Abstract and Applied Analysis

Advanced Theoretical and Applied Studies of Fractional Differential Equations

Guest Editors: Dumitru Baleanu, Juan J. Trujillo, and Bashir Ahmad

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Editorial Board

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Editorial

Advanced Theoretical and Applied Studies of Fractional Differential Equations

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Fractional calculus (fractional derivatives and fractional integrals together with their applications) is undergoing a rapid development, from both theoretical as well as applied viewpoints. Such a tool is an emergent topic, and within its framework new concepts and applications, which lead to a challenging insight, have appeared during the last few decades.

It may be the nonlocal property of fractional operators that could have motivated the rising of numerous new and important applications in many branches of applied sciences and engineering. Among other applications, modeling of the dynamics of processes through complex media using fractional calculus is an important one and has significantly contributed to the popularity of the subject.

Therefore, the goal of this special issue was focused on related topics with high current interest, both from theoretical and practical points of view.

We received 70 manuscripts and only 35 highest quality papers were accepted from the areas of mathematics, physics, engineering, biology, and other fields. This special issue contains the research papers on the existence theory of initial and boundary value problems of fractional order, numerical solutions of fractional differential equations, and modeling of real-world problems using fractional calculus.

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Research Article

The Positive Properties of Green's Function for Fractional Differential Equations and Its Applications

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We consider the properties of Green's function for the nonlinear fractional differential equation boundary value problem: $\mathbf{D}_{0+}^{\alpha}u(t)+f(t,u(t))+e(t)=0$, $0<t<1$, $u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0$, $u(1)=\beta u(\eta)$, where $n-1<\alpha\le n$, $n\ge 3$, $0<\beta\le 1$, $0\le\eta\le 1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. Here our nonlinearity f may be singular at $u=0$. As applications of Green's function, we give some multiple positive solutions for singular boundary value problems by means of Schauder fixed-point theorem.

1. Introduction

Fractional differential equations have been of great interest recently. This is due to the intensive development of the theory of fractional calculus itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in selfsimilar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science. For details, see [1–10].

It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations on terms of special functions. Recently, there are some papers dealing with the existence and multiplicity of solution to the nonlinear fractional differential equations boundary value problems, see [11–17].

Bai [14] investigated the existence and uniqueness of positive solutions for a nonlocal boundary value problem of fractional differential equation

$$\begin{gathered}\mathbf{D}_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\\ u(0)=0,\qquad u(1)=\beta u(\eta),\end{gathered}\tag{1}$$

by contraction map principle and fixed-point index theory, where $1<\alpha\le 2$, $0<\beta\eta^{\alpha-1}<1$, $0<\eta<1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. The function f is continuous on $[0,1]\times[0,\infty)$.

Li et al. [17] investigated the the existence and multiplicity results of positive solutions for the nonlinear differential equation of fractional order

$$\begin{gathered}\mathbf{D}_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\\ u(0)=0,\qquad \mathbf{D}_{0+}^{\beta}u(1)=a\mathbf{D}_{0+}^{\beta}u(\xi),\end{gathered}\tag{2}$$

by using some fixed-point theorems, where $1<\alpha\le 2$, $0<\beta\le 1$, $0\le a\le 1$, $\xi\in(0,1)$, $a\xi^{\alpha-\beta-2}\le 1-\beta$, $0\le\alpha-\beta-1$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative.

Xu and Fei [18] considered the properties of Green's function for the nonlinear fractional differential equation boundary value problem

$$\begin{gathered}\mathbf{D}_{0+}^{\alpha}u(t)+f(t,u(t))+e(t)=0,\quad 0<t<1,\\ u(0)=0,\qquad \mathbf{D}_{0+}^{\beta}u(1)=a\mathbf{D}_{0+}^{\beta}u(\xi),\end{gathered}\tag{3}$$

where $1<\alpha\le 2$, $0<\beta\le 1$, $0\le a\le 1$, $0<\xi<1$, $\alpha-\beta-1\ge 0$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. Here the nonlinearity f may be singular at $u=0$.

As applications of Green's function, they give some existence of positive solutions for singular boundary value problems by means of Schauder fixed-point theorem. Here they consider the case: $\gamma_* = 0, \gamma_* \geq 0, \gamma^* \leq 0$.

In this paper, we consider the singular boundary value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) + e(t) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \beta u(\eta), \tag{4}$$

where $n-1 < \alpha \leq n$, $n \geq 3$, $0 < \beta, \eta < 1$ is a real constant, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. We will deduce a property of Green's function. The result we establish in Section 2 can be stated as follow.

Theorem 1. *The Function $G(t,s)$ defined by* (12) *is continuous and satisfies*

$$\frac{Mt^{\alpha-1}s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} \leq G(t,s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, \quad \text{for } t,s \in [0,1], \tag{5}$$

where $0 < M = \min\{1-\beta\eta^{\alpha-1}, \beta\eta^{\alpha-2}(1-\eta), \beta\eta^{\alpha-1}\} < 1$.

In this paper, we give some existence of positive solutions for singular boundary value problems by means of Schauder fixed-point theorem for the case: $\gamma_* = 0$, $\gamma_* \geq 0$, $\gamma^* \leq 0$, $\gamma_* < 0 < \gamma^*$.

The paper is organized as follows. In Section 2, we state some known results and give a property of Green's function. In Section 3, using Schauder fixed-point theorem, the existence of positive solutions to singular problems are obtained.

2. Background Materials

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory.

Definition 2 (see [7]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \tag{6}$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 3 (see [7]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \tag{7}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0,\infty)$.

From the definition of Riemann-Liouville's derivative, we can obtain the statement.

Lemma 4 (see [7]). *Let $\alpha > 0$, if one assumes that $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0 \tag{8}$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, $C_i \in R$, $i = 1, 2, \ldots, N$, as unique solutions, where N is the smallest integer greater than or equal to α.

Lemma 5 (see [7]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then,*

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{9}$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$, N is the smallest integer greater than or equal to α.

Lemma 6. *Given $h \in C(0,1)$ the problem*

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \ n-1 < \alpha \leq n, \ n \geq 3,$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,$$
$$u(1) = \beta u(\eta), \quad 0 < \beta, \ \eta < 1 \tag{10}$$

is equivalent to

$$u(t) = \int_0^1 G(t,s) h(s)\, ds, \tag{11}$$

where

$$G(t,s) = \begin{cases} \left([t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} \right. \\ \quad \left. -(t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right)\right) \\ \quad \times\left(\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)\right)^{-1}, \\ \quad 0 \leq s \leq t \leq 1, \ s \leq \eta, \\ \left([t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right)\right) \\ \quad \times\left(\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)\right)^{-1}, \\ \quad 0 < \eta \leq s \leq t \leq 1, \\ \left([t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1}\right) \\ \quad \times\left(\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)\right)^{-1}, \\ \quad 0 \leq t \leq s \leq \eta < 1, \\ \left([t(1-s)]^{\alpha-1}\right) \\ \quad \times\left(\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)\right)^{-1}, \\ \quad 0 \leq t \leq s \leq 1, \ \eta \leq s. \end{cases} \tag{12}$$

Proof. We can apply Lemma 5 to reduce (10) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} h(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, \tag{13}$$

for some $C_1, C_2, \dots, C_n \in R$. Consequently, the general solution of (10) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}. \tag{14}$$

By $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, this is $C_2 = C_3 = \cdots = C_n = 0$.

On the other hand, $u(1) = \beta u(\eta)$ combining with

$$\begin{aligned} u(1) &= -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\, ds + C_1, \\ u(\eta) &= -\frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s)\, ds + C_1 \eta^{\alpha-1} \end{aligned} \tag{15}$$

yields

$$\begin{aligned} C_1 = {} & \frac{1}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \\ & - \frac{\beta}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds. \end{aligned} \tag{16}$$

Therefor, the unique solution of problem (10) is

$$\begin{aligned} u(t) = {} & -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \\ & + \frac{1}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \\ & - \frac{\beta}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(t(\eta-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds. \end{aligned} \tag{17}$$

For $t \le \eta$, we have

$$\begin{aligned} u(t) = {} & -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \\ & + \frac{1}{1-\beta\eta^{\alpha-1}} \left[\left(\int_0^t + \int_t^\eta + \int_\eta^1 \right) \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right] \\ & - \frac{\beta}{1-\beta\eta^{\alpha-1}} \left[\left(\int_0^t + \int_t^\eta \right) \frac{(t(\eta-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right] \\ = {} & \int_0^t \Big(\Big([t(1-s)]^{\alpha-1} - \beta [t(\eta-s)]^{\alpha-1} \\ & \qquad - (t-s)^{\alpha-1} \left(1-\beta\eta^{\alpha-1}\right) \Big) \\ & \qquad \times \left(\left(1-\beta\eta^{\alpha-1}\right) \Gamma(\alpha) \right)^{-1} \Big) h(s)\, ds \\ & + \int_t^\eta \frac{[t(1-s)]^{\alpha-1} - \beta [t(\eta-s)]^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds \\ & + \int_\eta^1 \frac{[t(1-s)]^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds \\ = {} & \int_0^1 G(t,s) h(s)\, ds. \end{aligned} \tag{18}$$

For $t \ge \eta$, we have

$$\begin{aligned} u(t) = {} & -\left[\left(\int_0^\eta + \int_\eta^t \right) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right] \\ & - \frac{\beta}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(t(\eta-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \\ & + \frac{1}{1-\beta\eta^{\alpha-1}} \left[\left(\int_0^\eta + \int_\eta^t + \int_t^1 \right) \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right] \\ = {} & \int_0^\eta \Big(\Big([t(1-s)]^{\alpha-1} - \beta [t(\eta-s)]^{\alpha-1} \\ & \qquad - (t-s)^{\alpha-1} \left(1-\beta\eta^{\alpha-1}\right) \Big) \\ & \qquad \times \left(\left(1-\beta\eta^{\alpha-1}\right) \Gamma(\alpha) \right)^{-1} \Big) h(s)\, ds \\ & + \int_\eta^t \frac{[t(1-s)]^{\alpha-1} - \left(1-\beta\eta^{\alpha-1}\right)(t-s)^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds \\ & + \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds \\ = {} & \int_0^1 G(t,s) h(s)\, ds. \end{aligned} \tag{19}$$

The proof is complete. □

Proof of Theorem 1. It is easy to prove that $G(t,s)$ is continuous on $[0,1] \times [0,1]$, here we omit it. In the following, we consider $(1-\beta\eta^{\alpha-1})\Gamma(\alpha)G(t,s)$. When $0 \le s < t \le 1$, $s < \eta$, let

$$\begin{aligned} g_1(t) = {} & [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1} (\eta-s)^{\alpha-1} \\ & - (t-s)^{\alpha-1} \left(1-\beta\eta^{\alpha-1}\right). \end{aligned} \tag{20}$$

We have

$$\begin{aligned} g_1(t) = {} & [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1} (\eta-s)^{\alpha-1} \\ & - (t-s)^{\alpha-1} \left(1-\beta\eta^{\alpha-1}\right) \\ = {} & [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1} t^{\alpha-1} \left(1-\frac{s}{\eta}\right)^{\alpha-1} \\ & - t^{\alpha-1} \left(1-\frac{s}{t}\right)^{\alpha-1} \left(1-\beta\eta^{\alpha-1}\right) \end{aligned}$$

$$\begin{aligned}
&\geq [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1}t^{\alpha-1}\left(1-\frac{s}{\eta}\right)^{\alpha-1} \\
&\quad - t^{\alpha-1}(1-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right) \\
&= t^{\alpha-1}\beta\eta^{\alpha-1}\left[(1-s)^{\alpha-1} - \left(1-\frac{s}{\eta}\right)^{\alpha-1}\right] \\
&= t^{\alpha-1}\beta\eta^{\alpha-1}\left[(1-s)^{\alpha-1} - \left(1-\frac{s}{\eta}\right)^{\alpha-2}\left(1-\frac{s}{\eta}\right)\right] \\
&\geq t^{\alpha-1}\beta\eta^{\alpha-1}(1-s)^{\alpha-2}\left[1-s-\left(1-\frac{s}{\eta}\right)\right] \\
&= t^{\alpha-1}\beta\eta^{\alpha-1}(1-s)^{\alpha-2}s\frac{1-\eta}{\eta} \\
&\geq \beta\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}s\frac{1-\eta}{\eta} \\
&\geq Mt^{\alpha-1}(1-s)^{\alpha-1}s.
\end{aligned} \tag{21}$$

When $0 < \eta \leq s < t \leq 1$, let

$$g_2(t) = [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right). \tag{22}$$

We have

$$\begin{aligned}
g_2(t) &= [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right) \\
&= [t(1-s)]^{\alpha-1} - t^{\alpha-1}\left(1-\frac{s}{t}\right)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right) \\
&\geq [t(1-s)]^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right) \\
&= \beta\eta^{\alpha-1}(t-ts)^{\alpha-1} \\
&\geq Mt^{\alpha-1}(1-s)^{\alpha-1}s.
\end{aligned} \tag{23}$$

When $0 \leq t \leq s \leq \eta < 1$, let $g_3(t) = [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta - s)^{\alpha-1}$, we have

$$\begin{aligned}
g_3(t) &= [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} \\
&= [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1}t^{\alpha-1}\left(1-\frac{s}{\eta}\right)^{\alpha-1} \\
&\geq [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1} \\
&= \left(1-\beta\eta^{\alpha-1}\right)t^{\alpha-1}(1-s)^{\alpha-1} \\
&\geq Mt^{\alpha-1}(1-s)^{\alpha-1}s.
\end{aligned} \tag{24}$$

When $0 \leq t \leq s \leq 1$, $\eta \leq s$, we have

$$\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)G(t,s) = [t(1-s)]^{\alpha-1} \geq Mt^{\alpha-1}(1-s)^{\alpha-1}s. \tag{25}$$

It is easy to see that $G(t,s) \leq t^{\alpha-1}(1-s)^{\alpha-1}/\Gamma(\alpha)(1-\beta\eta^{\alpha-1})$. Thus, the proof is complete. □

Let us fix some notations to be used in the following. "For a.e." means "for almost every". Given $a \in L^1(0,1)$, we write $a \succ 0$ if $a \geq 0$ for a.e. $t \in [0,1]$, and it is positive in a set of positive measure, we write $f \in \mathrm{Car}((0,1)\times(0,+\infty),(0,+\infty))$ if $f : (0,1)\times(0,+\infty) \to (0,+\infty)$ is a L^1-caratheodory function, that is, the map $x \mapsto f(t,x)$ is continuous for a.e. $t \in (0,1)$, and the map $t \mapsto f(t,x)$ is measurable for all $x \in (0,+\infty)$.

Let us define

$$\begin{aligned}
\gamma^* &= \sup_{t\in[0,1]}\int_0^1 \frac{G(t,s)}{t^{\alpha-1}}e(s)\,ds, \\
\gamma_* &= \inf_{t\in[0,1]}\int_0^1 \frac{G(t,s)}{t^{\alpha-1}}e(s)\,ds.
\end{aligned} \tag{26}$$

Then,

$$t^{\alpha-1}\gamma_* \leq \int_0^1 G(t,s)e(s)\,ds \leq t^{\alpha-1}\gamma^*. \tag{27}$$

3. Main Results

In this section, we establish the existence of positive solutions for equation

$$\begin{aligned}
&\mathbf{D}_{0+}^{\alpha}u(t) + f(t,u(t)) + e(t) = 0, \quad 0 < t < 1, \\
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \beta u(\eta),
\end{aligned} \tag{28}$$

where $n-1 < \alpha \leq n$, $n \geq 3$, $0 < \beta \leq 1$, $0 \leq \eta \leq 1$, $f \in \mathrm{Car}((0,1)\times(0,+\infty),(0,+\infty))$, $e(t) \in L^1[0,1]$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. The following is the first main result in this section.

Theorem 7. *Suppose that the following conditions are satisfied.*

(H_1) *For each $L > 0$, there exists a function $\phi_L \succ 0$ such that $f(t, t^{\alpha-1}x) \geq \phi_L(t)$ for a.e. $t \in (0,1)$, all $x \in (0,L]$.*

(H_2) *There exist $g(x)$, $h(x)$, and $k(t) \succ 0$, such that*

$$\begin{aligned}
&0 \leq f(t,x) \leq k(t)\{g(x)+h(x)\} \\
&\textit{for a.e.} t \in (0,1), \textit{all } x \in (0,\infty),
\end{aligned} \tag{29}$$

here

$$\begin{aligned}
&g : (0,+\infty) \\
&\quad \longrightarrow [0,+\infty) \textit{ is continuous and nonincreasing}, \\
&h : [0,+\infty) \\
&\quad \longrightarrow [0,+\infty) \textit{ is continuous, and } \frac{h}{g} \textit{ is nondecreasing}.
\end{aligned} \tag{30}$$

(H_3) *There exist two positive constants $R > r > 0$ such that*

$$R > \Phi_{R1} + \gamma_* \ge r > 0,$$
$$\int_0^1 k(s)\, g\left(rs^{\alpha-1}\right) ds < +\infty,$$
$$R \ge \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1 - \beta\eta^{\alpha-1}\right)} k(s)\, g\left(rs^{\alpha-1}\right) ds + \gamma^*, \tag{31}$$

and here

$$\Phi_{R1} = \int_0^1 \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \phi_R(s)\, ds. \tag{32}$$

Then, (28) *has at least one positive solution.*

Proof. Let $E = (C[0,1], \|\cdot\|)$, and Ω is a closed convex set defined as

$$\Omega = \left\{x \in C[0,1] : t^{\alpha-1} r \le x(t) \le t^{\alpha-1} R \ \forall t \in [0,1]\right\}, \tag{33}$$

here $E = C[0,1]$ is the Banach space of continuous functions defined on $[0,1]$ with the norm

$$\|x\| := \max_{t\in[0,1]} |x(t)|, \tag{34}$$

and $R > r > 0$ are positive constants to be given below.

Now, we define an operator $T : \Omega \to E$ by

$$(Tx)(t) := \int_0^1 G(t,s)\left[f(s, x(s)) + e(s)\right] ds. \tag{35}$$

Then, (28) is equivalent to the fixed-point problem

$$x = Tx. \tag{36}$$

Let R be the positive constant satisfying (H_3) and

$$\Phi_{R1} + \gamma_* \ge r. \tag{37}$$

Then, we have $R > r > 0$. Now, we prove $T(\Omega) \subset \Omega$.

In fact, for each $x \in \Omega$ and for all $t \in (0,1)$, by (H_1) and (H_3)

$$\begin{aligned}(Tx)(t) &\ge \int_0^1 t^{\alpha-1} \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \phi_R(s)\, ds \\ &\quad + \int_0^1 G(t,s)\, e(s)\, ds \\ &\ge t^{\alpha-1}\left[\Phi_{R1} + \gamma_*\right] \ge t^{\alpha-1} r.\end{aligned} \tag{38}$$

On the other hand, by conditions (H_2) and (H_3), we have

$$\begin{aligned}&(Tx)(t) \\ &\le \int_0^1 t^{\alpha-1} \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)\left(1 - a\xi^{\alpha-\beta-1}\right)} k(s) \\ &\qquad \times \left[g\left(s^{\alpha-1}\frac{x(s)}{s^{\alpha-1}}\right) + h\left(s^{\alpha-1}\frac{x(s)}{s^{\alpha-1}}\right)\right] ds \\ &\quad + \int_0^1 G(t,s)\, e(s)\, ds \\ &\le t^{\alpha-1}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s)\, g\left(s^{\alpha-1}\frac{x(s)}{s^{\alpha-1}}\right)\right. \\ &\qquad \left. \times \left\{1 + \frac{h\left(s^{\alpha-1}\left(x(s)/s^{\alpha-1}\right)\right)}{g\left(s^{\alpha-1}\left(x(s)/s^{\alpha-1}\right)\right)}\right\} ds + \gamma^*\right] \\ &\le t^{\alpha-1} \\ &\quad \times \left[\left(1 + \frac{h(R)}{g(R)}\right)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s)\, g\left(rs^{\alpha-1}\right) ds + \gamma^*\right] \\ &\le t^{\alpha-1} R.\end{aligned} \tag{39}$$

In conclusion, $T(\Omega) \subset \Omega$.

Finally, it is standard that $T : \Omega \to \Omega$ is a continuous and completely continuous operator. By a direct application of Schauder's fixed-point theorem, (28) has at least one positive solution $x(t) \in C[0,1]$, the proof is finished. □

Case 1 ($\gamma_* = 0$). As an application of Theorem 7, we consider the case $\gamma_* = 0$. The following corollary is a direct result of Theorem 7 with $r = \Phi_{R1}$.

Corollary 8. *Suppose that $f(t,x)$ satisfies conditions* (H_1)-(H_2). *Furthermore, assume the following.*

(H_3^*) *There exists a positive constant $R > 0$ such that*

$$R > \Phi_{R1} > 0,$$
$$\int_0^1 k(s)\, g\left[\left(\Phi_{R1}\right) s^{\alpha-1}\right] ds < +\infty,$$
$$\begin{aligned}R &\ge \left(1 + \frac{h(R)}{g(R)}\right) \\ &\quad \times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s)\, g\left[\left(\Phi_{R1}\right) s^{\alpha-1}\right] ds + \gamma^*,\end{aligned} \tag{40}$$

and here

$$\Phi_{R1} = \int_0^1 \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \phi_R(s)\, ds. \tag{41}$$

Then, (28) *has at least one positive solution.*

From now on, let us define

$$\beta_1 = \int_0^1 \frac{Ms^{1-\lambda(\alpha-1)}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} k(s)\, ds,$$
$$\beta_2 = \int_0^1 \frac{s^{-\lambda(\alpha-1)}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} k(s)\, ds. \tag{42}$$

Example 9. Suppose that the nonlinearity in (28) is

$$f(t,x) = \frac{k(t)}{x^{\lambda}}, \tag{43}$$

where $k \succ 0$, $0 < \lambda < 1$ and

$$\omega(\lambda) := \int_0^1 k(s)\, s^{-\lambda(\alpha-1)} ds < +\infty. \tag{44}$$

If $\gamma_* = 0$, then (28) has at least one positive solution.

Proof. We will apply Corollary 8. To this end, we take

$$\phi_L(t) = \frac{k(t)}{(t^{\alpha-1}L)^{\lambda}}, \qquad g(x) = \frac{1}{x^{\lambda}}, \qquad h(x) \equiv 0, \tag{45}$$

then (H_1) and (H_2) are satisfied since $\omega(\lambda) < +\infty$, and the existence condition (H_3^*) becomes

$$\left(\frac{R^{\lambda}}{\beta_1}\right)^{\lambda} \beta_2 + \gamma^* \le R, \quad R > \frac{\beta_1}{R^{\lambda}},\ \omega(\lambda) < +\infty, \tag{46}$$

for some $R > 0$. Since $0 < \lambda < 1$, we can choose $R > 0$ large enough such that (46) is satisfied, and the proof is finished. □

Example 10. Suppose that the nonlinearity in (28) is

$$f(t,x) = k(s)\left(x^{-\lambda} + \mu x^{\nu}\right), \tag{47}$$

where $0 < \lambda < 1$, $\nu \ge 0$ and $\mu \ge 0$ is a nonnegative parameter. For each $e(t)$ with $\gamma_* = 0, \omega(\lambda) < +\infty$,

(i) if $\lambda + \nu < 1 - \lambda^2$, then (28) has at least one positive solution for each $\mu \ge 0$.

(ii) If $\lambda + \nu \ge 1 - \lambda^2$, then (28) has at least one positive solution for each $0 \le \mu < \mu_1$, where μ_1 is some positive constant.

Proof. We will apply Corollary 8. To this end, we take

$$\phi_L(t) = \frac{k(t)}{(t^{\alpha-1}L)^{\lambda}}, \qquad g(x) = x^{-\lambda}, \qquad h(x) = \mu x^{\nu}. \tag{48}$$

Then, (H_1)-(H_2) are satisfied since $\omega(\lambda) < +\infty$. Now, the existence condition (H_3^*) becomes $\omega(\lambda) < +\infty$, and

$$\mu \le \frac{R^{1-\lambda^2}\beta_1^{\lambda} - \gamma^*\beta_1^{\lambda}R^{-\lambda^2} - \beta_2}{\beta_2 R^{\lambda+\nu}}, \tag{49}$$

for some $R > 0$ with $R^{1+\lambda} > \beta_1$. So, (28) has at least one positive solution for

$$0 < \mu < \mu_1 = \sup_{R > \beta_1^{1/1+\lambda}} \frac{R^{1-\lambda^2}\beta_1^{\lambda} - \gamma^*\beta_1^{\lambda}R^{-\lambda^2} - \beta_2}{\beta_2 R^{\lambda+\nu}}. \tag{50}$$

Note that $\mu_1 = \infty$ if $\lambda + \nu < 1 - \lambda^2$ and if $\lambda + \nu \ge 1 - \lambda^2$, set

$$l(R) := \frac{R^{1-\lambda^2}\beta_1^{\lambda} - \gamma^*\beta_1^{\lambda}R^{-\lambda^2} - \beta_2}{\beta_2 R^{\lambda+\nu}}, \tag{51}$$

then, we have

$$l'(R) = \frac{1}{R^{\lambda+\nu+1}\beta_2}\left[\left(1 - \lambda^2 - \lambda - \nu\right) R^{1-\lambda^2}\beta_1^{\lambda} + \left(\lambda^2 + \lambda + \nu\right)\gamma^*\beta_1^{\lambda}R^{-\lambda^2} + (\lambda+\nu)\beta_2\right]. \tag{52}$$

Let the function $l(R)$ possess a maximum at R_0, then

$$\left(\lambda^2 + \lambda + \nu - 1\right) R_0^{1-\lambda^2}\beta_1^{\lambda} = \left(\lambda^2 + \lambda + \nu\lambda\right)\gamma^*\beta_1^{\lambda}R_0^{-\lambda^2} + (\lambda+\nu)\beta_2, \tag{53}$$

so we have

$$\left(\lambda + \nu - 1 + \lambda^2\right)\beta_1^{\lambda}R_0 \ge (\lambda+\nu)\beta_2 R_0^{\lambda^2}, \tag{54}$$

it is easy to find that $R_0 > (\beta_2/\beta_1^{\lambda})^{1/(1-\lambda^2)}$ since $\lambda + \nu \ge 1 - \lambda^2$, and $0 < \lambda < 1$. Finally, it would remain to prove $R_0 > \beta_1^{1/1+\lambda}$. This is easily verified through elementary computations since $\beta_1 \le \beta_2$. We have the desired results (i) and (ii). □

Case 2 ($\gamma_* > 0$). The next result explores the case when $\gamma_* > 0$. In this case $r = \gamma_*$.

Corollary 11. *Suppose that $f(t,x)$ satisfies (H_2). Furthermore, assume the following.*

(H_4) *There exists $R > 0$, $\int_0^1 k(s)g(\gamma_* s^{\alpha-1})ds < +\infty$, such that*

$$\left(1 + \frac{h(R)}{g(R)}\right)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} k(s)\, g\left(\gamma_* s^{\alpha-1}\right) ds + \gamma^* \le R. \tag{55}$$

If $\gamma_* > 0$, then (28) *has at least one positive solution.*

Example 12. Suppose that the nonlinearity in (28) be (43) with $k \succ 0$, $\lambda > 0$. If $\gamma_* > 0$, $\omega(\lambda) < +\infty$, then (28) has at least one positive solution.

Proof. We will apply Corollary 11. Take $k(t)$, $g(x)$, and $h(x)$ as the same in the proof of Example 9. Then, (H_2) is satisfied, and the existence condition (H_4) is satisfied if we take $R > 0$ with $R \ge \beta_2/\gamma_*^{\lambda} + \gamma^*$, and $\omega(\lambda) < +\infty$. □

Example 13. Let the nonlinearity in (28) be (47) with $\lambda > 0$ and $\nu \ge 0$. For each $e(t)$ with $\gamma_* > 0$, $\omega(\lambda) < +\infty$,

(i) if $\lambda+\nu<1$, then (28) has at least one positive solution for each $\mu \geq 0$.

(ii) If $\lambda+\nu \geq 1$, then (28) has at least one positive solution for each $0 \leq \mu < \mu_2$, where μ_2 is some positive constant.

Proof. We will apply Corollary 11. To this end, we take $g(x)$, $h(x)$, and $k(t)$ as the same in the proof of Example 10, then (H_2) is satisfied, and the existence condition (H_4) becomes $\omega(\lambda) < +\infty$,

$$\mu \leq \frac{R\gamma_*^\lambda - \gamma^*\gamma_*^\lambda - \beta_2}{\beta_2 R^{\lambda+\nu}}, \tag{56}$$

for some $R>0$. So, (28) has at least one positive solution for

$$0 < \mu < \mu_2 = \sup_{R>0} \frac{R\gamma_*^\lambda - \gamma^*\gamma_*^\lambda - \beta_2}{\beta_2 R^{\lambda+\nu}}. \tag{57}$$

Note that $\mu_2 = \infty$ if $\lambda+\nu<1$ and $\mu_2 = \gamma_*^\lambda/\beta_2$ if $\lambda+\nu=1$, and if $\lambda+\nu>1$ set

$$l(R) := \frac{R\gamma_*^\lambda - \gamma^*\gamma_*^\lambda - \beta_2}{\beta_2 R^{\lambda+\nu}}. \tag{58}$$

The function $l(R)$ possesses a maximum at

$$R_0 := \frac{(\lambda+\nu)\left(\gamma^*\gamma_*^\lambda + \beta_2\right)}{(\lambda+\nu-1)\gamma_*^\lambda}, \tag{59}$$

then $\mu_2 = l(R_0)$. We have the desired results (i) and (ii). □

Case 3 ($\gamma^* \leq 0$). The next result considers the case $\gamma^* \leq 0$.

