisfied.

$(b_\nu(s)(\text{iv}))$ For $\mathfrak{u}, \mathfrak{z}_1, \mathfrak{z}_2, \cdots \mathfrak{z}_\nu, \mathfrak{w} \in \mathscr{M}$,

$$\begin{aligned}(b_\nu(s))^*(\mathfrak{u},\mathfrak{w}) &= m_\nu^b(\mathfrak{u},\mathfrak{w}) - m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b, \\ &\leq s\Big[\Big(m_\nu^b(\mathfrak{u},\mathfrak{z}_1) - m_{\nu_{\mathfrak{u},\mathfrak{z}_1}}^b\Big) + \Big(m_\nu^b(\mathfrak{z}_1,\mathfrak{z}_2) - m_{\nu_{\mathfrak{z}_1,\mathfrak{z}_2}}^b\Big) \\ &\quad + \cdots + \Big(m_\nu^b(\mathfrak{z}_\nu,\mathfrak{w}) - m_{\nu_{\mathfrak{z}_\nu,\mathfrak{w}}}^b\Big)\Big] - \Sigma_{i=1}^\nu m_\nu^b(\mathfrak{z}_i,\mathfrak{z}_i), \\ &\leq s\Big[\Big(m_\nu^b(\mathfrak{u},\mathfrak{z}_1) - m_{\nu_{\mathfrak{u},\mathfrak{z}_1}}^b\Big) + \Big(m_\nu^b(\mathfrak{z}_1,\mathfrak{z}_2) - m_{\nu_{\mathfrak{z}_1,\mathfrak{z}_2}}^b\Big) \\ &\quad + \cdots + \Big(m_\nu^b(\mathfrak{z}_\nu,\mathfrak{w}) - m_{\nu_{\mathfrak{z}_\nu,\mathfrak{w}}}^b\Big)\Big] \\ &= s[(b_\nu(s))^*(\mathfrak{u},\mathfrak{z}_1) + (b_\nu(s))^*(\mathfrak{z}_1,\mathfrak{z}_2) \\ &\quad + \cdots + (b_\nu(s))^*(\mathfrak{z}_\nu,\mathfrak{w})].\end{aligned} \tag{10}$$

Thus, $(\mathscr{M}, (b_\nu(s))^*)$ is a $b_\nu(s)$-metric.

Remark 13. Let $pb_\nu(s)$ be a partial $b_\nu(s)$ -metric on a set $\mathscr{M}$, then $(b_\nu(s))_1^* = pb_\nu(s)(\mathfrak{u},\mathfrak{w}) - pb_\nu(s)(\mathfrak{u},\mathfrak{u}) - pb_\nu(s)(\mathfrak{w},\mathfrak{w})$ is $b_\nu(s)$ -metric space.

To discuss the topology corresponding to M_ν^b-metric, the open ball with centre at $\mathfrak{u}$ and radius $\mathfrak{r} \in (0,\infty)$ is defined as $\mathscr{U}_{m_\nu^b}(\mathfrak{u},\varepsilon) = \{\mathfrak{w} \in \mathscr{M} : m_\nu^b(\mathfrak{u},\mathfrak{w}) < m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b + \varepsilon/s\}$.

Similarly, the closed ball with centre at $\mathfrak{u}$ and radius $\mathfrak{r} \in (0,\infty)$ is defined as $\mathscr{U}_{m_\nu^b}[\mathfrak{u},\varepsilon] = \{\mathfrak{w} \in \mathscr{M} : m_\nu^b(\mathfrak{u},\mathfrak{w}) \leq m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b + \varepsilon/s\}$.

Lemma 14. *In an M_ν^b -metric space $(\mathscr{M}, m_\nu^b)$, every open ball is an open set.*

Proof. Let $\mathfrak{w}_0 \in \mathscr{U}_{m_\nu^b}(\mathfrak{u},\mathfrak{r})$, then $m_\nu^b(\mathfrak{u},\mathfrak{w}_0) < m_{\nu_{\mathfrak{u},\mathfrak{w}_0}}^b + \mathfrak{r}/s$. Choose $\varepsilon/s = m_{\nu_{\mathfrak{u},\mathfrak{w}_0}}^b + \mathfrak{r}/s - m_\nu^b(\mathfrak{u},\mathfrak{w}_0) > 0$.

Again, let $\mathfrak{w}_1 \in \mathscr{U}_{m_\nu^b}(\mathfrak{w}_0,\varepsilon)$, so $m_\nu^b(\mathfrak{w}_1,\mathfrak{w}_0) < m_{\nu_{\mathfrak{w}_1,\mathfrak{w}_0}}^b + \varepsilon/s$ and choose $\varepsilon_1/s = m_{\nu_{\mathfrak{w}_1,\mathfrak{w}_0}}^b + \varepsilon/s - m_\nu^b(\mathfrak{w}_1,\mathfrak{w}_0) > 0$.

Proceeding as above, let $\mathfrak{w}_\nu \in \mathscr{U}_{m_\nu^b}(\mathfrak{w}_{\nu-1},\varepsilon_\nu)$, so $m_\nu^b(\mathfrak{w}_\nu, \mathfrak{w}_{\nu-1}) < m_{\nu_{\mathfrak{w}_\nu,\mathfrak{w}_{\nu-1}}}^b + \varepsilon_{\nu-1}/s$, choose $\varepsilon_\nu/s = m_{\nu_{\mathfrak{w}_\nu,\mathfrak{w}_{\nu-1}}}^b + \varepsilon_{\nu-1} - m_\nu^b(\mathfrak{u},\mathfrak{w}_1) > 0$.

Now, for $\mathfrak{u},\mathfrak{w}_0,\mathfrak{w}_1,\cdots,\mathfrak{w}_\nu$,

$$\begin{aligned}m_\nu^b(\mathfrak{u},\mathfrak{w}_\nu) - m_{\nu_{\mathfrak{u},\mathfrak{w}_\nu}} &\leq s\Big[\Big(m_\nu^b(\mathfrak{u},\mathfrak{w}_0) - m_{\nu_{\mathfrak{u},\mathfrak{w}_0}}\Big) + \Big(m_\nu^b(\mathfrak{w}_0,\mathfrak{w}_1) - m_{\nu_{\mathfrak{w}_0,\mathfrak{w}_1}}\Big) \\ &\quad + \cdots + \Big(m_\nu^b(\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu) - m_{\nu_{\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu}}\Big)\Big] - m_\nu^b(\mathfrak{w}_1,\mathfrak{w}_1) \\ &\quad - m_\nu^b(\mathfrak{w}_2,\mathfrak{w}_2) - \cdots - m_\nu^b(\mathfrak{w}_{\nu-1},\mathfrak{w}_{\nu-1}) \\ &\leq s\Big[\Big(m_\nu^b(\mathfrak{u},\mathfrak{w}_0) - m_{\nu_{\mathfrak{u},\mathfrak{w}_0}}\Big) + \Big(m_\nu^b(\mathfrak{w}_0,\mathfrak{w}_1) - m_{\nu_{\mathfrak{w}_0,\mathfrak{w}_1}}\Big) \\ &\quad + \cdots + \Big(m_\nu^b(\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu) - m_{\nu_{\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu}}\Big)\Big] \\ &= s\Big[\Big(\frac{\mathfrak{r}}{s} - \frac{\varepsilon}{s}\Big) + \Big(\frac{\varepsilon}{s} - \frac{\varepsilon_1}{s}\Big) + \cdots + \Big(\frac{\varepsilon_{\nu-1}}{s} - \frac{\varepsilon_\nu}{s}\Big)\Big] = \mathfrak{r} - \varepsilon_\nu.\end{aligned} \tag{11}$$

Hence, $\mathscr{U}_{m_\nu^b}(\mathfrak{w}_0,\varepsilon) \subseteq \mathscr{U}_{m_\nu^b}(\mathfrak{u},\mathfrak{r})$.

Theorem 15. *If $(\mathscr{M}, m_b^\nu)$ is an M_ν^b -metric space and τ_ν^b a topology generated by the open balls $\mathscr{U}_{m_\nu^b}(\mathfrak{u},\mathfrak{r})$, then $(\mathscr{M}, \tau_\nu^b)$ is a $\mathscr{T}_0$ -space.*

Proof. Let $(\mathscr{M}, \tau_\nu^b)$ be an M_ν^b-metric space and $\mathfrak{u},\mathfrak{w} \in \mathscr{M}$ are two distinct points. Then,

$$m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b \leq m_\nu^b(\mathfrak{u},\mathfrak{w}) \Longrightarrow \min\Big\{m_\nu^b(\mathfrak{u},\mathfrak{u}), m_\nu^b(\mathfrak{w},\mathfrak{w})\Big\} \leq m_\nu^b(\mathfrak{u},\mathfrak{w}), \tag{12}$$

i.e., $m_\nu^b(\mathfrak{u},\mathfrak{u}) \leq m_\nu^b(\mathfrak{u},\mathfrak{w})$ or $m_\nu^b(\mathfrak{w},\mathfrak{w}) \leq m_\nu^b(\mathfrak{u},\mathfrak{w})$.

Firstly, assume that $m_\nu^b(\mathfrak{u},\mathfrak{u}) = m_\nu^b(\mathfrak{u},\mathfrak{w})$,

$$\begin{aligned}\text{then } m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b &= m_\nu^b(\mathfrak{u},\mathfrak{u}) = m_\nu^b(\mathfrak{w},\mathfrak{w}) < m_\nu^b(\mathfrak{u},w) \Longrightarrow m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b - m_\nu^b(\mathfrak{u},\mathfrak{w}) \\ &= m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b - m_\nu^b(\mathfrak{u},\mathfrak{u}) < 0.\end{aligned} \tag{13}$$

For $s \geq 1$, if we chose $\varepsilon > 0$ such that $m_\nu^b(\mathfrak{u},\mathfrak{w}) = m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b + \varepsilon/s$, so $\mathfrak{w} \notin \mathscr{U}_{m_\nu^b}(\mathfrak{u},\varepsilon)$.

Next, assume that $m_\nu^b(\mathfrak{u},\mathfrak{u}) < m_\nu^b(\mathfrak{w},\mathfrak{w})$, then $m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b = m_\nu^b(\mathfrak{u},\mathfrak{u}) < m_\nu^b(\mathfrak{u},\mathfrak{w})$:

$$\Longrightarrow m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b - m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b = m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b - m_\nu^b(\mathfrak{u},\mathfrak{u}) < 0. \tag{14}$$

Again, for $s \geq 1$, if we chose $\varepsilon_1 > 0$ such that $m_\nu^b(\mathfrak{u},\mathfrak{w}) = m_{\nu_{\mathfrak{u},\mathfrak{w}}}^b + \varepsilon/s$, so $\mathfrak{w} \notin \mathscr{U}_{m_\nu^b}(\mathfrak{u},\varepsilon_1)$.

Similarly, for $m_\nu^b(\mathfrak{w},\mathfrak{w}) < m_\nu^b(\mathfrak{u},\mathfrak{u})$, one may easily find an open ball so that $\mathfrak{u} \in \mathscr{U}_{m_\nu^b}(\mathfrak{u},\varepsilon_1)$ and $\mathfrak{w} \notin \mathscr{U}_{m_\nu^b}(\mathfrak{u},\varepsilon_1)$, i.e., for two distinct points $\mathfrak{u},\mathfrak{w} \in \mathscr{M}$, there is a ball including the point $\mathfrak{u}$ but not including the other point $\mathfrak{w}$. Thus, $(\mathscr{M}, m_\nu^b)$ is a $\mathscr{T}_0$-space.

Now, we discuss the convergence of the sequence and introduce definitions related to it.

Definition 16.

(i) A sequence $\{\mathfrak{u}_n\}$ in $(\mathscr{M}, m_\nu^b)$ is m_ν^b-convergent to $\mathfrak{u} \in \mathscr{M}$ iff $\lim_{n\longrightarrow\infty} m_\nu^b(\mathfrak{u}_n, \mathfrak{u}) - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}}} = 0$

In other words, a sequence $\{\mathfrak{u}_n\}$ in a topological space $(\mathscr{M}, \tau_\nu^b)$ converges to a point $\mathfrak{u}$ in $\mathscr{M}$ if for each open ball $\mathscr{U}_{m_\nu^b}(\mathfrak{u}, \varepsilon)$ containing $\mathfrak{u}$ there exists k so that for each $n > k$, $\mathfrak{u}_n \in \mathscr{U}_{m_\nu^b}(\mathfrak{u}, \varepsilon)$.

(ii) A sequence $\{\mathfrak{u}_n\}$ in $(\mathscr{M}, m_\nu^b)$ is m_ν^b-Cauchy sequence iff $\lim_{n,m\longrightarrow\infty}(m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}_m}}^b)$ and $\lim_{n,m\longrightarrow\infty}(M_{\nu_{\mathfrak{u}_n,\mathfrak{u}_m}}^b - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}_m}}^b)$ exist and are finite

(iii) An M_ν^b-metric space is m_ν^b-complete if each m_ν^b-Cauchy sequence $\{\mathfrak{u}_n\}$ converges to a point $\mathfrak{u} \in \mathscr{M}$ so that $\lim_{n,m\longrightarrow\infty}(m_\nu^b(\mathfrak{u}_n, \mathfrak{u}) - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}}}^b) = 0$ and $\lim_{n,m\longrightarrow\infty}(M_{\nu_{\mathfrak{u}_n,\mathfrak{u}}}^b - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}}}^b) = 0$

Lemma 17. *Let $(\mathscr{M}, m_\nu^b)$ be an M_ν^b -metric space and $\mathscr{A}:\mathscr{M} \longrightarrow \mathscr{M}$ a self-map on $\mathscr{M}$ so that there exists $\eta \in [0, 1/s]$, satisfying*

$$m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) \le \eta m_\nu^b(\mathfrak{u}, \mathfrak{w}). \tag{15}$$

Consider the sequence $\{\mathfrak{u}_n\}$ defined by $\mathfrak{u}_{n+1} = \mathscr{A}\mathfrak{u}_n$. If $\mathfrak{u}_n \longrightarrow \mathfrak{u}$ as $n \longrightarrow \infty$, then $\mathscr{A}\mathfrak{u}_n \longrightarrow \mathscr{A}\mathfrak{u}$ as $n \longrightarrow \infty$.

Proof. If $m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}) = 0$, $m_{\nu_{\mathscr{A}\mathfrak{u}_n,\mathscr{A}\mathfrak{u}}}^b \le m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}) = 0 \Longrightarrow m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}) - m_{\nu_{\mathscr{A}\mathfrak{u}_n,\mathscr{A}\mathfrak{u}}}^b \longrightarrow 0$, i.e., $\mathscr{A}\mathfrak{u}_n \longrightarrow \mathscr{A}\mathfrak{u}_n$ as $n \longrightarrow \infty$.

Now, if $m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}) > 0$, then by (15),

$$m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}) \le \eta m_\nu^b(\mathfrak{u}_n, \mathfrak{u}). \tag{16}$$

There are two cases.

Case 1. If $m_\nu^b(\mathfrak{u}, \mathfrak{u}) \le m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n)$, then $m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n) \longrightarrow 0$ as $n \longrightarrow \infty \Longrightarrow m_\nu^b(\mathfrak{u}, \mathfrak{u}) = 0$.

Since $m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) \le m_\nu^b(\mathfrak{u}, \mathfrak{u})$, we have $m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) = 0$. Again, $\mathfrak{u}_n \longrightarrow \mathfrak{u}$ implies that $\lim_{n\longrightarrow\infty}(m_\nu^b(\mathfrak{u}_n, \mathfrak{u}) - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}}}^b) = 0$.

Since, $m_{\nu_{\mathfrak{u}_n,\mathfrak{u}}}^b = \min\{m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n), m_\nu^b(\mathfrak{u}, \mathfrak{u})\} = 0$, so $\lim_{n\longrightarrow\infty} m_\nu^b(\mathfrak{u}_n, \mathfrak{u}) = 0$.

Now, from equation (16),

$$\lim_{n\longrightarrow\infty} m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}) = 0, \text{ i.e.,} \mathscr{A}\mathfrak{u}_n \longrightarrow \mathscr{A}\mathfrak{u} \text{ as } n \longrightarrow \infty. \tag{17}$$

Case 2. If $m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n) \le m_\nu^b(\mathfrak{u}, \mathfrak{u})$, then $m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

So, as above $\lim_{n\longrightarrow\infty} m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}) = 0$, i.e., $\mathscr{A}\mathfrak{u}_n \longrightarrow \mathscr{A}\mathfrak{u}$ as $n \longrightarrow \infty$.

Now, we prove the first main result for a Kannan type contraction.

Theorem 18. *Let $(\mathscr{M}, m_\nu^b)$ be an M_ν^b -complete metric space with coefficient $s \ge 1$. Suppose, a self-map $\mathscr{A}:\mathscr{M} \longrightarrow \mathscr{M}$ satisfies*

$$m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) \le \mu\left[m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w})\right], \quad \mu < \frac{1}{2s}, \mathfrak{u}, \mathfrak{w} \in \mathscr{M}. \tag{18}$$

Then, $\mathscr{A}$ has a unique fixed point $\mathfrak{u}^*$ so that $m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*) = 0$ and the sequence of iterates $\{\mathscr{A}^n\mathfrak{u}_0\} \subseteq \mathscr{M}$ converges to $\mathfrak{u}^* \in \mathscr{M}$.

Proof. Starting from the given element $\mathfrak{u}_0 \in \mathscr{M}$, form the sequence $\{\mathfrak{u}_n\}$, where $\mathfrak{u}_n = \mathscr{A}\mathfrak{u}_{n-1}, n \in \mathbb{N}$. If $m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0, n \ge 0$, then $\mathscr{A}\mathfrak{u}_n = \mathfrak{u}_{n+1} = \mathfrak{u}_n$ and $m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n) = 0$, and this completes the proof.

Further, take $m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) > 0$, $n \ge 0$. For $\mathfrak{u} = \mathfrak{u}_n, \mathfrak{w} = \mathfrak{u}_{n+1}$, utilizing condition (18),

$$\begin{aligned} m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) &= m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}_{n+1}) \\ &\le \mu\left[m_\nu^b(\mathfrak{u}_n, \mathscr{A}\mathfrak{u}_n) + m_\nu^b(\mathfrak{u}_{n+1}, \mathscr{A}\mathfrak{u}_{n+1})\right] \\ &= \mu\left[m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) + m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})\right], \end{aligned}$$

$$\text{i.e.,}(1-\mu) m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) \le \mu m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}),$$

$$\text{i.e.,} m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) \le \frac{\mu}{1-\mu} m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}). \tag{19}$$

Let, $\xi_\nu^b(n) = m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}), n \ge 0, \lambda = \mu/(1-\mu) < 1$, then $\xi_\nu^b(n+1) \le \lambda \xi_\nu^b(n)$.

On repeating these steps, we get $\xi_\nu^b(n+1) \le \lambda^n \xi_\nu^b(0) \longrightarrow 0, n \longrightarrow \infty$.

Now,

$$\begin{aligned} m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}) &= m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}_n) \le \mu\left[m_\nu^b(\mathfrak{u}_n, \mathscr{A}\mathfrak{u}_n) + m_\nu^b(\mathfrak{u}_n, \mathscr{A}\mathfrak{u}_n)\right] \\ &= 2\mu m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 2\mu \xi_\nu^b(n) \\ &\le 2\lambda^{n-1}\xi_\nu^b(0) \longrightarrow 0, \text{as } n \longrightarrow \infty. \end{aligned} \tag{20}$$

First, we show that $\mathfrak{u}_n \ne \mathfrak{u}_m$, for $n \ne m$. Suppose, $\mathfrak{u}_n = \mathfrak{u}_m$, for $n > m$, then $\mathscr{A}\mathfrak{u}_n = \mathfrak{u}_{n+1} = \mathscr{A}\mathfrak{u}_m = \mathfrak{u}_{m+1}$.

Now, using inequality (18), for $\mathfrak{u} = \mathfrak{u}_n$ and $\mathfrak{w} = \mathfrak{u}_{n+1}$,

$$\begin{aligned}\xi_\nu^b(m) = m_\nu^b(\mathfrak{u}_m, \mathfrak{u}_{m+1}) = m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = \xi_\nu^b(n) \le \lambda \xi_\nu^b(n-1) \\ \le \lambda^2 \xi_\nu^b(n-2) \le \cdots \le \lambda^{n-m} \xi_\nu^b(m) < \xi_\nu^b(m),\end{aligned} \tag{21}$$

a contradiction. Thus, $\mathfrak{u}_n \ne \mathfrak{u}_m$, for $n \ne m$.

Now, we assert that $\{\mathfrak{u}_n\}$ is a Cauchy sequence in $(\mathscr{M}, m_\nu^b)$. We discuss two cases.

Case 1. First, let l be odd, i.e., $l = 2m+1$, for $n, m \in \mathbb{N}$. Now, by using $(m_\nu^b iv)$ for $n \le \nu \le n+l$,

$$\begin{aligned}
m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+l}) = m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+2m+1}) \\
&\le s\Big[m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) + m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) + \cdots + m_\nu^b(\mathfrak{u}_{n+\nu-1}, \mathfrak{u}_{n+\nu}) \\
&\quad + m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+2m+1})\Big] - m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}) - m_\nu^b(\mathfrak{u}_{n+2}, \mathfrak{u}_{n+2}) \\
&\quad - \cdots - m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+\nu}) \le s\left(\lambda^{n-1} + \lambda^n + \cdots \lambda^{n+\nu-2}\right)\xi_\nu^b(0) \\
&\quad - 2\left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+\nu-1}\right)\xi_\nu^b(0) + s m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+2m+1}) \\
&= s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - 2\frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) + s m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+2m+1}) \\
&\le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) + s^2\Big[m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+\nu+1}) \\
&\quad + m_\nu^b(\mathfrak{u}_{n+\nu+1}, \mathfrak{u}_{n+\nu+2}) + \cdots + m_\nu^b(\mathfrak{u}_{n+2\nu-1}, \mathfrak{u}_{n+2\nu}) + m_\nu^b(\mathfrak{u}_{n+2\nu}, \mathfrak{u}_{n+2m+1})\Big] \\
&\quad - s\Big[m_\nu^b(\mathfrak{u}_{n+\nu+1}, \mathfrak{u}_{n+\nu+1}) + m_\nu^b(\mathfrak{u}_{n+\nu+2}, \mathfrak{u}_{n+\nu+2}) + \cdots + m_\nu^b(\mathfrak{u}_{n+2\nu}, \mathfrak{u}_{n+2\nu})\Big] \\
&\le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) + s^2\left(\lambda^{n+\nu-1} + \lambda^{n+\nu}\right. \\
&\quad \left. + \cdots + \lambda^{n+2\nu-2}\right)\xi_\nu^b(0) + s^2 m_\nu^b(\mathfrak{u}_{n+2\nu}, \mathfrak{u}_{n+2m+1}) - s\left(\lambda^{n+\nu} + \lambda^{n+\nu+1}\right. \\
&\quad \left. + \cdots + \lambda^{n+2\nu-1}\right)\xi_\nu^b(0) \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda} \\
&\quad + s^2\left(\frac{\lambda^{n+\nu-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s\frac{\lambda^{n+\nu}(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \cdots \\
&\quad + s^{2m/\nu-1} m_\nu^b(\mathfrak{u}_{n+2m-\nu}, \mathfrak{u}_{n+2m+1}) \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) \\
&\quad - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) + s^2\left(\frac{\lambda^{n+\nu-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s\frac{\lambda^{n+\nu}(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + \cdots + s^{2m/\nu}\Big[m_\nu^b(\mathfrak{u}_{n+2m-\nu}, \mathfrak{u}_{n+2m-\nu+1}) + m_\nu^b(\mathfrak{u}_{n+2m-\nu+1}, \mathfrak{u}_{n+2m-\nu+2}) \\
&\quad + \cdots + m_\nu^b(\mathfrak{u}_{n+2m}, \mathfrak{u}_{n+2m+1})\Big] - s^{2m/\nu-1}\Big[m_\nu^b(\mathfrak{u}_{n+2m-\nu+1}, \mathfrak{u}_{n+2m-\nu+1}) \\
&\quad + \cdots + m_\nu^b(\mathfrak{u}_{n+2m}, \mathfrak{u}_{n+2m})\Big] \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + s^2\left(\frac{\lambda^{n+\nu-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s\frac{\lambda^{n+\nu}(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + \cdots + s^{2m/\nu}\left(\frac{\lambda^{n+2m-\nu-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s^{2m/\nu-1}\frac{\lambda^{n+2m-\nu}(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b \\
&\quad \cdot (0) \longrightarrow 0, \text{ as } n \longrightarrow \infty,
\end{aligned} \tag{22}$$

i.e., $\lim_{n,m\longrightarrow\infty} m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+2m+1}) = 0$.

Case 2. Now, let l is even, i.e., $l = 2m$ for $n, m \in \mathbb{N}$. Now, by using $(m_\nu^b iv)$ for $n \le \nu \le n+l$,

$$\begin{aligned}
m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+l}) = m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+2m}) \le s\Big[m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) + m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) \\
&\quad + \cdots + m_\nu^b(\mathfrak{u}_{n+\nu-1}, \mathfrak{u}_{n+\nu}) + m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+2m})\Big] - m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}) \\
&\quad - m_\nu^b(\mathfrak{u}_{n+2}, \mathfrak{u}_{n+2}) - \cdots - m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+\nu}) \\
&\le s\left(\lambda^{n-1} + \lambda^n + \cdots + \lambda^{n+\nu-2}\right)\xi_\nu^b(0) - \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+\nu-1}\right)\xi_\nu^b(0) \\
&\quad + s m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+2m}) = s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + s m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+2m}) \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + s^2\Big[m_\nu^b(\mathfrak{u}_{n+\nu}, \mathfrak{u}_{n+\nu+1}) + m_\nu^b(\mathfrak{u}_{n+\nu+1}, \mathfrak{u}_{n+\nu+2}) + \cdots + m_\nu^b(\mathfrak{u}_{n+2\nu-1}, \mathfrak{u}_{n+2\nu}) \\
&\quad + m_\nu^b(\mathfrak{u}_{n+2\nu}, \mathfrak{u}_{n+2m+1})\Big] - s\Big[m_\nu^b(\mathfrak{u}_{n+\nu+1}, \mathfrak{u}_{n+\nu+1}) + m_\nu^b(\mathfrak{u}_{n+\nu+2}, \mathfrak{u}_{n+\nu+2}) \\
&\quad + \cdots + m_\nu^b(\mathfrak{u}_{n+2\nu}, \mathfrak{u}_{n+2\nu})\Big] \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + s^2\left(\lambda^{n+\nu-1} + \lambda^{n+\nu} + \cdots + \lambda^{n+2\nu-2}\right) m_\nu^b(0) + s^2 m_\nu^b(\mathfrak{u}_{n+2\nu}, \mathfrak{u}_{n+2m}) \\
&\quad - s\left(\lambda^{n+\nu} + \lambda^{n+\nu+1} + \cdots + \lambda^{n+2\nu-1}\right)\xi_\nu^b(0) \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) \\
&\quad - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) + s^2\left(\frac{\lambda^{n+\nu-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s\frac{\lambda^{n+\nu}(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + \cdots + s^{2m/\nu-1} m_\nu^b(\mathfrak{u}_{n+2m-1-\nu}, \mathfrak{u}_{n+2m}) \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) \\
&\quad - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) + s^2\left(\frac{\lambda^{n+\nu-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s\frac{\lambda^{n+\nu}(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) \cdots \\
&\quad + s^{2m/\nu}\Big[m_\nu^b(\mathfrak{u}_{n+2m-1-\nu}, \mathfrak{u}_{n+2m-\nu}) + m_\nu^b(\mathfrak{u}_{n+2m-\nu}, \mathfrak{u}_{n+2m-\nu+1}) \\
&\quad + \cdots + m_\nu^b(\mathfrak{u}_{n+2m-1}, \mathfrak{u}_{n+2m})\Big] - s^{2m/\nu-1}\Big[m_\nu^b(\mathfrak{u}_{n+2m-\nu}, \mathfrak{u}_{n+2m-\nu}) \\
&\quad + \cdots + m_\nu^b(\mathfrak{u}_{n+2m-1}, \mathfrak{u}_{n+2m-1})\Big] \le s\left(\frac{\lambda^{n-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) \\
&\quad - \frac{\lambda^n(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b(0) + s^2\left(\frac{\lambda^{n+\nu-1}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s\frac{\lambda^{n+\nu}(1-\mu^\nu)}{1-\lambda}\xi_\nu^b(0) \\
&\quad + \cdots + s^{2m/\nu}\left(\frac{\lambda^{n+2m-\nu-2}(1-\lambda^\nu)}{1-\lambda}\right)\xi_\nu^b(0) - s^{2m/\nu-1}\frac{\lambda^{n+2m-\nu-1}(1-\lambda^\nu)}{1-\lambda}\xi_\nu^b \\
&\quad \cdot (0) \longrightarrow 0, \text{ as } n \longrightarrow \infty,
\end{aligned} \tag{23}$$

i.e., $\lim_{n,m\longrightarrow\infty} m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_{n+2m}) = 0$.

So, $\lim_{n,m\longrightarrow\infty}(m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}_m}}^b) = 0$.

Let $M_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) = m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n)$. Now,-

$M_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) - m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) \le M_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) = m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n) \le 2\lambda^{n-1}\xi_\nu^b(0) \longrightarrow 0$, as $n \longrightarrow \infty$.

Hence, $\lim_{n,m\longrightarrow\infty} M_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) - m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_m) = 0$.

Therefore, the sequence $\{\mathfrak{u}_n\}$ is m_ν^b-Cauchy in $\mathscr{M}$.

Since, $\mathscr{M}$ is m_ν^b-complete, there exists $\mathfrak{u}^* \in \mathscr{U}$ so that $\mathfrak{u}_n \longrightarrow \mathfrak{u}^*$. Now, we assert that $\mathscr{A}\mathfrak{u}^* = \mathfrak{u}^*$:

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\left(m_\nu^b(\mathfrak{u}_n, \mathfrak{u}^*) - m_{\nu_{\mathfrak{u}_n,\mathfrak{u}^*}}^b\right) \\
&\quad = 0 \Longrightarrow \lim_{n\longrightarrow\infty}\left(m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}^*) - m_{\nu_{\mathfrak{u}_{n+1},\mathfrak{u}^*}}^b\right) \\
&\quad = 0 \Longrightarrow \lim_{n\longrightarrow\infty}\left(m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathfrak{u}^*) - m_{\nu_{\mathscr{A}\mathfrak{u}_n,\mathfrak{u}^*}}^b\right) \\
&\quad = 0 \Longrightarrow m_\nu^b(\mathscr{A}\mathfrak{u}^*, \mathfrak{u}^*) - m_{\nu_{\mathscr{A}\mathfrak{u}^*,\mathfrak{u}^*}}^b = 0 \text{ (using Lemma 17)},
\end{aligned} \tag{24}$$

i.e., $m_\nu^b(\mathscr{A}\mathfrak{u}^*, \mathfrak{u}^*) = \min\{m_\nu^b(\mathscr{A}\mathfrak{u}^*, \mathscr{A}\mathfrak{u}^*), m_\nu^b(\mathfrak{u}, \mathfrak{u}^*)\} \Longrightarrow m_\nu^b(\mathscr{A}\mathfrak{u}^*, \mathfrak{u}^*) = m_\nu^b(\mathscr{A}\mathfrak{u}^*, \mathscr{A}\mathfrak{u}^*)$ or $m_\nu^b(\mathscr{A}\mathfrak{u}^*, \mathfrak{u}^*) = m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*)$.

Hence, $\mathscr{A}\mathfrak{u}^* = \mathfrak{u}^*$, i.e., $\mathfrak{u}^*$ is a fixed point of $\mathscr{A}$.

Now, we assert that $m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*) = 0$:

$$\begin{aligned} m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*) &= m_\nu^b(\mathscr{M}\mathfrak{u}^*, \mathscr{M}\mathfrak{u}^*) \leq \mu\left[m_\nu^b(\mathfrak{u}^*, \mathscr{A}\mathfrak{u}^*) + m_\nu^b(\mathfrak{u}^*, \mathscr{A}\mathfrak{u}^*)\right] \\ &= 2\mu m_\nu^b(\mathfrak{u}^*, \mathscr{A}\mathfrak{u}^*) = 2\mu m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*) < m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*), \end{aligned} \tag{25}$$

a contradiction. Hence, $m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*) = 0$.

Suppose, $\mathfrak{u}^*$ and $\mathfrak{w}^*$ are two different fixed points of $\mathscr{A}$, so

$$\begin{aligned} m_\nu^b(\mathfrak{u}^*, \mathfrak{w}^*) &= m_\nu^b(\mathscr{A}\mathfrak{u}^*, \mathscr{A}\mathfrak{w}^*) \leq \mu\left[m_\nu^b(\mathfrak{u}^*, \mathscr{A}\mathfrak{u}^*) + m_\nu^b(\mathfrak{w}^*, \mathscr{A}\mathfrak{w})^*\right] \\ &= \mu\left[m_\nu^b(\mathfrak{u}^*, \mathfrak{u}^*) + m_\nu^b(\mathfrak{w}^*, \mathfrak{w}^*)\right] = 0 \Longrightarrow m_\nu^b(\mathfrak{u}^*, \mathfrak{w}^*) = 0. \end{aligned} \tag{26}$$

Hence, $\mathfrak{u}^* = \mathfrak{w}^*$.

Now, we furnish two examples (one of continuous and another of discontinuous self-map) to validate Theorem 18 and demonstrate the fact that the continuity of a self-map is not indispensable for the survival of a unique fixed point in an M_ν^b-metric space for the Kannan type contraction. Also, the self-distance of a fixed point is zero, and the sequence of iterates $\{\mathscr{A}^n\mathfrak{u}_0\} \subseteq \mathscr{M}$ converges to a fixed point.

Example 19. Let $\mathscr{M} = [-10, 10]$ and an M_ν^b -metric $m_\nu^b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ be defined by $m_\nu^b(\mathfrak{u}, \mathfrak{w}) = ((|\mathfrak{u}|+|\mathfrak{w}|)/2)^2$, $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$. Then, $(\mathscr{M}, m_\nu^b)$ is a complete M_ν^b -metric space for $s = 3$. Define a self-map $\mathscr{A}$ on $\mathscr{M}$ as $\mathscr{A}\mathfrak{u} = 3/19\mathfrak{u}$, $\mathfrak{u} \in \mathscr{M}$. Observe that, for all $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$, we obtain

$$\begin{aligned} m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) &= \left(\frac{|\mathscr{A}\mathfrak{u}| + |\mathscr{A}\mathfrak{w}|}{2}\right)^2 = \left(\frac{3/19|\mathfrak{u}| + 3/19|\mathfrak{w}|}{2}\right)^2 \\ &= \frac{9}{361}\left(\frac{|\mathfrak{u}| + |\mathfrak{w}|}{2}\right)^2, \end{aligned}$$

$$\begin{aligned} m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w}) &= \left(\frac{|\mathfrak{u}| + |\mathscr{A}\mathfrak{u}|}{2}\right)^2 + \left(\frac{|\mathfrak{w}| + |\mathscr{A}\mathfrak{w}|}{2}\right)^2 \\ &= \left(\frac{|\mathfrak{u}| + 3/19|\mathfrak{u}|}{2}\right)^2 + \left(\frac{|\mathfrak{w}| + 3/19|\mathfrak{w}|}{2}\right)^2 \\ &= \frac{484}{361}\left(\frac{|\mathfrak{u}|}{2}\right)^2 + \frac{484}{361}\left(\frac{|\mathfrak{w}|}{2}\right)^2 \\ &\leq \frac{484}{361}\left(\frac{|\mathfrak{u}| + |\mathfrak{w}|}{2}\right)^2, \end{aligned} \tag{27}$$

i.e., $m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) \leq \mu[m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w})]$, $\mu = 9/484 < 1/2s$.

Thus, all the postulates of Theorem 18 are verified and $\mathscr{A}$ has a unique fixed point 0 in $\mathscr{M}$ and clearly, $m_\nu^b(0, 0) = 0$. Further, there exists a sequence $\{\mathscr{A}^n\mathfrak{u}_0\} \subseteq \mathscr{M}$, $\mathscr{A}^n\mathfrak{u}_0 = -n/(n^2 + n)$, $n \in \mathbb{N}$, which m_ν^b-converges to 0.

Example 20. Consider $\mathscr{M} = [0, 1]$ and an M_ν^b -metric $m_\nu^b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ defined by $m_\nu^b(\mathfrak{u}, \mathfrak{w}) = (|\mathfrak{u}| + |\mathfrak{w}|/2)^2$, $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$. Then, $(\mathscr{M}, m_\nu^b)$ is a complete M_ν^b -metric space for $s = 3$. Define a self-map $\mathscr{A}$ on $\mathscr{M}$ as

$$\mathscr{A}\mathfrak{u} = \begin{cases} \dfrac{\mathfrak{u}}{3}, & \mathfrak{u} \in [0, 1) \\ \dfrac{1}{9}, & \mathfrak{u} = 1 \end{cases}, \quad \mathfrak{u} \in \mathscr{M}. \tag{28}$$

Observe that, for all $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$, we obtain the following.

Case 21. When $\mathfrak{u}, \mathfrak{w} \in [0, 1)$,

$$m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) = \left(\frac{\mathscr{A}\mathfrak{u} + \mathscr{A}\mathfrak{w}}{2}\right)^2 = \left(\frac{\mathfrak{u}/3 + \mathfrak{w}/3}{2}\right)^2 = \left(\frac{\mathfrak{u} + \mathfrak{w}}{6}\right)^2,$$

$$\begin{aligned} m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w}) &= \left(\frac{\mathfrak{u} + \mathscr{A}\mathfrak{u}}{2}\right)^2 + \left(\frac{\mathfrak{w} + \mathscr{A}\mathfrak{w}}{2}\right)^2 \\ &= \left(\frac{\mathfrak{u} + \mathfrak{u}/3}{2}\right)^2 + \left(\frac{\mathfrak{w} + \mathfrak{w}/3}{2}\right)^2 \\ &= \left(\frac{4\mathfrak{u}}{6}\right)^2 + \left(\frac{4\mathfrak{w}}{6}\right)^2 \leq \frac{4}{9}(\mathfrak{u} + \mathfrak{w})^2. \end{aligned} \tag{29}$$

Case 22. When $\mathfrak{u} = \mathfrak{w} = 1$,

$$\begin{aligned} m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) &= \left(\frac{1/9 + 1/9}{2}\right)^2 = \frac{1}{81}, \\ m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w}) &= \left(\frac{1 + 1/9}{2}\right)^2 + \left(\frac{1 + 1/9}{2}\right)^2 = \frac{50}{81}. \end{aligned} \tag{30}$$

Case 23. When $\mathfrak{u} = 0, \mathfrak{w} = 1$,

$$\begin{aligned} m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) &= \left(\frac{0 + 1/9}{2}\right)^2 = \frac{1}{324}, \\ m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w}) &= \left(\frac{0 + 0}{2}\right)^2 + \left(\frac{1 + 1/9}{2}\right)^2 = \frac{25}{81}, \end{aligned} \tag{31}$$

i.e., $m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) \leq \mu[m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w})]$, $\mu = 1/16 < 1/2s$.

Thus, all the postulates of Theorem 18 are verified and $\mathscr{A}$ has a unique fixed point 0 in $\mathscr{M}$, and clearly, $m_\nu^b(0, 0) = 0$. Further, there exists a sequence $\{\mathscr{A}^n\mathfrak{u}_0\} \subseteq \mathscr{M}$, $\mathscr{A}^n\mathfrak{u}_0 = 1/(3n^2 + 2n)$, $n \in \mathbb{N}$, which m_ν^b-converges to 0.

Remark 24. Examples 19 and 20 demonstrate that Theorem 18 is an extension, generalization, and improvement of Asim et al. [10], Banach [21], Branciari [4], Kannan [22], Karahan and Isik [9], Mitrović and Radenović [7], Özgür et al. [8], Shukla [15, 16], and references therein to an M_v^b-metric space. It is fascinating to see that the continuity of a self-map is not an indispensable requirement for the survival of a unique fixed point satisfying (18) in this novel space.

Our next result extends the result of Reich [23] to M_v^b-metric space.

Theorem 25. *Let $(\mathcal{M}, m_v^b)$ be an M_v^b-complete metric space with coefficient $s \geq 1$. Suppose a self-map $\mathcal{A}:\mathcal{M} \longrightarrow \mathcal{M}$ satisfies*

$$m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w}) \leq \alpha m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u}) + \beta m_v^b(\mathfrak{w}, \mathcal{A}\mathfrak{w}) + \gamma m_v^b(\mathfrak{u}, \mathfrak{w}), \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma < \frac{1}{2s+1}, \mathfrak{u}, \mathfrak{w} \in \mathcal{M}. \tag{32}$$

Then, $\mathcal{A}$ has a unique fixed point $\mathfrak{u}^*$ so that $m_v^b(\mathfrak{u}^*, \mathfrak{u}^*) = 0$ and the sequence of iterates $\{\mathcal{A}^n\mathfrak{u}_0\} \subseteq \mathcal{M}$ converges to $\mathfrak{u}^* \in \mathcal{M}$.

Proof. It follows the pattern of Theorem 18.

Theorem 25 is an extension, improvement, and generalization of Banach [21] ($\alpha = \beta = 0$), Kannan [22] ($\alpha = \beta, \gamma = 0$), Reich [23], and Asim et al. [10] ($\alpha = \beta = 0$) to an M_v^b-metric space.

Now, we prove the result for a Hardy and Rogers type contraction [24], which includes all the results stated above as a special case.

Theorem 26. *Let $(\mathcal{M}, m_v^b)$ be an M_v^b-complete metric space. Suppose, a self-map $\mathcal{A}:\mathcal{M} \longrightarrow \mathcal{M}$ satisfies*

$$m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w}) \leq \alpha m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{w}) + \beta m_v^b(\mathfrak{w}, \mathcal{A}\mathfrak{u}) + \gamma m_v^b(\mathfrak{u}, \mathfrak{w}) + \delta m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u}) + \kappa m_v^b(\mathfrak{w}, \mathcal{A}\mathfrak{w}), \alpha, \beta, \gamma, \delta, \kappa \geq 0, \alpha + \beta + \gamma + \delta + \kappa < \frac{1}{2s+1}, \mathfrak{u}, \mathfrak{w} \in \mathcal{M}. \tag{33}$$

Then, $\mathcal{A}$ has a unique fixed point $\mathfrak{u}^*$ so that $m_v^b(\mathfrak{u}^*, \mathfrak{u}^*) = 0$ and the sequence of iterates $\{\mathcal{A}^n\mathfrak{u}_0\} \subseteq \mathcal{M}$ converges to $\mathfrak{u}^* \in \mathcal{M}$.

Proof. It follows the pattern of Theorem 18.

Theorem 26 is an extension, improvement, and generalization of Banach [21] ($\alpha = \beta = \delta = \kappa = 0$), Kannan [22] ($\alpha = \beta = \gamma = 0, \delta = \kappa$), Chatterjee [25] ($\gamma = \delta = \kappa = 0, \alpha = \beta$), Reich [23] ($\alpha = \beta = 0$), Hardy and Rogers [24] ($s = 1$), Asim et al. [10] ($\alpha = \beta = \delta = \kappa = 0$), and references therein to an M_v^b-metric space.

The following result is more fascinating as it is proved by altering the distances between the points, exploiting subadditive altering distance function [19].

Theorem 27. *Let $(\mathcal{M}, m_v^b)$ be a M_v^b-complete metric space. Suppose, we have a self-map $\mathcal{A}:\mathcal{M} \longrightarrow \mathcal{M}$ satisfying*

$$\psi\left(m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w})\right) \leq \mu\left[\psi\left(m_v^b(\mathfrak{u}, \mathfrak{w})\right) + \psi\left(m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u})\right) + \psi\left(m_v^b(\mathfrak{w}, \mathcal{A}\mathfrak{w})\right)\right], \quad \mu < \frac{1}{2s+1}, \mathfrak{u}, \mathfrak{w} \in \mathcal{M}. \tag{34}$$

Then, $\mathcal{A}$ has a unique fixed point $\mathfrak{u}^*$ so that $m_v^b(\mathfrak{u}^*, \mathfrak{u}^*) = 0$, and the sequence of iterates $\{\mathcal{A}^n\mathfrak{u}_0\} \subseteq \mathcal{M}$ converges to $\mathfrak{u}^* \in \mathcal{M}$ and for $\kappa = 2\mu/(1-\mu) < 1$,

$$M_v^b(\mathcal{A}^{n+1}\mathfrak{u}^*, \mathcal{A}^n\mathfrak{u}^*) \leq \kappa^n M_v^b(\mathfrak{u}^*, \mathcal{A}\mathfrak{u}^*), \quad n = 0, 1, 2, \cdots. \tag{35}$$

Proof. For an arbitrary $\mathfrak{u} \in \mathcal{M}$, let $\mathfrak{w} = \mathcal{A}\mathfrak{u}$. Then,

$$\begin{aligned}
\psi\left(m_v^b(\mathfrak{w}, \mathcal{A}\mathfrak{w})\right) &= \psi\left(m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w})\right) \\
&\leq \mu\left[\psi\left(m_v^b(\mathfrak{u}, \mathfrak{w})\right) + \psi\left(m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u})\right) + \psi\left(m_v^b(\mathfrak{w}, \mathcal{A}\mathfrak{w})\right)\right] \\
&= \mu\left[\psi\left(m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u})\right) + \psi\left(m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u})\right) + \psi\left(m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w})\right)\right](1-\mu)\psi\left(m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w})\right) \\
&\leq 2\mu\psi\left(m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u})\right)\psi\left(m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w})\right) \\
&\leq \frac{2\mu}{1-\mu}\psi\left(m_v^b(\mathfrak{u}, \mathcal{A}\mathfrak{u})\right)\psi\left(m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w})\right) \\
&\leq \kappa\psi\left(m_v^b(\mathfrak{u}, \mathfrak{w})\right), \quad \text{where } \kappa = \frac{2\mu}{1-\mu} \\
&< 1, \Longrightarrow m_v^b(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{w}) \leq \kappa m_v^b(\mathfrak{u}, \mathfrak{w}).
\end{aligned} \tag{36}$$

Now, starting from the given element $\mathfrak{u}_0 \in \mathcal{M}$, form the sequence $\{\mathfrak{u}_n\}$, where $\mathfrak{u}_n = \mathcal{A}\mathfrak{u}_{n-1}, n \in \mathbb{N}$. If $m_v^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0, n \geq 0$, then $\mathcal{A}\mathfrak{u}_n = \mathfrak{u}_{n+1} = \mathfrak{u}_n$ and $m_v^b(\mathfrak{u}_n, \mathfrak{u}_n) = 0$, and this completes the proof.

Further, take $m_v^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) > 0$, $n \geq 0$. For $\mathfrak{u} = \mathfrak{u}_n, \mathfrak{w} = \mathfrak{u}_{n+1}$, utilizing condition (36),

$$\begin{aligned}
m_v^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) &= m_v^b(\mathcal{A}\mathfrak{u}_n, \mathcal{A}\mathfrak{u}_{n+1}) \leq \kappa m_v^b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \\
&\leq \kappa^n m_v^b(\mathfrak{u}_0, \mathfrak{u}_1) \longrightarrow 0, \text{as}, n \longrightarrow \infty.
\end{aligned} \tag{37}$$

Also,

$$m_\nu^b(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}) = m_\nu^b(\mathscr{A}\mathfrak{u}_n, \mathscr{A}\mathfrak{u}_n) \le \kappa m_\nu^b(\mathfrak{u}_n, \mathfrak{u}_n) \le \kappa^n m_\nu^b(\mathfrak{u}_0, \mathfrak{u}_0) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{38}$$

So, following a similar pattern as that of Theorem 18, we may easily conclude that $\mathscr{A}$ has a unique fixed point $\mathfrak{u} \in \mathscr{M}$, where $m_\nu^b(\mathfrak{u}, \mathfrak{u}) = 0$.

Finally, by taking $\mathfrak{u} = \mathscr{A}^{n-1}\mathfrak{u}$, from (36),

$$m_\nu^b(\mathscr{A}^n\mathfrak{u}, \mathscr{A}^{n+1}\mathfrak{u}) \le \kappa m_\nu^b(\mathscr{A}^{n-1}\mathfrak{u}, \mathscr{A}^n\mathfrak{u}) \le \kappa^2 m_\nu^b(\mathscr{A}^{n-2}\mathfrak{u}, \mathscr{A}^{n-1}\mathfrak{u}) \cdots \le \kappa^n m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}), \quad n = 1, 2, 3, \cdots. \tag{39}$$

Now, we furnish two examples of a discontinuous self-map to validate Theorem 27.

Example 1. Let $\mathscr{M} = \mathbb{R}$ and an M_ν^b-metric $m_\nu^b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ be defined as $m_\nu^b(\mathfrak{u}, \mathfrak{w}) = \max\{|\mathfrak{u}|^2, |\mathfrak{w}|^2\} + |\mathfrak{u} - \mathfrak{w}|^2$, $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$ with $s = 3$.

Define a self-map $\mathscr{A}:\mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}$ by

$$\mathscr{A}\mathfrak{u} = \begin{cases} \dfrac{\mathfrak{u}}{9}, & \mathfrak{u} \in [-9, 9], \\ \dfrac{3\mathfrak{u}}{5}, & \text{otherwise}, \end{cases} \tag{40}$$

and $\psi : [0,\infty) \longrightarrow [0,\infty)$ as $\psi(\mathfrak{u}) = \mathfrak{u}/7$, then observe that, for all $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$, we obtain

$$\begin{aligned} \psi\left(m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w})\right) &= \psi\left(\max\{|\mathscr{A}\mathfrak{u}|^2, |\mathscr{A}\mathfrak{w}|^2\} + |\mathscr{A}\mathfrak{u} - \mathscr{A}\mathfrak{w}|^2\right) \\ &\le \psi\left(\frac{9}{25}\max\{|\mathfrak{u}|^2, |\mathfrak{w}|^2\} + |\mathfrak{u} - \mathfrak{w}|^2\right) \\ &< \frac{9}{25}\left[\psi\left(m_\nu^b(\mathfrak{u}, \mathfrak{w})\right) + \psi\left(m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u})\right) + \psi\left(m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w})\right)\right]. \end{aligned} \tag{41}$$

Thus, all the postulates of Theorem 27 are verified and $\mathscr{A}$ has a unique fixed point 0 in $\mathscr{M}$, and clearly, $m_\nu^b(0,0) = 0$. Further, there exists a sequence $\{\mathscr{A}^n\mathfrak{u}_0\} \subseteq \mathscr{M}$, $\mathscr{A}^n\mathfrak{u}_0 = n/(n^2 + 1)$, $n \in \mathbb{N}$, which m_ν^b-converges to 0.

Example 2. Let $\mathscr{M} = [-2, 2]$ and an M_ν^b *-metric* $m_\nu^b:\mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ be defined as

$$m_\nu^b(\mathfrak{u}, \mathfrak{w}) = \left(\frac{|\mathfrak{u}| + |\mathfrak{w}|}{2}\right)^2, \mathfrak{u}, \mathfrak{w} \in \mathscr{M} \text{ with } s = 3. \tag{42}$$

Define a self-map $\mathscr{A}:\mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}$ by

$$\mathscr{A}\mathfrak{u} = \begin{cases} 0, & \mathfrak{u} \in [-2, 0], \\ -1, & \text{otherwise}, \end{cases} \tag{43}$$

and $\psi:[0,\infty) \longrightarrow [0,\infty)$ as $\psi(\mathfrak{u}) = \mathfrak{u}/2$, then observe that, for all $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$, we obtain

$$\begin{aligned} &\psi\left(m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w})\right) \\ &< \mu\left[\psi\left(m_\nu^b(\mathfrak{u}, \mathfrak{w})\right) + \psi\left(m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u})\right) + \psi\left(m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w})\right)\right], \quad \mu = \frac{2}{11} < \frac{1}{2s+1}. \end{aligned} \tag{44}$$

Thus, all the postulates of Theorem 27 are verified and $\mathscr{A}$ has a unique fixed point 0 in $\mathscr{M}$, and clearly, $m_\nu^b(0,0) = 0$. Further, there exists a sequence $\{\mathscr{A}^n\mathfrak{u}_0\} \subseteq \mathscr{M}$, $\mathscr{A}^n\mathfrak{u}_0 = -3n/(n^2 + n)$, $n \in \mathbb{N}$, which m_ν^b-converges to 0.

Remark 28. Examples 1 and 2 demonstrate that Theorem 27 is an extension, generalization, and improvement of Asim et al. [10], Banach [21], Kannan [22], Hardy and Rogers [24], Karahan and Isik [9], Khan et al. [19], Mlaiki et al. [18], Mitrović and Radenović [7], Reich [23], and references therein to M_ν^b-metric space. It is fascinating to see that continuity of a self-map is not indispensable for the existence of a fixed point satisfying (34) in this novel space.

Corollary 29. *The conclusion of Theorem 27 is true even if we replace (34) by*

$$m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) \le \mu\left[m_\nu^b(\mathfrak{u}, \mathfrak{v}) + m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_\nu^b(\mathfrak{v}, \mathscr{A}\mathfrak{v})\right], \quad \mu < \frac{1}{2s+1}, \mathfrak{u}, \mathfrak{w} \in \mathscr{M}. \tag{45}$$

Proof. The proof follows, if we take, $\psi(\mathfrak{u}) = \mathfrak{u}$ in (34).

Corollary 30. *The conclusion of Theorem 27 is true even if we replace (34) by*

$$\psi\left(m_\nu^b(\mathscr{A}^j\mathfrak{u}, \mathscr{A}^j\mathfrak{w})\right) \le \mu\left[\psi\left(m_\nu^b(\mathfrak{u}, \mathfrak{w})\right) + \psi\left(m_\nu^b(\mathfrak{u}, \mathscr{A}^j\mathfrak{u})\right) + \psi\left(m_\nu^b(\mathfrak{w}, \mathscr{A}^j\mathfrak{w})\right)\right], \quad \mu < \frac{1}{2s+1}, \mathfrak{u}, \mathfrak{w} \in \mathscr{M}. \tag{46}$$

Proof. Applying Theorem 27 to the self-map $\mathscr{T} = \mathscr{A}^j$, we get $\mathscr{T}$ has a unique fixed point, say $\mathfrak{u}$, i.e., $\mathscr{T}\mathfrak{u} = \mathscr{A}^j\mathfrak{u} = \mathfrak{u}$.

Now,

$$\mathscr{T}\mathscr{A}\mathfrak{u} = \mathscr{A}^j\mathscr{A}\mathfrak{u} = \mathscr{A}^{j+1}\mathfrak{u} = \mathscr{A}\mathscr{A}^j\mathfrak{u} = \mathscr{A}\mathfrak{u}, \tag{47}$$

so $\mathscr{A}\mathfrak{u}$ is a unique fixed point of $\mathscr{T}$.

Corollary 31. *The conclusion of Theorem 27 remains true even if we replace (34) by*

$$\left(1+m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{w})\right)^{1/\mu} \le \left(1+m_\nu^b(\mathfrak{u},\mathfrak{w})\right)\left(1+m_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{u})\right) \cdot \left(1+m_\nu^b(\mathfrak{w},\mathscr{A}\mathfrak{w})\right), \quad \mu < \frac{1}{2s+1},\ \mathfrak{u},\mathfrak{w}\in\mathscr{M}. \tag{48}$$

Proof. The proof follows the pattern of Theorem 27 if we take $\psi(\mathfrak{u}) = \ln(1+\mathfrak{u})$ in (34).

Theorem 32. *Let $(\mathscr{M}, m_\nu^b)$ be an M_ν^b -complete metric space with coefficient $s\beta < 1$. Suppose $\mathscr{A}:\mathscr{M}\longrightarrow\mathscr{M}$ satisfies*

$$\psi\left(m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{w})\right) \le \alpha\psi\left(m_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{u})\right) + \beta\psi\left(m_\nu^b(\mathfrak{w},\mathscr{A}\mathfrak{w})\right) + \gamma\psi\left(m_\nu^b(\mathfrak{u},\mathfrak{w})\right), \quad \alpha+\beta+\gamma < 1,\ \mathfrak{u},\mathfrak{w}\in\mathscr{M}. \tag{49}$$

Then, $\mathscr{A}$ has a unique fixed point $\mathfrak{u}^*$ so that $m_\nu^b(\mathfrak{u}^*,\mathfrak{u}^*) = 0$, and the sequence of iterates $\{\mathscr{A}^n\mathfrak{u}_0\}\subseteq\mathscr{M}$ converges to $\mathfrak{u}^*\in\mathscr{M}$ and for $\delta = (\alpha+\beta)/(1-\gamma)$,

$$M_\nu^b\left(\mathscr{A}^{n+1}\mathfrak{u},\mathscr{A}^n\mathfrak{u}\right) \le \delta^n M_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{u}), \quad n = 0,1,2,\cdots. \tag{50}$$

Proof. The proof follows the pattern of Theorem 27.

Theorem 33. *The conclusion of Theorem 27 remains true even if we replace inequality (34) by*

$$\psi\left(m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{w})\right) \le \mu\left[\psi\left(m_\nu^b(\mathfrak{w},\mathscr{A}\mathfrak{w}) + \psi\left(m_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{w})\right)\right)\right], \quad \mu < \frac{1}{2s},\ \mathfrak{u},\mathfrak{w}\in\mathscr{M}. \tag{51}$$

Proof. The proof follows the pattern of Theorem 27.

We note that when $\psi(\mathfrak{u}) = \mathfrak{u}$, the above theorem is the same as Theorem 18.

4. Existence of a Fixed Circle/Disc

Following Özgür and Tas [26], we first familiarize fixed circle in an M_ν^b-metric space to establish fixed circle theorems. Next, we exploit m_ν^b-metric version (m_ν^b-Caristi map) of the classical Caristi map [27] to establish that the set of non-unique fixed points of a map includes a circle. Also, following Aydi et al. [28], we familiarize fixed disc to establish the greatest fixed disc in an M_ν^b-metric space.

Definition 34. We define a circle centred at $\mathfrak{u}_0$ and radius $\mathfrak{r}$ in an M_ν^b -metric space $(\mathscr{M}, m_\nu^b)$ as

$$\mathscr{C}(\mathfrak{u}_0,\mathfrak{r}) = \left\{\mathfrak{u}\in\mathscr{M} : m_\nu^b(\mathfrak{u}_0,\mathfrak{u}) = \mathfrak{r} + m_\nu^b(\mathfrak{u}_0,\mathfrak{u}_0), \quad \mathfrak{u}_0\in\mathscr{M}, \mathfrak{r}\in[0,\infty)\right\}. \tag{52}$$

Definition 35. We define a disc centred at $\mathfrak{u}_0$ and radius $\mathfrak{r}$ in an M_ν^b-metric space $(\mathscr{M}, m_\nu^b)$ as

$$\mathscr{D}(\mathfrak{u}_0,\mathfrak{r}) = \left\{\mathfrak{u}\in\mathscr{M} : m_\nu^b(\mathfrak{u}_0,\mathfrak{u}) \le \mathfrak{r} + m_\nu^b(\mathfrak{u}_0,\mathfrak{u}_0), \quad \mathfrak{u}_0\in\mathscr{M}, \mathfrak{r}\in[0,\infty)\right\}. \tag{53}$$

Geometrically, it is not necessary that a circle/disc defined in a M_ν^b-metric space is the same as the circle/disc in a Euclidean space.

Definition 36. Let $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ / $\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$ be a circle/disc centred at $\mathfrak{u}_0$ and radius $\mathfrak{r}$ in an M_ν^b -metric space $(\mathscr{M}, m_\nu^b)$. For a self-map $\mathscr{A}:\mathscr{M}\longrightarrow\mathscr{M}$ in an M_ν^b-metric space $(\mathscr{M}, m_\nu^b)$, if $m_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{u}) = m_\nu^b(\mathfrak{u},\mathfrak{u}) = m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})$,$\mathfrak{u}\in\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$/ $\mathfrak{u}\in\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$, then $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ / $\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$ is called the fixed circle/fixed disc of $\mathscr{A}$.

Theorem 37. *Let $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ be a circle in an M_ν^b-metric space $(\mathscr{M}, m_\nu^b)$. Define $\varsigma:\mathscr{M}\longrightarrow[0,\infty)$ as*

$$\varsigma(\mathfrak{u}) = m_\nu^b(\mathfrak{u},\mathfrak{u}_0) + m_\nu^b(\mathfrak{u},\mathfrak{u}), \quad \mathfrak{u}\in\mathscr{M}. \tag{54}$$

If there exists a self-map $\mathscr{A}:\mathscr{M}\longrightarrow\mathscr{M}$ so that

$$\begin{aligned} &(i)\ m_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{u}) \le \varsigma(\mathscr{A}\mathfrak{u}) - \varsigma(\mathfrak{u}) + m_\nu^b(\mathfrak{u},\mathfrak{u}),\\ &(ii)\ m_\nu^b(\mathscr{A}\mathfrak{u},\mathfrak{u}_0) \le \mathfrak{r} + m_\nu^b(\mathfrak{u}_0,\mathfrak{u}_0),\\ &(iii)\ m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u}) \le m_\nu^b(\mathfrak{u},\mathfrak{u}), \end{aligned} \tag{55}$$

(iv) If $m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{w}) \le \mu m_\nu^b(\mathfrak{u},\mathfrak{w})$, $\mathfrak{u}\in\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$, $\mathfrak{w}\in\mathscr{M}\setminus\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$, $\mu\in[0,1)$, then $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ is a unique fixed circle of $\mathscr{M}$

Proof. Let $\mathfrak{u}\in\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ be any arbitrary point. Using (i) and equation (54),

$$\begin{aligned} m_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{u}) &\le m_\nu^b(\mathscr{A}\mathfrak{u},\mathfrak{u}_0) + m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u}) - m_\nu^b(\mathfrak{u},\mathfrak{u}_0)\\ &\quad - m_\nu^b(\mathfrak{u},\mathfrak{u}) + m_\nu^b(\mathfrak{u},\mathfrak{u})\\ &= m_\nu^b(\mathscr{A}\mathfrak{u},\mathfrak{u}_0) + m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u}) - m_\nu^b(\mathfrak{u},\mathfrak{u}_0)\\ &\le \mathfrak{r} + m_\nu^b(\mathfrak{u}_0,\mathfrak{u}_0) + m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u}) - \mathfrak{r}\\ &\quad - m_\nu^b(\mathfrak{u}_0,\mathfrak{u}_0)(\text{using (ii)}), \end{aligned} \tag{56}$$

i.e., $m_\nu^b(\mathfrak{u},\mathscr{A}\mathfrak{u}) \le m_\nu^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})$.

But by the definition of an M_v^b-metric space, $m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) \leq m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u})$. So, $m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) = m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u})$. Using (iii) $m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) \leq m_v^b(\mathfrak{u}, \mathfrak{u})$. Again, by the definition of an M_v^b-metric space, $m_v^b(\mathfrak{u}, \mathfrak{u}) \leq m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u})$. Hence,

$$m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) = m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) = m_v^b(\mathfrak{u}, \mathfrak{u}), \tag{57}$$

i.e., $\mathfrak{u}$ is a fixed point of $\mathscr{A}$, $\forall \mathfrak{u} \in \mathscr{C}(\mathfrak{u}_0, \mathfrak{r})$. So, the self-map $\mathscr{A}$ fixes the circle $\mathscr{C}(\mathfrak{u}_0, \mathfrak{r})$, i.e., the set of nonunique fixed points of a map $\mathscr{A}$ includes a circle.

Let there exist two fixed circles, $\mathscr{C}(\mathfrak{u}_0, \mathfrak{r}_0)$ and $\mathscr{C}(\mathfrak{u}_1, \mathfrak{r}_1)$, $\mathfrak{r} \neq \mathfrak{r}_1$ of $\mathscr{A}$, i.e., $\mathscr{A}$ satisfies all the conditions (i) to (iii) for each of the circles $\mathscr{C}(\mathfrak{u}_0, \mathfrak{r}_0)$ and $\mathscr{C}(\mathfrak{u}_1, \mathfrak{r}_1)$. Let $\mathfrak{u} \in \mathscr{C}(\mathfrak{u}_0, \mathfrak{r}_0)$ and $\mathfrak{w} \in \mathscr{C}(\mathfrak{u}_1, \mathfrak{r}_1)$. Using (iv), $m_v^b(\mathfrak{u}, \mathfrak{w}) = m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) \leq \mu m_v^b(\mathfrak{u}, \mathfrak{w})$, a contradiction.

Hence, $\mathscr{C}(\mathfrak{u}_0, \mathfrak{r}_0)$ is a unique fixed circle of $\mathscr{A}$.

The following example illustrates Theorem 37.

Example 38. Let $\mathscr{M} = \mathbb{R}$ and an M_v^b-metric $m_v^b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ be defined as $m_v^b(\mathfrak{u}, \mathfrak{w}) = \max\{|\mathfrak{u}|^\alpha, |\mathfrak{w}|^\alpha\} + |\mathfrak{u} - \mathfrak{w}|^\alpha$, $\alpha > 1$, $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$ with $s = 3^{\alpha-1}$. The circle

$$\begin{aligned}\mathscr{C}(-2, 6) &= \left\{\mathfrak{u} \in \mathscr{M} : m_v^b(-2, \mathfrak{u}) = 6 + m_v^b(-2,-2)\right\}\\ &= \{\mathfrak{u} \in \mathscr{M} : \max\{|-2|^\alpha, |\mathfrak{u}|^\alpha\} + |-2 - \mathfrak{u}|^\alpha\\ &= 6 + |-2|^\alpha\} = \{\mathfrak{u} \in \mathscr{M} : \max\{2^\alpha, |\mathfrak{u}|^\alpha\} + |2 + \mathfrak{u}|^\alpha = 6 + 2^\alpha\}.\end{aligned} \tag{58}$$

Now, we discuss circles for different values of α:

(i) When $\alpha = 2$, $\max\{4, |\mathfrak{u}|^2\} + (2 + \mathfrak{u})^2 = 6 + 4$, then

$$\begin{gathered}\text{either } 4 + (2 + \mathfrak{u})^2 = 10 \text{ or } \mathfrak{u}^2 + (2 + \mathfrak{u})^2 = 10,\\ (2 + \mathfrak{u})^2 = 6 \text{ or } \mathfrak{u}^2 + 4 + \mathfrak{u}^2 + 4\mathfrak{u} = 10,\\ 2 + \mathfrak{u} = \pm\sqrt{6} \text{ or } 2\mathfrak{u}^2 + 4\mathfrak{u} - 6 = 0,\\ \mathfrak{u} = -2 \pm \sqrt{6} \text{ or } \mathfrak{u} = -3, \mathfrak{u} = 1\end{gathered} \tag{59}$$

(ii) When $\alpha = 3$, $\max\{4, |\mathfrak{u}|^3\} + (2 + \mathfrak{u})^3 = 6 + 8$, then

$$\begin{gathered}\text{either } 8 + |2 + \mathfrak{u}|^3 = 14 \text{ or } |\mathfrak{u}|^3 + |2 + \mathfrak{u}|^3 = 14 \pm (2 + \mathfrak{u})^3\\ = 6 \text{ or } \pm\mathfrak{u}^3 \pm (2 + \mathfrak{u})^3 = 14,\\ 2 + \mathfrak{u} = \pm(6)^{1/3} \text{ or } \mathfrak{u} = -2.40629, 0.6097, -1 \pm \sqrt{2},\\ \mathfrak{u} = -2 \pm 6^{1/3} \text{ or } \mathfrak{u} = -2.40629, 0.6097, -1 \pm \sqrt{2}\end{gathered} \tag{60}$$

Now, define a self-map $\mathscr{A} : \mathscr{M} \longrightarrow \mathscr{M}$ as

$$\mathscr{A}\mathfrak{u} = \begin{cases}\mathfrak{u}, & \mathfrak{u} \in \mathscr{C}(-2, 6),\\ -1 + \sqrt{2}, & \mathfrak{u} \notin \mathscr{C}(-2, 6).\end{cases} \tag{61}$$

Then, the self-map $\mathscr{A}$ verifies all the postulates of Theorem 37 except (iv) and fixes the circle $\mathscr{C}(-2, 6)$. Clearly, the set of fixed points of $\mathscr{A}$ contains a circle $\mathscr{C}(-2, 6)$. However, one may notice that there are more than one circle corresponding to each value of α. For instance, $\mathscr{C}(2, 6)$ is also a fixed circle of $\mathscr{A}$.

It is obvious that geometrically (i) implies that $\mathscr{A}\mathfrak{u}$ is in the exterior of a circle and (ii) implies that $\mathscr{A}\mathfrak{u}$ is in the interior of a circle. It means $\mathscr{A}(\mathscr{C}(\mathfrak{u}_0, \mathfrak{r})) \subseteq \mathscr{C}(\mathfrak{u}_0, \mathfrak{r})$ (see Example 38).

Theorem 39. *The conclusion of Theorem 37 is true even if we substitute (i) by (i)′ and (ii) by (ii)′:*

$$\begin{gathered}(i)'\ m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) \leq \varsigma(\mathfrak{u}) - \varsigma(\mathscr{A}\mathfrak{u}) + m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}),\\ (ii)'\ m_v^b(\mathscr{A}\mathfrak{u}, \mathfrak{u}_0) \geq \mathfrak{r} + m_v^b(\mathfrak{u}_0, \mathfrak{u}_0).\end{gathered} \tag{62}$$

Proof. Let $\mathfrak{u} \in \mathscr{C}(\mathfrak{u}_0, \mathfrak{r})$ be any arbitrary point. Using (i)′ and equation (54)

$$\begin{aligned}m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) &\leq m_v^b(\mathfrak{u}, \mathfrak{u}_0) + m_v^b(\mathfrak{u}, \mathfrak{u}) - m_v^b(\mathscr{A}\mathfrak{u}, \mathfrak{u}_0)\\ &\quad - m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) + m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) = \mathfrak{r} + m_v^b(\mathfrak{u}_0, \mathfrak{u}_0)\\ &\quad + m_v^b(\mathfrak{u}, \mathfrak{u}) - m_v^b(\mathscr{A}\mathfrak{u}, \mathfrak{u}_0) \leq \mathfrak{r} + m_v^b(\mathfrak{u}_0, \mathfrak{u}_0) + m_v^b(\mathfrak{u}, \mathfrak{u})\\ &\quad - \mathfrak{r} - m_v^b(\mathfrak{u}_0, \mathfrak{u}_0)\ \left(\text{using (ii)}'\right),\end{aligned} \tag{63}$$

$$\text{i.e.,} m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) \leq m_v^b(\mathfrak{u}, \mathfrak{u}). \tag{64}$$

But by the definition of an M_v^b-metric space, $m_v^b(\mathfrak{u}, \mathfrak{u}) \leq m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u})$.

So, $m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) = m_v^b(\mathfrak{u}, \mathfrak{u})$. Using (63) and (iii), $m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) \leq m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u})$. Also, by definition of an M_v^b-metric $m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) \leq m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u})$. So, $m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) = m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u})$, i.e., $m_v^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}) = m_v^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{u}) = m_v^b(\mathfrak{u}, \mathfrak{u}) \Longrightarrow \mathfrak{u}$ is a fixed point of $\mathscr{A}$,$\forall \mathfrak{u} \in \mathscr{C}(\mathfrak{u}_0, \mathfrak{r})$, i.e., the self-map $\mathscr{A}$ fixes the circle $\mathscr{C}(\mathfrak{u}_0, \mathfrak{r})$, i.e., the set of fixed points of a map $\mathscr{A}$ includes a circle.

Uniqueness of a fixed circle may be proved as in Theorem 37.

Example 40. Let $\mathscr{M} = \mathbb{R}$ and an M_v^b-metric $m_v^b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}^+$ be defined as $m_v^b(\mathfrak{u}, \mathfrak{w}) = (|\mathfrak{u}| + |\mathfrak{w}|/2)^2 + |\mathfrak{u} - \mathfrak{w}|^2$, $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$ with $s = 4$:

$$\begin{aligned}\mathscr{C}(-4, 11) &= \left\{\mathfrak{u} \in \mathscr{M} : m_v^b(-4, \mathfrak{u}) = 11 + m_v^b(-4,-4)\right\}\\ &= \left\{\mathfrak{u} \in \mathscr{M} : \left(\frac{4 + |\mathfrak{u}|}{2}\right)^2 + |-4 - \mathfrak{u}|^2 = 11 + \left(\frac{4 + 4}{2}\right)^2\right\},\end{aligned}$$

$$\Longrightarrow\left(\frac{4+|\mathfrak{u}|}{2}\right)^2+|4+\mathfrak{u}|^2=11+16,$$

$$\Longrightarrow(4+|\mathfrak{u}|)^2+4(4+\mathfrak{u})^2=108. \tag{65}$$

There arises two cases:

(i) $(4+\mathfrak{u})^2+4(4+\mathfrak{u})^2=108,(4+\mathfrak{u})^2=108/5=21.6$ $,4+\mathfrak{u}=\pm4.65\Longrightarrow\mathfrak{u}=0.65,\ \mathfrak{u}=-8.65$

(ii) $(4-\mathfrak{u})^2+4(4+\mathfrak{u})^2=108\ 4+\mathfrak{u}^2-8\mathfrak{u}+4(16+\mathfrak{u}^2+8\mathfrak{u})=108,5\mathfrak{u}^2+24\mathfrak{u}+68=108$ $,5\mathfrak{u}^2+24\mathfrak{u}-40=0\Longrightarrow\mathfrak{u}=-6.1,\ \mathfrak{u}=-1.3$

Define a self-map $\mathscr{A}:\mathscr{M}\longrightarrow\mathscr{M}$ as

$$\mathscr{A}\mathfrak{u}=\begin{cases}\mathfrak{u}, & \mathfrak{u}=\mathscr{C}(-4,11),\\ 0.65, & \mathfrak{u}\neq\mathscr{C}(-4,11).\end{cases} \tag{66}$$

Then, map $\mathscr{A}$ verifies all the postulates of Theorem 39 and fixes the unique circle $\mathscr{C}(-4,11)$, i.e., the set of non-unique fixed points of $\mathscr{A}=\{-8.65,-6.1,-1.3,0.65\}$contains a unique circle $\mathscr{C}(-4,11)$.

It is clear that geometrically (i)$'$ implies that $\mathscr{A}\mathfrak{u}$ is in the interior of a circle and (ii)$'$ implies that $\mathscr{A}\mathfrak{u}$ is in the exterior of a circle. It means $\mathscr{A}(\mathscr{C}(\mathfrak{u}_0,\mathfrak{r}))\subseteq\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ (see Example 40).

In the above theorems, we have assumed Banach contraction to demonstrate the uniqueness of fixed circle. So, it is significant to establish the uniqueness of the fixed circle using different contractive conditions. In the following, we establish uniqueness using a more general contractive condition.

Theorem 41. *Let $(\mathscr{M},m_v^b)$ be an M_v^b -metric space and $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ be a circle on $\mathscr{M}$. Let $\mathscr{A}:\mathscr{M}\longrightarrow\mathscr{M}$ be a self-map satisfying (i)-(iii), (i)$'$ and (ii)$'$ of Theorem37or39along with the contraction condition*

$$m_v^b(\mathscr{M}\mathfrak{u},\mathscr{M}\mathfrak{w})\le\mu\max\left\{m_v^b(\mathfrak{u},\mathfrak{w}),m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{w}),m_v^b(\mathfrak{w},\mathscr{M}\mathfrak{u}),\frac{1}{2}\left(m_v^b(\mathfrak{u},\mathscr{M}\mathfrak{u})+m_v^b(\mathfrak{w},\mathscr{M}\mathfrak{w})\right),\frac{1}{2}\left(m_v^b(\mathfrak{u},\mathscr{M}\mathfrak{w})+m_v^b(\mathfrak{w},\mathscr{M}\mathfrak{u})\right)\right\}\Big), \tag{67}$$

$\mathfrak{u}\in\mathscr{C}(\mathfrak{u}_0,\mathfrak{r}),\mathfrak{w}\in\mathscr{M}\setminus\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$, *where $\varsigma:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a continuous nondecreasing function and $\varsigma(t)<t$, then $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ is a unique fixed circle of $\mathscr{M}$.*

Proof. Let $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r}_0)$ and $\mathscr{C}(\mathfrak{u}_1,\mathfrak{r}_1)$ be two fixed circles of $\mathscr{M}$, i.e., $\mathscr{M}$ satisfies the conditions (i)-(iii), (i)$'$ and (ii)$'$ (Theorem 37 and 39) for both the circles $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r}_0)$ and $\mathscr{C}(\mathfrak{u}_1,\mathfrak{r}_1)$. Let $\mathfrak{u}\in\mathscr{C}(\mathfrak{u}_0,\mathfrak{r}_0)$ and $\mathfrak{w}\in\mathscr{C}(\mathfrak{u}_1,\mathfrak{r}_1)$. Using inequality (67),

$$\begin{aligned}m_v^b(\mathfrak{u},\mathfrak{w})=m_v^b(\mathscr{M}\mathfrak{u},\mathscr{M}\mathfrak{w})&\le\mu\Big(m_v^b(\mathfrak{u},\mathfrak{w}),m_v^b(\mathfrak{u},\mathfrak{w}),m_v^b(\mathfrak{u},\mathfrak{w}),\frac{1}{2}\\&\cdot\left(m_v^b(\mathfrak{u},\mathfrak{w})+m_v^b(u,w),\frac{1}{2}\left(m_v^b(u,w)+m_v^b(\mathfrak{u},\mathfrak{w})\right)\right)\\&\le\mu\left(m_v^b(\mathfrak{u},\mathfrak{w})\right)<m_v^b(\mathfrak{u},\mathfrak{w}),\end{aligned} \tag{68}$$

a contradiction. Hence, $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r}_0)$ is a unique fixed circle of $\mathscr{M}$.

Next, we establish the existence of a greatest fixed disc.

Theorem 42. *Let $\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$ be a disc in an M_v^b-metric space ($\mathscr{M},m_v^b$). Define $\varsigma:\mathscr{M}\longrightarrow[0,\infty)$ as in (54). If there exists a self-map $\mathscr{A}:\mathscr{M}\longrightarrow\mathscr{M}$ so that*

$$\begin{aligned}m_v^b(\mathscr{A}\mathfrak{u},\mathfrak{u}_0)&\le\mathfrak{r}+m_v^b(\mathfrak{u}_0,\mathfrak{u}_0),\\m_v^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})&\le m_v^b(\mathfrak{u},\mathfrak{u}),\end{aligned} \tag{69}$$

(i) $m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})\le\varsigma(\mathscr{A}\mathfrak{u})+\varsigma(\mathscr{A}\mathfrak{u})-2\mathfrak{r}-2m_v^b(\mathfrak{u}_0,\mathfrak{u}_0)$, then $\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$ is a fixed disc of $\mathscr{M}$

(ii) If $m_v^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{w})\le\mu m_v^b(\mathfrak{u},\mathfrak{w})$,$\mathfrak{u}\in\mathscr{D}(\mathfrak{u}_0,\mathfrak{r}),\mathfrak{w}\in\mathscr{M}\setminus\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$,$\mu\in[0,1)$, then $\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$ is a fixed disc of maximum radius $\mathfrak{r}$, i.e., there is no fixed disc $\mathscr{D}(\mathfrak{u}_0,\mathfrak{r}')$ of $\mathscr{A}$ having a radius greater than $\mathfrak{r}$

Proof. Let $\mathfrak{u}\in\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$ be any arbitrary point:

$$\begin{aligned}m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})&\le m_v^b(\mathfrak{u},\mathfrak{u}_0)+m_v^b(\mathfrak{u},\mathfrak{u})+m_v^b(\mathscr{A}\mathfrak{u},\mathfrak{u}_0)\\&\quad+m_v^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})-2\mathfrak{r}-2m_v^b(\mathfrak{u}_0,\mathfrak{u}_0)\\&\le\mathfrak{r}+m_v^b(\mathfrak{u}_0,\mathfrak{u}_0)+m_v^b(\mathfrak{u},\mathfrak{u})+\mathfrak{r}+m_v^b(\mathfrak{u}_0,\mathfrak{u}_0)\\&\quad-2\mathfrak{r}-2m_v^b(\mathfrak{u}_0,\mathfrak{u}_0)=m_v^b(\mathfrak{u},\mathfrak{u}),\end{aligned} \tag{70}$$

$$\text{i.e.,}m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})\le m_v^b(\mathfrak{u},\mathfrak{u}). \tag{71}$$

But by the definition of an M_v^b-metric space, $m_v^b(\mathfrak{u},\mathfrak{u})\le m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})$.

So, $m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})=m_v^b(\mathfrak{u},\mathfrak{u})$. Using (70) and (iii), $m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})\le m_v^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})$. Also, by definition of an M_v^b-metric $m_v^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})\le m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})$. So, $m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})=m_v^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})$, i.e., $m_v^b(\mathfrak{u},\mathscr{A}\mathfrak{u})=m_v^b(\mathscr{A}\mathfrak{u},\mathscr{A}\mathfrak{u})=m_v^b(\mathfrak{u},\mathfrak{u})\Longrightarrow\mathfrak{u}$ is a fixed point of $\mathscr{A},\forall\mathfrak{u}\in\mathscr{D}(\mathfrak{u}_0,\mathfrak{r})$, i.e., the self-map $\mathscr{A}$ fixes the disc

$\mathscr{D}(\mathfrak{u}_0, \mathfrak{r})$, i.e., the set of fixed points of a map $\mathscr{A}$ includes a disc.

Let there exist two fixed discs $\mathscr{D}(\mathfrak{u}_0, \mathfrak{r})$ and $\mathscr{D}(\mathfrak{u}_1, \mathfrak{r}')$, $\mathfrak{r} < \mathfrak{r}'$, of $\mathscr{A}$, i.e., $\mathscr{A}$ satisfies the conditions (i) to (iii) for each of the discs $\mathscr{D}(\mathfrak{u}_0, \mathfrak{r})$ and $\mathscr{D}(\mathfrak{u}_1, \mathfrak{r}')$.

Let $\mathfrak{u} \in \mathscr{D}(\mathfrak{u}_0, \mathfrak{r})$ and $\mathfrak{w} \in \mathscr{D}(\mathfrak{u}_1, \mathfrak{r}')$ such that $\mathfrak{w} \notin \mathscr{D}(\mathfrak{u}_0, \mathfrak{r})$.

Using (iv), $m_\nu^b(\mathfrak{u}, \mathfrak{w}) = m_\nu^b(\mathscr{A}\mathfrak{u}, \mathscr{A}\mathfrak{w}) \le \mu m_\nu^b(\mathfrak{u}, \mathfrak{w})$, a contradiction. Hence, $\mathscr{D}(\mathfrak{u}_0, \mathfrak{r}_0)$ is a fixed disc of $\mathscr{A}$ having maximum a radius $\mathfrak{r}$.

Remark 43. Following the pattern of Theorem 41, we may prove the existence of a greatest fixed disc of $\mathscr{A}$ having a maximum radius.

Remark 44. For more work on the set of nonunique fixed points forming a circle or a disc or an ellipse, one may refer to [26, 28–31] and references therein.

5. Application to Cantilever Beam

As an application of Theorem 18, we solve fourth-order differential equation arising in two point boundary value problem of a bending of an elastic beam. This problem is employed in the distortion of an elastic beam in equilibrium, of which one end is free whereas another is fixed. Let $\mathscr{I} = [0, 1]$ and $\mathscr{M} = C[\mathscr{I}, \mathbb{R}]$ be the set of all continuous functions on $[0, 1]$. Define M_ν^b-metric, $m_\nu^b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}$ as $m_\nu^b(\mathfrak{u}, \mathfrak{w}) = (|\mathfrak{u}|+|\mathfrak{w}|/2)^2$, $\mathfrak{u}, \mathfrak{w} \in \mathscr{M}$ and $s = 4$.

Theorem 45. *We consider the Cantilever beam problem*

$$\begin{aligned} \frac{d^4 u}{dt} &= \psi\left(t, u, \frac{du}{dt}, \frac{d^2u}{dt}, \frac{d^3u}{dt}\right), \\ u(0) = \frac{du}{dt}(0) &= \frac{d^2u}{dt^2}(1) = \frac{d^3u}{dt^3}(1) = 0, \quad t \in [0, 1], \end{aligned} \tag{72}$$

where $\psi : [0, 1] \times \mathbb{R}^4 \longrightarrow [0, 1]$ *is a continuous function. If*

$$\left|\psi\left(\xi, \mathfrak{u}, \frac{d\mathfrak{u}}{dt}\right)\right| < \sqrt{\mu}|\mathfrak{u}(\xi)|, \xi \in [0, 1], \mu \in \left(0, \frac{1}{8}\right), \tag{73}$$

then Problem (72) has a unique solution.

Proof. The Cantilever beam problem (72) may be rewritten as

$$\mathfrak{u}(t) = \int_0^1 \mathscr{G}(t, \xi)\psi\left(\xi, \mathfrak{u}(\xi), \frac{d\mathfrak{u}}{dt}(\xi)\right), \mathfrak{u} \in \mathscr{M}, \tag{74}$$

where the Green function

$$\mathscr{G}(t, \xi) = \begin{cases} \dfrac{1}{6}\xi^2(3t - \xi), & 0 \le \xi \le t \le 1, \\ \dfrac{1}{6}t^2(3\xi - t), & 0 \le t \le \xi \le 1. \end{cases} \tag{75}$$

Define a map $\mathscr{A} : \mathscr{M} \longrightarrow \mathscr{M}$ by

$$\mathscr{A}\mathfrak{u}(t) = \int_0^1 \mathscr{G}(t, \xi)\psi\left(\xi, \mathfrak{u}(\xi), \frac{d\mathfrak{u}}{dt}(\xi)\right) d\xi. \tag{76}$$

Now, $\mathfrak{u} \in \mathscr{M}$ is a solution of (72) iff it is a fixed point of $\mathscr{A}$:

$$\begin{aligned} m_\nu^b(\mathscr{A}\mathfrak{u}(t), \mathscr{A}\mathfrak{w}(t)) &= \left(\frac{|\mathscr{A}\mathfrak{u}(t)| + |\mathscr{A}\mathfrak{w}(t)|}{2}\right)^2 \\ &= \frac{1}{4}\left(\left|\int_0^1 \mathscr{G}(t, \xi)\psi\left(\xi, \mathfrak{u}(\xi), \frac{d\mathfrak{u}}{dt}(\xi)\right) d\xi\right| \right. \\ &\quad \left. + \left|\int_0^1 \mathscr{G}(t, \xi)\psi\left(\xi, \mathfrak{w}(\xi), \frac{d\mathfrak{w}}{dt}(\xi)\right) d\xi\right|\right)^2 \\ &\le \frac{1}{4}\left(\sqrt{\mu}|\mathfrak{u}(t)|\left|\int_0^1 \mathscr{G}(t, \xi) d\xi\right| + \sqrt{\mu}|\mathfrak{w}(t)|\left|\int_0^1 \mathscr{G}(t, \xi) d\xi\right|\right)^2 \\ &= \frac{\mu}{4}(|\mathfrak{u}(t)| + |\mathfrak{w}(t)|)^2\left|\int_0^1 \mathscr{G}(t, \xi) d\xi\right|^2 \\ &= \frac{\mu}{4}(|\mathfrak{u}(t)| + |\mathfrak{w}(t)|)^2\left|\int_0^t \frac{1}{6}\xi^2(3t - \xi) d\xi + \int_t^1 \frac{1}{6}t^2(3\xi - t) d\xi\right|^2 \\ &= \frac{\mu}{4}(|\mathfrak{u}(t)| + |\mathfrak{w}(t)|)^2\left|\frac{1}{8}t^4 + \frac{1}{12}\left(3t^2 - 2t^3 - t^4\right)\right|^2 \\ &\le \frac{\mu}{4}(|\mathfrak{u}(t)| + |\mathfrak{w}(t)|)^2\left(\frac{1}{8} + \frac{6}{12}\right)^2 = \frac{25\mu}{256}(|\mathfrak{u}(t)| + |\mathfrak{w}(t)|)^2, \end{aligned} \tag{77}$$

$$\begin{aligned} & m_\nu^b(\mathfrak{u}, \mathscr{A}\mathfrak{u}(t)) + m_\nu^b(\mathfrak{w}, \mathscr{A}\mathfrak{w}(t)) \\ &= \left(\frac{|\mathfrak{u}(t)| + |\mathscr{A}\mathfrak{u}(t)|}{2}\right)^2 + \left(\frac{|\mathfrak{w}(t)| + |\mathscr{A}\mathfrak{w}(t)|}{2}\right)^2 \\ &= \left(\frac{|\mathfrak{u}(t)| + \left|\int_0^1 \mathscr{G}(t, \xi)\psi(\xi, \mathfrak{u}(\xi), d\mathfrak{u}/dt(\xi)) d\xi\right|}{2}\right)^2 \\ &\quad + \left(\frac{|\mathfrak{u}(t)| + \left(|\int_0^1 \mathscr{G}(t, \xi)\psi(\xi, \mathfrak{w}(\xi), d\mathfrak{w}/dt(\xi)) d\xi\,|\right.}{2}\right)^2 \\ &= \left(\frac{|\mathfrak{u}(t)| + \sqrt{\mu}|\mathfrak{u}(t)||\int_0^1 \mathscr{G}(t, \xi) d\xi}{2}\right)^2 \\ &\quad + \left(\frac{|\mathfrak{w}(t)| + \mu|\mathfrak{w}(t)||\int_0^1 \mathscr{G}(t, \xi) d\xi}{2}\right)^2 \\ &\le \frac{1}{4}\left[\left(|\mathfrak{u}(t)| + \frac{5}{8}\sqrt{\mu}|\mathfrak{u}(t)|\right)^2 + \left(|\mathfrak{w}(t)| + \frac{5}{8}\sqrt{\mu}|\mathfrak{w}(t)|\right)^2\right] \\ &= \frac{1}{4}\left[\frac{\left(8 + 5\sqrt{\mu}\right)^2}{64}|\mathfrak{u}(t)|^2 + \frac{\left(8 + 5\sqrt{\mu}\right)^2}{64}|\mathfrak{w}(t)|^2\right] \\ &= \frac{\left(8 + 5\sqrt{\mu}\right)^2}{256}\left(|\mathfrak{u}(t)|^2 + |\mathfrak{w}\left(t|^2\right)\right) < \frac{\left(8 + 5\sqrt{\mu}\right)^2}{256}(|\mathfrak{u}| + |\mathfrak{w}(t)|)^2. \end{aligned} \tag{78}$$

Equations (77) and (78) imply that

$$m_\nu^b(\mathscr{A}\mathfrak{u}(t), \mathscr{A}\mathfrak{w}(t)) \le \mu\left[m_\nu^b(\mathfrak{u}(t), \mathscr{A}\mathfrak{u}(t)) + m_\nu^b(\mathfrak{w}(t), \mathscr{A}\mathfrak{w}(t))\right], \tag{79}$$

i.e., $\mathscr{A}$ satisfies Theorem 18, and hence, $\mathscr{A}$ has a unique fixed point, i.e., a Cantilever beam problem has a unique solution.

6. Conclusion

We have introduced M_v^b-metric as an improvement and generalization of an M_v-metric which need not be continuous and defined topological notions like open ball, closed ball, convergence of a sequence, Cauchy sequence, and completeness of a space to discuss the topology of M_v^b-metric and to create an environment for the survival of a unique fixed point in an M_v^b-metric space. Further, we have demonstrated that the collection of open ball, which forms a basis on M_v^b-metric space and generates a $\mathscr{T}_0$ topology on it. Also, we have introduced a notion of a fixed circle and a fixed disc to study the geometry of the set of nonunique fixed points of a discontinuous self-map. Our results are sharpened versions of the well-known results, wherein continuity of a self-map is not indispensable for the survival of a unique fixed point. Examples and applications to solve a Cantilever beam problem employed in distortion of an elastic beam in equilibrium substantiate the utility of these improvements and extensions. It is interesting to mention that the Cantilever structures permit overhanging constructions deprived of peripheral bracing.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this research.

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Research Article

On Convergence Theorems for Generalized Alpha Nonexpansive Mappings in Banach Spaces

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The present paper seeks to illustrate approximation theorems to the fixed point for generalized α-nonexpansive mapping with the Mann iteration process. Furthermore, the same results are established with the Ishikawa iteration process in the uniformly convex Banach space setting. The presented results expand and refine many of the recently reported results in the literature.

1. Introduction

Consider a Banach space (BS) X, together with its subset $D(\neq \phi)$. Let us also consider the following notations $\mathrm{Fix}(T)$, $\rightharpoonup$, and $\longrightarrow$ to represent the set of fixed points of T, weak convergence, and strong convergence, correspondingly.

A self-mapping T defined on a subset D is referred to as

(1) nonexpansive provided that $\|T(u) - T(v)\| \leq \|u - v\|$, for all $u, v \in D$

(2) quasi-nonexpansive provided that $\mathrm{Fix}(T) \neq \phi$, and for all $u \in D(T)$ and $v \in \mathrm{Fix}(T)$, the following assertion holds: $\|T(u) - v\| \leq \|u - v\|$

Notably, there is a relationship between a nonexpansive mapping and a quasi-nonexpansive mapping. That is, each nonexpansive mapping satisfying $\mathrm{Fix}(T) \neq \phi$ is quasi-nonexpansive; however, the opposite is not correct generally. Furthermore, the opposite is satisfied as shown in [1] when the linearity condition is added to the quasi-nonexpansive mapping. Thus, a linear quasi-nonexpansive mapping is nonexpansive. Yet, it can be straightforwardly verified that there exist nonlinear quasi-nonexpansive mappings which are continuous and are not nonexpansive; for example,

$$T(u) = \frac{u}{2} \sin \frac{1}{u}, \text{ on } \mathbb{R}^1, \text{ with } T(0) = 0. \tag{1}$$

Nonexpansive mapping and its generalization remain a central topic of interest in the fixed point (FP) theory among different mathematicians and mathematical theorists. Various considerations and a variety of in-depth investigations including generalizations to this mapping have been reported in the literature, in which we notice its development in different branches and under various conditions (see [2–7]). Browder in 1965 [8] and Kirk [9] have shown that self-nonexpansive mappings defined on a convex subset of a uniformly convex Banach space (UCBS) that is closed and bounded have fixed points. In 1974, Senter and Dotson [10] established a strong convergence fixed point theorem with regard to the Mann iteration of a nonexpansive mapping. Furthermore, in 1993, Xu and Tan [11] generalized the results of Reich [12] and Senter and Dotson [10] by using the Ishikawa iterative procedure instead of the Mann process.

Recently, the notion of α-nonexpansive mapping in BS was proposed by Aoyama and Kohsaka [13] in 2011. This notion was further partially extended to a generalized (glz) α-nonexpansive mapping by Pant and Shukla [14] in 2017 as follows: consider a BS X with its subset $D(\neq \phi)$, and the mapping $T : D \longrightarrow D$ is considered to be glz α-nonexpansive provided that $\exists \alpha \in [0, 1)$ such that $\forall u, v \in D$,

$$\begin{aligned} \frac{1}{2}\|u - T(u)\| \leq \|u - v\| \Rightarrow \|T(u) - T(v)\| \\ \leq \alpha\|T(u) - v\| + \alpha\|T(v) - u\| + (1 - 2\alpha)\|u - v\|. \end{aligned} \tag{2}$$

Also, in [14], they have obtained the existence of FP results and convergence theorems by using the iteration process defined by Agarwal et al. [15] that reads

$$\begin{cases} u_1 \in D, \\ u_{n+1} = (1-t_n)T(u_n) + t_n T(v_n), \\ v_n = (1-s_n)u_n + s_n T(u_n), n \in \mathbb{N}, \end{cases} \tag{3}$$

with $\{t_n\}$ and $\{s_n\}$ as sequences belonging to $(0, 1)$. This iteration is known as the S-iteration, and it is independent of both the Ishikawa and Mann iteration processes as demonstrated in [15].

Over the most recent forty years, both the Ishikawa and Mann iteration processes have been effectively utilized by different mathematicians to approximate FP of different types of nonexpansive mappings in BS.

In 1953, Mann [11] devised a methodology that is termed as the Mann iterative process for approximating FP of continuous transformation in BS that reads

$$\begin{cases} u_1 \in D, \\ u_{n+1} = t_n T(u_n) + (1-t_n)u_n, n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{t_n\}$ is a sequence belonging to $[0, 1]$.

Moreover, Ishikawa [16] in 1974 generalized the Mann iterative process from one- to two-step iterations; he also obtained an iterative process to approximate FP of pseducontractive compact mapping in the Hilbert space given below:

$$\begin{cases} u_1 \in D, \\ v_n = (1-s_n)u_n + s_n T(u_n), \\ u_{n+1} = (1-t_n)u_n + t_n T(v_n), n \in \mathbb{N}, \end{cases} \tag{5}$$

with $\{t_n\}$ and $\{s_n\}$ denoting sequences lying in $[0, 1]$ and satisfying some conditions.

Also, observe that the Mann iterative procedure is a particular case of the Ishikawa iteration by the choice of $s_n = 0$, $\forall n \in \mathbb{N}$.

More recently, Piri et al. [17] in 2019 have shown some interesting examples of the glz α-nonexpansive mapping and presented certain comparative convergence behaviors with regard to some powerful iteration procedures including the famous Mann and Ishikawa iterations among others.

As an application, fixed point theory of nonexpansive mapping and its generalization has many applications in different fields such as applications of nonexpansive mapping to solve an integral equation (see [18]) and to solve a variational inequality problem (see [19]). Also, there are applications of some classes of generalized nonexpansive mappings like quasi-nonexpansive mappings under contraction to find the minimum norm fixed point and generalized α-nonexpansive mappings to solve split feasibility problem (see [20, 21]).

However, the present paper is aimed at establishing certain strong and weak convergence theorems of FP for the glz α-nonexpansive mapping via the application of the Mann iteration. Similar results are also set to be established by the application of the Ishikawa iteration process in the sense of UCBS. Remarkably, these results happen to be an extension of the results presented in [1, 11].

2. Preliminaries

Recall that a BS X satisfies the Opial property [22] for every sequence $\{u_n\}$ in X such that $\{u_n\} \rightharpoonup p$; then, $\forall q \in X$ with $p \neq q$,

$$\liminf_{n \longrightarrow \infty} \|u_n - p\| < \liminf_{n \longrightarrow \infty} \|u_n - q\|. \tag{6}$$

For example, all Hilbert spaces, all finite dimensional BS, and $\ell^p (1 < p < \infty)$ have satisfied the Opial property, while $L_p[0, 2\pi] (p \neq 2)$ has not satisfied the Opial property [23].

A BS X is uniformly convex provided that for each ε, $(0 < \varepsilon \leq 2)$, $\exists \delta > 0$ such that for any u, $v \in X$ together with $\|u\| = \|v\| = 1$ and $\|u - v\| > \varepsilon$; then, $\|(u + v)/2\| \leq 1 - \delta$ is said to hold.

Let $\{u_n\}$ and $D(\neq \phi)$ be a bounded sequence and subset of a BS X, respectively. Then, $\forall u \in X$, we define

(i) the asymptotic radius of the bounded sequence $\{u_n\}$ at u as

$$r(u, \{u_n\}) = \lim_{n \longrightarrow \infty} \sup \|u_n - u\|, \tag{7}$$

(ii) the asymptotic radius of the bounded sequence $\{u_n\}$ relative to D as

$$r(D, \{u_n\}) = \inf \{r(u, \{u_n\}) \colon u \in D\}, \tag{8}$$

(iii) the asymptotic center of the bounded sequence $\{u_n\}$ relative to D as

$$A(D, \{u_n\}) = \{u \in D : r(u, \{u_n\}) = r(D, \{u_n\})\}. \tag{9}$$

We observe that $A(D, \{u_n\}) \neq \phi$. Moreover, if X is UCBS, then $A(D, \{u_n\})$ has exactly one point [23].

Let X^* denote a dual space of BS X. Recall that X possesses the Fréchet differentiable norm provided that for each v in the sphere (unit) S of X, there exists the following limit:

$$\lim_{n \longrightarrow \infty} \frac{\|v + t v_0\| - \|v\|}{t}, \tag{10}$$

which is attained uniformly for $v_0 \in S$.

Thus, as rightly given in [23], $\forall u, w \in X$,

$$\frac{1}{2}\|u\|^2 + \langle w, J(u)\rangle \le \frac{1}{2}\|u+w\|^2 \le \frac{1}{2}\|u\|^2 + \langle w, J(u)\rangle + g(\|w\|), \tag{11}$$

where $J(u) = \partial(1/2)\|u\|^2$ and g is a function (increasing) defined on $\mathbb{R}^+$ of which $\lim_{t\downarrow 0}(g(t)/t) = 0$.

Accordingly, we give an illustrative example for a glz α-nonexpansive mapping in what follows.

Example 1 [14]. Consider $D = [0, 4] \subset \mathbb{R}$ of which a usual norm is endowed on. Let $T : D \longrightarrow D$ be defined by

$$T(u) = \begin{cases} 0, & \text{if } u \neq 4, \\ 2, & \text{if } u = 4. \end{cases} \tag{12}$$

Therefore, T is indeed a glz α-nonexpansive mapping with $\alpha \ge 1/3$.

Definition 2. Mapping which satisfies condition (I) [10]. "Let X be a normed space and let $D \subseteq X$. A map $T : D \longrightarrow D$ satisfies condition (I) provided that there exists a nondecreasing function $h : [0,\infty) \longrightarrow [0,\infty)$ that satisfies $h(0) = 0$ and $h(t) > 0$, for every $t \in (0,\infty)$ such that $\|u - T(u)\| \ge h(d(u, \mathrm{Fix}(T)))$, for each $u \in D$, where $d(u, \mathrm{Fix}(T))$ denotes the distance of u from $\mathrm{Fix}(T)$."

Next, we state some important results that are essentially vital to the present work; these results were introduced in [14, 24] together with their proofs.

Proposition 3. *Consider a BS X together with its subset $D(\neq \phi)$. Let us also consider a glz α-nonexpansive mapping given by $T : D \longrightarrow D$ with a FP $v \in D$. Then, T is quasi-nonexpansive.*

Lemma 4. *Consider a BS X together with its subset $D(\neq \phi)$. Let us also consider a glz α-nonexpansive mapping given by $T : D \longrightarrow D$. Therefore, for every $u, v \in D$,*

$$\|u - T(v)\| \le \|u - v\| + \frac{(3+\alpha)}{(1-\alpha)}\|u - T(u)\|. \tag{13}$$

Proposition 5. Demiclosedness principle [14]. *"Consider a BS X together with the Opial property, and let $D(\neq \phi)$ be a closed subset of X. Let $T : D \longrightarrow D$ be a glz α-nonexpansive mapping. If $\{u_n\} \rightharpoonup z$ and $\lim_{n\longrightarrow\infty}\|T(u_n) - u_n\| = 0$, then $T(z) = z$. Meaning, $(I - T)$ is demiclosed at zero, with I denoting the identity mapping on X."*

The lemma below gives the convexity and closedness of the set of FP for the glz α-nonexpansive mapping.

Lemma 6 [14]. *"Consider a glz α-nonexpansive mapping $T : D \longrightarrow D$, where $D(\neq \phi)$ is a subset of a BS X. Then, $\mathrm{Fix}(T)$ is closed. In addition, if D is convex and X is strictly convex, then $\mathrm{Fix}(T)$ is also convex."*

In the sequel, the next lemma will be used to navigate the main results of the paper.

Lemma 7 [24]. *"Consider a UCBS X and $0 < a \le l_n \le b < 1$, $\forall n \in \mathbb{N}$. Moreover, consider the two sequences $\{u_n\}$ and $\{v_n\}$ such that $\limsup_{n\longrightarrow\infty}\|u_n\| \le r$, $\limsup_{n\longrightarrow\infty}\|v_n\| \le r$, and $\lim_{n\longrightarrow\infty}\|l_n u_n + (1 - l_n)v_n\| = r$ hold for some $r \ge 0$. Then, $\lim_{n\longrightarrow\infty}\|u_n - v_n\| = 0$."*

3. Main Results

This section starts off by investigating the weak and strong approximation FP for the glz α-nonexpansive mapping by using the Mann iteration process. Moreover, a similar examination will be looked at by the application of the Ishikawa iteration procedure.

3.1. Main Results for glz α-Nonexpansive with the Mann Iteration

Lemma 8. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\neq \phi)$ of a BS X. Let the sequence $\{u_n\}$ be defined by the Mann iteration (1), and assume ζ to be a FP of T; then, $\lim_{n\longrightarrow\infty}\|u_n - \zeta\|$ exists.*

Proof. By referring to the definition of the Mann iteration (1) and Proposition 3, we get

$$\begin{aligned} \|u_{n+1} - \zeta\| &= \|(1 - t_n)u_n + t_n T(u_n) - \zeta\|, \\ &= \|(1 - t_n)(u_n - \zeta) + t_n(T(u_n) - \zeta)\|, \\ &\le (1 - t_n)\|u_n - \zeta\| + t_n\|T(u_n) - \zeta\|, \\ &\le (1 - t_n)\|u_n - \zeta\| + t_n\|u_n - \zeta\|, = \|u_n - \zeta\|. \end{aligned} \tag{14}$$

Therefore, the sequence $\{\|u_n - \zeta\|\}$ is bounded and nonincreasing. Thus, we conclude that $\lim_{n\longrightarrow\infty}\|u_n - \zeta\|$ exists.

Theorem 9. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\neq \phi)$ of a UCBS X. Let the sequence $\{u_n\}$ with $u_1 \in D$ be defined by the Mann iteration (1). Then, $\mathrm{Fix}(T) \neq \phi$ iff the sequence $\{u_n\}$ is bounded and*

$$\lim_{n\longrightarrow\infty}\|T(u_n) - u_n\| = 0. \tag{15}$$

Proof. Consider a bounded sequence $\{u_n\}$, and $\lim_{n\longrightarrow\infty}\|T(u_n) - u_n\| = 0$. As X is UCBS, then $A(D, \{u_n\}) \neq \phi$ and it contains exactly one point.

Let $z \in A(D, \{u_n\})$, and we want to demonstrate that $\mathrm{Fix}(T) \neq \phi$.

Using the asymptotic radius definition as given above, we obtain

$$r(T(z), \{u_n\}) = \limsup_{n\longrightarrow\infty} \|u_n - T(z)\|. \quad (16)$$

Also, using Lemma 4, we get

$$\begin{aligned} r(T(z), \{u_n\}) &= \limsup_{n\longrightarrow\infty} \|u_n - T(z)\|, \\ &\le \limsup_{n\longrightarrow\infty} \|u_n - z\| + \frac{(3+\alpha)}{(1-\alpha)} \limsup_{n\longrightarrow\infty} \|u_n - T(u_n)\|, \\ &= \limsup_{n\longrightarrow\infty} \|u_n - z\|, = r(z, \{u_n\}). \end{aligned} \quad (17)$$

Hence, $T(z) \in A(D, \{u_n\})$. However, with regard to the uniqueness of the asymptotic center of $\{u_n\}$, we obtain $T(z) = z$. That means $z \in Fix(T)$, and thus, $\mathrm{Fix}(T) \neq \phi$.

Conversely, let $\mathrm{Fix}(T) \neq \phi$ and $w \in \mathrm{Fix}(T)$; then, from Lemma 8, $\lim_{n\longrightarrow\infty} \|u_n - w\|$ exists. Suppose

$$\lim_{n\longrightarrow\infty} \|u_n - w\| = r > . \quad (18)$$

Equation (18) and Proposition 3 yield

$$\limsup_{n\longrightarrow\infty} \|T(u_n) - w\| \le \limsup_{n\longrightarrow\infty} \|u_n - w\| = r. \quad (19)$$

Hence,

$$\limsup_{n\longrightarrow\infty} \|T(u_n) - w\| \le r. \quad (20)$$

From equations (18) and (20) and the definition of the Mann iteration (1), we get

$$\begin{aligned} r &= \lim_{n\longrightarrow\infty} \|u_{n+1} - w\| = \lim_{n\longrightarrow\infty} \|(1-t_n)u_n + t_n T(u_n) - w\|, \\ &= \lim_{n\longrightarrow\infty} \|(1-t_n)(u_n - w) + t_n(T(u_n) - w)\|. \end{aligned} \quad (21)$$

In view of equations (18), (20), and (21) and Lemma 7, we deduce that

$$\lim_{n\longrightarrow\infty} \|u_n - w - T(u_n) + w\| = 0. \quad (22)$$

Consequently,

$$\lim_{n\longrightarrow\infty} \|u_n - T(u_n)\| = 0. \quad (23)$$

In order to prove weak convergence of both the Mann and Ishikawa iterative processes to a FP for glz α-nonexpansive mapping, the following lemma is needed.

Lemma 10 [14]. *"Suppose that the conditions of Theorem 9 are fulfilled. Then,* $\lim_{n\longrightarrow\infty} \langle u_n, J(p_1 - p_2)\rangle$ *exists for any* $p_1, p_2 \in Fix(T)$*; in particular,* $\langle u_0 - v_0, J(p_1 - p_2)\rangle = 0$, $\forall u_0, v_0 \in \eta_w(u_n)$*, where* $\eta_w(u_n)$ *represents the set of all weak limit points of* $\{u_n\}$*."*

Theorem 11. Weak convergent theorem. *Consider a glz* α*-nonexpansive self-mapping* T *with* $Fix(T) \neq \phi$ *defined on a closed convex subset* $D(\neq \phi)$ *of a UCBS* X *which satisfies the Opial property or which has a Fréchet differentiable norm such that* $(I - T)$ *is demiclosed at zero. Let the sequence* $\{u_n\}$ *be defined by the Mann iteration (1) with* $u_1 \in D$ *such that a sequence* $\{t_n\}$ *in* $[0, 1]$ *and* $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$*. Then, the sequence* $\{u_n\}$ *converges weakly to a FP of* T*.*

Proof. Consider $\eta_w(u_n)$ to be the set of all weak limit points of $\{u_n\}$. Then, from the fact that $\mathrm{Fix}(T) \neq \phi$, $\{u_n\}$ is a bounded sequence and

$$\lim_{n\longrightarrow\infty} \|T(u_n) - u_n\| = 0, \quad (24)$$

from Theorem 9. Therefore, without loss of generality, let $p \in \eta_w(u_n)$, which means

$$u_n \rightharpoonup p \text{ as } n \longrightarrow \infty. \quad (25)$$

Now, we want to show that $\eta_w(u_n) \subset \mathrm{Fix}(T)$. From (24), (25), and Proposition 5, we have

$$\begin{aligned} &u_n \rightharpoonup p, \\ &(I - T)u_n \longrightarrow 0 \text{ as } n \longrightarrow \infty, \end{aligned} \quad (26)$$

then,

$$(I - T)p = 0 \text{ implies } p = T(p). \quad (27)$$

Thus, $p \in \mathrm{Fix}(T)$, and we deduce that $\eta_w(u_n)$ is a subset of $\mathrm{Fix}(T)$.

Now, to prove that the sequence $\{u_n\}$ converges weakly to a FP of T, it is sufficient to prove that $\eta_w(u_n)$ is a singleton set.

First, we assume X to fulfil the Opial property and suppose p_1 and $p_2 \in \eta_w(u_n)$ such that $p_1 \neq p_2$; then, by the reflexiveness of X, we have

$$\begin{aligned} p_1 &= \text{weak} - \lim_{k\longrightarrow\infty} u_{n_k}, \\ p_2 &= \text{weak} - \lim_{j\longrightarrow\infty} u_{n_j}, \end{aligned} \quad (28)$$

for some $n_k \uparrow \infty, n_j \uparrow \infty$.

By Lemma 8, $\lim_{n\longrightarrow\infty} \|u_n - p_1\|$ exists, since $p_1 \in \eta_w(u_n) \subset \mathrm{Fix}(T)$.

Using the Opial property on X, we get that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|u_n-p_1\| &= \lim_{k\longrightarrow\infty}\left\|u_{n_k}-p_1\right\| < \lim_{k\longrightarrow\infty}\left\|u_{n_k}-p_2\right\|,\\ &= \lim_{j\longrightarrow\infty}\left\|u_{n_j}-p_2\right\| < \lim_{j\longrightarrow\infty}\left\|u_{n_j}-p_1\right\|,\\ &= \lim_{n\longrightarrow\infty}\|u_n-p_1\|,\end{aligned} \tag{29}$$

arriving at a contradiction.

Consequently, $p_1=p_2$. Hence, $\eta_w(u_n)$ is a singleton. This proves our result for which X satisfies the Opial property.

Secondly, we assume X to have a Fréchert differentiable norm given that $(I-T)$ is demiclosed at zero.

Substituting f_1-f_2 and $t(u_n-f_1)$ for u and w, respectively, in

$$\langle w, J(u)\rangle + \frac{1}{2}\|u\|^2 \le \frac{1}{2}\|u+w\|^2 \le \langle w, J(u)\rangle + \frac{1}{2}\|u\|^2 + g(\|w\|), \tag{30}$$

where $f_1, f_2 \in \mathrm{Fix}(T)$ and $0<t<1$, we obtain

$$\begin{aligned}&\frac{1}{2}\|f_1-f_2\|^2 + t\langle u_n-f_1, J(f_1-f_2)\rangle,\\ &\le \frac{1}{2}\|tu_n+(1-t)(f_1-f_2)\|^2,\\ &\le \frac{1}{2}\|f_1-f_2\|^2 + t\langle u_n-f_1, J(f_1-f_2)\rangle + g(t\|u_n-f_1\|).\end{aligned} \tag{31}$$

By referring to Lemma 10, the limit $\lim_{n\longrightarrow\infty}\langle u_n-f_1, J(f_1-f_2)\rangle$ exists.

In particular, this implies that

$$\langle p_1-p_2, J(f_1-f_2)\rangle = 0, \tag{32}$$

for all $p_1, p_2 \in \eta_w(u_n)$ and $f_1, f_2 \in \mathrm{Fix}(T)$.

By replacing f_1, f_2 in (32) by p_1, p_2, respectively, we obtain $\|p_1-p_2\|^2 = \langle p_1-p_2, J(p_1-p_2)\rangle = 0$, since $J(p_1-p_2) = 0$.

Thus, $p_1=p_2$. This shows that $\eta_w(u_n)$ must be a singleton.

Theorem 12. Strong convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T)\neq\phi$ defined on a closed convex subset $D(\neq\phi)$ of a UCBS X. Then, for arbitrary $u_1\in D$, the sequence $\{u_n\}$ defined by the Mann iteration (1) converges strongly to a member of $Fix(T)$ provided that T satisfies condition (I).*

Proof. Since from Proposition 3, each glz α-nonexpansive mapping that possesses at least one FP is a quasi-nonexpansive mapping, then our conclusion follows from Theorem 2 in [10].

Theorem 13. Strong convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T)\neq\phi$ defined on a closed convex subset $D(\neq\phi)$ of a BS X. Let the sequence $\{u_n\}$ be defined by the Mann iteration (1). Then, the sequence $\{u_n\}$ converges strongly to a FP of T provided that*

$$\liminf_{n\longrightarrow\infty} d(u_n, Fix(T)) = 0. \tag{33}$$

Proof. Assume that the $\liminf_{n\longrightarrow\infty} d(u_n, \mathrm{Fix}(T)) = 0$, then $\exists\{w_n\}$ a subsequence of $\{u_n\}$ of which

$$\lim_{n\longrightarrow\infty} d(w_n, \mathrm{Fix}(T)) = 0. \tag{34}$$

By (34), suppose $\{w_{n_j}\}$ again to be a subsequence of $\{w_n\}$ of which

$$\left\|w_{n_j}-z_j\right\| \le \frac{1}{2^j}, \quad \forall j\ge 1, \tag{35}$$

such that $\{z_j\}$ is a sequence in $\mathrm{Fix}(T)$. Then, by Lemma 8, we have

$$\left\|w_{n_{j+1}}-z_j\right\| \le \left\|w_{n_j}-z_j\right\| \le \frac{1}{2^j}. \tag{36}$$

Now, we want to show that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$. By the triangular inequality and (36), we conclude that

$$\|z_{j+1}-z_j\| \le \left\|z_{j+1}-w_{n_{j+1}}\right\| + \left\|w_{n_{j+1}}-z_j\right\| < \frac{1}{2^{j-1}}. \tag{37}$$

A standard argument refers to the fact that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$. By Lemma 6, $\mathrm{Fix}(T)$ is a closed subset of the BS X. Thus, $\{z_j\}$ converges to a FP z. Then, we have

$$\left\|w_{n_j}-z\right\| \le \left\|w_{n_j}-z_j\right\| + \|z_j-z\|. \tag{38}$$

Assume $j\longrightarrow\infty$; this means that $\{w_{n_j}\}$ converges strongly to z. Accordingly, $\lim_{n\longrightarrow\infty}\|u_n-z\|$ exists for $z\in Fix(T)$ by Lemma 8. Therefore, the sequence $\{u_n\}$ converges strongly to z.

3.2. Main Results for glz α-Nonexpansive with the Ishikawa Iteration. Now, let us state and prove some lemmas that will be utilized to prove the results as follows.

Lemma 14. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\neq\phi)$ of a BS X. Let the sequence $\{u_n\}$ with $u_1\in D$ be defined by the Ishikawa iteration (2). Suppose that $\zeta\in Fix(T)$; then, the statements given below are true:*

(1) $\max\{\|u_{n+1}-\zeta\|, \|v_n-\zeta\|\} \le \|u_n-\zeta\|, \forall n\in\mathbb{N}$.

(2) $\lim_{n\longrightarrow\infty}\|u_n-\zeta\|$

Proof. (1) By definition of the Ishikawa iteration (2) and Proposition 3, we have

$$\begin{aligned}\|v_n-\zeta\| &= \|(1-s_n)u_n+s_nT(u_n)-\zeta\|,\\ &\le (1-s_n)\|u_n-\zeta\|+s_n\|T(u_n)-\zeta\|,\\ &\le (1-s_n)\|u_n-\zeta\|+s_n\|u_n-\zeta\|, = \|u_n-\zeta\|,\end{aligned} \tag{39}$$

hence

$$\|v_n-\zeta\| \le \|u_n-\zeta\|. \tag{40}$$

Again, using the definition of the Ishikawa iteration (2) and Proposition 3, one gets

$$\begin{aligned}\|u_{n+1}-\zeta\| &= \|(1-t_n)u_n+t_nT(v_n)-\zeta\|,\\ &\le (1-t_n)\|u_n-\zeta\|+t_n\|T(v_n)-\zeta\|,\\ &\le (1-t_n)\|u_n-\zeta\|+t_n\|v_n-\zeta\|,\\ &\le (1-t_n)\|u_n-\zeta\|+t_n\|u_n-\zeta\|, = \|u_n-\zeta\|,\end{aligned} \tag{41}$$

thus,

$$\|u_{n+1}-\zeta\| \le \|u_n-\zeta\|. \tag{42}$$

Now, from (40) and (42), we get that

$$\max\{\|u_{n+1}-\zeta\|, \|v_n-\zeta\|\} \le \|u_n-\zeta\|, \quad \forall n\in\mathbb{N}. \tag{43}$$

(2) Using (42), the sequence $\{\|u_n-\zeta\|\}$ is deduced to be nonincreasing and bounded. Thus, $\lim_{n\longrightarrow\infty}\|u_n-\zeta\|$ exists.

Theorem 15. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\neq\phi)$ of a UCBS X. Let the sequence $\{u_n\}$ with $u_1\in D$ be defined by the Ishikawa iteration (2). Then, $Fix(T)\neq\phi$ iff the sequence $\{u_n\}$ is bounded and also*

$$\lim_{n\longrightarrow\infty}\|T(u_n)-u_n\| = 0. \tag{44}$$

Proof. Consider a bounded sequence $\{u_n\}$ and $\lim_{n\longrightarrow\infty}\|T(u_n)-u_n\| = 0$. Therefore, we get $\mathrm{Fix}(T)\neq\phi$ by following the same steps of the analogous part in the proof of Theorem 9.

Conversely, assume $\mathrm{Fix}(T)\neq\phi$ and $u_0\in\mathrm{Fix}(T)$, so from Lemma 14, $\lim_{n\longrightarrow\infty}\|u_n-u_0\|$ exists. Suppose

$$\lim_{n\longrightarrow\infty}\|u_n-u_0\| = r. \tag{45}$$

From equation (45) and Lemma 14, we have

$$\limsup_{n\longrightarrow\infty}\|v_n-u_0\| \le \limsup_{n\longrightarrow\infty}\|u_n-u_0\| = r. \tag{46}$$

Hence,

$$\limsup_{n\longrightarrow\infty}\|v_n-u_0\| \le r. \tag{47}$$

From equation (47) and Proposition 3, we get

$$\limsup_{n\longrightarrow\infty}\|T(v_n)-u_0\| \le \limsup_{n\longrightarrow\infty}\|v_n-u_0\| \le r. \tag{48}$$

Thus,

$$\limsup_{n\longrightarrow\infty}\|T(v_n)-u_0\| \le r. \tag{49}$$

Now, by the definition of the Ishikawa iteration (2), one gets

$$r = \lim_{n\longrightarrow\infty}\|u_{n+1}-u_0\| = \lim_{n\longrightarrow\infty}\|(1-t_n)u_n+t_nT(v_n)-u_0\| \tag{50}$$

$$= \lim_{n\longrightarrow\infty}\|(1-t_n)(u_n-u_0)+t_n(T(v_n)-u_0)\|. \tag{51}$$

In view of equations (45), (49), and (50) and Lemma 7, one obtains

$$\lim_{n\longrightarrow\infty}\|T(v_n)-u_n\| = 0. \tag{52}$$

Again, by the definition of the Ishikawa iteration (2), we have

$$\|u_{n+1}-u_n\| = \|(1-t_n)u_n+t_nT(v_n)-u_n\| = t_n\|T(v_n)-u_n\|. \tag{53}$$

Now, letting $n\longrightarrow\infty$, we get

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| = t_n\lim_{n\longrightarrow\infty}\|T(v_n)-u_n\| = 0. \tag{54}$$

Hence,

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| = 0. \tag{55}$$

Then, we conclude

$$\begin{aligned}\|u_{n+1}-T(v_n)\| &= \|u_{n+1}-u_n+u_n-T(v_n)\|\\ &\le \|u_{n+1}-u_n\|+\|u_n-T(v_n)\|.\end{aligned} \tag{56}$$

By equations (52) and (55), we deduce

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-T(v_n)\| = 0. \tag{57}$$

We observe

$$\begin{aligned}\|u_{n+1}-u_0\| &= \|u_{n+1}-T(v_n)+T(v_n)-u_0\|\\ &\le \|u_{n+1}-T(v_n)\|+\|T(v_n)-u_0\|.\end{aligned} \tag{58}$$

Now, taking the liminf of the last inequality, we get that

$$r = \liminf_{n\longrightarrow\infty}\|u_{n+1}-u_0\| \le \liminf_{n\longrightarrow\infty}\|u_{n+1}-T(v_n)\| + \liminf_{n\longrightarrow\infty}\|v_n-u_0\|. \tag{59}$$

Thus,

$$r \leq \liminf_{n\longrightarrow\infty} \|v_n - u_0\|. \tag{60}$$

From equations (47) and (60), we get

$$\begin{aligned} r &= \lim_{n\longrightarrow\infty} \|v_n - u_0\| = \lim_{n\longrightarrow\infty} \|(1-s_n)u_n + s_n T(u_n) - u_0\|, \\ &= \lim_{n\longrightarrow\infty} \|(1-s_n)(u_n - u_0) + s_n(T(u_n) - u_0)\|. \end{aligned} \tag{61}$$

Finally, from equations (45), (47), and (60) and via Lemma 7, one concludes that

$$\lim_{n\longrightarrow\infty} \|u_n - u_0 - T(u_n) + u_0\| = 0. \tag{62}$$

Hence,

$$\lim_{n\longrightarrow\infty} \|T(u_n) - u_n\| = 0. \tag{63}$$

Theorem 16. Weak convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a UCBS X which satisfies the Opial property or which has a Fréchet differentiable norm such that $(I - T)$ is demiclosed at zero. So for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration (2) together with restricting $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ and $\sum_{n=1}^{\infty} t_n (1-t_n)s_n < \infty$ converges weakly to a FP of T.*

Proof. The methodology of the proof is identical to that of Theorem 11.

Theorem 17. Strong convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a UCBS X, and suppose in addition that T satisfies condition (I). So for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration given in (2) converges strongly to a FP p of T.*

Proof. Since $\mathrm{Fix}(T) \neq \phi$, Theorem 15 guarantees that the sequence $\{u_n\}$ is bounded and $\lim_{n\longrightarrow\infty} \|u_n - T(u_n)\| = 0$.

Also, from condition (I), we guarantee that

$$\|u_n - T(u_n)\| \geq h(d(u_n, \mathrm{Fix}(T))), \quad \forall n \geq 1. \tag{64}$$

Thus, $d(u_n, \mathrm{Fix}(T)) \longrightarrow 0$ as $n \longrightarrow \infty$ follows from equation (64). A standard argument happens when there exists $p \in \mathrm{Fix}(T)$ of which $u_n \longrightarrow p \in \mathrm{Fix}(T)$ as $n \longrightarrow \infty$.

Theorem 18. Strong convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a BS X. So for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration (2) converges strongly to a FP of T provided that*

$$\liminf_{n\longrightarrow\infty} d(u_n, Fix(T)) = 0. \tag{65}$$

Proof. Suppose that $\liminf_{n\longrightarrow\infty} d(u_n, \mathrm{Fix}(T)) = 0$, so a subsequence $\{w_n\}$ of the sequence $\{u_n\}$ exists of which $\lim_{n\longrightarrow\infty} d(w_n, \mathrm{Fix}(T)) = 0$. In view of the previous step, consider a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $\|w_{n_j} - z_j\| \leq 1/2^j, \forall j \geq 1$, where $\{z_j\}$ is a sequence in $\mathrm{Fix}(T)$.

So, with the help of Lemma 14, we guarantee that

$$\left\|w_{n_{j+1}} - z_j\right\| \leq \left\|w_{n_j} - z_j\right\| \leq \frac{1}{2^j}. \tag{66}$$

Now, we want to show that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$. From (66), we conclude that

$$\|z_{j+1} - z_j\| \leq \left\|z_{j+1} - w_{n_{j+1}}\right\| + \left\|w_{n_{j+1}} - z_j\right\| < \frac{1}{2^{j-1}}. \tag{67}$$

A standard argument proves that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$.

By referring to Lemma 6, we get that $\mathrm{Fix}(T)$ is a closed subset of BS X; then, $\{z_j\}$ converges to a FP z.

Now, we have

$$\left\|w_{n_j} - z\right\| \leq \left\|w_{n_j} - z_j\right\| + \|z_j - z\|. \tag{68}$$

Assume $j \longrightarrow \infty$; this means that $\{w_{n_j}\}$ converges strongly to z.

By Lemma 14, the limit $\lim_{n\longrightarrow\infty} \|u_n - z\|$ exists for $z \in \mathrm{Fix}(T)$.

Thus, $\{u_n\}$ converges strongly to z, where $z \in \mathrm{Fix}(T)$.

Theorem 19. Strong convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a UCBS X. Suppose also that the range of D under T is included in a subset of X that is compact (i.e., $T(D) \subseteq C \subseteq X$, where C is compact). Then, for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration (2) converges strongly to a FP of T.*

Proof. Given that $\mathrm{Fix}(T) \neq \phi$, Theorem 15 guarantees that the sequence $\{u_n\}$ is bounded and $\lim_{n\longrightarrow\infty} \|T(u_n) - u_n\| = 0$.

Now, from precompactness of $T(D)$, we conclude that $T(D) \subseteq C$, where C is compact.

Hence, $T\{u_n\} \subseteq C$ affirming $T(u_n) \longrightarrow u_n$ as $n \longrightarrow \infty$.

Then, $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$, so $\{u_{n_k}\} \longrightarrow z$ as $k \longrightarrow \infty$.

Again, from Theorem 15, one gets

$$\lim_{k\longrightarrow\infty} \left\|u_{n_k} - Tu_{n_k}\right\| = 0, \tag{69}$$

which implies $\|z - T(z)\| = 0$ which further implies $z \in \mathrm{Fix}(T)$. Since by Lemma 14, for $z \in \mathrm{Fix}(T)$, $\lim_{n\longrightarrow\infty} \|u_n - z\|$ exists, this means that the sequence $\{u_n\}$ converges strongly to $z \in \mathrm{Fix}(T)$.

4. Conclusion

In conclusion, the class of generalized α-nonexpansive mapping has been extensively examined in a uniformly convex Banach space setting. Results of the existence of the fixed point have been established and proven in Theorems 9 and 15 via the applications of the Mann and Ishikawa iterations, respectively. The established results corresponded to the results of Theorem 5.6 in [14].

Moreover, to approximate the fixed point of a generalized α-nonexpansive mapping, we made use of the Mann and Ishikawa iterations and proved strong convergence results. For instance, the established Theorems 12 and 13 via the Mann iteration came as a special state of Theorem 2 in [10] and corresponded to Theorem 5.9 in [14], correspondingly; while through the Ishikawa iteration, Theorems 17, 18, and 19 generalized Theorem 2.4 in [1] and corresponded to Theorem 5.9 in [14] and Theorem 2 in [11], respectively. Furthermore, with regard to weak convergence results, Theorems 11 and 16 for the Mann and Ishikawa iterations, respectively, generalized Theorem 2 in [12] and Theorem 2.3 in [1], respectively, by considering a generalized α-nonexpansive mapping instead of a nonexpansive mapping.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Random Fixed Point Theorems and Applications to Random First-Order Vector-Valued Differential Equations

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In this paper, we establish several random fixed point theorems for random operators satisfying some iterative condition w.r.t. a measure of noncompactness. We also discuss the case of monotone random operators in ordered Banach spaces. Our results extend several earlier works, including Itoh's random fixed point theorem. As an application, we discuss the existence of random solutions to a class of random first-order vector-valued ordinary differential equations with lack of compactness.

1. Introduction

Random fixed point theorems are stochastic generalizations of deterministic fixed point theorems. In recent years, random fixed point theorems have assumed importance due to their applications to random differential and integral equations (see for instance [1–9] and the references therein). Since Bharucha-Reid published his survey paper [10], numerous works of great importance have appeared and contributed strongly to the enrichment of the random fixed point theory. We quote, for instance, the contributions by Itoh [11], Reich [12], Rybinski [13], Sehgal and Singh [14], Sehgal and Waters [15], Papageorgiou [16], Tan and Yuan [17], Hussain et al. [18], and many others.

In 1979, Itoh [11] proved the random version of Sadovskii's well-known fixed point theorem [19]. Later on, Tan and Yuan [17] established an interesting result which serves as a bridge that links the random fixed point theory with the deterministic fixed point theory. Specifically, their result ensures the existence of a random fixed point for a continuous random operator under some compactness conditions provided that the corresponding deterministic fixed point problem is solvable. Recently, El-Ghabi and Taoudi [20] developed a new fixed point approach that combines the advantages of the strong topology with the advantages of the weak topology and which enables them to draw new and meaningful conclusions about random fixed points for a given random operator and to handle nonlinear random problems with lack of compactness.

In the present paper, we prove some new random fixed point theorems for (countably) convex-power condensing random operators in Banach spaces. Our results extend, generalize, and unify several known random fixed point results including the famous Itoh random fixed point theorem [11], Theorem 2.1. We should point out here that the concept of convex-power condensing operator was introduced in [21] as a generalization of the concept of condensing operator. We also prove some new random fixed point results for monotone (countably) convex-power condensing random operators in ordered Banach spaces. To illustrate our theoretical results, we investigate the solvability of a broad class of random first-order vector-valued ordinary differential equations. At this point, it is of significance to note that this class of random differential equations was already examined in [11] under some restriction on the constants used in the condition of measure of noncompactness and we have been successful in using our random fixed point results to remove this restriction (see Remark 34).

The paper is arranged as follows. In Section 2, we fix the notation and present some key tools that will be used to prove our main results. Section 3 is devoted to random fixed point theorems for (countably) convex-power condensing random operators and their corollaries. In Section 4, we prove some new random fixed point theorems for a class of monotone random operators in an ordered Banach space. Finally, in Section 5, we use some material from the previous sections to solve the random differential equation

$$\frac{d}{dt}u(t,w) = f(t,u(t,w),w), \quad (1)$$

where t belongs to a bounded and closed interval in the real line $\mathbb{R}$, ω belongs to a set Ω endowed with a σ-algebra, and the functions f, u have values in a Banach space E.

2. Preliminaries

Throughout this paper, (Ω, Σ) denotes a measurable space, where Ω is a nonempty set and Σ is a σ-algebra of subsets of Ω. Let $(E, \|.\|)$ be a Banach space, X be a nonempty subset of E and $\mathcal{B}_1$ be the closed unit ball of E. Let 2^X, $B(X)$, and $CD(X)$ denote, respectively, the collections of all nonempty subsets, bounded subsets, and closed subsets of X. In addition, let $K(X)$ and $WK(X)$ denote, respectively, the classes of compact and weakly compact subsets of X. We say that a mapping $T : X \longrightarrow E$ is hemicompact if each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X has a convergent subsequence whenever $\|x_n - T(x_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$. A mapping $F : \Omega \longrightarrow 2^E$ is said to be measurable if for every open subset U of E, $F^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \varnothing\} \in \Sigma$. It is important to stress here that, in the case where $F(\omega) \in K(E)$ for each $\omega \in \Omega$, F is measurable if and only if $F^{-1}(C) \in \Sigma$ for each $C \in CD(E)$ (cf. [22], Theorem 3]). A mapping $T : \Omega \times X \longrightarrow E$ is called a random operator if $T(.,x)$ is measurable for each $x \in X$. A mapping $\xi : \Omega \longrightarrow X$ is said to be:

(i) a deterministic fixed point of a random operator $T : \Omega \times X \longrightarrow E$ if $T(\omega, \xi(\omega)) = \xi(\omega)$ for every $\omega \in \Omega$,

(ii) a random fixed point of a random operator $T : \Omega \times X \longrightarrow E$ if it is measurable and $T(\omega, \xi(\omega)) = \xi(\omega)$ for every $\omega \in \Omega$,

Now, we present some basic facts regarding measures of (weak) noncompactness in Banach spaces, which we will be needed in the sequel. It is noteworthy that measures of noncompactness have played a substantial role in nonlinear functional analysis. They are very often used in the theory of functional equations, including ordinary differential equations, equations with partial derivatives, integral and integro-differential equations, and optimal control theory. We also highlight that the interplay between fixed point theory and measures of noncompactness is very powerful and fruitful (see for instance [19, 23–29] and the references therein).

Definition 1 (see [30–32]). Let E be a Banach space. A function $\psi : B(E) \longrightarrow [0,+\infty)$ is said to be a measure of (weak) noncompactness if it satisfies the following conditions:

(1) The family ker $(\psi) = \{M \in B(E): \psi(M) = 0\}$ is nonempty and ker (ψ) is contained in the set of relatively (weakly) compact sets of E.

(2) $M_1 \subset M_2 \Longrightarrow \psi(M_1) \le \psi(M_2)$ for all M_1, M_2 in $B(E)$.

(3) $\psi(\overline{co}(M)) = \psi(M)$ for each $M \in B(E)$, where $\overline{co}(M)$ denotes the closed convex hull of M

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \le \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for all $\lambda \in [0\,;1]$ and $M_1, M_2 \in B(E)$.

(5) If $(M_n)_{n\ge1}$ is a sequence of nonempty (weakly) closed subsets of E with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty} \psi(M_n) = 0$, then $M_\infty \coloneqq \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family ker (ψ) is said to be the kernel of the measure of (weak) noncompactness ψ. Note that the intersection set M_∞ belongs to ker (ψ) (see [33]). A measure of (weak) noncompactness ψ is said to be

(6) Full if $\psi(M) = 0$ if and only if M is a relatively (weakly) compact set.

(7) Nonsingular if

$$\psi(M \cup \{x\}) = \psi(M), \quad (2)$$

for all $M \in B(E)$ and $x \in E$

(8) Set additive or has the maximum property if

$$\psi(M_1 \cup M_2) = \max\,(\psi(M_1), \psi(M_2)) \text{for all } M_1, M_2 \in B(E) \quad (3)$$

(9) Homogeneous if

$$\psi(\lambda M) = |\lambda|\psi(M) \text{for all} \lambda \in \mathbb{R} \text{ and } M \in B(E) \quad (4)$$

(10) Subadditive if

$$\psi(M_1 + M_2) \le \psi(M_1) + \psi(M_2) \text{for all } M_1, M_2 \in B(E) \quad (5)$$

(11) Regular if ψ is full, homogeneous, nonsingular, subadditive, and set additive.

The first important measure of weak noncompactness was defined by De Blasi [34] as follows:

$$\beta(M) = \inf\,\{r > 0 : \text{there exists } W \in WK(E) \text{with } M \subset W + r\mathcal{B}_1\}, \quad (6)$$

for each M in $B(E)$. Notice that β is regular (see [34]).

Also, the most important regular measures of noncompactness include the Kuratowski measure of noncompactness defined by

$$\alpha(M) = \inf\left\{r > 0 : M \subseteq \bigcup_{i=1}^{n} M_i, \operatorname{diam}(M_i) \le r, \forall i\right\}, \quad (7)$$

here $\operatorname{diam}(M_i) = \sup\{\|x-y\|: x, y \in M_i\}$. For more details, we refer the reader to the interesting book by Banaś and Goebel [30].

Now, we give a short note about (countably) convex-power condensing mappings with respect to a measure of (weak) noncompactness. Let X be a nonempty closed convex subset of E and $x_0 \in X$. Let $T : X \longrightarrow X$ be a bounded mapping (i.e., T takes bounded sets into bounded ones). Following [33, 35], for any $M \subset X$, we set

$$\begin{gathered} T^{(1,x_0)}(M) = T(M), \\ T^{(n,x_0)}(M) = T\left(\overline{co}\left(T^{(n-1,x_0)}(M) \cup \{x_0\}\right)\right) \text{ for, } n = 2, 3,. \end{gathered} \tag{8}$$

Definition 2. Let E be a Banach space and ψ be a measure of (weak) noncompactness on E. Let X be a nonempty closed convex subset of E and $T : X \longrightarrow X$ be a bounded mapping. We say that T is (countably) ψ-convex-power condensing about $x_0 \in X$ and $n_0 \in \mathbb{N}$ if

$$\psi\left(T^{(n_0,x_0)}(M)\right) < \psi(M), \tag{9}$$

for any (countable) bounded subset M of X with $\psi(M) > 0$.

It is customary to say "(countably) ψ-condensing" in place of "ψ-convex-power condensing about x_0 and 1." We also say that T is (countably) ψ-convex-power condensing if there exist $x_0 \in X$ and $n_0 \in \mathbb{N}$ such that T is (countably) ψ-convex-power condensing about x_0 and n_0.

3. Random Fixed Point Theorems for (Countably) Convex-Power Condensing Random Operators in Banach Spaces

Throughout this section, (Ω, Σ) is a measurable space, E is a Banach space and X is a nonempty closed convex subset of E. Let $T : \Omega \times X \longrightarrow E$ be a random operator. We say that T is continuous (resp., uniformly continuous, hemicompact) if for each $\omega \in \Omega$, $T(\omega, .)$ is continuous (resp., uniformly continuous, hemicompact). If T has values in X, we say that T is (countably) ψ-convex-power condensing if $T(\omega, .)$ is (countably) ψ-convex-power condensing for every $\omega \in \Omega$. We say also that T is $k(.)$-lipschitzian (resp. a $k(.)$-contraction) if there exists a mapping $k : \Omega \longrightarrow [0,+\infty)$ (resp.: $\Omega \longrightarrow [0, 1)$) satisfying $\|T(\omega, x) - T(\omega, y)\| \le k(\omega)\|x - y\|$ for all $x, y \in X$ and all $\omega \in \Omega$.

The following result guarantees the existence of a random fixed point for a continuous random operator provided that the corresponding deterministic fixed point problem is solvable.

Theorem 3 (see [17], Theorem 2.3). *Let (Ω, Σ) be a measurable space and X be a nonempty separable complete subset of a Banach space E. Suppose that $T : \Omega \times X \longrightarrow CD(E)$ is a continuous hemicompact random operator. Then, T has a deterministic fixed point if and only if T has a random fixed point.*

Our main purpose in the immediate sequel is to prove some new random fixed point theorems for countably convex-power condensing mappings. Before proceeding further with the first main theorem, we obtain some auxiliary results.

Lemma 4. *Let E be a Banach space, X be a closed convex subset of E and ψ be a regular measure of noncompactness on E. Let $T : X \longrightarrow X$ be a uniformly continuous countably ψ-convex-power condensing operator. If $T(X)$ is bounded, then T is hemicompact.*

Proof. Let $\Delta = \{x_k\}_{k\ge 0}$ be a sequence of elements of X such that

$$\|T(x_k) - x_k\| \longrightarrow 0 \ as\ k \longrightarrow \infty. \tag{10}$$

We show that for each $n \in \mathbb{N}$, $\psi(\Delta) \le \psi(T^n(\Delta))$, where T^n is the n^{th} iterate of T. To see this, notice first that if T is uniformly continuous, then so is T^n for each integer n. Let $\varepsilon > 0$ be fixed and take any integer n. Then, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x - y\| < \delta$, we have $\|T^n(x) - T^n(y)\| < \varepsilon$. By virtue of (10), there exists an integer N such that for each $k \ge N$, we have $\|T(x_k) - x_k\| < \delta$, so that,

$$\left\|T^{n+1}(x_k) - T^n(x_k)\right\| = \|T^n(T(x_k)) - T^n(x_k)\| < \varepsilon, \tag{11}$$

whenever $k \ge N$. With this in mind, we can easily see that

$$\{T^n(x_k)\}_{k\ge N} \subseteq \left\{T^{n+1}(x_k)\right\}_{k\ge N} + \varepsilon\mathscr{B}_1 \subseteq T^{n+1}(\Delta) + \varepsilon\mathscr{B}_1, \tag{12}$$

where $\mathscr{B}_1$ is the closed unit ball of E. From the properties of the measure of noncompactness, it follows that

$$\psi\left(\{T^n(x_k)\}_{k\ge N}\right) \le \psi\left(T^{n+1}(\Delta)\right) + \varepsilon\psi(\mathscr{B}_1), \tag{13}$$

which amounts to

$$\psi(T^n(\Delta)) = \psi\left(\{T^n(x_k)\}_{k\ge N}\right) \le \psi\left(T^{n+1}(\Delta)\right) + \varepsilon\psi(\mathscr{B}_1). \tag{14}$$

The arbitrariness of ε yields $\psi(T^n(\Delta)) \le \psi(T^{n+1}(\Delta))$. Thus, $\{\psi(T^n(\Delta))\}_{n\in\mathbb{N}}$ is an increasing sequence and therefore

$$\psi(\Delta) = \psi\left(T^0(\Delta)\right) \le \psi(T^n(\Delta)), \tag{15}$$

for any integer $n \ge 0$. Let $a \in X$ and $n_0 \in \mathbb{N}$ such that T is countably ψ-convex-power condensing about a and n_0. By mathematical induction, we can see that

$$T^n(\Delta) \subseteq T^{(n,a)}(\Delta), \tag{16}$$

for any integer $n \ge 1$. Going back to (15), we get

$$\psi(\Delta) \le T^{(n_0,a)}(\Delta). \tag{17}$$

From our hypotheses, we know that $\psi(\Delta) = 0$ and so Δ is relatively compact. Thus, the sequence $\{x_k\}_{k\geq 0}$ has a convergent subsequence and consequently T is hemicompact.

Now, we are ready to state and prove the following deterministic fixed point theorem.

Theorem 5. *Let E be a Banach space and X be a closed convex subset of E. Let $T : X \longrightarrow X$ be a continuous mapping such that $T(X)$ is bounded. Assume further that there are an integer $n_0 \geq 1$ and a vector $x_0 \in X$ such that for any countable subset $C \subset X$, we have*

$$\left[C \subseteq \overline{co}\left(T^{(n_0,x_0)}(C) \cup \{x_0\}\right)\right] \Rightarrow [C \text{ is relatively compact}]. \tag{18}$$

Then, T has a fixed point.

Proof. Consider the iterative sequence (D_n) of sets given by

$$D_0 = \{x_0\}, D_n = \overline{co}(T(D_{n-1}) \cup \{x_0\}) \text{ for every } n \geq 1. \tag{19}$$

By mathematical induction, it is easy to show that $D_n \subseteq D_{n+1}$ and D_n is convex compact for each $n \geq 0$. Let $D = \bigcup_{n\geq 0} D_n$. Plainly, D is convex. Furthermore, observe that for each $n \in \mathbb{N}$, D_n is separable (since it is compact), and hence, there exists a countable set $C_n \subseteq D_n$ such that $\bar{C}_n = D_n$. Let us consider the countable subset $C = \bigcup_{n\geq 0} C_n$ of D. It is readily apparent that $D = \bigcup_{n\geq 0} \bar{C}_n \subseteq \overline{\bigcup_{n\geq 0} C_n} = C$, and so $\bar{D} = \bar{C}$.

We claim that for each $k \geq 1$,

$$C \subseteq \overline{co}\left(T^{(k,x_0)}(C) \cup \{x_0\}\right). \tag{20}$$

We proceed by induction. For the base case $k = 1$, we can easily verify that

$$\begin{aligned} C \subset \bar{D} \subseteq \overline{co}(T(D) \cup \{x_0\}) &= \overline{co}\left(T\left(\bar{D}\right) \cup \{x_0\}\right) \\ &= \overline{co}\left(T\left(\bar{C}\right) \cup \{x_0\}\right) \\ &= \overline{co}(T(C) \cup \{x_0\}) \\ &= \overline{co}\left(T^{(1,x_0)}(C) \cup \{x_0\}\right). \end{aligned} \tag{21}$$

Let $k \geq 1$ be fixed and suppose that

$$C \subseteq \overline{co}\left(T^{(k,x_0)}(C) \cup \{x_0\}\right). \tag{22}$$

Then,

$$T(C) \subseteq T\left(\overline{co}\left(T^{(k,x_0)}(C) \cup \{x_0\}\right)\right) = T^{(k+1,x_0)}(C), \tag{23}$$

so that

$$C \subset \overline{co}(T(C) \cup \{x_0\}) \subseteq \overline{co}\left(T^{(k+1,x_0)}(C) \cup \{x_0\}\right). \tag{24}$$

This proves our claim. From our hypotheses, we deduce that $\bar{C}$ (and so $\bar{D}$) is compact. Furthermore, it is not difficult to see that

$$T\left(\bar{D}\right) \subseteq \overline{\bigcup_{n\geq 0} T(D_n)} \subseteq \overline{\bigcup_{n\geq 0} T(\overline{co}(T(D_n) \cup \{x_0\})} = \bigcup_{n\geq 0} D_{n+1} \subseteq \bar{D}. \tag{25}$$

An appeal to Schauder's fixed point theorem yields a fixed point for T.

Remark 6. It is of significance to note that condition (18) is fulfilled whenever T is (countably) ψ-convex-power condensing about x_0 and n_0.

After these preparations, we are now ready to state the main result of this section. This result ensures the existence of a random fixed point for a uniformly continuous random operator under general compactness conditions.

Theorem 7. *Let E be a Banach space and ψ be a regular measure of noncompactness on E. Let X be a separable closed convex subset of E and let $T : \Omega \times X \longrightarrow X$ be a uniformly continuous and countably ψ-convex-power condensing random operator. Assume that $T(\omega, X)$ is bounded for each $\omega \in \Omega$, then, T has a random fixed point.*

Proof. Invoking Theorem 5, we infer that the random operator T has a deterministic fixed point. In addition, since for each $\omega \in \Omega$, $T(\omega,.) : X \longrightarrow X$ is a uniformly continuous countably ψ-convex-power condensing mapping and $T(\omega, X)$ is bounded, then by Lemma 4, T is hemicompact. The desired result follows from Theorem 3.

Next, we turn our attention to the case when the boundary condition is of Rothe type. Before making a formal statement of our next random fixed point result, we need some auxiliary results. We first recall the following lemma.

Lemma 8 (see [15], Lemma 2). *Let $f : \Omega \longrightarrow E$ and $l : \Omega \longrightarrow [0, 1]$ be measurable mappings. Then, for any $y \in E$, the mapping $h : \Omega \longrightarrow E$ defined by $h(\omega) = l(\omega) f(\omega) + (1 - l(\omega))y$ is measurable.*

Recall that, if U is a convex neighborhood of the origin in E, then the Minkowski functional p of U is defined on E by

$$p(x) = \inf \{\varepsilon > 0 : x \in \varepsilon U\}. \tag{26}$$

It is well known [36], Lemma 5.12.1 that p is continuous, subadditive, positively homogeneous, and satisfies

$$\{x \in E : p(x) < 1\} \subseteq U \subseteq \{x \in E : p(x) \leq 1\} \subseteq U. \tag{27}$$

Note also that if U has a nonempty interior, then p is uniformly continuous.

Now, we state the following Browder-Fan type result [37], which is crucial for our purposes.

Theorem 9. *Let X be a separable closed convex subset of a Banach space E with $Int(X) \neq \varnothing$ and let ψ be a regular measure of noncompactness on E. Let $x_0 \in Int(X)$, $n_0 : \Omega \longrightarrow \mathbb{N}$ be a mapping and $T : \Omega \times X \longrightarrow E$ be a uniformly continuous random operator. Assume that $T(\omega, X)$ is bounded and $T(\omega, .)$ is (countably) ψ-convex-power condensing about x_0 and $n_0(\omega)$ for each $\omega \in \Omega$. Then, there exists a measurable mapping $\xi : \Omega \longrightarrow X$ satisfying*

$$p(T(\omega, \xi(\omega)) - \xi(\omega)) = \min \{p(T(\omega, \xi(\omega)) - x\text{: } x \in X\}, \tag{28}$$

for each $\omega \in \Omega$, where p is the Minkowski functional of $(X - x_0)$. Further, if $p(T(\omega, \xi(\omega)) - x_0) \leq 1$ for some $\omega \in \Omega$, then $T(\omega, \xi(\omega)) = \xi(\omega)$.