Corollary 14. *Suppose that $f(t,x)$ satisfies (H_1)-(H_2). Furthermore, assume the following.*

(H_5) *There exist two positive constants $R>r>0$ such that*

$$R > \Phi_{R1} + \gamma_* \geq r > 0,$$

$$\int_0^1 k(s)\, g\left(rs^{\alpha-1}\right) ds < +\infty,$$

$$R \geq \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s)\, g\left(rs^{\alpha-1}\right) ds, \tag{60}$$

here

$$\Phi_{R1} = \int_0^1 \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \phi_R(s)\, ds. \tag{61}$$

Then, (28) *has at least one positive solution.*

Example 15. Suppose that the nonlinearity in (28) be (43) with $k \succ 0$, $\lambda > 0$. If $\gamma^* \leq 0$, $\omega(\lambda) < +\infty$,

$$\gamma_* \geq \left[\frac{\beta_1\lambda^2}{\beta_2^\lambda}\right]^{1/(1-\lambda^2)} \left(1 - \frac{1}{\lambda^2}\right), \tag{62}$$

then (28) has at least one positive solution.

Proof. We will apply Corollary 14. Take $k(t)$, $g(x)$ as the same in the proof of Example 9. Then, (H_2) is satisfied, and the existence condition (H_5) is satisfied if we take $R>r>0$ with

$$\frac{\beta_1}{R^\lambda} + \gamma_* \geq r, \quad R \geq \frac{\beta_2}{r^\lambda}, \tag{63}$$

and $\omega(\lambda) < +\infty$. If we fix $R = \beta_2/r^\lambda$, then the first inequality holds if r satisfies

$$\frac{\beta_1}{\beta_2^\lambda} r^{\lambda^2} + \gamma_* \geq r, \tag{64}$$

or equivalently

$$\gamma_* \geq l(r) := r - \frac{\beta_1}{\beta_2^\lambda} r^{\lambda^2}. \tag{65}$$

The function $l(r)$ possesses a minimum at

$$r_0 := \left[\frac{\beta_1}{\beta_2^\lambda}\lambda^2\right]^{1/(1-\lambda^2)}. \tag{66}$$

Taking $r = r_0$, then the first inequality in (63) holds if $\gamma_* \geq l(r_0)$, which is just condition (62). The second inequality holds directly from the choice of R, so it remains to prove that $R = \beta_2/r^\lambda > r_0$. This is easily verified through elementary computations. □

Example 16. Let the nonlinearity in (28) be (47) with $k \succ 0$, $\lambda > 0$ and $\nu \geq 0$. If $\gamma^* \leq 0$, $\omega(\lambda) < +\infty$,

$$\gamma_* \geq m_0^\lambda \left[\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} - \beta_1\right], \tag{67}$$

here m_0 is the unique solution of the equation

$$\begin{aligned}&\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\ &\quad \times \left[m^{1-\lambda^2} + \mu(1-\lambda-\nu)\, m^{1-\lambda-\nu-\lambda^2}\right] = \lambda^2\beta_1.\end{aligned} \tag{68}$$

Then (28) has at least one positive solution.

Proof. We will apply Corollary 14. To this end, we take $g(x)$, $h(x)$, and $k(t)$ as the same in the proof of Example 10, then (H_2) is satisfied, and the existence condition (H_5) is satisfied if we take $R>r>0$ with

$$\gamma_* \geq r - \beta_1 R^{-\lambda}, \qquad \left(1 + \mu R^{\lambda+\nu}\right)\beta_2 r^{-\lambda} \leq R, \tag{69}$$

and $\omega(\lambda) < +\infty$. If we fix $R = (1+\mu R^{\lambda+\nu})\beta_2 r^{-\lambda}$, then the first inequality holds if R satisfies

$$\gamma_* \geq \left[\frac{\left(1+\mu R^{\lambda+\nu}\right)\beta_2}{R}\right]^{1/\lambda} - \beta_1 R^{-\lambda}. \tag{70}$$

Let $m = 1/R$, then

$$\gamma_* \geq m^\lambda \left[\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} - \beta_1\right] =: F(m). \tag{71}$$

Then, we have

$$\begin{aligned}
F'(m) &= \lambda m^{\lambda-1}\left[\left(\beta_2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right)^{1/\lambda}-\beta_1\right] \\
&\quad + m^{\lambda}\left[\frac{1}{\lambda}\left(\beta_2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right)^{1/\lambda-1}\right. \\
&\qquad \times\left(\left(1-\lambda^2\right)\beta_2 m^{-\lambda^2}\right. \\
&\qquad\quad \left.\left.+\mu\left(1-\lambda-\nu-\lambda^2\right)\beta_2 m^{-\lambda-\nu-\lambda^2}\right)\right] \\
&= \frac{1}{\lambda}m^{\lambda-1}\left\{\lambda^2\left[\left(\beta_2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right)^{1/\lambda}-\beta_1\right]\right. \\
&\quad + \beta_2^{1/\lambda}\left[\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\qquad \times\left(\left(1-\lambda^2\right)m^{1-\lambda^2}\right. \\
&\qquad\quad \left.\left.\left.+\mu\left(1-\lambda-\nu-\lambda^2\right)m^{1-\lambda-\nu-\lambda^2}\right)\right]\right\} \\
&= \frac{1}{\lambda}m^{\lambda-1}\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\quad \times\left[\lambda^2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right. \\
&\qquad +\left(\left(1-\lambda^2\right)m^{1-\lambda^2}\right. \\
&\qquad\quad \left.\left.\left.+\mu\left(1-\lambda-\nu-\lambda^2\right)m^{1-\lambda-\nu-\lambda^2}\right)\right]-\lambda^2\beta_1\right\} \\
&= \frac{1}{\lambda}m^{\lambda-1}\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\quad \left.\times\left[m^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m^{1-\lambda-\nu-\lambda^2}\right]-\lambda^2\beta_1\right\}.
\end{aligned} \tag{72}$$

Let $F'(m)=0$, then we have

$$\begin{aligned}
&\frac{1}{\lambda}m^{\lambda-1}\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\quad \left.\times\left[m^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m^{1-\lambda-\nu-\lambda^2}\right]-\lambda^2\beta_1\right\}=0.
\end{aligned} \tag{73}$$

Now, let us define $\Phi(m)$ by

$$\begin{aligned}
\Phi(m) =: {}& \beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
&\times\left[m^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m^{1-\lambda-\nu-\lambda^2}\right].
\end{aligned} \tag{74}$$

It is easy to see that $\Phi(m)$ is a nondecreasing function for $m\in[0,+\infty)$ and $\Phi(m)\to+\infty$, as $m\to+\infty$. Thus, $\Phi(m)=\lambda^2\beta_1$ has a unique solution m_0 such that

$$\begin{aligned}
&\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
&\quad\times\left[m_0^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m_0^{1-\lambda-\nu-\lambda^2}\right]=\lambda^2\beta_1,
\end{aligned} \tag{75}$$

and $F(m_0)=\inf_{m>0}F(m)$.

So, it remains to prove that $R>r=[(1+\mu R^{\lambda+\nu})\beta_2/R]^{1/\lambda}$, that is,

$$m_0^{\lambda+1}\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}<1. \tag{76}$$

In fact, by (75), we have

$$\lambda^2\beta_1\geq\left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda-1}\left(\beta_2 m_0^{1-\lambda^2}\right)=\left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda}, \tag{77}$$

that is,

$$m_0\leq\left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}. \tag{78}$$

Also we have

$$\begin{aligned}
\lambda^2\beta_1 &\geq \beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
&\quad\times\left[\left(1-\lambda-\nu\right)m_0^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m_0^{1-\lambda-\nu-\lambda^2}\right] \\
&= \left(1-\lambda-\nu\right)\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda},
\end{aligned} \tag{79}$$

that is,

$$\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}\leq\frac{\lambda^2\beta_1}{1-\lambda-\nu}. \tag{80}$$

Thus, we have

$$\begin{aligned}
&m_0^{\lambda+1}\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \\
&\quad<\left(\left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}\right)^{\lambda+1}\frac{\lambda^2\beta_1}{1-\lambda-\nu} \\
&\quad=\left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda)}\frac{\lambda^2\beta_1}{1-\lambda-\nu} \\
&\quad=\lambda^{2\lambda/(1-\lambda)}\left(\frac{\beta_1}{\beta_2}\right)^{1/(1-\lambda)}\frac{\lambda^2}{1-\lambda-\nu}<1
\end{aligned} \tag{81}$$

since $0<\lambda,\nu<1$, and $1-\lambda-\nu-\lambda^2>0$.

We have the desired results. □

Case 4 ($\gamma_* < 0 < \gamma^*$).

Example 17. Suppose that the nonlinearity in (28) be (43) with $k > 0, \lambda > 0$. If $\gamma_* < 0 < \gamma^*, \omega(\lambda) < +\infty$,

$$\gamma_* \geq m_0^{\lambda}\left[\beta_2^{1/\lambda} m_0^{(1-\lambda^2)/\lambda} - \beta_1(1+\gamma^* m_0)^{-\lambda}\right], \tag{82}$$

and here m_0 is the unique solution of the equation

$$\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda}(1+\gamma^* m)^{\lambda+1} = \lambda^2\beta_1, \tag{83}$$

then (28) has at least one positive solution.

Proof. We will apply Theorem 7. Take $k(t), g(x)$ as the same in the proof of Example 9. Then, (H_2) is satisfied, and the existence condition (H_3) is satisfied if we take $R > r > 0$ with

$$\gamma_* \geq r - \beta_1 R^{-\lambda}, \qquad \beta_2 r^{-\lambda} + \gamma^* \leq R, \tag{84}$$

and $\omega(\lambda) < +\infty$. If we fix $R = \beta_2/r^{\lambda} + \gamma^*$, then the first inequality holds if R satisfies

$$\gamma_* \geq \left(\frac{\beta_2}{R-\gamma^*}\right)^{1/\lambda} - \beta_1 R^{-\lambda}. \tag{85}$$

Taking $m = 1/(R-\gamma^*)$, then for $m \in (0,+\infty)$, we have

$$\begin{aligned}\gamma_* &\geq (\beta_2 m)^{1/\lambda} - \frac{\beta_1 m^{\lambda}}{(1+\gamma^* m)^{\lambda}} \\ &= m^{\lambda}\left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \beta_1(1+\gamma^* m)^{-\lambda}\right] =: F(m).\end{aligned} \tag{86}$$

Then, we have

$$\begin{aligned}&F'(m)\\ &= \lambda m^{\lambda-1}\left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \beta_1(1+\gamma^* m)^{-\lambda}\right]\\ &\quad + m^{\lambda}\left[\frac{1-\lambda^2}{\lambda}\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda-1} + \lambda\beta_1\gamma^*(1+\gamma^* m)^{-\lambda-1}\right]\\ &= \frac{1}{\lambda} m^{\lambda-1}\left\{\lambda^2\left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \beta_1(1+\gamma^* m)^{-\lambda}\right]\right.\\ &\qquad + \lambda m\left[\frac{1-\lambda^2}{\lambda}\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda-1}\right.\\ &\qquad\qquad \left.\left.+\lambda\beta_1\gamma^*(1+\gamma^* m)^{-\lambda-1}\right]\right\}\\ &= \frac{1}{\lambda} m^{\lambda-1}\left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \lambda^2\beta_1(1+\gamma^* m)^{-\lambda-1}\right]\\ &= \frac{1}{\lambda} m^{\lambda-1}(1+\gamma^* m)^{-\lambda-1}\left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda}(1+\gamma^* m)^{\lambda+1} - \lambda^2\beta_1\right].\end{aligned} \tag{87}$$

Let $F'(m) = 0$, then we have

$$\frac{1}{\lambda} m^{\lambda-1}(1+\gamma^* m)^{-\lambda-1}\left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda}(1+\gamma^* m)^{\lambda+1} - \lambda^2\beta_1\right] = 0. \tag{88}$$

Now, let us define $\Phi(m)$ by

$$\Phi(m) =: \beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda}(1+\gamma^* m)^{\lambda+1}. \tag{89}$$

It is easy to see that $\Phi(m)$ is a nondecreasing function for $m \in [0,+\infty)$, and $\Phi(m) \to +\infty$, as $m \to +\infty$. Thus, $\Phi(m) = \lambda^2\beta_1$ has a unique solution m_0 such that

$$\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda}(1+\gamma^* m)^{\lambda+1} = \lambda^2\beta_1, \tag{90}$$

and $F(m_0) = \inf_{m>0} F(m)$.

So, it remains to prove that $R > r = [\beta_2/(R-\gamma^*)]^{1/\lambda}$, that is,

$$\beta_2^{1/\lambda} m_0^{1+1/\lambda} \leq 1 + \gamma^* m_0. \tag{91}$$

In fact, by (90), we have

$$\beta_2^{1/\lambda} m_0^{(1-\lambda^2)/\lambda} \leq \lambda^2\beta_1, \tag{92}$$

that is,

$$m_0 \leq \left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}. \tag{93}$$

Thus, we have

$$\begin{aligned}\beta_2^{1/\lambda} m_0^{1+1/\lambda} &< \beta_2^{1/\lambda}\left(\left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}\right)^{(1+\lambda)/\lambda} m_0^{1+1/\lambda}\\ &= \lambda^{2/(1-\lambda)}\left(\frac{\beta_1}{\beta_2}\right)^{1/(1-\lambda)} < 1 < 1 + \gamma^* m_0.\end{aligned} \tag{94}$$

The proof is complete. □

Example 18. Let the nonlinearity in (28) be (47) with $k > 0, \lambda > 0$ and $\nu \geq 0$. If $\gamma_* < 0 < \gamma^*, \omega(\lambda) < +\infty$,

$$\begin{aligned}\gamma_* \geq m_0^{\lambda}&\left[\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu(1+\gamma^* m_0)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}\right.\\ &\left.-\beta_1(1+\gamma^* m_0)^{-\lambda}\right],\end{aligned} \tag{95}$$

here m_0 is the unique solution of the equation

$$\begin{aligned}&\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\\ &\quad\times\left[m^{1-\lambda^2}(1+\gamma^* m)^{\lambda+1} + \mu(1-\lambda-\nu+\gamma^* m)\, m^{1-\lambda-\nu-\lambda^2}\right.\\ &\quad\left.\times(1+\gamma^* m)^{2\lambda+\nu}\right] = \lambda^2\beta_1.\end{aligned} \tag{96}$$

Then (28) has at least one positive solution.

Proof. We will apply Theorem 7. To this end, we take $g(x)$, $h(x)$, and $k(t)$ as the same in the proof of Example 10, then

(H_2) is satisfied, and the existence condition (H_3) is satisfied if we take $R > r > 0$ with

$$\gamma_* \geq r - \beta_1 R^{-\lambda}, \qquad \left(1 + \mu R^{\lambda+\nu}\right) \beta_2 r^{-\lambda} + \gamma^* \leq R, \tag{97}$$

and $\omega(\lambda) < +\infty$. If we fix $R = (1 + \mu R^{\lambda+\nu})\beta_2 r^{-\lambda} + \gamma^*$, then the first inequality holds if

$$\gamma_* \geq \left[\frac{\left(1 + \mu R^{\lambda+\nu}\right) \beta_2}{R - \gamma^*} \right]^{1/\lambda} - \beta_1 R^{-\lambda}. \tag{98}$$

Let $m = 1/(R - \gamma^*)$, then

$$\begin{aligned}\gamma_* \geq m^{\lambda} \Big[& \beta_2^{1/\lambda} \left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \\ & -\beta_1 (1+\gamma^*)^{-\lambda} \Big] =: F(m).\end{aligned} \tag{99}$$

Then, we have

$$\begin{aligned}
F'(m) &\\
= \lambda m^{\lambda-1} & \Big[\beta_2^{1/\lambda} \left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \\
& \quad -\beta_1 (1+\gamma^* m)^{-\lambda} \Big] \\
& + m^{\lambda} \Big[\frac{1}{\lambda} \beta_2^{1/\lambda} \left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
& \quad \times \Big(\left(1-\lambda^2\right) m^{-\lambda^2} + \mu (1+\gamma^* m)^{\lambda+\nu} \\
& \qquad \times \left(1-\lambda-\nu-\lambda^2\right) m^{-\lambda-\nu-\lambda^2} \\
& \qquad + \mu (\lambda+\nu) (1+\gamma^* m)^{\lambda+\nu-1} m^{1-\lambda-\nu-\lambda^2} \gamma^* \Big) \\
& \quad + \beta_1 \lambda (1+\gamma^* m)^{-\lambda-1} \gamma^* \Big] \\
= \frac{1}{\lambda} m^{\lambda-1} & \Big\{ \lambda^2 \beta_2^{1/\lambda} \Big[\left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \\
& \quad - \lambda^2 \beta_1 (1+\gamma^* m)^{-\lambda} \\
& \quad + \beta_2^{1/\lambda} \Big(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} \\
& \qquad \times m^{1-\lambda-\nu-\lambda^2} \Big)^{1/\lambda-1} \\
& \quad \times \Big[\left(1-\lambda^2\right) m^{1-\lambda^2} + \mu (1+\gamma^* m)^{\lambda+\nu} \\
& \qquad \times \left(1-\lambda-\nu-\lambda^2\right) m^{1-\lambda-\nu-\lambda^2} \\
& \qquad + \mu (\lambda+\nu) (1+\gamma^* m)^{\lambda+\nu-1} \\
& \qquad \times m^{1-\lambda-\nu-\lambda^2} m\gamma^* \Big] \Big] \\
& \quad + \lambda^2 \beta_1 (1+\gamma^* m)^{-\lambda-1} m\gamma^* \Big\} \\
= \frac{1}{\lambda} m^{\lambda-1} & \Big\{ \beta_2^{1/\lambda} \Big[\left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right) \Big]^{1/\lambda-1} \\
& \times \Big[\lambda^2 \left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right) \\
& \quad + \left(1-\lambda^2\right) m^{1-\lambda^2} \\
& \quad + \mu(1+\gamma^* m)^{\lambda+\nu} \left(1-\lambda-\nu-\lambda^2\right) m^{1-\lambda-\nu-\lambda^2} \\
& \quad + \mu (\lambda+\nu) (1+\gamma^* m)^{\lambda+\nu-1} m^{1-\lambda-\nu-\lambda^2} m\gamma^* \Big] \\
& \times \lambda^2 \beta_1 (1+\gamma^* m)^{-\lambda-1} m\gamma^* - \lambda^2 \beta_1 (1+\gamma^* m)^{-\lambda} \Big\} \\
= \frac{1}{\lambda} m^{\lambda-1} & (1+\gamma^* m)^{-\lambda-1} \\
& \times \Big\{ \beta_2^{1/\lambda} \left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
& \quad \times \Big[m^{1-\lambda^2} (1+\gamma^* m)^{\lambda+1} + \mu (1-\lambda-\nu+\gamma^* m) \\
& \qquad \times m^{1-\lambda-\nu-\lambda^2} (1+\gamma^* m)^{2\lambda+\nu} \Big] - \lambda^2 \beta_1 \Big\}.
\end{aligned} \tag{100}$$

Let $F'(m) = 0$, then we have

$$\begin{aligned}
& \frac{1}{\lambda} m^{\lambda-1} (1+\gamma^* m)^{-\lambda-1} \\
& \times \Big\{ \beta_2^{1/\lambda} \left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
& \quad \times \Big[m^{1-\lambda^2} (1+\gamma^* m)^{\lambda+1} + \mu (1-\lambda-\nu+\gamma^* m) m^{1-\lambda-\nu-\lambda^2} \\
& \qquad \times (1+\gamma^* m)^{2\nu+\nu} \Big] - \lambda^2 \beta_1 \Big\} = 0.
\end{aligned} \tag{101}$$

Now, let us define $\Phi(m)$ by

$$\begin{aligned}
& \Phi(m) \\
& =: \beta_2^{1/\lambda} \left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
& \quad \times \Big[m^{1-\lambda^2} (1+m\gamma^* m)^{\lambda+1} \\
& \qquad + \mu (1-\lambda-\nu+\gamma^* m) m^{1-\lambda-\nu-\lambda^2} (1+\gamma^* m)^{2\lambda+\nu} \Big].
\end{aligned} \tag{102}$$

It is easy to see that $\Phi(m)$ is a nondecreasing function for $m \in [0,+\infty)$, and $\Phi(m) \to +\infty$, as $m \to +\infty$. Thus, $\Phi(m) = \lambda^2\beta_1$ has a unique solution m_0 such that

$$\begin{aligned}&\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\\&\quad\times\left[m_0^{1-\lambda^2}(1+\gamma^* m)^{\lambda+1}\right.\\&\qquad\left.+\mu\left(1-\lambda-\nu+\gamma^* m\right) m_0^{1-\lambda-\nu-\lambda^2}\left(1+\gamma^* m_0\right)^{2\lambda+\nu}\right]\\&\quad=\lambda^2\beta_1,\end{aligned} \tag{103}$$

and $F(m_0) = \inf_{m>0} F(m)$.

So, it remains to prove that $R > r = [(1+\mu R^{\lambda+\nu})\beta_2/(R-\gamma^*)]^{1/\lambda}$, that is,

$$m_0^{\lambda+1}\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} < 1+\gamma^* m_0. \tag{104}$$

In fact, by (103), we have

$$\lambda^2\beta_1 \geq \left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda-1}\left(\beta_2 m_0^{1-\lambda^2}\right) = \left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda}, \tag{105}$$

that is,

$$m_0 \leq \left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}. \tag{106}$$

Also we have

$$\begin{aligned}&\lambda^2\beta_1\\&\geq \beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\\&\quad\times\left[(1-\lambda-\nu)\, m_0^{1-\lambda^2}\right.\\&\qquad\left.+\mu\,(1-\lambda-\nu)\, m_0^{1-\lambda-\nu-\lambda^2}(1+\gamma^* m)^{\lambda+\nu}\right]\\&= (1-\lambda-\nu)\,\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda},\end{aligned} \tag{107}$$

that is,

$$\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \leq \frac{\lambda^2\beta_1}{1-\lambda-\nu}. \tag{108}$$

Thus, we have

$$\begin{aligned}&m_0^{\lambda+1}\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}\\&\quad<\left(\left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}\right)^{\lambda+1}\frac{\lambda^2\beta_1}{1-\lambda-\nu}\\&\quad=\left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda)}\frac{\lambda^2\beta_1}{1-\lambda-\nu}\\&\quad=\lambda^{2\lambda/(1-\lambda)}\left(\frac{\beta_1}{\beta_2}\right)^{1/(1-\lambda)}\frac{\lambda^2}{1-\lambda-\nu} < 1 < 1+\gamma^* m,\end{aligned} \tag{109}$$

since $0 < \lambda, \nu < 1$ and $1-\lambda-\nu-\lambda^2 > 0$.

Thus, the proof is complete. □

Remark 19. It is easy to find that analogous results to Examples 10, 13, 16, and 18 for the general equation with the nonlinearity in (28) are

$$f(t,x) = \frac{b(t)}{x^\lambda} + \mu c(t)\, x^\nu + e(t), \tag{110}$$

with $b, c \succ 0$, but the notation becomes cumbersome. Here we consider the nonlinearity (47) only for the simplicity.

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Research Article

On the Laws of Total Local Times for h-Paths and Bridges of Symmetric Lévy Processes

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The joint law of the total local times at two levels for h-paths of symmetric Lévy processes is shown to admit an explicit representation in terms of the laws of the squared Bessel processes of dimensions two and zero. The law of the total local time at a single level for bridges is also discussed.

1. Introduction

Markov processes associated to heat semigroups generated by fractional derivatives are called *symmetric stable Lévy processes* (cf., e.g., [1]) or *Lévy flights* (cf., e.g., [2]). The purpose of the present paper is to study the laws of the total local times for h-paths and bridges of (one-dimensional) symmetric Lévy processes. We give an explicit representation (Theorem 16) of the joint law as a weighted sum of the law of the squared Bessel process of dimension two and the generalized excursion measure for the squared Bessel process of dimension zero. We also give an expression (Theorem 20) of the law of the total local time at a single level for bridges.

It is well known as one of the Ray-Knight theorems (see, e.g., [3, Chapter XI] and [4, Chapter 3]) that the total local time process with space parameter for a Bessel process of dimension three is a squared Bessel process of dimension two. Since the Bessel process of dimension three is the h-path process of a reflected Brownian motion, Theorem 16 may be considered to be a slight generalization of this result.

Eisenbaum and Kaspi [5] have proved that the total local time of a Markov process with discontinuous paths is no longer Markov. As an analogue of Ray-Knight theorems, Eisenbaum et al. [6] have recently characterized the law of the local time process with space parameter at inverse local time in terms of some Gaussian process whose covariance is given by the resolvent density of the potential kernel. Moreover, if the Lévy process is a symmetric stable process, then the corresponding Gaussian process is a fractional Brownian motion. Their results are based on a version of Feynman-Kac formulae, which characterizes the Laplace transform of the joint laws of total local times of Markov processes at several levels.

In this paper we first focus on the h-path process of a symmetric Lévy process, which has been introduced in the recent works [7–9] by Yano et al. The h-path process may be obtained as the process conditioned to avoid the origin during the whole time (see [10]). We will also start from a version of Feynman-Kac formulae and obtain an explicit representation of the joint law of the total local times at two levels. (For some discussions of the joint law of the total local times, see Blumenthal-Getoor [11, pages 221–226] and Pitman [12].) Unfortunately, we have no better result on the law of the total local time process with space parameter. The difficulty will be explained in Remark 3.

In comparison with the results by Pitman [13] and Pitman and Yor [14] about the Brownian and Bessel bridges, we also investigate the law of the total local time at a single point for bridges of symmetric Lévy process, which we call *Lévy bridges* in short, and also for bridges of the h-paths, which we call *h-bridges* in short. We will prove a version of Feynman-Kac formulae (Theorem 7) for Lévy bridges with the help of the general theorems by Fitzsimmons et al. [15]. As an application of the Feynman-Kac theorem, we will give an expression of the law of the total local time at a single level for the Lévy bridges, while, unfortunately, we do not have any nice formula for the h-bridges.

The present paper is organized as follows. In Section 2, we give two versions of Feynman-Kac formulae in general settings. In Section 3, we recall several formulae about squared Bessel processes and generalized excursion measures. In Section 4, we recall several facts about symmetric Lévy processes. In Section 5, we deal with the joint law of the total local times at two levels for the h-paths of symmetric Lévy processes. In Section 6, we study the laws of the total local times for the Lévy bridges and for the h-bridges.

2. Feynman-Kac Formulae

In order to study the laws of total local times, we prepare two versions of Feynman-Kac formulae, which describe their Laplace transforms. One is for transient Markov processes, and the other is for Markovian bridges.

Let $\mathbb{D}$ denote the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R} \cup \{\Delta\}$ with lifetime $\zeta = \zeta(\omega)$:

$$\forall t < \zeta, \quad \omega(t) \in \mathbb{R}, \qquad \forall t \geq \zeta, \quad \omega(t) = \Delta. \tag{1}$$

Let (X_t) denote the canonical process: $X_t(\omega) = \omega(t)$. Let $(\mathcal{F}_t)$ denote its natural filtration and $\mathcal{F}_\infty = \sigma(\cup_t \mathcal{F}_t)$. For $a \in \mathbb{R}$, we write $T_{\{a\}}$ for the first hitting time of the point a:

$$T_{\{a\}} = \inf\{t > 0 : X_t = a\}. \tag{2}$$

The set of all nonnegative Borel functions on $\mathbb{R}$ will be denoted by $\mathcal{B}_+(\mathbb{R})$.

Let $(\mathbf{P}_x : x \in \mathbb{R})$ denote the laws on $\mathbb{D}$ of a right Markov process. We assume that the transition kernels have jointly measurable densities $p_t(x,y)$ with respect to a reference measure $\mu(dy)$:

$$\mathbf{P}_x(X_t \in dy) = p_t(x,y)\,\mu(dy). \tag{3}$$

We define

$$u_q(x,y) = \int_0^\infty e^{-qt} p_t(x,y)\,dt, \quad q \geq 0, \tag{4}$$

which are resolvent densities if they are finite. We also assume that there exists a local time (L_t^x) such that

$$\int_0^t f(X_s)\,ds = \int f(y) L_t^y \mu(dy), \quad t > 0, \ f \in \mathcal{B}_+(\mathbb{R}) \tag{5}$$

holds with $\mathbf{P}_x$-probability one for any $x \in \mathbb{R}$.

2.1. Feynman-Kac Formula for Transient Markov Processes. In this section, we prove Feynman-Kac formula for transient Markov processes. We assume the following conditions:

(i) the process is transient;

(ii) $u_0(x,y) < \infty$ for any $x, y \in \mathbb{R}$ with $x \neq 0$ or $y \neq 0$.

Note that $u_0(0,0)$ may be infinite. We note that

$$\mathbf{P}_x(\forall y \in \mathbb{R}, L_\infty^y < \infty) = 1 \quad \text{for any } x \in \mathbb{R}. \tag{6}$$

By formula (5), it is easy to see that

$$\mathbf{P}_x[L_\infty^y] = u_0(x,y), \quad x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}. \tag{7}$$

We will prove a version of Feynman-Kac formulae following Marcus-Rosen's book [16] where it is assumed that $u_0(0,0) < \infty$.

For $t \geq 0$ and $x_1, x_2, \ldots, x_n \in \mathbb{R} \setminus \{0\}$, we set

$$J_t(\mathbf{x}) = \int_t^\infty dL_{t_1}^{x_1} \int_{t_1}^\infty dL_{t_2}^{x_2} \cdots \int_{t_{n-1}}^\infty dL_{t_n}^{x_n}, \tag{8}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$.

Theorem 1 (Kac's moment formula). *Let $x_0 \in \mathbb{R}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R} \setminus \{0\}$. Then we has*

$$\mathbf{P}_{x_0}[J_0(\mathbf{x})] = u_0(x_0,x_1)\,u_0(x_1,x_2)\cdots u_0(x_{n-1},x_n). \tag{9}$$

The proof is essentially the same to that of [16, Theorem 2.5.3], but we give it for completeness of the paper.

Proof. Note that

$$J_0(\mathbf{x}) = \int_0^\infty J_t(\mathbf{x}')\,dL_t^{x_1}, \tag{10}$$

where $\mathbf{x}' = (x_2, \ldots, x_n)$. Denote $\tau_l^{x_1} = \inf\{t > 0; L_t^{x_1} > l\}$. Since $J_t(\mathbf{x}') = J_0(\mathbf{x}') \circ \theta_t$, the strong Markov property yields that

$$\begin{aligned} \mathbf{P}_{x_0}[J_0(\mathbf{x})] &= \mathbf{P}_{x_0}\left[\int_0^\infty J_0(\mathbf{x}') \circ \theta_{\tau_l^{x_1}} 1_{\{\tau_l^{x_1} < \infty\}}\,dl\right] \\ &= \mathbf{P}_{x_0}\left[\int_0^\infty 1_{\{\tau_l^{x_1} < \infty\}} dl\right] \mathbf{P}_{x_1}\left[J_0(\mathbf{x}')\right] \\ &= \mathbf{P}_{x_0}[L_\infty^{x_1}]\,\mathbf{P}_{x_1}\left[J_0(\mathbf{x}')\right]. \end{aligned} \tag{11}$$

This yields (9) from (7). □

Theorem 2 (Feynman-Kac formula). *Let $x_1, \ldots, x_n \in \mathbb{R} \setminus \{0\}$. Set*

$$\begin{aligned} \Sigma &= \begin{pmatrix} u_0(x_1,x_1) & \cdots & u_0(x_1,x_n) \\ \vdots & \ddots & \vdots \\ u_0(x_n,x_1) & \cdots & u_0(x_n,x_n) \end{pmatrix}, \\ \Sigma^0 &= \begin{pmatrix} u_0(0,x_1) & \cdots & u_0(0,x_n) \\ \vdots & \ddots & \vdots \\ u_0(0,x_1) & \cdots & u_0(0,x_n) \end{pmatrix}. \end{aligned} \tag{12}$$

Then, for any diagonal matrix $\Lambda = (\lambda_i \delta_{i,j})_{i,j=1}^n$ with nonnegative entries, we have

$$\mathbf{P}_0\left[\exp\left\{-\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right\}\right] = \frac{\det\left(I + \left(\Sigma - \Sigma^0\right)\Lambda\right)}{\det(I + \Sigma\Lambda)}. \tag{13}$$

The proof is almost parallel to that of [16, Lemma 2.6.2], but we give it for completeness of the paper.

Proof. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. For $k \in \mathbb{N}$, we have

$$\mathbf{P}_0\left[\left(\sum_{j=1}^{n}\lambda_j L_{\infty}^{x_j}\right)^k\right] = \sum_{j_1,\ldots,j_k=1}^{n}\lambda_{j_1}\cdots\lambda_{j_k}\mathbf{P}_0\left[L_{\infty}^{x_{j_1}}\cdots L_{\infty}^{x_{j_k}}\right] \tag{14}$$

$$= k!\sum_{j_1,\ldots,j_k=1}^{n}\lambda_{j_1}\cdots\lambda_{j_k}\mathbf{P}_0\left[J_0\left(x_{j_1},\ldots,x_{j_k}\right)\right]. \tag{15}$$

It follows from Theorem 1 that

$$(15) = k!\sum_{j_1,\ldots,j_k=1}^{n} u_0\left(0,x_{j_1}\right)\lambda_{j_1} \cdot u_0\left(x_{j_1},x_{j_2}\right)\lambda_{j_2}\cdots u_0\left(x_{j_{k-1}},x_{j_k}\right)\lambda_{j_k} = k!\left\{\left(\widetilde{\Sigma}\widetilde{\Lambda}\right)^k\mathbf{1}\right\}_0, \tag{16}$$

where $\mathbf{1} = {}^{\top}(1,\ldots,1)$, $\{\mathbf{v}\}_0 = v_0$ for $\mathbf{v} = {}^{\top}(v_0, v_1, \ldots, v_n)$,

$$\widetilde{\Sigma} = \begin{pmatrix} 0 & u_0(0,x_1) & \cdots & u_0(0,x_n) \\ 0 & u_0(x_1,x_1) & \cdots & u_0(x_1,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & u_0(x_n,x_1) & \cdots & u_0(x_n,x_n) \end{pmatrix}, \qquad \widetilde{\Lambda} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \tag{17}$$

Hence, for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $|\lambda_i|$'s are small enough, we have

$$\mathbf{P}_0\left[\exp\left\{\sum_{i=1}^{n}\lambda_i L_{\infty}^{x_i}\right\}\right] = \sum_{k=0}^{\infty}\left\{\left(\widetilde{\Sigma}\widetilde{\Lambda}\right)^k\mathbf{1}\right\}_0 = \left\{\left(I-\widetilde{\Sigma}\widetilde{\Lambda}\right)^{-1}\mathbf{1}\right\}_0. \tag{18}$$

By Cramer's formula, we obtain

$$\left\{\left(I-\widetilde{\Sigma}\widetilde{\Lambda}\right)^{-1}\mathbf{1}\right\}_0 = \frac{\det\left(\left(I-\widetilde{\Sigma}\widetilde{\Lambda}\right)^{(1)}\right)}{\det\left(I-\widetilde{\Sigma}\widetilde{\Lambda}\right)} = \frac{\det\left(I-\left(\Sigma-\Sigma^0\right)\Lambda\right)}{\det\left(I-\Sigma\Lambda\right)}. \tag{19}$$

Here, for a matrix A, we denote by $A^{(1)}$ the matrix which is obtained by replacing each entry in the first column of A by number 1. Since Σ is nonnegative definite, we obtain the desired result (13) by analytic continuation. □

Remark 3. Eisenbaum et al. [6] have proved an analogue of Ray-Knight theorem for the total local time of a symmetric Lévy process killed at an independent exponential time. We may say that the key to the proof is that $\Sigma - \Sigma^0$ is a constant matrix which is positive definite. The difficulty in the case of the h-path process of a symmetric Lévy process is that the matrix $\Sigma - \Sigma^0$ no longer has such a nice property.

2.2. Feynman-Kac Formula for Markovian Bridges. In this section, we show Feynman-Kac formula for Markovian bridges. For this, we recall several theorems for Markovian bridges from Fitzsimmons et al. [15]. See [15] for details.

For $t > 0$, $x, y \in \mathbb{R}$, let $\mathbf{P}_{x,y}^t$ denote the bridge law, which serves as a version of the regular conditional distribution for $\{X_s;\ 0 \le s \le t\}$ under $\mathbf{P}_x$ given $X_{t-} = y$. In this section, we assume the following condition:

(i) $0 < p_t(x,y) < \infty$ for any $t > 0$, $x, y \in \mathbb{R}$.

We also assume that there exists a local time (L_t^x) such that

$$\int_0^s f(X_u)\,du = \int f(y)\,L_s^y\mu(dy), \quad 0 \le s \le t, \quad f \in \mathscr{B}_+(\mathbb{R}) \tag{20}$$

holds with $\mathbf{P}_{x,y}^t$-probability one for any $t > 0$ and $x, y \in \mathbb{R}$.

Theorem 4 (see [15, Lemma 1]). *Let $t > 0$, $x, y, z \in \mathbb{R}$. Then one has*

$$\mathbf{P}_{x,y}^t\left[\int_0^t f(s,X_s)\,dL_s^z\right] = \int_0^t ds\frac{p_s(x,z)\,p_{t-s}(z,y)}{p_t(x,y)}f(s,z) \tag{21}$$

for any nonnegative Borel function f.

We will also use the following conditioning formula.

Theorem 5 (see [15, Proposition 3]). *Let $t > 0$, $x, y, z \in \mathbb{R}$. Then one has*

$$\mathbf{P}_{x,y}^t\left[\int_0^t f(s,X_s)\,H_s\,dL_s^z\right] = \mathbf{P}_{x,y}^t\left[\int_0^t f(s,z)\,\mathbf{P}_{x,z}^s\left[H_s\right]dL_s^z\right] \tag{22}$$

for any nonnegative Borel function f and any nonnegative predictable process H_s.

For $s \ge 0$ and $z_1, \ldots, z_n \in \mathbb{R}$, we define

$$H_s\left(\mathbf{z}^{(n)}\right) = \int_0^s dL_{s_n}^{z_n}\int_0^{s_n} dL_{s_{n-1}}^{z_{n-1}}\cdots\int_0^{s_2} dL_{s_1}^{z_1}, \tag{23}$$

where $\mathbf{z}^{(n)} = (z_1, \ldots, z_n)$. The following theorem is a version of Kac's moment formulae.

Theorem 6. *For any $q > 0$, $n \in \mathbb{N}$ and for any $z_1, \ldots, z_n \in \mathbb{R}$, one has*

$$\int_0^\infty e^{-qt} p_t(x,y) \mathbf{P}^t_{x,y}\left[H_t\left(\mathbf{z}^{(n)}\right)\right] dt = u_q(x,z_1) \cdot \prod_{j=1}^{n-1} u_q(z_j, z_{j+1}) \cdot u_q(z_n, y). \tag{24}$$

Proof. Let us prove the claim by induction. For $n = 1$, the assertion follows from Theorem 4. Suppose that formula (24) holds for a given $n \geq 2$. Note that

$$H_t\left(\mathbf{z}^{(n+1)}\right) = \int_0^t H_s\left(\mathbf{z}^{(n)}\right) dL_s^{z_{n+1}}. \tag{25}$$

Since $H_s(\mathbf{z}^{(n)})$ is a nonnegative predictable process, Theorems 5 and 4 show that

$$\mathbf{P}^t_{x,y}\left[H_t\left(\mathbf{z}^{(n+1)}\right)\right] = \int_0^t ds \frac{p_s(x,z_{n+1}) p_{t-s}(z_{n+1},y)}{p_t(x,y)} \mathbf{P}^t_{x,z_{n+1}}\left[H_s\left(z^{(n)}\right)\right]. \tag{26}$$

Hence, we obtain

$$\begin{aligned}
&\int_0^\infty e^{-qt} p_t(x,y) \mathbf{P}^t_{x,y}\left[H_t\left(z^{(n+1)}\right)\right] dt \\
&= \int_0^\infty e^{-qt} dt \times \int_0^t p_s(x,z_{n+1}) p_{t-s}(z_{n+1},y) \mathbf{P}^s_{x,z_{n+1}}\left[H_s\left(\mathbf{z}^{(n)}\right)\right] ds \\
&= \int_0^\infty e^{-qs} p_s(x,z_{n+1}) \mathbf{P}^s_{x,z_{n+1}}\left[H_s\left(\mathbf{z}^{(n)}\right)\right] ds \times \int_0^\infty e^{-qt} p_t(z_{n+1},y) dt \\
&= u_q(x,z_1) \cdot \prod_{j=1}^{n-1} u_q(z_j,z_{j+1}) \cdot u_q(z_n,z_{n+1}) \cdot u_q(z_{n+1},y),
\end{aligned} \tag{27}$$

by the assumption of the induction. Now we have proved that formula (24) is valid also for $n+1$, which completes the proof. □

The following theorem is a version of Feynman-Kac formulae.

Theorem 7. *Let $z_1 = 0, z_2, \ldots, z_n \in \mathbb{R}$ and let $\lambda_1, \ldots, \lambda_n \geq 0$. Suppose that*

$$u_q(z_i, z_j) < \infty, \quad q > 0,\ i,j = 1, \ldots, n. \tag{28}$$

Let $\Sigma^{(q)}$ be the matrix with elements $\Sigma^{(q)}_{i,j} = u_q(z_i, z_j)$. Then, for any diagonal matrix $\Lambda = (\lambda_i \delta_{i,j})_{i,j=1}^n$ with nonnegative entries, one has

$$\int_0^\infty e^{-qt} p_t(0,0) \mathbf{P}^t_{0,0}\left[e^{-\sum_{j=1}^n \lambda_j L_t^{z_j}}\right] dt = \left\{\left(I + \Sigma^{(q)}\Lambda\right)^{-1} \Sigma^{(q)}\right\}_{1,1}. \tag{29}$$

Proof. We have

$$\begin{aligned}
&\int_0^\infty e^{-qt} p_t(0,0) \mathbf{P}^t_{0,0}\left[\left(\sum_{j=1}^n \lambda_j L_t^{z_j}\right)^k\right] dt \\
&= k! \sum_{j_1,\ldots,j_k=1}^n \lambda_{j_k} \cdots \lambda_{j_1} \times \int_0^\infty e^{-qt} p_t(0,0) \mathbf{P}^t_{0,0}\left[H_t\left(z_{j_k}, \ldots, z_{j_1}\right)\right] dt.
\end{aligned} \tag{30}$$

Using Theorem 6, we see that the above quantity is equal to

$$k! \sum_{j_1,\ldots,j_k=1}^n u_q\left(z_1, z_{j_1}\right) \lambda_{j_1} \cdot \prod_{i=1}^{k-1} u_q\left(z_{j_i}, z_{j_{i+1}}\right) \lambda_{j_{i+1}} \cdot u_q\left(z_{j_k}, z_1\right), \tag{31}$$

which amounts to $k!\{(\Sigma^{(q)}\Lambda)^k \Sigma^{(q)}\}_{1,1}$. Hence, for all $\lambda_1, \ldots, \lambda_n > 0$ sufficiently small, we obtain

$$\begin{aligned}
&\int_0^\infty e^{-qt} p_t(0,0) \mathbf{P}^t_{0,0}\left[\exp\left\{\sum_{j=1}^n \lambda_j L_t^{z_j}\right\}\right] dt \\
&= u_q(0,0) + \sum_{k=1}^\infty \left\{\left(\Sigma^{(q)}\Lambda\right)^k \Sigma^{(q)}\right\}_{1,1} \\
&= \left\{(I - \Sigma^{(q)}\Lambda)^{-1} \Sigma^{(q)}\right\}_{1,1}.
\end{aligned} \tag{32}$$

Since $\Sigma^{(q)}$ is nonnegative definite, we obtain the desired result (29) by analytic continuation. □

The following theorem is valid even if

$$u_q\left(0, z_j\right) = u_q\left(z_j, 0\right) = \infty, \quad q > 0, j = 1, \ldots, n. \tag{33}$$

Theorem 8. *Let $z_1, \ldots, z_n \in \mathbb{R} \setminus \{0\}$ and let $\lambda_1, \ldots, \lambda_n \geq 0$. Suppose that*

$$u_q\left(z_i, z_j\right) < \infty, \quad q > 0, i, j = 1, \ldots, n. \tag{34}$$

Let $\Sigma^{(q)}$ be the matrix with elements $\Sigma^{(q)}_{i,j} = u_q(z_i, z_j)$,

$$\mathbf{u}^{(q)} = \begin{pmatrix} u_q(0,z_1) \\ \vdots \\ u_q(0,z_n) \end{pmatrix}, \qquad \mathbf{v}^{(q)} = \begin{pmatrix} u_q(z_1,0) \\ \vdots \\ u_q(z_n,0) \end{pmatrix}, \tag{35}$$

and let Λ be the matrix with elements $\Lambda_{i,j} = \lambda_i \delta_{i,j}$. Then one has

$$\int_0^\infty e^{-qt} p_t(0,0) \mathbf{P}^t_{0,0} \left[1 - e^{-\sum_{j=1}^n \lambda_j L_t^{z_j}} \right] dt = {}^\top \mathbf{u}^{(q)} \Lambda \left(I + \Sigma^{(q)} \Lambda \right)^{-1} \mathbf{v}^{(q)}. \tag{36}$$

Proof. Using Theorem 6, we see that

$$\begin{aligned} &\int_0^\infty e^{-qt} p_t(0,0) \mathbf{P}^t_{0,0} \left[\left(\sum_{j=1}^n \lambda_j L_t^{z_j} \right)^k \right] dt \\ &= k! \sum_{j_1,\dots,j_k=1}^n u_q(0, z_{j_1}) \lambda_{j_1} \cdot \prod_{i=1}^{k-1} u_q(z_{j_i}, z_{j_{i+1}}) \lambda_{j_{i+1}} \\ &\cdot u_q(z_{j_k}, 0). \end{aligned} \tag{37}$$

Hence, we obtain

$$\begin{aligned} &\int_0^\infty e^{-qt} p_t(0,0) \mathbf{P}^t_{0,0} \left[\exp\left\{ \sum_{j=1}^n \lambda_j L_t^{z_j} \right\} - 1 \right] dt \\ &= \sum_{k=1}^\infty \left\{ {}^\top \mathbf{u}^{(q)} \Lambda \left(\Sigma^{(q)} \Lambda \right)^{k-1} \mathbf{v}^{(q)} \right\} \\ &= {}^\top \mathbf{u}^{(q)} \Lambda \left(I - \Sigma^{(q)} \Lambda \right)^{-1} \mathbf{v}^{(q)}. \end{aligned} \tag{38}$$

The rest of the proof is now obvious. □

3. Preliminaries: Squared Bessel Processes and Generalized Excursion Measures

In this section, we recall squared Bessel processes and generalized excursion measures.

First, we introduce several notations about squared Bessel processes, for which we follow [3, XI.1]. For $\delta \geq 0$, let $(\mathbf{Q}^\delta_z : z \geq 0)$ denote the law of the δ-dimensional squared Bessel process where the origin is a trap when $\delta = 0$. Then the Laplace transform of a one-dimensional marginal is given by

$$\mathbf{Q}^\delta_z \left[\exp\{-\lambda X_t\} \right] = \frac{1}{(1+2\lambda t)^{\delta/2}} \exp\left\{ -\frac{\lambda z}{1+2\lambda t} \right\}. \tag{39}$$

We may obtain the transition kernels $\mathbf{Q}^\delta_z(X_t \in dw)$ by the Laplace inversion.

(i) For $\delta > 0$ and $z > 0$, we have

$$\begin{aligned} &\mathbf{Q}^\delta_z(X_t \in dw) \\ &= \frac{1}{2t} \left(\frac{w}{z} \right)^{(1/2)(\delta/2-1)} \exp\left\{ -\frac{z+w}{2t} \right\} I_{\delta/2-1}\left(\frac{\sqrt{zw}}{t} \right) dw, \end{aligned} \tag{40}$$

where I_ν stands for the modified Bessel function of order ν.

(ii) For $\delta > 0$ and $z = 0$, we have

$$\mathbf{Q}^\delta_0(X_t \in dw) = \frac{1}{(2t)^{\delta/2} \Gamma(\delta/2)} w^{\delta/2-1} \exp\left\{ -\frac{w}{2t} \right\} dw, \tag{41}$$

where Γ stands for the gamma function.

(iii) For $\delta = 0$ and $z \geq 0$, we have

$$\begin{aligned} \mathbf{Q}^0_z(X_t \in dw) = {}& \exp\left\{ -\frac{z}{2t} \right\} \delta_0(dw) \\ &+ \frac{1}{2t} \left(\frac{w}{z} \right)^{-1/2} \exp\left\{ -\frac{z+w}{2t} \right\} \\ &\times I_1\left(\frac{\sqrt{zw}}{t} \right) dw. \end{aligned} \tag{42}$$

The squared Bessel process satisfies the scaling property: for $\delta \geq 0$, $z \geq 0$, and $c > 0$, it holds that

$$(cX_{t/c}) \text{ under } \mathbf{Q}^\delta_{z/c} \overset{\text{law}}{=} (X_t) \text{ under } \mathbf{Q}^\delta_z. \tag{43}$$

Second, we recall the notion of the generalized excursion measure. By formula (39), we have

$$\mathbf{Q}^4_0 \left[\frac{1}{X^2_{s+t}} ; X_{s+t} \in B \right] = \mathbf{Q}^4_0 \left[\frac{1}{X^2_s} \cdot \mathbf{Q}^0_{X_s}(X_t \in B) \right] \tag{44}$$

for $s, t > 0$ and $B \in \mathscr{B}([0,\infty))$. If we put $\mu_t(dx) = (1/x^2)\mathbf{Q}^4_0(X_t \in dx)$, we have

$$\mu_{s+t}(B) = \int \mu_s(dx) \mathbf{Q}^0_x(X_t \in B). \tag{45}$$

This shows that the family of laws $\{\mu_t : t > 0\}$ is an entrance law for $\{\mathbf{Q}^0_x : x > 0\}$. In fact, there exists a unique σ-finite measure $\mathbf{n}^{(0)}$ on $\mathbb{D}$ such that

$$\begin{aligned} &\mathbf{n}^{(0)}\left(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n \right) \\ &= \int_{B_1} \mu_{t_1}(dx_1) \int_{B_2} \mathbf{Q}^0_x \left(X_{t_2-t_1} \in dx_2 \right) \\ &\cdots \int_{B_n} \mathbf{Q}^0_x \left(X_{t_n-t_{n-1}} \in dx_n \right) \end{aligned} \tag{46}$$

for $0 < t_1 < \cdots < t_n$ and $B_1, \dots, B_n \in \mathscr{B}([0,\infty))$. Note that, to construct such a measure $\mathbf{n}^{(0)}$, we can not appeal to Kolmogorov's extension theorem, because the entrance laws have infinite total mass. However, we can actually construct $\mathbf{n}^{(0)}$ via the *agreement formula* (see Pitman-Yor [17, Cor. 3] with $\delta = 4$), or via the time change of a Brownian excursion (see Fitzsimmons-Yano [18, Theorem 2.5] with change of scales). We may call $\mathbf{n}^{(0)}$ the *generalized excursion measure* for the squared Bessel process of dimension 0. See the references above for several characteristic formulae of $\mathbf{n}^{(0)}$.

4. Symmetric Lévy Processes

Let us confine ourselves to one-dimensional symmetric Lévy processes. We recall general facts and state several results from [7].

In what follows, we assume that $(\mathbf{P}_x)$ is the law of a one-dimensional conservative Lévy process. Throughout the present paper, we assume the following conditions, which will be referred to as $(\mathbf{A})$, are satisfied:

(i) the process is symmetric;

(ii) the origin (and, consequently, any point) is regular for itself;

(iii) the process is not a compound Poisson.

Under the condition $(\mathbf{A})$, we have the following. The characteristic exponent is given by

$$\theta(\lambda) := -\log \mathbf{P}_0\left[e^{i\lambda X_1}\right] = \upsilon\lambda^2 + 2\int_0^\infty (1-\cos\lambda x)\,\nu(dx), \tag{47}$$

for some $\upsilon \geq 0$ and some positive Radon measure ν on $(0,\infty)$ such that

$$\int_{(0,\infty)} \min\left\{x^2, 1\right\} \nu(dx) < \infty. \tag{48}$$

The reference measure is $\mu(dx) = dx$ and we have

$$p_t(x,y) = p_t(y-x) = \frac{1}{\pi}\int_0^\infty (\cos\lambda(y-x))\,e^{-t\theta(\lambda)}d\lambda, \tag{49}$$

$$u_q(x,y) = u_q(y-x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda(y-x)}{q+\theta(\lambda)}\,d\lambda. \tag{50}$$

There exists a local time (L_t^x) such that

$$\int_0^t f(X_s)\,ds = \int f(y)\,L_t^y dy, \quad f \in \mathscr{B}_+(\mathbb{R}), \tag{51}$$

with $\mathbf{P}_x$-probability one for any $x \in \mathbb{R}$. Then it holds that

$$\mathbf{P}_x\left[\int_0^\infty e^{-qs}dL_s^y\right] = u_q(y-x), \quad x,y \in \mathbb{R}. \tag{52}$$

Let $\mathbf{n}$ denote the excursion measure associated to the local time L_t^0. We denote by $(\mathbf{P}_x^0 : x \in \mathbb{R}\setminus\{0\})$ the law of the process killed upon hitting the origin; that is,

$$\mathbf{P}_x^0(A;\zeta > t) = \mathbf{P}_x\left(A; T_{\{0\}} > t\right), \quad x \in \mathbb{R}\setminus\{0\}, \quad t > 0,\ A \in \mathscr{F}_t. \tag{53}$$

Then the excursion measure $\mathbf{n}$ satisfies the Markov property in the following sense: for any $t > 0$ and for any nonnegative $\mathscr{F}_t$-measurable functional Z_t and for any nonnegative $\mathscr{F}_\infty$-measurable functional F, it holds that

$$\mathbf{n}\left[Z_t F(X_{t+\cdot})\right] = \int \mathbf{n}\left[Z_t; X_t \in dx\right]\mathbf{P}_x^0\left[F(X)\right]. \tag{54}$$

We need the following additional conditions:

$(\mathbf{R})$ the process is recurrent;

$(\mathbf{T})$ the function $\theta(\lambda)$ is nondecreasing in $\lambda > \lambda_0$ for some $\lambda_0 > 0$.

Under the condition $(\mathbf{A})$, the condition $(\mathbf{R})$ is equivalent to

$$\int_0^\infty \frac{d\lambda}{\theta(\lambda)} = \infty. \tag{55}$$

All of the conditions $(\mathbf{A})$, $(\mathbf{R})$, and $(\mathbf{T})$ are obviously satisfied if the process is a symmetric stable Lévy process of index $\alpha \in (1,2]$:

$$\theta(\lambda) = |\lambda|^\alpha. \tag{56}$$

In what follows, we assume, as well as the condition $(\mathbf{A})$, that the conditions $(\mathbf{R})$ and $(\mathbf{T})$ are also satisfied.

The Laplace transform of the law of $T_{\{0\}}$ is given by

$$\mathbf{P}_z\left[e^{-qT_{\{0\}}}\right] = \frac{u_q(z)}{u_q(0)}, \tag{57}$$

see, for example, [19, pp. 64]. It is easy to see that the entrance law has the space density:

$$\rho(t,x) = \frac{\mathbf{n}(X_t \in dx)}{dx}. \tag{58}$$

In view of [7, Theorem 2.10], the law of the hitting time $T_{\{0\}}$ is absolutely continuous relative to the Lebesgue measure dt and the time density coincides with the space density of the entrance law:

$$\rho_x(t) = \frac{\mathbf{P}_x\left(T_{\{0\}} \in dt\right)}{dt} = \rho(t,x). \tag{59}$$

4.1. Absolute Continuity of the Law of the Inverse Local Time. Let $\tau(l)$ denote the inverse local time at the origin:

$$\tau(l) = \inf\left\{t > 0; L_t^0 > l\right\}. \tag{60}$$

We prove the absolute continuity of the law of inverse local time. Note that $\tau(l)$ is a subordinator such that

$$\mathbf{P}_0\left[e^{-q\tau(l)}\right] = e^{-l/u_q(0)}, \tag{61}$$

see, for example, [19, pp. 131].

Lemma 9. *For fixed $l > 0$, the law of $\tau(l)$ under $\mathbf{P}_0$ has a density $\gamma_l(t)$:*

$$\mathbf{P}_0(\tau(l) \in dt) = \gamma_l(t)\,dt. \tag{62}$$

Furthermore, $\gamma_l(t)$ may be chosen to be jointly continuous in $(l,t) \in (0,\infty)\times(0,\infty)$.

Proof. Following [7, Sec. 3.3], we define a positive Borel measure σ on $[0,\infty)$ as

$$\sigma(A) = \frac{1}{\pi}\int_0^\infty 1_A(\theta(\lambda))\,d\lambda, \quad A \in \mathscr{B}([0,\infty)). \tag{63}$$

Then we have $u_q(0) = \int_{[0,\infty)}(\sigma(d\xi)/(q+\xi))$ for $q > 0$, and hence there exists a Radon measure σ^* on $[0,\infty)$ with $\int_{[0,\infty)}(\sigma^*(d\xi)/(1+\xi)) < \infty$ such that

$$\frac{1}{qu_q(0)} = \int_{[0,\infty)} \frac{1}{q+\xi}\sigma^*(d\xi), \quad q > 0. \tag{64}$$

Hence, the Laplace exponent $1/u_q(0)$ may be represented as

$$\frac{1}{u_q(0)} = \int_0^\infty (1-e^{-qu})\nu(u)\,du, \tag{65}$$

where $\nu(u) = \int_{(0,\infty)} e^{-u\xi}\xi\sigma^*(d\xi)$. Since $\int_0^\infty (1\wedge u^2)\nu(u)du < \infty$, we may appeal to analytic continuation of both sides of formula (61) and obtain

$$\mathbf{P}_0\left[e^{i\lambda\tau(l)}\right] = \exp\left\{l\int_0^\infty \left(e^{i\lambda u}-1\right)\nu(u)\,du\right\}. \tag{66}$$

Following [20, Theorem 3.1], we may invert the Fourier transform of the law of $\tau(l)$ and obtain the desired result. □

4.2. h-Paths of Symmetric Lévy Processes. We follow [7] for the notations concerning h-paths of symmetric Lévy processes. For the interpretation of the h-paths as some kind of conditioning, see [10].