Proof. Let p be the Minkowski functional of $(X - x_0)$ and define the mapping $l : \Omega \times X \longrightarrow [0, 1]$ by

$$l(\omega, x) = [\max \{1, p(T(\omega, x) - x_0)\}]^{-1}, \tag{29}$$

for each $(\omega, x) \in \Omega \times X$. Consider the mapping $g : \Omega \times X \longrightarrow E$ defined by

$$g(\omega, x) = l(\omega, x)T(\omega, x) + (1 - l(\omega, x))x_0. \tag{30}$$

The reasoning in [15], Theorem 1 yields that g is a continuous random operator such that for each $\omega \in \Omega$, $g(\omega, X)$ is bounded. We will show that g satisfies all conditions of Theorem 7,. To this end, let $\omega \in \Omega$ be fixed. First notice that

$$p(g(\omega, x) - x_0) = l(\omega, x)p(T(\omega, x) - x_0) \leq 1, \tag{31}$$

for any $x \in X$, so that, $g(\Omega \times X) \subset X$. In addition, it is easy to check that

$$|l(\omega, x) - l(\omega, y)| \leq |p(T(\omega, x) - x_0) - p(T(\omega, y) - x_0)|, \tag{32}$$

for all $x, y \in E$. Since p and $T(\omega, .)$ are uniformly continuous, so is $l(\omega, .)$. Furthermore, for all $x, y \in X$, we obtain

$$\begin{aligned}
\|g(\omega, x) - g(\omega, y)\| &= \|l(\omega, x)(T(\omega, x) - x_0) \\
&\quad - l(\omega, y)(T(\omega, y) - x_0)\| \\
&= \|l(\omega, x)(T(\omega, x) - T(\omega, y)) + (l(\omega, x) \\
&\quad - l(\omega, y))(T(\omega, y) - x_0)\| \\
&\leq l(\omega, x)\|T(\omega, x) - T(\omega, y)\| + |l(\omega, x) \\
&\quad - l(\omega, y)|(\|T(\omega, y)\| + \|x_0\|) \\
&\leq \|T(\omega, x) - T(\omega, y)\| + |l(\omega, x) \\
&\quad - l(\omega, y)|\left(\|T(\omega, .)\|_\infty + \|x_0\|\right).
\end{aligned} \tag{33}$$

It follows that $g(\omega, .)$ is uniformly continuous. Next, we will illustrate that $g(\omega, .)$ is countably ψ-convex-power condensing about x_0 and $n_0(\omega)$. To see this, take any bounded countable subset M of X with $\psi(M) > 0$. We claim that $g^{(k,x_0)}(\omega, M) \subseteq \overline{co}(T^{(k,x_0)}(\omega, M) \cup \{x_0\})$ for each $k \geq 1$. Indeed, for $k = 1$, by (30), we have $g(\omega, M) \subset \overline{co}(T(\omega, M) \cup \{x_0\})$, so that

$$g^{(1,x_0)}(\omega, M) \subset \overline{co}\left(T^{(1,x_0)}(\omega, M) \cup \{x_0\}\right). \tag{34}$$

Let $k \geq 1$ be fixed and assume that $g^{(k,x_0)}(\omega, M) \subseteq \overline{co}(T^{(k,x_0)}(\omega, M) \cup \{x_0\})$.

Then,

$$\begin{aligned}
g^{(k+1,x_0)}(\omega, M) &= g\left(\overline{co}\left(g^{(k,x_0)}(\omega, M) \cup \{x_0\}\right)\right) \\
&\subseteq \overline{co}\left(T\left(\omega, \overline{co}\left(T^{(k,x_0)}(\omega, M) \cup \{x_0\}\right)\right) \cup \{x_0\}\right) \\
&= \overline{co}\left(T^{(k+1,x_0)}(\omega, M) \cup \{x_0\}\right).
\end{aligned} \tag{35}$$

This proves our claim. Thus,

$$\begin{aligned}
\psi\left(g^{(n_0(\omega),x_0)}(\omega, M)\right) &\leq \psi\left(\overline{co}\left(T^{(n_0(\omega),x_0)}(\omega, M) \cup \{x_0\}\right)\right) \\
&= \psi\left(T^{(n_0(\omega),x_0)}(\omega, M)\right) < \psi(M).
\end{aligned} \tag{36}$$

Consequently, $g(\omega, .)$ is countably ψ-convex-power condensing about x_0 and $n_0(\omega)$. Now, by applying Theorem 7, we infer that there is a measurable mapping $\xi : \Omega \longrightarrow X$ such that for each $\omega \in \Omega$, we have $g(\omega, \xi(\omega)) = \xi(\omega)$, that is,

$$l(\omega, \xi(\omega))T(\omega, \xi(\omega)) + (1 - l(\omega, \xi(\omega)))x_0 = \xi(\omega). \tag{37}$$

Now, for any $\omega \in \Omega$, either (a) $p(T(\omega, \xi(\omega)) - x_0) \leq 1$ or (b) $p(T(\omega, \xi(\omega)) - x_0) > 1$. In case of (a), it follows by (29) that $l(\omega, \xi(\omega)) = 1$ and hence by (37), $T(\omega, \xi(\omega)) = \xi(\omega)$. If (b) holds, then by (29), we have

$$l(\omega, \xi(\omega))p(T(\omega, \xi(\omega)) - x_0) = 1, \tag{38}$$

and since for every $x \in X$, $p(x - x_0) \leq 1$, it follows by (37) that for every $x \in X$, we have

$$T(\omega, \xi(\omega)) - \xi(\omega)) = (1 - l(\omega, \xi(\omega)))(T(\omega, \xi(\omega)) - x_0). \tag{39}$$

Then,

$$p(T(\omega, \xi(\omega)) - \xi(\omega)) \leq (p(T(\omega, \xi(\omega)) - x_0) - 1)p(\xi(\omega) - x_0). \tag{40}$$

Since $\xi(\omega) \in X$, the last inequality implies that

$$\begin{aligned}
p(T(\omega, \xi(\omega)) - \xi(\omega)) &\leq p(T(\omega, \xi(\omega)) - x_0) - 1 \\
&\leq p(T(\omega, \xi(\omega)) - x),
\end{aligned} \tag{41}$$

and

$$p(T(\omega, \xi(\omega)) - \xi(\omega)) = \min\{p(T(\omega, \xi(\omega)) - x) : x \in X\}. \tag{42}$$

Thus, (a) and (b) provide the desired conclusion.

With these preliminaries, we can proceed to the following interesting random fixed point theorem under Rothe-type boundary conditions.

Theorem 5. *Let X be a separable closed convex subset of a Banach space E with $Int(X) \neq \emptyset$ and let ψ be a regular measure of noncompactness on E. Let $n_0 : \Omega \longrightarrow \mathbb{N}$ be a mapping and $T : \Omega \times X \longrightarrow E$ be a uniformly continuous random operator. Let $x_0 \in Int(X)$ such that for each $\omega \in \Omega$, $T(\omega, X)$ is bounded and $T(\omega, .)$ is countably ψ -convex-power condensing about x_0 and $n_0(\omega)$. In addition, assume that $T(\Omega \times \partial X) \subseteq X$ (Rothe boundary condition). Then, T has a random fixed point.*

Proof. Let p be the Minkowski functional of $(X - x_0)$. By Theorem 9, there is a measurable mapping $\xi : \Omega \longrightarrow X$ satisfying (28). To prove that ξ is a random fixed point of T, it suffices to show that $p(T(\omega, \xi(\omega)) - x_0) \leq 1$ for each $\omega \in \Omega$. Suppose that for some $\omega \in \Omega$, we have $p(T(\omega, \xi(\omega)) - x_0) > 1$, then $T(\omega, \xi(\omega)) \notin X$. This implies that $\xi(\omega) \in \mathrm{Int}(X)$. Consequently, there is $0 < \eta < 1$ such that $z = \eta\xi(\omega) + (1-\eta)T(\omega, \xi(\omega)) \in \partial X$. Hence, by (28), we have

$$p(T(\omega, \xi(\omega)) - \xi(\omega)) \leq p(T(\omega, \xi(\omega)) - z) = \eta p(T(\omega, \xi(\omega)) - \xi(\omega)). \tag{43}$$

Thus, $p(T(\omega, \xi(\omega)) - \xi(\omega)) = 0$. This obviously yields that

$$p(T(\omega, \xi(\omega)) - x_0) \leq p(\xi(\omega) - x_0) \leq 1, \tag{44}$$

which contradicts the assumption.

Our next concern will be the existence of random fixed points for the sum of two random operators. We should mention here that the need for such random fixed point theorems arose out of the study of random differential equations. Specifically, the inversion of a random differential operator may yield the sum of a contraction and an operator satisfying some compactness conditions. Before, to state our next random fixed point result, we need to recall some basic facts.

Lemma 3 (see [20], Lemma 2.11). *Let E be a separable Banach space and $S : \Omega \times E \longrightarrow E$ be a $k(.)$-contraction random operator. Then,*

(i) *for each $x \in E$ there is a unique measurable function $\tau_x : \Omega \longrightarrow E$ such that for each $\omega \in \Omega : \tau_x(\omega) = S(\omega, \tau_x(\omega)) + x$*

(ii) *the mapping $\tau : \Omega \times E \longrightarrow E$ defined by $\tau(\omega, x) = \tau_x(\omega)$ is a Lipschitzian random operator*

Lemma 4 (see [20], Lemma 2.12). *Let X be a closed and convex subset of a separable Banach space E, $T : \Omega \times X \longrightarrow E$ be a continuous random operator, and $x : \Omega \longrightarrow X$ be measurable. Then, the mapping $\omega \longrightarrow T(\omega, x(\omega))$ is measurable.*

Now, we are in a position to state the following random fixed point theorem for the sum of two random operators. For convenient purposes, we list some necessary definitions below for completeness. Let X be a nonempty bounded closed convex subset of a Banach space E and $T : \Omega \times X \longrightarrow E$ and $S : \Omega \times E \longrightarrow E$ be two random operators. Following [24, 32, 38], we set for any $\omega \in \Omega$ and for any subset M of X:

$$\begin{aligned}\mathcal{F}^{(1,x_0)}(\omega, T, S, M) = \mathcal{F}(\omega, T, S, M) = \{x \in X : x = S(\omega, x)\\ + T(\omega, y), \text{for some } y \in M\}\end{aligned} \tag{45}$$

and

$$\begin{gathered}\mathcal{F}^{(n,x_0)}(\omega, T, S, M) = \mathcal{F}\left(\omega, T, S, \overline{co}\left(\mathcal{F}^{(n-1,x_0)}(\omega, T, S, M) \cup \{x_0\}\right)\right),\\ n = 2, 3, . \cdots\end{gathered} \tag{46}$$

In the case when $S = 0$, we have

$$\mathcal{F}^{(1,x_0)}(\omega, T, 0, M) = T^{(1,x_0)}(\omega, M) = T(\omega, M) \tag{47}$$

and

$$\begin{aligned}\mathcal{F}^{(n,x_0)}(\omega, T, 0, M) &= T\left(\omega, \overline{co}\left(\mathcal{F}^{(n-1,x_0)}(\omega, T, 0, M) \cup \{x_0\}\right)\right)\\ &= T^{(n,x_0)}(\omega, M).\end{aligned} \tag{48}$$

Theorem 6. *Let X be a nonempty bounded closed convex subset of a separable Banach space E and ψ be a regular measure of noncompactness on E. Let $T : \Omega \times X \longrightarrow E$ and $S : \Omega \times E \longrightarrow E$ be random operators satisfying the following conditions:*

(i) *T is uniformly continuous*

(ii) *there are mappings $n_0 : \Omega \longrightarrow N$ and $x_0 : \Omega \longrightarrow X$ such that*

$$\psi\left(\mathcal{F}^{(n_0(\omega), x_0(\omega))}(\omega, T, S, M)\right) < \psi M, \tag{49}$$

for every $\omega \in \Omega$ and for any countable subset M of X with $\psi(M) > 0$,

(iii) *S is a $k(.)$−contraction*

(iv) *for each $\omega \in \Omega$, $[x = T(\omega, y) + S(\omega, x)$ and $y \in X] \Longrightarrow x \in X$*

Then, $T + S$ has a random fixed point.

Proof. Consider the mapping $f : \Omega \times X \longrightarrow E$ defined by

$$f(\omega, x) = \tau(\omega, T(\omega, x)), \tag{50}$$

where τ is as described in Lemma 3. Referring to Lemma 4, we see that every $x \in X, f(.,x) = \tau(.,T(.,x))$ is measurable. Thus, f is a random operator. In addition, the fact that T is uniformly continuous and τ is Lipschitzian imply that f is uniformly continuous. Now, let $\omega \in \Omega$ be fixed. We claim that $f(\omega, X) \subseteq X$. Indeed, by virtue of Lemma 3 (i), we infer that for any $x \in X$ we have

$$\tau(\omega, T(\omega, x)) = S(\omega, \tau(\omega, T(\omega, x))) + T(\omega, x), \tag{51}$$

so that,

$$f(\omega, x) = S(\omega, f(\omega, x)) + T(\omega, x). \tag{52}$$

Hence, by (iv), we have $f(\omega, x) \in X$. This proves our claim.

Furthermore, for any subset M of X, by (52), we obtain

$$\begin{aligned} f^{(1,x_0(\omega))}(\omega, M) &= f(\omega, M) \\ &= \{x \in X : x = f(\omega, y) \text{ for some } y \in M\} \\ &\subseteq \{x \in X : x = S(\omega, x) \\ &\quad + T(\omega, y) \text{ for some } y \in M\} \\ &= \mathcal{F}^{(1,x_0(\omega))}(\omega, T, S, M). \end{aligned} \tag{53}$$

Let $n \geq 1$, suppose that

$$f^{(n,x_0(\omega))}(\omega, M) \subseteq \mathcal{F}^{(n,x_0(\omega))}(\omega, T, S, M). \tag{54}$$

We have

$$\begin{aligned} f^{(n+1,x_0(\omega))}(\omega, M) &= f\left(\omega, \overline{co}\left(f^{(n,x_0(\omega))}(\omega, M) \cup \{x_0(\omega)\}\right)\right) \\ &\subseteq f\left(\omega, \overline{co}\left(\mathcal{F}^{(n,x_0(\omega))}(\omega, T, S, M) \cup \{x_0(\omega)\}\right)\right) \\ &\subseteq \mathcal{F}\left(\omega, T, S, \overline{co}\left(\mathcal{F}^{(n,x_0(\omega))}(\omega, T, S, M) \cup \{x_0(\omega)\}\right)\right) \\ &= \mathcal{F}^{(n+1,x_0(\omega))}(\omega, T, S, M). \end{aligned} \tag{55}$$

Hence, for each $n \geq 1$, we have

$$f^{(n,x_0(\omega))}(\omega, M) \subseteq \mathcal{F}^{(n,x_0(\omega))}(\omega, T, S, M). \tag{56}$$

Thus, by (ii), we have

$$\psi\left(f^{(n_0(\omega),x_0(\omega))}(\omega, M)\right) < \psi(M), \tag{57}$$

for any countable subset M of X with $\psi(M) > 0$. As a result, $f(\omega, .)$ is countably ψ-convex-power condensing about $x_0(\omega)$ and $n_0(\omega)$. Consequently, f is countably ψ-convex-power condensing. Invoking Theorem 7, we see that f has a random fixed point $\xi : \Omega \longrightarrow X$ which is in turn a random fixed point of $T + S$.

Corollary 10. *Let X be a nonempty bounded closed convex subset of a separable Banach space E and ψ be a measure of noncompactness on E. Let $T : \Omega \times X \longrightarrow E$ and $S : \Omega \times E \longrightarrow E$ be random operators satisfying the following conditions:*

(i) T is uniformly continuous

(ii) there are mappings $n_0 : \Omega \longrightarrow \mathbb{N}$, $x_0 : \Omega \longrightarrow X$ and $\lambda : \Omega \longrightarrow [0, 1)$ such that

$$\psi\left(\mathcal{F}^{(n_0(\omega),x_0(\omega))}(\omega, T, S, M)\right) \leq \lambda(\omega)\psi(M), \tag{58}$$

for every $\omega \in \Omega$ and for any countable subset M of X

(iii) S is a $k(.)$–contraction

(iv) for each $\omega \in \Omega$, $[x = T(\omega, y) + S(\omega, x)$ and $y \in X] \Longrightarrow x \in X$. Then, $T + S$ has a random fixed point

4. Random Fixed Point Theorems for Monotone Random Operators

In this section, we prove some random fixed point theorems for monotone random operators in ordered real Banach spaces. We combine the advantages of the strong topology (continuity of random operators with respect to the strong topology) with the advantages of the weak topology (the random operators will satisfy some compactness conditions relative to the weak topology) to draw new conclusions about random fixed points for a given monotone random operator. Our results are random versions of the results in [39].

We start this section by recalling some definitions and auxiliary results which will be used further on. Throughout this section, (Ω, Σ) is a measurable space, E is a real Banach space, and X is a nonempty subset of E.

Definition 11. A subset P of E is called an order cone if it satisfies the following conditions:

(i) P is closed, nonempty and $P \neq \{0\}$,

(ii) $\forall a, b \in \mathbb{R}^+, \forall x, y \in P : ax + by \in P$

(iii) $|x \in P \text{ and } -x \in P| \Rightarrow x = 0$.

An order cone permits to define a partial order in E by

$$x \leq y \Leftrightarrow y - x \in P. \tag{59}$$

Conversely, let E be a real Banach space with a partial order compatible with the algebraic operations in E, that is,

$$\begin{gathered} x \geq 0 \text{ and } \lambda \in \mathbb{R}^+ \text{ implies } \lambda x \geq 0, \\ x_1 \leq y_1 \text{ and } x_2 \leq y_2 \text{ implies } x_1 + x_2 \leq y_1 + y_2. \end{gathered} \tag{60}$$

The positive cone of E is defined by

$$E^+ = \{x \in E : 0 \le x\}. \quad (61)$$

Let $x, y \in E$ with $x \le y$, the order interval $[x, y]$ is defined by

$$[x, y] = \{z \in E : x \le z \le y\} = (x + P) \cap (y - P). \quad (62)$$

Definition 12.

(i) A subset $X \subset E$ is said order bounded if there exist $u, v \in E$ such that $X \subset [u, v]$

(ii) The order cone is called normal if and only if there is a number $C > 0$ such that

$$\forall x, y \in E : (0 \le x \le y) \Rightarrow (\|x\| \le C\|y\|) \quad (63)$$

The least positive number C (if it exists) satisfying (63) is called a normal constant.

Remarks 13.

(1) If P is normal, then every order interval is norm bounded.

(2) Let K be a Hausdorff space and E be an ordered Banach space with normal cone P. We denote by $\mathscr{C}(K, E)$ the Banach space of all continuous E-valued functions on K equipped with the usual maximum norm. Plainly, $\mathscr{C}(K, E)$ is an ordered Banach space whose positive cone is given by

$$\mathscr{C}^+(K, E) = \{f \in \mathscr{C}(K, E): f(x) \in E^+, \forall x \in K\}, \quad (64)$$

and it is normal

Definition 14. Let X be a subset of E and $T : X \longrightarrow E$ be an operator.

(i) The operator T is said to be increasing if

$$\forall x, y \in X : x \le y \Rightarrow Tx \le Ty \quad (65)$$

(ii) The operator T is said to be decreasing if

$$\forall x, y \in X : x \le y \Rightarrow Ty \le Tx \quad (66)$$

Definition 15. Let E be an ordered real Banach space with a normal order cone P. A sequence $\{x_n\}$ is said to be totally ordered if for all $m, n \in \mathbb{N} : x_m \le x_n$ or $x_n \le x_m$.

The following lemmas are quite useful below.

Lemma 16 (see [27], Lemma 2.1). *Let E be an ordered real Banach space with a normal order cone P. Suppose that $\{x_n\}$ is a monotone sequence which has a subsequence $\{x_{nk}\}$ converging weakly to x_∞. Then, $\{x_n\}$ converges strongly to x_∞. Moreover, if $\{x_n\}$ is an increasing sequence then $x_n \le x_\infty$ for each $n \ge 1$; if $\{x_n\}$ is a decreasing sequence then $x_\infty \le x_n$ for each $n \ge 1$.*

Lemma 17 (see [39], Lemma 1.8). *Let E be an ordered real Banach space with a normal order cone P. Suppose that $\{x_n\}$ is a totally ordered sequence which is contained in a relatively weakly compact set, then it converges strongly in E.*

Let $u_0, v_0 : \Omega \longrightarrow X$ be two measurable mappings. By $u_0 \le v_0$ (resp., $u_0 < v_0$) on Ω, we mean $u_0(\omega) \le v_0(\omega)$ (resp., $u_0(\omega) < v_0(\omega)$) for every $\omega \in \Omega$. If $u_0 \le v_0$, the sector $[u_0, v_0]$ defined by

$$[u_0, v_0] = \{x : \Omega \to X : u_0 \le x \le v_0\}, \quad (67)$$

is called a random interval in X.

Let $T : \Omega \times E \longrightarrow E$ be a random operator. For all $n \in N$, $\omega \in \Omega$, and $x \in X$, we denote by $T^n(\omega, x)$ the value at x of the n^{th} iterate of the mapping $T(\omega, .)$. We say that a random operator $T : \Omega \times X \longrightarrow E$ satisfies the condition $(P(n_0))$ on $[u_0, v_0]$ if there is a mapping $n_0 : \Omega \to \mathbb{N}$ such that for each $\omega \in \Omega$,

$$\begin{cases} \text{if } V = \{x_n\} \text{is a monotone sequence of} [u_0(\omega), v_0(\omega)], \\ \text{such that } V = F \cup T^{n_0(\omega)}(\omega, V) \text{with } F \text{ is a finite set of} \\ \text{cardinal } \mathrm{n}_0(\omega), \text{ then } V \text{ is relatively weakly compact.} \end{cases} \quad (68)$$

We say that a random operator $T : \Omega \times X \longrightarrow E$ satisfies the condition $(P(1))$ on $[u_0, v_0]$ if T satisfies $(P(n_0))$ with $n_0(\omega) = 1$ for every $\omega \in \Omega$.

Definition 18. Let $[u_0, v_0]$ be a random interval in X with $u_0 \le v_0$. A random operator $T : \Omega \times X \longrightarrow E$ is called increasing (resp., decreasing) on $[u_0, v_0]$ if for each $\omega \in \Omega$, $T(\omega, .)$ is increasing (resp., decreasing) on $[u_0(\omega), v_0(\omega)]$.

Theorem 19. *Let E be an ordered separable real Banach space with a normal cone P and $T : \Omega \times E \longrightarrow E$ be a continuous random operator. Let $[u_0, v_0]$ be a random interval in E such that T is increasing on $[u_0, v_0]$ and satisfies the condition $(P(n_0))$. Assume that*

$$u_0(.) \le T(., u_0(.)), T(., v_0(.)) \le v_0(.). \quad (69)$$

Then, T has a random fixed point ξ in $[u_0, v_0]$ which can be obtained by monotone iterative procedure starting from u_0 or from v_0.

Proof. Consider the iterate sequence of mappings $u_n : \Omega \longrightarrow E$ defined by $u_n(\omega) = T(\omega, u_{n-1}(\omega))$, for all $\omega \in \Omega$ and all $n \in N$. Let $\omega \in \Omega$ be fixed. We can show by induction that

$$u_0(\omega) \le u_1(\omega) \le u_2(\omega) \le \cdots \le u_n(\omega) \cdots \le v_0(\omega). \tag{70}$$

Let us set $S(\omega) = \{u_n(\omega): n \in \mathbb{N}\}$. Then,

$$S(\omega) = T^{n_0(\omega)}(\omega, S(\omega)) \cup F(\omega), \tag{71}$$

with $F(\omega) = \{u_0(\omega), u_1(\omega), \cdots, u_{n_0(\omega)-1}(\omega)\}$ is a finite set of cardinal $n_0(\omega)$. Therefore, it follows from our hypotheses that $S(\omega)$ is relatively weakly compact. Referring to Lemma 17, we see that the sequence $\{u_n(\omega)\}$ converges strongly to some $\xi(\omega)$ in $[u_0(\omega), v_0(\omega)]$. The continuity of $T(\omega, .)$ yields that $T(\omega, \xi(\omega)) = \xi(\omega)$. Now, by using Lemma 4, we can show (via induction) that u_n is measurable for every $n \in \mathbb{N}$. The use of [8], Theorem 1.6 yields that $\xi : \Omega \longrightarrow E$ is measurable; and therefore, ξ is a random fixed point of T. Similarly, we can show that T has random fixed point obtained by monotone iterative procedure starting from v_0.

As a convenient specialization of Theorem 19, we state the following result.

Corollary 20. *Let E be an ordered separable real Banach space with a normal cone P and $T : \Omega \times E \longrightarrow E$ be a continuous random operator. Let $[u_0, v_0]$ be a random interval in E such that T is increasing on $[u_0, v_0]$ and satisfies the condition ($\mathcal{P}$ (1)). Assume that*

$$u_0(.) \le T(.,u_0(.)),T(.,v_0(.)) \le v_0(.). \tag{72}$$

Then, T has a random fixed point ξ in $[u_0, v_0]$ which can be obtained by monotone iterative procedure starting from u_0 or from v_0.

Proof. Apply Theorem 19 with $n0(\omega) = 1$ for every $\omega \in \Omega$.

Another consequence of Theorem 19 is the following.

Corollary 21. *Let E be an ordered separable real Banach space with a normal cone P and ψ be a nonsingular measure of weak noncompactness on E. Let $T : \Omega \times E \longrightarrow E$ be a continuous random operator and let $[u_0, v_0]$ be a random interval in E such that T is increasing on $[u_0, v_0]$ and*

$$u_0(.) \le T(.,u_0(.)),T(.,v_0(.)) \le v_0(.). \tag{73}$$

In addition, if there is a mapping $n_0 : \Omega \longrightarrow N$, such that for each $\omega \in \Omega$ and any monotone sequence $\Gamma = \{x_n\}$ in $[u_0(\omega), v_0(\omega)]$ with $\psi(\Gamma) > 0$, we have

$$\psi\left(T^{n_0(\omega)}(\omega, \Gamma)\right) < \psi(\Gamma). \tag{74}$$

Then, T has a random fixed point ξ in $[u_0, v_0]$ which can be obtained by monotone iterative procedure starting from u_0 or from v_0.

Proof. Let $\omega \in \Omega$ be fixed and let $V = \{x_n\}$ be a monotone sequence of $[u_0(\omega), v_0(\omega)]$ such that $V = F \cup T^{n0(\omega)}(\omega, V)$ with F is a finite set of cardinal $n_0(\omega)$. By (74), V is relatively weakly compact. Thus, $T(\omega, .)$ satisfies the condition ($\mathscr{P}(n_0)$) on $[u_0(\omega), v_0(\omega)]$. The arbitrariness of ω yields that T satisfies the condition $(P(n_0))$ on the random interval $[u_0, v_0]$. The desired result follows from Theorem 19.

Corollary 22. *Let E be an ordered separable real Banach space with a normal cone P and ψ be a measure of weak noncompactness on E.*

Let $T : \Omega \times X \longrightarrow E$ be a continuous random operator and let $[u_0, v_0]$ be a random interval in E such that T is increasing on $[u_0, v_0]$ and

$$u_0(.) \le T(.,u_0(.)),T(.,v_0(.)) \le v_0(.). \tag{75}$$

In addition, assume that there is a mapping $n_0 : \Omega \longrightarrow N$ such that for each $\omega \in \Omega$,

$$T^{n_0(\omega)}(\omega, [u_0(\omega), v_0(\omega)]) := \left\{T^{n_0(\omega)}(\omega, x(\omega)): x \in [u_0, v_0]\right\}, \tag{76}$$

is relatively weakly compact. Then, T has a random fixed point ξ in $[u_0, v_0]$ which can be obtained by monotone iterative procedure starting from u_0 or from v_0.

5. Applications to Random Differential Equations

Throughout this section, (Ω, Σ) denotes a measurable space and E is a separable real Banach space. Let $r, T > 0, x_0 \in E$, $\mathscr{B} = \{x \in E : \|x - x_0\| \le r\}$ and $I = [0, T]$. Denote by $C(I; E)$ the Banach space of all continuous mappings $u : I \longrightarrow E$ equipped with the supremum norm $\|u\|_\infty = \sup\{\|u(t)\|: t \in I\}$ and by $C^1(I; E)$ the Banach space of all continuously differentiable mappings from I to E. Let α (resp., α_C) be the Kuratowski measure of noncompactness on $E(\text{resp.}\mathscr{C}(I; E))$.

Definition 23. A mapping $u : I \times \Omega \longrightarrow E$ is said to satisfy the following conditions:

(1) (C, Ω) if for each $\omega \in \Omega$, $u(.,\omega)$ is continuous and for each $t \in I$, $u(t,.)$ is measurable

(2) (C^1, Ω) if $u(.,\omega) \in C^1(I; E)$ for every $\omega \in \Omega$ and $u(t, .)$ is measurable for every $t \in I$

If u satisfies condition (C, Ω), then u is considered a mapping of Ω into $\mathscr{C}(I, E)$. The following result, due to Itoh, discusses the measurability of u.

Proposition 24 (see [11], Proposition 4.2). *u satisfies condition (C, Ω) if and only if u is measurable as a mapping of Ω into $C(I; E)$.*

Now, we consider the initial random differential equation

$$\begin{cases} \dfrac{d}{dt}u(t,\omega) = f(t,u(t,\omega),\omega), & for (t,\omega) \in I \times \Omega, \\ u(0,\omega) = \eta(\omega), \end{cases} \tag{77}$$

where $f(.,.,\omega) \in C(I \times E, E)$ and the function $\eta : \Omega \longrightarrow E$ is measurable. Our main purpose in the immediate sequel is to show the existence of a random solution to Eq. (77). Before doing so, it is appropriate to clarify the definition of solution we will consider.

Definition 25. A mapping $u : I \times \Omega \longrightarrow E$ is said to be a random solution of (77) if u satisfies conditions (C^1, Ω) and (77).

For convenience of later reference, we list some necessary results below for completeness.

Lemma 26 (see [30]). *Let E be a Banach space and $C([0,T];E)$ be the space of continuous functions defined on $[0,T]$ with values in E. If $H \subset C([0,T];E)$ is bounded, then $\alpha(H(t)) \le \alpha_C(H)$ for any $t \in [0,T]$, where $H(t) = \{u(t)\colon u \in H\}$. Furthermore, if H is equicontinuous, then $t \longmapsto \alpha(H(t))$ is continuous on $[0,T]$,*

$$\begin{aligned} \alpha_C(H) &= \sup\{\alpha(H(t))\colon t \in [0,T]\}, \\ \alpha\left(\int_0^t H(s)ds\right) &\le \int_0^t \alpha(H(s))ds, \end{aligned} \tag{78}$$

for all $t \in [0,T]$, where $\int_0^t H(s)ds = \{\int_0^t x(s)ds : x \in H\}$.

Lemma 27 (see [21]). *Let E be a Banach space. If $H \subseteq C([0,T];E)$ is equicontinuous and $x_0 \in C([0,T];E)$, then $\overline{co}(H \cup \{x_0\})$ is also equicontinuous in $C([0,T];E)$.*

Lemma 28 (see [40]). *Let H be a bounded subset of E. Then, for each $\varepsilon > 0$, there exists a sequence $\{u_n\} \subseteq H$ such that*

$$\alpha(H) \le 2\alpha(\{u_n\}) + \varepsilon. \tag{79}$$

Lemma 29 (see [41], Lemma 2.7). *For each $n \in N$, $0 < \lambda < 1$, and $T > 0$, we put*

$$S_n = \left[\lambda^n + C_n^1 \lambda^{n-1} T + C_n^2 \lambda^{n-2} \frac{T^2}{2!} + \cdots + \frac{T^n}{n!}\right]. \tag{80}$$

Then, $\lim_{n \to \infty} S_n = 0$.

Before we proceed further, we present the following useful lemma.

Lemma 30. *Let E be a Banach space and X be a subset of E. Let $f : X \longrightarrow E$ be a mapping such that there exists a nonnegative constant $\gamma \ge 0$ satisfying*

$$\alpha(f(\Delta)) \le \gamma\alpha(\Delta), \tag{81}$$

for any countable subset Δ of X. Then, for any subset M of X, we have

$$\alpha(f(M))) \le 2\gamma\alpha(M). \tag{82}$$

Proof. Let M be a subset of X and $\varepsilon > 0$ be fixed. In view of Lemma 28, there is a sequence $\{y_n\}$ of $f(M)$ such that

$$\alpha(f(M)) \le 2\alpha(\{y_n\}) + \varepsilon. \tag{83}$$

Further, for each $n \in N$, there is $x_n \in M$ such that $y_n = f(x_n)$ and so

$$\alpha(\{y_n\}) = \alpha(f(\{x_n\})) \le \gamma\alpha(\{x_n\}) \le \gamma\alpha(M). \tag{84}$$

Linking (83) and (84), we obtain

$$\alpha(f(M)) \le 2\gamma\alpha(M) + \varepsilon. \tag{85}$$

The arbitrariness of ε yields that

$$\alpha(f(M)) \le 2\gamma\alpha(M). \tag{86}$$

If we replace the Kuratowski measure of noncompactness α by the Hausdorff measure of noncompactness χ in Lemma 30, we obtain the following result.

Lemma 31. *Let E be a Banach space and X be subset of E. Let $f : X \longrightarrow E$ be a mapping such that there exists a nonnegative constant $\gamma \ge 0$ satisfying*

$$\chi(f(\Delta)) \le \gamma\chi(\Delta), \tag{87}$$

for any countable subset Δ of X. Then, for any subset M of X, we have

$$\chi(f(M))) \le \gamma\chi(M). \tag{88}$$

Proof. By virtue of [42], Lemma 2.9, the reasoning in the proof of Lemma 30 yields the desired result.

Now, we are in a position to state the following existence result.

Theorem 32. *Let $f : I \times \mathscr{B} \times \Omega \to E$ and $\eta : \Omega \longrightarrow E$ be mappings satisfying the following assumptions:*

(i) For each $\omega \in \Omega$, $f(.,.,\omega)$ is uniformly continuous on $I \times \mathscr{B}$

(ii) For each $(t,x) \in I \times \mathscr{B}$, $f(t,x,.)$ is measurable

(iii) There exists a function $C : I \times \Omega \longrightarrow R^+$ such that

$$\alpha(f(t,D,\omega))) \le C(t,\omega)\alpha(D), \tag{89}$$

for all $\omega \in \Omega$, $t \in I$, and for any countable subset D of $\mathscr{B}$, where each $C(.,\omega) \in L^1([0,T];R^+)$

(iv) $m = \sup\{\|f(t,x,\omega)\|; t \in I, x \in \mathscr{B}, \omega \in \Omega\} < +\infty$,

(v) η *is measurable and*

$$\sup\{\|\eta(\omega) - x_0\| : \omega \in \Omega\} = r_0 < r \tag{90}$$

Then, there exists a random solution of (77) on $[0, T_1] \times \Omega$, with $T_1 = \min\{T, r - r_0/m\}$.

Proof. Let

$$K = \{u \in \mathscr{C}(I_1, E); \|u(s) - u(t)\| \le m|s-t|, \text{?and?} \|u(t) - x_0\| \le r, \forall s, t \in I_1\}, \tag{91}$$

where $I_1 = [0, T_1]$. Clearly, K is a nonempty bounded closed convex equicontinuous subset of $C(I1; E)$ and $u(t) \in \mathscr{B}$ for each $(u,t) \in K \times I_1$.

Define the mapping $W : \Omega \times K \longrightarrow K$ by

$$W(\omega, u)(t) = \eta(\omega) + \int_0^t f(s, u(s), \omega)\,ds, \tag{92}$$

for all $\omega \in \Omega$, $u \in K$, and $t \in I_1$. We show that W is a random operator. To see this, let $u \in K$ be fixed. Since the mapping $\omega \longrightarrow f(s, u(s), \omega)$ is measurable for each $s \in I_1$ and the mapping $s \longrightarrow f(s, u(s), \omega)$ is continuous for each $\omega \in \Omega$, then the mapping $F_{u:} I_1 \times \Omega \longrightarrow E$, $(s, \omega) \longrightarrow f(s, u(s), \omega)$ satisfies the condition (C, Ω). Thus, by Proposition 24, the mapping $\widehat{F}_u : \Omega \to \mathscr{C}(I_1; E)$ defined by $\widehat{F}_u(\omega) = F_u(\omega, .)$ for every $\omega \in \Omega$ is measurable. Further, observe that mapping

$$L: \mathcal{C}(I_1; E) \to \mathcal{C}(I_1; E) \text{defined by } L(v)(t) = \int_0^t v(s)\,ds \tag{93}$$

for all v $\in \mathcal{C}(I_1; E)$, and $t \in I_1$ is linear and hence, by [1], Theorem 2.14, $L \circ \widehat{F}_u$ is measurable. Therefore, $W(., u) = \eta + L \circ \widehat{F}_u$ is measurable. The arbitrariness of $u \in K$ yields that W is a random operator. Now, we will prove that W fulfills all conditions of Theorem 7. To achieve this, let $\omega \in \Omega$ be fixed. First, by (i) for $\varepsilon > 0$, there exists $\delta(\omega) > 0$ such that

$$\|f(t_1, x_1, \omega) - f(t_2, x_2, \omega)\| < \frac{\varepsilon}{T_1}, \tag{94}$$

for all $(t_1, x_1), (t_2, x_2) \in I_1 \times \mathscr{B}$ satisfying $\|x_1 - x_2\| < \delta(\omega)$ and $|t_1 - t_2| < \delta(\omega)$. Let $u, v \in K$ such that

$$\|v - u\|_\infty = \sup_{t \in I_1} \|v(t) - u(t)\| < \delta(\omega). \tag{95}$$

We obtain

$$\begin{aligned} &\|W(\omega, v)(t) - W(\omega, u)(t)\| \\ &\le \int_0^t \|f(s, v(s), \omega) - f(s, u(s), \omega)\|\,ds \\ &\le \frac{\varepsilon}{T_1} t < \varepsilon, \end{aligned} \tag{96}$$

for each $t \in I_1$. Hence,

$$\|W(\omega, v) - W(\omega, u)\|_\infty < \varepsilon. \tag{97}$$

Thus, $W(\omega, .)$ is uniformly continuous and, therefore, by arbitrariness of ω, W is uniformly continuous. Next, we claim that W is countably α_C-convex-power condensing. Let H be a countable subset of K and let $u \in H$. We see that for $\varepsilon > 0$ there exists $\delta(\omega) > 0$ such that

$$\|f(s, u(s), \omega) - f(t, u(t), \omega)\| < \varepsilon, \tag{98}$$

for all $(s,t) \in I_1^2$ satisfying $|s - t| < \delta(\omega)/\max\{1, m\}$. Then, $\|f(s, u(s), \omega) - f(t, u(t), \omega)\| \to 0$ as $t \to s$ independently to $u \in H$.

Hence, $\{f(., u(.), \omega) : u \in H\}$ is equicontinuous. By using Lemma 26 and assumption (iii), we obtain for each $t \in I_1$

$$\begin{aligned} \alpha(W(\omega, H)(t)) &= \alpha\left(\left\{\eta(\omega) + \int_0^t f(s, u(s), \omega)\,ds : u \in H\right\}\right) \\ &\le \int_0^t \alpha(f(s, H(s), \omega))\,ds \le \int_0^t C(s, \omega)\alpha(H(s))\,ds \\ &\le \left[\int_0^t C(s, \omega)\,ds\right]\alpha_C(H). \end{aligned} \tag{99}$$

Let $0 < \lambda < 1$, we know that there is a continuous function $\varphi(\omega)$: $I_1 \to \mathbb{R}^+$(with $M(\omega) = 1/2 \max\{|\varphi(\omega)(t)| : t \in I_1\}$) such that

$$\int_0^T |C(s, \omega) - \varphi(\omega)(s)|\,ds < \frac{\lambda}{2}. \tag{100}$$

So, for each $t \in I_1$, we have

$$\begin{aligned} \alpha(W(\omega, H)(t)) &\le \left[\left(\left(\int_0^t |C(s, \omega) - \varphi(\omega)(s)|\,ds \right.\right.\right. \\ &\left.\left. + \int_0^t |\varphi(\omega)(s)|\,ds\right)\right]\alpha_C(H) \\ &\le (\lambda + M(\omega)t)\alpha_C(H). \end{aligned} \tag{101}$$

Thus,

$$\alpha\left(W^{(1,\theta_0)}(\omega, H)(t)\right) \le (\lambda + M(\omega)t)\alpha_C(H), \text{for each } t \in I_1, \tag{102}$$

where $\theta_0 \in K$ is the mapping defined by $\theta_0(t) = x_0$ for every $t \in I_1$.

Now, since for $n \geq 1, \overline{co}(W^{(n,\theta_0)}(\omega, H) \cup \{\theta_0\}) \subset K$, then $f(.,\overline{co}(W^{(n,\theta_0)}(\omega, H) \cup \{\theta_0\})(.), \omega)$ is equicontinious. Thus, in view of (101), assumption (iii) and Lemmas 26–30, we obtain

$$\begin{aligned}\alpha\left(W^{(2,\theta_0)}(\omega, H)(t)\right) &= \alpha\left(W\left(\omega, \overline{co}\left(W^{(1,\theta_0)}\right)(\omega, H) \cup \{\theta_0\}(t)\right)\right) \\ &\leq \alpha\left(\int_0^t f\left(.,\overline{co}\left(W^{(1,\theta_0)}(\omega, H) \cup \{\theta_0\}\right)(.),\omega\right)(s)ds\right) \\ &\leq \int_0^t 2C(s,\omega)\alpha(\overline{co}(W(\omega, H) \cup \{\theta_0\}))(s)\Big)ds \\ &\leq 2\int_0^t (|C(s,\omega) - \varphi(\omega)(s)| + |\varphi(\omega)(s)|) \\ &(\lambda + M(\omega)s)\alpha_c(H)ds \\ &\leq \left[(\lambda + M(\omega)t)\lambda + M(\omega)\left(\lambda t + \frac{M(\omega)t^2}{2}\right)\right]\alpha_C(H) \\ &= \left(\lambda^2 + 2\lambda(M(\omega)t) + \frac{(M(\omega)t)^2}{2}\right)\alpha_C(H).\end{aligned} \tag{103}$$

Hence, by mathematical induction, for all $n \geq 1$ and $t \in I_1$, we obtain

$$\alpha\left(W^{(n,\theta_0)}(\omega, H)(t)\right) \leq \Gamma_n(t)\alpha_C(H), \tag{104}$$

where $\Gamma_n(t) = \lambda^n + C_n^1\lambda^{n-1}(M(\omega)t) + C_n^2\lambda^{n-2}((M(\omega)t)^2/2!) + \cdots + ((M(\omega)t)^n/n!)$.

Using the equicontinuity of the $W^{(n,\theta_0)}(\omega, H)$ on I_1, we get

$$\alpha_C\left(W^{(n,\theta_0)}(\omega, H)\right) \leq \Gamma_n(T_1)\alpha_C(H). \tag{105}$$

Since $0 < \lambda < 1$ and $M(\omega)T_1 > 0$, Lemma 29 yields that there exists a positive integer $n_0(\omega)$ such that $\Gamma_{n_0(\omega)}(T_1) < 1$. Consequently,

$$\alpha_C\left(W^{(n_0(\omega),\theta_0)}(\omega, H)\right) < \alpha_C(H), \tag{106}$$

for any countable subset H of K with $\alpha_C(H) > 0$. Thus, $W(\omega, .)$ is countably α_C-convex-power condensing about θ_0 and $n_0(\omega)$. Therefore, W is countably α_C-convex-power condensing. Now, Theorem 7 guarantees that W has a random fixed point $\xi : \Omega \longrightarrow K \subset \mathcal{C}(I_1; E)$. By Proposition 24, the mapping ξ satisfies condition $(\mathcal{C}, \Omega)$ as a mapping of $I_1 \times \Omega$ into E. Further, by (92), ξ satisfies condition (C^1, Ω) and ξ is a random solution of problem (77).

As a convenient specialization of Theorem 32, we consider the particular case when $\eta = 0$ and $x_0 = 0$. We, therefore, obtain the following result.

Corollary 33. *Let $f : I \times \mathcal{B} \times \Omega \longrightarrow E$ be a mapping satisfying the following assumptions:*

(i) For each $\omega \in \Omega, f(.,.,\omega)$ is uniformly continuous on $I \times B$

(ii) For each $(t, x) \in I \times B, f(t, x,.)$ is measurable

(iii) There exists a function $C : I \times \Omega \rightarrow \mathbb{R}^+$ such that

$$\alpha(f(t, D, \omega))) \leq C(t, \omega)\alpha(D), \tag{107}$$

for all $\omega \in \Omega$, $t \in I$, and for any countable subset D of $\mathcal{B}$, where each $C(.,\omega) \in L^1([0, T]; \mathbb{R}^+)$

(vi) $m = \sup \{\|f(t, x, \omega)\|; t \in I, x \in B, \omega \in \Omega\} < +\infty$,

Then, the random differential equation

$$\forall(t, \omega) \in I \times \Omega : \frac{d}{dt}u(t, \omega) = f(t, u(t, \omega), \omega), u(0, \omega) = 0, \tag{108}$$

has a random solution on $[0, T_1] \times \Omega$, with $T_1 = \min\{T, r/m\}$.

Remark 34.

(1) Theorem 32 improves [11], Theorem 4.3. In our considerations, mapping C is not subject to any restriction

(2) Theorem 32 remains true if we replace the Kuratowski measure of noncompactness by the Hausdorff measure of noncompactness

In the following, we present a new approach to solve (77) based on Theorem 19. To achieve this, let E be an ordered separable real Banach space with a normal cone P and β be the De Blasi measure of weak noncompactness on E and on $C(I; E)$.

Theorem 35. *Let $f : I \times E \times \Omega \longrightarrow E$ be a mapping satisfying the following assumptions:*

(i) For each $\omega \in \Omega$, $f(.,.,\omega)$ is continuous on $I \times E$ and maps bounded sets into bounded ones

(ii) For all $t \in I$ and $x \in E$, $f(t, x,.)$ is measurable

(iii) There exist two mappings $v_0, w_0 : I \times \Omega \longrightarrow E$ satisfying condition (C^1, Ω) and the following conditions hold

$$\begin{aligned}\forall\omega \in \Omega, t \in I, \frac{dv_0}{dt}(t, \omega) \leq f(t, v_0(t, \omega), \omega), f(t, w_0(t, \omega), \omega) \\ \leq \frac{dw_0}{dt}(t, \omega),\end{aligned} \tag{109}$$

(iv) There is a measurable mapping $M : \Omega \longrightarrow R^+ \setminus \{0\}$ satisfying

$$f(t,y,\omega)-f(t,x,\omega)\geq -M(\omega)(y-x), \tag{110}$$

whenever $v_0(t,\omega)\leq x\leq y\leq w_0(t,\omega)$, $(t,\omega)\in I\times\Omega$

(v) *there is a mapping* $\rho:\Omega\longrightarrow[0,+\infty)$ *such that for each* $\omega\in\Omega$, *for any monotone sequence* $V=\{u_n\}$ *of* $[v_0(.,\omega),w_0(.,\omega)]$ *and for all* $a,b\in I$ *with* $a\leq b$ *we have*

$$\beta(f([a,b]\times V\times\{\omega\}))\leq\rho(\omega)\beta(V([a,b])), \tag{111}$$

where $f([a,b]\times V\times\{\omega\}):=\{f(s,u_n(s),\omega)\,;\,s\in[a,b],n\geq 1\}$

Then, the random differential equation (77) has a random solution on $I\times\Omega$ which can obtained by monotone iterative procedure starting from v_0 or from w_0.

Remark 36. Set $g(t,x,\omega)=f(t,x,\omega)+M(\omega)x$, for $(t,x,\omega)\in I\times E\times\Omega$. Then,

(G_1) For each $(t,\omega)\in I\times\Omega$,

$$g(t,x,\omega)\leq g(t,y,\omega)\text{whenever } v_0(t,\omega)\leq x\leq y\leq w_0(t,\omega) \tag{112}$$

(G_2) For each $\omega\in\Omega$, for any monotone sequence $V=\{u_n\}$ in $[v_0(.,\omega),w_0(.,\omega)]$ and for all $a,b\in I$ with $a\leq b$, we have

$$\beta(g([a,b]\times V\times\{\omega\}))\leq\gamma(\omega)\beta(V([a,b])), \tag{113}$$

where $\gamma(\omega)=\rho(\omega)+M(\omega)$

Proof. The random problem (77) is equivalent to the random problem

$$\begin{aligned}\forall(t,\omega)\in I\times\Omega:&\frac{d}{dt}u(t,\omega)+M(\omega)u(t,\omega)\\&=g(t,u(t,\omega),\omega),\ u(0,\omega)=\eta(\omega),\end{aligned} \tag{114}$$

which is equivalent to the random problem

$$\begin{aligned}\forall(t,\omega)\in I\times\Omega:&\frac{d}{dt}\left(e^{M(\omega)t}u(t,\omega)\right)\\&=e^{M(\omega)t}g(t,u(t,\omega),\omega),\ u(0,\omega)=\eta(\omega).\end{aligned} \tag{115}$$

Let us write (115) as a random integral equation

$$\begin{aligned}\forall(t,\omega)\in I\times\Omega: u(t,\omega)=&e^{-M(\omega)t}\eta(\omega)\\&+\int_0^t e^{-M(\omega)(t-s)}g(s,u(s,\omega),\omega)ds.\end{aligned} \tag{116}$$

Define the mapping $\Phi:\Omega\times\mathcal{C}(I,E)\longrightarrow\mathcal{C}(I,E)$ by

$$\Phi(\omega,u)(t)=e^{-M(\omega)t}\eta(\omega)+\int_0^t e^{-M(\omega)(t-s)}g(s,u(s),\omega)ds, \tag{117}$$

for each $(t,\omega)\in I\times\Omega$ and any $u\in\mathcal{C}(I,E)$

We will show that Φ is a random operator. To this end, let $u\in\mathcal{C}(I,E)$ be fixed. Notice first that for each $t\in I$, the mapping $\Omega\longrightarrow[0,1]$, $\omega\longrightarrow e^{-M(\omega)t}$ is measurable, since $\exp:\mathbb{R}\longrightarrow\mathbb{R}^+\setminus\{0\}$, $x\longrightarrow e^x$ is continuous and M is measurable. Let $t\in I$ and $s\in[0,t]$. Using Lemma 8 together with assumptions (ii) and (iv) we infer that $\omega\longrightarrow e^{-M(\omega)t}\eta(\omega)$ and $\omega\longrightarrow e^{-M(\omega)(t-s)}g(s,u(s),\omega)$ are measurable. Furthermore, since $s\longrightarrow e^{-M(\omega)(t-s)}g(s,u(s),\omega)$ is continuous for each $\omega\in\Omega$, then the mapping $G_{u,t}:[0,t]\times\Omega\longrightarrow E$, $(s,\omega)\longrightarrow e^{-M(\omega)(t-s)}g(s,u(s),\omega)$ satisfies the condition (C,Ω). Thus, by Proposition 24, the mapping $\widehat{G}_{u,t}:\Omega\to\mathscr{C}([0,t]\,;E)$ defined by $\widehat{G}_{u,t}(\omega)=G_{u,t}(\omega,.)$ for every $\omega\in\Omega$, is measurable. Next, observe that the mapping $L_t:\mathcal{C}([0,t]\,;E)\longrightarrow\mathcal{C}([0,t]\,;E)$ defined by $L_t(v)(\tau)=\int_0^\tau v(s)ds$ for all $v\in\mathscr{C}(I;E)$ and $\tau\in[0,t]$ is linear. Hence, by [1], Theorem 2.14, $L_t\circ\widehat{G}_{u,t}$ is measurable. By virtue of Proposition 24, we see that the mapping $(\omega,\tau)\mapsto L_t\circ\widehat{G}_{u,t}(\omega)(\tau)$ satisfies condition (C,Ω).

Accordingly, for each $\tau\in[0,t]$, the mapping $\omega\mapsto L_t\circ\widehat{G}_{u,t}(\omega)(\tau)=\int_0^\tau e^{-M(\omega)(t-s)}g(s,u(s),\omega)ds$ is measurable. In particular, the mapping $\omega\mapsto L_t\circ\widehat{G}_{u,t}(\omega)(\tau)=\int_0^\tau e^{-M(\omega)(t-s)}g(s,u(s),\omega)ds$ is measurable.

Therefore,

$$\omega\mapsto\Phi(\omega,u)(\tau)=e^{-M(\omega)(t)}\eta(\omega)+\int_0^\tau e^{-M(\omega)(t-s)}g(s,u(s),\omega)ds, \tag{118}$$

is measurable. This is true for each $t\in I$. By a simple verification using assumption (i), we can show that $t\mapsto\Phi(\omega,u)(t)=e^{-M(\omega)t}\eta(\omega)+\int_0^t e^{-M(\omega)(t-s)}g(s,u(s),\omega)ds$ is continuous for each $\omega\in\Omega$. Then, the mapping $(\omega,t)\mapsto\Phi(\omega,u)(t)$ satisfies condition (C,Ω). Finally by invoking Proposition 24, we conclude that $\Phi(.,u)$: $\Omega\longrightarrow\mathcal{C}(I,E)$ is measurable. The arbitrariness of $u\in\mathcal{C}(I,E)$ yields that Φ is a random operator.

Now, we will prove that Φ fulfills all conditions of Theorem 19. This will be achieved in three steps:

Step 1. We show that Φ is continuous. To this end, fix $\omega\in\Omega$ and $u\in C(I,E)$. Let $\{u_n\}$ be a sequence in $C(I,E)$ which converges to u. Then, there is an integer $N\in\mathbb{N}$ such that

$$\|u_n-u\|_\infty<1\text{ whenever } n\geq N. \tag{119}$$

Hence,

$$\|u_n\|_\infty\leq\|u\|_\infty+1=r\text{ for each } n\geq N. \tag{120}$$

Now, we put

$$F_{\omega,n}(s) = g(s, u_n(s), \omega), F_\omega(s) = g(s, u(s), \omega), \tag{121}$$

and

$$K_\omega = \sup \{\|g(s, z, \omega)\| \,; s \in [0, T] \text{and} \|z\| \le r\} < \infty. \tag{122}$$

Note that, for all $n \ge N$ and $s \in [0, T]$ we have $\|F_{\omega,n}(s)\| \le K_\omega$. Furthermore, the continuity of $g(.,.,\omega)$ on $I \times E$ implies that

$$\|F_{\omega,n}(s) - F_\omega(s)\| \to 0 \text{ as } n \to \infty. \tag{123}$$

The dominated convergence theorem yields

$$\int_0^T \|F_{\omega,n}(s) - F_\omega(s)\| ds \to 0 \text{ as } n \to \infty. \tag{124}$$

On the other hand, for each $t \in [0, T]$,

$$\begin{aligned} \|\Phi(\omega, u_n)(t) - \Phi(\omega, u)(t)\| &\le \int_0^t \|g(s, u_n(s), \omega) - g(s, u(s), \omega)\| ds \\ &\le \int_0^T \|F_{\omega,n}(s) - F_\omega(s)\| ds. \end{aligned} \tag{125}$$

Hence,

$$\|\Phi(\omega, u_n) - \Phi(\omega, u)\|_\infty \le \int_0^T \|F_{\omega,n}(s) - F_\omega(s)\| ds. \tag{126}$$

Therefore, $\|\Phi(\omega, u_n) - \Phi(\omega, u)\|_\infty \to 0 \text{ as } n \to \infty$ and consequently $\Phi(\omega, .)$ is continuous at u. The arbitrariness of u implies that $\Phi(\omega, .)$ is continuous on $\mathcal{C}(I, E)$.

Step 2. We illuminate that Φ is increasing on the random interval

$$[v_0, w_0] = \{x : \Omega \to \mathscr{C}(I,E) \colon v_0(.,\omega) \le x(\omega) \le w_0(.,\omega) \text{for each } \omega \in \Omega\}. \tag{127}$$

To do this, let $\omega \in \Omega$ be fixed and $u, v \in C(I, E)$ with $v_0(.,\omega) \le u \le v \le w_0(.,\omega)$. Using (G_1) we obtain

$$g(t, u(t), \omega) \le g(t, v(t), \omega) \text{for each } t \in I. \tag{128}$$

Now, (117) becomes

$$\Phi(\omega, u)(t) \le \Phi(\omega, v)(t) \text{for each } t \in I. \tag{129}$$

Thus,

$$\Phi(\omega, u) \le \Phi(\omega, v). \tag{130}$$

Therefore, Φ is increasing on $[v_0, w_0]$.

Step 3. We show that the following conditions hold:

$$\forall \omega \in \Omega, v_0(.,\omega) \le \Phi(\omega, v_0(.,\omega)), \Phi(\omega, w_0(.,\omega)) \le w_0(.,\omega). \tag{131}$$

To this end, let $\omega \in \Omega$. We begin by setting

$$\begin{aligned} h_1(t, \omega) &= \frac{dv_0}{dt}(t, w) + M(\omega) v_0(t, \omega), \\ h_2(t, \omega) &= \frac{dw_0}{dt}(t, w) + M(\omega) w_0(t, \omega). \end{aligned} \tag{132}$$

Clearly, $h_1(.,\omega)$, $h_2(.,\omega) \in C(I, E)$ and for each $t \in I$, we have

$$\begin{aligned} h_1(t, \omega) = \frac{dv_0}{dt}(t, \omega) + M(\omega) v_0(t, \omega) \\ &\le f(t, v_0(t, \omega), \omega) + M(\omega) v_0(t, \omega) \\ &= g(t, v_0(t, \omega), \omega) \le g(t, \omega_0(t, \omega), \omega) \\ &= f(t, w_0(t, \omega), \omega) + M(\omega) w_0(t, \omega) \\ &\le \frac{dw_0}{dt}(t, \omega) + M(\omega) \omega_0(t, \omega) = h_2(t, \omega). \end{aligned} \tag{133}$$

Keeping in mind the fact that $(d/dt)(e^{M(\omega)t} v_0(t, \omega)) = e^{M(\omega)t} h_1(t, \omega)$ and $(d/dt)(e^{M(\omega)t} w_0(t)) = e^{M(\omega)t} h_2(t, \omega)$, we deduce that for each $t \in I$ have:

$$\begin{aligned} e^{M(\omega)t} v_0(t, \omega) &= v_0(0, \omega) + \int_0^t e^{M(\omega)s} h_1(s, \omega) ds \\ &\le \eta(\omega) + \int_0^t e^{M(\omega)s} g(t, v_0(s, \omega), \omega) ds \\ &\le \eta(\omega) + \int_0^t e^{M(\omega)s} g(t, w_0(s, \omega), \omega) ds \\ &\le w_0(0, \omega) + \int_0^t e^{M(\omega)s} h_2(s, \omega) ds \\ &= e^{M(\omega)t} w_0(t, \omega). \end{aligned} \tag{134}$$

Accordingly, $v_0(.,\omega) \le \Phi(\omega, v_0(.,\omega)) \le \Phi(\omega, w_0(.,\omega)) \le w_0(.,\omega)$.

Step 4. The reasoning in [39], Theorem 3.2 yields that each $\Phi(\omega, .)$ satisfies the condition $(\mathscr{P}(n_0))$. Therefore, the random operator Φ satisfies the condition $(\mathscr{P}(n_0))$.

Now, by applying Theorem 19, we infer that Φ has a random fixed point ξ which can obtained by monotone iterative procedure starting from v_0 (or w_0): $\Omega \longrightarrow \mathcal{C}(I, E)$. The mapping $\widehat{\xi} : I \times \Omega \longrightarrow E$ defined by $\widehat{\xi}(t, \omega) = \xi(\omega)(t)$ is measurable as mapping from Ω into $\mathscr{C}(I, E)$. Hence, by Proposition 24, it satisfies the condition (C, Ω). Further, ξ_b satisfies the random integral equation (116). Then, for every $\omega \in \Omega$, we have $\widehat{\xi}(.,\omega) \in \mathcal{C}^1(I, E)$ and so $\widehat{\xi}$ satisfies condition $(\mathscr{C}^1, \Omega)$. ξ is a random solution of problem (77).

Data Availability

The authors received no data support for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Iterative Solutions for Solving Variational Inequalities and Fixed-Point Problems

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In this paper, we are interested in variational inequalities and fixed-point problems in Hilbert spaces. We present an iterative algorithm for finding a solution of the studied variational inequalities and fixed-point problems. We show the strong convergence of the suggested algorithm.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $C \subset H$ be a nonempty closed convex set. Let $f : C \longrightarrow H$, $g : C \longrightarrow H$, $\varphi : C \longrightarrow C$, and $T : C \longrightarrow C$ be four nonlinear operators. Use $\mathrm{Fix}(T)$ to denote the fixed-point set of T.

In this paper, we will investigate the following variational inequalities and fixed-point problems of finding a point $u^\dagger$ such that

$$\begin{aligned} &u^\dagger \in \mathrm{GVI}(C, f, \varphi), \\ &\varphi\left(u^\dagger\right) \in \mathrm{VI}(C, g) \cap \mathrm{Fix}(T), \end{aligned} \tag{1}$$

where $\mathrm{GVI}(C, f, \varphi)$ denotes the solution set of the generalized variational inequality (shortly, GVI) which is to find a point $x^\dagger \in C$ such that

$$\left\langle f\left(x^\dagger\right), \varphi(x) - \varphi\left(x^\dagger\right) \right\rangle \geq 0, \quad \forall x \in C, \tag{2}$$

and $\mathrm{VI}(C, g)$ means the solution set of the variational inequality (shortly, VI) which is to find a point $x^\dagger \in C$ such that

$$\left\langle g\left(x^\dagger\right), x - x^\dagger \right\rangle \geq 0, \quad \forall x \in C. \tag{3}$$

Throughout, we use Θ to denote the solution set of problem (1), that is,

$$\Theta = \mathrm{GVI}(C, f, \varphi) \bigcap \varphi^{-1}(\mathrm{VI}(C, g) \cap \mathrm{Fix}(T)). \tag{4}$$

It is well known that variational inequalities play key roles and provide a useful mathematical framework, theory, and method for studying many valuable problems arising in water resources, finance, economics, medical images, and so on ([1–6]). A lot of work and a great deal of algorithms for solving GVI or VI have been introduced and investigated, see, e.g., [7–15]. Among them, a basic and important algorithm is the projected algorithm which generates a sequence $\{x_n\}$ with the form

$$x_{n+1} = \mathrm{proj}_C[x_n - \kappa_n f(x_n)], \quad n \geq 0, \tag{5}$$

where κ_n is step-size and $\mathrm{proj}_C : H \longrightarrow C$ is the orthogonal projection.

At the same time, we are also interested in the fixed-point problem of finding a point $u^\dagger$ such that $Tu^\dagger = u^\dagger$. Iterative solution for solving a fixed-point problem is an active research field, see, e.g., [16–24]. Recently, iterative algorithms for solving variational inequalities and fixed-point problems have been investigated extensively by many authors [25–33].

Motivated by the work in this direction, in this paper, we devote to research variational inequalities and fixed-point problem (1). We introduce an iterative algorithm for finding a solution of problem (1). We show the strong convergence of the suggested algorithm.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $f : C \longrightarrow H$ is said to be

(i) strongly monotone if

$$\langle f(u) - f(v), u - v\rangle \geq \lambda \|u - v\|^2, \quad \forall u, v \in C \tag{6}$$

(ii) ϑ-inverse strongly φ-monotone if there exists a constant $\vartheta > 0$ such that

$$\langle f(u) - f(v), \varphi(u) - \varphi(v)\rangle \geq \vartheta \|f(u) - f(v)\|^2, \quad \forall u, v \in C \tag{7}$$

(iii) relaxed (μ, ν)-cocoercive [34, 35], if there exist two constants $\mu > 0, \nu > 0$ such that

$$\langle f(u) - f(v), u - v\rangle \geq (-\mu)\|f(u) - f(v)\|^2 + \nu\|u - v\|^2, \quad \forall u, v \in C \tag{8}$$

An operator $T : C \longrightarrow C$ is said to be

(i) pseudocontractive [36] if

$$\|T(u^\dagger) - T(v^\dagger)\|^2 \leq \|u^\dagger - v^\dagger\|^2 + \|(I - T)u^\dagger - (I - T)v^\dagger\|^2, \quad \forall u^\dagger, v^\dagger \in C \tag{9}$$

(ii) L-Lipschitz if

$$\left\|T(u^\dagger) - T(v^\dagger)\right\| \leq L\left\|u^\dagger - u^\dagger\right\|, \quad \forall u^\dagger, v^\dagger \in C, \tag{10}$$

where $L > 0$ is a constant

If $L < 1$, then T is said to be L-contraction. If $L = 1$, then T is said to be nonexpansive.

An operator $A : H \longrightarrow 2^H$ is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \operatorname{dom}(A)$, $u \in A(x)$, and $v \in A(y)$. A monotone operator A on H is said to be maximal if its graph is not strictly contained in the graph of any other monotone operator on H.

For $\forall x^\dagger \in H$, there exists a unique point in C, denoted by $\operatorname{proj}_C[x^\dagger]$ satisfying

$$\left\|x^\dagger - \operatorname{proj}_C\left[x^\dagger\right]\right\| \leq \left\|x - x^\dagger\right\|, \quad \forall x \in C. \tag{11}$$

Moreover, proj_C is firmly nonexpansive, that is,

$$\left\|\operatorname{proj}_C\left[u^\dagger\right] - \operatorname{proj}_C\left[v^\dagger\right]\right\|^2 \leq \left\langle \operatorname{proj}_C\left[u^\dagger\right] - \operatorname{proj}_C\left[v^\dagger\right], u^\dagger - v^\dagger\right\rangle, \quad \forall u^\dagger, v^\dagger \in H. \tag{12}$$

Further, proj_C has the following property:

$$\left\langle u^\dagger - \operatorname{proj}_C\left[u^\dagger\right], x^\dagger - \operatorname{proj}_C\left[u^\dagger\right]\right\rangle \leq 0, \quad \forall u^\dagger \in H, x^\dagger \in C. \tag{13}$$

Lemma 1 ([37]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \longrightarrow C$ be an L -Lipschitz pseudocontractive operator. Then, $\forall x^\dagger \in C$ and $y^\dagger \in Fix(T)$, we have*

$$\begin{aligned} &\left\|(1-\varsigma)x^\dagger + \varsigma T\left[(1-\lambda)x^\dagger + \lambda T\left(x^\dagger\right)\right] - y^\dagger\right\|^2 \\ &\quad \leq \varsigma(\varsigma - \lambda)\left\|T\left[(1-\lambda)x^\dagger + \lambda T\left(x^\dagger\right)\right] - x^\dagger\right\|^2 + \left\|x^\dagger - y^\dagger\right\|^2, \end{aligned} \tag{14}$$

where $0 < \varsigma < \lambda < 1/(\sqrt{1 + L^2} + 1)$.

Lemma 2 ([24]). *Let C be a nonempty, convex, and closed subset of a Hilbert space H. Let $T : C \longrightarrow C$ be a continuous pseudocontractive operator. Then,*

(i) $Fix(T) \subset C$ is closed and convex

(ii) T is demiclosedness, i.e., $u_n \rightharpoonup \tilde{z}$ and $T(u_n) \longrightarrow z^\dagger$ imply that $T(\tilde{z}) = z^\dagger$

Lemma 3 ([23]). *Let $\{\varpi_n\} \subset [0,\infty)$, $\{\vartheta_n\} \subset (0, 1)$, and $\{\eta_n\}$ be real number sequences. Suppose the following conditions are satisfied:*

(i) $\varpi_{n+1} \leq (1 - \vartheta_n)\varpi_n + \eta_n, \forall n \geq 1$

(ii) $\sum_{n=1}^{\infty} \vartheta_n = \infty$

(iii) $\limsup_{n\longrightarrow\infty}(\eta_n/\vartheta_n) \leq 0$ or $\sum_{n=1}^{\infty} |\eta_n| < \infty$

Then, $\lim_{n\longrightarrow\infty}\varpi_n = 0$.

Lemma 4 ([38, 39]). *Let $\{x_n\}$ be a real number sequence. Assume there exists at least a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$x_{n_k} \leq x_{n_k+1}, \tag{15}$$

for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\mu$

$(n)\}$ *as*

$$\mu(n) = \max\left\{i \le n : x_{n_i} < x_{n_i+1}\right\}. \tag{16}$$

Then, $\mu(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and for all $n \ge N_0$, $\max\{x_{\mu(n)}, x_n\} \le x_{\mu(n)+1}$.

3. Main Results

In this section, we present our iterative algorithm and convergence theorem. Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that

(i) $\phi : C \longrightarrow C$ is a ρ-contractive operator

(ii) $\varphi : C \longrightarrow C$ is a weakly continuous and λ-strongly monotone operator such that its rang $R(\varphi) = C$

(iii) $f : C \longrightarrow H$ is a ϑ-inverse strongly φ-monotone operator

(iv) $g : C \longrightarrow H$ is an L_1-Lipschitz and relaxed (μ, ν)-cocoercive operator

(v) $T : C \longrightarrow C$ is an L_2-Lipschitz pseudocontractive operator with $L_2 > 1$

Let $\{\vartheta_n\}$, $\{\varsigma_n\}$, $\{\lambda_n\}$, and $\{\tau_n\}$ be four real number sequences in $[0, 1]$ and $\{\kappa_n\}$ and $\{\gamma_n\}$ be two real number sequences in $(0, \infty)$.

Now, we present our algorithm for solving problem (1).

Algorithm 5. Let $x_0 \in C$ be an initial value. Define the sequence $\{x_n\}$ by the following form:

$$\begin{cases} u_n = \vartheta_n\phi(x_n) + (1-\vartheta_n)\mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)], \\ y_n = (1-\varsigma_n)u_n + \varsigma_n T[(1-\lambda_n)u_n + \lambda_n T(u_n)], \\ z_n = \mathrm{proj}_C[y_n - \gamma_n g(y_n)], \\ \varphi(x_{n+1}) = (1-\tau_n)\varphi(x_n) + \tau_n z_n, \quad n \ge 0. \end{cases} \tag{17}$$

Theorem 6. *Suppose that $\Theta \ne \emptyset$. Assume that the following conditions are satisfied:*

(C1):$\lim_{n\longrightarrow\infty}\vartheta_n = 0$ and $\sum_{n=1}^{\infty}\vartheta_n = \infty$

(C2):$0 < a_1 < \varsigma_n < c_1 < \lambda_n < b_1 < 1/(\sqrt{1+L_2^2}+1)$ for all $n \ge 0$

(C3):$\nu > \mu L_1^2$ and $0 < a_2 \le \gamma_n \le b_2 < 2(\nu - \mu L_1^2)/L_1^2$ for all $n \ge 0$

(C4):$0 < \liminf_{n\longrightarrow\infty}\tau_n \le \limsup_{n\longrightarrow\infty}\tau_n < 1$

(C5):$0 < \rho < \lambda < 2\vartheta$ and $0 < \liminf_{n\longrightarrow\infty}\kappa_n \le \limsup_{n\longrightarrow\infty}\kappa_n < 2\vartheta$

Then, the sequence $\{x_n\}$ generated by (17) converges strongly to $u^\dagger \in \Theta$ verifying

$$\left\langle \phi(u^\dagger) - \varphi(u^\dagger), \varphi(x^\dagger) - \varphi(u^\dagger)\right\rangle \le 0, \quad \forall x^\dagger \in \Theta. \tag{18}$$

Proof. Since φ is λ-strongly monotone, we can get from (6) that

$$\|\varphi(u) - \varphi(v)\| \ge \lambda\|u - v\|, \quad \forall u, v \in C. \tag{19}$$

Thus, VI (18) has a unique solution, denoted by $u^\dagger$. It follows that $u^\dagger \in \mathrm{GVI}(C, f, \varphi)$ and $\varphi(u^\dagger) \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, g)$. Using inequality (13), we can obtain that $\varphi(u^\dagger) = \mathrm{proj}_C[\varphi(u^\dagger) - \kappa_n f(u^\dagger)]$ for all $n \ge 0$.

Since f is ϑ-inverse strongly φ-monotone, for any $u \in C$, we have

$$\begin{aligned} &\left\|(\varphi(u) - \kappa f(u)) - \left(\varphi(u^\dagger) - \kappa f(u^\dagger)\right)\right\|^2 \\ &\quad = \left\|\varphi(u) - \varphi(u^\dagger)\right\|^2 - 2\kappa\left\langle f(u) - f(u^\dagger), \varphi(u) - \varphi(u^\dagger)\right\rangle \\ &\qquad + \kappa^2\left\|f(u) - f(u^\dagger)\right\|^2 \le \left\|\varphi(u) - \varphi(u^\dagger)\right\|^2 \\ &\qquad - 2\kappa\vartheta\left\|f(u) - f(u^\dagger)\right\|^2 + \kappa^2\left\|f(u) - f(u^\dagger)\right\|^2 \\ &\quad \le \left\|\varphi(u) - \varphi(u^\dagger)\right\|^2 + \kappa(\kappa - 2\vartheta)\left\|f(u) - f(u^\dagger)\right\|^2. \end{aligned} \tag{20}$$

Based on (20), we deduce

$$\begin{aligned} &\left\|(\varphi(x_n) - \kappa_n f(x_n)) - \left(\varphi(u^\dagger) - \kappa_n f(u^\dagger)\right)\right\|^2 \\ &\quad \le \left\|\varphi(x_n) - \varphi(u^\dagger)\right\|^2 + \kappa_n(\kappa_n - 2\vartheta)\left\|f(x_n) - f(u^\dagger)\right\|^2 \\ &\quad \le \left\|\varphi(x_n) - \varphi(u^\dagger)\right\|^2, \end{aligned} \tag{21}$$

$$\begin{aligned} &\|\varphi(x_{n+1}) - \kappa_{n+1}f(x_{n+1}) - (\varphi(x_n) - \kappa_{n+1}f(x_n))\|^2 \\ &\quad \le \|\varphi(x_{n+1}) - \varphi(x_n)\|^2 + \kappa_{n+1}(\kappa_{n+1} - 2\vartheta)\|f(x_{n+1}) - f(x_n)\|^2. \end{aligned} \tag{22}$$

By (17), (19), and (21), we derive

$$\begin{aligned} \left\|u_n - \varphi(u^\dagger)\right\| &= \|\vartheta_n\phi(x_n) + (1-\vartheta_n)\mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)] \\ &\quad - \mathrm{proj}_C\left[\varphi(u^\dagger) - \kappa_n f(u^\dagger)\right]\| \\ &\le \left\|\vartheta_n\left(\phi(x_n) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\right)\right. \\ &\quad + (1-\vartheta_n)\left((\varphi(x_n) - \kappa_n f(x_n)) - \left(\varphi(u^\dagger)\right.\right. \\ &\quad \left.\left.\left. - \kappa_n f(u^\dagger)\right)\right)\right\| \\ &\le \vartheta_n\left\|\phi(x_n) - \phi(u^\dagger)\right\| + \vartheta_n\left\|\phi(u^\dagger) - \varphi(u^\dagger)\right. \\ &\quad \left. + \kappa_n f(u^\dagger)\right\| + (1-\vartheta_n)\|(\varphi(x_n) - \kappa_n f(x_n)) \\ &\quad - \left(\varphi(u^\dagger) - \kappa_n f(u^\dagger)\right)\| \le \vartheta_n\rho\left\|x_n - u^\dagger\right\| \\ &\quad + \vartheta_n\left\|\phi(u^\dagger) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\right\| \\ &\quad + (1-\vartheta_n)\left\|\varphi(x_n) - \varphi(u^\dagger)\right\| \le \vartheta_n\frac{\rho}{\lambda}\|\varphi(x_n) \\ &\quad - \varphi(u^\dagger)\| + \vartheta_n\left\|\phi(u^\dagger) - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\right\| \\ &\quad + (1-\vartheta_n)\left\|\varphi(x_n) - \varphi(u^\dagger)\right\| \\ &= \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\vartheta_n\right]\left\|\varphi(x_n) - \varphi(u^\dagger)\right\| + \vartheta_n\|\phi(u^\dagger) \\ &\quad - \varphi(u^\dagger) + \kappa_n f(u^\dagger)\| \le \left[1 - \left(1 - \frac{\rho}{\lambda}\right)\vartheta_n\right]\|\varphi(x_n) \\ &\quad - \varphi(u^\dagger)\| + \vartheta_n\left(\left\|\phi(u^\dagger) - \varphi(u^\dagger)\right\| + 2\vartheta\left\|f(u^\dagger)\right\|\right). \end{aligned} \tag{23}$$

According to (21) and (23), we obtain

$$\begin{aligned}\|u_n-\varphi(u^\dagger)\|^2 &\le \|\vartheta_n(\phi(x_n)-\varphi(u^\dagger)+\kappa_n f(u^\dagger))+(1-\vartheta_n)\\ &\quad\cdot((\varphi(x_n)-\kappa_n f(x_n))-(\varphi(u^\dagger)-\kappa_n f(u^\dagger)))\|^2\\ &\le \vartheta_n\|\phi(x_n)-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2\\ &\quad+(1-\vartheta_n)\|(\varphi(x_n)-\kappa_n f(x_n))-(\varphi(u^\dagger)-\kappa_n f(u^\dagger))\|^2\\ &\le \vartheta_n\|\phi(x_n)-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2+(1-\vartheta_n)\\ &\quad\cdot\left[\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\kappa_n(\kappa_n-2\vartheta)\|f(x_n)-f(u^\dagger)\|^2\right].\end{aligned}\tag{24}$$

Applying Lemma 1 to (17), we have

$$\begin{aligned}\|y_n-\varphi(u^\dagger)\|^2 &= \|(1-\varsigma_n)u_n+\varsigma_n T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-\varphi(u^\dagger)\|^2\\ &\le \|u_n-\varphi(u^\dagger)\|^2+\varsigma_n(\varsigma_n-\lambda_n)\|T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\|^2\\ &\le \|u_n-\varphi(u^\dagger)\|^2.\end{aligned}\tag{25}$$

Since g is relaxed (μ,ν)-cocoercive and L_1-Lipschitz, for all $u, \nu\in C$, we have

$$\begin{aligned}&\|(I-\gamma_n g)u-(I-\gamma_n g)\nu\|^2\\ &\quad=\|u-\nu\|^2-2\gamma_n\langle g(u)-g(\nu),u-\nu\rangle+\gamma_n^2\|g(u)-g(\nu)\|^2\\ &\quad\le\|u-\nu\|^2-2\gamma_n\left[-\mu\|g(u)-g(\nu)\|^2+\nu\|u-\nu\|^2\right]+\gamma_n^2\|g(u)-g(\nu)\|^2\\ &\quad\le\|u-\nu\|^2+2\gamma_n\mu L_1^2\|u-\nu\|^2-2\gamma_n\nu\|u-\nu\|^2+\gamma_n^2L_1^2\|u-\nu\|^2\\ &\quad=\left(1+2\gamma_n\mu L_1^2-2\gamma_n\nu+\gamma_n^2L_1^2\right)\|u-\nu\|^2.\end{aligned}\tag{26}$$

Since $0<\gamma_n<2(\nu-\mu L_1^2)/L_1^2$, $1+2\gamma_n\mu L_1^2-2\gamma_n\nu+\gamma_n^2L_1^2\le 1$. Thus, from (26), we obtain

$$\|(I-\gamma_n g)u-(I-\gamma_n g)\nu\|\le\|u-\nu\|,\quad\forall u,\nu\in C.\tag{27}$$

Hence,

$$\begin{aligned}\|z_n-\varphi(u^\dagger)\| &= \|\mathrm{proj}_C(I-\gamma_n g)y_n-\mathrm{proj}_C(I-\gamma_n g)\varphi(u^\dagger)\|\\ &\le\|(I-\gamma_n g)y_n-(I-\gamma_n g)\varphi(u^\dagger)\|\\ &\le\|y_n-\varphi(u^\dagger)\|.\end{aligned}\tag{28}$$

Combining (17), (23), (25), and (28), we obtain

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^\dagger)\| &\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|+\tau_n\|z_n-\varphi(u^\dagger)\|\\ &\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|+\tau_n\|u_n-\varphi(u^\dagger)\|\\ &\le (1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|+\tau_n\\ &\quad\cdot\left[1-\left(1-\frac{\rho}{\lambda}\right)\vartheta_n\right]\|\varphi(x_n)-\varphi(u^\dagger)\|+\tau_n\vartheta_n\\ &\quad\cdot(\|\phi(u^\dagger)-\varphi(u^\dagger)\|+2\vartheta\|f(u^\dagger)\|)\\ &=\left[1-\left(1-\frac{\rho}{\lambda}\right)\tau_n\vartheta_n\right]\|\varphi(x_n)-\varphi(u^\dagger)\|\\ &\quad+\left(1-\frac{\rho}{\lambda}\right)\tau_n\vartheta_n\frac{\|\phi(u^\dagger)-\varphi(u^\dagger)\|+2\vartheta\|f(u^\dagger)\|}{1-\rho/\lambda}.\end{aligned}\tag{29}$$

By induction, we have

$$\|\varphi(x_n)-\varphi(u^\dagger)\|\le\max\left\{\|\varphi(x_0)-\varphi(u^\dagger)\|,\frac{\|\phi(u^\dagger)-\varphi(u^\dagger)\|+2\vartheta\|f(u^\dagger)\|}{1-\rho/\lambda}\right\}.\tag{30}$$

It follows that

$$\begin{aligned}\|x_n-u^\dagger\| &\le\frac{1}{\lambda}\|\varphi(x_n)-\varphi(u^\dagger)\|\\ &\le\frac{1}{\lambda}\max\left\{\|\varphi(x_0)-\varphi(u^\dagger)\|,\frac{\|\phi(u^\dagger)-\varphi(u^\dagger)\|+2\vartheta\|f(u^\dagger)\|}{1-\rho/\lambda}\right\}.\end{aligned}\tag{31}$$

So, $\{\varphi(x_n)\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded. From (17), we have

$$\varphi(x_{n+1})-\varphi(x_n)=\tau_n(z_n-\varphi(x_n)),\quad n\ge 0.\tag{32}$$

It follows that

$$\langle\varphi(x_{n+1})-\varphi(x_n),\varphi(x_n)-\varphi(u^\dagger)\rangle=\tau_n\langle z_n-\varphi(x_n),\varphi(x_n)-\varphi(u^\dagger)\rangle.\tag{33}$$

Thanks to (33), we deduce

$$\begin{aligned}&\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(x_n)\|^2\\ &\quad=\tau_n\left[\|z_n-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|z_n-\varphi(x_n)\|^2\right].\end{aligned}\tag{34}$$

Combining (32) and (34), we obtain

$$\begin{aligned}&\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2\\ &\quad=\tau_n\left[\|z_n-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|z_n-\varphi(x_n)\|^2\right]\\ &\qquad+\tau_n^2\|z_n-\varphi(x_n)\|^2\\ &\quad=\tau_n\left[\|z_n-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2\right]-\tau_n(1-\tau_n)\|z_n\\ &\qquad-\varphi(x_n)\|^2\\ &\quad\le\tau_n\left[\|u_n-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2\right]-\tau_n(1-\tau_n)\|z_n\\ &\qquad-\varphi(x_n)\|^2.\end{aligned}\tag{35}$$

By virtue of (23), we get

$$\begin{aligned}\|u_n-\varphi(u^\dagger)\|^2 &\le\left[1-\left(1-\frac{\rho}{\lambda}\right)\vartheta_n\right]\|\varphi(x_n)-\varphi(u^\dagger)\|^2\\ &\quad+\left(1-\frac{\rho}{\lambda}\right)\vartheta_n\left(\frac{\|\phi(u^\dagger)-\varphi(u^\dagger)\|+2\vartheta\|f(u^\dagger)\|}{1-\rho/\lambda}\right)^2.\end{aligned}\tag{36}$$

Now, we consider two cases.

Case 1. There exists some integer $N_0>0$ such that $\{\|\varphi(x_n)-\varphi(u^\dagger)\|\}$ is decreasing when $n\ge N_0$. Then, $\lim_{n\longrightarrow\infty}\|\varphi(x_n)-\varphi(u^\dagger)\|$ exists. According to (35), (36), and $(C1)$, we have

$$\begin{aligned}\tau_n(1-\tau_n)\|z_n-\varphi(x_n)\|^2 &\le \|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2\\&\quad+\tau_n\left[\|u_n-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\varphi(u^\dagger)\|^2\right]\\&\le \|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2\\&\quad+\left(1-\frac{\rho}{\lambda}\right)\vartheta_n\left(\frac{\|\phi(u^\dagger)-\varphi(u^\dagger)\|+2\vartheta\|f(u^\dagger)\|}{1-\rho/\lambda}\right)^2\longrightarrow 0.\end{aligned} \tag{37}$$

This together with $(C4)$ implies that

$$\lim_{n\longrightarrow\infty}\|z_n-\varphi(x_n)\|=0. \tag{38}$$

Therefore, by (32), we have

$$\lim_{n\longrightarrow\infty}\|\varphi(x_{n+1})-\varphi(x_n)\|=0. \tag{39}$$

By (24), we have

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2 &= \|(1-\tau_n)(\varphi(x_n)-\varphi(u^\dagger))+\tau_n(z_n-\varphi(u^\dagger))\|^2\\&\le(1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|z_n-\varphi(u^\dagger)\|^2\\&\le(1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|u_n-\varphi(u^\dagger)\|^2\\&\le(1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)\\&\quad-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2+\tau_n(1-\vartheta_n)\kappa_n(\kappa_n-2\vartheta)\\&\quad\cdot\|f(x_n)-f(u^\dagger)\|^2+\tau_n(1-\vartheta_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2\\&\le\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\\&\quad+\kappa_n f(u^\dagger)\|^2+\tau_n(1-\vartheta_n)\kappa_n(\kappa_n-2\vartheta)\|f(x_n)\\&\quad-f(u^\dagger)\|^2.\end{aligned} \tag{40}$$

It results in that

$$\begin{aligned}&\tau_n(1-\vartheta_n)\kappa_n(2\vartheta-\kappa_n)\|f(x_n)-f(u^\dagger)\|^2\\&\quad\le\|\varphi(x_n)-\varphi(u^\dagger)\|^2-\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)\\&\qquad-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2\\&\quad\le\left(\|\varphi(x_n)-\varphi(u^\dagger)\|+\|\varphi(x_{n+1})-\varphi(u^\dagger)\|\right)\|\varphi(x_{n+1})-\varphi(x_n)\|\\&\qquad+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)+\kappa_n f(u^\dagger)\|^2\longrightarrow 0.\end{aligned} \tag{41}$$

Hence,

$$\lim_{n\longrightarrow\infty}\|f(x_n)-f(u^\dagger)\|=0. \tag{42}$$

Set $v_n=\varphi(x_n)-\kappa_n f(x_n)-(\varphi(u^\dagger)-\kappa_n f(u^\dagger))$ for all $n\ge 0$. Using (13) and (21), we have

$$\begin{aligned}&\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\|^2\\&\quad=\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\mathrm{proj}_C\left[\varphi(u^\dagger)-\kappa_n f(u^\dagger)\right]\|^2\\&\quad\le\left\langle v_n,\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\right\rangle\\&\quad=\frac{1}{2}\left\{\|v_n\|^2+\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\right.\\&\qquad-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\\&\qquad\left.-\kappa_n\left(f(x_n)-f(u^\dagger)\right)\|^2\right\}\\&\quad\le\frac{1}{2}\left\{\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\right.\\&\qquad-\varphi(u^\dagger)\|^2-\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2\\&\qquad-\kappa_n^2\|f(x_n)-f(u^\dagger)\|+2\kappa_n\langle\varphi(x_n)-\mathrm{proj}_C\\&\qquad\left.\times[\varphi(x_n)-\kappa_n f(x_n)],f(x_n)-f(u^\dagger)\rangle\right\}.\end{aligned} \tag{43}$$

It yields

$$\begin{aligned}&\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\|^2\\&\quad\le\|\varphi(x_n)-\varphi(u^\dagger)\|^2-\kappa_n^2\|f(x_n)-f(u^\dagger)\|-\|\varphi(x_n)\\&\qquad-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2+2\kappa_n\langle\varphi(x_n)\\&\qquad-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)],f(x_n)-f(u^\dagger)\rangle.\end{aligned} \tag{44}$$

In the light of (17) and (44), we have

$$\begin{aligned}\|u_n-\varphi(u^\dagger)\|^2&\le\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2+(1-\vartheta_n)\|\mathrm{proj}_C\\&\quad\cdot[\varphi(x_n)-\kappa_n f(x_n)]-\varphi(u^\dagger)\|^2\\&\le\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2+(1-\vartheta_n)\|\varphi(x_n)\\&\quad-\varphi(u^\dagger)\|^2+2\kappa_n\|\varphi(x_n)-\mathrm{proj}_C\\&\quad\cdot[\varphi(x_n)-\kappa_n f(x_n)]\|\|f(x_n)-f(u^\dagger)\\&\quad\cdot\|-(1-\vartheta_n)\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2.\end{aligned} \tag{45}$$

Based on (40) and (45), we obtain

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(u^\dagger)\|^2&\le(1-\tau_n)\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\|u_n-\varphi(u^\dagger)\|^2\\&\le\|\varphi(x_n)-\varphi(u^\dagger)\|^2+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2\\&\quad-\tau_n(1-\vartheta_n)\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2\\&\quad+2\tau_n\kappa_n\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|\|f(x_n)\\&\quad-f(u^\dagger)\|.\end{aligned} \tag{46}$$

Then,

$$\begin{aligned}&\tau_n(1-\vartheta_n)\|\varphi(x_n)-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|^2\\&\quad\le\left(\|\varphi(x_n)-\varphi(u^\dagger)\|+\|\varphi(x_{n+1})-\varphi(u^\dagger)\|\right)\|\varphi(x_{n+1})\\&\qquad-\varphi(x_n)\|+\tau_n\vartheta_n\|\phi(x_n)-\varphi(u^\dagger)\|^2+2\tau_n\kappa_n\|\varphi(x_n)\\&\qquad-\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]\|\|f(x_n)-f(u^\dagger)\|.\end{aligned} \tag{47}$$

According to $(C1)$, $(C4)$, (39), (42), and (47), we deduce

$$\lim_{n\longrightarrow\infty} \|\varphi(x_n) - \mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)]\| = 0. \tag{48}$$

Since $u_n - \phi(x_n) = (1-\vartheta_n)(\mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)] - \phi(x_n))$, from (38), (39), and (48), we have

$$\lim_{n\longrightarrow\infty} \|\varphi(x_n) - u_n\| = \lim_{n\longrightarrow\infty} \|\varphi(x_{n+1}) - u_n\| = \lim_{n\longrightarrow\infty} \|z_n - u_n\| = 0. \tag{49}$$

From (26) and (28), we get

$$\begin{aligned}\|z_n - \varphi(u^\dagger)\|^2 &\le \|y_n - \varphi(u^\dagger)\|^2 - 2\gamma_n\left[-\mu\|g(y_n) - g(\varphi(u^\dagger))\|^2\right.\\ &\quad \left.+ \nu\|y_n - \varphi(u^\dagger)\|^2\right] + \gamma_n^2\|g(y_n) - g(\varphi(u^\dagger))\|^2\\ &\le \|y_n - \varphi(u^\dagger)\|^2 + \left(2\gamma_n\mu + \gamma_n^2 - \frac{2\gamma_n\nu}{L_1^2}\right)\|g(y_n)\\ &\quad - g(\varphi(u^\dagger))\|^2.\end{aligned} \tag{50}$$

It follows that

$$\begin{aligned}\|\varphi(x_{n+1}) - \varphi(u^\dagger)\|^2 &\le (1-\tau_n)\|\varphi(x_n) - \varphi(u^\dagger)\|^2 + \tau_n\|u_n\\ &\quad - \varphi(u^\dagger)\|^2 + \tau_n\left(2\gamma_n\mu + \gamma_n^2 - \frac{2\gamma_n\nu}{L_1^2}\right)\\ &\quad \cdot \|g(y_n) - g(\varphi(u^\dagger))\|^2,\end{aligned} \tag{51}$$

which together with (49) implies that

$$\begin{aligned}&-\tau_n\left(2\gamma_n\mu + \gamma_n^2 - \frac{2\gamma_n\nu}{L_1^2}\right)\|g(y_n) - g(\varphi(u^\dagger))\|^2\\ &\quad\le (1-\tau_n)\left(\|\varphi(x_n) - \varphi(u^\dagger)\|^2 - \|\varphi(x_{n+1}) - \varphi(u^\dagger)\|^2\right)\\ &\qquad + \tau_n\left(\|u_n - \varphi(u^\dagger)\|^2 - \|\varphi(x_{n+1}) - \varphi(u^\dagger)\|^2\right) \longrightarrow 0.\end{aligned} \tag{52}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \|g(y_n) - g(\varphi(u^\dagger))\| = 0. \tag{53}$$

Since proj_C is firmly nonexpansive, from (12) and (28), we have

$$\begin{aligned}\|z_n - \varphi(u^\dagger)\|^2 &= \|\mathrm{proj}_C(I-\gamma_n g)y_n - \mathrm{proj}_C(I-\gamma_n g)\varphi(u^\dagger)\|^2\\ &\le \left\langle (I-\gamma_n g)y_n - (I-\gamma_n g)\varphi(u^\dagger), z_n - \varphi(u^\dagger)\right\rangle\\ &= \frac{1}{2}\left\{\|(I-\gamma_n g)y_n - (I-\gamma_n g)\varphi(u^\dagger)\|^2 + \|z_n - \varphi(u^\dagger)\|^2\right.\\ &\quad \left.- \|(I-\gamma_n g)y_n - (I-\gamma_n g)\varphi(u^\dagger) - (z_n - \varphi(u^\dagger))\|^2\right\}\\ &\le \frac{1}{2}\left\{\|y_n - \varphi(u^\dagger)\|^2 + \|z_n - \varphi(u^\dagger)\|^2 - \|y_n - z_n\right.\\ &\quad \left.- \gamma_n(g(y_n) - g(\varphi(u^\dagger)))\|^2\right\}\\ &\le \frac{1}{2}\left\{\|u_n - \varphi(u^\dagger)\|^2 + \|z_n - \varphi(u^\dagger)\|^2 - \|y_n - z_n\|^2\right.\\ &\quad - \gamma_n^2\|g(y_n) - g(\varphi(u^\dagger))\|^2\\ &\quad \left.+ 2\gamma_n\left\langle g(y_n) - g(\varphi(u^\dagger)), y_n - z_n\right\rangle\right\},\end{aligned} \tag{54}$$

which yields

$$\begin{aligned}\|z_n - \varphi(u^\dagger)\|^2 &\le \|u_n - \varphi(u^\dagger)\|^2 - \|y_n - z_n\|^2 + 2\gamma_n\|g(y_n)\\ &\quad - g(\varphi(u^\dagger))\|\|y_n - z_n\|.\end{aligned} \tag{55}$$

This together with (40) implies that

$$\begin{aligned}\|\varphi(x_{n+1}) - \varphi(u^\dagger)\|^2 &\le (1-\tau_n)\|\varphi(x_n) - \varphi(u^\dagger)\|^2 + \tau_n\|u_n\\ &\quad - \varphi(u^\dagger)\|^2 - \tau_n\|y_n - z_n\|^2 + 2\tau_n\gamma_n\|g(y_n)\\ &\quad - g(\varphi(u^\dagger))\|\|y_n - z_n\|.\end{aligned} \tag{56}$$

It follows that

$$\begin{aligned}\tau_n\|y_n - z_n\|^2 &\le (1-\tau_n)\|\varphi(x_n) - \varphi(u^\dagger)\|^2 + \tau_n\|u_n - \varphi(u^\dagger)\|^2\\ &\quad - \|\varphi(x_{n+1}) - \varphi(u^\dagger)\|^2 + 2\tau_n\gamma_n\|g(y_n)\\ &\quad - g(\varphi(u^\dagger))\|\|y_n - z_n\| \longrightarrow 0 (\text{by } (49) \text{ and } (53)).\end{aligned} \tag{57}$$

So,

$$\lim_{n\longrightarrow\infty} \|y_n - z_n\| = 0. \tag{58}$$

By (49) and (58), we have

$$\lim_{n\longrightarrow\infty} \|y_n - u_n\| = 0. \tag{59}$$

In view of (25) and (59), we get

$$\begin{aligned}&\varsigma_n(\lambda_n - \varsigma_n)\|T[(1-\lambda_n)u_n + \lambda_n T(u_n)] - u_n\|^2\\ &\quad\le \|u_n - \varphi(u^\dagger)\|^2 - \|y_n - \varphi(u^\dagger)\|^2\\ &\quad\le \|u_n - y_n\|\left(\|u_n - \varphi(u^\dagger)\| + \|y_n - \varphi(u^\dagger)\|\right) \longrightarrow 0.\end{aligned} \tag{60}$$

It follows from $(C2)$ and (60) that

$$\lim_{n\longrightarrow\infty} \|T[(1-\lambda_n)u_n + \lambda_n T(u_n)] - u_n\| = 0. \tag{61}$$

Since T is L_2-Lipschitz, we have

$$\begin{aligned}\|T(u_n)-u_n\| &\le \|T(u_n)-T[(1-\lambda_n)u_n+\lambda_n T(u_n)]\|+\|T\\ &\quad\cdot[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\| \le \lambda_n L_2\|T(u_n)\\ &\quad-u_n\|+\|T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\|.\end{aligned} \tag{62}$$

Hence,

$$\|T(u_n)-u_n\| \le \frac{1}{1-\lambda_n L_2}\|T[(1-\lambda_n)u_n+\lambda_n T(u_n)]-u_n\|. \tag{63}$$

Owing to $(C2)$, (61) and (63), we deduce

$$\lim_{n\longrightarrow\infty}\|T(u_n)-u_n\|=0. \tag{64}$$

Next, we show that $\limsup_{n\longrightarrow\infty}\langle\phi(u^\dagger)-\varphi(u^\dagger),u_n-\varphi(u^\dagger)\rangle\le 0$. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle\phi(u^\dagger)-\varphi(u^\dagger),u_n-\varphi(u^\dagger)\rangle=\lim_{i\longrightarrow\infty}\langle\phi(u^\dagger)-\varphi(u^\dagger),u_{n_i}-\varphi(u^\dagger)\rangle. \tag{65}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some point $z\in C$. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup z$. This implies that $\varphi(x_{n_i})\rightharpoonup\varphi(z)$ due to the weak continuity of φ. Thus, $u_{n_i}\rightharpoonup\varphi(z)$, $y_{n_i}\rightharpoonup\varphi(z)$, and $z_{n_i}\rightharpoonup\varphi(z)$. Applying Lemma 2 to (64) to deduce $\varphi(z)\in\mathrm{Fix}(T)$.

Now, we show that $\varphi(z)\in\mathrm{VI}(C,g)$. Let

$$S_1(v)=\begin{cases} g(v)+N_C v, & v\in C,\\ \varnothing, & v\notin C.\end{cases} \tag{66}$$

Since g is relaxed (μ,ν)-cocoercive, for all $x,y\in C$, we have

$$\begin{aligned}\langle g(x)-g(y),x-y\rangle &\ge (-\mu)\|g(x)-g(y)\|^2+\nu\|x-y\|^2\\ &\ge\left(\nu-\mu L^2\right)\|x-y\|^2\ge 0,\end{aligned} \tag{67}$$

which implies that g is monotone and so S_1 is maximal monotone. Let $(v,u)\in G(S_1)$. Owing to $u-g(v)\in N_C v$ and $z_n\in C$, we get

$$\langle v-z_n,u-g(v)\rangle\ge 0. \tag{68}$$

According to $z_n=\mathrm{proj}_C(I-\gamma_n g)y_n$, we obtain

$$\langle v-z_n,z_n-(I-\gamma_n g)y_n\rangle\ge 0. \tag{69}$$

Then,

$$\left\langle v-z_n,\frac{z_n-y_n}{\gamma_n}+g(y_n)\right\rangle\ge 0. \tag{70}$$

It follows that

$$\begin{aligned}\langle v-z_{n_i},u\rangle &\ge \langle v-z_{n_i},g(v)-g(z_{n_i})\rangle\\ &\quad+\left\langle v-z_{n_i},g(z_{n_i})-g\left(y_{n_i}\right)\right\rangle-\left\langle v-z_{n_i},\frac{z_{n_i}-y_{n_i}}{\gamma_{n_i}}\right\rangle\\ &\ge\left\langle v-z_{n_i},g(z_{n_i})-g\left(y_{n_i}\right)\right\rangle-\left\langle v-z_{n_i},\frac{z_{n_i}-y_{n_i}}{\gamma_{n_i}}\right\rangle.\end{aligned} \tag{71}$$

Since $z_{n_i}\rightharpoonup\varphi(z)$, $\|z_{n_i}-y_{n_i}\|\longrightarrow 0$, it follows from (71) that $\langle v-\varphi(z),u\rangle\ge 0$. Therefore, $\varphi(z)\in S_1^{-1}(0)$ and $\varphi(z)\in\mathrm{VI}(C,g)$.

Next, we prove $z\in\mathrm{GVI}(C,f,\varphi)$. Let

$$S_2(v)=\begin{cases} f(v)+N_C(v), & v\in C,\\ \varnothing, & v\in C.\end{cases} \tag{72}$$

It is known that S_2 is maximal φ-monotone. Let $(v,w)\in G(S_2)$. Since $w-f(v)\in N_C(v)$ and $x_n\in C$, we have $\langle\varphi(v)-\varphi(x_n),w-f(v)\rangle\ge 0$. Set $w_n=\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]$. Then,

$$\langle\varphi(v)-w_n,w_n-[\varphi(x_n)-\kappa_n f(x_n)]\rangle\ge 0. \tag{73}$$

It follows that

$$\left\langle\varphi(v)-w_n,\frac{w_n-\varphi(x_n)}{\kappa_n}+f(x_n)\right\rangle\ge 0. \tag{74}$$

Thus,

$$\begin{aligned}\langle\varphi(v)-\varphi(x_{n_i}),w\rangle &\ge \langle\varphi(v)-\varphi(x_{n_i}),f(v)-f(x_{n_i})\rangle\\ &\quad+\langle\varphi(v)-\varphi(x_{n_i}),f(x_{n_i})\rangle\\ &\quad-\left\langle\varphi(v)-w_{n_i},\frac{w_{n_i}-\varphi(x_{n_i})}{\kappa_{n_i}}\right\rangle\\ &\quad-\langle\varphi(v)-w_{n_i},f(x_{n_i})\rangle\\ &\ge-\left\langle\varphi(v)-w_{n_i},\frac{w_{n_i}-\varphi(x_{n_i})}{\kappa_{n_i}}\right\rangle\\ &\quad-\langle\varphi(x_{n_i})-w_{n_i},f(x_{n_i})\rangle.\end{aligned} \tag{75}$$

Since $\|\varphi(x_{n_i})-w_{n_i}\|\longrightarrow 0$ and $\varphi(x_{n_i})\rightharpoonup\varphi(z)$, we deduce that $\langle\varphi(v)-\varphi(z),w\rangle\ge 0$ by taking $i\longrightarrow\infty$ in (75). Thus, $z\in S_2^{-1}(0)$ by the maximal φ-monotonicity of S_2. Hence, $z\in\mathrm{GVI}(C,f,\varphi)$. Therefore, $z\in\varphi^{-1}(\mathrm{Fix}(T)\cap\mathrm{VI}(C,g))\bigcap\mathrm{GVI}(C,f,\varphi)=\Theta$.

From (49) and (65), we obtain

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle &= \lim_{i\longrightarrow\infty}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), \varphi\left(x_{n_i}\right)-\varphi\left(u^{\dagger}\right)\right\rangle \\ &= \left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), \varphi(z)-\varphi\left(u^{\dagger}\right)\right\rangle \le 0.\end{aligned} \tag{76}$$

By (17), we have

$$\begin{aligned}\|u_n-\varphi\left(u^{\dagger}\right)\|^2 &= \|\vartheta_n\left(\phi(x_n)-\varphi\left(u^{\dagger}\right)\right)+(1-\vartheta_n)\cdot\left(\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi\left(u^{\dagger}\right)\right)\|^2 \\ &\le (1-\vartheta_n)^2\|\mathrm{proj}_C[\varphi(x_n)-\kappa_n f(x_n)]-\varphi\left(u^{\dagger}\right)\|^2 + 2\vartheta_n\left\langle \phi(x_n)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle \\ &\le (1-\vartheta_n)^2\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 + 2\vartheta_n\left\langle \phi(x_n)-\phi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle \\ &\quad + 2\vartheta_n\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle \\ &\le (1-\vartheta_n)^2\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 + 2\vartheta_n\frac{\rho}{\lambda}\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|\|u_n-\varphi\left(u^{\dagger}\right)\|+2\vartheta_n\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle \\ &\le (1-\vartheta_n)^2\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 + \vartheta_n\frac{\rho}{\lambda}\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 + \vartheta_n\frac{\rho}{\lambda}\|u_n-\varphi\left(u^{\dagger}\right)\|^2 + 2\vartheta_n \cdot \left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle.\end{aligned} \tag{77}$$

It follows that

$$\begin{aligned}\|u_n-\varphi\left(u^{\dagger}\right)\|^2 &\le \left[1-\frac{2(1-\rho/\lambda)\vartheta_n}{1-\vartheta_n\rho/\lambda}\right]\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 + \frac{\vartheta_n^2}{1-\vartheta_n\rho/\lambda}\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 \\ &\quad + \frac{2\vartheta_n}{1-\vartheta_n\rho/\lambda}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle.\end{aligned} \tag{78}$$

Set $M=\sup_n\|\varphi(x_n)-\varphi(u^{\dagger})\|^2$. Therefore,

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi\left(u^{\dagger}\right)\|^2 &\le (1-\tau_n)\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2+\tau_n\|u_n-\varphi\left(u^{\dagger}\right)\|^2 \\ &\le \left[1-\frac{2(1-\rho/\lambda)\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda}\right]\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 + \frac{\vartheta_n^2\tau_n}{1-\vartheta_n\rho/\lambda}\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 \\ &\quad + \frac{2\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle \\ &= \left[1-\frac{2(1-\rho/\lambda)\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda}\right]\|\varphi(x_n)-\varphi\left(u^{\dagger}\right)\|^2 + \frac{2(1-\rho/\lambda)\vartheta_n\tau_n}{1-\vartheta_n\rho/\lambda} \\ &\quad \times\left\{\frac{\vartheta_n}{2(1-\rho/\lambda)}M+\frac{1}{1-\rho/\lambda}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_n-\varphi\left(u^{\dagger}\right)\right\rangle\right\}.\end{aligned} \tag{79}$$

By Lemma 3 and (79), we conclude that $\varphi(x_n)\longrightarrow\varphi(u^{\dagger})$ and $x_n\longrightarrow u^{\dagger}$.

Case 2. For any N, there exists an integer $n_0>N$ such that $\|\varphi(x_{n_0})-\varphi(u^{\dagger})\|\le\|\varphi(x_{n_0+1})-\varphi(u^{\dagger})\|$. Let $\psi_n=\{\|\varphi(x_n)-\varphi(u^{\dagger})\|^2\}$. Then, we have $\psi_{n_0}\le\psi_{n_0+1}$. Let $\{\mu_n\}$ be an integer sequence defined by, for all $n\ge n_0$,

$$\mu(n)=\max\left\{l\in\mathbb{N}\mid n_0\le l\le n, \psi_l\le\psi_{l+1}\right\}. \tag{80}$$

Note that $\mu(n)$ is nondecreasing and satisfies $\lim_{n\longrightarrow\infty}\mu(n)=\infty$ and $\psi_{\mu(n)}\le\psi_{\mu(n)+1}, \forall n\ge n_0$.

Similarly, we can deduce

$$\limsup_{n\longrightarrow\infty}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_{\mu(n)}-\varphi\left(u^{\dagger}\right)\right\rangle \le 0 \tag{81}$$

$$\begin{aligned}\psi_{\mu(n)+1} &\le \left[1-\frac{2(1-\rho/\lambda)\vartheta_\mu(n)\tau_\mu(n)}{1-\vartheta_\mu(n)\rho/\lambda}\right]\psi_{\mu(n)}+\frac{2(1-\rho/\lambda)\vartheta_\mu(n)\tau_\mu(n)}{1-\vartheta_\mu(n)\rho/\lambda} \\ &\quad \times\left\{\frac{\vartheta_\mu(n)}{2(1-\rho/\lambda)}M+\frac{1}{1-\rho/\lambda}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_\mu(n)-\varphi\left(u^{\dagger}\right)\right\rangle\right\}.\end{aligned} \tag{82}$$

Note that $\psi_{\mu(n)}\le\psi_{\mu(n)+1}$. By (82), we have

$$\psi_{\mu(n)}\le\frac{\vartheta_\mu(n)}{2(1-\rho/\lambda)}M+\frac{1}{1-\rho/\lambda}\left\langle \phi\left(u^{\dagger}\right)-\varphi\left(u^{\dagger}\right), u_\mu(n)-\varphi\left(u^{\dagger}\right)\right\rangle. \tag{83}$$

Based on (81) and (83), we derive

$$\limsup_{n\longrightarrow\infty}\psi_{\mu(n)}\le 0, \tag{84}$$

and thus,

$$\lim_{n\longrightarrow\infty}\psi_{\mu(n)}=0. \tag{85}$$

From (81) and (82), we can deduce

$$\limsup_{n\longrightarrow\infty}\psi_{\mu(n)+1}\le\limsup_{n\longrightarrow\infty}\psi_{\mu(n)}. \tag{86}$$

This together with (85) implies that

$$\lim_{n\longrightarrow\infty}\psi_{\mu(n)+1}=0. \tag{87}$$

By Lemma 4, we obtain

$$0\le\psi_n\le\max\left\{\psi_{\mu(n)},\psi_{\mu(n)+1}\right\}. \tag{88}$$

Therefore, $\psi_n\longrightarrow 0$. That is, $\varphi(x_n)\longrightarrow\varphi(u^{\dagger})$ and thus $x_n\longrightarrow u^{\dagger}$. This completes the proof.

In Algorithm 5, choose $\varphi=I$, identity operator, and $f: C\longrightarrow H$ is a ϑ-inverse strongly monotone operator. Then, we have the following algorithm and corollary.

Algorithm 7. Let $x_0 \in C$ be an initial value. Define the sequence $\{x_n\}$ by the following form:

$$\begin{cases} u_n = \vartheta_n \phi(x_n) + (1-\vartheta_n)\mathrm{proj}_C[x_n - \kappa_n f(x_n)], \\ y_n = (1-\varsigma_n)u_n + \varsigma_n T[(1-\lambda_n)u_n + \lambda_n T(u_n)], \\ z_n = \mathrm{proj}_C[y_n - \gamma_n g(y_n)], \\ x_{n+1} = (1-\tau_n)x_n + \tau_n z_n, \quad n \geq 0. \end{cases} \tag{89}$$

Corollary 8. *Suppose that $\Theta_1 := VI(C, f) \cap VI(C, g) \cap Fix(T) \neq \emptyset$. Assume that conditions (C1)-(C5) are satisfied. Then, the sequence $\{x_n\}$ generated by (89) converges strongly to $v^\dagger = proj_{\Theta_1}\phi(v^\dagger)$.*

Algorithm 9. Let $x_0 \in C$ be an initial value. Define the sequence $\{x_n\}$ by the following form:

$$\begin{cases} u_n = \vartheta_n \phi(x_n) + (1-\vartheta_n)\mathrm{proj}_C[\varphi(x_n) - \kappa_n f(x_n)], \\ z_n = \mathrm{proj}_C[u_n - \gamma_n g(u_n)], \\ \varphi(x_{n+1}) = (1-\tau_n)\varphi(x_n) + \tau_n z_n, \quad n \geq 0. \end{cases} \tag{90}$$

Corollary 10. *Suppose that $\Theta_2 := GVI(C, f, \varphi) \cap \varphi^{-1}(VI(C, g)) \neq \emptyset$. Assume that conditions (C1) and (C3)-(C5) are satisfied. Then, the sequence $\{x_n\}$ generated by (90) converges strongly to $u^\dagger \in \Theta_2$ verifying*

$$\left\langle \phi\left(u^\dagger\right) - \varphi\left(u^\dagger\right), \varphi\left(x^\dagger\right) - \varphi\left(u^\dagger\right) \right\rangle \leq 0, \quad \forall x^\dagger \in \Theta_2. \tag{91}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Orthogonal Stability and Nonstability of a Generalized Quartic Functional Equation in Quasi-β-Normed Spaces

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In this work, we examine the generalized Hyers-Ulam orthogonal stability of the quartic functional equation in quasi-β-normed spaces. Moreover, we prove that this functional equation is not stable in a special condition by a counterexample.

1. Introduction

In this paper, $\mathbb{R}$ and $\mathbb{C}$ denote sets of all real numbers and complex numbers, respectively.

In the fall of 1940, Ulam [1] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows:

Ulam's question: let $(G_1, *)$, $(G_2, \star)$ be two groups and $d : G_2 \times G_2 \rightarrow [0,\infty)$ be a metric. Given $\delta > 0$, does there exist $\varepsilon > 0$ such that if a function $g : G_1 \rightarrow G_2$ satisfies the inequality

$$d(g(x * y), g(x)\star g(y)) \leq \delta, \tag{1}$$

for all $x, y \in G_1$, then there is a homomorphism $h : G_1 \rightarrow G_2$ with

$$d(g(x), h(x)) \leq \varepsilon \text{ for } all\, x \in G_1? \tag{2}$$

In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. This result was generalized by Aoki [3] for additive mappings.

During the past few years, several mathematicians have published on various generalizations and applications of generalized Hyers-Ulam stability to a number of functional equations and mappings (see, for instance, [4–15]).

In [16], Xu et al. obtained the general solution and investigated the Ulam stability problem for the quintic functional equation in quasi-β-normed spaces via fixed point method. This method is different from the direct method, initiated by Hyers in [2]. And also, Eskandani et al. [17, 18] obtained the general solution for the mixed additive and quadratic functional equation and a cubic functional equation and established its generalized Hyers-Ulam stability in quasi-β-normed spaces.

The Ulam-type stability result for the quartic functional equation

$$\begin{aligned} &F(x_1 + 2x_2) + F(x_1 - 2x_2) + 6F(x_1) \\ &\quad = 4[F(x_1 + x_2) + F(x_1 - x_2) + 6F(x_2)], \end{aligned} \tag{3}$$

was first developed by Rassias [19]. Subsequently, Sahoo and Chung [20] determined the general solution of (3) without assuming any regularity conditions on the unknown function. In fact, they proved that the function $f:\mathbb{R}\to\mathbb{R}$ is a solution of (3) if and only if $f(x)=A(x,x,x,x)$, where the function $A:\mathbb{R}^4\to\mathbb{R}$ is symmetric and additive in each variable. Since the solution of (3) is even, we can rewrite (3) as

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{4}$$

Lee et al. [21] obtained the general solution of (4) and proved the Hyers-Ulam-Rassias stability of this equation. It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (4), which is called a quartic functional equation, and every solution of the quartic functional equation is said to be a quartic function. In [22] Ravi et al. have investigated the generalized Hyers-Ulam *product-sum* stability of functional equations and have the following theorem.

Theorem 1. *Let $f:E\to F$ be a mapping which satisfies the inequality*

$$\begin{aligned}&\left\|f(mx+y)+f(mx-y)-2f(x+y)-2f(x-y)-2(m^2-2)f(x)+2f(y)\right\|_F\\&\leq\varepsilon\left(\|x\|_E^p\|y\|_E^p+\|x\|_E^{2p}+\|y\|_E^{2p}\right),\end{aligned} \tag{5}$$

for all $x,y\in E$ with $x\perp y$, where ε and p are constants with ε, $p>0$ and either $m>1$, $p<1$ or $m<1$, $p>1$ with $m\neq 0$, $m\neq\pm 1$, $m\neq\pm\sqrt{2}$, and $-1\neq|m|^{p-1}<1$. Then, the limit $\lim_{n\to\infty} m^{-2n}f(m^n x)$ exists for all $x\in E$, and $Q:E\to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x)-Q(x)\|_F\leq\frac{\varepsilon}{2|m^2-m^2p|}\|x\|_E^{2p}, \tag{6}$$

for all $x\in E$.

In 1982, Rassias [23] provided generalizations of the Hyers-Ulam stability theorem which allows the Cauchy difference controlled by a product of different powers of norm. And then, the result of the Rassias theorem has been generalized by Gavruta [24] by replacing the unbounded Cauchy difference by a generalized control function. Also, Rassias (see [23, 25–28]) solved the Ulam problem for different mappings. In addition, Ravi et al. considered the mixed product-sum of powers of norms control function [22]. Note that the mixed *product-sum* function was introduced by Ravi et al. in 2008-2009 ([22, 29–31]).

In this paper, we examine the generalized Hyers-Ulam orthogonal stability of the quartic functional equation as

$$\begin{aligned}\phi\left(\sum_{a=1}^{m}v_a\right)&=\sum_{1\leq a<b<c<d\leq m}\phi(v_a+v_b+v_c+v_d)+(-m+4)\\&\cdot\sum_{1\leq a<b<c\leq m}\phi(v_a+v_b+v_c)+\left(\frac{m^2-7m+12}{2}\right)\\&\cdot\sum_{\substack{1=a;\\a\neq b}}^{m}\phi(v_a+v_b)-\sum_{a=1}^{m}\phi(2v_a)\\&+\left(\frac{-m^3+9m^2-26m+120}{6}\right)\sum_{a=1}^{m}\left(\frac{\phi(v_a)+\phi(-v_a)}{2}\right),\end{aligned} \tag{7}$$

where m is a positive integer with $\mathbb{N}-\{0,1,2,3,4\}$. It is easy to see that the function $\phi(v)=av^4$ is a solution of the functional equation (7).

2. Orthogonal Hyers-Ulam Stability

Lemma 2 (see [32]. *Let E and F be real vector spaces. If the mapping $\phi:E\to F$ satisfies the functional equation (7) for all $v_1,v_2,\cdots,v_m\in E$ with $v_i\perp v_j$; $i\neq j=1,2,\cdots,m$, then ϕ is quartic.*

Remark 3. Let E be a linear space and $\phi:\mathbb{R}\to E$ be a function satisfies (7). Then, the following two assertions hold:

(1) $\phi(r^{k/4}v)=r^k\phi(v)$ for all $v\in\mathbb{R}$, $r\in\mathbb{Q}$ and k integers.

(2) $\phi(v)=v^4\phi(1)$ for all $v\in\mathbb{R}$ if ϕ is continuous.

Here, let us consider E to be a linear space over $\mathbb{K}$ and F is a (β,p)-Banach space with p-norm $\|\cdot\|_Y$.

Let K be the modulus concavity of $\|\cdot\|_F$.

For our convenience, we use the abbreviation for a function $\phi:E\to F$:

$$\begin{aligned}\Delta\phi(v_1,v_2,\cdots,v_m)&=\phi\left(\sum_{1\leq a\leq n}v_a\right)-\sum_{1\leq a<b<c<d\leq m}\phi(v_a+v_b+v_c+v_d)\\&-(-m+4)\sum_{1\leq a<b<c\leq m}\phi(v_a+v_b+v_c)\\&-\left(\frac{m^2-7m+12}{2}\right)\sum_{\substack{1=a;\\a\neq b}}^{m}\phi(v_a+v_b)\\&+\sum_{a=1}^{m}\phi(2v_a)-\left(\frac{-m^3+9m^2-26m+120}{6}\right)\\&\cdot\sum_{a=1}^{m}\left(\frac{\phi(v_a)+\phi(-v_a)}{2}\right),\end{aligned} \tag{8}$$

for all $v_1,v_2,\cdots,v_m\in E$.

Theorem 4. *Let a function $\phi : E \to F$ which there exists $\psi : E^m \to [0,\infty)$ such that*

$$\|\Delta\phi(v_1, v_2,\cdots,v_m)\|_F \le \psi(v_1, v_2,\cdots,v_m), v_1, v_2, \cdots, v_m \in E, \tag{9}$$

with $v_i \perp v_j$, $i \ne j = 1, 2, \cdots, m$, and the contractively subadditive function ψ and a constant L fulfilling $2^{(1-4\beta)}L < 1$. Then, there exists a unique mapping $Q_4 : E \to F$ which is quartic such that

$$\|\phi(v) - Q_4(v)\|_F \le \frac{K}{\sqrt[p]{2^{4\beta p} - (2L)^p}} \psi(v, 0,\cdots,0), \tag{10}$$

for all $v \in E$.

Proof. Setting $(v_1, v_2,\cdots,v_m)$ by $(v, 0,\cdots,0)$ in (9), we have

$$\left\|\phi(2v) - 2^4\phi(v)\right\|_F \le \psi(v, 0,\cdots,0), \tag{11}$$

for all $v \in E$. Replacing v in (11) by $2^m v$ and dividing by $2^{4(m+1)\beta}$ in (11) we attain

$$\left\|\frac{\phi(2^{m+1}v)}{2^{4(m+1)}} - \frac{\phi(2^m v)}{2^{4m}}\right\|_F \le \frac{K}{2^{4(m+1)\beta}} \psi(2^m v, 0,\cdots,0), \quad v \in E, m > 0. \tag{12}$$

We have

$$\begin{aligned}\left\|\frac{\phi(2^{m+1}v)}{2^{4(m+1)}} - \frac{\phi(2^i v)}{2^{4i}}\right\|_F^p &\le \sum_{a=i}^{m} \left\|\frac{\phi(2^{a+1}v)}{2^{4(a+1)}} - \frac{\phi(2^a v)}{2^{4a}}\right\|_F^p \\ &\le \sum_{a=i}^{m} \frac{K}{2^{4(a+1)\beta p}} \psi^p(2^a v, 0,\cdots,0) \\ &\le \frac{K\psi^p(v, 0,\cdots,0)}{2^{4p\beta}} \sum_{a=i}^{m} \left(2^{(1-4\beta)}L\right)^{ap},\end{aligned} \tag{13}$$

for all $v \in E$ and $m \ge i > 0$. Clearly, F is complete, the Cauchy sequence $\{\phi(2^m v)/2^{4m}\}$ converges for every $v \in E$. Next, we define a mapping $Q_4 : E \to F$ by

$$Q_4(v) := \lim_{m\to\infty} \frac{\phi(2^m v)}{2^{4m}}, \tag{14}$$

for all $v \in E$. Letting $i = 0$ and taking $m \to \infty$ in (13), we obtain (10). Next, we want to prove that Q_4 is quartic. From (9) and (14) that

$$\begin{aligned}\|\Delta Q_4(v_1, v_2,\cdots,v_m)\|_F^p &= \lim_{m\to\infty} \left\|\frac{\Delta\phi(2^m v_1, 2^m v_2,\cdots,2^m v_m)}{2^{4m}}\right\|_F^p \\ &\le \lim_{m\to\infty} \frac{K}{2^{4m\beta p}} \|\Delta\phi(2^m v_1, 2^m v_2,\cdots,2^m v_m)\|_F^p \\ &\le \lim_{m\to\infty} \frac{K}{2^{4m\beta p}} \psi^p(2^m v_1, 2^m v_2,\cdots,2^m v_m) \\ &\le \lim_{m\to\infty} \frac{K(2L)^{mp}}{2^{4m\beta p}} \psi^p(v_1, v_2,\cdots,v_m) \\ &\le \lim_{m\to\infty} \left(K2^{(1-4\beta)}L\right)^{mp} \psi^p(v_1, v_2,\cdots,v_m) = 0,\end{aligned} \tag{15}$$

for all $v_1, v_2, \cdots, v_m \in E$ with $v_i \perp v_j$, $i \ne j = 1, 2, \cdots, m$. Therefore, by Lemma 2, we conclude that Q_4 is quartic. Next, to show that the function Q_4 is unique.

Let us consider another quartic function $R_4 : E \to F$ which fulfils the inequality (10) we get

$$\begin{aligned}\|Q_4(v) - R_4(v)\| &= \lim_{m\to\infty} \frac{1}{2^{4m\beta}} \|\phi(2^m v) - R_4(2^m v)\|_F \\ &\le \lim_{m\to\infty} \frac{K\psi(2^m v, 0,\cdots,0)}{2^{4m\beta}\sqrt[p]{2^{4\beta p} - (2L)^p}} \\ &\le \lim_{m\to\infty} \frac{K\left(2^{(1-4\beta)}L\right)^m}{\sqrt[p]{2^{4\beta p} - (2L)^p}} \psi(v, 0,\cdots,0) = 0.\end{aligned} \tag{16}$$

This shows that $Q_4 = R_4$; therefore, Q_4 is unique mapping. This ends the proof of the theorem.

Corollary 5. *If $\beta = 1$ and τ be a positive real number and a function $\phi : E \to F$ for which*

$$\|\Delta\phi(v_1, v_2,\cdots,v_m)\|_F \le \tau, \tag{17}$$

for all $v_1, v_2, \cdots, v_m \in E$ with $v_i \perp v_j$, $i \ne j = 1, 2, \cdots, m$. Then, there exists $Q_4 : E \to F$which is a unique quartic mapping that fulfils

$$\|\phi(v) - Q_4(v)\|_F \le \frac{K\tau}{\sqrt[p]{2^{4p} - (2L)^p}}, \quad v \in E. \tag{18}$$

The following theorem is obtained by replacing the expansive superadditive instead of the contractive subadditive in Theorem 4.

Theorem 6. *Let a function $\phi : E \to F$ in which exists a mapping $\psi : E^m \to [0,\infty)$ such that*

$$\|\Delta\phi(v_1, v_2,\cdots,v_m)\|_F \le \psi(v_1, v_2,\cdots,v_m), \tag{19}$$

for all $v_1, v_2, \cdots, v_m \in E$ with $v_i \perp v_j$, $i \ne j = 1, 2, \cdots, m$, and the expansively superadditive function ψ and a constant L fulfilling $2^{(4\beta-1)}L < 1$. Then, there exists a unique mapping $Q_4 : E \to F$ which is quartic which fulfils

$$\|\phi(v)-Q_4(v)\|_F \le \frac{KL}{\sqrt[p]{2^p-(2^{4\beta}L)^p}}\psi(v,0,\cdots,0), \qquad (20)$$

for all $v \in E$.

With the upcoming theorems, we establish the stability of equation (7) by using an idea of Gavruta in [24].

Theorem 7. *Let a mapping $\psi : E^m \to [0,\infty)$ such that*

$$\lim_{m\to\infty}\frac{1}{2^{4m}}\psi(2^m v_1, 2^m v_2,\cdots,2^m v_m)=0, \qquad (21)$$

for all $v_1, v_2, \cdots, v_m \in E$ with $v_i \perp v_j$, $i \neq j = 1, 2, \cdots, m$, and

$$\tilde{\psi}_{Q_4}(v) := \sum_{a=0}^{\infty}\frac{K}{2^{4a\beta p}}\psi^p(2^a v, 0,\cdots,0) < \infty, \quad v \in E. \qquad (22)$$

If $\phi : E \to F$ is a mapping which fulfils

$$\|\Delta\phi(v_1, v_2,\cdots,v_m\|_F \le \psi(v_1, v_2,\cdots,v_m), \quad v_1, v_2, \cdots, v_m \in E, \qquad (23)$$

with $v_i \perp v_j$, $i \neq j = 1, 2, \cdots, m$, then there exists a unique mapping $Q_4 : E \to F$ which is quartic which satisfies

$$\|\phi(v)-Q_4(v)\|_F \le \frac{K}{2^{4\beta}}\left[\tilde{\psi}_{Q_4}(v)\right]^{1/p}, \qquad (24)$$

for all $v \in E$.

Proof. From equation (11) in Theorem 4, we get

$$\left\|\phi(2v)-2^4\phi(v)\right\| \le \psi(v,0,\cdots,0), \quad v \in E. \qquad (25)$$

Replacing v through $2^m v$ in inequality (25) and dividing by $2^{4\beta(m+1)}$, we obtain

$$\left\|\frac{\phi(2^{m+1}v)}{2^{4(m+1)}}-\frac{\phi(2^m v)}{2^{4m}}\right\|_F \le \frac{K}{2^{4\beta(m+1)}}\psi(2^m v, 0,\cdots,0), \quad v \in E, m > 0. \qquad (26)$$

Already, we know that F is a (β, p)-Banach space; we obtain

$$\begin{aligned}\left\|\frac{\phi(2^{m+1}v)}{2^{4(m+1)}}-\frac{\phi(2^i v)}{2^{4i}}\right\|_F^p &\le \sum_{a=i}^{m}\left\|\frac{\phi(2^{a+1}v)}{2^{4(a+1)}}-\frac{\phi(2^a v)}{2^{4a}}\right\|_F^p \\ &\le \frac{K}{2^{4\beta p}}\sum_{a=i}^{m}\frac{1}{2^{4a\beta p}}\psi^p(2^a v, 0,\cdots,0),\end{aligned} \qquad (27)$$

for all $v \in E$ with $m \ge i > 0$. From inequalities (22) and (27) that the sequence $\{\phi(2^m v)/2^{4m}\}$ is Cauchy in F for every $v \in E$. We know that if F is complete, the sequence $\{\phi(2^m v)/2^{4m}\}$ converges for every $v \in E$. Now, we can define a mapping $Q_4 : E \to F$ by

$$Q_4(v) := \lim_{m\to\infty}\frac{\phi(2^m v)}{2^{4m}}, \qquad (28)$$

for all $v \in E$. Letting $i = 0$ and taking $m \to \infty$ in (27), we obtain the result (24). The remaining proof is the same as the proof of Theorem 4.

Theorem 8. *Let $\psi : E^m \to [0,\infty)$ be a mapping such that*

$$\lim_{m\to\infty}2^{4m}\psi\left(\frac{v_1}{2^m}, \frac{v_2}{2^m},\cdots,\frac{v_m}{2^m}\right)=0, \quad v_1, v_2, \cdots, v_m \in E, \qquad (29)$$

with $v_i \perp v_j$, $i \neq j = 1, 2, \cdots, m$, and

$$\psi_{Q_4}(v, 0,\cdots,0) := \sum_{a=0}^{\infty}2^{4a\beta p}\psi^p\left(\frac{v}{2^{a+1}}, 0,\cdots,0\right) < \infty, \qquad (30)$$

for all $v \in E$. If $\phi : E \to F$ fulfils

$$\|\Delta\phi(v_1, v_2,\cdots,v_m)\|_F \le \psi(v_1, v_2,\cdots,v_m), \quad v_1, v_2, \cdots, v_m \in E, \qquad (31)$$

with $v_i \perp v_j$, $i \neq j = 1, 2, \cdots, m$. Then, there exists a unique function $Q_4 : E \to F$ which is quartic which fulfils

$$\|\phi(v)-Q_4(v)\|_F \le \frac{K}{2^{4\beta}}\psi^{1/p}(v, 0,\cdots,0), \quad v \in E. \qquad (32)$$

Proof. From equation (11), we get

$$\left\|\phi(2v)-2^4\phi(v)\right\| \le \psi(v,0,\cdots,0), \quad v \in E. \qquad (33)$$

Setting v by $v/2^{m+1}$ in (33) and multiply by $2^{4\beta m}$, we have

$$\left\|2^{4(m+1)}\phi\left(\frac{v}{2^{m+1}}\right)-2^{4m}\phi\left(\frac{v}{2^m}\right)\right\|_F \le 2^{4\beta m}\psi\left(\frac{v}{2^{m+1}}, 0,\cdots,0\right), \quad v \in E, m > 0, \qquad (34)$$

we have

$$\begin{aligned}\left\|2^{4(m+1)}\phi\left(\frac{v}{2^{m+1}}\right)-2^{4i}\phi\left(\frac{v}{2^i}\right)\right\|_F^p &\le \sum_{a=i}^{m}\left\|2^{4(a+1)}\phi\left(\frac{v}{2^{a+1}}\right)-2^{4a}\phi\left(\frac{v}{2^a}\right)\right\|_F^p \\ &\le \sum_{a=i}^{m}2^{4p\beta a}\psi^p\left(\frac{v}{2^{a+1}}, 0,\cdots,0\right), \quad v \in E, m \ge i > 0.\end{aligned} \qquad (35)$$

Then, we conclude from (42) and (34) that the sequence $\{2^{4m}\phi(v/2^m)\}$ is Cauchy in F for every $v \in E$.

As F is complete, the sequence $\{2^{4m}\phi(v/2^m)\}$ converges for every $v \in E$. Next, we define a mapping $Q_4 : E \to F$ by

$$Q_4(v) := \lim_{m\to\infty}2^{4m}\phi\left(\frac{v}{2^m}\right), \qquad (36)$$

for all $v \in E$. Letting $i = 0$ and taking $m \to \infty$ in (34), we obtain (32). The remaining proof is the same as the proof of Theorem 4.

Corollary 9. *Let s, t be the positive real numbers such that $s + t < 4\beta$ or $s + t > 4$. If a mapping $\phi : E \to F$ satisfies the inequality*

$$\|\Delta\phi(v_1, v_2, \cdots, v_m\|_F \leq \prod_{a=1}^{m} \|v_a\|_E^{(s+t)} + \sum_{a=1}^{m} \|v_a\|_E^{m(s+t)}, \quad (37)$$

for all $v_1, v_2, \cdots, v_m \in E$ with $v_i \perp v_j, i \neq j = 1, 2, \cdots, m$, then there exists a unique quartic mapping $Q_4 : E \to F$ which satisfies

$$\|\phi(v) - Q_4(v)\|_F \leq \frac{K\|v\|_E^{m(s+t)}}{\sqrt[p]{|2^{4\beta p} - 2^{(m(s+t))p}|}}, \quad (38)$$

for all $v \in E$.

Corollary 10. *Let s, t be the positive real numbers such that $s + t < 4\beta$ or $s + t > 4$. If a mapping $\phi : E \to F$ satisfies the inequality*

$$\|\Delta\phi(v_1, v_2, \cdots, v_m\|_F \leq \prod_{a=1}^{m} \|v_a\|_E^{(s+t)}, \quad (39)$$

for all $v_1, v_2, \cdots, v_m \in E$ with $v_i \perp v_j, i \neq j = 1, 2, \cdots, m$, then the mapping $\phi : E \to F$ is quartic.

3. Counterexample

Here, we proved the nonstability of equation (7) in a special condition by a counterexample which is a modified idea of Gajda [9].

Example 11. Let a mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(v) = \sum_{m=0}^{\infty} \frac{\chi(2^m v)}{2^{4m}}, \quad (40)$$

where

$$\chi(v) = \begin{pmatrix} \Theta v^4, & -1 < v < 1, \\ \Theta, & \text{otherwise}, \end{pmatrix} \quad (41)$$

then the function $\phi : \mathbb{R} \to \mathbb{R}$ fulfils

$$\|\Delta\phi(v_1, v_2, \cdots, v_m)\| \leq \left(\frac{-m^3 + 12m^2 - 53m + 198}{6}\right)\left(\frac{4096}{15}\right)\Theta\left(\sum_{a=1}^{m} |v_a|^4\right), \quad (42)$$

for all $v_1, v_2, \cdots, v_m \in \mathbb{R}$, but there does not exist a quartic mapping $Q_4 : \mathbb{R} \to \mathbb{R}$ such that

$$\|\phi(v) - Q_4(v)\| \leq \varepsilon|v|^4, \quad v \in \mathbb{R}, \quad (43)$$

where$\Theta$and$\varepsilon$are constants.

Proof. Clearly, ϕ is bounded by $(16/15)\theta$ on $\mathbb{R}$. If $\sum_{a=1}^{m} |v_a|^4 \geq 1/2^4$ or 0, then the left side of (29) is less than $((-m^3 + 12m^2 - 53m + 198)/6)(16/15)\theta$, and thus, (29) is true.

Next, we assume that

$$0 < \sum_{a=1}^{m} |v_a|^4 < \frac{1}{2^4}, \quad (44)$$

then there exists an integer i such that

$$\frac{1}{2^{4(i+2)}} \leq \sum_{a=1}^{m} |v_a|^4 < \frac{1}{2^{4(i+1)}}. \quad (45)$$

So that $2^i \mid v_1 \mid < 1/2, 2^i|v_2| < 1/2, \cdots, 2^i|v_m| < 1/2$ and $2^m v_1, 2^m v_2, \cdots, 2^m v_m \in (-1, 1)$ for every $m = 0, 1, 2, \cdots, i - 1$. For $m = 0, 1, \cdots, i - 1$,

$$\begin{aligned} \chi\left(\sum_{a=1}^{m} 2^m v_a\right) &- \sum_{1\leq a<b<c<d\leq m} \chi(2^m(v_a + v_b + v_c + v_d)) \\ &- (-m + 4) \sum_{1\leq a<b<c\leq m} \chi(2^m(v_a + v_b + v_c)) \\ &- \left(\frac{m^2 - 7m + 12}{2}\right) \sum_{\substack{1=a; \\ a\neq b}}^{m} \chi(2^m(v_a + v_b)) \\ &+ \sum_{a=1}^{m} \chi(2^m(2v_a)) - \left(\frac{-m^3 + 9m^2 - 26m + 120}{6}\right) \\ &\cdot \sum_{a=1}^{m} \left(\frac{\chi(2^m v_a) + \chi(-2^m v_a)}{2}\right) = 0. \end{aligned} \quad (46)$$

Next,

$$\begin{aligned} |\Delta\phi(v_1, v_2, \cdots, v_m)| &\leq \sum_{a=i}^{\infty} \frac{1}{2^{4a}} |\Delta\chi(2^a v_1, 2^a v_2, \cdots, 2^a v_m)| \\ &\leq \sum_{a=i}^{\infty} \frac{1}{2^{4a}} |\chi\left(\sum_{a=1}^{m} 2^m v_a\right) \\ &- \sum_{1\leq a<b<c<d\leq m} \chi(2^m(v_a + v_b + v_c + v_d)) \\ &- (-m + 4) \sum_{1\leq a<b<c\leq m} \chi(2^m(v_a + v_b + v_c)) \\ &- \left(\frac{m^2 - 7m + 12}{2}\right) \sum_{\substack{1=a; \\ a\neq b}}^{m} \chi(2^m(v_a + v_b)) \\ &+ \sum_{a=1}^{m} \chi(2^m(2v_a)) - \left(\frac{-m^3 + 9m^2 - 26m + 120}{6}\right) \end{aligned}$$

$$\begin{aligned}&\sum_{a=1}^{m}\left(\frac{\chi(2^m v_a)+\chi(-2^m v_a)}{2}\right)| \\ &\quad\le \sum_{a=i}^{\infty}\frac{1}{2^{4a}}\left(\frac{-m^3+12m^2-53m+198}{6}\right)\Theta \\ &\quad\le \left(\frac{-m^3+12m^2-53m+198}{6}\right)\frac{2^{4(1-i)}}{15}\Theta.\end{aligned} \tag{47}$$

It follows from (43) that

$$|\Delta\phi(v_1, v_2,\cdots,v_m)| \le \left(\frac{-m^3+12m^2-53m+198}{6}\right)\frac{4096}{15}\Theta\left(\sum_{a=1}^{m}|v_a|^4\right), \tag{48}$$

for all $v_1, v_2, \cdots, v_m \in \mathbb{R}$. Thus, ϕ satisfies (29) for all $v_1, v_2, \cdots, v_m \in \mathbb{R}$ with $v_i \perp v_j$, $i \neq j = 1, 2, \cdots, m$.

Assume that there is a contrary mapping$Q_4 : \mathbb{R} \to \mathbb{R}$ which is quartic which fulfils (42). We know that, for every $v \in \mathbb{R}$, ϕ is bounded and continuous and Q_4 is bounded on any open interval containing the origin which is continuous at the origin.

In the view of Remark 3, Q_4 must be $Q_4(v) = av^4$, $v \in \mathbb{R}$. Thus, we have

$$|\phi(v)| \le (\varepsilon+|a\,|)|v|^4, \quad v \in \mathbb{R}. \tag{49}$$

But we can select an integer $i > 0$ with $i\Theta > \varepsilon + |a\,|$. If $v \in (0, 1/2^{i-1})$, then $2^m v \in (0, 1)$ for any $m = 0, 1, \cdots, i-1$, and for v, we obtain

$$\phi(v) = \sum_{m=0}^{\infty}\frac{\chi(2^m v)}{2^{4m}} \ge \sum_{m=0}^{i-1}\frac{\Theta(2^m v)^4}{2^{4m}} = i\Theta v^4 > (\varepsilon+|a\,|)v^4, \tag{50}$$

which contradicts.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version of the manuscript.

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Research Article

Analytical Solution for Differential and Nonlinear Integral Equations via F_{ϖ_e}-Suzuki Contractions in Modified ϖ_e-Metric-Like Spaces

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The aim of this manuscript is to present a new space, namely, a modified ϖ_e-metric-like space, and we establish some related fixed point results using extended F_{ϖ_e}-Suzuki and generalized F_{ϖ_e}-Suzuki contractions on the mentioned space. Here, we support our theoretical consequences in two ways: the first one consists of presenting illustrative examples and the second one consists of finding analytical solutions for some integral and differential equations in the context of the mentioned space.

1. Introduction and Elementary Discussions

Solving second-order differential equations after converting to integral equations using Green's function has become commonplace for many academic researchers because of its importance in theoretical and practical applications. One aspect of the solution of the differential and integral equations is the analytical solution, which in turn mainly supports the numerical solution used in solving dynamical systems.

There are many methods for obtaining analytical solutions, including the fixed point technique. For instance, many works have focused their attention on solving the Fredholm integral equation [1] analytically and numerically by this technique [2, 3].

Among the generalizations of the Banach principle [4], the notion of F-contractions was initiated by Wardkowski [5].

After two years, Piri and Kuman [6] made a slight change in the principle of Wardkowski and called it an F-Suzuki contraction. It has contributed significantly to upholding the reputation of the fixed point theory in many fields and has become a great weight in the functional analysis.

Definition 1. A mapping $A : \daleth \to \daleth$ defined on the metric space $(\daleth, d)$ is named as an

(i) F-contraction if there are $F \in \Pi$ and $\vartheta > 0$ so that

$$d(A\ell, A\hbar) > 0 \Rightarrow \vartheta + F(d(A\ell, A\hbar)) \le F(d(\ell, \hbar)) \text{ for all } \ell, \hbar \in \daleth \tag{1}$$

(ii) F-Suzuki contraction if there are $F \in \Pi$ and $\vartheta > 0$ so that

$$\frac{1}{2}d(\ell, A\ell) < d(\ell, \hbar) \Rightarrow \vartheta + F(d(A\ell, A\hbar)) \leq F(d(\ell, \hbar)) \text{ for all } \ell, \hbar \in \daleth, \tag{2}$$

where Σ is the class of functions $F : (0,+\infty) \to \mathbb{R}$ so that

$(F_1)$for all $\mho, \Omega \in \mathbb{R}^+$ such that $\mho<\Omega, F(\mho) < F(\Omega)$

$(F_2)$for each positive real sequence $\{\mho_p\}$, $\lim_{p\to\infty}\mho_p = 0$ iff $\lim_{n\to\infty}F(\mho_p) = -\infty$

(F_3)there is $\omega \in (0, 1)$ so that $\lim_{\mho\to 0^+}\mho^{\omega}F(\mho) = 0$

There are developments to translate fixed point theorems into nonlinear integral equations and differential equations (for related works and developments, see [7–20]).

The nonlinear mapping in Banach contraction needs to be continuous. It is not applicable in the discontinuous case. In the past, Kannan [21]. was able to overcome this shortcoming by giving a fixed point result without the mapping being continuous. Variant works appeared to resolve this problem by adding conditions to the spaces (see [22–25]).

Among modern spaces, a b-metric-like space was introduced by Alghmandi et al. [26] as an extension of a b-metric, which was presented by Bakhtin [27], and a metric-like, which was presented by Amini-Harandi [28]. In [26], some fixed point theorems have been provided. In recent works, many contributions on fixed point results involving different contractive conditions are given (see [29–36]).

At the beginning of 2019, Parvaneh and Kadelburg [37] generalized the b-metric-like space by replacing the coefficient located in the third condition by a strictly increasing continuous function. They named it an extended b-metric-like space and studied on it some fixed point sequences for JSHR-contractive type mappings with some applications.

It is noted that in this section, we did not address definitions and mathematical theorems for two reasons: first, there are large basics related to the mentioned spaces, and second, access to the main results is direct.

According to the previous results, in this paper, we present fixed point consequences by using F_{ϖ_e}-Suzuki contractions in the class of modified ϖ_e-metric-like spaces. Under the framework of the mentioned space, we apply the theoretical results to find an analytical solution for nonlinear integral equations. On the other hand, some important examples to justify our theorems are discussed.

2. An Extended F_{ϖ_e}-Suzuki Contraction

We begin this section with definitions of metric-like and b-metric-like spaces.

Definition 2 (see [26]). Let $\daleth$ be a nonempty set. A function $\Delta : \daleth^2 \to \mathbb{R}^+$ is named as metric-like on $\daleth$, if for all $\mho_1, \mho_2, \mho_3 \in \daleth$:

$(\daleth_1)\Delta(\mho_1, \mho_2) = 0 \Rightarrow \mho_1 = \mho_2$

$(\daleth_2)\Delta(\mho_1, \mho_2) = \Delta(\mho_2, \mho_1)$

$(\daleth_3)\Delta(\mho_1, \mho_3) \leq \Delta(\mho_1, \mho_2) + \Delta(\mho_2, \mho_3)$

Definition 3 (see [26]). A b -metric-like on a nonempty set $\daleth$ is a function $\varpi : \daleth^2 \to \mathbb{R}^+$ so that for all $\mho_1, \mho_2, \mho_3 \in \daleth$ and a constant $s \geq 1$:

$(\varpi_1)\varpi(\mho_1, \mho_2) = 0 \Rightarrow \mho_1 = \mho_2$

$(\varpi_2)\varpi(\mho_1, \mho_2) = \varpi(\mho_2, \mho_1)$

$(\varpi_3)\varpi(\mho_1, \mho_3) \leq s[\varpi(\mho_1, \mho_2) + \varpi(\mho_2, \mho_3)]$

Here, $(\daleth, \varpi)$ is named as a b-metric-like space (with constant s).

For examples about metric-like and b-metric-like spaces, see [33–35].

Now, we will generalize Definition 3 as follows.

Definition 4. Let $\daleth$ be a nonempty set and $s : \daleth \times \daleth \to [1,\infty)$. A function $\varpi_e : \daleth^2 \to [0,\infty)$ is called a modified ϖ_e -metric-like if, for all $\mho_1, \mho_2, \mho_3 \in \daleth$:

$(\varpi_{e1})\varpi_e(\mho_1, \mho_2) = 0 \Rightarrow \mho_1 = \mho_2$

$(\varpi_{e2})\varpi_e(\mho_1, \mho_2) = \varpi_e(\mho_2, \mho_1)$

$(\varpi_{e3})\varpi_e(\mho_1, \mho_3) \leq s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]$

Here, $(\daleth, \varpi_e)$ is named as a modified extended b-metric-like space (simply, a modified ϖ_e-metric-like space).

Note that the class of modified ϖ_e-metric-like spaces is larger than the class of b-metric-like spaces by replacing the constant $s \geq 1$ of Definition 3 by a nonconstant function $s : \daleth \times \daleth \to [1,\infty)$ of Definition 4.

Example 5. Let $\daleth = [0,\infty)$. Define $\varpi_e : \daleth^2 \to [0,\infty)$ by

$$\varpi_e(\kappa, \mu) = \begin{cases} 0, & \text{if } \kappa = \mu = 0, \\ \dfrac{\mu}{1+\mu}, & \text{if } \kappa = 0,\ \mu \neq 0, \\ \dfrac{\kappa}{1+\kappa}, & \text{if } \mu = 0,\ \kappa \neq 0, \\ \kappa + \mu, & \text{if } \kappa \neq 0,\ \mu \neq 0. \end{cases} \tag{3}$$

Consider $s : \daleth^2 \to [1,\infty)$ as $s(\kappa, \mu) = 2 + 2\kappa + 2\mu$.

First, (ϖ_{e1}) and (ϖ_{e2}) are obvious. We need to prove (ϖ_{e3}) . For this, let $\mho_1, \mho_2$, and $\mho_3$ in $\daleth$. We state the following cases.

Case 1. $\mho_1 = \mho_3 = 0$. Here, (ϖ_{e3}) holds.

Case 2. $\mho_1 = 0$ and $\mho_3 \neq 0$. Then,

$$\begin{aligned} \varpi_e(\mho_1, \mho_3) &= \frac{\mho_3}{1+\mho_3}, \\ s(\mho_1, \mho_3) &= 2 + 2\mho_3. \end{aligned} \tag{4}$$

Subcase 1. $\mho_2 = 0$. We have

$$\begin{aligned} \varpi_e(\mho_1, \mho_3) = \frac{\mho_3}{1+\mho_3} &\leq (2+2\mho_3)\left[0 + \frac{\mho_3}{1+\mho_3}\right] \\ &= s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]. \end{aligned} \tag{5}$$

Subcase 2. $\mho_2 \neq 0$. We have

$$\begin{aligned}\varpi_e(\mho_1,\mho_3) = \frac{\mho_3}{1+\mho_3} &\leq (2+2\mho_3)\left[\frac{\mho_2}{1+\mho_2}+\mho_2+\mho_3\right] \\ &= s(\mho_1,\mho_3)[\varpi_e(\mho_1,\mho_2)+\varpi_e(\mho_2,\mho_3)].\end{aligned} \tag{6}$$

Case 3. $\mho_3 = 0$ and $\mho_1 \neq 0$. Proceeding similarly as in Case 2, (ϖ_{e3}) holds.

Case 4. $\mho_1 \neq 0$ and $\mho_3 \neq 0$. Then,

$$\begin{aligned}\varpi_e(\mho_1,\mho_3) &= \mho_1+\mho_3, \\ s(\mho_1,\mho_3) &= 2+2\mho_1+2\mho_3.\end{aligned} \tag{7}$$

Subcase 1. $\mho_2 = 0$. We have

$$\begin{aligned}\varpi_e(\mho_1,\mho_3) = \mho_1+\mho_3 &\leq (2+2\mho_1+2\mho_3)\left[\frac{\mho_1}{1+\mho_1}+\frac{\mho_3}{1+\mho_3}\right] \\ &= s(\mho_1,\mho_3)[\varpi_e(\mho_1,\mho_2)+\varpi_e(\mho_2,\mho_3)].\end{aligned} \tag{8}$$

Subcase 2. $\mho_2 \neq 0$. We have

$$\begin{aligned}\varpi_e(\mho_1,\mho_3) = \mho_1+\mho_3 &\leq (2+2\mho_1+2\mho_3)[\mho_1+\mho_2+\mho_2+\mho_3] \\ &= s(\mho_1,\mho_3)[\varpi_e(\mho_1,\mho_2)+\varpi_e(\mho_2,\mho_3)].\end{aligned} \tag{9}$$

On the other hand, $(\daleth, \varpi_e)$ is not a b-metric-like space. We argue by contradiction by assuming that $(\daleth, \varpi_e)$ is a b-metric-like space with a coefficient $s \geq 1$ (a constant). Then, for any real $\mu > 0$, we have

$$\varpi_e(\mu,\mu+1) \leq s[\varpi_e(\mu,0)+\varpi_e(0,\mu+1)]. \tag{10}$$

That is,

$$2\mu+1 \leq s\left[\frac{\mu}{1+\mu}+\frac{\mu+1}{2+\mu}\right]. \tag{11}$$

Letting $\mu \to \infty$, we get $+\infty \leq 2s$, which is a contradiction.

Example 6. Let $\daleth = \{0,1,2\}$. Define $\varpi_e : \daleth^2 \to [0,\infty)$ and $s : \daleth \times \daleth \to [1,\infty)$ as follows:

$$\begin{aligned}\varpi_e(0,0) = \varpi_e(1,1) = \varpi_e(2,2) &= 0, \\ \varpi_e(0,1) = \varpi_e(1,0) &= 12, \\ \varpi_e(0,2) = \varpi_e(2,0) &= 1, \\ \varpi_e(1,2) = \varpi_e(2,1) &= 3,\end{aligned} \tag{12}$$

and $s(\kappa,\mu) = 2+\kappa+\mu$.

First, we show that ϖ_e is a modified ϖ_e-metric-like space. Trivially, the conditions (ϖ_{e1}) and (ϖ_{e2}) hold. For (ϖ_{e3}), we get

$$\varpi_e(0,1) = 12\,;\, s(0,1)[\varpi_e(0,2)+\varpi_e(2,1)] = 12. \tag{13}$$

Thus,

$$\varpi_e(0,1) \leq s(0,1)[\varpi_e(0,2)+\varpi_e(2,1)]. \tag{14}$$

Again,

$$\begin{aligned}\varpi_e(1,2) = 3\,;\, s(1,2)[\varpi_e(1,0)+\varpi_e(0,2)] &= 65, \\ \varpi_e(0,2) = 1\,;\, s(0,2)[\varpi_e(0,1)+\varpi_e(1,2)] &= 60.\end{aligned} \tag{15}$$

Hence, for all $\mho_1,\mho_2,\mho_3 \in \daleth$, $\varpi_e(\mho_1,\mho_3) \leq s(\mho_1,\mho_3)[\varpi_e(\mho_1,\mho_2)+\varpi_e(\mho_2,\mho_3)]$. Then, $(\daleth,\varpi_e)$ is a modified ϖ_e-metric-like space, but it is not a b-metric-like space because if we take $s = 2$ in the inequality (13), we get

$$\varpi_e(0,1) = 12\,;\, 2[\varpi_e(0,2)+\varpi_e(2,1)] = 8. \tag{16}$$

Thus,

$$\varpi_e(\mho_1,\mho_3) \nleq s[\varpi_e(\mho_1,\mho_2)+\varpi_e(\mho_2,\mho_3)]. \tag{17}$$

Definition 7. Let $\{\mho_\iota\}$ be a sequence in the modified ϖ_e-metric-like space $(\daleth,\varpi_e)$.

(a) If $\lim_{\iota\to\infty}\varpi_e(\mho_\iota,\mho) = \varpi_e(\mho,\mho)$, then $\{\mho_\iota\}$ is convergent to $\mho$

(b) $\{\mho_\iota\}$ is called Cauchy if $\lim_{\iota,j\to\infty}\varpi_e(\mho_\iota,\mho_j)$ exists and is finite

(c) If for each Cauchy sequence $\{\mho_\iota\}$, there is $\mho\in\daleth$, so that $\lim_{\iota,j\to\infty}\varpi_e(\mho_\iota,\mho_j) = \varpi_e(\mho,\mho) = \lim_{\iota\to\infty}\varpi_e(\mho_\iota,\mho)$; therefore, $(\daleth,\varpi_e)$ is said to be complete

Definition 8. A nonlinear self-mapping A on a modified ϖ_e-metric-like space $(\daleth,\varpi_e)$ is named as an extended F_{ϖ_e}-Suzuki contraction if there are $F_{\varpi_e} \in \Pi$ *and* $\vartheta > 0$ so that for $\mho,\mu \in \daleth$, the following condition holds:

$$\begin{aligned}\frac{1}{2}\varpi_e(\mho,A\mho) < \varpi_e(\mho,\mu) \Rightarrow \vartheta + F_{\varpi_e}(\varpi_e(A\mho,A\mu)) \\ \leq F_{\varpi_e}(\varpi_e(\mho,\mu)),\end{aligned} \tag{18}$$

such that $\lim_{n,m\to\infty}s(\mho_\iota,\mho_j) < 1/\eta$ for all $\mho_\circ \in \daleth$, where $0 < \eta < 1$. We consider here $\mho_\iota = A^\iota\mho_\circ$, $\iota = 1,2,\cdots$, where Π is the set of continuous functions $F_{\varpi_e} : \mathbb{R}^+ \to \mathbb{R}$ so that

$(\mathfrak{I}_1)$For all $j,\ell \in \mathbb{R}^+$ with $j < \ell$,$F_{\varpi_e}(j) < F_{\varpi_e}(\ell)$

$(\mathfrak{I}_2)$For each positive real sequence $\{j_p\}$, $\lim_{p\to\infty}j_p = 0$ iff $\lim_{p\to\infty}F_{\varpi_e}(j_p) = -\infty$

$(\mathfrak{I}_3)$There is $\eta \in (0,1)$ so that $\lim_{j\to 0^+}j_\eta F_{\varpi_e}(j) = 0$

Now, we introduce our first theorem.

Theorem 9. *Let $(\daleth, \varpi_e)$ be a complete modified ϖ_e -metric-like space and A be an extended F_{ϖ_e} -Suzuki contraction mapping, then A admits a unique fixed point.*

Proof. Let $\mho_\circ \in \daleth$ and $\{\mho_\iota\}_{\iota=1}^{\infty}$ defined by $\mho_{\iota+1} = A\mho_\iota = A^{\iota+1}$ $\mho_\circ$. If there is $\iota \in \mathbb{N}$ so that $\varpi_e(\mho_\iota, A\mho_\iota) = 0$. It completes the proof. Otherwise, assume that $0 < \varpi_e(A_\iota, \Gamma\mho_\iota) = \varpi_e(\mho_\iota, \mho_{\iota+1}) = \varpi_e^\iota$; therefore, for all $\iota \in \mathbb{N}$,

$$\frac{1}{2}\varpi_e(\mho_\iota, A\mho_\iota) < \varpi_e(\mho_\iota, A\mho_\iota), \tag{19}$$

it yields or

$$\vartheta + F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho_\iota\right)\right) \le F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho_\iota)), \tag{20}$$

$$F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho_\iota\right)\right) \le F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho_\iota)) - \vartheta. \tag{21}$$

By the same method, one gets

$$\begin{aligned} F_{\varpi_e}(\varpi_e^\iota) &= F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho_\iota)) = F_{\varpi_e}\left(\varpi_e\left(A\mho_{\iota-1}, A^2\mho_{\iota-1}\right)\right) \\ &\le F_{\varpi_e}(\varpi_e(\mho_{\iota-1}, A\mho_{\iota-1})) - \vartheta \le F_{\varpi_e}(\varpi_e(\mho_{\iota-2}, A\mho_{\iota-2})) \\ &\quad - 2\vartheta \vdots \le F_{\varpi_e}(\varpi_e(\mho_\circ, A\mho_\circ)) - \iota\vartheta \text{ for all } \iota \ge 1. \end{aligned} \tag{22}$$

Taking $\iota \to \infty$ in (22), we have

$$\lim_{\iota\to\infty} F_{\varpi_e}(\varpi_e^\iota) = -\infty. \tag{23}$$

So, by $(\mathfrak{I}_2)$, we obtain

$$\lim_{\iota\to\infty} \varpi_e^\iota = 0. \tag{24}$$

Applying $(\mathfrak{I}_3)$, there is $\eta \in (0, 1)$ so that

$$\lim_{\iota\to\infty} (\varpi_e^\iota)^\eta F_{\varpi_e}(\varpi_e^\iota) = 0. \tag{25}$$

By (22), one writes for all $\iota \ge 1$,

$$(\varpi_e^\iota)^\eta\left(F_{\varpi_e}(\varpi_e^\iota) - F_{\varpi_e}(\varpi_e^\circ)\right) \le -\iota\vartheta(\varpi_e^\iota)^\eta \le 0. \tag{26}$$

Considering (24) and (25) and passing $\iota \to \infty$ in (26), one gets

$$\lim_{\iota\to\infty} \iota(\varpi_e^\iota)^\eta = 0. \tag{27}$$

By (27), there exists $\iota_1 \in \mathbb{N}$ so that $\iota(\varpi_e^\iota)^\eta \le 1$ for all $\iota \ge \iota_1$, or

$$\varpi_e^\iota \le \frac{1}{\iota^{1/\eta}} \text{ for all } \iota \ge \iota_1. \tag{28}$$

Consider the integers $m > \iota$. Applying (ϖ_3) and (28), one writes

$$\begin{aligned} \varpi_e(\mho_\iota, \mho_m) &\le s(\mho_\iota, \mho_m)[\varpi_e(\mho_\iota, \mho_{\iota+1}) + \varpi_e(\mho_{\iota+1}, \mho_m)] \\ &\le s(\mho_\iota, \mho_m)\varpi_e(\mho_\iota, \mho_{\iota+1}) + s(\mho_\iota, \mho_m)s(\mho_{\iota+1}, \mho_m) \\ &\quad \cdot [\varpi_e(\mho_{\iota+1}, \mho_{\iota+2}) + \varpi_e(\mho_{\iota+2}, \mho_m)] \\ &\le s(\mho_\iota, \mho_m)\varpi_e(\mho_\iota, \mho_{\iota+1}) + s(\mho_\iota, \mho_m)s(\mho_{\iota+1}, \mho_m) \\ &\quad \cdot \varpi_e(\mho_{\iota+1}, \mho_{\iota+2}) + \cdots + s(\mho_\iota, \mho_m)s(\mho_{\iota+1}, \mho_m)s \\ &\quad \cdot (\mho_{\iota+2}, \mho_m) \cdots s(\mho_{m-2}, \mho_m)s(\mho_{m-1}, \mho_m) \\ &\quad \cdot \varpi_e(\mho_{m-1}, \mho_m) \le s(\mho_1, \mho_m)s(\mho_2, \mho_m) \cdots s(\mho_\iota, \mho_m) \\ &\quad \cdot \varpi_e(\mho_\iota, \mho_{\iota+1}) + s(\mho_1, \mho_m)s(\mho_2, \mho_m) \cdots s(\mho_{\iota+1}, \mho_m) \\ &\quad \cdot \varpi_e(\mho_{\iota+1}, \mho_{\iota+2}) + \cdots + s(\mho_1, \mho_m)s(\mho_2, \mho_m) \cdots \\ &\quad \cdot s(\mho_{m-1}, \mho_m)\varpi_e(\mho_{m-1}, \mho_m) = \sum_{\iota=1}^{\infty} \varpi_e(\mho_\iota, \mho_{\iota+1}) \\ &\quad \cdot \prod_{j=1}^{\iota} s(\mho_j, \mho_m) \le \sum_{\iota=1}^{\infty} \frac{1}{\iota^{1/\eta}} \prod_{j=1}^{\iota} s(\mho_j, \mho_m) \\ &\le \sum_{\iota=1}^{\infty} \frac{1}{\iota^{1/\eta}} \frac{1}{\eta} = \frac{1}{\eta} \sum_{\iota=1}^{\infty} \frac{1}{\iota^{1/\eta}}. \end{aligned} \tag{29}$$

Recall that $\sum_{\iota=1}^{\infty} 1/\iota^{1/\eta}$ converges, so $\varpi_e(\mho_\iota, \mho_m) \to 0$. Therefore, $\{\mho_\iota\}$ is a Cauchy sequence in the complete modified ϖ_e-metric-like space $(\daleth, \varpi_e)$; hence, there is $\mho^* \in \daleth$ such that $\mho_\iota \to \mho^*$, as $\iota \to \infty$. That is,

$$\lim_{\iota,m\to\infty} \varpi_e(\mho_\iota, \mho_m) = \lim_{\iota\to\infty} \varpi_e(\mho_\iota, \mho^*) = \varpi_e(\mho^*, \mho^*) = 0. \tag{30}$$

Next, if F_{ϖ_e} is continuous, then two short cases arise.

Case 1. For each $\iota \in \mathbb{N}$, there exists $j_\iota \in \mathbb{N}$ such that $\mho_{j_\iota} = A\mho^*$ and $j_\iota > j_{\iota-1}$, where $j_\circ = 0$. Therefore, one gets $\mho^* = \lim_{\iota\to\infty}$ $\mho_{j_\iota} = \lim_{\iota\to\infty} A\mho^* = A\mho^*$.

Case 2. There is $\iota_\circ \in \mathbb{N}$ so that for all $\iota \ge \iota_\circ, \mho_\iota \ne A\mho^*$. It is clear that $1/2\varpi_e(\mho_\iota, A\mho^*) < \varpi_e(\mho_\iota, A\mho^*)$ for all $\iota \ge \iota_\circ$.

By (18), we have

$$\begin{aligned} \vartheta + F_{\varpi_e}\left(\varpi_e\left(\mho_{\iota+1}, A^2\mho^*\right)\right) &= \vartheta + F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho^*\right)\right) \\ &\le F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho^*)). \end{aligned} \tag{31}$$

Since F_{ϖ_e} is continuous, we obtain at the limit $\iota \to \infty$,or

$$\vartheta + F_{\varpi_e}\left(\varpi_e\left(\mho^*, A^2\mho^*\right)\right) \le F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)), \tag{32}$$

$$F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)) \le F_{\varpi_e}(\varpi_e(\mho^*, \mho^*)) - \vartheta, \tag{33}$$

which is a contradiction due to $(\mathfrak{I}_1)$. Then, $\varpi_e(\mho^*, A\mho^*) = 0$, which means that $\mho^* = A\mho^*$.

The two cases above lead to the existence of a fixed point of A, i.e., $\mho^* = A\mho^*$.

Now, assume that $\mho_1^*$ and $\mho_2^*$ are so that $\mho_1^* = A\mho_1^* \ne \mho_2^* = A\mho_2^*$. We have $1/2\varpi_e(\mho_1^*, \mho_2^*) < \varpi_e(\mho_1^*, \mho_2^*)$, which implies by (18) that

$$\vartheta + F_{\varpi_e}(\varpi_e(\mho_1^*, \mho_2^*)) = \vartheta + F_{\varpi_e}(\varpi_e(A\mho_1^*, A\mho_2^*)) \le F_{\varpi_e}(\varpi_e(\mho_1^*, \mho_2^*)). \tag{34}$$

It is again a contradiction.

The following examples verify all required hypotheses of Theorem 9.

Example 10. Let $\daleth = [0,\infty)$. Define $\varpi_e : \daleth^2 \to \mathbb{R}$ *by* $\varpi_e(\mho, \aleph) = (\mho + \aleph)^2$ and $s : \daleth \times \daleth \to [1,\infty)$ by $s(\mho, \aleph) = 1 + \mho + \aleph$, for all $\mho, \aleph \in \daleth$. Here, ϖ_e is a modified extended ϖ_e -metric-like space. Define $A : \daleth \to \daleth$ *as* $A\mho = (1/3)\mho$, for all $\mho \in \daleth$. It is clear that

$$\begin{aligned}\frac{1}{2}\varpi_e(\mho, A\mho) &= \frac{1}{2}\varpi_e\left(\mho, \frac{1}{3}\mho\right) = \frac{1}{2}\left(\mho + \frac{1}{3}\mho\right)^2 \\ &= \frac{16}{18}\mho^2 \le \mho^2 \le (\mho + \mu)^2 = \varpi_e(\mho, \mu).\end{aligned} \tag{35}$$

Consider, for all $\mho, \mu \in \daleth$,

$$\begin{aligned}F_{\varpi_e}(\varpi_e(A\mho, A\mu)) &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3}\mho, \frac{1}{3}\mu\right)\right) = F_{\varpi_e}\left(\left(\frac{1}{3}\mho + \frac{1}{3}\mu\right)^2\right) \\ &= F_{\varpi_e}\left(\frac{1}{9}(\mho + \mu)^2\right).\end{aligned} \tag{36}$$

Also,

$$F_{\varpi_e}(\varpi_e(\mho, \mu)) = F_{\varpi_e}\left((\mho + \mu)^2\right). \tag{37}$$

Let the function $F_{\varpi_e} \in \Pi$ be defined by $F_{\varpi_e}(\ell) = \ln(\ell)$, for $\ell > 0$. Then,

$$\begin{aligned}&F_{\varpi_e}(\varpi_e(A\mho, A\mu)) - F_{\varpi_e}(\varpi_e(\mho, \mu)) \\ &\quad = \ln\left(\frac{1}{9}(\mho + \mu)^2\right) - \ln\left((\mho + \mu)^2\right) = \ln\left(\frac{1/9(\mho + \mu)^2}{(\mho + \mu)^2}\right) \\ &\quad = \ln\left(\frac{1}{9}\right) = -2.197 \le -2.\end{aligned} \tag{38}$$

Therefore, A is an extended F_{ϖ_e}-Suzuki contraction mapping with $\vartheta = 2$. Moreover, if $\mho_m = \{1/(m+1)\} \in \daleth$, we have

$$\lim_{i,m\to\infty} s(\mho_m, \mho_i) = \lim_{i,m\to\infty}\left(1 + \frac{1}{m+1} + \frac{1}{i+1}\right) = 1 < \frac{1}{\eta}, \tag{39}$$

for $\eta \in (0,1)$. So, all hypotheses of Theorem 9 are satisfied, and A has 0 as a unique fixed point.

Example 11. Let $\daleth = \{1/3^{\iota-1} : \iota \in \mathbb{N}\} \cup \{0\}$. Suppose that $\varpi_e : \daleth \times \daleth \to 0,\infty)$ and $s : \daleth \times \daleth \to [1,\infty)$ are functions defined by $\varpi_e(\mho, \mu) = (\max\{\mho, \mu\})^2$ *and* $s(\mho, \mu) = 1 + \mu + \mho$, respectively, for all $\mu, \mho \in \daleth$. Then, the pair $(\daleth, \varpi_e)$ is a complete modified ϖ_e -metric-like space. Define a nonlinear mapping $A : \daleth \to \daleth$ *by*

$$A\mho = \begin{cases}\left\{\dfrac{1}{3^{2\iota}}\right\}, & \text{if}\, \mho \in \left\{\dfrac{1}{3^{2\iota-1}}\, ; \iota \in \mathbb{N}\right\}, \\ 0, & \text{if}\, \mho = 0.\end{cases} \tag{40}$$

We shall prove that a mapping A is an extended F_{ϖ_e} -Suzuki contraction with $F_{\varpi_e}(\ell) = \ln(\ell)$ for $\ell > 0$ and $\vartheta > 0$, by showing the following cases.

Case 1. Let $\mho = 1/3^{2\iota-1}$ and $\mu = 1/3^{2m-1}$, for $m > \iota \ge 1$, one can write

$$\begin{aligned}\frac{1}{2}\varpi_e(\mho, A\mho) &= \frac{1}{2}\varpi_e\left(\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2\iota}}\right) = \frac{1}{2}\left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2\iota}}\right\}\right)^2 \\ &= \frac{1}{2}\left(\frac{1}{3^{2\iota-1}}\right)^2 < \left(\frac{1}{3^{2\iota-1}}\right)^2 \\ &= \left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2m-1}}\right\}\right)^2 = \varpi_e(\mho, \mu).\end{aligned} \tag{41}$$

Consider

$$\begin{aligned}F_{\varpi_e}(\varpi_e(A\mho, A\mu)) &= F_{\varpi_e}\left(\varpi_e\left(A\frac{1}{3^{2\iota-1}}, A\frac{1}{3^{2m-1}}\right)\right) \\ &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota}}, \frac{1}{3^{2m}}\right)\right) \\ &= F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota}}, \frac{1}{3^{2m}}\right\}\right)^2\right) \\ &= F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota}}\right)^2\right) = \ln\left(\frac{1}{3^{2\iota}}\right)^2 = 2\ln\left(\frac{1}{3^{2\iota}}\right).\end{aligned} \tag{42}$$

Also,

$$\begin{aligned}F_{\varpi_e}(\varpi_e(\mho, \mu)) &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2m-1}}\right)\right) \\ &= F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2m-1}}\right\}\right)^2\right) \\ &= F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota-1}}\right)^2\right) = 2\ln\left(\frac{1}{3^{2\iota-1}}\right).\end{aligned} \tag{43}$$

By subtracting (42) and (43), we find that

$$\begin{aligned}&F_{\varpi_e}(\varpi_e(A\mho, A\mu)) - F_{\varpi_e}(\varpi_e(\mho, \mu)) \\ &\quad = 2\left(\ln\left(\frac{1}{3^{2\iota}}\right) - \ln\left(\frac{1}{3^{2\iota-1}}\right)\right) = 2\left(\ln\left(\frac{1}{3^{2\iota}} \times 3^{2\iota} \cdot 3^{-1}\right)\right) \\ &\quad = -2\ln 3 < -2.\end{aligned} \tag{44}$$

Case 2. Let $\mho = 1/3^{2\iota-1}$ and $\mu = 0$. We have

$$\frac{1}{2}\varpi_e(\mho, A\mho) = \frac{1}{2}\left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2\iota}}\right\}\right)^2 = \frac{1}{2}\left(\frac{1}{3^{2\iota-1}}\right)^2 < \left(\frac{1}{3^{2n-1}}\right)^2 = \left(\max\left\{\frac{1}{3^{2\iota-1}}, 0\right\}\right)^2 = \varpi_e(\mho, \mu). \tag{45}$$

Suppose that

$$F_{\varpi_e}(\varpi_e(A\mho, A\mu)) = F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota}}, 0\right)\right) = F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota}}, 0\right\}\right)^2\right) = F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota}}\right)^2\right) = \ln\left(\frac{1}{3^{2\iota}}\right)^2 = 2\ln\left(\frac{1}{3^{2\iota}}\right); \tag{46}$$

also,

$$F_{\varpi_e}(\varpi_e(\mho, \mu)) = F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota-1}}, 0\right)\right) = F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota-1}}, 0\right\}\right)^2\right) = F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota-1}}\right)^2\right) = 2\ln\left(\frac{1}{3^{2\iota-1}}\right). \tag{47}$$

By subtracting (46) and (47), we have the same inequality (44).

Case 3. Let $\mho = 0$ and $\mu = 1/3^{2m-1}$. The proof follows immediately as Case 2. Thus, A is an extended F_{ϖ_e}-Suzuki contraction mapping with $\vartheta = 2$. Here, 0 is the unique fixed point.

3. An Extended Generalized F_{ϖ_e}-Suzuki Contraction

Definition 12. A self-mapping A on a modified extended b-metric-like space $(\daleth, \varpi_e)$ is called an extended generalized F_{ϖ_e}-Suzuki contraction if there are $F_{\varpi_e} \in \Pi$ and $\vartheta > 0$ such that, if for all $\mho, \aleph \in \daleth$, the following hypothesis is satisfied

$$\frac{1}{2}\varpi_e(\mho, A\mho) < \varpi_e(\mho, \aleph) \Rightarrow \vartheta + F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1 + \varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1 + \varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph) + \varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right). \tag{48}$$

Remark 13.

(i) Every extended F_{ϖ_e}-Suzuki contraction is an extended generalized F_{ϖ_e}-Suzuki contraction

(ii) Suppose that A is an extended generalized F_{ϖ_e}-Suzuki contraction, by Definition 12, for all $\mho, \aleph \in \daleth$, we get $A\mho \neq A\aleph$ and $1/2\varpi_e(\mho, A\mho) < \varpi_e(\mho, \aleph)$. Thus,

$$F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) < \vartheta + F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1 + \varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1 + \varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph) + \varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right) \tag{49}$$

By condition $(\mathfrak{I}_1)$, for all $\mho, \aleph \in \daleth$ with $A\mho \neq A\aleph$, we have

$$\varpi_e(A\mho, A\aleph) \le \max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1 + \varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1 + \varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph) + \varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}. \tag{50}$$

Note that the inverse of the above remark is generally incorrect.

Example 14. Let $\daleth = [0,\infty)$. Define $\varpi_e : \daleth^2 \to \mathbb{R}$ *by* $\varpi_e(\mho, \aleph) = (\mho + \aleph)^2$ and $s : \daleth \times \daleth \to [1,\infty)$ *by* $s(\mho, \aleph) = 1 + \mho + \aleph$, for all $\mho, \aleph \in \daleth$. Here, ϖ_e is a modified extended ϖ_e-metric-like space. Define $A : \daleth \to \daleth$ as

$$A\mho = \begin{cases} 0, & \text{if } 0 \le \mho < 1, \\ \dfrac{1}{2}, & \text{if} \mho \ge 1. \end{cases} \tag{51}$$

Note that A is not an extended F_{ϖ_e}-Suzuki contraction. Indeed, for $0 \le \mho < 1$ and $\aleph = 1$, we can write $1/2\varpi_e(\mho, A\mho) = 1/2\varpi_e(\mho, 0) = 1/2\mho^2 < (\mho + 1)^2 = \varpi_e(\mho, \aleph)$ and

$$\begin{aligned} &\max\left\{\varpi_e(\mho, 1), \frac{\varpi_e(\mho, A\mho)}{1 + \varpi_e(\mho, A\mho)}, \frac{\varpi_e(1, A1)}{1 + \varpi_e(1, A1)}, \frac{\varpi_e(\mho, A1) + \varpi_e(1, A\mho)}{4(2 + \mho)}\right\} \\ &= \max\left\{\varpi_e(\mho, 1), \frac{\varpi_e(\mho, 0)}{1 + \varpi_e(\mho, 0)}, \frac{\varpi_e(1, A1)}{1 + \varpi_e(1, A1)}, \frac{\varpi_e(\mho, A1) + \varpi_e(1, 0)}{4(2 + \mho)}\right\} \ge \varpi_e(1, A1) = \varpi_e\left(1, \frac{1}{2}\right) \\ &= \frac{9}{4} > \frac{1}{4} = \varpi_e(A\mho, A\aleph). \end{aligned} \tag{52}$$

Let the function $F_{\varpi_e} \in \Pi$ be defined by $F_{\varpi_e}(\ell) = \ln(\ell)$, for $\ell > 0$. Then,

$$
\begin{aligned}
&F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) - F_{\varpi_e}(\varpi_e(\mho, \aleph)) \\
&\quad \le F_{\varpi_e}(\varpi_e(A\mho, A1)) - F_{\varpi_e}(\varpi_e(1, A1)) \\
&\quad = F_{\varpi_e}\left(\frac{1}{4}\right) - F_{\varpi_e}\left(\frac{9}{4}\right) = \ln\left(\frac{1}{4} \times \frac{4}{9}\right) = -\ln(9) < -2.
\end{aligned}
\tag{53}
$$

Therefore, A is an extended generalized F_{ϖ_e}-Suzuki contraction (for $\vartheta = 2$).

The following theorem is the main consequence of this part.

Theorem 15. *Let $(\daleth, \varpi_e)$ be an extended b -metric-like space and A be an extended generalized F_{ϖ_e} -Suzuki contraction self-mapping, then A has a unique fixed point, provided that $\lim_{i,m} s(\mho_i, \mho_m) \le 1/\eta$, for $0 < \eta < 1$.*

Proof. By the first lines of proof of Theorem 9, we build a sequence $\{\mho_\iota\}_{\iota=1}^{\infty}$ as $\mho_{\iota+1} = A\mho_\iota = A^{\iota+1}\mho_\circ$. Here, we consider $\iota \in \mathbb{N} \cup \{0\}$, $0 < \varpi_e(\mho_\iota, A\mho_\iota) = \varpi_e(\mho_\iota, \mho_{\iota+1})$, so

$$
\frac{1}{2}\varpi_e(\mho_\iota, A\mho_\iota) < \varpi_e(\mho_\iota, A\mho_\iota). \tag{54}
$$

Applying conditions (48) and (ϖ_{e3}), we get

$$
\begin{aligned}
&\vartheta + F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) = \vartheta + F_{\varpi_e}(\varpi_e(A\mho_\iota, A(A\mho_\iota))) \\
&\quad \le F_{\varpi_e}\Bigg(\max\Bigg\{\varpi_e(\mho_\iota, A\mho_\iota), \frac{\varpi_e(\mho_\iota, A\mho_\iota)}{1+\varpi_e(\mho_\iota, A\mho_\iota)}, \\
&\qquad \frac{\varpi_e(A\mho_\iota, A^2\mho_\iota)}{1+\varpi_e(A\mho_\iota, A^2\mho_\iota)}, \frac{\varpi_e(\mho_\iota, A^2\mho_\iota) + \varpi_e(A\mho_\iota, A\mho_\iota)}{4s(\mho_\iota, A\mho_\iota)}\Bigg\}\Bigg) \\
&\quad = F_{\varpi_e}\Bigg(\max\Bigg\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \frac{\varpi_e(\mho_\iota, \mho_{\iota+1})}{1+\varpi_e(\mho_\iota, \mho_{\iota+1})}, \frac{\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{1+\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}, \\
&\qquad \frac{\varpi_e(\mho_\iota, \mho_{\iota+2}) + \varpi_e(\mho_{\iota+1}, \mho_{\iota+1})}{4s(\mho_\iota, \mho_{\iota+1})}\Bigg\}\Bigg) \\
&\quad \le F_{\varpi_e}\Bigg(\max\Bigg\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \frac{\varpi_e(\mho_\iota, \mho_{\iota+1})}{1+\varpi_e(\mho_\iota, \mho_{\iota+1})}, \frac{\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{1+\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}, \\
&\qquad \frac{s(\mho_\iota, \mho_{\iota+1})[\varpi_e(\mho_\iota, \mho_{\iota+1}) + \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})] + 2s(\mho_\iota, \mho_{\iota+2})\varpi_e(\mho_\iota, \mho_{\iota+1})}{4s(\mho_\iota, \mho_{\iota+2})}\Bigg\}\Bigg) \\
&\quad = F_{\varpi_e}\Bigg(\max\Bigg\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \frac{\varpi_e(\mho_\iota, \mho_{\iota+1})}{1+\varpi_e(\mho_\iota, \mho_{\iota+1})}, \frac{\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{1+\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}, \\
&\qquad \frac{3\varpi_e(\mho_\iota, \mho_{\iota+1}) + \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{4}\Bigg\}\Bigg) \\
&\quad \le F_{\varpi_e}(\max\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})\}).
\end{aligned}
\tag{55}
$$

Now, if $\varpi_e(\mho_\iota, \mho_{\iota+1}) < \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})$, then

$$
F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) \le F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) - \vartheta, \tag{56}
$$

which is a contradiction due to $(\mathfrak{I}_1)$, so we should write

$$
F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) \le F_{\varpi_e}(\varpi_e(\mho_\iota, \mho_{\iota+1})) - \vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{57}
$$

By the same manner,

$$
F_{\varpi_e}(\varpi_e(\mho_\iota, \mho_{\iota+1})) \le F_{\varpi_e}(\varpi_e(\mho_{\iota-1}, \mho_\iota)) - \vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{58}
$$

From (57) and (58), one can write

$$
F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) \le F_{\varpi_e}(\varpi_e(\mho_{\iota-1}, \mho_\iota)) - 2\vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{59}
$$

Repeating the same scenario, we have

$$
F_{\varpi_e}(\varpi_e(\mho_\iota, \mho_{\iota+1})) \le F_{\varpi_e}(\varpi_e(\mho_\circ, \mho_1)) - \iota\vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{60}
$$

The proof of Theorem 9, namely, relations (22)–(29), yields that $\{\mho_\iota\}$ is Cauchy sequence in $(\daleth, \varpi_e)$, which is complete; hence, there is $\mho^* \in \daleth$ so that $\mho_\iota \to \mho^*$ as $\iota \to \infty$. That is,

$$
\lim_{\iota,m\to\infty} \varpi_e(\mho_\iota, \mho_m) = \lim_{\iota\to\infty} \varpi_e(\mho_\iota, \mho^*) = \varpi_e(\mho^*, \mho^*) = 0. \tag{61}
$$

Now, if A is continuous, by (24) we get

$$
\varpi_e(A\mho^*, \mho^*) = \lim_{\iota\to\infty} \varpi_e(A\mho_\iota, \mho_\iota) = \lim_{\iota\to\infty} \varpi_e(\mho_{\iota+1}, \mho_\iota) = 0. \tag{62}
$$

Thus, $A\mho^* = \mho^*$; that is, $\mho^*$ is a fixed point of A. Next, in the case that F_{ϖ_e} is continuous, we claim that

$$
\varpi_e(\mho_m, \mho^*) \le \varpi_e(\mho^*, A\mho_m), \quad \forall m \in \mathbb{N} \cup \{0\}. \tag{63}
$$

By the fact $1/2\varpi(\mho_m, A\mho_m) < \varpi(\mho_m, A\mho_m)$ and using (48), we obtain that

$$
\begin{aligned}
&\vartheta + F_{\varpi_e}(\varpi(A\mho_m, A^2\mho_m)) \le F_{\varpi_e}\Bigg(\max\Bigg\{\varpi_e(\mho_m, A\mho_m), \frac{\varpi_e(\mho_m, A\mho_m)}{1+\varpi_e(\mho_m, A\mho_m)}, \\
&\qquad \frac{\varpi_e(A\mho_m, A^2\mho_m)}{1+\varpi_e(A\mho_m, A^2\mho_m)}, \frac{\varpi_e(\mho_m, A^2\mho_m) + \varpi_e(A\mho_m, A\mho_m)}{4s(\mho_m, A\mho_m)}\Bigg\}\Bigg) \\
&\quad \le F\Bigg(\max\Bigg\{\varpi_e(\mho_m, A\mho_m), \frac{\varpi_e(\mho_m, A\mho_m)}{1+\varpi_e(\mho_m, A\mho_m)}, \frac{\varpi_e(A\mho_m, A^2\mho_m)}{1+\varpi_e(A\mho_m, A^2\mho_m)}, \\
&\qquad \frac{s(\mho_m, A\mho_m)[\varpi_e(\mho_m, A\mho_m) + \varpi_e(A\mho_m, A^2\mho_m)] + 2s(\mho_m, A\mho_m)\varpi_e(\mho_m, A\mho_m)}{4s(\mho_m, A\mho_m)}\Bigg\}\Bigg) \\
&\quad \le F\Bigg(\max\Bigg\{\varpi_e(\mho_m, A\mho_m), \frac{\varpi_e(\mho_m, A\mho_m)}{1+\varpi_e(\mho_m, A\mho_m)}, \frac{\varpi_e(A\mho_m, A^2\mho_m)}{1+\varpi_e(A\mho_m, A^2\mho_m)}, \\
&\qquad \frac{3\varpi_e(\mho_m, A\mho_m) + \varpi_e(A\mho_m, A^2\mho_m)}{4}\Bigg\}\Bigg) \\
&\quad \le F(\max\{\varpi_e(\mho_m, A\mho_m), \varpi_e(A\mho_m, A^2\mho_m)\}).
\end{aligned}
\tag{64}
$$

If $\varpi_e(\mho_m, A\mho_m) < \varpi_e(A\mho_m, A^2\mho_m)$, then we have

$$F_{\varpi_e}\left(\varpi\left(A\mho_m, A^2\mho_m\right)\right) \le F_{\varpi_e}\left(\varpi\left(A\mho_m, A^2\mho_m\right)\right) - \vartheta, \quad (65)$$

a contradiction due to $(\mathfrak{I}_1)$. So, we should write

$$F_{\varpi_e}\left(\varpi\left(A\mho_m, A^2\mho_m\right)\right) \le F_{\varpi_e}(\varpi_e(\mho_m, A\mho_m)) - \vartheta. \quad (66)$$

Since F_{ϖ_e} is continuous and strictly increasing, it follows that

$$\varpi_e\left(A\mho_m, A^2\mho_m\right) < \varpi_e(\mho_m, A\mho_m). \quad (67)$$

Now, to ensure the existence of a fixed point, two cases arise as follows.

Case 1. For each $\iota \in \mathbb{N}$, there is $j_\iota \in \mathbb{N}$ so that $\mho_{j_\iota} = A\mho^*$ and $j_\iota > j_{\iota-1}$, where $j_\circ = 1$. Then, we have $\mho^* = \lim_{\iota\to\infty}\mho_{j_\iota} = \lim_{\iota\to\infty}A\mho^* = A\mho^*$, i.e., $\mho^*$ is a fixed point of A.

Case 2. There is $\iota_\circ \in \mathbb{N}$ such that for all $\iota \ge \iota_\circ, \mho_{\iota+1} \ne A\mho^*$. It is clear that $1/2\varpi_e(\mho_\iota, A\mho^*) < \varpi_e(\mho_\iota, A\mho^*)$ for all $\iota \ge \iota_\circ$.

By (48) and (63), we get

$$\begin{aligned}\vartheta + F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho^*\right)\right) &\le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, A\mho^*), \frac{\varpi_e(\mho_\iota, A\mho_\iota)}{1+\varpi_e(\mho_\iota, A\mho_\iota)}, \frac{\varpi_e\left(A\mho^*, A^2\mho^*\right)}{1+\varpi_e\left(A\mho^*, A^2\mho^*\right)}, \frac{\varpi_e\left(\mho_\iota, A^2\mho^*\right)+\varpi_e(A\mho^*, A\mho_\iota)}{4s(\mho_\iota, A\mho^*)}\right\}\right)\\ &\le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, A\mho^*), \frac{\varpi_e(\mho_\iota, A\mho_\iota)}{1+\varpi_e(\mho_\iota, A\mho_\iota)}, \frac{\varpi_e\left(A\mho^*, A^2\mho^*\right)}{1+\varpi_e\left(A\mho^*, A^2\mho^*\right)}, \frac{2\varpi_e(\mho_\iota, A\mho^*)+\varpi_e\left(A\mho^*, A^2\mho^*\right)+\varpi_e(\mho_\iota, A\mho_\iota)}{4}\right\}\right)\\ &\le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, A\mho^*), \varpi_e(\mho_\iota, A\mho_\iota), \varpi_e\left(A\mho^*, A^2\mho^*\right)\right\}\right)\\ &< F_{\varpi_e}(\max\{\varpi_e(\mho_\iota, A\mho^*), \varpi_e(\mho_\iota, A\mho_\iota), \varpi_e(\mho^*, A\mho^*)\}).\end{aligned} \quad (68)$$

Since F_{ϖ_e} is continuous, we find at the limit $\iota \to \infty$, or

$$\vartheta + F_{\varpi_e}\left(\varpi_e\left(\mho^*, A^2\mho^*\right)\right) \le F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)), \quad (69)$$

$$F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)) \le F_{\varpi_e}(\varpi_e(\mho^*, \mho^*)) - \vartheta, \quad (70)$$

which is a contradiction. So, $\varpi_e(\mho^*, A\mho^*) = 0$, which leads to $\mho^* = A\mho^*$.

The two cases above ensure the existence of a fixed point of A.

To ensure the uniqueness, suppose that $\mho^*, \upsilon^*$ are distinct fixed points of A. Hence, $1/2\varpi_e(\mho^*, \upsilon^*) < \varpi_e(\mho^*, \upsilon^*)$, which implies that

$$\begin{aligned}\vartheta + F_{\varpi_e}(\varpi_e(\mho^*, \upsilon^*)) &= \vartheta + F_{\varpi_e}(\varpi_e(A\mho^*, A\upsilon^*))\\ &\le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho^*, \upsilon^*), \frac{\varpi_e(\mho^*, A\mho^*)}{1+\varpi_e(\mho^*, A\mho^*)}, \frac{\varpi_e(\upsilon^*, A\upsilon^*)}{1+\varpi_e(\upsilon^*, A\upsilon^*)}, \frac{\varpi_e(\mho^*, A\upsilon^*)+\varpi_e(\upsilon^*, A\mho^*)}{4s(\mho^*, \upsilon^*)}\right\}\right)\\ &\le F_{\varpi_e}(\varpi_e(\mho^*, \upsilon^*)),\end{aligned} \quad (71)$$

which is a contradiction again. Hence, the fixed point is unique.

In the following, we justify all required hypotheses of Theorem 15.

Example 16. Suppose that $\daleth = [0,\infty)$. Define functions $\varpi_e : \daleth^2 \to \mathbb{R}$ and $s : \daleth \times \daleth \to [1,\infty)$ *by* $\varpi_e(\mho, \aleph) = (\mho+\aleph)^2$ and $s(\mho, \aleph) = 1 + \mho + \aleph$, respectively. Then, $(\daleth, \varpi_e)$ is a complete modified ϖ_e -metric-like space. Define $A : \daleth \to \daleth$ by

$$A\mho = \begin{cases} 0, & \text{if } \mho\in\left[0, \dfrac{1}{4}\right),\\ \left\{\dfrac{1}{4^\iota}\right\}, & \text{if } \mho\in\left[\dfrac{1}{4}, \infty\right), \iota \in \daleth. \end{cases} \quad (72)$$

Define the function $F_{\varpi_e} \in \Pi$ by $F_{\varpi_e}(\ell) = \ln(\ell)$ for $\ell > 0$ and $\vartheta > 0$. We state the following.

Case 1. Let $\mho = 1/4^{\iota-1}$ and $\aleph = 1/4^{m-1}$, for $m > \iota \ge 2$. Now, for $\iota = 1$ and $m = 2$, we have $\mho = 1$ and $\aleph = 1/4$. Therefore,

$$\begin{aligned}\frac{1}{2}\varpi_e(\mho, A\mho) &= \frac{1}{2}\varpi_e(1, A1) = \frac{1}{2}\left(1+\frac{1}{4}\right)^2 = \frac{25}{32} < \frac{25}{16}\\ &= \left(1+\frac{1}{4}\right)^2 = \varpi_e(\mho, \aleph).\end{aligned} \quad (73)$$

Let

$$\begin{aligned}F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) &= F_{\varpi_e}\left(\varpi_e\left(A1, A\frac{1}{4}\right)\right) = F_{\varpi_e}\left(\varpi_e\left(\frac{1}{4}, \frac{1}{16}\right)\right)\\ &= F_{\varpi_e}\left(\frac{5}{16}\right)^2 = 2\ln\left(\frac{5}{16}\right) = -2.326,\end{aligned} \quad (74)$$

as well as,

$$\begin{aligned}&F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right)\\ &= F_{\varpi_e}\left(\max\left\{\varpi_e\left(1, \frac{1}{4}\right), \frac{\varpi_e(1, A1)}{1+\varpi_e(1, A1)}, \frac{\varpi_e(1/4, A(1/4))}{1+\varpi_e(1/4, A(1/4))}, \frac{\varpi_e(1, A(1/4))+\varpi_e(1/4, A1)}{4s(1, 1/4)}\right\}\right) = F_{\varpi_e}\left(\max\left\{\frac{25}{16}, \frac{25}{41}, \frac{25}{281}, \frac{353}{2304}\right\}\right)\\ &= F_{\varpi_e}\left(\frac{25}{16}\right) = \ln\left(\frac{25}{16}\right) = 0.0446.\end{aligned} \quad (75)$$

So, we get

$$F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) - F_{\varpi_e}\left(\max\left\{\varpi_e(\mho,\aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho,\aleph)}\right\}\right) = -2.326 - 0.044 = -2.37 < -2. \tag{76}$$

Case 2. Let $\mho = 1/4$ and $\aleph = 0$. So, we get

$$\frac{1}{2}\varpi_e(\mho, A\mho) = \frac{1}{2}\varpi_e\left(\frac{1}{4}, \frac{1}{16}\right) = \frac{25}{512} < \frac{1}{16} = \left(\frac{1}{4}+0\right)^2 = \varpi_e(\mho,\aleph). \tag{77}$$

Consider

$$F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) = F_{\varpi_e}\left(\varpi_e\left(A\frac{1}{4}, A0\right)\right) = F_{\varpi_e}\left(\varpi_e\left(\frac{1}{16}, 0\right)\right) = F_{\varpi_e}\left(\frac{1}{16}\right)^2 = 2\ln\left(\frac{1}{16}\right) = -5.545. \tag{78}$$

Additionally,

$$\begin{aligned} &F_{\varpi_e}\left(\max\left\{\varpi_e(\mho,\aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho,\aleph)}\right\}\right) \\ &= F_{\varpi_e}\left(\max\left\{\varpi_e\left(\frac{1}{4}, 0\right), \frac{\varpi_e(1/4, A(1/4))}{1+\varpi_e(1/4, A(1/4))}, \frac{\varpi_e(0, A0)}{1+\varpi_e(0, A0)}, \frac{\varpi_e(1/4, A0)+\varpi_e(0, A(1/4))}{4s(1/4, 0)}\right\}\right) \\ &= F_{\varpi_e}\left(\max\left\{\frac{1}{16}, \frac{25}{281}, 0, \frac{17}{1280}\right\}\right) = F_{\varpi_e}\left(\frac{25}{281}\right) \\ &= \ln\left(\frac{25}{281}\right) = -2.419. \end{aligned} \tag{79}$$

Subtracting the two relations, we have

$$F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) - F_{\varpi_e}\left(\max\left\{\varpi_e(\mho,\aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho,\aleph)}\right\}\right) = -5.545 + 2.419 = -3.126 < -2. \tag{80}$$

From the above, we deduce that the mapping A is an extended generalized F_{ϖ_e}-Suzuki contraction with $\vartheta = 2$. Moreover, if $\mho_m = \{1/4^m\} \in \daleth$, we have

$$\lim_{i,m\to\infty} s(\mho_m, \mho_i) = \lim_{i,m\to\infty}\left(1 + \frac{1}{4^m} + \frac{1}{4^i}\right) = 1 < \frac{1}{\eta}, \tag{81}$$

for $\eta \in (0, 1)$. Hence, the requirements of Theorem 15 hold; therefore, A has a unique fixed point. Here, it is 0.

4. Supportive Applications

This part is considered as the strength of the paper, where we use the results presented in Theorems 9 and 15 to get the analytical solutions both of the Fredholm integral equation and the second-order differential equation, respectively. For this purpose, we will divide this section into two parts as follows.

4.1. Analytical Solution of Fredholm Integral Equation. Let the Fredholm integral equation given by

$$\mho(\eta) = \int_u^v \Phi(\eta, \zeta, \mho(\zeta))d\zeta, \tag{82}$$

for all $\eta, \zeta \in [u, v]$, where $F_{\varpi_e} : [u, v] \to \mathbb{R}$ and $\Phi : [u, v] \times u, v] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $\daleth = C([u, v], \mathbb{R})$ be the set of all real continuous functions defined on $[u, v]$, endowed with

$$\varpi_e(\mho, \aleph) = \left(\|\mho + \aleph\|_\infty\right)^2 \text{ for all} \mho, \aleph \in \daleth, \tag{83}$$

where $\|\mho\|_\infty = \sup_{\eta\in[u,v]}\{|\mho(\eta)|e^{-\eta\vartheta}\}$ with $s(\mho, \aleph) = 1 + |\mho| + |\aleph|$, where $s : \daleth \times \daleth \to [1,\infty)$. Note that $(\daleth, \varpi_e)$ is a complete modified ϖ_e-metric-like space.

Now, the following is the main result of this part.

Theorem 17. *Let A be self-mapping on the complete modified ϖ_e -metric-like space $(\daleth, \varpi_e)$. Assume that*

(i) for each $\eta, \zeta \in [u, v]$ and $\mho, \aleph \in \daleth$,

$$\frac{1}{2}\left(\left\|\mho(\eta) + \int_u^v \Phi(\eta, \zeta, \mho(\zeta))d\zeta\right\|_\infty\right)^2 \le \left(\|\mho(\eta)\|_\infty + \|\aleph(\eta)\|_\infty\right)^2 \tag{84}$$

(ii) for all $\eta, \zeta \in [u, v]$, there is a constant $\vartheta \in \mathbb{R}^+$ such that

$$|\Phi(\eta, \zeta, \mho(\zeta)) + \Phi(\eta, \zeta, \aleph(\zeta))| \le \frac{e^{-\vartheta/2}}{(v-u)}(|\mho(\zeta) + \aleph(\zeta)|) \tag{85}$$

Then, there exists a solution of the problem (82).

Proof. Consider the nonlinear self-mapping $A : \daleth \rightarrow \daleth$ given as

$$A\mho(\eta) = \int_u^v \Phi(\eta, \zeta, \mho(\zeta))d\zeta. \tag{86}$$

Clearly, if $\mho^* = A\mho^*$, then it is a solution of the problem (82).

Let $\mho, \aleph \in \daleth$, so, by condition (i), we deduce that $1/2\varpi_e(\mho(\eta), A\mho(\eta)) < \varpi_e(\mho(\eta), \aleph(\eta))$. After applying the condition (ii), for any $\mho(\eta), \aleph(\eta) \in \daleth$, we can write

$$\begin{aligned}
|A\mho(\eta) + A\aleph(\eta)|^2 &\leq \left(\int_u^v |\Phi(\eta, \zeta, \mho(\zeta)) + \Phi(\eta, \zeta, \aleph(\zeta))| d\zeta\right)^2 \\
&\cdot \left(\int_u^v \frac{e^{-\vartheta/2}}{(v-u)} (|\mho(\zeta) + \aleph(\zeta)|) d\zeta\right)^2 \\
&\leq \left(\int_u^v \frac{e^{-\vartheta/2}}{v-u} \times \sqrt{e^{-2\vartheta\eta} \times e^{2\vartheta\eta}} \times (|\mho(\zeta) + \aleph(\zeta)|) d\zeta\right)^2 \\
&\leq \frac{e^{-\vartheta}}{(v-u)^2} \times \varpi_e(\mho, \aleph) \times e^{2\vartheta\eta} \left(\int_u^v d\zeta\right)^2 \leq e^{-\vartheta} \varpi_e(\mho, \aleph) \times e^{2\vartheta\eta},
\end{aligned} \tag{87}$$

so we have

$$\left(|A\mho(\eta) + A\aleph(\eta)| \times e^{-\vartheta\eta}\right)^2 \leq e^{-\vartheta}\varpi_e(\mho, \aleph), \tag{88}$$

which leads to

$$\left(\|A\mho(\eta) + A\aleph(\eta)\|_\infty\right)^2 \leq e^{-\vartheta}\varpi_e(\mho, \aleph). \tag{89}$$

It yields that

$$\varpi_e(A\mho(\eta), A\aleph(\eta)) \leq e^{-\vartheta}\varpi_e(\mho, \aleph). \tag{90}$$

Taking $F_{\varpi_e}(\ell) = \ln(\ell)$ for $\ell > 0$, one gets or

$$\ln(\varpi_e(A\mho(\eta), A\aleph(\eta))) \leq \ln\left(e^{-\vartheta}\varpi_e(\mho, \aleph)\right), \tag{91}$$

$$\vartheta + \ln(\varpi_e(A\mho(\eta), A\aleph(\eta))) \leq \ln(\varpi_e(\mho, \aleph)). \tag{92}$$

Equivalently,

$$\vartheta + F_{\varpi_e}(\varpi_e(A\mho(\eta), A\aleph(\eta))) \leq F_{\varpi_e}(\varpi_e(\mho, \aleph)). \tag{93}$$

By Theorem 9, A admits a fixed point, which is a solution of the problem (82).

4.2. Analytical Solution of Second-Order Differential Equation. Consider the second-order differential equation given as follows:

$$\begin{cases} \mho''(\eta) = -\Phi(\eta, \mho(\eta)), & \eta \in [0, \gamma], \\ \mho(0) = \mho(\eta) = 0, \end{cases} \tag{94}$$

where $\Phi : [0, \gamma] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Resolving the problem (94) is equivalent to resolving the following integral equation:

$$\mho(\eta) = \int_0^\eta \phi(\eta, \zeta)\Phi(\zeta, \mho(\zeta))d\zeta, \quad \forall \eta \in [0, \gamma], \tag{95}$$

where ϕ is Green's function defined by

$$\phi(\eta, \zeta) = \begin{cases} \eta(1-\zeta), & \text{if } 0 \leq \eta \leq \zeta \leq \gamma, \\ \zeta(1-\eta), & \text{if } 0 \leq \zeta \leq \eta \leq \gamma, \end{cases} \tag{96}$$

and Φ is a function as in Theorem 17. Hence, if $\mho \in C([0, \gamma])$, then $\mho$ is a solution of the problem (94) if and only if $\mho$ is a solution of the problem (95).

Let $\daleth = C([0, \gamma], \mathbb{R})$ be the set of all continuous functions defined on $[0, \gamma]$, and define a norm $\|\mho\|_\vartheta = \max_{\eta \in 0, \gamma]}\{|\mho(\eta)| e^{-1/2\eta\vartheta}\}$, for arbitrary $\eta \geq 1$. Obviously, $\|\cdot\|_\vartheta$ is equivalent to the maximum norm $\|\cdot\|$ on $\daleth$, and $\daleth$ is endowed with the extended generalized ϖ_{e_ϑ}-metric-like as

$$\varpi_{e_\vartheta}(\mho, \aleph) = \left(\|\mho + \aleph\|_\vartheta\right)^2 = \max_{\eta \in 0, \gamma]}\left\{|\mho(\eta) + \aleph(\eta)|^2 e^{-\eta\vartheta}\right\} \text{ for all } \mho, \aleph \in \daleth \text{ and } e^{\eta\vartheta} \geq 1. \tag{97}$$

Then, $(\daleth, \varpi_e)$ is a complete modified ϖ_e-metric-like space with $s(\mho, \aleph) = 1 + |\mho| + |\aleph|$. Our main theorem is as follows.

Theorem 18. *Suppose that $(\daleth, \varpi_e)$ is a complete modified b -metric-like space and A is a nonlinear self-mapping on $\daleth$, then (95) possesses a unique solution $\mho \in C([0, \gamma], \mathbb{R})$, if*

(a1) for each $\eta, \zeta \in [0, \gamma]$ and $\mho, \aleph \in \daleth$,

$$\frac{1}{2}\left\|\mho(\eta) + \int_0^\eta \phi(\eta, \zeta)\Phi(\zeta, \mho(\zeta))d\zeta\right\|_\vartheta \leq \|\mho + \aleph\|_\vartheta \tag{98}$$

(a2) $\Phi \in C([0, \gamma] \times \mathbb{R})$ and $\phi \in C([0, \gamma] \times [0, \gamma])$

(a3) Φ satisfies

$$|\Phi(\zeta, \mho(\zeta)) + \Phi(\zeta, \aleph(\zeta))|^2 \leq \vartheta e^{-\vartheta}\psi(\mho, \aleph), \tag{99}$$

for all $\zeta\in[0,\gamma]$ *and* $\mho, \aleph \in \mathbb{R}$, *where*

$$\psi(\mho,\aleph)=\max\left\{|\mho+\aleph|^2,\frac{|\mho+A\mho|^2}{1+|\mho+A\mho|^2},\frac{|\aleph+A\aleph|^2}{1+|\aleph+A\aleph|^2},\frac{|\mho+A\aleph|^2+|\aleph+A\mho|^2}{4(1+|\mho|+|\aleph|)}\right\} \tag{100}$$

(a4) $\max\int_0^\eta \phi(\eta,\zeta)d\zeta\le 1$, *for all* $\eta\in[0,\gamma]$

Proof. Consider on the set $\daleth$, the mapping A as

$$A\mho(\eta)=\int_0^\eta \phi(\eta,\zeta)\Phi(\zeta,\mho(\zeta))d\zeta, \tag{101}$$

for all $\eta\in[0,\gamma]$ and $\mho\in\daleth$. The solution of (95) is also a fixed point of A on $\daleth$. By condition (a1) and the definition of A, we can write $1/2\varpi_{e_\vartheta}(\mho(\eta),A\mho(\eta))<\varpi_{e_\vartheta}(\mho(\eta),\aleph(\eta))$.

Let $\mho,\aleph\in\daleth$. By the hypotheses (a2)-(a4), we have

$$\begin{aligned}(|A\mho(\eta)+A\aleph(\eta)|)^2&=\left|\int_0^\eta \phi(\eta,\zeta)[\Phi(\zeta,\mho(\zeta))+\Phi(\zeta,\aleph(\zeta))]d\zeta\right|^2\\
&\le\int_0^\eta|\phi(\eta,\zeta)|^2|\Phi(\zeta,\mho(\zeta))+\Phi(\zeta,\aleph(\zeta))|^2d\zeta\\
&\le\int_0^\eta|\Phi(\zeta,\mho(\zeta))+\Phi(\zeta,\aleph(\zeta))|^2d\zeta\\
&\le\int_0^\eta\vartheta e^{-\vartheta}\psi(\mho(\zeta),\aleph(\zeta))d\zeta\\
&\le\vartheta e^{-\vartheta}\int_0^\eta e^{\zeta\vartheta}\max\left\{|\mho+\aleph|^2e^{-\zeta\vartheta},\frac{|\mho+A\mho|^2e^{-2\zeta\vartheta}}{1+|\mho+A\mho|^2e^{-\zeta\vartheta}},\right.\\
&\qquad\left.\frac{|\aleph+A\aleph|^2e^{-2\zeta\vartheta}}{1+|\aleph+A\aleph|^2e^{-\zeta\vartheta}},\frac{|\mho+A\aleph|^2+|\aleph+A\mho|^2}{4(1+|\mho|+|\aleph|)}e^{-\zeta\vartheta}\right\}d\zeta\\
&\le\vartheta e^{-\vartheta}\int_0^\eta e^{\zeta\vartheta}\max\left\{\varpi_{e_\vartheta}(\mho,\aleph),\frac{\varpi_{e_\vartheta}(\mho,A\mho)}{1+\varpi_{e_\vartheta}(\mho,A\mho)},\right.\\
&\qquad\left.\frac{\varpi_{e_\vartheta}(\aleph,A\aleph)}{1+\varpi_{e_\vartheta}(\aleph,A\aleph)},\frac{\varpi_{e_\vartheta}(\mho,A\aleph)+\varpi_{e_\vartheta}(\aleph,A\mho)}{4s(\mho,\aleph)}\right\}d\zeta\\
&=\vartheta e^{-\vartheta}\psi(\mho,\aleph)\int_0^\eta e^{\zeta\vartheta}d\zeta=\vartheta e^{-\vartheta}\psi(\mho,\aleph)\times\left(\frac{e^{\eta\vartheta}}{\vartheta}-1\right)\\
&\le\vartheta e^{-\vartheta}\psi(\mho,\aleph)\times\frac{e^{\eta\vartheta}}{\vartheta}\le e^{-\vartheta}\psi(\mho,\aleph)e^{\eta\vartheta}.\end{aligned} \tag{102}$$

Hence, for all $\mho,\aleph\in\daleth$,

$$(|A\mho(\eta)+A\aleph(\eta)|)^2\times e^{-\eta\vartheta}\le e^{-\vartheta}\psi(\mho,\aleph), \tag{103}$$

which yields

$$\varpi_{e_\vartheta}(A\mho,A\aleph)=\max_{\eta\in0,\gamma]}\left\{(|A\mho(\eta)+A\aleph(\eta)|)^2\times e^{-\eta\vartheta}\right\}\le e^{-\vartheta}\psi(\mho,\aleph). \tag{104}$$

That is,

$$\vartheta+\ln\varpi_{e_\vartheta}(A\mho,A\aleph)\le\ln\psi(\mho,\aleph). \tag{105}$$

Defining the function $F_{\varpi_e}(\alpha)=\ln(\alpha),\alpha>0$ in (105), such that $F_{\varpi_e}\in\Pi$, we have

$$\vartheta+F_{\varpi_e}(\varpi_e(A\mho,A\aleph))\le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho,\aleph),\frac{\varpi_e(\mho,A\mho)}{1+\varpi_e(\mho,A\mho)},\frac{\varpi_e(\aleph,A\aleph)}{1+\varpi_e(\aleph,A\aleph)},\frac{\varpi_e(\mho,A\aleph)+\varpi_e(\aleph,A\mho)}{4s(\mho,\aleph)}\right\}\right). \tag{106}$$

Hence, all requirements of Theorem 15 hold and A is an extended generalized F-Suzuki contraction; hence, Γ possesses a fixed point $\mho\in\daleth$, which is a solution of the problem (95).

5. Conclusion

A modified ϖ_e-metric-like space is presented, and related fixed point results via it are discussed. Nontrivial examples are conducted for supporting the mentioned space and theorems. Thereafter, by using a fixed point technique, a simple and efficient solution for the integral and differential equations is found in the setting of a modified ϖ_e-metric-like space. A lot of authors connected fixed point techniques and classical integral equations in various abstract spaces such as metric spaces, b-metric spaces, and partial metric spaces. We also follow the same method in the new space. In the literature, our obtained applications are an extension and/or a generalization of many existing classical integral and differential equations. The observed results of this paper open new framework research avenues for

(i) fixed point techniques for solving Volterra-Fredholm integral equation in a modified ϖ_e-metric-like space

(ii) collocation-type methods for Volterra-Hammerstein integral equations in modified ϖ_e-metric-like spaces

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Research Article

Fixed Point Results for $\mathscr{C}$-Contractive Mappings in Generalized Metric Spaces with a Graph

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In this paper, we establish fixed point theorems for Chatterjea contraction mappings on a generalized metric space endowed with a graph. Our results extend, generalize, and improve many of existing theorems in the literature. Moreover, some examples and an application to matrix equations are given to support our main result.

1. Introduction

Fixed point theorems for contraction mappings and their generalizations play a crucial role in the determination of the existence and uniqueness of solutions of certain problems in mathematics and applied sciences, such as variational and linear inequalities, mathematical models, optimization, and mathematical economics. In 1922, Banach [1] proved the contraction principle, today named after him, which states any contraction on a complete metric space has a unique fixed point. In 1972, Chatterjea [2] proved that a self-mapping on a complete metric space X has a unique fixed point whenever there exists $0 \le k < 1/2$ such that

$$d(Tx, Ty) \le k[d(x, Ty) + d(y, Tx)], \text{for all } x, y \in X. \tag{1}$$

On the other hand, different generalizations of the usual notion of metric space were proposed by a number of mathematicians (see [3, 4]). Recently, Jleli and Samet [5] introduced a new concept of generalized metric space that, in fact, recovers various topological spaces. The class of such metric spaces is larger than the class of standard metric spaces, than the class of b-metric spaces, than that of dislocated metric spaces, than that of dislocated b-metric spaces, and than the class of modular spaces with the Fatou property. The interested reader is referred to [5] for further details.

This work is the continuation of [6]. Motivated by the ideas recently introduced in [7–12]), we extend the Chatterjea fixed point theorem to the setting of generalized metric spaces with a graph. As corollaries, we obtain Chatterjea fixed point theorems in the setting of partially ordered metric spaces. Furthermore, we generalize the common fixed point result given in [13]. We provide an example to illustrate our main result.

2. Preliminaries

We recall the definition of generalized metric space and some related topological concepts, as introduced firstly by Jleli and Samet in [5].

Definition 1 [5]. Let X be a nonempty set and $\mathscr{D} : X \times X \longrightarrow [0, +\infty]$ be a given mapping.

For every $x \in X$, define the set

$$\mathscr{C}(\mathscr{D}, X, x) = \left\{ \{x_n\} \subset X : \lim_{n\longrightarrow\infty} \mathscr{D}(x_n, x) = 0 \right\}. \tag{2}$$

We say that $\mathscr{D}$ is a generalized metric on X if it satisfies the following conditions:

($\mathscr{D}_1$) For every $(x, y) \in X \times X$, $\mathscr{D}(x, y) = 0$ implies $x = y$

($\mathscr{D}_2$) For every $(x, y) \in X \times X$, $\mathscr{D}(x, y) = \mathscr{D}(y, x)$

($\mathcal{D}_3$) There exists $C > 0$ such that for all $(x, y) \in X \times X$, if there exists $\{x_n\} \in \mathcal{C}(\mathcal{D}, X, x)$, then

$$\mathcal{D}(x, y) \leq C\lim\sup_{n\longrightarrow\infty}\mathcal{D}(x_n, y). \tag{3}$$

The pair $(X, \mathcal{D})$ is called a generalized metric space.

Definition 2 [5].

(i) A sequence $\{x_n\}$ in a generalized metric space $(X, \mathcal{D})$ is said to be $\mathcal{D}$-convergent to $x \in X$ if $\{x_n\} \in \mathcal{C}(\mathcal{D}, X, x)$

(ii) A sequence $\{x_n\}$ in a generalized metric space $(X, \mathcal{D})$ is said to be a $\mathcal{D}$-Cauchy sequence if $\lim_{m,n\longrightarrow\infty}\mathcal{D}(x_n, x_m) = 0$

(iii) The space $(X, \mathcal{D})$ is said to be $\mathcal{D}$-complete if every $\mathcal{D}$-Cauchy sequence in X is $\mathcal{D}$-convergent to some element in X

(iv) The space $(X, \mathcal{D})$ is said to be $\mathcal{D}$-compact if every sequence in X has a $\mathcal{D}$-convergent subsequence to some element in X

The basic concepts, notation, and terminology related to graph theory can be found, for example, in [14, 15]. A directed graph or digraph G consists of a nonempty set $V(G)$, whose elements are called the vertices of G, and a set $E(G) \subset V(G) \times V(G)$, called the set of directed edges of G. The diagonal of the cartesian product $V(G) \times V(G)$ will be denoted by Δ. A digraph is said to be reflexive if $E(G)$ contains all loops, i.e., if $\Delta \subset E(G)$. G is said to be transitive if, for any $x, y, z \in V(G)$

$$[(x, y) \in E(G)\text{and}(y, z) \in E(G)](x, z) \in E(G). \tag{4}$$

Given a digraph $G = (V, E)$, a directed path in G is a sequence of vertices.

$a_0,a_1,\ldots,a_n \cdots$, with $(a_i, a_{i+1}) \in E(G)$ for each $i \in \mathbb{N}$. A finite path $(a_0, a_1, \cdots, a_n)$ is said to have length n. The transitive closure of G is the digraph G' such that $V(G') = V(G)$ and that (i, j) is an edge in G' if there is a directed path from i to j in G.

We say that a vertex x in $V(G)$ is isolated if for any vertex y in $V(G)$ such that $x \neq y$, neither $(x, y) \in E(G)$ nor $(y, x) \in E(G)$.

In the sequel, given a graph G, G^{-1} will stand for it, that is, for the graph obtained from G by reversing the direction of its edges. Thus,

$$E\left(G^{-1}\right) = \{(x, y) \in X \times X : (y, x) \in E(G)\}. \tag{5}$$

In addition, $\tilde{G}$ will stand for the undirected graph obtained from G by ignoring the direction of its edges. In other words,

$$E\left(\tilde{G}\right) = E(G) \cup E\left(G^{-1}\right). \tag{6}$$

Throughout this paper, the triplet $(X, \mathcal{D}, G)$ will stand for the generalized metric space $(X, \mathcal{D})$ endowed with a reflexive digraph G such that $V(G) = X$. In [16], Alfuraidan et al. introduced the idea of G-monotonicity of sequences and the G-completeness of the metric space. Specifically,

Definition 3 [16]. Let G be a digraph. A sequence $\{x_n\} \in V(G)$ is said to be

(i) G-increasing, if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$

(ii) G-decreasing, if $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$

(iii) G-monotone, if it is either G-increasing or G-decreasing

The preceding notion of G-completeness can naturally be extended to the setting of generalized metric spaces as follows:

Definition 4. A generalized metric space $(X, \mathcal{D})$ is said to be G-complete if any $\mathcal{D}$-Cauchy, G-monotone sequence $\{x_n\} \subset V(G)$ is $\mathcal{D}$-convergent to an element in $V(G)$.

Remark 5. It is shown in [16] (Example 3.3) that G-completeness is finer than usual completeness.

The following definitions of some useful types of continuity are borrowed from [11].

Definition 6. A self-mapping T on the generalized metric space X is called

(i) Subsequentially continuous, if for every sequence $\{x_n\} \subset X$, $\mathcal{D}$-convergent to $x \in X$, there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $\{Tx_{n_q}\}$ $\mathcal{D}$-converges to Tx (as $q \longrightarrow \infty$)

(ii) Orbitally G-continuous, if for all $x, y \in V(G)$ and any sequence $\{k_n\}$ of positive integers

$$\left\{T^{k_n}x\right\} \longrightarrow y \text{ and } \left(T^{k_n}x, T^{k_{n+1}}x\right) \in E\left(\tilde{G}\right) \text{ imply } \left\{T\left(T^{k_n}x\right)\right\} \longrightarrow Ty. \tag{7}$$

The following property, initially introduced in [17] for partially ordered sets and in [11] for metric spaces with a graph, is often assumed to relax continuity assumptions.

Property (JNRL). The digraph G is said to satisfy the property (JNRL), if for any G-monotone increasing (decreasing) sequence $\{x_n\}$, which $\mathcal{D}$-converges to some $x \in V(G)$, it holds that $(x_n, x) \in E(G)$ ($(x, x_n) \in E(G)$), for any $n \in \mathbb{N}$.

Let $(X, \mathcal{D}, G)$ be a generalized metric space endowed with a reflexive graph. Motivated by [11, 18], we define G-Chatterjea mappings on a generalized metric space $(X, \mathcal{D})$ with a graph, as follows:

Definition 7. A mapping $T : X \longrightarrow X$ is said to be a G-Chatterjea mapping if the following conditions are satisfied:

(i) T is G-monotone (edge-preserving), that is, if:

$$(Tx, Ty) \in E(G), \text{ for every } ((x, y) \in E(G), \tag{8}$$

(ii) There exists $k \in [0, 1/2)$ such that for every $(x, y) \in E(G)$,

$$\mathcal{D}(Tx, Ty) \le k(\mathcal{D}(Tx, y) + \mathcal{D}(x, Ty)). \tag{9}$$

Remark 8. It follows immediately from the above definition that:

(i) If T is a G-Chatterjea mapping, then T is both a G^{-1}-Chatterjea and a $\tilde{G}$-Chatterjea mapping

(ii) Any Chatterjea mapping is a G_0-Chatterjea mapping, where the complete graph G_0 is defined by $V(G_0) = X$ and $E(G_0) = X \times X$

The following example shows that a G-Chatterjea mapping is not necessarily a Chatterjea mapping.

Example 1. Let $X = \{0, 1, 2, 3\}$. Consider the function $\mathcal{D}$ defined on X by $\mathcal{D}(x, y) = (x - y)^2$. It can be shown that $\mathcal{D}$ is a generalized metric with constant $C \ge 2$.

Consider the mapping $f : X \longrightarrow X$ defined by

$$\begin{cases} f(0) = 1 \\ f(1) = f(2) = 0, \\ f(3) = 1. \end{cases} \tag{10}$$

Since $\mathcal{D}(f(0), f(2)) = 1$ and $\mathcal{D}(f(0), 2)) + \mathcal{D}((0, f(2)) = 1$, f is not a Chatterjea mapping.

On the other hand, consider the digraph G with $V(G) = X$ and edges

$$E(G) = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 3), (1, 2), (1, 3), (2, 3)\}. \tag{11}$$

It can be easily seen that f is a G-Chatterjea mapping with constant $k \in [1/9, 1/2)$.

3. Main Results

In this section, we extend the fixed point theorems for G-Chatterjea mappings to the setting of a generalized metric space with a digraph.

Let $T : X \longrightarrow X$ be a mapping. Let $x_0 \in X$. Let $G[\mathbb{O}_T(x_0)]$ be the subgraph of G induced on the orbit $\mathbb{O}_T(x_0) := \{T^n x_0 : n \in \mathbb{N}\}$. The following technical lemmas are necessary for the proof of the main result in this work.

Lemma 9. *Let $T : X \longrightarrow X$ be a G-monotone mapping and suppose that there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ (respectively, $(Tx_0, x_0) \in E(G)$) and that the subgraph $G[\mathbb{O}_T(x_0)]$ is transitive. Then, $\{T^n x_0\}$ is a G-increasing (respectively, G-decreasing) sequence and $(T^m x_0, T^n x_0) \in E(G)$ (respectively, $(T^n x_0, T^m x_0) \in E(G)$) for any $m, n \in \mathbb{N}$ such that $m \le n$.*

Proof. Without loss of generality, assume that $(x_0, Tx_0) \in E(G)$. Since T is G-monotone, it follows that $(Tx_0, T^2 x_0) \in E(G)$. Induction on n yields$(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$. Therefore, $\{T^n x_0\}$ is a G-monotone increasing sequence. Since $(T^m x_0, T^{m+1} x_0), (T^{m+1} x_0, T^{m+2} x_0), \cdots, (T^{n-1} x_0, T^n x_0) \in E(G)$ and $G[\mathbb{O}_T(x_0)]$ is transitive, it follows that $(T^m x_0, T^n x_0) \in E(G)$.

The following notation will be used in the sequel:

$$\delta(\mathcal{D}, T, x_0) := \sup \left\{ \mathcal{D}\left(T^i x_0, x_0\right) : i \in \mathbb{N}^* \right\}. \tag{12}$$

Lemma 10. *Under the assumptions of Lemma 9, if T is a G-Chatterjea mapping with constant $k \in [0, 1/2)$, then*

(i) For every $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$, we have

$$\mathcal{D}(T^n x_0, T^m x_0) \le \delta_0 \left(\sum_{m}^{n+m-1} k^j \begin{pmatrix} j-1 \\ m-1 \end{pmatrix} + \sum_{n}^{n+m-1} k^j \begin{pmatrix} j-1 \\ n-1 \end{pmatrix} \right), \tag{13}$$

(ii) For every $(m, n) \in \mathbb{N}^{*2}$ such that $m \le n$, we have

$$\mathcal{D}(T^n x_0, T^m x_0) \le \frac{\delta_0}{1-2k} (2k)^m, \tag{14}$$

where $\delta_0 = \delta(\mathcal{D}, T, x_0)$.

Proof.

(i) The proof of this statement follows from the application of two-dimensional induction on $p = n + m$, for every $p \ge 2$

Since

$$\mathcal{D}\left(T^1 x_0, T^1 x_0\right) \le k\mathcal{D}(Tx_0, x_0) + k\mathcal{D}(x_0, Tx_0) \le \delta_0 (2k), \tag{15}$$

it is clear that the inequality (13) holds for $p = 2$ with $(m, n) = (1, 1)$.

Assume next that inequality (13) holds for any $(m', n') \in \mathbb{N}^{*2}$ be chosen in such a way that $n' + m' = p$; let $(m, n) \in \mathbb{N}^{*2}$ with $n + m = p + 1$.

Since T is a G-Chatterjea mapping and $(T^{n-1}x_0, T^{m-1}x_0) \in E(\tilde{G})$, it holds that

$$\mathscr{D}(T^n x_0, T^m x_0) \le k\left(\mathscr{D}\left(T^n x_0, T^{m-1}x_0\right) + \mathscr{D}\left(T^{n-1}x_0, T^m x_0\right)\right). \tag{16}$$

Since $n + (m-1) = p$ and $(n-1) + m = p$, the inductive hypothesis yields

$$\begin{aligned}
\mathscr{D}(T^n x_0, T^m x_0) &\le k\delta_0\left(\sum_{j=m}^{n+m-2} k^j \binom{j-1}{m-1} + \sum_{j=n-1}^{n+m-2} k^j \binom{j-1}{n-2}\right.\\
&\quad\left. + \sum_{j=m-1}^{n+m-2} k^j \binom{j-1}{m-2} + \sum_{j=n}^{n+m-2} k^j \binom{j-1}{n-1}\right)\\
&\le k\delta_0\left(\sum_{j=m}^{n+m-2} k^j \left(\binom{j-1}{m-1} + \binom{j-1}{m-2}\right) + k^{m-1}\right.\\
&\quad\left. + \sum_{j=n}^{n+m-2} k^j \left(\binom{j-1}{n-1} + \binom{j-1}{n-2}\right) + k^{n-1}\right)\\
&\le k\delta_0\left(\sum_{j=m}^{n+m-2} k^j \binom{j}{m-1} + \sum_{j=n}^{n+m-2} k^j\right.\\
&\quad\left. \cdot \binom{j}{n-1} + k^{m-1} + k^{n-1}\right)\\
&\le k\delta_0\left(\sum_{j=m-1}^{n+m-2} k^j \binom{j}{m-1} + \sum_{j=n-1}^{n+m-2} k^j \binom{j}{n-1}\right)\\
&\le \delta_0\left(\sum_{j=m}^{n+m-1} k^j \binom{j-1}{m-1} + \sum_{j=n}^{n+m-1} k^j \binom{j-1}{n-1}\right),
\end{aligned} \tag{17}$$

that inequality (13) holds for $(n, m) \in \mathbb{N}^*$ such that $n + m = p + 1$.

(ii) Let $n, m \in \mathbb{N}^*$; assume that $m \le n$. Since $\binom{j-1}{m-1} \le 2^{j-1}$ for any $j \in m, n+m-1$ and $\binom{j-1}{n-1} \le 2^{j-1}$ for any $j \in n, n+m-1$, it follows that

$$\sum_{j=m}^{n+m-1} k^j \binom{j-1}{m-1} \le \frac{1}{2}\sum_{j=m}^{n+m-1} (2k)^j \le \frac{1}{2(1-2k)}(2k)^m, \tag{18}$$

and that

$$\sum_{j=n}^{n+m-1} k^j \binom{j-1}{n-1} \le \frac{1}{2(1-2k)}(2k)^n \le \frac{1}{2(1-2k)}(2k)^m. \tag{19}$$

It follows from inequality (13) that

$$\mathscr{D}(T^n x_0, T^m x_0) \le \frac{\delta_0}{1-2k}(2k)^m. \tag{20}$$

Theorem 11. *Let $(X, \mathscr{D}, G)$ be a generalized, G-complete metric space and $T : X \longrightarrow X$ be a G-Chatterjea mapping with constant $k \in [0, 1/2)$. Suppose that there exists $x_0 \in X$ such that $\delta(\mathscr{D}, T, x_0) < \infty$, that $(x_0, Tx_0) \in E(\tilde{G})$, and that the subgraph $G[O_T(x_0)]$ is transitive. Under these assumptions, the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover, if one of the following conditions $(i)-(iii)$ holds, namely*

(i) T is subsequentially continuous

(ii) T is orbitally G-continuous

(iii) G satisfies property (JNRL) and $\mathscr{D}(x_0, T\omega) < \infty$

then ω is a fixed point of T.

Proof. Without loss of generality, it may be assumed that $(x_0, Tx_0) \in E(G)$. Select $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $m \le n$. From Lemma 9, it is clear that $(T^m x_0, T^n x_0) \in E(G)$. If T is a G-Chatterjea mapping, Lemma 10 yields

$$\mathscr{D}(T^n x_0, T^m x_0) \le \delta_0/(1-2k)(2k)^m. \tag{21}$$

Thus, $\{T^n x_0\}$ is a $\mathscr{D}$-Cauchy sequence. Since $(X, \mathscr{D}, G)$ is G-complete, the sequence $\{T^n x_0\}$ $\mathscr{D}$-converges to some $\omega \in X$.

(i) It follows from the subsequential continuity assumption on T that there exists a subsequence $\{T^{n_q}x_0\}$ such that $\{T^{n_q+1}x_0\}$ $\mathscr{D}$-converges to $T\omega$ as $n_q \longrightarrow \infty$. The uniqueness of the limit yields $T\omega = \omega$

(ii) Assume that T is orbitally G-continuous. Since $\{T^n x_0\}$ $\mathscr{D}$-converges to ω and $(T^n x_0, T^{n+1}x_0) \in E(G)$, it follows that $T(T^n x_0) \longrightarrow T\omega$. Likewise, $T(T^n x_0) = T^{n+1}x_0 \longrightarrow \omega$. Hence, $\omega = T\omega$

(iii) Assume that G satisfies Property (JNRL) and that $\mathscr{D}(x_0, T\omega) < \infty$. Since $\{T^n x_0\}$ is G-increasing and it $\mathscr{D}$-converges to $\omega \in X$, it follows that $(T^n x_0, \omega) \in E(G)$, for any $n \in \mathbb{N}$

Select $n \in \mathbb{N}$, $n \ge 1$. If T is a G-Chatterjea mapping, then necessarily

$$\begin{aligned}
\mathscr{D}(T^n x_0, T\omega) &\le k\mathscr{D}\left(T^{n-1}x_0, T\omega\right) + k\mathscr{D}(T^n x_0, \omega)\\
&\le k^2\mathscr{D}\left(T^{n-2}x_0, T\omega\right) + k^2\mathscr{D}\left(T^{n-1}x_0, \omega\right)\\
&\quad + k\mathscr{D}(T^n x_0, \omega).
\end{aligned} \tag{22}$$

It follows by induction on n, that for any $n \ge 1$,

$$\mathscr{D}(T^n x_0, T\omega) \le k^n \mathscr{D}(x_0, T\omega) + \sum_{j=1}^{n} k^j \mathscr{D}\left(T^{n+1-j}x_0, \omega\right). \tag{23}$$

Let $j \in 1, n$. Since $\{T^p x_0\}_{p \geq n}$ $\mathcal{D}$-converges to ω using $(\mathcal{D}_3)$, it follows that

$$\mathcal{D}\left(T^{n+1-j}x_0, \omega\right) \leq C\limsup_{p \longrightarrow \infty} \mathcal{D}\left(T^{n+1-j}x_0, T^p x_0\right). \quad (24)$$

Applying Lemma 10, we obtain

$$\mathcal{D}\left(T^{n+1-j}x_0, \omega\right) \leq \frac{C\delta_0}{1-2k} \limsup_{p \longrightarrow \infty} (2k)^{n+1-j} \leq \frac{C\delta_0}{1-2k} (2k)^{n+1-j}. \quad (25)$$

Then,

$$k^j \mathcal{D}\left(T^{n+1-j}x_0, \omega\right) \leq \frac{C\delta_0}{1-2k} (k)^j (2k)^{n+1-j} \leq \frac{C\delta_0}{1-2k} (2k)^{n+1} \frac{1}{2^j}. \quad (26)$$

Hence

$$\sum_{j=1}^{n} k^j \mathcal{D}\left(T^{n+1-j}x_0, \omega\right) \leq \frac{C\delta_0}{1-2k} (2k)^{n+1} \left(\sum_{j=1}^{n} \frac{1}{2^j}\right) \leq \frac{C\delta_0}{1-2k} (2k)^{n+1}. \quad (27)$$

Finally, inequality (23) becomes

$$\mathcal{D}(T^n x_0, T\omega) \leq k^n \mathcal{D}(x_0, T\omega) + \frac{C\delta_0}{1-2k} (2k)^{n+1}. \quad (28)$$

Since $\mathcal{D}(x_0, T\omega) < \infty$, it follows that $\lim_{n \longrightarrow \infty} \mathcal{D}(T^n x_0, T\omega) = 0$. Therefore, $\{T^n x_0\}$ $\mathcal{D}$-converges to $T\omega$. Uniqueness of the limit yields $T\omega = \omega$.

Proposition 12. *Suppose that T is G -Chatterjea. If T has two fixed points ω and ω' in X, such that $\mathcal{D}(\omega, \omega') < \infty$ and $(\omega, \omega') \in E(G)$, then $\omega = \omega'$.*

Proof. Suppose that $\omega, \omega' \in X$ are two fixed points of T such that $\mathcal{D}(\omega, \omega') < \infty$. Since T is a G-Chatterjea mapping, we have

$$\mathcal{D}\left(\omega, \omega'\right) = \mathcal{D}\left(T\omega, T\omega'\right) \leq k\left(\mathcal{D}\left(T\omega, \omega'\right) + \mathcal{D}\left(\omega, T\omega'\right)\right), \quad (29)$$

which implies that

$$\mathcal{D}\left(\omega, \omega'\right) \leq 2k\mathcal{D}\left(\omega, \omega'\right). \quad (30)$$

Hence

$$(1-2k)\mathcal{D}\left(\omega, \omega'\right) \leq 0. \quad (31)$$

Therefore, $\mathcal{D}(\omega, \omega') = 0$, i.e., $\omega = \omega'$.

The following example illustrates Theorem 11.

Example 2. Let X be the open interval $(-1, 1)$. Consider the function $\mathcal{D}$ defined on X as follows:

$$\mathcal{D}(x, y) = \begin{cases} 2(|x| + |y|) & \text{if either of } x = 0 \text{ or } y = 0, \\ \dfrac{|x| + |y|}{3} & \text{otherwise.} \end{cases} \quad (32)$$

It can be easily verified that (D_1) and (D_2) hold. For the validity of (D_3), observe first that for all $x \neq 0$, we have $\mathcal{C}(\mathcal{D}, X, x) = \varnothing$. If $x = 0$, then there exists a sequence $\{x_n\}$ such that $\lim_{n \longrightarrow \infty} \mathcal{D}(x_n, x) = 0$. Consider the sets $P := \{n \in \mathbb{N} : x_n \neq 0\}$ and $Q := \{n \in \mathbb{N} : x_n = 0\}$. We distinguish three cases:

If P is finite, then there exists $C \geq 1$ such that for any $y \in X$ it holds that

$$\mathcal{D}(0, y) = 2|y| \leq 2C|y| = C\limsup_{n \longrightarrow \infty} \mathcal{D}(x_n, y). \quad (33)$$

If Q is finite, then there exists $C \geq 6$ such that for any $y \in X$

$$\mathcal{D}(0, y) = 2|y| \leq C\frac{|y|}{3} = C\limsup_{n \longrightarrow \infty} \mathcal{D}(x_n, y). \quad (34)$$

If P and Q are infinite, there exist two increasing functions $\varphi, \psi : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $x_{\varphi(n)} \neq 0$, $x_{\psi(n)} = 0$, and $\{x_n\} = \{x_{\varphi(n)}\} \cup \{x_{\psi(n)}\}$. Then, for any $y \in X$

$$\limsup_{n \longrightarrow \infty} \mathcal{D}\left(x_{\varphi(n)}, y\right) = \frac{|y|}{3} \text{ and } \limsup_{n \longrightarrow \infty} \mathcal{D}\left(x_{\psi(n)}, y\right) = 2|y|. \quad (35)$$

Thus, $\mathcal{D}$ is a generalized metric with $C \geq 6$. Note that X is not a $\mathcal{D}$-compact space. Indeed, let $\{x_n\}_{n \in \mathbb{N}^*}$ be a sequence of X such that $x_n = 1 - 1/n$ and suppose that there exists a subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ wich $\mathcal{D}$-converges to an element x in X. Since $\lim_{n \longrightarrow \infty} \mathcal{D}(x_{\varphi(n)}, x) = 0$, we have

$$\lim_{n \longrightarrow \infty} \left|x_{\varphi(n)}\right| + |x| = 0. \quad (36)$$

Thus, $|x| = -1$. Contradiction.

Consider the graph G on X consisting of the transitive closure of the graph represented in Figure 1.

Note that

$$E(G) = \Delta \bigcup \left\{ \left(\frac{(-1)^n}{2^n}, 0\right), \left(\frac{(-1)^m}{2^m}, \frac{(-1)^n}{2^n}\right) : n, m \in \mathbb{N}^* \text{ and } n > m \right\}. \quad (37)$$

Let us prove that the space $(X, \mathcal{D})$ is G-complete. Let $\{x_n\}$ be a G-monotone, $\mathcal{D}$-Cauchy sequence in X. We have two cases:

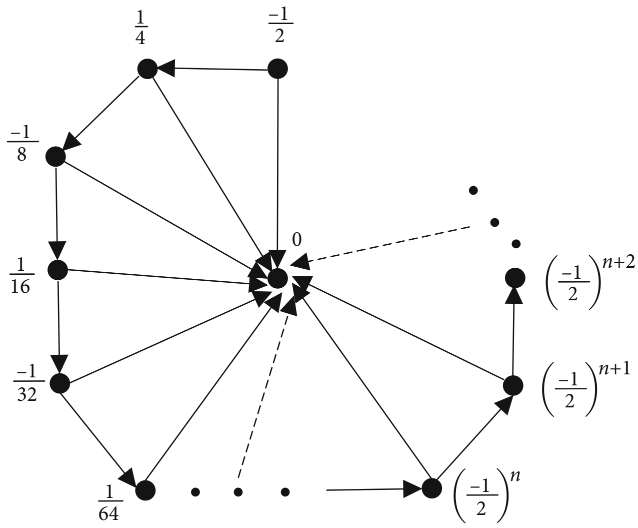

FIGURE 1: All loops and isolated vertices are not represented.

Case 1. If there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$, for any $n \geq n_0$. We have $\lim_{n\longrightarrow\infty} \mathscr{D}(x_n, 0) = 0$. Therefore, the sequence $\{x_n\}\mathscr{D}$-converges to 0.

Case 2. If there exists $n_0 \in \mathbb{N}$ and a nondecreasing sequence $\{p_n\} \subset \mathbb{N}^*$ such that $x_n = (-1/2)^{p_n}$, for any $n \geq n_0$. We have $\lim_{n\longrightarrow\infty} \mathscr{D}(x_n, 0) = \lim_{n\longrightarrow\infty} (1/3)(1/2)^{p_n} = 0$. Therefore, the sequence $\{x_n\}$ is $\mathscr{D}$-convergent to 0.

(Note that in the case where there exists a nonzero element α in X such that $x_n = \alpha$ for any $n \in \mathbb{N}$, the sequence $\{x_n\}$ is not $\mathscr{D}$-Cauchy).

Now, consider the self-mapping T on X defined by

$$Tx = \begin{cases} -\dfrac{x}{2} & \text{if } x \in X \cap \mathbb{Q}, \\ \dfrac{x}{5} & \text{otherwise.} \end{cases} \tag{38}$$

One can easily see that:

$$\begin{cases} \left(T\dfrac{(-1)^n}{2^n}, T0\right) = \left(\dfrac{(-1)^{n+1}}{2^{n+1}}, 0\right) \in E(G), \text{for any } n \in \mathbb{N}^*, \\ \left(T\dfrac{(-1)^m}{2^m}, T\dfrac{(-1)^n}{2^n}\right) = \left(\dfrac{(-1)^{m+1}}{2^{m+1}}, \dfrac{(-1)^{n+1}}{2^{n+1}}\right) \in E(G), \text{for any } n, m \in \mathbb{N}^* \text{ with } n > m. \end{cases} \tag{39}$$

It is therefore apparent that T is G-monotone. For $x_0 = -(1/2)$, we have that $(x_0, Tx_0) \in E(G)$, $G[\mathbb{O}_T(x_0)]$ is transitive and

$$\delta(T, \mathscr{D}, x_0) = \sup\left\{\mathscr{D}\left(\frac{(-1)^i}{2^i}, \frac{-1}{2}\right) : i \in \mathbb{N} \text{ and } i \geq 2\right\} = \frac{1}{4}. \tag{40}$$

Pick $x, y \in X$ such that $(x, y) \in E(G)$. If $x = y \in X \cap \mathbb{Q}$, then

$$\mathscr{D}(Tx, Tx) = \frac{|x|}{3} \leq k|x| = k(\mathscr{D}(Tx, x) + \mathscr{D}(x, Tx)). \tag{41}$$

Observe that if $x = y \in X \setminus \mathbb{Q}$, then

$$\mathscr{D}(Tx, Tx) = \frac{2|x|}{15} \leq k\frac{12|x|}{15} = k(\mathscr{D}(Tx, x) + \mathscr{D}(x, Tx)). \tag{42}$$

On the other hand, if $(x, y) = ((-1)^n/2^n, 0)$, then

$$\mathscr{D}\left(T\frac{(-1)^n}{2^n}, T0\right) = \frac{1}{2^n} \leq k\frac{3}{2^n} = k\left(\mathscr{D}\left(T\frac{(-1)^n}{2^n}, 0\right) + \mathscr{D}\left(\frac{(-1)^n}{2^n}, T0\right)\right). \tag{43}$$

Finally, in the case $(x, y) = ((-1)^m/2^m, (-1)^n/2^n))$, then

$$\begin{aligned} \mathscr{D}\left(T\frac{(-1)^m}{2^m}, T\frac{(-1)^n}{2^n}\right) &= \frac{2^n + 2^m}{3 \times 2^{n+m+1}} \leq k\frac{2^n + 2^m}{2^{n+m+1}} \\ &= k\left(\mathscr{D}\left(T\frac{(-1)^m}{2^m}, \frac{(-1)^n}{2^n}\right) + \mathscr{D}\left(\frac{(-1)^m}{2^m}, T\frac{(-1)^n}{2^n}\right)\right). \end{aligned} \tag{44}$$

In all cases, $\mathscr{D}(Tx, Ty) \leq k(\mathscr{D}(Tx, y) + \mathscr{D}(y, Tx))$. Thus, T is a G-Chatterjea mapping with constant $k \in [1/3, 1/2)$.

The sequence $\{T^n x_0\} = \{(-1)^{n+1}/2^{n+1}\}$ is G-increasing, $\mathscr{D}$-convergent to 0, and furthermore, $((-1)^{n+1}/2^{n+1}, 0) \in E(G)$. Hence, G satisfies Property (JNRL). On account of Theorem 11, T has a fixed point, namely 0.

Next, we present a version of Theorem 11 in the setting of a partially ordered generalized metric space. Let $(X, \mathscr{D}, \leq)$ be a generalized metric space endowed with a partial order. We define the directed graph $G_\leq$ on X as follows: $V(G_\leq) = X$ and $E(G_\leq) = \{(x, y) \in X \times X : x \leq y\}$. In this setting, we say that $T : X \longrightarrow X$ is a monotone Chatterjea mapping if it is a $G_\leq$-Chatterjea mapping. We also say that T is orbitally monotone continuous if T is orbitally $G_\leq$-continuous. The generalized metric space $(X, \mathscr{D}, \leq)$ satisfies the Property (JNRL) if whenever $\{x_n\}$ is a decreasing (respectively increasing) sequence such that $x_n \longrightarrow x$ in X, then for all $n \in \mathbb{N}$, $x \leq x_n$ (respectively $x_n \leq x$).

Theorem 13. *Let $(X, \mathscr{D}, \leq)$ be a generalized D-complete metric space endowed with a partial order and $T : X \longrightarrow X$ be a monotone Chatterjea mapping with constant $k \in [0, 1/2)$. Suppose that there exists $x_0 \in X$ such that $\delta(\mathscr{D}, T, x_0) < \infty$ and that either $x_0 \leq Tx_0$ or $Tx_0 \leq x_0$. Then, the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover, if any one of the conditions $(i) - (iii)$ in Theorem 11 holds, then ω is a fixed point of T.*

Proof. Since the subgraph $G_{\leq}[\mathbb{O}_T(x_0)]$ is transitive, Theorem 13 is a direct consequence of Theorem 11.

We remark that Theorem 3.9 in [19] is a corollary of the preceding theorem, from which it can be derived simply by removing the ordering.

Corollary 14. *Let $(X, \mathscr{D})$ be a D-complete generalized metric space and $T : X \longrightarrow X$ be a Chatterjea contraction with constant $k \in [0, 1/2)$. Suppose that there exists $x_0 \in X$ such that $\delta(\mathscr{D}, T, x_0) < \infty$. Then, $\{T^n x_0\}$ converges to some $\omega \in X$. If $\mathscr{D}(x_0, T\omega) < \infty$, then ω is a fixed point of T with $\mathscr{D}(\omega, \omega) = 0$. Moreover, if $\omega' \in X$ is another fixed point of T such that $\mathscr{D}(\omega, \omega') < \infty$, then $\omega = \omega'$.*

Proof. Taking $G = G_0$, where G_0 is the complete graph, i.e., $V(G_0) = X$ and $E(G_0) = X \times X$, the proof follows from Theorem 11 and Proposition 12.

We next set to show that the fixed point result given in Theorem 11 is, in fact, a generalization of the analogue common fixed point theorem established in [13]. To this effect, we state and prove the following lemma, introduced by Haghi et al. in [20].

Lemma 15. *Let X be a nonempty set and $f : X \longrightarrow X$ a function. Then, there exists a subset $E \subset X$ such that $f(E) = f(X)$. Moreover, $f : E \longrightarrow X$ is one-to-one.*

Let $T, S : X \longrightarrow X$ be two self mappings. We recall the definition of G-Chatterjea S-contraction and the property (P) given in [13].

Definition 16. We say that T is G-Chatterjea S-contraction if there exists $k \in [0, 1/2)$ such that for every $x, y \in V(G)$, it holds that

$$(Sx, Sy) \in E\left(\tilde{G}\right) \mathscr{D}(Tx, Ty) \leq k(\mathscr{D}(Tx, Sy) + \mathscr{D}(Sx, Ty)). \tag{45}$$

We recall that x^* is said to be a point of coincidence of T and S, if there exists a in X such that $x^* = Ta = Sa$.

Property (P). The digraph G is said to satisfy the property (P) for T and S, if whenever x^*, y^* are points of coincidence of T and S in $V(G)$, then $(x^*, y^*) \in E(\tilde{G})$ and $\mathscr{D}(x^*, y^*) < \infty$.

Suppose that $T(X) \subseteq S(X)$. If $x_0 \in X$ is arbitrary, we can choose a point x_1 in X such that $Tx_0 = Sx_1$. Proceeding in this manner, assuming that x_n in X is given, we can define $x_{n+1} \in X$ by the recurrence relation

$$Tx_n = Sx_{n+1}, n = 0, 1, 2,. \cdots \tag{46}$$

By $C(T, S)$, we denote the set of all elements x_0 of X such that $(Sx_n, Sx_m) \in E(\tilde{G})$, for $n, m = 1, 2, \cdots$. The following notation will be used in the sequel:

$$\delta(\mathscr{D}, S, T, x_0) = \sup\left\{\mathscr{D}\left(Sx_p, Sx_1\right): p \geq 2\right\}. \tag{47}$$

Corollary 17. *Let $(X, \mathscr{D})$ be a generalized metric space endowed with a reflexive digraph G. Assume that $V(G) = X$, that G has no parallel edges, and that it satisfies the (JNRL) property. Let T and S be two self mappings on X such that T is a G-Chatterjea S-contraction, $S(X)$ is a $\mathscr{D}$-complete subspace of X and that $T(X) \subseteq S(X)$.*

(1) Suppose that there exists $x_0 \in C(T, S)$ such that $\delta(\mathscr{D}, S, T, x_0) < \infty$. Then, the sequence $\{Sx_n\}$ defined by (46) $\mathscr{D}$-converges to $x^* = Sa$, with $a \in X$. Moreover, if $\mathscr{D}(Tx_0, Ta) < \infty$, then x^* is a point of coincidence of T and S in X

(2) In addition, T and S have a unique point of coincidence in X if the digraph G has the property (P) for T and S. Finally, if T and S are weakly compatible, then T and S have a unique common fixed point in X

Proof. By Lemma 15, there exists $X_0 \subset X$ such that $S(X_0) = S(X) = Y$; moreover, $S : X_0 \longrightarrow X$ is one-to-one. Define the mapping $F : Y \longrightarrow Y$ as

$$F(Sx) = Tx. \tag{48}$$

Since S is one-to-one on X_0, F is well defined.

Let $u, v \in Y$. There exist $x, y \in X$ such that $u = Sx$ and $v = Sy$. If $(u, v) \in E(\tilde{G})$, then $(Sx, Sy) \in E(\tilde{G})$. Since T is a G-Chatterjea S-contraction, there exists $k \in [0, 1/2)$ such that

$$\mathscr{D}(Tx, Ty) \leq k(\mathscr{D}(Tx, Sy) + \mathscr{D}(Sx, Ty)), \tag{49}$$

i.e., $\mathscr{D}(F(Sx), F(Sy)) \leq k(\mathscr{D}(F(Sx), Sy) + \mathscr{D}(Sx, F(Sy)))$. Then

$$\mathscr{D}(Fu, Fv) \leq k(\mathscr{D}(Fu, v) + \mathscr{D}(u, Fv)). \tag{50}$$

Consequently, F is a G-Chatterjea mapping on Y.

Suppose that there exists $x_0 \in C(T, S)$ such that $\delta(\mathscr{D}, S, T, x_0) < \infty$. Setting $y_0 = Sx_1$, it is clear that $Fy_0 = F(Sx_1) = Tx_1 = Sx_2$. It follows easily by induction that $F^p y_0 = Sx_{p+1}$, for any $p \in \mathbb{N}$. Moreover,

$$\begin{aligned}\delta(\mathscr{D}, F, y_0) &= \sup\left\{\mathscr{D}\left(F^i y_0, y_0\right): i \geq 1\right\}\\ &= \sup\left\{\mathscr{D}\left(F^{p-1} y_0, y_0\right): p \geq 2\right\}\\ &= \sup\left\{\mathscr{D}\left(Sx_p, Sx_1\right): p \geq 2\right\} = \delta(\mathscr{D}, S, T, x_0) < \infty.\end{aligned} \tag{51}$$

From $(Sx_n, Sx_m) \in E(\tilde{G})$ for $n, m = 1, 2, \cdots$, it follows that $(F^i y_0, F^j y_0) \in E(\tilde{G})$ for any i, j in $\mathbb{N}$. Hence, $G[O_F(y_0)]$ is transitive. Furthermore, $(Sx_1, Sx_2) = (y_0, Fy_0) \in E(\tilde{G})$.

By virtue of Theorem 11, the sequence $\{Sx_n\} = \{F^{n-1} y_0\}$ $\mathscr{D}$-converges to $x^* = Sa$ with $a \in Y \subset X$.

Moreover, we have

$$\mathscr{D}(y_0, Fx^*) = \mathscr{D}(Sx_1, F(Sa)) = \mathscr{D}(Tx_0, Ta) < \infty, \tag{52}$$

and since G satisfies property (JNRL), on account of Theorem 11, x^* is a fixed point of F. Hence $Ta = F(Sa) = Fx^* = x^* = Sa$, and x^* is a point of coincidence of T and S in X, as claimed.

Assume next that there exists another point of coincidence $y^* \in S(X)$, that $b \in X$, and that $y^* = Sb = Tb = F(Sb) = Fy^*$. Since the digraph G has the property (P) for T and S, then $(x^*, y^*) \in E(\tilde{G})$ and $\mathscr{D}(x^*, y^*) < \infty$. By Proposition 12, necessarily $x^* = y^*$, which implies that T and S have a unique point of coincidence in X. It follows from [21] (Proposition 1.4) that if T and S are weakly compatible, then T and S have a unique common fixed point in X.

4. Application

In this section, we study the existence and uniqueness of solution for the following general nonlinear matrix equation in the set of all $n \times n$ Hermitian-positive definite matrices $\mathscr{P}(n)$:

$$X^q - A^* \mathscr{F}(X)^s A = B, q > \sqrt{2}, s \in (0,1], X \in \mathscr{P}(n), \qquad (53)$$

where A is $n \times n$ nonsingular matrix, A^* is the Hermitian transpose of the matrix A, the matrix B is $n \times n$ positive define matrix, and $\mathscr{F} : \mathscr{E}(n) \longrightarrow \mathscr{E}(n)$ is a self-adjoint operator such that $\mathscr{E}(n)$ is a nonempty subset of $\mathscr{P}(n)$. This type of matrix equation arises in control theory, ladder networks, dynamic programming, stochastic filtering and statistics, etc.

For $M, N \in \mathscr{P}(n)$, we denote

$$M \prec N \Leftrightarrow N - M \text{ is positive definite}. \qquad (54)$$

We denote by $\|.\|$ the spectral norm $\|A\| = \sqrt{\rho(A^*A)} = \|A^*\|$, where $\rho(A^*A)$ is the largest eigenvalue of A^*A. We recall that the Thompson metric is defined on $\mathscr{P}(n)$ by:

$$d : \mathscr{P}(n) \times \mathscr{P}(n) \longrightarrow \mathbb{R}_+, \qquad (55)$$

such that

$$\begin{aligned} d(A,B) &= \max\left\{\ln\left(\mathscr{W}\left(\frac{A}{B}\right)\right), \ln\left(\mathscr{W}\left(\frac{B}{A}\right)\right)\right\} \\ &= \left\|\ln\left(A^{-(1/2)}BA^{-(1/2)}\right)\right\|, \end{aligned} \qquad (56)$$

where $\mathscr{W}(A/B) = \inf\{\lambda > 0 : A \le \lambda B\} = \lambda \max (A^{-(1/2)}B A^{-(1/2)})$. It is easy to verify that $(\mathscr{P}(n), d)$ is a complete metric space (see [22]). In the sequel, we consider the space $\mathscr{P}(n)$ endowed by the Thompson generalized metric $\mathscr{D}$ defined by

$$\mathscr{D}(A,B) = \left\|\ln\left(A^{-(1/2)}BA^{-(1/2)}\right)\right\|^2, \qquad (57)$$

for any $A, B \in \mathscr{P}(n)$. In the following lemmas, we extend some properties of the Thompson metric given in [23] to the Thompson generalized metric space.

Lemma 18. *Let $\mathscr{D} : \mathscr{P}(n) \times \mathscr{P}(n) \longrightarrow \mathbb{R}_+$ be a Thompson generalized metric on the open convex cone $\mathscr{P}(n)$; then, for any $A,B \in \mathscr{P}(n)$ and nonsingular matrix M, we have the following conditions:*

(i) $\mathscr{D}(A,B) = \mathscr{D}(A^{-1}, B^{-1}) = \mathscr{D}(M^*AM, M^*BM)$, where A^{-1}, B^{-1} are the inversion of matrices A and B, respectively

(ii) $\mathscr{D}(A^r, B^r) \le r^2\mathscr{D}(A,B), r \in [-1,1]$

(iii) $\mathscr{D}(M^*A^rM, M^*B^rM) \le r^2\mathscr{D}(A,B), r \in [-1,1]$

Proof. Let d be the Thompson metric defined by (56). Since $\mathscr{D}(A,B) = (d(A,B))^2$, we get (i), (ii), and (iii) by the invariant under the matrix inversion, congruence transformations for nonsingular matrix M, and the nonpositive curvature property of the Thompson metric d.

Lemma 19. *For any $A, B, C, D \in \mathscr{P}(n)$,*

$$\mathscr{D}(A+C, B+D) \le \max\{\mathscr{D}(A,B), \mathscr{D}(C,D)\}. \qquad (58)$$

Especially, $\mathscr{D}(A+C, B+C) \le \mathscr{D}(A,B)$.

Proof. Let d be the Thompson metric defined by (56). By using [23] (Lemma 2.1), we have $d(A+C, B+D) \le \max\{d(A,B), d(C,D)\}$. Since $\mathscr{D}(A,B) = (d(A,B))^2$, we deduce our result.

We endow $\mathscr{P}(n)$ by the graph G defined by:

$$V(G) = \mathscr{P}(n) \text{ and } E(G) = \Delta \cup \{(M,N) \in \mathscr{P}(n) \times \mathscr{P}(n), M \prec N\}. \qquad (59)$$

We give a graphical version of [24] (Lemma 4.3) in $\mathscr{P}(n)$ endowed with the graph G.

Lemma 20. *For any $A, B \in \mathscr{P}(n)$, if $(A,B) \in E(G)$, then $(A^r, B^r) \in E(G)$ for all $r \in]0,1]$, and $(B^r, A^r) \in E(G)$ for all $r \in [-1,0[$.*

Proof. Let $A, B \in \mathscr{P}(n)$ such that $(A,B) \in E(G)$. If $A = B$ then $(A^r, B^r) \in E(G)$ for all $r \in]0,1]$. If $A \prec B$, then by using the Löwner-Heinz inequality [25, 26], we get $A^r \prec B^r$. Thus, $(A^r, B^r) \in E(G)$ for all $r \in]0,1]$.

Theorem 21. *Let $X_0 \in \mathscr{P}(n)$ and $\mathscr{E}(n) = \{X \in \mathscr{P}(n) : (X_0, X) \in E(G)\}$. If the operator $\mathscr{F}$ is nondecreasing and for all $X, Y \in \mathscr{E}(n)$ such that $(X,Y) \in E(G)$, we have:*

$$\begin{aligned} &\left\|\ln\left(\mathscr{F}(X)^{-(1/2)}F(Y)F(X)^{-(1/2)}\right)\right\| \\ &\quad \le \left\|\ln\left(X^{-(1/2)}(B + A^*\mathscr{F}(Y)^sA)^{1/q}X^{-(1/2)}\right)\right\| \end{aligned} \qquad (60)$$

and $(X_0^q, B) \in E(G)$; then, the matrix equation (53) has a unique solution.

Proof. Let $T : \mathscr{E}(n) \longrightarrow \mathscr{E}(n)$ be a mapping defined by

$$T(X) = \left(B + A^* \mathscr{F}(X)^s A\right)^{1/q}, X \in \mathscr{E}(n). \quad (61)$$

Let $X \in \mathscr{E}(n)$. Since $(X_0, X) \in E(G)$ and F is nondecreasing,

$$(B + A^* F(X_0)^s A, B + A^* \mathscr{F}(X)^s A) \in E(G). \quad (62)$$

As $(X_0^q, B) \in E(G)$ and $A^* F(X_0)^s A \in \mathscr{P}(n)$, then $(X_0^q, B + A^* F(X_0)^s A) \in E(G)$. By Lemma 20, we have

$$\left(X_0, (B + A^* F(X)^s A)^{1/q}\right) = (X_0, T(X)) \in E(G). \quad (63)$$

Thus, $T(X) \in \mathscr{E}(n)$ and so T is well defined.

Let $X, Y \in \mathscr{E}(n)$ such that $(X, Y) \in E(G)$, we have $(T(Y))^q - (T(X))^q = A^*(F(Y)^s - F(X)^s)A$, then $((T(X))^q, (T(Y))^q) \in E(G)$. Since $0 \le 1/q < 1$, by Lemma 20 we have $(T(X), T(Y)) \in E(G)$.

Let X, Y be two elements in $\mathscr{E}(n)$ such that $(X, Y) \in E(G)$. By using Lemmas 18 and 19 we have

$$\begin{aligned}
\mathscr{D}(F(X), F(Y)) &\ge \frac{1}{s^2}\mathscr{D}(F(X)^s, F(Y)^s) \ge \frac{1}{s^2}\mathscr{D}(A^* F(X)^s A, A^* F(Y)^s A) \\
&\ge \frac{1}{s^2}\mathscr{D}(B + A^* F(X)^s A, B + A^* F(Y)^s A) \\
&\ge \frac{1}{s^2}\mathscr{D}(T(X)^q, T(Y)^q) \ge \left(\frac{s}{q}\right)^2 \mathscr{D}(T(X), T(Y)).
\end{aligned} \quad (64)$$

Thus,

$$\mathscr{D}(T(X), T(Y)) \le \left(\frac{s}{q}\right)^2 \mathscr{D}(F(X), F(Y)). \quad (65)$$

If $\|\ln(\mathscr{F}(X)^{-(1/2)} F(Y) F(X)^{-(1/2)})\| \le \|\ln(X^{-(1/2)}(B + A^* \mathscr{F}(Y)^s A)^{1/q} X^{-(1/2)})\|$, then

$$\mathscr{D}(F(X), F(Y)) \le \mathscr{D}(X, T(Y)). \quad (66)$$

From (65) and (66), we get

$$\mathscr{D}(T(X), T(Y)) \le \left(\frac{s}{q}\right)^2 (\mathscr{D}(X, T(Y)) + \mathscr{D}(Y, T(X))). \quad (67)$$

Thus, there exists $k = (s/q)^2 \in [0, 1/2)$ such that T is a G-Chatterjea mapping on $\mathscr{E}(n)$.

Since $(X_0^q, B) \in E(G)$, $(X_0^q, B + A^* F(X_0)^s A) \in E(G)$. Thus, $(X_0, T(X_0)) \in E(G)$.

Next, we show that $\mathscr{E}(n)$ satisfies the Property (JNRL) for the generalized metric $\mathscr{D}$. In fact, let $(X_k)_k$ be a nondecreasing sequence of $\mathscr{E}(n)$ which converges to $X \in \mathscr{E}(n)$. If the set $\{k \in \mathbb{N} : X_k = X_0\}$ is infinite, there exists a nondecreasing function $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $X_{\phi(k)} = X_0$, $\forall k \in \mathbb{N}$, then $X = X_0 \in \mathscr{E}(n)$. If not, $X_k \ne X_0$, for large integer k. Fix $m \in \mathbb{N}$ arbitrary. For all $k > m$,

$$(X_m, X_k) \in E(G) \Longrightarrow X_k - X_m \in \mathscr{P}(n) \Longrightarrow X_k \in \mathscr{P}(n) + X_m. \quad (68)$$

Since $\mathscr{P}(n) + X_m$ is closed, $X \in \mathscr{P}(n) + X_m$. Thus, $(X_m, X) \in E(G)$, for all $m \in \mathbb{N}$. Thus, according to Theorem 11, we can show that there exists $X^* \in \mathscr{E}(n)$ such that $T(X^*) = X^*$ which is a solution of the matrix equation (53).

If equation (53) has another solution Y^* such that $(X^*, Y^*) \in E(\tilde{G})$, then using Proposition 12, we have $Y^* = X^*$.

5. Conclusion

Summarizing the present work enhances the area in many directions

(1) Using Theorem 13, we can improve the following results

  (i) Theorem 2 in [27]

  (ii) Theorem 2.12 in [18]

  (iii) Theorem 3.9 in [19]

  (iv) Theorem 8 in [13]

(2) Establish or improve Chatterjea fixed point theorems in the setting of standard metric spaces, dislocated metric spaces, b-metric spaces, and modular spaces with the Fatou property, also in these spaces endowed with a partial order and more generally with a graph

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Common Fixed Point Theorems for Contractive Mappings of Integral Type in G-Metric Spaces and Applications

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Two common fixed point theorems for weakly compatible mappings satisfying contractive conditions of integral type in G-metric spaces are demonstrated. The results obtained in this paper generalize and differ from a few results in the literature and are used to prove the existence and uniqueness of common bounded and continuous solutions for certain functional equations and nonlinear Volterra integral equations. A nontrivial example is included.

1. Introduction

The Banach fixed point theorem which was first presented by Banach in 1922 is a significant result in fixed point theory. Because of its importance in proving the existence of solutions for functional equations, nonlinear Volterra integral equations and nonlinear integro-differential equations, this result has been extended in many different directions (see, e.g., [1–22] and the references cited therein). In particular, Rhoades [12] and Branciari [4] generalized the Banach fixed point theorem and gave the following fixed point theorems, respectively.

Theorem 1 (see [12]). *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(fx, fy) \le d(x, y) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{1}$$

where $\varphi \in \Phi_4$. Then, f has a unique fixed point in X.

Theorem 2 (see [4]). *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{2}$$

where $\varphi \in \Phi_1$ and $c \in [0, 1)$ is a constant. Then, f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

In 2013, Gupta and Mani [21] obtained the existence and uniqueness of a fixed point for contractive mappings of an integral type in complete metric spaces by using iterative approximations. In 2007, Kumar et al. [6] proved a common fixed point theorem for a pair of compatible mappings satisfying a contractive inequality of integral type, which improves Theorem 2.

Theorem 3 (see [6]). *Let (X, d) be a complete metric space and $f, g : X \to X$ be compatible mappings such that*

$$\begin{gathered} f(X) \subseteq g(X), g \text{ is continuous}, \\ \int_0^{d(fx,fy)} \varphi(t)dt \le c\int_0^{d(gx,gy)} \varphi(t)dt, \quad \forall x, y \in X, \end{gathered} \tag{3}$$

where $\varphi \in \Phi_1$ and $c \in [0, 1)$ is a constant. Then, f and g have a unique common fixed point in X.

In 2006, Mustafa and Sims [9] introduced a new concept of generalized metric space called G-metric space. From then on, lots of research works have been carried out on generalizing contractive conditions for different contractive mappings satisfying various known properties in G-metric spaces [1–3, 5, 10, 11, 13, 15, 19, 20]. In 2018, Gupta et al. [19] proved some fixed point theorems for the functions satisfying ϕ -contraction and mixed g-monotone property in G-metric spaces. In 2015, Gupta and Deep [20] gave a few common fixed point theorems using the property E.A. in the setting of G-metric and fuzzy metric spaces by taking a set of three conditions for self-mappings. In 2011, Aydi [1] proved a fixed point theorem for mappings satisfying a (ψ, ϕ)-weakly contractive condition in G-metric spaces.

Theorem 4 (see [1]). *Let (X, G) be a complete G-metric space and f be a mapping from X into itself satisfying*

$$\psi(G(fx, fy, fz)) \leq \psi(G(x, y, z)) - \phi(G(x, y, z)), \quad \forall x, y, z \in X, \tag{4}$$

where $\psi, \phi \in \Phi_2$. Then, f has a unique fixed point $u \in X$ and f is G-continuous at u.

In 2012, Aydi [2] obtained the following common fixed point theorem for a pair of mappings involving a contractive condition of integral type in G-metric spaces.

Theorem 5 (see [2]). *Let (X, G) be a G-metric space and f, g be two mappings from X into itself such that*

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt \leq \alpha \int_0^{G(gx,gy,gz)} \varphi(t)dt, \quad \forall x, y, z \in X, \tag{5}$$

where $\varphi \in \Phi_1$ and $\alpha \in [0, 1)$ is a constant. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

The objective of this paper is both to introduce two new classes of contractive mappings of integral type in the setting of G-metric spaces and to prove the existence and uniqueness of points of coincidence and common fixed points for these mappings. Our results extend Theorem 5, are different from Theorem 4, and are used to show solvability of the functional equations arising in dynamic programming and nonlinear Volterra integral equations. A nontrivial example is given.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R}^+ = [0,+\infty)$, and $\mathbb{R} = (-\infty, +\infty)$. Put

$$\Phi_1 = \Big\{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is Lebesgue integrable and summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^{\epsilon} \varphi(t)dt > 0, \forall \epsilon > 0\Big\},$$

$$\Phi_2 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a continuous and nondecreasing function such that } \varphi(t) = 0 \text{ if and only if } t = 0\},$$

$$\Phi_3 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ satisfies that } \liminf_{n\to\infty} \varphi(a_n) > 0 \leftrightarrow \liminf_{n\to\infty} a_n > 0 \text{ for each } \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+\},$$

$$\Phi_4 = \Big\{\varphi \mid \varphi \in \Phi_2 \text{ and} \lim_{t\to+\infty} \varphi(t) = +\infty\Big\},$$

$$\Phi_5 = \{\varphi \mid \varphi \in \Phi_2 \text{ and} \, \varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2), \quad \forall t_1, t_2 \in \mathbb{R}^+\}. \tag{6}$$

Definition 6 (see [9]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$,

$(G2)$ $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

$(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables,

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

The function G is called a G-metric in X, and the pair (X, G) is called a G-metric space.

Definition 7 (see [9]). Let (X, G) be a G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of X. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0, \tag{7}$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$G(x, x_n, x_m) < \varepsilon, \quad \forall m, n \geq N. \tag{8}$$

The point x is called the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and write$x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Lemma 8 (see [9]). *Let (X, G) be a G-metric space. Then, the following statements are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ is G-convergent to x,

(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(3) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Definition 9 (see [9]). Let (X, G) be a G-metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called G-Cauchy if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$G(x_l, x_n, x_m) < \varepsilon, \quad \forall l, n, m \geq N, \tag{9}$$

that is, $G(x_l, x_n, x_m) \to 0$as$l, n, m \to \infty$.

Lemma 10 (see [9]). *Let (X, G) be a G-metric space. Then, the following statements are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ is G-Cauchy,

(2) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Definition 11 (see [9]). A G-metric space (X, G) is called G -complete if every G-Cauchy sequence is G-convergent in (X, G).

Lemma 12 (see [9]). *Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Lemma 13 (see [9]). *Let (X, G) be a G-metric space. Then,*

$$\begin{gathered} G(x, y, y) \leq 2G(y, x, x), \quad \forall x, y \in X, \\ |G(x, y, z) - G(x, y, a)| \leq \max\{G(a, z, z), G(z, a, a)\}, \quad \forall x, y, z, a \in X. \end{gathered} \tag{10}$$

Definition 14 (see [14]). Let$S$ and T be self-mappings of a nonempty set X.

(1) A point $x \in X$ is said to be a fixed point of T if $Tx = x$.

(2) A point $x \in X$ is said to be a coincidence point of S and T if $Tx = Sx$ and $w = Sx = Tx$ is said to be a point of coincidence of S and T.

(3) A point $x \in X$ is said to be a common fixed point of S and T if $x = Tx = Sx$.

Definition 15. A pair of self-mappings f and g in a G-metric space (X, G) are said to be weakly compatible if for any $x \in X$, the equality $fx = gx$ gives that $fgx = gfx$.

Lemma 16 (see [14]). *Let X be a nonempty set and $f, g : X \to X$ be weakly compatible mappings. If f and g have a unique point of coincidence $w \in X$, then w is the unique common fixed point of f and g.*

Lemma 17 (see [7]). *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then,*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt. \tag{11}$$

Lemma 18 (see [8]). *Let $\varphi \in \Phi_3$. Then, $\varphi(t) > 0$ if and only if $t > 0$.*

3. Main Results

Now, we study the existence and uniqueness of points of coincidence and common fixed points for contractive mappings (12) and (51) below in G-metric spaces, respectively.

Theorem 19. *Let (X, G) be a G-metric space, f and $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} \psi\left(\int_0^{G(fx,fy,fz)} \varphi(t)dt\right) &\leq \psi\left(\int_0^{M_1(x,y,z)} \varphi(t)dt\right) \\ &\quad - \phi\left(\int_0^{M_1(x,y,z)} \varphi(t)dt\right), \quad \forall x, y, z \in X, \end{aligned} \tag{12}$$

where $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and

$$\begin{aligned} M_1(x, y, z) = \max\Bigg\{ & G(gx, gy, gz), \frac{[1 + G(gx, gx, fx)]G(gy, gy, fz)}{2 + G(gx, fx, fy)}, \\ & \frac{[1 + G(gx, gx, fx)]G(gz, gz, fy)}{2 + G(gx, fx, fz)}, \\ & \frac{[1 + G(gy, gy, fy)]G(gz, gz, fx)}{2 + G(gy, fy, fz)}, \\ & \frac{[1 + G(gy, gy, fy)]G(gx, gx, fz)}{2[1 + G(gy, fx, fy)]}, \\ & \frac{[1 + G(gz, gz, fz)]G(gy, gy, fx)}{2[1 + G(gz, fy, fz)]}, \\ & \frac{[1 + G(gz, gz, fz)]G(gx, gx, fy)}{2[1 + G(gz, fx, fz)]}, \\ & \frac{[1 + G(gx, gy, gz)]G(fx, fy, fz)}{1 + G(gz, fy, fy) + G(gx, gy, gz)}, \\ & \frac{[1 + G(gx, gy, gz)]G(fx, fy, fz)}{1 + G(gy, fx, fx) + G(gx, gy, gz)}, \\ & \frac{[1 + G(gx, gy, gz)]G(fx, fy, fz)}{1 + G(gx, fz, fz) + G(gx, gy, gz)} \Bigg\}. \end{aligned} \tag{13}$$

If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subset of X, then f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible mappings, then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$, it follows that there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X satisfying

$$fx_n = gx_{n+1}, \quad \forall n \in \mathbb{N}_0. \tag{14}$$

Put $G_n = G(fx_n, fx_{n+1}, fx_{n+2})$ for all $n \in \mathbb{N}_0$. Assume that $fx_{n_0} = fx_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$. It is clear that $gx_{n_0+1} = fx_{n_0+1}$, that is, fx_{n_0+1} is a point of coincidence of f and g. Assume that $fx_n \neq fx_{n+1}$ for all $n \in \mathbb{N}_0$. Clearly, $G_n > 0$ for all $n \in \mathbb{N}_0$. By virtue of $(G3)$–$(G5)$ and (14), we observe that

$$\begin{aligned}
&\frac{[1+G(gx_{n+1},gx_{n+1},fx_{n+1})]G(gx_n,gx_n,fx_{n+2})}{2[1+G(gx_{n+1},fx_n,fx_{n+1})]}\\
&=\frac{[1+G(fx_n,fx_n,fx_{n+1})]G(fx_{n-1},fx_{n-1},fx_{n+2})}{2[1+G(fx_n,fx_n,fx_{n+1})]}\\
&=\frac{G(fx_{n-1},fx_{n-1},fx_{n+2})}{2}=\frac{G(fx_{n+2},fx_{n-1},fx_{n-1})}{2}\\
&\le\frac{G(fx_{n+2},fx_n,fx_n)+G(fx_n,fx_{n-1},fx_{n-1})}{2}\\
&=\frac{G(fx_n,fx_n,fx_{n+2})+G(fx_{n-1},fx_{n-1},fx_n)}{2}\\
&\le\frac{G(fx_n,fx_{n+2},fx_{n+1})+G(fx_{n-1},fx_n,fx_{n+1})}{2}\\
&=\frac{G_{n-1}+G_n}{2},\quad \forall n\in\mathbb{N}.
\end{aligned}\tag{15}$$

Set

$$\begin{aligned}
m_1(x_n,x_{n+1},x_{n+2})=\max\Bigg\{&\frac{[1+G(gx_n,gx_n,fx_n)]G(gx_{n+1},gx_{n+1},fx_{n+2})}{2+G(gx_n,fx_n,fx_{n+1})},\\
&\frac{[1+G(gx_n,gx_n,fx_n)]G(gx_{n+2},gx_{n+2},fx_{n+1})}{2+G(gx_n,fx_n,fx_{n+2})},\\
&\frac{[1+G(gx_{n+1},gx_{n+1},fx_{n+1})]G(gx_{n+2},gx_{n+2},fx_n)}{2+G(gx_{n+1},fx_{n+1},fx_{n+2})},\\
&\frac{[1+G(gx_{n+1},gx_{n+1},fx_{n+1})]G(gx_n,gx_n,fx_{n+2})}{2[1+G(gx_{n+1},fx_n,fx_{n+1})]},\\
&\frac{[1+G(gx_{n+2},gx_{n+2},fx_{n+2})]G(gx_{n+1},gx_{n+1},fx_n)}{2[1+G(gx_{n+2},fx_{n+1},fx_{n+2})]},\\
&\frac{[1+G(gx_{n+2},gx_{n+2},fx_{n+2})]G(gx_n,gx_n,fx_{n+1})}{2[1+G(gx_{n+2},fx_n,fx_{n+2})]}\Bigg\},\quad \forall n\in\mathbb{N},
\end{aligned}\tag{16}$$

which together with (15) yields that

$$\begin{aligned}
m_1(x_n,x_{n+1},x_{n+2})=\max\Bigg\{&\frac{[1+G(fx_{n-1},fx_{n-1},fx_n)]G(fx_n,fx_n,fx_{n+2})}{2+G(fx_{n-1},fx_n,fx_{n+1})},\\
&\frac{[1+G(fx_{n-1},fx_{n-1},fx_n)]G(fx_{n+1},fx_{n+1},fx_{n+1})}{2+G(fx_{n-1},fx_n,fx_{n+2})},\\
&\frac{[1+G(fx_n,fx_n,fx_{n+1})]G(fx_{n+1},fx_{n+1},fx_n)}{2+G(fx_n,fx_{n+1},fx_{n+2})},\\
&\frac{[1+G(fx_n,fx_n,fx_{n+1})]G(fx_{n-1},fx_{n-1},fx_{n+2})}{2[1+G(fx_n,fx_n,f_{n+1})]},\\
&\frac{[1+G(fx_{n+1},fx_{n+1},fx_{n+2})]G(fx_n,fx_n,fx_n)}{2[1+G(fx_{n+1},fx_{n+1},fx_{n+2})]},\\
&\frac{[1+G(fx_{n+1},fx_{n+1},fx_{n+2})]G(fx_{n-1},fx_{n-1},fx_{n+1})}{2[1+G(fx_{n+1},fx_n,fx_{n+2})]}\Bigg\}\\
\le\max\Bigg\{&\frac{(1+G_{n-1})G_n}{2+G_{n-1}},0,\frac{(1+G_n)G_{n-1}}{2+G_n},\frac{G_{n-1}+G_n}{2},0,\\
&\frac{(1+2G_n)G_{n-1}}{2(1+G_n)}\Bigg\}\le\max\{G_{n-1},G_n\},\quad \forall n\in\mathbb{N}.
\end{aligned}\tag{17}$$

In light of $(G1)$, $(G3)$, $(G5)$, and (13)–(17), we get that

$$\begin{aligned}
M_1(x_n,x_{n+1},x_{n+2})&=\max\Bigg\{G(gx_n,gx_{n+1},gx_{n+2}),m_1(x_n,x_{n+1},x_{n+2}),\\
&\quad\frac{[1+G(gx_n,gx_{n+1},gx_{n+2})]G(fx_n,fx_{n+1},fx_{n+2})}{1+G(gx_{n+2},fx_{n+1},fx_{n+1})+G(gx_n,gx_{n+1},gx_{n+2})},\\
&\quad\frac{[1+G(gx_n,gx_{n+1},gx_{n+2})]G(fx_n,fx_{n+1},fx_{n+2})}{1+G(gx_{n+1},fx_n,fx_n)+G(gx_n,gx_{n+1},gx_{n+2})},\\
&\quad\frac{[1+G(gx_n,gx_{n+1},gx_{n+2})]G(fx_n,fx_{n+1},fx_{n+2})}{1+G(gx_n,fx_{n+2},fx_{n+2})+G(gx_n,gx_{n+1},gx_{n+2})}\Bigg\}\\
&=\max\Bigg\{G(fx_{n-1},fx_n,fx_{n+1}),m_1(x_n,x_{n+1},x_{n+2}),\\
&\quad\frac{[1+G(fx_{n-1},fx_n,fx_{n+1})]G(fx_n,fx_{n+1},fx_{n+2})}{1+G(fx_{n+1},fx_{n+1},fx_{n+1})+G(fx_{n-1},fx_n,fx_{n+1})},\\
&\quad\frac{[1+G(fx_{n-1},fx_n,fx_{n+1})]G(fx_n,fx_{n+1},fx_{n+2})}{1+G(fx_n,fx_n,fx_n)+G(fx_{n-1},fx_n,fx_{n+1})},\\
&\quad\frac{[1+G(fx_{n-1},fx_n,fx_{n+1})]G(fx_n,fx_{n+1},fx_{n+2})}{1+G(fx_{n-1},fx_{n+2},fx_{n+2})+G(fx_{n-1},fx_n,fx_{n+1})}\Bigg\}\\
&=\max\Bigg\{G_{n-1},m_1(x_n,x_{n+1},x_{n+2}),G_n,G_n,\\
&\quad\frac{(1+G_{n-1})G_n}{1+G(fx_{n-1},fx_{n+2},fx_{n+2})+G_{n-1}}\Bigg\}\\
&=\max\{G_{n-1},G_n\},\quad \forall n\in\mathbb{N}.
\end{aligned}\tag{18}$$

Now we assert that $G_n\le G_{n-1}$, $\forall n\in\mathbb{N}$. Suppose that there exists some $n_0\in\mathbb{N}$ satisfying $G_{n_0}>G_{n_0-1}$. It follows from (12), $(\varphi,\psi,\phi)\in\Phi_1\times\Phi_2\times\Phi_3$ and Lemma 18, we infer that

$$\begin{aligned}
\psi\left(\int_0^{G_{n_0}}\varphi(t)\,dt\right)&=\psi\left(\int_0^{G(fx_{n_0},fx_{n_0+1},fx_{n_0+2})}\varphi(t)\,dt\right)\\
&\le\psi\left(\int_0^{M_1(x_{n_0},x_{n_0+1},x_{n_0+2})}\varphi(t)\,dt\right)\\
&\quad-\phi\left(\int_0^{M_1(x_{n_0},x_{n_0+1},x_{n_0+2})}\varphi(t)\,dt\right)\\
&=\psi\left(\int_0^{G_{n_0}}\varphi(t)\,dt\right)-\phi\left(\int_0^{G_{n_0}}\varphi(t)\,dt\right)\\
&<\psi\left(\int_0^{G_{n_0}}\varphi(t)\,dt\right),
\end{aligned}\tag{19}$$

which is a contradiction. Therefore, $G_n\le G_{n-1}$ for all $n\in\mathbb{N}$ and

$$M_1(x_n,x_{n+1},x_{n+2})=G_{n-1},\quad \forall n\in\mathbb{N}.\tag{20}$$

It is apparent that the sequence $\{G_n\}_{n\in\mathbb{N}_0}$ is nonincreasing and bounded, which implies that there exists r with

$$\lim_{n\to\infty}G_n=r\ge 0.\tag{21}$$

Now, we demonstrate that $r=0$. Suppose that $r>0$. On account of (12), (20), and (21), $(\varphi,\psi,\phi)\in\Phi_1\times\Phi_2\times\Phi_3$ and Lemma 17, we deduce that

$$\begin{aligned}\psi\left(\int_0^r \varphi(t)dt\right) &= \limsup_{n\to\infty}\psi\left(\int_0^{G_n}\varphi(t)dt\right)\\ &= \limsup_{n\to\infty}\psi\left(\int_0^{G(fx_n,fx_{n+1},fx_{n+2})}\varphi(t)dt\right)\\ &\le \limsup_{n\to\infty}\left[\psi\left(\int_0^{M_1(x_n,x_{n+1},x_{n+2})}\varphi(t)dt\right)\right.\\ &\quad\left.-\phi\left(\int_0^{M_1(x_n,x_{n+1},x_{n+2})}\varphi(t)dt\right)\right]\\ &= \limsup_{n\to\infty}\left[\psi\left(\int_0^{G_{n-1}}\varphi(t)dt\right)-\phi\left(\int_0^{G_{n-1}}\varphi(t)dt\right)\right]\\ &\le \limsup_{n\to\infty}\psi\left(\int_0^{G_{n-1}}\varphi(t)dt\right)-\liminf_{n\to\infty}\phi\left(\int_0^{G_{n-1}}\varphi(t)dt\right)\\ &= \psi\left(\int_0^r\varphi(t)dt\right)-\liminf_{n\to\infty}\phi\left(\int_0^{G_{n-1}}\varphi(t)dt\right)\\ &< \psi\left(\int_0^r\varphi(t)dt\right),\end{aligned} \tag{22}$$

which is impossible. Thus, $r = 0$. That is,

$$\lim_{n\to\infty} G_n = 0. \tag{23}$$

It follows from $(G3)$, $(G4)$, and (23) that

$$0 \le G(fx_{n-1}, fx_{n-1}, fx_n) \le G(fx_{n-1}, fx_n, fx_{n+1}) \to 0 \text{ as } n \to \infty,$$

$$0 \le G(fx_{n-1}, fx_n, fx_n) = G(fx_n, fx_n, fx_{n-1}) \le G(fx_n, fx_{n-1}, fx_{n-2}) \to 0 \text{ as } n \to \infty, \tag{24}$$

which yield that

$$\lim_{n\to\infty} G(fx_{n-1}, fx_{n-1}, fx_n) = \lim_{n\to\infty} G(fx_{n-1}, fx_n, fx_n) = 0. \tag{25}$$

Next, we verify that $\{fx_n\}_{n\in\mathbb{N}_0}$ is a G-Cauchy sequence. Suppose that $\{fx_n\}_{n\in\mathbb{N}_0}$ is not a G-Cauchy sequence. It follows from Lemma 10 that there exist a constant $\varepsilon > 0$ and two subsequences $\{fx_{m(k)}\}_{k\in\mathbb{N}}$ and $\{fx_{n(k)}\}_{k\in\mathbb{N}}$ of $\{fx_n\}_{n\in\mathbb{N}_0}$ such that $n(k)$ is minimal in the sense that

$$k < m(k) < n(k) < m(k+1) \text{ and } G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right) > \varepsilon, \quad \forall k \in \mathbb{N}, \tag{26}$$

which means that $G(fx_{m(k)}, fx_{m(k)}, fx_{n(k)-1}) \le \varepsilon$ for all $k \in \mathbb{N}$.

By means of $(G3)$–$(G5)$ and Lemma 13, we deduce that

$$\begin{aligned}\varepsilon < G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right) &= G\left(fx_{n(k)}, fx_{m(k)}, fx_{m(k)}\right)\\ &\le G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right)\\ &\quad + G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{m(k)}\right)\\ &\le \varepsilon + G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right)\\ &\le \varepsilon + G_{n(k)-1}, \quad \forall k \in \mathbb{N},\end{aligned} \tag{27}$$

$$\begin{aligned}&\left|G\left(fx_{n(k)}, fx_{m(k)-1}, fx_{m(k)}\right) - G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)\right|\\ &\le \max\left\{G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)}\right), G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{m(k)-1}\right)\right\}\\ &\le 2G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)+1}\right) = 2G_{m(k)-1}, \quad \forall k \in \mathbb{N},\end{aligned} \tag{28}$$

$$\begin{aligned}&\left|G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)}\right) - G\left(fx_{n(k)}, fx_{m(k)-1}, fx_{m(k)}\right)\right|\\ &\le \max\left\{G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)}\right), G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{m(k)-1}\right)\right\}\\ &\le 2G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)+1}\right) = 2G_{m(k)-1}, \quad \forall k \in \mathbb{N},\end{aligned} \tag{29}$$

$$\begin{aligned}&\left|G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right) - G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)}\right)\right|\\ &\le \max\left\{G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{n(k)}\right), G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right)\right\}\\ &\le 2G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{n(k)+1}\right) = 2G_{n(k)-1}, \quad \forall k \in \mathbb{N},\end{aligned} \tag{30}$$

$$\begin{aligned}&\left|G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)-1}\right) - G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)\right|\\ &\le \max\left\{G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right), G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{n(k)}\right)\right\}\\ &\le 2G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{n(k)+1}\right) = 2G_{n(k)-1}, \quad \forall k \in \mathbb{N}\end{aligned} \tag{31}$$

$$\begin{aligned}&\left|G\left(fx_{n(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right) - G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)-1}\right)\right|\\ &\le \max\left\{G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{m(k)-1}\right), G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)}\right)\right\}\\ &\le 2G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)+1}\right) = 2G_{m(k)-1}, \quad \forall k \in \mathbb{N}.\end{aligned} \tag{32}$$

Letting $k \to \infty$ in (27)–(32) and using (23) and (25), we obtain that

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right) = \lim_{k\to\infty} G\left(fx_{n(k)}, fx_{m(k)-1}, fx_{m(k)}\right)\\ &= \lim_{k\to\infty} G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)}\right) = \lim_{k\to\infty} G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right)\\ &= \lim_{k\to\infty} G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)-1}\right) = \lim_{k\to\infty} G\left(fx_{n(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right).\end{aligned} \tag{33}$$

In view of $(G3)$–$(G5)$ and Lemma 13, we infer that

$$\begin{aligned} G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right) &= G\left(fx_{m(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right) \\ &\le G\left(fx_{m(k)}, fx_{n(k)}, fx_{n(k)}\right) \\ &\quad + G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right) \\ &\le G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{n(k)}\right) \\ &\quad + G\left(fx_{n(k)+1}, fx_{n(k)}, fx_{n(k)-1}\right) \\ &= G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{n(k)}\right) \\ &\quad + G_{n(k)-1}, \quad \forall k \in N, \end{aligned} \tag{34}$$

$$\begin{aligned} G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{n(k)}\right) &= G\left(fx_{n(k)}, fx_{m(k)-1}, fx_{m(k)}\right) \\ &\le G\left(fx_{n(k)}, fx_{m(k)}, fx_{m(k)}\right) \\ &\quad + G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{m(k)}\right) \\ &\le G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{m(k)}\right) \\ &\quad + G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{m(k)}\right) \\ &\le G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right) \\ &\quad + G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right) \\ &\quad + G\left(fx_{m(k)}, fx_{m(k)-1}, fx_{m(k)}\right) \\ &\le G_{n(k)-1} + 2G_{m(k)-1} \\ &\quad + G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{35}$$

Taking $k \to \infty$ in (34) and (35) and utilizing (23) and (33), we conclude that

$$\varepsilon = \lim_{k\to\infty} G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right). \tag{36}$$

On the basis of (G3)–(G5) and Lemma 13, we arrive at

$$\begin{aligned} G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{n(k)}\right) &= G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{m(k)}\right) \\ &\le G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right) \\ &\quad + G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right) \\ &\le G_{n(k)-1} + G \\ &\quad \cdot \left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right), \quad \forall k \in \mathbb{N}, \end{aligned} \tag{37}$$

$$\begin{aligned} G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right) &\le G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{n(k)}\right) \\ &\quad + G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{m(k)}\right) \\ &\le 2G\left(fx_{n(k)}, fx_{n(k)-1}, fx_{n(k)-1}\right) \\ &\quad + G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{n(k)}\right) \\ &\le 2G_{n(k)-1} + G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{n(k)}\right), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{38}$$

Letting $k \to \infty$ in (37) and (38) and using (23) and (36), we deduce that

$$\varepsilon = \lim_{k\to\infty} G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{n(k)}\right). \tag{39}$$

On account of $(G3)$–$(G5)$ and Lemma 13, we receive that

$$\begin{aligned} G\left(fx_{n(k)}, fx_{n(k)}, fx_{m(k)-1}\right) &= G\left(fx_{m(k)-1}, fx_{n(k)}, fx_{n(k)}\right) \\ &\le G\left(fx_{m(k)-1}, fx_{n(k)-1}, fx_{n(k)-1}\right) \\ &\quad + G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{n(k)}\right) \\ &\le G\left(fx_{n(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right) \\ &\quad + 2G_{n(k)-1}, \quad \forall k \in \mathbb{N}, \end{aligned} \tag{40}$$

$$\begin{aligned} G\left(fx_{n(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right) &\le G\left(fx_{n(k)-1}, fx_{m(k)-1}, fx_{m(k)-1}\right) \\ &\quad + G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right) \\ &\le G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{m(k)-1}\right) \\ &\quad + G_{m(k)-1} \le G\left(fx_{n(k)-1}, fx_{n(k)}, fx_{n(k)}\right) \\ &\quad + G\left(fx_{n(k)}, fx_{n(k)}, fx_{m(k)-1}\right) + G_{m(k)-1} \\ &\quad \cdot \le 2G_{n(k)-1} + G\left(fx_{n(k)}, fx_{n(k)}, fx_{m(k)-1}\right) \\ &\quad + G_{m(k)-1}, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{41}$$

Taking $k \to \infty$ in (40) and (41) and using (23) and (33), we conclude that

$$\varepsilon = \lim_{k\to\infty} G\left(fx_{n(k)}, fx_{n(k)}, fx_{m(k)-1}\right). \tag{42}$$

Making use of (12)–(14), (23), (25), (33), (36), (39), (42), $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, and Lemma 12, we obtain that

$$\begin{aligned}
&\lim_{k\to\infty} M_1\left(x_{m(k)}, x_{m(k)}, x_{n(k)}\right) \\
&= \lim_{k\to\infty} \max\left\{G\left(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\right),\right. \\
&\quad \frac{\left[1+G\left(gx_{m(k)}, gx_{m(k)}, fx_{m(k)}\right)\right]G\left(gx_{m(k)}, gx_{m(k)}, fx_{n(k)}\right)}{2+G\left(gx_{m(k)}, fx_{m(k)}, fx_{m(k)}\right)}, \\
&\quad \frac{\left[1+G\left(gx_{m(k)}, gx_{m(k)}, fx_{m(k)}\right)\right]G\left(gx_{n(k)}, gx_{n(k)}, fx_{m(k)}\right)}{2+G\left(gx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}, \\
&\quad \frac{\left[1+G\left(gx_{m(k)}, gx_{m(k)}, fx_{m(k)}\right)\right]G\left(gx_{n(k)}, gx_{n(k)}, fx_{m(k)}\right)}{2+G\left(gx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}, \\
&\quad \frac{\left[1+G\left(gx_{m(k)}, gx_{m(k)}, fx_{m(k)}\right)\right]G\left(gx_{m(k)}, gx_{m(k)}, fx_{n(k)}\right)}{2\left[1+G\left(gx_{m(k)}, fx_{m(k)}, fx_{m(k)}\right)\right]}, \\
&\quad \frac{\left[1+G\left(gx_{n(k)}, gx_{n(k)}, fx_{n(k)}\right)\right]G\left(gx_{m(k)}, gx_{m(k)}, fx_{m(k)}\right)}{2\left[1+G\left(gx_{n(k)}, fx_{m(k)}, fx_{n(k)}\right)\right]}, \\
&\quad \frac{\left[1+G\left(gx_{n(k)}, gx_{n(k)}, fx_{n(k)}\right)\right]G\left(gx_{m(k)}, gx_{m(k)}, fx_{m(k)}\right)}{2\left[1+G\left(gx_{n(k)}, fx_{m(k)}, fx_{n(k)}\right)\right]}, \\
&\quad \frac{\left[1+G\left(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\right)\right]G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}{1+G\left(gx_{n(k)}, fx_{m(k)}, fx_{m(k)}\right)+G\left(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\right)}, \\
&\quad \frac{\left[1+G\left(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\right)\right]G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}{1+G\left(gx_{m(k)}, fx_{m(k)}, fx_{m(k)}\right)+G\left(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\right)}, \\
&\quad \left.\frac{\left[1+G\left(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\right)\right]G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}{1+G\left(gx_{m(k)}, fx_{n(k)}, fx_{n(k)}\right)+G\left(gx_{m(k)}, gx_{m(k)}, gx_{n(k)}\right)}\right\} \\
&= \lim_{k\to\infty} \max\left\{G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right),\right. \\
&\quad \frac{\left[1+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right)\right]G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)}\right)}{2+G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)}\right)}, \\
&\quad \frac{\left[1+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right)\right]G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right)}{2+G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{n(k)}\right)}, \\
&\quad \frac{\left[1+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right)\right]G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{m(k)}\right)}{2+G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{n(k)}\right)}, \\
&\quad \frac{\left[1+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right)\right]G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)}\right)}{2\left[1+G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)}\right)\right]}, \\
&\quad \frac{\left[1+G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{n(k)}\right)\right]G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right)}{2\left[1+G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{n(k)}\right)\right]}, \\
&\quad \frac{\left[1+G\left(fx_{n(k)-1}, fx_{n(k)-1}, fx_{n(k)}\right)\right]G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{m(k)}\right)}{2\left[1+G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{n(k)}\right)\right]}, \\
&\quad \frac{\left[1+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right)\right]G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}{1+G\left(fx_{n(k)-1}, fx_{m(k)}, fx_{m(k)}\right)+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right)}, \\
&\quad \frac{\left[1+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right)\right]G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}{1+G\left(fx_{m(k)-1}, fx_{m(k)}, fx_{m(k)}\right)+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right)}, \\
&\quad \left.\frac{\left[1+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right)\right]G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)}{1+G\left(fx_{m(k)-1}, fx_{n(k)}, fx_{n(k)}\right)+G\left(fx_{m(k)-1}, fx_{m(k)-1}, fx_{n(k)-1}\right)}\right\} \\
&= \max\left\{\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2+\varepsilon}, \frac{\varepsilon}{2+\varepsilon}, \frac{\varepsilon}{2}, 0, 0, \frac{(1+\varepsilon)\varepsilon}{1+2\varepsilon}, \varepsilon, \frac{(1+\varepsilon)\varepsilon}{1+2\varepsilon}\right\} = \varepsilon.
\end{aligned} \tag{43}$$

Making use of (12), (33), (43), $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, and Lemma 17, we conclude that

$$\begin{aligned}
\psi\left(\int_0^{\varepsilon} \varphi(t)\,dt\right) &= \limsup_{k\to\infty} \psi\left(\int_0^{G\left(fx_{m(k)}, fx_{m(k)}, fx_{n(k)}\right)} \varphi(t)\,dt\right) \\
&\le \limsup_{k\to\infty} \left[\psi\left(\int_0^{M_1\left(x_{m(k)}, x_{m(k)}, x_{n(k)}\right)} \varphi(t)\,dt\right)\right. \\
&\quad \left. - \phi\left(\int_0^{M_1\left(x_{m(k)}, x_{m(k)}, x_{n(k)}\right)} \varphi(t)\,dt\right)\right] \\
&\le \limsup_{k\to\infty} \psi\left(\int_0^{M_1\left(x_{m(k)}, x_{m(k)}, x_{n(k)}\right)} \varphi(t)\,dt\right) \\
&\quad - \liminf_{k\to\infty} \phi\left(\int_0^{M_1\left(x_{m(k)}, x_{m(k)}, x_{n(k)}\right)} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{\varepsilon} \varphi(t)\,dt\right) - \liminf_{k\to\infty} \phi \\
&\quad \cdot \left(\int_0^{M_1\left(x_{m(k)}, x_{m(k)}, x_{n(k)}\right)} \varphi(t)\,dt\right) \\
&< \psi\left(\int_0^{\varepsilon} \varphi(t)\,dt\right),
\end{aligned} \tag{44}$$

which is ridiculous. Thus, $\{fx_n\}_{n\in\mathbb{N}_0}$ is a G-Cauchy sequence. Since $g(X)$ is complete, it follows that there exists $w \in g(X)$ such that

$$\lim_{n\to\infty} fx_n = w. \tag{45}$$

In light of Lemma 8 and $w \in g(X)$, there exists $a \in X$ satisfying $ga = w$ and

$$\lim_{n\to\infty} G(fx_n, fx_n, ga) = \lim_{n\to\infty} G(fx_n, ga, ga) = 0. \tag{46}$$

Next, we prove $ga = fa$. Suppose that $ga \ne fa$. In view of (12), (13), (25), (46), $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, and Lemmas 12 and 17, we obtain that

$$\begin{aligned}\lim_{n\to\infty} M_1(x_n,x_n,a) = \lim_{n\to\infty} \max\Big\{&G(gx_n,gx_n,ga),\\ &\frac{[1+G(gx_n,gx_n,fx_n)]G(gx_n,gx_n,fa)}{2+G(gx_n,fx_n,fx_n)},\\ &\frac{[1+G(gx_n,gx_n,fx_n)]G(ga,ga,fx_n)}{2+G(gx_n,fx_n,fa)},\\ &\frac{[1+G(gx_n,gx_n,fx_n)]G(ga,ga,fx_n)}{2+G(gx_n,fx_n,fa)},\\ &\frac{[1+G(gx_n,gx_n,fx_n)]G(gx_n,gx_n,fa)}{2[1+G(gx_n,fx_n,fx_n)]},\\ &\frac{[1+G(ga,ga,fa)]G(gx_n,gx_n,fx_n)}{2[1+G(ga,fx_n,fa)]},\\ &\frac{[1+G(ga,ga,fa)]G(gx_n,gx_n,fx_n)}{2[1+G(ga,fx_n,fa)]},\\ &\frac{[1+G(gx_n,gx_n,ga)]G(fx_n,fx_n,fa)}{1+G(ga,fx_n,fx_n)+G(gx_n,gx_n,ga)},\\ &\frac{[1+G(gx_n,gx_n,ga)]G(fx_n,fx_n,fa)}{1+G(gx_n,fx_n,fx_n)+G(gx_n,gx_n,ga)},\\ &\frac{[1+G(gx_n,gx_n,ga)]G(fx_n,fx_n,fa)}{1+G(gx_n,fa,fa)+G(gx_n,gx_n,ga)}\Big\}\\ = \lim_{n\to\infty} \max\Big\{&G(fx_{n-1},fx_{n-1},ga),\\ &\frac{[1+G(fx_{n-1},fx_{n-1},fx_n)]G(fx_{n-1},fx_{n-1},fa)}{2+G(fx_{n-1},fx_n,fx_n)},\\ &\frac{[1+G(fx_{n-1},fx_{n-1},fx_n)]G(ga,ga,fx_n)}{2+G(fx_{n-1},fx_n,fa)},\\ &\frac{[1+G(fx_{n-1},fx_{n-1},fx_n)]G(ga,ga,fx_n)}{2+G(fx_{n-1},fx_n,fa)},\\ &\frac{[1+G(fx_{n-1},fx_{n-1},fx_n)]G(fx_{n-1},fx_{n-1},fa)}{2[1+G(fx_{n-1},fx_n,fx_n)]},\\ &\frac{[1+G(ga,ga,fa)]G(fx_{n-1},fx_{n-1},fx_n)}{2[1+G(ga,fx_n,fa)]},\\ &\frac{[1+G(ga,ga,fa)]G(fx_{n-1},fx_{n-1},fx_n)}{2[1+G(ga,fx_n,fa)]},\\ &\frac{[1+G(fx_{n-1},fx_{n-1},ga)]G(fx_n,fx_n,fa)}{1+G(ga,fx_n,fx_n)+G(fx_{n-1},fx_{n-1},ga)},\\ &\frac{[1+G(fx_{n-1},fx_{n-1},ga)]G(fx_n,fx_n,fa)}{1+G(fx_{n-1},fx_n,fx_n)+G(fx_{n-1},fx_{n-1},ga)},\\ &\frac{[1+G(fx_{n-1},fx_{n-1},ga)]G(fx_n,fx_n,fa)}{1+G(fx_{n-1},fa,fa)+G(fx_{n-1},fx_{n-1},ga)}\Big\}\\ = \max\Big\{&0,\frac{G(ga,ga,fa)}{2},0,0,\\ &\frac{G(ga,ga,fa)}{2},0,0,G(ga,ga,fa),G(ga,ga,fa),\\ &\frac{G(ga,ga,fa)}{1+G(ga,fa,fa)}\Big\} = G(ga,ga,fa)\end{aligned} \tag{47}$$

$$\begin{aligned}\psi\left(\int_0^{G(ga,ga,fa)}\varphi(t)\,dt\right) &= \limsup_{n\to\infty}\psi\left(\int_0^{G(fx_n,fx_n,fa)}\varphi(t)\,dt\right)\\ &\le \limsup_{n\to\infty}\left[\psi\left(\int_0^{M_1(x_n,x_n,a)}\varphi(t)\,dt\right)\right.\\ &\quad\left.-\phi\left(\int_0^{M_1(x_n,x_n,a)}\varphi(t)\,dt\right)\right]\\ &\le \limsup_{n\to\infty}\psi\left(\int_0^{M_1(x_n,x_n,a)}\varphi(t)\,dt\right)-\liminf_{n\to\infty}\phi\left(\int_0^{M_1(x_n,x_n,a)}\varphi(t)\,dt\right)\\ &= \psi\left(\int_0^{G(ga,ga,fa)}\varphi(t)\,dt\right)-\liminf_{n\to\infty}\phi\left(\int_0^{M_1(x_n,x_n,a)}\varphi(t)\,dt\right)\\ &< \psi\left(\int_0^{G(ga,ga,fa)}\varphi(t)\,dt\right),\end{aligned} \tag{48}$$

which is absurd. Consequently, $w = ga = fa$, that is, w is a point of coincidence of f and g.