We define

$$\begin{aligned} h(x) &= \lim_{q\to 0+}\left\{u_q(0)-u_q(x)\right\} \\ &= \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda x}{\theta(\lambda)}d\lambda, \quad x\in\mathbb{R}. \end{aligned} \tag{67}$$

The second equality follows from (50). Then the function h satisfies the following:

(i) $h(x)$ is continuous;

(ii) $h(0)=0$, $h(x)>0$ for all $x\in\mathbb{R}\setminus\{0\}$;

(iii) $h(x)\to\infty$ as $|x|\to\infty$ (since the condition (**R**) is satisfied).

See [7, Lemma 4.2] for the proof. Moreover, the function h is harmonic with respect to the killed process:

$$\begin{aligned} \mathbf{P}_x^0\left[h(X_t)\right] &= h(x) \quad \text{if } x\in\mathbb{R}\setminus\{0\},\ t>0, \\ \mathbf{n}\left[h(X_t)\right] &= 1 \quad \text{if } t>0. \end{aligned} \tag{68}$$

See [7, Theorems 1.1 and 1.2] for the proof. We define the h-path process $(\mathbf{P}_x^h : x\in\mathbb{R})$ by the following local equivalence relations:

$$d\mathbf{P}_x^h\big|_{\mathcal{F}_t} = \begin{cases} \dfrac{h(X_t)}{h(x)}d\mathbf{P}_x^0\Big|_{\mathcal{F}_t} & \text{if } x\in\mathbb{R}\setminus\{0\}, \\ h(X_t)\,d\mathbf{n}\big|_{\mathcal{F}_t} & \text{if } x=0. \end{cases} \tag{69}$$

Remark that, from the strong Markov properties of (X_t) under $\mathbf{P}_x^0$ and $\mathbf{n}$, the family $\{\mathbf{P}_x^h|_{\mathcal{F}_t}; t\ge 0\}$ is consistent, and hence the probability measure $\mathbf{P}_x^h$ is well defined.

The h-path process is then symmetric; more precisely, the transition kernel has a symmetric density $p_t^h(x,y)$ with respect to the measure $h(y)^2dy$. Here the density $p_t^h(x,y)$ is given by

$$\begin{aligned} p_t^h(x,y) &= \frac{1}{h(x)h(y)}\left\{p_t(y-x)-\frac{p_t(x)p_t(y)}{p_t(0)}\right\} \\ &\qquad \text{if } x,y\in\mathbb{R}\setminus\{0\}, \\ p_t^h(x,0) &= p_t^h(0,x) = \frac{\rho(t,x)}{h(x)} \\ &\qquad \text{if } x\in\mathbb{R}\setminus\{0\}, \\ p_t^h(0,0) &= \int_{(0,\infty)} e^{-t\xi}\xi\sigma^*(d\xi). \end{aligned} \tag{70}$$

By (65), we see that $p_t^h(0,0)$ is characterized by

$$\int_0^\infty \left(1-e^{-qt}\right)p_t^h(0,0)\,dt = \frac{1}{u_q(0)}, \quad q>0. \tag{71}$$

See [7, Section 5] for the details. The h-path process also satisfies the following conditions:

(i) the process is conservative;

(ii) any point is regular for itself;

(iii) the process is transient (since the condition (**T**) is satisfied).

We can easily prove regularity of any point by the local equivalence (69). See [7, Theorem 1.4] for the proof of transience.

The resolvent density of the h-path process with respect to $h(y)^2dy$ is given by

$$\begin{aligned} u_q^h(x,y) &= \frac{1}{h(x)h(y)}\left\{u_q(x-y)-\frac{u_q(x)u_q(y)}{u_q(0)}\right\}, \\ &\qquad q>0,\ x,y\in\mathbb{R}\setminus\{0\}, \\ u_q^h(x,0) &= u_q^h(0,x) = \frac{1}{h(x)}\cdot\frac{u_q(x)}{u_q(0)}, \quad q>0,\ x\in\mathbb{R}\setminus\{0\}. \end{aligned} \tag{72}$$

We remark here that, since $\lim_{q\to\infty}u_q(0)=0$, we see, by (71), that

$$u_q^h(0,0)=\infty. \tag{73}$$

The Green function $u_0^h(x,y) = \lim_{q\to 0+}u_q^h(x,y)$ exists and is given by

$$\begin{aligned} u_0^h(x,y) &= \frac{h(x)+h(y)-h(x-y)}{h(x)h(y)}, \quad x,y\in\mathbb{R}\setminus\{0\}, \\ u_0^h(x,0) &= u_0^h(0,x) = \frac{1}{h(x)}, \quad x\in\mathbb{R}\setminus\{0\}. \end{aligned} \tag{74}$$

See [7, Section 5.3] for the proof. Since $u_0^h(x,y) \geq 0$, we have

$$h(x) + h(y) - h(x-y) \geq 0, \quad x, y \in \mathbb{R}. \tag{75}$$

It follows from the local equivalence (69) that there exists a local time (L_t^x) such that

$$\int_0^t f(X_s)\, ds = \int f(y) L_t^y\, h(y)^2\, dy, \quad t > 0, f \in \mathscr{B}_+(\mathbb{R}) \tag{76}$$

with $\mathbf{P}_x^h$-probability one for any $x \in \mathbb{R}$. We have

$$\mathbf{P}_x^h\left[L_\infty^y\right] = u_0^h(x,y), \quad x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}. \tag{77}$$

Example 10. If the process is the symmetric stable process of index $\alpha \in (1,2]$, then the harmonic function $h(x)$ may be computed as

$$h(x) = C(\alpha)|x|^{\alpha-1}, \tag{78}$$

where $C(\alpha)$ is given as follows (see [9, Appendix]):

$$C(\alpha) = \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^\alpha}\, d\lambda = \frac{1}{2\Gamma(\alpha)\sin(\pi(\alpha-1)/2)}. \tag{79}$$

5. The Laws of the Total Local Times for h-Paths

In this section, we state and prove our main theorems concerning the laws of the total local times of h-paths.

5.1. Laplace Transform Formula for h-Paths. In this section, we prove Laplace transform formula for h-paths at two levels.

Lemma 11. *For $x, y \in \mathbb{R} \setminus \{0\}$ and $\lambda_1, \lambda_2 \geq 0$, one has*

$$\begin{aligned}\mathbf{P}_0^h&\left[\exp\left\{-\lambda_1 h(x) L_\infty^x - \lambda_2 h(y) L_\infty^y\right\}\right] \\ &= \frac{1+\lambda_1+\lambda_2+D\lambda_1\lambda_2}{1+2\lambda_1+2\lambda_2+4E\lambda_1\lambda_2},\end{aligned} \tag{80}$$

where

$$\begin{aligned} D = D(x,y) &= h(x-y)\cdot\frac{h(x)+h(y)-h(x-y)}{h(x)h(y)} \geq 0, \\ E = E(x,y) &= 1 - \frac{(h(x)+h(y)-h(x-y))^2}{4h(x)h(y)} \geq 0. \end{aligned} \tag{81}$$

Proof. Let us apply Theorem 2 with

$$\begin{aligned} I + &\left(\Sigma - \Sigma^0\right)\Lambda \\ &= \begin{pmatrix} 1+\lambda_1 & \dfrac{h(y)-h(x-y)}{h(x)}\lambda_2 \\ \dfrac{h(x)-h(x-y)}{h(y)}\lambda_1 & 1+\lambda_2 \end{pmatrix}, \\ I + &\Sigma\Lambda \\ &= \begin{pmatrix} 1+2\lambda_1 & \dfrac{h(x)+h(y)-h(x-y)}{h(x)}\lambda_2 \\ \dfrac{h(x)+h(y)-h(x-y)}{h(y)}\lambda_1 & 1+2\lambda_2 \end{pmatrix}. \end{aligned} \tag{82}$$

Then we obtain (80) by an easy computation.

By (75), we have $D \geq 0$. Since

$$E = \frac{h(x)h(y)}{4}\det\begin{pmatrix} u_0^h(x,x) & u_0^h(x,y) \\ u_0^h(y,x) & u_0^h(y,y) \end{pmatrix}, \tag{83}$$

we obtain $E \geq 0$ by nonnegative definiteness of the above matrix. The proof is now complete. □

5.2. The Law of L_∞^x. Using formula (80), we can determine the law of L_∞^x; see [16, Example 3.10.5] for the formula in a more general case.

Theorem 12. *For any $x \in \mathbb{R} \setminus \{0\}$, one has*

$$\mathbf{P}_0^h\left(h(x)L_\infty^x \in dl\right) = \frac{1}{2}\left\{\delta_0(dl) + e^{-l/2}\frac{dl}{2}\right\}, \tag{84}$$

where δ_0 stands for the Dirac measure concentrated at 0. Consequently, one has

$$\mathbf{P}_0^h\left(L_\infty^x = 0\right) = \frac{1}{2}. \tag{85}$$

Proof. Letting $\lambda_2 = 0$ in Lemma 11, we have

$$\mathbf{P}_0^h\left[\exp\left\{-\lambda_1 h(x) L_\infty^x\right\}\right] = \frac{1+\lambda_1}{1+2\lambda_1} = \frac{1}{2}\left(1+\frac{1}{1+2\lambda_1}\right), \tag{86}$$

which proves the claim. □

Remark 13. Since $L_\infty^x = 0$ if and only if $T_{\{x\}} = \infty$, the identity (85) is equivalent to

$$\mathbf{P}_0^h\left(T_{\{x\}} = \infty\right) = \frac{1}{2}. \tag{87}$$

This formula may also be obtained from the following formula (see [9, Proposition 5.10]):

$$\mathbf{n}\left(T_{\{x\}} < \zeta\right) = \frac{1}{2h(x)}. \tag{88}$$

Suppose that, in the definition (69), we may replace the fixed time t with the stopping time $T_{\{x\}}$. Then we have

$$\begin{aligned} \mathbf{P}_0^h\left(T_{\{x\}} < \infty\right) &= \mathbf{n}\left[h\left(X_{T_{\{x\}}}\right); T_{\{x\}} < \zeta\right] \\ &= h(x)\,\mathbf{n}\left(T_{\{x\}} < \zeta\right) = \frac{1}{2}. \end{aligned} \tag{89}$$

5.3. The Probability That Two Levels Are Attained. Let us discuss the probability that the total local times at two given levels are positive.

Theorem 14. *Let $x, y \in \mathbb{R} \setminus \{0\}$ such that $x \neq y$. Then one has $E > 0$ and*

$$\mathbf{P}_0^h\left(L_\infty^x > 0, L_\infty^y > 0\right) = \mathbf{P}_0^h\left(L_\infty^x = L_\infty^y = 0\right) = \frac{D}{4E}, \tag{90}$$

$$\mathbf{P}_0^h\left(L_\infty^x > 0, L_\infty^y = 0\right) = \mathbf{P}_0^h\left(L_\infty^x = 0, L_\infty^y > 0\right) = \frac{1}{2} - \frac{D}{4E}. \tag{91}$$

Consequently, one has $D \leq 2E$.

Proof. Letting $\lambda_1 = \lambda_2 = \lambda \geq 0$ in formula (80), we have

$$\mathbf{P}_0^h\left[\exp\left\{-\lambda h(x) L_\infty^x - \lambda h(y) L_\infty^y\right\}\right] = \frac{1 + 2\lambda + D\lambda^2}{1 + 4\lambda + 4E\lambda^2}. \tag{92}$$

If E were zero, then D would be positive, and hence the right-hand side of (92) would diverge as $\lambda \to \infty$, which contradicts the fact that the left-hand side of (92) is bounded in $\lambda > 0$. Hence, we obtain $E > 0$.

Taking the limit as $\lambda \to \infty$ in both sides of formula (92), we have

$$\mathbf{P}_0^h\left(L_\infty^x = L_\infty^y = 0\right) = \frac{D}{4E}, \tag{93}$$

which is nothing else but the second equality of (90). By formula (85), we obtain

$$\begin{aligned} \mathbf{P}_0^h\left(L_\infty^x = 0, L_\infty^y > 0\right) &= \mathbf{P}_0^h\left(L_\infty^x = 0\right) \\ &\quad - \mathbf{P}_0^h\left(L_\infty^x = L_\infty^y = 0\right) = \frac{1}{2} - \frac{D}{4E}. \end{aligned} \tag{94}$$

Thus we obtain (91). Therefore, we obtain

$$\mathbf{P}_0^h\left(L_\infty^x > 0, L_\infty^y > 0\right) = 1 - \frac{D}{4E} - 2\left\{\frac{1}{2} - \frac{D}{4E}\right\} = \frac{D}{4E}, \tag{95}$$

which is nothing else but the first equality of (90). The proof is now complete. □

5.4. Joint Law of L_∞^x and L_∞^y. Let us discuss the joint law of L_∞^x and L_∞^y for $x, y \in \mathbb{R} \setminus \{0\}$ such that $x \neq y$.

By Lemma 11, we know that $D \geq 0$. First, we discuss the case of $D = 0$.

Theorem 15. *Suppose that $h(x) + h(y) - h(x - y) = 0$. Then*

$$\begin{aligned} &\mathbf{P}_0^h\left(h(x) L_\infty^x \in dl_1, h(y) L_\infty^y \in dl_2\right) \\ &\quad = \frac{1}{2}\left\{e^{-l_1/2}\frac{dl_1}{2} \cdot \delta_0\left(dl_2\right) + \delta_0\left(dl_1\right) \cdot e^{-l_2/2}\frac{dl_2}{2}\right\}. \end{aligned} \tag{96}$$

Proof. Since $D = 0$ and $E = 1$, formula (80) implies

$$\begin{aligned} &\mathbf{P}_0^h\left[\exp\left\{-\lambda_1 h(x) L_\infty^x - \lambda_2 h(y) L_\infty^y\right\}\right] \\ &\quad = \frac{1 + \lambda_1 + \lambda_2}{1 + 2\lambda_1 + 2\lambda_2 + 4\lambda_1\lambda_2}. \end{aligned} \tag{97}$$

We may rewrite the right-hand side as

$$\frac{1}{2}\left(\frac{1}{1 + 2\lambda_1} + \frac{1}{1 + 2\lambda_2}\right), \tag{98}$$

which proves the claim. □

Second, we discuss the case of $D > 0$.

Theorem 16. *Suppose that $h(x) + h(y) - h(x - y) > 0$. Set*

$$\begin{aligned} m &= m(x, y) = h(x) + h(y) - h(x - y), \\ M &= M(x, y) = \frac{4h(x)\,h(y)}{h(x) + h(y) - h(x - y)}. \end{aligned} \tag{99}$$

Then one has $E > 0$ and $0 < m < M$. For any $A, B \in \mathscr{B}([0, \infty))$, one has

$$\mathbf{P}_0^h\left(h(x) L_\infty^x \in A,\ h(y) L_\infty^y \in B\right) = \sum_{k=1}^{4} C_k \Phi_k(A \times B), \tag{100}$$

where $C_k = C_k(x, y)$, $k = 1, 2, 3, 4$ are constants given as

$$C_1 = \frac{D}{4E}, \qquad C_3 = C_4 = \frac{1}{2E}\left(1 - \frac{D}{2E}\right), \qquad C_2 = 1 - C_1 - C_3 - C_4 \tag{101}$$

and Φ_k, $k = 1, 2, 3, 4$ are positive measures on $[0, \infty)^2$ such that

$$\Phi_1(A \times B) = \delta_0(A)\,\delta_0(B), \tag{102}$$

$$\begin{aligned} \Phi_2(A \times B) &= \mathbf{Q}_0^2\left(\frac{X_m}{m} \in A, \frac{X_M}{M} \in B\right) \\ &= \mathbf{Q}_0^2\left(\frac{X_M}{M} \in A, \frac{X_m}{m} \in B\right), \end{aligned} \tag{103}$$

$$\begin{aligned} \Phi_3(A \times B) &= \Phi_4(B \times A) \\ &= \mathbf{Q}_0^2\left[\frac{X_m}{m} \in A; \mathbf{Q}_{X_m}^0\left(\frac{X_{M-m}}{M} \in B\right)\right]. \end{aligned} \tag{104}$$

Remark 17. The expression (104) coincides with

$$\mathbf{n}^{(0)}\left[2mX_m; \frac{X_m}{m} \in A, \frac{X_M}{M} \in B\right], \tag{105}$$

where $\mathbf{n}^{(0)}$ is the generalized excursion measure introduced in Section 3.

The proof of Theorem 16 will be given in the next section.

Remark 18. In the case where $\alpha = 2$, the process $((X_t/\sqrt{2}), \mathbf{P}_0^h)$ is the symmetrized three-dimensional Bessel process. In other words, if we set

$$\Omega_+ = \{w \in \mathbb{D} : \forall t > 0, w(t) > 0\},$$
$$\Omega_- = \{w \in \mathbb{D} : \forall t > 0, w(t) < 0\}, \tag{106}$$

then we have

$$\mathbf{P}_0^h(\Omega_+) = \mathbf{P}_0^h(\Omega_-) = \frac{1}{2} \tag{107}$$

and the processes $((X_t/\sqrt{2}), \mathbf{P}_0^h(\cdot \mid \Omega_+))$ and $((-X_t/\sqrt{2}), \mathbf{P}_0^h(\cdot \mid \Omega_-))$ are one-sided three-dimensional Bessel processes. Hence, the Ray-Knight theorem implies that the process $(x^2 L_\infty^x : x \geq 0)$ conditional on Ω_+ is the squared Bessel process of dimension two. Let us check that Theorems 15 and 16 are consistent with this fact. Since $h(x) = |x|/2$, we have

$$h(x) + h(y) - h(x-y) = \frac{|x| + |y| - |x-y|}{2} = \begin{cases} \min\{|x|, |y|\} & \text{if } xy > 0, \\ 0 & \text{if } xy < 0. \end{cases} \tag{108}$$

If $x > 0 > y$, then we should look at Theorem 15 which implies that

$$\mathbf{P}_0^h(L_\infty^x \in A, L_\infty^y \in B \mid \Omega_+) = \mathbf{Q}_0^2(X_1 \in A) \cdot \delta_0(B). \tag{109}$$

If $x, y > 0$, then we should look at Theorem 16. Note that

$$m(x,y) = \min\{x,y\}, \qquad M(x,y) = \max\{x,y\},$$
$$D = \frac{2|x-y|}{\max\{x,y\}}, \qquad E = \frac{|x-y|}{\max\{x,y\}}, \qquad \frac{D}{4E} = \frac{1}{2} \tag{110}$$

and that

$$C_1 = C_2 = \frac{1}{2}, \qquad C_3 = C_4 = 0. \tag{111}$$

Hence, Theorem 16 implies that

$$\mathbf{P}_0^h\left(x^2 L_\infty^x \in A, y^2 L_\infty^y \in B \mid \Omega_+\right) = \mathbf{Q}_0^2\left(X_x \in A, X_y \in B\right). \tag{112}$$

5.5. Proof of Theorem 16. We give the proof of Theorem 16. We divide the proofs into several steps.

Step 1. Since

$$0 < D \leq 2E = 2\left(1 - \frac{m}{M}\right), \tag{113}$$

we have $0 < m < M$.

Step 2. Let us compute the Laplace transform:

$$\mathbf{Q}_0^2\left[\exp\left\{-\lambda_1 \frac{X_m}{m} - \lambda_2 \frac{X_M}{M}\right\}\right]. \tag{114}$$

By the Markov property, the right-hand side is equal to

$$\mathbf{Q}_0^2\left[\exp\left\{-\lambda_1 \frac{X_m}{m}\right\} \mathbf{Q}_{X_m}^2\left[\exp\left\{-\lambda_2 \frac{X_{M-m}}{M}\right\}\right]\right]. \tag{115}$$

By formula (39), this expectation is equal to

$$\frac{1}{1 + 2(\lambda_2/M)(M-m)} \times \mathbf{Q}_0^2\left[\exp\left\{-\left(\frac{\lambda_1}{m} + \frac{\lambda_2/M}{1 + 2(\lambda_2/M)(M-m)}\right) X_m\right\}\right]. \tag{116}$$

Again by formula (39), this expectation is equal to

$$\frac{1}{1 + 2(\lambda_2/M)(M-m)} \cdot \frac{1}{1 + 2(\lambda_1/m + (\lambda_2/M)/(1 + 2(\lambda_2/M)(M-m)))m}. \tag{117}$$

Simplifying this quantity with $E = 1 - m/M$, we see that

$$\iint e^{-\lambda_1 l_1 - \lambda_2 l_2} \Phi_2(dl_1 \times dl_2) = \frac{1}{1 + 2\lambda_1 + 2\lambda_2 + 4E\lambda_1\lambda_2}. \tag{118}$$

Note that this expression is invariant under interchange between λ_1 and λ_2, which proves the second equality of (103).

Step 3. Let us compute the Laplace transform:

$$\mathbf{Q}_0^2\left[\exp\left\{-\lambda_1 \frac{X_m}{m}\right\} \mathbf{Q}_{X_m}^0\left[\exp\left\{-\lambda_2 \frac{X_{M-m}}{M}\right\}\right]\right]. \tag{119}$$

By formula (39), this expectation is equal to

$$\mathbf{Q}_0^2\left[\exp\left\{-\left(\frac{\lambda_1}{m} + \frac{\lambda_2/M}{1 + 2(\lambda_2/M)(M-m)}\right) X_m\right\}\right]. \tag{120}$$

Using the equality between (116) and (118), we see that

$$\iint e^{-\lambda_1 l_1 - \lambda_2 l_2} \Phi_3(dl_1 \times dl_2) = \frac{1 + 2E\lambda_2}{1 + 2\lambda_1 + 2\lambda_2 + 4E\lambda_1\lambda_2}. \tag{121}$$

Now we also obtain

$$\iint e^{-\lambda_1 l_1 - \lambda_2 l_2} \Phi_4(dl_1 \times dl_2) = \frac{1 + 2E\lambda_1}{1 + 2\lambda_1 + 2\lambda_2 + 4E\lambda_1\lambda_2}. \tag{122}$$

Step 4. Noting that

$$\iint e^{-\lambda_1 l_1 - \lambda_2 l_2} \Phi_1(dl_1 \times dl_2) = 1, \tag{123}$$

we sum up formulae (123), (118), (121), and (122), and we obtain

$$\sum_{k=1}^{4} C_k \iint e^{-\lambda_1 l_1 - \lambda_2 l_2} \Phi_k \left(dl_1 \times dl_2\right) = \frac{1 + \lambda_1 + \lambda_2 + D\lambda_1\lambda_2}{1 + 2\lambda_1 + 2\lambda_2 + 4E\lambda_1\lambda_2}. \tag{124}$$

By Lemma 11, we see that the right-hand side coincides with the Laplace transform of the joint law of (L_∞^x, L_∞^y) under $\mathbf{P}_0^h$. By the uniqueness of Laplace transforms, we obtain the desired conclusion.

6. The Laws of Total Local Times for Bridges

In this section, we study the total local time of Lévy bridges and h-bridges.

6.1. The Laws of the Total Local Times for Lévy Bridges. Let us work with the Lévy bridge $\mathbf{P}_{0,0}^t$ and its local time $(L_s^z : 0 \le s \le t)$ such that

$$\int_0^s f\left(X_u\right) du = \int f(z) L_s^z dz, \quad 0 \le s \le t, f \in \mathscr{B}_+ (\mathbb{R}) \tag{125}$$

with $\mathbf{P}_{0,0}^t$-probability one. Let us study the law of the total local time L_t^z under $\mathbf{P}_{0,0}^t$.

Theorem 19. *For $t > 0$, it holds that*

$$\mathbf{P}_{0,0}^t \left(L_t^0 \in dl\right) = \frac{\gamma_l(t)}{p_t(0)} dl. \tag{126}$$

Proof. Using Theorem 7 with $n = 1$ and $z_1 = 0$, we have

$$\int_0^\infty e^{-qt} p_t(0) \mathbf{P}_{0,0}^t \left[e^{-\lambda L_t^0}\right] dt = \frac{1}{1/u_q(0) + \lambda}. \tag{127}$$

By formula (61), we have

$$\frac{1}{1/u_q(0) + \lambda} = \int_0^\infty e^{-(1/u_q(0) - \lambda) l} dl = \int_0^\infty \mathbf{P}_0 \left[e^{-q\tau(l)}\right] e^{-\lambda l} dl. \tag{128}$$

Hence, using Lemma 9, we obtain (126) by the Laplace inversion. The proof is now complete. □

Theorem 20. *For any $z \in \mathbb{R} \setminus \{0\}$, one has*

$$\mathbf{P}_{0,0}^t \left(L_t^z = 0\right) = \frac{p_t(0) - \left(\rho_z * \rho_z * p_\cdot(0)\right)(t)}{p_t(0)}, \tag{129}$$

$$\mathbf{P}_{0,0}^t \left(L_t^z \in dl\right) = \frac{\left(\rho_z * \rho_z * \gamma_l\right)(t)}{p_t(0)} dl, \quad \textit{for } l \neq 0, \tag{130}$$

where the symbol $*$ stands for the convolution operation.

Proof. Using Theorem 7 with $n = 2$, $\lambda_1 = 0$, $\lambda_2 = \lambda$, $z_1 = 0$, and $z_2 = z$, we have

$$\int_0^\infty e^{-qt} p_t(0) \mathbf{P}_{0,0}^t \left[e^{-\lambda L_t^z}\right] dt = u_q(0) \left(1 - \frac{u_q(z)^2}{u_q(0)^2}\right) + \frac{u_q(z)^2}{u_q(0)^2} \frac{1}{1/u_q(0) + \lambda}. \tag{131}$$

On the one hand, it follows from (57) that

$$u_q(0) \left(1 - \frac{u_q(z)^2}{u_q(0)^2}\right) = \left(\int_0^\infty e^{-qt} p_t(0) dt\right) \left(1 - \mathbf{P}_z \left[e^{-qT_{\{0\}}}\right]^2\right). \tag{132}$$

This implies (129). On the other hand, by (61) and (57), we have

$$\frac{u_q(z)^2}{u_q(0)^2} \frac{1}{1/u_q(0) + \lambda} = \mathbf{P}_z \left[e^{-qT_{\{0\}}}\right]^2 \int_0^\infty \mathbf{P}_0 \left(e^{-q\tau(l)}\right) e^{-\lambda l} dl. \tag{133}$$

This implies (130). The proof is now complete. □

6.2. The Laws of the Total Local Times for h-Bridges. Let us work with the h-bridge $\mathbf{P}_{0,0}^{h,t}$ and its local time (L_t^z) such that

$$\int_0^s f\left(X_u\right) du = \int f(z) L_s^z h(z)^2 dz, \quad 0 \le s \le t, \ f \in \mathscr{B}_+ (\mathbb{R}) \tag{134}$$

with $\mathbf{P}_{0,0}^{h,t}$-probability one. We give the Laplace transform formula for the law of the total local time L_t^z under $\mathbf{P}_{0,0}^{h,t}$.

Lemma 21. *For $z \in \mathbb{R} \setminus \{0\}$ and $\lambda \ge 0$, one has*

$$\int_0^\infty e^{-qt} p_t^h(0,0) \mathbf{P}_{0,0}^{h,t} \left[1 - e^{-\lambda h(z)^2 L_t^z}\right] dt = \frac{\left(u_q(z)^2 / u_q(0)^2\right) \lambda}{1 + u_q(0) \left\{1 - u_q(z)^2 / u_q(0)^2\right\} \lambda}. \tag{135}$$

Proof. Using Theorem 8 with $n = 1$, $\lambda_1 = \lambda$, and $z_1 = z$, we have

$$\int_0^\infty e^{-qt} p_t^h(0,0) \mathbf{P}_{0,0}^{h,t} \left[1 - e^{-\lambda L_t^z}\right] dt = \frac{u_q^h(0,z)^2 \lambda}{1 + u_q^h(z,z) \lambda}. \tag{136}$$

By formulae (72), we obtain the desired formula. □

7. Concluding Remark

We gave an explicit formula which describes the joint distribution of the total local times at two levels and we discussed several formulae related to the law of the total local times. However, we could not obtain any better result on the law of the total local time with space parameter. As we noted in Remark 3, a difficulty arises in the case of h-paths, which comes from the asymmetry of the matrix $\Sigma - \Sigma^0$. We also remark that we have no better result related to the law of total local time in the case where the Markov process is asymmetric. We left the further study of the law of the total local time for asymmetric Markov process with space parameter for future work.

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Research Article

Fractional Difference Equations with Real Variable

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We independently propose a new kind of the definition of fractional difference, fractional sum, and fractional difference equation, give some basic properties of fractional difference and fractional sum, and give some examples to demonstrate several methods of how to solve certain fractional difference equations.