Lastly, we certify that f and g have a unique point of coincidence in X. Assume that there exists $b \in X$ with $fb = gb \ne fa$. In terms of (13), ($G2$), and Lemma 13, we receive that

$$\begin{aligned}M_1(a,a,b) = \max\Big\{&G(ga,ga,gb),\frac{[1+G(ga,ga,fa)]G(ga,ga,fb)}{2+G(ga,fa,fa)},\\ &\frac{[1+G(ga,ga,fa)]G(gb,gb,fa)}{2+G(ga,fa,fb)},\frac{[1+G(ga,ga,fa)]G(gb,gb,fa)}{2+G(ga,fa,fb)},\\ &\frac{[1+G(ga,ga,fa)]G(ga,ga,fb)}{2[1+G(ga,fa,fa)]},\frac{[1+G(gb,gb,fb)]G(ga,ga,fa)}{2[1+G(gb,fa,fb)]},\\ &\frac{[1+G(gb,gb,fb)]G(ga,ga,fa)}{2[1+G(gb,fa,fb)]},\frac{[1+G(ga,ga,gb)]G(fa,fa,fb)}{1+G(gb,fa,fa)+G(ga,ga,gb)},\\ &\frac{[1+G(ga,ga,gb)]G(fa,fa,fb)}{1+G(ga,fa,fa)+G(ga,ga,gb)},\frac{[1+G(ga,ga,gb)]G(fa,fa,fb)}{1+G(ga,fb,fb)+G(ga,ga,gb)}\Big\}\\ = \max\Big\{&G(fa,fa,fb),\frac{G(fa,fa,fb)}{2},\frac{G(fb,fb,fa)}{2+G(fa,fa,fb)},\\ &\frac{G(fb,fb,fa)}{2+G(fa,fa,fb)},\frac{G(fa,fa,fb)}{2},0,0,\\ &\frac{[1+G(fa,fa,fb)]G(fa,fa,fb)}{1+2G(fa,fa,fb)},\quad G(fa,fa,fb),\\ &\frac{[1+G(fa,fa,fb)]G(fa,fa,fb)}{1+G(fa,fb,fb)+G(fa,fa,fb)}\Big\} = G(fa,fa,fb) > 0.\end{aligned} \tag{49}$$

According to (12), (49), $(\varphi,\psi,\phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, and Lemma 18, we gain that

$$\begin{aligned}\psi\left(\int_0^{G(fa,fa,fb)}\varphi(t)\,dt\right) &\le \psi\left(\int_0^{M_1(a,a,b)}\varphi(t)\,dt\right)-\phi\left(\int_0^{M_1(a,a,b)}\varphi(t)\,dt\right)\\ &= \psi\left(\int_0^{G(fa,fa,fb)}\varphi(t)\,dt\right)-\phi\left(\int_0^{G(fa,fa,fb)}\varphi(t)\,dt\right)\\ &< \psi\left(\int_0^{G(fa,fa,fb)}\varphi(t)\,dt\right),\end{aligned} \tag{50}$$

which is contradictive. Therefore, f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible mappings, by Lemma 16, we know that f and g have a unique common fixed point in X. This completes the proof.

Similar to the argument of Theorem 19, we derive the following result and omit its proof.

Theorem 20. *Let (X,G) be a G-metric space, f and $g: X \to X$ be two mappings satisfying*

$$\psi\left(\int_0^{G(fx,fy,fz)}\varphi(t)dt\right) \le \psi\left(\int_0^{M_2(x,y,z)}\varphi(t)dt\right) - \phi\left(\int_0^{M_2(x,y,z)}\varphi(t)dt\right), \quad \forall x,y,z\in X, \tag{51}$$

where $(\varphi,\psi,\phi)\in\Phi_1\times\Phi_2\times\Phi_3$ and

$$\begin{aligned} M_2(x,y,z) = \max\Bigg\{ & G(gx,gy,gz), \frac{[1+G(gx,fx,fx)]G(gz,fy,fy)}{2+G(gx,gy,fx)}, \\ & \frac{[1+G(gy,fy,fy)]G(gz,fx,fx)}{2+G(gx,gy,fy)}, \frac{[1+G(gz,fz,fz)]G(gx,fy,fy)}{2+G(gy,gz,fz)}, \\ & \frac{[1+G(gx,fx,fx)]G(gy,fz,fz)}{2[1+G(gx,gz,fx)]}, \frac{[1+G(gy,fy,fy)]G(gx,fz,fz)}{2[1+G(gy,gz,fy)]}, \\ & \frac{[1+G(gz,fz,fz)]G(gy,fx,fx)}{2[1+G(gx,gz,fz)]}, \frac{[1+G(gx,gy,gz)]G(fx,fy,fz)}{1+G(gz,gz,fy)+G(gx,gy,gz)}, \\ & \frac{[1+G(gx,gy,gz)]G(fx,fy,fz)}{1+G(gy,gy,fx)+G(gx,gy,gz)}, \frac{[1+G(gx,gy,gz)]G(fx,fy,fz)}{1+G(gx,gx,fz)+G(gx,gy,gz)}\Bigg\}. \end{aligned} \tag{52}$$

If $f(X)\subseteq g(X)$ and $g(X)$ is a complete subset of X, then f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible mappings, then f and g have a unique common fixed point in X.

Remark 21. In case $\psi(t)=t$, $\phi(t)=(1-\lambda)t$, $\forall t\in R^+$ and $\lambda\in(0,1)$ is a constant, then Theorems 19 and 20 reduce to results, which include Theorem 5 as a special case. The following example shows that Theorems 19 and 20 generalize substantially Theorem 5 and differ from Theorem 4.

Example 22. Let $X=[0,2]$. Define $f,g: X\to X$, $\varphi,\psi,\phi,\eta: \mathbb{R}^+\to\mathbb{R}^+$, and $G: X\times X\times X\to R^+$ by

$$fx = \begin{cases} 0, & \forall x\in[0,1], \\ \frac{1}{6}, & \forall x\in(1,2], \end{cases} \qquad gx = \begin{cases} x, & \forall x\in[0,1], \\ \frac{7}{6}, & \forall x\in(1,2], \end{cases}$$

$$\varphi(t)=2t, \psi(t)=3t, \phi(t)=t, \eta(t)=2t, \quad \forall t\in\mathbb{R}^+,$$

$$G(x,y,z) = |x-y|+|y-z|+|z-x|, \forall x,y,z\in X. \tag{53}$$

Clearly, (X,G) is a G-metric space, $(\varphi,\psi,\phi)\in\Phi_1\times\Phi_2\times\Phi_3$, f and g are weakly compatible mappings, $f(X)\subseteq g(X)$ and $g(X)$ is complete, $\eta(t)=\psi(t)-\phi(t)$, $\forall t\in\mathbb{R}^+$ and η is nondecreasing in $\mathbb{R}^+$. Let $x,y,z\in X$. In order to verify (12) and (51), we consider the following seven cases:

Case 1. $x,y,z\in[0,1]$ or $x,y,z\in(1,2]$. It follows that

$$\begin{aligned} \psi\left(\int_0^{G(fx,fy,fz)}\varphi(t)dt\right) &= 0 \le \eta\left(\int_0^{M_i(x,y,z)}\varphi(t)dt\right) = \psi\left(\int_0^{M_i(x,y,z)}\varphi(t)dt\right) \\ &\quad - \phi\left(\int_0^{M_i(x,y,z)}\varphi(t)dt\right), \quad \forall i\in\{1,2\}. \end{aligned} \tag{54}$$

Case 2. $x,y\in[0,1]$ and $z\in(1,2]$. It follows that

$$\begin{aligned} M_1(x,y,z) &\ge \frac{[1+G(gy,gy,fy)]G(gz,gz,fx)}{2+G(gy,fy,fz)} \\ &= \frac{[1+G(y,y,0)]G(7/6,7/6,0)}{2+G(y,0,1/6)} \\ &= \frac{(7/3)(1+2y)}{2+y+1/6+|y-1/6|} \ge 1, \end{aligned}$$

$$\begin{aligned} M_2(x,y,z) &\ge \frac{[1+G(gy,fy,fy)]G(gz,fx,fx)}{2+G(gx,gy,fy)} \\ &= \frac{[1+G(y,0,0)]G(7/6,0,0)}{2+G(x,y,0)} \\ &= \frac{(7/3)(1+2y)}{2+x+y+|x-y|} \ge \frac{7}{12}. \end{aligned} \tag{55}$$

It is easy to see that

$$\begin{aligned} & M_i(x,y,z) \ge \frac{7}{12}, \\ \psi\left(\int_0^{G(fx,fy,fz)}\varphi(t)dt\right) &= 3\int_0^{1/3} 2t dt = \frac{1}{3} < 2\cdot\frac{49}{144} \\ &= \eta\left(\int_0^{7/12}\varphi(t)dt\right) \le \eta\left(\int_0^{M_i(x,y,z)}\varphi(t)dt\right) \\ &= \psi\left(\int_0^{M_i(x,y,z)}\varphi(t)dt\right) \\ &\quad - \phi\left(\int_0^{M_i(x,y,z)}\varphi(t)dt\right), \quad \forall i\in\{1,2\}. \end{aligned} \tag{56}$$

Case 3. $x,z\in[0,1]$ and $y\in(1,2]$. It follows that

$$\begin{aligned} M_1(x,y,z) &\ge \frac{[1+G(gz,gz,fz)]G(gy,gy,fx)}{2[1+G(gz,fy,fz)]} \\ &= \frac{[1+G(z,z,0)]G(7/6,7/6,0)}{2[1+G(z,1/6,0)]} \\ &= \frac{(7/3)(1+2z)}{2(1+1/6+z+|z-1/6|)} \ge \frac{7}{8}, \end{aligned}$$

$$\begin{aligned} M_2(x,y,z) &\ge \max\left\{\frac{[1+G(gx,fx,fx)]G(gy,fz,fz)}{2[1+G(gx,gz,fx)]}, \frac{[1+G(gz,fz,fz)]G(gy,fx,fx)}{2[1+G(gx,gz,fz)]}\right\} \\ &= \max\left\{\frac{[1+G(x,0,0)]G(7/6,0,0)}{2[1+G(x,z,0)]}, \frac{[1+G(z,0,0)]G(7/6,0,0)}{2[1+G(x,z,0)]}\right\} \\ &= \max\left\{\frac{(7/3)(1+2x)}{2(1+x+z+|x-z|)}, \frac{(7/3)(1+2z)}{2(1+x+z+|x-z|)}\right\} \\ &\ge \frac{7}{6}. \end{aligned} \tag{57}$$

It is obvious that

$$M_i(x,y,z) \geq \frac{7}{8},$$

$$\begin{aligned}\psi\left(\int_0^{G(fx,fy,fz)}\varphi(t)\,dt\right) &= 3\int_0^{1/3} 2t\,dt = \frac{1}{3} < 2\cdot\frac{49}{64} \\ &= \eta\left(\int_0^{7/8}\varphi(t)\,dt\right) \leq \eta\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right) \\ &= \psi\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right) \\ &\quad - \phi\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right), \quad \forall i\in\{1,2\}.\end{aligned} \tag{58}$$

Case 4. $y,z\in[0,1]$ and $x\in(1,2]$. It follows that

$$\begin{aligned}M_1(x,y,z) &\geq \frac{[1+G(gy,gy,fy)]G(gx,gx,fz)}{2[1+G(gy,fx,fy)]} \\ &= \frac{[1+G(y,y,0)]G(7/6,7/6,0)}{2[1+G(y,1/6,0)]} \\ &= \frac{(7/3)(1+2y)}{2(1+y+1/6+|y-1/6|)} \geq \frac{7}{8},\end{aligned}$$

$$\begin{aligned}M_2(x,y,z) &\geq \frac{[1+G(gz,fz,fz)]G(gx,fy,fy)}{2+G(gy,gz,fz)} \\ &= \frac{[1+G(z,0,0)]G(7/6,0,0)}{2+G(y,z,0)} = \frac{(7/3)(1+2z)}{2+y+z+|y-z|} \\ &\geq \frac{7}{12}.\end{aligned} \tag{59}$$

It is apparent that

$$M_i(x,y,z) \geq \frac{7}{12},$$

$$\begin{aligned}\psi\left(\int_0^{G(fx,fy,fz)}\varphi(t)\,dt\right) &= 3\int_0^{1/3} 2t\,dt = \frac{1}{3} < 2\cdot\frac{49}{144} \\ &= \eta\left(\int_0^{7/12}\varphi(t)\,dt\right) \leq \eta\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right) \\ &= \psi\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right) \\ &\quad - \phi\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right), \quad \forall i\in\{1,2\}.\end{aligned} \tag{60}$$

Case 5. $x\in[0,1]$ and $y,z\in(1,2]$. It follows that

$$\begin{aligned}M_1(x,y,z) &\geq \frac{[1+G(gz,gz,fz)]G(gy,gy,fx)}{2[1+G(gz,fy,fz)]} \\ &= \frac{[1+G(7/6,7/6,1/6)]G(7/6,7/6,0)}{2[1+G(7/6,1/6,1/6)]} \\ &= \frac{(7/3)(1+2)}{2(1+2)} = \frac{7}{6},\end{aligned}$$

$$\begin{aligned}M_2(x,y,z) &\geq \frac{[1+G(gy,fy,fy)]G(gz,fx,fx)}{2+G(gx,gy,fy)} \\ &= \frac{[1+G(7/6,1/6,1/6)]G(7/6,0,0)}{2+G(x,7/6,1/6)} \\ &= \frac{(7/3)(1+2)}{2+1+|x-7/6|+|x-1/6|} \geq \frac{21}{13}.\end{aligned} \tag{61}$$

It is distinct that

$$M_i(x,y,z) \geq \frac{7}{6},$$

$$\begin{aligned}\psi\left(\int_0^{G(fx,fy,fz)}\varphi(t)\,dt\right) &= 3\int_0^{1/3} 2t\,dt = \frac{1}{3} < 2\cdot\frac{49}{36} \\ &= \eta\left(\int_0^{7/6}\varphi(t)\,dt\right) \leq \eta\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right) \\ &= \psi\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right) \\ &\quad - \phi\left(\int_0^{M_i(x,y,z)}\varphi(t)\,dt\right), \quad \forall i\in\{1,2\}.\end{aligned} \tag{62}$$

Case 6. $y\in[0,1]$ and $x,z\in(1,2]$. It follows that

$$\begin{aligned}M_1(x,y,z) &\geq \frac{[1+G(gy,gy,fy)]G(gx,gx,fz)}{2[1+G(gy,fx,fy)]} \\ &= \frac{[1+G(y,y,0)]G(7/6,7/6,1/6)}{2[1+G(y,1/6,0)]} \\ &= \frac{2(1+2y)}{2(1+y+1/6+|y-1/6|)} \geq \frac{3}{4},\end{aligned}$$

$$\begin{aligned}M_2(x,y,z) &\geq \frac{[1+G(gy,fy,fy)]G(gz,fx,fx)}{2+G(gx,gy,fy)} \\ &= \frac{[1+G(y,0,0)]G(7/6,1/6,1/6)}{2+G(7/6,y,0)} \\ &= \frac{2(1+2y)}{2+y+7/6+|y-7/6|} \geq \frac{6}{13}.\end{aligned} \tag{63}$$

It is clear that

$$M_i(x,y,z) \geq \frac{6}{13}, \psi\left(\int_0^{G(fx,fy,fz)} \varphi(t)dt\right) = 3\int_0^{1/3} 2tdt$$
$$= \frac{1}{3} < 2 \cdot \frac{36}{169} = \eta\left(\int_0^{6/13} \varphi(t)dt\right) \leq \eta\left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right)$$
$$= \psi\left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right) - \phi\left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right), \quad \forall i \in \{1,2\}. \tag{64}$$

Case 7. $z \in [0,1]$ and $x, y \in (1,2]$. It follows that

$$M_1(x,y,z) \geq \frac{[1+G(gy,gy,fy)]G(gx,gx,fz)}{2[1+G(gy,fx,fy)]}$$
$$= \frac{[1+G(7/6,7/6,1/6)]G(7/6,7/6,0)}{2[1+G(7/6,1/6,1/6)]}$$
$$= \frac{(7/3)(1+2)}{2(1+2)} = \frac{7}{6},$$

$$M_2(x,y,z) \geq \frac{[1+G(gy,fy,fy)]G(gx,fz,fz)}{2[1+G(gy,gz,fy)]}$$
$$= \frac{[1+G(7/6,1/6,1/6)]G(7/6,0,0)}{2[1+G(7/6,z,1/6)]}$$
$$= \frac{(7/3)(1+2)}{2(1+1+|z-7/6|+|z-1/6|)} \geq \frac{21}{20}. \tag{65}$$

It is easy to obtain that

$$M_i(x,y,z) \geq \frac{21}{20},$$

$$\psi\left(\int_0^{G(fx,fy,fz)} \varphi(t)dt\right) = 3\int_0^{1/3} 2tdt = \frac{1}{3} < 2 \cdot \frac{441}{400}$$
$$= \eta\left(\int_0^{21/20} \varphi(t)dt\right) \leq \eta\left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right)$$
$$= \psi\left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right)$$
$$- \phi\left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right), \quad \forall i \in \{1,2\}. \tag{66}$$

Therefore, (12) and (51) hold. That is, the conditions of Theorems 19 and 20 are fulfilled. It follows from each of Theorems 19 and 20 that f and g have a unique common fixed point in X.

However, Theorem 4 cannot be applied to testify the existence of fixed points of the mapping f in X. Suppose that there exists $\psi, \phi \in \Phi_2$ satisfying the conditions of Theorem 4. In virtue of (4), we infer that

$$\psi\left(\frac{1}{3}\right) = \psi\left(G\left(f1, f1, f\frac{7}{6}\right)\right) \leq \psi\left(G\left(1, 1, \frac{7}{6}\right)\right)$$
$$- \phi\left(G\left(1, 1, \frac{7}{6}\right)\right) = \psi\left(\frac{1}{3}\right) - \phi\left(\frac{1}{3}\right) < \psi\left(\frac{1}{3}\right), \tag{67}$$

which is a contradiction.

Now we claim that Theorem 5 is useless in proving the existence of common fixed points of the mappings f and g in X. Suppose that there exist $\alpha \in [0,1)$ and $\varphi \in \Phi_1$ satisfying the conditions of Theorem 5. Taking advantage of (5), we receive that

$$\int_0^{1/3} \varphi(t)dt = \int_0^{G(f1,f2,f2)} \varphi(t)dt \leq \alpha \int_0^{G(g1,g2,g2)} \varphi(t)dt$$
$$= \alpha \int_0^{1/3} \varphi(t)dt < \int_0^{1/3} \varphi(t)dt, \tag{68}$$

which is impossible.

4. Applications

In this section, we study the existence and uniqueness of common solutions for the below functional equations (72) and nonlinear Volterra integral equations (94) by using the results obtained in Section 3.

Let U and V denote two Banach spaces; $S \subseteq U$ and $D \subseteq V$ signify the state and decision spaces, respectively. $B(S)$ indicates the Banach space of all bounded functions in S with norm

$$\|h\| = \sup\{|h(x)| : x \in S\}, \quad \forall h \in B(S). \tag{69}$$

Define $G : (B(S))^3 \to \mathbb{R}^+$ by

$$G(x,y,z) = \max\{\|x-y\|, \|y-z\|, \|z-x\|\}, \quad \forall x,y,z \in B(S). \tag{70}$$

It is clear that $(B(S), G)$ is a complete G-metric space.

Consider the functional equations arising in dynamic programming:

$$f(x) = \sup_{y \in D}\{u(x,y) + H(x,y,f(a(x,y)))\}, \quad \forall x \in S, \tag{71}$$

$$g(x) = \sup_{y \in D}\{v(x,y) + L(x,y,g(b(x,y)))\}, \quad \forall x \in S, \tag{72}$$

where$u, v : S \times D \to \mathbb{R}$, $a, b : S \times D \to S$, and $H, L : S \times D \times \mathbb{R} \to \mathbb{R}$ are mappings. Put

$$fh(x) = \sup_{y \in D}\{u(x,y) + H(x,y,h(a(x,y)))\}, \quad \forall (x,h) \in S \times B(S), \tag{73}$$

$$gh(x) = \sup_{y \in D} \{v(x,y) + L(x,y,h(b(x,y)))\}, \quad \forall (x,h) \in S \times B(S). \tag{74}$$

Theorem 23. *Let $u, v : S \times D \to \mathbb{R}$, $a, b : S \times D \to S$, and $H, L : S \times D \times \mathbb{R} \to \mathbb{R}$ satisfy that*

(C1) u, v, H, and L are bounded;

(C2) $fgh = gfh$ for each $h \in B(S)$ with $fh = gh$;

(C3) $f(B(S)) \subseteq g(B(S))$ and $g(B(S))$ is complete;

(C4) for each $i \in \{1,2,3\}$, $x \in S$, $y \in D$, and $h_1, h_2, h_3 \in B(S)$

$$\psi\left(\int_0^{|H(x,y,h_i(a(x,y)))-H(x,y,h_{i+1}(a(x,y)))|} \varphi(t)dt\right) \le \psi\left(\int_0^{M_1^*} \varphi(t)dt\right) - \phi\left(\int_0^{M_1^*} \varphi(t)dt\right), \tag{75}$$

where $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_5 \times \Phi_3$, $h_4 = h_1$ and

$$\begin{aligned} M_1^*(h_1,h_2,h_3) = \max \Big\{ & G(gh_1, gh_2, gh_3), \\ & \frac{[1 + G(gh_1, gh_1, fh_1)]G(gh_2, gh_2, fh_3)}{2 + G(gh_1, fh_1, fh_2)}, \\ & \frac{[1 + G(gh_1, gh_1, fh_1)]G(gh_3, gh_3, fh_2)}{2 + G(gh_1, fh_1, fh_3)}, \\ & \frac{[1 + G(gh_2, gh_2, fh_2)]G(gh_3, gh_3, fh_1)}{2 + G(gh_2, fh_2, fh_3)}, \\ & \frac{[1 + G(gh_2, gh_2, fh_2)]G(gh_1, gh_1, fh_3)}{2[1 + G(gh_2, fh_1, fh_2)]}, \\ & \frac{[1 + G(gh_3, gh_3, fh_3)]G(gh_2, gh_2, fh_1)}{2[1 + G(gh_3, fh_2, fh_3)]}, \\ & \frac{[1 + G(gh_3, gh_3, fh_3)]G(gh_1, gh_1, fh_2)}{2[1 + G(gh_3, fh_1, fh_3)]}, \\ & \frac{[1 + G(gh_1, gh_2, gh_3)]G(fh_1, fh_2, fh_3)}{1 + G(gh_3, fh_2, fh_2) + G(gh_1, gh_2, gh_3)}, \\ & \frac{[1 + G(gh_1, gh_2, gh_3)]G(fh_1, fh_2, fh_3)}{1 + G(gh_2, fh_1, fh_1) + G(gh_1, gh_2, gh_3)}, \\ & \frac{[1 + G(gh_1, gh_2, gh_3)]G(fh_1, fh_2, fh_3)}{1 + G(gh_1, fh_3, fh_3) + G(gh_1, gh_2, gh_3)} \Big\}. \end{aligned} \tag{76}$$

Then, the functional equations (72) have a unique common bounded solution $h^* \in B(S)$.

Proof. By virtue of (C1) and (73), we obtain that fh and gh are bounded for each $h \in B(S)$, which yields that f and g are self mappings in $B(S)$. It follows from $\varphi \in \Phi_1$ that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_C \varphi(t)dt < \varepsilon, \quad \forall C \subset \mathbb{R}^+ \text{ with } m(C) \le \delta, \tag{77}$$

where m denotes the Lebesgue measure. Put $x \in S$ and $h_1, h_2, h_3 \in B(S)$. (73) means that there exist $y_1, y_2 \in D$ such that

$$fh_1(x) < u(x,y_1) + H(x,y_1,h_1(a(x,y_1))) + \delta, \tag{78}$$

$$fh_2(x) < u(x,y_2) + H(x,y_2,h_2(a(x,y_2))) + \delta, \tag{79}$$

$$fh_1(x) \ge u(x,y_2) + H(x,y_2,h_1(a(x,y_2))), \tag{80}$$

$$fh_2(x) \ge u(x,y_1) + H(x,y_1,h_2(a(x,y_1))). \tag{81}$$

In terms of (78) and (81), we gain that

$$\begin{aligned} fh_1(x) - fh_2(x) &< H(x,y_1,h_1(a(x,y_1))) \\ &\quad - H(x,y_1,h_2(a(x,y_1))) + \delta \le |H(x,y_1,h_1(a(x,y_1))) \\ &\quad - H(x,y_1,h_2(a(x,y_1)))| + \delta. \end{aligned} \tag{82}$$

On account of (79) and (80), we derive that

$$\begin{aligned} fh_2(x) - fh_1(x) &< H(x,y_2,h_2(a(x,y_2))) \\ &\quad - H(x,y_2,h_1(a(x,y_2))) + \delta \le |H(x,y_2,h_2(a(x,y_2))) \\ &\quad - H(x,y_2,h_1(a(x,y_2)))| + \delta. \end{aligned} \tag{83}$$

In light of (82) and (83), we get that

$$|fh_1(x) - fh_2(x)| < \max\{T_1, T_2\} + \delta, \tag{84}$$

where

$$\begin{aligned} T_1 &= |H(x,y_1,h_1(a(x,y_1))) - H(x,y_1,h_2(a(x,y_1)))|, \\ T_2 &= |H(x,y_2,h_2(a(x,y_2))) - H(x,y_2,h_1(a(x,y_2)))|. \end{aligned} \tag{85}$$

It follows from (75), (84), and $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_5 \times \Phi_3$ that

$$\begin{aligned} \psi\left(\int_0^{|fh_1(x)-fh_2(x)|} \varphi(t)dt\right) &\le \psi\left(\int_0^{\max\{T_1,T_2\}+\delta} \varphi(t)dt\right) \\ &= \max\left\{\psi\left(\int_0^{T_1+\delta} \varphi(t)dt\right), \psi\left(\int_0^{T_2+\delta} \varphi(t)dt\right)\right\} \\ &= \max\left\{\psi\left(\int_0^{T_1} \varphi(t)dt + \int_{T_1}^{T_1+\delta} \varphi(t)dt\right), \psi\right. \\ &\quad \left.\cdot \left(\int_0^{T_2} \varphi(t)dt + \int_{T_2}^{T_2+\delta} \varphi(t)dt\right)\right\} \\ &\le \max\left\{\psi\left(\int_0^{T_1} \varphi(t)dt\right) + \psi\left(\int_{T_1}^{T_1+\delta} \varphi(t)dt\right), \psi\right. \\ &\quad \left.\cdot \left(\int_0^{T_2} \varphi(t)dt\right) + \psi\left(\int_{T_2}^{T_2+\delta} \varphi(t)dt\right)\right\} \\ &\le \max\left\{\psi\left(\int_0^{T_1} \varphi(t)dt\right), \psi\left(\int_0^{T_2} \varphi(t)dt\right)\right\} \\ &\quad + \max\left\{\psi\left(\int_{T_1}^{T_1+\delta} \varphi(t)dt\right), \psi\left(\int_{T_2}^{T_2+\delta} \varphi(t)dt\right)\right\} \\ &\le \psi\left(\int_0^{M_1^*} \varphi(t)dt\right) - \phi\left(\int_0^{M_1^*} \varphi(t)dt\right) + \psi(\varepsilon). \end{aligned} \tag{86}$$

Taking $\varepsilon \to 0^+$ in the above inequalities and using $\psi \in \Phi_5$

and (70), we infer that

$$\psi\left(\int_0^{|fh_1(x)-fh_2(x)|}\varphi(t)dt\right)\le\psi\left(\int_0^{M_1^*}\varphi(t)dt\right)-\phi\left(\int_0^{M_1^*}\varphi(t)dt\right),$$

$$\psi\left(\int_0^{\|fh_1-fh_2\|}\varphi(t)dt\right)\le\psi\left(\int_0^{M_1^*}\varphi(t)dt\right)-\phi\left(\int_0^{M_1^*}\varphi(t)dt\right). \tag{87}$$

Similarly, we deduce that

$$\psi\left(\int_0^{\|fh_2-fh_3\|}\varphi(t)dt\right)\le\psi\left(\int_0^{M_1^*}\varphi(t)dt\right)-\phi\left(\int_0^{M_1^*}\varphi(t)dt\right),$$
$$\psi\left(\int_0^{\|fh_3-fh_1\|}\varphi(t)dt\right)\le\psi\left(\int_0^{M_1^*}\varphi(t)dt\right)-\phi\left(\int_0^{M_1^*}\varphi(t)dt\right). \tag{88}$$

It follows that

$$\begin{aligned}\psi\left(\int_0^{G(fh_1,fh_2,fh_3)}\varphi(t)dt\right)&=\psi\left(\int_0^{\max\{\|fh_1-fh_2\|,\|fh_2-fh_3\|,\|fh_3-fh_1\|\}}\varphi(t)dt\right)\\&=\max\left\{\psi\left(\int_0^{\|fh_1-fh_2\|}\varphi(t)dt\right),\psi\right.\\&\quad\left.\cdot\left(\int_0^{\|fh_2-fh_3\|}\varphi(t)dt\right),\psi\left(\int_0^{\|fh_3-fh_1\|}\varphi(t)dt\right)\right\}\\&\le\psi\left(\int_0^{M_1^*}\varphi(t)dt\right)\\&\quad-\phi\left(\int_0^{M_1^*}\varphi(t)dt\right),\quad\forall h_1,h_2,h_3\in B(S).\end{aligned} \tag{89}$$

Consequently, the conditions of Theorem 19 are satisfied. It follows from Theorem 19 that f and g have a unique common fixed point $h^*\in B(S)$, that is, the functional equations (72) have a unique common bounded solution $h^*\in B(S)$. This completes the proof.

As in the proof of Theorem 23, we obtain similarly the following result and omit its proof.

Theorem 24. *Let $u,v:S\times D\to\mathbb{R}$, $a,b:S\times D\to S$, and H, $L:S\times D\times\mathbb{R}\to\mathbb{R}$ satisfy (C1)–(C3) and (C5) for each $i\in\{1,2,3\}$, $x\in S$, $y\in D$ and $h_1,h_2,h_3\in B(S)$*

$$\psi\left(\int_0^{|H(x,y,h_i(a(x,y)))-H(x,y,h_{i+1}(a(x,y)))|}\varphi(t)dt\right)\le\psi\left(\int_0^{M_2^*}\varphi(t)dt\right)-\phi\left(\int_0^{M_2^*}\varphi(t)dt\right), \tag{90}$$

where $(\varphi,\psi,\phi)\in\Phi_1\times\Phi_5\times\Phi_3$, $h_4=h_1$ and

$$\begin{aligned}M_2^*(h_1,h_2,h_3)=\max\Bigg\{&G(gh_1,gh_2,gh_3),\\&\frac{[1+G(gh_1,fh_1,fh_1)]G(gh_3,fh_2,fh_2)}{2+G(gh_1,gh_2,fh_1)},\\&\frac{[1+G(gh_2,fh_2,fh_2)]G(gh_3,fh_1,fh_1)}{2+G(gh_1,gh_2,fh_2)},\\&\frac{[1+G(gh_3,fh_3,fh_3)]G(gh_1,fh_2,fh_2)}{2+G(gh_2,gh_3,fh_3)},\\&\frac{[1+G(gh_1,fh_1,fh_1)]G(gh_2,fh_3,fh_3)}{2[1+G(gh_1,gh_3,fh_1)]},\\&\frac{[1+G(gh_2,fh_2,fh_2)]G(gh_1,fh_3,fh_3)}{2[1+G(gh_2,gh_3,fh_2)]},\\&\frac{[1+G(gh_3,fh_3,fh_3)]G(gh_2,fh_1,fh_1)}{2[1+G(gh_1,gh_3,fh_3)]},\\&\frac{[1+G(gh_1,gh_2,gh_3)]G(fh_1,fh_2,fh_3)}{1+G(gh_3,gh_3,fh_2)+G(gh_1,gh_2,gh_3)},\\&\frac{[1+G(gh_1,gh_2,gh_3)]G(fh_1,fh_2,fh_3)}{1+G(gh_2,gh_2,fh_1)+G(gh_1,gh_2,gh_3)},\\&\frac{[1+G(gh_1,gh_2,gh_3)]G(fh_1,fh_2,fh_3)}{1+G(gh_1,gh_1,fh_3)+G(gh_1,gh_2,gh_3)}\Bigg\}.\end{aligned} \tag{91}$$

Then, the functional equations (72) have a unique common bounded solution $h^*\in B(S)$.

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions in $[0,T]$ with norm

$$\|x\|=\sup\{|x(t)|:t\in[0,T]\},\quad\forall x\in C([0,T],R). \tag{92}$$

Put $X=C([0,T],\mathbb{R})$ and define $G:X\times X\times X\to\mathbb{R}^+$ by

$$G(x,y,z)=\|x-y\|+\|y-z\|+\|z-x\|,\quad\forall x,y,z\in X. \tag{93}$$

It is obvious that (X,G) is a complete G-metric space. Consider the nonlinear Volterra integral equations:

$$\begin{aligned}x(t)&=p_1(t)+\int_0^t K_1(t,s,x(s))ds,\forall t\in[0,T],y(t)\\&=p_2(t)+\int_0^t K_2(t,s,y(s))ds,\forall t\in[0,T],\end{aligned} \tag{94}$$

where $T>0$ is a constant, $p_1,p_2:[0,T]\to\mathbb{R}$ and $K_1,K_2:[0,T]^2\times\mathbb{R}\to\mathbb{R}$ are given functions.

Put

$$fx(t)=p_1(t)+\int_0^t K_1(t,s,x(s))ds,\quad\forall(t,x)\in[0,T]\times X, \tag{95}$$

$$gx(t) = p_2(t) + \int_0^t K_2(t, s, x(s))ds, \quad \forall (t, x) \in [0, T] \times X. \tag{96}$$

Theorem 25. *Let $K_1, K_2 : [0, T]^2 \times \mathbb{R} \to \mathbb{R}$ and $p_1, p_2 : [0, T] \to R$ satisfy that*

(d1) K_1, K_2, p_1, and p_2 are continuous;

(d2) $fgx = gfx$ for each $x \in X$ with $fx = gx$;

(d3) $f(X) \subseteq g(X)$ and $g(X)$ is complete;

(d4) there exists a continuous function $Q : [0, T] \times [0, T] \to \mathbb{R}^+$ such that

$$|K_1(t, s, x(s)) - K_1(t, s, y(s))| \le Q(t, s)|gx(s) - gy(s)|, \quad \forall t, s \in [0, T], x, y \in X. \tag{97}$$

(d5) $\sup_{t\in[0,T]} \int_0^t Q(t, s)ds \le 1/2$.

Then, the nonlinear Volterra integral equations (94) have a unique common continuous solution in X.

Proof. Define $\varphi, \psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi(t) = 1, \psi(t) = 2t, \phi(t) = t, \quad \forall t \in R^+. \tag{98}$$

Clearly, $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. On account of $(d1)$ and (95), we deduce that for each $x \in X$, fx and gx are continuous functions in $[0, T]$, that is, f and g are mappings from X into itself. By means of (93), (95), (97), and $(d5)$, we obtain that

$$\begin{aligned}
\psi\left(\int_0^{G(fx,fy,fz)} \varphi(t)dt\right) &= 2\int_0^{G(fx,fy,fz)} \varphi(t)dt = 2G(fx, fy, fz) \\
&= 2(\|fx - fy\| + \|fy - fz\| + \|fz - fx\|) \\
&\le 2\left(\sup_{t\in[0,T]} \int_0^t |K_1(t, s, x(s)) - K_1(t, s, y(s))|ds\right. \\
&\quad + \sup_{t\in[0,T]} \int_0^t |K_1(t, s, y(s)) - K_1(t, s, z(s))|ds \\
&\quad \left. + \sup_{t\in[0,T]} \int_0^t |K_1(t, s, z(s)) - K_1(t, s, x(s))|ds\right) \\
&\le 2\left(\sup_{t\in[0,T]} \int_0^t Q(t, s)|gx(s) - gy(s)|ds\right. \\
&\quad + \sup_{t\in[0,T]} \int_0^t Q(t, s)|gy(s) - gz(s)|ds \\
&\quad \left. + \sup_{t\in[0,T]} \int_0^t Q(t, s)|gz(s) - gx(s)|ds\right) \\
&\le 2\left(\sup_{t\in[0,T]} \int_0^t Q(t, s)ds\right)(\|gx - gy\| \\
&\quad + \|gy - gz\| + \|gz - gx\|) \le G(gx, gy, gz) \\
&\le M_i(x, y, z) = \psi\left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right) - \phi \\
&\quad \cdot \left(\int_0^{M_i(x,y,z)} \varphi(t)dt\right), \quad \forall x, y, z \in X, i \in \{1, 2\},
\end{aligned} \tag{99}$$

where $M_1(x, y, z)$ and $M_2(x, y, z)$ are defined by (13) and (52), respectively. That is, the conditions of Theorems 19 and 20 are fulfilled. Therefore, each of Theorems 19 and 20 guarantees that f and g have a unique common fixed point $x \in X$, which is a unique common continuous solution of the nonlinear Volterra integral equations (94) in X. This completes the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Research Article

A Study of a Nonlinear Ordinary Differential Equation in Modular Function Spaces Endowed with a Graph

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In this paper, we prove by means of a fixed-point theorem an existence result of the Cauchy problem associated to an ordinary differential equation in modular function spaces endowed with a reflexive convex digraph.

1. Introduction

It is well known that fixed-point theory is a powerful tool that was frequently exploited to prove existence of solutions of differential equations not only in Banach spaces but also in a wider range of spaces, particularly in Orlicz and Musielak-Orlicz spaces [1, 2] and more generally in modular function spaces.

The Orlicz spaces were introduced in the early 1930s when the lack of flexibility of classical Lebesgue function spaces L^p, in fact the lack of stability under some differential operators, leads Orlicz and Birnbaum to consider the space

$$L\varphi = \left\{ f : \mathbb{R} \longrightarrow \mathbb{R} : \int_{\mathbb{R}} \varphi(\lambda |f(x)|) dx \longrightarrow 0 \text{ as } \lambda \longrightarrow 0 \right\}, \tag{1}$$

where $\varphi : \mathbb{R}_+ \mathbb{R}_+$ is a convex increasing function such that $\lim_{x\to\infty} \varphi(x) = \infty$ (the convexity of φ was subsequently very often omitted).

Later, in the end of the 1950s, Orlicz and Musielak considered the space

$$L\phi = \left\{ f \in X : \int_{\Omega} \phi(x, \lambda |f(x)|) d\mu \longrightarrow 0 \text{ as } \lambda \longrightarrow 0 \right\}, \tag{2}$$

where $(\Omega, \sum, \mu)$ is a measure space, X is the set of all real-valued (or complex-valued) Σ-measurable, μ-almost everywhere finite functions on Ω, and $\phi : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Carathéodory function which means that it is Σ-measurable for first variable, nondecreasing continuous mapping on the second variable and such that $\phi(x, 0) = 0$, $\phi(x, u) > 0$ if $u > 0$.

The theory of modular function spaces was initiated by Kozlowski [3–5], and those spaces were a sort of spaces situated in between the Musielak-Orlicz and modular ones that were both more concrete of ordinary modular spaces, as treating about functions sets, and offering much more flexibility than the Musielak-Orlicz spaces.

Furthermore, in [6, 7] jointly with Khamsi, Kozlowski has initiated fixed-point results in modular function spaces. Recently, a new direction has been developed, combining fixed-point results and graph theory; see, for instance, [8–10].

In the same vein, Kozlowski in [11] managed to prove the existence of solutions of the following differential equation of type:

$$(\text{O.D.E}) \begin{cases} u(0) = f, \\ u'(t) + (I - T)u(t) = 0, \ \forall t \in [0, A], \end{cases} \tag{3}$$

where $u(t)$ has values in modular function spaces and T satisfies nonexpansiveness assumption.

In this work, we intend to solve the equation (O.D.E) in modular function spaces endowed with a digraph, where nonexpansiveness of T is restricted to connected points, which is a far more general result than the one obtained by Kozlowski. We will first establish a fixed-point result that will be employed to prove the existence of solutions of (O.D.E) with less restraint conditions over T.

2. Preliminaries

We begin by recalling some elementary notions about graphs; see [12] for further properties.

Definition 1. A directed graph or digraph G is determined by a nonempty set $V(G)$ of its vertices and the set $E(G) \subset V(G) \times V(G)$ of its directed edges. A digraph is reflexive if each vertex has a loop. Given a digraph $G = (V, E)$.

(i) If whenever $(x, y) \in E(G) \Rightarrow (y, x) \notin E(G)$, then the digraph G is called an oriented graph

(ii) A digraph G is transitive whenever $[(x, y) \in E(G)$ and $(y, z) \in E(G)] \Rightarrow (x, z) \in E(G)$, for any $x, y, z \in V(G)$

(iii) A dipath of G is a sequence $a_0, a_1, \cdots, a_n, \cdots$ with $(a_i, a_i + 1) \in E(G)$ for each $i \in \mathbb{N}$

(iv) A finite dipath of length n from x to y is a sequence of $n + 1$ vertices $(a_0, a_1, \cdots, a_n)$ with $(a_i, a_{i+1}) \in E(G)$ and $x = a_0, y = a_n$

(v) A closed directed path of length $n > 1$ from x to y, i.e., $x = y$, is called a directed cycle

(vi) A digraph is connected if there is a finite (di)path joining any two of its vertices and it is weakly connected if $G^{\sim}$ is connected

(vii) $[x]_G$ is the set of all vertices which are contained in some path beginning at x (i.e., $y \in [x]_G \Leftrightarrow$ there exist $(a_0, a_1, \cdots, a_n)$ with $(a_i, a_{i+1}) \in E(G)$ and $x = a_0, y = a_n$)

We also need to introduce some properties of modular function spaces and tools that will be often used later. For more details, one can consult [3–5, 13, 14].

Let Ω be a nonempty set and $\mathcal{P}$ a nontrivial δ-ring of subsets of Ω and let Σ be the smallest σ-algebra of subsets of Ω such that Σ contains $\mathcal{P}$ such that $E \cap A \in \mathcal{P}$ for every $E \in \mathcal{P}$ and $A \in \sum$; and $K_n \uparrow \Omega$ where $K_n \in \mathcal{P}$, for all n.

$\mathcal{E}$ is the linear space of $\mathcal{P}$-simple functions, and M_∞ is the set of measurable functions. We denote by 1_A the characteristic function of A, where $A \subset \Omega$.

Definition 2 [11]. An even convex function $\rho : M_\infty \longrightarrow [0,+\infty]$ is called regular convex function pseudomodular if

(i) $\rho(0) = 0$

(ii) ρ is monotone, i.e., if for $f, g \in M_\infty, f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$, then $\rho(f) \leq \rho(g)$

(iii) ρ is orthogonally subadditive, i.e., $\rho(f.1_{A\cup B}) = \rho(f.1_A) + \rho(f.1_B)$ whenever $A, B \in \sum$ and $A \cap B = \phi$ and $f \in \sum$

(iv) ρ has the Fatou property, i.e., if $(|f_n(\omega)|)_n \uparrow |f(\omega)|$, for all $\omega \in \Omega, f_n, f \in M_\infty$ then $\rho(f_n) \uparrow \rho(f)$

(v) ρ is order continuous in $\mathcal{E}$, i.e., $g_n \in \mathcal{E}$, and $|g_n| \downarrow 0$ implies $\rho|g_n| \downarrow 0$

Let ρ be a regular convex function pseudomodular, the following notions are borrowed from [11].

(i) A set $A \in \sum$ is said to be ρ-null if $\rho(g.1_A) = 0, \forall g \in \mathcal{E}$

(ii) A property (P) is said to hold ρ almost everywhere if the exceptional set is ρ-null

(iii) We will identify pair of measurable sets whose symmetric difference is ρ-null, as well as pair of measurable function differing only on a ρ-null set

(iv) $M(\Omega, \sum, \mathcal{P}, \rho) = \{f \in M_\infty : |f(\omega)| < \infty \rho - \text{a.e.}\}$ briefly noted M

(v) ρ is said to be a regular convex function modular if $\rho(f) = 0$ implies $f = 0 \rho - a.e$

(vi) We denote by $\mathfrak{R}$ the set of all nonzero regular convex function modulars on Ω

Definition 3 [11]. Let $\rho \in \mathfrak{R}$.

(a) We say that $(f_n)_n \in L_\rho \rho$ converges to f and write $f_n \longrightarrow f(\rho)$, if $\rho(f_n - f) \longrightarrow 0$, and a sequence $(f_n)_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \longrightarrow 0$ as $(n, m) \longrightarrow \infty$

(b) A set $B \subset L_\rho$ is called ρ-closed, if for any sequence $(f_n)_n \in B, f_n \longrightarrow f(\rho)$ implies $f \in B$

(c) A set $B \subset L_\rho$ is called ρ-bounded, if its diameter $\delta_\rho(B) = \sup \{\rho(f - g): f, g \in B\}$ is finite

(d) A set $B \subset L_\rho$ is called ρ-compact, if for any sequence $(f_n)_n \in B$ there exists a subsequence $(f_{k_n})_n$ and $f \in B$ such that $f_{k_n} \rho$ -converges to f

(e) A set $B \subset L_\rho$ is called ρ-a.e.-closed, if for any sequence $(f_n)_n \in B, f_n \longrightarrow f$, ρ-a.e. implies $f \in B$

(f) A set $B \subset L_\rho$ is called ρ-a.e.compact, if for any sequence $(f_n)_n \in B$ there exists a subsequence $(f_{n_k})_k$ and $f \in B$ such that $f_{n_k} \longrightarrow f$, ρ-a.e

Definition 4. [11]. Let $\rho \in \mathfrak{R}$.

The modular function space is the vector space $L_\rho(\Omega, \sum)$ or briefly $L_\rho$is defined

$$L_\rho = \left\{ f \in \mathcal{M} : \lim_{\lambda \to 0} \rho(\lambda . f) = 0 \right\}. \tag{4}$$

The map $\|\cdot\|_\rho : L_\rho \longrightarrow [0, +\infty)$ defined by

$$\|f\|_\rho = \inf \left\{ \alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1 \right\} \tag{5}$$

is called norm of Luxembourg on L_ρ.

The following properties play a prominent role in the study of modular function spaces.

Definition 5 [11]. Let $\rho \in \mathfrak{R}$.

We say that ρ has the Δ_2 -property, if $\rho(2f_n) \longrightarrow 0$ whenever $\rho(f_n) \longrightarrow 0$, $((f_n)_n \in L_\rho)$. We say that ρ has the Δ_2 -type condition, if there exists $k \in [0, +\infty)$ such that $\rho(2f) \leq k\rho(f)$, for any $f \in L_\rho$.

The following definitions and results could be found in [6].

Theorem 6. *Let $\rho \in \mathfrak{R}$.*

(i) $(L_\rho, \|.\|_\rho)$ is a complete normed space, and $L\rho$ is ρ -complete

(ii) $\|f_n\|_\rho \longrightarrow 0$ iff $\rho(\alpha f_n) \longrightarrow 0$ for every $\alpha > 0$

(iii) If $\rho(f_n - f) \longrightarrow 0$ there exists $(f_{n_k})_k$ subsequence of $(f_n)_n$ such that $f_{n_k} \longrightarrow f$, ρ-a.e

(iv) If $f_n \longrightarrow f \rho - a.e$, then $\rho(f) \leq \liminf_{n \to +\infty} \rho(f_n)$ (the Fatou property)

(v) If ρ has the Δ_2 -property and $\rho(\alpha f_n) \longrightarrow 0$ for $\alpha > 0$, then $\|f_n\|_\rho \longrightarrow 0$

Definition 7. Let $\rho \in \mathfrak{R}$, we define

$$\begin{gathered} L^o_\rho = \left\{ f \in L_\rho : \rho(f,.) \text{ is order continuous} \right\}, \\ E_\rho = \left\{ f \in L_\rho : \lambda f \in L^0_\rho, \forall \lambda > 0 \right\}. \end{gathered} \tag{6}$$

Theorem 8. *Let $\rho \in \mathfrak{R}$, then E_ρ is a $\|.\|_\rho$ -closed subspace of L_ρ . Moreover, E_ρ is the $\|.\|_\rho$ closure of $\mathscr{E}$ the set of all $(\mathscr{P})$ simple functions.*

Definition 9. We say that a set $C \subset L_\rho$ possesses the Vitali property if $C \subset E_\rho$, and for any $g \subset L_\rho$ and $(g_n)_n \subset C$ with $g_n \longrightarrow g(\rho)$, there exists a subsequence $(g_{n_k})_k$ of $(g_n)_n$ such that for every $\alpha > 0$ the subadditive measures $\rho(\alpha g_{n_k}, .)$ are order equicontinuous. That is, if $(E_p)_p \subset \Sigma$ such that $(E_p)_p \downarrow \varnothing$ then $\forall \alpha > 0 \lim_{\rho \to \infty} \sup_{k\epsilon \mathrm{N}} \rho(\alpha g_{nk}, E_p) 0$.

The following statement characterizes sets with the Vitali property as subsets of E_ρ where the ρ convergence is equivalent to the $\|\bullet\|_\rho$ convergence.

Theorem 10. *Let $\rho \in \mathfrak{R}$. A set $C \in L_\rho$ has the Vitali property if and only if the following conditions are satisfied:*

(i) $C \subset E_\rho$

(ii) If $g \in L_\rho$ and $(g_n)_n \subset C$ and $(g_n)_n \longrightarrow g(\rho)$, then $\|g_n - g\|_\rho \longrightarrow 0$

Definition 11. A convex function modular $\rho \in \mathfrak{R}$ is said separable if $\forall f \in \mathcal{E}$, $(\|f1_{(.)}\|_\rho)$ is a separable set function for each $f \in \mathcal{E}$, which means that there exists a countable $\mathscr{A} \subset \mathscr{P}$ such that to every $A \in \mathscr{P}$ there corresponds a sequence $(A_k)_k$ of elements of A with

$$\forall \alpha > 0, \rho\left(\alpha f, A \Delta A_k\right) \underset{k \to \infty}{\longrightarrow} 0. \tag{7}$$

We recall this important result, which states that if ρ is separable, then $(L_\rho, \|.\|_\rho)$ is a separable Banach space; it is then a Polish space.

Theorem 12. *Let $\rho \in \mathfrak{R}$. The space $(L_\rho, \|.\|_\rho)$ is separable if and only if ρ is separable.*

Remark 13. Let Z be a separable linear subspace of E_ρ, $\|.\|_\rho$ and let $C \subset Z$ have the Vitali property. Assume that the function $u : [a, b] \longrightarrow C$ $(a, b \in \mathbb{R})$, is ρ-continuous. Then, u is Bochner integrable function with respect to the Lebesgue measure m on $[a, b]$. That is, if for $\tau := t_0 < t_1 < \cdots < t_m$ a subdivision of $[a, b]$, we define $|\tau| = \sup_{0 \leq i \leq m-1} (t_{i+1} - t_i)$ (called the step of τ), then

$$\lim_{|\tau| \to 0} \sum_{i=0}^{m-1} (t_{i+1} - t_i) u(t_i) \text{ exists}, \tag{8}$$

and we write

$$\int_a^b u(s) ds = \lim_{|\tau| \to 0} \sum_{i=0}^{m-1} (t_{i+1} - t_i) u(t_i). \tag{9}$$

3. Main Results

Definition 14. Let $C \subset L_\rho$, $a, b, \in \mathbb{R}$ and $C \subset L_\rho$.

(i) A function $u : [a, b] \longrightarrow C$ is said to be continuous if $u(t_n) \longrightarrow u(t)(\rho)$ provided $t_n \longrightarrow t$. We denote by $C([a.b], C)$ the set of all these continuous functions

(ii) A mapping $T : C \longrightarrow C$ is said to be ρ-continuous if $T(f_n) \longrightarrow T(f)(\rho)$ provided $f_n \longrightarrow f(\rho)$

Proposition 15. *Let $\rho \in \mathfrak{R}$ be separable and $C \subset E_\rho$ be a nonempty convex and ρ-closed set that has the Vitali property. Let $T : C \longrightarrow C$ be a ρ-continuous mapping and $f \in C$. Then, for every $u \in \mathcal{C}([0, A], C)$ the mapping $\phi(u)$ defined by*

$$\phi(u)(t) = e^{-t} f + \int_0^t e^{s-t} T(u(s)) ds \text{ for every } t \in [0, A] \quad (10)$$

takes values in C and is continuous, i.e., $\phi(u) \in \mathcal{C}([0, A], C)$.

Proof. Let $u \in C([0, A], C)$. We will first prove that $\phi(u)(t) \in C$ for every $t \in (0, A)$. Let $t \in (0, A)$, for $\tau := t_0 < t_1 < \cdots < t_m$, a subdivision of $[0, t]$, we define

$$S_\tau(T(u))(t) = \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} T(u(t)). \quad (11)$$

By definition of Bochner integral

$$\int_0^t e^{s-t} T(u(s)) ds = \lim_{|\tau| \to 0} S_\tau(T(u))(t). \quad (12)$$

We have $f \in C$, $T(u(t_i)) \in C$ for every $i \in \{1, \cdots, n\}$ and

$$e^{-t} + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} \leq e^{-t} + \int_0^t e^{s-t} ds = 1. \quad (13)$$

Using convexity of C, we get

$$e^{-t} f + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} T(u(t_i)) + \left(1 - \left(e^{-t} + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t}\right)\right) f \in C. \quad (14)$$

Since C is ρ-closed, it is also closed with respect to $\|.\|_\rho$. Thus,

$$\lim_{|\tau| \to 0} \left[e^{-t} f + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} T(u(t_i)) + \left(1 - \left(e^{-t} + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t}\right)\right) f \in C \right]. \quad (15)$$

Observing that

$$\lim_{|\tau| \to 0} \left(1 - \left(e^{-t} + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t}\right)\right) = 1 - e^{-t} - \int_0^t e^{s-t} ds = 0, \quad (16)$$

we conclude

$$\phi(u)(t) = e^{-t} f + \int_0^t e^{s-t} T(u(s)) ds \in C. \quad (17)$$

Furthermore, as C has the Vitali property, T is continuous with respect to the norm of Luxembourg, and then $\phi(u)$ is continuous as $t \mapsto \int_0^t e^{s-t} T(u(s)) ds$ is continuous; it is even differentiable.

The following notion of convex digraph was already introduced in [15].

Definition 16. Let $C \in L_\rho$ be a convex subset and G a digraph $(E(g) = C)$, we say that G is convex if $\forall \lambda \in [0, 1]$ and $\forall f, g, u, v, \in C$ such that $f \in [u]_G$ and $g \in [v]_G$; then

$$\lambda f + (1 - \lambda) g \in [\lambda u + (1 - \lambda) v]_G. \quad (18)$$

Definition 17. Let $\rho \in \mathfrak{R}$ and G a digraph $E(G) \subset L_\rho$. We say that G has the (P_0) property, if for all $(f_n)_n, (g_n)_n, f, g \in L_\rho$, $g \in [f]_G$ provided $g_n \in [f_n]_G$ for every $n \in \mathbb{N}$ and $f_n \longrightarrow f(\rho)$, $g_n \longrightarrow g(\rho)$.

Definition 18. Let $C \subset L_\rho$, G a digraph $E(G) = C$ and $T : C \longrightarrow C$. We say that T is G-monotone ρ-nonexpansive if for all $f, g \in C$ such that $g \in [f]_G$ we have

$$\begin{gathered} T(g) \in [T(f)]_G, \\ \rho(T(f) - T(g)) \leq \rho(f - g). \end{gathered} \quad (19)$$

The following lemma will play a preponderant role in the proof of the next theorem; its proof can be found in [11].

Lemma 19. *Let $\rho \in \mathfrak{R}$ be separable. Let $x, y : [0, A] \longrightarrow L_p$ two Bochner integrable $\|.\|_{\bar{\rho}}$ bounded functions, where $A > 0$. Then, for every $t \in [0, A]$, we have*

$$\rho\left(e^{-t} y(t) + \int_0^t e^{s-t} x(s) ds\right) \leq e^{-t} \rho(y(t)) + \left(1 - e^{-t}\right) \sup_{s \in [0,t]} (\rho(x(s))). \quad (20)$$

Now, we are able to state the main result.

Theorem 20. *Let $\rho \in \mathfrak{R}$ be separable, $C \subset E_\rho$ a nonempty convex, ρ-bounded, ρ-closed set that has the Vitali property, and let $G\,(E(G) = C)$ be a reflexive, convex digraph, with property (P_0). Let $T : C \longleftarrow C$ be a ρ-continuous and G-monotone ρ-nonexpansive mapping and suppose there exists $f \in C$ such that $T(f) \in [f]_G$; then, the mapping*

$$\begin{gathered} \phi : \mathcal{C}([0, A], C) \longrightarrow \mathcal{C}([0, A], C) \\ u \mapsto \phi(u) \end{gathered} \quad (21)$$

has a fixed point, where $\phi(u)(t) = e^{-t} f + \int_0^t e^{s-t} T(u(s)) ds$, for every $t \in [0, A]$.

Proof. Note that the mapping φ is well defined by Proposition 15. We define the sequence $(u_n)_n$ by

$$\begin{cases} u_0(t) = f, & \forall t \in [0, A], \\ u_{n+1} = \phi(u_n), & \forall n \in \mathbb{N}. \end{cases} \tag{22}$$

It is easy to see that $(u_n)_n \subset \mathcal{C}([0, A], C)$. We will prove by induction over $n \in \mathbb{N}$ that

$$\forall t \in [0, A], \forall n, p \in \mathbb{N}, \rho\left(u_{n+p}(t) - u_n(t)\right) \leq \left(1 - e^{A}\right)^{n+1} \delta_\rho(C). \tag{23}$$

For $n = 0$, it comes for every t in $[0, A]$,

$$u_p(t) - u_0(t) = \int_0^t e^{s-t} T\left(u_{p-1}\right)(s) ds - \left(1 - e^{-t}\right) f = \int_0^t e^{s-t} \left(T\left(u_{p-1}\right)(s) - f\right) ds. \tag{24}$$

Lemma 19 applied for $y = 0$ and $x(t) = T(u_{p-1}(t)) - f$ for every $t \in [0, A]$ gives

$$\forall t \in [0, A], \rho\left(u_p(t) - u_0\right) \leq \left(1 - e^{-t}\right) \sup_{s \in [0,t]} \rho\left(T\left(u_{p-1}(s)\right) - f\right) \leq \left(1 - e^{-A}\right) \delta_\rho(C). \tag{25}$$

We suppose now that for all $t \in [0, A]$ and for all $p \in \mathbb{N}$

$$\rho\left(u_{n+p}(t) - u_n(t)\right) \leq \left(1 - e^{-A}\right)^{n+1} \delta_\rho(C). \tag{26}$$

Let us first prove that $u_{n+1}(t) \in [u_n(t)]_G, \forall n \in \mathbb{N}$ and $\forall t \in [0, A]$. By induction on n, for $n = 0$, we have $\forall t \in [0, A]$,

$$u_1(t) = e^{-t} f + \left(1 - e^{-t}\right) T(f) \in [u_0(t)]_G = [f]_G, \tag{27}$$

as G is convex and $T(f) \in [f]_G$. If we suppose that for every $t \in [0, A]$, $u_{n+1}(t) \in [u_n(t)]_G$, then for $\tau := t_0 < t_1 \cdots < t_m$ a subdivision of $[0, t]$. Set for $k \geq 1$

$$u_k^\tau = e^{-t} f + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} T(u_{k-1}(t_i)), \tag{28}$$

then

$$u_{n+2}^\tau = e^{-t} f + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} T(u_{n+1}(t_i)), \tag{29}$$

as G is convex and $T(u_{n+1}(t_i)) \in [T(u_n(t_i))]_G$ for every $i = 1, \cdots, m$ we have

$$u_{n+2}^\tau + \left(1 - \left(e^{-t} + \sum_{i=0}^{m=1} (t_{i+1} - t_i) e^{t_i - t}\right)\right) f \in \left[u_{n+1}^\tau + \left(1 - \left(e^{-t} \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t}\right)\right) f\right]_G, \tag{30}$$

since G has the (P_0) property it follows that

$$\lim_{|\tau| \to 0} u_{n+2}^\tau + \left(1 - \left(e^{-t} + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t}\right)\right) f \in \left[\lim_{|\tau| \to 0} u_{n+1}^\tau + \left(1 - \left(e^{-t} + \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t}\right)\right) f\right]_G, \tag{31}$$

which is exactly $u_{n+2}(t) \in [u_{n+1}(t)]_G$. And then $u_{n+1}(t) \in [u_n(t)]_G$ and $u_{n+p}(t) \in [u_n(t)]_G$ for every $n, p \in \mathbb{N}$ and $t \in [0, A]$.

Now, as $\rho(u_{n+1+p}(t) - u_{n+1}(t)) = \rho(\int_0^t e^{s-t} (T(u_{n+p}(s)) - T(u_n(s))) ds)$, applying again Lemma 19 for $y(t) = 0, x(t) = T(u_{n+p}(s)) - T(u_n(s))$, we get

$$\rho\left(u_{n+1+p}(t) - u_{n+1}(t)\right) \leq \left(1 - e^{-t}\right) \sup_{s \in [0,t]} \rho\left(T\left(u_{n+p}(s)\right) - T(u_n(s))\right), \forall t \in [0, A], \tag{32}$$

but $\rho(T(u_{n+p}(s)) - T(u_n(s))) \leq \rho(u_{n+p}(s) - u_n(s)), \forall s \in [0, t]$ (as $u_{n+p}(s) \in [u_n(s)]_G$ and T is G-monotone ρ-nonexpansive); consequently,

$$\rho\left(u_{n+1+p}(t) - u_{n+1}(t)\right) \leq \left(1 - e^{t}\right) \sup_{s \in [0,t]} \rho\left(\left(u_{n+p}(s)\right) - \left(u_n(s)\right)\right), \forall t \in [0, A], \tag{33}$$

by the inductive assumption, we get

$$\rho\left(u_{n+1+p}(t) - u_{n+1}(t)\right) \leq \left(1 - e^{-t}\right)\left(1 - e^{-A}\right)^{n+1} \delta_p(C), \forall t \in [0, A], \tag{34}$$

i.e.,

$$\rho\left(u_{n+1+p}(t) - u_{n+1}(t)\right) \leq \left(1 - e^{-A}\right)^{n+2} \delta_\rho(C), \forall t \in [0, A]. \tag{35}$$

Using inequality (23), it is clear that for every $t \in [0, A]$, $(u_n(t))_n$ is a ρ-Cauchy sequence in C. Since C is ρ-closed in L_ρ, then it is ρ-complete and then $(u_n(t))_n \rho-$ converges to some $u(t) \in C$, and thus, it converges to $u(t)$ with respect to $\|.\|\rho$. We also have $\forall t \in [0, A]$, and $\forall n \in \mathbb{N}, u(t) \in [u_n(t)]_G$ as G has (P_0) property. Indeed, for $n \in \mathbb{N}, u_{n+p}(t) \in [u_n(t)]G$, $\forall p \in \mathbb{N}$ and when $p \longrightarrow \infty, u(t) \in [u_n(t)]_G$.

Now, for $t \in [0, A]$, let $\tau := t_0 < t_1 < \cdots < t_m$ be a subdivision of $[0, t]$. We have

$$\begin{aligned} &\rho(S_\tau(T(u_n(t))) - S_\tau(T(u(t)))) \\ &\quad = \rho\left(\sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} (T(u_n(t_i)) - T(u(t_i)))\right), \end{aligned} \tag{36}$$

and by convexity of ρ

$$\rho(S_\tau(T(u_n(t))) - S_\tau(T(u(t)))) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} \rho(T(u_n(t_i)) - T(u(t_i))). \tag{37}$$

Since $u(t_i) \in [u_n(t_i)]_G$ for every $i = 0, 1, \cdots, m$ and T is G-monotone ρ-nonexpansive

$$\rho(S_\tau(T(u_n(t))) - S_\tau(T(u(t)))) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) e^{t_i - t} \rho(u_n(t_i) - u(t_i)) \xrightarrow[n\to\infty]{} 0, \tag{38}$$

that is, $\lim_{n\to\infty} \rho(S_\tau(T(u_n(t))) - S_\tau(T(u(t)))) = 0$.

Hence,

$$\lim_{n\to\infty} \| S_\tau(T(u_n(t))) - S_\tau(T(u(t))) \|_\rho = 0. \tag{39}$$

Now, for every $t \in [0, A]$ and $n \in \mathbb{N}$

$$\begin{aligned} \left\| S_\tau\left(T(u(t)) - u(t) + e^{-f}\right) \right\|_\rho &\leq \| S_\tau(T(u(t)) - S_\tau(T(u_n(t)))) \|_\rho \\ &+ \left\| S_\tau(T(u_n(t))) - \int_0^t e^{s-t} T(u_n(s)) ds \right\|_\rho \\ &+ \left\| \int_0^t e^{s-t} T(u_n(s)) ds - u(t) + e^{-t} f \right\|_\rho . \end{aligned} \tag{40}$$

From $\| \int_0^t e^{s-t} T(u_n(s)) ds - u(t) + e^{-t} f \|_\rho = \| u_{n+1}(t) - u(t) \|_\rho$, we get

$$\lim_{|\tau|\to 0} \left\| S_\tau(T(u(t))) - u(t) + e^{-t} f \right\|_\rho = 0, \tag{41}$$

i.e., $s \mapsto e^{(s-t)} T(u(s))$ is Bochner integrable and $\int_0^t e^{s-t} T(u(s))ds = u(t) - e^{-t} f$. Finally, we get for every $t \in [0, A]$

$$\phi(u)(t) = e^{-t} f + \int_0^t e^{s-t} T(u(s)) ds = u(t), \tag{42}$$

that is, u is a fixed point of ϕ.

A similar result can be obtained without assuming the Vitali property, but we need to assume that ρ has the Δ_2-property.

Theorem 21. *Let $\rho \in \mathfrak{R}$ be separable and has the Δ_2-property, $C \subset L_\rho$ a nonempty convex, ρ-bounded, ρ-closed set, and let $G\,(E(G) = C)$ be a reflexive, convex digraph, with property (P_0). Let $T : C \longrightarrow C$ be a ρ-continuous and G-monotone ρ-nonexpansive mapping, and suppose that there exist $f \in C$ such that $T(f) \in [f]_G$; then, the mapping*

$$\begin{aligned} \phi : \mathcal{C}([0, A], C) &\longrightarrow \mathcal{C}([0, A], C) \\ u &\mapsto \phi(u) \end{aligned} \tag{43}$$

has a fixed point where $\phi(u)(t) = e^{-t} f + \int_0^t e^{s-t} T(u(s))ds$,, for every $t \in [0, A]$.

Proof. Since ρ has the Δ_2-property, the ρ-convergence is equivalent to the convergence with respect to $\|.\|_\rho$ all over in L_ρ; the proof of this corollary runs along similar lines to the proof of Theorem 20.

The last result is devoted to prove the existence of solution of the equation (O.D.E).

Theorem 22. *Let $\rho \in \mathfrak{R}$ be separable, $C \subset E_\rho$ a nonempty convex, ρ-bounded, ρ-closed set that has the Vitali property, and let $G(E(G) = C)$ be a reflexive, convex digraph, with property (P_0). Let $T : C \longrightarrow C$ be a ρ-continuous and G-monotone ρ-nonexpansive mapping, and suppose that there exists $f \in C$ such that $T(f) \in [f]_G$; then, the differential equation*

$$(O.D.E) \begin{cases} u(0) = f, \\ u'(t) + (I - T)u(t) = 0, \ \forall t \in [0, A], \end{cases} \tag{44}$$

where $u : [0, A] \longrightarrow C$, $A > 0$, has a solution.

Proof. The application ϕ defined above has a fixed point $u \in \mathcal{C}([0, A], C)$, that is,

$$u(t) = e^{-t} f + \int_0^t e^{s-t} T(u(s)) ds, \forall \in [0, A], \tag{45}$$

then u is differentiable and

$$\begin{aligned} u'(t) &= -e^{-t} f - \int_0^t e^{s-t} T(u(s)) ds + e^{-t}\left(e^t T(u(t))\right) \\ &= T(u(t)) - u(t), \forall t \in [0, A], \end{aligned} \tag{46}$$

that is, u is the solution of (O.D.E).

Note that the result of Theorem 22 remains true if ρ has the Δ_2-property instead of C having the Vitali property.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Existence and Uniqueness of Solutions for Fractional Boundary Value Problems under Mild Lipschitz Condition

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This paper deals with the following boundary value problem $\begin{cases} D^\alpha u(t) = f(t, u(t)), t \in (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0, \end{cases}$ where $3 < \alpha \le 4$, D^α is the Riemann-Liouville fractional derivative, and the nonlinearity f, which could be singular at both $t = 0$ and $t = 1$, is required to be continuous on $(0,1) \times \mathbb{R}$ satisfying a mild Lipschitz assumption. Based on the Banach fixed point theorem on an appropriate space, we prove that this problem possesses a unique continuous solution u satisfying $|u(t)| \le c\omega(t)$, for $t \in [0,1]$ and $c > 0$, where $\omega(t) \coloneqq t^{\alpha-2}(1-t)^2$.

1. Introduction

Higher order fractional differential equations subject to two-point boundary value problems occur naturally when modeling various phenomena in the applied sciences (see, for example, [1–4] and references therein). The study of existence, uniqueness, and qualitative properties of the solutions of such problems subject to various type of boundary conditions become an active area of research (see, for instance, [5–13] and references therein).

In [10], the authors considered the following problem

$$\begin{cases} D^\alpha u(t) = f(t, u(t)), & t \in (0,1), \\ u(0) = u(1) = u'(0) = u(1) = 0, \end{cases} \tag{1}$$

where $3 < \alpha \le 4$,D^α is the standard Riemann–Liouville fractional derivative.

By reducing problem (1) to an equivalent Fredholm integral equation and using some fixed-point theorems, they have proved the existence, multiplicity, and uniqueness of positive solutions. In their approach, properties of the corresponding Green's function are used.

In [6], the authors proved the existence and uniqueness of positive solutions of the problem

$$\begin{cases} D^\alpha u(t) = p(t)u^\sigma(t), & t \in (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0, \end{cases} \tag{2}$$

where $\sigma \in (-1,1)$,$3 < \alpha \le 4$,D^α is the standard Riemann–Liouville fractional derivative. Their approach relies on properties of Karamata regular variation functions and the Schauder fixed point theorem.

Recently, in [13], Zou and He, by using the Banach fixed point theorem on an appropriate space, they have proved the existence and uniqueness of a solution to the following problem

$$\begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = u(1) = 0, \end{cases} \tag{3}$$

where $2 < \alpha \leq 3, D^{\alpha}$ denotes the standard Riemann–Liouville fractional derivative and f satisfying:

(C1) $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$ and $f(s,0) \in L^1((0,1))$.

(C2) There exists $p \in C((0,1),[0,\infty))$ such that

$$|f(t,u) - f(t,v)| \leq p(t)|u - v|, \forall t \in (0,1), u, v \in \mathbb{R}$$
$$0 < \int_0^1 p(s)ds < \infty. \tag{4}$$

In this paper, following the approach used in [13], we will address the existence and uniqueness of a solution to the following fractional problem involving fractional boundary derivatives

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)), t \in (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0, \end{cases} \tag{5}$$

where $3 < \alpha \leq 4$ and D^{α} denotes the standard Riemann–Liouville fractional derivative. The nonlinear term f is required to satisfy:

(H1) $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 s(1-s)^{\alpha-3}|f(s,0)|ds < \infty$.

(H2) There exists $p \in C((0,1),[0,\infty))$ such that

$$|f(t,u) - f(t,v)| \leq p(t)|u - v|, \forall t \in (0,1), u, v \in \mathbb{R}$$
$$0 < L_{p,\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}p(s)ds < \infty. \tag{6}$$

Note that the nonlinearity f is allowed to be singular at $t = 0$ or/and $t = 1$.

Remark 1. It is worth mentioning that problem (5) is different form problem (3) and that conditions (*H1*) and (*H2*) are weaker then (*C1*) and (*C2*). Indeed, for example, the function $f(t,u) := (1-t)^{-\alpha/3}\cos u$ with $\alpha \in (3,4]$ satisfies (*H1*) and (*H2*) (see Example 8) but not (*C1*) and (*C2*).

To simplify our statement, for $\alpha \in (3,4]$, we fix the following notation:

(i) $G_{\alpha}(t,s)$ denotes the Green's function of the operator $u \to D^{\alpha}u$ with boundary conditions $u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0$, which is given (see [6], Lemma 5) for $t, s \in [0,1]$ by

$$G_{\alpha}(t,s) = \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-2}(1-s)^{\alpha-2}[s - t + (\alpha-2)s(1-t)] + (\max(t-s,0))^{\alpha-1}\right). \tag{7}$$

(ii) Let L be the minimum positive constant such that

$$\int_0^1 G_{\alpha}(t,s)\omega(t)p(s)ds \leq L\omega(t), \tag{8}$$

where

$$\omega(t) := t^{\alpha-2}(1-t)^2, \text{for } t \in [0,1]. \tag{9}$$

We will prove that L satisfies

$$\alpha(\alpha-2)L_{p,\alpha+1} \leq L \leq \gamma L_{p,\alpha}, \tag{10}$$

where $\gamma := \max((\alpha-2)^2, \alpha-1)$.

(iii) $C_{\omega}([0,1]) := \{u \in C([0,1])$ there is $\sigma > 0$ such that $|u(t)| \leq \sigma\omega(t), t \in [0,1]\}$

Note that $C_{\omega}([0,1])$ is a Banach space equipped with the $\|.\|_{\omega}$-norm

$$\|u\|_{\omega} := \inf\{\sigma > 0 : |u(t)| \leq \sigma\omega(t), t \in [0,1]\} = \sup_{t\in(0,1)} \frac{|u(t)|}{\omega(t)}. \tag{11}$$

Our main result is the following.

Theorem 2. *Assume that (H1) and (H2) hold and $L < 1$. Then, problem (5) has a unique solution u in $C_{\omega}([0,1])$.*

In addition, for any $u_0 \in C_{\omega}([0,1])$, the iterative sequence $u_k(t) := \int_0^1 G_{\alpha}(t,s)f(s,u_{k-1}(s))ds$ converges to u with respect to the $\|.\|_{\omega}$-norm, and we have

$$\|u_k - u\|_{\omega} \leq \frac{L^k}{1-L}\|u_1 - u_0\|_{\omega}. \tag{12}$$

Remark 3. Assume that (*H1*) and (*H2*) hold. If $\gamma L_{p,\alpha} < 1$, then from (10), it follows that $L < 1$.

Note that the condition $\gamma L_{p,\alpha} < 1$ is satisfied by a large class of functions$p$ including singular ones. As examples, we have

(i) If p is a positive continuous function on $(0,1)$ with $\sup_{t\in(0,1)} |p(t)| \leq 1$, then

$$\gamma L_{p,\alpha} \leq \frac{\gamma}{\Gamma(\alpha)}\mathscr{B}(\alpha,\alpha) < 1. \tag{13}$$

(ii) If $p(s) \equiv (1-s)^{-\alpha/3}$, then

$$\gamma L_{p,\alpha} = \frac{\gamma}{\Gamma(\alpha)}\mathscr{B}\left(\alpha, \frac{2\alpha}{3}\right) < 1. \tag{14}$$

(iii) If $p(s) \equiv s^{-\alpha/3}(1-s)^{-\alpha/3}$, then

$$\gamma L_{p,\alpha} = \frac{\gamma}{\Gamma(\alpha)} \mathscr{B}\left(\frac{2\alpha}{3}, \frac{2\alpha}{3}\right) < 1, \quad (15)$$

where $\mathscr{B}$ denotes the Beta function.

The paper is organized as follows. In Section 2, we recall basic properties of the Green's function $G_\alpha(t,s)$, and we prove that L satisfies the range estimates (10). In Section 3, we prove our main result. To illustrate our existence result, some examples and approximations are given.

2. Preliminaries

For the convenient of the reader, we recall the following definition.

Definition 4 ([2, 14]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $g : (0,\infty) \to \mathbb{R}$ is defined as

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} g(s)ds, \quad (16)$$

provided that the right-hand side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

The next key lemma is useful. For the proof, we refer the reader to [6].

Lemma 5. *Let $G_\alpha(t,s)$ be the Green's function given by (7). Then*

(i) $G_\alpha(t,s)$ is a nonnegative continuous function on $[0,1] \times [0,1]$

(ii) For all $t, s \in [0,1]$, we have

$$(\alpha-2)H_\alpha(t,s) \le G_\alpha(t,s) \le \gamma H_\alpha(t,s), \quad (17)$$

where $H_\alpha(t,s) := (1/\Gamma(\alpha)) t^{\alpha-3}(1-t)s(1-s)^{\alpha-3} \min(t,s)(1-\max(t,s))$.

(iii) Let g be a function such that the map $t \to t^2(1-t)^{\alpha-2} g(t)$ is continuous and integrable on $(0,1)$. Then, $Vg(t) := \int_0^1 G_\alpha(t,s)g(s)ds$ is the unique solution in $C([0,1])$ of the boundary value problem

$$\begin{cases} D^\alpha u(t) = g(t), t \in (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0. \end{cases} \quad (18)$$

Lemma 6. *Let $p \in C((0,1), [0,\infty))$ such that $0 < L_{p,\alpha} < \infty$. Then*

$$\alpha(\alpha-2)L_{p,\alpha+1} \le L \le \gamma L_{p,\alpha}, \quad (19)$$

where $\gamma := \max((\alpha-2)^2, \alpha-1)$, and L is the constant defined in (8).

Proof. Consider the set

$$E = \left\{ a > 0 : \int_0^1 G_\alpha(t,s)\omega(s)p(s)ds \le a\omega(t), t \in [0,1] \right\}, \quad (20)$$

where $\omega(t) := t^{\alpha-2}(1-t)^2, t \in [0,1]$.

From the upper inequality in (17), we obtain

$$\int_0^1 G_\alpha(t,s)\omega(s)p(s)ds \le \gamma \int_0^1 H_\alpha(t,s)\omega(s)p(s)ds \le \gamma L_{p,\alpha}\omega(t). \quad (21)$$

It follows that $E \ne \varnothing$ and $L \le \gamma L_{p,\alpha}$.

On the other hand, from the lower inequality in (17), we deduce for any $a \in E$ that

$$\begin{aligned} a\omega(t) &\ge \frac{(\alpha-2)}{\Gamma(\alpha)} t^{\alpha-3}(1-t) \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1} \min(t,s) \\ &\quad \cdot (1-\max(t,s))p(s)ds \\ &\ge \frac{(\alpha-2)}{\Gamma(\alpha)} t^{\alpha-3}(1-t) \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1} ts(1-t)(1-s)p(s)ds \\ &= \alpha(\alpha-2)\omega(t)L_{p,\alpha+1}. \end{aligned} \quad (22)$$

Hence, for each $a \in E$,

$$a \ge \alpha(\alpha-2)L_{p,\alpha+1}. \quad (23)$$

Therefore, $L \ge \alpha(\alpha-2)L_{p,\alpha+1}$. That is $L \in [\alpha(\alpha-2)L_{p,\alpha+1}, \gamma L_{p,\alpha}]$.

3. Main Results

Assume that $(H1)$ and $(H2)$ hold and $L < 1$. We prove that problem (5) has a unique solution u in $C_\omega([0,1])$. In addition, for any $u_0 \in C_\omega([0,1])$, the iterative sequence $u_k(t) := \int_0^1 G_\alpha(t,s)f(s,u_{k-1}(s))ds$ converges to u with respect to the $\|.\|_\omega$-norm, and we have

$$\|u_k - u\|_\omega \le \frac{L^k}{1-L} \|u_1 - u_0\|_\omega. \quad (24)$$

Consider the operator T defined by

$$Tu(t) := \int_0^1 G_\alpha(t,s)f(s,u(s))ds, t \in [0,1], u \in C_\omega([0,1]). \quad (25)$$

We claim that T is a contraction operator from $(C_\omega([0,1]), \|.\|_\omega)$ into itself.

Indeed, let $u \in C_\omega([0,1])$ and $\sigma > 0$ such that $|u(t)| \le \sigma\omega(t)$, for all $t \in [0,1]$.

Since by Lemma 5 (ii), $0 \le G_\alpha(t,s) \le (\gamma/\Gamma(\alpha)) s(1-s)^{\alpha-3}$, it follows from (*H2*) that

$$\begin{aligned} &|G_\alpha(t,s) f(s,u(s))| \\ &\quad\le \frac{\gamma}{\Gamma(\alpha)} s(1-s)^{\alpha-3}(|f(s,u(s)) - f(s,0)| + |f(s,0)|) \\ &\quad\le \frac{\gamma}{\Gamma(\alpha)} s(1-s)^{\alpha-3}(p(s)|u(s)| + |f(s,0)|) \\ &\quad\le \frac{\gamma}{\Gamma(\alpha)} \left(\sigma s^{\alpha-1}(1-s)^{\alpha-1}p(s) + s(1-s)^{\alpha-3}|f(s,0)|\right). \end{aligned} \tag{26}$$

Using the fact that $G_\alpha(t,s)$ is continuous on $[0,1]\times[0,1]$, we deduce by (*H1*),(*H2*), and the dominated convergence theorem that $Tu \in C([0,1])$.

On the other hand, from Lemma 6 (ii), we have

$$0 \le G_\alpha(t,s) \le \frac{\gamma}{\Gamma(\alpha)} s(1-s)^{\alpha-3}\omega(t). \tag{27}$$

By using (27) and similar arguments as above, we obtain

$$|Tu(t)| \le \gamma\left[\sigma L_{p,\alpha} + \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-3}|f(s,0)|ds\right]\omega(t). \tag{28}$$

So, $T(C_\omega([0,1])) \subset C_\omega([0,1])$.

Let $u, v \in C_\omega([0,1])$. By using (*H2*) and (8), we obtain for $t \in [0,1]$,

$$\begin{aligned} &|Tu(t) - Tv(t)| \\ &\quad\le \int_0^1 G_\alpha(t,s)|f(s,u(s)) - f(s,v(s))|ds \\ &\quad\le \int_0^1 G_\alpha(t,s)p(s)|u(s)-v(s)|ds \\ &\quad\le \|u-v\|_\omega \int_0^1 G_\alpha(t,s)p(s)\omega(s)ds \le L\|u-v\|_\omega\omega(t). \end{aligned} \tag{29}$$

Hence

$$\|Tu - Tv\|_\omega \le L\|u-v\|_\omega. \tag{30}$$

Since $L<1$, we deduce that T is a contraction operator in $C_\omega([0,1])$. Hence, there exists a unique $u \in C_\omega([0,1])$ satisfying

$$u(t) = \int_0^1 G_\alpha(t,s) f(s,u(s))ds, t \in [0,1]. \tag{31}$$

We need to prove that u is a solution of problem (5). Indeed, it is clear that $s \to s^2(1-s)^{\alpha-2} f(s,u(s))$ is continuous on $(0,1)$.

On the other hand, by writing

$$|f(s,u(s))| \le |f(s,u(s)) - f(s,0)| + |f(s,0)| \le p(s)|u(s)| + |f(s,0)|, \tag{32}$$

we deduce from (*H1*), (*H2*), and $u \in C_\omega([0,1])$ that the function $s \to s^2(1-s)^{\alpha-2} f(s,u(s))$ is integrable on $((0,1))$.

Hence, from (31) and Lemma 5 (iii), we conclude that u is the unique solution of problem (5).

Finally, it is well known that for any $u_0 \in C_\omega([0,1])$, the iterative sequence $u_k := T(u_{k-1})$ converges to u, and we have

$$\|u_k - u\|_\omega \le \frac{L^k}{1-L}\|u_1 - u_0\|_\omega. \tag{33}$$

Corollary 7. *Let $3<\alpha\le 4$ and f be a function satisfying (H1) and (H3) for all $t \in (0,1)$ and $u, v \in \mathbb{R}$*

$$|f(t,u) - f(t,v)| \le L_0|u-v|, \tag{34}$$

where $0 < L_0 < \Gamma(2\alpha)/\max((\alpha-2)^2, \alpha-1)\Gamma(\alpha)$.

Then, problem (5) has a unique solution u in $C_\omega([0,1])$.

Proof. The conclusion follows from Lemma 6 and Theorem 2.

Example 8. Let $3<\alpha\le 4, \gamma = \max((\alpha-2)^2, \alpha-1)$ and consider the following singular fractional problem:

$$\begin{cases} D^\alpha u(t) = (1-t)^{-\alpha/3}\cos u, & t \in (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0. \end{cases} \tag{35}$$

To verify that hypotheses (*H1*) and (*H2*) are satisfied, set $f(t,u) := (1-t)^{-\alpha/3}\cos u$, for all $t \in (0,1)$ and $u \in \mathbb{R}$.

We have $f \in C((0,1)\times\mathbb{R},\mathbb{R})$ and $\int_0^1 s(1-s)^{\alpha-3}|f(s,0)|ds = \int_0^1 s(1-s)^{(2\alpha/3)-3}ds < \infty$. So condition (*H1*) is satisfied.

On the other hand, we have for all $t \in (0,1)$ and $u, v \in \mathbb{R}$

$$|f(t,u) - f(t,v)| \le p(t)|u-v|, \tag{36}$$

where $p(t) = (1-t)^{-\alpha/3}$ and

$$0 < L_{p,\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{(2\alpha/3)-1}ds < \infty. \tag{37}$$

Furthermore, since by Lemma 6 and Remark 3 (ii), $L \le \gamma L_{p,\alpha} < 1$, we deduce from Theorem 2 that problem (35) has a unique solution $u \in C_\omega([0,1])$.

In particular, for $\alpha = 7/2$, this solution is approximated (see Figure 1) by the iterative sequence $u_k(t) := \int_0^1 G_{7/2}(t,s)(1-s)^{-7/6}\cos(u_{k-1}(s))ds$ with $u_0(t) = t^{3/2}(1-t)^2, t \in [0,1]$.

The sequences of functions $u_0(t), u_1(t), u_2(t), u_3(t)$, and $u_4(t)$ are illustrated in Figure 1.

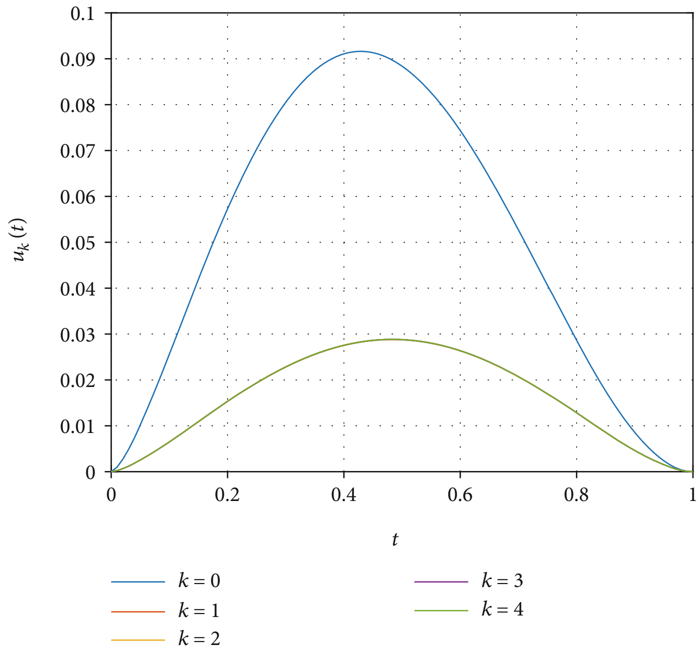

FIGURE 1: The approximation of the solution of problem (35).

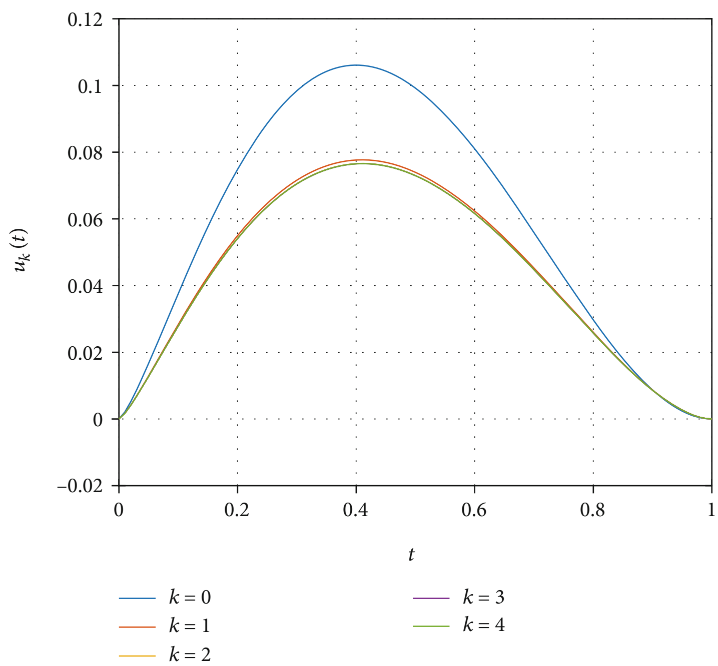

FIGURE 2: The approximation of the solution of problem (38).

Example 9. Consider the following singular fractional problem:

$$\begin{cases} D^{10/3}u(t) = t^{-10/9}(1-t)^{-10/9}(1+u), t \in (0,1), \\ u(0) = u(1) = D^{1/3}u(0) = u'(1) = 0. \end{cases} \quad (38)$$

By using similar arguments as in the previous example, we verify that conditions (*H1*) and (*H2*) are fulfilled.

Hence, by applying Theorem 2, problem (38) has a unique solution $u \in C_\omega([0,1])$.

Furthermore, the iterative sequence defined by $u_k(t) := \int_0^1 G_{10/3}(t,s)s^{-10/9}(1-s)^{-10/9}(1+u_{k-1}(s))ds$ and $u_0(t) := t^{4/3}(1-t)^2$ converges to u. Some iterations are depicted in Figure 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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Research Article

Fixed Points for Contractive Mappings of Integral Type Involving ω-Distance and Applications

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In this paper, we use ω-distance to prove the existence, uniqueness, and iterative approximations of fixed points for a few contractive mappings of integral type in complete metric spaces. The proved results are used to investigate the solvability of certain nonlinear integral equations. Four examples are given.

1. Introduction

The researchers [1–17] attained various generalizations of the well-known Banach contraction principle. In 2001, Rhoades [15] certified several fixed point results for the weakly contractive mappings. Branciari [1] introduced the notion of integral type contraction and established a nice fixed point result for the mapping. Many authors investigated the existence of fixed points for a lot of contractive mappings of integral type, for example, see [11–14]. Particularly, Liu et al. [14] obtained several fixed point theorems for contractive mappings of integral type in complete metric spaces.

Kada et al. [8] introduced the concept of ω-distance in metric spaces and proved a few fixed point theorems for some contractive mappings by using ω-distance. It is clear that the results in [8] extended the Caristi's fixed point theorem, Ekeland's ε-variational's principle, and the nonconvex minimization theorem. The researchers in [3, 5–7, 9, 10, 17] got several fixed point results for certain contractive mappings with respect to ω-distance.

In this paper, we prove the existence, uniqueness, and iterative approximations of fixed points for several kinds of mappings, which satisfy some contractive conditions of integral type with respect to ω-distance in complete metric spaces. We also construct four illustrative examples and give applications of the obtained results in nonlinear Fredholm and Volterra integral equations, respectively. Our results generalize or differ from the corresponding fixed point theorems in [1, 14, 15].

2. Preliminaries

Let $\mathbb{N}$ denotes the set of all positive integers, $R=(-\infty, +\infty)$, $R^+=[0,+\infty)$, $N_0=N\cup\{0\}$ and

$$\Phi_1=\left\{w \mid w: R^+\to R^+ \text{ is Lebesgue integrable, summable on each compact subset of } R^+ \text{ and } \int_0^\varepsilon w(r)dr>0 \text{ for each } \varepsilon>0\right\};$$

$$\Phi_2=\left\{w \mid w\in\Phi_1 \text{ and satisfies that } \int_0^u w(r)dr<\int_0^v w(r)dr \text{ for each } u, v\in R^+ \text{ with } u<v\right\};$$

$$\Phi_3=\left\{w \mid w: R^+\to R^+ \text{ is nondecreasing, continuous, } w(0)=0,\ \lim_{t\to+\infty} w(t)=+\infty \text{ and } w(t)>0 \text{ for each } t>0\right\};$$

$$\Phi_4 = \{w \mid w : R^+ \to R^+ \text{ is lower semi} - \text{continuous}, w(0) = 0 \text{ and } w(t) > 0 \text{ for each } t > 0\}. \quad (1)$$

Recall that a self-mapping f in a metric space X is called orbitally continuous if $\lim_{n\to\infty} f^n x = u$ implies $\lim_{n\to\infty} f^{n+1} x = fu$ for each $\{f^n x\}_{n\in N_0} \subseteq X$ and $u \in X$.

3. Fixed Point Results with respect to ω -Distance

In this section, using ω-distance, we give four fixed point theorems for the contractive mappings (2), (38), (82), and (114) below.

Theorem 1. *Let p be a ω-distance in a complete metric space (X, d) and let $f : X \to X$ satisfy that*

$$\int_0^{p(fx,fy)} w(r)dr \le \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,y))} w(r)dr, \forall x, y \in X. \quad (2)$$

Here, $(w, \psi) \in \Phi_1 \times \Phi_4$. Then, f possesses a unique fixed point $u \in X$ such that $p(u, u) = 0$,

$$\lim_{n\to\infty} p(f^n x_0, u) = 0 \text{ and } \lim_{n\to\infty} f^n x_0 = u \text{ for each } x_0 \in X. \quad (3)$$

Proof. Firstly, we claim the existence of fixed points of f in X. Put $x_0 \in X$ and $x_n = f^n x_0$ for each $n \in N_0$. Now, we need to think over two situations as follows:

Case 2. $x_{n_0} = x_{n_0-1}$ for some $n_0 \in N$. Clearly, x_{n_0-1} is a fixed point of f and $\lim_{n\to\infty} f^n x_0 = x_{n_0-1}$. Suppose that $p(x_{n_0-1}, x_{n_0-1}) > 0$. Making use of (2) and $(w, \psi) \in \Phi_1 \times \Phi_4$, we obtain that

$$\begin{aligned} \int_0^{p(x_{n_0-1},x_{n_0-1})} w(r)dr &= \int_0^{p(fx_{n_0-1},fx_{n_0-1})} w(r)dr \\ &\le \int_0^{p(x_{n_0-1},x_{n_0-1})} w(r)dr \\ &\quad - \int_0^{\psi(p(x_{n_0-1},x_{n_0-1}))} w(r)dr \\ &< \int_0^{p(x_{n_0-1},x_{n_0-1})} w(r)dr, \end{aligned} \quad (4)$$

which is ridiculous. Hence, $p(x_{n_0-1}, x_{n_0-1}) = 0$, which means that

$$\lim_{n\to\infty} p(f^n x_0, x_{n_0-1}) = p(x_{n_0-1}, x_{n_0-1}) = 0\,; \quad (5)$$

Case 3. $x_n \ne x_{n-1}$ for all $n \in N$. Suppose that

$$p(x_{n_0-1}, x_{n_0}) = 0 \text{ for some } n_0 \in N. \quad (6)$$

(2), (6), and $(w, \psi) \in \Phi_1 \times \Phi_4$ ensure that

$$\begin{aligned} 0 \le \int_0^{p(x_{n_0},x_{n_0+1})} w(r)dr &= \int_0^{p(fx_{n_0-1},fx_{n_0})} w(r)dr \\ &\le \int_0^{p(x_{n_0-1},x_{n_0})} w(r)dr - \int_0^{\psi(p(x_{n_0-1},x_{n_0}))} w(r)dr = 0, \end{aligned} \quad (7)$$

that is,

$$\int_0^{p(x_{n_0},x_{n_0+1})} w(r)dr = 0. \quad (8)$$

The above equation and $w \in \Phi_1$ give that

$$p(x_{n_0}, x_{n_0+1}) = 0. \quad (9)$$

Combining (6), (9), and (p_1), we know that

$$0 \le p(x_{n_0-1}, x_{n_0+1}) \le p(x_{n_0-1}, x_{n_0}) + p(x_{n_0}, x_{n_0+1}) = 0, \quad (10)$$

that is,

$$p(x_{n_0-1}, x_{n_0+1}) = 0. \quad (11)$$

Because of Lemma 1 in [8], (6), and (11), we deduce that $x_{n_0} = x_{n_0+1}$, which is contradictive, and, hence,

$$p(x_{n-1}, x_n) > 0, \forall n \in N. \quad (12)$$

By means of $(w, \psi) \in \Phi_1 \times \Phi_4$, (2) and (12), we have

$$\begin{aligned} \int_0^{p(x_n,x_{n+1})} w(r)dr &= \int_0^{p(fx_{n-1},fx_n)} w(r)dr \\ &\le \int_0^{p(x_{n-1},x_n)} w(r)dr - \int_0^{\psi(p(x_{n-1},x_n))} w(r)dr \\ &< \int_0^{p(x_{n-1},x_n)} w(r)dr, \forall n \in N, \end{aligned} \quad (13)$$

which together with (12) and $w \in \Phi_1$ ensures that

$$0 < p(x_n, x_{n+1}) < p(x_{n-1}, x_n), \forall n \in N. \quad (14)$$

We see from (14) that $\{p(x_n, x_{n+1})\}_{n\in N_0}$ is a positive and strictly decreasing sequence. It follows that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = c \text{ for some } c \ge 0. \quad (15)$$

Now, we claim that $c = 0$. Otherwise, $c > 0$. In view of Lemma 2.1 in [12], (2), (15), and $(w, \psi) \in \Phi_1 \times \Phi_4$, we see that

$$\begin{aligned}
\int_0^c w(r)dr &= \limsup_{n\to\infty}\int_0^{p(x_n,x_{n+1})} w(r)dr \\
&= \limsup_{n\to\infty}\int_0^{p(fx_{n-1},fx_n)} w(r)dr \\
&\le \limsup_{n\to\infty}\left(\int_0^{p(x_{n-1},x_n)} w(r)dr - \int_0^{\psi(p(x_{n-1},x_n))} w(r)dr\right) \\
&\le \limsup_{n\to\infty}\int_0^{p(x_{n-1},x_n)} w(r)dr \\
&\quad - \liminf_{n\to\infty}\int_0^{\psi(p(x_{n-1},x_n))} w(r)dr \\
&\le \int_0^c w(r)dr - \int_0^{\psi(c)} w(r)dr < \int_0^c w(r)dr.
\end{aligned} \tag{16}$$

It is ridiculous. Therefore, $c = 0$. Consequently,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{17}$$

In the same way, we have

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{18}$$

Now, we proceed to show that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{19}$$

Suppose that there exists a real number $\varepsilon > 0$ such that for every $k \in N$, there exist $m(k), n(k) \in N$ satisfying

$$p\left(x_{n(k)}, x_{m(k)}\right) > \varepsilon, m(k) > n(k) > k, \forall k \in N. \tag{20}$$

For each $k \in N$, $m(k)$ denotes the least integer exceeding $n(k)$ and satisfying (20). Obviously,

$$p\left(x_{n(k)}, x_{m(k)}\right) > \varepsilon \text{ and } p\left(x_{n(k)}, x_{m(k)-1}\right) \le \varepsilon, \forall k \in N. \tag{21}$$

On account of (p_1) and (21), we obtain that

$$\begin{aligned}
\varepsilon &< p\left(x_{n(k)}, x_{m(k)}\right) \le p\left(x_{n(k)}, x_{n(k)-1}\right) \\
&\quad + p\left(x_{n(k)-1}, x_{m(k)-1}\right) + p\left(x_{m(k)-1}, x_{m(k)}\right) \\
&\le p\left(x_{n(k)}, x_{n(k)-1}\right) + p\left(x_{n(k)-1}, x_{n(k)}\right) \\
&\quad + p\left(x_{n(k)}, x_{m(k)-1}\right) + p\left(x_{m(k)-1}, x_{m(k)}\right) \\
&\le p\left(x_{n(k)}, x_{n(k)-1}\right) + p\left(x_{n(k)-1}, x_{n(k)}\right) \\
&\quad + \varepsilon + p\left(x_{m(k)-1}, x_{m(k)}\right), \forall k \in N.
\end{aligned} \tag{22}$$

Letting k tend to infinity in (22) and taking advantage of (17), (18), and (21), we have

$$\lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)}\right) = \lim_{k\to\infty} p\left(x_{n(k)-1}, x_{m(k)-1}\right) = \varepsilon. \tag{23}$$

In light of Lemma 2.1 in [12], (2), (23), and $(w, \psi) \in \Phi_1 \times \Phi_4$, we deduct that

$$\begin{aligned}
\int_0^\varepsilon w(r)dr &= \limsup_{k\to\infty}\int_0^{p\left(x_{n(k)},x_{m(k)}\right)} w(r)dr \\
&= \limsup_{k\to\infty}\int_0^{p\left(fx_{n(k)-1},fx_{m(k)-1}\right)} w(r)dr \\
&\le \limsup_{k\to\infty}\left(\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_{n(k)-1},x_{m(k)-1}\right)\right)} w(r)dr\right) \\
&\le \limsup_{k\to\infty}\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \liminf_{k\to\infty}\int_0^{\psi\left(p\left(x_{n(k)-1},x_{m(k)-1}\right)\right)} w(r)dr \\
&\le \int_0^\varepsilon w(r)dr - \int_0^{\psi(\varepsilon)} w(r)dr < \int_0^\varepsilon w(r)dr.
\end{aligned} \tag{24}$$

It is ridiculous. Of course, (19) is true.

Assume that $\varepsilon > 0$ and δ denotes the real number appearing in (3) of [8]. By means of (19), we infer that there is $N \in N$ satisfying

$$p(x_N, x_n) < \delta, p(x_N, x_m) < \delta, \forall n, m > N, \tag{25}$$

which ensures that

$$d(x_n, x_m) \le \varepsilon, \forall n, m > N. \tag{26}$$

So $\{x_n\}_{n\in N_0}$ is a Cauchy sequence. Completeness of X means that

$$\lim_{n\to\infty} x_n = u \text{ for some } u \in X. \tag{27}$$

According to (19), we are aware of the fact that for any $\varepsilon > 0$ there is $M \in N$ with

$$p(x_n, x_m) < \varepsilon, \forall m > n > M, \tag{28}$$

which together with (27) gives that

$$0 \le p(x_n, u) \le \liminf_{m\to\infty} p(x_n, x_m) \le \varepsilon, \forall n > M. \tag{29}$$

It follows that

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{30}$$

Taking account of (2), (30), $(w, \psi) \in \Phi_1 \times \Phi_4$, and Lemma 2.1 in [12], we infer that

$$\begin{aligned}0 &\le \limsup_{n\to\infty}\int_0^{p(fx_n,fu)} w(r)dr \\ &\le \limsup_{n\to\infty}\left(\int_0^{p(x_n,u)} w(r)dr - \int_0^{\psi(p(x_n,u))} w(r)dr\right) \\ &\le \limsup_{n\to\infty}\int_0^{p(x_n,u)} w(r)dr - \liminf_{n\to\infty}\int_0^{\psi(p(x_n,u))} w(r)dr \\ &\le 0 - \int_0^{\psi(0)} w(r)dr = 0.\end{aligned} \tag{31}$$

It follows that

$$\lim_{n\to\infty}\int_0^{p(fx_n,fu)} w(r)dr = 0. \tag{32}$$

Lemma 2.2 in [12] and the above equation give that

$$\lim_{n\to\infty} p(fx_n, fu) = \lim_{n\to\infty} p(x_{n+1}, fu) = 0. \tag{33}$$

We get from (1) in [8] and (17) that

$$0 \le p(x_n, fu) \le p(x_n, x_{n+1}) + p(x_{n+1}, fu) \to 0 \text{ as} n \to \infty. \tag{34}$$

Clearly,

$$\lim_{n\to\infty} p(x_n, fu) = 0. \tag{35}$$

Applying (30), (35), and Lemma 1 in [8], we gain that $u = fu$.

Secondly, we prove that $p(u,u) = 0$. Suppose that $p(u,u) > 0$. In view of $(w,\psi) \in \Phi_1 \times \Phi_4$ and (2), we receive that

$$\begin{aligned}0 &< \int_0^{p(u,u)} w(r)dr = \int_0^{p(fu,fu)} w(r)dr \\ &\le \int_0^{p(u,u)} w(r)dr - \int_0^{\psi(p(u,u))} w(r)dr \\ &< \int_0^{p(u,u)} w(r)dr,\end{aligned} \tag{36}$$

which is not possible. Hence, $p(u,u) = 0$.

Thirdly, we assert the uniqueness of fixed points of f in X. Assume that f possesses two fixed points $u, v \in X$. We know, analogous to the proof of (36), that $p(u,u) = p(v,v) = 0$. Assume that $p(u,v) > 0$. Due to $(w,\psi) \in \Phi_1 \times \Phi_4$ and (2), we deduce that

$$\begin{aligned}0 &< \int_0^{p(u,v)} w(r)dr = \int_0^{p(fu,fv)} w(r)dr \\ &\le \int_0^{p(u,v)} w(r)dr - \int_0^{\psi(p(u,v))} w(r)dr \\ &< \int_0^{p(u,v)} w(r)dr,\end{aligned} \tag{37}$$

which is ridiculous. Consequently, $p(u,v) = 0$. Using Lemma 1 in [8] and $p(u,u) = 0$, we obtain that $u = v$.

Theorem 4. *Let p be a ω-distance in a complete metric space (X,d) and let $f : X \to X$ satisfy that*

$$\int_0^{p(fx,fy)} w(r)dr \le \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,fy))} w(r)dr, \forall x, y \in X. \tag{38}$$

Here, $(w,\psi) \in \Phi_2 \times \Phi_4$. Then, f possesses a unique fixed point $u \in X$ such that $p(u,u) = 0$,

$$\lim_{n\to\infty} p(f^n x_0, u) = 0 \text{ and } \lim_{n\to\infty} f^n x_0 = u \text{ for each } x_0 \in X. \tag{39}$$

Proof. Firstly, we show that f possesses fixed points in X. Let $x_0 \in X$ and $x_n = f^n x_0$ for each $n \in N_0$. Now, we divide the proof into two steps.

Step 5. Put $x_{n_0} = x_{n_0-1}$ for some $n_0 \in N$. In addition, x_{n_0-1} is a fixed point of f and $\lim_{n\to\infty} f^n x_0 = x_{n_0-1}$. Suppose that $p(x_{n_0-1}, x_{n_0-1}) > 0$. Owing to (38) and $(w,\psi) \in \Phi_2 \times \Phi_4$, we acquire that

$$\begin{aligned}\int_0^{p\left(x_{n_0-1},x_{n_0-1}\right)} w(r)dr &= \int_0^{p\left(fx_{n_0-1},fx_{n_0-1}\right)} w(r)dr \\ &\le \int_0^{p\left(x_{n_0-1},x_{n_0-1}\right)} w(r)dr \\ &\quad - \int_0^{\psi\left(p\left(x_{n_0-1},fx_{n_0-1}\right)\right)} w(r)dr \\ &= \int_0^{p\left(x_{n_0-1},x_{n_0-1}\right)} w(r)dr \\ &\quad - \int_0^{\psi\left(p\left(x_{n_0-1},x_{n_0-1}\right)\right)} w(r)dr \\ &< \int_0^{p\left(x_{n_0-1},x_{n_0-1}\right)} w(r)dr,\end{aligned} \tag{40}$$

which is ridiculous. Hence, $p(x_{n_0-1}, x_{n_0-1}) = 0$, which means that

$$\lim_{n\to\infty} p\left(f^n x_0, x_{n_0-1}\right) = p\left(x_{n_0-1}, x_{n_0-1}\right) = 0\,; \tag{41}$$

Step 6. $x_n \ne x_{n-1}$ for all $n \in N$. Assume that

$$p\left(x_{n_0}, x_{n_0-1}\right) = 0 \text{ for some } n_0 \in N. \tag{42}$$

Using $(w,\psi)\in\Phi_2\times\Phi_4$, (38), and (42), we conclude that

$$\begin{aligned}0&\le\int_0^{p(x_{n_0+1},x_{n_0})}w(r)dr=\int_0^{p(fx_{n_0},fx_{n_0-1})}w(r)dr\\&\le\int_0^{p(x_{n_0},x_{n_0-1})}w(r)dr-\int_0^{\psi(p(x_{n_0},fx_{n_0-1}))}w(r)dr\\&=\int_0^{p(x_{n_0},x_{n_0-1})}w(r)dr-\int_0^{\psi(p(x_{n_0},x_{n_0}))}w(r)dr\le0.\end{aligned}\tag{43}$$

It becomes that

$$\int_0^{p(x_{n_0+1},x_{n_0})}w(r)dr=0.\tag{44}$$

Thus, $w\in\Phi_2$ and the above equation guarantee that

$$p(x_{n_0+1},x_{n_0})=0.\tag{45}$$

As a result of (42), (45), and (1) in [8], we deduce that

$$0\le p(x_{n_0+1},x_{n_0-1})\le p(x_{n_0+1},x_{n_0})+p(x_{n_0},x_{n_0-1})=0,\tag{46}$$

in other words,

$$p(x_{n_0+1},x_{n_0-1})=0.\tag{47}$$

From (45), (47), and Lemma 1 in [8], we obtain that $x_{n_0}=x_{n_0-1}$, which is impossible, and, hence,

$$p(x_n,x_{n-1})>0,\forall n\in N.\tag{48}$$

Suppose that there exists $q\in N$ with

$$p(x_{q+1},x_q)>p(x_q,x_{q-1}).\tag{49}$$

In light of (38), (48), (49), and $(w,\psi)\in\Phi_2\times\Phi_4$, we infer that

$$\begin{aligned}0&<\int_0^{p(x_q,x_{q-1})}w(r)dr<\int_0^{p(x_{q+1},x_q)}w(r)dr\\&=\int_0^{p(fx_q,fx_{q-1})}w(r)dr\le\int_0^{p(x_q,x_{q-1})}w(r)dr\\&\quad-\int_0^{\psi(p(x_q,fx_{q-1}))}w(r)dr=\int_0^{p(x_q,x_{q-1})}w(r)dr\\&\quad-\int_0^{\psi(p(x_q,x_q))}w(r)dr\le\int_0^{p(x_q,x_{q-1})}w(r)dr.\end{aligned}\tag{50}$$

It is ridiculous. By means of (48), we get that

$$0<p(x_{n+1},x_n)\le p(x_n,x_{n-1}),\forall n\in N.\tag{51}$$

Thus, (51) means that the sequence $\{p(x_{n+1},x_n)\}_{n\in N_0}$ is both positive and decreasing. Consequently,

$$\lim_{n\to\infty}p(x_{n+1},x_n)=v\text{ for some }v\ge0.\tag{52}$$

Assume that

$$p(x_{j+2},x_j)>p(x_{j+1},x_{j-1})\text{ for some }j\in N.\tag{53}$$

In terms of (38), (53), and $(w,\psi)\in\Phi_2\times\Phi_4$, we attain that

$$\begin{aligned}0&\le\int_0^{p(x_{j+1},x_{j-1})}w(r)dr<\int_0^{p(x_{j+2},x_j)}w(r)dr\\&=\int_0^{p(fx_{j+1},fx_{j-1})}w(r)dr\le\int_0^{p(x_{j+1},x_{j-1})}w(r)dr\\&\quad-\int_0^{\psi(p(x_{j+1},fx_{j-1}))}w(r)dr=\int_0^{p(x_{j+1},x_{j-1})}w(r)dr\\&\quad-\int_0^{\psi(p(x_{j+1},x_j))}w(r)dr\le\int_0^{p(x_{j+1},x_{j-1})}w(r)dr,\end{aligned}\tag{54}$$

which is ridiculous. Hence,

$$0\le p(x_{n+2},x_n)\le p(x_{n+1},x_{n-1}),\quad\forall n\in N.\tag{55}$$

We get from (55) that the sequence $\{p(x_{n+2},x_n)\}_{n\in N_0}$ is both nonnegative and nonincreasing. Thus,

$$\lim_{n\to\infty}p(x_{n+2},x_n)=b\text{ for some }b\ge0.\tag{56}$$

Assume that $v>0$. In view of (38), (52), (56), $(w,\psi)\in\Phi_2\times\Phi_4$, and Lemma 2.1 in [12], we gain that

$$\begin{aligned}0&\le\int_0^b w(r)dr=\limsup_{n\to\infty}\int_0^{p(x_{n+2},x_n)}w(r)dr\\&=\limsup_{n\to\infty}\int_0^{p(fx_{n+1},fx_{n-1})}w(r)dr\\&\le\limsup_{n\to\infty}\left(\int_0^{p(x_{n+1},x_{n-1})}w(r)dr-\int_0^{\psi(p(x_{n+1},fx_{n-1}))}w(r)dr\right)\\&=\limsup_{n\to\infty}\left(\int_0^{p(x_{n+1},x_{n-1})}w(r)dr-\int_0^{\psi(p(x_{n+1},x_n))}w(r)dr\right)\\&\le\limsup_{n\to\infty}\int_0^{p(x_{n+1},x_{n-1})}w(r)dr-\liminf_{n\to\infty}\int_0^{\psi(p(x_{n+1},x_n))}w(r)dr\\&\le\int_0^b w(r)dr-\int_0^{\psi(v)}w(r)dr<\int_0^b w(r)dr.\end{aligned}\tag{57}$$

It is impossible. Thus, (18) is true. Suppose that (6) holds.

Taking advantage of (6), (38), and $(w,\psi)\in\Phi_2\times\Phi_4$, we have

$$\begin{aligned}0&\le\int_0^{p(x_{n_0},x_{n_0+1})}w(r)dr=\int_0^{p(fx_{n_0-1},fx_{n_0})}w(r)dr\\&\le\int_0^{p(x_{n_0-1},x_{n_0})}w(r)dr-\int_0^{\psi(p(x_{n_0-1},fx_{n_0}))}w(r)dr\\&=\int_0^{p(x_{n_0-1},x_{n_0})}w(r)dr-\int_0^{\psi(p(x_{n_0-1},x_{n_0+1}))}w(r)dr\le 0.\end{aligned}\tag{58}$$

It follows that

$$\int_0^{p(x_{n_0},x_{n_0+1})}w(r)dr=0.\tag{59}$$

Thus, the above equation and $w\in\Phi_2$ give (9). Using (6), (9), and (1) in [8], we conclude that

$$0\le p(x_{n_0-1},x_{n_0+1})\le p(x_{n_0-1},x_{n_0})+p(x_{n_0},x_{n_0+1})=0,\tag{60}$$

that is, (11) holds. By virtue of (6), (11), and Lemma 1 in [8], we obtain that $x_{n_0}=x_{n_0+1}$, which is impossible. As a result, (12) holds. Assume that there exists $q\in N$ with

$$p(x_q,x_{q+1})>p(x_{q-1},x_q).\tag{61}$$

We know from (12), (38), (61), and $(w,\psi)\in\Phi_2\times\Phi_4$ that

$$\begin{aligned}0&<\int_0^{p(x_{q-1},x_q)}w(r)dr<\int_0^{p(x_q,x_{q+1})}w(r)dr\\&=\int_0^{p(fx_{q-1},fx_q)}w(r)dr\le\int_0^{p(x_{q-1},x_q)}w(r)dr\\&\quad-\int_0^{\psi(p(x_{q-1},fx_q))}w(r)dr=\int_0^{p(x_{q-1},x_q)}w(r)dr\\&\quad-\int_0^{\psi(p(x_{q-1},x_{q+1}))}w(r)dr\le\int_0^{p(x_{q-1},x_q)}w(r)dr,\end{aligned}\tag{62}$$

which is ridiculous. By means of (12), we obtain that

$$0<p(x_n,x_{n+1})\le p(x_{n-1},x_n),\forall n\in N.\tag{63}$$

With the help of (63), there is a real number $c\ge 0$ satisfying (15). Assume that $c>0$. Let $\limsup_{n\to\infty}p(x_n,x_n)=w$. Clearly, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N_0}$ with

$$\lim_{k\to\infty}p(x_{n_k},x_{n_k})=w.\tag{64}$$

Note that (38) and $(w,\psi)\in\Phi_2\times\Phi_4$ infer that

$$\begin{aligned}0&\le\int_0^{p(x_{n+1},x_n)}w(r)dr=\int_0^{p(fx_n,fx_{n-1})}w(r)dr\\&\le\int_0^{p(x_n,x_{n-1})}w(r)dr-\int_0^{\psi(p(x_n,fx_{n-1}))}w(r)dr\\&=\int_0^{p(x_n,x_{n-1})}w(r)dr-\int_0^{\psi(p(x_n,x_n))}w(r)dr\\&\le\int_0^{p(x_n,x_{n-1})}w(r)dr,\forall n\in N.\end{aligned}\tag{65}$$

Letting $n\to\infty$ in (65) and utilizing (18), $w\in\Phi_2$ and Lemma 2.2 in [12], we make a conclusion that

$$\lim_{n\to\infty}\left(\int_0^{p(x_n,x_{n-1})}w(r)dr-\int_0^{\psi(p(x_n,x_n))}w(r)dr\right)=0.\tag{66}$$

It follows that

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\psi(p(x_n,x_n))}w(r)dr&=\lim_{n\to\infty}\int_0^{p(x_n,x_{n-1})}w(r)dr\\&\quad-\lim_{n\to\infty}\left(\int_0^{p(x_n,x_{n-1})}w(r)dr-\int_0^{\psi(p(x_n,x_n))}w(r)dr\right)=0.\end{aligned}\tag{67}$$

In view of (18), (38), (64), (65), and (67), $(w,\psi)\in\Phi_2\times\Phi_4$ and Lemma 2.2 in [12], we deduct that

$$\begin{aligned}0&=\limsup_{k\to\infty}\int_0^{p(x_{n_k+1},x_{n_k})}w(r)dr=\limsup_{k\to\infty}\int_0^{p(fx_{n_k},fx_{n_k-1})}w(r)dr\\&\le\limsup_{k\to\infty}\left(\int_0^{p(x_{n_k},x_{n_k-1})}w(r)dr-\int_0^{\psi(p(x_{n_k},fx_{n_k-1}))}w(r)dr\right)\\&=\limsup_{k\to\infty}\left(\int_0^{p(x_{n_k},x_{n_k-1})}w(r)dr-\int_0^{\psi(p(x_{n_k},x_{n_k}))}w(r)dr\right)\\&\le\limsup_{k\to\infty}\int_0^{p(x_{n_k},x_{n_k-1})}w(r)dr-\liminf_{k\to\infty}\int_0^{\psi(p(x_{n_k},x_{n_k}))}w(r)dr\\&\le 0-\int_0^{\psi(w)}w(r)dr,\end{aligned}\tag{68}$$

which together with $(w,\psi)\in\Phi_2\times\Phi_4$ yields that $w=0$. It is obvious that

$$\lim_{n\to\infty}p(x_n,x_n)=0.\tag{69}$$

Using Lemma 2.2 in [12], (15), (38), (69), and $(w,\psi)\in\Phi_2\times\Phi_4$, we find that

$$\begin{aligned}0 &= \limsup_{n\to\infty}\int_0^{p(x_{n+1},x_{n+1})} w(r)dr = \limsup_{n\to\infty}\int_0^{p(fx_n,fx_n)} w(r)dr\\ &\le \limsup_{n\to\infty}\left(\int_0^{p(x_n,x_n)} w(r)dr - \int_0^{\psi(p(x_n,fx_n))} w(r)dr\right)\\ &= \limsup_{n\to\infty}\left(\int_0^{p(x_n,x_n)} w(r)dr - \int_0^{\psi(p(x_n,x_{n+1}))} w(r)dr\right)\\ &\le \limsup_{n\to\infty}\int_0^{p(x_n,x_n)} w(r)dr - \liminf_{n\to\infty}\int_0^{\psi(p(x_n,x_{n+1}))} w(r)dr\\ &\le 0 - \int_0^{\psi(c)} w(r)dr,\end{aligned} \tag{70}$$

which together with $(w,\psi)\in\Phi_2\times\Phi_4$ implies that $c=0$. Thus, (17) holds.

Now, we assert that (19) holds. Otherwise, there is a real number $\varepsilon>0$ such that for arbitrary $k\in N$, there are $m(k)$, $n(k)\in N$ with (20) and (21). On account of (1) in [8] and (3.12), we gain that

$$\begin{aligned}\varepsilon &< p\left(x_{n(k)},x_{m(k)}\right) \le p\left(x_{n(k)},x_{n(k)-1}\right) + p\left(x_{n(k)-1},x_{m(k)}\right)\\ &\le p\left(x_{n(k)},x_{n(k)-1}\right) + p\left(x_{n(k)-1},x_{m(k)-1}\right) + p\left(x_{m(k)-1},x_{m(k)}\right)\\ &\le p\left(x_{n(k)},x_{n(k)-1}\right) + p\left(x_{n(k)-1},x_{n(k)}\right) + p\left(x_{n(k)},x_{m(k)-1}\right)\\ &\quad + p\left(x_{m(k)-1},x_{m(k)}\right) \le p\left(x_{n(k)},x_{n(k)-1}\right) + p\left(x_{n(k)-1},x_{n(k)}\right)\\ &\quad + \varepsilon + p\left(x_{m(k)-1},x_{m(k)}\right), \forall k\in N.\end{aligned} \tag{71}$$

Letting $k\to\infty$ in (71) and making use of (17), (18), and (21), we require that

$$\begin{aligned}\lim_{k\to\infty} p\left(x_{n(k)},x_{m(k)}\right) &= \lim_{k\to\infty} p\left(x_{n(k)-1},x_{m(k)-1}\right)\\ &= \lim_{k\to\infty} p\left(x_{n(k)-1},x_{m(k)}\right) = \varepsilon.\end{aligned} \tag{72}$$

Taking notice of (38), (72), $(w,\psi)\in\Phi_2\times\Phi_4$, and Lemma 2.1 in [12], we receive that

$$\begin{aligned}\int_0^{\varepsilon} w(r)dr &= \limsup_{k\to\infty}\int_0^{p\left(x_{n(k)},x_{m(k)}\right)} w(r)dr\\ &= \limsup_{k\to\infty}\int_0^{p\left(fx_{n(k)-1},fx_{m(k)-1}\right)} w(r)dr\\ &\le \limsup_{k\to\infty}\left(\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_{n(k)-1},fx_{m(k)-1}\right)\right)} w(r)dr\right)\\ &= \limsup_{k\to\infty}\left(\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_{n(k)-1},x_{m(k)}\right)\right)} w(r)dr\right)\\ &\le \limsup_{k\to\infty}\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \liminf_{k\to\infty}\int_0^{\psi\left(p\left(x_{n(k)-1},x_{m(k)}\right)\right)} w(r)dr\\ &\le \int_0^{\varepsilon} w(r)dr - \int_0^{\psi(\varepsilon)} w(r)dr < \int_0^{\varepsilon} w(r)dr.\end{aligned} \tag{73}$$

It is ridiculous. That is, (19) is true.

We deduce, similar to the proof of Theorem 1, that (27) holds. It follows from (19) that for every real number $\varepsilon>0$ there is $M\in N$ with

$$p(x_n,x_m) < \varepsilon, \forall m>n>M, \tag{74}$$

which together with (2) in [8] and (27) gives that

$$0 \le p(x_n,u) \le \liminf_{m\to\infty} p(x_n,x_m) \le \varepsilon, \forall n>M, \tag{75}$$

that is, (30) holds. In terms of (30), (38), $(w,\psi)\in\Phi_2\times\Phi_4$, and Lemma 2.1 in [12], we get that

$$\begin{aligned}0 &\le \limsup_{n\to\infty}\int_0^{p(fx_n,fu)} w(r)dr\\ &\le \limsup_{n\to\infty}\left(\int_0^{p(x_n,u)} w(r)dr - \int_0^{\psi(p(x_n,fu))} w(r)dr\right)\\ &\le \limsup_{n\to\infty}\int_0^{p(x_n,u)} w(r)dr - \liminf_{n\to\infty}\int_0^{\psi(p(x_n,fu))} w(r)dr\\ &\le 0 - \liminf_{n\to\infty}\int_0^{\psi(p(x_n,fu))} w(r)dr \le 0.\end{aligned} \tag{76}$$

It follows that

$$\lim_{n\to\infty}\int_0^{p(fx_n,fu)} w(r)dr = 0. \tag{77}$$

Thus, Lemma 2.2 in [12] and the above equation ensure that

$$\lim_{n\to\infty} p(fx_n,fu) = \lim_{n\to\infty} p(x_{n+1},fu) = 0. \tag{78}$$

In light of (1) in [8] and (17), we arrive at

$$0 \le p(x_n,fu) \le p(x_n,x_{n+1}) + p(x_{n+1},fu) \to 0 \text{ as} n\to\infty, \tag{79}$$

that is to say, (35) holds. By virtue of (30), (35) and Lemma 1 in [8], we have $u=fu$.

Secondly, we assert that $p(u,u)=0$. Assume that $p(u,u)>0$. Owing to (38) and $(w,\psi)\in\Phi_2\times\Phi_4$, we deduce that

$$0<\int_0^{p(u,u)}w(r)dr=\int_0^{p(fu,fu)}w(r)dr\le\int_0^{p(u,u)}w(r)dr-\int_0^{\psi(p(u,fu))}w(r)dr=\int_0^{p(u,u)}w(r)dr-\int_0^{\psi(p(u,u))}w(r)dr<\int_0^{p(u,u)}w(r)dr. \tag{80}$$

It is ridiculous. Hence, $p(u,u)=0$.

Thirdly, we show the uniqueness of fixed points of f in X. Assume that f possesses two fixed points $u,v\in X$. We get, similar to the proof of (80), that $p(u,u)=p(v,v)=0$. Assume that $p(u,v)>0$. Taking account of $(w,\psi)\in\Phi_2\times\Phi_4$ and (38), we get that

$$0<\int_0^{p(u,v)}w(r)dr=\int_0^{p(fu,fv)}w(r)dr\le\int_0^{p(u,v)}w(r)dr-\int_0^{\psi(p(u,fv))}w(r)dr=\int_0^{p(u,v)}w(r)dr-\int_0^{\psi(p(u,v))}w(r)dr<\int_0^{p(u,v)}w(r)dr, \tag{81}$$

which is ridiculous. Therefore, $p(u,v)=0$. It follows from $p(u,u)=0$ and Lemma 1 in [8] that $u=v$.

Theorem 7. *Let p be a ω-distance in a complete metric space (X,d) and let $f:X\to X$ satisfy that*

$$\int_0^{p(fx,fy)}w(r)dr\le\int_0^{p(x,y)}w(r)dr-\int_0^{\psi(p(fx,y))}w(r)dr,\forall x,y\in X, \tag{82}$$

here $(w,\psi)\in\Phi_2\times\Phi_4$. Then, f possesses a unique fixed point $u\in X$ satisfying $p(u,u)=0$,

$$\lim_{n\to\infty}p(f^nx_0,u)=0\ and\ \lim_{n\to\infty}f^nx_0=u\ for\ each\ x_0\in X. \tag{83}$$

Proof. Firstly, we demonstrate that f possesses fixed points in X. Let $x_0\in X$ and $x_n=f^nx_0$ for each $n\in N_0$. Now, we consider two cases below:

Case 8. $x_{n_0}=x_{n_0-1}$ for some $n_0\in N$. Since x_{n_0-1} is a fixed point of f, it follows that $\lim_{n\to\infty}f^nx_0=x_{n_0-1}$. Assume that $p(x_{n_0-1},x_{n_0-1})>0$. Due to (82) and $(w,\psi)\in\Phi_2\times\Phi_4$, we have

$$\begin{aligned}\int_0^{p(x_{n_0-1},x_{n_0-1})}w(r)dr&=\int_0^{p(fx_{n_0-1},fx_{n_0-1})}w(r)dr\le\int_0^{p(x_{n_0-1},x_{n_0-1})}w(r)dr-\int_0^{\psi(p(fx_{n_0-1},x_{n_0-1}))}w(r)dr\\&=\int_0^{p(x_{n_0-1},x_{n_0-1})}w(r)dr-\int_0^{\psi(p(x_{n_0-1},x_{n_0-1}))}w(r)dr\\&<\int_0^{p(x_{n_0-1},x_{n_0-1})}w(r)dr,\end{aligned} \tag{84}$$

which is absurd. Hence, $p(x_{n_0-1},x_{n_0-1})=0$ and

$$\lim_{n\to\infty}p(f^nx_0,x_{n_0-1})=p(x_{n_0-1},x_{n_0-1})=0. \tag{85}$$

Case 9. $x_n\ne x_{n-1}$ for all $n\in N$. Assume that (6) holds. In view of (6), (82), and $(w,\psi)\in\Phi_2\times\Phi_4$, we obtain that

$$\begin{aligned}0&\le\int_0^{p(x_{n_0},x_{n_0+1})}w(r)dr=\int_0^{p(fx_{n_0-1},fx_{n_0})}w(r)dr\\&\le\int_0^{p(x_{n_0-1},x_{n_0})}w(r)dr-\int_0^{\psi(p(fx_{n_0-1},x_{n_0}))}w(r)dr\\&=\int_0^{p(x_{n_0-1},x_{n_0})}w(r)dr-\int_0^{\psi(p(x_{n_0},x_{n_0}))}w(r)dr\le0,\end{aligned} \tag{86}$$

which means that

$$\int_0^{p(x_{n_0},x_{n_0+1})}w(r)dr=0. \tag{87}$$

Combining $w\in\Phi_2$ and the above equation, we get (3.3). We gain from (6), (9), and (p_1) that

$$0\le p(x_{n_0-1},x_{n_0+1})\le p(x_{n_0-1},x_{n_0})+p(x_{n_0},x_{n_0+1})=0, \tag{88}$$

in other words, (11) sets up. In terms of (6), (11), and Lemma 1 in [8], we know immediately that $x_{n_0}=x_{n_0+1}$, which is absurd, and, hence, (12) is true. Assume that there is $q\in N$ satisfying (61). We conclude from (12), (61), (82), and $(w,\psi)\in\Phi_2\times\Phi_4$ that

$$\begin{aligned}0&<\int_0^{p(x_{q-1},x_q)}w(r)dr<\int_0^{p(x_q,x_{q+1})}w(r)dr=\int_0^{p(fx_{q-1},fx_q)}w(r)dr\\&\le\int_0^{p(x_{q-1},x_q)}w(r)dr-\int_0^{\psi(p(fx_{q-1},x_q))}w(r)dr\\&=\int_0^{p(x_{q-1},x_q)}w(r)dr-\int_0^{\psi(p(x_q,x_q))}w(r)dr\le\int_0^{p(x_{q-1},x_q)}w(r)dr,\end{aligned} \tag{89}$$

which is impossible. Hence, (63) is true. It follows from (63) that there is a real number $c\ge0$ with (15). Assume that there is $j \in N$ with

$$p\left(x_j, x_{j+2}\right) > p\left(x_{j-1}, x_{j+1}\right). \tag{90}$$

In terms of (82), (90), and $(w, \psi) \in \Phi_2 \times \Phi_4$, we infer that

$$\begin{aligned} 0 &\le \int_0^{p\left(x_{j-1}, x_{j+1}\right)} w(r)dr < \int_0^{p\left(x_j, x_{j+2}\right)} w(r)dr = \int_0^{p\left(fx_{j-1}, fx_{j+1}\right)} w(r)dr \\ &\le \int_0^{p\left(x_{j-1}, x_{j+1}\right)} w(r)dr - \int_0^{\psi\left(p\left(fx_{j-1}, x_{j+1}\right)\right)} w(r)dr \\ &= \int_0^{p\left(x_{j-1}, x_{j+1}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_j, x_{j+1}\right)\right)} w(r)dr \le \int_0^{p\left(x_{j-1}, x_{j+1}\right)} w(r)dr. \end{aligned} \tag{91}$$

It is absurd, and, hence,

$$0 \le p(x_n, x_{n+2}) \le p(x_{n-1}, x_{n+1}), \forall n \in N. \tag{92}$$

(92) means that $\{p(x_n, x_{n+2})\}_{n \in N_0}$ is both nonnegative and nonincreasing. Consequently,

$$\lim_{n\to\infty} p(x_n, x_{n+2}) = z \text{ for some constant } z \ge 0. \tag{93}$$

Assume that $c > 0$. Owing to (15), (82), (93), $(w, \psi) \in \Phi_2 \times \Phi_4$, and Lemma 2.1 in [12], we get that

$$\begin{aligned} 0 &\le \int_0^{z} w(r)dr = \limsup_{n\to\infty} \int_0^{p(x_n, x_{n+2})} w(r)dr \\ &= \limsup_{n\to\infty} \int_0^{p(fx_{n-1}, fx_{n+1})} w(r)dr \\ &\le \limsup_{n\to\infty} \left(\int_0^{p(x_{n-1}, x_{n+1})} w(r)dr - \int_0^{\psi(p(fx_{n-1}, x_{n+1}))} w(r)dr \right) \\ &= \limsup_{n\to\infty} \left(\int_0^{p(x_{n-1}, x_{n+1})} w(r)dr - \int_0^{\psi(p(x_n, x_{n+1}))} w(r)dr \right) \\ &\le \limsup_{n\to\infty} \int_0^{p(x_{n-1}, x_{n+1})} w(r)dr - \liminf_{n\to\infty} \int_0^{\psi(p(x_n, x_{n+1}))} w(r)dr \\ &\le \int_0^{z} w(r)dr - \int_0^{\psi(c)} w(r)dr < \int_0^{z} w(r)dr. \end{aligned} \tag{94}$$

It is contradictive. Thus, (17) is true. Suppose that (42) holds. We infer from (42), (82), and $(w, \psi) \in \Phi_2 \times \Phi_4$ that

$$\begin{aligned} 0 &\le \int_0^{p\left(x_{n_0+1}, x_{n_0}\right)} w(r)dr = \int_0^{p\left(fx_{n_0}, fx_{n_0-1}\right)} w(r)dr \\ &\le \int_0^{p\left(x_{n_0}, x_{n_0-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(fx_{n_0}, x_{n_0-1}\right)\right)} w(r)dr \\ &= \int_0^{p\left(x_{n_0}, x_{n_0-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_{n_0+1}, x_{n_0-1}\right)\right)} w(r)dr \le 0, \end{aligned} \tag{95}$$

that is,

$$\int_0^{p\left(x_{n_0+1}, x_{n_0}\right)} w(r)dr = 0. \tag{96}$$

Thus, (45) follows from the above equation and $w \in \Phi_2$. We deduct from (42), (45), and (1) in [8] that

$$0 \le p\left(x_{n_0+1}, x_{n_0-1}\right) \le p\left(x_{n_0+1}, x_{n_0}\right) + p\left(x_{n_0}, x_{n_0-1}\right) = 0, \tag{97}$$

that is to say, (47) holds. Thus, $x_{n_0} = x_{n_0-1}$ is easily obtained from (45), (47), and Lemma 1 in [8], which is ridiculous. As a result, (48) holds. Suppose that there exists $q \in N$ satisfying (49). By virtue of (48), (49), (82), and $(w, \psi) \in \Phi_2 \times \Phi_4$, we know that

$$\begin{aligned} 0 &< \int_0^{p\left(x_q, x_{q-1}\right)} w(r)dr < \int_0^{p\left(x_{q+1}, x_q\right)} w(r)dr = \int_0^{p\left(fx_q, fx_{q-1}\right)} w(r)dr \\ &\le \int_0^{p\left(x_q, x_{q-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(fx_q, x_{q-1}\right)\right)} w(r)dr \\ &= \int_0^{p\left(x_q, x_{q-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_{q+1}, x_{q-1}\right)\right)} w(r)dr \\ &\le \int_0^{p\left(x_q, x_{q-1}\right)} w(r)dr, \end{aligned} \tag{98}$$

which is ridiculous. By means of (48), we have (51). It follows from (51) that the sequence $\{p(x_{n+1}, x_n)\}_{n \in N_0}$ is both positive and decreasing, which yields (52) for some a constant $\nu \ge 0$. Suppose that $\nu > 0$. Put $\limsup_{n\to\infty} p(x_n, x_n) = w$. Obviously, there is a subsequence $\{x_{n_k}\}_{k \in N}$ of $\{x_n\}_{n \in N_0}$ with (64). Using (82), and $(w, \psi) \in \Phi_2 \times \Phi_4$, we deduce that

$$\begin{aligned} 0 &\le \int_0^{p(x_n, x_{n+1})} w(r)dr = \int_0^{p(fx_{n-1}, fx_n)} w(r)dr \\ &\le \int_0^{p(x_{n-1}, x_n)} w(r)dr - \int_0^{\psi(p(fx_{n-1}, x_n))} w(r)dr \\ &= \int_0^{p(x_{n-1}, x_n)} w(r)dr - \int_0^{\psi(p(x_n, x_n))} w(r)dr \\ &\le \int_0^{p(x_{n-1}, x_n)} w(r)dr, \quad \forall n \in N. \end{aligned} \tag{99}$$

Letting $n \to \infty$ in (99) and using (17), $w \in \Phi_2$ and Lemma 2.2 in [12], we find that

$$\lim_{n\to\infty} \left(\int_0^{p(x_{n-1}, x_n)} w(r)dr - \int_0^{\psi(p(x_n, x_n))} w(r)dr \right) = 0. \tag{100}$$

It follows that

$$\lim_{n\to\infty}\int_0^{\psi(p(x_n,x_n))} w(r)dr = \lim_{n\to\infty}\int_0^{p(x_{n-1},x_n)} w(r)dr - \lim_{n\to\infty}\left(\int_0^{p(x_{n-1},x_n)} w(r)dr - \int_0^{\psi(p(x_n,x_n))} w(r)dr\right) = 0. \tag{101}$$

We attain from Lemma 2.2 in [12], (17), (64), (82), (99), (101), and $(w,\psi)\in\Phi_2\times\Phi_4$ that

$$\begin{aligned} 0 &= \limsup_{k\to\infty}\int_0^{p\left(x_{n_k},x_{n_k+1}\right)} w(r)dr = \limsup_{k\to\infty}\int_0^{p\left(fx_{n_k-1},fx_{n_k}\right)} w(r)dr \\ &\le \limsup_{k\to\infty}\left(\int_0^{p\left(x_{n_k-1},x_{n_k}\right)} w(r)dr - \int_0^{\psi\left(p\left(fx_{n_k-1},x_{n_k}\right)\right)} w(r)dr\right) \\ &= \limsup_{k\to\infty}\left(\int_0^{p\left(x_{n_k-1},x_{n_k}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_{n_k},x_{n_k}\right)\right)} w(r)dr\right) \\ &\le \limsup_{k\to\infty}\int_0^{p\left(x_{n_k-1},x_{n_k}\right)} w(r)dr - \liminf_{k\to\infty}\int_0^{\psi\left(p\left(x_{n_k},x_{n_k}\right)\right)} w(r)dr \\ &\le 0 - \int_0^{\psi(w)} w(r)dr. \end{aligned} \tag{102}$$

It follows that $\int_0^{\psi(w)} w(r)dr \le 0$. By virtue of $(w,\psi)\in\Phi_2\times\Phi_4$, we have $\int_0^{\psi(w)} w(r)dr = 0$ and $w=0$. It means that (69) holds. In view of (52), (69), (82), $(w,\psi)\in\Phi_2\times\Phi_4$, and Lemma 2.2 in [12], we infer that

$$\begin{aligned} 0 &= \limsup_{n\to\infty}\int_0^{p(x_{n+1},x_{n+1})} w(r)dr = \limsup_{n\to\infty}\int_0^{p(fx_n,fx_n)} w(r)dr \\ &\le \limsup_{n\to\infty}\left(\int_0^{p(x_n,x_n)} w(r)dr - \int_0^{\psi(p(fx_n,x_n))} w(r)dr\right) \\ &= \limsup_{n\to\infty}\left(\int_0^{p(x_n,x_n)} w(r)dr - \int_0^{\psi(p(x_{n+1},x_n))} w(r)dr\right) \\ &\le \limsup_{n\to\infty}\int_0^{p(x_n,x_n)} w(\mathrm{r})dr - \liminf_{n\to\infty}\int_0^{\psi(p(x_{n+1},x_n))} w(r)dr \\ &\le 0 - \int_0^{\psi(v)} w(r)dr, \end{aligned} \tag{103}$$

which yields that $\int_0^{\psi(v)} w(r)dr \le 0$. Using $(w,\psi)\in\Phi_2\times\Phi_4$, we obtain that $\int_0^{\psi(v)} w(r)dr = 0$ and $v=0$. Thus, (3.9) holds.

Now, we prove that (19) holds. Suppose that there is an $\varepsilon>0$ such that for arbitrary $k\in N$, (20), (21), and (22) hold for some $m(k), n(k)\in N$. As $k\to\infty$ in (3.13) and by virtue of (17), (18), and (21), we acquire that

$$\begin{aligned} \lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)}\right) &= \lim_{k\to\infty} p\left(x_{n(k)-1}, x_{m(k)-1}\right) \\ &= \lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)-1}\right) = \varepsilon. \end{aligned} \tag{104}$$

In terms of Lemma 2.1 in [12], (82), (104), and $(w,\psi)\in\Phi_2\times\Phi_4$, we are aware of the fact that

$$\begin{aligned} \int_0^{\varepsilon} w(r)dr &= \limsup_{k\to\infty}\int_0^{p\left(x_{n(k)},x_{m(k)}\right)} w(r)dr \\ &= \limsup_{k\to\infty}\int_0^{p\left(fx_{n(k)-1},fx_{m(k)-1}\right)} w(r)dr \\ &\le \limsup_{k\to\infty}\left(\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(fx_{n(k)-1},x_{m(k)-1}\right)\right)} w(r)dr\right) \\ &= \limsup_{k\to\infty}\left(\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \int_0^{\psi\left(p\left(x_{n(k)},x_{m(k)-1}\right)\right)} w(r)dr\right) \\ &\le \limsup_{k\to\infty}\int_0^{p\left(x_{n(k)-1},x_{m(k)-1}\right)} w(r)dr - \liminf_{k\to\infty}\int_0^{\psi\left(p\left(x_{n(k)},x_{m(k)-1}\right)\right)} w(r)dr \\ &\le \int_0^{\varepsilon} w(r)dr - \int_0^{\psi(\varepsilon)} w(r)dr < \int_0^{\varepsilon} w(r)dr, \end{aligned} \tag{105}$$

which is absurd. Thus, (19) is true.

We infer, similar to the proof of Theorem 1, that (27) holds. It follows from (19) that for each $\varepsilon>0$ there is $M\in N$ with

$$p(x_n,x_m) < \varepsilon, \forall m>n>M, \tag{106}$$

which together with (2) in [8] and (27) gets that

$$0 \le p(x_n,u) \le \liminf_{m\to\infty} p(x_n,x_m) \le \varepsilon, \forall n>M, \tag{107}$$

that is, (30) holds. On account of Lemma 2.1 in [12], (30), (82), and $(w,\psi)\in\Phi_2\times\Phi_4$, we deduct that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty}\int_0^{p(fx_n,fu)} w(r)dr \le \limsup_{n\to\infty}\left(\int_0^{p(x_n,u)} w(r)dr - \int_0^{\psi(p(fx_n,u))} w(r)dr\right) \\ &\le \limsup_{n\to\infty}\int_0^{p(x_n,u)} w(r)dr - \liminf_{n\to\infty}\int_0^{\psi(p(x_{n+1},u))} w(r)dr \le 0 - \int_0^{\psi(0)} w(r)dr = 0, \end{aligned} \tag{108}$$

in other words,

$$\lim_{n\to\infty}\int_0^{p(fx_n,fu)} w(r)dr = 0. \tag{109}$$

Lemma 2.2 in [12] and the above equation give that

$$\lim_{n\to\infty} p(fx_n, fu) = \lim_{n\to\infty} p(x_{n+1}, fu) = 0. \tag{110}$$

In light of (1) in [8] and (17), we attain that

$$0 \le p(x_n, fu) \le p(x_n, x_{n+1}) + p(x_{n+1}, fu) \to 0 \text{as} n \to \infty, \tag{111}$$

that is to say, (35) holds. Using (30), (35), and Lemma 1 in [8], we have $u = fu$.

Secondly, we prove that $p(u,u) = 0$. Assume that $p(u,u) > 0$. Because of (82) and $(w,\psi) \in \Phi_2 \times \Phi_4$, we deduce that

$$\begin{aligned}0 &< \int_0^{p(u,u)} w(r)dr = \int_0^{p(fu,fu)} w(r)dr \le \int_0^{p(u,u)} w(r)dr - \int_0^{\psi(p(fu,u))} w(r)dr\\ &= \int_0^{p(u,u)} w(r)dr - \int_0^{\psi(p(u,u))} w(r)dr < \int_0^{p(u,u)} w(r)dr,\end{aligned} \tag{112}$$

which is impossible. Hence, $p(u,u) = 0$.

Thirdly, we assert the uniqueness of fixed points of f in X. Assume that f possesses two fixed points $u, v \in X$. We deduce, similar to the proof of (112), that $p(u,u) = p(v,v) = 0$. Assume that $p(u,v) > 0$. On account of (82) and $(w,\psi) \in \Phi_2 \times \Phi_4$, we get that

$$\begin{aligned}0 &< \int_0^{p(u,v)} w(r)dr = \int_0^{p(fu,fv)} w(r)dr \le \int_0^{p(u,v)} w(r)dr - \int_0^{\psi(p(fu,v))} w(r)dr\\ &= \int_0^{p(u,v)} w(r)dr - \int_0^{\psi(p(u,v))} w(r)dr < \int_0^{p(u,v)} w(r)dr,\end{aligned} \tag{113}$$

which is ridiculous. Therefore, $p(u,v) = 0$. Using $p(u,u) = 0$ and Lemma 1 in [8], we infer immediately that $u = v$.

We have, similar to the proof of Theorem 1, the result below and omit its proof.

Theorem 10. *Let p be a ω-distance in a complete metric space (X,d) and let $f : X \to X$ satisfy that*

$$\int_0^{p(fx,fy)} w(r)dr \le \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(fx,fy))} w(r)dr, \forall x,y \in X. \tag{114}$$

Here, $(w,\psi) \in \Phi_1 \times \Phi_4$. Then, f possesses a unique fixed point $u \in X$ such that $p(u,u) = 0$,

$$\lim_{n\to\infty} p(f^n x_0, u) = 0 \text{ and } \lim_{n\to\infty} f^n x_0 = u \text{ for each } x_0 \in X. \tag{115}$$

4. Four Examples

Now, we give four examples to explain the fixed point results obtained in Section 3.

Remark 11. Letting $p(x,y) = d(x,y), \forall x,y \in X$, we deduce that Theorem 1 reduces to Theorem 2.1 in [14], which generalizes Theorem 1 in [15]. On the other hand, the example below proves that Theorem 1 extends indeed these results in [14, 15] and differs from Theorem 2.1 in [1].

Example 12. Let $X = [0,6]$, $d(x,y) = |x - y|$ and $p(x,y) = y/2$, $\forall x,y \in X$. Let $f : X \to X$, w and $\psi : R^+ \to R^+$ be defined by, respectively,

$$fx = 2x - x^2, \forall x \in \left(\frac{6}{5}, 2\right], 0, \forall x \in \left[0, \frac{6}{5}\right] \cup (2,6] \tag{116}$$

and

$$w(r) = 2, \quad \psi(r) = \frac{1}{3}r^2, \quad \forall r \in R^+. \tag{117}$$

It follows that p is a ω-distance in X and $(w,\psi) \in \Phi_1 \times \Phi_4$. Put $x,y \in X$. In order to check (2), we consider two cases below:

Case 13. $(x,y) \in X \times (6/5, 2]$. It follows that

$$\begin{aligned}\int_0^{p(fx,fy)} w(r)dr &= \int_0^{y-\frac{1}{2}y^2} 2dr = 2y - y^2 \le y - \frac{1}{6}y^2 = \int_0^{\frac{y}{2}} 2dr - \int_0^{\frac{1}{12}y^2} 2dr\\ &= \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,y))} w(r)dr.\end{aligned} \tag{118}$$

Case 14. $(x,y) \in X \times [0, 6/5] \cup (2,6]$. Note that

$$\int_0^{p(fx,fy)} w(r)dr = 0 \le y - \frac{1}{6}y^2 = \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,y))} w(r)dr. \tag{119}$$

That is, (2) is true. Hence, the conditions of Theorem 1 are fulfilled. Thus, Theorem 1 ensures that f possesses a unique fixed point in X. Now, we need to prove that Theorem 2.1 in [1], Theorem 2.1 in [14], and Theorem 1 in [15] are useless in checking the existence of fixed points for the mapping f in X.

If there is $\psi \in \Phi_3$ satisfying the conditions of Theorem 1 in [15], we know that

$$\frac{21}{25} = d\left(f\frac{7}{5}, f1\right) \le d\left(\frac{7}{5}, 1\right) - \psi\left(d\left(\frac{7}{5}, 1\right)\right) < \frac{2}{5}, \quad (120)$$

which is absurd.

If there are $c \in (0, 1)$ and $w \in \Phi_1$ satisfying the conditions of Theorem 2.1 in [1], we attain that

$$\begin{aligned} 0 &< \int_0^{\frac{3}{4}} w(r)dr = \int_0^{d\left(f\frac{3}{4}, f\frac{3}{2}\right)} w(r)dr \\ &\le c\int_0^{d\left(\frac{3}{4},\frac{3}{2}\right)} w(r)dr < \int_0^{\frac{3}{4}} w(r)dr, \end{aligned} \quad (121)$$

which is ridiculous.

If there is $(w, \psi) \in \Phi_1 \times \Phi_4$ satisfying the conditions of Theorem 2.1 in [14], we conclude that

$$\begin{aligned} 0 &< \int_0^{\frac{24}{25}} w(r)dr = \limsup_{y\to\frac{6}{5}^+} \int_0^{\left|0-\left(2y-y^2\right)\right|} w(r)dr \\ &= \limsup_{y\to\frac{6}{5}^+} \int_0^{d\left(f\frac{6}{5}, fy\right)} w(r)dr \\ &\le \limsup_{y\to\frac{6}{5}^+} \left(\int_0^{d\left(\frac{6}{5},y\right)} w(r)dr - \int_0^{\psi\left(d\left(\frac{6}{5},y\right)\right)} w(r)dr\right) \\ &\le \limsup_{y\to\frac{6}{5}^+} \int_0^{d\left(\frac{6}{5},y\right)} w(r)dr - \liminf_{y\to\frac{6}{5}^+} \int_0^{\psi\left(d\left(\frac{6}{5},y\right)\right)} w(r)dr \\ &\le 0 - \int_0^{\psi(0)} w(r)dr = 0, \end{aligned} \quad (122)$$

which is impossible.

Remark 15. Examples 16, 21, and 26 explain that Theorems 4, 7, and 10 are different from Theorem 2.1 in [14].

Example 16. Let $X = R^+$, $d(x, y) = |x - y|$ and $p(x, y) = x + y$, $\forall x, y \in X$. Let $f : X \to X$, w and $\psi : R^+ \to R^+$ be defined by, respectively,

$$fx = \frac{x}{3}, \forall x \in [0, 1], \ \frac{x}{4}, \forall x \in (1,+\infty), \quad (123)$$

and

$$w(r) = 2r, \psi(r) = \frac{r}{2}, \forall r \in R^+. \quad (124)$$

Evidently, p is a ω-distance in X and $(w, \psi) \in \Phi_2 \times \Phi_4$. Let $x, y \in X$. For the sake of verifying (38), we take into account the following four possible cases:

Case 17. $(x, y) \in [0, 1] \times [0, 1]$. Note that

$$\begin{aligned} \int_0^{p(fx,fy)} w(r)dr &= \int_0^{\frac{x+y}{3}} 2tdt = \frac{1}{9}(x+y)^2 \le (x+y)^2 - \frac{1}{4}\left(x+\frac{y}{3}\right)^2 \\ &= \int_0^{x+y} 2tdt - \int_0^{\psi\left(x+\frac{y}{3}\right)} 2tdt \\ &= \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,fy))} w(r)dr. \end{aligned} \quad (125)$$

Case 18. $(x, y) \in [0, 1] \times (1,+\infty)$. Obviously,

$$\begin{aligned} \int_0^{p(fx,fy)} w(r)dr &= \int_0^{\frac{x}{3}+\frac{y}{4}} 2tdt = \left(\frac{x}{3}+\frac{y}{4}\right)^2 \le (x+y)^2 - \frac{1}{4}\left(x+\frac{y}{4}\right)^2 \\ &= \int_0^{x+y} 2tdt - \int_0^{\psi\left(x+\frac{y}{4}\right)} 2tdt \\ &= \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,fy))} w(r)dr. \end{aligned} \quad (126)$$

Case 19. $(x, y) \in (1,+\infty) \times [0, 1]$. Notice that

$$\begin{aligned} \int_0^{p(fx,fy)} w(r)dr &= \int_0^{\frac{x}{4}+\frac{y}{3}} 2tdt = \left(\frac{x}{4}+\frac{y}{3}\right)^2 \le (x+y)^2 - \frac{1}{4}\left(x+\frac{y}{3}\right)^2 \\ &= \int_0^{x+y} 2tdt - \int_0^{\psi\left(x+\frac{y}{3}\right)} 2tdt \\ &= \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,fy))} w(r)dr. \end{aligned} \quad (127)$$

Case 20. $(x, y) \in (1,+\infty) \times (1,+\infty)$. It follows that

$$\begin{aligned} \int_0^{p(fx,fy)} w(r)dr &= \int_0^{\frac{x+y}{4}} 2tdt = \frac{1}{16}(x+y)^2 \\ &\le (x+y)^2 - \frac{1}{4}\left(x+\frac{y}{4}\right)^2 \\ &= \int_0^{x+y} 2tdt - \int_0^{\psi\left(x+\frac{y}{4}\right)} 2tdt \\ &= \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,fy))} w(r)dr. \end{aligned} \quad (128)$$

That is to say, (38) is true. Therefore, the conditions of Theorem 4 are fulfilled. Consequently, Theorem 4 means that f possesses a unique fixed point in X. However, we cannot use Theorem 2.1 in [14] to show the existence of fixed points for the mapping f in X. Or else, there is $(w, \psi) \in \Phi_1$

$\times \Phi_4$ with

$$0<\int_0^{\frac{1}{12}} w(r)dr=\limsup_{y\to 1^+}\int_0^{\left|\frac{1}{3}-\frac{y}{4}\right|} w(r)dr$$
$$=\limsup_{y\to 1^+}\int_0^{d(f1,fy)} w(r)dr$$
$$\leq \limsup_{y\to 1^+}\left(\int_0^{d(1,y)} w(r)dr-\int_0^{\psi(d(1,y))} w(r)dr\right) \tag{129}$$
$$\leq \limsup_{y\to 1^+}\int_0^{d(1,y)} w(r)dr-\liminf_{y\to 1^+}\int_0^{\psi(d(1,y))} w(r)dr$$
$$\leq 0-\int_0^{\psi(0)} w(r)dr=0,$$

which is absurd.

Example 21. Let $X=R^+$, $d(x,y)=|x-y|$ and $p(x,y)=x+y$, $\forall x,y\in X$. Let $f:X\to X$, w and $\psi:R^+\to R^+$ be defined by, respectively,

$$fx=0,\forall x\in\left[0,\sqrt{3}\right],\ \frac{\sqrt{3}}{4}x,\forall x\in\left(\sqrt{3},+\infty\right), \tag{130}$$

and

$$w(r)=4r,\quad \psi(r)=\frac{\sqrt{3}}{2}r,\forall r\in R^+. \tag{131}$$

Obviously, p is a ω-distance in X and $(w,\psi)\in\Phi_2\times\Phi_4$. Let $x,y\in X$. To demonstrate (82), we consider four cases below:

Case 22. $(x,y)\in[0,\sqrt{3}]\times[0,\sqrt{3}]$. Evidently,

$$\int_0^{p(fx,fy)} w(r)dr=0\leq 2(x+y)^2-\frac{3}{2}y^2$$
$$=\int_0^{p(x,y)} w(r)dr-\int_0^{\psi(p(fx,y))} w(r)dr. \tag{132}$$

Case 23. $(x,y)\in[0,\sqrt{3}]\times(\sqrt{3},+\infty)$. Clearly,

$$\int_0^{p(fx,fy)} w(r)dr=\int_0^{\frac{\sqrt{3}}{4}y} 4rdr=\frac{3}{8}y^2\leq 2(x+y)^2-\frac{3}{2}y^2$$
$$=\int_0^{x+y} 4rdr-\int_0^{\psi(y)} 4rdr \tag{133}$$
$$=\int_0^{p(x,y)} w(r)dr-\int_0^{\psi(p(fx,y))} w(r)dr.$$

Case 24. $(x,y)\in(\sqrt{3},+\infty)\times[0,\sqrt{3}]$. It is obvious that

$$\int_0^{p(fx,fy)} w(r)dr=\int_0^{\frac{\sqrt{3}}{4}x} 4rdr=\frac{3}{8}x^2$$
$$\leq 2(x+y)^2-\frac{3}{2}\left(\frac{\sqrt{3}}{4}x+y\right)^2$$
$$=\int_0^{x+y} 4rdr-\int_0^{\psi\left(\frac{\sqrt{3}}{4}x+y\right)} 4rdr \tag{134}$$
$$=\int_0^{p(x,y)} w(r)dr-\int_0^{\psi(p(fx,y))} w(r)dr.$$

Case 25. $(x,y)\in(\sqrt{3},+\infty)\times(\sqrt{3},+\infty)$. Clearly

$$\int_0^{p(fx,fy)} w(r)dr=\int_0^{\frac{\sqrt{3}}{4}(x+y)} 4rdr=\frac{3}{8}(x+y)^2$$
$$\leq 2(x+y)^2-\frac{3}{2}\left(\frac{\sqrt{3}}{4}x+y\right)^2$$
$$=\int_0^{x+y} 4rdr-\int_0^{\psi\left(\frac{\sqrt{3}}{4}x+y\right)} 4tdt \tag{135}$$
$$=\int_0^{p(x,y)} w(r)dr-\int_0^{\psi(p(fx,y))} w(r)dr.$$

In other words, (82) is true, and consequently, the conditions of Theorem 7 are fulfilled. Thus, Theorem 7 yields that f possesses a unique fixed point in X. Next, we testify that Theorem 2.1 in [14] is unapplicable in ensuring the existence of fixed points for the mapping f in X.

If there is $(w,\psi)\in\Phi_1\times\Phi_4$ satisfying the conditions of Theorem 2.1 in [14], we have

$$0<\int_0^{\frac{3}{4}} w(r)dr=\limsup_{y\to\sqrt{3}^+}\int_0^{\left|0-\frac{\sqrt{3}}{4}y\right|} w(r)dr$$
$$=\limsup_{y\to\sqrt{3}^+}\int_0^{d\left(f\sqrt{3},fy\right)} w(r)dr$$
$$\leq\limsup_{y\to\sqrt{3}^+}\left(\int_0^{d\left(\sqrt{3},y\right)} w(r)dr-\int_0^{\psi\left(d\left(\sqrt{3},y\right)\right)} w(r)dr\right)$$
$$\leq\limsup_{y\to\sqrt{3}^+}\int_0^{d\left(\sqrt{3},y\right)} w(r)dr-\liminf_{y\to\sqrt{3}^+}\int_0^{\psi\left(d\left(\sqrt{3},y\right)\right)} w(r)dr$$
$$\leq 0-\int_0^{\psi(0)} w(r)dr=0, \tag{136}$$

which is ridiculous.

Example 26. Let $X=R^+$, $d(x,y)=|x-y|$ and $p(x,y)=y$, $\forall x,y\in$ X. Let $f:X\to X$, w and $\psi:R^+\to R^+$ be defined by, respectively,

$$fx = 0, \forall x \in [0, 25], \ \frac{x}{5}, \forall x \in (25, +\infty), \tag{137}$$

and

$$w(r) = \frac{4}{5}r^3, \quad \psi(r) = 5\sqrt{r}, \forall r \in R^+. \tag{138}$$

It is easy to see that p is a ω-distance in X and $(w, \psi) \in \Phi_1 \times \Phi_4$. Let $x, y \in X$. To prove (114), we have to consider two cases below:

Case 27. $(x, y) \in X \times [0, 25]$. Apparently,

$$\int_0^{p(fx,fy)} w(r)dr = 0 \le \frac{1}{5}y^4 = \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(fx,fy))} w(r)dr. \tag{139}$$

Case 28. $(x, y) \in X \times (25, +\infty)$. It is easy to demonstrate that

$$\begin{aligned}\int_0^{p(fx,fy)} w(r)dr &= \int_0^{\frac{y}{5}} \frac{4}{5}r^3 dr = \frac{1}{5^5}y^4 \le \frac{1}{5}y^4 - 5y^2 \\ &= \int_0^{y} \frac{4}{5}r^3 dr - \int_0^{\psi\left(\frac{y}{5}\right)} \frac{4}{5}r^3 dr \\ &= \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(fx,fy))} w(r)dr.\end{aligned}$$

Hence, (114) is true, and the conditions of Theorem 10 are satisfied. Thus, Theorem 10 guarantees that f possesses a unique fixed point in X. Then, we certify that Theorem 2.1 in [14] is unfulfilled in showing the existence of fixed points for the mapping f in X. Otherwise, there is $(w, \psi) \in \Phi_1 \times \Phi_4$ satisfying the conditions Theorem 2.1 in [14]. It means that

$$\begin{aligned}0 &< \int_0^5 w(r)dr = \limsup_{y \to 25^+} \int_0^{|0 - \frac{y}{5}|} w(r)dr \\ &= \limsup_{y \to 25^+} \int_0^{d(f25,fy)} w(r)dr \\ &\le \limsup_{y \to 25^+} \left(\int_0^{d(25,y)} w(r)dr - \int_0^{\psi(d(25,y))} w(r)dr \right) \\ &\le \limsup_{y \to 25^+} \int_0^{d(25,y)} w(r)dr - \liminf_{y \to 25^+} \int_0^{\psi(d(25,y))} w(r)dr \\ &\le 0 - \int_0^{\psi(0)} w(r)dr = 0,\end{aligned} \tag{141}$$

which is ridiculous.

5. Applications

In this section, we utilize Theorems 1 and 7 to investigate the solvability of the nonlinear Fredholm and Volterra integral equations below, respectively,

$$x(t) = h(t) + \int_a^b K(t, r, x(r))dr, \forall t \in [a, b], \tag{142}$$

$$x(t) = h(t) + \int_a^t K(t, r, x(r))dr, \forall t \in [a, b], \tag{143}$$

where a and b are constants in R with $a < b$, $h : [a, b] \to R$ and $K : [a, b]^2 \times R \to R$ are given functions.

We assume that $C([a, b], R)$ denotes the Banach space of all continuous functions $x : [a, b] \to R$ with the norm $\|x\| = \sup_{t \in [a,b]} |x(t)|$. Let $X = C([a, b], R)$ and

$$d(x, y) = \sup_{r \in [a,b]} |x(r) - y(r)|, \forall x, y \in X. \tag{144}$$

Obviously, (X, d) is a complete metric space. Define two mappings T and S as follows:

$$(Tx)(t) = h(t) + \int_a^b K(t, r, x(r))dr, \forall (t, x) \in [a, b] \times X, \tag{145}$$

$$(Sx)(t) = h(t) + \int_a^t K(t, r, x(r))dr, \forall (t, x) \in [a, b] \times X. \tag{146}$$

Theorem 29. *Let $h : [a, b] \to R$ and $K : [a, b]^2 \times R \to R$ satisfy that*

(a1) h and K are continuous;

(a2) there is $\psi \in \Phi_4$ with

$$\left| \frac{h(t)}{b - a} + K(t, r, y(r)) \right| \le \frac{\sup_{r \in [a,b]} |y(r)| - \psi\left(\sup_{r \in [a,b]} |y(r)| \right)}{b - a}, \forall (t, r, y) \in [a, b]^2 \times X. \tag{147}$$

Then, Eq. (142) possesses a unique solution in X.

Proof. Define two functions $w : R^+ \to R^+$ and $p : X \times X \to R^+$ by

$$w(r) = \frac{1}{2}, \forall r \in R^+, p(x, y) = \sup_{t \in [a,b]} |y(t)|, \forall x, y \in X. \tag{148}$$

Obviously, p is a ω-distance and $w \in \Phi_1$. It follows from $(a1)$ and (145) that for arbitrary $x \in X$, Tx is continuous in $[a, b]$, which means that T maps X into itself. Taking account of (145) and $(a2)$, we get that

$$\begin{aligned}\int_0^{|(Ty)(t)|} w(r)dr &= \frac{1}{2}|(Ty)(t)| = \frac{1}{2}|h(t) + \int_a^b K(t, r, y(r))dr| \\ &\le \frac{1}{2}\int_a^b |\frac{h(t)}{b-a} + K(t, r, y(r))|dr \\ &\le \frac{1}{2}\int_a^b \frac{\sup_{r\in[a,b]} |y(r)| - \psi\left(\sup_{r\in[a,b]} |y(r)|\right)}{b-a} dr \\ &= \frac{1}{2}\int_a^b \frac{p(x,y) - \psi(p(x,y))|}{b-a} ds \\ &= \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,y))} w(r)dr, \forall (t, x, y) \\ &\in [a,b] \times X^2.\end{aligned} \tag{149}$$

It follows that

$$\begin{aligned}\int_0^{p(Tx,Ty)} w(r)dr &= \int_0^{\sup_{t\in[a,b]} |(Ty)(t)|} w(r)dr \\ &\le \int_0^{p(x,y)} w(r)dr - \int_0^{\psi(p(x,y))} w(r)dr \\ &= \int_0^{p(x,y)} w(r)dr \\ &\quad - \int_0^{\psi(p(Tx,y))} w(r)dr, \forall x, y \in X.\end{aligned} \tag{150}$$

That is, (2) and (82) hold. It follows from each of Theorems 1 and 7 that T possesses a unique fixed point $x \in X$, that is, Eq. (142) has a unique solution $x \in X$.

We get, similar to the proof of Theorem 29, the following result and omit its proof.

Theorem 30. *Let $h : [a,b] \to R$ and $K : [a,b]^2 \times R \to R$ satisfy $(a1)$ and $(a2)$. Then, Eq. (143) possesses a unique solution in X.*

6. Conclusion

By using ω-distance, we prove several fixed point results for a few contractive mappings of integral type, some of which are used to investigate the existence and uniqueness of solutions for certain nonlinear Fredholm and Volterra integral equations, respectively. Four examples are provided to testify that our results extend or differ from some known results in the literature.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Existence Theorems on Advanced Contractions with Applications

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In this research article, by introducing a mapping φ defined on $[0,\infty)^4$, with some axioms, we define two generalized contractions called $F^{\varphi}_{H^+}$-contractions and $\varphi H^+_{\mathscr{P}}$-contractions. We investigate their mutual relation and establish an existence theorem addressing $F^{\varphi}_{H^+}$-contractions with some applications.

1. Introduction

Frechet gave an abstraction to the notion of distance in Euclidean spaces by introducing metric spaces. Partial metrics (denoted by $\mathscr{P}$) were introduced in [1] as a generalization of the notion of metric to allow nonzero self-distance for the purpose of modeling partial objects in reasoning about data flow networks. The self-distance $\mathscr{P}(\alpha, \alpha)$ is to be understood as a quantification of the extent to which α is unknown. Matthews [1] proved an analogue of Banach's fixed point theorem in partial metric spaces. This remarkable fixed point theorem led many researchers to investigate fixed points of self-mappings in partial metric spaces (see [2–7]).

The investigation of fixed points of multivalued or set-valued mappings was started by Nadler [8]. For this purpose, Nadler introduced a metric function H to measure distance between two nonempty closed and bounded sets. This metric function is also known as the Hausdorff metric in literature. Aydi et al. [5] generalized the Hausdorff metric to the partial Hausdorff metric and hence generalized the Nadler fixed point theorem. Nazam et al. [7] established various fixed point results using the partial Hausdorff metric. Recently, Pathak et al. [9] introduced another metric function H^+ to measure the distance between two nonempty closed and bounded sets and hence proved some fixed point results. Nashine et al. [10] also proved some fixed points theorems on H^+-multivalued contractions and their application to homotopy theory. For recent research in this direction, see [11–13].

In 1922, Banach introduced the Banach Contraction Principle in his PhD thesis. Since then, there has been a trend to generalize and apply it to show the existence of the solutions to various mathematical models (both linear and nonlinear). A large number of research articles contain many useful generalizations of Banach Contraction Principle. In one such attempt, Wardowski [14] introduced F-contractions, where F represents the class of nonlinear real-valued functions satisfying three axioms (F_1, F_2, F_3). The concept of F-contractions proved to be a useful addition in fixed point theory (see for instant [15–18] and references therein). The advancement in the study of F-contraction is in progress, and in this direction recently, Abbas et al. [19] introduced the Presic-type F-contraction and established a fixed point theorem for such kind of mappings. Tomar et al. [20] provided an existence theorem for six self-mappings under the notion of F-contraction. Durmaz et al. [17] studied F-contraction under the effect of a partial order. Sgroi et al. [21] extended the notion of F-contraction to multivalued F-contraction by combining the ideas of Wardowski and Nadler. Durmaz et al. [22] generalized the results given in [16, 17, 21] by introducing (α, F)-contraction. Similarly, Piri et al. [23] proved some theorems on the F-Suzuki type inequalities under some weaker conditions, and Shukla et al. [24] established a common fixed point theorem for weak F-contraction under 0-complete partial metric spaces. Recently, Karapinar et al. [25] presented a survey paper which encompasses almost all the results addressing F-contractions.

The motivation to write this article is the contents of the article [26]. In [26], authors introduced a function (called auxiliary function) defined on $[0,\infty)^4$ satisfying some axioms and used it to establish a fixed point theorem. It was then shown that the Banach fixed point theorem, Kannan fixed point theorem, Chatterjea fixed point theorem, Reich fixed point theorem, Hardy and Rogers fixed point theorem, and Círíc type fixed point theorems are particular cases of this fixed point theorem. Since all the above mentioned fixed point theorems have been generalized using the notion of F-contraction both in metric spaces and partial metric spaces (see [25]), we develop a general fixed point theorem, representing them all, in the Hausdorff partial metric spaces.

This article is organized as follows. In Section 2, some basic notions are given. In Section 3, we give highlights of Hausdorff p-ms. In Section 4, we introduce the $F^{\varphi}_{H^+}$-contractions and $\varphi H^+_{\mathscr{P}}$-contractions and investigate the relations between them. We also study the existence theorem and its consequences. And in Section 5, we derive two results regarding applications by applying the existence theorem given in Section 4. The presented existence theorem generalizes, improves, and extends the results established by Pathak et al. [9].

2. Basic Notions

Let partial metric spaces be denoted by p-m-s.

Matthews [1], while working on networking topologies, noticed the nonzero self-distance (loop is the best example to understand his point). The self-distance played a key role in introduction of p-m-s. Matthews [1] defined the p-m-s as follows: let $\mathfrak{I}$ be a nonempty set, and the function $\mathscr{P}:\mathfrak{I}\times\mathfrak{I}\to[0,\infty)$ is said to be a partial metric (p-m) on $\mathfrak{I}$ if for all $\alpha,\beta,\gamma\in\mathfrak{I}$, the axioms ($p_1$)-($p_4$) are satisfied.

(p_1) $\alpha=\beta\Leftrightarrow\mathscr{P}(\alpha,\alpha)=\mathscr{P}(\alpha,\beta)=\mathscr{P}(\beta,\beta)$

(p_2) $\mathscr{P}(\alpha,\alpha)\le\mathscr{P}(\alpha,\beta)$

(p_3) $\mathscr{P}(\alpha,\beta)=\mathscr{P}(\beta,\alpha)$

(p_4) $\mathscr{P}(\alpha,\gamma)\le\mathscr{P}(\alpha,\beta)+\mathscr{P}(\beta,\gamma)-\mathscr{P}(\beta,\beta)$.

Some examples of $(\mathfrak{I},\mathscr{P})$ are as follows. The function $\mathscr{P}:\mathfrak{I}^2\to[0,\infty)$ defined by

(1) $\mathscr{P}(\alpha,\beta)=|\alpha-\beta|+C\,;C\ge 0$ for all $\alpha,\beta\in\mathfrak{I}$ is a $(\mathfrak{I},\mathscr{P})$

(2) $\mathscr{P}(\alpha,\beta)=\max\{\alpha,\beta\}$, is a $(\mathfrak{I},\mathscr{P})$

(3) $\mathscr{P}(\alpha,\beta)=e^{|\alpha-\beta|}+\max\{\alpha,\beta\}$, is a $(\mathfrak{I},\mathscr{P})$.

It is noted that $\mathscr{P}(\alpha,\beta)=0$ implies $\alpha=\beta$. The p-m function $\mathscr{P}$ is continuous. If $\mathscr{P}$ is a p-m then the function $d_{\mathscr{P}}:\mathfrak{I}\times\mathfrak{I}\to[0,\infty)$ defined by

$$d_{\mathscr{P}}(\alpha,\beta)=2\mathscr{P}(\alpha,\beta)-[\mathscr{P}(\alpha,\alpha)+\mathscr{P}(\beta,\beta)]\text{ for all }\alpha,\beta\in\mathfrak{I}\tag{1}$$

defines a metric on $\mathfrak{I}$. A T_0 topology can be defined on $(\mathfrak{I},\mathscr{P})$ with $\mathscr{P}$-open balls being its elements. The $\mathscr{P}$-open ball centered at σ_0 having radius ε is defined by $O_{\mathscr{P}}(\sigma_0,\varepsilon)=\{\sigma\in\mathfrak{I}:\mathscr{P}(\sigma_0,\sigma)<\mathscr{P}(\sigma_0,\sigma_0)+\varepsilon\}$. A set G is said to be bounded in $(\mathfrak{I},\mathscr{P})$ if there exist $\sigma_0\in\mathfrak{I}$ and $\Delta\ge 0$ such that $\mathscr{P}(\sigma_0,\eta)<\mathscr{P}(\eta,\eta)+\Delta$ for all $\eta\in G$. Also it is easy to write $\eta\in\bar{G}$ (closure of G) $\Leftrightarrow\mathscr{P}(\eta,G)=\mathscr{P}(\eta,\eta)$ and G is closed in $(\mathfrak{I},\mathscr{P})$ if and only if $G=\bar{G}$. If $\mathscr{P}(\sigma,\sigma)=\lim_{n\to\infty}\mathscr{P}(\sigma,\sigma_n)$; then we say that $\{\sigma_n\}$ converges to σ and conversely. If $\lim_{n,m\to\infty}\mathscr{P}(\sigma_n,\sigma_m)$ is finite, then the sequence $\{\sigma_n\}$ is said to be Cauchy, and in particular, if this Cauchy sequence converges in $(\mathfrak{I},\mathscr{P})$, then we say that the p-m-s $(\mathfrak{I},\mathscr{P})$ is complete. Lemma 1 provides fundamental rules to work in the p-m-s.

Lemma 1 [1].

(1) If the sequence σ_n is Cauchy sequence in $(\mathfrak{I},\mathscr{P})$, then it is Cauchy sequence in the metric space $(\mathfrak{I},d_{\mathscr{P}})$ and conversely

(2) The completeness of $(\mathfrak{I},\mathscr{P})$ implies the completeness of $(\mathfrak{I},d_{\mathscr{P}})$ and conversely

(3) $\lim_{n\to\infty}d_{\mathscr{P}}(\sigma,\sigma_n)=0\Leftrightarrow\lim_{n\to\infty}\mathscr{P}(\sigma,\sigma_n)=\mathscr{P}(\sigma,\sigma)=\lim_{n,m\to\infty}\mathscr{P}(\sigma_n,\sigma_m)$, provided $(\mathfrak{I},\mathscr{P})$ is complete.

Remark 1. There are sequences which converge in p-m-s but not in metric spaces. Indeed, for the sequence $\{1/n:n\in\mathbb{N}\}$ in $\mathfrak{I}=[0,1]$ and p-m $\mathscr{P}$ defined by $\mathscr{P}(\rho,\varsigma)=|\rho-\varsigma|+C$ $(C\ge 0)\ \forall\rho,\varsigma\in\mathfrak{I}$, it is easy to check that the sequence $\{1/n\}$ converges to 0 with respect to $\mathscr{P}$ but does not converge to 0 with respect to metric d defined by $d(\rho,\varsigma)=\mathscr{P}(\rho,\varsigma)$ if $\rho\ne\varsigma$ and 0 otherwise.

3. Hausdorff Partial Metric

Let the set of nonempty closed and bounded subsets of $(\mathfrak{I},\mathscr{P})$ be denoted by $CB_{\mathscr{P}}(\mathfrak{I})$. Let $\mathscr{P}(\sigma,A)=\inf\{\mathscr{P}(\sigma,a):a\in A\}$, $A\in CB_{\mathscr{P}}(\mathfrak{I})$. Let $\Delta_{\mathscr{P}}:CB_{\mathscr{P}}(\mathfrak{I})\times CB_{\mathscr{P}}(\mathfrak{I})\to[0,\infty)$ be defined by $\Delta_{\mathscr{P}}(X,Y)=\sup\{\mathscr{P}(a,Y):a\in X\}$. Let $H_{\mathscr{P}}:CB_{\mathscr{P}}(\mathfrak{I})\times CB_{\mathscr{P}}(\mathfrak{I})\to[0,\infty)$ be defined by

$$H_{\mathscr{P}}(X,Y)=\max\{\Delta_{\mathscr{P}}(X,Y),\Delta_{\mathscr{P}}(Y,X)\}.\tag{2}$$

Let $H^+_{\mathscr{P}}:CB_{\mathscr{P}}(\mathfrak{I})\times CB_{\mathscr{P}}(\mathfrak{I})\to[0,\infty)$ be defined by

$$H^+_{\mathscr{P}}(X,Y)=\frac{1}{2}\{\Delta_{\mathscr{P}}(X,Y)+\Delta_{\mathscr{P}}(Y,X)\}.\tag{3}$$

Since $\max\{\sigma,\varsigma\}\ge 1/2(\sigma+\varsigma)$, $H_{\mathscr{P}}(X,Y)\ge H^+_{\mathscr{P}}(X,Y)$ for all $X,Y\in CB_{\mathscr{P}}(\mathfrak{I})$. A comprehensive study of the distance $H^+(X,Y)$ with reference to metric d was presented by Pathak et al. in [9]. We claim that

(a) $H^+_{\mathscr{P}}(X,Y)$ and $H_{\mathscr{P}}(X,Y)$ are topological equivalent

(b) the mapping $H^+_{\mathscr{P}}:CB_{\mathscr{P}}(\mathfrak{I})\times CB_{\mathscr{P}}(\mathfrak{I})\to[0,\infty)$ defines a p-m on $CB_{\mathscr{P}}(\mathfrak{I})$

(c) if the p-m-s $(\mathfrak{I}, \mathscr{P})$ is complete then $(CB_{\mathscr{P}}(\mathfrak{I}), H^+_{\mathscr{P}})$ is also complete and vice versa

(d) the mapping $H^+_{\mathscr{P}} : CB_{\mathscr{P}}(\mathfrak{I}) \times CB_{\mathscr{P}}(\mathfrak{I}) \to [0,\infty)$ is continuous.

Proposition 1 [7]. *Let $(\mathfrak{I}, \mathscr{P})$ be p-m-s. For any $J, K, L \in CB_{\mathscr{P}}(\mathfrak{I})$, we have the following:*

(1) $\Delta_{\mathscr{P}}(J, J) = \sup \{\mathscr{P}(u, v): u, v \in J\}$

(2) $\Delta_{\mathscr{P}}(J, K) = \Delta_{\mathscr{P}}(K, J)$

(3) $\Delta_{\mathscr{P}}(J, K) = 0 \Rightarrow J \subseteq K$

(4) $\Delta_{\mathscr{P}}(J, L) \le \Delta_{\mathscr{P}}(J, K) + \Delta_{\mathscr{P}}(K, L) - \inf_{k\in K}\mathscr{P}(k, k)$.

Proposition 2. *Let $(\mathfrak{I}, \mathscr{P})$ be p-m-s. For any $J, K, L \in CB_{\mathscr{P}}(\mathfrak{I})$, we have the following:*

(1) $H^+_{\mathscr{P}}(J, K) = 0$ implies $J = K$

(2) $H^+_{\mathscr{P}}(J, J) \le H^+_{\mathscr{P}}(J, K)$

(3) $H^+_{\mathscr{P}}(J, K) = H^+_{\mathscr{P}}(K, J)$

(4) $H^+_{\mathscr{P}}(J, L) \le H^+_{\mathscr{P}}(J, K) + H^+_{\mathscr{P}}(K, L) - \inf_{k\in K}\mathscr{P}(k, k)$.

Proof. Following the arguments given in ([5], Proposition 2.2 and Proposition 2.3), we get the result. We omit its details.

$H_{\mathscr{P}}$-**contraction**: Let $(\mathfrak{I}, \mathscr{P})$ be a p-m-s, the mapping $T : \mathfrak{I} \to CB_{\mathscr{P}}(\mathfrak{I})$ is called an $H_{\mathscr{P}}$-contraction, if there exists $k < 1$ such that $H_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\}) \le k\mathscr{P}(\sigma, \varsigma)$ for all $\sigma, \varsigma \in \mathfrak{I}$ (see [7]).

$H^+_{\mathscr{P}}$-**contraction**: Let $(\mathfrak{I}, \mathscr{P})$ be a p-m-s; the mapping $T : \mathfrak{I} \to CB_{\mathscr{P}}(\mathfrak{I})$ is called an $H^+_{\mathscr{P}}$-contraction, if (1) there exists $k < 1$ such that $H^+_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\}) \le k\mathscr{P}(\sigma, \varsigma)$ for all $\sigma, \varsigma \in \mathfrak{I}$; (2) for all $\sigma \in \mathfrak{I}$, $\{\varsigma\} \in T(\sigma)$, $\varepsilon > 0$ there exists $\{\xi\} \in T(\varsigma)$ such that $\mathscr{P}(\varsigma, \xi) \le H^+_{\mathscr{P}}(T(\sigma), T(\varsigma)) + \varepsilon$ (see [27]).

Since $H_{\mathscr{P}}(X, Y) \ge H^+_{\mathscr{P}}(X, Y)$ for all $X, Y \in CB_{\mathscr{P}}(\mathfrak{I})$, $H_{\mathscr{P}}$-contraction implies $H^+_{\mathscr{P}}$-contraction but not conversely (see Example 1).

Example 1. Let $\mathfrak{I} = \{0, 1/7, 1\}$. *Define the function* $\mathscr{P} : \mathfrak{I} \times \mathfrak{I} \to [0,\infty)$ *by*

$$\mathscr{P}(\sigma, \varsigma) = \max \{\sigma, \varsigma\} \text{ for all } \sigma, \varsigma \in \mathfrak{I}. \tag{4}$$

Then $(\mathfrak{I}, \mathscr{P})$ is a p-m-s. Let $T : \mathfrak{I} \to CB_{\mathscr{P}}(\mathfrak{I})$ be defined by

$$T(\sigma) = \begin{cases} \{0\} & \text{if } \sigma = 0 \\ \left\{0, \dfrac{1}{7}\right\} & \text{if } \sigma = \dfrac{1}{7} \\ \{0, 1\} & \text{if } \sigma = 1. \end{cases} \tag{5}$$

We have three cases (Case 1: $\sigma = 0, \varsigma = 1/7$, Case 2: $\sigma = 0, \varsigma = 1$, and Case 3: $\sigma = 1/7, \varsigma = 1$).

Case 1. If $\sigma = 0, \varsigma = 1/7$, then $\mathscr{P}(\sigma, \varsigma) = 1/7$, $H_{\mathscr{P}}(T(0), R(1/7)) = 1/7$, and $H^+_{\mathscr{P}}(T(0), R(1/7)) = 1/14$. This clearly shows that

$$H^+_{\mathscr{P}}\left(T(0), R\left(\frac{1}{7}\right)\right) \le L\mathscr{P}\left(0, \frac{1}{7}\right) \text{holds for all } L \ge \frac{1}{2}, \tag{6}$$

whereas

$$H_{\mathscr{P}}\left(T(0), R\left(\frac{1}{7}\right)\right) > L\mathscr{P}\left(0, \frac{1}{7}\right) \text{for any } L < 1. \tag{7}$$

Case 2. If $\sigma = 0, \varsigma = 1$, then $\mathscr{P}(\sigma, \varsigma) = 1$, $H_{\mathscr{P}}(T(0), R(1)) = 1$, and $H^+_{\mathscr{P}}(T(0), R(1)) = 1/2$. This clearly shows that

$$H^+_{\mathscr{P}}(T(0), R(1)) \le L\mathscr{P}(0, 1) \text{ holds for all } L \ge \frac{1}{2}, \tag{8}$$

whereas

$$H_{\mathscr{P}}(T(0), R(1)) > L\mathscr{P}(0, 1) \text{ for any } L < 1. \tag{9}$$

Case 3. If $\sigma = 1/7, \varsigma = 1$, then $\mathscr{P}(\sigma, \varsigma) = 1$, $H_{\mathscr{P}}(R(1/7), R(1)) = 1$, and $H^+_{\mathscr{P}}(R(1/7), R(1)) = 4/7$. This clearly shows that

$$H^+_{\mathscr{P}}\left(R\left(\frac{1}{7}\right), R(1)\right) \le L\mathscr{P}\left(\frac{1}{7}, 1\right) \text{ holds for all } L \ge \frac{1}{2}, \tag{10}$$

whereas

$$H_{\mathscr{P}}\left(R\left(\frac{1}{7}\right), R(1)\right) > L\mathscr{P}\left(\frac{1}{7}, 1\right) \text{ for any } L < 1. \tag{11}$$

Note: the inequality $\mathscr{P}(\varsigma, \xi) \le \mathrm{H}^+_{\mathscr{P}}(\mathrm{T}(\sigma), \mathrm{T}(\varsigma)) + \varepsilon$ also holds for each case, and for all $\sigma \in \mathfrak{I}$, $\varsigma \in \mathrm{T}(\sigma)$, $\xi \in \mathrm{T}(\varsigma)$.

4. Fixed Points of $F^{\varphi}_{H^+}$-Contraction

Let $T : \mathfrak{I} \to \mathfrak{I}$ be a self-mapping defined on nonempty set $\mathfrak{I}$. The problem "to find $\sigma^* \in \mathfrak{I}$ such that $\sigma^* = T(\sigma^*)$" is called fixed point problem. If $T : \mathfrak{I} \to CB(\mathfrak{I})$, then the fixed point problem turns into the form "to find $\sigma^* \in \mathfrak{I}$ such that $\sigma^* \in T(\sigma^*)$." For the solution of fixed point problem, generally, a Picard iterative sequence ($\{\sigma_n\}$such that $\sigma_{n+1} = T(\sigma_n)$) is proved to be a Cauchy sequence subject to contractive condition and completeness of the underlying abstract metric space leads to such σ^*. In this section, at first, we introduce and compare $F^{\varphi}_{H^+}$-contraction and $\varphi H^+_{\mathscr{P}}$-contraction, and secondly, we obtain a theorem assuring unique fixed point of $F^{\varphi}_{H^+}$-contraction. We proceed with definitions of functions F and φ associated with some axioms.

Wardowski [14] considered a nonlinear function $F : (0, \infty) \to \mathbb{R}$ with the following axioms: (F_1): F is strictly increasing. (F_2): For each sequence $\{\sigma_n\}$ of positive numbers, $\lim_{n\to\infty}\sigma_n = 0$ if and only if $\lim_{n\to\infty}F(\sigma_n) = -\infty$. (F_3): For each sequence $\{\sigma_n\}$ of positive numbers $\lim_{n\to\infty}\sigma_n = 0$, there exists $\theta \in (0, 1)$ such that $\lim_{\sigma_n\to 0^+}(\sigma_n)^{\theta}F(\sigma_n) = 0$. Let $\mathscr{F} = \{F : (0,\infty) \to \mathbb{R} |\, F \text{ satisfies} (F_1) - (F_3)\}$.

The collection $\mathcal{F}$ is nonempty: $f(\sigma) = \ln(\sigma)$, $g(\sigma) = \sigma + \ln(\sigma)$, $h(\sigma) = \ln(\sigma^2 + \sigma)$, and $k(\sigma) = -1/\sqrt{\sigma}$ are members of this collection.

Let us consider the function $\varphi : [0,\infty)^4 \to [0,\infty)$ satisfying the following axioms:

(C_1) φ is continuous and non-decreasing in each coordinate

(C_2) if there exist $\sigma, \varsigma \in [0,\infty)$ such that $\sigma < \varsigma$ then $\varphi(\varsigma, \varsigma, \sigma, \varsigma) \le \varsigma$

(C_3) if there exists $\sigma \in [0,\infty)$ such that $\sigma \le \varphi(0, 0, \sigma, (\sigma/2))$ then $\varphi(0, 0, \sigma, (\sigma/2)) = \sigma$.

Let $\mathscr{C}_\varphi = \{\varphi : [0,\infty)^4 \to [0,\infty)|_\varphi \text{ satisfies } (C_1) - (C_3)\}$. The following examples show that the set $\mathscr{C}_\varphi$ is nonempty:

(1) $\varphi_a(\sigma, \varsigma, \omega, \theta) = \max\{\sigma, \varsigma, \omega, \theta\}$

(2) $\varphi_b(\sigma, \varsigma, \omega, \theta) = \theta$

(3) $\varphi_c(\sigma, \varsigma, \omega, \theta) = \max\{\sigma, \varsigma, \omega\}$

(4) $\varphi_d(\sigma, \varsigma, \omega, \theta) = \max\{\varsigma, \omega\}$

(5) $\varphi_e(\sigma, \varsigma, \omega, \theta) = \sigma$

(6) $\varphi_f(\sigma, \varsigma, \omega, \theta) = 1/2(\varsigma + \omega)$

(7) $\varphi_g(\sigma, \varsigma, \omega, \theta) = \max\{\sigma, (\varsigma + \omega/2), \theta\}$

(8) $\varphi_Z(\sigma, \varsigma, \omega, \theta) = a\sigma + b(\varsigma + \omega) + 2c\theta$, $a + 2b + 2c = 1$

(9) $\varphi_i(\sigma, \varsigma, \omega, \theta) = a\sigma + b\varsigma + c\omega$, $a + b + c = 1$.

Definition 1. Let $T : \mathfrak{I} \to P(\mathfrak{I})$ *and* $\alpha : \mathfrak{I} \times \mathfrak{I} \to [0,\infty)$ be two functions. A mapping T is said to be strictly α -admissible if for each $\sigma \in \mathfrak{I}$ *and* $\varsigma \in T(\sigma)$ with $\alpha(\sigma, \varsigma) > 1$, there exists $\omega \in T(\varsigma)$ such that $\alpha(\varsigma, \omega) > 1$.

Definition 2. Let $(\mathfrak{I}, \mathscr{P})$ be a p-m-s and let $\alpha : \mathfrak{I} \times \mathfrak{I} \to [0, \infty)$ be a function. The space $(\mathfrak{I}, \mathscr{P})$ is said to be strictly α -regular if for any sequence $\{\sigma_n\} \subset \mathfrak{I}$ such that $\alpha(\sigma_n, \sigma_{n+1}) > 1$ for all $n \in \mathbb{N}$ and $\sigma_n \to \sigma$ *as* $n \to \infty$, we have $\alpha(\sigma_n, \sigma) > 1$ for all $n \in \mathbb{N}$.

Definition 3. Let $(\mathfrak{I}, \mathscr{P})$ be a p-m-s. A mapping $T : \mathfrak{I} \to CB_{\mathscr{P}}(\mathfrak{I})$ is said to be a $\varphi H^+_{\mathscr{P}}$ -contraction if there exist $k \in [0, 1)$ and $\varphi \in \mathscr{C}_\varphi$ such that

$$\alpha(\sigma, \varsigma) H^+_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\}) \le k\varphi \begin{pmatrix} \mathscr{P}(\sigma, \varsigma), \mathscr{P}(\sigma, T(\sigma) \setminus \{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma) \setminus \{\varsigma\}), \\ \\ \dfrac{\mathscr{P}(\varsigma, T(\sigma) \setminus \{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma) \setminus \{\varsigma\})}{2} \end{pmatrix}, \tag{12}$$

for all $\sigma, \varsigma \in \mathfrak{I}$.

Let $\mathscr{A}^* = \{(\sigma, \varsigma) \in \mathfrak{I}^2 \mid \alpha(\sigma, \varsigma) \ge 1 \text{and} H^+(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\}) > 0\}$.

Definition 4. Let $(\mathfrak{I}, \mathscr{P})$ be a p-m-s. A mapping $T : \mathfrak{I} \to CB_{\mathscr{P}}(\mathfrak{I})$ is said to be an $F^{\varphi}_{H^+}$-contraction if

(a) there exist $\varphi \in \mathscr{C}_\varphi$, $\tau > 0$ and $F \in \mathscr{F}$ such that

$$\tau + F(\alpha(\sigma, \varsigma) H^+_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\})) \le F\left(\varphi\left(\begin{matrix} \mathscr{P}(\sigma, \varsigma), \mathscr{P}(\sigma, T(\sigma) \setminus \{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma) \setminus \{\varsigma\}), \\ \\ \dfrac{\mathscr{P}(\varsigma, T(\sigma) \setminus \{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma) \setminus \{\varsigma\})}{2} \end{matrix}\right)\right), \tag{13}$$

for all $\sigma, \varsigma \in \mathscr{A}^*$.

(b) For every $\varepsilon > 0$, $\sigma \in \mathfrak{I}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma, \xi) \le H^+_{\mathscr{P}}(T(\sigma), T(\varsigma)) + \varepsilon. \tag{14}$$

Remark 2. In particular if $\mathscr{P}(\sigma, \sigma) = 0$, then for $\varphi_e \in \mathscr{C}_\varphi$, the inequality (13) turns into H^+ -contraction [9] for $F(\sigma) = \ln(\sigma)$.

Proposition 3. *Every $\varphi H^+_{\mathscr{P}}$-contraction is an $F^{\varphi}_{H^+}$-contraction, but the converse may not be true.*

Proof. Let $T : \mathfrak{I} \to CB_{\mathscr{P}}(\mathfrak{I})$ be a $\varphi H^+_{\mathscr{P}}$-contraction defined on $(\mathfrak{I}, \mathscr{P})$; then for all $\sigma, \varsigma \in \mathfrak{I}$ there exist $k \in [0, 1)$ and $\varphi \in \mathscr{C}_\varphi$ such that

$$\begin{aligned} &\alpha(\sigma, \varsigma) H^+_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\}) \\ &\le k\varphi\bigg(\mathscr{P}(\sigma, \varsigma), \mathscr{P}(\sigma, T(\sigma) \setminus \{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma) \setminus \{\varsigma\}), \\ &\quad \frac{\mathscr{P}(\varsigma, T(\sigma) \setminus \{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma) \setminus \{\varsigma\})}{2}\bigg). \end{aligned} \tag{15}$$

This can be written as

$$\begin{aligned} &\ln\left(\frac{1}{k}\right) + \ln(\alpha(\sigma, \varsigma) H^+_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\})) \\ &\le \ln\bigg(\mathscr{P}(\sigma, \varsigma), \mathscr{P}(\sigma, T(\sigma) \setminus \{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma) \setminus \{\varsigma\}), \\ &\quad \frac{\mathscr{P}(\varsigma, T(\sigma) \setminus \{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma) \setminus \{\varsigma\})}{2}\bigg) \\ &\quad \cdot \bigg(\varphi(\mathscr{P}(\sigma, \varsigma), \mathscr{P}(\sigma, T(\sigma) \setminus \{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma) \setminus \{\varsigma\}), \\ &\quad \frac{\mathscr{P}(\varsigma, T(\sigma) \setminus \{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma) \setminus \{\varsigma\})}{2}\bigg)\bigg). \end{aligned} \tag{16}$$

Let $F \in \mathscr{F}$ be defined by $F(\sigma) = \ln(\sigma)$ for all $\sigma > 0$ and put $\tau = \ln(1/k)$. The inequality (16) leads to

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\})) \leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\\ \\ \dfrac{\mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right)\right). \quad (17)$$

The following example (Example 2) shows that an $F^{\varphi}_{H^{+}}$-contraction needs not to be a $\varphi H^{+}_{\mathscr{P}}$-contraction.

Example 2. Let $\varphi_a \in \mathscr{C}_{\varphi}\tau = 1$ and $F \in \mathscr{F}$ defined by $F(\sigma) = \ln(\sigma) + \sigma$ where $\varphi_a = \varphi_a(u, r, s, t)$: $(u = \mathscr{P}(\sigma,\varsigma)$, $r = \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}), s = \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}), t = \mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})/4)$. Let $\mathfrak{I} = \{0, 1, 2, \cdots\}$ equipped with p-m $\mathscr{P} : \mathfrak{I}\times\mathfrak{I} \to [0,\infty)$ defined by

$$\mathscr{P}(\sigma,\varsigma) = \sigma + \varsigma \text{ forall } \sigma \leq \varsigma. \quad (18)$$

Then, $(\mathfrak{I}, \mathscr{P})$ is a p-m-s. Define the mapping $T : \mathfrak{I} \to 2^{\mathfrak{I}}$ by

$$R(\sigma) = \begin{cases}\{0\} & \text{if} \sigma \in \{0, 1\};\\ \{0, \sigma - 1\} & \text{if} \sigma \geq 2,\end{cases}$$
$$\alpha(\sigma,\varsigma) = \begin{cases} 0 & \text{if} \sigma, \varsigma \in (-\infty, 0);\\ e^{\mathscr{P}(\sigma,\varsigma)} & \text{if} \sigma, \varsigma \in \{0, 1, 2, \cdots\}.\end{cases} \quad (19)$$

The mapping T is α-admissible, closed, and bounded. We show that this mapping satisfies inequality (13) for all $\sigma, \varsigma \in \mathfrak{I}$. We observe that $H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) > 0$ if and only if $\sigma \geq 2$ and $\varsigma > 0$. Also for all $\sigma, \varsigma \in \mathfrak{I}$ with $\varsigma \in T(\sigma)$ and taking $\zeta = 0 \in T(\varsigma)$, we have

$$\begin{aligned}\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) &= \alpha(\sigma,\varsigma)\mathscr{P}(\varsigma,\zeta)\\ &= \alpha(\sigma,\varsigma)\varsigma < \alpha(\sigma,\varsigma)(\sigma+\varsigma) = \alpha(\sigma,\varsigma)\mathscr{P}(\sigma,\varsigma),\end{aligned} \quad (20)$$

and thus

$$\begin{aligned}&\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) - \varphi_a(u, r, s, t)\\ &\leq \alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) - \mathscr{P}(\sigma,\varsigma) \leq -2.\end{aligned} \quad (21)$$

Consequently,

$$\frac{\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma))}{\varphi_a(u, r, s, t)} e^{\alpha(\sigma,\varsigma)H^{+}(T(\sigma),T(\varsigma)) - \varphi_a(u,r,s,t)} \leq e^{-1}. \quad (22)$$

Hence,

$$1 + F(\alpha(\varsigma,\omega)H^{+}_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\})) \leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\\ \\ \dfrac{\mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right)\right). \quad (23)$$

Similarly, for every member of $\mathscr{C}_{\varphi}$, the mapping T is $F^{\varphi}_{H^{+}}$-contraction. However, the mapping T is not $\varphi H^{+}_{\mathscr{P}}$-contraction: for $\varphi_e \in \mathscr{C}_{\varphi}$ and $\sigma \neq \varsigma = 0$, we have

$$\alpha(\sigma, 0)H^{+}_{\mathscr{P}}(T(\sigma), T(0)) \leq k\varphi_e(u, r, s, t) \Rightarrow e^{\sigma}(\sigma - 1) \leq k\sigma, \quad (24)$$

which then gives $e^{\sigma}(\sigma-1)/\sigma \leq k$, and $\lim_{\sigma\to\infty} e^{\sigma}(\sigma-1)/\sigma \leq k$ implies $k \geq \infty$, a contradiction. Hence, T is not $\varphi H^{+}_{\mathscr{P}}$-contraction for this particular member of $\mathscr{C}_{\varphi}$. Similarly, for $\varphi_b(u, r, s, t) = t \in \mathscr{C}_{\varphi}$ and $\sigma \neq \varsigma = 1$, we have

$$\alpha(\sigma, 1)H^{+}_{\mathscr{P}}(T(\sigma), T(1)) \leq k\varphi_b(u, r, s, t) \text{ does not exist.} \quad (25)$$

Hence, T is not $\varphi H^{+}_{\mathscr{P}}$-contraction for this member of $\mathscr{C}_{\varphi}$. The mapping T has similar nature for other members of $\mathscr{C}_{\varphi}$.

The following theorem (Theorem 1) gives the proof of all particular problems corresponding to members of $\mathscr{C}_{\varphi}$ in one attempt.

Theorem 1. *Let $(\mathfrak{I}, \mathscr{P})$ be a complete p-m-s and $T : \mathfrak{I} \to BC_{\mathscr{P}}(\mathfrak{I})$ be an $F^{\varphi}_{H^{+}}$-contraction such that*

(1) T is a strictly α-admissible mapping

(2) $\exists\sigma_0$ and $\sigma_1 \in T(\sigma_0)$ in $\mathfrak{I}$ such that $\alpha(\sigma_0, \sigma_1) > 1$

(3) $\mathfrak{I}$ is a strictly α-regular space

(4) F is continuous.

Then, there exists $x^* \in \mathfrak{I}$ such that $x^* \in T(x^*)$.

Proof. By assumption (2), there exist σ_0 and $\sigma_1 \in T(\sigma_0)$ in $\mathscr{A}$ such that $\alpha(\sigma_0, \sigma_1) > 1$. Note that if $\sigma_0 \in T(\sigma_0)$, then σ_0 is a fixed point of T, and if $\sigma_1 \in T(\sigma_1)$, then σ_1 is a fixed point of T as required. We proceed by assuming $\sigma_0 \notin T(\sigma_0)$ and $\sigma_1 \notin T(\sigma_1)$; thus, $\sigma_0, \sigma_1 \in \mathscr{A}^*$. Given $\alpha(\sigma_0, \sigma_1) > 1$ and $T(\sigma_0), T(\sigma_1)$ are nonempty, closed, and bounded sets, so, by Definition 4(b), there exists $\sigma_2 \in T(\sigma_1)$ such that

$$\mathscr{P}(\sigma_1, \sigma_2) \leq H^{+}_{\mathscr{P}}(T(\sigma_0), T(\sigma_1)) + \varepsilon. \quad (26)$$

Letting $\varepsilon = (\alpha(\sigma_0, \sigma_1) - 1)H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1))$, we have

$$\begin{aligned}\mathscr{P}(\sigma_1, \sigma_2) &\le H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1)) \\ &\quad + (\alpha(\sigma_0, \sigma_1) - 1)H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1)) \\ &= \alpha(\sigma_0, \sigma_1)H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1)).\end{aligned} \tag{27}$$

By (F_1), (13) and (C_1), we have

$$\begin{aligned}F(\mathscr{P}(\sigma_1, \sigma_2)) &\le F(\alpha(\sigma_0, \sigma_1)H^+(T(\sigma_0), T(\sigma_1))) \\ &\le F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_0, T(\sigma_0)), \mathscr{P}(\sigma_1, T(\sigma_1)), \\ \\ \dfrac{\mathscr{P}(\sigma_1, T(\sigma_0)) + \mathscr{P}(\sigma_0, T(\sigma_1))}{2}\end{array}\right)\right) \\ &\quad - \tau, \le F\left(\varphi\left(\mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_1, \sigma_2), \frac{\mathscr{P}(\sigma_1, \sigma_1) + \mathscr{P}(\sigma_0, \sigma_2)}{2}\right)\right) - \tau.\end{aligned} \tag{28}$$

By the triangular inequality, we have

$$\mathscr{P}(\sigma_0, \sigma_2) + \mathscr{P}(\sigma_1, \sigma_1) \le \mathscr{P}(\sigma_0, \sigma_1) + \mathscr{P}(\sigma_1, \sigma_2). \tag{29}$$

We claim that $\mathscr{P}(\sigma_1, \sigma_2) < \mathscr{P}(\sigma_0, \sigma_1)$. On the contrary, if $\mathscr{P}(\sigma_1, \sigma_2) \ge \mathscr{P}(\sigma_0, \sigma_1)$, then due to (29), we get $\mathscr{P}(\sigma_0, \sigma_2) \le 2\mathscr{P}(\sigma_1, \sigma_2)$. The inequality (28) implies

$$F(\mathscr{P}(\sigma_1, \sigma_2)) < F(\varphi(\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, \sigma_2))). \tag{30}$$

By (C_2), we have $\varphi(\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, \sigma_2)) \le \mathscr{P}(\sigma_1, \sigma_2)$, and by axiom (F_1), the inequality (30) reduces to

$$F(\mathscr{P}(\sigma_1, \sigma_2)) < F(\mathscr{P}(\sigma_1, \sigma_2)). \tag{31}$$

This is an absurdity. This indicates that our claim is valid. Thus, $\mathscr{P}(\sigma_1, \sigma_2) < \mathscr{P}(\sigma_0, \sigma_1)$. Let $\mathscr{P}_n = \mathscr{P}(\sigma_n, \sigma_{n+1})$ for all positive integers n, and by inequality (28) we obtain

$$F(\mathscr{P}_1) \le F(\varphi(\mathscr{P}_0, \mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_0)) - \tau. \tag{32}$$

Applying (C_2) and (F_1) consecutively, we have

$$F(\mathscr{P}_1) \le F(\mathscr{P}_0) - \tau. \tag{33}$$

Since T is a strictly α-admissible mapping, $\alpha(\sigma_0, \sigma_1) > 1$ implies $\alpha(\sigma_1, \sigma_2) > 1$; thus, $\sigma_1, \sigma_2 \in \mathscr{A}^*$ (assume $\sigma_2 \notin T(\sigma_2)$). Since, $T(\sigma_1), T(\sigma_2)$ are nonempty, closed, and bounded sets. By Definition 4(b), there exists $\sigma_3 \in T(\sigma_2)$ such that

$$\mathscr{P}(\sigma_2, \sigma_3) \le H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)) + \varepsilon. \tag{34}$$

Letting $\varepsilon = (\alpha(\sigma_1, \sigma_2) - 1)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2))$, we have

$$\begin{aligned}\mathscr{P}(\sigma_2, \sigma_3) &\le H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)) \\ &\quad + (\alpha(\sigma_1, \sigma_2) - 1)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)) \\ &= \alpha(\sigma_1, \sigma_2)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)).\end{aligned} \tag{35}$$

By (F_1), (13) and (C_1), we have

$$\begin{aligned}F(\mathscr{P}(\sigma_2, \sigma_3)) &\le F(\alpha(\sigma_1, \sigma_2)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2))) \\ &\le F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, T(\sigma_1)), \mathscr{P}(\sigma_2, T(\sigma_2)), \\ \\ \dfrac{\mathscr{P}(\sigma_2, T(\sigma_1)) + \mathscr{P}(\sigma_1, T(\sigma_2))}{2}\end{array}\right)\right) \\ &\quad - \tau, \le F\left(\varphi\left(\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_2, \sigma_3), \frac{\mathscr{P}(\sigma_2, \sigma_2) + \mathscr{P}(\sigma_1, \sigma_3)}{2}\right)\right) - \tau.\end{aligned} \tag{36}$$

By the triangular inequality, we have

$$\mathscr{P}(\sigma_1, \sigma_3) + \mathscr{P}(\sigma_2, \sigma_2) \le \mathscr{P}(\sigma_1, \sigma_2) + \mathscr{P}(\sigma_2, \sigma_3). \tag{37}$$

We claim that $\mathscr{P}(\sigma_2, \sigma_3) < \mathscr{P}(\sigma_1, \sigma_2)$. On the contrary, if $\mathscr{P}(\sigma_2, \sigma_3) \ge \mathscr{P}(\sigma_1, \sigma_2)$, then by (37), we get $\mathscr{P}(\sigma_1, \sigma_3) \le 2\mathscr{P}(\sigma_2, \sigma_3)$. The inequality (36) implies

$$F(\mathscr{P}(\sigma_2, \sigma_3)) < F(\varphi(\mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_2, \sigma_3))) - \tau. \tag{38}$$

By (C_2), $\varphi(\mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_2, \sigma_3)) \le \mathscr{P}(\sigma_2, \sigma_3)$. By (F_1) and (38), we have

$$F(\mathscr{P}(\sigma_2, \sigma_3)) < F(\mathscr{P}(\sigma_2, \sigma_3)). \tag{39}$$

This is an absurdity. Thus, $\mathscr{P}(\sigma_2, \sigma_3) < \mathscr{P}(\sigma_1, \sigma_2)$. By (36), we obtain

$$F(\mathscr{P}_2) \le F(\varphi(\mathscr{P}_1, \mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_1)) - \tau. \tag{40}$$

Again applying the condition (C_2) followed by (F_1), we have

$$F(\mathscr{P}_2) \le F(\mathscr{P}_1) - \tau \le F(\mathscr{P}_0) - 2\tau. \tag{41}$$

Similarly, there exists $\sigma_4 \in T(\sigma_3)(\sigma_3 \notin T(\sigma_3))$, such that

$$F(\mathscr{P}_3) \le F(\mathscr{P}_2) - \tau \le F(\mathscr{P}_0) - 3\tau. \tag{42}$$

Thus, we are able to construct an iterative sequence $\{\sigma_n\} \subset X$ such that

$$\begin{gathered}\sigma_n \in T(\sigma_{n-1}), \sigma_{n-1} \notin T(\sigma_{n-1}), \alpha(\sigma_{n-1}, \sigma_n) > 1, \\ \mathscr{P}_n < \mathscr{P}_{n-1} \text{ for all } n \in \mathbb{N} \text{ and}\end{gathered} \tag{43}$$

$$F(\mathscr{P}_n)) \le F(\mathscr{P}_0)) - n\tau. \tag{44}$$

By (44), we obtain $\lim_{n\to\infty} F(\mathscr{P}_n) = -\infty$, by (F_2) we have $\lim_{n\to\infty} \mathscr{P}_n = 0$, and by (F_3), there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} ((\mathscr{P}_n)^\kappa F(\mathscr{P}_n)) = 0. \tag{45}$$

Following (44), for all $n \in \mathbb{N}$, we obtain

$$(\mathscr{P}_n)^\kappa (F(\mathscr{P}_n) - F(\mathscr{P}_0)) \le -(\mathscr{P}_n)^\kappa n\tau \le 0. \tag{46}$$

Letting $n \to \infty$, in (46), we have $\lim_{n\to\infty}(n(\mathscr{P}_n)^\kappa) = 0$; thus, there exists $n_1 \in \mathbb{N}$, such that $n(\mathscr{P}_n)^\kappa \le 1$ for all $n \ge n_1$, that is $\mathscr{P}_n \le (1/n^{1/\kappa}) \text{for all} n \ge n_1$.

For $m > n \ge n_1$,

$$\begin{aligned}\mathscr{P}(\sigma_n, \sigma_m) &\le \mathscr{P}(\sigma_n, \sigma_{n+1}) + \mathscr{P}(\sigma_{n+1}, \sigma_{n+2}) + \mathscr{P}(\sigma_{n+2}, \sigma_{n+3}) \\ &\quad + \cdots + \mathscr{P}(\sigma_{m-1}, \sigma_m) \\ &\le \sum_{i=n}^{m-1} \mathscr{P}(\sigma_i, \sigma_{i+1}) \le \sum_{i=n}^{\infty} \mathscr{P}(\sigma_i, \sigma_{i+1}) \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.\end{aligned} \tag{47}$$

Since the series $\sum_{i=n}^{\infty}(1/i^{1/\kappa})$ on the right hand side is convergent and by definition of metric $d_{\mathscr{P}}$ defined on $\mathfrak{I}$, we know that $d_{\mathscr{P}}(\sigma_n, \sigma_m) \le 2\mathscr{P}(\sigma_n, \sigma_m)$; thus, $\lim_{n,m\to\infty} d_{\mathscr{P}}(\sigma_n, \sigma_m) = 0$. This implies $\{\sigma_n\}$ is a Cauchy sequence in $(\mathfrak{I}, d_{\mathscr{P}})$. Since $(\mathfrak{I}, \mathscr{P})$ is complete, so by Lemma 1(2), the metric space $(\mathfrak{I}, d_{\mathscr{P}})$ is complete. Thus, there exists $x^* \in \mathfrak{I}$ such that $\sigma_n \to x^*$ as $n \to \infty$ with respect to metric $d_{\mathscr{P}}$. Then Lemma 1(3) implies

$$\lim_{n\to\infty} \mathscr{P}(x^*, \sigma_n) = \mathscr{P}(x^*, x^*) = \lim_{n,m\to\infty} \mathscr{P}(\sigma_n, \sigma_m). \tag{48}$$

This shows that $\{\sigma_n\}$ is a Cauchy sequence in $(\mathfrak{I}, \mathscr{P})$. Now, we show that $x^* \in T(x^*)$, and to do so, we claim that $\mathscr{P}(x^*, T(x^*)) = 0$. If on the other hand $\mathscr{P}(x^*, T(x^*)) > 0$, then there exists $n_1 \in \mathbb{N}$ such that $\mathscr{P}(\sigma_n, T(x^*)) > 0$ for each $n \ge n_1$. By assumption (3), $\alpha(\sigma_n, x^*) > 1$. By (13),

$$\begin{aligned}F(\mathscr{P}(\sigma_{n+1}, T(x^*))) &\le F(\alpha(\sigma_n, x^*)H^+(T(\sigma_n), T(x^*))) \\ &\le F\left(\varphi\left(\mathscr{P}(\sigma_n, x^*), \mathscr{P}(\sigma_n, T(\sigma_n)), \mathscr{P}(x^*, T(x^*)),\right.\right. \\ &\quad \left.\left.\frac{\mathscr{P}(x^*, T(\sigma_n)) + \mathscr{P}(\sigma_n, T(x^*))}{2}\right)\right) \\ &\quad - \tau \le F\left(\varphi\left(\mathscr{P}(\sigma_n, x^*), \mathscr{P}(\sigma_n, \sigma_{n+1}),\right.\right. \\ &\quad \left.\left.\mathscr{P}(x^*, T(x^*)), \frac{\mathscr{P}(x^*, \sigma_{n+1}) + \mathscr{P}(\sigma_n, T(x^*))}{2}\right)\right) - \tau.\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}F(\mathscr{P}(\sigma_{n+1}, T(x^*))) < F\left(\varphi\left(\mathscr{P}(\sigma_n, x^*), \mathscr{P}(\sigma_n, \sigma_{n+1}), \mathscr{P}(x^*, T(x^*)),\right.\right. \\ \left.\left.\frac{\mathscr{P}(x^*, \sigma_{n+1}) + \mathscr{P}(\sigma_n, T(x^*))}{2}\right)\right).\end{aligned} \tag{50}$$

Since φ is a coordinate-wise continuous function, letting $n \to \infty$ in the above inequality, we obtain

$$F(\mathscr{P}(x^*, T(x^*))) < F\left(\varphi\left(0, 0, \mathscr{P}(x^*, T(x^*)), \frac{\mathscr{P}(x^*, T(x^*))}{2}\right)\right). \tag{51}$$

By (C_3), we have

$$F(\mathscr{P}(x^*, T(x^*))) < F(\mathscr{P}(x^*, T(x^*))). \tag{52}$$

This is an absurdity and consequently $\mathscr{P}(x^*, T(x^*)) = 0$; thus, we have $\mathscr{P}(x^*, T(x^*)) = \mathscr{P}(x^*, x^*)$ which implies that $x^* \in \overline{T(x^*)} = T(x^*)$. Hence, x^* is a fixed point of T.

The following example explains Theorem 1.

Example 3. Consistent with ([28], Example 3.3), let $\varphi_a \in \mathscr{C}_\varphi$ where

$$\begin{aligned}\varphi_a = \varphi_a(u, r, s, t)&: \left(u = \mathscr{P}(\sigma, \varsigma), r = \mathscr{P}(\sigma, T(\sigma)), s = \mathscr{P}(\varsigma, T(\varsigma)),\right. \\ &\left. t = \frac{\mathscr{P}(\varsigma, T(\sigma)) + \mathscr{P}(\sigma, T(\varsigma))}{2}\right),\end{aligned} \tag{53}$$

$\tau = 1$ and $F \in \mathscr{F}$ defined by $F(\sigma) = \ln(\sigma) + \sigma$. Let $\mathfrak{I} = \{0, 1, 2, \cdots\}$ equipped with p-m $\mathscr{P} : \mathfrak{I} \times \mathfrak{I} \to [0,\infty)$ defined by

$$\mathscr{P}(\sigma, \varsigma) = \sigma + \varsigma \text{for all} \sigma \ne \varsigma. \tag{54}$$

Then, $(\mathfrak{I}, \mathscr{P})$ is a complete p-m-s. Define the mapping $T : \mathfrak{I} \to 2^{\mathfrak{I}}$ by

$$\begin{aligned}R(\sigma) &= \begin{cases}\{0\} & \text{if } \sigma \in \{0, 1\}; \\ \{0, \sigma - 1\} & \text{if } \sigma \ge 2,\end{cases} \\ \alpha(\sigma, \varsigma) &= \begin{cases}0 & \text{if } \sigma, \varsigma \in (-\infty, 0); \\ e^{\mathscr{P}(\sigma,\varsigma)} & \text{if } \sigma, \varsigma \in \{0, 1, 2, \cdots\}.\end{cases}\end{aligned} \tag{55}$$

The mapping T is strict α-admissible, closed, and bounded. We show that T is $F_{H^+}^{\varphi}$-contraction. We observe that $H^+(T(\sigma), T(\varsigma)) > 0$ if and only if $\sigma \ge 2$ and $\varsigma > 0$. Also for all $\sigma, \varsigma \in \mathfrak{I}$ with $\varsigma \in T(\sigma)$ and taking $\zeta = 0 \in T(\varsigma)$, we have

$$\begin{aligned}\alpha(\sigma, \varsigma)H^+(T(\sigma), T(\varsigma)) &= e^{\mathscr{P}(\sigma,\varsigma)}\mathscr{P}(\varsigma, \zeta) \\ &= e^{\mathscr{P}(\sigma,\varsigma)}\varsigma < e^{\mathscr{P}(\sigma,\varsigma)}(\sigma + \varsigma) \\ &= e^{\mathscr{P}(\sigma,\varsigma)}\mathscr{P}(\sigma, \varsigma), \text{ and thus,}\end{aligned}$$

$$\begin{aligned}&e^{\mathscr{P}(\sigma,\varsigma)}H^+(T(\sigma), T(\varsigma)) - \varphi_a(u, r, s, t) \\ &\quad \le e^{\mathscr{P}(\sigma,\varsigma)}H^+(T(\sigma), T(\varsigma)) - \mathscr{P}(\sigma, \varsigma) \le -2.\end{aligned} \tag{56}$$

Consequently,

$$\frac{e^{\mathscr{P}(\sigma,\varsigma)}H^+(T(\sigma), T(\varsigma))}{\varphi_a(u, r, s, t)} e^{e^{\mathscr{P}(\sigma,\varsigma)}H^+(T(\sigma),T(\varsigma)) - \varphi_a(u,r,s,t)} \le e^{-1}. \tag{57}$$

Hence,

$$1 + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma),T(\varsigma))) \le F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)),\mathscr{P}(\varsigma,T(\varsigma)),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma))+\mathscr{P}(\sigma,T(\varsigma))}{2}\end{array}\right)\right). \tag{58}$$

Similarly, for every member of $\mathscr{C}_{\varphi}$, the mapping T satisfies all assumptions in Theorem 1. As it is clear from Proposition 3 that $F^{\varphi}_{H^{+}}$-contraction needs not to be $\varphi H^{+}_{\mathscr{P}}$-contraction, and hence, it is not $H^{+}_{\mathscr{P}}$-contraction. Consequently, $F^{\varphi}_{H^{+}}$-contraction needs not to be H^{+}-contraction. Thus, the results in [9, 10, 27] are not applicable in this case.

Remark 3. In the following section, we obtain the corollaries of Theorem 1. To simplify the expression of the corollaries, we consider the three conditions below.

Let

(A1) there exist σ_0 in $\mathfrak{J}$ such that $\alpha(\sigma_0, T(\sigma_0)) > 1$

(A2) $\mathfrak{J}$ be a strictly α-regular space

(A3) F be continuous.

Corollary 1. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{J}\to BC_{\mathscr{P}}(\mathfrak{J})$ be a strictly α-admisible mapping. Assume that*

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \le F(\mathscr{P}(\sigma,\varsigma)), \tag{59}$$

for all $\sigma,\varsigma\in\mathscr{A}^{*}$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{J}$, and $\varsigma\in T(\sigma)$, there exists $\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{60}$$

Then, the mapping T has a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi:[0,\infty)^4\to[0,\infty)$ by

$$\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right) = \mathscr{P}(\sigma,\varsigma) \tag{61}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 2. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{J}\to BC_{\mathscr{P}}(\mathfrak{J})$ be a strictly α-admisible mapping. Assume that*

$$\begin{aligned}&\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\\ &\quad\le F(\max\{\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\}),\end{aligned} \tag{62}$$

for all $\sigma,\varsigma\in\mathscr{A}^{*}$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{J}$, and $\varsigma\in T(\sigma)$, $\exists\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{63}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi:[0,\infty)^4\to[0,\infty)$ by

$$\begin{aligned}&\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right)\\ &\quad= \max\{\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\}\end{aligned} \tag{64}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 3. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{J}\to BC_{\mathscr{P}}(\mathfrak{J})$ be a strictly α-admissible mapping. Assume that*

$$\begin{aligned}&\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\\ &\quad\le F(\max\{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\}),\end{aligned} \tag{65}$$

for all $\sigma,\varsigma\in\mathscr{A}^{*}$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{J}$, and $\varsigma\in T(\sigma)$, there exists $\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{66}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi:[0,\infty)^4\to[0,\infty)$ by

$$\begin{aligned}&\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right)\\ &\quad= \max\{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\}\end{aligned} \tag{67}$$

and following the steps given in the proof of Theorem 1, we obtain the result.

Corollary 4. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{J}\to BC_{\mathscr{P}}(\mathfrak{J})$ be a strictly α-admissible mapping. Assume that*

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \le F\left(\max\left\{\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right\}\right), \tag{68}$$

for all $\sigma,\varsigma\in\mathscr{A}^{*}$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{J}$, and $\varsigma\in T(\sigma)$, there exists $\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{69}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}&\varphi\Bigg(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\&\quad\frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg)\\&=\max\Bigg\{\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\&\quad\frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg\},\end{aligned} \tag{70}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 5. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{I}\to BC_{\mathscr{P}}(\mathfrak{I})$ be a strictly α-admissible mapping. Assume that*

$$\begin{aligned}&\tau+F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\\&\quad\le F\left(\frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right),\end{aligned} \tag{71}$$

for all $\sigma,\varsigma\in\mathscr{A}^*$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{I}$, and $\varsigma\in T(\sigma)$, there exists $\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{72}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}&\varphi\Bigg(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\&\quad\frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg)\\&=\frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{aligned} \tag{73}$$

in the proof of Theorem 1, we get the result.

Corollary 6. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{I}\to BC_{\mathscr{P}}(\mathfrak{I})$ be a strictly α-admissible mapping. Assume that*

$$\begin{aligned}&\tau+F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\\&\quad\le F\left(\frac{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right),\end{aligned} \tag{74}$$

for all $\sigma,\varsigma\in\mathscr{A}^*$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{I}$, and $\varsigma\in T(\sigma)$, there exists $\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{75}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}&\varphi\Bigg(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\&\quad\frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg)\\&=\frac{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{aligned} \tag{76}$$

in the proof of Theorem 1, we get the result.

Corollary 7. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{I}\to BC_{\mathscr{P}}(\mathfrak{I})$ be strictly α-admissible mapping. Assume that*

$$\begin{aligned}&\tau+F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\\&\le F\left(\max\left\{\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\dfrac{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})}{2},\\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right\}\right),\end{aligned} \tag{77}$$

for all $\sigma,\varsigma\in\mathscr{A}^*$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{I}$, and $\varsigma\in T(\sigma)$, there exists $\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{78}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}&\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)),\mathscr{P}(\varsigma,T(\varsigma)),\frac{\mathscr{P}(\varsigma,T(\sigma))+\mathscr{P}(\sigma,T(\varsigma))}{2}\right)\\&=\max\left\{\mathscr{P}(\sigma,\varsigma),\frac{\mathscr{P}(\sigma,T(\sigma))+\mathscr{P}(\varsigma,T(\varsigma))}{2},\frac{\mathscr{P}(\varsigma,T(\sigma))+\mathscr{P}(\sigma,T(\varsigma))}{2}\right\}\end{aligned} \tag{79}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 8. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T:\mathfrak{I}\to BC_{\mathscr{P}}(\mathfrak{I})$ be strictly α-admisible mapping. Assume that there exist $a\ge 0$, $b\ge 0$, $c\ge 0$ satisfying $a+2b+2c=1$, such that*

$$\begin{aligned}&\tau+F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\\&\le F\left(\begin{array}{c}a\mathscr{P}(\sigma,\varsigma)+b(\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}))\\ +c(\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\}))\end{array}\right),\end{aligned} \tag{80}$$

for all $\sigma,\varsigma\in\mathscr{A}^*$, and for every $\varepsilon>0$, $\sigma\in\mathfrak{I}$, and $\varsigma\in T(\sigma)$, *there exists $\xi \in T(\varsigma)$ such that*

$$\mathscr{P}(\varsigma, \xi) \leq H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) + \varepsilon. \tag{81}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}\varphi\Bigg(&\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\\ &\frac{\mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg)\\ &= a\mathscr{P}(\sigma,\varsigma) + b(\mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\})\\ &\quad + 2c\frac{\mathscr{P}(\varsigma, T(\sigma)\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\{\varsigma\})}{2},\end{aligned} \tag{82}$$

in the proof of Theorem 1, we obtain the result.

Corollary 9. *Let $(\mathfrak{I}, \mathscr{P})$ be a complete p-m-s and $T : \mathfrak{I} \to B C_{\mathscr{P}}(\mathfrak{I})$ be a strictly α-admissible mapping. Assume that there exist $a \geq 0$, $b \geq 0$, $c \geq 0$ satisfying $a + b + c = 1$, such that*

$$\begin{aligned}\tau + F(&\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\}))\\ &\leq F(a\mathscr{P}(\sigma,\varsigma) + b\mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}) + c\mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\})),\end{aligned} \tag{83}$$

for all $\sigma, \varsigma \in \mathscr{A}^*$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{I}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma, \xi) \leq H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) + \varepsilon. \tag{84}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. If we define $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}\varphi\Bigg(&\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\\ &\frac{\mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg)\\ &= a\mathscr{P}(\sigma,\varsigma) + b\mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}) + c\mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\end{aligned} \tag{85}$$

in the proof of Theorem 1, then the result follows.

Let

$$\mathfrak{I}^* = \left\{(\sigma,\varsigma) \in \mathfrak{I}^2 \mid \alpha(\sigma,\varsigma) > 1 \text{and} \mathscr{P}(T(\sigma), T(\varsigma)) > 0\right\}. \tag{86}$$

For a single-valued self-mapping, Theorem 1 can be stated as follows:

Theorem 2. *Let $(\mathfrak{I}, \mathscr{P})$ be a complete p-m-s and $T : \mathfrak{I} \to \mathfrak{I}$ be a φF-contraction, that is, there exist $\varphi \in \mathscr{C}_{\varphi}$ and $F \in \mathscr{F}$ such that*

$$\begin{aligned}\tau + F(&\alpha(\sigma,\varsigma)\mathscr{P}(T(\sigma), T(\varsigma)))\\ &\leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)), \mathscr{P}(\varsigma, T(\varsigma)),\\ \\ \dfrac{\mathscr{P}(\varsigma, T(\sigma)) + \mathscr{P}(\sigma, T(\varsigma))}{2}\end{array}\right)\right),\end{aligned} \tag{87}$$

for all $\sigma, \varsigma \in \mathfrak{I}^*$ and

(1) T is a strictly α-admissible mapping

(2) there exists σ_0 in $\mathfrak{I}$ such that $\alpha(\sigma_0, T(\sigma_0)) > 1$

(3) $\mathfrak{I}$ is a strictly α-regular space

(4) F is continuous.

Then, T admits a fixed point.

We omit its proof as it is a mere repetition of the proof of Theorem 1 with some minor modifications.

5. Applications of Theorem 2

5.1. Applications to Fractional Differential Equations. Lacroix (1819) introduced and investigated several applicable properties of fractional differentials. Recently, various new models involving Caputo-Fabrizio derivative (CFD) were discovered and analyzed in [29–31]. We investigate one of these models in p-m-s. We introduce some notations as follows:

Let $\mathscr{C}_{0,1} = \{f \mid f : [0,1] \to \mathbb{R} \text{ and } f \text{ is continuous}\}$. Define the metric function $d : \mathscr{C}_{0,1} \times \mathscr{C}_{0,1} \to [0,\infty)$ by

$$d(f,g) = \|f - g\|_{\infty} = \max_{\nu \in [0,1]} |f(\nu) - g(\nu)|, \text{for all } f, g \in \mathscr{C}_{0,1}. \tag{88}$$

Then, the space $(\mathscr{C}_{0,1}, d)$ is a complete metric space. The function $\alpha : \mathscr{C}_{0,1} \times \mathscr{C}_{0,1} \to (1,\infty)$ by

$$\alpha(r,t) = e^{\|r+t\|_{\infty}} \text{ for all } r, t \in \mathscr{C}_{0,1}. \tag{89}$$

Let $K_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We shall investigate the following CFDE:

$$^{C}D^{\beta} f(\nu) = K_1(\nu, f(\nu)), \tag{90}$$

with boundary conditions

$$\sigma(0) = 0, I\sigma(1) = \sigma'(0). \tag{91}$$

Here, $^{C}D^{\beta}$ denotes CFD of order β defined by

$$^{C}D^{\beta} K_1(\nu) = \frac{1}{\Gamma(n-\beta)} \int_0^{\nu} (\nu - \eta)^{n-\beta-1} K_1^{n}(\eta)\Big)\, d\eta, \tag{92}$$

where

$$n-1<\beta<n \text{and} n=[\beta]+1, \tag{93}$$

and $I^{\beta}K_1$ is given by

$$I^{\beta}K_1(\nu)=\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta)\,d\eta, \text{with} \beta>0. \tag{94}$$

Then, the equation (90) can be modified to

$$\begin{aligned} f(\nu)&=\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,f(\eta))\,d\eta\\ &+\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,f(u))\,du d\eta, \end{aligned} \tag{95}$$

Theorem 3. *Equation (90) admits a solution in $\mathscr{C}_{0,1}$ provided*

(I) *there exists $\tau>0$ such that for all $\sigma,\varsigma\in\mathscr{C}_{0,1}$, we have*

$$|K_1(\eta,\sigma(\eta))-K_1(\eta,\varsigma(\eta))|\le\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}|\sigma(\eta)-\varsigma(\eta)| \tag{96}$$

(II) *there exists $\sigma_0\in\mathscr{C}_{0,1}$ such that for all $\nu\in[0,1]$, we have*

$$\begin{aligned} \sigma_0(\nu)&\le\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma_0(\eta))\,d\eta\\ &+\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\sigma_0(u))\,du d\eta. \end{aligned} \tag{97}$$

Proof. Consistent with the notations introduced above and defining the mapping $R:\mathscr{C}_{0,1}\to\mathscr{C}_{0,1}$ by

$$\begin{aligned} R(\sigma(\nu))&=\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma(\eta))\,d\eta\\ &+\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\sigma(u))\,du d\eta. \end{aligned} \tag{98}$$

By (II), there exists $\sigma_0\in\mathscr{C}_{0,1}$ such that $\sigma_n=t^n(\sigma_0))$. The continuity of function K_1 leads to the continuity of mapping t on $\mathscr{C}_{0,1}$. It is easy to verify the assumptions (1)-(4) in Theorem 2. In the following, we verify the contractive condition (87) of Theorem 2.

$$|R(\sigma(\nu))-R(\varsigma(\nu))|=\left|\begin{array}{l}\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma(\eta))\,d\eta\\ -\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\varsigma(\eta))\,d\eta\\ +\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\sigma(u))\,du d\eta\\ -\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\varsigma(u))\,du d\eta\end{array}\right| \text{implies}$$

$$\begin{aligned} |R(\sigma(\nu))-R(\varsigma(\nu))|&\le\left|\int_0^{\nu}\left(\frac{1}{\Gamma(\beta)}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma(\eta))\right.\right.\\ &\left.\left.-\frac{1}{\Gamma(\beta)}(\nu-\eta)^{\beta-1}K_1(\eta,\varsigma(\eta))\right)d\eta\right|\\ &+\left|\int_0^1\int_0^{\eta}\left(\frac{2}{\Gamma(\beta)}(\eta-u)^{\beta-1}K_1(\eta,\sigma(\eta))\right.\right.\\ &\left.\left.-\frac{2}{\Gamma(\beta)}(\eta-u)^{\beta-1}K_1(u,\varsigma(u))\right)du d\eta\right|\\ &\le\frac{1}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}\cdot\int_0^{\nu}(\nu-\eta)^{\beta-1}(\sigma(\eta)-\varsigma(\eta))\,d\eta\\ &+\frac{2}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}\cdot\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}(\varsigma(u)-\sigma(u))\,du d\eta\\ &\le\frac{1}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}\cdot d(\sigma,\varsigma)\cdot\int_0^{\nu}(\nu-\eta)^{\beta-1}\,d\eta\\ &+\frac{2}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta)\cdot\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)\Gamma(\alpha)\cdot\Gamma(\beta+1)}\cdot d(\sigma,\varsigma)\\ &\cdot\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}\,du d\eta\le\left(\frac{e^{-\tau}\Gamma(\beta)\cdot\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)\Gamma(\beta)\cdot\Gamma(\beta+1)}\right)\\ &\cdot d(\sigma,\varsigma)+2e^{-\tau}B(\beta+1,1)\frac{\Gamma(\beta)\cdot\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)\Gamma(\beta)\cdot\Gamma(\beta+1)}\\ &\cdot d(\sigma,\varsigma)\le\frac{e^{-\tau}}{4\alpha(\sigma,\varsigma)}d(\sigma,\varsigma)+\frac{e^{-\tau}}{2\alpha(\sigma,\varsigma)}d(\sigma,\varsigma), \end{aligned} \tag{99}$$

where B is the beta function. The last inequality can be written by that

$$\alpha(\sigma,\varsigma)d(R(\sigma),R(\varsigma))\le e^{-\tau}d(\sigma,\varsigma). \tag{100}$$

Let us define the metric d on $\mathscr{C}_{0,1}$ by

$$d(\sigma,\varsigma)=\begin{cases}\mathscr{P}(\sigma,\varsigma)=\|\sigma-\varsigma\|_{\infty}+l(l\ge 0) & \text{if } \sigma\ne\varsigma\\ 0 & \text{if } \sigma=\varsigma.\end{cases} \tag{101}$$

Thus, (100) can be written as

$$\alpha(\sigma,\varsigma)\mathscr{P}(R(\sigma),R(\varsigma))\le e^{-\tau}\mathscr{P}(\sigma,\varsigma). \tag{102}$$

Define the functions $\varphi\in\mathscr{C}_{\varphi}$ and F by

$$\varphi_a\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,t(\sigma)),\mathscr{P}(\varsigma,t(\varsigma)),\frac{\mathscr{P}(\varsigma,t(\sigma))+\mathscr{P}(\sigma,t(\varsigma))}{2}\right)$$
$$=\max\left\{\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,t(\sigma)),\mathscr{P}(\varsigma,t(\varsigma)),\frac{\mathscr{P}(\varsigma,t(\sigma))+\mathscr{P}(\sigma,t(\varsigma))}{2}\right\},\tag{103}$$

$F(\sigma(\nu))=\ln(\sigma(\nu))$ for all $\sigma,\varsigma\in\mathscr{C}_{0,1}$. Under these definitions, the inequality (102) gets the form

$$\tau+F(\alpha(\sigma,\varsigma)\mathscr{P}(R(\sigma),R(\varsigma)))\le F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)),\mathscr{P}(\varsigma,T(\varsigma)),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma))+\mathscr{P}(\sigma,T(\varsigma))}{2}\end{array}\right)\right).\tag{104}$$

Hence, by Theorem 2, the self-mapping t admits a fixed point, and hence, the equation (90) has a solution.

5.2. Applications to the Matrix Equations. In this section, by Theorem 2, we shall investigate study the existence of the solutions to

$$X=\mathbb{D}+\frac{1}{m+\theta}\left(\sum_{i=1}^{m}\mathbb{W}_i^*X\mathbb{W}_i+\sum_{i=1}^{m}\mathbb{G}_i^*X\mathbb{G}_i\right),\tag{105}$$

where $\theta\in(0,1)$, $\mathbb{D}\in\mathscr{P}^{(m)}$ (set of $m\times m$ positive definite matrices), and $\mathbb{W}_i,\mathbb{G}_i$ are arbitrary $m\times m$ matrices for each i and are entries of block matrices given by

$$\mathbb{W}=\begin{bmatrix}\mathbb{W}_1\\ \mathbb{W}_2\\ \mathbb{W}_3\\ \vdots\\ \mathbb{W}_m\end{bmatrix}\mathbb{G}=\begin{bmatrix}\mathbb{G}_1\\ \mathbb{G}_2\\ \mathbb{G}_3\\ \vdots\\ \mathbb{G}_m\end{bmatrix}.\tag{106}$$

Let $\mathbb{W}_Z\in\mathscr{Z}^{(m)}$ (set of $m\times m$ Hermitian matrices) be an arbitrary matrix; then, its eigenvalues $e_1,e_2,e_3,\cdots e_m$ are real. Moreover, if $\mathbb{W}_Z\in\mathscr{Z}_+^{(m)}$, then the eigenvalues are nonnegative. Let the functional $\|.\|_{tr}:\mathscr{Z}^{(m)}\to\mathbb{R}$ be defined by

$$\|\mathbb{W}_Z\|_{tr}=\sum_{i=1}^{m}|e_i|.\tag{107}$$

Let $X\in\mathscr{P}^{(m)}$ be arbitrary and define $\|\mathbb{W}_Z\|_{tr,X}=\|X^{1/2}\mathbb{W}_ZX^{1/2}\|_{tr}$. By ([32], Theorem IX.2.2), $(\mathscr{Z}^{(m)},\|.\|_{tr,X})$ is a Banach space (see also [33–35]). Hence, $(\mathscr{Z}^{(m)},d)$ is a complete metric space. The induced metric $d:\mathscr{Z}^{(m)}\times\mathscr{Z}^{(m)}\to\mathbb{R}$ is defined by

$$d(\mathbb{W}_Z,\mathbb{G}_Z)=\|\mathbb{W}_Z-\mathbb{G}_Z\|_{tr,X}\text{forall}\mathbb{W}_Z,\mathbb{G}_Z\in\mathscr{Z}^{(m)}.\tag{108}$$

To establish the existence result we need the following lemma.

Lemma 2 [35]. *If $\mathbb{W}_Z,\mathbb{G}_Z\in\mathscr{Z}_+^{(m)}$, then*

$$0\le Tr(\mathbb{W}_Z\mathbb{G}_Z)\le\|\mathbb{W}_Z\|Tr(\mathbb{G}_Z).\tag{109}$$

Define the operator $\mathscr{E}:\mathscr{Z}^{(m)}\to\mathscr{Z}^{(m)}$ by

$$\mathscr{E}(U)=\mathbb{G}+\frac{1}{m+\theta}\left(\sum_{i=1}^{m}\mathbb{W}_i^*U\mathbb{W}_i+\sum_{i=1}^{m}\mathbb{G}_i^*U\mathbb{G}_i\right),\text{forall}U\in\mathscr{Z}^{(m)}.\tag{110}$$

Remark 4. Since $\mathscr{E}(U)-\mathbb{G}\in\mathscr{P}^{(m)}$ for all $U\in\mathscr{Z}^{(m)}$, in particular, we have $\mathscr{E}(\mathbb{G})-\mathbb{G}\in\mathscr{P}^{(m)}$. The operator $\mathscr{E}$ is continuous on $\mathscr{Z}^{(m)}$.

The solution of the matrix equation (105) is the fixed point of the operator $\mathscr{E}$.

Theorem 4. *Let X and Y be two positive definite matrices such that $\sum_{i=1}^{m}\mathbb{W}_i^*X\mathbb{W}_i\prec 1/2X$ and $\sum_{i=1}^{m}\mathbb{G}_i^*X\mathbb{G}_i\prec 1/2X$. Then, the operator $\mathscr{E}$ has a fixed point in $\mathscr{Z}^{(m)}$.*

Proof. Let U and V be any matrices in $\mathscr{P}^{(m)}$. We observe that the operator $\mathscr{E}$ and the space $(\mathscr{Z}^{(m)},\|.\|_{tr,X})$ fulfill the assumptions (1)-(4) in Theorem 2. To prove that $\mathscr{E}$ is an φ F-contraction, we proceed with

$$\begin{aligned}
&\|\mathscr{E}(V)-\mathscr{E}(U)\|_{tr,X}=tr\left(X^{1/2}(\mathscr{E}(V)-\mathscr{E}(U))X^{1/2}\right)\\
&=tr\left(\frac{1}{m+\theta}\sum_{i=1}^{m}\left\{X^{1/2}(\mathbb{W}_i^*(V-U)\mathbb{W}_i+\mathbb{G}_i^*(V-U)\mathbb{G}_i)X^{1/2}\right\}\right)\\
&=tr\left(\frac{1}{m+\theta}\sum_{i=1}^{m}\left\{X^{1/2}(\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}\right\}+\frac{1}{m+\theta}\sum_{i=1}^{m}\left\{X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_i)X^{1/2}\right\}\right)\\
&=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\left(X^{1/2}\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}+X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_iX^{1/2}\right)=\frac{1}{m+\theta}\sum_{i=1}^{m}\\
&\cdot\left\{tr\left(X^{1/2}\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}\right)+tr\left(X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_iX^{1/2}\right)\right\}=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\cdot\left(X^{1/2}\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}\right)+\frac{1}{m+\theta}\sum_{i=1}^{m}tr\left(X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_iX^{1/2}\right)\\
&=\frac{1}{m+\theta}\sum_{i=1}^{m}tr(\mathbb{W}_iX\mathbb{W}_i^*(V-U))+\frac{1}{m+\theta}\sum_{i=1}^{m}tr(\mathbb{G}_iX\mathbb{G}_i^*(V-U))=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\cdot\left(\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}X^{-1/2}\right)+\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\cdot\left(\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}X^{-1/2}\right)=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\cdot\left(X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)+\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\cdot\left(X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)=\frac{1}{m+\theta}tr\\
&\cdot\left(\sum_{i=1}^{m}X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)+\frac{1}{m+\theta}tr\\
&\cdot\left(\sum_{i=1}^{m}X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)\le\frac{1}{m+\theta}\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}\right\|\\
&\cdot\|V-U\|_{tr,X}+\frac{1}{m+\theta}\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}\right\|\|V-U\|_{tr,X}\text{ by Lemma2}\\
&\cdot=\frac{1}{m+\theta}\left(\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}\right\|+\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}\right\|\right)\|V-U\|_{tr,X}.
\end{aligned}\tag{111}$$

Given $\sum_{i=1}^{m} \mathbb{W}_i^* X \mathbb{W}_i \prec 1/2X$, $\sum_{i=1}^{m} \mathbb{G}_i^* X \mathbb{G}_i \prec 1/2X$, and letting K be a number such that

$$K = \left\| \sum_{i=1}^{m} X^{-1/2} \mathbb{W}_i X \mathbb{W}_i^* X^{-1/2} \right\| + \left\| \sum_{i=1}^{m} X^{-1/2} \mathbb{G}_i X \mathbb{G}_i^* X^{-1/2} \right\|$$

< 1, we have

$$\|\mathscr{E}(V) - \mathscr{E}(U)\|_{tr,X} \leq \frac{K}{m+\theta} \|V - U\|_{tr,X}. \quad (112)$$

Thus,

$$\frac{m+\theta}{K} d(\mathscr{E}(V), \mathscr{E}(U)) \leq K d(V, U). \quad (113)$$

We define $\alpha : \mathscr{Z}^{(m)} \times \mathscr{Z}^{(m)} \to (1,\infty)$ by

$$\alpha(U, V) = m + \theta \text{ for all } U, V \in \mathscr{Z}^{(m)} \text{ and } \theta \in (0, 1), \quad (114)$$

and the metric d on $\mathscr{Z}^{(m)}$ by

$$d(\mathbb{W}_i, \mathbb{G}_i) = \begin{cases} \mathscr{P}(\mathbb{W}_i, \mathbb{G}_i) & \text{if } \mathbb{W}_i \neq \mathbb{G}_i;\ \sum_{i=1}^{m} \mathbb{W}_i^* X \mathbb{W}_i \prec \frac{1}{2}X \text{ and } \sum_{i=1}^{m} \mathbb{G}_i^* X \mathbb{G}_i \prec \frac{1}{2}X \\ 0 & \text{if } \mathbb{W}_i = \mathbb{G}_i, \end{cases} \quad (115)$$

In view of the metric defined above, the inequality (113) can be written as

$$\begin{aligned} &\frac{(m+\theta)\mathscr{P}(\mathscr{E}(V), \mathscr{E}(U))}{K} \leq \mathscr{P}(V, U) \\ &\ln\left(\frac{\alpha(U, V)\mathscr{P}(\mathscr{E}(V), \mathscr{E}(U))}{K}\right) \leq \ln\left(\mathscr{P}(V, U)\right). \end{aligned} \quad (116)$$

Define the functions $\varphi \in \mathscr{C}_\varphi$ and F by

$$\begin{aligned} &\varphi_a\left(\mathscr{P}(U, V), \mathscr{P}(U, \mathscr{E}(U)), \mathscr{P}(V, \mathscr{E}(V)), \frac{\mathscr{P}(V, \mathscr{E}(U)) + \mathscr{P}(U, \mathscr{E}(V))}{2}\right) \\ &= \max\left\{\mathscr{P}(U, V), \mathscr{P}(U, \mathscr{E}(U)), \mathscr{P}(V, \mathscr{E}(V)), \frac{\mathscr{P}(V, \mathscr{E}(U)) + \mathscr{P}(U, T(V))}{2}\right\}, \end{aligned} \quad (117)$$

$F(\sigma) = \ln(\sigma)$ for all $\sigma \in (0,\infty)$, respectively. Under these definitions, we have

$$\begin{aligned} &\tau + F(\alpha(U, V)\mathscr{P}(\mathscr{E}(V), \mathscr{E}(U))) \leq F(\mathscr{P}(V, U)) \text{put} \tau = \ln\left(K^{-1}\right) \\ &\leq F\left(\max\left\{\mathscr{P}(U, V), \mathscr{P}(U, \mathscr{E}(U)), \mathscr{P}(V, \mathscr{E}(V)), \frac{\mathscr{P}(V, \mathscr{E}(U)) + \mathscr{P}(U, \mathscr{E}(V))}{2}\right\}\right). \end{aligned} \quad (118)$$

By Theorem 2, the operator $\mathscr{E}$ has a fixed point, and hence, the matrix equation (105) has a solution.

Remark 5. The numerical explanation of the conditions $\sum_{i=1}^{m} \mathbb{W}_i^* X \mathbb{W}_i \prec 1/2X$ and $\sum_{i=1}^{m} \mathbb{G}_i^* X \mathbb{G}_i \prec 1/2X$ imposed in Theorem 4 for $i = 2$ and taking 4×4 matrices is as follows:

$$\begin{aligned} \text{let} \mathbb{W}_1 &= \begin{bmatrix} 0.1 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.1 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.1 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.1 \end{bmatrix} \mathbb{W}_2 \\ &= \begin{bmatrix} 0.5 & -0.02 & -0.02 & -0.02 \\ -0.02 & 0.5 & -0.02 & -0.02 \\ -0.02 & -0.02 & 0.5 & -0.02 \\ -0.02 & -0.02 & -0.02 & 0.5 \end{bmatrix}. \end{aligned} \quad (119)$$

Then, for a matrix

$$X = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}, \quad (120)$$

we have

$$\sum_{i=1}^{2} \mathbb{W}_i^* X \mathbb{W}_i = \begin{bmatrix} 0.2662 & 0.0479 & 0.0479 & 0.0479 \\ 0.0479 & 0.2662 & 0.0479 & 0.0479 \\ 0.0479 & 0.0479 & 0.2662 & 0.0479 \\ 0.0479 & 0.0479 & 0.0479 & 0.2662 \end{bmatrix} \prec \frac{1}{2}X.$$

$$\begin{aligned} \text{Similarly, let } \mathbb{G}_1 &= \begin{bmatrix} 0.01 & 0.001 & 0.01 & 0.01 \\ 0.001 & 0.01 & 0.01 & 0.001 \\ 0.01 & 0.001 & 0.001 & 0.01 \\ 0.001 & 0.01 & 0.001 & 0.001 \end{bmatrix} \mathbb{G}_2 \\ &= \begin{bmatrix} 0.1413 & 0.008294 & 0.1413 & 0.1413 \\ 0.008294 & 0.0997 & 0.008294 & 0.1413 \\ 0.1413 & 0.008294 & 0.1413 & 0.0997 \\ 0.1109 & 0.1413 & 0.008294 & 0.0997 \end{bmatrix}. \end{aligned} \quad (121)$$

Then, for a matrix

$$X = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}, \quad (122)$$

we have

$$\sum_{i=1}^{2} \mathbb{G}_i^* X \mathbb{G}_i = \begin{bmatrix} 0.0744 & 0.0359 & 0.0570 & 0.0.0760 \\ 0.0359 & 0.0376 & 0.0191 & 0.0491 \\ 0.0570 & 0.0191 & 0.0502 & 0.0579 \\ 0.0760 & 0.0491 & 0.0579 & 0.0.0946 \end{bmatrix} \prec \frac{1}{2} X. \tag{123}$$

6. Conclusion

The introduced contractions encompass the F-contractions and multivalued contractions and hence the Banach contractions, Kannan contractions, Chatterjea contractions, Reich contractions, Hardy-Rogers contractions, and Ciric-type contractions (both metric and p-m versions). It is a real generalization of Matthews contractions and F-contractions. The theorems give general criteria for the existence of the uniqueness of the fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work.

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Research Article

Some Results on Iterative Proximal Convergence and Chebyshev Center

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In this paper, we prove a sufficient condition that every nonempty closed convex bounded pair (M, N) in a reflexive Banach space B satisfying Opial's condition has proximal normal structure. We analyze the relatively nonexpansive self-mapping T on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$, to show that Ishikawa's and Halpern's iteration converges to the best proximity point. Also, we prove that under relatively isometry self-mapping T on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$, Ishikawa's iteration converges to the best proximity point in the collection of all Chebyshev centers of N relative to M. Some illustrative examples are provided to support our results.

1. Introduction and Preliminaries

Let M and N be nonempty subsets of a Banach space B. A mapping $T : M \cup N \to M \cup N$ satisfying $\|Ts - Tt\| \le \|s - t\|$ (respectively, $\|Ts - Tt\| = \|s - t\|$) for all $s \in M$, $t \in N$ is called relatively nonexpansive mapping (respectively, relatively isometry mapping). For more results on relatively nonexpansive (respectively, relatively isometry) mappings, readers can see the research papers in [1, 2] and references therein.

For any two nonempty bounded subsets M and N of a Banach space B, we denote some notations as follows:

$$\begin{aligned} R(s, N) &:= \sup \{\|s - t\| : t \in N\}, \\ M_0 &:= \{s \in M : \|s - t\| = \operatorname{dist}(M, N) \text{for some } t \in N\}, \\ N_0 &:= \{t \in N : \|s - t\| = \operatorname{dist}(M, N) \text{ for some } s \in M\}, \end{aligned} \tag{1}$$

where $\operatorname{dist}(M, N) := \inf \{\|s - t\| : s \in M \text{ and } t \in N\}$. Here, it is to note that if $M \cap N \neq \varnothing$, then $M_0 = N_0 = M \cap N$.

Let M be a nonempty convex subset of a normed linear space X, and let $T : M \to M$ be a mapping with $\operatorname{Fix}(T) \neq \varnothing$, where $\operatorname{Fix}(T) = \{s \in M : Ts = s\}$. A set M is said to have approximate fixed point property (AFPP) if the nonexpansive mapping T has an approximate fixed point sequence, that is, a sequence $\{p_n\}$ in M satisfies $\lim_{n \to +\infty} \|Tp_n - p_n\| = 0$.

Definition 1 [3]. A normed space X is said to be uniformly convex (or uniformly rotund) if and only if for every $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that $(\|s + t\|/2) \le 1 - \delta$ whenever s, $t \in X$ implies $\|s\| = 1$, $\|t\| = 1$, and $\|s - t\| \ge \varepsilon$.

Definition 2 [4]. A nonempty convex subset M of a Banach space B is said to have normal structure if for any nonempty convex closed bounded subset S of M with $\operatorname{diam}(S) > 0$ there exists $s \in S$ such that $R(s, S) < \operatorname{diam}(S)$, where $\operatorname{diam}(S) = \operatorname{diam}(S, S) = \sup \{R(s, S): s \in S\}$.

Eldred et al. [1] introduced the notions of proximal pair and proximal normal structure.

Definition 3 [1]. *A nonempty pair (M, N) of a normed linear space X is known as a proximal pair if, for every $(s, t) \in M \times N$*

,*there exists* $(s', t') \in M \times N$ *such that*

$$\|s - t'\| = \operatorname{dist}(M, N) = \|s' - t\|. \quad (2)$$

A nonempty convex pair (M, N) in a Banach space B is said to have proximal normal structure if $(M_1, N_1) \subseteq (M, N)$ is a closed bounded convex pair for which $\operatorname{dist}(M_1, N_1) = \operatorname{dist}(M, N)$ and $\operatorname{diam}(M_1, N_1) > \operatorname{dist}(M_1, N_1)$, there exists $(s_1, t_1) \in M_1 \times N_1$ such that

$$\begin{aligned} R(s_1, N_1) &< \operatorname{diam}(M_1, N_1) \\ &= \sup\{R(s_1, N_1) \colon s_1 \in M_1\} \text{ and } R(t_1, M_1) \\ &< \operatorname{diam}(M_1, N_1). \end{aligned} \quad (3)$$

Here, it is to note that every nonempty convex weakly compact pair in a uniformly convex Banach space has proximal normal structure. If $M = N$, then proximal normal structure becomes normal structure of Definition 2.

Definition 4 [5]. *A proximal pair* (M, N) *in a Banach space* B *is known as a proximal parallel pair if*

(1) for every element (s, t) in $M \times N$, there exists a unique element (s_1, t_1) in $M \times N$ such that $\|s - t_1\| = \|t - s_1\| = \operatorname{dist}(M, N)$ and

(2) $N = M + h$, where h is a unique element in B

Further, Espinola [5] proved the following lemma.

Lemma 5 [5]. *If* (M, N) *is a nonempty proximal pair in a strictly convex Banach space* B *, then proximal pair* (M, N) *is a proximal parallel pair.*

Definition 6 [6]. *The nonempty proximal parallel pair* (M, N) *in a Banach space* B *is said to have rectangle property if for any* $s, t \in M$,

$$\|s + h - t\| = \|t + h - s\|, \quad (4)$$

where $h \in B$ and $N = M + h$.

Eldred et al. [1] proved the following result.

Theorem 7 [1]. *Let* (M, N) *be a nonempty closed bounded convex pair in a uniformly convex Banach space* B *. Let* T *be a relatively nonexpansive self-mapping on* $M \cup N$ *satisfying* $T(M) \subseteq M$ *and* $T(N) \subseteq N$ *. Let* $s_0 \in M$ *be an initial point, and define a sequence (Krasnoselskii's iteration formula) by* $s_{n+1} = (s_n + Ts_n/2)$, $n \geq 0$ *. Then,* $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$ *. If* $T(M)$ *is a subset of some compact set in* B *, then the limit point of* $\{s_n\}$ *under the norm topology is the best proximity point of* T.

It is ascertained that the geometric property, that is, proximal normal structure, was used in the following result of Eldred et al. [1].

Theorem 8 [1]. *Let* B *be a strictly convex Banach space, and let* (M, N) *be a nonempty weakly compact convex pair having proximal normal structure. Let* T *be a self-mapping on* $M \cup N$ *satisfying*

$$T(M) \subseteq M, T(N) \subseteq N \text{ and} \|Ts - Tt\| \leq \|s - t\| \text{for all } s \in M, t \in N, \quad (5)$$

then T *has fixed points* $s \in M$, $t \in N$, *and* $\|s - t\| = dist(M, N)$.

Definition 9 [7]. *Let* M *be a nonempty convex subset of a real Hilbert space* H *, and let* T *be a self-mapping on* M *. Let* $s_0 \in M$ *be an initial point, and* $\{s_n\}$ *is a sequence defined by*

$$s_{n+1} = (1 - \xi_n)s_n + \xi_n T t_n, t_n = (1 - \eta_n)s_n + \eta_n T s_n, \quad (6)$$

where $0 \leq \xi_n \leq 1$, $0 \leq \eta_n \leq 1$, $n \geq 0$.

The iterative sequence defined in (6) is called Ishikawa's iteration. If $\eta_n = 0$, then Ishikawa's iteration sequence reduces to Mann's iteration sequence. Eldred and Praveen [8] generalized and extended Theorem 7 of Eldred et al. [1] by using Mann's iteration method.

Definition 10 [9]. *Let* M *be a nonempty convex subset of a real Hilbert space* H *, and let* T *be a self-mapping on* M *. Fix* $u \in M$ *. Let* $p_0 \in M$ *be an initial point, and a sequence* $\{p_n\}$ *is defined by*

$$p_{n+1} = \xi_n u + (1 - \xi_n)Tp_n, \quad 0 \leq \xi_n \leq 1, n \geq 0. \quad (7)$$

The iterative sequence defined in (7) is called Halpern's iteration.

The following interesting result will be used extensively in the sequel.

Proposition 11 [10]. *Let* X *be a uniformly convex normed linear space,* $0 < \alpha < 1$ *, and* $\varepsilon > 0$ *. For any* $r > 0$ *, if* $s, t \in X$ *are such that* $\|s\| \leq r$, $\|t\| \leq r$, $\|s - t\| \geq \varepsilon$ *, then there exists* $\delta = \delta(\varepsilon/r) > 0$ *such that*

$$\|\alpha s + (1 - \alpha)t\| \leq \left(1 - 2\delta\left(\frac{\varepsilon}{r}\right) \min\{\alpha, 1 - \alpha\}\right) r. \quad (8)$$

Almezel et al. [11] modified the result of Xu [12] in the following way.

Lemma 12 [11, 12]. *Let* $\{x_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$x_{n+1} \leq (1 - \eta_n)x_n + \eta_n \nu_n, n \geq 0, \quad (9)$$

where $\{\eta_n\}$ *and* $\{\nu_n\}$ *satisfy the following conditions.*

(1) $\{\eta_n\}$ *is a sequence in* $]0, 1[$, *where* $\sum_{n=1}^{+\infty} \eta_n = +\infty$

(2) $\{\nu_n\}$ *is a sequence in* $\mathbb{R}$; *either* $\limsup_{n\to+\infty} \eta_n \leq 0$ *or* $\sum_{n=0}^{+\infty} |\eta_n \nu_n| < +\infty$

Then, $x_n \to 0$ *as* $n \to +\infty$.

Let M and N be nonempty bounded subsets of a Banach space B. The number $R(M,N) := \inf\{R(s,N): s \in M\}$ is the Chebyshev radius of N relative to M and $C_M(N) := \{s \in M : R(s,N) = R(M,N)\}$ is the set of all Chebyshev centers of N relative to M. Since the function R is convex and continuous on X, R is lower semicontinuous with respect to the weak topology. Consequently, if M is a nonempty weakly compact convex set, then $C_M(N)$ is a nonempty convex weakly compact subset of M. Rajesh and Veeramani [2] proved the following proposition.

Proposition 13 [2]. *Let* (M,N) *be a nonempty convex weakly compact proximal parallel pair in a Banach space* B *. Let the nonempty pair* (M,N) *have the rectangle property. Then,* $R(s,N) = R(s+h,M)$ *for* $s \in M$ *, and* $R(t,M) = R(t-h,N)$ *for* $t \in N$ *. Moreover,* $C_N(M) = C_M(N) + h$.

Definition 14 [13]. *Let B be a Banach space. We say that B satisfies Opial's condition if for any sequence* $\{p_n\}$ *in B converges weakly to some* s *, then* $\limsup_{n\to+\infty}\|p_n - p\| > \limsup_{n\to+\infty}\|p_n - s\|$ *for all* $p \neq s \in B$ *. If a reflexive Banach space B satisfies Opial's condition, then B has a normal structure.*

Proposition 15 (demiclosed principle [13]). *Let B be a Banach space, and let M be a nonempty weakly compact subset of B . Also, let T be a nonexpansive self-mapping on M with Fix* $(T) \neq \varnothing$ *. If a sequence* $\{p_n\}$ *in M converges weakly to s and a sequence* $\{(I-T)p_n\}$ *converges strongly to* p *, then* $(I-T)s = p$ *. Moreover, if* $p = 0$ *, then* $I - T$ *is demiclosed at zero.*

We need the following result of Dutta and Veeramani [14] to prove Proposition 17.

Theorem 16 [14]. *If a nonempty convex pair* (M,N) *in a Banach space B does not have a proximal normal structure, then there exist sequences* $\{s_n\} \subset M$, $\{t_n\} \subset N$ *such that* $\|s_n - t_n\| = dist(M,N)$ *for all* n, $\|s_m - t_n\| > dist(M,N)$ *for some* m, n *and*

$$\lim_{n\to+\infty} dist(s_{n+1}, conv\{t_1, t_2, \cdots, t_n\}) = diam(\{s_n\}, \{t_n\}), \tag{10}$$

or

$$\lim_{n\to+\infty} dist(t_{n+1}, conv\{s_1, s_2, \cdots, s_n\}) = diam(\{s_n\}, \{t_n\}), \tag{11}$$

where $diam(\{s_n\},\{t_n\}) = diam(\{s_1, \cdots, s_n, \cdots\}, \{t_1, \cdots, t_n, \cdots\})$.

2. Opial's Condition and Ishikawa's Iteration for Relatively Nonexpansive Mappings

The geometrical property, that is, the proximal normal structure, is the sufficient condition for the existence of the best proximity [1]. For details about the best proximity point, one can see research papers in [1, 2, 5, 15–19]. We now prove the following result, which shows that the above condition can be dropped if a reflexive Banach space satisfies Opial's condition.

Proposition 17. *Every closed bounded convex pair* (M,N) *in reflexive Banach space B satisfying Opial's condition has proximal normal structure.*

Proof. Suppose the pair (M,N) does not have a proximal normal structure. Then, by Theorem 16, there exist sequences $\{s_n\} \subset M$, $\{t_n\} \subset N$ such that $\|s_n - t_n\| = \mathrm{dist}(M,N)$ for all n, $\|s_m - t_n\| > \mathrm{dist}(M,N)$ for some m, n, and $\lim_{n\to+\infty} \mathrm{dist}(t_{n+1}, \mathrm{conv}\{s_1, s_2, \cdots, s_n\}) = R(\{s_n\},\{t_n\})$. Let the sequence $\{s_n\}$ converges weakly to 0. Therefore, $0 \in \overline{\mathrm{conv}}\{s_1, s_2, \cdots, s_n\}$.

Suppose $s \in \mathrm{conv}\{s_1, s_2, \cdots, s_n\}$, then $\lim_{n\to+\infty}\|s - t_n\| = R(\{s_n\},\{t_n\})$, and the same holds as $s \in \overline{\mathrm{conv}}\{s_1, s_2, \cdots, s_n\}$. Therefore, when taking $s = 0$, we get $\lim_{n\to+\infty}\|t_n\| = R(\{s_n\},\{t_n\})$, and $\lim_{n\to+\infty}\|s_1 - t_n\| = R(\{s_n\},\{t_n\})$, which is a contradiction, hence the result.

After analyzing the theorems, definitions, lemma, and propositions mentioned above, we have some impressive new results herewith.

Theorem 18. *Let* (M,N) *be a nonempty convex closed bounded proximal pair of B , a uniformly convex Banach space. Let T be a relatively nonexpansive self-mapping on* $M \cup N$ *satisfying* $T(M) \subseteq M$ *and* $T(N) \subseteq N$ *. Let* $s_0 \in M$ *be an initial point, and a sequence* $\{s_n\}$ *is defined as*

$$\begin{aligned} s_{n+1} &= (1-\xi_n)s_n + \xi_n T t_n, t_n = (1-\eta_n)s_n \\ &\quad + \eta_n T s_n, \theta < \eta_n \le \xi_n < 1-\theta, 0 < \theta \le \frac{1}{2}, \lim_{n\to+\infty}\xi_n\eta_n = 0. \end{aligned} \tag{12}$$

Then, $\lim_{n\to+\infty}\|Ts_n - s_n\| = 0$*. If* $T(M)$ *is a subset of a compact set, then the limit point of* $\{s_n\}$ *under the norm topology is the best proximity point of* T.

Proof. If $\mathrm{dist}(M,N) = 0$, then it is not necessary to discuss. Suppose $\mathrm{dist}(M,N) > 0$, then by applying the result of Theorem 8, there exists $t \in N$ such that $Tt = t$. Since

$$\begin{aligned} \|s_{n+1} - t\| &\le (1-\xi_n)\|s_n - t\| + \xi_n\|Tt_n - t\| \\ &\le (1-\xi_n)\|s_n - t\| + \xi_n(1-\eta_n)\|s_n - t\| \\ &\quad + \xi_n\eta_n\|Ts_n - t\| \le \|s_n - t\|, \end{aligned} \tag{13}$$

$\{\|s_n - t\|\}$ is a nonincreasing sequence, there exists $k > 0$ such that $\lim_{n\to+\infty} \|s_n - t\| = k$.

Suppose $\lim_{n\to+\infty} \|Ts_n - s_n\| \neq 0$, then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that

$$\|s_{n_i} - Ts_{n_i}\| \geq \varepsilon > 0. \tag{14}$$

Let $\alpha \in]0, 1[$ and ε_1 such that $\varepsilon/\alpha > k$ and $0 < \varepsilon_1 < \min\{(\varepsilon/\alpha) - k, (k\delta(\alpha)/(1 - \delta(\alpha)))\}$. Since B is a uniformly convex Banach space, the modulus of convexity function $\delta(.)$ is strictly increasing and continuous. Hence, $0 < \delta(\alpha) < \delta(\varepsilon/(k + \varepsilon_1))$. So, we can choose a small positive number $\varepsilon_1 > 0$ such that $(1 - a\delta(\varepsilon/(k + \varepsilon_1)))(k + \varepsilon_1) < k$, where $a > 0$.

Let $\|s_{n_i} - t\| \leq k + \varepsilon_1$ and $\|t_{n_i} - t\| \leq k + \varepsilon_1$ for some i. Now,

$$\begin{aligned} \|t - Tt_{n_i}\| &\leq \|t - t_{n_i}\| = \left\|t - \left\{\left(1 - \eta_{n_i}\right)s_{n_i} + \eta_{n_i} Ts_{n_i}\right\}\right\| \\ &= \left\|\left(1 - \eta_{n_i}\right)\left(t - s_{n_i}\right) + \eta_{n_i}\left(t - Ts_{n_i}\right)\right\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right) \min\left\{\eta_{n_i}, 1 - \eta_{n_i}\right\}\right)(k + \varepsilon_1) \\ &\leq \left(1 - a_1\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1), \end{aligned} \tag{15}$$

where $0 < a_1 \leq 2 \min\{\eta_{n_i}, 1 - \eta_{n_i}\}$. Further

$$\begin{aligned} \|t - s_{n_{i+1}}\| &= \left\|t - \left\{(1 - \xi_{n_i})s_{n_i} + \xi_{n_i} Tt_{n_i}\right\}\right\| \\ &= \left\|(1 - \xi_{n_i})\left(t - s_{n_i}\right) + \xi_{n_i}\left(t - Tt_{n_i}\right)\right\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right) \min\left\{\xi_{n_i}, 1 - \xi_{n_i}\right\}\right)(k + \varepsilon_1) \\ &\leq \left(1 - a_2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1), \end{aligned} \tag{16}$$

where $0 < a_2 \leq 2 \min\{\xi_{n_i}, 1 - \xi_{n_i}\}$.

By choosing $\varepsilon_1 > 0$ as small as we wish, we get

$$\max\left\{\left(1 - a_1\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1), \left(1 - a_2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1)\right\} < k, \tag{17}$$

which is a contradiction. Hence, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$ and $\lim_{n\to+\infty} \|s_{n+1} - s_n\| = 0$.

If $T(M)$ is compact, then the sequence $\{s_n\}$ has a subsequence $\{s_{n_i}\}$ such that $\lim_{i\to+\infty} s_{n_i} = s \in M$. Since (M, N) is a proximal pair, there exists $v \in N$ such that $\|s - v\| = \operatorname{dist}(M, N)$.

Now, we have $\lim_{i\to+\infty} \|s_{n_i} - v\| = \operatorname{dist}(M, N)$, and $\{\|s_n - v\|\}$ is a nonincreasing sequence; it implies that $\lim_{n\to+\infty} \|s_n - v\| = \operatorname{dist}(M, N)$. This shows that $\lim_{n\to+\infty} s_n = s \in M$. By strict convexity of the norm, $\lim_{n\to+\infty} \|Ts_n - v\| = \operatorname{dist}(M, N)$ and $\|Ts_n - Tv\| \leq \|s_n - v\| \to \operatorname{dist}(M, N)$ as $n \to +\infty$ give $Tv = v$. Since $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$, it follows that $Ts = s$.

We obtain the following result from Theorem 18 by taking $\eta_n = 0$ for $n \in \mathbb{N}$.

Corollary 19 [8]. *Let (M, N) be a nonempty convex closed bounded proximal pair of B , a uniformly convex Banach space, and let T be a relatively nonexpansive self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$. Let $s_0 \in M$ be an initial point, and a sequence $\{s_n\}$ is defined as*

$$s_{n+1} = (1 - \xi_n)s_n + \xi_n Ts_n, \varepsilon < \xi_n < 1 - \varepsilon, 0 < \varepsilon \leq \frac{1}{2}(\text{Mann's iteration}). \tag{18}$$

Then, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$. Moreover, if $T(M)$ is a subset of a compact set, then the limit point of $\{s_n\}$ under norm topology is the best proximity point of T.

3. Halpern's Iteration and Relatively Nonexpansive Mapping

Let M be a nonempty subset of a real Hilbert space H, and let $P_M : H \to 2^M$ be the nearest point projection mapping from H onto M that is, $P_M(s) := \{s' \in M : \|s' - s\| = \operatorname{dist}(s, M)\}$. If M is nonempty convex closed, then P_M is nonexpansive giving unique image for all s in H, and hence by Kolmogorov's criterion $\langle P_M t - s, P_M t - t \rangle \leq 0$ for all $t \in X, s \in M$. Here, we use the following notation $M_{\mathrm{Fix}T} = \{s \in M : Ts = s\}$.

Theorem 20. *Let (M, N) be a nonempty closed bounded convex proximal pair of a real Hilbert space H, and let T be a relatively nonexpansive self-mapping on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. Let $0 < \xi_n < 1$, and $s_0 \in M$ be an initial point. A sequence $\{s_n\}$ is defined as*

$$s_{n+1} = \xi_n u + (1 - \xi_n)Ts_n, \tag{19}$$

where $u \in M$ such that $\|s_n - t\| \geq \|u - t\|$ for all $t \in N$.

If $\lim_{n\to+\infty} \xi_n = 0$, $\sum_{n=1}^{+\infty} \xi_n = +\infty$, and either $\sum_{n=1}^{+\infty} |\xi_{n+1} - \xi_n| < +\infty$ or $\lim_{n\to+\infty} (\xi_n/\xi_{n+1}) = 1$, then the sequence $\{s_n\}$ under the norm topology converges to $s \in M_{\mathrm{Fix}T}$, closest to point u such that $\|s - t\| = \operatorname{dist}(M, N)$ for some $t \in N_{\mathrm{Fix}T}$.

Proof. By applying Theorem 8, it is found that there exists $t \in N$ such that $Tt = t$. Now, we have

$$\begin{aligned} \|s_{n+1} - t\| &= \|\xi_n u + (1 - \xi_n)Ts_n - t\| \leq \xi_n\|u - t\| \\ &\quad + (1 - \xi_n)\|s_n - t\| \leq \xi_n\|u - t\| \\ &\quad + (1 - \xi_n)\|s_n - t\| \leq \xi_n\|s_n - t\| \\ &\quad + (1 - \xi_n)\|s_n - t\| (\text{since}\|s_n - t\| \geq \|u - t\|). \end{aligned} \tag{20}$$

Hence $\{\|s_n - t\|\}$ is nonincreasing and $\lim_{n\to+\infty}\|s_n - t\| = k > 0$.

Suppose $\lim_{n\to+\infty}\|Ts_n - s_n\| \neq 0$, then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $\|s_{n_i} - Ts_{n_i}\| \geq \varepsilon > 0$. Since H is a Hilbert space (and hence uniformly convex space), it is possible to choose a small positive number $\varepsilon_1 > 0$, such that $(1 - a\delta(\varepsilon/(k+\varepsilon_1)))(k+\varepsilon_1) < k$, where $a > 0$.

Let $\|s_{n_i} - t\| \leq k + \varepsilon_1$ for some i. Now,

$$\begin{aligned}\|t - s_{n_{i+1}}\| &= \|t - \{\xi_{n_i}u + (1-\xi_{n_i})Ts_{n_i}\}\| \\ &= \|(1-\xi_{n_i})(t - Ts_{n_i}) + \xi_{n_i}(t-u)\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\min\{\xi_{n_i}, 1-\xi_{n_i}\}\right)(k+\varepsilon_1) \\ &\leq \left(1 - a_1\delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\right)(k+\varepsilon_1),\end{aligned} \tag{21}$$

where $0 < a_1 \leq 2\min\{\xi_{n_i}, 1-\xi_{n_i}\}$.

By choosing $\varepsilon_1 > 0$ as small as we wish, we have

$$\left(1 - a_1\delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\right)(k+\varepsilon_1) < k, \tag{22}$$

which is a contradiction. Hence, $\lim_{n\to+\infty}\|Ts_n - s_n\| = 0$ and $\lim_{n\to+\infty}\|s_{n+1} - s_n\| = 0$.

Let $\{s_{n_i}\}$ be a subsequence of $\{s_n\}$ such that

$$\limsup_{n\to+\infty} < s_n - s, s - u > \; = \limsup_{i\to+\infty} < s_{n_i} - s, s - u > . \tag{23}$$

Without loss of generality, we assume that subsequence $\{s_{n_i}\}$ converges weakly to $p \in M$ such that $\|p - t\| = \mathrm{dist}(M, N)$ for some $t \in N_{\mathrm{Fix}T}$. Since $\lim_{n\to+\infty}\|Ts_n - s_n\| = 0$, by applying the demiclosed principle, we have $p \in M_{\mathrm{Fix}T}$. Hence, by applying Kolmogorov's criterion, we have

$$\limsup_{i\to+\infty} < s_{n_i} - s, s - u > \; = \; < p - s, s - u > \geq 0. \tag{24}$$

Now, we have

$$\begin{aligned}\|s_{n+1} - s\|^2 &= <\xi_n u + (1-\xi_n)Ts_n - s, s_{n+1} - s> \\ &\leq \xi_n s < u - s, s_{n+1} - s > + (1-\xi_n)\|s_n - s\|.\|s_{n+1} - s\| \\ &\leq \frac{(1-\xi_n)}{2}\left(\|s_n - s\|^2 + \|s_{n+1} - s\|^2\right) \\ &\quad + \xi_n < u - s, s_{n+1} - s > .\end{aligned} \tag{25}$$

Hence,

$$\begin{aligned}\Rightarrow \|s_{n+1} - s\|^2 &\leq \frac{2\xi_n}{1+\xi_n} < u - s, s_{n+1} - s > + \left(1 - \frac{2\xi_n}{1+\xi_n}\right)\|s_n - s\|^2 \\ &= (1-\eta_n)\|s_n - s\|^2 + \xi_n\nu_n,\end{aligned} \tag{26}$$

where $\eta_n = 2\xi_n/(1+\xi_n)$ and $\nu_n = (2/(1+\xi_n)) < u - s, s_{n+1} - s >$.

Since $\sum_{n=1}^{+\infty}\eta_n = +\infty$ and $\limsup_{n\to+\infty}\nu_n \leq 0$, by Lemma 12, we have $\lim_{n\to+\infty} s_n = s \in M_{\mathrm{Fix}T}$, closest to point u so that $\|s - t\| = \mathrm{dist}(M, N)$ for some $t \in N_{\mathrm{Fix}T}$.

We obtain the following corollary from Theorem 20 when $M = N$.

Corollary 21 [9]. *Let M be nonempty closed bounded convex subsets of a real Hilbert space H and T be a nonexpansive self-mapping on M. Let $s_0 \in M$ be an initial point, and $\{s_n\}$ is a sequence defined as*

$$s_{n+1} = \xi_n u + (1-\xi_n)Ts_n, \tag{27}$$

where $u \in M$ and $0 < \xi_n < 1$ (Halpern's iteration).

If $\lim_{n\to+\infty}\xi_n = 0$, $\sum_{n=1}^{+\infty}\xi_n = +\infty$, and either $\sum_{n=1}^{+\infty}|\xi_{n+1} - \xi_n| < +\infty$ or $\lim_{n\to+\infty}\xi_n/\xi_{n+1} = 1$, then the sequence $\{s_n\}$ under the norm topology converges to $s \in M_{\mathrm{Fix}T}$, closest to point u.

4. Ishikawa's Iteration and Chebyshev Centre

Lim et al. [20] proved the following interesting theorem in the year 2003, by using the geometrical property, viz., normal structure.

Theorem 22 [20]. *Let B be a Banach space, and let T be an isometry self-mapping on M, a nonempty weakly compact convex subset of B. It is assumed that M has a normal structure. Then, there exists $s \in C(M) = C_M(M)$, the set of all Chebyshev centers of M such that $Ts = s$.*

Let (M, N) be a nonempty convex weakly compact proximal parallel pair in a Banach space B. Suppose the pair (M, N) has the rectangle property. Let $T : M \cup N \to M \cup N$ be a relatively isometry mapping satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. It is ascertained that $T(C_M(N)) \subseteq C_M(N)$ if and only if $R(s, N) = R(Ts, N) = R(M, N)$ for all $s \in C_M(N)$. Similarly, $T(C_N(M)) \subseteq C_N(M)$ if and only if $R(t, M) = R(Tt, M) = R(M, N)$ for all $t \in C_N(M)$. It is affirmed that $C_N(M) = C_M(N) + h$ for some $h \in B$ (for details, see [2, 21, 22]). We establish the following result.

Lemma 23. *Let (M, N) be a nonempty weakly compact convex proximal pair in a strictly convex Banach space B. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. If $s \in M$ and $\{T^n s\}$ has a Cauchy subsequence in M, then $R(s, N) = R(Ts, N)$. Similarly, if $t \in$*

N and $\{T^n t\}$ has a Cauchy subsequence in N, then $R(t, M) = R(Tt, M)$.

Proof. Let $s \in M$. Then,

$$R(s, N) = R(Ts, TN) \le R(Ts, N). \tag{28}$$

Let $(s, t) \in M \times N$ such that $\|s - t\| = \mathrm{dist}(M, N)$. Suppose $\|T^{j_n}s - T^{i_n}t\| < \mathrm{dist}(M, N) + 1/n$, where $i_n, j_n \in \mathbb{Z}^+$, with $i_n < j_n$, for every $n \in \mathbb{Z}^+$. Since T is a relatively isometry mapping, we get $\lim_{n\to+\infty} T^{j_n - i_n}s = s$.

Let $\{a_n\}$ be a nondecreasing subsequence of $\{j_n - i_n\}$. Since R is a nonnegative continuous real valued function, then the sequence $\{R(T^{a_n}s, N)\}$ is nondecreasing, and $\lim_{n\to+\infty} T^{a_n}s = s$. Therefore, $\lim_{n\to+\infty} R(T^{a_n}s, N) = R(s, N)$. Thus,

$$R(Ts, N) \le R(T^{a_1}s, N) \le \lim_{n\to+\infty} R(T^{a_n}s, N) = R(s, N). \tag{29}$$

From, (28) and (29), we have $R(s, N) = R(Ts, N)$. Similarly, we can show that $R(t, M) = R(Tt, M)$.

Lemma 24. *Let (M, N) be a nonempty weakly compact convex proximal parallel pair in a strictly convex Banach space B. It is assumed that the pair (M, N) has the rectangle property. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. If $(C_M(N), C_N(M))$ is nonempty and contained in a totally bounded proximal parallel pair (M_1, N_1) of (M, N) such that $T(M_1) \subseteq M_1$ and $T(N_1) \subseteq N_1$, then $T(C_M(N)) \subseteq C_M(N)$ and $T(C_N(M)) \subseteq C_N(M)$.*

Proof. Let $s \in C_M(N)$, where $C_M(N) \subseteq M_1$, $T(M_1) \subseteq M_1$, and $C_N(M) = C_M(N) + h$, for some $h \in B$. Then, $\{T^n s : n \in \mathbb{Z}^+\} \subseteq M_1$, and $\{T^n(s + h): n \in \mathbb{Z}^+\} \subseteq N_1$.

As (M_1, N_1) is a totally bounded proximal pair, the sequences $\{T^n(s)\}$ and $\{T^n(s + h)\}$, respectively, have Cauchy subsequences in M_1 and N_1. So, by Lemma 23, we have $R(s, N) = R(M, N) = R(Ts, N)$.

Hence, $T(C_M(N)) \subseteq C_M(N)$. Similarly, $T(C_N(M)) \subseteq C_N(M)$.

Example 25. Let $X = (\mathbb{R}^2, \|.\|_1)$. Let $M = \{(s, 10 - 10s): s \in [0, 1]\}$, and $N = M + h$, where $h = (0, 1) \in X$. Let $(s, 10 - 10s) \in M$ and $(1 + t, 10 - 10t) \in N$, where $s, t \in [0, 1]$. Now, we have

$$\begin{aligned}\|(1 + t, 10 - 10t) - (s, 10 - 10s)\|_1 &= \|(1 + t - s, 10s - 10t)\|_1 \\ &= 1 + t - s + |10s - 10t| \\ &= \begin{cases} 1 + 9s - 9t, & \text{if } t \le s, \\ 1 + 11t - 11s, & \text{if } s \le t. \end{cases}\end{aligned} \tag{30}$$

In particular, take $(1, 0), (0, 10) \in M$, and $(2, 0), (1, 10) \in N$, we have $\|\{(1, 0) + (1, 0)\} - (0, 10)\|_1 = 12$ and $\|\{(0, 10) + (1, 0)\} - (1, 0)\|_1 = 10$.

It shows that there exists a proximal parallel pair (M, N) with $\mathrm{dist}(M, N) = \|h\|_1 = 1$ which does not satisfy the rectangle property.

Theorem 26. *Let (M, N) be a nonempty totally bounded convex closed proximal pair in a uniformly convex (and hence reflexive) Banach space B. It is also assumed that the pair (M, N) has the rectangle property. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$. Let $s_0 \in C_M(N)$ be an initial point, and a sequence $\{s_n\}$ is defined as*

$$\begin{aligned} s_{n+1} &= (1 - \xi_n)s_n + \xi_n Tt_n,\ t_n = (1 - \eta_n)s_n + \eta_n Ts_n,\ 0 \le \eta_n \\ &\le \xi_n < 1,\ \lim_{n\to+\infty} \xi_n\eta_n = 0, -\theta, 0 < \theta \le \frac{1}{2} \end{aligned} \tag{31}$$

Then, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$. If $T(C_M(N))$ is a subset of a compact set, then the limit point $s \in C_M(N)$ of the sequence $\{s_n\}$ under norm topology is the best proximity point of T.

Proof. It is easy to see that $(C_M(N), C_N(M))$ is a nonempty convex weakly compact proximal parallel pair having the rectangle property in a uniformly convex Banach space B.

Since (M, N) is totally bounded and T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$, by applying Lemma 24, we have $T(C_M(N)) \subseteq C_M(N)$ and $T(C_N(M)) \subseteq C_N(M)$.

Now, by Theorem 8, there exist $s \in C_M(N)$ and $t \in C_N(M)$ such that $Ts = s$, $Tt = t$, and $\|s - t\| = \mathrm{dist}(M, N)$.

By applying Theorem 18, it is found that the sequence $\{s_n\}$ under norm topology converges to $Ts = s \in C_M(N)$, such that $\|s - t\| = \mathrm{dist}(M, N)$ for some $t \in N_{\mathrm{Fix}T}$.

We obtain the following result from Theorem 26 if $\eta_n = 0$ for $n \in \mathbb{N}$.

Theorem 27. *Let (M, N) be a nonempty totally bounded convex closed proximal pair in a uniformly convex (and hence reflexive) Banach space B. It is also assumed that (M, N) has the rectangle property. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$. Let $s_0 \in C_M(N)$ be an initial point, and a sequence $\{s_n\}$ is defined as*

$$s_{n+1} = (1 - \xi_n)s_n + \xi_n Ts_n,\ \varepsilon < \xi_n < 1 - \varepsilon,\ 0 < \varepsilon \le \frac{1}{2}. \tag{32}$$

Then, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$. If $T(C_M(N))$ is a subset of a compact set, then the limit point $s \in C_M(N)$ of $\{s_n\}$ under the norm topology is the best proximity point of T.

Proof. The result is similar to that of Theorem 26.

Example 28. Let $X = (\mathbb{R}^2, \|.\|)$, a Euclidean space. Let

$$\begin{aligned} M &= \{(s, t): s = -2, -1 \le t \le 1\}, \\ N &= \{(s, t): s = 2, -1 \le t \le 1\}. \end{aligned} \tag{33}$$

Here, (M,N) is a proximal parallel pair having the rectangle property, $R(M,N)=\sqrt{17}$, $C_M(N)=\{(-2,0)\}$, $C_N(M)=\{(2,0)\}$, and $C_N(M)=C_M(N)+h$, where $h=(4,0)$.

Define

$$T: M \to M \text{ by } T(s,t)=(T_1 s, T_2 t)=(-2,-t), \tag{34}$$

where $T_1:\{-2\}\to\{-2\}$ and $T_2:[-1,1]\to[-1,1]$.

Let $(s,t)\in M$, and $(s',t')\in N$. Then

$$\begin{aligned}\|T(s,t)-T\left(s',t'\right)\| &= \|(-2,-t)-\left(2,-t'\right)\| \\ &= \|\left(-4,t'-t\right)\| \\ &= \sqrt{(-4)^2+\left(t'-t\right)^2} \\ &= \|(s,t)-\left(s',t'\right)\|.\end{aligned} \tag{35}$$

Hence, T is a relatively isometry (and hence relatively nonexpansive) mapping on $M\cup N$ satisfying $T(N)\subseteq N$ and $T(M)\subseteq M$.

From Theorem 18, we take the initial point $(s,t)\in M$ and set $s_1=(1-\xi_0)s+\xi_0 T_1 s'_0$ and $s'_0=(1-\eta_0)s+\eta_0 T_1 s$. We have $T_1 s=-2$. Since $s=-2$, we obtain $s'_0=-2$ which implies $s_1=-2$.

Similarly, set $s_2=(1-\xi_1)s_1+\xi_1 T_1 s'_1$ and $s'_1=(1-\eta_1)s_1+\eta_1 T_1 s_1$. Since $s_1=-2$, we obtain $s'_1=-2$ which implies $s_2=-2$. In general, we obtain $s_{n+1}=-2$. Therefore, $s_n\to -2$ as $n\to+\infty$.

Again, set $t_1=(1-\xi_0)t+\xi_0 T_2 t'_0$ and $t'_0=(1-\eta_0)t+\eta_0 T_2 t$. Since $T_2 t=-t$, we obtain $t'_0=(1-2\eta_0)t$ which implies $t_1=(1-2\xi_0+2\xi_0\eta_0)t$. Similarly, set $t_2=(1-\xi_1)t_1+\xi_1 T_2 t'_1$ and $t'_1=(1-\eta_1)t_1+\eta_1 T_2 t_1$. Since $T_2 t_1=-t_1$, we obtain

$$\begin{aligned}t'_1 &= (1-\eta_1)t_1+\eta_1 T_2 t_1 \\ &= (1-\eta_1)(1-2\xi_0+2\xi_0\eta_0)t+\eta_1 T_2[(1-2\xi_0+2\xi_0\eta_0)t] \\ &= (1-\eta_1)(1-2\xi_0+2\xi_0\eta_0)t-\eta_1(1-2\xi_0+2\xi_0\eta_0)t \\ &= (1-2\eta_1)(1-2\xi_0+2\xi_0\eta_0)t,\end{aligned} \tag{36}$$

which implies

$$\begin{aligned}t_2 &= (1-\xi_1)t_1+\xi_1 T_2 t'_1 \\ &= (1-\xi_1)(1-2\xi_0+2\xi_0\eta_0)t-\xi_1(1-2\eta_1)(1-2\xi_0+2\xi_0\eta_0)t \\ &= (1-2\xi_0+2\xi_0\eta_0)(1-2\xi_1+2\xi_1\eta_1)t.\end{aligned} \tag{37}$$

In general, $t_{n+1}=(1-2\xi_0+2\xi_0\eta_0)(1-2\xi_1+2\xi_1\eta_1)\cdots(1-2\xi_n+2\xi_n\eta_n)t$. Therefore, $t_n\to 0$ as $n\to+\infty$. Hence, $\lim_{n\to+\infty}(s_n,t_n)=(-2,0)$, a fixed point of T. In a similar way, if $(s',t')\in N$, then $\lim_{n\to+\infty}(s'_n,t'_n)=(2,0)$, a fixed point of T and $\|(-2,0)-(2,0)\|=\mathrm{dist}(M,N)$.

From Theorem 26, if we take the initial point $(x,y)\in C_M(N)$, then it is trivial that $\lim_{n\to+\infty}(s_n,t_n)=(-2,0)$, a fixed point of T. In a similar way, if $(s',t')\in C_N(M)$, then $\lim_{n\to+\infty}(s'_n,t'_n)=(2,0)$, a fixed point of T and $\|(-2,0)-(2,0)\|=\mathrm{dist}(M,N)$.

5. Open Problem

Let (M,N) be a nonempty weak compact convex pair in a Banach space (or Hilbert space) B. Can Ishikawa's iteration and Halpern's iteration converge to the best proximity point of relatively nonexpansive (or relatively isometry) mapping $T: M\cup N\to M\cup N$ satisfying $T(M)\subseteq N$ and $T(N)\subseteq M$?

6. Conclusion

If a reflexive Banach space satisfies Opial's condition, then every bounded convex pair (M,N) has a proximal normal structure. Also, we show that Ishikawa's and Halpern's iterative sequences converge to the best proximity point. Finally, we show that Ishikawa's iterative sequence converges to the best proximity point, which is a Chebyshev center.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally in writing this article.

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Research Article

Kannan Prequasi Contraction Maps on Nakano Sequence Spaces

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In this article, we explore the concept of the prequasi norm on Nakano special space of sequences (sss) such that its variable exponent in $(0, 1]$. We evaluate the sufficient setting on it with the definite prequasi norm to configuration prequasi Banach and closed (sss). The Fatou property of different prequasi norms on this (sss) has been investigated. Moreover, the existence of a fixed point of Kannan prequasi norm contraction maps on the prequasi Banach (sss) and the prequasi Banach operator ideal constructed by this (sss) and $s-$ numbers have been examined.

1. Introduction

Ideal maps and summability theorems [1–6] are extremely significant in mathematical models and have more achievements, such as ideal transformations, normal series, fixed point theory, geometry of Banach spaces, and approximation theory. By $\mathfrak{R}^{\mathcal{N}}$, we mark the spaces of all sequences of real numbers. We denote the space of all bounded linear maps from a Banach space Z into a Banach space M by $\mathscr{L}(Z, M)$, and if $Z = M$, we indicate $\mathscr{L}(Z)$, the d-th s number by $s_d(W)$ [7], the d-th approximation number by $\alpha_d(W)$, and $e_d = \{0, 0, \cdots, 1, 0, 0, \cdots\}$, where 1 shows at the d^{th} place, for every $d \in \mathcal{N} = \{0, 1, 2, \cdots\}$.

Notations 1. The sets S_A, $S_A(Z, M)$, S_A^{app} and $S_A^{\text{app}}(Z, M)$, (cf. [8]) denote

$$\begin{aligned} S_A &:= \{S_A(Z,M)\}, \quad \text{where } S_A(Z,M) \\ &:= \left\{W \in \mathscr{L}(Z,M):\ \left((s_d(W))_{d=0}^{\infty} \in A\right\}. \quad \text{Also} \\ S_A^{\text{app}} &:= \left\{S_A^{\text{app}}(Z,M)\right\}, \quad \text{where } S_A^{\text{app}}(Z,M) \\ &:= \left\{W \in \mathscr{L}(Z,M):\ \left((\alpha_d(W))_{d=0}^{\infty} \in A\right\}. \end{aligned} \tag{1}$$

Let $r = (r_a) \in (0, 1]^{\mathcal{N}}$, the Nakano sequence space defined and studied in [9–11] is denoted by:

$$\ell(r) = \left\{ \nu = (\nu_a) \in \mathfrak{R}^{\mathcal{N}} : \phi(\mu\nu) < \infty, \quad \text{for any} \quad \mu > 0 \right\}, \tag{2}$$

where $\phi(\nu) = \sum_{a\in\mathcal{N}} |\nu_a|^{r_a}$. And $(\ell(r), \|.\|)$ is a Banach space, however, $\|\nu\| = \inf\{\kappa > 0 : \phi(\nu/\kappa) \le 1\}$. Faried and Bakery [8] assumed the hypothesis of prequasi operator ideal that is more established than the quasi operator ideal. Bakery and Abou Elmatty [9] demonstrated the strictly inclusion of the prequasi operator ideal $S_{\ell(r)}^{\text{app}}$, for inconsistent powers. It was a small prequasi operator ideal. As the literature of the Banach fixed point theorem [12], many mathematicians created on many actions. Haghi et al. [13, 14] showed that some generalizations in fixed point theory are not real generalizations and investigated some fixed point generalizations to partial metric spaces, which are obtained from the corresponding results in metric spaces. Kannan [15] presented a representation of a class of operators with the same fixed point actions as contractions nevertheless that fails to be continuous. They only try to illustrate Kannan maps [16] in modular vector spaces. The target of this paper is to appraise the concept of prequasi norm on $\ell(r)$. The Fatou property of different prequasi norms on this (sss) has been examined. We are delving the sufficient set-up on $\ell(r)$ equipped with the definite prequasi norm to pattern prequasi Banach and closed (sss). The existence of a fixed point of Kannan prequasi norm contraction mapping on the prequasi Banach (sss) has been given. Finally, the existence of a fixed point of Kannan prequasi norm contraction mapping on the prequasi Banach operator ideal $S_{(\ell(r))_\phi}$ has been made current.

2. Definitions and Preliminaries

Definition 2 (see [2]). The linear space of sequences $\mathfrak{A}$ is detailed as a special space of sequences (sss), if

$$\{e_a\}_{a\in\mathcal{N}} \subseteq \mathfrak{A}, \quad (3)$$

(1) $\mathfrak{A}$ is solid, i.e., let $v=(v_a)\in\mathfrak{R}^{\mathcal{N}}$, $t=(t_a)\in\mathfrak{A}$, and $|v_a|\le|t_a|$, for every $a\in\mathcal{N}$, then $v\in\mathfrak{A}$

(2) $(v_{[a/2]})_{a=0}^{\infty}\in\mathfrak{A}$, where $[a/2]$ marks the integral part of $a/2$, if $(v_a)_{a=0}^{\infty}\in\mathfrak{A}$

Definition 3 (see [8]). A subclass $\mathfrak{A}_\phi$ of $\mathfrak{A}$ is definite a premodular (sss), if there is $\phi\in[0,\infty)^{\mathfrak{A}}$ verifying the set-up:

(i) For $v\in\mathfrak{A}$, $v=\theta\Leftrightarrow\phi(v)=0$ with $\phi(v)\ge 0$, where θ is the zero vector of $\mathfrak{A}$

(ii) For every $v\in\mathfrak{A}$ and $\eta\in\mathfrak{R}$, we have $B\ge 1$ for which $\phi(\eta v)\le B|\eta|\phi(v)$,

(iii) $\phi(v+t)\le J(\phi(v)+\phi(t))$, for each $v,t\in\mathfrak{A}$, for some $J\ge 1$

(iv) For $a\in\mathcal{N}$ and $|v_a|\le|t_a|$, then $\phi((v_a))\le\phi((t_a))$

(v) The inequality, $\phi((v_a))\le\phi((v_{[a/2]}))\le J_0\phi((v_a))$ holds, for some $J_0\ge 1$

(vi) Assume F be the space of finite sequences, then $\bar{F}=\mathfrak{A}_\phi$

(vii) There is $\varsigma>0$ such that $\phi(\beta,0,0,0,\cdots)\ge\varsigma|\beta|\phi(1,0,0,0,\cdots)$, for every $\beta\in\mathfrak{R}$

Definition 4 (see [17]). Suppose $\mathfrak{A}$ be a (sss). The function $\phi\in[0,\infty)^{\mathfrak{A}}$ is called prequasi norm on $\mathfrak{A}$, if it provides the conditions (i), (ii), and (iii) of Definition 3.