1. Introduction

Fractional calculus is an emerging field recently drawing attention from both theoretical and applied disciplines. During the last two decades, it has been successfully applied to several fields [1–6], and it is well known that there is a large quantity of research on what is usually called integer-order difference equations [7, 8]. However, discrete fractional calculus and fractional difference equations represent a very new area for scientists. A pioneering work has been done by Atici et al. [9–12], Anastassiou [13, 14], Bastos et al. [15], Abdeljawad et al. [16–20], and Cheng [21–23], and so forth. In this paper, limited to the length of the paper, we will introduce some of our basic works about discrete fractional calculus and fractional difference equations. Some proofs and results of the theorems and examples in Sections 3–5 are well proved by a more concise method. We refer to the monographer [23] for more further results. In [23] we also aim at presenting some basic properties about discrete fractional calculus and, in a systematic manner, results including the existence and uniqueness of solutions for the Cauchy Type and Cauchy problems, involving nonlinear fractional difference equations, explicit solutions of linear difference equations and linear difference system by their deduction to Volterra sum equation and by using operational methods, applications of Z-transform, R-transform, N-transform, Adomian decomposition method, method of undetermined coefficients, Jordan matrix theory method, and by discrete Mittag-Leffler function and discrete Green' function, and a theory of so-called sequential linear fractional difference equations, as well as some introduction for discrete fractional difference variational problem, and so forth.

2. Integer-Order Difference and Sum with Real Variable

Let us start from sum and difference of the integer order. Define

$$ {}_h\sum_{s=a}^{t} x(s) \triangleq [x(a) + x(a+h) + x(a+2h) + \cdots + x(t)], \tag{2.1}$$

where $t = a + jh,\ j \in N_0 = \{0, 1, 2, \ldots\}$.

Definition 2.1. Let a, t be real numbers, and let h be a positive number, we call

$$ {}_a\nabla_h^{-1} x(t) = {}_h\sum_{s=a}^{t} x(s)h \tag{2.2}$$

one-order backward sum of $x(t)$, where $t = a + jh,\ j \in N_0 = \{0, 1, 2, \ldots\}$. We call

$$ {}_a\nabla_h^{-k} x(t) = {}_a\nabla_h^{-1}\left({}_a\nabla_h^{-(k-1)} x(t)\right) \tag{2.3}$$

k-order backward sum of $x(t)$, where k is a positive integer number.

Definition 2.2. Let a, t be real numbers, and let h be a positive number, we call

$$ {}_a\Delta_h^{-1} x(t) = {}_h\sum_{s=a}^{t-h} x(s)h \tag{2.4}$$

one-order forward sum of $x(t)$, where $t = a + jh,\ j \in N_1 = \{1, 2, \ldots\}$. We call

$$ {}_a\Delta_h^{-k} x(t) = {}_a\Delta_h^{-1}\left({}_a\Delta_h^{-(k-1)} x(t)\right) \tag{2.5}$$

k-order forward difference of $x(t)$, where k is a positive integer number.

Definition 2.3. Let t be a real number, and let h be a positive number, we call

$$ \nabla_h x(t) = \frac{x(t) - x(t-h)}{h} \tag{2.6}$$

one-order backward difference of $x(t)$, where h is step. We call

$$ \nabla_h^k x(t) = \nabla_h\left(\nabla_h^{k-1} x(t)\right) \tag{2.7}$$

k-order backward difference of $x(t)$, where k is a positive integer number.

Similarly, we can define forward difference as follows.

Definition 2.4. Let t be a real number, and let h be a positive number, we call

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \tag{2.8}$$

one-order forward difference of $x(t)$, where h is step. We call

$$\Delta_h^k x(t) = \Delta_h \left(\Delta_h^{k-1} x(t) \right) \tag{2.9}$$

k-order forward difference of $x(t)$, where k is a positive integer number.

Theorem 2.5. *The following two equalities hold:*

(1) $\nabla_h ({}_a\nabla_h^{-1} x(t)) = x(t)$,

(2) $\Delta_h ({}_a\Delta_h^{-1} x(t)) = x(t)$.

Definition 2.6. If k, t are real numbers, and let h be a positive number, define

$$t_h^{\overline{k}} = h^k \frac{\Gamma(t/h + k)}{\Gamma(t/h)}, \quad (k \in R) \tag{2.10}$$

rising factorial function, and set $t_h^{\overline{0}} = 1$. If k is a positive integer number, then we have

$$t_h^{\overline{k}} = t(t+h)(t+2h) \cdots (t+(k-1)h). \tag{2.11}$$

Definition 2.7. Let k, t be real numbers, and let h be a positive number, define

$$t_h^{(k)} = h^k \frac{\Gamma(t/h + 1)}{\Gamma(t/h + 1 - k)}, \quad (k \in R) \tag{2.12}$$

down factorial function, and set $t_h^{(0)} = 1$. If k is an positive integer number, then

$$t_h^{(k)} = t(t-h)(t-2h) \cdots (t-(k-1)h). \tag{2.13}$$

In Definitions 2.6 and 2.7, if $h = 1$, we can simply denote $t_h^{\overline{k}}$, $t_h^{(k)}$ as $t^{\overline{k}}$, $t^{(k)}$.

Definition 2.8. For any $k, \gamma \in R$, $h > 0$, we define

$$\begin{bmatrix} \gamma \\ k \end{bmatrix} \triangleq \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(k+1)}, \qquad \begin{bmatrix} \gamma \\ k \end{bmatrix}_h \triangleq h^{\gamma} \begin{bmatrix} \gamma \\ \dfrac{k}{h} \end{bmatrix}. \tag{2.14}$$

If $k \in N$, $h = 1$, then it is easy to see that

$$\begin{bmatrix} \gamma \\ k \end{bmatrix} = \frac{\gamma(\gamma+1)\cdots(\gamma+k-1)}{k!}. \tag{2.15}$$

If we let $t/h = \tilde{t}$, or $t = \tilde{t}h$, then we clearly have the following.

Theorem 2.9. *Assume that $k \in R$, $h > 0$, $t/h = \tilde{t}$; then*

$$t_h^{\overline{k}} = h^k \tilde{t}^{\overline{k}}; \qquad t_h^{(k)} = h^k \tilde{t}^{(k)}. \tag{2.16}$$

Theorem 2.10. *Let $k \in R$, $h > 0$, then, following equality holds:*

$$\lim_{h\to 0} t_h^{\overline{k}} = \lim_{h\to 0} t_h^{(k)} = t^k. \tag{2.17}$$

3. Fractional Sum and Difference with Real Variable

Before giving the definitions of fractional sum ${}_a\nabla_h^{-\gamma} x(t)$, $\gamma > 0$, let us revisit the calculation of the sum of the integer order. By Definition 2.1, we have

$${}_a\nabla_h^{-1} x(t) = {}_h\sum_{s=a}^{t} x(s)h, \quad t = a + jh,\ j \in N_0, \tag{3.1}$$

then

$$\begin{aligned}
{}_a\nabla_h^{-2} x(t) &= {}_a\nabla_h^{-1}\left[{}_a\nabla_h^{-1} x(t)\right] = {}_h\sum_{s=a}^{t} {}_a\nabla_h^{-1} x(s)h = h\left[{}_h\sum_{s=a}^{t}\, {}_h\sum_{r=a}^{s} x(r)h\right] \\
&= h^2\left[{}_h\sum_{r=a}^{t}\, {}_h\sum_{s=r}^{t} x(r)\right] = h^2\left[{}_h\sum_{r=a}^{t} \frac{t-r+h}{h} x(r)\right] = {}_h\sum_{r=a}^{t} (t-r+h)_h^{\overline{1}} x(r)h, \\
{}_a\nabla_h^{-3} x(t) &= {}_a\nabla_h^{-1}\left[{}_a\nabla_h^{-2} x(t)\right] = {}_h\sum_{s=a}^{t} {}_a\nabla_h^{-2} x(s)h = h^2\left[{}_h\sum_{s=a}^{t}\, {}_h\sum_{r=a}^{s} \frac{t-r+h}{h} x(r)h\right] \\
&= \frac{h^3}{2}\left[{}_h\sum_{r=a}^{t} \left(\frac{t-r+h}{h}\right)\left(\frac{t-r+2h}{h}\right) x(r)\right] = \frac{1}{2!}\left[{}_h\sum_{r=a}^{t} (t-r+h)_h^{\overline{2}} x(r)h\right] \ldots.
\end{aligned} \tag{3.2}$$

By recursive, it is not hard to obtain

$$\begin{aligned}
{}_a\nabla_h^{-m} x(t) &= \frac{h^m}{(m-1)!}\left[{}_h\sum_{s=a}^{t} \left(\frac{t-s+h}{h}\right)\left(\frac{t-s+2h}{h}\right)\cdots\left(\frac{t-s+(m-1)h}{h}\right) x(s)\right] \\
&= \frac{1}{\Gamma(m)}\left[{}_h\sum_{s=a}^{t} (t-s+h)_h^{\overline{m-1}} x(s)h\right] = \frac{1}{\Gamma(m)}\left[{}_h\sum_{s=a}^{t} (t-\rho_h(s))_h^{\overline{m-1}} x(s)h\right],
\end{aligned} \tag{3.3}$$

where $\rho_h(s) = s - h$.

Obviously, the right side of formula (3.3) is also meaningful for all real $m > 0$, so we define fractional sum as follows.

Definition 3.1. Let $\gamma > 0$, $a \in R$, $h > 0$, $t = a + kh$, $k \in N_0$, we call

$$ {}_a\nabla_h^{-\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \left[{}_h\sum_{s=a}^{t} (t - \rho_h(s))_h^{\overline{\gamma-1}} x(s) h \right] \tag{3.4} $$

γ order fractional sum of $x(t)$.

For any positive number order fractional difference, we take the following.

Definition 3.2. Let $\mu > 0$, and assume that $m - 1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\nabla_h^{\mu} x(t) = \nabla_h^{m} \left({}_a\nabla_h^{-(m-\mu)} x(t) \right) \tag{3.5} $$

as μ order *R-L* type backward fractional difference. Meantime, define

$$ {}_a^C\nabla_h^{\mu} x(t) = \left({}_a\nabla_h^{-(m-\mu)} \right) \nabla_h^{m} x(t) \tag{3.6} $$

as μ order Caputo type backward fractional difference.

If we start from Definition 2.2,

$$ {}_a\Delta_h^{-1} x(t) = {}_h\sum_{s=a}^{t-h} x(s) h, \quad t = a + jh, \ j \in N_1, \tag{3.7} $$

completely in a similar way, we get positive integer m-order forward sum

$$ {}_a\Delta_h^{-m} x(t) = \frac{1}{\Gamma(m)} \left[{}_h\sum_{s=a}^{t-mh} (t - \sigma_h(s))_h^{(m-1)} x(s) h \right], \tag{3.8} $$

where $\sigma_h(s) = s + h$.

The right side of (3.8) is meaningful for all real $m > 0$, so we can define forward fractional sum as follows.

Definition 3.3. Let $\gamma > 0$, $a \in R$, $h > 0$, $t = a + \gamma h + kh$, $k \in N_0$, define

$$ {}_a\Delta_h^{-\gamma} x(t) = \frac{1}{\Gamma(\nu)} \left[{}_h\sum_{s=a}^{t-\gamma h} (t - \sigma_h(s))_h^{(\gamma-1)} x(s) h \right] \tag{3.9} $$

as γ order fractional sum of $x(t)$, where $\sigma_h(s) = s + h$.

Definition 3.4. Let $\mu > 0$, and assume that $m-1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\Delta_h^{\mu} x(t) = \Delta_h^m \left({}_a\Delta_h^{-(m-\mu)} x(t) \right) \tag{3.10} $$

as μ order *R-L* type forward fractional difference. Meantime, define

$$ {}_a^C\Delta_h^{\mu} x(t) = \left({}_a\Delta_h^{-(m-\mu)} \right) \Delta_h^m x(t) \tag{3.11} $$

as μ order Caputo type forward fractional difference.

In Definitions 3.1–3.4, if step $h = 1$, it is a kind of important situation. At this time, we simply denote ${}_a\nabla_h^{-\gamma}$, ${}_a\Delta_h^{-\gamma}$; ∇_h^{μ}, Δ_h^{μ} as ${}_a\nabla^{-\gamma}$, ${}_a\Delta^{-\gamma}$; ∇^{μ}, Δ^{μ}. When $h = 1$, backward fractional sum is defined as follows.

Definition 3.5. Let $\gamma > 0$, and define

$$ {}_a\nabla^{-\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\gamma-1}} x(s) \tag{3.12} $$

as γ order fractional sum of $x(t)$, where $t = a \bmod (1)$, $\rho(s) = s-1$.

For any positive number order fractional difference, we can take the following way.

Definition 3.6. Let $\mu > 0$ and assume that $m-1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\nabla^{\mu} x(t) = \nabla^m \left({}_a\nabla^{-(m-\mu)} x(t) \right) \tag{3.13} $$

as μ order *R-L* type backward fractional difference. Meantime, define

$$ {}_a^C\nabla^{\mu} x(t) = \left({}_a\nabla^{-(m-\mu)} \right) \nabla^m x(t) \tag{3.14} $$

as μ order Caputo type backward fractional difference.

We can define forward fractional sum as follows.

Definition 3.7. Let $\gamma > 0$, and define

$$ {}_a\Delta^{-\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-\sigma(s))^{(\gamma-1)} x(s) \tag{3.15} $$

as γ order forward fractional sum of $x(t)$, where $t-\gamma = a \bmod (1)$, $\sigma(s) = s+1$.

Definition 3.8. Let $\mu > 0$, and assume that $m-1 < \mu < m$, where m denotes a positive integer. Define

$$ {}_a\Delta^{\mu} x(t) = \Delta^m \left({}_a\Delta^{-(m-\mu)} x(t) \right) \tag{3.16}$$

as μ order *R-L* type forward fractional difference. Meantime, define

$$ {}_a^C\Delta^{\mu} x(t) = \left({}_a\Delta^{-(m-\mu)} \right) \Delta^m x(t) \tag{3.17}$$

as μ order Caputo type forward fractional difference.

By Definition 2.8, it is easy to calculate

$$\begin{gathered} \begin{bmatrix} \gamma \\ t-s \end{bmatrix} = \frac{1}{\Gamma(\gamma)} (t-\rho(s))^{\overline{\gamma-1}}, \\ \begin{bmatrix} \gamma \\ t-\gamma-s \end{bmatrix} = \frac{1}{\Gamma(\gamma)} (t-\sigma(s))^{\gamma-1}. \end{gathered} \tag{3.18}$$

By Theorem 2.9 we have

$$\begin{gathered} \frac{(t-s+h)_h^{\overline{\gamma-1}}}{\Gamma(\gamma)} = h^{\gamma-1} \frac{((t-s)/h+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} = h^{\gamma-1} \begin{bmatrix} \gamma \\ \dfrac{t-s}{h} \end{bmatrix}, \\ \frac{(t-s-h)_h^{(\gamma-1)}}{\Gamma(\gamma)} = h^{\gamma-1} \frac{((t-s)/h-1)^{(\gamma-1)}}{\Gamma(\gamma)} = h^{\gamma-1} \begin{bmatrix} \gamma \\ \dfrac{t-s}{h} - \gamma \end{bmatrix}. \end{gathered} \tag{3.19}$$

Therefore, if we adopt Definition 2.8, then Definitions 3.1, 3.3, 3.5, and 3.7 can be rewritten as follows.

Definition 3.9. Assume that $\gamma > 0$, let $a \in R$, $h > 0$, $t = a + kh$, $k \in N_0$, and define

$$ {}_a\nabla_h^{-\gamma} x(t) = h \sum_{s=a}^{t} \begin{bmatrix} \gamma \\ t-s \end{bmatrix}_h x(s) \tag{3.20}$$

as γ order backward fractional sum of $x(t)$.

Definition 3.10. Assume that $\gamma > 0$, let $a \in R$, $h > 0$, $t = a + \gamma h + kh$, $k \in N_0$, and define

$$ {}_a\Delta_h^{-\gamma} x(t) = h \sum_{s=a}^{t-\gamma h} \begin{bmatrix} \gamma \\ t-s-\gamma h \end{bmatrix}_h x(s) \tag{3.21}$$

as γ order forward fractional sum of $x(t)$.

Definition 3.11. Assume that $\gamma > 0$, $t, a \in R$, and $t = a \bmod (1)$, and define

$$ {}_a\nabla^{-\gamma} x(t) = \sum_{s=a}^{t} \begin{bmatrix} \gamma \\ t-s \end{bmatrix} x(s) \tag{3.22} $$

as γ order backward fractional sum of $x(t)$.

Definition 3.12. Assume that $\gamma > 0$, $t, a \in R$, and $t - \gamma = a \bmod (1)$, and define

$$ {}_a\Delta^{-\gamma} x(t) = \sum_{s=a}^{t-\gamma} \begin{bmatrix} \gamma \\ t-\gamma-s \end{bmatrix} x(s) \tag{3.23} $$

as γ order forward fractional sum of $x(t)$.

Set $a/h = \tilde{a}$, $t/h = \tilde{t}$, or $a = \tilde{a}h, t = \tilde{t}h$, and set $x(t) = x(\tilde{t}h) = y(\tilde{t})$; then by Theorem 2.9 and Definitions 3.1–3.4, one obtains the following.

Theorem 3.13. *For any $\gamma, \mu > 0$, the following equalities hold:*

(1) ${}_a\nabla_h^{-\gamma} x(t) = h^{\gamma}[{}_{\tilde{a}}\nabla^{-\gamma} y(\tilde{t})]$; ${}_a\Delta_h^{-\gamma} x(t) = h^{\gamma}[{}_{\tilde{a}}\Delta^{-\gamma} y(\tilde{t})]$,

(2) ${}_a\nabla_h^{\mu} x(t) = h^{-\mu}[{}_{\tilde{a}}\nabla^{\mu} y(\tilde{t})]$; ${}_a\Delta_h^{\mu} x(t) = h^{-\mu}[{}_{\tilde{a}}\Delta^{\mu} y(\tilde{t})]$,

(3) ${}_a^C\nabla_h^{\mu} x(t) = h^{-\mu}[{}_{\tilde{a}}^C\nabla^{\mu} y(\tilde{t})]$; ${}_a^C\Delta_h^{\mu} x(t) = h^{-\mu}[{}_{\tilde{a}}^C\Delta^{\mu} y(\tilde{t})]$.

From Theorem 3.13 we can see, by stretching $t = \tilde{t}h$, the functions ${}_a\nabla_h^{-\gamma} x(t)$ and ${}_a\nabla_h^{\mu} x(t)$, with common step h, can be convert into the functions ${}_a\nabla^{-\gamma} y(\tilde{t})$ and ${}_a\nabla^{\mu} y(\tilde{t})$ with step $h = 1$, respectively. In essence, nothing arises much different, but the latter is more convenient in research.

In view of Definitions 3.1–3.4 and Theorem 2.10, if we let $h \to 0$, then we can obtain the following.

Corollary 3.14. *Assume that $x(t)$ is integrable, then:*

(1) $\lim_{h\to 0}({}_a\nabla_h^{-\gamma} x(t)) = \lim_{h\to 0}({}_a\nabla_h^{-\gamma} x(t)) = (1/\Gamma(\gamma)) \int_a^t (t-s)^{\gamma-1} x(t)ds \triangleq D_t^{-\gamma} x(t)$,

(2) $\lim_{h\to 0}({}_a\nabla_h^{\mu} x(t)) = \lim_{h\to 0}({}_a\nabla_h^{\mu} x(t)) = D^m({}_aD_t^{-(m-\mu)} x(t)) \triangleq {}_a D_t^{\mu} x(t)$,

(3) $\lim_{h\to 0}({}_a^C\nabla_h^{\mu} x(t)) = \lim_{h\to 0}({}_a^C\nabla_h^{\mu} x(t)) = D^m({}_aD_t^{-(m-\mu)} x(t)) \triangleq {}_a^C D_t^{\mu} x(t)$.

4. Some Basic Properties

We sometimes only list some basic results here, for more detailed results and their proofs can been seen in monographer [23].

Theorem 4.1. *Assume that the following function is well defined; then*

(1) $\nabla_h t_h^{\overline{\gamma}} = \gamma t_h^{\overline{\gamma-1}}$, $\Delta_h t_h^{(\gamma)} = \gamma t_h^{(\gamma-1)}$,

(2) $(t+\gamma h) t_h^{\overline{\gamma}} = t_h^{\overline{\gamma+1}}$, $(t-\gamma h) t_h^{(\gamma)} = t_h^{(\gamma+1)}$, $\gamma \in R$,

(3) *If* $0 < \gamma < 1$, *then* $t_h^{\overline{\alpha\gamma}} \le (t_h^{\overline{\alpha}})^{\gamma}$, $t_h^{(\alpha\gamma)} \ge (t_h^{(\alpha)})^{\gamma}$,

(4) $t_h^{\overline{\alpha+\beta}} = (t+\beta)_h^{\overline{\alpha}} t_h^{\overline{\beta}}$, $t_h^{(\alpha+\beta)} = (t-\beta)_h^{(\alpha)} t_h^{(\beta)}$,

(5) *Let* $0 < t \le r$, *if* $\gamma > 0$, *then* $t_h^{\overline{\gamma}} \le r_h^{\overline{\gamma}}$, $t_h^{(\gamma)} \le r_h^{\gamma}$; *If* $\gamma < 0$, *then* $t_h^{\overline{\gamma}} \ge r_h^{\overline{\gamma}}$, $t_h^{(\gamma)} \ge r_h^{\gamma}$.

Theorem 4.2. *Let* $0 \le m-1 < \gamma \le m, m \in N$, *where* $x(t)$ *is defined in* $N_{h,a} = \{a, a+h, a+2h, \ldots\}$, *then*

(1) ${}_a\nabla_h^{-\gamma} x(t) = {}_a\Delta_h^{-\gamma} x(t+\gamma h)$, $t \in N_{h,a}$,

(2) ${}_a\nabla^{\gamma} x(t) = {}_a\Delta^{\gamma} x(t-\gamma h)$, $t \in N_{h,m+a}$.

Theorem 4.3. *Let* $0 \le m-1 < \gamma \le m$, $m \in N$, $x(t)$ *is defined in* $N_{h,a} = \{a, a+h, a+2h, \ldots\}$, *then*

(1) ${}_a\Delta_h^{-\gamma} x(t) = {}_a\nabla_h^{-\gamma} x(t-\gamma h), t \in N_{h,a+\gamma}$,

(2) ${}_a\Delta_h^{\gamma} x(t) = {}_a\nabla_h^{\gamma} x(t+\gamma h), t \in N_{h,a-\gamma+m}$.

Theorem 4.4. *For any real* γ, *the following equality holds:*

(1) ${}_a\nabla_h^{-\gamma}\nabla_h x(t) = \nabla_h({}_a\nabla_h^{-\gamma})x(t) - ((t-a-1)_h^{\overline{\gamma-1}}/\Gamma(\gamma))x(a-h)$,

(2) ${}_a\Delta_h^{-\gamma}\Delta_h x(t) = \Delta_h({}_a\Delta_h^{-\gamma})x(t) - ((t-a)_h^{(\gamma-1)}/\Gamma(\gamma))x(a)$.

Theorem 4.5. *For any real* γ *and* $p > 0$, *the following equality holds:*

(1) ${}_a\nabla_h^{-\gamma}\nabla_h^{p} x(t) = \nabla_h^{p}({}_a\nabla_h^{-\gamma}x(t)) - \sum_{k=0}^{p-1}((t-a+1)_h^{\overline{\gamma-p+k}}/\Gamma(\gamma+k-p+1))\nabla_h^{k}x(a-h)$,

(2) ${}_a\Delta_h^{-\gamma}\Delta_h^{p} x(t) = \Delta_h^{p}({}_a\Delta_h^{-\gamma}x(t)) - \sum_{k=0}^{p-1}((t-a)_h^{(\gamma-p+k)}/\Gamma(\gamma+k-p+1))\Delta_h^{k}x(a)$.

Theorem 4.6. *Let* $p, \gamma > 0$, *then*

(1) $\nabla_h^{p}({}_a\nabla_h^{-\gamma}x(t)) = {}_a\nabla_h^{-(\gamma-p)}x(t)$,

(2) $\Delta_h^{p}({}_a\Delta_h^{-\gamma}x(t)) = {}_a\Delta_h^{-(\gamma-p)}x(t)$.

In the previous theorems, we only need to consider the simplest case $h = 1$, but actually the methods of proof and conclusions can also be extended for general step $h > 0$. In fact, we only need do a stretching transformation and then make use of Theorem 2.9.

Next, we discusses fractional sum transform such as: Z transform, N transform, R transform, and some properties of these transforms.

Definition 4.7. Let $f(t)$ be defined in $N_0 = \{0, 1, 2, \ldots\}$, we call

$$f(t) = \sum_{t=0}^{\infty} f(t) z^{-t} \tag{4.1}$$

is a Z transform of $f(t)$, denote it by $Z[f(t)]$.

Definition 4.8. Let $f(t)$ be defined in $N_{t_0} = \{t_0, t_0+1, t_0+2, \ldots\}$, $t_0 \in R$, and define N transform as follows:

$$N_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} (1-s)^{t-1} f(t). \tag{4.2}$$

If the domain of the function $f(t)$ is N_1, then we use the notation $N(f(t))$.

If we set $t - t_0 = n \in N_0$, define

$$f_n^{\{t_0\}} = f(n + t_0) = f(t), \qquad f_{n-1}^{\{t_0\}} = f(n - 1 + t_0) = f(t - 1), \dots, \\ f_0^{\{t_0\}} = f(0 + t_0) = f(t_0). \tag{4.3}$$

Then, $f(t_0), f(t_0 + 1), \dots, f(t), \dots$ can be regarded as a sequence

$$f_0^{\{t_0\}}, f_1^{\{t_0\}}, \dots f_n^{\{t_0\}}, \dots. \tag{4.4}$$

Under this definition, N transform can be simply rewritten as

$$\begin{aligned} N_0\big(f(t)\big)(s) &= \sum_{t=t_0}^{\infty}(1 - s)^{t-1} f(s) \\ &= \sum_{n=0}^{\infty}(1 - s)^{n+t_0-1} f(n + t_0) \\ &= (1 - s)^{t_0-1}\sum_{n=0}^{\infty}(1 - s)^n f_n^{\{t_0\}}. \end{aligned} \tag{4.5}$$

Set $z = 1/(1 - s)$, then we have

$$N_0\big(f(t)\big)(s) = z^{1-t_0}\sum_{n=0}^{\infty} f_n^{\{t_0\}} z^{-n} = z^{1-t_0} F(z), \tag{4.6}$$

where $F(z)$ is Z transform of sequence $f_n^{\{t_0\}}$.

If $t_0 = 1$, then

$$N\big(f(t)\big) = F(z), \quad \left(z = \frac{1}{1 - s}\right). \tag{4.7}$$

Theorem 4.9. *For any $\gamma \in R \setminus \{\dots, -2, -1, 0\}$, then*

(1) $N(t^{\overline{\gamma-1}})(s) = \Gamma(\gamma)/s^{\gamma}$, $|1 - s| < 1$,

(2) $N(t^{\overline{\gamma-1}}\alpha^{-t})(s) = \alpha^{\gamma-1}\Gamma(\gamma)/(s + \alpha - 1)^{\gamma}$, $|1 - s| < \alpha$.

Proof. (1) Making use of (4.7), we get

$$N\left(\frac{t^{\overline{\gamma-1}}}{\Gamma(\gamma)}\right) = F(z), \tag{4.8}$$

where $F(z)$ is Z transform of sequence $f_n^{\{1\}} = f(n+1)$,

$$f_n^{\{1\}} = f(n+1) = \frac{(n+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} = \begin{bmatrix} \gamma \\ n \end{bmatrix}. \tag{4.9}$$

Since (see [21–23])

$$F\left(\begin{bmatrix} \gamma \\ n \end{bmatrix}\right) = \left(\frac{z-1}{z}\right)^{-\gamma} = \frac{1}{s^{\gamma}}, \quad (|z| > 1, |1-s| < 1), \tag{4.10}$$

hence

$$N\left(\frac{t^{\overline{\gamma-1}}}{\Gamma(\gamma)}\right) = \frac{1}{s^{\gamma}}, \quad (|1-s| < 1). \tag{4.11}$$

(2) It is only to use

$$\sum_{t=1}^{\infty}(1-s)^{t-1}t^{\overline{\gamma-1}}\alpha^{-t} = \frac{1}{\alpha}\sum_{t=1}^{\infty}\left(1-\frac{s+\alpha-1}{\alpha}\right)^{t-1}t^{\overline{\gamma-1}}, \tag{4.12}$$

then the proof of (2) follows from the proof of (1). □

Theorem 4.10. *Let $f(t)$ and $g(t)$ be defined in N_a, and define convolution of $f(t)$, $g(t)$ as follows:*

$$(h * g)_a(t) = \sum_{s=a}^{t} h(t-\rho(s))g(s). \tag{4.13}$$

For $h(t) = t^{\overline{\gamma-1}}/\Gamma(\gamma)$, then

$$(h * g)_a(t) = \frac{1}{\Gamma(\gamma)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\gamma-1}}g(s) = {}_a\nabla^{-\gamma}g(t). \tag{4.14}$$

Theorem 4.11. *Let f, g be defined in N_a, then*

$$N_a(f * g) = N_1(f)N_a(g). \tag{4.15}$$

Theorem 4.12. *For any real γ, one has*

$$N_a\left({}_a\nabla^{-\gamma}f(t)\right) = s^{-\gamma}N_a(f(t)). \tag{4.16}$$

Theorem 4.13. *For $0 < \gamma \le 1$, one has*

$$N_{a+1}\left({}_a\nabla^{-\gamma}f(t)\right) = s^{\gamma}N_a(f(t))(s) - (1-s)^{\alpha-1}f(a), \tag{4.17}$$

where f is defined in N_a.