Theorem 5 (see [17]). *Pick up $\mathfrak{A}$ be a premodular (sss), then it is prequasi normed (sss).*

Theorem 6 (see [17]). *$\mathfrak{A}$ is a prequasi normed (sss), if it is quasinormed (sss).*

Definition 7 (see [3]). Let $\mathscr{L}$ be the class of all bounded linear operators between any two arbitrary Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is named an operator ideal, if every vector $\mathscr{U}(\mathrm{Z},\mathrm{M})=\mathscr{U}\cap\mathscr{L}(\mathrm{Z},\mathrm{M})$ verifies the next setting:

(i) $I_\Gamma\in\mathscr{U}$ where Γ denotes Banach space of one dimension

(ii) The space $\mathscr{U}(Z,M)$ is linear over $\mathfrak{R}$

(iii) Assume $W\in\mathscr{L}(Z_0,Z)$, $X\in\mathscr{U}(Z,M)$, and $Y\in\mathscr{L}(M,M_0)$, then, $YXW\in\mathscr{U}(Z_0,M_0)$, where Z_0 and M_0 are normed spaces (see [18, 19])

The theory of prequasi operator ideal, which is more general than the quasi operator ideal.

Definition 8 (see [8]). A function $\phi\in[0,\infty)^{\mathscr{U}}$ is named a prequasi norm on the ideal $\mathscr{U}$ if the following setting includes

(1) Assume $W\in\mathscr{U}(Z,M)$, $\phi(W)\ge 0$, and $\phi(W)=0\Leftrightarrow W=0$

(2) There is $D\ge 1$ so as to $\phi(\eta W)\le D|\eta|\phi(W)$, for every $W\in\mathscr{U}(Z,M)$ and $\eta\in\mathfrak{R}$

(3) There is $J\ge 1$ such that $\phi(W_1+W_2)\le J[\phi(W_1)+\phi(W_2)]$, for each $W_1,W_2\in\mathscr{U}(Z,M)$,

(4) There is $\sigma\ge 1$ for to if $W\in\mathscr{L}(Z_0,Z)$, $X\in\mathscr{U}(Z,M)$ and $Y\in\mathscr{L}(M,M_0)$, then $\phi(YXW)\le\sigma\|Y\|\phi(X)\|W\|$

Theorem 9 (see [20]). *Pick up $\mathfrak{A}_\phi$ be a premodular (sss), then $\phi(W)=\phi(s_a(W))_{a=0}^{\infty}$ be a prequasi norm on $S_{\mathfrak{A}_\phi}$.*

Theorem 10 (see [9]). *Suppose Z and M be Banach spaces, and $\mathfrak{A}_\phi$ be a premodular (sss), then $(S_{\mathfrak{A}_\phi},\phi)$ be a prequasi Banach operator ideal, such that $\phi(W)=\phi((s_a(W))_{a=0}^{\infty})$.*

Theorem 11 (see [8]). *ϕ is a prequasi norm on the ideal $\mathscr{U}$, if ϕ is a quasinorm on the ideal $\mathscr{U}$.*

The agreeable inequality [21] will be used in the consequence: Suppose $(r_a)\in(0,1]^{\mathcal{N}}$ and $v_a,t_a\in\mathfrak{R}$, for every $a\in\mathcal{N}$, then $|v_a+t_a|^{r_a}\le|v_a|^{r_a}+|t_a|^{r_a}$.

3. Main Results

3.1. Prequasi Normed (sss). We illustrate the adequate set-up on $\ell(r)$ equipped with a prequasi norm ϕ to generate prequasi Banach and closed (sss).

Definition 12. (a) $\{v_a\}_{a\in\mathcal{N}}\subseteq(\ell(r))_\phi$ is ϕ-convergent to $v\in(\ell(r))_\phi\Leftrightarrow\lim_{a\longrightarrow\infty}\phi(v_a-v)=0$. If the ϕ-limit exists, then it is unique

(b) $\{v_a\}_{a\in\mathcal{N}}\subseteq(\ell(r))_\phi$ is ϕ-Cauchy, if $\lim_{a,b\longrightarrow\infty}\phi(v_a-v_b)=0$

(c) $\Lambda\subseteq(\ell(r))_\phi$ is ϕ-closed, if for all ϕ-converging $\{v_a\}_{a\in\mathcal{N}}\subset\Lambda$ to v, then $v\in\Lambda$

Theorem 13. $(\ell(r))_\phi$, *where* $\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}$, *for all* $v\in\ell(r)$, *is a premodular (sss), if* $(r_a)_{a\in\mathcal{N}}\in(0,1]^{\mathcal{N}}$ *is an increasing.*

Proof. First, we have to prove $\ell(r)$ is a (sss):

(1) Suppose $v,t\in\ell(r)$. Since $(r_a)\in(0,1]^{\mathcal{N}}$, we have

$$\phi(v+t)=\sum_{a\in\mathcal{N}}|v_a+t_a|^{r_a}\le\sum_{a\in\mathcal{N}}|v_a|^{r_a}+\sum_{a\in\mathcal{N}}|t_a|^{r_a}=\phi(v)+\phi(t)<\infty, \tag{4}$$

so $v+t\in\ell(r)$.

(2) Assume $\eta\in\mathfrak{R}$ and $v\in\ell(r)$. As $(r_a)\in(0,1]^{\mathcal{N}}$, one has

$$\phi(\eta v)=\sum_{a\in\mathcal{N}}|\eta v_a|^{r_a}\le\sup_a|\eta|^{r_a}\sum_{a\in\mathcal{N}}|v_a|^{r_a}\le D|\eta|\phi(v)<\infty. \tag{5}$$

Hence, $\eta v\in\ell(r)$. So, by using Parts (1) and (2), we get $\ell(r)$ is linear. Also $e_a\in\ell(r)$, for all $a\in\mathcal{N}$, since $\phi(e_a)=\sum_{j=0}^{\infty}|e_a(j)|^{r_j}=1$.

(3) Let $|v_a|\le|t_a|$, for every $a\in\mathcal{N}$ and $t\in\ell(r)$. One can see

$$\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}\le\sum_{a\in\mathcal{N}}|t_a|^{r_a}=\phi(t)<\infty, \tag{6}$$

we have $v\in\ell(r)$. This implies the sequence space $\ell(r)$ is solid.

(4) Suppose $(v_a)\in\ell(r)$ and (r_a) be an increasing sequence, one has

$$\begin{aligned}\phi\left(\left(v_{[a/2]}\right)\right)&=\sum_{a\in\mathcal{N}}\left|v_{[a/2]}\right|^{r_a}=\sum_{a\in\mathcal{N}}|v_a|^{r_{2a}}+\sum_{a\in\mathcal{N}}|v_a|^{r_{2a+1}}\\&\le 2\sum_{a\in\mathcal{N}}|v_a|^{r_a}=2\phi((v_a)),\end{aligned} \tag{7}$$

then $(v_{[a/2]})\in\ell(r)$. Secondly, we show that the functional ϕ on $\ell(r)$ is a premodular:

(i) Evidently, $\phi(v)\ge0$ and $\phi(v)=0\Leftrightarrow v=\theta$

(ii) We have $D=\max\{1,\sup_a|\eta|^{r_a-1}\}\ge1$ such that $\phi(\eta v)\le D|\eta|\phi(v)$, for every $v\in\ell(r)$ and $\eta\in\mathfrak{R}\setminus\{0\}$. For $\eta=0$, there is $D\ge1$ such that $\phi(\eta v)\le D|\eta|\phi(v)$, for every $v\in\ell(r)$

(iii) We have $J\ge1$ so that $\phi(v+t)\le J(\phi(v)+\phi(t))$, for every $v,t\in\ell(r)$

(iv) Clearly, since $\ell(r)$ is solid

(v) From (49), we have $J_0=2\ge1$

(vi) Clearly, $\bar{F}=\ell(r)$

(vii) There is $0<\varsigma\le|\beta|^{r_0-1}$, for $\beta\ne0$ or $\varsigma>0$, for $\beta=0$ such that $\phi(\beta,0,0,0,\cdots)\ge\varsigma|\beta|\phi(1,0,0,0,\cdots)$

Theorem 14. *Assume* $(r_a)\in(0,1]^{\mathcal{N}}$ *be an increasing, then* $(\ell(r))_\phi$ *be a prequasi Banach (sss), where* $\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}$, *for every* $v\in\ell(r)$.

Proof. Let the set-up be verified. From Theorem 13, the space $(\ell(r))_\phi$ is a premodular (sss). By Theorem 5, the space $(\ell(r))_\phi$ is a prequasi normed (sss). To prove that $(\ell(r))_\phi$ is a prequasi Banach (sss), assume $v^p=(v_a^p)_{a=0}^{\infty}$ be a Cauchy sequence in $(\ell(r))_\phi$. Hence, for every $\varepsilon\in(0,1)$, we have $p_0\in\mathcal{N}$ such that for all $p,q\ge p_0$, one has

$$\phi(v^p-v^q)=\sum_{a\in\mathcal{N}}|v_a^p-v_a^q|^{r_a}<\varepsilon. \tag{8}$$

Therefore, for $p,q\ge p_0$ and $a\in\mathcal{N}$, we get $|v_a^p-v_a^q|<\varepsilon$. So (v_a^q) is a Cauchy sequence in $\mathfrak{R}$, for constant $a\in\mathcal{N}$. Which implies $\lim_{q\longrightarrow\infty}v_a^q=v_a^0$, for fixed $a\in\mathcal{N}$. Hence, $\phi(v^p-v^0)<\varepsilon$, for every $p\ge p_0$. Then, to show that $v^0\in\ell(r)$, we have $\phi(v^0)=\phi(v^0-v^p+v^p)\le\phi(v^p-v^0)+\phi(v^p)<\infty$. So $v^0\in\ell(r)$. This explains that $(\ell(r))_\phi$ is a prequasi Banach (sss).

Theorem 15. *Pick up* $(r_a)\in(0,1]^{\mathcal{N}}$ *be an increasing, then* $(\ell(r))_\phi$ *be a prequasi closed (sss), where* $\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}$, *for every* $v\in\ell(r)$.

Proof. Assume the conditions be verified. From Theorem 13, the space $(\ell(r))_\phi$ be a premodular (sss). By Theorem 5, the space $(\ell(r))_\phi$ is a prequasi normed (sss). To show that $(\ell(r))_\phi$ is a prequasi closed (sss), suppose $v^p=(v_a^p)_{a=0}^{\infty}\in(\ell(r))_\phi$ and $\lim_{p\longrightarrow\infty}\phi(v^p-v^0)=0$, then for all $\varepsilon\in(0,1)$, we have $p_0\in\mathcal{N}$ so that for all $p\ge p_0$, we have $\varepsilon>\phi(v^p-v^0)=\sum_{a\in\mathcal{N}}|v_a^p-v_a^0|^{r_a}$. Hence, for $p\ge p_0$ and $a\in\mathcal{N}$, one has $|v_a^p-v_a^0|<\varepsilon$. Therefore, (v_a^p) is a convergent sequence in $\mathfrak{R}$, for fixed $a\in\mathcal{N}$. Hence, $\lim_{p\longrightarrow\infty}v_a^p=v_a^0$, for constant $a\in\mathcal{N}$. Finally, to prove that $v^0\in\ell(r)$, we obtain

$$\phi\left(v^0\right)=\phi\left(v^0-v^p+v^p\right)\le\phi\left(v^p-v^0\right)+\phi\left(v^p\right)<\infty, \tag{9}$$

hence, $v^0\in\ell(r)$. This gives that $(\ell(r))_\phi$ is a prequasi closed (sss).

Example 16. The functional $\phi(\mathrm{v})=\sum_{\mathrm{a}\in\mathcal{N}}|\mathrm{v_a}|^{\mathrm{a}+1/\mathrm{a}+2}$ is a prequasi norm (not a quasinorm) on Nakano special space of sequences $\ell((\mathrm{a}+1/\mathrm{a}+2)_{\mathrm{a}=0}^{\infty})$.

Example 17. The functional $\phi(\mathrm{v})=[\sum_{\mathrm{a}\in\mathcal{N}}|\mathrm{v_a}|^{\mathrm{a}+1/2\mathrm{a}+4}]^4$ is a prequasi norm (not a quasinorm) on Nakano special space of sequences $\ell((\mathrm{a}+1/2\mathrm{a}+4)_{\mathrm{a}=0}^{\infty})$.

Example 18. The functional $\phi(\mathrm{v})=\sum_{\mathrm{a}\in\mathcal{N}}|\mathrm{v_a}|^{\mathrm{r}}$ is a prequasi norm (not a norm) on r-absolutely summable sequences of real numbers ℓ_{r}, for all $0<\mathrm{r}\le1$.

Example 19. For $(r_a) \in (0,1]^{\mathcal{N}}$, the functional $\phi(v) = \inf\{\kappa > 0 : \sum_{a\in\mathcal{N}} |v_a/\kappa|^{r_a} \leq 1\}$ is a prequasi norm (a quasinorm and a norm) on Nakano special space of sequences $\ell(r)$.

4. The Fatou Property

We investigate here the Fatou property of different prequasi norms ϕ on $\ell(r)$.

Definition 20. A prequasi norm ϕ on $\ell(r)$ provides the Fatou property, if for all sequence $\{t^a\} \subseteq (\ell(r))_\phi$ with $\lim_{a\longrightarrow\infty}\phi(t^a - t) = 0$ and any $v \in (\ell(r))_\phi$ then $\phi(v-t) \leq \sup_j \inf_{a\geq j}\phi(v - t^a)$.

Theorem 21. *The function $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$ provides the Fatou property, if $(r_a) \in (0,1]^{\mathcal{N}}$ is an increasing, for all $v \in \ell(r)$.*

Proof. Let the set-up be satisfied and $\{t^b\} \subseteq (\ell(r))_\phi$ with $\lim_{b\longrightarrow\infty}\phi(t^b - t) = 0$. Since the space $(\ell(r))_\phi$ is a prequasi closed space, then $t \in (\ell(r))_\phi$. So for every $v \in (\ell(r))_\phi$, one has

$$\begin{aligned}\phi(v-t) &= \sum_{a\in\mathcal{N}} |v_a - t_a|^{r_a} \leq \sum_{a\in\mathcal{N}} \left|v_a - t_a^b\right|^{r_a} + \sum_{a\in\mathcal{N}} \left|t_a^b - t_a\right|^{r_a} \\ &\leq \sup_j \inf_{b\geq j}\phi\left(v - t^b\right).\end{aligned} \tag{10}$$

Theorem 22. *The function $\phi(v) = [\sum_{a\in\mathcal{N}} |v_a|^{r_a}]^{1/\inf_a r_a}$ does not fulfill the Fatou property, for all $v \in \ell(r)$, if $(r_a) \in (0,1]^{\mathcal{N}}$ with $\inf_a r_a > 0$.*

Proof. Suppose the set-up be confirmed and $\{t^b\} \subseteq (\ell(r))_\phi$ with $\lim_{b\longrightarrow\infty}\phi(t^b - t) = 0$. Since the space $(\ell(r))_\phi$ is a prequasi closed space, then $t \in (\ell(r))_\phi$. Then, for each $v \in (\ell(r))_\phi$, we get

$$\begin{aligned}\phi(v-t) &= \left[\sum_{a\in\mathcal{N}} |v_a - t_a|^{r_a}\right]^{1/\inf_a r_a} \\ &\leq 2^{1/\inf_a r_a - 1}\left(\left[\sum_{a\in\mathcal{N}} \left|v_a - t_a^b\right|^{r_a}\right]^{1/\inf_a r_a} + \left[\sum_{a\in\mathcal{N}} \left|t_a^b - t_a\right|^{r_a}\right]^{1/\inf_a r_a}\right) \\ &\leq 2^{1/\inf_a r_a - 1}\sup_j \inf_{b\geq j}\phi\left(v - t^b\right).\end{aligned} \tag{11}$$

So, ϕ does not indulge the Fatou property.

5. Kannan Prequasi ϕ-Contraction Operator

Now, we explain the definition of Kannan ϕ-contraction mapping on the prequasi normed (sss). We study the sufficient setting on $(\ell(r))_\phi$ constructed with definite prequasi norm so that there is one and only one fixed point of Kannan prequasi norm contraction mapping.

Definition 23. An operator $W : \mathfrak{A}_\phi \longrightarrow \mathfrak{A}_\phi$ is called a Kannan ϕ-contraction, if there is $\xi \in [0, 1/2)$, so that $\phi(Wv - Wt) \leq \xi(\phi(Wv - v) + \phi(Wt - t))$, for all $v, t \in \mathfrak{A}_\phi$.

An element $v \in \mathfrak{A}_\phi$ is named a fixed point of W, if $W(v) = v$.

Theorem 24. *Assume $(r_a) \in (0,1]^{\mathcal{N}}$ be an increasing, and $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ be Kannan ϕ-contraction mapping, where $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$, for all $v \in \ell(r)$, then W has one fixed point.*

Proof. Let the setting be satisfied. For each $v \in \ell(r)$, then $W^p v \in \ell(r)$. As W is a Kannan ϕ-contraction operator, one has

$$\begin{aligned}\phi\left(W^{p+1}v - W^p v\right) &\leq \xi\left(\phi\left(W^{p+1}v - W^p v\right) + \phi\left(W^p v - W^{p-1}v\right)\right) \\ &\Rightarrow \phi\left(W^{p+1}v - W^p v\right) \leq \frac{\xi}{1-\xi}\phi\left(W^p v - W^{p-1}v\right) \\ &\leq \left(\frac{\xi}{1-\xi}\right)^2 \phi\left(W^{p-1}v - W^{p-2}v\right) \\ &\leq \leq \left(\frac{\xi}{1-\xi}\right)^p \phi(Wv - v).\end{aligned} \tag{12}$$

So, for all $p, q \in \mathcal{N}$ with $q > p$, one can see

$$\begin{aligned}\phi\left(W^p v - W^q v\right) &\leq \xi\left(\phi\left(W^p v - W^{p-1}v\right) + \phi\left(W^q v - W^{q-1}v\right)\right) \\ &\leq \xi\left(\left(\frac{\xi}{1-\xi}\right)^{p-1} + \left(\frac{\xi}{1-\xi}\right)^{q-1}\right)\phi(Wv - v).\end{aligned} \tag{13}$$

Therefore, $\{W^p v\}$ is a Cauchy sequence in $(\ell(r))_\phi$. As the space $(\ell(r))_\phi$ is prequasi Banach space. Hence, there is $t \in (\ell(r))_\phi$ so that $\lim_{p\longrightarrow\infty} W^p v = t$. To prove that $Wt = t$. Since ϕ has the Fatou property, we have

$$\begin{aligned}\phi(Wt - t) &\leq \sup_i \inf_{p\geq i}\phi\left(W^{p+1}v - W^p v\right) \\ &\leq \sup_i \inf_{p\geq i}\left(\frac{\xi}{1-\xi}\right)^p \phi(Wv - v) = 0,\end{aligned} \tag{14}$$

hence, $Wt = t$. Then, t is a fixed point of W. To show that the fixed point is unique. Let we have two distinctive fixed points $b, t \in (\ell(r))_\phi$ of W. So, we have

$$\phi(b - t) \leq \phi(Wb - Wt) \leq \xi(\phi(Wb - b) + \phi(Wt - t)) = 0. \tag{15}$$

Therefore, $b = t$.

Corollary 25. *Let $(r_a) \in (0,1]^{\mathcal{N}}$ be an increasing, and $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ be Kannan ϕ-contraction mapping, where $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$, for all $v \in \ell(r)$, then W has unique fixed point b with*

$$\phi(W^p v - b) \le \xi\left(\frac{\xi}{1-\xi}\right)^{p-1} \phi(Wv - v). \tag{16}$$

Proof. Pick up the conditions be satisfied. By Theorem 24, we have a unique fixed point b of W. Hence, one has

$$\begin{aligned}\phi(W^p v - b) &= \phi(W^p v - Wb) \le \xi\left(\phi\left(W^p v - W^{p-1} v\right) + \phi(Wb - b)\right)\\ &= \xi\left(\frac{\xi}{1-\xi}\right)^{p-1}\phi(Wv - v).\end{aligned} \tag{17}$$

Definition 26. Suppose $\mathfrak{A}_\phi$ be a prequasi normed (sss), $W : \mathfrak{A}_\phi \longrightarrow \mathfrak{A}_\phi$ and $b \in \mathfrak{A}_\phi$. The operator W is called ϕ-sequentially continuous at b, if and only if, when $\lim_{a\longrightarrow\infty}\phi(v_a - b) = 0$, then $\lim_{a\longrightarrow\infty}\phi(Wv_a - Wb) = 0$.

Theorem 27. *Pick up $(r_a) \in (0,1]^{\mathcal{N}}$ with $\inf_a r_a > 0$, and $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$, where $\phi(v) = [\sum_{a\in\mathcal{N}} |v_a|^{r_a}]^{1/\inf_a r_a}$, for all $v \in \ell(r)$. The point $g \in (\ell(r))_\phi$ is the only fixed point of W, if the following conditions are satisfied:*

(a) W is Kannan ϕ-contraction mapping

(b) W is ϕ-sequentially continuous at $g \in (\ell(r))_\phi$

(c) There is $v \in (\ell(r))_\phi$ so that the sequence of iterates $\{W^p v\}$ has a subsequence $\{W^{p_i} v\}$ converging to g

Proof. Let the set-up be verified. Suppose g be not a fixed point of W, then $Wg \neq g$. By the set-up (b) and (c), we have

$$\lim_{p_i\longrightarrow\infty}\phi(W^{p_i}v - g) = 0 \text{ and } \lim_{p_i\longrightarrow\infty}\phi\left(W^{p_i+1}v - Wg\right) = 0. \tag{18}$$

As the operator W is Kannan ϕ-contraction, one has

$$\begin{aligned}0 < \phi(Wg - g) &= \phi\left(\left(Wg - W^{p_i+1}v\right) + \left(W^{p_i}v - g\right) + \left(W^{p_i+1}v - W^{p_i}v\right)\right)\\ &\le 2^{2/\inf_a r_a - 2}\phi\left(W^{p_i+1}v - Wg\right) + 2^{2/\inf_a r_a - 2}\phi\left(W^{p_i}v - g\right)\\ &\quad + 2^{1/\inf_a r_a - 1}\xi\left(\frac{\xi}{1-\xi}\right)^{p_i-1}\phi(Wv - v).\end{aligned} \tag{19}$$

As $p_i \longrightarrow \infty$, we have a contradiction. Therefore, g is a fixed point of W. To prove that the fixed point g is unique. Assume we have two different fixed points $g, b \in (\ell(r))_\phi$ of W. So, one can see

$$\phi(g - b) \le \phi(Wg - Wb) \le \xi(\phi(Wg - g) + \phi(Wb - b)) = 0. \tag{20}$$

Therefore, $g = b$.

Example 28. Let $W : (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi \longrightarrow (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi$, where $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{a+1/2a+4}$, for all $v \in \ell((a + 1/2a + 4)_{a=0}^{\infty})$ and

$$W(v) = \begin{cases} \dfrac{v}{18}, & \phi(v) \in [0,1),\\ \dfrac{v}{20}, & \phi(v) \in [1,\infty).\end{cases} \tag{21}$$

Since for all $v_1, v_2 \in (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi$ with $\phi(v_1), \phi(v_2) \in [0,1)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{18} - \frac{v_2}{18}\right) \le \frac{1}{\sqrt[4]{17}}\left(\phi\left(\frac{17v_1}{18}\right) + \phi\left(\frac{17v_2}{18}\right)\right)\\ &= \frac{1}{\sqrt[4]{17}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{22}$$

For all $v_1, v_2 \in (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi$ with $\phi(v_1), \phi(v_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{20} - \frac{v_2}{20}\right) \le \frac{1}{\sqrt[4]{19}}\left(\phi\left(\frac{19v_1}{20}\right) + \phi\left(\frac{19v_2}{20}\right)\right)\\ &= \frac{1}{\sqrt[4]{19}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{23}$$

For all $v_1, v_2 \in (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi$ with $\phi(v_1) \in [0,1)$ and $\phi(v_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{18} - \frac{v_2}{20}\right) \le \frac{1}{\sqrt[4]{17}}\phi\left(\frac{17v_1}{18}\right) + \frac{1}{\sqrt[4]{19}}\phi\left(\frac{19v_2}{20}\right)\\ &\le \frac{1}{\sqrt[4]{17}}\left(\phi\left(\frac{17v_1}{18}\right) + \phi\left(\frac{19v_2}{20}\right)\right)\\ &= \frac{1}{\sqrt[4]{17}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{24}$$

Therefore, the map W is Kannan ϕ-contraction mapping. Since ϕ satisfies the Fatou property. By Theorem 24, the map W has a unique fixed point $\theta \in (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi$.

Let $\{v^{(n)}\} \subseteq (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi$ be such that $\lim_{n\longrightarrow\infty}\phi(v^{(n)} - v^{(0)}) = 0$, where $v^{(0)} \in (\ell((a + 1/2a + 4)_{a=0}^{\infty}))_\phi$ with

$\phi(\nu^{(0)}) = 1$. Since the prequasi norm ϕ is continuous, we have

$$\lim_{n\longrightarrow\infty} \phi\left(W\nu^{(n)} - W\nu^{(0)}\right) = \lim_{n\longrightarrow\infty} \phi\left(\frac{\nu^{(n)}}{18} - \frac{\nu^{(0)}}{20}\right) = \phi\left(\frac{\nu^{(0)}}{180}\right) > 0. \tag{25}$$

Hence, W is not ϕ-sequentially continuous at $\nu^{(0)}$. So, the map W is not continuous at $\nu^{(0)}$.

If $\phi(\nu) = \left[\sum_{a\in\mathcal{N}} |\nu_a|^{a+1/2a+4}\right]^4$, for all $\nu \in \ell((a+1/2a+4)_{a=0}^{\infty})$. Since for all $\nu_1, \nu_2 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\phi(\nu_1)$, $\phi(\nu_2) \in [0, 1)$, we have

$$\begin{aligned}\phi(W\nu_1 - W\nu_2) &= \phi\left(\frac{\nu_1}{18} - \frac{\nu_2}{18}\right) \le \frac{8}{17}\left(\phi\left(\frac{17\nu_1}{18}\right) + \phi\left(\frac{17\nu_2}{18}\right)\right) \\ &= \frac{8}{17}(\phi(W\nu_1 - \nu_1) + \phi(W\nu_2 - \nu_2)).\end{aligned} \tag{26}$$

For all $\nu_1, \nu_2 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\phi(\nu_1), \phi(\nu_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(W\nu_1 - W\nu_2) &= \phi\left(\frac{\nu_1}{20} - \frac{\nu_2}{20}\right) \le \frac{8}{19}\left(\phi\left(\frac{19\nu_1}{20}\right) + \phi\left(\frac{19\nu_2}{20}\right)\right) \\ &= \frac{8}{19}(\phi(W\nu_1 - \nu_1) + \phi(W\nu_2 - \nu_2)).\end{aligned} \tag{27}$$

For all $\nu_1, \nu_2 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\phi(\nu_1) \in [0, 1)$ and $\phi(\nu_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(W\nu_1 - W\nu_2) &= \phi\left(\frac{\nu_1}{18} - \frac{\nu_2}{20}\right) \le \frac{8}{17}\phi\left(\frac{17\nu_1}{18}\right) + \frac{8}{19}\phi\left(\frac{19\nu_2}{20}\right) \\ &\le \frac{8}{17}\left(\phi\left(\frac{17\nu_1}{18}\right) + \phi\left(\frac{19\nu_2}{20}\right)\right) \\ &= \frac{8}{17}(\phi(W\nu_1 - \nu_1) + \phi(W\nu_2 - \nu_2)).\end{aligned} \tag{28}$$

Therefore, the map W is Kannan ϕ-contraction mapping and $W^p(\nu) = \begin{cases} \nu/18^p, & \phi(\nu) \in [0, 1), \\ \nu/20^p, & \phi(\nu) \in [1,\infty). \end{cases}$

It is clear that W is ϕ-sequentially continuous at $\theta \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ and $\{W^p\nu\}$ has a subsequence $\{W^{p_i}\nu\}$ converging to θ. By Theorem 27, the point $\theta \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ is the only fixed point of W.

Example 29. Let $W : (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi \longrightarrow (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$, where $\phi(\nu) = \left[\sum_{a\in\mathcal{N}} |\nu_a|^{a+1/2a+4}\right]^4$, *for all* $\nu \in \ell((a+1/2a+4)_{a=0}^{\infty})$ and

$$W(\nu) = \begin{cases} \frac{1}{18}(1+\nu_0, \nu_1, \nu_2, \cdots), & \nu_0 \in \left(-\infty, \frac{1}{17}\right), \\ \frac{1}{17}(1, 0, 0, 0, \cdots), & \nu_0 = \frac{1}{17}, \\ \frac{1}{18}(1, 0, 0, 0, \cdots), & \nu_0 \in \left(\frac{1}{17}, \infty\right). \end{cases} \tag{29}$$

Since for all $\nu, t \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0, t_0 \in (-\infty, 1/17)$, we have

$$\begin{aligned}\phi(W\nu - Wt) &= \phi\left(\frac{1}{18}(\nu_0 - t_0, \nu_1 - t_1, \nu_2 - t_2, \cdots)\right) \\ &\le \frac{8}{17}\left(\phi\left(\frac{17\nu}{18}\right) + \phi\left(\frac{17t}{18}\right)\right) \\ &\le \frac{8}{17}(\phi(W\nu - \nu) + \phi(Wt - t)).\end{aligned} \tag{30}$$

For all $\nu, t \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0, t_0 \in (1/17, \infty)$, then for any $\varepsilon > 0$, we have

$$\phi(W\nu - Wt) = 0 \le \varepsilon(\phi(W\nu - \nu) + \phi(Wt - t)). \tag{31}$$

For all $\nu, t \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0 \in (-\infty, 1/17)$ and $t_0 \in (1/17, \infty)$, we have

$$\begin{aligned}\phi(W\nu - Wt) &= \phi\left(\frac{\nu}{18}\right) \le \frac{1}{17}\phi\left(\frac{17\nu}{18}\right) = \frac{1}{17}\phi(W\nu - \nu) \\ &\le \frac{1}{17}(\phi(W\nu - \nu) + \phi(Wt - t)).\end{aligned} \tag{32}$$

Therefore, the map W is Kannan ϕ-contraction mapping. It is clear that W is ϕ-sequentially continuous at $1/17e_0 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ and there is $\nu \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0 \in (-\infty, 1/17)$ such that the sequence of iterates $\{W^p\nu\} = \{\sum_{n=1}^{p} 1/18^n e_0 + 1/18^p \nu\}$ has a subsequence $\{W^{p_i}\nu\} = \{\sum_{n=1}^{p_i} 1/18^n e_0 + 1/18^{p_i}\nu\}$ converging to $1/17e_0$. Then, W has one fixed point $1/17e_0 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$. Note that W is not continuous at $1/17e_0 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$.

6. Kannan Contraction Maps on Prequasi Ideal

We account the being present of a fixed point of Kannan prequasi norm contraction operator on the prequasi Banach operator ideal investigated by $(\ell(r))_\phi$ and $s-$ numbers.

Theorem 30. *Let Z and M be Banach spaces, and $(r_a) \in (0, 1]^{\mathcal{N}}$ be an increasing, then $(S_{(\ell(r))_\phi}, \Phi)$, where $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$ be a prequasi Banach operator ideal.*

Proof. Pick up the conditions be verified. By Theorem 13, the space $(\ell(r))_\phi$ is a premodular (sss). Therefore, from Theorem

9, one has $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$ is a prequasi norm on $S_{(\ell(r))_\phi}$. So, from Theorem 10, we obtain the space $(S_{(\ell(r))_\phi}, \Phi)$ is a prequasi Banach operator ideal.

Theorem 31. *Pick up Z and M be Banach spaces, and $(r_a)\in(0,1]^{\mathcal{N}}$ be an increasing, then $(S_{(\ell(r))_\phi}, \Phi)$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$ be a prequasi closed operator ideal.*

Proof. By Theorem 13, the space $(\ell(r))_\phi$ is a premodular (sss). Therefore, from Theorem 9, we have $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$ is a prequasi norm on $S_{(\ell(r))_\phi}$. Assume $W_q\in S_{(\ell(r))_\phi}(Z,M)$, for each $q\in\mathcal{N}$ and $\lim_{q\longrightarrow\infty}\Phi(W_q-W)=0$. Hence, we have $\varsigma>0$ and since $\mathscr{L}(Z,M)\supseteq S_{(\ell(r))_\phi}(Z,M)$, we have

$$\begin{aligned}\Phi(W_q-W)&=\phi\left(\left(s_a(W_q-W)\right)_{a=0}^{\infty}\right)\geq\phi(s_0(W_q-W),0,0,0,\cdots)\\&=\phi(\|W_q-W\|,0,0,0,\cdots)\geq\varsigma\|W_q-W\|.\end{aligned}\tag{33}$$

So $(W_q)_{q\in\mathcal{N}}$ is convergent in $\mathscr{L}(Z,M)$, i.e., $\lim_{q\longrightarrow\infty}\|W_q-W\|=0$ and as $(s_a(W_q))_{a=0}^{\infty}\in(\ell(r))_\phi$, for every $q\in\mathcal{N}$ and $(\ell(r))_\phi$ is a premodular (sss). Therefore, we get

$$\begin{aligned}\Phi(W)&=\phi((s_a(W))_{a=0}^{\infty})=\phi\left(\left(s_a\left(W-W_q+W_q\right)\right)_{a=0}^{\infty}\right)\\&\leq\phi\left(\left(s_{[a/2]}\left(W-W_q\right)\right)_{a=0}^{\infty}\right)+\phi\left(\left(s_{[a/2]}\left(W_q\right)_{a=0}^{\infty}\right)\right)\\&\leq\phi\left(\left(W_q-W\right)_{a=0}^{\infty}\right)+2\phi\left(\left(s_a\left(W_q\right)_{a=0}^{\infty}\right)\right)<\varepsilon,\end{aligned}\tag{34}$$

we have $(s_a(W))_{a=0}^{\infty}\in(\ell(r))_\phi$, so $W\in S_{(\ell(r))_\phi}(Z,M)$.

Definition 32. A prequasi norm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$, provides the Fatou property if for every sequence $\{W_a\}_{a\in\mathcal{N}}\subseteq S_{\mathfrak{A}_\phi}(Z,M)$ with $\lim_{a\longrightarrow\infty}\Phi(W_a-W)=0$ and all $V\in S_{\mathfrak{A}_\phi}(Z,M)$, then

$$\Phi(V-W)\leq\sup_a\inf_{i\geq a}\Phi(V-W_i).\tag{35}$$

Theorem 33. *The prequasi norm $\Phi(W)=\sum_{a\in\mathcal{N}}|s_a(W)|^{r_a}$, for all $W\in S_{(\ell(r))_\phi}(Z,M)$ does not satisfy the Fatou property, if $(r_a)\in(0,1]^{\mathcal{N}}$ is increasing.*

Proof. Let the setting be provided and $\{W_p\}_{p\in\mathcal{N}}\subseteq S_{(\ell(r))_\phi}(Z,M)$ with $\lim_{p\longrightarrow\infty}\Phi(W_p-W)=0$. Since the space $S_{(\ell(r))_\phi}$ is a prequasi closed ideal, so $W\in S_{(\ell(r))_\phi}(Z,M)$. Therefore, for every $V\in S_{(\ell(r))_\phi}(Z,M)$, we have

$$\begin{aligned}\Phi(V-W)&=\sum_{a\in\mathcal{N}}|s_a(V-W)|^{r_a}\leq\sum_{a\in\mathcal{N}}\left|s_{[a/2]}(V-W_i)\right|^{r_a}\\&\quad+\sum_{a\in\mathcal{N}}\left|s_{[a/2]}(W_i-W)\right|^{r_a}\\&\leq 2\sup_p\inf_{i\geq p}\sum_{a\in\mathcal{N}}|s_a(V-W_i)|^{r_a}.\end{aligned}\tag{36}$$

Hence, Φ does not support the Fatou property.

Now, we introduce the definition of Kannan Φ-contraction operator on the prequasi operator ideal.

Definition 34. For the prequasi norm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$. An operator $G:S_{\mathfrak{A}_\phi}(Z,M)\longrightarrow S_{\mathfrak{A}_\phi}(Z,M)$ is called a Kannan Φ -contraction, if we have $\xi\in[0,1/2)$ so that $\Phi(GW-GA)\leq\xi(\Phi(GW-W)+\Phi(GA-A))$, for all $W,A\in S_{\mathfrak{A}_\phi}(Z,M)$.

Definition 35. For the prequasi norm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$, $G:S_{\mathfrak{A}_\phi}(Z,M)\longrightarrow S_{\mathfrak{A}_\phi}(Z,M)$ and $B\in S_{\mathfrak{A}_\phi}(Z,M)$. The operator G is called Φ -sequentially continuous at B , if and only if, when $\lim_{p\longrightarrow\infty}\Phi(W_p-B)=0$, then $\lim_{p\longrightarrow\infty}\Phi(GW_p-GB)=0$.

Theorem 36. *Set up $(r_a)\in(0,1]^{\mathcal{N}}$ be an increasing and $G:S_{(\ell(r))_\phi}(Z,M)\longrightarrow S_{(\ell(r))_\phi}(Z,M)$, where $\Phi(W)=\sum_{a\in\mathcal{N}}|s_a(W)|^{r_a}$, for every $W\in S_{(\ell(r))_\phi}(Z,M)$. The point $A\in S_{(\ell(r))_\phi}(Z,M)$ is the unique fixed point of G, if the following set up are satisfied:*

(a) G is Kannan Φ-contraction mapping

(b) G is Φ-sequentially continuous at a point $A\in S_{(\ell(r))_\phi}(Z,M)$

(c) There is $B\in S_{(\ell(r))_\phi}(Z,M)$ such that the sequence of iterates $\{G^pB\}$ has a subsequence $\{G^{p_i}B\}$ converging to A

Proof. Let the conditions be verified. If A is not a fixed point of G, then $GA\neq A$. From the setting (b) and (c), we have

$$\lim_{p_i\longrightarrow\infty}\Phi(G^{p_i}B-A)=0\text{ and }\lim_{p_i\longrightarrow\infty}\Phi\left(G^{p_i+1}B-GA\right)=0.\tag{37}$$

Since G is Kannan Φ-contraction mapping, one can see

$$\begin{aligned}0<\Phi(GA-A)&=\Phi\left(\left(GA-G^{p_i+1}B\right)+\left(G^{p_i}B-A\right)+\left(G^{p_i+1}B-G^{p_i}B\right)\right)\\&\leq 2\Phi\left(G^{p_i+1}B-GA\right)+4\Phi\left(G^{p_i}B-A\right)\\&\quad+4\xi\left(\frac{\xi}{1-\xi}\right)^{p_i-1}\Phi(GB-B).\end{aligned}\tag{38}$$

As $p_i \longrightarrow \infty$, this implies a contradiction. Therefore, A is a fixed point of G. To show that the fixed point A is unique. Let we have two different fixed points $A, D \in S_{(\ell(r))_\phi}(Z, M)$ of G. Hence, one has

$$\Phi(A-D) \leq \Phi(GA-GD) \leq \xi(\Phi(GA-A)+\Phi(GD-D)) = 0. \tag{39}$$

Therefore, $A = D$.

Example 37. Let Z and M be Banach spaces, $G : S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}(Z, M) \longrightarrow S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}(Z, M)$, where $\Phi(W) = \sum_{a\in\mathcal{N}} (s_a(W))^{a+1/a+2}$, for every $W \in S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}(Z, M)$ and

$$G(W) = \begin{cases} \dfrac{W}{26}, & \Phi(W) \in [0,1), \\ \dfrac{W}{37}, & \Phi(W) \in [1,\infty). \end{cases} \tag{40}$$

Since for all $W_1, W_2 \in S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W_1), \Phi(W_2) \in (0, 1]$, we have

$$\begin{aligned}\Phi(GW_1-GW_2) &= \Phi\left(\frac{W_1}{26}-\frac{W_2}{26}\right) \leq \frac{2}{5}\left(\Phi\left(\frac{25W_1}{26}\right)+\Phi\left(\frac{25W_2}{26}\right)\right) \\ &= \frac{2}{5}(\Phi(GW_1-W_1)+\Phi(GW_2-W_2)).\end{aligned} \tag{41}$$

For all $W_1, W_2 \in S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W_1), \Phi(W_2) \in [1,\infty)$, we have

$$\begin{aligned}\Phi(GW_1-GW_2) &= \Phi\left(\frac{W_1}{37}-\frac{W_2}{37}\right) \leq \frac{1}{3}\left(\Phi\left(\frac{36W_1}{37}\right)+\Phi\left(\frac{36W_2}{37}\right)\right) \\ &= \frac{1}{3}(\Phi(GW_1-W_1)+\Phi(GW_2-W_2)).\end{aligned} \tag{42}$$

For all $W_1, W_2 \in S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W_1) \in [0, 1)$ and $\Phi(W_2) \in [1,\infty)$, we have

$$\begin{aligned}\Phi(GW_1-GW_2) &= \Phi\left(\frac{W_1}{26}-\frac{W_2}{37}\right) \leq \frac{2}{5}\Phi\left(\frac{25W_1}{26}\right)+\frac{1}{3}\Phi\left(\frac{36W_2}{37}\right) \\ &\leq \frac{2}{5}\left(\Phi\left(\frac{25W_1}{26}\right)+\Phi\left(\frac{36W_2}{37}\right)\right) \\ &= \frac{2}{5}(\Phi(GW_1-W_1)+\Phi(GW_2-W_2)).\end{aligned} \tag{43}$$

Therefore, the map W is Kannan Φ-contraction mapping and $G^p(W) = \begin{cases} W/26^p, & \Phi(W) \in [0,1), \\ W/37^p, & \Phi(W) \in [1,\infty). \end{cases}$

It is clear that G is Φ-sequentially continuous at the zero operator $\Theta \in S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}$ and $\{G^p W\}$ has a subsequence $\{G^{p_i} W\}$ converging to Θ. By Theorem 36, the zero operator $\Theta \in S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}$ is the only fixed point of G. Let $\{W^{(n)}\} \subseteq S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}$ be such that $\lim_{n\longrightarrow\infty}\Phi(W^{(n)} - W^{(0)}) = 0$, where $W^{(0)} \in S_{(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W^{(0)}) = 1$. Since the prequasi norm Φ is continuous, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\Phi\left(GW^{(n)}-GW^{(0)}\right) &= \lim_{n\longrightarrow\infty}\Phi\left(\frac{W^{(n)}}{26}-\frac{W^{(0)}}{37}\right) \\ &= \Phi\left(\frac{11W^{(0)}}{962}\right) > 0.\end{aligned} \tag{44}$$

Hence G is not Φ-sequentially continuous at $W^{(0)}$. So, the map G is not continuous at $W^{(0)}$.

7. Application to the Existence of Solutions of Summable Equations

Summable equations like (45) were studied by Salimi et al. [22], Agarwal et al. [23], and Hussain et al. [24]. In this section, we search for a solution to (45) in $(\ell(r))_\phi$, where $(r_a) \in (0,1]^{\mathcal{N}}$ be an increasing and $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$, for all $v \in \ell(r)$. Consider the summable equations

$$v_a = p_a + \sum_{m=0}^{\infty} A(a,m) f(m, v_m), \tag{45}$$

and let $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ defined by

$$W(v_a)_{a\in\mathcal{N}} = \left(p_a + \sum_{m=0}^{\infty} A(a,m) f(m, v_m)\right)_{a\in\mathcal{N}}. \tag{46}$$

Theorem 38. *The summable equations ((45)) has a solution in $(\ell(r))_\phi$, if $A : \mathcal{N}^2 \longrightarrow \mathfrak{R}, f : \mathcal{N}\times\mathfrak{R} \longrightarrow \mathfrak{R}, p : \mathcal{N} \longrightarrow \mathfrak{R}$, and for all $a \in \mathcal{N}$, there is $\xi \in [0, 1/2)$, so that*

$$\begin{aligned}&\left|\sum_{m\in\mathcal{N}} A(a,m)(f(m,v_m)-f(m,t_m))\right|^{r_a} \\ &\quad\leq \xi\left[\left|p_a - v_a + \sum_{m=0}^{\infty} A(a,m)f(m,v_m)\right|^{r_a} + \left|p_a - t_a + \sum_{m=0}^{\infty} A(a,m)f(m,t_m)\right|^{r_a}\right].\end{aligned} \tag{47}$$

Proof. Let the conditions be verified. Consider the mapping $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ defined by (46). We have

$$\begin{aligned}\phi(Wv - Wt) &= \sum_{a\in\mathcal{N}} |Wv_a - Wt_a|^{r_a} \\ &= \sum_{a\in\mathcal{N}} \left| \sum_{m\in\mathcal{N}} A(a,m)[f(m,v_m) - f(m,t_m)] \right|^{r_a} \\ &\le \xi\left(\sum_{a\in\mathcal{N}} \left[\left| p_a - v_a + \sum_{m=0}^{\infty} A(a,m) f(m,v_m) \right|^{r_a} \right.\right. \\ &\quad \left.\left. + \sum_{a\in\mathcal{N}} \left| p_a - t_a + \sum_{m=0}^{\infty} A(a,m) f(m,t_m) \right|^{r_a} \right] \right) \\ &= \xi(\phi(Wv - v) + \phi(Wt - t)).\end{aligned} \tag{48}$$

Then, from Theorem 24, we have a solution of equation (45) in $(\ell(r))_\phi$.

Example 39. Given the sequence space $(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi$, *where* $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{a+1/a+2}$, *for all* $v \in (\ell((a+1/a+2)_{a=0}^{\infty}))$. *Consider the summable equations*

$$v_a = e^{-(3a+6)} + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m! + 1} \right)^q, \tag{49}$$

where $q > 2$ and let $W : (\ell((a+1/a+2)_{a=0}^{\infty}))_\phi \longrightarrow (\ell((a+1/a+2)_{a=0}^{\infty}))_\phi$ defined by

$$W(v_a)_{a\in\mathcal{N}} = \left(e^{-(3a+6)} + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m! + 1} \right)^q \right)_{a\in\mathcal{N}}. \tag{50}$$

It is easy to see that

$$\begin{aligned}&\left| \sum_{m=0}^{\infty} (-1)^a \left(\frac{v_a}{a^2 + m! + 1} \right)^q ((-1)^m - (-1)^m) \right|^{a+1/a+2} \\ &\le \frac{1}{3}\left[\left| e^{-(3a+6)} - v_a + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m! + 1} \right)^q \right|^{a+1/a+2} \right. \\ &\quad \left. + \left| e^{-(3a+6)} - t_a + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m! + 1} \right)^q \right|^{a+1/a+2} \right].\end{aligned} \tag{51}$$

By Theorem 38, the summable equations (49) has a solution in $(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi$.

Data Availability

Not applicable.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Coincidence Best Proximity Point Results in Branciari Metric Spaces with Applications

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This paper is aimed at studying the uniqueness of coincidence best proximity point for $(\vartheta, \alpha^+, \mathfrak{g})$-proximal contractions in complete Branciari metric space. Throughout this article, discontinuity of the Branciari metric space is used and we obtained the desired results without assuming it as a continuous. Some examples are provided to validate the results proved herein. As an application, we derive the best proximity point results in the setup of complete Branciari metric space endowed with graph. Further, our results extend and generalize the existing ones in literature.

1. Introduction and Preliminaries

Let $\mathfrak{F} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping, where $\mathcal{X}$ be any nonempty set. "An element $q^* \in \mathcal{X}$ is a fixed point of $\mathfrak{F}$ if q^* satisfies the equation $\mathfrak{F}q^* = q^*$ (known as a fixed point equation) or $\mathfrak{d}(q^*, \mathfrak{F}q^*) = 0$." A collection of all "fixed points" of $\mathfrak{F}$ will be represented as $\mathfrak{F}(\mathcal{X})$, that is,

$$\mathfrak{F}(\mathcal{X}) = \{q^* \in \mathcal{X} : \mathfrak{d}(q^*, \mathfrak{F}q^*) = 0\}. \tag{1}$$

In this direction, Banach [1] gives the existence and uniqueness of the "fixed point" of the self mapping $\mathfrak{F}$, if mapping $\mathfrak{F}$ is a contraction and $(\mathcal{X}, \mathfrak{d})$ is a complete, but it becomes more interesting, if $\mathfrak{F}$ is a nonself mapping then it is not necessary that the operator equation $\mathfrak{F}q^* = q^*$ has a solution. In this situation, we can find a point $q^* \in \mathcal{X}$ which is closest to $\mathfrak{F}q^*$ and we have the following minimization/optimization problem

$$\min_{q \in \mathcal{X}} \mathfrak{d}(q, \mathfrak{F}q). \tag{2}$$

Now, consider $\mathcal{X} = (\mathcal{X}, \mathfrak{d})$ be a metric space, $\mathcal{Q}$ and $\mathcal{P}$ are nonempty subsets of $\mathcal{X}$, and consider a mapping $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$, we can find a point q^* in $\mathcal{Q}$ such that $\mathfrak{d}(q^*, \mathfrak{F}q^*)$ is minimum. In other words, we have to minimize $\mathfrak{d}(q^*, \mathfrak{F}q^*)$ for all q^* in $\mathcal{Q}$ and $\mathfrak{F}q^*$ in $\mathcal{P}$. It is important to see that the $\min_{q \in \mathcal{Q}} \mathfrak{d}(q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P})$, where $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \inf \{\mathfrak{d}(q, p): q \in \mathcal{Q}, p \in \mathcal{P}\}$ which cannot be further reduced. If such point q^* in $\mathcal{Q}$ exists then q^* is called an "approximate fixed point" of $\mathfrak{F}$ [2].

Later, several authors studied the results dealing with "approximate fixed points" in different spaces (for detail, see [3–14]).

The best proximity point of the mapping $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ is actually "a point $q^* \in \mathcal{X}$ such that $\mathfrak{d}(q^*, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P})$." Note that if $\mathcal{Q} \cap \mathcal{P} \neq \phi$ then $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0$; in this case, every "approximate fixed point" becomes "fixed point" of the mapping $\mathfrak{F}$. From this perspective, we can say that "the best proximity point results" are natural generalization of "fixed point results."

The concept of "coincidence best proximity point" was introduced in [5] for a pair of mappings in metric space. "A point $q^* \in \mathcal{Q}$ is called the coincidence best proximity point of a pair of mappings $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ and $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ if $\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P})$." We denote the set of all "coincidence best proximity points" of a pair of mappings $\mathfrak{F}$ and $\mathfrak{g}$ by $\mathfrak{F}\mathfrak{g}(\mathcal{Q})$, that is,

$$\mathfrak{F}\mathfrak{g}(\mathcal{Q}) = \{q^* \in \mathcal{Q} : \mathfrak{d}(\mathfrak{g}q^*, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P})\}. \tag{3}$$

This is an extension of a "best proximity point problem." If $\mathfrak{g}$ is an identity mapping on $\mathcal{Q}$ then a "coincidence best proximity point" will reduce to a "best proximity point" of mapping $\mathfrak{F}$.

Recently, Branciari [15] defined a Branciari (by some author generalized/rectangular) metric space. Branciari metric space generalizes the deterministic metric space in a natural way. Few examples are also provided to show that a Branciari metric space is not a metric space.

Definition 1 (see [15]). Let $(\mathcal{X}, \mathfrak{d})$ be a nonempty set. Any mapping $\mathfrak{d} : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ is a Branciari metric on $\mathcal{X}$, if for all $q, p \in \mathcal{X}$,$u, v \in \mathcal{X} \setminus \{q, p\}$, and the following conditions are satisfied:

(1) $\mathfrak{d}(q, p) = 0$ if and only if $q = p$

(2) $\mathfrak{d}(q, p) = \mathfrak{d}(p, q)$

(3) $\mathfrak{d}(q, p) \leq \mathfrak{d}(q, u) + \mathfrak{d}(u, v) + \mathfrak{d}(v, p)$

Then, the pair $(\mathcal{X}, \mathfrak{d})$ is called a Branciari metric space.

Definition 2 (see [15]). Let $(\mathcal{X}, \mathfrak{d})$ be a Branciari metric space. Then, a sequence $\{q_n\}$ in $(\mathcal{X}, \mathfrak{d})$ is as follows:

(a) Convergent sequence which converges to $q \in \mathcal{X}$ if and only if $q_n \longrightarrow q$ as $n \longrightarrow \infty$. In this case, we can write

$$\lim_{n\longrightarrow\infty} q_n = q. \tag{4}$$

(b) Cauchy sequence if and only if $\mathfrak{d}(q_n, q_m) \longrightarrow 0$ as n, $m \longrightarrow \infty$.

(c) A Branciari metric space $(\mathcal{X}, \mathfrak{d})$ is complete if every Cauchy sequence in $\mathcal{X}$ converges to some element in $\mathcal{X}$.

Lemma 3 (see [16]). *Let $(\mathcal{X}, \mathfrak{d})$ be a Branciari metric space and $\{q_n\}$ be a Cauchy sequence in $\mathcal{X}$ such that $q_m \neq q_n$ whenever $m \neq n$. Then, the sequence $\{q_n\}$ can converge to utmost one point.*

In a Branciari metric space, if a sequence is both Cauchy and convergent, then pathologies provided in an example [17] cannot happen, as shown in the following Lemma.

Lemma 4 (see [18]). *Suppose that $\{q_n\}$ is a Cauchy sequence in a Branciari metric space $(\mathcal{X}, \mathfrak{d})$ with $\lim_{n\longrightarrow\infty}\mathfrak{d}(q_n, q) = 0$, where $q \in \mathcal{X}$. Then, $\lim_{n\longrightarrow\infty}\mathfrak{d}(q_n, u) = \mathfrak{d}(q, u)$, for all $u \in \mathcal{X}$. In particular, the sequence $\{q_n\}$ cannot converge to u if $u \neq q$.*

Recently, Jleli and Samet [4] introduced the concept of ϑ-contraction and proved a fixed point result for such mappings in the setup of Branciari metric spaces.

Definition 5 (see [4]). Let Δ_ϑ be the set of all functions $\vartheta : (0, \infty) \longrightarrow (1,\infty)$ satisfying the following conditions:

(θ_1) ϑ is increasing

(θ_2) for any sequence $\{\kappa_n\}$ in $(0, \infty)$,$\lim_{n\longrightarrow\infty}\kappa_n = 0^+$ if and only if $\lim_{n\longrightarrow\infty}\vartheta(\kappa_n) = 1$

(θ_3) there exist $Y \in (0, 1)$ and $\ell \in (0,\infty]$ such that $\lim_{\kappa\longrightarrow 0^+} \vartheta(\kappa) - 1/\kappa^Y = \ell$

Definition 6 (see [4]). Let $(\mathcal{X}, \mathfrak{d})$ be a complete Branciari metric space and $\vartheta \in \Delta_\vartheta$. A mapping $\mathfrak{F} : \mathcal{X} \longrightarrow \mathcal{X}$ is called ϑ-contraction if

$$\vartheta(\mathfrak{d}(\mathfrak{F}(q), \mathfrak{F}(p))) \leq [\vartheta(\mathfrak{d}(q, p))]^\varsigma, \tag{5}$$

for any q,$p \in \mathcal{X}$, where $\mathfrak{d}(\mathfrak{F}(q), \mathfrak{F}(p)) > 0$ and $0 \leq \varsigma < 1$.

Theorem 7 (see [4]). *Let $(\mathcal{X}, \mathfrak{d})$ be a Branciari metric space and $\mathfrak{F} : \mathcal{X} \longrightarrow \mathcal{X}$ be a ϑ-contraction. Then, $\mathfrak{F}$ has a unique fixed point in $\mathcal{X}$.*

Following definitions are also needed in the sequel.

Definition 8. Let $(\mathcal{X}, \mathfrak{d})$ be a Branciari metric space and $\mathcal{Q}$ and $\mathcal{P}$ be two nonempty subsets of $\mathcal{X}$. Define

$$\begin{aligned} \mathfrak{d}(\mathcal{Q}, \mathcal{P}) &= \inf \{\mathfrak{d}(q, p) : q \in \mathcal{Q}, p \in \mathcal{P}\}, \\ \mathcal{Q}_0 &= \{q \in \mathcal{Q} : \mathfrak{d}(q, p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \text{ for some } p \in \mathcal{P}\}, \\ \mathcal{P}_0 &= \{q \in \mathcal{P} : \mathfrak{d}(q, p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \text{for some } p \in \mathcal{Q}\}. \end{aligned} \tag{6}$$

Definition 9 (see [19]). Let $\mathcal{Q}_0$ be a nonempty subset of $\mathcal{Q}$ and $\mathcal{P}_0$ be a nonempty subset of $\mathcal{P}$; then, the pair $(\mathcal{Q}, \mathcal{P})$ satisfies the "weak P-property," if

$$\left.\begin{aligned} \mathfrak{d}(q_1, p_1) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \\ \mathfrak{d}(q_2, p_2) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \end{aligned}\right\} \text{implies } \mathfrak{d}(q_1, q_2) \leq \mathfrak{d}(p_1, p_2), \tag{7}$$

for all $q_1, q_2 \in \mathcal{Q}_0$ and $p_1, p_2 \in \mathcal{P}_0$.

Definition 10. A mapping $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ is called the following:

(1) [20] An "isometry," if

$$\mathfrak{d}(\mathfrak{g}q, \mathfrak{g}p) = \mathfrak{d}(q, p) \tag{8}$$

(2) [21] An "expansive mapping," if

$$\mathfrak{d}(\mathfrak{g}q, \mathfrak{g}p) \geq \mathfrak{d}(q, p), \tag{9}$$

for any $q, p \in \mathcal{Q}$

Proposition 11 (see [22]). *A self mapping $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ is said to satisfy α_R property if there exist a mapping $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow$*

$-\infty, \infty)$ *such that*

$$\alpha(\mathfrak{g}q, \mathfrak{g}p) \geq 0 \text{ implies that } \alpha(q,p) \geq 0, \quad (10)$$

for all $q, p \in \mathcal{Q}$.

Definition 12 (see [3]). Let $(\mathcal{X}, \mathfrak{d})$ be a metric space, $\mathcal{Q}$ is a nonempty subset of ,$\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty, \infty)$, and $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ is a mapping. Then, $\mathcal{Q}$ is said to be α-regular, if $\{q_n\}$ is a sequence in $\mathcal{Q}$ such that $\alpha(q_n, q_{n+1}) \geq 0$ and $q_n \longrightarrow q \in \mathcal{Q}$ as $n \longrightarrow \infty$ then $\alpha(q_n, q) \geq 0$ for all $n \in N$.

Now we have the following definitions:

Definition 13. If $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty, \infty)$ then mapping $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ is called "triangular proximal α^+-admissible," if

(T1)

$$\left.\begin{array}{l} \alpha(q,p) \geq 0 \\ \mathfrak{d}(u, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \\ \mathfrak{d}(v, \mathfrak{F}p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \end{array}\right\} \text{implies that } \alpha(u,v) \geq 0, \quad (11)$$

(T2)

$$\left.\begin{array}{l} \alpha(w,z) \geq 0 \\ \alpha(z,r) \geq 0 \end{array}\right\} \text{implies that } \alpha(w,r) \geq 0, \quad (12)$$

for all $q, p, u, v, w, z, r \in \mathcal{Q}$.

Example 1. Consider $\mathcal{X} = \mathcal{E}$ with the usual metric $\mathfrak{d} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{E}$ defined by $\mathfrak{d}(q,p) = |q-p|$. Let $\mathcal{Q}$ and $\mathcal{P}$ be subsets of $\mathcal{X}$ defined by

$$\begin{aligned} \mathcal{Q} &= \left\{\frac{1}{6}, \frac{3}{6}, \frac{5}{6}, \frac{7}{6}, \frac{9}{6}, \frac{11}{6}, \frac{13}{6}\right\}, \\ \mathcal{P} &= \left\{0, \frac{2}{6}, \frac{4}{6}, \frac{6}{6}\right\}. \end{aligned} \quad (13)$$

It is easy to see that $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 1/6$. Let $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be the mapping defined as

$$\mathfrak{F}(q) = \begin{cases} \dfrac{6}{6}, \text{if } q = \dfrac{3}{6}. \\ 0, \text{else} \end{cases} \quad (14)$$

Also, define $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty,\infty)$ by

$$\alpha(q,p) = \begin{cases} p - q \text{ if } p > q \\ q - p \text{ if } q \geq p \end{cases} \text{for all } q, p \in \mathcal{Q}. \quad (15)$$

Case 1. If we take $q \in \{1/6, 5/6, 7/6, 9/6, 11/6, 13/6\}$,$p = 3/6$, $v = 7/6$, and $u = 1/6$ in $\mathcal{Q}$, we get

$$\begin{aligned} &\alpha(q,p) \geq 0, \\ &\mathfrak{d}(u, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \\ &\mathfrak{d}(v, \mathfrak{F}p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \end{aligned} \quad (16)$$

which implies that

$$\alpha(u,v) \geq 0. \quad (17)$$

If we take $w, z, \kappa \in \mathcal{Q}$, we get

$$\left.\begin{array}{l} \alpha(w,z) \geq 0 \\ \alpha(z,\kappa) \geq 0 \end{array}\right\} \text{implies } \alpha(w,\kappa) \geq 0. \quad (18)$$

Case 2. If we take $q \in \{1/6, 5/6, 7/6, 9/6, 11/6, 13/6\}$,$p = 3/6$, $v = 5/6$, and $u = 1/6$ in $\mathcal{Q}$, we get

$$\begin{aligned} &\alpha(q,p) \geq 0, \\ &\mathfrak{d}(u, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \\ &\mathfrak{d}(v, \mathfrak{F}p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \end{aligned} \quad (19)$$

which implies that

$$\alpha(u,v) \geq 0. \quad (20)$$

If we take $w, z, \kappa \in \mathcal{Q}$, we get

$$\left.\begin{array}{l} \alpha(w,z) \geq 0 \\ \alpha(z,\kappa) \geq 0 \end{array}\right\} \text{implies } \alpha(w,\kappa) \geq 0. \quad (21)$$

Definition 14. If $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ and $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ then pair of mappings $(\mathfrak{F}, \mathfrak{g})$ satisfies

(1) $(\vartheta, \alpha^+, \mathfrak{g})$-proximal contraction, if

$$\begin{gathered} \left.\begin{array}{l} \alpha(q,p) \geq 0 \\ \mathfrak{d}(\mathfrak{g}u, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \\ \mathfrak{d}(\mathfrak{g}v, \mathfrak{F}p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \end{array}\right\} \text{implies} \\ \alpha(q,p) + \vartheta[\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)] \leq [\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1} [\vartheta(\mathfrak{d}(\mathfrak{g}u, \mathfrak{g}v))]^{\varsigma_2} \end{gathered} \quad (22)$$

(2) $(\vartheta, \alpha^+, \mathfrak{g})$-generalized proximal contraction, if

$$\left.\begin{array}{l}\alpha(q,p)\geq 0\\ \mathfrak{d}(\mathfrak{g}u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(\mathfrak{g}v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies}\quad \alpha(q,p)+\vartheta[\mathfrak{d}(\mathfrak{g}u,\mathfrak{g}v)]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u,v))]^{\varsigma_2}, \tag{23}$$

where , $p,u,v\in\mathcal{Q}$,$\vartheta\in\Delta_\vartheta$, and $\varsigma_1,\varsigma_2\geq 0$ with $0\leq\varsigma_1+\varsigma_2<1$

Definition 15. If $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ and $\mathfrak{g}:\mathcal{Q}\longrightarrow\mathcal{Q}$ then pair of mappings $(\mathfrak{F},\mathfrak{g})$ satisfies

(1) $(\vartheta,\mathfrak{g})$-proximal contraction, if

$$\left.\begin{array}{l}\mathfrak{d}(\mathfrak{g}u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(\mathfrak{g}v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies }\vartheta[\mathfrak{d}(\mathfrak{F}q,\mathfrak{F}p)]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{g}u,\mathfrak{g}v))]^{\varsigma_2} \tag{24}$$

(2) "$(\vartheta,\mathfrak{g})$-generalized proximal contraction," if

$$\left.\begin{array}{l}\mathfrak{d}(\mathfrak{g}u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(\mathfrak{g}v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies }\vartheta[\mathfrak{d}(\mathfrak{g}u,\mathfrak{g}\mathfrak{y})]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u,v))]^{\varsigma_2}, \tag{25}$$

where , $p,u,v\in\mathcal{Q}$,$\vartheta\in\Delta_\vartheta$, and $\varsigma_1,\varsigma_2\geq 0$ with $0\leq\varsigma_1+\varsigma_2<1$

Definition 16. Let $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ be a mapping satisfying

(1) (ϑ,α^+)-proximal contraction, if

$$\left.\begin{array}{l}\alpha(q,p)\geq 0\\ \mathfrak{d}(u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies}\quad \alpha(q,p)+\vartheta[\mathfrak{d}(\mathfrak{F}q,\mathfrak{F}p)]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u,v))]^{\varsigma_2} \tag{26}$$

(2) (ϑ,α^+)-generalized proximal contraction, if

$$\left.\begin{array}{l}\alpha(q,p)\geq 0\\ \mathfrak{d}(u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies}\quad \alpha(q,p)+\vartheta[\mathfrak{d}(u,v)]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u,v))]^{\varsigma_2}, \tag{27}$$

where q,p,u,$v\in\mathcal{Q}$,$\vartheta\in\Delta_\vartheta$, and $\varsigma_1,\varsigma_2\geq 0$ with$0\leq\varsigma_1+\varsigma_2<1$.

Definition 17. Let $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ be a mapping satisfying

(1) ʿϑ-proximal contraction, if

$$\left.\begin{array}{l}\mathfrak{d}(u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies }\vartheta[\mathfrak{d}(\mathfrak{F}q,\mathfrak{F}p)]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u,v))]^{\varsigma_2} \tag{28}$$

(2) ϑ-generalized proximal contraction, if

$$\left.\begin{array}{l}\mathfrak{d}(u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies }[\vartheta(\mathfrak{d}(u,v))]^{1-\varsigma_2}\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}, \tag{29}$$

where , $p,u,v\in\mathcal{Q}$,$\vartheta\in\Delta_\vartheta$, and $\varsigma_1,\varsigma_2\geq 0$with $0\leq\varsigma_1+\varsigma_2<1$.

Remark 18. By taking $\mathfrak{g}=I_{\mathcal{Q}}$ (identity, mapping over $\mathcal{Q}$) then every $(\vartheta,\alpha^+,\mathfrak{g})$-proximal contraction will reduce to (ϑ,α^+) -proximal contraction and $(\vartheta,\alpha^+,\mathfrak{g})$-generalized proximal contraction will reduce to (ϑ,α^+)-generalized proximal contraction.

Remark 19. By taking $\alpha(q,p)=0$, every $(\vartheta,\alpha^+,\mathfrak{g})$-proximal contraction will reduce to $(\vartheta,\mathfrak{g})$-proximal contraction and $(\vartheta,\alpha^+,\mathfrak{g})$-generalized proximal contraction will reduce to $(\vartheta,\mathfrak{g})$-generalized proximal contraction.

Remark 20. By taking $\mathfrak{g}=I_{\mathcal{Q}}$ (identity, mapping over $\mathcal{Q}$) and $\alpha(q,p)=0$ then every $(\vartheta,\alpha^+,\mathfrak{g})$-proximal contraction will reduce to ϑ-proximal contraction and $(\vartheta,\alpha^+,\mathfrak{g})$-generalized proximal contraction will reduce to ϑ-generalized proximal contraction.

Note that now and onwards in this article, we assumed that $\mathcal{Q}$ and $\mathcal{P}$ are nonempty, distinct, and disjoint subsets of a complete Branciari metric space $(\mathcal{X},\mathfrak{d})$ and $\mathcal{X}=(\mathcal{X},\mathfrak{d})$ will represent a "complete Branciari metric space" until otherwise it is stated. Also, note that if $(\mathcal{Q},\mathcal{P})$ is nonempty, weakly compact, and convex pair in Banach space $\mathcal{X}$, then $\mathcal{Q}_0$ and $\mathcal{P}_0$ are nonempty ([23]).

2. Main Results

Now we state and prove our main result which runs as follows.

Theorem 21. *Let $\alpha:\mathcal{Q}\times\mathcal{Q}\longrightarrow -\infty,\infty)$, $(\mathfrak{F},\mathfrak{g})$ be a pair of mappings satisfying $(\vartheta,\alpha^+,\mathfrak{g})$-proximal contraction, where $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ be a triangular proximal α^+-admissible and $\mathfrak{g}:\mathcal{Q}\longrightarrow\mathcal{Q}$ be a one to one expansive mapping satisfying α_R property. Moreover, if $\mathcal{Q}_0$ is a nonempty α-regular closed set in $\mathcal{X}$,$\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathcal{P}_0$,$\mathcal{Q}_0\subseteq\mathfrak{g}(\mathcal{Q}_0)$, and $(\mathcal{Q},\mathcal{P})$ satisfy the weak P-property. Then, there exists a coincidence best proximity point of the pair of mappings $(\mathfrak{F},\mathfrak{g})$ provided that there exists $q_0,q_1\in\mathcal{Q}_0$ such that*

$$\mathfrak{d}(\mathfrak{g}q_1,\mathfrak{F}q_0)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\ \text{and}\ \alpha(q_0,q_1)\geq 0. \tag{30}$$

Moreover, if $\alpha(q,p)\geq 0$, for every $q,p\in\mathcal{Q}$ such that $\mathfrak{d}(\mathfrak{g}q,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathscr{P})=\mathfrak{d}(\mathfrak{g}p,\mathfrak{F}p)$ then q^* is the unique coincidence best proximity point of the pair of mappings $(\mathfrak{F},\mathfrak{g})$.

Proof. Let $q_0,q_1\in\mathcal{Q}_0$ such that $\mathfrak{d}(\mathfrak{g}q_1,\mathfrak{F}q_0)=\mathfrak{d}(\mathcal{Q},\mathscr{P})$ and $\alpha(q_0,q_1)\geq 0$. As $\mathfrak{F}q_0\in\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathscr{P}_0$ and $\mathfrak{g}q_1\in\mathcal{Q}_0\subseteq\mathfrak{g}(\mathcal{Q}_0)$, there exists $q_2\in\mathcal{Q}_0$ such that $\mathfrak{d}(\mathfrak{g}q_2,\mathfrak{F}q_1)=\mathfrak{d}(\mathcal{Q},\mathscr{P})$. Since $\mathfrak{F}$ is a triangular proximal α^+-admissible, we have $\alpha(\mathfrak{g}q_1,\mathfrak{g}q_2)\geq 0$. Since $\mathfrak{g}$ satisfies α_R property, hence $\alpha(q_1,q_2)\geq 0$. Similarly, by $\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathscr{P}_0$ and $\mathcal{Q}_0\subseteq\mathfrak{g}(\mathcal{Q}_0)$, there exists a point $q_3\in\mathcal{Q}_0$ such that $\mathfrak{d}(\mathfrak{g}q_3,\mathfrak{F}q_2)=\mathfrak{d}(\mathcal{Q},\mathscr{P})$. Since $\mathfrak{F}$ is a triangular proximal α^+-admissible, this further implies that $\alpha(\mathfrak{g}q_2,\mathfrak{g}q_3)\geq 0$. Following the same arguments, we have $\alpha(q_2,q_3)\geq 0$. Continuing this way, we can obtain a sequence $\{q_n\}$ in $\mathcal{Q}_0$ such that

$$\mathfrak{d}(\mathfrak{g}q_n,\mathfrak{F}q_{n-1})=\mathfrak{d}(\mathcal{Q},\mathscr{P})=\mathfrak{d}(\mathfrak{g}q_{n+1},\mathfrak{F}q_n),\ \alpha(q_n,q_{n+1})\geq 0,\ \text{for all } n\in\mathbb{N}\cup\{0\}. \tag{31}$$

If $q_n\neq q_{n+1}$, for all $n\in\mathbb{N}\cup\{0\}$, the pair of mappings $(\mathfrak{F},\mathfrak{g})$ is $(\vartheta,\alpha^+,\mathfrak{g})$-proximal contraction, and we have

$$\alpha(q_n,q_{n-1})+\vartheta(\mathfrak{d}(\mathfrak{F}q_{n-1},\mathfrak{F}q_n))\leq[\vartheta(\mathfrak{d}(q_{n-1},q_n))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+1},\mathfrak{g}q_n))]^{\varsigma_2}. \tag{32}$$

Since, $\mathfrak{F}$ is a triangular proximal α^+-admissible, we obtain that

$$\vartheta(\mathfrak{d}(\mathfrak{F}q_{n-1},\mathfrak{F}q_n))\leq[\vartheta(\mathfrak{d}(q_{n-1},q_n))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+1},\mathfrak{g}q_n))]^{\varsigma_2}. \tag{33}$$

Since pair $(\mathcal{Q},\mathscr{P})$ satisfies the weak P-property and $\mathfrak{g}$ is one to one on $\mathcal{Q}_0$ then we have

$$\begin{aligned}\vartheta(\mathfrak{d}(\mathfrak{g}q_n,\mathfrak{g}q_{n+1}))&\leq\vartheta(\mathfrak{d}(\mathfrak{F}q_{n-1},\mathfrak{F}q_n))\\&\leq[\vartheta(\mathfrak{d}(q_{n-1},q_n))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+1},\mathfrak{g}q_n))]^{\varsigma_2},\end{aligned} \tag{34}$$

which further implies that

$$[\vartheta(\mathfrak{d}(\mathfrak{g}q_n,\mathfrak{g}q_{n+1}))]^{1-\varsigma_2}\leq[\vartheta(\mathfrak{d}(q_{n-1},q_n))]^{\varsigma_1}, \tag{35}$$

since $\mathfrak{g}$ is an expansive and ϑ is an increasing mapping; hence,

$$[\vartheta(\mathfrak{d}(q_n,q_{n+1}))]^{1-\varsigma_2}\leq[\vartheta(\mathfrak{d}(\mathfrak{g}q_n,\mathfrak{g}q_{n+1}))]^{1-\varsigma_2}\leq[\vartheta(\mathfrak{d}(q_{n-1},q_n))]^{\varsigma_1}. \tag{36}$$

After simplification, we have

$$\vartheta(\mathfrak{d}(q_n,q_{n+1}))\leq[\vartheta(\mathfrak{d}(q_{n-1},q_n))]^{(\varsigma_1/1-\varsigma_2)}. \tag{37}$$

Further, we can write as

$$[\vartheta(\mathfrak{d}(q_n,q_{n+1}))]\leq[\vartheta(\mathfrak{d}(q_{n-2},q_{n-1}))]^{(\varsigma_1/1-\varsigma_2)^2}\leq[\vartheta(\mathfrak{d}(q_0,q_1))]^{(\varsigma_1/1-\varsigma_2)^n}, \tag{38}$$

which implies that

$$\vartheta(\mathfrak{d}(q_n,q_{n+1}))\leq[\vartheta(\mathfrak{d}(q_0,q_1))]^{\nu^n},\ \text{for all } n\in\mathbb{N}, \tag{39}$$

where $\nu=\varsigma_1/1-\varsigma_2<1$. Taking limit $n\longrightarrow\infty$ in the above inequality, we have

$$\lim_{n\longrightarrow\infty}\vartheta(\mathfrak{d}(q_n,q_{n+1}))=1, \tag{40}$$

which implies that

$$\lim_{n\longrightarrow\infty}\mathfrak{d}(q_n,q_{n+1})=0. \tag{41}$$

Since $\vartheta\in\Delta_\vartheta$ then there exist $0<Y<1$ and $0<\ell\leq\infty$, such that the following limit holds true:

$$\lim_{n\longrightarrow\infty}\frac{\vartheta(\mathfrak{d}(q_n,q_{n+1}))-1}{[\mathfrak{d}(q_n,q_{n+1})]^Y}=\ell. \tag{42}$$

Assume that $\ell<\infty$ and $C=\ell/2$. Thus, there exist $n_0\in\mathbb{N}$, such that

$$\left|\frac{\vartheta(\mathfrak{d}(q_n,q_{n+1}))-1}{[\mathfrak{d}(q_n,q_{n+1})]^Y}-\ell\right|\leq C,\ \text{for all } n\geq n_0. \tag{43}$$

Hence, we have

$$\frac{\vartheta(\mathfrak{d}(q_n,q_{n+1}))-1}{[\mathfrak{d}(q_n,q_{n+1})]^Y}\geq\ell-C=\frac{\ell}{2}=C,\ \text{for all } n\geq n_0. \tag{44}$$

Further, we can write as

$$n[\mathfrak{d}(q_n,q_{n+1})]^Y\leq\omega D[\vartheta(\mathfrak{d}(q_n,q_{n+1}))-1],\ \text{for all } n\geq n_0, \tag{45}$$

where $\mathfrak{d}=1/C$. If $\ell=\infty$ then there exists $n_0\in\mathbb{N}$, such that

$$\frac{\vartheta(\mathfrak{d}(q_n,q_{n+1}))-1}{[\mathfrak{d}(q_n,q_{n+1})]^Y}\geq C,\ \text{for all } n\geq n_0, \tag{46}$$

which implies that

$$n[\mathfrak{d}(q_n,q_{n+1})]^Y\leq\omega D[\vartheta(\mathfrak{d}(q_n,q_{n+1}))-1],\ \text{for all } n\geq n_0, \tag{47}$$

where $D=1/C$. Hence, in all cases, there exist $D>0$ and $n_0\in\mathbb{N}$, such that

$$n[\mathfrak{d}(q_n,q_{n+1})]^Y\leq\omega D[\vartheta(\mathfrak{d}(q_n,q_{n+1}))-1],\ \text{for all } n\geq n_0. \tag{48}$$

From inequalities (39) and (48), we have

$$n[\mathfrak{d}(q_n,q_{n+1})]^Y\leq\omega D\left[\left(\vartheta(\mathfrak{d}(q_0,q_1))\right)^{\nu^n}-1\right],\ \text{for all } n\geq n_0. \tag{49}$$

Then, by taking limit as $n\longrightarrow\infty$ on both sides of the above inequality, we obtain

$$n[\mathfrak{d}(q_n, q_{n+1})]^{Y} = 0. \tag{50}$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\mathfrak{d}(q_n, q_{n+1}) \leq \frac{1}{n^{1/Y}}, \text{ for all } n \geq n_0. \tag{51}$$

Now suppose that $q_n = q_m$ for all $n, m \in \mathbb{N}$ and $n = m$. If $q_n \neq q_{n+2}$, for all $n \in \mathbb{N} \cup \{0\}$ then

$$\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{F}q_{n-1}) = \mathfrak{d}(\mathfrak{g}q_{n+2}, \mathfrak{F}q_{n+1}) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{52}$$

Since $\mathfrak{F}$ is a triangular proximal α^+-admissible $\alpha(q_{n-1}, q_n) \geq 0$ and $\alpha(q_n, q_{n+1}) \geq 0$, by (T2) of Definition 13, we have

$$\alpha(q_{n-1}, q_{n+1}) \geq 0. \tag{53}$$

Again, since the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ is $(\vartheta, \alpha^+, \mathfrak{g})$-proximal contraction, we have

$$\begin{aligned} \vartheta(\mathfrak{d}(\mathfrak{F}q_{n-1}, \mathfrak{F}q_{n+1})) &\leq \alpha(q_{n-1}, q_{n+1}) + \vartheta(\mathfrak{d}(\mathfrak{F}q_{n-1}, \mathfrak{F}q_{n+1})) \\ &\leq [\vartheta(\mathfrak{d}(q_{n-1}, q_{n+1}))]^{\varsigma_1} [\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+2}, \mathfrak{g}q_n))]^{\varsigma_2}. \end{aligned} \tag{54}$$

The pair $(\mathcal{Q}, \mathcal{P})$ satisfies the weak P-property; $\mathfrak{g}$ is one to one on $\mathcal{Q}_0$, and ϑ is increasing; then,

$$\vartheta(\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{g}q_{n+2})) \leq \vartheta(\mathfrak{d}(\mathfrak{F}q_{n-1}, \mathfrak{F}q_{n+1})). \tag{55}$$

Then, from inequalities (54) and (55), we have

$$\vartheta(\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{g}q_{n+2})) \leq [\vartheta(\mathfrak{d}(q_{n-1}, q_{n+1}))]^{\varsigma_1} [\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+2}, \mathfrak{g}q_n))]^{\varsigma_2}, \tag{56}$$

which further implies that

$$[\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+2}, \mathfrak{g}q_n))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(q_{n-1}, q_{n+1}))]^{\varsigma_1}. \tag{57}$$

Since $\mathfrak{g}$ is an expansive mapping and ϑ is an increasing mapping, then we have

$$[\vartheta(\mathfrak{d}(q_{n+2}, q_n))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+2}, \mathfrak{g}q_n))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(q_{n-1}, q_{n+1}))]^{\varsigma_1}. \tag{58}$$

After further simplifications, we have

$$\begin{aligned} \vartheta(\mathfrak{d}(q_{n+2}, q_n)) &\leq [\vartheta(\mathfrak{d}(q_{n-1}, q_{n+1}))]^{(\varsigma_1/1-\varsigma_2)} \\ &\leq [\vartheta(\mathfrak{d}(q_{n-2}, q_n))]^{(\varsigma_1/1-\varsigma_2)^2} \\ &\leq [\vartheta(\mathfrak{d}(q_0, q_2))]^{(\varsigma_1/1-\varsigma_2)^n}, \end{aligned} \tag{59}$$

which implies that

$$\vartheta(\mathfrak{d}(q_{n+2}, q_n)) \leq [\vartheta(\mathfrak{d}(q_0, q_2))]^{(\varsigma_1/1-\varsigma_2)^n}, \text{ for all } n \in \mathbb{N}, \tag{60}$$

where $\nu = \varsigma_1/1 - \varsigma_2 < 1$. Taking the limit $n \longrightarrow \infty$ in above inequality, we have

$$\lim_{n \longrightarrow \infty} \vartheta(\mathfrak{d}(q_{n+2}, q_n)) = 1. \tag{61}$$

Similarly, from condition (ϑ_2), we have

$$\lim_{n \longrightarrow \infty} \mathfrak{d}(q_n, q_{n+2}) = 0. \tag{62}$$

Similarly, from condition (ϑ_3), there exists $n_1 \in \mathbb{N}$, such that

$$\mathfrak{d}(q_n, q_{n+2}) \leq \frac{1}{n^{1/Y}}, \text{ for all } n \geq n_1. \tag{63}$$

Now we have the following cases:

Case 1. If $m > 2$ and m is odd. Consider $m = 2\kappa + 1, \kappa \geq 1$, using (51), and we obtain

$$\begin{aligned} \mathfrak{d}(q_n, q_{n+m}) &\leq \mathfrak{d}(q_n, q_{n+1}) + \mathfrak{d}(q_{n+1}, q_{n+2}) + \mathfrak{d}(q_{n+2}, q_{n+m}) \\ &\leq \mathfrak{d}(q_n, q_{n+1}) + \mathfrak{d}(q_{n+1}, q_{n+2}) + \mathfrak{d}(q_{n+2}, q_{n+3}) \\ &\quad + \mathfrak{d}(q_{n+3}, q_{n+4}) + \mathfrak{d}(q_{n+4}, q_{n+m}) \\ &\leq \mathfrak{d}(q_n, q_{n+1}) + \mathfrak{d}(q_{n+1}, q_{n+2}) + \mathfrak{d}(q_{n+2}, q_{n+3}) + \cdots \\ &\quad + \mathfrak{d}(q_{n+m-1}, q_{n+m}) \leq \frac{1}{n^{1/Y}} + \frac{1}{(n+1)^{1/Y}} \\ &\quad + \frac{1}{(n+2)^{1/Y}} + \cdots + \frac{1}{(n+m-1)^{1/Y}} \\ \mathfrak{d}(q_n, q_{n+m}) &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/Y}}, \end{aligned} \tag{64}$$

for all $m \geq n \geq N$, where $N = \max\{n_0, n_1\}$.

Case 2. If $m > 2$ is even. Consider $m = 2\kappa, \kappa \geq 2$, using (51) and (63), and we obtain

$$\begin{aligned} \mathfrak{d}(q_n, q_{n+m}) &\leq \mathfrak{d}(q_n, q_{n+2}) + \mathfrak{d}(q_{n+2}, q_{n+3}) + \mathfrak{d}(q_{n+3}, q_{n+m}) \\ &\leq \mathfrak{d}(q_n, q_{n+2}) + \mathfrak{d}(q_{n+2}, q_{n+3}) + \mathfrak{d}(q_{n+3}, q_{n+4}) \\ &\quad + \mathfrak{d}(q_{n+4}, q_{n+5}) + \mathfrak{d}(q_{n+5}, q_{n+m}) \\ &\leq \mathfrak{d}(q_n, q_{n+2}) + \mathfrak{d}(q_{n+2}, q_{n+3}) + \mathfrak{d}(q_{n+3}, q_{n+4}) + \cdots \\ &\quad + \mathfrak{d}(q_{n+m-1}, q_{n+m}) \leq \frac{1}{n^{1/Y}} + \frac{1}{(n+2)^{1/Y}} + \cdots \\ &\quad + \frac{1}{(n+m-1)^{1/Y}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/Y}}, \end{aligned} \tag{65}$$

for all $m \geq n \geq N$, where $N = \max\{n_0, n_1\}$. Thus, by combining all the cases, we have

$$\mathfrak{d}(q_n, q_{n+m}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/Y}}, \text{ for all } m \geq n \geq N, \tag{66}$$

where $N = \max\{n_0, n_1\}$. Then, by P-series test, $\sum_{i=n}^{\infty} (1/i^{1/Y})$

converges as $1/Y > 1$. We deduce that $\{q_n\}$ is a Cauchy sequence in $\mathcal{Q}_0 \subseteq \mathcal{Q} \subset \mathcal{X}$. By the completeness of space $\mathcal{X}$ and $\mathcal{Q}_0$ is closed, there exists $q^* \in \mathcal{Q}_0$ such that $q_n \longrightarrow q^* \in \mathcal{Q}_0$. Since $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$, we have $q^* = \mathfrak{g}q^*$. Since $\mathcal{Q}_0$ is α-regular then $\alpha(q_n, q^*) \geq 0$. Since $q^* \in \mathcal{Q}_0, \mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$ then $\mathfrak{F}q^* \in \mathcal{P}_0$; thus, there exists a point $z \in \mathcal{Q}_0$ such that $z=q^*$ and

$$\mathfrak{d}(\mathfrak{g}z, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{67}$$

Since the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ is $(\vartheta, \alpha^+, \mathfrak{g})$-proximal contraction and by using weak P-property, we obtain

$$\begin{aligned}\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+1}, \mathfrak{g}z)) &\leq \vartheta(\mathfrak{d}(\mathfrak{F}q_n, \mathfrak{F}q^*)) \leq \alpha(q_n, q^*) + \vartheta(\mathfrak{d}(\mathfrak{F}q_n, \mathfrak{F}q^*)) \\ &\leq [\vartheta(\mathfrak{d}(q_n, q^*))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+1}, \mathfrak{g}z))]^{\varsigma_2}.\end{aligned} \tag{68}$$

Further, the above inequality can be written as follows:

$$[\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+1}, \mathfrak{g}z))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(q_n, q^*))]^{\varsigma_1}. \tag{69}$$

Since $\mathfrak{g}$ is an expansive mapping, we have

$$\begin{aligned}[\vartheta(\mathfrak{d}(q_{n+1}, z))]^{1-\varsigma_2} &\leq [\vartheta(\mathfrak{d}(\mathfrak{g}q_{n+1}, \mathfrak{g}z))]^{1-\varsigma_2} \\ &\leq [\vartheta(\mathfrak{d}(q^*, q_n))]^{\varsigma_1} < [\vartheta(\mathfrak{d}(q^*, q_n))]^{1-\varsigma_2}.\end{aligned} \tag{70}$$

This implies that

$$[\vartheta(\mathfrak{d}(q_{n+1}, z))] < [\vartheta(\mathfrak{d}(q^*, q_n))]. \tag{71}$$

As ϑ is increasing, we have

$$\mathfrak{d}(q_{n+1}, z) < \mathfrak{d}(q^*, q_n). \tag{72}$$

Then, by rectangular property, (51) and (72), we have

$$\begin{aligned}\mathfrak{d}(q, z) &\leq \mathfrak{d}(q^*, q_n) + \mathfrak{d}(q_n, q_{n+1}) + \mathfrak{d}(q_{n+1}, z) \\ &\leq \mathfrak{d}(q^*, q_n) + \frac{1}{n^{1/Y}} + \mathfrak{d}(q^*, q_n).\end{aligned} \tag{73}$$

Taking limit $n \longrightarrow \infty$ in the above inequality, we conclude that $z = q^*$. Hence,

$$\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{74}$$

Thus, q^* is a coincidence best proximity point of pair of mappings $(\mathfrak{F}, \mathfrak{g})$.

Uniqueness. Now we have to show that q^* is a unique coincidence best proximity point of pair of mappings $(\mathfrak{F}, \mathfrak{g})$. Suppose that q^* and w^* be two coincidence best proximity points of a pair of mappings $(\mathfrak{F}, \mathfrak{g})$, that is,

$$\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(\mathfrak{g}w^*, \mathfrak{F}w^*). \tag{75}$$

Since $\alpha(q, p) \geq 0$, for every $q, p \in \mathcal{Q}$, and by using properties of $\mathfrak{F}$ and $\mathfrak{g}$ and reasoning as above, we obtain that

$$\begin{aligned}[\vartheta(\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{g}w^*))] &\leq \vartheta(\mathfrak{d}(\mathfrak{F}q^*, \mathfrak{F}w^*)) \leq \alpha(q^*, w^*) \\ &\quad + \vartheta(\mathfrak{d}(\mathfrak{F}q^*, \mathfrak{F}w^*)) \leq [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_1} \\ &\quad \cdot [\vartheta(\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{g}w^*))]^{\varsigma_2}[\vartheta(\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{g}w^*))]^{1-\varsigma_2} \\ &\leq [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_1} < [\vartheta(\mathfrak{d}(q^*, w^*))]^{1-\varsigma_2}.\end{aligned} \tag{76}$$

Further, we have

$$[\vartheta(\mathfrak{d}(q^*, w^*))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{g}w^*))]^{1-\varsigma_2} < [\vartheta(\mathfrak{d}(q^*, w^*))]^{1-\varsigma_2}, \tag{77}$$

which is a contradiction. Therefore, $q^* = w^*$. Hence, pair of mappings $(\mathfrak{F}, \mathfrak{g})$ has a unique coincidence best proximity point.

If $\mathfrak{g}$ is an isometry in Theorem 21 then it yields the following theorem.

Theorem 22. *Let $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty, \infty)$, $(\mathfrak{F}, \mathfrak{g})$ be a pair of mappings satisfying $(\vartheta, \alpha^+, \mathfrak{g})$-proximal contraction, where $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a triangular proximal α^+-admissible and $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a one to one isometry mapping satisfying α_R property. Moreover, if $\mathcal{Q}_0$ is a nonempty α-regular closed set in $\mathcal{X}, \mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0, \mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$, and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a coincidence best proximity point of the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(\mathfrak{g}q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \text{ and } \alpha(q_0, q_1) \geq 0. \tag{78}$$

Moreover, if $\alpha(q, p) \geq 0$, for every $q, p \in \mathcal{Q}$ such that $\mathfrak{d}(\mathfrak{g}q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(\mathfrak{g}p, \mathfrak{F}p)$ then q^* is the unique coincidence best proximity point of the pair of mappings $(\mathfrak{F}, \mathfrak{g})$.

Proof. The result follows from Theorem 21 by choosing $\mathfrak{g}$ as an isometry mapping instead of expansive mapping, and the remaining proof follows under the same lines.

Corollary 23. *Let $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a $(\vartheta, \mathfrak{g})$-proximal contraction and $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a one to one expansive mapping satisfying q_R property. Moreover, if $\mathcal{Q}_0$ is a nonempty closed set in $\mathcal{X}$, $\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$, $\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$, and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P -property. Then, there exists a unique coincidence best proximity point of pair of mappings $(\mathfrak{F}, \mathfrak{g})$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(\mathfrak{g}q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{79}$$

Proof. The result follows from Theorem 21 by choosing $\alpha(q, p) = 0$, and the remaining proof follows under the same lines.

Corollary 24. *Let $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty, \infty)$ and $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a triangular proximal α^+-admissible and (ϑ, α^+)-proximal contraction. Moreover, if $\mathcal{Q}_0$ is a nonempty α-regular closed* *set, $\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$ and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a best proximity point of $\mathfrak{F}$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \text{ and } \alpha(q_0, q_1) \geq 0. \tag{80}$$

Moreover, if $\alpha(q, p) \geq 0$, for every $q, p \in \mathcal{Q}$ such that $\mathfrak{d}(q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(p, \mathfrak{F}p)$ then q^* is a unique best proximity point of $\mathfrak{F}$.

Proof. The result follows from Theorem 21 by choosing $\mathfrak{g} = I_{\mathcal{Q}}$, and the remaining proof follows under the same lines.

Corollary 25. *Let $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a ϑ-proximal contraction. Moreover, if $\mathcal{Q}_0$ is a nonempty closed set, $\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$ and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a unique best proximity point of $\mathfrak{F}$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{81}$$

Proof. The result follows from Theorem 21 by choosing $\mathfrak{g} = I_{\mathcal{Q}}$ and $\alpha(q, p) = 0$, and the remaining proof follows under the same lines.

To support the Corollary 25, we provide the following example.

Example 2. Let $\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathfrak{d} : \mathcal{X} \times \mathcal{X} \longrightarrow 0, \infty)$ which is defined as

$$\begin{gathered}
\mathfrak{d}(q, p) = \mathfrak{d}(p, q) \text{ and } \mathfrak{d}(q, q) = 0, \text{ for all } q, p \in \mathcal{X}, \\
\mathfrak{d}(5, 7) = 1, \\
\mathfrak{d}(2, 6) = \mathfrak{d}(1, 3) = 2, \mathfrak{d}(0, 2) = 3, \\
\mathfrak{d}(3, 7) = \mathfrak{d}(2, 4) = \mathfrak{d}(3, 5) = 4, \mathfrak{d}(0, 4) = \mathfrak{d}(1, 5) = 5, \\
\mathfrak{d}(2, 7) = \mathfrak{d}(4, 7) = \mathfrak{d}(6, 1) = \mathfrak{d}(6, 5) = \mathfrak{d}(6, 7) = \mathfrak{d}(2, 5) = 7, \\
\mathfrak{d}(4, 6) = \mathfrak{d}(0, 8) = \mathfrak{d}(2, 8) = \mathfrak{d}(4, 8) = \mathfrak{d}(6, 8) = \mathfrak{d}(1, 8) = \mathfrak{d}(4, 1) = 8, \\
\mathfrak{d}(0, 1) = \mathfrak{d}(2, 3) = \mathfrak{d}(4, 5) = \mathfrak{d}(6, 7) = \mathfrak{d}(0, 5) = \mathfrak{d}(0, 7) = \mathfrak{d}(2, 1) = 7, \\
\mathfrak{d}(0, 6) = \mathfrak{d}(1, 7) = \mathfrak{d}(6, 3) = \mathfrak{d}(4, 3) = \mathfrak{d}(0, 3) = \mathfrak{d}(3, 8) = \mathfrak{d}(5, 8) = \mathfrak{d}(7, 8) = 8.
\end{gathered} \tag{82}$$

Then, $(\mathcal{X}, \mathfrak{d})$ is a Branciari metric space. Since

$$\mathfrak{d}(0, 6) \leq \mathfrak{d}(0, 2) + \mathfrak{d}(2, 6) \tag{83}$$

then $(\mathcal{X}, \mathfrak{d})$ is not a metric space.

Suppose that $\mathcal{Q} = \{2, 4, 6\}$ and $\mathcal{P} = \{1, 3, 7\}$ are subsets of Branciari metric space $(\mathcal{X}, \mathfrak{d})$. Note that $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 7$, $\mathcal{Q} = \mathcal{Q}_0$, and $\mathcal{P} = \mathcal{P}_0$. It is easy to see that $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Define a mapping $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ as follows:

$$\mathfrak{F}(2) = 3, \mathfrak{F}(4) = 1, \mathfrak{F}(6) = 3. \tag{84}$$

Clearly, $\mathfrak{F}$ has no fixed point; also, note that $\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$. Define a function $\vartheta : (0,\infty) \longrightarrow (1,\infty)$ as follows:

$$\vartheta(\kappa) = e^{\sqrt{\kappa}}. \tag{85}$$

Hence, $\mathfrak{F}$ satisfies the conditions of ϑ-proximal contractive mapping. Further, by taking $u = 2, v = 6, q = 6, p = 4 \in \mathcal{Q}$, $\varsigma_1 = 1/2$, and $\varsigma_2 \geq 0$ in such a way that $0 \leq \varsigma_1 + \varsigma_2 < 1$, we have

$$\mathfrak{d}(2, \mathfrak{F}(6)) = \mathfrak{d}(6, \mathfrak{F}(4)) = 7 = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{86}$$

This implies that $\mathfrak{F}$ satisfies

$$\vartheta[\mathfrak{d}(\mathfrak{F}(q), \mathfrak{F}(p))] \leq [\vartheta(\mathfrak{d}(q, p))]^{1/2} [\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}. \tag{87}$$

Hence, $\mathfrak{F}$ is ϑ-proximal contraction for all $u, v, q, p \in \mathcal{Q}$. Thus, all conditions of Corollary 25 hold true; after simple calculation, we can find $q = 2$ is a unique best proximity point of $\mathfrak{F}$.

Remark 26. If we take $\mathcal{X} = \mathcal{Q} = \mathcal{P}$ and $\varsigma_2 = 0$ in the Example 2, we obtained the example of the main Theorem (1.7) in [4].

Theorem 27. *Let $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty, \infty)$, $(\mathfrak{F}, \mathfrak{g})$ be a pair of mappings satisfying $(\vartheta, \alpha^+, \mathfrak{g})$-generalized proximal contraction, where $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a triangular proximal α^+-admissible and $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a one to one expansive mapping satisfying α_R property. Moreover, if $\mathcal{Q}_0$ is a nonempty α-regular closed set in $\mathcal{X}$,$\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$,$\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$, and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a coincidence best proximity point of the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(\mathfrak{g}q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \text{ and } \alpha(q_0, q_1) \geq 0. \tag{88}$$

Moreover, if $\alpha(q, p) \geq 0$, for every $q, p \in \mathcal{Q}$ such that $\mathfrak{d}(\mathfrak{g}q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(\mathfrak{g}p, \mathfrak{F}p)$ then q^* is the unique coincidence best proximity point of the pair of mappings $(\mathfrak{F}, \mathfrak{g})$.

Proof. Following the arguments similar to those given in the proof of Theorem 21, we obtain a sequence $\{q_n\}$ in $\mathcal{Q}_0$ such that

$$\begin{gathered}
\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{F}q_{n-1}) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \\
\mathfrak{d}(\mathfrak{g}q_{n+1}, \mathfrak{F}q_n) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \alpha(q_{n+1}, q_n) \geq 0, \text{for all } n \in \mathbb{N} \cup \{0\}.
\end{gathered} \tag{89}$$

If $q_n \neq q_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$, also the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ is $(\vartheta, \alpha^+, \mathfrak{g})$-generalized proximal contraction and $\mathfrak{g}$ is an expansive mapping, we have

$$\begin{aligned}
\vartheta(\mathfrak{d}(q_n, q_{n+1})) \leq \vartheta(\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{g}q_{n+1})) &\leq \alpha(q_n, q_{n-1}) + \vartheta(\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{g}q_{n+1})) \\
&\leq [\vartheta(\mathfrak{d}(q_{n-1}, q_n))]^{\varsigma_1} [\vartheta(\mathfrak{d}(q_{n+1}, q_n))]^{\varsigma_2}.
\end{aligned} \tag{90}$$

Further, the above inequality becomes

$$[\vartheta(\mathfrak{d}(q_n, q_{n+1}))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(q_n, q_{n-1}))]^{\varsigma_1}. \tag{91}$$

Again by using the arguments similar to those given in the proof of Theorem 21, we obtain

$$\mathfrak{d}(q_n, q_{n+1}) \leq \frac{1}{n^{1/Y}}, \text{ for all } n \geq n_0. \tag{92}$$

Now suppose that $q_n = q_m$ for every $n, m \in \mathbb{N}$ such that $n = m$. If $q_n \neq q_{n+2}$, for all $n \in \mathbb{N} \cup \{0\}$, since the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ is $(\vartheta, \alpha^+, \mathfrak{g})$-generalized proximal contraction and $\mathfrak{g}$ is an expansive mapping, we have

$$\begin{aligned} [\vartheta(\mathfrak{d}(q_n, q_{n+2}))] &\leq \vartheta(\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{g}q_{n+2})) \\ &\leq [\vartheta(\mathfrak{d}(q_{n-1}, q_{n+1}))]^{\varsigma_1} [\vartheta(\mathfrak{d}(q_{n+2}, q_n))]^{\varsigma_2}. \end{aligned} \tag{93}$$

Further, the above inequality becomes

$$[\vartheta(\mathfrak{d}(q_n, q_{n+2}))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(q_{n-1}, q_{n+1}))]^{\varsigma_1}. \tag{94}$$

By using the arguments similar to those given in the proof of Theorem 21, we deduce that $\{q_n\}$ is a Cauchy sequence in $\mathcal{Q}_0 \subseteq \mathcal{Q} \subset \mathcal{X}$ and

$$\mathfrak{d}(\mathfrak{g}z, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{95}$$

Since the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ is $(\vartheta, \alpha^+, \mathfrak{g})$-generalized proximal contraction and $\mathfrak{g}$ is an expansive mapping, we have

$$\begin{aligned} \vartheta[\mathfrak{d}(q_n, z)] &\leq \vartheta[\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{g}z)] \leq \alpha(q_n, q^*) + \vartheta[\mathfrak{d}(\mathfrak{g}q_n, \mathfrak{g}z)] \\ &\leq [\vartheta(\mathfrak{d}(q_{n-1}, q^*))]^{\varsigma_1} [\vartheta(\mathfrak{d}(z, q_n))]^{\varsigma_2} \\ \vartheta[\mathfrak{d}(q_n, z)]^{1-\varsigma_2} &\leq [\vartheta(\mathfrak{d}(q_{n-1}, q^*))]^{\varsigma_1} < [\vartheta(\mathfrak{d}(q_{n-1}, q^*))]^{1-\varsigma_2}. \end{aligned} \tag{96}$$

By using the arguments similar to those given in the proof of Theorem 21, we conclude that q^* is a coincidence best proximity point of pair of mappings $(\mathfrak{F}, \mathfrak{g})$.

Uniqueness. Now we have to show that q^* is a unique coincidence best proximity point of pair of mappings $(\mathfrak{F}, \mathfrak{g})$. Suppose that q^* and w^* be two coincidence best proximity points of a pair of mappings $(\mathfrak{F}, \mathfrak{g})$, that is,

$$\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(\mathfrak{g}w^*, \mathfrak{F}w^*). \tag{97}$$

Since $\alpha(q, p) \geq 0$, for every $q, p \in \mathcal{Q}$ and by using properties of $\mathfrak{F}$ and $\mathfrak{g}$ and reasoning as above, we obtain that

$$\begin{aligned} \vartheta(\mathfrak{d}(q^*, w^*)) &\leq \vartheta(\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{g}w^*)) \leq \alpha(q^*, w^*) + \vartheta(\mathfrak{d}(\mathfrak{g}q^*, \mathfrak{g}w^*)) \\ &\leq [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_1} [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_2} \\ &= [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_1+\varsigma_2} < \vartheta(\mathfrak{d}(q^*, w^*)), \end{aligned} \tag{98}$$

which is a contradiction. Therefore, $q^* = w^*$. Hence, pair of mappings $(\mathfrak{F}, \mathfrak{g})$ has a unique coincidence best proximity point.

If $\mathfrak{g}$ is an isometry then the preceding theorem yields the following Theorem.

Theorem 28. *Let $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty, \infty)$, $(\mathfrak{F}, \mathfrak{g})$ be a pair of mappings satisfying $(\vartheta, \alpha^+, \mathfrak{g})$-generalized proximal contraction, where $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a triangular proximal α^+-admissible and $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a one to one isometry mapping satisfying α_R property. Moreover, if a $\mathcal{Q}_0$ is nonempty α-regular closed set in $\mathcal{X}$,$\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$,$\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$, and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a coincidence best proximity point of the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(\mathfrak{g}q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \text{ and } \alpha(q_0, q_1) \geq 0. \tag{99}$$

Moreover, if $\alpha(q, p) \geq 0$, for every $q, p \in \mathcal{Q}$ such that $\mathfrak{d}(\mathfrak{g}q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(\mathfrak{g}p, \mathfrak{F}p)$ then q^* is the unique coincidence best proximity point of the pair of mappings $(\mathfrak{F}, \mathfrak{g})$.

Proof. The result follows from Theorem 27 by choosing $\mathfrak{g}$ as an isometry mapping instead of an expansive mapping, and the remaining proof follows under the same lines.

Corollary 29. *Let $(\mathfrak{F}, \mathfrak{g})$ be a pair of mappings satisfying $(\vartheta, \alpha^+, \mathfrak{g})$-generalized proximal contraction and $\mathfrak{g} : \mathcal{Q} \longrightarrow \mathcal{Q}$ be an expansive mapping satisfying α_R property. Moreover, if $\mathcal{Q}_0$ is nonempty closed set in $\mathcal{X}$,$\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$,$\mathcal{Q}_0 \subseteq \mathfrak{g}(\mathcal{Q}_0)$, and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a unique coincidence best proximity point of the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(\mathfrak{g}q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{100}$$

Proof. The result follows from Theorem 27 by choosing $\alpha(q, p) = 0$, and the remaining proof follows under the same lines.

Corollary 30. *Let $\alpha : \mathcal{Q} \times \mathcal{Q} \longrightarrow -\infty, \infty)$, $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a (ϑ, α^+)-generalized proximal contraction. Moreover, if $\mathcal{Q}_0$ is a nonempty α-regular closed set in $\mathcal{X}$,$\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$ and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a best proximity point of mapping $\mathfrak{F}$ provided that there exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) \text{ and } \alpha(q_0, q_1) \geq 0. \tag{101}$$

Moreover, if $\alpha(q, p) \geq 0$, for every $q, p \in \mathcal{Q}$ such that $\mathfrak{d}(q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(p, \mathfrak{F}p)$ then q^* is the unique best proximity point of the mapping $\mathfrak{F}$.

Proof. The result follows from Theorem 27 by choosing $\mathfrak{g} = I_{\mathcal{Q}}$, and the remaining proof follows under the same lines.

Corollary 31. *Let $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ be a ϑ-generalized proximal contraction. Moreover, if $\mathcal{Q}_0$ is a nonempty closed set, $\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$ and $(\mathcal{Q}, \mathcal{P})$ satisfy the weak P-property. Then, there exists a unique best proximity point of $\mathfrak{F}$ provided that there* *exists $q_0, q_1 \in \mathcal{Q}_0$ such that*

$$\mathfrak{d}(q_1, \mathfrak{F}q_0) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{102}$$

Proof. The result follows from Theorem 27 by choosing $\mathfrak{g} = I_{\mathcal{Q}}$ and $\alpha(q, p) = 0$, and the remaining follows under the same lines.

To support the Corollary 31, we provide the following example.

Example 3. Let $\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathfrak{d} : \mathcal{X} \times \mathcal{X} \longrightarrow 0, \infty)$ which is defined as

$$\mathfrak{d}(q, p) = \mathfrak{d}(p, q) \text{ and } \mathfrak{d}(q, q) = 0, \text{ for all } q, p \in \mathcal{X}, \tag{103}$$

where

$$\begin{gathered}
\mathfrak{d}(5, 7) = 1, \\
\mathfrak{d}(2, 6) = \mathfrak{d}(1, 3) = 2, \mathfrak{d}(0, 2) = 3, \\
\mathfrak{d}(3, 7) = \mathfrak{d}(2, 4) = \mathfrak{d}(3, 5) = 4, \mathfrak{d}(0, 4) = \mathfrak{d}(1, 5) = 5, \\
\mathfrak{d}(2, 7) = \mathfrak{d}(6, 1) = \mathfrak{d}(6, 5) = \mathfrak{d}(6, 7) = \mathfrak{d}(2, 5) = 7, \\
\mathfrak{d}(4, 6) = \mathfrak{d}(0, 8) = \mathfrak{d}(4, 7) = \mathfrak{d}(2, 8) = \mathfrak{d}(4, 8) = \mathfrak{d}(6, 8) = \mathfrak{d}(1, 8) = \mathfrak{d}(4, 1) = 8, \\
\mathfrak{d}(0, 1) = \mathfrak{d}(2, 3) = \mathfrak{d}(4, 5) = \mathfrak{d}(6, 7) = \mathfrak{d}(0, 5) = \mathfrak{d}(0, 7) = \mathfrak{d}(2, 1) = 7, \\
\mathfrak{d}(0, 6) = \mathfrak{d}(1, 7) = \mathfrak{d}(6, 3) = \mathfrak{d}(4, 3) = \mathfrak{d}(0, 3) = \mathfrak{d}(3, 8) = \mathfrak{d}(5, 8) = \mathfrak{d}(7, 8) = 8.
\end{gathered} \tag{104}$$

Then, $(\mathcal{X}, \mathfrak{d})$ is a Branciari metric space. Since

$$\mathfrak{d}(0, 6) \leq \mathfrak{d}(0, 2) + \mathfrak{d}(2, 6), \tag{105}$$

then $(\mathcal{X}, \mathfrak{d})$ is not a metric space. Suppose that $\mathcal{Q} = \{2, 4, 6\}$ and $\mathcal{P} = \{3, 5, 7\}$. Note that $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 7, \mathcal{Q} = \mathcal{Q}_0$, and $\mathcal{P} = \mathcal{P}_0$. Define a mapping $\mathfrak{F} : \mathcal{Q} \longrightarrow \mathcal{P}$ as

$$\mathfrak{F}(2) = 3, \mathfrak{F}(4) = 7, \mathfrak{F}(6) = 3. \tag{106}$$

Clearly, $\mathfrak{F}$ has no fixed point; also, note that $\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$. Define a function $\vartheta : (0,\infty) \longrightarrow (1,\infty)$ as

$$\vartheta(\kappa) = e^{\sqrt{\kappa}}. \tag{107}$$

Hence, $\mathfrak{F}$ satisfies the conditions of ϑ-generalized proximal contractive mapping. Further, by taking $u = 2, v = 6$, $q = 6, p = 4 \in \mathcal{Q}, \varsigma_1 = 1/2$, and $\varsigma_2 \geq 0$ in such a way that $0 \leq \varsigma_1 + \varsigma_2 < 1$, we have

$$\mathfrak{d}(u, \mathfrak{F}q) = \mathfrak{d}(2, \mathfrak{F}(6)) = \mathfrak{d}(6, \mathfrak{F}(4)) = \mathfrak{d}(v, \mathfrak{F}p) = 7. \tag{108}$$

Further, $\mathfrak{F}$ satisfies

$$\vartheta[\mathfrak{d}(u, v)] \leq [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1} [\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}. \tag{109}$$

Hence, $\mathfrak{F}$ is ϑ-generalized proximal contraction for all $u, v, q, p \in \mathcal{Q}$. Thus, all conditions of Corollary 31 hold true; after simple calculation, we can find $q = 2$ is a unique best proximity point of $\mathfrak{F}$.

3. Application to Coincidence Point and Fixed Point Theory

If we take $\mathcal{Q} = \mathcal{P} = \mathcal{X}$, then from Definition 13, triangular proximal α^+-admissible implies

$$\begin{aligned}
\mathfrak{d}(u, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0 \text{ implies } u = \mathfrak{F}q, \\
\mathfrak{d}(v, \mathfrak{F}p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0 \text{ implies } v = \mathfrak{F}p,
\end{aligned} \tag{110}$$

which becomes

(P1)

$$\alpha(q, p) \geq 0 \text{ implies } \alpha(\mathfrak{F}q, \mathfrak{F}p) \geq 0, \tag{111}$$

(P2)

$$\left.\begin{aligned} \alpha(q, z) \geq 0 \\ \alpha(z, p) \geq 0 \end{aligned}\right\} \text{implies } \alpha(q, p) \geq 0, \text{ for all } q, p \in \mathcal{X}. \tag{112}$$

Remark 32. Note that, for self mapping, every triangular proximal α^+-admissible self mapping is triangular α^+-admissible mapping.

Remark 33. If $\mathfrak{F}, \mathfrak{g} : \mathcal{X} \longrightarrow \mathcal{X}$ then $(\vartheta, \mathfrak{g})$-proximal contraction becomes

$$[\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p))] \leq [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1} [\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p))]^{\varsigma_2}, \tag{113}$$

and $(\vartheta, \mathfrak{g})$-generalized proximal contraction becomes

$$\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) \leq [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1} [\vartheta(\mathfrak{d}(v, u))]^{\varsigma_2}, \tag{114}$$

where $\vartheta \in \Delta_\vartheta, \varsigma \in (0, 1), \mathfrak{g}u = \mathfrak{F}q$, and $\mathfrak{g}v = \mathfrak{F}p$, for all $q, p \in \mathcal{X}$.

Definition 34. A self mapping $\mathfrak{F}$ satisfying inequality (113) is called $(\vartheta, \mathfrak{F})$-contraction, and the mapping which satisfies inequality (114) is called $(\vartheta, \mathfrak{F})$-generalized contraction.

Remark 35. If $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [-\infty,\infty)$ and a self mapping $\mathfrak{F}$ on $\mathcal{X}$ is (ϑ, α^+)-proximal contraction and (ϑ, α^+)-generalized proximal contraction then $\alpha(q, p) \geq 0$ implies that

$$\alpha(q, p) + \vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) \leq [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1} [\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p))]^{\varsigma_2}, \tag{115}$$

where $\vartheta \in \Delta_\vartheta, \varsigma \in (0, 1), u = \mathfrak{F}q$, and $v = \mathfrak{F}p$, for all $q, p \in \mathcal{X}$.

Definition 36. A self mapping $\mathfrak{F}$ on $\mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [-\infty,\infty)$ satisfying inequality (115) is called $(\vartheta, \alpha^+, \mathfrak{F})$-contraction.

Corollary 37. *Let $\mathcal{X}$ be a complete Branciari metric space. Let $\mathfrak{F} : \mathcal{X} \longrightarrow \mathcal{X}$ be a $(\vartheta, \mathfrak{g})$-proximal contraction. Then, the pair of mappings $(\mathfrak{F}, \mathfrak{g})$ has a coincidence point.*

Proof. Let $\mathcal{Q} = \mathcal{P} = \mathcal{X}$. We show that $\mathfrak{F}$ satisfies $(\vartheta, \mathfrak{F})$-contraction. Then, we have

$$\begin{aligned} \mathfrak{d}(\mathfrak{g}u, \mathfrak{F}q) &= \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0, \\ \mathfrak{d}(\mathfrak{g}v, \mathfrak{F}p) &= \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0, \end{aligned} \tag{116}$$

for all $q, p, u, v \in \mathcal{X}$, since $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0$ implies that $\mathfrak{g}u = \mathfrak{F}q$ and $\mathfrak{g}v = \mathfrak{F}p$ and since $\mathfrak{F}$ satisfies condition (113). So,

$$\begin{aligned} \vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) &\le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{g}q, \mathfrak{g}p))]^{\varsigma_2} \\ &= [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p))]^{\varsigma_2}, \end{aligned} \tag{117}$$

which implies that

$$\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) \le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p))]^{\varsigma_2}, \tag{118}$$

which further implies that $\mathfrak{F}$ is $(\vartheta, \mathfrak{F})$-contraction. As $\mathfrak{d}(\mathfrak{g}q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0$ then q is a coincidence point of $\mathfrak{F}$ and $\mathfrak{g}$.

Corollary 38. *Let $\mathcal{X}$ be a complete Branciari metric space and pair of mapping $\mathfrak{F}, \mathfrak{g} : \mathcal{X} \longrightarrow \mathcal{X}$ be a $(\vartheta, \mathfrak{g})$-generalized proximal contraction. Then, pair of mappings $(\mathfrak{F}, \mathfrak{g})$ has a coincidence point.*

Proof. Let $\mathcal{Q} = \mathcal{P} = \mathcal{X}$. We show that $\mathfrak{F}$ satisfies $(\vartheta, \mathfrak{F})$-generalized contraction. Then, we have

$$\begin{aligned} \mathfrak{d}(\mathfrak{g}u, \mathfrak{F}q) &= \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0, \\ \mathfrak{d}(\mathfrak{g}v, \mathfrak{F}p) &= \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0, \end{aligned} \tag{119}$$

for all $q, p, u, v \in \mathcal{X}$, since $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0$ implies that $\mathfrak{g}u = \mathfrak{F}q$ and $\mathfrak{g}v = \mathfrak{F}p$ and since $\mathfrak{F}$ satisfies condition (114). So,

$$\begin{aligned} \vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) = \vartheta(\mathfrak{d}(\mathfrak{g}u, \mathfrak{g}v)) &\le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(v, u))]^{\varsigma_2}, \\ &\le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}, \end{aligned} \tag{120}$$

which implies that

$$\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) \le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}, \tag{121}$$

which further implies that $\mathfrak{F}$ is $(\vartheta, \mathfrak{F})$-generalized contraction. As $\mathfrak{d}(\mathfrak{g}q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0$ then q is a coincidence point of $\mathfrak{F}$ and $\mathfrak{g}$.

If $\mathfrak{g}$ is an identity mapping then we will have the following corollary.

Corollary 39. *Let $\mathcal{X}$ be a complete Branciari metric space, $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [-\infty,\infty)$, and $\mathfrak{F} : \mathcal{X} \longrightarrow \mathcal{X}$ be a $(\vartheta, \alpha^+, \mathfrak{F})$-contraction. If $\{q_n\}$ is a sequence in $\mathcal{X}$ such that $\alpha(q_n, q_{n+1}) \ge 0$ and $q_n \longrightarrow q \in \mathcal{X}$ as $n \longrightarrow \infty$ then $\alpha(q_n, q) \ge 0$ for all $n \in \mathbb{N}$. Then, there exists a fixed point of $\mathfrak{F}$ provided that there exists $q_0 \in \mathcal{X}$ such that $\alpha(q_0, \mathfrak{F}q_0) \ge 0$. Moreover, if $\alpha(q, p) \ge 0$, for all $q, p \in \mathfrak{F}(\mathcal{X})$ then q^* is the unique fixed point of the mapping $\mathfrak{F}$.*

Proof. Let $\mathcal{Q} = \mathcal{P} = \mathcal{X}$. We show that $\mathfrak{F}$ satisfies (ϑ, α^+)-proximal contraction and (ϑ, α^+)-generalized proximal contraction. Then, we have

$$\begin{aligned} &\alpha(q, p) \ge 0, \\ &\mathfrak{d}(u, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0, \\ &\mathfrak{d}(v, \mathfrak{F}p) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0, \end{aligned} \tag{122}$$

for all $q, p, u, v \in \mathcal{X}$, since $\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0$ implies that $u = \mathfrak{F}q$ and $v = \mathfrak{F}p$ and since $\mathfrak{F}$ satisfies condition (115). So,

$$\begin{aligned} \alpha(q, p) + \vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) &= \alpha(q, p) + \vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) \\ &\le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p))]^{\varsigma_2} \\ &= [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}, \end{aligned} \tag{123}$$

which implies that

$$\alpha(q, p) + \vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) \le [\vartheta(\mathfrak{d}(\mathrm{q}, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}, \tag{124}$$

which further implies that $\mathfrak{F}$ is (ϑ, α^+)-proximal contraction. Also, we have

$$\begin{aligned} \alpha(q, p) + \vartheta(\mathfrak{d}(u, v)) &\le \alpha(q, p) + \vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p)) \\ &\le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{F}q, \mathfrak{F}p))]^{\varsigma_2} \\ &\le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}, \end{aligned} \tag{125}$$

which implies that

$$\alpha(q, p) + \vartheta(\mathfrak{d}(u, v)) \le [\vartheta(\mathfrak{d}(q, p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(u, v))]^{\varsigma_2}, \tag{126}$$

which further implies that $\mathfrak{F}$ is (ϑ, α^+)-generalized proximal contraction. Since $\mathfrak{F}$ is triangular proximal α^+-admissible, by Remark 32, $\mathfrak{F}$ is a triangular α^+-admissible mapping. If there exists $q \in \mathcal{X}$ such that $\alpha(q, \mathfrak{F}q) \ge 0$ and $\mathfrak{d}(q, \mathfrak{F}q) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = 0$, it gives that q is a fixed point of $\mathfrak{F}$.

Uniqueness. Let $\alpha(q, p) \ge 0$ for all $q, p \in \mathfrak{F}(\mathcal{X})$. Now we will show that q^* is a unique fixed point of $\mathfrak{F}$. On contrary, suppose that w^* be another fixed point of mapping $\mathfrak{F}$ with $q^* = w^*$. Hence,

$$\mathfrak{d}(\mathfrak{g}w^*, \mathfrak{F}w^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{127}$$

Then, by using the properties of $\mathfrak{F}$, we obtain that

$$\begin{aligned} \vartheta(\mathfrak{d}(q^*, w^*)) = \vartheta(\mathfrak{d}(\mathfrak{F}q^*, \mathfrak{F}w^*)) &\le \alpha(q^*, w^*) + \vartheta(\mathfrak{d}(\mathfrak{F}q^*, \mathfrak{F}w^*)) \\ &\le [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_1}[\vartheta(\mathfrak{d}(\mathfrak{F}q^*, \mathfrak{F}w^*))]^{\varsigma_2} \\ &= [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_1}[\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_2} \\ &= [\vartheta(\mathfrak{d}(q^*, w^*))]^{\varsigma_1+\varsigma_2} < [\vartheta(\mathfrak{d}(q^*, w^*))], \end{aligned} \tag{128}$$

which is a contradiction. Therefore, $q^* = w^*$. Hence, the mapping $\mathfrak{F}$ has a unique fixed point.

Remark 40. By taking $\alpha(q,p)=0$ and $\varsigma_2=0$, for all $q,p\in\mathcal{X}$ in the above Corollary 39, we obtain the main Theorem (1.7) in [4] as Corollary.

4. Application to Graph Theory

Let $\mathcal{X}$ be a nonempty set and define a set $\Delta=\{(q,p)\in\mathcal{X}\times\mathcal{X}, q,p\in\mathcal{X}\}$. A graph $\mathfrak{g}$ is a pair (V,E), where $V=V(\mathfrak{g})$ is a set of vertices coinciding with $\mathcal{X}$ and $E=E(\mathfrak{g})$ is a set of its edges such that $\Delta\subset E(\mathfrak{g})$. Moreover, we suppose that the graph $\mathfrak{g}$ is without parallel edges. By reversing the direction of edges in $\mathfrak{g}$, we obtain a graph $\mathfrak{g}^{-1}$ whose edge set and vertex set are defined as follows:

$$E\left(\mathfrak{g}^{-1}\right)=\left\{(q,p)\in\mathcal{X}^2 : (p,q)\in E(\mathfrak{g})\right\} \text{ and } V\left(\mathfrak{g}^{-1}\right)=V(\mathfrak{g}). \tag{129}$$

Consider the graph $\tilde{\mathfrak{g}}$ comprising of all vertices and edges of $\mathfrak{g}$ and $\mathfrak{g}^{-1}$, that is,

$$E(\tilde{\mathfrak{g}})=E(\mathfrak{g})\cup E\left(\mathfrak{g}^{-1}\right). \tag{130}$$

We denote the undirected graph by $\tilde{\mathfrak{g}}$ obtained by ignoring the direction of edges of $\mathfrak{g}$.

Definition 41 (see [24]).

(1) A subgraph H of a graph $\mathfrak{g}$ is consisting upon a subset of edges of graph $\mathfrak{g}$ and associated vertices.

(2) Let q and p be two vertices in a graph $\mathfrak{g}$. A path of length n (where $n\in\mathbb{N}\cup\{0\}$) in $\mathfrak{g}$ from q to p is a sequence $(q_i)_{i=0}^n$ of $n+1$ distinct vertices such that $q_0=q$,$q_n=p$, and $(q_i,q_{i+1})\in E(\mathfrak{g})$ for $i=1,2,\cdots,n$.

(3) A graph $\mathfrak{g}$ is called a connected graph if there exists a path between any two vertices of graph $\mathfrak{g}$, and it is said to be a weakly connected graph if $\tilde{\mathfrak{g}}$ is connected.

(4) A path is called elementary if no vertices appear more than one time in it.

Throughout this section, we suppose that $(\mathcal{X},\mathfrak{d})$ be a *Branciari metric space* and a graph $\mathfrak{g}$ may be transformed to a weighed graph by appointing to each edge the distance given by the Branciari metric between its vertices. In order to apply the rectangular inequality to the vertices of the graph $\mathfrak{g}$, we will consider the graph of length greater than 2, which signifies that between two vertices, we will obtain a path between two vertices.

Definition 42. Let $(\mathcal{Q},\mathcal{P})$ be a pair of nonempty subsets of a Branciari metric space $\mathcal{X}$ and $\mathfrak{g}$ be a directed graph without parallel edges such that $V(\mathfrak{g})=\mathcal{X}$. A mapping $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ is said to be $(\vartheta,\mathfrak{g})$-proximal contraction if for all $q,p\in\mathcal{Q}$, $q\neq p$ with $(q,p)\in E(\mathfrak{g})$ such that

$$\left.\begin{array}{l}\mathfrak{d}(u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies } [\vartheta(\mathfrak{d}(\mathfrak{F}q,\mathfrak{F}p))]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(v,u))]^{\varsigma_2},$$
$$(u,v)\in E(\mathfrak{g}), \tag{131}$$

where $,p,u,v\in\mathcal{Q}$,$\vartheta\in\Delta_\vartheta$, and $\varsigma_1,\varsigma_2\geq 0$ with $0<\varsigma_1+\varsigma_2<1$.

Definition 43. Let $(\mathcal{Q},\mathcal{P})$ be a pair of nonempty subsets of a Branciari metric space $\mathcal{X}$ and $\mathfrak{g}$ be a directed graph without parallel edges such that $V(\mathfrak{g})=\mathcal{X}$. A mapping $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ is said to be $(\vartheta,\mathfrak{g})$-generalized proximal contraction if for all $q,p\in\mathcal{Q}$,$q\neq p$ with $(q,p)\in E(\mathfrak{g})$ such that

$$\left.\begin{array}{l}\mathfrak{d}(u,\mathfrak{F}q)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\\ \mathfrak{d}(v,\mathfrak{F}p)=\mathfrak{d}(\mathcal{Q},\mathcal{P})\end{array}\right\}\text{implies } [\vartheta(\mathfrak{d}(u,v))]\leq[\vartheta(\mathfrak{d}(q,p))]^{\varsigma_1}[\vartheta(\mathfrak{d}(v,u))]^{\varsigma_2},$$
$$(u,v)\in E(\mathfrak{g}), \tag{132}$$

where $,p,u,v\in\mathcal{Q}$,$\vartheta\in\Delta_\vartheta$, and $\varsigma_1,\varsigma_2\geq 0$ with $0<\varsigma_1+\varsigma_2<1$.

Corollary 44. *Let $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ a $(\vartheta,\mathfrak{g})$-proximal contraction. If $\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathcal{P}_0$ and $\mathcal{Q}_0$ is nonempty closed subset in $\mathcal{X}$, and $(\mathcal{Q},\mathcal{P})$ satisfy the weak P-property. Also, there exists $q_0,q_1\in\mathcal{Q}_0$ such that there exists an elementary path between them and*

$$\mathfrak{d}(q_1,\mathfrak{F}q_0)=\mathfrak{d}(\mathcal{Q},\mathcal{P}). \tag{133}$$

Moreover, if there exists a path $(\mathfrak{v}_i^{\varsigma})_{i,\varsigma=0}^N\subseteq\mathcal{Q}_0$ in $\mathfrak{g}$ between any two elements q and p then $\mathfrak{F}$ has a unique best proximity point.

Proof. Let $q_0,q_1\in\mathcal{Q}_0$ such that $\mathfrak{d}(q_1,\mathfrak{F}q_0)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$. A path $(\mathfrak{v}_0^i)_{i=0}^N$ in $\mathfrak{g}$ is a sequence containing points of $\mathcal{Q}_0$. Consequently, $\mathfrak{v}_0^0=q_0$, $\mathfrak{v}_0^N=q_1$ and $(\mathfrak{v}_0^i,\mathfrak{v}_0^{i+1})\in E(\mathfrak{g})$ for all $0\leq i\leq N$. Given that $\mathfrak{v}_0^1\in\mathcal{Q}_0$, since $\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathcal{P}_0$ and from definition of $\mathcal{Q}_0$, there exists $\mathfrak{v}_1^1\in\mathcal{Q}_0$ such that $\mathfrak{d}(\mathfrak{v}_1^1,\mathfrak{F}\mathfrak{v}_0^1)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$. Similarly, for each $i=2,\cdots,N$, there exists $\mathfrak{v}_1^i\in\mathcal{Q}_0$ such that $\mathfrak{d}(\mathfrak{v}_1^i,\mathfrak{F}\mathfrak{v}_0^i)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$. As $(\mathfrak{v}_0^i)_{i=0}^N$ is a path in $\mathfrak{g}$ then $(\mathfrak{v}_0^0,\mathfrak{v}_0^1)=(q_0,\mathfrak{v}_0^1)\in E(\mathfrak{g})$. From the above argument, we have $\mathfrak{d}(q_1,\mathfrak{F}q_0)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$ and $\mathfrak{d}(\mathfrak{v}_1^1,\mathfrak{F}\mathfrak{v}_0^1)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$. Since, $\mathfrak{F}$ is $(\vartheta,\mathfrak{g})$-proximal contraction, it follows that $(q_1,\mathfrak{v}_1^1)\in E(\mathfrak{g})$. In similar manner, we have the following:

$$\left(\mathfrak{v}_1^{i-1},\mathfrak{v}_1^i\right)\in E(\mathfrak{g}),\ \text{for all } i=1,2,\cdots,N. \tag{134}$$

Let $q_2=\mathfrak{v}_1^N$ then $(\mathfrak{v}_1^i)_{i=0}^N$ is a path from $q_1=\mathfrak{v}_1^0$ to $q_2=\mathfrak{v}_1^N$. For each $i=1,2,3,\cdots,N$, as $\mathfrak{v}_1^i\in\mathcal{Q}_0$ and $\mathfrak{F}\mathfrak{v}_1^i\in\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathcal{P}_0$ then by definition of $\mathcal{P}_0$, there exists $\mathfrak{v}_2^i\in\mathcal{Q}_0$ such that $\mathfrak{d}(\mathfrak{v}_2^i,\mathfrak{F}\mathfrak{v}_1^i)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$. In addition, we have $\mathfrak{d}(q_2,\mathfrak{F}q_1)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$.

As above mentioned, we have

$$\left(q_2, \mathfrak{v}_2^1\right) \in E(\mathfrak{g}) \text{ and } \left(\mathfrak{v}_2^{i-1}, \mathfrak{v}_2^i\right) \in E(\mathfrak{g}), \text{ for all } i = 1, 2 \cdots, N. \tag{135}$$

Similarly, by $\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$, there exists a point $q_3 \in \mathcal{Q}_0$ where $q_3 = \mathfrak{v}_2^N$. Then, $(\mathfrak{v}_2^i)_{i=0}^N$ is a path from $\mathfrak{v}_2^0 = q_2$ and $\mathfrak{v}_2^N = q_3$. Continuing this way, we can obtain a sequence $\{q_n\}_{n\in\mathbb{N}}$ where $q_{n+1} \in [q_n]_{\mathfrak{g}}^N$ and $\mathfrak{d}(q_{n+1}, \mathfrak{F}q_n) = \mathfrak{d}(\mathcal{Q}, \mathcal{P})$ by producing a path $(\mathfrak{v}_n^i)_{i=0}^N$ from $q_n = \mathfrak{v}_{\mathrm{n}}^0$ and $q_{n+1} = \mathfrak{v}_n^N$ in such a way that

$$\mathfrak{d}\left(\mathfrak{v}_{n+1}^i, \mathfrak{F}\mathfrak{v}_n^i\right) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}), \tag{136}$$

for all $i = 0, 1, \cdots, N, n \in \mathbb{N}$. Thus, we have

$$\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{F}\mathfrak{v}_{n-1}^{i-1}\right) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}\left(\mathfrak{v}_n^i, \mathfrak{F}\mathfrak{v}_{n-1}^i\right), \text{ for all } i = 1, 2, \cdots, N, n \in \mathbb{N}. \tag{137}$$

Next, we claim that $\mathfrak{d}(q_n, q_{n+1}) \leq C/n^{1/Y}$, where C is a constant. To prove the claim, we need to consider the following two cases where $(\mathfrak{v}_n^i)_{i=0,1,\cdots,N}$ is a path from q_n to q_{n+1}. Note that for all $i = 0, 1, \cdots, N(\mathfrak{v}_n^i)_{i=0,1,\cdots,N}$ are different owing to the fact that the considered path $(\mathfrak{v}_n^i)$ is elementary. Then, we can apply the rectangular inequality.

Case 1. $N = 2k + 1$ (N is odd). For any positive integer n, we get

$$\begin{aligned}
\mathfrak{d}(q_n, q_{n+1}) &= \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^N\right) = \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^{2k+1}\right) \\
&\leq \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^1\right) + \mathfrak{d}\left(\mathfrak{v}_n^1, \mathfrak{v}_n^2\right) + \mathfrak{d}\left(\mathfrak{v}_n^2, \mathfrak{v}_n^{2k+1}\right) \\
&\leq \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^1\right) + \mathfrak{d}\left(\mathfrak{v}_{\mathrm{n}}^1, \mathfrak{v}_n^2\right) + \cdots + \mathfrak{d}\left(\mathfrak{v}_n^{2k}, \mathfrak{v}_n^{2k+1}\right) \\
&\leq \sum_{i=1}^{2k+1} \mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right).
\end{aligned} \tag{138}$$

Since $\mathfrak{F}$ is $(\vartheta, \mathfrak{g})$-proximal contraction, we have

$$\sum_{i=1}^{2k+1} \vartheta\left(\mathfrak{d}\left(\mathfrak{F}\mathfrak{v}_{n-1}^{i-1}, \mathfrak{F}\mathfrak{v}_{n-1}^i\right)\right) \leq \sum_{i=1}^{2k+1} \left(\left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{i-1}, \mathfrak{v}_{n-1}^i\right)\right)\right]^{\varsigma_1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{i-1}, \mathfrak{v}_{n-1}^i\right)\right)\right]^{\varsigma_2}\right). \tag{139}$$

Since pair of subsets $(\mathcal{Q}, \mathcal{P})$ satisfies the weak P-property then we have

$$\sum_{i=1}^{2k+1} \mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right) \leq \sum_{i=1}^{2k+1} \mathfrak{d}\left(\mathfrak{F}\mathfrak{v}_{n-1}^{i-1}, \mathfrak{F}\mathfrak{v}_{n-1}^i\right), \text{ for all } n \in \mathbb{N}. \tag{140}$$

Then, from above inequalities, we have

$$\begin{aligned}
\sum_{i=1}^{2k+1} \vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) &\leq \sum_{i=1}^{2k+1} \left(\left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{i-1}, \mathfrak{v}_{n-1}^i\right)\right)\right]^{\varsigma_1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{i-1}, \mathfrak{v}_{n-1}^i\right)\right)\right]^{\varsigma_2}\right) \\
&\quad \cdot \sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right)\right]^{1-\varsigma_2} \\
&\leq \sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{i-1}, \mathfrak{v}_{n-1}^i\right)\right)\right]^{\varsigma_1}.
\end{aligned} \tag{141}$$

After simplification, we have

$$\sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right)\right] \leq \sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{i-1}, \mathfrak{v}_{n-1}^i\right)\right)\right]^{(\varsigma_1/1-\varsigma_2)}. \tag{142}$$

By induction, it follows that for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right)\right] \leq \sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_0^{i-1}, \mathfrak{v}_0^i\right)\right)\right]^{(\varsigma_1/1-\varsigma_2)^n}, \tag{143}$$

which implies that

$$\sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right)\right] \leq \sum_{i=1}^{2k+1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_0^{i-1}, \mathfrak{v}_0^i\right)\right)\right]^{\nu^n} \text{ for all } n \in \mathbb{N}, \tag{144}$$

where $\nu = \varsigma_1/1 - \varsigma_2 < 1$. Taking limit $n \longrightarrow \infty$ in the above inequality, we have

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{2k+1} \vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) = 1, \tag{145}$$

which implies that

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{2k+1} \mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right) = 0. \tag{146}$$

Since $\vartheta \in \Delta_\vartheta$ then there exist $0 < Y < 1$ and $0 < \ell \leq \infty$, such that the following limit holds true:

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{2k+1} \frac{\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) - 1}{\left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y} = \ell. \tag{147}$$

Assume that $\ell < \infty$ and $C = \ell/2$. Thus, there exists $n_0 \in \mathbb{N}$, such that

$$\left| \sum_{i=1}^{2k+1} \frac{\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) - 1}{\left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y} - \ell \right| \leq C, \text{ for all } n \geq n_0. \tag{148}$$

Hence, we have

$$\sum_{i=1}^{2k+1} \frac{\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) - 1}{\left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y} \geq \ell - C = \frac{\ell}{2} = C, \text{ for all } n \geq n_0, \tag{149}$$

and so we obtain

$$n \sum_{i=1}^{2k+1} \left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y \leq \omega D \left[\sum_{i=1}^{2k+1} \vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) - 1\right], \text{ for all } n \geq n_0, \tag{150}$$

where $\mathfrak{d} = 1/C$. If $\ell = \infty$ then there exists $n_0 \in \mathbb{N}$, such that

$$\sum_{i=1}^{2k+1} \frac{\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) - 1}{\left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y} \geq C, \text{ for all } n \geq n_0, \tag{151}$$

which implies

$$n \sum_{i=1}^{2k+1} \left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y \leq \omega D \left[\sum_{i=1}^{2k+1} \vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) - 1\right], \text{ for all } n \geq n_0, \tag{152}$$

where $\mathfrak{d} = 1/C$. Hence, in all cases, there exist $\mathfrak{d} > 0$ and $n_0 \in \mathbb{N}$, such that

$$n \sum_{i=1}^{2k+1} \left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y \leq \omega D \left[\sum_{i=1}^{2k+1} \vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) - 1\right], \text{ for all } n \geq n_0. \tag{153}$$

From inequalities (144) and (153), we have

$$n \sum_{i=1}^{2k+1} \left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y \leq \omega D \left[\sum_{i=1}^{2k+1} \left(\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_0^{i-1}, \mathfrak{v}_0^i\right)\right)\right)^{\nu^n} - 1\right], \text{ for all } n \geq n_0, \tag{154}$$

and taking limit as $n \longrightarrow \infty$ on both sides of the above inequality, we obtain

$$n \sum_{i=1}^{2k+1} \left[\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right]^Y = 0. \tag{155}$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{2k+1} \mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right) \leq \frac{1}{n^{1/Y}}, \text{ for all } n \geq n_0. \tag{156}$$

By putting inequality (156) in inequality (138), we get

$$\mathfrak{d}(q_n, q_{n+1}) \leq \frac{C}{n^{1/Y}}, \tag{157}$$

where $C = 1$.

Case 2. $N = 2k$ (N is even).

$$\begin{aligned} \mathfrak{d}(q_n, q_{n+1}) &= \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^N\right) = \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^{2k}\right) \leq \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^1\right) + \mathfrak{d}\left(\mathfrak{v}_n^1, \mathfrak{v}_n^2\right) \\ &\quad + \mathfrak{d}\left(\mathfrak{v}_n^2, \mathfrak{v}_n^{2k}\right) \leq \mathfrak{d}\left(\mathfrak{v}_n^0, \mathfrak{v}_n^1\right) + \mathfrak{d}\left(\mathfrak{v}_n^1, \mathfrak{v}_n^2\right) + \cdots + \mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right) \\ &= \sum_{i=1}^{2k} \mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right) - \mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k-1}\right) - \mathfrak{d}\left(\mathfrak{v}_n^{2k-1}, \mathfrak{v}_n^{2k}\right) \\ &\quad + \mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right) \leq \sum_{i=1}^{2k} \mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right) + \mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right). \end{aligned} \tag{158}$$

By the same arguments used in Case 1, we deduce that

$$\sum_{i=1}^{2k} \mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right) \leq \frac{1}{n^{1/Y}}, \text{ for all } n \geq n_0. \tag{159}$$

Indeed, from (136), we have

$$\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{F}\mathfrak{v}_{n-1}^{2k-2}\right) = \mathfrak{d}\left(\mathfrak{v}_n^{2k}, \mathfrak{F}\mathfrak{v}_{n-1}^{2k}\right) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{160}$$

Since $\mathfrak{F}$ is $(\vartheta, \mathfrak{g})$-proximal contraction, we have

$$\vartheta\left(\mathfrak{d}\left(\mathfrak{F}\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{F}\mathfrak{v}_{n-1}^{2k}\right)\right) \leq \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{v}_{n-1}^{2k}\right)\right)\right]^{\varsigma_1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right)\right)\right]^{\varsigma_2}. \tag{161}$$

The pair $(\mathcal{Q}, \mathcal{P})$ satisfies weak P-property; then, we can write

$$\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right) \leq \mathfrak{d}\left(\mathfrak{F}\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{F}\mathfrak{v}_{n-1}^{2k}\right). \tag{162}$$

Since ϑ is increasing then the above inequality becomes

$$\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right)\right) \leq \vartheta\left(\mathfrak{d}\left(\mathfrak{F}\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{F}\mathfrak{v}_{n-1}^{2k}\right)\right). \tag{163}$$

Then, from inequalities (161) and (163), we have

$$\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right)\right) \leq \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{v}_{n-1}^{2k}\right)\right)\right]^{\varsigma_1} \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right)\right)\right]^{\varsigma_2}. \tag{164}$$

After further simplifications,

$$\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right)\right)\Big]^{1-\varsigma_2} \leq \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{v}_{n-1}^{2k}\right)\right)\right]^{\varsigma_1}. \tag{165}$$

By induction, it follows that for all $n \in \mathbb{N}$,

$$\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right)\right)\Big] \leq \vartheta\left(\mathfrak{d}\left(\mathfrak{v}_0^{2k-2}, \mathfrak{v}_0^{2k}\right)\right)\Big]^{(\varsigma_1/1-\varsigma_2)^n}, \text{ for all } n \in \mathbb{N}, \tag{166}$$

where $\nu = \varsigma_1/1 - \varsigma_2$. Taking the limit $n \longrightarrow \infty$ in above inequality, we have

$$\lim_{n\longrightarrow\infty}\left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2},\mathfrak{v}_n^{2k}\right)\right)\right]=1. \tag{167}$$

Similarly, from condition (ϑ_2), we have

$$\lim_{n\longrightarrow\infty}\mathfrak{d}\left(\mathfrak{v}_n^{2k-2},\mathfrak{v}_n^{2k}\right)=0. \tag{168}$$

Similarly, from condition (ϑ_3), there exists $n_1\in\mathbb{N}$, such that

$$\mathfrak{d}\left(\mathfrak{v}_n^{2k-2},\mathfrak{v}_n^{2k}\right)\le\frac{1}{n^{1/Y}},\ \text{for all } n\ge n_1. \tag{169}$$

Then, from inequalities (158), (159), and (169), we have

$$\mathfrak{d}(q_n,q_{n+1})\le\frac{C}{n^{1/Y}},\ \text{for all } n\ge n_1. \tag{170}$$

where $C=2$. Let us prove that $\{q_n\}$ is a Cauchy sequence. Let $n,m\in N$ such that $m>n$. We suppose that m is odd $(m=2k+1)$ since the case $m=2k$ is similar. Note that $q_n=\mathfrak{v}_0^n$,$q_{n+1}=\mathfrak{v}_n^N$, and $\mathfrak{v}_0^n\ne\mathfrak{v}_n^N$ for all n since the path $(\mathfrak{v}_n^i)_{i=0,1,\cdots,N}$ is elementary. Then, by using the rectangular metric, we obtain

$$\begin{aligned}\mathfrak{d}(q_n,q_m)&\le\mathfrak{d}(q_n,q_{n+1})+\mathfrak{d}(q_{n+1},q_{n+2})\\&\quad+\mathfrak{d}(q_{n+2},q_{n+3})+\cdots+\mathfrak{d}(q_{m-1},q_m)\\&\le\frac{C}{n^{1/Y}}+\frac{C}{(n+1)^{1/Y}}+\frac{C}{(n+2)^{1/Y}}+\cdots\\&\quad+\frac{C}{(n+m-2)^{1/Y}}\le\sum_{i=n}^{\infty}\frac{C}{i^{1/Y}},\end{aligned} \tag{171}$$

for all $m\ge n\ge N$, where $N=\max\{n_0,n_1\}$. Hence, we obtain

$$\mathfrak{d}(q_n,q_m)\le\sum_{i=n}^{\infty}\frac{C}{i^{1/Y}},\ \text{for all } n\ge N, m\in\mathbb{N}. \tag{172}$$

We deduce that $\{q_n\}$ is a Cauchy sequence in $\mathcal{Q}_0\subseteq\mathcal{Q}\subset\mathcal{X}$. By using the completeness of Branciari metric space $\mathcal{X}$ and closeness of $\mathcal{Q}_0$, there exists $q^*\in\mathcal{Q}_0$ such that $q_n\longrightarrow q^*\in\mathcal{Q}_0$. Since $q^*\in\mathcal{Q}_0$,$\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathcal{P}_0$ then $\mathfrak{F}q^*\in\mathcal{P}_0$, thus there exists a point $z\in\mathcal{Q}_0$ such that z=q^* and

$$\mathfrak{d}(z,\mathfrak{F}q^*)=\mathfrak{d}(\mathcal{Q},\mathcal{P}). \tag{173}$$

As $\mathfrak{F}$ is $(\vartheta,\mathfrak{g})$-proximal contraction, we obtain

$$\vartheta(\mathfrak{d}(\mathfrak{F}q^*,\mathfrak{F}q_n))\le[\vartheta(\mathfrak{d}(q^*,q_n))]^{\varsigma_1}[\vartheta(\mathfrak{d}(q_{n+1},z))]^{\varsigma_2}. \tag{174}$$

Then, by using the weak P-property of subsets $(\mathcal{Q},\mathcal{P})$ of Branciari metric space,we have

$$\vartheta(\mathfrak{d}(z,q_{n+1}))\le\vartheta(\mathfrak{d}(\mathfrak{F}q^*,\mathfrak{F}q_n)). \tag{175}$$

Hence, we have

$$\vartheta(\mathfrak{d}(z,q_{n+1}))\le[\vartheta(\mathfrak{d}(q^*,q_n))]^{\varsigma_1}[\vartheta(\mathfrak{d}(q_{n+1},z))]^{\varsigma_2}. \tag{176}$$

Further,

$$[\vartheta(\mathfrak{d}(q_{n+1},z))]^{1-\varsigma_2}\le[\vartheta(\mathfrak{d}(q^*,q_n))]^{\varsigma_1}<[\vartheta(\mathfrak{d}(q^*,q_n))]^{1-\varsigma_2}. \tag{177}$$

This implies that

$$[\vartheta(\mathfrak{d}(q_{n+1},z))]<[\vartheta(\mathfrak{d}(q^*,q_n))]. \tag{178}$$

As ϑ is increasing, we have

$$\mathfrak{d}(q_{n+1},z)<\mathfrak{d}(q^*,q_n). \tag{179}$$

Then, by rectangular property and using above inequality and inequality (170), we have

$$\begin{aligned}\mathfrak{d}(q^*,z)&\le\mathfrak{d}(q^*,q_n)+\mathfrak{d}(q_n,q_{n+1})+\mathfrak{d}(q_{n+1},z)\\&<\mathfrak{d}(q^*,q_n)+\frac{C}{n^{1/Y}}+\mathfrak{d}(q^*,q_n).\end{aligned} \tag{180}$$

Now, by taking limit as $n\longrightarrow\infty$, we have $z=q^*$.

$$\mathfrak{d}(\mathfrak{g}q^*,\mathfrak{F}q^*)=\mathfrak{d}(\mathcal{Q},\mathcal{P}). \tag{181}$$

Thus, q^* is a coincidence best proximity point of pair of mappings $(\mathfrak{F},\mathfrak{g})$.

Uniqueness. We now show that q^* is a unique best proximity point of $\mathfrak{F}$. Let w be another best proximity point of $\mathfrak{F}$ with q^*=w. Hence,

$$\mathfrak{d}(\mathcal{Q},\mathcal{P})=\mathfrak{d}(w,\mathfrak{F}w). \tag{182}$$

Then, by using properties of $\mathfrak{F}$ and $\mathfrak{g}$ and reasoning as above, we obtain that

$$\begin{aligned}[\vartheta(\mathfrak{d}(q^*,w))]&\le\vartheta(\mathfrak{d}(\mathfrak{F}q^*,\mathfrak{F}w))\le[\vartheta(\mathfrak{d}(q^*,w))]^{\varsigma_1}[\vartheta(\mathfrak{d}(q^*,w))]^{\varsigma_2}\\&\le[\vartheta(\mathfrak{d}(q^*,w))]^{\varsigma_1+\varsigma_2}<[\vartheta(\mathfrak{d}(q^*,w))],\end{aligned} \tag{183}$$

which is a contradiction. Therefore, $q^*=w$. Hence, $\mathfrak{F}$ has a unique best proximity point.

Corollary 45. *Let $\mathfrak{F}:\mathcal{Q}\longrightarrow\mathcal{P}$ be a $(\vartheta,\mathfrak{g})$-generalized proximal contraction. Moreover, $\mathcal{Q}_0$ is a nonempty closed subset in Branciari metric space $\mathcal{X}$ and $\mathfrak{F}(\mathcal{Q}_0)\subseteq\mathcal{P}_0$. If there exists a path $(z^i)_{i=0}^{\mathfrak{v}}\subseteq\mathcal{Q}_0$ in $\mathfrak{g}$ between two elements q and p then there exists a unique best proximity of $\mathfrak{F}$ provided that there exists $q_0,q_1\in\mathcal{Q}_0$ such that there is an elementary path in $\mathcal{Q}_0$ between them and*

$$\mathfrak{d}(q_1,\mathfrak{F}q_0)=\mathfrak{d}(\mathcal{Q},\mathcal{P}). \tag{184}$$

Proof. Let $q_0,q_1\in\mathcal{Q}_0$ such that $\mathfrak{d}(q_1,\mathfrak{F}q_0)=\mathfrak{d}(\mathcal{Q},\mathcal{P})$. A path

$(\mathfrak{v}_0^i)_{i=0}^N$ in $\mathfrak{g}$ is a sequence containing points of $\mathcal{Q}_0$. Following arguments similar to those given in the proof of Corollary 44, we obtain

$$\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{F}\mathfrak{v}_{n-1}^{i-1}\right) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}\left(\mathfrak{v}_n^i, \mathfrak{F}\mathfrak{v}_{n-1}^i\right), \text{for all } i = 1, 2, \cdots, N, n \in \mathbb{N} \cup \{0\}. \tag{185}$$

Since $\mathfrak{F}$ is $(\vartheta, \mathfrak{g})$-generalized proximal contraction, we have

$$\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{i-1}, \mathfrak{v}_n^i\right)\right) \leq \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{i-1}, \mathfrak{v}_{n-1}^i\right)\right)\right]^{\varsigma_1}\left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^i, \mathfrak{v}_n^{i-1}\right)\right)\right]^{\varsigma_2}. \tag{186}$$

Again by using the arguments similar to those given in the proof of Corollary 44, we have

$$\mathfrak{d}(q_n, q_{n+1}) \leq \frac{C}{n^{1/Y}}, \text{for all } n \geq n_0, \tag{187}$$

where $C = 1$. Again by using the arguments similar to those given in the proof of Corollary 44 and since $\mathfrak{F}$ is $(\vartheta, \mathfrak{g})$ -proximal contraction, we have

$$\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_n^{2k}\right)\right) \leq \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{v}_{n-1}^{2k}\right)\right)\right]^{\varsigma_1}\left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k}, \mathfrak{v}_n^{2k-2}\right)\right)\right]^{\varsigma_2} \cdot \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_n^{2k-2}, \mathfrak{v}_{\mathrm{n}}^{2k}\right)\right)\right]^{1-\varsigma_2} \leq \left[\vartheta\left(\mathfrak{d}\left(\mathfrak{v}_{n-1}^{2k-2}, \mathfrak{v}_{n-1}^{2k}\right)\right)\right]^{\varsigma_1}. \tag{188}$$

By using the arguments similar to those given in the proof of Corollary 44, we deduce that $\{q_n\}$ is a Cauchy sequence in $\mathcal{Q}_0 \subseteq \mathcal{Q} \subset \mathcal{X}$. By using the completeness of space $\mathcal{X}$ and $\mathcal{Q}_0$ is closed, there exists $q^* \in \mathcal{Q}_0$ such that $q_n \longrightarrow q^* \in \mathcal{Q}_0$. Since $q^* \in \mathcal{Q}_0$,$\mathfrak{F}(\mathcal{Q}_0) \subseteq \mathcal{P}_0$ then $\mathfrak{F}q^* \in \mathcal{P}_0$; thus, there exists a point $z \in \mathcal{Q}_0$ such that z=q^* and

$$\mathfrak{d}(z, \mathfrak{F}q^*) = \mathfrak{d}(\mathcal{Q}, \mathcal{P}). \tag{189}$$

As $\mathfrak{F}$ is $(\vartheta, \mathfrak{g})$-generalized proximal contraction, we have

$$\vartheta(\mathfrak{d}(z, q_{n+1})) \leq [\vartheta(\mathfrak{d}(q^*, q_n))]^{\varsigma_1}[\vartheta(\mathfrak{d}(q_{n+1}, z))]^{\varsigma_2}[\vartheta(\mathfrak{d}(z, q_{n+1}))]^{1-\varsigma_2} \leq [\vartheta(\mathfrak{d}(q^*, q_n))]^{\varsigma_1} < [\vartheta(\mathfrak{d}(q^*, q_n))]^{1-\varsigma_2}. \tag{190}$$

Following arguments similar to those given in the proof of Corollary 44, we obtain that q^* is the best proximity point of the mapping $\mathfrak{F}$.

Uniqueness. Now we have to show that q^* is a unique best proximity point of $\mathfrak{F}$. Let w be another best proximity point of $\mathfrak{F}$ with q^*=w. Hence,

$$\mathfrak{d}(\mathcal{Q}, \mathcal{P}) = \mathfrak{d}(w, \mathfrak{F}w). \tag{191}$$

Then, by using properties of $\mathfrak{F}$ and $\mathfrak{g}$ and reasoning as above, we obtain that

$$[\vartheta(\mathfrak{d}(q^*, w))] \leq [\vartheta(\mathfrak{d}(q^*, w))]^{\varsigma_1}[\vartheta(\mathfrak{d}(q^*, w))]^{\varsigma_2} \leq [\vartheta(\mathfrak{d}(q^*, w))]^{\varsigma_1+\varsigma_2} < [\vartheta(\mathfrak{d}(q^*, w))], \tag{192}$$

which is a contradiction. Therefore, $q^* = w$. Hence, $\mathfrak{F}$ has a unique best proximity point.

5. Conclusion

In this paper, we define $(\vartheta, \alpha^+, \mathfrak{g})$-proximal contraction and provide the existence results for coincidence best proximity point in Branciari metric space. The important aspect of Branciari metric space is that it is not continuous; we dealt with discontinuity of Branciari metric space and obtained the desired results. As an application, we derive the coincidence point and fixed point results for some self mappings. We also introduce the notion of $(\vartheta, \mathfrak{g})$-proximal contraction and provide an application to graph theory in the setting of Branciari metric space. Some examples are also provided to illustrate the novelty of the result proved herein.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors have read and agreed to the published version of the manuscript.

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Research Article

Best Proximity Coincidence Point Results for (α, D)-Proximal Generalized Geraghty Mappings in JS-Metric Spaces

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We introduce a type of Geraghty contractions in a JS-metric space X, called (α, D)-proximal generalized Geraghty mappings. By using the triangular-(α, D)-proximal admissible property, we obtain the existence and uniqueness theorem of best proximity coincidence points for these mappings together with some corollaries and illustrative examples. As an application, we give a best proximity coincidence point result in X endowed with a binary relation.

1. Introduction and Preliminaries

Let $T : A \to B$ be a map where A and B are two nonempty subsets of a metric space X. It is known that if T is a non-self-map, the equation $Tx = x$ does not always have a solution, and it clearly has no solution when A and B are disjoint. However, it is possible to determine an approximate solution x^* such that the error is $d(x^*, Tx^*) = d(A, B)$. Such point x^* is called a best proximity point of T. The best proximity point theorem was first studied in [1]. Then, there has been a wide range of research in this framework. Many researchers have studied and generalized the result in many aspects (for example, see [2–15]). For some recent articles regarding these points, see [16, 17] where Geraghty type mappings were studied and [18] where cyclic and noncyclic nonexpansive mappings were considered.

One of the well-known generalizations of the Banach contraction principle is the result given by Geraghty [19] which enriches the principle by considering the class of mappings $\theta : [0,\infty) \to [0, 1)$ such that $t_n \to 0$ when $\theta(t_n) \to 1$. By including 1 in the ranges of those mappings θ, Ayari [20] provided a new result on the existence and uniqueness of the best proximity point for α-proximal Geraghty mappings.

The concept of the best proximity coincidence point, which is an extension of a best proximity point problem, was mentioned in [21] (see also [22]) where some results of mappings in generalized metric spaces were presented. A point a is called a best proximity coincidence point of the pair (g, T), where g is a self-map on A, if $d(ga, Ta) = d(A, B)$. Clearly, if g is the identity map, then each best proximity coincidence point of the pair (g, T) is a best proximity point for T.

A large number of results concerning these point problems in various metric spaces have been investigated since then. Hussain and the coauthors contributed several interesting results and generalizations in [23–25], including the recent article [26] where best proximity point results for Suzuki-Edelstein proximal contractions were studied. (See also, [27–31] for his work.)

Zhang and Su [32] weakened the P-property, called the weak P-property, and improved a best proximity point theorem for Geraghty nonself-contractions. In 2018, Komal et al. [33] obtained best proximity coincidence point theorems for α-Geraghty contractions (g, T) in metric spaces by using the weak P-property where g is an isometry.

The concept of generalized metric spaces (or JS-metric spaces) was introduced in [34] in 2015. It is a generalization of standard metric spaces covering many topological structures.

Let X be a nonempty set, and let $D : X \times X \to [0,\infty]$ be a function. For each $x \in X$, we set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0 \right\}. \tag{1}$$

Definition 1 (see [34]). *A function $D : X \times X \to [0,\infty]$ is called a generalized metric on X if it satisfies the following conditions.*

(D_1)For any $x, y \in X$, $D(x, y) = 0$ implies $x = y$.

(D_2)For any $x, y \in X$, $D(x, y) = D(y, x)$.

(D_3)There exists a constant $C_X > 0$ such that

$$D(x, y) \le C_x \limsup_{n\to\infty} D(x_n, y), \tag{2}$$

whenever $x, y \in X$ and $\{x_n\} \in C(D, X, x)$.

In this case, we say that (X, D) is a generalized metric space. It is, however, usually called a JS-metric space.

Remark 2. We note that, in general, results of best proximity points using the weak P -property in usual metric spaces might not be attained in the setting of JS -metric spaces. For example, $D(x, x)$ is not necessarily equal to 0, and $D(x_n, y_n)$ might not converge to $D(x, y)$ when $x_n \to x$ and $y_n \to y$.

Let $X \coloneqq (X, D)$ be a JS-metric space. We now discuss the convergence and the continuity in these spaces.

Definition 3 (see [34]). *Let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to D -converge to $x \in X$ if $\{x_n\} \in C(D, X, x)$. Moreover, $\{x_n\}$ is called a D -Cauchy sequence if $\lim_{m,n\to\infty} D(x_n, x_m) = 0$. Finally, (X, D) is said to be D -complete if each D -Cauchy sequence in X is a D -convergent sequence in X.*

Proposition 4 (see [34]). *For any $x, y \in X$, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, then $x = y$.*

Definition 5 (see [34]). *A function $f : X \to X$ is said to be D -continuous at a point $x_0 \in X$ if for any $\{x_n\} \in C(D, X, x_0)$, $\{f x_n\} \in C(D, X, f x_0)$. In addition, f is said to be D -continuous on X if it is D -continuous at each point in X.*

The concept of α-admissible mapping was introduced by Samet et al. [35] in 2012. The notion of triangular α-admissible mappings was then given by Karapinar [36]. Recently, Khemphet [37] extended the concept as follows.

Definition 6 (see [37]). *Let (X, D) be a generalized metric space, and let f and g be self-mappings on X . Given that $\alpha : X \times X \to [0,\infty)$ is a function, f is said to be triangular- (α, D) -admissible w.r.t. g if, for all $x, y, z \in X$, the following conditions hold.*

(i) If $\alpha(gx, gy) \ge 1$, then $\alpha(fx, fy) \ge 1$ and $D(gx, gy) < \infty$.

(ii) If $\alpha(x, z) \ge 1$ and $\alpha(z, y) \ge 1$, then $\alpha(x, y) \ge 1$.

In this article, we introduce a type of Geraghty contractions which will be called (α, D)-proximal generalized Geraghty mappings. These maps are motivated by the work of Khemphet [37]. Using the weak P-property in the setting of JS-metric space, we establish a result on the existence and uniqueness of the best proximity coincidence point for these mappings. Examples showing the validity of the main result and some corollaries are listed. Finally, by applying our main result, we obtain a best proximity coincidence point result in X endowed with a binary relation. Note that some other results of best proximity points in X endowed with binary relations can be deduced from our result.

2. Main Results

Throughout this article, let $X \coloneqq (X, D)$ be a JS-metric space, and let A and B be nonempty disjoint subsets of X. Also, we require the following notations:

$$\begin{aligned} D(A, B) &\coloneqq \inf \{D(a, b) : a \in A, b \in B\}, \\ A_0 &\coloneqq \{a \in A : \text{there exists } b \in B \text{ such that } D(a, b) = D(A, B)\}, \\ B_0 &\coloneqq \{b \in B : \text{there exists } a \in A \text{ such that } D(a, b) = D(A, B)\}. \end{aligned} \tag{3}$$

Clearly, if one of A_0 and B_0 is nonempty, then so is the other.

Definition 7 (see [21]). *Let $T : A \to B$ and $S : A \to A$ be mappings. An element $x^* \in A$ is said to be a best proximity coincidence point of the pair (S, T) if $D(Sx^*, Tx^*) = D(A, B)$. The set of all best proximity coincidence points of the pair (S, T) is denoted by $BC(S, T)$.*

Definition 8 (see [32]). *Suppose that A_0 is nonempty. The pair (A, B) is said to have the weak P -property if and only if $D(x_1, y_1) = D(x_2, y_2) = D(A, B)$ implies $D(x_1, x_2) \le D(y_1, y_2)$, where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.*

Definition 9. Let $T : A \to B$ and $S : A \to A$ be mappings. The pair (S, T) is said to be triangular- (α, D) -proximal admissible if the following conditions hold.

(i) If $\alpha(Sx_1, Sx_2) \ge 1$ and $D(Su_1, Tx_1) = D(Su_2, Tx_2) = D(A, B)$, then $\alpha(Su_1, Su_2) \ge 1$ and $D(Su_1, Su_2) < \infty$ for all $x_1, x_2, u_1, u_2 \in A$.

(ii) If $\alpha(x, z) \ge 1$ and $\alpha(z, y) \ge 1$, then $\alpha(x, y) \ge 1$, for all $x, y, z \in X$.

We consider the class of mappings Θ which is a slight generalization of the well-known class of $[0,1)$-valued functions introduced by Geraghty [19]:

$$\Theta := \{\theta : [0,\infty] \to [0,1]: \theta(t_n) \to 1 \text{ implies } t_n \to 0\}. \quad (4)$$

Now, we introduce a class of our contractions as follows.

Definition 10. Let $T : A \to B$ and $S : A \to A$ be mappings. Given that $\alpha : X \times X \to [0,\infty)$ is a function, the pair (S,T) is said to be an (α, D) -proximal generalized Geraghty mapping if the following conditions hold.

(i) (S,T) is triangular-(α,D)-proximal admissible.

(ii) There is $\theta \in \Theta$ such that for all $x, y, u, v \in A$, if $D(Su, Tx) = D(Sv, Ty) = D(A,B)$ and $\alpha(Sx, Sy) \geq 1$, then

$$\alpha(Sx, Sy)D(Tx, Ty) \leq \theta(M(x,y,u,v))M(x,y,u,v), \quad (5)$$

where $M(x,y,u,v) = \max\{D(Sx,Sy), D(Sx,Su), D(Sy,Sv)\}$.

We first give a useful lemma.

Lemma 11. *Let $\alpha : X \times X \to [0,\infty)$ be a function. Let $T : A \to B$ and $S : A \to A$ be two mappings such that (S,T) is an (α, D) -proximal generalized Geraghty mapping, and let (A, B) have the weak P -property. If $\alpha(Sx, Sy) \geq 1$ for all $x, y \in BC(S,T)$, then $Sx = Sy$.*

Proof. Let $x, y \in BC(S,T)$, we have that

$$D(Sx, Tx) = D(Sy, Ty) = D(A,B). \quad (6)$$

From the assumption, $\alpha(Sx, Sx) \geq 1$, $\alpha(Sy, Sy) \geq 1$, and $\alpha(Sx, Sy) \geq 1$. Since $\alpha(Sx, Sy) \geq 1$, (S,T) is triangular-(α,D)-proximal admissible and (6), we have that $D(Sx, Sy) < \infty$. Also, since $D(Sx, Tx) = D(Sx, Tx) = D(A,B)$, $\alpha(Sx, Sx) \geq 1$ and (S,T) is triangular-(α,D)-proximal admissible, then $D(Sx, Sx) < \infty$. Similarly, we can show that $D(Sy, Sy) < \infty$.

Note that

$$\begin{aligned} M(x,x,x,x) &= \max\{D(Sx,Sx), D(Sx,Sx), D(Sx,Sx)\} \\ &= D(Sx,Sx) < \infty. \end{aligned} \quad (7)$$

Since (S,T) is an (α,D)-proximal generalized Geraghty mapping, and (A,B) has the weak P-property,

$$\begin{aligned} D(Sx,Sx) &\leq \alpha(Sx,Sx)D(Sx,Sx) \leq \alpha(Sx,Sx)D(Tx,Tx) \\ &\leq \theta(D(Sx,Sx))D(Sx,Sx), \end{aligned} \quad (8)$$

for some $\theta \in \Theta$. From the property of θ, we can conclude that $D(Sx, Sx) = 0$. Similarly, we also have that $D(Sy, Sy) = 0$. Then,

$$\begin{aligned} M(x,y,x,y) &:= \max\{D(Sx,Sy), D(Sx,Sx), D(Sy,Sy)\} \\ &= D(Sx,Sy) < \infty. \end{aligned} \quad (9)$$

Since $\alpha(Sx, Sy) \geq 1$, we have that

$$\begin{aligned} D(Sx,Sy) &\leq \alpha(Sx,Sy)D(Sx,Sy) \leq \alpha(Sx,Sy)D(Tx,Ty) \\ &\leq \theta(M(x,y,x,y))M(x,y,x,y) \\ &= \theta(D(Sx,Sy))D(Sx,Sy), \end{aligned} \quad (10)$$

for some $\theta \in \Theta$. Thus, $D(Sx, Sy) = 0$ which implies that $Sx = Sy$.

Theorem 12. *Let $\emptyset \neq A_0 \subseteq S(A_0)$, and let $(S(A_0), D)$ be D -complete. Given that $\alpha : X \times X \to [0,\infty)$ is a function, and let $T : A \to B$ and $S : A \to A$ be mappings such that (S,T) is an (α, D) -proximal generalized Geraghty mapping. Suppose that the following conditions hold.*

(i) $T(A_0) \subseteq B_0$ and the pair (A,B) has the weak P -property.

(ii) There exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A,B)$, $\alpha(Sy, Sx) \geq 1$ and $D(Sy, Sx) < \infty$.

(iii) For $\{Sx_n\} \in C(D, S(A_0), Sx^*)$ such that $\alpha(Sx_n, Sx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there is a subsequence $\{Sx_{n_k}\}$ with $\alpha(Sx_{n_k}, Sx^*) \geq 1$ for all $k \in \mathbb{N}$.

Then, there exists $x^* \in A_0$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if $\alpha(Sx^*, Sy^*) \geq 1$ for all $x^*, y^* \in BC(S,T)$ and S is injective, then (S,T) has a unique best proximity coincidence point.

Proof. From (ii), there exist $x_0, x_1 \in A_0$ such that

$$D(Sx_1, Tx_0) = D(A,B), \alpha(Sx_0, Sx_1) \geq 1, D(Sx_0, Sx_1) < \infty. \quad (11)$$

Since $T(A_0) \subseteq B_0$, $A_0 \subseteq S(A_0)$, and (S,T) is triangular-(α, D)-proximal admissible, there exists $x_2 \in A_0$ such that

$$D(Sx_2, Tx_1) = D(A,B), \alpha(Sx_1, Sx_2) \geq 1, D(Sx_1, Sx_2) < \infty. \quad (12)$$

Continuing in this way, we obtain a sequence $\{Sx_n\} \subseteq S(A_0)$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} D(Sx_n, Tx_{n-1}) = D(A,B) = D(Sx_{n+1}, Tx_n), \alpha(Sx_{n-1}, Sx_n) \\ \geq 1, D(Sx_{n-1}, Sx_n) < \infty. \end{aligned} \quad (13)$$

Using the weak P-property to (13), for n and $n+1$, we have that

$$D(Sx_n, Sx_{n+1}) \leq D(Tx_{n-1}, Tx_n) \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $Sx_{n_0} = Sx_{n_0+1}$, then from (13),

$$D\left(Sx_{n_0+1}, Tx_{n_0}\right) = D\left(Sx_{n_0}, Tx_{n_0}\right) = D(A, B). \tag{15}$$

Now suppose that $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By the definition of D, $D(Sx_n, Sx_{n+1}) \neq 0$. We will first show that $\lim_{n\to\infty} D(Sx_{n-1}, Sx_n) = 0$. Let $n \in \mathbb{N}$. Since (S, T) is an (α, D) -proximal generalized Geraghty mapping together with (13) and (14), we obtain that

$$\begin{aligned} D(Sx_n, Sx_{n+1}) &\leq D(Tx_{n-1}, Tx_n) \\ &\leq \alpha(Sx_{n-1}, Sx_n) D(Tx_{n-1}, Tx_n) \\ &\leq \theta(M(x_{n-1}, x_n, x_n, x_{n+1})) M(x_{n-1}, x_n, x_n, x_{n+1}) \\ &\leq M(x_{n-1}, x_n, x_n, x_{n+1}), \end{aligned} \tag{16}$$

where

$$M(x_{n-1}, x_n, x_n, x_{n+1}) = \max\left\{D(Sx_{n-1}, Sx_n), D(Sx_n, Sx_{n+1})\right\}. \tag{17}$$

If $M(x_{n-1}, x_n, x_n, x_{n+1}) = D(Sx_n, Sx_{n+1})$, then by (16),

$$\begin{aligned} D(Sx_n, Sx_{n+1}) &\leq \theta(D(Sx_n, Sx_{n+1})) D(Sx_n, Sx_{n+1}) \\ &\leq D(Sx_n, Sx_{n+1}). \end{aligned} \tag{18}$$

Since $D(Sx_n, Sx_{n+1}) > 0$ for all $n \geq 0$,

$$1 \leq \theta(D(Sx_n, Sx_{n+1})) \leq 1, \tag{19}$$

and thus,

$$\lim_{n\to\infty} \theta(D(Sx_n, Sx_{n+1})) = 1. \tag{20}$$

By the definition of θ, $\lim_{n\to\infty} D(Sx_n, Sx_{n+1}) = 0$.

If $M(x_{n-1}, x_n, x_n, x_{n+1}) = D(Sx_{n-1}, Sx_n)$, we again have that

$$D(Sx_n, Sx_{n+1}) \leq \theta(D(Sx_{n-1}, Sx_n)) D(Sx_{n-1}, Sx_n) \leq D(Sx_{n-1}, Sx_n). \tag{21}$$

Since n is arbitrary, $\{D(Sx_n, Sx_{n+1})\}$ is nonnegative and nonincreasing. Therefore, $\{D(Sx_n, Sx_{n+1})\}$ converges to $s \geq 0$. Suppose on the contrary that $s > 0$. From (21),

$$\frac{D(Sx_n, Sx_{n+1})}{D(Sx_{n-1}, Sx_n)} \leq \theta(D(Sx_{n-1}, Sx_n)) \leq 1. \tag{22}$$

It follows that $\lim_{n\to\infty} \theta(D(Sx_{n-1}, Sx_n)) = 1$. Since $\theta \in \Theta$, we have that $\lim_{n\to\infty} D(Sx_{n-1}, Sx_n) = 0$ which is a contradiction. Thus, s must be 0 and that

$$\lim_{n\to\infty} D(Sx_{n-1}, Sx_n) = 0. \tag{23}$$

Next, we shall show that $\{Sx_n\}$ is a D-Cauchy sequence. Suppose that this is not the case. Then, there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there are subsequences $\{Sx_{n_k}\}$ and $\{Sx_{m_k}\}$ of $\{Sx_n\}$ satisfying $D(Sx_{n_k}, Sx_{m_k}) \geq \varepsilon$ for $m_k \geq n_k \geq k$.

Since (S, T) is triangular-(α, D)-proximal admissible, it is easy to see that

$$\alpha(Sx_n, Sx_m) \geq 1 \text{ and } D(Sx_n, Sx_m) < \infty \text{ when } m \geq n \text{ for all } m, n \in \mathbb{N}. \tag{24}$$

It follows from (13) and (24) that for any $k \in \mathbb{N}$,

$$\begin{aligned} \alpha\left(Sx_{n_k-1}, Sx_{m_k-1}\right) \geq 1 \text{ and } & D\left(Sx_{n_k}, Tx_{n_k-1}\right) \\ &= D(A, B) = D\left(Sx_{m_k}, Tx_{m_k-1}\right). \end{aligned} \tag{25}$$

Since (S, T) is an (α, D)-proximal generalized Geraghty mapping and (A, B) has the weak P-property, we obtain that

$$\begin{aligned} D(Sx_{n_k}, Sx_{m_k}) &\leq D(Tx_{n_k-1}, Tx_{m_k-1}) \\ &\leq \alpha\left(Sx_{n_k-1}, Sx_{m_k-1}\right) D\left(Tx_{n_k-1}, Tx_{m_k-1}\right) \\ &\leq \theta\left(M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right)\right) M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right), \end{aligned} \tag{26}$$

where

$$\begin{aligned} & M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right) \\ &\quad = \max\left\{D\left(Sx_{n_k-1}, Sx_{m_k-1}\right), D\left(Sx_{n_k-1}, Sx_{n_k}\right), D\left(Sx_{m_k-1}, Sx_{m_k}\right)\right\}. \end{aligned} \tag{27}$$

If $M(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k})$ is either $D(Sx_{n_k-1}, Sx_{n_k})$ or $D(Sx_{m_k-1}, Sx_{m_k})$, then, by (23), $\lim_{k\to\infty} D(Sx_{n_k}, Sx_{m_k}) = 0$. This contradicts the assumption that $\{Sx_n\}$ is not D-Cauchy. Thus, $M(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}) = D(Sx_{n_k-1}, Sx_{m_k-1})$.

As a consequence,

$$D\left(Sx_{n_k}, Sx_{m_k}\right) \leq \theta\left(D\left(Sx_{n_k-1}, Sx_{m_k-1}\right)\right) D\left(Sx_{n_k-1}, Sx_{m_k-1}\right). \tag{28}$$

By repeating the same steps, it follows that

$$D\left(Sx_{n_k-i}, Sx_{m_k-i}\right) \leq \theta\left(D\left(Sx_{n_k-i-1}, Sx_{m_k-i-1}\right)\right) D\left(Sx_{n_k-i-1}, Sx_{m_k-i-1}\right), \tag{29}$$

where $i = 0, 1, 2, \cdots, n_k - 1$. Therefore,

$$D\left(Sx_{n_k}, Sx_{m_k}\right) \leq \prod_{i=1}^{n_k} \theta\left(D\left(Sx_{n_k-i}, Sx_{m_k-i}\right)\right) D\left(Sx_0, Sx_{m_k-n_k}\right). \tag{30}$$

Let $i_k \in \{1, 2, \cdots, n_k\}$ such that

$$\theta(D(Sx_{n_k-i_k}, Sx_{m_k-i_k})) = \max\{\theta(D(Sx_{n_k-i}, Sx_{m_k-i})): 1 \le i \le n_k\}. \quad (31)$$

Define

$$\eta = \limsup_{k\to\infty}\{\theta(D(Sx_{n_k-i_k}, Sx_{m_k-i_k}))\}. \quad (32)$$

If $\eta < 1$, $\lim_{k\to\infty} D(Sx_{n_k}, Sx_{m_k}) = 0$ which is impossible. Thus, $\eta = 1$. Without loss of generality, we may assume that $\lim_{k\to\infty} \theta(D(Sx_{n_k-i_k}, Sx_{n_k+m_k-i_k})) = 1$. By the definition of θ, $\lim_{k\to\infty} D(Sx_{n_k-i_k}, Sx_{n_k+m_k-i_k}) = 0$. Then, there exists $k_0 \in \mathbb{N}$ such that

$$D\left(Sx_{n_{k_0}-i_{k_0}}, Sx_{n_{k_0}+m_{k_0}-i_{k_0}}\right) < \frac{\varepsilon}{2}. \quad (33)$$

Now,

$$\varepsilon \le D\left(Sx_{n_{k_0}}, Sx_{m_{k_0}}\right) \le \prod_{j=1}^{i_{k_0}} \theta\left(D\left(Sx_{n_{k_0}-j}, Sx_{m_{k_0}-j}\right)\right) D\left(Sx_{n_{k_0}-i_{k_0}}, Sx_{m_{k_0}-i_{k_0}}\right) < \frac{\varepsilon}{2}, \quad (34)$$

which is a contradiction. Therefore, $\{Sx_n\}$ is a D-Cauchy sequence.

Since $(S(A_0), D)$ is D-complete, there exists $x^* \in A_0$ such that

$$\lim_{n\to\infty} D(Sx_n, Sx^*) = 0. \quad (35)$$

Equivalently,

$$\{Sx_n\} \in C(D, S(A_0), Sx^*). \quad (36)$$

Since $A_0 \subseteq S(A_0)$ and $T(A_0) \subseteq B_0$, it follows that there exists $a \in A_0$ such that

$$D(Sa, Tx^*) = D(A, B). \quad (37)$$

By (13) and (iii), there is a subsequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$ such that $\alpha(Sx_{n_k}, x^*) \ge 1$ for all $k \in \mathbb{N}$. From (13), we have that

$$D(Sx_{n_k+1}, Tx_{n_k}) = D(A, B) \quad \text{for all } k \in \mathbb{N}. \quad (38)$$

By the weak P-property, (37) and (38), we obtain that $D(Sx_{n_k+1}, Sa) \le D(Tx_{n_k}, Tx^*)$.

Since $\alpha(Sx_{n_k}, x^*) \ge 1$ and (S, T) is an (α, D)-proximal generalized Geraghty mapping,

$$\begin{aligned} D(Sx_{n_k+1}, Sa) \le D(Tx_{n_k}, Tx^*) &\le \alpha(Sx_{n_k}, Sx^*) D(Tx_{n_k}, Tx^*) \\ &\le \theta(M(x_{n_k}, x^*, x_{n_k+1}, a)) M(x_{n_k}, x^*, x_{n_k+1}, a) \\ &\le M(x_{n_k}, x^*, x_{n_k+1}, a), \quad \text{for all } k \ge 1, \end{aligned} \quad (39)$$

where

$$M(x_{n_k}, x^*, x_{n_k+1}, a) = \max\{D(Sx_{n_k}, Sx^*), D(Sx_{n_k}, Sx_{n_k+1}), D(Sx^*, Sa)\}. \quad (40)$$

By (23) and (35), we immediately have that

$$\lim_{k\to\infty} M(x_{n_k}, x^*, x_{n_k+1}, a) = D(Sx^*, Sa) \ge 0. \quad (41)$$

If $D(Sx^*, Sa) > 0$, by letting $k \to \infty$ in (39),

$$1 \le \lim_{k\to\infty} \theta(M(x_{n_k}, x^*, x_{n_k+1}, a)) \le 1. \quad (42)$$

We subsequently have that

$$\lim_{k\to\infty} \theta(M(x_{n_k}, x^*, x_{n_k+1}, a)) = 1. \quad (43)$$

By the property of θ,

$$\lim_{k\to\infty} M(x_{n_k}, x^*, x_{n_k+1}, a) = D(Sx^*, Sa) = 0, \quad (44)$$

which is a contradiction. It follows that $D(Sx^*, Sa)$ must be equal to 0, and thus $Sx^* = Sa$. Therefore, from (37), there exists $x^* \in A$ such that

$$D(Sx^*, Tx^*) = D(A, B). \quad (45)$$

Suppose further that $x^*, y^* \in BC(S, T)$ and $\alpha(x^*, y^*) \ge 1$. By Lemma 11, $Sx^* = Sy^*$. Since S is injective, $x^* = y^*$. The proof is now completed.

Example 13. Let $X = [-3, 3]$ be equipped with the JS-metric D given by

$$D(x, y) = \begin{cases} |x| + |y|, & x \ne 0 \text{ and } y \ne 0, \\ \left|\frac{x}{2}\right|, & y = 0, \\ \left|\frac{y}{2}\right|, & x = 0. \end{cases} \quad (46)$$

Choose $A = [-2, 0]$ and $B = [0, 1]$. Let $T : A \to B$ be a mapping defined by

$$T(x) = -\frac{x}{3}, \quad \text{for all } x \in A, \quad (47)$$

and let a mapping $S : A \to A$ be defined by

$$S(x) = \frac{x}{2}, \quad \text{for all } x \in A. \quad (48)$$

It is not difficult to see that $D(A, B) = 0$ and (A, B) has the weak P-property. Next, define the map $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \ne 0 \text{ or } y = 0, \\ 0, & \text{otherwise}, \end{cases} \quad (49)$$

for all $x, y \in X$. Since $A_0 = \{0\} = B_0$, then $T(A_0) = \{0\} \subseteq B_0 = \{0\}$ and $A_0 = \{0\} \subseteq S(A_0) = \{0\}$. Also, there is $0 \in A_0$ satisfying

$$D(S(0), T(0)) = D(0, 0) = 0 = D(A, B), \alpha(0, 0) \geq 1. \quad (50)$$

We will first show that (S, T) is triangular-(α, D)-proximal admissible.

Let $x_1, x_2, u_1, u_2 \in A$ such that $\alpha(Sx_1, Sx_2) \geq 1$ and

$$D(Su_1, Tx_1) = D(Su_2, Tx_2) = D(A, B). \quad (51)$$

Then, $Sx_1 \neq 0$ or $Sx_2 = 0$ and

$$D\left(\frac{u_1}{2}, -\frac{x_1}{3}\right) = D\left(\frac{u_2}{2}, -\frac{x_2}{3}\right) = 0. \quad (52)$$

Assume that $\alpha(Su_1, Su_2) \neq 1$, then $u_1/2 = Su_1 = 0$ and $u_2/2 = Su_2 \neq 0$.

Since $\alpha(Sx_1, Sx_2) \geq 1$, we consider the following two cases.

Case 1. If $Sx_2 \neq 0$, then $Sx_1 \neq 0$, and thus,

$$\left|-\frac{x_1}{6}\right| = D\left(0, -\frac{x_1}{3}\right) = 0. \quad (53)$$

Then $x_1 = 0$. This implies that $Sx_1 = 0$ which is impossible.

Case 2. If $Sx_2 = 0 = x_2/2$, then

$$D\left(Su_2, -\frac{x_2}{3}\right) = D\left(\frac{u_2}{2}, 0\right) = \left|\frac{u_2}{4}\right| = 0. \quad (54)$$

This implies that $u_2 = 0$ and $Su_2 = 0$ which is impossible. Thus, $\alpha(Su_1, Su_2) \geq 1$.

Next, assume that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$. Then, we can see that $y = 0$ if $z = 0$, and $x \neq 0$ if $z \neq 0$. Hence, $x \neq 0$ or $y = 0$, and thus, $\alpha(x, y) \geq 1$. This means that (S, T) is triangular-(α, D)-proximal admissible.

We note that there is a map $\theta \in \Theta$ defined by $\theta(t) = 2/3$.

Now, for x, y satisfying $\alpha(Sx, Sy) \geq 1$, we have that $Sx \neq 0$ or $Sy = 0$. We consider the following two cases.

Case 1. If $Sy = 0$, then $y = 0$ and

$$\begin{aligned}\alpha(Sx, Sy)D(Tx, Ty) &= \alpha(Sx, 0)D(Tx, T(0)) = D\left(-\frac{x}{3}, 0\right) \\ &= \left|-\frac{x}{6}\right| = \frac{2}{3}\left|\frac{x}{4}\right| = \frac{2}{3}D(Sx, Sy) \\ &\leq \theta(M(x, y, u, v))M(x, y, u, v).\end{aligned} \quad (55)$$

Case 2. If $Sy \neq 0$, then $Sx \neq 0$, and thus, $x \neq 0$ and $y \neq 0$. We obtain that

$$\begin{aligned}\alpha(x, y)D(Tx, Ty) &= D(Tx, Ty) = D\left(-\frac{x}{3}, -\frac{y}{3}\right) = \left|-\frac{x}{3} - \frac{y}{3}\right| \\ &\leq \frac{2}{3}|x| + |y| = \frac{2}{3}D(Sx, Sy) \\ &\leq \theta(M(x, y, u, v))M(x, y, u, v).\end{aligned} \quad (56)$$

Therefore, (S, T) is an (α, D)-proximal generalized Geraghty mapping.

Finally, we will show that assumption (iii) in Theorem 12 holds. Let $a \in A_0$ and $\{Sx_n\} \in C(D, S(A_0), Sa)$ such that $\alpha(Sx_n, Sx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then,

$$Sx_n \neq 0 \text{ or } Sx_{n+1} = 0 \text{ for each } n \in \mathbb{N}. \quad (57)$$

If $Sx_n \neq 0$ for all $n \in \mathbb{N}$, then $\alpha(Sx_n, Sa) \geq 1$ for all $n \in \mathbb{N}$. Assume that there exists $n_0 \in \mathbb{N}$ such that $Sx_{n_0} = 0$. By (57), $Sx_k = 0$ for all $k \geq n_0$. Suppose that $Sa \neq 0$. Then,

$$D(Sx_k, Sa) = D(0, a) = \left|\frac{a}{2}\right| \neq 0 \quad \text{for all } k \geq n_0. \quad (58)$$

This contradicts with the fact that $\{Sx_n\} \in C(D, S(A_0), Sa)$. Thus, $Sa = 0$ and so $\alpha(Sx_n, Sa) \geq 1$. We also have that $(S(A_0), D)$ is D-complete. Therefore, by Theorem 12, (S, T) has a best proximity coincidence point, which is 0.

Example 14. Let $X = \mathbb{R}^2$ be equipped with the JS -metric D given by

$$D((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| + |y_1 - y_2|, & (x_1, x_2) \neq (0, 0), (y_1, y_2) \neq (0, 0), \\ |x_1 - x_2|, & (y_1, y_2) = (0, 0), \\ \dfrac{|y_1 - y_2|}{2}, & (x_1, x_2) = (0, 0). \end{cases} \quad (59)$$

We consider the disjoint subsets A and B of X given by $A = \{(-1, y) ; 0 \leq y \leq 1\}$ and $B = \{(1, y) ; 0 \leq y \leq 1\}$. We can check that $D(A, B) = 2$ and the pair (A, B) has the weak P-property.

Let $T : A \to B$ be a map defined by

$$T(-1, y) = (1, \ln(1 + y)), \quad \text{for all } (-1, y) \in A, \quad (60)$$

and let $S : A \to A$ be a map defined by

$$S(-1, y) = (-1, y), \quad \text{for all } (-1, y) \in A. \quad (61)$$

Then, we consider a map $\alpha : X \times X \to [0,\infty)$ given by

$$\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1, & \text{if } x_1 \leq x_2, y_1 \geq y_2, \\ 0, & \text{otherwise}, \end{cases} \quad (62)$$

for all $x = (x_1, y_1), y = (x_2, y_2) \in X$.

Next, we will show that (S, T) is triangular-(α, D)-proximal admissible. Let $x, y, u, v \in A$ such that $x = (-1, \widehat{x})$, $y = (-1, \widehat{y})$, $u = (-1, \widehat{u})$, and $v = (-1, \widehat{v})$ satisfying $\alpha(Sx, Sy) \geq 1$ and

$$D(Su, Tx) = D(Sv, Ty) = D(A, B). \tag{63}$$

Consequently, $\hat{x} \geq \hat{y}$ and

$$D((-1, \hat{u}), (-1, \ln(1+\hat{x}))) = D((-1, \hat{v}), (-1, \ln(1+\hat{y}))) = 2. \tag{64}$$

It follows that $\hat{u} = \ln(1+\hat{x}))$ and $\hat{v} = \ln(1+\hat{y})$. Since $\hat{x} \geq \hat{y}$, then $\hat{u} \geq \hat{v}$, and thus, $\alpha(Su, Sv) \geq 1$.

Assume that $\alpha(x, y) \geq 1$ and $\alpha(y, u) \geq 1$. Then, we can see that $\hat{x} \geq \hat{y}$ and $\hat{y} \geq \hat{u}$. Therefore, $\hat{x} \geq \hat{u}$, and thus, $\alpha(x, u) \geq 1$. This means that (S, T) is triangular-(α, D)-proximal admissible.

We choose a map $\theta \in \Theta$ which is defined by

$$\theta(t) = \begin{cases} 1, & t = 0, \\ \dfrac{\ln(1+t)}{t}, & t > 0. \end{cases} \tag{65}$$

Let $x, y, u, v \in A$ such that $x = (-1, \hat{x}), y = (-1, \hat{y}), u = (-1, \hat{u})$, and $v = (-1, \hat{v})$ satisfying $\alpha(Sx, Sy) \geq 1$. If $x = y$, then we are done. Suppose that $x \neq y$. It follows that $D(x, y) > 0$, and so, $M(x, y, u, v) > 0$. Thus,

$$\begin{aligned} \alpha(Sx, Sy)D(Tx, Ty) &= D(Tx, Ty) \\ &= D((1, \ln(1+\hat{x})), (1, \ln(1+\hat{y}))) \\ &= |\ln(1+\hat{x}) - \ln(1+\hat{y})| \\ &= \left|\ln\left(\frac{1+\hat{y}+\hat{x}-\hat{y}}{1+\hat{y}}\right)\right| \leq \ln(1+|\hat{x}-\hat{y}|) \\ &= \ln(1+D(x, y)) \leq \ln(1+M(x, y, u, v)) \\ &= \left[\frac{\ln(1+M(x, y, u, v))}{M(x, y, u, v)}\right] M(x, y, u, v) \\ &= \theta(M(x, y, u, v))M(x, y, u, v). \end{aligned} \tag{66}$$

Therefore, (S, T) is an (α, D)-proximal generalized Geraghty mapping.

Since $A_0 = A = \{(-1, y); 0 \leq y \leq 1\}$ and $B_0 = B = \{(1, y); 0 \leq y \leq 1\}$,

$$\begin{aligned} T(A_0) &= \{(1, y); 0 \leq y \leq \ln 2\} \subseteq B_0, \\ A_0 &= \{(-1, 0)\} \subseteq S(A_0) = A_0. \end{aligned} \tag{67}$$

Also, $(S(A_0), D)$ is D-complete, and there is $(-1, 0) \in A_0$ satisfying

$$\begin{aligned} D(S(-1, 0), T(-1, 0)) &= D((-1, 0), (1, 0)) = 2 = D(A, B), \\ \alpha(S(-1, 0), T(-1, 0)) &= \alpha((-1, 0), (1, 0)) \geq 1. \end{aligned} \tag{68}$$

We have left to that show assumption (iii) in Theorem 12 holds. Let $a = (-1, \hat{a}) \in A_0$ and $\{Sx_n\} \in C(D, S(A_0), Sa)$ such that $\alpha(Sx_n, Sx_{n+1}) = \alpha((-1, y_n), (-1, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then, $y_n \geq y_{n+1}$ for all n. Since $S(A_0) = A_0 = \{(-1, y); 0 \leq y \leq 1\}$ and $\{y_n\}$ is nonincreasing which $\{Sx_n\} \in C(D, S(A_0), Sa) = C(D, A_0, a)$. It follows that $y_n \geq \hat{a}$ for all $n \in \mathbb{N}$. Then, $\alpha(Sx_n, Sa) \geq 1$ for all $n \in \mathbb{N}$. Therefore, by Theorem 12, (S, T) has a best proximity coincidence point.

Next, we present a corollary of our result. The following definition is required.

Definition 15. Let $T : A \to B$ and $S : A \to A$ be mappings. Let $\alpha : X \times X \to [0,\infty)$ be a function. Then, the pair (S, T) is said to be an (α, D) -proximal mapping if the following conditions hold.

(i) The pair (S, T) is triangular-(α, D)-proximal admissible.

(ii) There exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$, if $D(Su, Tx) = D(Sv, Ty) = D(A, B)$ and $\alpha(Sx, Sy) \geq 1$, then

$$D(Tx, Ty) \leq kD(x, y). \tag{69}$$

By putting $\theta(t) = k$, where $k \in [0, 1)$ in Theorem 12, we have the following result.

Corollary 16. *Let $A_0 \subseteq S(A_0)$ and $(S(A_0), D)$ be D -complete. Given that $\alpha : X \times X \to [0,\infty)$ is a function, and let $T : A \to B$ and $S : A \to A$ be mappings such that (S, T) is an (α, D) -proximal mapping. Suppose that the following conditions hold.*

(i) $T(A_0) \subseteq B_0$ and the pair (A, B) has the weak P -property.

(ii) There exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A, B)$ and $\alpha(Sy, Sx) \geq 1$ and $D(Sy, Sx) < \infty$.

(iii) For $\{Sx_n\} \in C(D, S(A_0), Sx^*)$, if $\alpha(Sx_n, Sx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then there is a subsequence $\{Sx_{n_k}\}$ with $\alpha(Sx_{n_k}, Sx^*) \geq 1$ for all $k \in \mathbb{N}$.

Then, there exists $x^* \in A$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if $\alpha(Sx^*, Sy^*) \geq 1$ for all $x^*, y^* \in BC(S, T)$, then (S, T) has a unique best proximity coincidence point.

3. Consequence

We will apply our result on the best proximity coincidence point on a JS-metric space endowed with a binary relation R.

Let $T : A \to B$ and $S : A \to A$ be mappings. The pair (S, T) is said to be (R, D)-proximally comparative if $SxRSy$ and $D(Su_1, Tx) = D(Su_2, Ty) = D(A, B) \Rightarrow Su_1 R Su_2$ and $D(Su_1, Su_2) < \infty$ for all $x, y, u_1, u_2 \in A$.

Definition 17. Let $T : A \to B$ and $S : A \to A$ be mappings. The pair (S, T) is said to be an (R, D) -proximally comparative generalized Geraghty mapping if the following hold.

(1) The pair (S, T) is (R, D)-proximally comparative.

(2) There exists $\theta \in \Theta$ such that for all $x, y, u, v \in A$, if $D(Su, Tx) = D(Sv, Ty) = D(A, B)$ and $SxRSy$, then

$$D(Tx, Ty) \leq \theta(M(x, y, u, v))M(x, y, u, v), \tag{70}$$

where $M(x, y, u, v) = \max\{D(Sx, Sy), D(Sx, Su), D(Sy, Sv)\}$.

Corollary 18. *Let X be endowed with a symmetric, transitive binary relation R. Let $T : A \to B$ and $S : A \to A$ be mappings such that $\emptyset \neq A_0 \subseteq S(A_0)$ and $(S(A_0), D)$ be D -complete. If (S, T) is an (R, D) -proximally comparative generalized Geraghty mapping and the following conditions hold:*

(i) $T(A_0) \subseteq B_0$ and the pair (A, B) has the weak P -property;

(ii) there exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A, B)$ and $SyRSx$ and $D(Sy, Sx) < \infty$;

(iii) for $\{Sx_n\} \in C(D, S(A_0), Sx^*)$, if Sx_nRSx_{n+1} for all $n \in \mathbb{N}$, then there is a subsequence $\{Sx_{n_k}\}$ with $Sx_{n_k}R\,Sx^*$ for all $k \in \mathbb{N}$,

then there exists $x^* \in A_0$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if Sx^*RSy^* for all $x^*, y^* \in BC(S, T)$ and S is injective, then (S, T) has a unique best proximity coincidence point.

Proof. Define

$$\alpha(x, y) = \begin{cases} 1, & \text{if } xRy, \\ 0, & \text{otherwise}, \end{cases} \tag{71}$$

for all $x, y \in X$. We can see that the hypotheses of Theorem 12 hold which imply that there is $x^* \in A$ such that $D(Sx^*, Tx^*) = D(A, B)$. Let $x^*, y^* \in BC(S, T)$. Then, Sx^*RSy^* which implies that $\alpha(Sx^*, Sy^*) \geq 1$. Again, by Theorem 12, $x^* = y^*$.

4. Conclusion and Open Questions

We have introduced new classes of Geraghty's type mappings called (α, D)-proximal generalized Geraghty mappings. Then, we investigated some conditions for this type of mappings to have a best proximity coincidence point in JS-metric spaces using the weak P-property. The question is whether one can extend Theorem 12 to the framework of common best proximity point in a JS-metric space X. Can we also extend the result when X is other generalized metric spaces?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

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Research Article

On the Well-Posedness of a Fractional Model of HIV Infection

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In this paper, we are concerned with the well-posedness of a fractional model of human immunodeficiency virus infection. Namely, using Grönwall's lemma and Perov's fixed point theorem, we obtain sufficient conditions for which the considered model admits a unique solution.

1. Introduction

The human immunodeficiency virus (HIV) is one of the world's leading infectious diseases. A big number of people have died around the globe due to this disease. HIV infects vital cells in the human immune system, such as CD4+ T cells. In this way, the body becomes progressively more susceptible to opportunistic infections, leading to the development of AIDS (acquired immunodeficiency syndrome) (see, e.g., [1, 2]).

Mathematical models play an important role in understanding the dynamics of HIV infection. The dynamical models for HIV usually consist of systems of ordinary differential equations which range from two-component models (see, e.g., [3, 4]) to three-component (see e.g. [5–7]) and four-component models (see, e.g., [8]). In particular, in [3], the following two-cell model was proposed to describe the HIV infection:

$$\begin{cases} u'(t) = \kappa - \sigma u(t) - \delta u(t)v(t), \\ v'(t) = \delta u(t)v(t) - \tau v(t), \end{cases} \tag{1}$$

where u is the density of uninfected CD4+ T cells, v is the density of virus-producing cells, κ is the rate of production of CD4+ T cells, σ is their per capita death rate, δ is the rate of infection of CD4+ T cells, and τ is the rate of disappearance of infected cells.

Due to the importance of fractional calculus in modeling real-world phenomena (see, e.g., [9–12] and the references therein), a great attention was paid to the study of fractional models of HIV infection of CD4+ T cells (see, e.g., [13–17] and the references therein).

In this paper, a fractional model of (1) is investigated. Namely, we are concerned with the system of fractional differential equations

$$\begin{cases} \left({}^C D_0^{\alpha,\psi} u\right)(t) = I_0^{\mu,\psi}(\kappa - \sigma u - \delta uv)(t), & 0 \le t \le T, \\ \left({}^C D_0^{\beta,\psi} v\right)(t) = I_0^{\nu,\psi}(\delta uv - \tau v)(t), & 0 \le t \le T, \end{cases} \tag{2}$$

subjet to the initial conditions

$$(u(0), v(0)) = (u_0, v_0), \tag{3}$$

where $T > 0$, $u_0, v_0 \ge 0$, $0 < \alpha, \beta < 1$, $\mu \ge 1 - \alpha$, $\nu \ge 1 - \beta$, $\psi \in C^1([0, T])$, $\psi' > 0$, ${}^C D_0^{\ell,\psi}$, $\ell \in \{\alpha, \beta\}$ is the ψ-Caputo fractional derivative of order ℓ, $I_0^{\varkappa,\psi}$, $\varkappa \in \{\mu, \nu\}$ is the ψ-Riemann-Liouville fractional integral of order u, and $\kappa, \sigma, \delta, \tau > 0$. Using Grönwall's lemma and Perov's fixed point theorem, we derive sufficient conditions for which system (2) subject to the initial conditions (3) admits a unique solution. Moreover, we provide a numerical algorithm that converges uniformly to the unique solution. Notice that when $\psi(t)=t$, $(\alpha,\mu)\to(1^-,0^+)$ and $(\beta,\nu)\to(1^-,0^+)$, (2) reduces to (1).

The rest of the paper is organized as follows. In Section 2, we recall some notions on fractional calculus and Perov's fixed point theorem. In Section 3, we state and prove our main results. In Section 4, some special cases are discussed.

2. Some Preliminaries

Let $(a,b)\in\mathbb{R}^2$ be such that $a<b$.

Given $f\in C([a,b])$ and $\theta>0$, the (left-sided) Riemann-Liouville fractional integral of order θ of f is defined by (see [10])

$$\left(I_a^\theta f\right)(t)=\begin{cases}\dfrac{1}{\Gamma(\theta)}\displaystyle\int_a^t (t-s)^{\theta-1}f(s)\,ds & \text{if} \quad a<t\le b,\\ 0 & \text{if} \quad t=a,\end{cases} \tag{4}$$

where Γ is the Gamma function. It can be easily seen that

$$f\in C([a,b])\Longrightarrow I_a^\theta f\in C([a,b]). \tag{5}$$

Lemma 1 (see [10], Property 2.6). *Let $f\in C([a,b])$ and $\theta,\eta>0$. Then,*

$$\left(I_a^\theta I_a^\eta f\right)(t)=\left(I_a^{\theta+\eta}f\right)(t),\quad a\le t\le b. \tag{6}$$

Given $f\in C^1([a,b])$ and $0<\theta<1$, the (left-sided) Caputo fractional derivative of order θ of f is defined by (see [10])

$$\left({}^C D_a^\theta f\right)(t)=\left(I_a^{1-\theta}f'\right)(t),\quad a\le t\le b. \tag{7}$$

Lemma 2 (see [10], Lemma 2.22). *Let $f\in C^1([a,b])$ and $0<\theta<1$. Then,*

$$\left(I_a^\theta {}^C D_a^\theta f\right)(t)=f(t)-f(a),\quad a\le t\le b. \tag{8}$$

Lemma 3 (see [10], Lemma 2.21). *Let $f\in C([a,b])$ and $0<\theta<1$. Then,*

$$\left({}^C D_a^\theta I_a^\theta f\right)(t)=f(t), a\le t\le b. \tag{9}$$

Let $T>0$ be fixed. We introduce the set of functions

$$\Psi=\left\{\psi\in C^1([0,T]):\ \psi'(t)>0 \text{ for all } 0\le t\le T\right\}. \tag{10}$$

Given $f\in C([0,T])$, $\theta>0$, and $\psi\in\Psi$, the (left-sided) ψ-Riemann-Liouville fractional integral of order θ of f is defined by (see [10, 18])

$$\left(I_0^{\theta,\psi}f\right)(t)=\begin{cases}\dfrac{1}{\Gamma(\theta)}\displaystyle\int_0^t(\psi(t)-\psi(s))^{\theta-1}\psi'(s)f(s)\,ds & \text{if} \quad 0<t\le T,\\ 0 & \text{if} \quad t=0.\end{cases} \tag{11}$$

It can be easily seen that

$$f\in C([0,T])\Rightarrow I_0^{\theta,\psi}f\in C([0,T]). \tag{12}$$

Remark 4. *In the special case $\psi(t)=t$, one observes that $I_0^{\theta,\psi}=I_0^\theta$.*

Remark 5. *Using the change of variable $z=\psi(s)$, one deduces from ((11)) that*

$$\left(I_0^{\theta,\psi}f\right)(t)=\left(I_{\psi(0)}^\theta f\circ\psi^{-1}\right)(\psi(t)), 0\le t\le T. \tag{13}$$

Using Lemma 1 and Remark 5, one deduces the following result.

Lemma 6. *Let $f\in C([0,T])$, $\theta,\eta>0$ and $\psi\in\Psi$. Then*

$$\left(I_0^{\theta,\psi}I_0^{\eta,\psi}f\right)(t)=\left(I_0^{\theta+\eta,\psi}f\right)(t), 0\le t\le T. \tag{14}$$

Given $f\in C^1([0,T])$, $0<\theta<1$, and $\psi\in\Psi$, the (left-sided) ψ-Caputo fractional derivative of order θ of f is defined by (see [18])

$$\left({}^C D_0^{\theta,\psi}f\right)(t)=\left(I_0^{1-\theta,\psi}\frac{f'}{\psi'}\right)(t), 0\le t\le T. \tag{15}$$

Remark 7. *In the special case $\psi(t)=t$, one observes that ${}^C D_0^{\theta,\psi}={}^C D_0^\theta$.*

Remark 8. *Using the change of variable $z=\psi(s)$, one deduces from ((15)) that*

$$\left({}^C D_0^{\theta,\psi}f\right)(t)=\left({}^C D_{\psi(0)}^\theta f\circ\psi^{-1}\right)(\psi(t)), 0\le t\le T. \tag{16}$$

Using Lemma 2, Remark 5, and Remark 8, one deduces the following result.

Lemma 9. *Let $f\in C^1([0,T])$, $0<\theta<1$ and $\psi\in\Psi$. Then*

$$\left(I_0^{\theta,\psi}{}^C D_0^{\theta,\psi}f\right)(t)=f(t)-f(0), 0\le t\le T. \tag{17}$$

Similarly, using Lemma 3, Remark 5, and Remark 8, one deduces the following result.

Lemma 10. *Let $f\in C([0,T])$, $0<\theta<1$ and $\psi\in\Psi$. Then,*

$$\left({}^C D_0^{\theta,\psi}I_0^{\theta,\psi}f\right)(t)=f(t), 0\le t\le T. \tag{18}$$

Now, we recall some concepts on fixed point theory that will be used later. Let n be a positive natural number and define the partial order $\circ_n$ in $\mathbb{R}^n$ by

$$y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \preceq_n z=\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \Leftrightarrow y_i \le z_i, i=1,2,\cdots,n, \tag{19}$$

for all $y, z \in \mathbb{R}^n$. We denote by $0_{\mathbb{R}^n}$ the zero vector in $\mathbb{R}^n$, i.e.,

$$0_{\mathbb{R}^n}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{20}$$

Let $\mathcal{X}$ be a nonempty set and $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$ be a given mapping. We say that d is a vector-valued metric on $\mathcal{X}$ (see, e.g., [19]), if for all $x, y, z \in \mathcal{X}$,

$$\begin{aligned} &0_{\mathbb{R}^n} \preceq_n d(x,y), \\ &d(x,y)=0_{\mathbb{R}^n} \Leftrightarrow x=y, \\ &d(x,y)=d(y,x), \\ &d(x,z) \preceq_n d(x,y)+d(y,z). \end{aligned} \tag{21}$$

In this case, the pair $(\mathcal{X}, d)$ is called a generalized metric space. In such spaces, the notions of convergent sequence, Cauchy sequence, and completeness are similar to those for usual metric spaces.

Let us denote by $M_n(\mathbb{R}_+)$ the set of square matrices of size n with nonnegative coefficients. Given $M \in M_n(\mathbb{R}_+)$, we denote by $\rho(M)$ its spectral radius.

Lemma 11 (Perov's fixed point theorem). *Let $(\mathcal{X}, d)$ be a complete generalized metric space and $F : \mathcal{X} \to \mathcal{X}$ be a given mapping. Suppose that there exists $\mathcal{M} \in M_n(\mathbb{R}_+)$ with $\rho(M) < 1$ such that*

$$d(F(x), F(y)) \preceq_n \mathcal{M} d(x,y), \tag{22}$$

for all $x, y \in \mathcal{X}$. Then,

(i) the mapping F admits a unique fixed point in $\mathcal{X}$, say x^*

(ii) for all $x_0 \in \mathcal{X}$, the sequence $\{x_m\} \subset \mathcal{X}$ defined by $x_{m+1}=F(x_m)$ converges to x^*

We end this section by recalling the Grönwall's lemma.

Lemma 12. *(see [20]) Let $K \ge 0$ and f, g be nonnegative functions on $[0, T]$ such that $f \in L^\infty(0,T)$ and $g \in L^1(0,T)$. If*

$$f(t) \le K + \int_0^t g(s) f(s)\, ds, 0 \le t \le T, \tag{23}$$

then,

$$f(t) \le K \exp\left(\int_0^t g(s)\, ds\right), 0 \le t \le T. \tag{24}$$

3. Main Results

Problem (2) and (3) is investigated under the following assumptions:

(i) $T>0$ and $\psi \in \Psi$, where Ψ is the functional space defined by (10).

(ii) $u_0, v_0 \ge 0$ and $\kappa, \sigma, \delta, \tau > 0$

(iii) $0<\alpha<1$ and $\mu \ge 1-\alpha$

(iv) $0<\beta<1$ and $\nu \ge 1-\beta$

3.1. Integral Formulation of Problem (2) and (3). Let

$$\begin{aligned} V &= \{(u,v) \in C([0,T]) \times C([0,T]) : u, v \ge 0\} \\ W &= \left\{(u,v) \in C^1([0,T]) \times C^1([0,T]) : u, v \ge 0\right\}. \end{aligned} \tag{25}$$

Suppose that $(u, v) \in W$ is a solution to problem (2) and (3). Using the first equation in (2), one obtains

$$\left(I_0^{\alpha,\psi\,C} D_0^{\alpha,\psi} u\right)(t) = \left(I_0^{\alpha,\psi} I_0^{\mu,\psi}(\kappa - \sigma u - \delta u v)\right)(t), 0 \le t \le T. \tag{26}$$

Hence, by Lemma 6 and Lemma 9, it holds that

$$u(t) - u(0) = I_0^{\alpha+\mu,\psi}(\kappa - \sigma u - \delta u v)(t), 0 \le t \le T. \tag{27}$$

Using the initial conditions (3), one obtains

$$\begin{aligned} u(t) = u_0 + \frac{1}{\Gamma(\alpha+\mu)} \int_0^t (\psi(t)-\psi(s))^{\alpha+\mu-1} \psi'(s) \\ \cdot (\kappa - \sigma u(s) - \delta u(s) v(s))\, ds, 0 \le t \le T. \end{aligned} \tag{28}$$

Similarly, using the second equation in (2), one obtains

$$\left(I_0^{\beta,\psi\,C} D_0^{\beta,\psi} v\right)(t) = \left(I_0^{\beta,\psi} I_0^{\nu,\psi}(\delta u v - \tau v)\right)(t), 0 \le t \le T, \tag{29}$$

which yields

$$v(t) - v(0) = I_0^{\beta+\nu,\psi}(\delta u v - \tau v)(t), 0 \le t \le T. \tag{30}$$

By the initial conditions (3), it holds that

$$v(t)=v_0+\frac{1}{\Gamma(\beta+\nu)}\int_0^t(\psi(t)-\psi(s))^{\beta+\nu-1}\psi'(s)\cdot(\delta u(s)v(s)-\tau v(s))\,ds,\ 0\le t\le T. \quad (31)$$

Therefore, one deduces that, if $(u,v)\in W$ is a solution to problem (2) and (3), then $(u,v)\in V$ is a solution to the system of integral equations

$$\begin{cases} u(t)=u_0+\dfrac{1}{\Gamma(\alpha+\mu)}\displaystyle\int_0^t(\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)(\kappa-\sigma u(s)-\delta u(s)v(s))\,ds, \\ v(t)=v_0+\dfrac{1}{\Gamma(\beta+\nu)}\displaystyle\int_0^t(\psi(t)-\psi(s))^{\beta+\nu-1}\psi'(s)(\delta u(s)v(s)-\tau v(s))\,ds, \end{cases} \quad (32)$$

for all $0\le t\le T$.

Conversely, suppose that $(u,v)\in V$ is a solution to (32). By assumptions (iii) and (iv), one deduces that $(u,v)\in W$. Moreover, by (32), one has $u(0)=u_0$ and $v(0)=v_0$. On the other hand, using the first equation in (32), Lemma 6, and Lemma 10, one obtains

$$\begin{aligned}\left({}^{C}D_0^{\alpha,\psi}u\right)(t)&=\left({}^{C}D_0^{\alpha,\psi}I_0^{\alpha+\mu,\psi}(\kappa-\sigma u-\delta uv)\right)(t)\\&=\left({}^{C}D_0^{\alpha,\psi}I_0^{\alpha,\psi}I_0^{\mu,\psi}(\kappa-\sigma u-\delta uv)\right)(t)\\&=I_0^{\mu,\psi}(\kappa-\sigma u-\delta uv)(t).\end{aligned} \quad (33)$$

Similarly, using the second equation in (32), one obtains

$$\begin{aligned}\left({}^{C}D_0^{\beta,\psi}v\right)(t)&=\left({}^{C}D_0^{\beta,\psi}I_0^{\beta+\nu,\psi}(\delta uv-\tau v)\right)(t)\\&=\left({}^{C}D_0^{\beta,\psi}I_0^{\beta,\psi}I_0^{\nu,\psi}(\delta uv-\tau v)\right)(t)\\&=I_0^{\nu,\psi}(\delta uv-\tau v)(t).\end{aligned} \quad (34)$$

Therefore, one deduces that, if $(u,v)\in V$ is a solution to the system of integral equation (32), then $(u,v)\in W$ is a solution to problem (2) and (3).

From the above study, the following result holds.

Lemma 13. *The following statements are equivalent:*

(I) $(u,v)\in W$ is a solution to problem (2) and (3).

(II) $(u,v)\in V$ is a solution to the system of integral equations (32).

By the above lemma, the study of problem (2) and (3) in W reduces to the study of the system of integral equation (32) in V.

3.2. Uniqueness. In this part, using Grönwall's lemma, we shall prove that the system of integral equations (32) admits at most one solution $(u,v)\in V$.

Proposition 14. *Suppose that the assumptions (i)–(iv) are satisfied. Then the system of integral equation ((32)) admits at most one solution $(u,v)\in V$.*

Proof. Suppose that $(u_1,v_1),(u_2,v_2)\in V$ are two solutions to (32). Then, for all $0\le t\le T$, one has

$$\begin{aligned}u_2(t)-u_1(t)&=u_0+\frac{1}{\Gamma(\alpha+\mu)}\int_0^t(\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)\\&\quad\cdot(\kappa-\sigma u_2(s)-\delta u_2(s)v_2(s))\,ds-u_0\\&\quad-\frac{1}{\Gamma(\alpha+\mu)}\int_0^t(\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)\\&\quad\cdot(\kappa-\sigma u_1(s)-\delta u_1(s)v_1(s))\,ds\\&=\frac{\sigma}{\Gamma(\alpha+\mu)}\int_0^t(\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)\\&\quad\cdot(u_1(s)-u_2(s))\,ds\\&\quad+\frac{\delta}{\Gamma(\alpha+\mu)}\int_0^t(\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)\\&\quad\cdot(u_1(s)v_1(s)-u_2(s)v_2(s))\,ds,\end{aligned} \quad (35)$$

which yields

$$\begin{aligned}|u_2(t)-u_1(t)|&\le\sigma C_1\int_0^t\psi'(s)|u_1(s)-u_2(s)|ds\\&\quad+\delta C_1\int_0^t\psi'(s)|u_1(s)v_1(s)-u_2(s)v_2(s)|ds\\&=\sigma C_1\int_0^t\psi'(s)|u_1(s)-u_2(s)|ds\\&\quad+\delta C_1\int_0^t\psi'(s)|u_1(s)(v_1(s)-v_2(s))\\&\quad+v_2(s)(u_1(s)-u_1(s))|\,ds\\&\le\sigma C_1\int_0^t\psi'(s)|u_1(s)-u_2(s)|ds\\&\quad+\delta C_1C_2\int_0^t\psi'(s)|v_1(s)-v_2(s)|ds\\&\quad+\delta C_1C_3\int_0^t\psi'(s)|u_1(s)-u_2(s)|\,ds\\&=C_1(\sigma+\delta C_3)\int_0^t\psi'(s)|u_1(s)-u_2(s)|\,ds\\&\quad+\delta C_1C_2\int_0^t\psi'(s)|v_1(s)-v_2(s)|\,ds,\end{aligned} \quad (36)$$

where

$$C_1=\frac{(\psi(T)-\psi(0))^{\alpha+\mu-1}}{\Gamma(\alpha+\mu)},\ C_2=\max_{0\le x\le T}u_1(x),\ C_3=\max_{0\le x\le T}v_2(x). \quad (37)$$

Hence, one deduces that

$$|u_2(t) - u_1(t)| \le C_4 \int_0^t \psi'(s)(|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)|)\, ds, \tag{38}$$

where

$$C_4 = C_1 \max\{\sigma + \delta C_3, \delta C_2\}. \tag{39}$$

Similarly, for all $0 \le t \le T$, one has

$$\begin{aligned} v_2(t) - v_1(t) &= v_0 + \frac{1}{\Gamma(\beta+\nu)} \int_0^t (\psi(t) - \psi(s))^{\beta+\nu-1} \psi'(s) \\ &\cdot (\delta u_2(s) v_2(s) - \tau v_2(s))\, ds - v_0 \\ &- \frac{1}{\Gamma(\beta+\nu)} \int_0^t (\psi(t) - \psi(s))^{\beta+\nu-1} \psi'(s) \\ &\cdot (\delta u_1(s) v_1(s) - \tau v_1(s))\, ds \\ &= \frac{\delta}{\Gamma(\beta+\nu)} \int_0^t (\psi(t) - \psi(s))^{\beta+\nu-1} \psi'(s) \\ &\cdot (u_2(s) v_2(s) - u_1(s) v_1(s))\, ds \\ &+ \frac{\tau}{\Gamma(\beta+\nu)} \int_0^t (\psi(t) - \psi(s))^{\beta+\nu-1} \psi'(s) \\ &\cdot (v_1(s) - v_2(s))\, ds, \end{aligned} \tag{40}$$

which yields

$$\begin{aligned} |v_2(t) - v_1(t)| + \tau C_5 \int_0^t \psi'(s) |v_1(s) - v_2(s)| ds &\le \delta C_5 C_6 \int_0^t \psi'(s) |v_1(s) - v_2(s)| ds \\ &+ \delta C_5 C_7 \int_0^t \psi'(s) |u_2(s) - u_1(s)| ds \\ &+ \tau C_5 \int_0^t \psi'(s) |v_1(s) - v_2(s)|\, ds \\ &= C_5(\delta C_6 + \tau) \int_0^t \psi'(s) |v_1(s) - v_2(s)| ds \\ &+ \delta C_5 C_7 \int_0^t \psi'(s) |u_2(s) - u_1(s)| ds, \end{aligned} \tag{41}$$

where

$$C_5 = \frac{(\psi(T) - \psi(0))^{\beta+\nu-1}}{\Gamma(\beta+\nu)}, C_6 = \max_{0\le x\le T} u_2(x), C_7 = \max_{0\le x\le T} v_1(x). \tag{42}$$

Therefore, one obtains

$$|v_2(t) - v_1(t)| \le C_8 \int_0^t \psi'(s)(|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)|)\, ds, \tag{43}$$

where

$$C_8 = C_5 \max\{\tau + \delta C_6, \delta C_7\}. \tag{44}$$

Next, combining (38) with (43), it holds that

$$\begin{aligned} |u_2(t) - u_1(t)| + |v_2(t) - v_1(t)| &\le (C_4 + C_8) \int_0^t \psi'(s) \\ &\cdot (|u_2(s) - u_1(s)| + |v_2(s) - v_2(s)|)\, ds. \end{aligned} \tag{45}$$

Finally, using Grönwall ‘s lemma (see Lemma 12), it holds that $(u_1, v_1) = (u_2, v_2)$.

3.3. Well-Posedness. We first fix some notations. Let

$$\begin{aligned} a_T &= \frac{(\psi(T) - \psi(0))^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)}, \\ b_T &= \frac{(\psi(T) - \psi(0))^{\beta+\nu}}{\Gamma(\beta+\nu+1)}. \end{aligned} \tag{46}$$

We introduce the function $f : [0,\infty) \to [0,\infty)$ defined by

$$\begin{aligned} f(r) = a_T\sigma + b_T\tau + (a_T + b_T)\delta r + \big[((a_T\sigma - b_T\tau) \\ + (a_T - b_T)\delta r)^2 + 4a_T b_T \delta^2 r^2\big]^{1/2}, r \ge 0. \end{aligned} \tag{47}$$

We denote by $\|\cdot\|_\infty$ the norm in $C([0, T])$ defined by

$$\|\xi\|_\infty = \max_{0\le x\le T} |\xi(x)|, \xi \in C([0, T]). \tag{48}$$

In addition to the assumptions (i)–(iv), suppose that

(v) $a_T \le \tau/\delta\kappa$ and $b_T \le 1/(\sqrt{(\sigma^2 + 4\delta\kappa)} - \sigma)$

$$4\tau \le \sqrt{\sigma^2 + 4\delta\kappa} - \sigma \tag{49}$$

(vi) There exists

$$\frac{2\tau}{\delta} \le r \le \frac{\sqrt{\sigma^2 + 4\delta\kappa} - \sigma}{2\delta} \tag{50}$$

such that

$$\frac{\tau}{\delta} \le u_0 \le \frac{r}{2}, 0 \le v_0 \le \frac{r}{2}, \quad f(r) < 2. \tag{51}$$

Theorem 15. *Suppose that the assumptions (i)–(vii) are satisfied. Then, the system of integral equation ((32)) admits one and only one solution* $(u^*, v^*) \in V$ *. Moreover, for all* $f_0, g_0 \in V$ *satisfying* $u_0 \le f_0$, $\|f_0\|_\infty \le r$ *and* $\|g_0\|_\infty \le r$ *, the sequence* $\{(f_n, g_n)\} \subset V$ *defined by*

$$\begin{cases} f_{n+1}(t) = u_0 + \dfrac{1}{\Gamma(\alpha+\mu)}\displaystyle\int_0^t (\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)(\kappa - \sigma f_n(s) - \delta f_n(s) g_n(s))\, ds, \\ g_{n+1}(t) = v_0 + \dfrac{1}{\Gamma(\beta+\nu)}\displaystyle\int_0^t (\psi(t)-\psi(s))^{\beta+\nu-1}\psi'(s)(\delta f_n(s) g_n(s) - \tau g_n(s))\, ds, \end{cases} \tag{52}$$

for all $0 \le t \le T$*, converges uniformly to* (u^*, v^*).

Proof. We first introduce the functional space

$$V_r = \{(u, v) \in V : u_0 \le u, \|u\|_\infty \le r, \|v\|_\infty \le r\}. \tag{53}$$

Let $F : V_r \to C([0,T]) \times C([0,T])$ be the mapping defined by

$$F(u,v)(t) = (F_1(u,v)(t), F_2(u,v)(t)), \quad 0 \le t \le T, \tag{54}$$

where

$$\begin{aligned} F_1(u,v)(t) &= u_0 + \frac{1}{\Gamma(\alpha+\mu)}\int_0^t (\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s) \\ &\quad \cdot (\kappa - \sigma u(s) - \delta u(s) v(s))\, ds, \\ F_2(u,v)(t) &= v_0 + \frac{1}{\Gamma(\beta+\nu)}\int_0^t (\psi(t)-\psi(s))^{\beta+\nu-1}\psi'(s) \\ &\quad \cdot (\delta u(s) v(s) - \tau v(s))\, ds. \end{aligned} \tag{55}$$

We shall prove that

$$F(V_r) \subset V_r. \tag{56}$$

Let $(u, v) \in V_r$. Then, for all $s \in [0, T]$, one has

$$\kappa - \sigma u(s) - \delta u(s) v(s) \ge \kappa - \sigma r - \delta r^2 := P(r). \tag{57}$$

On the other hand, an elementary calculation shows that the polynomial function $P(r)$ admits two roots

$$r_1 = \frac{-\sigma - \sqrt{\sigma^2 + 4\delta\kappa}}{2\delta} < 0 < r_2 = \frac{\sqrt{\sigma^2 + 4\delta\kappa} - \sigma}{2\delta}. \tag{58}$$

Since $-\delta < 0$ and (by (vii))

$$0 < r \le \frac{\sqrt{\sigma^2 + 4\delta\kappa} - \sigma}{2\delta} = r_2, \tag{59}$$

one obtains

$$P(r) \ge 0, \tag{60}$$

which yields

$$\kappa - \sigma u(s) - \delta u(s) v(s) \ge 0, 0 \le s \le T. \tag{61}$$

Therefore, since $\psi \in \Psi$ and $u_0 \ge 0$ (see (vii)), one deduces that

$$F_1(u, v) \ge u_0 \ge 0. \tag{62}$$

Moreover, for all $s \in [0, T]$, one has

$$\delta u(s) v(s) - \tau v(s) = v(s)(\delta u(s) - \tau) \ge v(s)(\delta u_0 - \tau). \tag{63}$$

Since by (vii), $u_0 \ge \tau/\delta$, one obtains

$$\delta u(s) v(s) - \tau v(s) \ge 0, \quad 0 \le s \le T. \tag{64}$$

Using the above inequality and the fact that $v_0 \ge 0$ (by (vii)), one deduces that

$$F_2(u, v) \ge 0. \tag{65}$$

On the other hand, by (62), for all $0 \le t \le T$, one has

$$\begin{aligned} |F_1(u,v)(t)| &= F_1(u,v)(t) \\ &\le u_0 + \frac{\kappa}{\Gamma(\alpha+\mu)}\int_0^t (\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)\, ds \\ &= u_0 + \kappa\frac{(\psi(t)-\psi(0))^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)} \le u_0 + \kappa a_T. \end{aligned} \tag{66}$$

Since $a_T \le \tau/\delta\kappa$ (by (v)) and $\tau/\delta \le u_0 \le r/2$ (by (vii)), one has

$$u_0 + \kappa a_T \le \frac{r}{2} + \frac{\tau}{\delta} \le \frac{r}{2} + \frac{r}{2} = r. \tag{67}$$

Hence, it follows from (66) and (67) that

$$\|F_1(u, v)\|_\infty \le r. \tag{68}$$

Similarly, by (65), for all $0 \le t \le T$, one has

$$\begin{aligned} |F_2(u,v)(t)| &= F_2(u,v)(t) \\ &\le v_0 + \frac{\delta r^2}{\Gamma(\beta+\nu)}\int_0^t (\psi(t)-\psi(s))^{\beta+\nu-1}\psi'(s)\, ds \\ &= v_0 + \delta r^2\frac{(\psi(t)-\psi(0))^{\beta+\nu}}{\Gamma(\beta+\nu+1)} \le v_0 + \delta r^2 b_T. \end{aligned} \tag{69}$$

Since $v_0 \le r/2$ (by (vii)), $b_T \le 1/(\sqrt{(\sigma^2 + 4\delta\kappa)} - \sigma)$ (by (v)) and $r \le (\sqrt{(\sigma^2 + 4\delta\kappa)} - \sigma)/2\delta$ (by (vii)), one has

$$v_0 + \delta r^2 b_T \le \frac{r}{2} + \delta r^2 \frac{1}{2\delta r} = \frac{r}{2} + \frac{r}{2} = r. \tag{70}$$

Therefore, it follows from (69) and (70) that

$$\|F_2(u,v)\|_\infty \le r. \tag{71}$$

Hence, by (62), (65), (68), and (71), one deduces that $F(u,v) \in V_r$. This proves (56).

Consider now the self-mapping,

$$F : V_r \to V_r. \tag{72}$$

By the definition of F, one observes that, if $(u,v) \in V_r$ is a fixed point of F; then, $(u,v) \in V$ is a solution to the system of integral equation (32). In order to prove that F admits a fixed point in V_r, we shall use Perov's fixed point theorem (see Lemma 11). Namely, we define the vector-valued metric $d : V_r \times V_r \to \mathbb{R}^2$ by

$$d((u_1,v_1),(u_2,v_2)) = \begin{pmatrix} \|u_1-u_2\|_\infty \\ \|v_1-v_2\|_\infty \end{pmatrix}, (u_1,v_1),(u_2,v_2) \in V_r. \tag{73}$$

Notice that (V_r, d) is a complete generalized metric space. On the other hand, for all $(u_1,v_1),(u_2,v_2) \in V_r$ and $0 \le t \le T$, one has

$$\begin{aligned} &|F_1(u_2,v_2)(t) - F_1(u_1,v_1)(t)| \\ &\quad \le \frac{\sigma}{\Gamma(\alpha+\mu)} \int_0^t (\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)|u_1(s)-u_2(s)|ds \\ &\qquad + \frac{\delta}{\Gamma(\alpha+\mu)} \int_0^t (\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s)|u_1(s)v_1(s) \\ &\qquad - u_2(s)v_2(s)|ds \le \sigma a_T\|u_1-u_2\|_\infty \\ &\qquad + \frac{\delta}{\Gamma(\alpha+\mu)} \int_0^t (\psi(t)-\psi(s))^{\alpha+\mu-1}\psi'(s) \\ &\qquad \times (|u_1(s)||v_1(s)-v_2(s)| + |v_2(s)||u_2(s)-u_1(s)|)\,ds \\ &\quad \le \sigma a_T\|u_1-u_2\|_\infty + \delta a_T r(\|v_2-v_1\|_\infty + \|u_2-u_1\|_\infty), \end{aligned} \tag{74}$$

which yields

$$\begin{aligned} \|F_1(u_2,v_2) - F_1(u_1,v_1)\|_\infty &\le a_T(\delta r+\sigma)\|u_1-u_2\|_\infty \\ &\quad + \delta a_T r\|v_1-v_2\|_\infty. \end{aligned} \tag{75}$$

Similarly, one has

$$\begin{aligned} &|F_2(u_2,v_2)(t) - F_2(u_1,v_1)(t)| \\ &\quad \le \frac{\delta}{\Gamma(\beta+\nu)} \int_0^t (\psi(t)-\psi(s))^{\beta+\nu-1}\psi'(s) \\ &\qquad \times (|u_2(s)||v_2(s)-v_1(s)| + |v_1(s)||u_2(s)-u_1(s)|)\,ds \\ &\qquad + \frac{\tau}{\Gamma(\beta+\nu)} \int_0^t (\psi(t)-\psi(s))^{\beta+\nu-1}\psi'(s)|v_1(s)-v_2(s)|ds \\ &\quad \le \delta b_T r(\|v_2-v_1\|_\infty + \|u_2-u_1\|_\infty) + \tau b_T\|v_1-v_2\|_\infty, \end{aligned} \tag{76}$$

which yields

$$\begin{aligned} &\|F_2(u_2,v_2) - F_2(u_1,v_1)\|_\infty \\ &\quad \le \delta b_T r\|u_1-u_2\|_\infty + b_T(\delta r+\tau)\|v_1-v_2\|_\infty. \end{aligned} \tag{77}$$

Hence, it follows from (75) and (77) that for all $(u_1,v_1),(u_2,v_2) \in V_r$,

$$d(F(u_1,v_1),F(u_2,v_2)) \circ_2 \mathcal{M} d((u_1,v_1),(u_2,v_2)), \tag{78}$$

where

$$\mathcal{M} = \begin{pmatrix} a_T(\delta r+\sigma) & \delta a_T r \\ \delta b_T r & b_T(\delta r+\tau) \end{pmatrix}. \tag{79}$$

On the other hand, the characteristic polynomial of $\mathcal{M}$ is given by

$$Q(\lambda) = [a_T(\delta r+\sigma) - \lambda][b_T(\delta r+\tau) - \lambda] - \delta^2 a_T b_T r^2. \tag{80}$$

An elementary calculation shows that the polynomial function Q admits two roots (which are the eigenvalues of $\mathcal{M}$) given by

$$\begin{aligned} \lambda_1(r) &= \frac{a_T(\sigma+\delta r) + b_T(\tau+\delta r)}{2} \\ &\quad - \frac{1}{2}\left[((a_T\sigma - b_T\tau) + (a_T-b_T)\delta r)^2 + 4a_T b_T \delta^2 r^2\right]^{\frac{1}{2}}, \\ \lambda_2(r) &= \frac{f(r)}{2}. \end{aligned} \tag{81}$$

It can be easily seen that

$$0 \le \lambda_1(r) \le \lambda_2(r), \tag{82}$$

which yields

$$\rho(\mathcal{M}) = \frac{f(r)}{2}. \tag{83}$$

Since $f(r) < 2$ (by (vii)), it holds that $\rho(\mathcal{M}) < 1$. Therefore, by Lemma 11, the mapping F admits a fixed point $(u^*, v^*) \in V_r \subset V$, which is a solution to the system of integral equation (32). Moreover, for all $(f_0, g_0) \in V_r$, the Picard sequence $\{(f_n, g_n)\}$ defined by $(f_{n+1}, g_{n+1}) = F(f_n, g_n) = (F_1(f_n, g_n), F_2(f_n, g_n))$ converges uniformly to (u^*, v^*). On the other hand, by Proposition 14, we know that (32) admits at least one solution in V. Hence, one deduces that (u^*, v^*) is the only solution to (32) in V.

4. Some Special Cases

Some special cases of Theorem 15 are discussed in this section. We first consider the case $\alpha + \mu = \beta + \nu := \xi$. In this case, one has

$$a_T = b_T = \frac{(\psi(T) - \psi(0))^{\xi}}{\Gamma(\xi + 1)} := c_T. \tag{84}$$

Corollary 16. *Suppose that the following conditions are satisfied:*

(A_1) $T > 0, \psi \in \Psi$

(A_2) $0 < \alpha, \beta < 1, \alpha + \mu = \beta + \nu := \xi \geq 1$

(A_3) $\kappa, \sigma, \delta, \tau > 0$

(A_4) $4\tau \leq \sqrt{\sigma^2 + 4\delta\kappa} - \sigma$

(A_5) $u_0 = \dfrac{\tau}{\delta}, 0 \leq v_0 \leq u_0$

$$(A_6)\ C_T < \min\left\{\frac{\tau}{\delta\kappa}, \frac{1}{\sqrt{\sigma^2 + 4\delta\kappa} - \sigma}, \frac{2}{\sigma + 5\tau + \sqrt{(\sigma - \tau)^2 + 16\tau^2}}\right\} \tag{85}$$

Then, the system of integral equation (32) admits one and only one solution $(u^*, v^*) \in V$. Moreover, for all $f_0, g_0 \in V$ satisfying $u_0 \leq f_0$, $\|f_0\|_\infty \leq 2\tau/\delta$ and $\|g_0\|_\infty \leq 2\tau/\delta$, the sequence $\{(f_n, g_n)\} \subset V$ defined by (52) converges uniformly to (u^*, v^*).

Proof. The results follow immediately by taking $r = 2\tau/\delta$ and $\alpha + \mu = \beta + \nu$ in Theorem 15.

Consider now the special case of Corollary 16 when $\psi(t) = t$.

Corollary 17. *Suppose that the following conditions are satisfied:*

(B_1) $0 < \alpha, \beta < 1, \alpha + \mu = \beta + \nu := \xi \geq 1$

(B_2) $\kappa, \sigma, \delta, \tau > 0$

(B_3) $4\tau \leq \sqrt{\sigma^2 + 4\delta\kappa} - \sigma$

(B_4) $u_0 = \dfrac{\tau}{\delta}, 0 \leq v_0 \leq u_0$

$$(B_4)\ 0 < T < \left(\Gamma(\xi + 1) \min\left\{\frac{\tau}{\delta\kappa}, \frac{1}{\sqrt{\sigma^2 + 4\delta\kappa} - \sigma}, \frac{2}{\sigma + 5\tau + \sqrt{(\sigma - \tau)^2 + 16\tau^2}}\right\}\right)^{1/\xi} \tag{86}$$

Then, the system of integral equation (32) admits one and only one solution $(u^*, v^*) \in V$. Moreover, for all $f_0, g_0 \in V$ satisfying $u_0 \leq f_0$, $\|f_0\|_\infty \leq 2\tau/\delta$ and $\|g_0\|_\infty \leq 2\tau/\delta$, the sequence $\{(f_n, g_n)\} \subset V$ defined by (52) converges uniformly to (u^*, v^*).

Proof. In this case, one has

$$C_T = \frac{T^{\xi}}{\Gamma(\xi + 1)}. \tag{87}$$

Therefore, the assumption (A_6) in Corollary 16 reduces to the assumption (B_5). Hence, by Corollary 16, the desired results follow.

We take now $\psi(t) = t$ and consider the limit cases $(\alpha, \mu) \to (1^-, 0^+)$ and $(\beta, \nu) \to (1^-, 0^+)$. Notice that in this case, the system of integral equation (32) reduces to

$$\begin{cases} u(t) = u_0 + \displaystyle\int_0^t (\kappa - \sigma u(s) - \delta u(s)v(s))\, ds, & 0 \leq t \leq T \\ v(t) = v_0 + \displaystyle\int_0^t (\delta u(s)v(s) - \tau v(s))\, ds, & 0 \leq t \leq T, \end{cases} \tag{88}$$

which corresponds to the integral representation of the system of ordinary differential equations (1) subject to the initial conditions (3).

Taking $\xi = 1$ in Corollary 17, one deduces the following results.

Corollary 18. *Suppose that the following conditions are satisfied:*

(C_1) $\kappa, \sigma, \delta, \tau > 0$

(C_2) $4\tau \leq \sqrt{\sigma^2 + 4\delta\kappa} - \sigma$

(C_3) $u_0 = \dfrac{\tau}{\delta}, 0 \leq v_0 \leq u_0$

$$(C_4)\ 0 < T < \min\left\{\frac{\tau}{\delta\kappa}, \frac{1}{\sqrt{\sigma^2 + 4\delta\kappa} - \sigma}, \frac{2}{\sigma + 5\tau + \sqrt{(\sigma - \tau)^2 + 16\tau^2}}\right\} \tag{89}$$

Then, the system of integral equation (88) admits one and only one solution $(u^*, v^*) \in V$. Moreover, for all $f_0, g_0 \in V$ satisfying $u_0 \leq f_0$, $\|f_0\|_\infty \leq 2\tau/\delta$ and $\|g_0\|_\infty \leq 2\tau/\delta$, the sequence $\{(f_n, g_n)\} \subset V$ defined by

$$\begin{cases} f_{n+1}(t) = \mu_0 + \displaystyle\int_0^t (\kappa - \sigma f_n(s) - \delta f_n(s) g_n(s))\, ds, & 0 \leq t \leq T, \\ g_{n+1}(t) = v_0 + \displaystyle\int_0^t (\delta f_n(s) g_n(s) - \tau g_n(s))\, ds, & 0 \leq t \leq T, \end{cases} \tag{90}$$

converges uniformly to (u^*, v^*).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Discussion on Geraghty Type Hybrid Contractions

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In this manuscript, we define the notion of Geraghty type hybrid contractions in the setting of b-metric spaces. We prove the existence of a fixed point for such mappings whenever b-metric space is complete. Our observed results not only unify several existing results but also extend some known results.

1. Introduction and Preliminaries

The distance notion is one of the ancient and most basic concepts in the history of mathematics. In modern mathematics history, this notion was formally formulated by Frechét as "L-space." Later, it was redefined as "metric space" by Hausdorff. After then, this concept has been extended and generalized in several ways. From all these generalizations of metric notions, the b-metric is the most interesting.

In order to introduce the subject clearly, we first fix the basic concepts and notations. A function δ, defined from $X \times X$ (where X is a nonempty set) to nonnegative reals, is said to be a distance function, if it is symmetrical, that is $\delta(u, \nu) = \delta(\nu, u)$, for every $u, \nu \in X$ and $\delta(u, \nu) = 0$ if and only if $u = \nu$.

Moreover, a distance function δ is a (standard) metric in case that

$$\delta(u, \nu) \leq \delta(u, \nu) + \delta(\nu, \nu), \text{ for all } u, \nu, \nu \in X. \quad (1)$$

As we mentioned above, the distance notion, as well as the notion of the metric, has been extended and generalized in several directions. One of the outstanding generalizations of metric notion is named b-metric. Indeed, the concept of b-metric was considered by distinct authors, in various periods of the time, involving Bakhtin [1] and Czerwik [2]. Later, many researchers were interested in this topic, and thus, a series of interesting results were obtained, see, e.g., [3–19].

Definition 1. A distance function b on X is said to be a b-metric over constant $s \geq 1$ if the inequality (weighted triangle inequality)

$$b(u, \nu) \leq s[\mathrm{b}(u, \nu) + \mathrm{b}(\nu, \nu)], \quad (2)$$

holds for *all* $u, \nu, \nu \in X$.

In what follows, we consider that (X, b, s) denotes a b-metric space.

An immediate observation is that the notion of b-metric is more general than the concept of metric; for instance, when $s = 1$, we recover the notion of metric space. Moreover, we mention that a b-metric is not necessarily continuous, see, e.g., [20, 21].

Example 2. The function b on $X = [0,\infty)$, where $b(u, \nu) = |u - \nu|^q$, $q > 1$, is a b-metric over $s = 2^q$, but not a metric.

Definition 3. On a b-metric space (X, b, s), let $\{u_n\}$ be a sequence in X.

(a) The sequence $\{u_n\}$ is convergent in (X, b, s) to u, if for every $e > 0$, there exists $n_0 \in \mathbb{N}$ such that $b(u_n, u) < e$ for all $n > n_0$. (We denote by $u_n \to u$ as $n \to \infty$ or $\lim_{n\to\infty} u_n = u$.)

(b) The sequence $\{u_n\}$ is Cauchy, if for every $e > 0$, there exists $n_0 \in \mathbb{N}$ such that $b(u_n, u_{n+l}) < e$ for all $n > n_0, l > 0$

(c) If every Cauchy sequence in X converges to a point $u \in X$, then the triplet (X, b, s) is said to be complete

In short, (X^*, b, s) denotes a complete b-metric space over s.

Recently, Mitrovic et al. [22] introduced the following type of contractions.

Definition 4 (see [22]). Let (X, b, s) and $T : X \to X$ be a self-mapping. We say that T is a (r, a) -weight type contraction, if there exists $\kappa \in [0, 1)$ such that

$$b(Tu, Tv) \le \kappa \cdot M^r(T, u, v, a), \tag{3}$$

where $r \ge 0$, $a = (a_1, a_2, a_3)$,$a_i \ge 0, i = 1, 2, 3$ such that $a_1 + a_2 + a_3 = 1$ and

$$M^r(T, u, v, a) = \begin{cases} [a_1(b(u, v))^r + a_2(b(u, Tu))^r + a_3(b(v, Tv))^r]^{1/r}, & r > 0 \\ (b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{a_3}, & r = 0, \end{cases} \tag{4}$$

for all $u, v \in X \setminus \mathrm{Fix}(T)$, where $\mathrm{Fix}(T) = \{\varpi \in X, T\varpi = \varpi\}$.

In 1973, Geraghty [23] introduced a class of auxiliary functions to refine the Banach contraction principle. Let $\mathcal{G}$ be the set defined as

$$\mathcal{G} = \left\{\beta_b : [0,\infty) \to [0, 1) \mid \lim_{n\to\infty} \beta_b(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0\right\}. \tag{5}$$

Theorem 5 (see Geraghty [23]). *On a complete metric space (X, d), a mapping $T : X \to X$ admits a unique fixed point provided that there exists a function $\beta \in \mathcal{G}$ such that*

$$d(Tu, Tv) \le \beta(d(u, v))d(u, v), \textit{ for any } u, v \in X. \tag{6}$$

2. Main Results

Let the set $\mathcal{G}_b = \{\beta_b : [0,\infty) \to [0, (1/s)) \mid \limsup_{n\to\infty} \beta_b(t_n) = (1/s)$ implies $\lim_{n\to\infty} t_n = 0.\}$.

Definition 6. On (X, b, s), a mapping $T : X \to X$ is called Geraghty type hybrid contraction, if there exists $\beta_b \in \mathcal{G}_b$ such that

$$b(Tu, Tv) \le \beta_b(M^r(T, u, v, a))M^r(T, u, v, a), \tag{7}$$

where $r \ge 0$, $a = (a_1, a_2, a_3) \in [0,\infty) \times [0,\infty) \times [0,\infty)$, with $a_1 < 1$, $a_1 + a_2 + a_3 = 1$ and

$$M^r(T, u, v, a) = \begin{Bmatrix} [a_1(b(u, v))^r + a_2(b(u, Tu))^r + a_3(b(v, Tv))^r]^{1/r}, & r > 0, & u, v \in X \\ (b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{a_3}, & r = 0, & u, v \in X \setminus Fix(T). \end{Bmatrix} \tag{8}$$

Theorem 7. *On (X^*, b, s), a Geraghty type hybrid contraction $T : X \to X$ admits a unique fixed point $\varpi \in X$ if one of the following hypotheses is satisfied:*

(i) T is continuous at ϖ

(ii) $a_2 < 1$

(iii) $a_3 < 1$

Moreover, for any $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to ϖ.

Proof. We take an arbitrary point $u_0 \in X$. Starting from this initial point, we shall construct a recursive sequence $\{u_n\}$ with the following formula:

$$u_{n+1} = Tu_n \text{ for all } n \ge 0. \tag{9}$$

It is evident that if there exists k_0 such that $u_{k_0} = u_{k_0+1}$, then u_{k_0} becomes a fixed point of T. Therefore, from now, we assume that

$$u_n \ne u_{n+1} \text{ for all } n \ge 0. \tag{10}$$

We shall discuss all possible situations.

Case 8. Suppose that $r > 0$. From (7), we have that

$$b(Tu_n, Tu_{n-1}) \le \beta_b(M^r(T, u_n, u_{n-1}, a))M^r(T, u_n, u_{n-1}, a), \tag{11}$$

where

$$M^r(T, u_n, u_{n-1}, a) = [a_1(b(u_n, u_{n-1}))^r + a_2(b(u_n, u_{n+1}))^r + a_3(b(u_{n-1}, u_n))^r]^{1/r}. \tag{12}$$

It yields

$$b(u_{n+1}, u_n) \le \beta_b\left([(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r}\right) \cdot [(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r} \tag{13}$$

which is equivalent to

$$\frac{b(u_{n+1},u_n)}{[(a_1+a_3)(b(u_n,u_{n-1}))^r+a_2b(u_n,u_{n+1})]^r]^{1/r}} \leq \beta_b\left([(a_1+a_3)(b(u_n,u_{n-1}))^r+a_2b(u_n,u_{n+1})^r]^{1/r}\right)<\frac{1}{s}. \tag{14}$$

Therefore, we have

$$b(u_{n+1},u_n)<\left[\frac{(a_1+a_3)}{s^r-a_2}\right]^{1/r}b(u_n,u_{n-1})\leq b(u_n,u_{n-1}). \tag{15}$$

It yields $\{b(u_{n+1},u_n)\}$ is nonincreasing sequence bounded by 0. Thus, the sequence $\{b(u_{n+1},u_n)\}$ converges to a nonnegative real number, say L. We assert that L is 0. On the one hand, by taking the lim sup of all sides of (13), we deduce that

$$L\leq \limsup_{n\to\infty}\beta_b\left([(a_1+a_3)(b(u_n,u_{n-1}))^r+a_2b(u_n,u_{n+1})^r]^{1/r}\right)L<\frac{1}{s}L. \tag{16}$$

Suppose on the contrary, that $L>0$, we obtain

$$\frac{1}{s}\leq 1\leq \limsup_{n\to\infty}\beta_b\left([(a_1+a_3)(b(u_n,u_{n-1}))^r+a_2b(u_n,u_{n+1})^r]^{1/r}\right)<\frac{1}{s}. \tag{17}$$

Thus, the limit $\lim_{n\to\infty}[(a_1+a_3)(b(u_n,u_{n-1}))^r+a_2b(u_n,u_{n+1})^r]^{1/r}=0$. Consequently, $L=0$.

We assert that the sequence $\{u_n\}$ is b-Cauchy.

On contrary, supposing that the sequence $\{u_n\}$ is not b-Cauchy, we can find $e>0$ and two sequences of positive integers $\{q_i\}$ and $\{p_i\}$, $p_i>q_i\geq i$ such that

$$b\left(u_{q_i},u_{p_i}\right)\geq e \text{ and } b\left(u_{q_i},u_{p_i-1}\right)<e, \tag{18}$$

$$\begin{aligned}
&e\leq \liminf_{i\to\infty}b\left(u_{q_i},u_{p_i}\right)\leq \limsup_{i\to\infty}b\left(u_{q_i},u_{p_i}\right)\leq se\\
&\frac{e}{s}\leq \liminf_{i\to\infty}b\left(u_{q_i+1},u_{p_i}\right)\leq \limsup_{i\to\infty}b\left(u_{q_i+1},u_{p_i}\right)\leq s^2e\\
&\frac{e}{s}\leq \liminf_{i\to\infty}b\left(u_{q_i},u_{p_i+1}\right)\leq \limsup_{i\to\infty}b\left(u_{q_i},u_{p_i+1}\right)\leq s^2e\\
&\frac{e}{s^2}\leq \liminf_{i\to\infty}b\left(u_{q_i+1},u_{p_i+1}\right)\leq \limsup_{i\to\infty}b\left(u_{q_i+1},u_{p_i+1}\right)\leq s^3e.
\end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned}
\frac{e}{s}&\leq b\left(u_{q_i+1},u_{p_i}\right)=b\left(Tu_{q_i},Tu_{p_i-1}\right)\\
&\leq \beta_b\left(M^r\left(T,u_{q_i},u_{p_i-1},a\right)\right)M^r\left(T,u_{q_i},u_{p_i-1},a\right)\\
&<\frac{1}{s}M^r\left(T,u_{q_i},u_{p_i-1},a\right)
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
M^r\left(T,u_{q_i},u_{p_i-1},a\right)&=\left[a_1\left(b\left(u_{q_i},u_{p_i-1}\right)\right)^r+a_2\left(b\left(u_{q_i},Tu_{q_i}\right)\right)^r\right.\\
&\quad\left.+a_3\left(b\left(u_{p_i-1},Tu_{p_i-1}\right)\right)^r\right]^{1/r}\\
&=\left[a_1\left(b\left(u_{q_i},u_{p_i-1}\right)\right)^r+a_2\left(b\left(u_{q_i},u_{q_i+1}\right)\right)^r\right.\\
&\quad\left.+a_3\left(b\left(u_{p_i-1},u_{p_i}\right)\right)^r\right]^{1/r}
\end{aligned} \tag{21}$$

Taking lim sup of (21), we find

$$\limsup_{i\to\infty}M^r\left(T,u_{q_i},u_{p_i-1},a\right)\leq a_1^{1/r}e<e. \tag{22}$$

If we combine the observed inequalities above, in particular, (20) and (22), we have

$$\begin{aligned}
\frac{e}{s}&\leq \limsup_{i\to\infty}b\left(u_{q_i+1},u_{p_i}\right)\\
&\leq \limsup_{i\to\infty}\beta_b\left(M^r\left(T,u_{q_i},u_{p_i-1},a\right)\right)M^r\left(T,u_{q_i},u_{p_i-1},a\right)\\
&<e\limsup_{i\to\infty}\beta_b\left(M^r\left(T,u_{q_i},u_{p_i-1},a\right)\right)<\frac{e}{s},
\end{aligned} \tag{23}$$

since $a_1<1$. It implies that

$$\frac{1}{s}\leq \limsup_{i\to\infty}\beta_b\left(M^r\left(T,u_{q_i},u_{p_i-1},a\right)\right)\leq\frac{1}{s}. \tag{24}$$

Since $\beta_b\in\mathscr{G}_b$, we conclude $\lim_{i\to\infty}M^r(T,u_{q_i},u_{p_i-1},a)=0$. Attendantly,

$$\lim_{i\to\infty}n\left(u_{q_i},u_{p_i-1}\right)=0. \tag{25}$$

Under these observations, by employing the weighted triangle axiom together with (18), we get

$$e\leq b\left(u_{q_i},u_{p_i}\right)\leq s\left[b\left(u_{q_i},u_{p_i-1}\right)+b\left(u_{p_i-1},u_{p_i}\right)\right]\to 0asi\to\infty. \tag{26}$$

Therefore, $\{u_n\}$ is a b-Cauchy sequence in (X^*,b,s), so we can find a point $\varpi\in X$ such that

$$\lim_{n\to\infty}u_n=\varpi. \tag{27}$$

We assert now that this point, ϖ is a fixed point of T.

(i) Assuming that the mapping T is continuous at $\varpi\in X$, since $\lim_{n\to\infty}\mathrm{b}(\varpi,u_{n+1})=0$, we have

$$\lim_{n\to\infty} b(Tu_n, T\varpi) = b(T\varpi, T\varpi) = 0 \tag{28}$$

and from (49), we get $b(\varpi, T\varpi) = 0$, i.e., $T\varpi = \varpi$.

For the other cases, we consider the inequality,

$$b(\varpi, T\varpi) \le s[b(\varpi, u_{n+1}) + b(Tu_n, T\varpi)], \tag{29}$$

for any $n \in \mathbb{N}$.

(ii) Suppose that $a_2 < 1$. If $T\varpi \ne \varpi$, we have

$$\begin{aligned}
0 < b(T\varpi, \varpi) &\le s[b(T\varpi, u_{n+1}) + b(u_{n+1}, \varpi)] \\
&= s[b(T\varpi, Tu_n) + b(u_{n+1}, \varpi)] \\
&\le s[\beta_b(M^r(T, \varpi, u_n, a))M^r(T, \varpi, u_n, a) + b(u_{n+1}, \varpi)] \\
&< M^r(T, \varpi, u_n, a) + sb(u_{n+1}, \varpi) \\
&= [a_1(b(\varpi, u_n))^r + a_2(b(\varpi, T\varpi))^r + a_3(b(u_n, Tu_n))^r]^{1/r} \\
&\quad + sb(u_{n+1}, \varpi)
\end{aligned} \tag{30}$$

and when $n \to \infty$, we get

$$0 < b(T\varpi, \varpi) \le (a_2)^{1/r} b(T\varpi, \varpi). \tag{31}$$

Since $a_2 < 1$, we get a contradiction, that is, $T\varpi = \varpi$.

(iii) Suppose that $a_3 < 1$. Assuming that $b(T\varpi, \varpi) > 0$, we have

$$\begin{aligned}
0 < b(\varpi, T\varpi) &\le s[b(\varpi, u_{n+1}) + b(u_{n+1}, T\varpi)] \\
&= s[b(\varpi, u_{n+1}) + b(Tu_n, T\varpi)] \\
&\le s[b(\varpi, u_{n+1}) + \beta_b(M^r(T, u_n, \varpi, a))M^r(T, u_n, \varpi, a)] \\
&< sb(\varpi, u_{n+1}) + M^r(T, u_n, \varpi, a)] \le sb(\varpi, u_{n+1}) \\
&\quad + [a_1(b(u_n, \varpi))^r + a_2(b(u_n, u_{n+1}))^r + a_3(b(\varpi, T\varpi))^r]^{1/r}.
\end{aligned} \tag{32}$$

Taking $n \to \infty$, we have

$$0 < b(T\varpi, \varpi) \le (a_3)^{1/r} b(T\varpi, \varpi), \tag{33}$$

which is a contradiction, since $a_3 < 1$. Therefore, $T\varpi = \varpi$.

Case 9. $r = 0$. Here, (7) and (8) become

$$b(Tu, Tv) \le \beta_b(I(T, u, v, a))I(T, u, v, a), \tag{34}$$

for every $u, v \in X$, where

$$I(T, u, v, a) := (b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{1-a_1-a_2} \tag{35}$$

$\kappa \in [0, 1)$ and $a_1, a_2 \in (0, 1)$.

As in the proof of the case $r > 0$, we shall consider a recursive sequence $\{u_n = Tu_{n-1}\}$, starting with an arbitrary point $u \in X$ where $u_0 = u$. By using the same argument of this part of the proof, we presume that

$$u_n \ne u_{n+1} \text{ for all } n \ge 0. \tag{36}$$

Employing $u = u_{n-1}$ and $v = u_n$ in (34), we find that

$$\begin{aligned}
b(u_n, u_{n+1}) = b(Tu_{n-1}, Tu_n) &\le \beta_b((b(u_{n-1}, u_n))^{a_1} \\
&\quad \cdot (b(u_{n-1}, Tu_{n-1}))^{a_2}(b(u_n, Tu_n))^{1-a_1-a_2}) \\
&\quad \cdot (b(u_{n-1}, u_n))^{a_1}(b(u_{n-1}, Tu_{n-1}))^{a_2} \\
&\quad \cdot (b(u_n, Tu_n))^{1-a_1-a_2} \\
&= \beta_b((b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, u_{n+1}))^{1-a_1-a_2}) \\
&\quad \cdot (b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, u_{n+1}))^{1-a_1-a_2} \\
&< \frac{1}{s}(b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, Tu_n))^{1-a_1-a_2} \\
&\le (b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, Tu_n))^{1-a_1-a_2}
\end{aligned} \tag{37}$$

It yields that

$$\begin{aligned}
(b(u_n, u_{n+1}))^{a_1+a_2} \le (b(u_{n-1}, u_n))^{a_1+a_2} iff b(u_n, u_{n+1}) \\
\le b(u_{n-1}, u_n),
\end{aligned} \tag{38}$$

for each $n \in \mathbb{N}$. Attendantly, we deduce that the sequence of nonnegative numbers $\{b(u_{n-1}, u_n)\}$ is a nonincreasing sequence. Ergo, there is a real number $L \ge 0$ such that $\lim_{n\to\infty} b(u_{n-1}, u_n) = L$.

As in the previous case, we assert that $L = 0$. Supposing on the contrary, that $L > 0$, by taking lim sup in (37), we derive that

$$\begin{aligned}
L \le \limsup_{n\to\infty} \beta_b((b(u_{n-1}, u_n))^{a_1}(b(u_{n-1}, Tu_{n-1}))^{a_2} \\
\cdot (b(u_n, Tu_n))^{1-a_1-a_2})L.
\end{aligned} \tag{39}$$

Since $L > 0$, we obtain

$$\begin{aligned}
\frac{1}{s} \le 1 \le \limsup_{n\to\infty} \beta_b((b(u_{n-1}, u_n))^{a_1}(b(u_{n-1}, u_n))^{a_2} \\
\cdot (b(u_n, u_{n+1}))^{1-a_1-a_2})\frac{1}{s}.
\end{aligned} \tag{40}$$

Thus, $\lim_{n\to\infty}((b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, u_{n+1}))^{1-a_1-a_2}) = 0$ and consequently, $L = 0$.

We claim that $\{u_n\}$ is a b-Cauchy sequence. On contrary, if we suppose that $\{u_n\}$ is not a b-Cauchy sequence, that is, we can find $e > 0$ and the sequences $\{q_i\}$, $\{p_i\}$ of positive integers with $p_i > q_i \ge i$ such that

$$b\left(u_{q_i}, u_{p_i}\right) \ge e \text{ and } b\left(u_{q_i}, u_{p_i-1}\right) < e. \tag{41}$$

By the weighted triangle inequality, we have

$$e \le b\left(u_{q_i}, u_{p_i}\right) \le s\left[b\left(u_{q_i}, u_{q_i+1}\right) + b\left(u_{q_i+1}, u_{p_i}\right)\right]. \tag{42}$$

Since

$$\begin{aligned} b\left(u_{q_i+1}, u_{p_i}\right) &= b\left(Tu_{q_i}, Tu_{p_i-1}\right) \\ &\le \beta_b\left(I\left(T, u_{q_i}, u_{p_i-1}, a\right)\right) I\left(T, u_{q_i}, u_{p_i-1}, a\right) \\ &< \frac{1}{s} I\left(T, u_{q_i}, u_{p_i-1}, a\right) \end{aligned} \tag{43}$$

where

$$\begin{aligned} I\left(T, u_{q_i}, u_{p_i-1}, a\right) &= \left(b\left(u_{q_i}, u_{p_i-1}\right)\right)^{a_1} \left(b\left(u_{q_i}, Tu_{q_i}\right)\right)^{a_2} \\ &\quad \cdot \left(b\left(u_{p_i-1}, Tu_{p_i-1}\right)\right)^{1-a_1-a_2} \\ &= \left(b\left(u_{q_i}, u_{p_i-1}\right)\right)^{a_1} \left(b\left(u_{q_i}, u_{q_i+1}\right)\right)^{a_2} \\ &\quad \cdot \left(b\left(u_{p_i-1}, u_{p_i}\right)\right)^{1-a_1-a_2}, \end{aligned} \tag{44}$$

taking lim sup of (43), we find

$$\limsup_{i\to\infty} I\left(T, u_{q_i}, u_{p_i-1}, a\right) = 0 \text{ and hence } \limsup_{i\to\infty} b\left(u_{q_i+1}, u_{p_i}\right) = 0. \tag{45}$$

If we combine the observed inequalities above, in particular, (42) and (45), we get that we get

$$e \le b\left(u_{q_i}, u_{p_i}\right) \le s\left[b\left(u_{q_i}, u_{q_i+1}\right) + b\left(u_{q_i+1}, u_{p_i}\right)\right] \longrightarrow 0, as i \longrightarrow \infty. \tag{46}$$

Therefore, the sequence $\{u_n\}$ is b-Cauchy in (X^*, b, s), so it is convergent at a point $\varpi \in X$, that is

$$\lim_{n\to\infty} u_n = \varpi. \tag{47}$$

Now, we assert that ϖ is a fixed point of T.

If the assumption (i) holds, since $\lim_{n\to\infty} b(\varpi, u_{n+1}) = 0$ we get

$$\lim_{n\to\infty} b(Tu_n, T\varpi) = b(T\varpi, T\varpi) = 0 \tag{48}$$

and from the inequality

$$b(\varpi, T\varpi) \le s[b(\varpi, u_{n+1}) + b(Tu_n, T\varpi)], \tag{49}$$

for any $n \in \mathbb{N}$, we obtain $b(\varpi, T\varpi) = 0$, i.e., $T\varpi = \varpi$.

Suppose that $a_2 < 1$ or $a_3 < 1$. Assuming that $T\varpi \ne \varpi$, we have

$$\begin{aligned} 0 < b(T\varpi, \varpi) &\le s[b(T\varpi, u_{n+1}) + b(u_{n+1}, \varpi)] \\ &= s[b(T\varpi, Tu_n) + b(u_{n+1}, \varpi)] \\ &\le s[\beta_b(I(T, \varpi, u_n, a)) I(T, \varpi, u_n, a) + b(u_{n+1}, \varpi)] \\ &< I(T, \varpi, u_n, a) + sb(u_{n+1}, \varpi) \\ &= \left[(b(\varpi, u_n))^{a_1} (b(\varpi, T\varpi))^{a_2} (b(u_n, Tu_n))^{1-a_1-a_2}\right] \\ &\quad + sb(u_{n+1}, \varpi). \end{aligned} \tag{50}$$

At the limit as $n \longrightarrow \infty$, we have $b(T\varpi, \varpi) = 0$. So, $T\varpi = \varpi$.

Example 10. We shall derive several distinct contractions from Definition 6. Some examples are given below. Let T be a self-mapping on X.

(1) If $r = 2$, $a = (1/3, 1/3, 1/3)$, we obtain the following condition

$$\begin{aligned} b(Tu, T\nu) &\le \beta_b\left(\frac{1}{\sqrt{3}}\left[b^2(u,\nu) + b^2(u,Tu) + b^2(\nu,T\nu)\right]^{1/2}\right) \\ &\quad \cdot \left(\frac{1}{\sqrt{3}}\left[b^2(u,\nu) + b^2(u,Tu) + b^2(\nu,T\nu)\right]^{1/2}\right) \end{aligned} \tag{51}$$

(2) If $r = 1$, $a = (1/3, 1/3, 1/3)$, we obtain the following condition

$$\begin{aligned} b(Tu, T\nu) &\le \beta_b\left(\frac{1}{3}[b(u,\nu) + b(u,Tu) + b(\nu,T\nu)]\right) \\ &\quad \cdot \left(\frac{1}{3}[b(u,\nu) + b(u,Tu) + b(\nu,T\nu)]\right) \end{aligned} \tag{52}$$

(3) If $r = 0$, $a = (0, a_1, 1 - a_1)$ with $a_1 \in (0, 1)$, we obtain

$$\begin{aligned} b(Tu, T\nu) &\le \beta_b\left((b(u,Tu))^{a_1} b(\nu,T\nu))^{1-a_1}\right) \\ &\quad \cdot (b(u,Tu))^{a_1} (b(\nu,T\nu))^{1-a_1} \end{aligned} \tag{53}$$

which means that T is an interpolative Kannan type Geraghty-contraction;

(4) If $r = 0$, $a = (a_1, a_2, 1 - a_1 - a_2)$ with $a_1, a_2 \in (0, 1)$, we have

$$\begin{aligned} b(Tu, T\nu) &\le \beta_b\left((b(u,\nu))^{a_1} (b(u,Tu))^{a_2} (b(\nu,T\nu))^{1-a_1-a_2}\right) \\ &\quad \cdot (b(u,\nu))^{a_1} (b(u,Tu))^{a_2} (b(\nu,T\nu))^{1-a_1-a_2} \end{aligned} \tag{54}$$

that is T is an interpolative Reich-Rus-C$'$iric$'$ type Geraghty-contraction.

Related to these examples, we can establish some consequences, by choosing proper values for r, a_1, a_2, a_3 in Theorem 15.

Corollary 11. *Let (X^*, b, s) and a self-mapping T on X. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left(\frac{\left[b^2(u,v) + b^2(u,Tu) + b^2(v,Tv)\right]^{1/2}}{\sqrt{3}}\right) \cdot \frac{\left[b^2(u,v) + b^2(u,Tu) + b^2(v,Tv)\right]^{1/2}}{\sqrt{3}}, \tag{55}$$

for all $u, v \in X$ then T admits a unique fixed point $\varpi \in X$.

Corollary 12. *Let (X^*, b, s) and a self-mapping T on X. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left(\frac{b(u,v) + b(u,Tu) + b(v,Tv)}{3}\right) \cdot \frac{b(u,v) + b(u,Tu) + b(v,Tv)}{3}, \tag{56}$$

for all $u, v \in X$ then T admits a unique fixed point $\varpi \in X$.

Corollary 13. *Let (X^*, b, s) be a complete b -metric space, a self-mapping T on X and $a_1 \in (0, 1)$. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left((b(u,Tu))^{a_1}(b(v,Tv))^{1-a_1}\right) \cdot \left((b(u,Tu))^{a_1}(b(v,Tv))^{1-a_1}\right), \tag{57}$$

for all $u, v \in XFix(T)$, then T admits a fixed point $\varpi \in X$.

Corollary 14. *Let (X^*, b, s) be a complete b-metric space, a self-mapping T on X and $a_1, a_2 \in (0, 1)$. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left((b(u,v))^{a_1}(b(u,Tu)^{a_2}(b(v,Tv))^{1-a_1-a_2}\right) \cdot \left((b(u,v))^{a_1}(b(u,Tu)^{a_2}(b(v,Tv))^{1-a_1-a_2}\right), \tag{58}$$

for all $u, v \in X \setminus Fix(T)$, then T admits a fixed point $\varpi \in X$.

3. Immediate Consequences

By letting $\beta_b(t) = \kappa$, we shall observe the Definition 4, [22].

Theorem 15 (see [22]). *Let (X^*, b, s). A (r, a) -weight type contraction mapping $T : X \to X$ admits a fixed point $\varpi \in X$ if one of the following holds:*

(i) T is continuous at such point ϖ

(ii) $s^r a_2 < 1$

(iii) $s^r a_3 < 1$

Moreover, for any $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to ϖ.

We list the following corollaries.

Corollary 16. *On the complete b -metric space (X^*, b, s) let $T : X \to X$ be a mapping. If there exists $\kappa \in [0, 1)$ such that*

$$b(Tu, Tv) \le \kappa \cdot b^{a_1}(u,v) \cdot b^{a_2}(u,Tu) \cdot b^{a_3}(v,Tv), \tag{59}$$

for all $u, v \in X \setminus Fix(T)$, $a_1, a_2, a_3 \ge 0$ and $\sum_{i=1}^{3} a_i = 1$, then T has a fixed point $\varpi \in X$. $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to ϖ.

Proof. Put in Theorem 15, $r = 0$ and $a = (a_1, a_2, a_3)$.

Corollary 17. *On the complete b -metric let $T : X \to X$ be a mapping such that*

$$b(Tu, Tv) \le \kappa \sqrt[3]{b(u,v) b(u,Tu) b(v,Tv)}, \tag{60}$$

for all $u, v \in X \setminus Fix(T)$, where $\kappa \in [0, 1)$. Then, T has a fixed point $\varpi \in X$.

Proof. Put in Theorem 15, $r = 0$ and $a = (1/3, 1/3, 1/3)$.

Corollary 18. *Let (X^*, b, s) be a complete b -metric space and $T : X \to X$ be a mapping such that for every $u, v \in X \setminus Fix(T)$*

$$b(Tu, Tv) \le \frac{\kappa}{3}[b(u,v) + b(u,Tu) + b(v,Tv)], \tag{61}$$

where $\kappa \in [0, 1)$. The mapping T has a fixed point ϖ provided that one of the following hold:

(i) T is continuous at $\varpi \in X$

(ii) $s < 3$

Then, T has a fixed point ϖ. Moreover for any $u_0 \in X$, the sequence $\{T^n u_0\}$ converges to ϖ.

Proof. Let $r = 1$ and $a = (1/3, 1/3, 1/3)$ in Theorem 15.

Corollary 19. *Let (X^*, b, s) be a complete b -metric space and $T : X \to X$ be a mapping such that*

$$b(Tu, Tv) \le \frac{\kappa}{\sqrt{3}}\left[b^2(u,v) + b^2(u,Tu) + b^2(v,Tv)\right]^{1/2}, \tag{62}$$

for all $u, v \in XFix(T)$, where $\kappa \in [0, 1)$. Assume that one of the following conditions hold:

(i) T is continuous at $\varpi \in X$

(ii) $s^2 < 3$

Then, T has a fixed point ϖ *and for any* $u_0 \in X$, *the sequence* $\{T^n u_0\}$ *converges to* ϖ.

Proof. Take $r = 2$ and $a = (1/3, 1/3, 1/3)$ in Theorem 15.

Corollary 20 (see [24]). *Let* (X^*, b, s), *T be a self-mapping on X and* $a_1 \in (0, 1)$. *If there exists a function* $\beta_b \in \mathscr{G}_b$ *such that*

$$b(Tu, T\nu) \leq \kappa \cdot b^{a_1}(u, Tu)\, b^{1-a_1}(\nu, T\nu), \qquad (63)$$

for all $u, \nu \in inX \setminus Fix(T)$; *then, T admits a unique fixed point* $\varpi \in X$.

Proof. Choose $\beta_b(t) = \kappa$ in Corollary 13.

Corollary 21 (see [25]). *Let* (X^*, b, s), *T be a self-mapping on X and* $a_1, a_2 \in (0, 1)$. *If there exists* $\kappa \in (0, 1)$ *such that*

$$b(Tu, T\nu) \leq \kappa \cdot b^{a_1}(u, \nu)\, b^{a_2}(u, Tu)\, b^{1-a_1-a_2}(\nu, T\nu), \qquad (64)$$

for all $u, \nu \in inX \setminus Fix(T)$; *then, T admits a unique fixed point* $\varpi \in X$.

Proof. Choose $\beta_b(t) = \kappa$ in Corollary 14.

4. Conclusions

In this paper, we combine linear and nonlinear contractions to unify and extend the several existing results. This approach may bring new frames to the topic of metric fixed point theory. In particular, interpolative contraction may extend several results in the setting of Banach space.

We also mention that, for the case $s = 1$, we find a series of results known in the context of metric spaces, see, e.g., [25–36].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Best Proximity Point Theorems for Cyclic Contractions Mappings in Banach Algebras

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In this paper, we present some new best proximity point theorems for three operators acting in Banach algebras. An application is given to show the usefulness and the applicability of the obtained results.

1. Introduction

The study of functional integral equations and differential equations is the main object of research in nonlinear functional analysis. These equations occur in physical, biological, and economic problems. Some of these equations can be formulated into nonlinear operator equations:

$$x = A(x) \cdot B(x) + C(x) \tag{NOE}$$

in suitable Banach algebras.

Recently, many authors are interested on the study of equation (NOE) and obtained some interesting results (see for instance [1–5]). In 2010, Ben Amar et al. [6] proved some existence fixed point theorems which allowed them to solve equation (NOE) where the involved operators are weakly sequentially continuous.

Let X be a Banach algebra with a norm $\|\cdot\|$. Let (Ω_1, Ω_2) be a pair of nonempty subsets of X. Given two mappings A and C defined on X and an operator $B : \Omega_1 \cup \Omega_2 \longrightarrow X$. Under suitable conditions, we define the operator $T := ((I - C)/A)^{-1} \circ B : \Omega_1 \cup \Omega_2 \longrightarrow \Omega_1 \cup \Omega_2$ such that $T(\Omega_1) \subset \Omega_2$ and $T(\Omega_2) \subset \Omega_1$. If $\Omega_1 \cap \Omega_2$ is nonempty, then the mapping T restricted to $\Omega_1 \cap \Omega_2$ is a self mapping. Then, a solution of equation (NOE) is a fixed point of T. Furthermore, if the fixed point equation $x = A(x).B(x) + C(x)$ does not possess a solution it is natural to explore to find an $x^* \in \Omega_1$ satisfying

$$\|x^* - T(x^*)\| = \mathrm{dist}(\Omega_1, \Omega_2), \tag{1}$$

where $\mathrm{dist}(\Omega_1, \Omega_2) = \inf \{\|x - y\| : x \in \Omega_1, y \in \Omega_2\}$. This point $x^* \in \Omega_1$ is said to be the best proximity point of T. Note that a point $x \in \Omega_1 \cup \Omega_2$ is the best proximity point of T if x is a solution of the minimization problem

$$\min_{x \in \Omega_1 \cup \Omega_2} d(x, Tx). \tag{2}$$

The best proximity point notion can be viewed as a generalization of fixed point, since most fixed point theorems can be derived as corollaries of the best proximity point theorems.

The first result of this kind is due to Fan (see [7], Theorem 2) which is stated in normed spaces for continuous mappings. In [8], Eldred and Veeramani introduced the concept of cyclic contraction mappings and gave the best proximity point results for this class of mappings. They also gave an algorithm to reach this best proximity point where the space is uniformly convex. Furthermore, in [9], Taghafi and Shahzad proved the existence of the best proximity point for a cyclic contraction mapping in a reflexive Banach space.

For noncyclic mappings, i.e., $T(\Omega_1) \subset \Omega_1$ and $T(\Omega_2) \subset \Omega_2$, Gabelah and Künzi in [10] established some best proximity point results in the framework of complete CAT(0) spaces. In addition, they gave an approach to reach this best proximity point by means of an algorithm. Regarding the relationship between the noncyclic and cyclic results, the authors in [11] proved that the existence of best proximity points for cyclic nonexpansive mappings is equivalent to the existence of best proximity pairs for noncyclic nonexpansive mappings in the setting of strictly convex Banach spaces. For more on the best proximity point results, the interested reader can consult [12–18].

The paper is organized as follows. After some preliminaries, in Section 3, we prove the existence of the best proximity point where the involving operators are $\mathscr{D}$-Lipschitzs and cyclic contraction (see Theorem 1). Also, an example is given to illustrate the obtained result. In Theorem 1, we consider the case where X is a uniformly convex Banach algebra. In Section 4, we show the applicability of our result (Theorem 1) to the theory of nonlinear integral equations:

$$x(t) = K(t, x(t)) + (Tx)(t) \cdot \left(q_2 + \int_0^t g(s, y(s)) ds \right) \quad \text{(FIS1)}$$

$$y(t) = K(t, y(t)) + (Ty)(t) \cdot \left(q_1 + \int_0^t f(s, x(s)) ds \right), \quad \text{(FIS2)}$$

where Ω_1 and Ω_2 two subsets of the Banach algebra $E = \mathscr{C}(J, X)$ of all continuous functions from J to X.

2. Preliminaries

Definition 1. An algebra X is a vector space endowed with an internal composition law noted by $(\cdot)$ i.e.,

$$\begin{cases} (\cdot)\colon X \times X \longrightarrow X \\ (x, y) \longrightarrow x \cdot y \end{cases} \quad (3)$$

which is associative and bilinear.

A normed algebra is an algebra endowed with a norm satisfying the following property for all $x, y \in X$; $\|x \cdot y\| \leq \|x\| \|y\|$. A complete normed algebra is called a Banach algebra.

Definition 2. Let X be a Banach space with norm $\|\cdot\|$. A mapping $T : X \longrightarrow X$ is called $\mathscr{D}$ -Lipschitz if there exists a continuous nondecreasing function $\Phi_T : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

$$\|Tx - Ty\| \leq \Phi_T(x - y) \quad (4)$$

for all $x, y \in X$ with $\Phi_T(0) = 0$. In the special case when $\Phi_T(r) = \alpha r$ for some $\alpha > 0$, T is called lipschitzian mapping with a Lipschitz constant α.

Definition 3. Let $(X, \|\cdot\|)$ be a Banach space. We say that X is uniformly convex if for every $\varepsilon > 0$,

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1 \text{and} \|x - y\| \geq \varepsilon \right\} > 0, \quad (5)$$

The function δ is known as the modulus of uniform convexity of X. Note that any uniformly convex Banach space is reflexive.

Theorem 4 (see [17]). *Let X be a uniformly convex Banach space. Let Ω_1 be a nonempty closed bounded convex subset of X such that Ω_1^0 is compact, and Ω_2 be a nonempty closed convex subset of X. Let $T : \Omega_1 \cup \Omega_2 \longrightarrow \Omega_1 \cup \Omega_2$ be a relatively nonexpansive mapping. Then, there exists $x^* \in \Omega_1$ such that $\|x^* - T(x^*)\| = dist(\Omega_1, \Omega_2)$.*

The authors in [8] introduced the following notion of cyclic contraction.

Definition 5. Let Ω_1 and Ω_2 be nonempty subsets of a metric space X. A mapping $T : \Omega_1 \cup \Omega_2 \longrightarrow \Omega_1 \cup \Omega_2$ is said to be a cyclic contraction if it satisfies:

(1) $T(\Omega_1) \subset \Omega_2$ and $T(\Omega_2) \subset \Omega_1$

(2) for some $k \in (0, 1)$, $d(T(x), T(y)) \leq kd(x, y) + (1 - k)\text{dist}(\Omega_1, \Omega_2)$, for all $x \in \Omega_1$, $y \in \Omega_2$

Since $\text{dist}(\Omega_1, \Omega_2) \leq d(x, y)$, for $x \in \Omega_1$ and $y \in \Omega_1$, $d(T(x), T(y)) \leq d(x, y)$ for all $x \in \Omega_1 y \in \Omega_2$, i.e., T is relatively nonexpansive.

We conclude this section by recalling some best proximity point results for this class of mappings.

Theorem 6 (see [8]). *Let Ω_1 and Ω_2 be nonempty closed subsets of a complete metric space X. Let $T : \Omega_1 \cup \Omega_2 \longrightarrow \Omega_1 \cup \Omega_2$ be a cyclic contraction mapping and $x_0 \in \Omega_1$. Define $x_{n+1} = T(x_n)$, $n \in \mathbb{N}$. Suppose $\{x_{2n} : n \in \mathbb{N}\}$ has a convergent subsequence in Ω_1, then there exists $x \in \Omega_1$ such that $d(x, T(x)) = dist(\Omega_1, \Omega_2)$.*

Theorem 7 (see [8, 19]). *Let Ω_1 and Ω_2 be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $T : \Omega_1 \cup \Omega_2 \longrightarrow \Omega_1 \cup \Omega_2$ is a cyclic contraction mapping. Then, T has a unique best proximity point in Ω_1. Further, if $x_0 \in \Omega_1$ and $x_{n+1} = T(x_n)$, then the sequence $(x_{2n})_{n\geq 0}$ converges to the best proximity point.*

3. Main Results

We start this section by introducing the notion of (α, β)-monotone property for a pair of functions.

Definition 8. Let $(\alpha, \beta) \in (\mathbb{R}_+)^2$ and $\phi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two mappings. We say that the pair (ϕ, ψ) has the property (α, β) -monotone if

(i) $\phi(0) = 0 = \psi(0)$,

(ii) $I - \alpha \cdot \phi - \beta \cdot \psi$ is nondecreasing on $\mathbb{R}_+$ and $\lim_{r \to +\infty}(r - \alpha\phi(r) - \beta\psi(r)) = +\infty$

Remark 9. If (ϕ, ψ) has the property (α, β) -monotone and ϕ, ψ are continuous, then the mapping $I - \alpha \cdot \phi - \beta \cdot \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is invertible.

Example 1. Let $\phi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the mappings defined by:

$$\phi(r) = \frac{r}{r+1} \text{ and } \psi(r) = \frac{r}{2}, \text{ for all } r \in \mathbb{R}_+. \tag{6}$$

So, the mapping $I - 1/2\phi - \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ has the property $(1/2, 1)$-monotone.

Recall that an operator A from a Banach algebra X is said to be regular on X if A maps X into the set of all invertible elements of X.

Theorem 10. *Let (Ω_1, Ω_2) be a nonempty closed pair of a Banach algebra X. Let $A, C : X \longrightarrow X$ and $B : \Omega_1 \cup \Omega_2 \longrightarrow X$ be three operators which satisfy the following conditions:*

(1) A is regular on X and $\|A\| < 1$

(2) A and C are $\mathcal{D}$-Lipschitzs with the $\mathcal{D}$-functions Φ_A and Φ_C, respectively, $B(\Omega_1 \cup \Omega_2)$ is bounded with bound M, and $M\Phi_A(r) + \Phi_C(r) \le (1 - \|A\|)r$ for all $r > 0$, and $I - M\Phi_A - \Phi_C$ is nondecreasing

(3) B is cyclic contraction mapping on $\Omega_1 \cup \Omega_2$

(4) suppose there exists a sequence $(x_n)_{n\ge0}$ of $\Omega_1 \cup \Omega_2$ such that $x_0 \in \Omega_1$, $B(x_n) = ((I - C)/A)(x_{n+1})$ and the sequence $(x_{2n})_{n\ge0}$ has a convergent subsequence in Ω_1,

(5) $$\begin{cases} y = A(y) \cdot B(x) + C(y),\ x \in \Omega_1 \Rightarrow y \in \Omega_2 \\ x = A(x) \cdot B(y) + C(x),\ y \in \Omega_2 \Rightarrow x \in \Omega_1 \end{cases}$$

Then, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\begin{aligned} &\left\| \frac{x - A(x).B(x) - C(x)}{A(x)} \right\| \\ &\quad = dist(\Omega_1, \Omega_2) = \left\| \frac{y - A(y) \cdot B(y) - C(y)}{A(y)} \right\|. \end{aligned} \tag{7}$$

Proof. Let y be fixed in $\Omega_1 \cup \Omega_2$ and let us define the mapping F_y on X by

$$F_y(x) = A(x) \cdot B(y) + C(x), \text{ for all } x \in X. \tag{8}$$

Let $x_1, x_2 \in X$. The use of assumption (ii) leads to

$$\begin{aligned} \left\| F_y(x_1) - F_y(x_2) \right\| &\le \|A(x_1) \cdot B(y) - A(x_2) \cdot B(y)\| \\ &\quad + \|C(x_1) - C(x_2)\| \\ &\le \|A(x_1) - A(x_2)\| \|B(y)\| \\ &\quad + \|C(x_1) - C(x_2)\| \\ &\le M\Phi_A(\|x_1 - x_2\|) \\ &\quad + \Phi_C(\|x_1 - x_2\|) \end{aligned} \tag{9}$$

Now an application of Boyd and Wong's fixed point theorem [20], Theorem 1 leads to the existence of a unique point $x_y \in X$ such that $F_y(x_y) = x_y$. Hence, the operator $T := ((I - C)/A)^{-1}B : \Omega_1 \cup \Omega_2 \longrightarrow X$ is well defined.

Moreover, assumption (v) implies that $T(\Omega_1) \subset \Omega_2$ and $T(\Omega_2) \subset \Omega_1$. Indeed, let $x \in \Omega_1$ and $y \in X$ such that $y = A(y) .B(x) + C(y)$, so $T(x) = ((I - C)/A)^{-1}B(x) = y \in \Omega_2$. Similarly, for all $y \in \Omega_2$, $T(y) \in \Omega_1$. Hence, T is cyclic on $\Omega_1 \cup \Omega_2$.

T is cyclic contraction on $\Omega_1 \cup \Omega_2$. Indeed, let $(x, y) \in \Omega_1 \times \Omega_2$, the use of assumption (ii) and (iii) and the fact that $T(z) = A(T(z)) \cdot B(z) + C(T(z))$ for all $z \in \Omega_1 \cup \Omega_2$ leads to

$$\begin{aligned} \|T(x) - T(y)\| &\le \|A(T(x)) \cdot B(x) - A(T(y)) \cdot B(y)\| \\ &\quad + \|C(T(x) - C(T(y)\| \\ &\le \|A(T(x)) - A(T(y))\| \|B\| \\ &\quad + \|B(x) - B(y)\| \|A\| \\ &\quad + \|C(T(x) - C(T(y)\| \\ &\le \|A\|(k\|x - y\| + (1 - k)\mathrm{dist}(\Omega_1, \Omega_2)) \\ &\quad + M\Phi_A(\|T(x) - T(y)\|) \\ &\quad + \Phi_C(\|T(x) - T(y)\| \end{aligned} \tag{10}$$

Since (Φ_A, Φ_C) has the property $(M, 1)$-monotone, we have

$$\begin{aligned} \|T(x) - T(y)\| &\le (I - M\Phi_A - \Phi_C)^{-1} \\ &\quad \cdot (\|A\|(k\|x - y\| + (1 - k)\mathrm{dist}(A, B))) \\ &\le k\|x - y\| + (1 - k)\mathrm{dist}(\Omega_1, \Omega_2). \end{aligned} \tag{11}$$

By (iv), there exists a sequence $(x_n)_{n\ge0}$ of $\Omega_1 \cup \Omega_2$ such that $x_0 \in \Omega_1$, $T(x_n) = x_{n+1}$ and the sequence $(x_{2n})_{n\ge0}$ has a convergent subsequence in Ω_1.

Thus, by Theorem 6, there exists $(x_1, y_1) \in \Omega_1 \times \Omega_2$ such that $\|x_1 - Tx_1\| = \mathrm{dist}(\Omega_1, \Omega_2) = \|y_1 - Ty_1\|$.

Let $y = ((I - C)/A)^{-1}(B(x_1)) \in \Omega_2$. By (iii), B is cyclic contraction on $\Omega_1 \cup \Omega_2$, $B(y) \in \Omega_1$and $((I - C)/A)(y) \in \Omega_2$, so

$$\begin{aligned}\operatorname{dist}(\Omega_1,\Omega_2) &\le \left\|\frac{y-A(y)\cdot B(y)-C(y)}{A(y)}\right\| \\ &= \left\|\left(\frac{I-C}{A}\right)(y)-B(y)\right\| \le \|B(x_1)-B(y)\| \\ &\le \|x_1-y\| = \left\|x_1-\left(\frac{I-C}{A}\right)^{-1}(B(x_1))\right\| \\ &= \operatorname{dist}(\Omega_1,\Omega_2).\end{aligned} \tag{12}$$

Similarly, $\|(x-A(x)\cdot B(x)-C(x))/A(x)\| = \operatorname{dist}(\Omega_1,\Omega_2)$, where $x=((I-C)/A)^{-1}(B(y_1))\in\Omega_1$.

Example 2. Let $X=\mathbb{R}$ endowed with the usual norm $|\cdot|$ and let $\Omega_1=[1/6,1/4]$, $\Omega_2=[3/8,1]$.

(i) Let B the function defined on $\Omega_1\cup\Omega_2$ by

$$Bx=\begin{cases}\dfrac{3}{8}\,\text{if}\,x\in\Omega_1\\ \dfrac{1}{4}\,\text{if}\,x\in\Omega_2.\end{cases} \tag{13}$$

Let $(x,y)\in\Omega_1\times\Omega_2$, we have $|x-y|=y-x\ge 3/8-1/4=1/8$.

$$\begin{aligned}|Bx-By| &= \left|\frac{3}{8}-\frac{1}{4}\right| = \frac{1}{8} = \operatorname{dist}(\Omega_1,\Omega_2) \\ &\le k\operatorname{dist}(\Omega_1,\Omega_2)+(1-k)\operatorname{dist}(\Omega_1,\Omega_2) \\ &\le k|x-y|+(1-k)\operatorname{dist}(\Omega_1,\Omega_2),\ \text{where}\,k\in[0,1[.\end{aligned} \tag{14}$$

Thus, B is cyclic contraction on $\Omega_1\cup\Omega_2$ and $M=|B(\Omega_1\cup\Omega_2)|=3/8$.

(ii) Let A the function defined on $\mathbb{R}$ by $Ax=1/3$, for all $x\in\mathbb{R}$. The function A is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function $\Phi_A=0$, and $\|A\|=1/3$

(iii) Let C the function defined on X by

$$Cx=\begin{cases}\dfrac{1}{6} & \text{if}\,x\in X\setminus\Omega_2\\ \dfrac{1}{4} & \text{if}\,x\in\Omega_2.\end{cases} \tag{15}$$

For each $(x,y)\in\Omega_1\times\Omega_2$,

$$|Cx-Cy|=\left|\frac{1}{6}-\frac{1}{4}\right|=\frac{1}{12}=\frac{2}{3}\cdot\frac{1}{8}\le\frac{2}{3}\cdot|x-y|. \tag{16}$$

The function C is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function defined by $\Phi_C(r)=2/3.r$, for all $r\in\mathbb{R}_+$. We have $I-M\Phi_A-\Phi_C=I-\Phi_C : r\mapsto 2/3r$ is nondecreasing, and for all $r>0$,

$$(M\Phi_A+\Phi_C)(r)=\frac{2}{3}r\le(1-\|A\|)r. \tag{17}$$

(iv) Let $x\in\mathbb{R}$ and $y\in\Omega_2$. Suppose $x=Ax\cdot By+Cx$ and $y=Ay\cdot Bx+Cy$. We have

$$\begin{aligned}x &= Ax\cdot By+Cx=\left(\frac{1}{3}\right)\cdot\left(\frac{1}{4}\right)+\frac{1}{6}=\frac{1}{4}\in\Omega_1,\\ y &= Ay\cdot Bx+Cy=\left(\frac{1}{3}\right)\cdot\left(\frac{3}{8}\right)+\frac{1}{4}=\frac{3}{8}\in\Omega_2.\end{aligned} \tag{18}$$

(v) For all $y\in\Omega_2$, $((I-C)/A)^{-1}(y)=(y/3)+1/4$, so for each $x\in\Omega_1$, $((I-C)/A)^{-1}(Bx)=((I-C)/A)^{-1}(3/8)=3/8$. Thus, for any sequence $(x_n)_{n\ge0}$ of $\Omega_1\cup\Omega_2$ such that $x_0\in\Omega_1$, $B(x_n)=((I-C)/A)(x_{n+1})$, the sequence $(x_{2n})_{n\ge0}$ has a convergent subsequence

Hence, by Theorem 1, there exists $(x,y)\in\Omega_1\times\Omega_2$ such that

$$\begin{aligned}&\left\|\frac{x-A(x).B(x)-C(x)}{A(x)}\right\| \\ &\quad=\operatorname{dist}(\Omega_1,\Omega_2)=\left\|\frac{y-A(y)\cdot B(y)-C(y)}{A(y)}\right\|.\end{aligned} \tag{19}$$

where $(x,y)=(1/4,3/8)$.

Theorem 11. *Let (Ω_1,Ω_2) be a nonempty closed pair of a Banach algebra X. Let $B:\Omega_1\cup\Omega_2\longrightarrow X$ and $C:X\longrightarrow X$ be two operators which satisfy the following conditions:*

(1) C is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function Φ_C, $\Phi_C(r)<r$, for all $r>0$

(2) B is cyclic contraction mapping on $\Omega_1\cup\Omega_2$

(3) $(I-C)^{-1}$ is relatively nonexpansive mapping on $\Omega_1\cup\Omega_2$

(4) suppose there exists a sequence $(x_n)_{n\ge0}$ of $\Omega_1\cup\Omega_2$ such that $x_0\in\Omega_1$, $B(x_n)=(I-C)(x_{n+1})$ and the sequence $(x_{2n})_{n\ge0}$ has a convergent subsequence in Ω_1

(5) $\begin{cases}y=B(x)+C(y),\ x\in\Omega_1\Rightarrow y\in\Omega_2\\ x=B(y)+C(x),\ y\in\Omega_2\Rightarrow x\in\Omega_1\end{cases}$

Then, there exists $(x,y)\in\Omega_1\times\Omega_2$ such that

$$\|x-B(x)-C(x)\|=dist(\Omega_1,\Omega_2)=\|y-B(y)-C(y)\|. \tag{20}$$

Proof. By (i), we show that $T := (I-C)^{-1} \cdot B : \Omega_1 \cup \Omega_2 \longrightarrow X$ is well defined. Moreover, the use of assumptions (ii), (iii), and (v) shows that T is cyclic contraction on $\Omega_1 \cup \Omega_2$. Indeed, by (v), it is cyclic. Let $(x, y) \in \Omega_1 \times \Omega_2$, the use of assumptions (ii) and (iii) leads to $(B(x), B(y)) \in \Omega_2 \times \Omega_1$,

$$\begin{aligned} \|T(x) - T(y)\| &= \left\|(I-C)^{-1}B(x) - (I-C)^{-1}B(y)\right\| \\ &\leq \|B(x) - B(y)\| \leq k\|x-y\| \\ &\quad + (1-k)\operatorname{dist}(\Omega_1, \Omega_2). \end{aligned} \tag{21}$$

By (iv), there exists a sequence $(x_n)_{n\geq 0}$ of $\Omega_1 \cup \Omega_2$ such that $x_0 \in \Omega_1$, $T(x_n) = x_{n+1}$ and the sequence $(x_{2n})_{n\geq 0}$ has a convergent subsequence in Ω_1.

Thus, by Theorem 6, there exists $(x_1, y_1) \in \Omega_1 \times \Omega_2$ such that

$$\|x_1 - Tx_1\| = \operatorname{dist}(\Omega_1, \Omega_2) = \|y_1 - Ty_1\|. \tag{22}$$

Let $y = (I-C)^{-1}(B(x_1)) \in \Omega_2$. We have $B(y) \in \Omega_1$ and $(I-C)(y) \in \Omega_2$. By (iii), B is cyclic contraction on $\Omega_1 \cup \Omega_2$, so

$$\begin{aligned} \operatorname{dist}(\Omega_1, \Omega_2) &\leq \|y - B(y) - C(y)\| = \|(I-C)(y) - B(y))\| \\ &\leq \|B(x_1) - B(y))\| \leq k\|x_1 - y\| \\ &\quad + (1-k)\operatorname{dist}(\Omega_1, \Omega_2) \leq \|x_1 - y\| \\ &= \left\|x_1 - (I-C)^{-1}(B(x_1))\right\| = \operatorname{dist}(\Omega_1, \Omega_2) \end{aligned} \tag{23}$$

Similarly, we get $\|x - A(x) \cdot B(x) - C(x)\| = \operatorname{dist}(\Omega_1, \Omega_2)$, where $x = (I-C)^{-1}(B(y_1)) \in \Omega_1$.

Remark 12. Under the same hypotheses of the previous theorem where C = 0, we obtain the classical result. That is, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\|x - B(x)\| = \operatorname{dist}(\Omega_1, \Omega_2) = \|y - B(y)\|. \tag{24}$$

Theorem 13. *Let (Ω_1, Ω_2) be a nonempty closed convex pair of a uniformly convex Banach algebra X. Let $A, C : X \longrightarrow X$ and $B : \Omega_1 \cup \Omega_2 \longrightarrow X$ be three operators which satisfy the following conditions:*

(1) A is regular on X and $\|A\| < 1$

(2) A and C are $\mathscr{D}$-Lipschitzs with the $\mathscr{D}$-functions Φ_A and Φ_C, respectively, $B(\Omega_1 \cup \Omega_2)$ is bounded with bound M, and $M\Phi_A(r) + \Phi_C(r) \leq (1-\|A\|)r$ for all $r > 0$ and $I - M\Phi_A - \Phi_C$ is nondecreasing

(3) B is cyclic contraction on $\Omega_1 \cup \Omega_2$

(4) $$\begin{cases} y = A(y) \cdot B(x) + C(y), x \in \Omega_1 \Rightarrow y \in \Omega_2 \\ x = A(x) \cdot B(y) + C(x), y \in \Omega_2 \Rightarrow x \in \Omega_1. \end{cases}$$

Then, there exists a unique $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\begin{aligned} &\left\|\frac{x - A(x).B(x) - C(x)}{A(x)}\right\| \\ &\quad = dist(\Omega_1, \Omega_2) = \left\|\frac{y - A(y) \cdot B(y) - C(y)}{A(y)}\right\|. \end{aligned} \tag{25}$$

Further, if $x_0 \in \Omega_1$ and $B(x_n) = ((I-C)/A)(x_{n+1})$, $n \in \mathbb{N}$, then the sequence $(x_{2n})_{n\geq 0}$ converges to the best proximity point.

Proof. By (i), (ii), (iii), and (v), we show that $T := ((I-C)/A)^{-1} \cdot B : \Omega_1 \cup \Omega_2 \longrightarrow X$ is well defined and cyclic contraction on $\Omega_1 \cup \Omega_2$.

Thus, by Theorem 7 and (iv), there exists a unique $(x, y) \in A \times B$ such that

$$\begin{aligned} &\left\|\frac{x - A(x) \cdot B(x) - C(x)}{A(x)}\right\| \\ &\quad = \operatorname{dist}(\Omega_1, \Omega_2) = \left\|\frac{y - A(y) \cdot B(y) - C(y)}{A(y)}\right\|, \end{aligned} \tag{26}$$

and if $x_0 \in \Omega_1$ with $B(x_n) = ((I-C)/A)(x_{n+1})$, i.e., $T(x_n) = x_{n+1}$ for all $n \in \mathbb{N}$, then the sequence $(x_{2n})_{n\geq 0}$ converges to the best proximity point.

4. Application

Let $E = \mathscr{C}(J, \mathbb{R})$ the Banach algebra of all continuous functions from $J = [0, 1]$ to $\mathbb{R}$, endowed with the sup-norm $\|\cdot\|_\infty$, defined by

$$\|x\|_\infty = \sup\{|x(t)|, t \in [0, 1]\}, \tag{27}$$

for each $x \in C(J, \mathbb{R})$. Let $(q_1, q_2) \in \mathbb{R}^2$ and suppose $q_1 < 0 < q_2$.

We consider the closed and nonempty sets

$$\begin{aligned} \Omega_1 &= \{x \in E : x(t) \geq q_2, \forall t \in J\} \\ \Omega_2 &= \{y \in E : y(t) \leq q_1, \forall t \in J\}. \end{aligned} \tag{28}$$

For any $(x, y) \in \Omega_1 \times \Omega_2$ and for all $t \in J$, we have

$$\|x - y\|_\infty \geq |x(t) - y(t)| = x(t) - y(t) \geq |q_1 - q_2|, \tag{29}$$

so $\operatorname{dist}(\Omega_1, \Omega_2) = |q_1 - q_2|$. We consider the following two nonlinear functional integral equations

$$x(t) = K(t, x(t)) + (Tx)(t) \cdot \left(q_2 + \int_0^t g(s, y(s))ds\right) \quad \text{(FIS1)}$$

$$y(t) = K(t, y(t)) + (Ty)(t) \cdot \left(q_1 + \int_0^t f(s, x(s))ds\right), \quad \text{(FIS2)}$$

where $(x, y) \in \Omega_1 \times \Omega_2$ and $t \in [0, 1]$.