Theorem 4.14. *Let $\mu \in R \setminus \{\ldots,-2,-1,0\}$, $\gamma > 0$, then*

$$ {}_1\nabla^{-\gamma}\left(\frac{t^{\overline{\mu}}}{\Gamma(\mu+1)}\right) = \frac{t^{\overline{\mu+\gamma}}}{\Gamma(\mu+\gamma+1)}. \tag{4.18}$$

Theorem 4.15. *Let f be a real function, $\mu, \gamma > 0$, then*

$$ {}_a\nabla^{-\gamma}\left[{}_a\nabla^{-\mu} f(t)\right] = {}_a\nabla^{-(\mu+\gamma)} f(t) = {}_a\nabla^{-\mu}\left[{}_a\nabla^{-\gamma} f(t)\right]. \tag{4.19}$$

Definition 4.16. Let $f(t)$ be defined in N_{t_0}, and define R transform as follows:

$$R_{t_0}(f(t)) = \sum_{t=t_0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1} f(t). \tag{4.20}$$

In Definition 4.16, if we set $t - t_0 = n \in N_0$, and define:

$$\begin{gathered} f_n^{\{t_0\}} = f(n+t_0) = f(t), \qquad f_{n-1}^{\{t_0\}} = f(n-1+t_0) = f(t-1), \ldots, \\ f_0^{\{t_0\}} = f(0+t_0) = f(t_0), \end{gathered} \tag{4.21}$$

then, $f(t_0), f(t_0+1), \ldots, f(t), \ldots$ can be regarded as a sequence

$$f_0^{\{t_0\}}, f_1^{\{t_0\}}, \ldots f_n^{\{t_0\}}, \ldots. \tag{4.22}$$

Under this definition, R transform can be simply rewritten as

$$\begin{aligned} R_{t_0}(f(t))(s) &= \sum_{t=t_0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1} f(t) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{s+1}\right)^{n+t_0+1} f(n+t_0) \\ &= \left(\frac{1}{s+1}\right)^{t_0+1}\sum_{n=0}^{\infty}\left(\frac{1}{1+s}\right)^{n} f_n^{\{t_0\}}. \end{aligned} \tag{4.23}$$

Set $z = 1 + s$, then

$$R_{t_0}(f(t))(s) = z^{-1-t_0}\sum_{n=0}^{\infty} f_n^{\{t_0\}} z^{-n} = z^{-1-t_0}F(z), \tag{4.24}$$

where $F(z)$ is a Z transform of sequence $f_n^{\{t_0\}}$.

Theorem 4.17. *For any $\gamma \in R \setminus \{\dots, -2, -1, 0\}$, then*

(1) $R_{\gamma-1}(t^{(\gamma-1)})(s) = \Gamma(\gamma)/s^{\gamma}$,

(2) $R_{\gamma-1}(t^{(\gamma-1)}\alpha^{t})(s) = \alpha^{\gamma-1}\Gamma(\gamma)/(s+1-\alpha)^{\gamma}$.

Proof. (1) let $t_0 = \gamma - 1$, then

$$R_{\gamma-1}\left(\frac{t^{(\gamma-1)}}{\Gamma(\gamma)}\right) = z^{-\gamma}F(z), \tag{4.25}$$

where $F(z)$ is a Z transform of sequence $f_n^{\{\gamma-1\}}$. Since

$$f_n^{\{\gamma-1\}} = f(n+\gamma-1) = \frac{(n+\gamma-1)^{(\gamma-1)}}{\Gamma(\gamma)} = \begin{bmatrix}\gamma\\ n\end{bmatrix}, \tag{4.26}$$

and (see [22, 23])

$$F\left(\begin{bmatrix}\gamma\\ n\end{bmatrix}\right) = \left(\frac{z-1}{z}\right)^{-\gamma}, \tag{4.27}$$

hence

$$R_{\gamma-1}\left(\frac{t^{(\gamma-1)}}{\Gamma(\gamma)}\right) = (z-1)^{-\gamma} = s^{-\gamma}, \quad (|1+s|<1), \tag{4.28}$$

or

$$R_{\gamma-1}\left(\frac{t^{(\gamma-1)}}{\Gamma(\gamma)}\right) = \frac{1}{s^{\gamma}}, \quad (|1+s|<1). \tag{4.29}$$

(2) The proof of (2) follows from the proof of (1). □

Definition 4.18. Define convolution of $h(t)$ and $g(t)$ as follows:

$$(h*g)(t) = \sum_{s=a}^{t-\gamma} h(t-\sigma(s))g(s). \tag{4.30}$$

If $h(t) = t^{(\gamma-1)}/\Gamma(\gamma)$, then

$$(h*g)_a(t) = \frac{1}{\Gamma(\gamma)}\sum_{s=a}^{t-\gamma}(t-\rho(s))^{(\gamma-1)}g(s) = {}_a\Delta^{-\gamma}g(t). \tag{4.31}$$

Theorem 4.19. *For any $\gamma \in R \setminus \{\dots,-2,-1,0\}$, then*

$$R_{\gamma+a}(h * g) = R_{\gamma-1}(h) R_a(g). \tag{4.32}$$

Theorem 4.20. *Let $\mu > 0$, $m-1 < \mu \leq m \in N_1$, and let $f(t)$ be defined in $N_{\mu-m} = \{\mu-m, \mu-m+1,\dots\}$, then*

$$R_0(\Delta^{\mu} f(t))(s) = s^{\mu} R_{\mu-m}(f(t))(s) - \sum_{k=0}^{m-1} s^{m-k-1} \Delta^{k-m+\mu} f(t)\Big|_{t=0}. \tag{4.33}$$

Theorem 4.21. *Let $\mu \in R \setminus \{\dots,-2,-1,0\}$, $\gamma > 0$, then*

$$\Delta^{-\gamma}\left(\frac{t^{(\mu)}}{\Gamma(\mu+1)}\right) = \frac{t^{(\mu+\gamma)}}{\Gamma(\mu+\gamma+1)}. \tag{4.34}$$

Theorem 4.22. *Let f be a real function, $\mu, \gamma > 0$, then for all $t = \mu+\gamma \bmod (1)$, one has*

$$\Delta^{-\gamma}\left[\Delta^{-\mu} f(t)\right] = \Delta^{-(\mu+\gamma)} f(t) = \Delta^{-\mu}\left[\Delta^{-\gamma} f(t)\right]. \tag{4.35}$$

5. The Solution of the Fractional Difference Equations with Real Variable

In this section, we give examples to demonstrate the solving method of fractional difference equations and reveal the inner relationship between fractional differential equations and fractional differential equations.

Theorem 5.1. *Let $\mu \in R$, $\gamma \in R$, then*

(1) $\nabla^{\gamma} t^{\overline{\mu}} = \mu^{(\gamma)} t^{\overline{\mu-\gamma}}$, $\Delta^{\gamma} t^{(\mu)} = \mu^{(\gamma)} t^{(\mu-\gamma)}$,

(2) $\Delta^{\gamma} t^{\overline{\mu}} = \mu^{(\gamma)} (t+\gamma)^{\overline{\mu-\gamma}}$, $\nabla^{\gamma} t^{(\mu)} = \mu^{(\gamma)} (t-\gamma)^{(\mu-\gamma)}$.

Proof. (1) The proof of (1) directly follows from Theorem 4.1 and Theorem 4.2.

(2) By Theorem 4.2 and (1), we have

$$\begin{aligned} \Delta^{\gamma} t^{\overline{\mu}} &= \nabla^{\gamma} (t+\gamma)^{\overline{\mu}} = \mu^{(\gamma)} (t+\gamma)^{\overline{\mu-\gamma}}, \\ \nabla^{\gamma} t^{(\mu)} &= \Delta^{\gamma} (t-\gamma)^{(\mu)} = \mu^{(\gamma)} (t-\gamma)^{(\mu-\gamma)}. \end{aligned} \tag{5.1}$$

□

Example 5.2. Consider Euler type fractional difference equations

$$t^{\overline{2\alpha}} \Delta^{2\alpha} x(t) + a t^{\overline{\alpha}} \Delta^{\alpha} x(t) + b x(t) = 0, \quad (0 < \alpha < 1). \tag{5.2}$$

Set $x(t) = t^{\overline{\gamma}}$, and take it into previous equation, we get

$$t^{\overline{2\alpha}}\gamma^{(2\alpha)}(t+2\alpha)^{\overline{\gamma-2\alpha}} + at^{\overline{\alpha}}\gamma^{(\alpha)}(t+\alpha)^{\overline{\gamma-\alpha}} + bt^{\overline{\gamma}} = 0. \tag{5.3}$$

By Theorem 4.1 (4), we obtain

$$\gamma^{(2\alpha)}t^{\overline{\gamma}} + a\gamma^{(\alpha)}t^{\overline{\gamma}} + bt^{\overline{\gamma}} = 0, \tag{5.4}$$

and get indicator equation

$$\gamma^{(2\alpha)} + a\gamma^{(\alpha)} + b = 0. \tag{5.5}$$

Therefore, we can transform Euler type fractional difference equations into its indicator equation.

Example 5.3. Consider initial value problem of homogeneous linear γ order ($0 < \gamma \leq 1$) fractional difference equation with constant coefficient

$$\begin{gathered} \nabla^{\gamma} y(t) + a\nabla^{0} y(t) = 0, \quad t \in N_0, \\ \nabla^{\gamma-1}(t)\Big|_{t=-1} = a_0. \end{gathered} \tag{5.6}$$

Note that $\nabla^{\gamma-1}y(t)$ is defined in $N_{-1} = \{-1, 0, 1, 2, \ldots\}$, since

$$\begin{aligned} {}_{-1}\nabla^{\gamma-1} f(t)\Big|_{t=-1} &= \frac{1}{\Gamma(1-\gamma)} \sum_{s=-1}^{t} (t-\rho(s))^{\overline{-\gamma}} y(s) \\ &= \frac{1^{\overline{-\gamma}}}{\Gamma(1-\gamma)} y(-1) = y(-1). \end{aligned} \tag{5.7}$$

Therefore, initial problem of (5.6) is equivalent to initial problem

$$\begin{gathered} \nabla^{\gamma} y(t) + a\nabla^{0} y(t) = 0, \quad t \in N, \\ y(-1) = a_0. \end{gathered} \tag{5.8}$$

The solution of initial problem of (5.6) is equivalent to the solution of sum equations

$$y(t) = \frac{(t+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} a_0 + a\sum_{s=0}^{t} (t-\rho(s))^{\overline{\gamma-1}} y(s). \tag{5.9}$$

We use approximation method to solve these sum equations. Set

$$\begin{aligned} y_0(t) &= \frac{(t+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} a_0, \\ y_m(t) &= y_0(t) + \frac{a}{\Gamma(\gamma)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{\gamma-1}} y_{m-1}(s) \\ &= y_0(t) + a\nabla^{-\gamma} y_{m-1}(t), \quad m = 1, 2, \ldots. \end{aligned} \tag{5.10}$$

Applying power law (Theorem 4.22), we get

$$y_1(t) = y_0(t) + a\nabla^{-\gamma} y_0(t) = a_0 \left(\frac{(t+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)} + a \frac{(t+1)^{\overline{2\gamma-1}}}{\Gamma(2\gamma)} \right). \tag{5.11}$$

Applying power law repeatedly, and by recursion, we obtain

$$y_m(t) = a_0 \sum_{i=0}^{m} \frac{a^i t^{\overline{i\gamma+\gamma-1}}}{\Gamma((i+1)\gamma)}, \quad m = 0, 1, 2, \ldots. \tag{5.12}$$

Let $m \to \infty$, then

$$y(t) = a_0 \sum_{i=0}^{\infty} \frac{a^i (t+1)^{\overline{i\gamma+\gamma-1}}}{\Gamma((i+1)\gamma)} = a_0 \sum_{i=0}^{\infty} a^i \begin{bmatrix} i\gamma+\gamma \\ t \end{bmatrix}. \tag{5.13}$$

Example 5.4. Let $\gamma = 1/q$, $q \in N$, we call

$$\nabla^{\gamma} y(t) - a\nabla^{0} y(t) = 0, \quad t \in N_0, \tag{5.14}$$

the fractional difference equation of order $(1, q)$.

In order to solve this equation, we need to introduce some special functions.

Definition 5.5. Define function

$$\Lambda(t, \gamma, \lambda) = {}_a\nabla^{-\gamma} \lambda^t, \quad \gamma \in R, \tag{5.15}$$

where $t = a$ mod (1). Sometimes denote it $\Lambda(\gamma, \lambda)$ or $\Lambda(t, \gamma, \lambda; a)$.

In view of Theorems 4.2 and 4.3, we can establish the following theorem.

Theorem 5.6. *Assume the following function is well defined; then*

(1) $\Lambda(t, \gamma, \lambda) = (1 - 1/\lambda)\Lambda(t, \gamma+1, \lambda) + (t-a+1)^{\overline{\gamma}}/\Gamma(\gamma+1)$,

(2) $\nabla\Lambda(t, \gamma+1, \lambda) = \Lambda(t, \gamma, \lambda)$,

(3) $\nabla^{p}\Lambda(t, \gamma+t, \lambda) = \Lambda(t, \gamma, \lambda)$, *where* $p = 0, 1, 2, \ldots$,

(4) $\nabla^{\mu}\Lambda(t,\gamma,\lambda) = \Lambda(t,\gamma-\mu,\lambda)$, *where* $p-1<\mu\le p$,

(5) $\nabla^{-\mu}\Lambda(t,\gamma,\lambda) = \Lambda(t,\gamma+\mu,\lambda)$.

Now we will use the method of undetermined coefficients to solve Example 5.4. By Theorem 5.6, we notice that

$$
\begin{aligned}
\nabla^{\gamma}\Lambda(t,0,\lambda) &= \Lambda(t,-\gamma,\lambda), \\
\nabla^{\gamma}\Lambda(t,-\gamma,\lambda) &= \Lambda(t,-2\gamma,\lambda), \\
&\vdots \\
\nabla^{\gamma}\Lambda(t,-(q-2)\gamma,\lambda) &= \Lambda(t,-(q-1)\gamma,\lambda), \\
\nabla^{\gamma}\Lambda(t,-(q-1)\gamma,\lambda) &= \Lambda(t,-1,\lambda) = \left(1-\frac{1}{\lambda}\right)\Lambda(t,0,\lambda).
\end{aligned}
\tag{5.16}
$$

The significance of these applications is that if we apply the operator ∇^{γ} to

$$
\Lambda(t,0,\lambda), \Lambda(t,-\gamma,\lambda), \dots, \Lambda(t,-(q-1)\gamma,\lambda),
\tag{5.17}
$$

then we get a cyclic permutation of the same functions. That is, no new functions are introduced. Therefore, we will choose a linear combination of these functions as a candidate for a solution of (5.14). Say

$$
\begin{aligned}
y(t) = {}& b_0\Lambda(t,0,\lambda) + b_1\Lambda(t,-\gamma,\lambda) \\
&+ \dots + b_{q-2}\Lambda(t,-(q-2)\gamma,\lambda) + b_{q-1}\Lambda(t,-(q-1)\gamma,\lambda).
\end{aligned}
\tag{5.18}
$$

Then

$$
\begin{aligned}
\nabla^{\gamma}y(t) = {}& b_0\Lambda(t,-\gamma,\lambda) + b_1\Lambda(t,-2\gamma,\lambda) \\
&+ \dots + b_{q-2}\Lambda(t,-(q-1)\gamma,\lambda) + b_{q-1}\left(1-\frac{1}{\lambda}\right)\Lambda(t,0,\lambda).
\end{aligned}
\tag{5.19}
$$

Taking $y(t)$, $\nabla^{\gamma}y(t)$ into the left side of (5.14), we obtain

$$
\begin{aligned}
\nabla^{\gamma}y(t) - ay(t) = {}& \left[b_{q-1}\left(1-\frac{1}{\lambda}\right) - ab_0\right]\Lambda(t,0,\lambda) \\
&+ (b_0 - ab_1)\Lambda(t,-\gamma,\lambda) + \cdots + (b_{q-2} - ab_{q-1})\Lambda(t,-(q-1)\gamma,\lambda).
\end{aligned}
\tag{5.20}
$$

In order to make the right side equate zero, set

$$
b_k = c\alpha^{-k}, \quad (k = 1,2,\dots,q-1).
\tag{5.21}
$$

Then

$$b_{k-1} - ab_k = c\left(\alpha^{-k+1} - a\alpha^{-k}\right) = c\alpha^{-k}(\alpha - a). \tag{5.22}$$

If we let α be a root of the indicial equation

$$P(x) = x - a = 0, \tag{5.23}$$

or $\alpha = a$, then we have

$$b_{k-1} - ab_k = ca^{-k}P(a) = 0 \quad (k = 1, 2, \ldots, q-1). \tag{5.24}$$

Since we also need

$$0 = b_{q-1}\left(1 - \frac{1}{\lambda}\right) - ab_0 = ac\left[\left(1 - \frac{1}{\lambda}\right)a^{-q} - 1\right], \tag{5.25}$$

so let us set

$$\left(1 - \frac{1}{\lambda}\right) = a^q, \quad \lambda = \frac{1}{1 - \alpha^q}. \tag{5.26}$$

Since c is an arbitrary number, set $c = a^{q-1}$, then

$$b_k = a^{q-1-k}. \tag{5.27}$$

Therefore, we obtain a solution of fractional difference of order $(1, q)$ as

$$\begin{aligned} y(t) &= \sum_{k=0}^{q-1} b_k \Lambda(t, -k\gamma, \lambda) \\ &= \sum_{k=0}^{q-1} a^{q-1-k} \Lambda\left(t, -k\gamma, \frac{1}{1 - a^q}\right) \triangleq \lambda_a(t). \end{aligned} \tag{5.28}$$

The fractional difference equation of order $(1, q)$ in Example 5.4 can be solved by the method of N_0 transform. Make N_1 transform to the following equation:

$$\nabla^{\gamma} y(t) - a\nabla^0 y(t) = 0. \tag{5.29}$$

We have

$$\begin{aligned} s^{\gamma}N_0\big(f(t)\big) - (1-s)^{-1}f(0) + aN_1\big(f(t)\big) &= 0,\\ N_1\big(f(t)\big) = \sum_{t=1}^{\infty}(1-s)^{t-1}f(t)&\\ = \sum_{t=0}^{\infty}(1-s)^{t-1}f(t) - (1-s)^{-1}f(0).& \end{aligned} \tag{5.30}$$

Taking them into previos equation, we get

$$s^{\gamma}N_0\big(f(t)\big) - (1-a)(1-s)^{-1}f(0) - aN_0\big(f(t)\big) = 0, \tag{5.31}$$

and we have

$$\begin{aligned} N_0\big(f(t)\big) &= (1-a)y(0)\frac{1}{(1-s)(s^{\gamma}-a)}\\ &= (1-a)y(0)\frac{\sum_{k=0}^{q-1}a^{q-1-k}s^{k\gamma}}{(1-s)(s^{\gamma}-a)\sum_{k=0}^{q-1}a^{q-1-k}s^{k\gamma}}\\ &= (1-a)y(0)\frac{\sum_{k=0}^{q-1}a^{q-1-k}s^{k\gamma}}{(1-s)(s-a^{q})}. \end{aligned} \tag{5.32}$$

In [23], we have the following

Theorem 5.7. *The following equality holds:*

(1) $N_0(\Lambda(t,0,\lambda)) = N_0(\lambda^t) = 1/(1-s)\cdot 1/(1-(1-s)\lambda)$,

(2) $N_0(\Lambda(t,-k\gamma,\lambda)) = N_0(\nabla^{k\gamma}\lambda^t) = 1/(1-s)\cdot s^{k\gamma}/(1-(1-s)\lambda)\cdot\ (k=1,2,\ldots,q-1)$.

Set $\lambda = 1/(1-a^q)$, *then*

$$\begin{aligned} N_0\Lambda\left(t,0,\frac{1}{1-a^q}\right) &= \frac{1}{1-s}\cdot\frac{1-a^q}{s-a^q},\\ N_0\Lambda\left(t,-k\gamma,\frac{1}{1-a^q}\right) &= \frac{s^{k\gamma}}{1-s}\cdot\frac{1-a^q}{s-a^q}.\quad (k=1,2,\ldots,q-1). \end{aligned} \tag{5.33}$$

By Theorem 5.7 and (5.33), *we know that*

$$y(t) = (1-a)y(0)\sum_{k=0}^{q-1}a^{q-1-k}\Lambda\left(t,-k\gamma,\frac{1}{1-a^q}\right) \tag{5.34}$$

is a solution of (5.14). *Except a constant, the solution* $y(t)$ *is the same as the solution* (5.28), *where*

$$y(t) = \lambda_a(t), \tag{5.35}$$

which is solved by the method of undetermined coefficients before.

6. Relationship between the Fractional Difference Equations and the Fractional Differential Equations

In this section, we only give an example to demonstrate the relationship between integers order difference equations and integral order differential equation.

Let us recall the definition of fractional sum when step $h = 1$

$$ {}_a\nabla_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\gamma-1}} f(s), \tag{6.1}$$

where $t \in N_a = \{a, a+1, a+2, \ldots\}$. If we set

$$ \begin{gathered} t-a = n \in N_0, \qquad s-a = r \in N_0, \\ f_r^{\{a\}} = f(r+a) = f(s), \qquad f_n^{\{a\}} = f(n+a) = f(t), \end{gathered} \tag{6.2}$$

then

$$ \begin{aligned} \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\gamma-1}} f(s) &= \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{n+a} (n+a-\rho(s))^{\overline{\gamma-1}} f(s) \\ &= \frac{1}{\Gamma(\gamma)} \sum_{r=0}^{n} (n+a-(r+a+1))^{\overline{\gamma-1}} f(r+a) \\ &= \frac{1}{\Gamma(\gamma)} \sum_{r=0}^{n} (n-r+1)^{\overline{\gamma-1}} f_r^{\{a\}} = {}_0\nabla_n^{-\gamma} f_n^{\{a\}}. \end{aligned} \tag{6.3}$$

And it is easy to prove that

$$ {}_a\nabla_t^{\mu} f(t) = {}_0\nabla_n^{\mu} f_n^{\{a\}}, \quad (\mu > 0). \tag{6.4}$$

Therefore, we have the following.

Theorem 6.1. *Let $t \in N_a$, and set $t-a = n \in N_0$, $f_n^{\{a\}} = f(n+a) = f(t)$, then*

$$ {}_a\nabla_t^{-\gamma} f(t) = {}_0\nabla_n^{-\gamma} f_n^{\{a\}}; \quad {}_a\nabla_t^{\mu} f(t) = {}_0\nabla_n^{\mu} f_n^{\{a\}}, \quad (\mu, \gamma > 0). \tag{6.5}$$

Example 6.2. (1) Set $\gamma = 1/q$, $q \in N, n \in N$, and solve the fractional difference equation of order $(1, q)$,

$$ \nabla^{\gamma} x(n) - \alpha x(n) = 0. \tag{6.6}$$

(2) Let $t \in R$, and solve the equation

$$ \nabla^{\gamma} x(t) - \alpha x(t) = 0. \tag{6.7}$$

(3) Let $h \in R^{+}, t \in R$, and solve the equation

$$\nabla_h^{\gamma} x(t) - \alpha x(t) = 0. \tag{6.8}$$

(4) If we let $h \rightarrow 0$, we ask whether the limit solution of (6.8) is equivalent to that of the following fractional differential equation? Consider

$$D^{\gamma} x(t) - \alpha x(t) = 0, \quad (t \in R). \tag{6.9}$$

Solution 1. (1) By a result in Chapter 7 of book [23], the solution of (6.6) is

$$x(n) = \lambda_{\alpha}(n) = \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda_n \left[-k\gamma, \left(\frac{1}{1-\alpha^q} \right)^n \right]. \tag{6.10}$$

(2) Set $t - t_0 = n \in N_0$, and define

$$\begin{gathered} x_n^{\{t_0\}} = x(n+t_0) = x(t), \qquad x_{n-1}^{\{t_0\}} = x(n-1+t_0) = x(t-1), \ldots, \\ x_0^{\{t_0\}} = x(0+t_0) = x(t_0). \end{gathered} \tag{6.11}$$

Hence, we can regard the following $x(t_0), x(t_0+1), \ldots, x(t), \ldots$ as a sequence

$$x_0^{\{t_0\}}, x_1^{\{t_0\}}, \ldots x_n^{\{t_0\}}, \ldots. \tag{6.12}$$

Under this definition, (6.7) is actually equivalent to the following integer variable difference equation:

$$\nabla^{\gamma} x_n^{\{t_0\}} - \alpha x_n^{\{t_0\}} = 0. \tag{6.13}$$

By (1), we know that its solution is

$$\begin{aligned} x_n^{\{t_0\}} &= \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda_n \left[-k\gamma, \left(\frac{1}{1-\alpha^q} \right)^n \right] \\ &= \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda \left[t, -k\gamma, \left(\frac{1}{1-\alpha^q} \right)^t \right]. \end{aligned} \tag{6.14}$$

That is

$$x(t) = \sum_{k=0}^{q-1} \alpha^{q-k-1} \Lambda \left[t, -k\gamma, \left(\frac{1}{1-\alpha^q} \right)^t \right] \triangleq \lambda_{\alpha}(t). \tag{6.15}$$

(3) Set $t = sh, x(t) = x(sh) = y(s)$, then (6.8) is equivalent to

$$h^{-\gamma}\nabla^{\gamma}y(s) - \alpha y(s) = 0. \tag{6.16}$$

By (2), we obtain that the solution of (6.16) is

$$\begin{aligned} x(t) &= y(s) = \lambda_{\alpha h^{\gamma}}(s) \\ &= \sum_{k=0}^{q-1}(\alpha h^{\gamma})^{q-k-1}\Lambda\left[s, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{s}\right]. \end{aligned} \tag{6.17}$$

Since

$$\Lambda\left[s, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{s}\right] = h^{k\gamma}\Lambda_h\left[t, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{t/h}\right], \tag{6.18}$$

hence we have

$$x(t) = \sum_{k=0}^{q-1}(\alpha h^{\gamma})^{q-k-1}h^{k\gamma}\Lambda_h\left[t, -k\gamma, \left(\frac{1}{1-(\alpha h^{\gamma})^{q}}\right)^{t/h}\right]. \tag{6.19}$$

(4) Let $h \longrightarrow 0$, and since

$$\begin{gathered} \left(\frac{1}{1-\alpha^{q}h}\right)^{t/h} \longrightarrow e^{\alpha^{q}}, \\ h^{k\gamma}\Lambda_h\left[t, -k\gamma, \left(\frac{1}{1-\alpha^{q}h}\right)^{t/h}\right] = h^{k\gamma}\nabla_h^{k\gamma}\left(\frac{1}{1-\alpha^{q}h}\right)^{t/h} \longrightarrow D^{k\gamma}e^{\alpha^{q}} = E(-k\gamma, \alpha^{q}). \end{gathered} \tag{6.20}$$

We then obtain

$$x(t) = e_{\alpha}(t) = \sum_{k=0}^{q-1}\alpha^{q-k-1}E(-k\gamma, \alpha^{q}), \tag{6.21}$$

and this is exactly the solution of (6.9). (See Chapter 5 in monographer [2]).

Remark 6.3. If we take $\gamma = 1/2$, $q = 2$, then the followong occurs.
(1) The solution of (6.19) reduces to

$$\begin{aligned} x(t) &= \alpha\left(\frac{1}{1-\alpha^{2}}\right)^{t} + \nabla^{1/2}\left(\frac{1}{1-\alpha^{2}}\right)^{t} \\ &= \alpha F\left(t, 0, \frac{1}{1-\alpha^{2}}\right) + F\left(t, -\frac{1}{2}, \frac{1}{1-\alpha^{2}}\right), \end{aligned} \tag{6.22}$$

and this result is consistent with the solution (5.28) or (5.34) in Example 5.4 in Section 5.

(2) The solution (6.21) reduces to

$$x(t) = \alpha h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h} + h^{1/2}\nabla_h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h}$$
$$= h^{1/2}\left[\alpha\left(\frac{1}{1-\alpha^q h}\right)^{t/h} + \nabla_h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h}\right]. \tag{6.23}$$

Let $h \to 0$, then

$$\left[\alpha\left(\frac{1}{1-\alpha^q h}\right)^{t/h} + \nabla_h^{1/2}\left(\frac{1}{1-\alpha^q h}\right)^{t/h}\right] \tag{6.24}$$

tend to

$$\alpha e^{\alpha^q t} + D^{1/2} e^{\alpha^q t} = e_\alpha(t). \tag{6.25}$$

The results perfectly coincide with the monographer [2].

From Theorem 6.1, we see that if we take t as $a, a+1, a+2, \ldots$, it is only a sequence with step 1, but the initial time is not zero but a. If we make a translation variable transformation, set $t = n + a$, $n \in N_0$, then we can change the definition of fractional sum and fractional difference with real variable into the definition of fractional sum and difference with integer variable. But, no doubt, it will be more convenient for us to study fractional sum and difference with integer variable.

7. Conclusion

This work reveals some results in discrete fractional calculus and fractional h-difference equations. This study also provides a reference for researchers in this area. First, this paper gives the definition of the fractional h-difference from the difference of integer order. Then some integral transforms are proposed, that is, Z transform, N transform, and R transform. These integral transforms are applied to linear fractional h-difference equations, and approximate solutions are obtained. At last, the study explains the relationship between the fractional difference equations and the fractional differential equations.

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Research Article

An Adaptive Pseudospectral Method for Fractional Order Boundary Value Problems

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An adaptive pseudospectral method is presented for solving a class of multiterm fractional boundary value problems (FBVP) which involve Caputo-type fractional derivatives. The multiterm FBVP is first converted into a singular Volterra integrodifferential equation (SVIDE). By dividing the interval of the problem to subintervals, the unknown function is approximated using a piecewise interpolation polynomial with unknown coefficients which is based on shifted Legendre-Gauss (ShLG) collocation points. Then the problem is reduced to a system of algebraic equations, thus greatly simplifying the problem. Further, some additional conditions are considered to maintain the continuity of the approximate solution and its derivatives at the interface of subintervals. In order to convert the singular integrals of SVIDE into nonsingular ones, integration by parts is utilized. In the method developed in this paper, the accuracy can be improved either by increasing the number of subintervals or by increasing the degree of the polynomial on each subinterval. Using several examples including Bagley-Torvik equation the proposed method is shown to be efficient and accurate.

1. Introduction

Due to the development of the theory of fractional calculus and its applications, such as in the fields of physics, Bode's analysis of feedback amplifiers, aerodynamics and polymer rheology, and so forth, many works on the basic theory of fractional calculus and fractional order differential equations have been established [1–3].

In general, the analytical solutions for most of the fractional differential equations are not readily attainable, and thus the need for finding efficient computational algorithms for obtaining numerical solutions arises. Recently, there have been many papers dealing with the solutions of initial value and boundary value problems for linear and nonlinear fractional differential equations. These methods include finite difference approximation method [4], collocation method [5, 6], the Adomian decomposition method [7, 8], variational iteration method [9–12], operational matrix methods [13–16], and homotopy methods [17, 18]. In [19] suitable spline functions of polynomial form are derived and used to solve linear and nonlinear fractional differential equations. The authors of [20] have investigated the existence and multiplicity of positive solutions of a nonlinear fractional differential equation initial value problem. Furthermore, some physical and geometrical interpretations of fractional operators and fractional differential equations have been of concern to many authors [12, 21, 22].

In the present paper, we intend to introduce an efficient adaptive pseudospectral method for multiterm fractional boundary value problems (FBVP) of the form

$$F\left(x, y(x), D^{\alpha_1} y(x), \ldots, D^{\alpha_m} y(x)\right) = 0, \quad x \in [0, L], \tag{1.1}$$

subject to

$$H_r\left(y(\xi_0), \ldots, y^{(l)}(\xi_0), \ldots, y(\xi_l), \ldots, y^{(l)}(\xi_l)\right) = 0, \quad r = 0, 1, \ldots, l, \tag{1.2}$$

where F can be nonlinear in general, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$, $l < \alpha_m \leq l + 1$, $L \in \mathbb{R}$, H_r are linear functions, the points ξ_0, ξ_1, …, ξ_l lie in $[0, L]$, and D^{α_q} denotes the Caputo-fractional derivative of order α_q, defined as follows [23]:

$$D^{\alpha_q} y(x) = \frac{1}{\Gamma\left(n_q - \alpha_q\right)} \int_0^x \frac{y^{(n_q)}(t)}{(x - t)^{\alpha_q + 1 - n_q}} \mathrm{d}t, \qquad n_q = \left[\alpha_q\right] + 1, \ q = 1, 2, \ldots, m, \tag{1.3}$$

where $[\alpha_q]$ denotes the integer part of the real number α_q. For details about the mathematical properties of fractional derivatives, see [2].

In this method, the multi-term FBVP is first converted into a singular Volterra integrodifferential equation (SVIDE). By dividing the interval of the problem to subintervals, the unknown function is approximated using a piecewise interpolation polynomial with unknown coefficients which is based on shifted Legendre-Gauss (ShLG) collocation points. Then the problem is reduced to a system of algebraic equations using collocation. Further, some additional conditions are considered to maintain the continuity of the approximate solution and its first l derivatives at the interface of subintervals. The singular integrals of SVIDE are converted into nonsingular ones by utilizing integration by parts and thus greatly improve the accuracy and convergence rate of the approximate solution. The main characteristics of the method are that it converts the FBVP into a system of algebraic equations which greatly simplifies it. In addition, in the method developed in this paper, the accuracy can be improved either by increasing the number of subintervals or by increasing the degree of the polynomial on each subinterval. The present adaptive pseudospectral method can be implemented for FBVPs defined in large domains. Moreover, this new algorithm also works well even for some solutions having oscillatory behavior. Numerical examples including Bagley-Torvik equation subject to boundary conditions are also presented to illustrate the accuracy of the present scheme. Finally, in order to have a physical understanding of fractional differential equations, the derivation of Bagley-Torvik equation is given in the appendix.

The outline of this paper is as follows. In Section 2, some basic properties of Legendre and shifted Legendre polynomials, which are required for our subsequent development, are first presented. Piecewise polynomials interpolation based on ShLG points and its convergence properties are then investigated, and finally the adaptive pseudospectral method for FBVPs is explained. Section 3 is devoted to some numerical examples. In Section 4, a brief conclusion is given. The appendix is given which consists of the derivation of Bagley-Torvik equation.

2. The Adaptive Pseudospectral Method for FBVPs

In this section we drive the adaptive pseudospectral method based on ShLG collocation points and apply it to solve the nonlinear multi-term FBVP (1.1)-(1.2).

2.1. Review of Legendre and Shifted Legendre Polynomials

The Legendre polynomials, $P_i(z)$, $i = 0, 1, 2, \ldots$, are the eigenfunctions of the singular Sturm-Liouville problem

$$\left[\left(1 - z^2\right)P_i'(z)\right]' + i(i+1)P_i(z) = 0. \tag{2.1}$$

Also, they are orthogonal with respect to L^2 inner product on the interval $[-1, 1]$ with the weight function $w(z) = 1$, that is,

$$\int_{-1}^{1} P_i(z)P_j(z)\mathrm{d}z = \frac{2}{2i+1}\delta_{ij}, \tag{2.2}$$

where δ_{ij} is the Kronecker delta. The Legendre polynomials satisfy the recursion relation

$$P_{i+1}(z) = \frac{2i+1}{i+1}zP_i(z) - \frac{i}{i+1}P_{i-1}(z), \tag{2.3}$$

where $P_0(z) = 1$ and $P_1(z) = z$. If $P_i(z)$ is normalized so that $P_i(1) = 1$, then, for any i, the Legendre polynomials in terms of power of z are

$$P_i(z) = \frac{1}{2^i}\sum_{k=0}^{[i/2]}(-1)^k\binom{i}{k}\binom{2i-2k}{i}z^{i-2k}, \tag{2.4}$$

where $[i/2]$ denotes the integer part of $i/2$.

The Legendre-Gauss (LG) collocation points $-1 < z_1 < z_2 < \cdots < z_{N-1} < 1$ are the roots of $P_{N-1}(z)$. No explicit formulas are known for the LG points; however, they are computed numerically using existing subroutines. The LG points have the property that

$$\int_{-1}^{1} p(z)\mathrm{d}z = \sum_{j=1}^{N-1} w_j p(z_j), \tag{2.5}$$

is exact for polynomials of degree at most $2N-3$, where

$$w_j = \frac{2}{\left(1-z_j^2\right)\left[P'_{N-1}(z_j)\right]^2}, \quad j = 1,2,\ldots,N-1, \tag{2.6}$$

are LG quadrature weights. For more details about Legendre polynomials, see [24].

The shifted Legendre polynomials on the interval $x \in [a,b]$ are defined by

$$\widehat{P}_i(x) = P_i\left(\frac{1}{b-a}(2x-a-b)\right), \quad i = 0,1,2,\ldots, \tag{2.7}$$

which are obtained by an affine transformation from the Legendre polynomials. The set of shifted Legendre polynomials is a complete $L^2[a,b]$-orthogonal system with the weight function $w(x) = 1$. Thus, any function $f \in L^2[a,b]$ can be expanded in terms of shifted Legendre polynomials.

The ShLG collocation points $a < x_1 < x_2 < \cdots < x_{N-1} < b$ on the interval $[a,b]$ are obtained by shifting the LG points, z_j, using the transformation

$$x_j = \frac{1}{2}\left((b-a)z_j + a + b\right), \quad j = 1,2,\ldots,N-1. \tag{2.8}$$

Thanks to the property of the standard LG quadrature, it follows that for any polynomial p of degree at most $2N-3$ on (a,b),

$$\begin{aligned}\int_a^b p(x)\mathrm{d}x &= \frac{b-a}{2}\int_{-1}^{1} p\left(\frac{1}{2}[(b-a)z+a+b]\right)\mathrm{d}z \\ &= \frac{b-a}{2}\sum_{j=1}^{N-1} w_j p\left(\frac{1}{2}\left[(b-a)z_j+a+b\right]\right) = \sum_{j=1}^{N-1}\widehat{w}_j p(x_j),\end{aligned} \tag{2.9}$$

where $\widehat{w}_j = ((b-a)/2)w_j$, $1 \leqslant j \leqslant N-1$ are ShLG quadrature weights.

2.2. Function Approximation

Suppose that the interval $[0,L]$ is divided into K subintervals $I_k = [(k-1)h, kh]$, $k = 1,2,\ldots,K$, where $h = L/K$. Let $y_k(x)$ be the solution of the problem in (1.1)-(1.2) in the subinterval I_k. Consider now the ShLG collocation points $(k-1)h < x_{k1} < \cdots < x_{k,N-1} < kh$ on the kth subinterval I_k, $k = 1, 2, \ldots, K$, obtained using (2.8). Obviously,

$$x_{kj} = \frac{h}{2}\left(z_j + 2k - 1\right), \quad j = 1, 2, \ldots, N-1. \tag{2.10}$$

Also, consider two additional *noncollocated* points $x_{k0} = (k-1)h$ and $x_{kN} = kh$. Let us define

$$\Re_{kN} = \operatorname{Span}\{L_{k0}(x), L_{k1}(x), \ldots, L_{kN}(x)\}, \quad x \in I_k, \tag{2.11}$$

where

$$L_{ki}(x) = \prod_{l=0, l \neq i}^{N} \frac{x - x_{kl}}{x_{ki} - x_{kl}}, \quad i = 0, 1, \ldots, N, \tag{2.12}$$

is a basis of Lagrange interpolating polynomials on the subinterval I_k that satisfy $L_{ki}(x_{kj}) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. The $L^2(I_k)$-orthogonal projection $I_N : L^2(I_k) \to \Re_{kN}$ is a mapping in a way that for any $y_k \in L^2(I_k)$

$$\langle I_N(y_k) - y_k, \phi_k \rangle = 0, \quad \forall \phi_k \in \Re_{kN}, \tag{2.13}$$

or equivalently

$$I_N(y_k)(x) = \sum_{i=0}^{N} y_{ki} L_{ki}(x), \quad x \in I_k, \tag{2.14}$$

where $y_{ki} = y_k(x_{ki})$.

Here, it can be easily seen that for $i = 0, 1, \ldots, N$ and $k = 1, 2, \ldots, K$, we have

$$L_{ki}(x) = L_{1i}(x - x_{k0}), \quad x \in I_k. \tag{2.15}$$

Thus, by utilizing (2.15) for (2.14), the approximation of $y_k(x)$ within each subinterval I_k can be restated as

$$y_k(x) \approx I_N(y_k)(x) = \sum_{i=0}^{N} y_{ki} L_{1i}(x - x_{k0}) = Y_k^T \cdot L_k(x), \quad x \in I_k, \tag{2.16}$$

where Y_k and $L_k(x)$ are $(N+1) \times 1$ matrices given by $Y_k = [y_{k0}, \ldots, y_{kN}]^T$ and $L_k(x) = [L_{10}(x - x_{k0}), \ldots, L_{1N}(x - x_{k0})]^T$. It is important to observe that the series (2.16) includes the Lagrange polynomials associated with the *noncollocated* points $x_{k0} = (k-1)h$ and $x_{kN} = kh$. Moreover, it is seen from (2.15)-(2.16) that, in the present adaptive scheme, it is only needed to produce the basis of Lagrange polynomials $L_{1i}(x)$ at the first subinterval.

By $n_m + 1$ times (n_m is defined in (1.3)) differentiating of (2.16), we obtain

$$y_k^{(r)}(x) \approx Y_k^T \cdot L_k^{(r)}(x), \quad x \in I_k, \tag{2.17}$$

where $L_k^{(r)}(x) = (d^r/dx^r)L_k(x)$.

2.3. Convergence Rate

For $N \geqslant 1$ we introduce the piecewise polynomials space

$$\Psi_{I_k}^N = \left\{ y \in C^0([0,L]) : y_k = y|_{I_k} \in \mathbb{P}_N(I_k) \right\}, \tag{2.18}$$

which is the space of the continuous functions over $[0,L]$ whose restrictions on each subinterval I_k are polynomials of degree $\leqslant N$. Then, for any continuous function y in $[0,L]$, the piecewise interpolation polynomial $\varphi_N(y)$ coincides on each subinterval I_k with the interpolating polynomial $I_N(y)$ of $y_k = y|_{I_k}$ at the ShLG points.

In [25], with the aid of the formulas (5.4.33), (5.4.34) of [24], we prove the convergence properties of piecewise interpolation polynomial based on shifted Legendre-Gauss-Radau points in the norms of the Sobolev spaces. Accordingly, the following results for the convergence based on ShLG points hold.

Theorem 2.1. *Suppose that $y \in H^v(0,L)$ with $v \geqslant 1$. Then*

$$\left\| y - \varphi_N(y) \right\|_{L^2(0,L)} \leqslant cN^{-v} |y|_{H^{0;v;N;h}(0,L)}, \tag{2.19}$$

and, for $1 \leqslant u \leqslant v$, if $h \leqslant 1$, then

$$\left\| y - \varphi_N(y) \right\|_{H^u(0,L)} \leqslant cN^{2u-(1/2)-v} |y|_{H^{u;v;N;h}(0,L)}, \tag{2.20}$$

and if $h > 1$, then

$$\left\| y - \varphi_N(y) \right\|_{H^u(0,L)} \leqslant cN^{2u-(1/2)-v} |y|_{H^{0;v;N;h}(0,L)}. \tag{2.21}$$

Note that c denotes a positive constant that depends on v, but which is independent of the function y and integer N. Moreover, we introduce the seminorm of $H^v(0,L)$, $0 \leqslant u \leqslant v$, $N \geqslant 0$, $h > 0$, as

$$|y|_{H^{u;v;N;h}(0,L)} = \left(\sum_{l=\min\{v,N+1\}}^{v} h^{2l-2u} \left\| y^{(l)} \right\|_{L^2(0,L)}^2 \right)^{1/2}. \tag{2.22}$$

Remark 2.2. Whenever $N \geqslant v - 1$, using (2.19)–(2.22), we get

$$\left\| y - \varphi_N(y) \right\|_{L^2(0,L)} \leqslant cN^{-v} h^v \left\| y^{(v)} \right\|_{L^2(0,L)}, \tag{2.23}$$

and, for $u \geqslant 1$, if $h \leqslant 1$, then

$$\left\| y - \varphi_N(y) \right\|_{H^u(0,L)} \leqslant cN^{2u-(1/2)-v} h^{v-u} \left\| y^{(v)} \right\|_{L^2(0,L)}, \tag{2.24}$$

and if $h > 1$, then

$$\left\| y - \varphi_N(y) \right\|_{H^u(0,L)} \leqslant cN^{2u-(1/2)-v} h^{v} \left\| y^{(v)} \right\|_{L^2(0,L)}. \tag{2.25}$$

Equations (2.23)–(2.25) show that if y is infinitely smooth on $[0, L]$ and $h \leqslant 1$, the convergence rate of $\varphi_N(y)$ to y is faster than h to the power of $N + 1 - u$ and any power of $1/N$, which is superior to that for the global collocation method over $[0, L]$. Thus, the bigger the subinterval length the slower the convergence rate.

2.4. Problem Replacement and the Solution Technique

Consider the multi-term FBVP in (1.1)-(1.2). With substituting the definition of the Caputo-derivative (1.3) into (1.1), we can convert (1.1) into an equivalent SVIDE as

$$F\left(x, y(x), \frac{1}{\Gamma(n_1 - \alpha_1)} \int_0^x \frac{y^{(n_1)}(t)}{(x-t)^{\alpha_1+1-n_1}} \mathrm{d}t, \dots, \frac{1}{\Gamma(n_m - \alpha_m)} \int_0^x \frac{y^{(n_m)}(t)}{(x-t)^{\alpha_m+1-n_m}} \mathrm{d}t \right) = 0. \tag{2.26}$$

The problem is to find $y(x)$, $x \in [0, L]$, satisfying (2.26) and (1.2).

The generally nonlinear SVIDE in (2.26) is given in subinterval I_k, $k = 1, 2, \dots, K$ as follows:

$$\begin{aligned} F\Bigg(x, y_k(x), & \frac{1}{\Gamma(n_1 - \alpha_1)} \left[\sum_{s=1}^{k-1} \Lambda_{\alpha_1,s}(x) + \int_{(k-1)h}^{x} \frac{y_k^{(n_1)}(t)}{(x-t)^{\alpha_1+1-n_1} \mathrm{d}t} \right] \\ & , \dots, \frac{1}{\Gamma(n_m - \alpha_m)} \left[\sum_{s=1}^{k-1} \Lambda_{\alpha_m,s}(x) + \int_{(k-1)h}^{x} \frac{y_k^{(n_m)}(t)}{(x-t)^{\alpha_m+1-n_m}} \mathrm{d}t \right] \Bigg) = 0, \end{aligned} \tag{2.27}$$

where $x \in I_k$ and

$$\Lambda_{\alpha_q,s}(x) = \int_{(s-1)h}^{sh} \frac{y_s^{(n_q)}(t)}{(x-t)^{\alpha_q+1-n_q}} \mathrm{d}t, \quad q = 1, 2, \dots, m; \; s = 1, 2, \dots, k-1. \tag{2.28}$$

It is important to note that, at the first subinterval, the summations in (2.27) are automatically discarded. For approximating the functions $\Lambda_{\alpha_q,s}(x)$, $q = 1, \dots, m$; $s = 1, \dots, k-1$, with the aid of (2.17) and the Gaussian integration formula in the subinterval I_s, given by (2.9), we obtain

$$\Lambda_{\alpha_q,s}(x) \simeq \frac{h}{2}\sum_{j=1}^{N-1} w_j \frac{Y_s^{T} \cdot L_s^{(n_q)}(x_{sj})}{(x - x_{sj})^{\alpha_q+1-n_q}}, \tag{2.29}$$

where the ShLG collocation points x_{sj} on the subinterval I_s are defined by (2.10).

From (2.27), (2.16), and (2.17), for $k = 1, 2, \ldots, K$ we have

$$\begin{aligned} &F\left(x, Y_k^{T} \cdot L_k(x), \frac{1}{\Gamma(n_1 - \alpha_1)}\left[\sum_{s=1}^{k-1}\Lambda_{\alpha_1,s}(x) + \int_{x_{k0}}^{x} \frac{Y_k^{T} \cdot L_k^{(n_1)}(t)}{(x-t)^{\alpha_1+1-n_1}}\mathrm{d}t\right]\right. \\ &\quad \left., \ldots, \frac{1}{\Gamma(n_m - \alpha_m)}\left[\sum_{s=1}^{k-1}\Lambda_{\alpha_m,s}(x) + \int_{x_{k0}}^{x} \frac{Y_k^{T} \cdot L_k^{(n_m)}(t)}{(x-t)^{\alpha_m+1-n_m}}\mathrm{d}t\right]\right) = 0, \quad x \in I_k. \end{aligned} \tag{2.30}$$

We now collocate (2.30) at collocation points x_{kj}, $k = 1, 2, \ldots, K$ and $j = 1, \ldots, N - l$ as

$$\begin{aligned} &F\left(x_{kj}, Y_k^{T} \cdot L_k(x_{kj}), \frac{1}{\Gamma(n_1 - \alpha_1)}\left[\sum_{s=1}^{k-1}\Lambda_{\alpha_1,s}(x_{kj}) + \int_{x_{k0}}^{x_{kj}} \frac{Y_k^{T} \cdot L_k^{(n_1)}(t)}{(x_{kj}-t)^{\alpha_1+1-n_1}}\mathrm{d}t\right]\right. \\ &\quad \left., \ldots, \frac{1}{\Gamma(n_m - \alpha_m)}\left[\sum_{s=1}^{k-1}\Lambda_{\alpha_m,s}(x_{kj}) + \int_{x_{k0}}^{x_{kj}} \frac{Y_k^{T} \cdot L_k^{(n_m)}(t)}{(x_{kj}-t)^{\alpha_m+1-n_m}}\mathrm{d}t\right]\right) = 0. \end{aligned} \tag{2.31}$$

The integrals involved in (2.31) are singular. In order to convert them into nonsingular integrals, using integration by parts and with the aid of (2.10) we obtain

$$\begin{aligned} &F\left(x_{kj}, y_{kj}, \frac{1}{\Gamma(n_1 - \alpha_1)}\right. \\ &\quad \times \left[\sum_{s=1}^{k-1}\Lambda_{\alpha_1,s}(x_{kj}) + \frac{1}{n_1 - \alpha_1}\left(x_{1j}^{n_1-\alpha_1} Y_k^{T} \cdot L_k^{(n_1)}(x_{k0}) + \int_{x_{k0}}^{x_{kj}} \frac{Y_k^{T} \cdot L_k^{(n_1+1)}(t)}{(x_{kj}-t)^{\alpha_1-n_1}}\mathrm{d}t\right)\right], \\ &\quad \ldots, \frac{1}{\Gamma(n_m - \alpha_m)} \\ &\quad \left. \times \left[\sum_{s=1}^{k-1}\Lambda_{\alpha_m,s}(x_{kj}) + \frac{1}{n_m - \alpha_m}\left(x_{1j}^{n_m-\alpha_m} Y_k^{T} \cdot L_k^{(n_m)}(x_{k0}) + \int_{x_{k0}}^{x_{kj}} \frac{Y_k^{T} \cdot L_k^{(n_m+1)}(t)}{(x_{kj}-t)^{\alpha_m-n_m}}\mathrm{d}t\right)\right]\right) = 0. \end{aligned} \tag{2.32}$$

In order to use the Gaussian integration formula in the subinterval I_k for (2.32), we transfer the t-interval $[x_{k0}, x_{kj}]$ into the τ-interval I_k by means of the transformation $\tau = (h/x_{1j})(t - x_{k0}) + x_{k0}$. Using this transformation, the Gaussian integration formula and (2.10), we have

$$\begin{aligned}\int_{x_{k0}}^{x_{kj}} \frac{Y_k^T \cdot L_k^{(n_q+1)}(t)}{(x_{kj}-t)^{\alpha_q-n_q}}dt &= \frac{x_{1j}}{h}\int_{x_{k0}}^{x_{kN}} \frac{Y_k^T \cdot L_k^{(n_q+1)}((x_{1j}/h)(\tau - x_{k0}) + x_{k0})}{(x_{1j}-(x_{1j}/h)(\tau - x_{k0}))^{\alpha_q-n_q}}d\tau \\ &\simeq \frac{x_{1j}}{2}\sum_{p=1}^{N-1} w_p \frac{Y_k^T \cdot L_k^{(n_q+1)}((x_{1j}x_{1p}/h) + x_{k0})}{(x_{1j}-(x_{1j}\ x_{1p}/h))^{\alpha_q-n_q}} := G_q(x_{1j}).\end{aligned} \tag{2.33}$$

By (2.33), (2.32) may be approximated as

$$\begin{aligned}F\Bigg(&x_{kj}, y_{kj}, \frac{1}{\Gamma(n_1-\alpha_1)}\left[\sum_{s=1}^{k-1}\Lambda_{\alpha_1,s}(x_{kj}) + \frac{1}{n_1-\alpha_1}\left(x_{1j}^{n_1-\alpha_1}Y_k^T \cdot L_k^{(n_1)}(x_{k0}) + G_1(x_{1j})\right)\right] \\ &,\ldots, \frac{1}{\Gamma(n_m-\alpha_m)}\left[\sum_{s=1}^{k-1}\Lambda_{\alpha_m,s}(x_{kj}) + \frac{1}{n_m-\alpha_m}\left(x_{1j}^{n_m-\alpha_m}Y_k^T \cdot L_k^{(n_m)}(x_{k0}) + G_m(x_{1j})\right)\right]\Bigg) = 0.\end{aligned} \tag{2.34}$$

In addition, substituting (2.16) and (2.17) into the boundary conditions (1.2) yields

$$H_r\left(Y_{\rho_0}^T \cdot L_{\rho_0}(\xi_0), \ldots, Y_{\rho_0}^T \cdot L_{\rho_0}^{(l)}(\xi_0), \ldots, Y_{\rho_l}^T \cdot L_{\rho_l}(\xi_l), \ldots, Y_{\rho_l}^T \cdot L_{\rho_l}^{(l)}(\xi_l)\right) = 0, \quad r = 0, 1, \ldots, l, \tag{2.35}$$

where $\xi_r \in I_{\rho_r}$. Besides, it is required that the approximate solution and its first l derivatives be continuous at the interface of subintervals, that is,

$$Y_k^T \cdot L_k^{(r)}(x_{kN}) = Y_{k+1}^T \cdot L_{k+1}^{(r)}(x_{k+1,0}), \quad k = 1, 2, \ldots, K-1, \ r = 0, 1, \ldots, l. \tag{2.36}$$

Equation (2.34) for $k = 1, \ldots, K$, $j = 1, \ldots, N - l$ together with (2.35)-(2.36) gives a system of equations with $K(N+1)$ set of algebraic equations, which can be solved to find the unknowns of the vectors Y_k, $k = 1, 2, \ldots, K$. Consequently, the unknown functions $y(x)|_{I_k} \simeq y_k(x)$ given in (2.16) can be calculated.

3. Numerical Examples

In this section we give the computational results of numerical experiments with the method based on preceding sections to support our theoretical discussion.

Example 3.1. In this example, we consider the Bagley-Torvik equation [26]

$$Ay''(x) + BD^{\alpha}y(x) + Cy(x) = f(x), \quad x \in [0, L], \tag{3.1}$$

with boundary conditions

$$y(0) = c_0, \qquad y(L) = c_1, \tag{3.2}$$

where $A, B, C \in \mathbb{R}$ and $A \neq 0$. Bagley-Torvik equation involving fractional derivative of order $1/2$ or $3/2$ arises in the modeling of the motion of a rigid plate in a Newtonian fluid and a gas in a fluid. Since the Bagley-Torvik equation is a prototype fractional differential equation with two derivatives and represents a general form of the fractional problems, its solution can give many ideas about the solution of similar problems in fractional differential equations. Podlubny [2] has investigated the solution of Bagley-Torvik equation (3.1) and for $\alpha = 3/2$ gave the analytical solution with homogeneous initial conditions by using Green's function, as follows:

$$\begin{gathered}
y(x) = \int_0^x G_3(x-t)f(t)\mathrm{d}t, \\
G_3(x) = \frac{1}{A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{C}{A}\right)^k x^{2k+1}E^{(k)}_{1/2,2+(3k/2)}\left(-\frac{B}{A}\sqrt{x}\right), \\
E^{(k)}_{\lambda,\mu}(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k}E_{\lambda,\mu}(z) = \sum_{j=0}^{\infty}\frac{(j+k)!z^j}{j!\Gamma(\lambda j + \lambda k + \mu)}, \quad k = 0, 1, 2, \ldots,
\end{gathered} \tag{3.3}$$

where $E_{\lambda,\mu}$ is the Mittag-Leffler function in two parameters and the G_3 three-term Green's function. However, in practice, these equations can not be evaluated easily for different functions $f(x)$. Several other authors have proposed different techniques for the solution of this equation. A review of the solution techniques for Bagley-Torvik equation can be found in [27].

Here, we solve (3.1) with two-point boundary conditions (3.2) by using the adaptive pseudospectral method. For comparison purposes and in order to demonstrate the efficiency of our method, we investigate the following cases. Further, for completeness, the derivation of Bagley-Torvik equation is given in the appendix.

Case 1. In (3.1)-(3.2) set $\alpha = 3/2$, $A = B = C = 1$, $f(x) = x^2 + 2 + 4\sqrt{x/\pi}$, $L = 5$, $c_0 = 0$, and $c_1 = 25$. It is readily verified that the exact solution of this case is $y(x) = x^2$. Using the adaptive pseudospectral method in Section 2 with $K = 1$ and $N = 2$, the unknowns y_{ki}, $k = 1, \ldots, K$, $i = 0, \ldots, N$ in (2.16) are found to be

$$y_{10} = 0, \qquad y_{11} = 6.25, \qquad y_{12} = 5, \tag{3.4}$$

which lead to the exact solution $y(x) = x^2$. This case was solved in [6] using a collocation-shooting method. Their computed maximum absolute error and L_2 error norm were 2.00×10^{-14} and 3.78×10^{-12}, respectively, which show that our method is more efficient.

Table 1: Maximum absolute errors for Example 3.1, Case 2.