The integral equations (FIS1)–(FIS2) may be written, respectively, as:

$$\begin{aligned} x(t) &= Ax(t).By(t) + Cx(t) \\ y(t) &= Ay(t).Bx(t) + Cy(t), \end{aligned} \quad (30)$$

where $(x, y) \in \Omega_1 \times \Omega_2$ and $t \in J$. To simplify the notations, we put

$$\begin{aligned} Ax(t) &= Tx(t), \\ Bx(t) &= \begin{cases} q_1 + \int_0^t f(s, x(s))\, ds \text{if} x \in \Omega_1 \\ q_2 + \int_0^t g(s, x(s))\, ds \text{if} x \in \Omega_2, \end{cases} \\ Cx(t) &= K(t, x(t)). \end{aligned} \quad (31)$$

The goal of this section is to apply our main result to investigate the existence of an optimum solution (x, y) of the (FIS1)–(FIS2) problem in the sense that the pair (x, y) satisfies:

$$\left\| \frac{x - A(x) \cdot B(x) - C(x)}{A(x)} \right\| = \left\| \frac{y - A(y) \cdot B(y) - C(y)}{A(y)} \right\| = \operatorname{dist}(\Omega_1, \Omega_2). \quad (32)$$

Note that, if x is a solution of (FIS1) and y is a solution of (FIS2), then the pair (x, y) need not form an optimum solution see [21], pp 27-31 for more details.

We consider the following assumptions:

(i)

(a) The function $K(\cdot, x(\cdot))$: $J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, for all $x \in E$

(b) There is a continuous function $\delta : J \longrightarrow [0, +\infty)$ with bound $\Delta = \sup_{t \in J} |\delta(t)|$ such that $|K(t, x(t)) - K(t, y(t))| \leq \delta(t)|x(t) - y(t)|$, for all $x, y \in E$ and $t \in [0, 1]$

(ii)

(a) The functions $f, g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are measurables and

$$N = \max \left\{ \sup_{x \in E} \|f(\cdot, x(\cdot))\|_\infty, \sup_{x \in E} \|g(\cdot, x(\cdot))\|_\infty \right\} < \infty \quad (33)$$

(b) Let $(x, y) \in \Omega_1 \times \Omega_2$. For all $s \in J$,

$$|f(s, x(s)) - g(s, y(s))| \leq \alpha(|x(s) - y(s)| - |q_1 - q_2|) \quad (34)$$

(c) For all $s \in J$, $f(s, y(s)) \leq 0 \leq g(s, x(s))$

(iii)

(a) $T : E \longrightarrow E$ is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function Φ_T, such that Φ_T is differentiable on $\mathbb{R}_+$ and $|\Phi_A{}'| < 1 - \Delta/Q + N$, $N \neq 0$, where $Q = \max\{|q_1|, |q_2|\}$

(b) T is regular on $\mathscr{C}(J, \mathbb{R})$, and $\|T\| = \sup_{x \in E} \|Tx\|_\infty < 1$

(iv) The family $\{B(x): x \in \Omega_1 \cup \Omega_2\}$ is equicontinuous and closed

Theorem 14. *Assume the hypotheses $(\mathscr{H}_1) - (\mathscr{H}_4)$ hold. $\Phi_T(r) \leq (1 - \|T\| - \Delta/Q + N)r$ for all $r > 0$ and $T(x) > 0$ for all $x \in E$. Moreover,*

$$\begin{cases} K(\cdot, x(\cdot))q_2 \text{ if } x \in \Omega_1 \\ K(\cdot, x(\cdot))q_1 \text{ if } x \in \Omega_2. \end{cases} \quad (35)$$

Then, there exists an optimum solution $(x, y) \in \Omega_1 \times \Omega_2$ for (FIS1)–(FIS2) problem.

Proof. (1)

(i) By $(\mathscr{H}_1)$, we have $|K(\cdot, x(\cdot)) - K(\cdot, y(\cdot))|_\infty \leq \Delta \|x - y\|_\infty$ for all $x, y \in E$, so $\|Cx - Cy\|_\infty \leq \Phi_C(\|x - y\|_\infty)$ for all $x, y \in E$, where $\Phi_C(r) = \Delta r$, for all $r \geq 0$

(ii) -Let $x \in \Omega_1$, We have, for all $t \in J$

$$|(Bx)(t)| = \left| q_1 + \int_0^t f(s, x(s))ds \right| \leq Q + N. \quad (36)$$

Similarly, for all $y \in \Omega_2$ and $t \in J$ we get $|(By)(t)| \leq Q + N$. Hence, $M = \|B(\Omega_1 \cup \Omega_2)\| \leq Q + N < \infty$.

(iii) -Furthermore, by hypothesis we have $\Phi_A(r) \le (1-\|A\|-\Delta/Q+N)r$, for all $r>0$. So

$$M\Phi_A(r)+\Phi_C(r)=M\Phi_A(r)+\Delta r \le (Q+N)\Phi_A(r)+\Delta r \le (1-\|A\|)r, \text{ for all } r>0. \tag{37}$$

(iv) We show that $I-M\Phi_A-\Phi_C$ is nondecreasing. Let $r, r' \in \mathbb{R}_+$ such that $r<r'$.

Since, ϕ_A is nondecreasing and differentiable on $\mathbb{R}_+$, with $|\Phi_A{}'| < 1-\Delta/Q+N$, so

$$0 \le \frac{\Phi_A\left(r'\right)-\Phi_A(r)}{r'-r} < \frac{1-\Delta}{Q+N} \le \frac{1-\Delta}{M}, \tag{38}$$

thus,

$$(I-M\Phi_A-\Phi_C)(r) < (I-M\Phi_A-\Phi_C)\left(r'\right). \tag{39}$$

That is, $I-M\Phi_A-\Phi_C$ is nondecreasing.

(2)

(i) Let $t, t' \in J$ such that $t<t'$,

$$\left|B(x)(t)-B(x)\left(t'\right)\right| = \left|\int_t^{t'} f(s,x(s))\, s\right| \le \int_t^{t'} |f(s,x(s))|ds \le N\left|t-t'\right| \tag{40}$$

Thus, $B(x)$ is Lipschitzian, so $B(x)\in E$. Let $x\in\Omega_1$ and $t\in J$. By $(\mathscr{H}_2)$, (iii), We have

$$Bx(t)=q_1+\int_0^t f(s,x(s))ds \le q_1. \tag{41}$$

Hence, $B(x)\in\Omega_2$. Similarly, we get $B(\Omega_2)\subset\Omega_1$.

(ii) -Let $(x,y)\in\Omega_2\times\Omega_1$ and $t\in J$

$$|Bx(t)-By(t)| = \left|(q_1-q_2)+\int_0^t (f(s,x(s))-g(s,y(s)))ds\right|. \tag{42}$$

By $(\mathscr{H}_2)$, (iii), we have $f(s,x(s))-g(s,y(s))\le 0$, for all $s\in J$, so

$$q_1-q_2+\int_{J_2\cap[0,t]} (f(s,x(s))-g(s,y(s)))ds \le 0. \tag{43}$$

Then,

$$\begin{aligned}|Bx(t)-By(t)| &= q_2-q_1+\int_0^t (g(s,x(s))-f(s,y(s)))ds\\ &\le |q_1-q_2|+\int_0^t \alpha(|x(s)-y(s)|-|q_1-q_2|)ds\\ &\le |q_1-q_2|+\int_0^t \alpha(\|x-y\|_\infty-|q_1-q_2|)ds\\ &\le (1-\alpha)|q_1-q_2|+\alpha\|x-y\|_\infty.\end{aligned} \tag{44}$$

Thus, $\|Bx-By\|_\infty \le \alpha\|x-y\|_\infty+(1-\alpha)\mathrm{dist}(\Omega_1,\Omega_2)$ which shows that B is cyclic contraction.

(3) Let $y\in\Omega_2$ and $x\in E$ such that $x=A(x)\cdot B(y)+C(x)$. We show that $x\in\Omega_1$. We have, for all $t\in J$, $B(y)(t)\ge 0$, $A(x)(t)\ge 0$ and $C(x)(t)\ge q_2$, so

$$x(t)=(Ax)(t).(By)(t)+(Cx)(t)\ge C(x)(t)\ge q_2. \tag{45}$$

Let $x\in\Omega_1$ and $y\in E$ such that $y=A(y)\cdot B(x)+C(y)$. We show that $y\in\Omega_2$. We have, for all $t\in J$, $B(y)(t)\le 0$, $A(x)(t)\ge 0$ and $C(x)(t)\le q_1$, so

$$y(t)=(Ay)(t).(Bx)(t)+(Cy)(t)\le C(x)(t)\le q_1. \tag{46}$$

(4) As $\|B(\Omega_1\cup\Omega_2)\|<\infty$, so the family $\{B(x): x\in\Omega_1\cup\Omega_2\}$ is uniformly bounded; by $(\mathscr{H}_4)$, this family is equicontinuous. Therefore, by Arzela-Ascoli's theorem, $\{B(x): x\in\Omega_1\cup\Omega_2\}$ lies in a compact subset of $\Omega_1\cup\Omega_2$. Let $(x_n)_{n\ge 0}$ be a sequence of $\Omega_1\cup\Omega_2$ such that $(I-C/A)^{-1}B(x_n)=x_{n+1}$, i.e., $B(x_n)=(I-C/A)(x_{n+1})$. We have $(B(x_{2n-1}))_{n\ge1}$ has a convergent subsequence $(B(x_{2\sigma(n)-1}))_{n\ge1}$. Let $u=\lim_{n\to+\infty}B(x_{2\sigma(n)-1})$. As $B(\Omega_1\cup\Omega_2)$ is closed, there exists $z\in\Omega_1\cup\Omega_2$ such that $B(z)=u$. We obtain, for each $n\in\mathbb{N}^*$

$$\begin{aligned}&\left\|T\left(x_{2\sigma(n)-1}\right)-T(z)\right\|_\infty\\ &\le \left\|A\left(T\left(x_{2\sigma(n)-1}\right)\right)\cdot B\left(x_{2\sigma(n)-1}\right)-A(T(z))\cdot B(z)\right\|_\infty\\ &\quad+\left\|C\left(T\left(x_{2\sigma(n)-1}\right)\right)-C(T(z))\right\|_\infty\\ &\le \left\|A\left(T\left(x_{2\sigma(n)-1}\right)\right)-A(T(z))\right\|_\infty\|B\|\\ &\quad+\left\|B\left(x_{2\sigma(n)-1}\right)-B(z)\right\|_\infty\|A\|\\ &\quad+\left\|C\left(T\left(x_{2\sigma(n)-1}\right)\right)-C(T(z))\right\|_\infty\\ &\le \|A\|\left\|B\left(x_{2\sigma(n)-1}\right)-B(z)\right\|_\infty\\ &\quad+M\Phi_A\left(\left\|T\left(x_{2\sigma(n)-1}\right)-T(z)\right\|_\infty\right)\\ &\quad+\Phi_C\left(\left\|T\left(x_{2\sigma(n)-1}\right)-T(z)\right\|_\infty\right).\end{aligned} \tag{47}$$

Thus,

$$\begin{aligned}\left\|T\left(x_{2\sigma(n)-1}\right)-T(z)\right\|_{\infty} &\\ \leq (I-M\Phi_A-\Phi_C)^{-1}\left(\|A\|\left\|B\left(x_{2\sigma(n)-1}\right)-B(z)\right\|_{\infty}\right) &\\ \leq \left\|B\left(x_{2\sigma(n)-1}\right)-B(z)\right\|_{\infty}. &\end{aligned} \tag{48}$$

Hence,

$$\left\|x_{2\sigma(n)}-T(z)\right\|_{\infty} \leq \left\|B\left(x_{2\sigma(n)-1}\right)-B(z)\right\|_{\infty}. \tag{49}$$

Which prove that the sequence $(x_{2\sigma(n)})_{n\geq 0}$ is convergent.

Thus, by Theorem 1, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\begin{aligned}\left\|\frac{x-A(x)\cdot B(x)-C(x)}{A(x)}\right\| &\\ = \operatorname{dist}(\Omega_1,\Omega_2) = \left\|\frac{y-A(y)\cdot B(y)-C(y)}{A(y)}\right\|. &\end{aligned} \tag{50}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

On Some Metric Inequalities and Applications

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We derive a new inequality in metric spaces and provide its geometric interpretation. Some applications of our result are given, including metric inequalities in Lebesgue spaces, matrices inequalities, multiplicative metric inequalities, and partial metric inequalities. Our main result is a generalization of that obtained by Dragomir and Gosa.

1. Introduction

In [1], Dragomir and Gosa established the following interesting inequality in metric spaces.

Theorem 1. *(Dragomir-Gosa inequality).*
Let $\mathscr{V}$ be a nonempty set equipped with a metric δ. Let $N \geq 2$ be a natural number, $\{\xi_i\}_{i=1}^N \subset [0,\infty)$, $\sum_{i=1}^N \xi_i = 1$, and $\{\vartheta_i\}_{i=1}^N \subset \mathscr{V}$. Then,

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \xi_i \xi_j \delta(\vartheta_i, \vartheta_j) \leq \inf_{\vartheta \in \mathscr{V}} \sum_{i=1}^{N} \xi_i \delta(\vartheta_i, \vartheta). \tag{1}$$

Moreover, the inequality is optimal, in the sense that the multiplicative coefficient $C = 1$ on the right-hand side of (1) (in front of inf*) cannot be replaced by a smaller real number.*

In the special case when $\xi_i = 1/N$, for all $i \in \{1, 2, \cdots, N\}$, (1) reduces to

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \delta(\vartheta_i, \vartheta_j) \leq N \inf_{\vartheta \in \mathscr{V}} \sum_{i=1}^{N} \delta(\vartheta_i, \vartheta). \tag{2}$$

Inequality (2) can be considered a polygonal-type inequality. Namely, it has the following geometric interpretation: let Ω be a polygon in a metric space with N vertices and ϑ be an arbitrary point in the space. Then, the sum of all edges and diagonals of Ω is less than N-times the sum of the distances from ϑ to the vertices of Ω.

In the same paper [1], the authors presented some interesting applications of inequality (1) to normed linear spaces and pre-Hilbert spaces.

In this paper, motivated by the above-mentioned work, a generalization of inequality (1) is obtained and its geometric interpretation is provided. Moreover, some applications of our result are given, including metric inequalities in Lebesgue spaces, matrices inequalities, multiplicative metric inequalities, and partial metric inequalities.

2. Main Result and Some Consequences

We first recall briefly the notion of metric spaces (see, e.g., [2]). Let $\mathscr{V}$ be a nonempty set and $\delta : \mathscr{V} \times \mathscr{V} \longrightarrow [0,\infty)$ be a given function. We say that δ is a metric on $\mathscr{V}$ if the following conditions hold: for all $\vartheta, \rho, \kappa \in \mathscr{V}$,

(i) $\delta(\vartheta, \rho) = 0$, if and only if $\vartheta = \rho$

(ii) $\delta(\vartheta, \rho) = \delta(\rho, \vartheta)$

(iii) $\delta(\vartheta,\rho) \le \delta(\vartheta,\kappa) + \delta(\kappa,\rho)$

In this case, we say that $(\mathscr{V},\delta)$ is a metric space.

Let $\mathbb{N}$ be the set of positive natural numbers. Our main result is the following:

Theorem 2. *Let $\mathscr{V}$ be a nonempty set equipped with a metric δ. Let $m, N \in \mathbb{N}$, $N \ge 2$, $\{\xi_i\}_{i=1}^N \subset [0,\infty)$, $\sum_{i=1}^N \xi_i = 1$, and $\{\vartheta_i\}_{i=1}^N \subset \mathscr{V}$. Then,*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \xi_i\xi_j\delta(\vartheta_i,\vartheta_j)^m \le \frac{1}{2}\inf_{\vartheta\in\mathscr{V}}\left[2\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^m + \sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^k\right)\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta,\vartheta_i)^{m-k}\right)\right]. \tag{3}$$

Moreover, the inequality is optimal, in the sense that the multiplicative coefficient 1/2 on the right-hand side of (3) cannot be replaced by a smaller real number.

Proof. By the triangle inequality, for all $\vartheta \in \mathscr{V}$, one has

$$\delta(\vartheta_i,\vartheta_j) \le \delta(\vartheta_i,\vartheta) + \delta(\vartheta,\vartheta_j), \quad i, j \in \{1,2,\cdots,N\}, \tag{4}$$

which yields

$$\delta(\vartheta_i,\vartheta_j)^m \le \left(\delta(\vartheta_i,\vartheta) + \delta(\vartheta,\vartheta_j)\right)^m. \tag{5}$$

On the other hand, by the binomial theorem, one has

$$\left(\delta(\vartheta_i,\vartheta) + \delta(\vartheta,\vartheta_j)\right)^m = \sum_{k=0}^{m}\binom{m}{k}\delta(\vartheta_i,\vartheta)^k\delta(\vartheta,\vartheta_j)^{m-k}, \tag{6}$$

where

$$\binom{m}{k} = \frac{m!}{(m-k)!k!}. \tag{7}$$

Hence, combining (5) with (6), one obtains

$$\delta(\vartheta_i,\vartheta_j)^m \le \sum_{k=0}^{m}\binom{m}{k}\delta(\vartheta_i,\vartheta)^k\delta(\vartheta,\vartheta_j)^{m-k}. \tag{8}$$

Multiplying the above inequality by $\xi_i\xi_j$ and taking the sum over i and j, it holds that

$$\sum_{i,j=1}^{N}\xi_i\xi_j\delta(\vartheta_i,\vartheta_j)^m \le \sum_{i,j=1}^{N}\xi_i\xi_j\sum_{k=0}^{m}\binom{m}{k}\delta(\vartheta_i,\vartheta)^k\delta(\vartheta,\vartheta_j)^{m-k}. \tag{9}$$

Notice that due to the symmetry of δ and the fact that $\delta(u,u)=0$, $u\in\mathscr{V}$, one has

$$\sum_{i,j=1}^{N}\xi_i\xi_j\delta(\vartheta_i,\vartheta_j)^m = 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\xi_i\xi_j\delta(\vartheta_i,\vartheta_j)^m. \tag{10}$$

Furthermore, using that $\sum_{i=1}^N \xi_i = 1$, one obtains

$$\begin{aligned}\sum_{i,j=1}^{N}\xi_i\xi_j\sum_{k=0}^{m}\binom{m}{k}\delta(\vartheta_i,\vartheta)^k\delta(\vartheta,\vartheta_j)^{m-k} &= \sum_{k=0}^{m}\binom{m}{k}\sum_{i,j=1}^{N}\xi_i\xi_j\delta(\vartheta_i,\vartheta)^k\delta(\vartheta,\vartheta_j)^{m-k}\\ &= \sum_{k=0}^{m}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^k\right)\left(\sum_{j=1}^{N}\xi_j\delta(\vartheta,\vartheta_j)^{m-k}\right)\\ &= \sum_{k=0}^{m}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^k\right)\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta,\vartheta_i)^{m-k}\right)\\ &= 2\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^m + \sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^k\right)\\ &\quad\times\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta,\vartheta_i)^{m-k}\right),\end{aligned} \tag{11}$$

i.e.,

$$\sum_{i,j=1}^{N}\xi_i\xi_j\sum_{k=0}^{m}\binom{m}{k}\delta(\vartheta_i,\vartheta)^k\delta(\vartheta,\vartheta_j)^{m-k} = 2\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^m + \sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^k\right)\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta,\vartheta_i)^{m-k}\right). \tag{12}$$

Hence, it follows from (9), (10), and (12) that

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\xi_i\xi_j\delta(\vartheta_i,\vartheta_j)^m \le \frac{1}{2}\left[2\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^m + \sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^k\right)\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta,\vartheta_i)^{m-k}\right)\right]. \tag{13}$$

Notice that the above inequality holds for all $\vartheta\in\mathscr{V}$. So, taking the infimum over $\vartheta\in\mathscr{V}$, (3) follows.

Suppose now that there exists a certain constant $M > 0$ such that

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \xi_i\xi_j\delta(\vartheta_i,\vartheta_j)^m \le M\left[2\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^m + \sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta_i,\vartheta)^k\right)\left(\sum_{i=1}^{N}\xi_i\delta(\vartheta,\vartheta_i)^{m-k}\right)\right]. \quad (14)$$

Taking $N = 2$, $\vartheta_1 \neq \vartheta_2$, $\xi_1 = \lambda$, and $\xi_2 = 1-\lambda$, where $\lambda \in (0,1)$, one obtains

$$\begin{aligned}\lambda(1-\lambda)\delta(\vartheta_1,\vartheta_2)^m \le{}& 2M[\lambda\delta(\vartheta_1,\vartheta)^m + (1-\lambda)\delta(\vartheta_2,\vartheta)^m] \\ &+ M\sum_{k=1}^{m-1}\binom{m}{k}\left[\lambda\delta(\vartheta_1,\vartheta)^k + (1-\lambda)\delta(\vartheta_2,\vartheta)^k\right]\left[\lambda\delta(\vartheta_1,\vartheta)^{m-k} + (1-\lambda)\delta(\vartheta_2,\vartheta)^{m-k}\right],\end{aligned} \quad (15)$$

for all $\vartheta \in \mathscr{V}$. In particular, for $\vartheta = \vartheta_1$, one deduces that

$$\lambda(1-\lambda)\delta(\vartheta_1,\vartheta_2)^m \le 2M(1-\lambda)\delta(\vartheta_1,\vartheta_2)^m + M(1-\lambda)^2\delta(\vartheta_1,\vartheta_2)^m\sum_{k=1}^{m-1}\binom{m}{k}, \quad (16)$$

which yields (since $\vartheta_1 \neq \vartheta_2$)

$$\lambda \le 2M + M(1-\lambda)\sum_{k=1}^{m-1}\binom{m}{k}. \quad (17)$$

Taking the limit as $\lambda \longrightarrow 1^-$ in the above inequality, it holds that $M \ge 1/2$. The proof is then complete.

Remark 3. Taking $m = 1$ in Theorem 2, (3) reduces to (1).

Corollary 4. *Let $\mathscr{V}$ be a nonempty set equipped with a metric δ. Let $m, N \in \mathbb{N}$, $N \ge 2$, and $\{\vartheta_i\}_{i=1}^N \subset \mathscr{V}$. Then,*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\delta(\vartheta_i,\vartheta_j)^m \le \inf_{\vartheta\in\mathscr{V}}\left[N\sum_{i=1}^{N}\delta(\vartheta_i,\vartheta)^m + \frac{1}{2}\sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\delta(\vartheta_i,\vartheta)^k\right)\times\left(\sum_{i=1}^{N}\delta(\vartheta,\vartheta_i)^{m-k}\right)\right]. \quad (18)$$

Proof. Using (3) with

$$\xi_i = \frac{1}{N}, \quad i \in \{1,2,\cdots,N\}, \quad (19)$$

(18) follows.

Remark 5. Taking $m = 1$ in Corollary 4, (18) reduces to (2).

In the special case $m = 2$, one deduces form Corollary 4 the following result.

Corollary 6. *Let $\mathscr{V}$ be a nonempty set equipped with a metric δ. Let $N \in \mathbb{N}$, $N \ge 2$, and $\{\vartheta_i\}_{i=1}^N \subset \mathscr{V}$. Then,*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\delta(\vartheta_i,\vartheta_j)^2 \le \inf_{\vartheta\in\mathscr{V}}\left[N\sum_{i=1}^{N}\delta(\vartheta_i,\vartheta)^2 + \left(\sum_{i=1}^{N}\delta(\vartheta_i,\vartheta)\right)^2\right]. \quad (20)$$

Inequality (20) has the following geometric interpretation.

Corollary 7. *Let Ω be a polygon in a metric space with N vertices, and ϑ be an arbitrary point in the space. Then, the sum of the squares of all edges and diagonals of Ω is less than N-times the sum of the squares of the distances from ϑ to the vertices of Ω plus the square of the sum of the distances from ϑ to the vertices of Ω.*

Given $\rho > 0$ and $\vartheta \in \mathscr{V}$, where $(\mathscr{V},\delta)$ is a metric space, we denote by $\bar{B_\vartheta}(\rho)$ the closed ball in $\mathscr{V}$ with center ϑ and radius ρ, namely,

$$\bar{B_\vartheta}(\rho) = \{u \in \mathscr{V} : \delta(\vartheta,u) \le \rho\}. \quad (21)$$

Corollary 8. *Let $\mathscr{V}$ be a nonempty set equipped with a metric δ. Let $m, N \in \mathbb{N}$, $N \ge 2$, $\{\xi_i\}_{i=1}^N \subset [0,\infty)$, $\sum_{i=1}^N \xi_i = 1$, and $\{\vartheta_i\}_{i=1}^N \subset \mathscr{V}$. Suppose that there exist $\rho > 0$ and $\vartheta \in \mathscr{V}$ such that $\{\vartheta_i\}_{i=1}^N \subset \bar{B_\vartheta}(\rho)$. Then,*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\xi_i\xi_j\delta(\vartheta_i,\vartheta_j)^m \le 2^{m-1}\rho^m. \quad (22)$$

Proof. Since $\{\vartheta_i\}_{i=1}^N \subset \bar{B_\vartheta}(\rho)$, one has

$$\delta(\vartheta_i,\vartheta) \le \rho, \quad i \in \{1,2,\cdots,N\}. \quad (23)$$

Hence, using (3), (23), and the fact that $\sum_{i=1}^N \xi_i = 1$, one deduces that

$$\begin{aligned}
&\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \xi_i \xi_j \delta(\vartheta_i, \vartheta_j)^m \\
&\le \frac{1}{2}\left[2\sum_{i=1}^{N} \xi_i \delta(\vartheta_i, \vartheta)^m + \sum_{k=1}^{m-1} \binom{m}{k} \left(\sum_{i=1}^{N} \xi_i \delta(\vartheta_i, \vartheta)^k\right)\right. \\
&\quad \left. \times \left(\sum_{i=1}^{N} \xi_i \delta(\vartheta, \vartheta_i)^{m-k}\right)\right] \le \frac{1}{2}\left[2\rho^m + \rho^m \sum_{k=1}^{m-1} \binom{m}{k}\right] \\
&= \frac{\rho^m}{2}\left[2 + \sum_{k=0}^{m} \binom{m}{k} - 2\right] = 2^{m-1}\rho^m.
\end{aligned} \tag{24}$$

The proof is complete.

Remark 9. Taking $m = 1$ in Corollary 8, one obtains Corollary 2 in [1].

3. Applications

3.1. A Metric Inequality in Lebesgue Spaces. Consider a measure space $(\chi, \mathscr{M}, \mu)$ and a real number $r \in [1,\infty)$. We denote by $L^r(\chi, \mathscr{M}, \mu)$ the space of measurable functions f such that

$$\int_\chi |f|^r \, d\mu < \infty. \tag{25}$$

Proposition 10. *Let $m, N \in \mathbb{N}$, $N \ge 2$, $\{\xi_i\}_{i=1}^N \subset [0,\infty)$, $\sum_{i=1}^N \xi_i = 1$, and $\{f_i\}_{i=1}^N \subset L^r(\chi, \mathscr{M}, \mu)$. Then,*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \xi_i \xi_j \left(\int_\chi \left|f_i - f_j\right|^r d\mu\right)^{m/r} \le \frac{1}{2} \inf_{f \in L^r(\chi,\mathscr{M},\mu)} \Lambda(f), \tag{26}$$

where

$$\begin{aligned}
\Lambda(f) &= 2\sum_{i=1}^{N} \xi_i \left(\int_\chi |f_i - f|^r d\mu\right)^{m/r} \\
&\quad + \sum_{k=1}^{m-1} \binom{m}{k} \left(\sum_{i=1}^{N} \xi_i \left(\int_\chi |f_i - f|^r d\mu\right)^{k/r}\right) \\
&\quad \cdot \left(\sum_{i=1}^{N} \xi_i \left(\int_\chi |f_i - f|^r d\mu\right)^{(m-k)/r}\right),
\end{aligned} \tag{27}$$

for all $f \in L^r(\chi, \mathscr{M}, \mu)$. Moreover, this inequality is optimal, in the sense that the multiplicative coefficient 1/2 on the right-hand side of (26) cannot be replaced by a smaller real number.

Proof. Consider the distance function

$$\delta(f, g) = \left(\int_\chi |f - g|^r d\mu\right)^{1/r}, \quad f, g \in L^r(\chi, \mathscr{M}, \mu). \tag{28}$$

Then, $(L^r(\chi, \mathscr{M}, \mu), \delta)$ is a metric space (see, e.g., [3]). Hence, using (3) with δ as defined above, (26) follows. The optimality of (26) follows from Theorem 2.

3.2. A Matrix Inequality. We denote by $\mathscr{M}_n(\mathbb{R})$ the set of square matrices of size $n \in \mathbb{N}$, $n \ge 2$, with real number coefficients. Let $M \in \mathscr{M}_n(\mathbb{R})$. We denote by $\rho(M)$ the spectral radius of M, namely,

$$\rho(M) = \max\{|\lambda_i| : i = 1, 2, \cdots, n\}, \tag{29}$$

where $\lambda_i, i \in \{1, 2, \cdots, n\}$, are the eigenvalues of M. We denote by $\sigma_{\max}(M)$ the largest singular value of M, namely,

$$\sigma_{\max}(M) = \sqrt{\rho(MM^t)}, \tag{30}$$

where M^t is the transpose of M. For more details on matrix analysis, see, for example, [4, 5].

Proposition 11. *Let $m, N \in \mathbb{N}$, $N \ge 2$, $\{\xi_i\}_{i=1}^N \subset [0,\infty)$, $\sum_{i=1}^N \xi_i = 1$, and $\{M_i\}_{i=1}^N \subset \mathscr{M}_n(\mathbb{R})$. Then,*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \xi_i \xi_j \left[\sigma_{\max}(M_i - M_j)\right]^m \le \frac{1}{2} \inf_{M \in \mathscr{M}_n(\mathbb{R})} \mu(M), \tag{31}$$

where

$$\begin{aligned}
\mu(M) &= 2\sum_{i=1}^{N} \xi_i [\sigma_{\max}(M_i - M)]^m \\
&\quad + \sum_{k=1}^{m-1} \binom{m}{k} \left(\sum_{i=1}^{N} \xi_i [\sigma_{\max}(M_i - M)]^k\right) \\
&\quad \cdot \left(\sum_{i=1}^{N} \xi_i [\sigma_{\max}(M - M_i)]^{m-k}\right).
\end{aligned} \tag{32}$$

Moreover, the inequality is optimal, in the sense that the multiplicative coefficient 1/2 on the right-hand side of (31) cannot be replaced by a smaller real number.

Proof. Consider the function $\delta : \mathscr{M}_n(\mathbb{R}) \times \mathscr{M}_n(\mathbb{R}) \longrightarrow [0,\infty)$ defined by

$$\delta(A, B) = \sigma_{\max}(A - B), \quad A, B \in \mathscr{M}_n(\mathbb{R}). \tag{33}$$

Then, δ is a metric on $\mathscr{M}_n(\mathbb{R})$ (see, e.g., [5]). Hence, using (3) with δ as defined above, (31) follows. The optimality of (31) follows from Theorem 2.

3.3. A Multiplicative Metric Inequality. We first recall the notion of multiplicative metric spaces (see [6]). A multiplicative metric on a nonempty set $\mathscr{V}$ is a function $\sigma : \mathscr{V} \times \mathscr{V} \longrightarrow [1, \infty)$ satisfying the following properties: for all $\vartheta, \rho, \kappa \in \mathscr{V}$,

(i) $\sigma(\vartheta, \rho) = 1$, if and only if $\vartheta = \rho$

(ii) $\sigma(\vartheta, \rho) = \sigma(\rho, \vartheta)$

(iii) $\sigma(\vartheta, \rho) \le \sigma(\vartheta, \kappa)\sigma(\kappa, \rho)$

In this case, we say that $(\mathcal{V}, \sigma)$ is a multiplicative metric space.

Example 12. Let $\sigma : (0,\infty) \times (0,\infty) \longrightarrow [1,\infty)$ be the function defined by

$$\sigma(\vartheta, \rho) = \begin{cases} \vartheta\rho^{-1}, & \text{if } \vartheta \ge \rho, \\ \rho\vartheta^{-1}, & \text{if } \vartheta < \rho. \end{cases} \tag{34}$$

Then σ is a multiplicative metric on $(0, \infty)$.

Proposition 13. *Let $\mathcal{V}$ be a nonempty set equipped with a multiplicative metric σ. Let $m, N \in \mathbb{N}$, $N \ge 2$, $\{\xi_i\}_{i=1}^N \subset [0,\infty)$, $\sum_{i=1}^N \xi_i = 1$, and $\{\vartheta_i\}_{i=1}^N \subset \mathcal{V}$. Then,*

$$\begin{aligned}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \xi_i\xi_j\left[\ln \sigma(\vartheta_i, \vartheta_j)\right]^m \le \frac{1}{2}\inf_{\vartheta\in\mathcal{V}}\Bigg[2\sum_{i=1}^{N} \xi_i[\ln \sigma(\vartheta_i, \vartheta)]^m \\ + \sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i[\ln \sigma(\vartheta_i, \vartheta)]^k\right)\left(\sum_{i=1}^{N}\xi_i[\ln \sigma(\vartheta, \vartheta_i)]^{m-k}\right)\Bigg].\end{aligned} \tag{35}$$

Moreover, the inequality is optimal, in the sense that the multiplicative coefficient 1/2 on the right-hand side of (35) cannot be replaced by a smaller real number.

Proof. Consider the function $\delta : \mathcal{V} \times \mathcal{V} \longrightarrow [0,\infty)$ defined by

$$\delta(\vartheta, \rho) = \ln \sigma(\vartheta, \rho), \quad \vartheta, \rho \in \mathcal{V}. \tag{36}$$

It can be easily seen that δ is a metric on $\mathcal{V}$. Then, using (3) with δ as defined above, (35) follows. The optimality of (35) follows from Theorem 2.

3.4. A Partial Metric Inequality. We first recall briefly some notions on partial metric spaces (see, e.g., [7–10]).

A partial metric on a nonempty set $\mathcal{V}$ is a function $\eta : \mathcal{V} \times \mathcal{V} \longrightarrow [0,\infty)$ satisfying the following properties: for all $\vartheta, \rho, \kappa \in \mathcal{V}$,

$$\begin{aligned} &\eta(\vartheta, \vartheta) = \eta(\rho, \rho) = \eta(\vartheta, \rho) \Leftrightarrow \vartheta = \rho, \\ &\eta(\vartheta, \vartheta) \le \eta(\vartheta, \rho), \\ &\eta(\vartheta, \rho) = \eta(\rho, \vartheta), \\ &\eta(\vartheta, \rho) \le \eta(\vartheta, \kappa) + \eta(\kappa, \rho) - \eta(\kappa, \kappa). \end{aligned} \tag{37}$$

In this case, we say that $(\mathcal{V}, \eta)$ is a partial metric space.

Example 14. Let $\eta : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be the function defined by

$$\eta(\vartheta, \rho) = \max\{\vartheta, \rho\}, \quad \vartheta, \rho \ge 0. \tag{38}$$

Then, η is a partial metric on $[0, \infty)$.

Proposition 15. *Let $\mathcal{V}$ be a nonempty set equipped with a partial metric η. Let $m, N \in \mathbb{N}$, $N \ge 2$, $\{\xi_i\}_{i=1}^N \subset [0,\infty)$, $\sum_{i=1}^N \xi_i = 1$, and $\{\vartheta_i\}_{i=1}^N \subset \mathcal{V}$. Then,*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \xi_i\xi_j\left[2\eta(\vartheta_i, \vartheta_j) - \eta(\vartheta_i, \vartheta_i) - \eta(\vartheta_j, \vartheta_j)\right]^m \le \frac{1}{2}\inf_{\vartheta\in\mathcal{V}}\Xi(\vartheta), \tag{39}$$

where

$$\begin{aligned}\Xi(\vartheta) = 2\sum_{i=1}^{N} \xi_i[2\eta(\vartheta_i, \vartheta) - \eta(\vartheta_i, \vartheta_i) - \eta(\vartheta, \vartheta)]^m \\ + \sum_{k=1}^{m-1}\binom{m}{k}\left(\sum_{i=1}^{N}\xi_i[2\eta(\vartheta_i, \vartheta) - \eta(\vartheta_i, \vartheta_i) - \eta(\vartheta, \vartheta)]^k\right) \\ \times\left(\sum_{i=1}^{N}\xi_i[2\eta(\vartheta_i, \vartheta) - \eta(\vartheta_i, \vartheta_i) - \eta(\vartheta, \vartheta)]^{m-k}\right),\end{aligned} \tag{40}$$

for all $\vartheta \in \mathcal{V}$. Moreover, the inequality is optimal, in the sense that the multiplicative coefficient 1/2 on the right-hand side of (39) cannot be replaced by a smaller real number.

Proof. Consider the function $\delta : \mathcal{V} \times \mathcal{V} \longrightarrow [0,\infty)$ defined by

$$\delta(\vartheta, \rho) = 2\eta(\vartheta, \rho) - \eta(\vartheta, \vartheta) - \eta(\rho, \rho), \quad \vartheta, \rho \in \mathcal{V}. \tag{41}$$

It can be easily seen that δ is a metric on $\mathcal{V}$. Then, using (3) with δ as defined above, (39) follows. The optimality of (39) follows from Theorem 2.

4. Conclusion

A new inequality in metric spaces is proved. This inequality is a generalization of that derived by Dragomir and Gosa [1]. Moreover, we provided a geometric interpretation of our main result (see Corollary 7) and discussed some special cases including Lebesgue spaces, matrices inequalities, multiplicative metric inequalities, and partial metric inequalities.

Data Availability

The data used to support the study can be available upon request.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Some Generalized Fixed Point Results with Applications to Dynamic Programming

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The aim of this paper is to introduce some generalized contractions and prove certain new fixed point results for self-mappings satisfying these contractions in the setting of $\mathscr{F}$-metric space. As an application of our results, we investigate the problem of dynamic programming related to the multistage process which formulates the problems of computer programming and mathematical optimization. We also provide an example to support the validity of our main results.

1. Introduction

Because of utility and applications of metric fixed point theory in mathematics and related fields like social sciences, physical sciences, computer sciences, and engineering have grown in many directions. More general theorems have been introduced along with the provision of many useful tools to solve problems arising in several diverse areas of research.

Recently, Jlei and Samet [1] initiated a generalized metric space named as $\mathscr{F}$-metric space and showed a generalization of the Banach contraction principle. Meanwhile, researchers have picked keen interests in extending results in this generalized metric space; see for instance, [2–5]. In this paper, we define some generalized contractions and establish some results in the context of $\mathscr{F}$-metric spaces.

2. Preliminaries

Here, we record some requisites definitions and results for the purpose of the next sections.

Definition 1 (see [1]). Let $\mathscr{F}$ be the class of functions $f : (0, +\infty) \longrightarrow \mathbb{R}$ satisfying these assertions:

$(\mathscr{F}_1) 0 < s < \iota = \Longrightarrow f(s) \leq f(\iota)$

$(\mathscr{F}_2) \forall \{\iota_n\} \subseteq R^+, \lim_{n\to\infty} \iota_n = 0 \Leftrightarrow \lim_{n\to\infty} f(\iota_n) = -\infty$

Example 2. If $f : (0,+\infty) \longrightarrow \mathbb{R}$ are defined as.

(1) $f(\varrho) = \ln(\varrho)$

(2) $f(\varrho) = \varrho + \ln(\varrho)$

(3) $f(\varrho) = -1/\sqrt{\varrho}$

(4) $f(\varrho) = \ln(\varrho^2 + \varrho)$

for $\varrho > 0$, then $f \in \mathscr{F}$.

Definition 3 (see [1]). Let $\mathscr{M} \neq \varnothing$, and $d_{\mathscr{F}} : \mathscr{M} \times \mathscr{M} \longrightarrow 0, +\infty)$. Assume that $\exists (f, \tau) \in \mathscr{F} \times [0,+\infty)$ such that

(D_1) for all $\zeta, \xi \in \mathscr{M}$, $d_{\mathscr{F}}(\zeta, \xi) = 0 \Leftrightarrow \zeta = \xi$

(D_2) $d_{\mathscr{F}}(\zeta, \xi) = d_{\mathscr{F}}(\xi, \zeta)$, for all $(\zeta, \xi) \in \mathscr{M} \times \mathscr{M}$

(D_3) for every $(\zeta, \xi) \in \mathscr{M} \times \mathscr{M}$, for every $N \in \mathbb{N}$, $N \geq 2$, and $(\mathfrak{w}_i)_{i=1}^N \subset \mathscr{M}$, with $(\mathfrak{w}_1, \mathfrak{w}_N) = (\zeta, \xi)$, we have

$$d_{\mathscr{F}}(\zeta, \xi) > 0 \Longrightarrow f(d_{\mathscr{F}}(\zeta, \xi)) \leq f\left(\sum_{i=1}^{N-1} d_{\mathscr{F}}(\zeta_i, \zeta_{i+1})\right) + \tau. \tag{1}$$

Then, $(\mathscr{M}, d_{\mathscr{F}})$ is called an $\mathscr{F}$-metric space.

Example 4 (see [1]). Let $d_{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be defined by

$$d_{\mathcal{F}}(\zeta, \xi) = \begin{cases} (\zeta - \xi)^2 \text{ if } (\zeta, \xi) \in [0, 3] \times [0, 3], \\ |\zeta - \xi| \text{ if } (\zeta, \xi) \in [0, 3] \times [0, 3], \end{cases} \quad (2)$$

with $f(\varrho) = \ln(\varrho)$ and $\tau = \ln(3)$. Then, $(\mathbb{R}, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space.

Theorem 5. *[1].*

Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space and $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$. Assume that these conditions are satisfied:

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(ii) $\exists 0 < \lambda < 1$ such that

$$d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) \le \lambda d_{\mathcal{F}}(\zeta, \xi). \quad (3)$$

Then, $\exists \zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$. Furthermore, for any $\zeta_0 \in \mathcal{M}$, $\{\zeta_n\} \subset \mathcal{M}$ defined by

$$\zeta_{n+1} = \mathcal{J}(\zeta_n), \ n \in \mathbb{N}, \quad (4)$$

is $\mathcal{F}$-convergent to ζ^*.

Afterwards, many researchers [2–9] worked in this space.

In this article, we give the notions of twisted (α, β) -admissible and twisted $(\alpha, \beta) - \varphi$-rational contractions in the setting of $\mathcal{F}$-metric spaces and prove some new theorems.

3. Main Results

In 2012, Samet et al. [10] introduced the concepts of α-admissibility mappings and α-φ-contraction in complete metric spaces.

Definition 6 (see [10]). Let $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ and $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow 0,+\infty)$. Then, $\mathcal{J}$ is said to be α -admissible if $\zeta, \xi \in \mathcal{M}$,

$$\alpha(\zeta, \xi) \ge 1 \Longrightarrow \alpha(\mathcal{J}\zeta, \mathcal{J}\xi) \ge 1. \quad (5)$$

According to Samet et al. [10], Ψ represents the class of all nondecreasing functions $\varphi : [0,+\infty) \longrightarrow 0,+\infty)$ such that $\sum_{n=1}^{\infty} \varphi^n(\iota) < +\infty$ for all $\iota > 0$, where φ^n is the nth iterate of φ.

Lemma 7 (see [10]). *If $\varphi \in \Psi$, then*

(i) $(\varphi^n(\iota))_{n\in\mathbb{N}} \longrightarrow 0$ as $n \longrightarrow \infty, \forall \iota \in (0,+\infty)$

(ii) $\varphi(\iota) < \iota$ for all $\iota > 0$

(iii) $\varphi(\iota) = 0$ iff $\iota = 0$

Now, we give the concept of twisted (α, β)-admissible in the $\mathcal{F}$-metric space as follows.

Definition 8. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space, $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ and $\alpha, \beta : \mathcal{M} \times \mathcal{M} \longrightarrow 0,+\infty)$. Then, $\mathcal{J}$ is said to be twisted (α, β) -admissible if

$$\begin{cases} \alpha(\zeta, \xi) \ge 1 \\ \beta(\zeta, \xi) \ge 1 \end{cases} \Longrightarrow \begin{cases} \alpha(\mathcal{J}\zeta, \mathcal{J}\xi) \ge 1, \\ \beta(\mathcal{J}\zeta, \mathcal{J}\xi) \ge 1, \end{cases} \quad (6)$$

for $\zeta, \xi \in \mathcal{M}$.

Now, we state our main result.

Theorem 9. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space and $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ be twisted (α, β) -admissible. Suppose that the following assertions are satisfied:*

(a) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(b) there exists $\zeta_0 \in \mathcal{M}$ such that $\alpha(\zeta_0, \mathcal{J}(\zeta_0)) \ge 1$ and $\beta(\zeta_0, \mathcal{J}(\zeta_0)) \ge 1$

(c) $\mathcal{J}$ is continuous. If any one of these assertions hold:

$$\alpha(\zeta, \xi)\beta(\zeta, \xi) d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) \le \varphi(\mathcal{R}(\zeta, \xi)) \quad (7)$$

(i) $\exists 0 < l \le 1$ such that

$$(\alpha(\zeta, \xi)\beta(\zeta, \xi) + l)^{d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi))} \le (1 + l)^{\varphi(\mathcal{R}(\zeta, \xi))} \quad (8)$$

(ii) $\exists l \ge 1$ such that

$$(d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) + l)^{\alpha(\zeta,\xi)\beta(\zeta,\xi)} \le \varphi(\mathcal{R}(\zeta, \xi)) + l \quad (9)$$

where

$$\mathcal{R}(\zeta, \xi) = \max \left\{ d_{\mathcal{F}}(\zeta, \xi), \frac{d_{\mathcal{F}}(\zeta, \mathcal{J}(\zeta)) d_{\mathcal{F}}(\xi, \mathcal{J}(\xi))}{1 + d_{\mathcal{F}}(\zeta, \xi)} \right\}, \quad (10)$$

for all $\zeta, \xi \in \mathcal{M}$, then $\exists \zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$.

Proof. Let $\zeta_0 \in \mathcal{M}$ such that $\alpha(\zeta_0, \mathcal{J}(\zeta_0)) \ge 1$ and $\beta(\zeta_0, \mathcal{J}(\zeta_0)) \ge 1$. Generate $\{\zeta_n\}$ in $\mathcal{M}$ by $\zeta_{n+1} = \mathcal{J}(\zeta_n), \forall n \in \mathbb{N}$. If $\zeta_{n+1} = \zeta_n$ for some $n \in \mathbb{N}$, then $\zeta^* = \zeta_n$ is a fixed point of $\mathcal{J}$. So we suppose that $\zeta_{n+1} \ne, \forall n \in \mathbb{N}$. Then as $\mathcal{J}$ is twisted (α, β)-admissible, we get $\alpha(\zeta_0, \zeta_1) = \alpha(\zeta_0, \mathcal{J}(\zeta_0)) \ge 1$ implies $\alpha(\zeta_1, \zeta_2) = \alpha(\mathcal{J}(\zeta_0), \mathcal{J}(\zeta_1)) \ge 1$ and $\beta(\zeta_1, \zeta_2) = \beta(\mathcal{J}(\zeta_0), \mathcal{J}(\zeta_1)) \ge 1$. By induction, we get $\alpha(\zeta_n, \zeta_{n+1}) \ge 1$ and $\beta(\zeta_n, \zeta_{n+1}) \ge 1$ for all $n \in \mathbb{N}$. Suppose the inequality (7) holds. So with $\zeta = \zeta_{n-1}$ and $\xi = \zeta_n$, we have

$$\begin{aligned} d_{\mathcal{F}}(\zeta_n, \zeta_{n+1}) &= d_{\mathcal{F}}(\mathcal{J}(\zeta_{n-1}), \mathcal{J}(\zeta_n)) \\ &\le \alpha(\zeta_{n-1}, \zeta_n)\beta(\zeta_{n-1}, \zeta_n) d_{\mathcal{F}}(\mathcal{J}(\zeta_{n-1}), \mathcal{J}(\zeta_n)) \\ &\le \varphi(\mathcal{R}(\zeta_{n-1}, \zeta_n)), \end{aligned} \quad (11)$$

where

$$\mathscr{R}(\zeta_{n-1},\zeta_n)=\max\left\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),\frac{d_{\mathscr{F}}(\zeta_{n-1},\mathscr{J}(\zeta_{n-1}))d_{\mathscr{F}}(\zeta_n,\mathscr{J}(\zeta_n))}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right\}=\max\left\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right\},\tag{12}$$

If $\max\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))\}=(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))$, then from (11), we obtain

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi\left(\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right)<\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\le d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}),\tag{13}$$

a contradiction. Hence, $\max\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))\}=d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)$; therefore, (11) becomes

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)).\tag{14}$$

Consequently, we get

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi^n(d_{\mathscr{F}}(\zeta_0,\zeta_1)),\tag{15}$$

$\forall n\in\mathbb{N}$.

Assume inequality (8) holds and $\exists 0<l\le 1$ such that

$$(1+l)^{d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}\le(\alpha(\zeta_{n-1},\zeta_n)\beta(\zeta_{n-1},\zeta_n)+l)^{d_{\mathscr{F}}(\mathscr{J}(\zeta_{n-1}),\mathscr{J}(\zeta_n))}\le(1+l)^{\varphi(\mathscr{R}(\zeta_{n-1},\zeta_n))},\tag{16}$$

which implies that

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi(\mathscr{R}(\zeta_{n-1},\zeta_n)),\tag{17}$$

where

$$\mathscr{R}(\zeta_{n-1},\zeta_n)=\max\left\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),\frac{d_{\mathscr{F}}(\zeta_{n-1},\mathscr{J}(\zeta_{n-1}))d_{\mathscr{F}}(\zeta_n,\mathscr{J}(\zeta_n))}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right\}=\max\left\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right\}.\tag{18}$$

If $\max\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))\}=(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))$, then from (17), we obtain

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi\left(\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right)<\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\le d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}),\tag{19}$$

a contradiction. Thus, $\max\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))\}=d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)$; therefore, (17) becomes

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)).\tag{20}$$

Consequently, we get

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi^n(d_{\mathscr{F}}(\zeta_0,\zeta_1)).\tag{21}$$

Assume inequality (9) holds and $\exists l\le 1$ such that

$$(d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})+l)\le(d_{\mathscr{F}}(\mathscr{J}(\zeta_{n-1}),\mathscr{J}(\zeta_n))+l)^{\alpha(\zeta_{n-1},\zeta_n)\beta(\zeta_{n-1},\zeta_n)}\le\varphi(\mathscr{R}(\zeta_{n-1},\zeta_n))+l,\tag{22}$$

which implies that

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi(\mathscr{R}(\zeta_{n-1},\zeta_n)),\tag{23}$$

where

$$\mathscr{R}(\zeta_{n-1},\zeta_n)=\max\left\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),\frac{d_{\mathscr{F}}(\zeta_{n-1},\mathscr{J}(\zeta_{n-1}))d_{\mathscr{F}}(\zeta_n,\mathscr{J}(\zeta_n))}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right\}=\max\left\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right\},\tag{24}$$

If $\max\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))\}=(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))$, then from (23), we obtain

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi\left(\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\right)<\frac{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)}\le d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}),\tag{25}$$

a contradiction. Thus, $\max\{d_{\mathscr{F}}(\zeta_{n-1},\zeta_n),(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)d_{\mathscr{F}}(\zeta_n,\zeta_{n+1}))/(1+d_{\mathscr{F}}(\zeta_{n-1},\zeta_n))\}=d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)$; therefore, (23) becomes

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi(d_{\mathscr{F}}(\zeta_{n-1},\zeta_n)).\tag{26}$$

Consequently, we get

$$d_{\mathscr{F}}(\zeta_n,\zeta_{n+1})\le\varphi^n(d_{\mathscr{F}}(\zeta_0,\zeta_1)),\tag{27}$$

Let $(f,\tau)\in\mathcal{F}\times 0,+\infty)$ be such that (D_3) is satisfied. Let $\varepsilon>0$ be fixed. By $(\mathcal{F}_2)$, $\exists\delta>0$ such that

$$0<\iota<\delta\Longrightarrow f(\iota)<f(\delta)-\tau. \tag{28}$$

Let $n(\varepsilon)\in\mathbb{N}$ such that $0<\sum_{n\geq n(\varepsilon)}\varphi^n(d_{\mathcal{F}}(\zeta_0,\zeta_1))<\delta$. Hence, by (27), $(\mathcal{F}_1)$, and $(\mathcal{F}_2)$, we have

$$\begin{aligned} f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\zeta_i,\zeta_{i+1})\right) &\leq f\left(\sum_{i=n}^{m-1}\varphi^i(d_{\mathcal{F}}(\zeta_0,\zeta_1))\right)\\ &\leq f\left(\sum_{n\geq n(\varepsilon)}\varphi^n(d_{\mathcal{F}}(\zeta_0,\zeta_1))\right)<f(\varepsilon)-\tau, \end{aligned} \tag{29}$$

for $m>n\geq n(\varepsilon)$. Using (D_3) and (29), we get $d_{\mathcal{F}}(\zeta_n,\zeta_m)>0$, $m>n\geq n(\varepsilon)$ implies

$$f(d_{\mathcal{F}}(\zeta_n,\zeta_m))\leq f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\zeta_i,\zeta_{i+1})\right)+\tau<f(\varepsilon), \tag{30}$$

which implies by $(\mathcal{F}_1)$ that $d_{\mathcal{F}}(\zeta_n,\zeta_m)<\varepsilon, m>n\geq n(\varepsilon)$. This shows that $\{\zeta_n\}$ is $\mathcal{F}$-Cauchy. As $(\mathcal{M},d_{\mathcal{F}})$ is $\mathcal{F}$-complete, $\exists$ $\zeta^*\in\mathcal{M}$ such that $\{\zeta_n\}$ is $\mathcal{F}$-convergent to ζ^*. As $\mathcal{J}$ is continuous, so we have $\mathcal{J}(\zeta^*)=\lim_{n\to\infty}\mathcal{J}(\zeta_n)\ll\lim_{n\to\infty}\zeta_{n+1}=\zeta^*$. Thus, $\exists\zeta^*\in\mathcal{M}$ such that $\mathcal{J}\zeta^*=\zeta^*$.

In the next result, we omit the continuity of $\mathcal{J}$ and use an adjunctive condition on $\mathcal{M}$.

Theorem 10. *Let $(\mathcal{M},d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space and $\mathcal{J}:\mathcal{M}\longrightarrow\mathcal{M}$ be twisted (α,β) -admissible. Suppose that the following assertions are satisfied:*

(a) $(\mathcal{M},d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(b) $\exists\zeta_0\in\mathcal{M}$ such that $\alpha(\zeta_0,\mathcal{J}(\zeta_0))\geq 1$ and $\beta(\zeta_0,\mathcal{J}(\zeta_0))\geq 1$

(c) If $\{\zeta_n\}$ is a sequence in $\mathcal{M}$ such that $\alpha(\zeta_n,\zeta_{n+1})\geq 1$ and $\beta(\zeta_n,\zeta_{n+1})\geq 1,\forall n$ and $\zeta_n\longrightarrow\zeta^*\in\mathcal{M}$ as $n\longrightarrow\infty$, then $\alpha(\zeta_n,\zeta^*)\geq 1$ and $\beta(\zeta_n,\zeta^*)\geq 1,\forall n\in\mathbb{N}$

If any one of these assertions hold:

$$\alpha(\zeta,\xi)\beta(\zeta,\xi)d_{\mathcal{F}}(\mathcal{J}(\zeta),\mathcal{J}(\xi))\leq\varphi(\mathcal{R}(\zeta,\xi)), \tag{31}$$

(i) $\exists 0<l\leq 1$ such that

$$(\alpha(\zeta,\xi)\beta(\zeta,\xi)+l)^{d_{\mathcal{F}}(\mathcal{J}(\zeta),\mathcal{J}(\xi))}\leq(1+l)^{\varphi(\mathcal{R}(\zeta,\xi))} \tag{32}$$

(ii) $\exists l\geq 1$ such that

$$(d_{\mathcal{F}}(\mathcal{J}(\zeta),\mathcal{J}(\xi))+l)^{\alpha(\zeta,\xi)\beta(\zeta,\xi)}\leq\varphi(\mathcal{R}(\zeta,\xi))+l \tag{33}$$

where

$$\mathcal{R}(\zeta,\xi)=\max\left\{d_{\mathcal{F}}(\zeta,\xi),\frac{d_{\mathcal{F}}(\zeta,\mathcal{J}(\zeta))d_{\mathcal{F}}(\xi,\mathcal{J}(\xi))}{1+d_{\mathcal{F}}(\zeta,\xi)}\right\}, \tag{34}$$

for all $\zeta,\xi\in\mathcal{M}$, then $\exists\zeta^*\in\mathcal{M}$ such that $\mathcal{J}\zeta^*=\zeta^*$.

Proof. >Let $\zeta_0\in\mathcal{M}$ such that $\alpha(\zeta_0,\mathcal{J}(\zeta_0))\geq 1$ and $\beta(\zeta_0,\mathcal{J}(\zeta_0))\geq 1$. Proceeding as in the proof of Theorem 9, we have $\zeta^*\in\mathcal{M}$ such that $\{\zeta_n\}$ is $\mathcal{F}$-convergent to ζ^*, i.e.,

$$\lim_{n\to\infty} d_{\mathcal{F}}(\zeta_n,\zeta^*)=0. \tag{35}$$

Suppose that $d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*)>0$ and inequality (31) holds. By $(\mathcal{F}_1)$ and (D_3), we have

$$\begin{aligned} d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta_{n+1}) &= d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\zeta_n))\\ &\leq\alpha(\zeta^*,\zeta_n)\beta(\zeta^*,\zeta_n)d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\zeta_n))\\ &\leq\varphi\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\zeta_n),\frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\mathcal{J}(\zeta_n))}{1+d_{\mathcal{F}}(\zeta^*,\zeta_n)}\right\}\right). \end{aligned} \tag{36}$$

Similarly, if inequality (32) holds. So $\exists 0<l\leq 1$ such that

$$\begin{aligned} (1+l)^{d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta_{n+1})} &= (1+l)^{d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\zeta_n))}\\ &\leq(\alpha(\zeta^*,\zeta_n)\beta(\zeta^*,\zeta_n)+l)^{d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\zeta_n))}\\ &\leq(1+l)^{\varphi(\mathcal{R}(\zeta^*,\zeta_n))}, \end{aligned} \tag{37}$$

which implies that

$$d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta_{n+1})\leq\varphi(\mathcal{R}(\zeta^*,\zeta_n)), \tag{38}$$

that is

$$d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta_{n+1})\leq\varphi\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\zeta_n),\frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\mathcal{J}(\zeta_n))}{1+d_{\mathcal{F}}(\zeta^*,\zeta_n)}\right\}\right). \tag{39}$$

Also if inequality (33) holds, then $\exists l\geq 1$ such that

$$\begin{aligned} (d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta_{n+1})+l) &\leq(d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\zeta_n))+l)^{\alpha(\zeta^*,\zeta_n)\beta(\zeta^*,\zeta_n)}\\ &\leq\varphi(\mathcal{R}(\zeta^*,\zeta_n))+ld_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta_{n+1})\\ &\leq\varphi(\mathcal{R}(\zeta^*,\zeta_n)), \end{aligned} \tag{40}$$

that is

$$d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta_{n+1})\leq\varphi\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\zeta_n),\frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\mathcal{J}(\zeta_n))}{1+d_{\mathcal{F}}(\zeta^*,\zeta_n)}\right\}\right). \tag{41}$$

Thus, for all cases, by $(\mathcal{F}_1)$ and (D_3), we have

$$\begin{aligned} f(d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*)) &\le f(d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\zeta_n)) + d_{\mathcal{F}}(\mathcal{J}(\zeta_n),\zeta^*)) + \tau \\ &\le f\left(\varphi\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\zeta_n), \frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\mathcal{J}(\zeta_n))}{1+d_{\mathcal{F}}(\zeta^*,\zeta_n)}\right\}\right)\right. \\ &\quad \left.+ d_{\mathcal{F}}(\zeta_{n+1},\zeta *)\right) + \tau, \\ &< f\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\zeta_n), \frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathcal{F}}(\zeta^*,\zeta_n)}\right\}\right. \\ &\quad \left.+ d_{\mathcal{F}}(\zeta_{n+1},\zeta *)\right) + \tau, \end{aligned} \tag{42}$$

for $n \in \mathbb{N}$. If $\max\{d_{\mathcal{F}}(\zeta^*,\zeta_n), (d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\zeta_{n+1})/1 + d_{\mathcal{F}}(\zeta^*,\zeta_n))\} = d_{\mathcal{F}}(\zeta^*,\zeta_n)$, then

$$f(d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*)) \le f(d_{\mathcal{F}}(\zeta^*,\zeta_n) + d_{\mathcal{F}}(\zeta_{n+1},\zeta *)) + \tau, \tag{43}$$

Letting $n \longrightarrow \infty$ and utilizing $(\mathcal{F}_2)$ and (35), we get

$$\begin{aligned} &\lim_{n\to\infty} f(d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*)) \\ &\quad \le \lim_{n\to\infty} f(d_{\mathcal{F}}(\zeta^*,\zeta_n) + d_{\mathcal{F}}(\zeta_{n+1},\zeta *)) + \tau = -\infty, \end{aligned} \tag{44}$$

which implies that $d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*) = 0$, a contradiction.

If $\max\{d_{\mathcal{F}}(\zeta^*,\zeta_n), (d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\zeta_{n+1}))/(1 + d_{\mathcal{F}}(\zeta^*,\zeta_n))\} = (d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\zeta_{n+1}))/(1 + d_{\mathcal{F}}(\zeta^*,\zeta_n))$, then

$$f(d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*)) \le f\left(\frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathcal{F}}(\zeta^*,\zeta_n)} + d_{\mathcal{F}}(\zeta_{n+1},\zeta *)\right) + \tau. \tag{45}$$

Letting $n \longrightarrow \infty$ and utilizing $(\mathcal{F}_2)$ and (35), we get

$$\begin{aligned} &\lim_{n\to\infty} f(d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*)) \\ &\quad \le \lim_{n\to\infty} f\left(\frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\zeta_n,\zeta_{n+1})}{1+d_{\mathcal{F}}(\zeta^*,\zeta_n)} + d_{\mathcal{F}}(\zeta_{n+1},\zeta *)\right) + \tau = -\infty, \end{aligned} \tag{46}$$

which implies that $d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*) = 0$, a contradiction. Thus, we have $d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\zeta^*) = 0$, i.e., $\mathcal{J}(\zeta^*) = \zeta^*$.

For the uniqueness of the fixed point, we take the following property:

(P) $\alpha(\zeta,\xi) \ge 1$ and $\beta(\zeta,\xi) \ge 1$ for all fixed points $\zeta,\xi \in \mathcal{M}$ of $\mathcal{J}$

Theorem 11. *If we add the property (P) in supposition of Theorem 10,then we get that the fixed point of the mapping $\mathcal{J}$ is unique.*

Proof. Let $\zeta^*, \widehat{\zeta} \in \mathcal{M}$ be such that $\mathcal{J}\zeta^* = \zeta^*$ and $\mathcal{J}\widehat{\zeta} = \widehat{\zeta}$ such that $\zeta^* = \widehat{\zeta}$. Then, by hypothesis (P), we have $\alpha(\zeta^*,\widehat{\zeta}) \ge 1$ and $\beta(\zeta^*,\widehat{\zeta}) \ge 1$. Suppose (i) holds. Then,

$$\begin{aligned} d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}) &= d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\widehat{\zeta})) \\ &\le \alpha(\zeta^*,\widehat{\zeta})\beta(\zeta^*,\widehat{\zeta})d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\widehat{\zeta})) \\ &\le \varphi\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}), \frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\widehat{\zeta},\mathcal{J}(\widehat{\zeta}))}{1+d_{\mathcal{F}}(\zeta^*,\widehat{\zeta})}\right\}\right) \\ &= \varphi(d_{\mathcal{F}}(\zeta^*,\widehat{\zeta})) < d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}), \end{aligned} \tag{47}$$

a contradiction. Hence, $\exists\zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$. Suppose (ii) holds. Then, there exists $0 < l \le 1$ such that

$$\begin{aligned} &\left(\alpha(\zeta^*,\widehat{\zeta})\beta(\zeta^*,\widehat{\zeta}) + l\right)^{d_{\mathcal{F}}(\zeta^*,\widehat{\zeta})} \\ &\quad = \left(\alpha(\zeta^*,\widehat{\zeta})\beta(\zeta^*,\widehat{\zeta}) + l\right)^{d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\widehat{\zeta}))} \\ &\quad \le (1+l)^{\varphi(\mathcal{R}(\zeta^*,\widehat{\zeta}))}, \end{aligned} \tag{48}$$

which implies that

$$\begin{aligned} d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}) &\le \varphi\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}), \frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\widehat{\zeta},\mathcal{J}(\widehat{\zeta}))}{1+d_{\mathcal{F}}(\zeta^*,\widehat{\zeta})}\right\}\right) \\ &= \varphi(d_{\mathcal{F}}(\zeta^*,\widehat{\zeta})) < d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}), \end{aligned} \tag{49}$$

a contradiction. Hence, $\exists\zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$. Suppose (iii) holds. Then, there exists $l \ge 1$ such that

$$\begin{aligned} (d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}) + l) &\le (d_{\mathcal{F}}(\mathcal{J}(\zeta^*),\mathcal{J}(\widehat{\zeta})) + l)^{\alpha(\zeta^*,\widehat{\zeta})\beta(\zeta^*,\widehat{\zeta})} \\ &\le \varphi(\mathcal{R}(\zeta^*,\widehat{\zeta})) + l, \end{aligned} \tag{50}$$

which implies that

$$\begin{aligned} d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}) &\le \varphi\left(\max\left\{d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}), \frac{d_{\mathcal{F}}(\zeta^*,\mathcal{J}(\zeta^*))d_{\mathcal{F}}(\widehat{\zeta},\mathcal{J}(\widehat{\zeta}))}{1+d_{\mathcal{F}}(\zeta^*,\widehat{\zeta})}\right\}\right) \\ &= \varphi(d_{\mathcal{F}}(\zeta^*,\widehat{\zeta})) < d_{\mathcal{F}}(\zeta^*,\widehat{\zeta}), \end{aligned} \tag{51}$$

a contradiction. Hence, $\exists\zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$.

Corollary 12. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space and $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ be α -admissible. Assume that these assertions hold:*

(a) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(b) $\exists \zeta_0 \in \mathcal{M}$ such that $\alpha(\zeta_0, \mathcal{J}(\zeta_0)) \geq 1$

(c) $\mathcal{J}$ is continuous or if $\{\zeta_n\}$ is a sequence in $\mathcal{M}$ such that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for all n and $\zeta_n \longrightarrow \zeta^* \in \mathcal{M}$ as $n \longrightarrow \infty$, then $\alpha(\zeta_n, \zeta^*) \geq 1, \forall n \in \mathbb{N}$. If any one of these assertions hold:

$$\alpha(\zeta, \xi) d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) \leq \varphi(\mathcal{R}(\zeta, \xi)) \tag{52}$$

(i) $\exists 0 < l \leq 1$ such that

(1)

$$(\alpha(\zeta, \xi) + l)^{d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi))} \leq (1 + l)^{\varphi(\mathcal{R}(\zeta, \xi))} \tag{53}$$

(ii) $\exists l \geq 1$ such that

(2)

$$(d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) + l)^{\alpha(\zeta, \xi)} \leq \varphi(\mathcal{R}(\zeta, \xi)) + l \tag{54}$$

where

$$\mathcal{R}(\zeta, \xi) = \max \left\{ d_{\mathcal{F}}(\zeta, \xi), \frac{d_{\mathcal{F}}(\zeta, \mathcal{J}(\zeta)) d_{\mathcal{F}}(\xi, \mathcal{J}(\xi))}{1 + d_{\mathcal{F}}(\zeta, \xi)} \right\} \tag{55}$$

$\forall \zeta, \xi \in \mathcal{M}$, then $\exists \zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$.

Proof. Taking $\beta(\zeta, \xi) = 1$ for all $\zeta, \xi \in \mathcal{M}$ in Theorem 10.

Corollary 13. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space and $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ be α -admissible mapping such that*

$$\alpha(\zeta, \xi) d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) \leq \varphi(\mathcal{R}(\zeta, \xi)), \tag{56}$$

where

$$\mathcal{R}(\zeta, \xi) = \max \left\{ d_{\mathcal{F}}(\zeta, \xi), \frac{d_{\mathcal{F}}(\zeta, \mathcal{J}(\zeta)) d_{\mathcal{F}}(\xi, \mathcal{J}(\xi))}{1 + d_{\mathcal{F}}(\zeta, \xi)} \right\}, \tag{57}$$

$\forall \zeta, \xi \in \mathcal{M}$. *Suppose that the following assertions are satisfied:*

(a) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(b) $\exists \zeta_0 \in \mathcal{M}$ such that $\alpha(\zeta_0, \mathcal{J}(\zeta_0)) \geq 1$

(c) $\mathcal{J}$ is continuous or if $\{\zeta_n\}$ is a sequence in $\mathcal{M}$ such that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for all n and $\zeta_n \longrightarrow \zeta^* \in \mathcal{M}$ as $n \longrightarrow \infty$, then $\alpha(\zeta_n, \zeta^*) \geq 1, \forall n \in \mathbb{N}$. Then $\exists \zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$

Proof. If only (i) holds in Corollary 12.

Corollary 14. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space and $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ be (α, β) -admissible mapping such that*

$$\alpha(\zeta, \xi) \beta(\zeta, \xi) d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) \leq \varphi(d_{\mathcal{F}}(\zeta, \xi)), \tag{58}$$

$\forall \zeta, \xi \in \mathcal{M}$. *Assume that these assertions hold:*

(a) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(b) $\exists \zeta_0 \in \mathcal{M}$ such that $\alpha(\zeta_0, \mathcal{J}(\zeta_0)) \geq 1$ and $\beta(\zeta_0, \mathcal{J}(\zeta_0)) \geq 1$

(c) $\mathcal{J}$ is continuous or if $\{\zeta_n\}$ is a sequence in $\mathcal{M}$ such that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ and $\beta(\zeta_n, \zeta_{n+1}) \geq 1, \forall n$ and $\zeta_n \longrightarrow \zeta^* \in \mathcal{M}$ as $n \longrightarrow \infty$, then $\alpha(\zeta_n, \zeta^*) \geq 1$ and $\beta(\zeta_n, \zeta^*) \geq 1, \forall n \in \mathbb{N}$. Then, $\exists \zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$

Proof. If only hypothesis (i) holds in Theorem 10 and $\max \{d_{\mathcal{F}}(\zeta, \xi), (d_{\mathcal{F}}(\zeta, \mathcal{J}(\zeta)) d_{\mathcal{F}}(\xi, \mathcal{J}(\xi)))/(1 + d_{\mathcal{F}}(\zeta, \xi))\} = d_{\mathcal{F}}(\zeta, \xi)$.

Following is Boyd and Wong type in the setting of $\mathcal{F}$ -metric space which is a consequence of Corollary 14.

Corollary 15. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ -metric space and $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ be a self-mapping such that*

$$d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) \leq \varphi(d_{\mathcal{F}}(\zeta, \xi)) \tag{59}$$

$\forall \zeta, \xi \in \mathcal{M}$. *Assume that these assertions hold:*

(a) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(b) $\mathcal{J}$ is continuous. Then, $\exists \zeta^* \in \mathcal{M}$ such that $\mathcal{J}\zeta^* = \zeta^*$

Proof. Taking $\alpha(\zeta, \xi) = \beta(\zeta, \xi) = 1$ for all $\zeta, \xi \in \mathcal{M}$ in Corollary 14.

Example 16. Let $\mathcal{M} = \mathbb{R}$ and $\mathcal{F}$ -metric $d_{\mathcal{F}}$ given by

$$d_{\mathcal{F}}(\zeta, \xi) = \begin{cases} 2^{|\zeta - \xi|}, & \text{if } \zeta \neq \xi, \\ 0, \text{ if } \zeta = \xi. \end{cases} \tag{60}$$

Take $f(\varrho) = -1/\varrho$ and $\tau = 1$. Define $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$\mathcal{J}(\zeta) = \begin{cases} 2\zeta, & \text{if } \zeta > 1, \\ \dfrac{\zeta}{2}, & \text{if } 0 \leq \zeta \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{61}$$

Now, we define $\alpha, \beta : \mathcal{M} \times \mathcal{M} \longrightarrow [0,+\infty)$ by

$$\alpha(\zeta, \xi) = \beta(\zeta, \xi) = \begin{cases} 1 \text{ if } \zeta, \xi \in [0,1], \\ 0, \text{ if } \zeta, \xi \in [0,1]. \end{cases} \tag{62}$$

Evidently, $\mathcal{J}$ is twisted (α, β)-φ rational contraction of type (i) with $\varphi(\iota) = k\iota, \forall \iota \geq 0$ and $k \in (0,1)$. Actually, $\forall \zeta, \xi \in \mathcal{M}$, we have

$$d_{\mathcal{F}}(\mathcal{J}(\zeta), \mathcal{J}(\xi)) \leq k\left(\max\left\{d_{\mathcal{F}}(\zeta, \xi), \frac{d_{\mathcal{F}}(\zeta, \mathcal{J}(\zeta))d_{\mathcal{F}}(\xi, \mathcal{J}(\xi))}{1 + d_{\mathcal{F}}(\zeta, \xi)}\right\}\right). \tag{63}$$

All the conditions of Theorem 9 are satisfied. Hence, $\mathcal{J}0 = 0$ which is unique.

4. Applications in Dynamic Programming

In this section, we now establish the solution of functional equations arising from dynamic programming related to multistage process [11, 12] as an application of Theorem 10. Recall that a dynamic programming problem is a decision-making problem in n variables in which the problem being subdivided into n subproblems (stages), each being a decision-making problem in one variable only. The decision is the "goodness" of a selected alternative depending on satisfying the optimal policy of the problem. The state of the system at any stage is regarded as the information that links the stages together, such that the optimal decisions for the remaining stages can be made. The state allows us to consider each stage separately and guarantees that the solution is feasible for all the stages. This setting formulates the problems of mathematical optimization and computer programming which are converted into the problems of functional equations

$$\mu(\zeta) = \sup_{\xi \in D}\{f(\zeta, \xi) + K(\zeta, \xi, \mu(g(\zeta, \xi)))\}, \zeta \in W, \tag{64}$$

where $f : W \times D \longrightarrow \mathbb{R}$ and $K : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : W \times D \rightarrow W$,$W$ and D are Banach spaces, precisely, W is a state space and D is a decision space.

Let $B(W)$ represent the set of all bounded real-valued functions defined on W. For $h \in B(W)$, consider

$$\|h\| = \sup_{\zeta \in W}|h(\zeta)|. \tag{65}$$

Clearly, $(B(W), \|h\|)$ is a Banach space. We endow $B(W)$ with the $\mathcal{F}$-metric (with $f(\varrho) = \ln \varrho$ and $a = 0$) defined by

$$d_{\mathcal{F}}(h, k) = \sup_{\iota \in W}|h(\iota) - k(\iota)|. \tag{66}$$

Evidently, $(B(W), d_{\mathcal{F}})$ is a $\mathcal{F}$-complete $\mathcal{F}$-metric space. To show the existence of a solution of (64), we define $\mathcal{J} : B(W) \longrightarrow B(W)$ by

$$\mathcal{J}(h)(\zeta) = \sup_{\xi \in D}\{f(\zeta, \xi) + K(\zeta, \xi, \zeta(g(\zeta, \xi)))\}, \tag{67}$$

$\forall h \in B(W)$ and $\zeta \in W$. Clearly, $\mathcal{J}$ is well-defined as f and K are bounded.

Theorem 17. *Let $\mathcal{J} : B(W) \longrightarrow B(W)$ is given by (67) and assume that these assertions are satisfied:*

Assume that there exists $\Theta : B(W) \times B(W) \longrightarrow \mathbb{R}$ such that

(i) $\Theta(h, k) \geq 0 \Rightarrow \Theta(\mathcal{J}(k), \mathcal{J}(h)) \geq 0$, where $h, k \in B(W)$

(ii) $|K(\zeta, \xi, h(\zeta)) - K(\zeta, \xi, k(\zeta))| \leq \varphi(|h(\zeta) - k(\zeta)|)$, where $\varphi \in \Psi$, $h, k \in B(W)$, $\Theta(h, k) \geq 0$, $\zeta \in W$ and $\xi \in D$

(iii) if $\{h_n\}$ is a sequence in $B(W)$ such that $\Theta(h_n, h_{n+1}) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $h_n \longrightarrow h^*$ as $n \longrightarrow +\infty$, then $\Theta(h_n, h^*) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$

(iv) there exists $h_0 \in B(W)$ such that $\Theta(h_0, \mathcal{J}(h_0)) \geq 0$

Then, functional equation (64) has a unique and bounded solution.

Proof. Note that $(B(W), d_{\mathcal{F}})$ is a $\mathcal{F}$-complete $\mathcal{F}$- metric space. Let ε be an arbitrary positive number and $h_1, h_2 \in B(W)$ such that $\Theta(h_1, h_2) \geq 0$, then there exist $\xi_1, \xi_2 \in D$ such that

$$\mathcal{J}(h_1)(\zeta) < f(\zeta, \xi_1) + K(\zeta, \xi_1, h_1(g(\zeta, \xi_1))) + \varepsilon, \tag{68}$$

$$\mathcal{J}(h_2)(\zeta) < f(\zeta, \xi_2) + K(\zeta, \xi_2, h_2(g(\zeta, \xi_2))) + \varepsilon, \tag{69}$$

$$\mathcal{J}(h_2)(\zeta) \geq f(\zeta, \xi_2) + K(\zeta, \xi_2, h_2(g(\zeta, \xi_2))), \tag{70}$$

$$\mathcal{J}(h_1)(\zeta) \geq f(\zeta, \xi_1) + K(\zeta, \xi_1, h_1(g(\zeta, \xi_1))). \tag{71}$$

Now, from (68) and (71), it follows easily that

$$\begin{aligned} \mathcal{J}(h_1)(\zeta) - \mathcal{J}(h_2)(\zeta) &< K(\zeta, \xi_1, h_1(g(\zeta, \xi_1))) - K(\zeta, \xi_1, h_2(g(\zeta, \xi_1))) + \\ &\leq |K(\zeta, \xi_1, h_1(g(\zeta, \xi_1))) - K((\zeta, \xi_1, h_2(g(\zeta, \xi_1)))| + \\ &\leq \varphi(|h_1(\zeta) - h_2(\zeta)|) + \varepsilon. \end{aligned} \tag{72}$$

Hence, we get

$$\mathcal{J}(h_1)(\zeta) - \mathcal{J}(h_2)(\zeta) < \varphi(h_1(\zeta) - h_2(\zeta)) + \varepsilon. \tag{73}$$

Similarly, from (69) and (70), we obtain

$$\mathcal{J}(h_2)(\zeta) - \mathcal{J}(h_1)(\zeta) < \varphi(h_1(\zeta) - h_2(\zeta)) + \varepsilon. \tag{74}$$

Therefore, from (73) and (74), we have

$$|\mathscr{J}(h_1)(\zeta) - \mathscr{J}(h_2)(\zeta)| < \varphi(h_1(\zeta) - h_2(\zeta)) + \varepsilon. \tag{75}$$

that implies,

$$d(\mathscr{J}(h_1), \mathscr{J}(h_2)) \leq \varphi(d(h_1, h_2)) + \varepsilon. \tag{76}$$

Since $\varepsilon > 0$ is arbitrary, then

$$d(\mathscr{J}(h_1), \mathscr{J}(h_2)) \leq \varphi(d(h_1, h_2)). \tag{77}$$

Define

$$\alpha(h,k) = \beta(h,k) = \begin{cases} 1 \text{ if } \Theta(h,k) \geq 0, \text{ where } h, k \in B(W), \\ 0, \text{ otherwise.} \end{cases} \tag{78}$$

Consequently, we have

$$\alpha(h_1, h_2)\beta(h_1, h_2)d(\mathscr{J}(h_1), \mathscr{J}(h_2)) \leq \varphi(d(h_1, h_2)), \tag{79}$$

that is, $\mathscr{J}$ is a twisted $(\alpha, \beta) - \varphi$-contractive mapping of type (i) with $\alpha(h,k) = \beta(h,k) = 1$ for all $h, k \in B(W)$. Thus, by Theorem 10, $\mathscr{J}$ has a fixed point. Thus, all the suppositions of Theorem 10 are satisfied. Therefore, there exists h, such that $\mathscr{J}(h) = h$, which is the bounded solution of the functional Equation (64).

5. Conclusion

In the present paper, we have defined the notions of twisted (α, β)-admissible and twisted $(\alpha, \beta) - \varphi$-rational contractions in the context of $\mathscr{F}$-metric spaces and established some generalized fixed point results. As an application of our main results, we investigate the problem of dynamic programming related to multistage process which reduces to the problem of solving the functional equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and approved the final manuscript.

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Research Article

On Local Weak Solutions for Fractional in Time SOBOLEV-Type Inequalities

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We consider two fractional in time nonlinear Sobolev-type inequalities involving potential terms, where the fractional derivatives are defined in the sense of Caputo. For both problems, we study the existence and nonexistence of nontrivial local weak solutions. Namely, we show that there exists a critical exponent according to which we have existence or nonexistence.

1. Introduction and Main Results

We consider the fractional in time Sobolev-type inequalities

$$\begin{cases} \partial_t^\sigma + \Delta_\eta(t,z) + \Delta_\eta(t,z)\left(1+|z|^2\right)^{\rho/2}|\eta(t,z)|^p \le 0, (t,z) \in (0,\infty)\times\mathbb{R}^N, \\ \eta(0,z) = \eta_0(z), z\in\mathbb{R}^N \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t^{\sigma+1} + \Delta_\eta(t,z) + \Delta_\eta(t,z)\left(1+|z|^2\right)^{\rho/2}|\eta(t,z)|^p \le 0, (t,z) \in (0,\infty)\times\mathbb{R}^N, \\ \eta(0,z) = \eta_0(z), \partial_t\eta(0,z) = \eta_1(z), z\in\mathbb{R}^N, \end{cases} \tag{2}$$

where $N \ge 1$, $p > 1$, and $\rho > -2$. Here, $\sigma \in (0,1)$ and ∂_t^λ, $\lambda \in \{\sigma, \sigma+1\}$ is the derivative of fractional order λ in the sense of Caputo. Namely, we are concerned with the existence and nonexistence of nontrivial local weak solutions to problems (1) and (2). We shall establish that there exists a critical exponent $p_c > 1$ that depends on N and ρ such that if $p \in (1, p_c)$, then problems (1) and (2) admit no nontrivial local weak solutions (i.e., we have an instantaneous blow-up), while if $p \in (p_c, \infty)$, then the considered problems admit local solutions for some initial values. In the proofs of our results, we use the test function method with some integral estimates. For more details about the test function method and its applications to partial differential equations, we refer to [1–3] and the references therein.

The absence of solutions (complete blow-up phenomenon) was observed in [4] for the following elliptic inequality with a singular potential term

$$-\Delta_\eta \ge \frac{|\eta|^2}{|z|^2} in \mathscr{D}'(g\Omega\setminus\{0\}), \eta \ge 0 a.e., \tag{3}$$

where Ω is a smooth bounded domain in R^N containing 0. In the same reference, an instantaneous blow-up result was obtained for the parabolic analogue of (3), namely

$$\partial_t\eta - \Delta_\eta \ge \frac{|\eta|^2}{|z|^2} in\, \mathscr{D}'((0,T)\times\Omega\{0\}), \eta \ge 0 a.e., \tag{4}$$

Notice that the method in [4] is based on comparison principles. In [5], using the test function method and avoiding the maximum principle, instantaneous blow-up results were obtained for certain classes of elliptic and parabolic inequalities including as special cases (3) and (4). For more results on instantaneous blow-up for nonlinear evolution equations, we refer to [6–9] and the references therein.

The investigation of instantaneous blow-up for linear Sobolev-type equations was first considered in [10]. Namely, the following problem was studied

$$\partial_t(\eta_{zz}+\eta)=\eta_{zz}, \eta(0,z)=\eta_0(z), \eta(t,0)=\eta(t,L), L>0. \quad (5)$$

In the limit case $\sigma=1$ and $\rho=0$, (1) and (2) (with equalities instead of inequalities), reduce, respectively, to

$$\begin{cases} \partial_t+\Delta_\eta(t,z)+\Delta_\eta(t,z)+|\eta(t,z)|^p=0, (t,z)\in(0,\infty)\times\mathbb{R}^N, \\ \eta(0,z)=\eta_0(z), z\in\mathbb{R}^N \end{cases} \quad (6)$$

$$\begin{cases} \partial_{tt}+\Delta_\eta(t,z)+\Delta_\eta(t,z)+|\eta(t,z)|^p=0, (t,z)\in(0,\infty)\times\mathbb{R}^N, \\ \eta(0,z)=\eta_0(z), \partial_t\eta(0,z)=\eta_1(z), z\in\mathbb{R}^N. \end{cases} \quad (7)$$

The nonexistence of local weak solutions to (6) and (7) was considered in [11] by the use of the test function method. When $N\in\{1,2\}$, it was proved that for all $p>1$, (6) and (7) admit no nontrivial local weak solutions. If $N\geq 3$, it was shown that if $1<p\leq N/N-2$, then (6) and (7) admit no nontrivial local weak solutions, while if $P>N/N-2$, then local solutions exist. Our aim in this paper is to study the instantaneous blow-up for the fractional in time versions of (6) and (7) with the potential term $V(z)=(1+|z|^2)^{\rho/2}$.

For existence results for stationary problems involving potential terms, see for example [12] and the references therein.

Notice that the study of fractional in time Sobolev-type equations was first considered in [13], where the nonexistence of global weak solutions was investigated.

Before mentioning our main results, let us define the meaning of solutions to (1) and (2).

Let $T\in(0,\infty)$. Given $\kappa>0$ and $\omega\in C([0,T])$, $T>0$, we define the fractional integral operators

$$(\mathcal{J}_0^\kappa\omega)(t)=\frac{1}{\Gamma(\kappa)}\int_0^t(t-\varsigma)^{\kappa-1}\omega(\varsigma)d\varsigma, 0<t\leq T \quad (8)$$

and

$$(\mathcal{J}_T^\kappa\omega)(t)=\frac{1}{\Gamma(\kappa)}\int_t^T(\varsigma-t)^{\kappa-1}\omega(\varsigma)d\varsigma, 0<t\leq T. \quad (9)$$

Given $\omega_i\in C([0,T])$, $i=1,2$, one has (see e.g., [14, 15])

$$\int_0^T(\mathcal{J}_0^\kappa\omega_1)(t)\omega_2(t)dt=\int_0^T(\mathcal{J}_T^\kappa\omega_2)(t)\omega_1(t)dt. \quad (10)$$

For $\sigma\in(0,1)$, the derivatives of fractional orders σ and $\sigma+1$ in the Caputo sense are defined, respectively, by

$$\begin{aligned} (\partial_t^\sigma\omega)(t)&=\left(\mathcal{J}_0^{1-\sigma}\omega'\right)(t), 0<t\leq T, \omega\in C^1([0,T]) \\ \left(\partial_t^{\sigma+1}\omega\right)(t)&=\left(\mathcal{J}_0^{1-\sigma}\omega''\right)(t), 0<t\leq T, \omega\in C^2([0,T]). \end{aligned} \quad (11)$$

Using the above notions and property (10), we define local weak solutions to problem (1) as follows.

Definition 1. Let $\eta_0\in L^1_{loc}(\mathbb{R}^N)$. We say that η is a local weak solution to (1), if there exists $\eta\in L^p_{loc}(\Lambda_{T,N})$ and satisfies

$$\begin{aligned} &\int_{\Lambda T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt-\int_{\mathscr{R}^N}\eta_0(z)\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)dz \\ &\quad\leq-\int_{\Lambda T,N}\eta\Delta\xi dzdt+\int_{\Lambda T,N}\eta\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]dzdt, \end{aligned} \quad (12)$$

for all $\xi\in C\infty(\Lambda_{T,N})$, $\xi\geq 0$, with $supp_z(\xi)\subset\subset R^N$. Here, $\Lambda_{T,N}$ is the product set $[0,T]\times R^N$. Local weak solutions to problem (2) are defined as follows.

Definition 2. Let $\eta_0,\eta_1\in L^1{}_{loc}(\mathbb{R}^N)$. We say that η is a local weak solution to (2), if there exists $T\in(0,\infty)$ such that $\eta\in L^p_{loc}(\Lambda_{T,N})$ and satisfies

$$\begin{aligned} &\int_{\Lambda_{T,N}}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt+\int_{\mathbb{R}^N}\eta 0(z)\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right]dz \\ &\quad-\int_{\mathbb{R}^N}\eta_1(z)\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)dz\leq-\int_{\Lambda_{T,N}}\eta\Delta\xi dzdt \\ &\quad-\int_{\Lambda^{T,N}}\eta\Delta\left[\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]dzdt, \end{aligned} \quad (13)$$

for all $\xi\in C\infty(\Lambda_{T,N})$, $\xi\geq 0$, with $\mathrm{supp}_z(\xi)\subset\subset R^N$ and ∂t $(JT1-\sigma\xi)(T,\cdot)\equiv 0$.

Now, we state our results. We first define the (critical) exponent

$$p_c=\begin{cases}\infty & if\ N\in\{1,2\}, \\ \dfrac{N+\rho}{N-2} & if\ N\geq 3.\end{cases} \quad (14)$$

Theorem 3. *Let $N\geq 1$, $\rho>-2$, and $\sigma\in(0,1)$.*

(i) If $\eta_0\in L^1(R^N)$ and $1<p<p_c$, then problem (1) admits no nontrivial local weak solution

(ii) If $N\geq 3$ and $p>p_c$, then problem (1) admits local solutions for some $\eta_0>0$

Theorem 4. *Let $N \geq 1$, $\rho > -2$, and $\sigma \in (0, 1)$.*

(i) *If $\eta_0, \eta_1 \in L^1(\mathbb{R}^N)$ and $1 < p < p_c$, then problem (2) admits no nontrivial local weak solution*

(ii) *If $N \geq 3$ and $p > p_c$, then problem (2) admits local solutions for some $\eta_0 > 0$ and $\eta 1 \equiv 0$*

The next section contains some preliminary estimates that will be useful in the proofs of our results. Section 3 is devoted to the proofs of Theorems 3 and 4.

2. Preliminaries

For $S, T \in (0,\infty)$, let

$$\alpha(t) = \left(1 - T^{-1}t\right)^m, t \in [0, T] \tag{15}$$

and

$$\beta(z) = F\left(\frac{|z|^2}{S^2}\right)^m, z \in \mathbb{R}^N, \tag{16}$$

where $m \gg 1$ (i.e., sufficiently large) is a natural number and $F \in C^\infty(R_+)$ satisfies

$$0 \leq F \leq 1, F_{|[0,1]} \equiv 1, F_{|[2,\infty)} \equiv 0. \tag{17}$$

Let us introduce the function

$$\xi(t, z) = \alpha(t)\beta(z), \quad (t, z) \in \Lambda_{T,N}. \tag{18}$$

Clearly, one has $\xi \in C^\infty(\Lambda_{T,N}), \xi \geq 0$ *and* $\sup p_z(\xi) \subset \subset \mathbb{R}^N$.

Lemma 5. *The function α defined by (15) satisfies the following properties:*

$$\left(\mathcal{I}_T^k \alpha\right)(t) = \frac{m!}{\Gamma(\kappa + m + 1)} T^{-m}(T - t)^{\kappa+m}, t \in [0, T], \tag{19}$$

$$\alpha_t\left(\mathcal{I}_T^k \alpha\right)(t) = \frac{m!}{\Gamma(\kappa + m)} T^{-m}(T - t)^{\kappa+m-1}, t \in [0, T], \tag{20}$$

$$\partial_{tt}\left(\mathcal{I}_T^k \alpha\right)(t) = \frac{m!}{\Gamma(\kappa + m + 1)} T^{-m}(T - t)^{\kappa+m-2}, t \in [0, T], \tag{21}$$

where $\kappa \in (0, 1)$.

Proof. One has

$$\left(\mathcal{I}_T^k \alpha\right)(t) = \frac{1}{\Gamma(\kappa)} T^{-m} \int_t^T (\varsigma - t)^{\kappa-1}(T - \varsigma)^m d\varsigma. \tag{22}$$

Taking $\tau = T - \varsigma/T - t$, the above integral reduces to

$$\begin{aligned}\left(\mathcal{I}_T^k \alpha\right)(t) &= \frac{1}{\Gamma(\kappa)} T^{-m}(T - t)^{\kappa+m} \int_0^1 (1 - \tau)^{\kappa-1}\tau^m d\tau \\ &= \frac{1}{\Gamma(\kappa)} T^{-m}(T - t)^{\kappa+m} \frac{\Gamma(\kappa)\Gamma(m + 1)}{\Gamma(\kappa + m + 1)}, \\ &= \frac{m!}{\Gamma(\kappa + m + 1)} T^{-m}(T - t)^{\kappa+m},\end{aligned} \tag{23}$$

which proves (19). Next, (20) and (21) follow by differentiating (19).

Lemma 6. *For sufficiently large S, one has*

$$\int_{\mathbb{R}^N} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)} |\Delta\beta(z)|^{p/p-1} dz \leq CS^{N-2p+p/p-1}. \tag{24}$$

Here, $C > 0$ is a constant (independent on S).

Proof. Using(16) and (17), one obtains

$$\begin{aligned}&\int_{\mathbb{R}^N} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)} |\Delta\beta(z)|^{p/p-1} dz \\ &\quad = \int_{S^l<|z|<\sqrt{2S^l}} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)} |\Delta\beta(z)|^{p/p-1} dz.\end{aligned} \tag{25}$$

On the other hand, an elementary calculation shows that

$$|\Delta\beta(z)| \leq CS^{-2} F\left(\frac{|z|^2}{S^2}\right)^{m-2}, S < |z| < \sqrt{2S}. \tag{26}$$

Here and below, $C > 0$ is a constant independent on S, whose value may change from line to line. Hence, one deduces that

$$\begin{aligned}&\int_{\mathbb{R}^N} \beta(z)^{-1/p-1}\left(1 + |z|^2\right)^{-\rho/2(p-1)} |\Delta\beta(z)|^{p/p-1} \\ &\quad \leq CS^{-2p/p-1} \int_{S<|z|<\sqrt{2S}} \left(1 + |z|^2\right)^{-\rho/2(p-1)} F\left(\frac{|z|^2}{S^2}\right)^{m-2p/p-1} dz \\ &\quad \leq CS^{-2p/p-1} \int_{S<|z|<\sqrt{2S}} \left(1 + |z|^2\right)^{-\rho/2(p-1)} dz \\ &\quad \leq CS^{-2p/p-1} \int_{r=S}^{\sqrt{2S}} \left(1 + r^2\right)^{-\rho/2(p-1)} r^{N-1} dr \\ &\quad \leq CS^{-2p/p-1}\left(1 + S^2\right)^{-\rho/2(p-1)} S^{N-1}S = CS^{N-2p+\rho/p-1},\end{aligned} \tag{27}$$

which proves the desired result.

The following result is obvious.

Lemma 7. *For $T > 0$, one has*

$$\int_0^T \alpha(t)dt = \frac{T}{m+1}. \tag{28}$$

Using (20), one obtains easily the following result.

Lemma 8. *For $T > 0$, one has*

$$\int_0^T \alpha(t)^{-1/p-1}|\alpha_t(\mathcal{J}_T^{\kappa}\alpha)(t)|dt = CT^{1+(\kappa-1)p/p-1}, \tag{29}$$

where $\kappa \in (0, 1)$.

Using (21), the following result follows.

Lemma 9. *For $T > 0$, one has*

$$\int_0^T \alpha(t)^{-1/p-1}|\partial_{tt}(\mathcal{J}_T^{\kappa}\alpha)(t)|dt = CT^{1+(\kappa-2)p/p-1}, \tag{30}$$

where $\kappa \in (0, 1)$.

3. Proofs of the Main Results

Proof of Theorem 10.

(i) Suppose that $\eta \in L^p_{loc}(\Lambda_{T,N})$ is a nontrivial local wessak solution to (1) for some fixed $T \in (0,\infty)$. Then, using (12) with ξ is the function defined by (18), one obtains

$$\begin{aligned}\int_{\Lambda T,N}&\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt - \int_{\mathbb{R}^N}\eta_0(z)\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)dz \\ &\le \int_{\Lambda T,N}|\eta||\Delta\xi|dzdt + \int_{\Delta T,N}|\eta|\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|dzdt.\end{aligned} \tag{31}$$

Next, using ε-Young inequality with $0 < \varepsilon < 1/2$, one obtains

$$\begin{aligned}\int_{\Lambda T,N}|\eta||\Delta\xi|dzdt &\le \varepsilon\int_{\Lambda T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt + C \\ &+ \int_{\Delta T,N}\xi^{-1/P-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}|\Delta\xi|^{p/p-1}dzdt.\end{aligned} \tag{32}$$

Similarly, one has

$$\begin{aligned}\int_{\Lambda T,N}|\eta|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]dzdt &\le \varepsilon\int_{\Lambda T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt \\ &+ C\int_{\Delta T,N}\xi^{-1/P-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}|\Delta \\ &\cdot\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]|^{p/p-1}dzdt.\end{aligned} \tag{33}$$

It follows from (31), (32), and (33) that

$$\begin{aligned}(1-2\varepsilon)&\int_{\Lambda T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt + \int_{\mathbb{R}^N}\eta_0(z)\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)dz \\ &\le C\left(\int_{\Lambda T,N}\xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}|\Delta\xi|^{p/p-1}dzdt\right. \\ &\left.+\int_{\Lambda T,N}\xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1}dzdt\right).\end{aligned} \tag{34}$$

On the other hand, by (18), one has

$$\begin{aligned}\int_{\Lambda T,N}&\xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}|\Delta\xi|^{p/p-1}dzdt = \left(\int_0^T\alpha(t)dt\right) \\ &\cdot\left(\int_{\mathbb{R}^N}\beta(z)^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}|\Delta\beta(z)|^{p/p-1}dz\right).\end{aligned} \tag{35}$$

Hence, using Lemmas 6 and 7, for sufficiently large S, one obtains

$$\int_{\Lambda T,N}\xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}|\Delta\xi|^{p/p-1}dzdt \le CTS^{N-2p+\rho/p-1}. \tag{36}$$

Again, by (18), one has

$$\begin{aligned}\int_{\Lambda T,N}&\xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1}dzdt \\ &= \left(\int_0^T\alpha(t)^{-1/p-1}\left|\partial_t\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(t)\right|dt\right) \\ &\cdot\left(\int_{\mathbb{R}^N}\beta(z)^{-1/p-1}\left(1+|z|^2\right)^{-p/2(p-1)}|\Delta\beta(z)|^{p/p-1}dz\right).\end{aligned} \tag{37}$$

Therefore, using Lemma 6 and Lemma 8 with $\kappa = 1-\sigma$, for sufficiently large S, one obtains

$$\begin{aligned}\int_{\Lambda T,N}&\xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)}\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1}dzdt \\ &\le CT^{1-\sigma p/p-1}S^{N-2p+\rho/p-1}.\end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz &= \left|\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(0)\right| \\ &\cdot\int_{\mathbb{R}^N}|\eta_0(z)||\Delta\beta(z)|dz.\end{aligned} \tag{39}$$

Hence, using (19) (with $\kappa = 1-\sigma$) and (26), one deduces that

$$\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz \le CT^{1-\sigma}S^{-2}\int_{\mathbb{R}^N}|\eta_0(z)|\left|F\left(\frac{|z|^2}{S^2}\right)\right|^{m-2}dz. \tag{40}$$

Notice that since $\eta_0 \in L^1(\mathbb{R}^N)$, by (17), one deduces from the above estimate that

$$\lim_{S\to\infty}\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz = 0. \tag{41}$$

Next, it follows from (34), (36), (38), and (40) that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\alpha(t)F\left(\frac{|z|^2}{S^2}\right)^m dzdt\\&\le C\big(TS^{N-2p+\rho/p-1}+T^{1-\sigma p/p-1}S^{N-2-+\rho/p-1}\\&\quad+\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right|dz\big).\end{aligned} \tag{42}$$

Notice that since $1<p<p_c$, then $N-2p+p/p-1<0$. Hence, passing to the infimum limit as $S\longrightarrow\infty$ in (42), using (17), (41), and Fatou's lemma, one deduces that

$$\int_0^T\int_{\mathbb{R}^N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\alpha(t)dzdt = 0, \tag{43}$$

which contradicts the fact that η is nontrivial.

(ii) Let

$$\frac{\rho+2}{p-1}\le a<N-2 \tag{44}$$

$$\varkappa = [a(N-a-2)]^{1/p-1}. \tag{45}$$

Notice that since $p>p_c$, the set of values of a satisfying (44) is nonempty. Consider the function

$$w(z)=\varkappa\left(1+|z|^2\right)^{-a/2}, z\in\mathbb{R}^N. \tag{46}$$

An elementary calculation shows that

$$\Delta w(z)=\varkappa a\left(1+|z|^2\right)^{-a/2-2}\left[N+(N-a-2)|z|^2\right]. \tag{47}$$

Hence, using (44), (45), and (46), one obtains

$$\begin{aligned}-\Delta w(z)&=\left(1+|z|^2\right)^{\rho/2}|w(z)|^p=\varkappa a\left(1+|z|^2\right)^{-a/2-2}\\&\quad\cdot\left[N+N-a-2|z|^2\right]-\varkappa^p\left(1+|z|^2\right)^{\rho/2-ap/2}\\&>\varkappa a(N-a-2)\left(1+|z|^2\right)^{-a/2-1}-\varkappa^p\left(1+|z|^2\right)^{\rho/2-ap/2}\\&=\varkappa a(N-a-2)\left(1+|z|^2\right)^{-a/2-1}\\&\quad\cdot\left[1-\frac{\varkappa^{p-1}}{a(N-a-2)}\left(1+|z|^2\right)^{\rho/2-ap/2+a/2+1}\right]\\&=\varkappa a(N-a-2)\left(1+|z|^2\right)^{-a/2-2}\\&\quad\cdot\left[1-\left(1+|z|^2\right)^{\rho+2/2-a(p-1)/2}\right]>0.\end{aligned} \tag{48}$$

This shows that

$$\eta(t,z)=w(z), t\ge 0, z\in\mathbb{R}^N, \tag{49}$$

is a global solution (so local solution) to (1) with $\eta_0 = w>0$.

Proof of Theorem 11.

(i) Suppose that $\eta\in L_{loc}(\Lambda_{T,N})$ is a nontrivial local weak solution to (2) for some fixed $T\in(0,\infty)$. Then, using (13) with ξ is the function defined by (18), and one obtains

$$\begin{aligned}\int_{\Lambda T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt&\le\int_{\Lambda T,N}|\eta||\Delta\xi|dzdt\\&\quad+\int_{\Lambda T,N}|\eta|\left|\Delta\left[\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|dzdt\\&\quad+\int_{\mathbb{R}^N}|\eta_0(z)|\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right]\right|dz\\&\quad+\int_{\mathbb{R}^N}|\eta_1(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right|dz.\end{aligned} \tag{50}$$

Notice that by (20) (with $\kappa=1-\sigma$), one has $\partial_t(\mathcal{J}_T^{1-\sigma}\xi)(T,\cdot)\equiv 0$. Hence, by Definition 2, the choice of the test function ξ defined by (18) is allowed. Next, following the same arguments used in the proof of part (i) of Theorem 3, by the use of ε-Young inequality, one obtains

$$\int_{\Delta T,N}\left(1+|z|^2\right)^{\rho/2}|\eta|^p\xi dzdt\le C\sum_{j=1}^{5}A_j(S), \tag{51}$$

where

$$
\begin{aligned}
A_1(S) &= \int_{\Lambda T,N} \xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)} |\Delta\xi|^{p/p-1} dzdt,\\
A_2(S) &= \int_{\Lambda T,N} \xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)} \left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1} dzdt,\\
A_3(S) &= \int_{\Lambda T,N} \xi^{-1/p-1}\left(1+|z|^2\right)^{-\rho/2(p-1)} \left|\Delta\left[\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\xi\right)\right]\right|^{p/p-1} dzdt,\\
A_4(S) &= \int_{\mathbb{R}^N} |\eta_0(z)|\left|\Delta\left[\partial_t\left(\mathcal{J}_T^{1-\sigma}\xi\right)(0,z)\right]\right| dz,\\
A_5(S) &= \int_{\mathbb{R}^N} |\eta_1(z)|\left|\Delta\left(\mathcal{J}_T^{1-\sigma}\xi(0,z)\right)\right| dz.
\end{aligned}
\tag{52}
$$

Notice that by (18), one has

$$
\begin{aligned}
A_3(S) = &\left(\int_0^T \alpha(t)^{-1/p-1}\left|\partial_{tt}\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(t)\right| dt\right)\\
&\cdot\left(\int_{\mathbb{R}^N} \beta(z)^{-1/p-1}\left(1+|z|^2\right)^{-p/2(p-1)} |\Delta\beta(z)|^{p/p-1} dz\right).
\end{aligned}
\tag{53}
$$

Hence, using Lemma 6 and Lemma 9 with $\kappa = 1-\sigma$, for sufficiently large S, one deduces that

$$
A_3(S) \le C T^{-(\sigma p+1)/p-1} S^{N-2p+\rho/p-1}. \tag{54}
$$

Furthermore, by (19) and (20) (with $\kappa = 1-\sigma$), one has

$$
\begin{aligned}
A_4(S) &= \left|\partial_t\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(0)\right| \int_{\mathbb{R}^N} |\eta_0(z)||\Delta\beta(z)| dz\\
&= C T^{-\sigma} \int_{\mathbb{R}^N} |\eta_0(z)||\Delta\beta(z)| dz
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
A_5(S) &= \left|\left(\mathcal{J}_T^{1-\sigma}\alpha\right)(0)\right| \int_{\mathbb{R}^N} |\eta_1(z)||\Delta\beta(z)| dz\\
&= C T^{1-\sigma} \int_{\mathbb{R}^N} |\eta_1(z)||\Delta\beta(z)| dz.
\end{aligned}
\tag{56}
$$

Next, using (36), (38), (51), (54), (55), and (56), for sufficiently large S, one obtains

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N} \left(1+|z|^2\right)^{\rho/2} |\eta|^p \alpha(t) F\left(\frac{|z|^2}{S^2}\right)^m dzdt\\
&\quad \le C\left[\left(T + T^{1-\sigma p/p-1} + T^{-(\sigma p+1)/p-1}\right) S^{N-2p+\rho/p-1}\right.\\
&\quad\quad \left. + \int_{\mathbb{R}^N} \left(T^{-\sigma}|\eta_0(z)| + T^{1-\sigma}|\eta_1(z)|\right) |\Delta\beta(z) dz|\right]
\end{aligned}
\tag{57}
$$

Notice that by (26), since $\eta_0, \eta_1 \in L^1(\mathbb{R}^N)$, one has

$$
\lim_{S\to\infty} \int_{\mathbb{R}^N} \left(T^{-\sigma}|\eta_0(z)| + T^{1-\sigma}|\eta_1(z)|\right) |\Delta\beta(z)| dz = 0. \tag{58}
$$

Therefore, passing to the infimum limit as $S \longrightarrow \infty$ in (57) and using Fatou's lemma, since $1 < p < p_c$, it holds that

$$
\int_0^T\int_{\mathbb{R}^N} \left(1+|z|^2\right)^{\rho/2} |\eta|^p \alpha(t) dzdt = 0, \tag{59}
$$

which contradicts the fact that η is nontrivial.

(ii) From the proof of part (ii) of Theorem 3, one deduces that the function $\eta(t,z) = w(z)$, where w is defined by (46), is a global solution (so local solution) to (2) with $\eta_0 = w > 0$ and $\eta_1 \equiv 0$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work.

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Research Article

3D Dynamic Programming Approach to Functional Equations with Applications

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This article demonstrates the graphical existence of a single fixed point while imposing the contractive condition of Chatterjea type F-contraction on F-metric space (briefly as F-MS). We present two examples that verify the validity of the results given in the paper. The paper further explains the subsistence of the fixed point even if the contractive condition is valid only for a closed ball inside the space rather than imposing it on the whole F-MS. Moreover, the application of the mentioned results in finding a single solution of functional equations is described that is widely used in computer programming and optimization.

1. Introduction and Preliminaries

After Banach presented his renowned Banach Contraction Principle, his idea was generalized by various authors into many interesting generalizations (see [1–8]). Wardowski [9] extended his idea to a more generalized form which he named as F-contraction. An additional strictly increasing function F with certain other restrictions was used to modify the Banach theorem. He investigated the fixed point of the contraction and explained the generality of his theorem with the help of concrete examples. This idea was furthered by Klim and Wardowski [10] into set-valued maps using a dynamic process instead of the ordinary Picard sequence. Later, Nazam et al. [11] extended Wardowski's theorem into the form of Kannan's theorem and hence proved the theorem for noncontinuous maps. He also described that a fixed point for such maps can be iterated even if the contractive inequality holds true only for a subset closed ball of the MS. The notion of F-contraction was extended by other authors as well (see [12–16]).

This article relaxes the map F by eliminating one of its restrictions, (F3) and hence iterates a fixed point for it. The investigation is carried out for single as well as set-valued maps. Our work is new which extends the preexisting theorems of Wardowski and their consequent results. This paper demonstrates the main idea of this research with the help of graphs which unifies work from the previous research carried out on the topic.

Some basic definitions are given below which will be needed in a sequel.

Definition 1 (see [17]). Assume $\mathscr{G}$ is a set of functions $g:(0,+\infty)\longrightarrow R$ satisfying the following conditions:

(F1) g is a nondecreasing function, i.e., $0<u<v\Longrightarrow g(u)\le g(v)$,

(F2) for any sequence $t_n\subset(0,+\infty)$, we have

$$\lim_{n\to\infty} t_n=0 \Leftrightarrow \lim_{n\to\infty} g(t_n)=-\infty. \tag{1}$$

Definition 2 (see [17]). Assume a nonempty set E and $d : E \times E \longrightarrow [0, \infty)$ is a map. Assume that there is some $(g, \sigma) \in \mathscr{G} \times [0,+\infty)$ such that

(d1) $(a, x) \in E \times E$, $d(a, x) = 0 \Leftrightarrow a = x$, $(a, x) \in E \times E$, $d(a, x) = d(x, a)$,

(d2) $(a, x) \in E \times E, d(a, x) = d(x, a)$,

(d3) for every $(a, x) \in E \times E$, for each $N' \in N, N' \geq 2$, and for every $(t_i)_{i=1}^{n} \subset E$ with $(t_1, t_{N'}) = (a, x)$, we have $(a, x) > 0 \Longrightarrow g(d(a, x)) \leq g(\sum_{i=1}^{N'-1} d(t_i, t_{i+1})) + \sigma$.

Then d is called an F-metric on A, while (E, d) is named as F-MS.

Example 3 (see [17]). Let $E = N$ and $d : E \times E \longrightarrow (0, \infty)$ be defined by

$$d(a, x) = \begin{cases} (a-x)^2, & \text{if } (a, x) \in [0, 3] \times [0, 3], \\ |a - x|, & \text{if } (a, x) \notin [0, 3] \times [0, 3], \end{cases} \tag{2}$$

for all $(a, x) \in E \times E$. Then, d is an F-MS.

Example 4 (see [17]). Let $E = N$ and $d : E \times E \longrightarrow (0, \infty)$ is defined as

$$d(a, x) = \begin{cases} 0, & \text{if } a = x, \\ e^{|a-x|}, & \text{if } a \neq x, \end{cases} \tag{3}$$

for all $(a, x) \in E \times E$. Then, d is an F-metric on E.

Definition 5 (see [17]). Let $(a_n) \in E$. If

(i) $\lim_{n\to\infty} d(a_n, a) = 0$ for some $a \in E$. Then (a_n) is F-convergent to a

(ii) $\lim_{n,m\to\infty} d(a_n, a_m) = 0$ then the sequence (a_n) is F-Cauchy

(iii) For each $(a_n) \subset E$ implies (a_n) is F-convergent. Then, the space (E, d) is known as F-complete

Definition 6 (see [17]). Let (E, d) be an F-MS. A subset O of E is said to be F-open if, for every $a \in O$, there is some $r > 0$ such that $B(a, r) \subset O$, where

$$B(a, r) = \{x \in E : d(a, x) < r\}. \tag{4}$$

We say that a subset C of E is F-closed if $E \setminus C$ is F-open.

Definition 7 (see [17]). Let B be a nonempty subset of E and d be an F-metric, then, the following statements are equivalent:

(i) B is F-closed

(ii) For any sequence $(a_n) \subset B$, we have

$$\lim_{n\to\infty} d(a_n, a) = 0, a \in E \Longrightarrow a \in B. \tag{5}$$

Theorem 8 (see [17]). *Assume (E, d) is an F-complete F-MS, and let $h : E \longrightarrow E$ be a given map. Let there is some $k \in (0, 1)$ such that*

$$d(h(a), h(x)) \leq kd(a, x), (a, x) \in E \times E. \tag{6}$$

Then, $h a^* = a^*$ for at most one $a^* \in E$. Moreover, for any $a_0 \in E$, the sequence $(a_n) \subset E$ defined by $a_{n+1} = h(a_n)$, $n \in N$ is F-convergent to a^*.

Theorem 9 (see [8]). *Assume that E is a complete MS with metric d, and consider $h : E \longrightarrow E$ be a function such that*

$$d(h(a), h(x)) \leq \alpha d(a, x) + \beta d(a, h(x)) + \gamma d(a, h(x)), \tag{7}$$

for all $a, x \in E$, where α, β, and γ are nonnegative numbers satisfying $\alpha + \beta + \gamma < 1$. Then, h has a unique fixed point.

Lemma 10 (see [18]). *Let $(B(W), \|\cdot\|)$ is a Banach space and d is a metric defined by*

$$d(J, h) = \|J - h\| = \max_{a\in W} |J(a) - h(a)|, \quad J, h \in B(W). \tag{8}$$

Then, $(B(W), \|\cdot\|)$ is an F-MS.

2. Common Fixed Points Results of Reich Type F-Contractions

This section of the paper investigates the fixed point of single-valued F-contractions for two maps and single map in F-MS.

Theorem 11. *Assume $F \in \mathscr{G}$, (X, d) is an F-complete F-MS and $S, T : X \longrightarrow X$ are self-mappings. Assume that for non-negative functions α and β with $\max_{a,x\in X}\{\alpha(a, x) + 2\beta(a, x)\} < 1$, there is some $\tau > \sigma > 0$ such that*

$$\tau + F(d(Sa, Tx)) \leq F[\alpha(a, x)d(a, x) + \beta(a, x)\{d(x, Sa) + d(a, Tx)\}], \tag{9}$$

with $\min\{d(Sa, Tx), d(a, x)\} > 0$, for all $(a, x) \in X \times X$. Then, $Sy = Ty = y$ for some y in X.

Proof. Choose an arbitrary point a_0 and iterate a sequence (a_n) by

$$Sa_{2c} = a_{2c+1} \text{ and } Ta_{2c+1} = a_{2c+2} ; c = 0, 1, 2, \cdots \tag{10}$$

Using (9) and (10), we can write

$$\begin{aligned}
&\tau + F(d(a_{2c+1}, a_{2c+2})) \\
&\quad = \tau + F(d(Sa_{2c}, Ta_{2c+1})) \\
&\quad \le F[\alpha(a_{2c}, a_{2c+1})d(a_{2c}, a_{2c+1}) + \beta(a_{2c}, a_{2c+1}) \\
&\qquad \cdot \{d(a_{2c+1}, Sa_{2c}) + d(a_{2c}, Ta_{2c+1})\}] \\
&\quad = F[\alpha(a_{2c}, a_{2c+1})d(a_{2c}, a_{2c+1}) + \beta(a_{2c}, a_{2c+1})(a_{2c}, a_{2c+2})] \\
&\quad \le F[\alpha(a_{2c}, a_{2c+1})d(a_{2c}, a_{2c+1}) + \beta(a_{2c}, a_{2c+1}) \\
&\qquad \cdot \{(a_{2c}, a_{2c+1}) + (a_{2c+1}, a_{2c+2})\}] + \sigma.
\end{aligned} \tag{11}$$

Using (F1), we have

$$\begin{aligned}
d(a_{2c+1}, a_{2c+2}) &< \frac{\alpha(a_{2c}, a_{2c+1}) + \beta(a_{2c}, a_{2c+1})}{1 - \beta(a_{2c}, a_{2c+1})} d(a_{2c}, a_{2c+1}) \\
&= \lambda_1 d(a_{2c}, a_{2c+1}), \\
&\quad \text{say } \frac{\alpha(a_{2c}, a_{2c+1}) + \beta(a_{2c}, a_{2c+1})}{1 - \beta(a_{2c}, a_{2c+1})} = \lambda_1.
\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}
d(a_{2c+2}, a_{2c+3}) &< \frac{\alpha(a_{2c+1}, a_{2c+2}) + \beta(a_{2c+1}, a_{2c+2})}{1 - \beta(a_{2c+1}, a_{2c+2})} d(a_{2c+1}, a_{2c+2}) \\
&= \lambda_2 d(a_{2c+1}, a_{2c+2}).
\end{aligned} \tag{13}$$

Hence, for $\lambda = \max\{\lambda_1, \lambda_2, \cdots, \lambda_l\}$, we have

$$d(a_n, a_{n+1}) < \lambda d(a_{n-1}, a_n) \quad \text{for all} \quad n \in N, \tag{14}$$

which yields

$$\begin{aligned}
d(a_n, a_{n+1}) &< \lambda d(a_{n-1}, a_n) < \lambda^2 d(a_{n-2}, a_{n-1}) < \cdots \\
&< \lambda^n d(a_0, a_1), \; n \in N.
\end{aligned} \tag{15}$$

Using (15), we can write

$$\begin{aligned}
\sum_{k=n}^{m-1} d(a_k, a_{k+1}) &< \lambda^n \left[1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1}\right] d(a_0, a_1) \\
&\le \frac{\lambda^n}{1 - \lambda} d(a_0, a_1), m > n.
\end{aligned} \tag{16}$$

Since $\lim_{n \to \infty} (\lambda^n / 1 - \lambda) d(a_0, a_1) = 0$, for any $\delta > 0$, $\exists$ some $n' \in N$ such that

$$0 < \frac{\lambda^n}{1 - \lambda} d(a_0, a_1) < \delta, n \ge n'. \tag{17}$$

Further, let $(g, \sigma) \in \mathcal{G} \times [0, \infty)$ satisfies (d3) and $\epsilon > 0$ is fixed. By (F2), there is some $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow g(t) < g(\epsilon) - \sigma. \tag{18}$$

By (17) and (18), we write

$$g\left(\sum_{k=n}^{m-1} d(a_k, a_{k+1})\right) \le g\left(\frac{\lambda^n}{1 - \lambda} d(a_0, a_1)\right) < g(\epsilon) - \sigma, m > n \ge n'. \tag{19}$$

Using the above equation and (d3), we have

$$d(a_n, a_m) > 0, m > n > n' \Longrightarrow g(d(a_n, a_m)) < g(\epsilon). \tag{20}$$

This shows

$$d(a_n, a_m) < \epsilon, m > n \ge n'. \tag{21}$$

Hence, (a_n) is F-Cauchy in X. Since (X, d) is F-complete, $\exists\, y \in X$ such that (a_n) is F-convergent to y, i.e.,

$$\lim_{n \to \infty} d(a_n, y) = 0. \tag{22}$$

Now, assume that $d(Sy, y) > 0$. Then,

$$\tau + F(d(Sy, a_{2c+2})) \le F\left[\begin{array}{c} \alpha(y, a_{2c+1})d(y, a_{2c+1}) \\ +\beta(y, a_{2c+1})\{d(a_{2c+1}, Sy) + d(y, a_{2c+2})\} \end{array}\right]. \tag{23}$$

By (F1) and letting $c \longrightarrow \infty$, we have

$$(1 - \beta(y, a_{2c+1}))d(Sy, y) < 0, \tag{24}$$

which is a contradiction, as $\beta(a, x)$ is nonnegative. Hence, $d(Sy, y) = 0$, i.e., $Sy = y$.

Following the same steps, we get $Ty = y$. Hence, $Ty = Sy = y$.

Uniqueness: assume another common fixed z of the maps S and T and $y \ne z$.Then

$$\begin{aligned}
\tau + F(d(y, z)) &= F(d(Sy, Tz)) \\
&\le F\left[\begin{array}{c} \alpha(y, z)d(y, z) \\ +\beta(y, z)\{d(z, Sy) + d(y, Tz)\} \end{array}\right] \\
&= F\left[\begin{array}{c} \alpha(y, z)d(y, z) \\ +\beta(y, z)\{d(z, y) + d(y, z)\} \end{array}\right].
\end{aligned} \tag{25}$$

Using (F1), we get $(1 - \alpha(y, z) - 2\beta(y, z))\, d(y, z) < 0$, which is a contradiction. Hence, $y = z$.

Example 12. Assume that $X = X_c \coloneqq \{6c + 2/3, c \in N\}$,

$$d(X_c, X_k) = \begin{cases} 0, & \text{if } X_c = X_k, \\ e^{|X_c - X_k|}, & \text{if } X_c \ne X_k, \end{cases} \qquad F(X_c) = \ln(X_c), \tag{26}$$

and $S, T : X \longrightarrow X$ are defined by

$$TX_c = \begin{cases} X_1, & \text{if } c = 1, 2, \\ X_{c-1}, & \text{if } c > 2, \end{cases} \tag{27}$$

and

$$SX_n = \begin{cases} X_1, & \text{if } c = 1, \\ X_2, & \text{if } c = 2, \\ X_{c-2}, & \text{if } c > 4. \end{cases} \tag{28}$$

On the other hand, we define $\alpha, \beta : X \times X \longrightarrow [0, \infty)$ by

$$\alpha(X_c, X_k) = \begin{cases} 0 & \text{if } X_c = X_k, \\ e^{-1/2} & \text{if } X_c \neq X_k, \end{cases}$$
$$\beta(X_c, X_k) = \begin{cases} e^{-2}, & \text{if } X_k = SX_c \quad \text{and} \quad X_c = TX_k, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

One can verify that F fulfill conditions (F1) and (F2) and that d is an F-metric. Assume $c \neq k$, then

$$\begin{aligned} F(d(SX_c, TX_k)) &= \ln\left(e^{|X_{c-2}-X_{k-1}|}\right) = \ln\left(e^{|2(c-k)-2|}\right) \\ &< \ln\left(e^{-1/2}.e^{|2(c-k)|}\right) = F(\alpha(X_c, X_k)d(X_c, X_k)) \\ &< F\left[\begin{array}{c} \alpha(X_c, X_k)d(X_c, X_k) \\ +\beta(X_c, X_k)\{d(X_k, SX_c) + d(X_c, TX_k)\} \end{array}\right], \end{aligned} \tag{30}$$

whenever $\min\{d(SX_c, TX_k), d(X_c, X_k)\} > 0$. One can verify that $\max_{X_c, X_k}\{\alpha(X_c, X_k) + 2\beta(X_c, X_k)\} < 1$. For $\tau \in (\sigma, \ln(e^{-1/2}.e^{|2(c-k)|}/e^{|2(c-k)-2|})) = (\sigma, \ln(\sqrt[3]{e}))$, the inequality (9) is true. Furthermore, X_1 is a unique point such that $SX_1 = TX_1 = X_1$.

Choosing $\alpha(a, x) = 0$ in the previous result, a result of Chatterjea type F-contraction is obtained.

Corollary 13. *Assume $F \in \mathcal{G}$,(X, d) is an F-complete F-MS and $S, T : X \longrightarrow X$ are self-maps. Assume that for $\beta : X \times X \longrightarrow [0, 1)$, there is some $\tau > \sigma$ such that*

$$\tau + F(d(Sa, Tx)) \leq F\left[\frac{\beta(a, x)}{2}(d(x, Sa) + d(a, Tx))\right], \tag{31}$$

with $\min\{d(Sa, Tx), d(a, Sa), d(x, Tx)\} > 0$, *for all* $(a, x) \in X \times X$. *Then,* $Ty = Sy = y$ *for a unique* y *in* X.

Substituting S with T in Corollary 13, we obtain the following result.

Corollary 14. *Assume that (X, d) is an F-complete F-MS, $F \in \mathcal{G}$ and $T : X \longrightarrow X$ is a self-map. Assume that for $\beta : X \times X \longrightarrow [0, 1)$, there is some $\tau > \sigma$ such that*

$$\tau + F(d(Ta, Tx)) \leq F\left[\frac{\beta(a, x)}{2}(d(a, Tx) + d(x, Ta))\right], \tag{32}$$

with $\min\{d(Ta, Tx), d(a, Ta) + d(x, Tx)\} > 0$, *for all* $(a, x) \in X \times X$. *Then, T has at most one fixed point in X.*

3. Investigation of Fixed Points of F-Contractions on F-Closed Balls

This section of the paper investigates a fixed point of single-valued F-contractions for two maps and single map imposed only on a F-closed subset of F-MS.

Definition 15. Assume an F-MS (X, d) which is F-complete, $F \in \mathcal{G}$, and $S, T : X \longrightarrow X$ are self-maps, let α,β be non-negative functions with $\max_{a,x \in X}\{\alpha(a, x) + 2\beta(a, x)\} < 1$. Then, T is named as Reich type F-contraction on $B(a_0, r) \subseteq X$ if there is some $\tau > \sigma$ satisfying

$$\begin{aligned} &\tau + F(d(Sa, Tx)) \\ &\leq F\left[\begin{array}{c} \alpha(a, x)d(a, x) \\ +\beta(a, x)\{d(x, Sa) + d(a, Tx)\} \end{array}\right], \forall a, x \in B(a_0, r). \end{aligned} \tag{33}$$

Theorem 16. *Assume $(g, \sigma) \in \mathcal{G} \times [0, \infty)$, an F-MS (X, d) which is F-complete and T is a Reich type F-contraction on $B(a_0, r)$. Assume that for $a_0 \in X$ and $r > 0$, the conditions given below are fulfilled:*

(a) $B(a_0, r)$ is F-closed

(b) $d(a_0, a_1) \leq (1 - \lambda)r$, for $a_1 \in X$ and $\lambda = \alpha(a, x) + \beta(a, x)/1 - \beta(a, x)$

(c) $\exists\ 0 < \epsilon < r$ such as $g((1 - \lambda^{k+1})r) \leq g(\epsilon) - \sigma$, where $k \in N$

Then, $Ty^* = Sy = y$ for some y in $B(a_0, r)$.

Proof. Choose and arbitrary point a_0 and iterate a sequence (a_n) by

$$Ta_{2c} = a_{2c+1} \text{and } Sa_{2c+1} = a_{2c+2}\ ; c = 0, 1, 2, \cdots. \tag{34}$$

Using mathematical induction, we show that a_n is in $B(a_0, r)$ for all $n \in N$. By hypothesis

$$d(a_0, a_1) < r. \tag{35}$$

Therefore, $a_1 \in B(a_0, r)$. Assume $a_2, \cdots, a_k \in B(a_0, r)$ for some $k \in N$. Now, if $a_{2c+1} \leq a_k$, then by (33), we can write

$$\begin{aligned}
&\tau + F(d(a_{2c}, a_{2c+1})) \\
&\quad = \tau + F(d(Sa_{2c-1}, Ta_{2c})) \\
&\quad \leq F\left[\begin{array}{c} \alpha(a_{2c-1}, a_{2c})d(a_{2c-1}, a_{2c}) \\ +\beta(a_{2c-1}, a_{2c})\{d(a_{2c}, Sa_{2c-1}) + d(a_{2c-1}, Ta_{2c})\} \end{array}\right] \\
&\quad = F[\alpha(a_{2c-1}, a_{2c})d(a_{2c-1}, a_{2c}) + \beta(a_{2c-1}, a_{2c})d(a_{2c-1}, a_{2c+1})] \\
&\quad \leq F\left[\begin{array}{c} \alpha(a_{2c-1}, a_{2c})d(a_{2c-1}, a_{2c}) \\ +\beta(a_{2c-1}, a_{2c})\{d(a_{2c-1}, a_{2c}) + d(a_{2c}, a_{2c+1})\} \end{array}\right] + \sigma.
\end{aligned} \tag{36}$$

From (F1), we have

$$\begin{aligned}
d(a_{2c}, a_{2c+1}) &< \frac{\alpha(a_{2c-1}, a_{2c}) + \beta(a_{2c-1}, a_{2c})}{1 - \beta(a_{2c-1}, a_{2c})} d(a_{2c-1}, a_{2c}) \\
&= \lambda_1 d(a_{2c-1}, a_{2c}).
\end{aligned} \tag{37}$$

On the other hand, if $a_{2c} \leq a_k$

$$\begin{aligned}
d(a_{2c-1}, a_{2c}) &< \frac{\alpha(a_{2c-1}, a_{2c}) + \beta(a_{2c-1}, a_{2c})}{1 - \beta(a_{2c-1}, a_{2c})} d(a_{2c-2}, a_{2c-1}) \\
&= \lambda_2 d(a_{2c-2}, a_{2c-1}).
\end{aligned} \tag{38}$$

Continuity this way, for $\lambda = \max\{\lambda_1, \lambda_2, \cdots, \lambda_l\}$, we deduce from inequality (37) and (38) that

$$d(a_{2c}, a_{2c+1}) < \lambda d(a_{2c-1}, a_{2c}) < \cdots < \lambda^{2c} d(a_0, a_1), \tag{39}$$

$$d(a_{2c-1}, a_{2c}) < \lambda d(a_{2c-2}, a_{2c-1}) < \cdots < \lambda^{2c-1} d(a_0, a_1). \tag{40}$$

From (39) and (40), we write

$$d(a_k, a_{k+1}) \leq \lambda^k d(a_0, a_1) \quad \text{for some} \quad k \in N. \tag{41}$$

Now, using (41), we have

$$\begin{aligned}
g(d(a_0, a_{k+1})) &\leq g\left(\sum_{i=1}^{k+1} d(a_{i-1}, a_i)\right) + \sigma \\
&= g(d(a_0, a_1) + \cdots + d(a_k, a_{k+1})) + \sigma \\
&\leq g\left[\left(1 + \lambda + \lambda^2 + \cdots + \lambda^k\right) d(a_0, a_1)\right] + \sigma \\
&= g\left[\frac{1 - \lambda^{k+1}}{1 - \lambda} d(a_0, a_1)\right] + \sigma.
\end{aligned} \tag{42}$$

From (b) and (c), we obtain

$$g(d(a_0, a_{k+1})) \leq g\left(\left(1 - \lambda^{k+1}\right) r\right) + \sigma \leq g(\epsilon) < g(r). \tag{43}$$

Hence by (F1), we deduce that

$$a_{k+1} \in B(a_0, r). \tag{44}$$

Therefore, $a_n \in B(a_0, r)$ for all $n \in N$. Now for $p \in N$,we have by (33)

$$\begin{aligned}
\tau + F\left(d\left(a_{2p+1}, a_{2p+2}\right)\right) &= \tau + F\left(d\left(Sa_{2p}, Ta_{2p+1}\right)\right) \\
&\leq F\left[\alpha\left(a_{2p}, a_{2p+1}\right) d\left(a_{2p}, a_{2p+1}\right)\right. \\
&\quad + \beta\left(a_{2p}, a_{2p+1}\right)\left\{d\left(a_{2p+1}, Sa_{2p}\right)\right. \\
&\quad \left.\left. + d\left(a_{2p}, Ta_{2p+1}\right)\right\}\right] \\
&= F\left[\alpha\left(a_{2p}, a_{2p+1}\right) d\left(a_{2p}, a_{2p+1}\right)\right. \\
&\quad \left. + \beta\left(a_{2p}, a_{2p+1}\right) d\left(a_{2p}, a_{2p+2}\right)\right] \\
&\leq F\left[\alpha\left(a_{2p}, a_{2p+1}\right) d\left(a_{2p}, a_{2p+1}\right)\right. \\
&\quad \left. + \beta\left(a_{2p}, a_{2p+1}\right)\left\{\begin{array}{c} d\left(a_{2p}, a_{2p+1}\right) \\ +d\left(a_{2p+1}, a_{2p+2}\right) \end{array}\right\}\right] + \sigma.
\end{aligned} \tag{45}$$

Using (a) and repeating the steps done in heorem 11, we get to the conclusion that (a_n) is F-Cauchy to a point y in $B(a_0, r)$. Proceeding in a similar way as in heorem 11, we obtain that $y = Sy = Ty$

Substituting S by T in the previous theorem, the following result is obtained.

Corollary 17. *Assume $(g, \sigma) \in \mathcal{G} \times [0, \infty)$, $(F, \tau) \in \mathcal{G} \times (0, \infty)$, (X, d) is an F-complete F-MS and $T : X \longrightarrow X$ is a self-map. Let $\alpha, \beta : X \times X \longrightarrow [0, \infty)$ such that $\max_{a,x \in a}\{\alpha(a, x) + 2\beta(a, x)\} < 1$. Assume that for $a_0 \in X$ and $r > 0$, the below conditions are fulfilled:*

(a) $B(a_0, r) \subseteq X$ is F-closed

(b) $d(a_0, a_1) \leq (1 - \lambda)r$, for $a_1 \in X$ and $\lambda = \alpha(a, x) + \beta(a, x)/1 - \beta(a, x)$

(c) $\exists 0 < \epsilon < r$ such as $g((1 - \lambda^{k+1})r) \leq g(\epsilon) - \sigma$, where $k \in N$

(d) $\tau + F(d(Ta, Tx)) \leq F[\alpha(a, x)d(a, x) + \beta(a, x)(d(x, Ta) + d(a, Tx))]$, $\forall a, x \in B(a_0, r)$

with $\min\{d(Ta, Tx), d(a, x)\} > 0$.Then there is a unique y in $B(a_0, r)$ such that $Ty = y$.

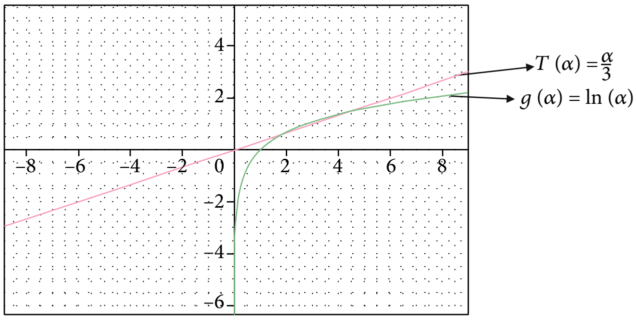

Figure 1

Example 18. Let $X = R_0^+$ and $F(a) = g(a) = \ln a$. Define $T : X \longrightarrow X$ by

$$Ta = \begin{cases} \frac{a}{3}, \text{ if } a \in [0, 1], \\ a^2, \text{ if } a \in (1, \infty). \end{cases} \tag{46}$$

See Figure 1. Define d by

$$d(a, x) = \begin{cases} (a - x)^2, & \text{if} \quad (a, x) \in [0, 1] \times [0, 1], \\ |a - x|, & \text{if} \quad (a, x) \notin [0, 1] \times [0, 1]. \end{cases} \tag{47}$$

One can verify that F fulfill conditions (F1) and (F2) and that d is and F-metric.

On the other hand, define $\alpha, \ : X \times X \longrightarrow [0, \infty)$ by

$$\alpha(a, x) = \begin{cases} 0, & \text{if} \quad a = x, \\ e^{-3/7}, & \text{if} \quad a \neq x, \end{cases}$$
$$\beta(a, x) = \begin{cases} e^{-2}, & \text{if } x = Ta \quad \text{and} \quad a = Tx, \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

Fix $a_0 = r = 1/2$, then $B(a_0, r) = [0, 1]$. Clearly, $B(a_0, r)$ is F-closed hence is satisfied. Now, since $a_0 \neq a_1$, therefore, $\alpha(a, x) = e^{-3/7}$ and $\beta(a, x) = 0$, which implies that $\lambda = \alpha(a, x)$ and

$$d(a_0, a_1) = d(a_0, Ta_0) = \left(\frac{1}{2} - \frac{1}{6}\right)^2 < \left(1 - e^{-3/7}\right)\frac{1}{2} = (1 - \lambda)r. \tag{49}$$

Therefore, condition (b) is obeyed. Moreover, assume $k = 1$, then $g((1 - \lambda^{k+1})r) = \ln((1 - (e^{-3/7})^2)(1/2)) = \ln(2/5) - \ln(2/7) = g(\varepsilon) - \sigma$ is satisfied. i.e., $\varepsilon = (2/5) \leq (1/2) = r$ and $\sigma = \ln(2/7)$. In a similar way, for each $k \in N$, $\exists$ some $0 < \epsilon < r$ and σ satisfying condition (c). Now checking for condition (d), we have two cases:

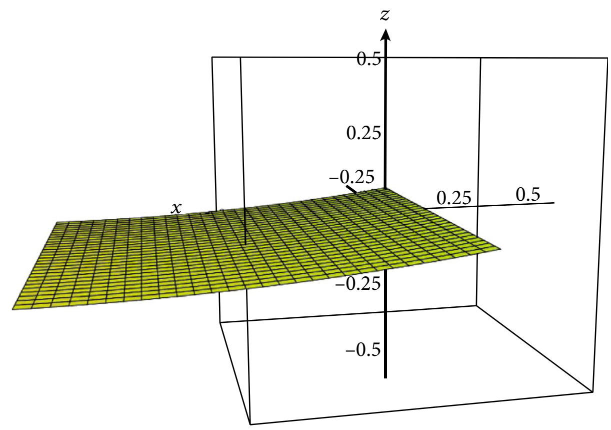

Figure 2

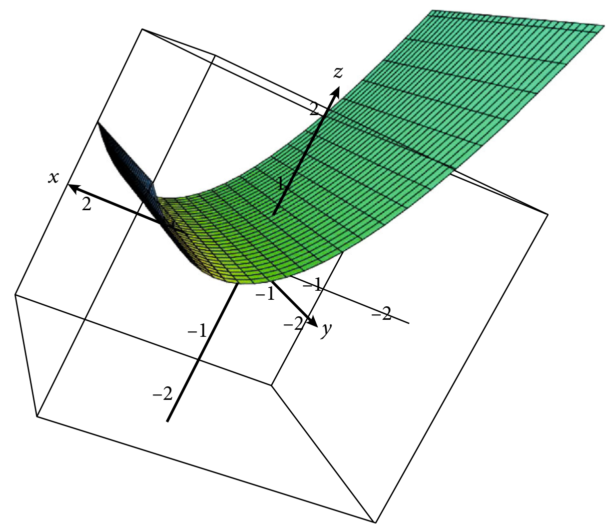

Figure 3

Case 19. If $(a, x) \in B(a_0, r) \times B(a_0, r)$, then

$$\begin{aligned} F(d(Ta, Tx)) &= \ln\left(\left(\frac{a}{3} - \frac{x}{3}\right)^2\right) < \ln\left(e^{-3/7}(a - x)^2\right) \\ &= F[\alpha(a, x)d(a, x) + \beta(a, x)\{d(a, Ta) + d(x, Tx)\}]. \end{aligned} \tag{50}$$

Figures 2–4 illustrate this inequality, where

$$\begin{aligned} \tau &\in \left(\frac{2}{7}, \ln\left(e^{-3/7}(a - x)^2\right) - \ln\left(\left(\frac{a}{3} - \frac{x}{3}\right)^2\right)\right) \\ &= \left(\frac{2}{7}, \ln\frac{e^{-3/7}}{1/9}\right) = \left(\frac{2}{7}, \ln\left(9e^{-3/7\frac{3}{7}}\right)\right). \end{aligned} \tag{51}$$

Therefore, for all $(a, x) \in B(a_0, r) \times B(a_0, r)$, condition (d) is also satisfied.

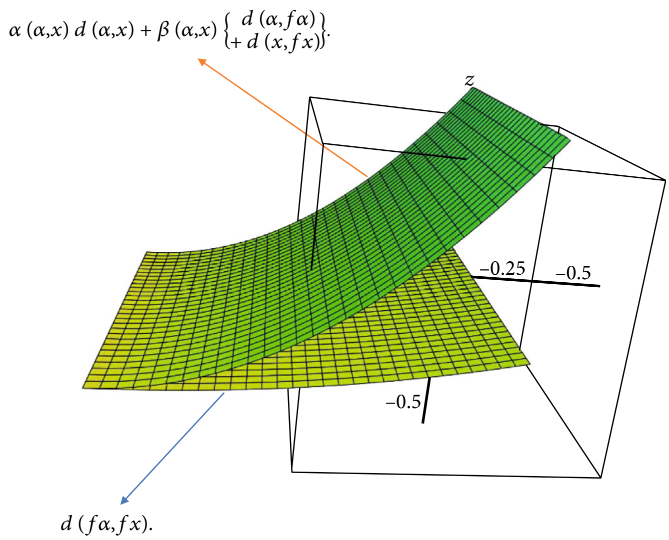

FIGURE 4

Case 20. If $(a, x) \notin B(a_0, r) \times B(a_0, r)$, e.g., $a = 2$ and $x = 3$, then

$$\begin{aligned} F(d(Ta, Tx)) &= \ln \left(\left|2^2 - 3^2\right|\right) > \ln \left(e^{-3/7}|(2-3)|\right) \\ &= F(\alpha(a, x)d(a, x)) \\ &= F\begin{bmatrix} \alpha d(a, x) \\ +\beta d(a, Ta) + \gamma d(x, Tx) \end{bmatrix}.F(d(Ta, Tx)) \\ &= \ln \left(\left|2^2 - 3^2\right|\right) > \mathrm{In}\left(e^{-3/7}|2-3|\right) \\ &= F(\alpha(a, x)d(a, x)) \\ &= F\begin{bmatrix} \alpha d(a, x) \\ +\beta d(a, Ta) + \gamma d(x, Tx) \end{bmatrix}. \end{aligned} \tag{52}$$

Hence, condition (b) holds only for $B(a_0, r)$ and not on $X \times X$. Moreover, $0 \in B(a_0, r)$ is the fixed point of T. Given below Figure 1 shows two maps $f(a) = a/3$ and $g(a) = \ln(a)$

Figures 2 and 3 are 3D graphs of the functions $z = d(fa, fx) = ((a/3) - (x/3))^2$ and $z = d(a, x) = (a - x)^2$, respectively.

Multiplying $\alpha(a, x) = e^{-3/7}$ to $d(a, x)$ of the contractive inequality and combining Figures 2 and 3, we get the below 3D graph which clearly demonstrate that the graph of $\alpha(a, x)d(a, x)$ is dominating the graph of $d(fa, fx)$.

$$\alpha(a, x)d(a, x) + \beta(a, x)\left\{\begin{array}{c} d(a, fa) \\ +d(x, fx) \end{array}\right\}. \tag{53}$$

$$d(fa, fx).$$

As we see in Figure 1, *lnx* is an increasing function. Therefore, it will not change the inequality, i.e., the right side of the inequality will still be dominant. Note that z -axis represents the values of the function g, and it can be observed that for every value of a and x, $F(d(fa, fx)) < F(\alpha(a, x)d(a, x))$ and hence satisfy the inequality of the above example.

Corollary 21. *Assume $(g, \sigma) \in \mathcal{G} \times [0, \infty)$, $(F, \tau) \in \mathcal{G} \times (0, \infty)$ such that $\sigma < \tau$ and (X, d) is an F-complete F-MS. Let S. $T : X \longrightarrow X$ are self-maps and $k : X \times X \longrightarrow [0, 1)$. Assume that for $a_0 \in X$ and $r > 0$, the below conditions are fulfilled:*

(a) $B(a_0, r) \subseteq X$ is F-closed

(b) $\tau + F(d(Sa, Tx)) \leq F[k(a, x)/2(d(a, Sa) + d(x, Tx))]$, for all $a, x \in B(a_0, r)$

(c) $d(a_0, a_1) \leq (1 - \lambda)r$, for $a_1 \in X$ and $\lambda = k(a, x)/2 - k(a, x)$

(d) $\exists\ 0 < \epsilon < r$ such as $g((1 - \lambda^{n+1})r) \leq g(\epsilon) - \sigma$, where $n \in N$

Then $Sy = Ty = y$ for a unique y in $B(a_0, r)$.

4. Application to Functional Equations

This section discusses the application of our results in finding a common solution of functional equations that are used in dynamic programming.

The study of dynamic programming splits into two parts. A state space is a set of parameters of various states, i.e., initial states, transitional states, and action states. On the other hand, a decision space is a series of actions taking place for finding the possible solution to the indicated problem. The problem of dynamic program is transformed into functional equations:

$$g(x) = \max_{y \in X} \{H(x, y) + J(x, y, g(\eta(x, y)))\} \quad \text{for} \quad x \in A, \tag{54}$$

$$f(x) = \max_{y \in X}\{H(x,y) + K(x,y,g(\eta(x,y)))\} \quad \text{for} \quad x \in A, \tag{55}$$

where U and V are Banach spaces such that $A \subseteq U$ and $X \subseteq V$ and

$$\begin{aligned} \eta &: A \times X \longrightarrow A, \\ H &: A \times X \longrightarrow R, \\ J, K &: A \times X \times R \longrightarrow R. \end{aligned} \tag{56}$$

Assume A and X are state space and decision spaces, respectively. Assume $W(A)$ denotes a set of all-bounded real-valued maps on A. Let $h \in W(A)$ and say $\|h\| = \max_{x \in A}|h(x)|$. Then, $(W(A), \|.\|)$ is a Banach space and d is the metric defined as

$$d(h,k) = \max_{x \in A}|h(x) - k(x)|. \tag{57}$$

Suppose the following conditions are satisfied:

(C_1): H, J, and K are bounded.

(C_2): For $x \in A$ and $h \in W(A)$, define $P, Q : W(A) \longrightarrow W(A)$ by

$$\begin{aligned} Ph(x) &= \max_{y \in X}\{H(x,y) + J(x,y,h(\eta(x,y)))\} \text{for } x \in A, \\ Qh(x) &= \max_{y \in X}\{H(x,y) + K(x,y,h(\eta(x,y)))\} \text{for } x \in A. \end{aligned} \tag{58}$$

Observe that the functions H, J, and K are bounded hence P and Q are well-defined.

(C_3): For $\sigma < \tau : R_+ \longrightarrow R_+$, $(x,y) \in A \times X$, $h, k \in W(A)$ and $t \in A$, we have

$$|J(x,y,h(t)) - K(x,y,k(t))| \le e^{-\tau}M(h,k), \tag{59}$$

where

$$M(h,k) = \alpha d(h,k) + \beta d(k,Ph) + \gamma d(h,Qk) \tag{60}$$

for $\alpha, \beta, \gamma \in [0, \infty)$ such that $\alpha + \beta + \gamma < 1$, where $\min\{d(Ph, Qk), d(h,k)\} > 0$.

Based on the above hypothesis, we present the below theorem.

Theorem 22. *Let $(C_1) - (C_3)$ are satisfied, then at most one bounded common solution exists for Equations (54) and (55).*

Proof. We know by Lemma 10 that $(W(A), d)$ is an F-complete F-MS, d is stated by (57) and (C_1) say that P and Q are self-maps on $W(A)$. Choose any positive number ω and $h_1, h_2 \in W(A)$. Take $x \in A$ and $y_1, y_2 \in X$ such that

$$Ph_c < H(x,y_c) + J(x, y_c, h_c(\eta(x,y_c))) + \omega, \tag{61}$$

$$Qh_c < H(x,y_c) + K(x, y_c, h_c(\eta(x,y_c))) + \omega, \tag{62}$$

$$Ph_1 \ge H(x,y_2) + J(x, y_2, h_1(\eta(x,y_2))), \tag{63}$$

$$Qh_2 \ge H(x,y_1) + K(x, y_1, h_2(\eta(x,y_1))). \tag{64}$$

Then using (61) and (64), we get

$$\begin{aligned} &Ph_1(x) - Qh_2(x) \\ &\quad < J(x, y_1, h_1(\eta(x,y_1))) - J(x, y_1, h_2(\eta(x,y_1))) + \omega \\ &\quad \le |K(x, y_1, h_1(\eta(x,y_1))) - K(x, y_1, h_2(\eta(x,y_1)))| + \omega \\ &\quad \le e^{-\tau}M(h_1(x), h_2(x)) + \omega. \end{aligned} \tag{65}$$

Similarly, by (62) and (63), we get

$$Qh_2(x) - Ph_1(x) < e^{-\tau}M(h_1(x), h_2(x)) + \omega. \tag{66}$$

Combining the above two inequalities, we get

$$|Ph_1(x) - Qh_2(x)| < e^{-\tau}M(h_1(x), h_2(x)) + \omega, \tag{67}$$

for all $\omega > 0$. Hence,

$$d(Ph_1(x), Qh_2(x)) \le e^{-\tau}M(h_1(x), h_2(x)), \tag{68}$$

that is,

$$d(Ph_1, Qh_2) \le e^{-\tau}M(h_1, h_2), \tag{69}$$

for each $x \in A$. Applying logarithms on both sides, we get

$$\ln(d(Ph_1, Qh_2)) \le \ln(e^{-\tau}M(h_1, h_2)). \tag{70}$$

This shows that $F : R_+ \longrightarrow R$ defined as $F(x) = \ln x$ is a member of $\mathscr{G}$, and

$$\tau + F(d(Ph_1, Qh_2)) \le F(M(h_1, h_2)). \tag{71}$$

As every condition in Theorem 11 is fulfilled, therefore using heorem 11 P and T have a unique common and bounded solution of the Equations (54) and (55).

5. Conclusion

This article instigated the establishment of fixed point result of Reich type F-contractions, while imposing the contractive conditions on both the whole F-MS as well as only on a subset (F-closed ball) of the F-MS. However, the constants in the inequality of Reich type contractive conditions are replaced by real-valued functions. The validity of the inequality is verified graphically, making the results clearer and more certain. At last, the use of our results in assuring the existence of a solution to the functional equation is described.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no competing interests

Authors' Contributions

All the authors contributed equally to the research.

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