	$N = 10$	$N = 20$	$N = 30$	$N = 40$
$K = 1$	3.5×10^{-5}	3.8×10^{-6}	1.5×10^{-6}	4.4×10^{-7}
$K = 2$	6.9×10^{-6}	7.1×10^{-7}	2.0×10^{-7}	8.4×10^{-8}

Table 2: Comparison of solutions for Example 3.1, Case 3.

x	GTCM [27]	Present method	Analytical [2]
0	0	0	0
0.1	0.03648555	0.03648741	0.03648748
0.2	0.14063472	0.14063951	0.14063962
0.3	0.30747623	0.30748449	0.30748463
0.4	0.53327129	0.53328396	0.53328411
0.5	0.81473561	0.81475679	0.81475695
0.6	1.14880581	1.14883734	1.14883742
0.7	1.53252126	1.53256541	1.53256543
0.8	1.96297499	1.96302931	1.96302925
0.9	2.43745598	2.43733391	2.43733397
1	2.95407000	2.95258388	2.95258388

Case 2. Set $\alpha = 3/2$, $A = B = C = 1$, $f(x) = (15/4)\sqrt{x} + (15/8)\sqrt{\pi x} + x^{5/2} + 1$, $L = 1$, $c_0 = 1$, and $c_1 = 2$. The exact solution of this case, which was considered in [5], is $y(x) = x^{5/2} + 1$. In Table 1 the maximum absolute errors for different values of K and N are presented. We see from Table 1 that, as stated in Section 2.3, the more rapid convergence rate is obtained with smaller subinterval length.

Case 3. For comparison, the same coefficients as considered in [27] have been used here. Set $\alpha = 3/2$, $A = 1$, $B = C = 1/2$, $f(x) = 8$, $L = 1$, $c_0 = 0$, and $c_1 = 2.95258388$. Table 2 shows the comparison of solutions of this case by the present method (with $K = 2$, $N = 40$), GTC method [27] and the analytical solution [2], and the good agreement of our adaptive pseudospectral solution with analytical solution.

Case 4. Set $\alpha = 1/2$, $A = B = C = 1$, $f(x) = 8$, $L = 1$, $c_0 = 0$, and $c_1 = 3.10190571$. The numerical solutions obtained by the present method (with $K = 2$, $N = 40$), fractional finite difference method (FDM), the Adomian decomposition method (ADM), and the variational iteration method (VIM) from [28] are given in Table 3. The exact solution refers to the closed form series solution given in [28]. Table 3 shows the excellent agreement of our adaptive pseudospectral solution with the exact solution.

Example 3.2. As a multi-term equation, consider the linear multi-term FBVP described by

$$\begin{gathered}\sqrt{\pi x}D^{5/2}y(x) + 10\Gamma\left(\frac{2}{3}\right)\sqrt[3]{x}D^{4/3}y(x) + \Gamma\left(\frac{3}{4}\right)\sqrt[4]{x}D^{1/4}y(x) - \frac{40}{77}y(x) = 12x + 54x^2 + \frac{8}{7}x^3, \\ y(0) = y'(0) = 0, \quad y(1) = 1.\end{gathered} \tag{3.5}$$

Table 3: Comparison of solutions for Example 3.1, Case 4.

x	FDM [28]	ADM [28]	VIM [28]	Present method	Exact
0	0	0	0	0	0
0.1	0.039473	0.039874	0.039874	0.03975004	0.03975003
0.2	0.157703	0.158512	0.158512	0.15703584	0.15703582
0.3	0.352402	0.353625	0.353625	0.34736999	0.34736998
0.4	0.620435	0.622083	0.622083	0.60469514	0.60469515
0.5	0.957963	0.960047	0.960047	0.92176757	0.92176764
0.6	1.360551	1.363093	1.363093	1.29045651	1.29045656
0.7	1.823267	1.826257	1.826257	1.70200794	1.70200797
0.8	2.340749	2.344224	2.344224	2.14728692	2.14728693
0.9	2.907324	2.911278	2.911278	2.61700100	2.61700101
1	3.517013	3.521462	3.521462	3.10190571	3.10190571

Table 4: Comparison of maximum absolute errors for Example 3.2.

	Present method				Method [13]	
	$N = 4$	$N = 8$	$N = 16$	$N = 24$	J	Error
$K = 1$	6.9×10^{-4}	6.4×10^{-5}	4.1×10^{-6}	9.9×10^{-7}	4	1.5×10^{-3}
$K = 2$	2.2×10^{-4}	1.2×10^{-5}	1.3×10^{-6}	2.5×10^{-7}	5	6.1×10^{-4}
$K = 3$	8.1×10^{-5}	4.7×10^{-6}	4.0×10^{-7}	9.8×10^{-8}	6	1.8×10^{-4}
$K = 4$	4.5×10^{-5}	2.8×10^{-6}	2.4×10^{-7}	5.9×10^{-8}	7	7.2×10^{-5}

The exact solution to this problem is $y(x) = x^3$. Since this problem is a third-order equation, it can demonstrate the effect of the continuity conditions (2.36) on the approximate solution. Table 4 compares the maximum absolute errors obtained using the present method for different values of K and N with the errors reported in [13] using operational matrix of fractional derivatives using B-spline functions. Note that in [13], for each value of J, the obtained algebraic system is of order 2^J+1, while in the present method the obtained algebraic system is of order $K(N+1)$. It is important to see that our method provides more accurate results with solving lower-order algebraic systems. Further, it is seen that in the present method the accuracy can be improved either by increasing the number of subintervals or by increasing the number of collocation points within each subinterval.

Example 3.3. Consider the nonlinear multi-term FBVP described by

$$y''(x) + \Gamma\left(\frac{4}{5}\right)\sqrt[5]{x^6}D^{6/5}y(x) + \frac{11}{9}\Gamma\left(\frac{5}{6}\right)\sqrt[6]{x}D^{1/6}y(x) - \left(y'(x)\right)^2 = 2 + \frac{1}{10}x^2,$$
$$y(0) = 1, \qquad y(1) = 2. \tag{3.6}$$

The exact solution to this problem is $y(x) = x^2 + 1$. In Table 5, we compare the maximum absolute errors obtained using the present adaptive method for different values of K and N with the errors reported in [13] using operational matrix of fractional derivatives using B-spline functions.

Table 5: Comparison of maximum absolute errors for Example 3.3.

	Present method				Method [13]	
	$N = 10$	$N = 20$	$N = 30$	$N = 40$	J	Error
$K = 1$	1.3×10^{-6}	8.9×10^{-8}	2.0×10^{-8}	6.7×10^{-9}	4	1.2×10^{-3}
$K = 2$	7.5×10^{-7}	5.3×10^{-8}	1.7×10^{-8}	4.9×10^{-9}	5	3.3×10^{-4}
$K = 3$	6.5×10^{-7}	4.7×10^{-8}	1.4×10^{-8}	3.6×10^{-9}	6	8.1×10^{-5}
$K = 4$	5.0×10^{-7}	3.6×10^{-8}	7.9×10^{-9}	2.8×10^{-9}	7	2.1×10^{-5}

Table 6: Comparison of absolute errors for Example 3.4.

α		$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
1.1	Present method	2.7×10^{-10}	5.0×10^{-10}	2.1×10^{-9}	6.4×10^{-10}	1.3×10^{-9}
	Method [16]	2.9×10^{-4}	6.0×10^{-3}	8.4×10^{-3}	5.8×10^{-3}	3.4×10^{-3}
1.3	Present method	2.6×10^{-9}	9.0×10^{-9}	2.4×10^{-8}	1.5×10^{-8}	9.4×10^{-9}
	Method [16]	2.0×10^{-4}	3.0×10^{-3}	4.5×10^{-3}	3.0×10^{-3}	4.0×10^{-3}
1.5	Present method	6.3×10^{-9}	3.8×10^{-8}	1.0×10^{-7}	7.9×10^{-8}	2.9×10^{-8}
	Method [16]	9.7×10^{-5}	1.5×10^{-3}	2.4×10^{-3}	1.5×10^{-3}	4.7×10^{-3}
1.7	Present method	3.1×10^{-8}	1.6×10^{-8}	2.0×10^{-7}	1.8×10^{-7}	7.9×10^{-8}
	Method [16]	4.8×10^{-5}	7.5×10^{-4}	1.3×10^{-3}	6.0×10^{-4}	5.4×10^{-3}
1.9	Present method	2.1×10^{-7}	3.7×10^{-7}	2.0×10^{-7}	1.2×10^{-7}	2.2×10^{-7}
	Method [16]	2.9×10^{-5}	3.7×10^{-4}	6.8×10^{-4}	9.2×10^{-5}	6.2×10^{-3}
2.0	Present method	7.0×10^{-14}	9.0×10^{-14}	6.8×10^{-14}	3.7×10^{-13}	2.1×10^{-13}
	Method [16]	2.4×10^{-5}	2.6×10^{-4}	4.9×10^{-4}	7.0×10^{-5}	6.6×10^{-3}

Example 3.4. Consider the following nonlinear FBVP:

$$\begin{gathered} D^\alpha y(x) + a y^n(x) = f(x), \quad 1 < \alpha \leqslant 2, \\ y(0) = c_0, \qquad y(1) = c_1. \end{gathered} \tag{3.7}$$

For comparison, we choose $n = 2$, $a = e^{-2\pi}$, $f(x) = (105\sqrt{\pi}/16\Gamma((9/2) - \alpha))x^{(7/2)-\alpha} + e^{-2\pi}x^7$, $c_0 = 0$, and $c_1 = 1$. It is readily verified that the exact solution is $y(x) = x^{7/2}$. In Table 6, the absolute errors obtained using the present adaptive pseudospectral method for $K = 4$ and $N = 40$ and different values of α are compared with the errors obtained in [16] using Legendre wavelets, which show that the present method provides more accurate numerical results.

Example 3.5. In this example, to show the applicability of the present method for larger interval, we consider the nonlinear FBVP described by

$$\begin{gathered} D^\alpha y(x) + y^2(x) = E_\alpha^2(-x^\alpha) - E_\alpha(-x^\alpha), \quad 1 < \alpha \leqslant 2, \\ y(0) = 1, \qquad y(10) = E_\alpha(-10^\alpha). \end{gathered} \tag{3.8}$$

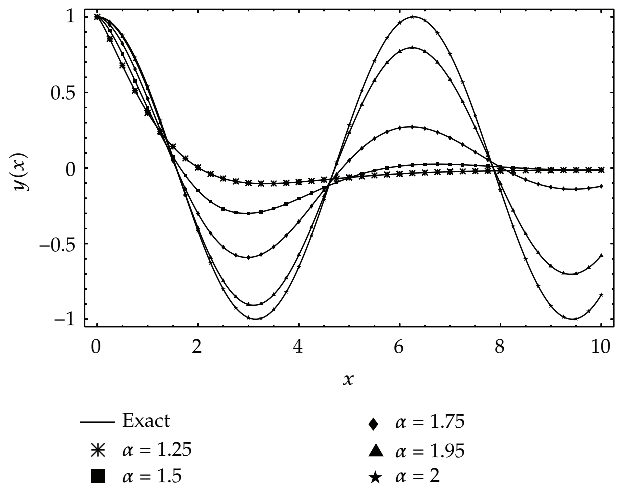

Figure 1: Comparison of $y(x)$ for $K = 20$, $N = 30$ with exact solutions for Example 3.5.

Table 7: Maximum absolute errors for $\alpha = 1.75$ for Example 3.5.

	$N = 10$	$N = 20$	$N = 30$
$K = 5$	3.5×10^{-3}	8.1×10^{-4}	3.5×10^{-4}
$K = 10$	1.7×10^{-3}	3.9×10^{-4}	1.8×10^{-4}
$K = 20$	9.4×10^{-4}	2.1×10^{-4}	9.1×10^{-5}

The exact solution of this problem is given by $y(x) = E_\alpha(-x^\alpha)$, where $E_\alpha(z) = \sum_{k=0}^{\infty}(z^k/\Gamma(\alpha k+1))$ is the Mittag-Leffler function.

In Table 7, the maximum absolute errors in the interval $[0, 10]$ for $\alpha = 1.75$ and different values of K and N are presented, which shows the efficiency of the present method for FBVPs in large domains. Also, the numerical results for $y(x)$ by adaptive pseudospectral method for $K = 20$, $N = 30$ and $\alpha = 1.25, 1.5, 1.75, 1.95$, and 2 together with the exact solutions are plotted in Figure 1, which indicates that the numerical results are in high agreement with the exact ones. Moreover, Figure 1 demonstrates the efficiency of the present method for solutions having oscillatory behavior. For $\alpha = 2$, the exact solution is given as $y(x) = \cos(x)$. Note that as α approaches 2, the numerical solution converges to the analytical solution; that is, in the limit, the solution of the fractional differential equations approaches to that of the integer-order differential equations.

Example 3.6. Finally consider the nonlinear multi-term FBVP described by

$$ay''(x) + bD^{\alpha_2}y(x) + c\left(D^{\alpha_1}y(x)\right)^2 + ey^3(x) = f(x), \quad 0 \leqslant x \leqslant 2,$$
$$y(0) = 0, \qquad y(2) = \frac{8}{3}, \tag{3.9}$$

where $a, b, c, e \in \mathbb{R}$, $0 < \alpha_1 \leqslant 1$, $1 < \alpha_2 \leqslant 2$ and $f(x) = 2ax + (2b/\Gamma(4-\alpha_2))x^{3-\alpha_2} + c((2/\Gamma(4-\alpha_1))x^{3-\alpha_1})^2 + e((1/3)x^3)^3$. The exact solution to this problem is $y(x) = (1/3)x^3$.

For $a = b = c = e = 1$, $\alpha_1 = 0.555$, and $\alpha_2 = 1.455$ the maximum absolute errors obtained using the adaptive pseudospectral method are given in Table 8. Also, for $a = 0.1$, $b = c = e = 0.5$, $\alpha_1 = 0.219$, and $\alpha_2 = 1.965$ the maximum absolute errors are given in Table 9. Again, it is seen that in the present adaptive pseudospectral method the accuracy is improved either by increasing the number of subintervals or by increasing the number of collocation points within each subinterval.

4. Conclusion

In this work a new adaptive pseudospectral method based on ShLG collocation points has been proposed for solving the multi-term FBVPs. We converted the original FBVP into a SIVDE and then reduced it to a system of algebraic equations using collocation. The difficulty in SIVDE, due to the singularity, is overcome here by utilizing integration by parts. By considering some additional conditions, the continuity of the approximate solution and its first l derivatives is kept. It was also shown that the accuracy can be improved either by increasing the number of subintervals or by increasing the number of collocation points in subintervals. Moreover, this method is valid for large-domain calculations. The achieved results are compared with exact solutions and with the solutions obtained by some other numerical methods, which demonstrate the convergence, validity, and accuracy of the proposed method.

Appendix

The Derivation of Bagley-Torvik Equation

Here, in order to give a physical understanding of fractional differential equations, the derivation of Bagley-Torvik equation, which describes the modeling of the motion of a rigid plate in a Newtonian fluid, is given.

Consider a half-space Newtonian viscous fluid in which certain motions are induced by the general transverse motion of an infinite plate. The equation of motion of the half-space fluid is the diffusion equation:

$$\rho \frac{\partial v(z,t)}{\partial t} = \mu \frac{\partial^2 v(z,t)}{\partial z^2}, \tag{A.1}$$

where ρ is the fluid density, μ is the viscosity, and $v(z,t)$ describes the transverse fluid velocity as a function of z and t. Taking the Laplace transform of (A.1) and using the properties of the Laplace transform, one obtains

$$\rho s L[v(z,t)] - \rho v(z,t=0) = \mu \frac{\partial^2}{\partial z^2} L[v(z,t)]. \tag{A.2}$$

Torvik and Bagley assumed the initial velocity profile in the fluid to be zero and thus (A.2) reduces to

$$\rho s L[v(z,t)] = \mu \frac{\partial^2}{\partial z^2} L[v(z,t)]. \tag{A.3}$$

Since the Laplace transformation is evaluated with respect to the time variable, only the following representation for the velocity profile with respect to the depth z can be used:

$$v(z,t) = v(t)e^{\lambda z}, \tag{A.4}$$

thus

$$\begin{aligned} L[v(z,t)] &= e^{\lambda z}L[v(t)], \\ \frac{\partial^2}{\partial z^2}L[v(z,t)] &= \lambda^2 e^{\lambda z}L[v(t)]. \end{aligned} \tag{A.5}$$

With insertion of (A.5) in (A.3) the following algebraic equation for the unknown parameter λ is obtained:

$$\lambda = \sqrt{\frac{s\rho}{\mu}}. \tag{A.6}$$

Next, the shear stress relationship of the Newtonian fluid given as

$$\sigma(z,t) = \mu\frac{\partial v(z,t)}{\partial z} \tag{A.7}$$

can be transformed into the Laplace domain using the above results:

$$L[\sigma(z,t)] = \mu\sqrt{\frac{s\rho}{\mu}}e^{\sqrt{(s\rho/\mu)}z}L[v(t)] = \sqrt{\mu\rho}\sqrt{s}L[v(z,t)]. \tag{A.8}$$

Equation (A.8) can be restated as

$$L[\sigma(z,t)] = \sqrt{\mu\rho}\frac{s}{\sqrt{s}}L[v(z,t)]. \tag{A.9}$$

Now, the following two transforms can be identified in (A.9):

$$\begin{aligned} sL[v(z,t)] &= L\left[\frac{\partial v(z,t)}{\partial t}\right], \\ \frac{1}{\sqrt{s}} &= L\left[\frac{1}{\Gamma(1/2)\sqrt{t}}\right]. \end{aligned} \tag{A.10}$$

With substituting (A.10) into (A.9), one obtains

$$L[\sigma(z,t)] = \sqrt{\mu\rho}L\left[\frac{1}{\Gamma(1/2)\sqrt{t}}\right] \cdot L[\dot{v}(z,t)]. \tag{A.11}$$

Table 8: Maximum absolute errors for $a = b = c = e = 1, \alpha_1 = 0.555, \alpha_2 = 1.455$ for Example 3.6.

	$N = 10$	$N = 20$	$N = 30$
$K = 1$	1.4×10^{-3}	1.7×10^{-4}	5.0×10^{-5}
$K = 2$	3.3×10^{-4}	4.1×10^{-5}	8.3×10^{-6}
$K = 4$	2.2×10^{-5}	3.0×10^{-6}	9.1×10^{-7}

Table 9: Maximum absolute errors for $a = 0.1, b = c = e = 0.5, \alpha_1 = 0.219, \alpha_2 = 1.965$ for Example 3.6.

	$N = 10$	$N = 20$	$N = 30$
$K = 1$	1.9×10^{-2}	1.7×10^{-3}	6.6×10^{-4}
$K = 2$	2.4×10^{-3}	4.2×10^{-4}	1.7×10^{-4}
$K = 4$	3.2×10^{-4}	5.8×10^{-5}	2.4×10^{-5}

The product of two transforms in (A.11) corresponds to the following convolution when evaluating the inverse transformation:

$$\sigma(z,t) = \sqrt{\mu\rho}\frac{1}{\Gamma(1/2)}\int_0^t \frac{\dot{v}(z,\tau)}{(t-\tau)^{1/2}}\mathrm{d}\tau = \sqrt{\mu\rho}D_t^{1/2}v(z,t), \tag{A.12}$$

which introduces a fractional derivative of degree $\alpha = 1/2$ within the shear stress-velocity relationship of a half-space Newtonian fluid.

Finally, consider a rigid plate of mass m immersed into an infinite Newtonian fluid. The plate is held at a fixed point by means of a spring of stiffness k. It is assumed that the motions of the spring do not influence the motion of the fluid, and that the surface A of the plate is very large, such that the stress-velocity relationship in (A.12) is valid on both sides of the plate. Equilibrium of all forces acting on the plate gives

$$my''(t) + ky(t) + 2A\sigma(z=0,t) = 0. \tag{A.13}$$

By substituting (A.12) one obtains

$$my''(t) + ky(t) + 2A\sqrt{\mu\rho}D_t^{1/2}v(z=0,t) = 0. \tag{A.14}$$

With $v(z = 0, t) = y'(t)$, a fractional differential equation of degree $\alpha = 3/2$ follows for the displacement of a rigid plate immersed into an infinite Newtonian fluid, as follows:

$$my''(t) + ky(t) + 2A\sqrt{\mu\rho}D_t^{3/2}y(t) = 0. \tag{A.15}$$

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Research Article

Numerical Solutions to Fractional Perturbed Volterra Equations

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In the paper, a class of perturbed Volterra equations of convolution type with three kernel functions is considered. The kernel functions $g_\alpha = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, $\alpha \in [1,2]$, correspond to the class of equations interpolating heat and wave equations. The results obtained generalize our previous results from 2010.

1. Introduction

We study perturbed Volterra equations of the form

$$u(x,t) = u(x,0) + \int_0^t \left[g_\alpha(t-s) + \left(g_\alpha * k\right)(t-s)\right] \Delta u(x,s)ds + \int_0^t b(t-s)u(x,s)ds, \tag{1.1}$$

where $x \in \mathbb{R}^d, t > 0, g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, Γ is the gamma function, $g_\alpha * k$ denotes the convolution, $\alpha \in [1,2]$, $b, k \in L^1_{\text{loc}}(\mathbb{R}_+;\mathbb{R})$, and Δ is the Laplace operator.

The perturbation approach to Volterra equations of convolution type has been used by many authors, see, for example, [1]. Such approach may be applied to more general, not necessary convolution equations, too. Recently, perturbed Volterra equations, deterministic and stochastic as well, have been studied for instance by Karczewska and Lizama [2]. The authors consider the class of equations with three kernel functions which satisfy some scalar auxiliary equations. Such condition enables to construct the family of resolvent operators admitted by the Volterra equations. In consequence, the resolvent approach to the considered Volterra equations can be used. Unfortunately, the resolvent approach proposed by Karczewska and Lizama may be used to (1.1) for some particular kernel functions b, k and g_α, $\alpha \in (0, 1/2)$ for $t > 1$, only. Hence, in our case, the method proposed in [2] cannot be applied for (1.1).

Motivation for the study of the fractional integro-differential equations comes from several problems appearing in physics, biology, and/or engineering. There are many phenomena well modeled by deterministic or stochastic fractional equations, see, for example, [3–7]. Several results on stochastic Volterra equations, fractional as well, particularly on the existence of strong solutions to those equations have been obtained by one of us [8–14].

Equation (1.1) is an interesting example of using the so-called fractional calculus in the theory of "classical" equations. Let us emphasize that (1.1) is a generalization of the equations which interpolate the heat and wave equations [15, 16]. Two convolutions appearing in (1.1) with the kernel functions b and k, respectively, represent some perturbation acting on the Volterra equation of convolution type.

Fractional calculus is a generalization of ordinary differentiation and itegration to arbitrary order [4, 17–19]. There is an increasing interest in applications of fractional calculus in many fields of mathematics [20], mechanics [5, 21, 22], physics [23, 24], and even in biology [6, 25]. A thorough and comprehensive survey of analytical and numerical methods used in solving many problems with applications of fractional calculus is contained in a recent monograph by Baleanu et al. [17].

Spectral methods belong to frequently used tools to obtain approximate solutions to complicated problems like fluid dynamic equations, weather predictions, and many others (see e.g., the monograph of Canuto et al. [26]). Recently, these methods have been used as a tool for calculation of fractional derivatives and integrals [27] and to solve Volterra equations with fractional time [28]. In general, spectral methods consist in representation of the solution to the equation under consideration in a finite subspace whereas the exact solution belongs to space of infinite dimension. The method presented in the present paper belongs to that class.

The paper is organized as follows. In Section 2, a general idea of Galerkin method to integral equations is presented and approximation by the use of finite dimensional Hilbert space is explained. Section 3 presents a system of linear equations obtained from (1.1) by a discrete formulation enabling for numerical solutions. The detailed form of matrices appearing in that approximation is presented for one dimensional case in Section 3.1 and for two spatial dimension case in Section 3.2. The set of basis functions is represented in Section 3.3 and numerical methods used to solve large-scale sparse linear systems are discussed in Section 3.4, as well. Examples of numerical solutions to (1.1) are exhibited and discussed in detail in Section 4, whereas error estimations for the precision of approximate results are given in Section 5.

2. Galerkin Method

Let $\{\phi_i : i = 1, 2, \ldots, \infty\}$ represent a set of orthonormal functions on the interval $[0, t]$, spanning a Hilbert space H.

Definition 2.1. Let $f, g \in H$. The number

$$\langle f(t), g(t) \rangle := \int_0^t f(\tau) g(\tau) \Theta(\tau) d\tau, \tag{2.1}$$

where Θ is a weight function, is called the *scalar product* of functions f, g on the interval $[0, t]$.

Let us recall that two functions are orthonormal when

$$\forall_{i,j} \quad \langle \phi_i(t), \phi_j(t) \rangle = \delta_{ij}, \tag{2.2}$$

where δ_{ij} is the Kronecker delta.

We are looking for an approximate solution to (1.1) as an element of the subspace H_{n_ϕ}, spanned on n_ϕ first basic functions $\{\phi_j : j = 1, 2, \ldots, n_\phi\}$

$$u_{n_\phi}(x,t) = \sum_{j=1}^{n_\phi} c_j(x)\phi_j(t). \tag{2.3}$$

For simplicity of notations, let us consider (1.1) in one spatial dimension only. Inserting (2.3) into (1.1), one obtains

$$\begin{aligned} u_{n_\phi}(x,t) = u(x,0) &+ \int_0^t [a(t-s) + (a*k)(t-s)]\Delta u_{n_\phi}(x,s)ds \\ &+ \int_0^t b(t-s)u_{n_\phi}(x,s)ds + \epsilon_{n_\phi}(x,t), \end{aligned} \tag{2.4}$$

where function ϵ_{n_ϕ} represents the approximation error function. From (2.3) and (2.4), one gets

$$\begin{aligned} \epsilon_{n_\phi}(x,t) = \sum_{j=1}^{n_\phi} c_j(x)\phi_j(t) &- \int_0^t [a(t-s) + (a*k)(t-s)] \sum_{j=1}^{n_\phi} \frac{d^2}{dx^2} c_j(x)\phi_j(s)ds \\ &- \int_0^t b(t-s)\sum_{j=1}^{n_\phi} c_j(x)\phi_j(s)ds - u(x,0). \end{aligned} \tag{2.5}$$

Definition 2.2. The *Galerkin approximation* of (1.1) is the function $u_{n_\phi} \in H_{n_\phi}$, such that $\epsilon_{n_\phi} \perp H_{n_\phi}$, that is,

$$\forall_{j=1,2,\ldots,n} \quad \left\langle \epsilon_{n_\phi}(x,t), \phi_j(t) \right\rangle = 0. \tag{2.6}$$

It follows from Definitions 2.2 and 2.1 and (2.5) that

$$
\begin{aligned}
0 = & \int_0^t \left[\sum_{j=1}^{n_\phi} c_j(x)\phi_j(\tau)\right]\phi_i(\tau)\Theta(\tau)d\tau - \int_0^t u(x,0)\phi_i(\tau)\Theta(\tau)d\tau \\
& - \int_0^t \left[\int_0^\tau [a(\tau - s) + (a * k)(\tau - s)]\sum_{j=1}^{n_\phi}\frac{d^2}{dx^2}c_j(x)\phi_j(s)ds\right]\phi_i(\tau)\Theta(\tau)d\tau \\
& - \int_0^t \left[\int_0^\tau b(\tau - s)\sum_{j=1}^{n_\phi}c_j(x)\phi_j(s)ds\right]\phi_i(\tau)\Theta(\tau)d\tau \quad \text{for } i = 1, 2, \dots, n_\phi.
\end{aligned}
\tag{2.7}
$$

Therefore

$$
\begin{aligned}
\int_0^t u(x,0)\phi_i(\tau)\Theta(\tau)d\tau = & \int_0^t \left[\sum_{j=1}^{n_\phi} c_j(x)\phi_j(\tau)\right]\phi_i(\tau)\Theta(\tau)d\tau \\
& - \int_0^t \left[\int_0^\tau [a(\tau - s) + (a * k)(\tau - s)]\sum_{j=1}^{n_\phi}\frac{d^2}{dx^2}c_j(x)\phi_j(s)ds\right]\phi_i(\tau)\Theta(\tau)d\tau \\
& - \int_0^t \left[\int_0^\tau b(\tau - s)\sum_{j=1}^{n_\phi}c_j(x)\phi_j(s)ds\right]\phi_i(\tau)\Theta(\tau)d\tau, \quad i = 1, 2, \dots, n_\phi.
\end{aligned}
\tag{2.8}
$$

Using (2.2), (2.8) can be written in an abbreviated form

$$
g_i(x) = c_i(x) - \sum_{j=1}^{n_\phi} a_{ij}\frac{d^2}{dx^2}c_j(x) - \sum_{j=1}^{n_\phi} b_{ij}c_j(x),
\tag{2.9}
$$

where

$$
g_i(x) = u(x,0)\int_0^t \phi_i(\tau)\Theta(\tau)d\tau,
\tag{2.10}
$$

$$
a_{ij} = \int_0^t \left[\int_0^\tau [a(\tau - s) + (a * k)(\tau - s)]\phi_j(s)ds\right]\phi_i(\tau)\Theta(\tau)d\tau,
\tag{2.11}
$$

$$
b_{ij} = \int_0^t \left[\int_0^\tau b(\tau - s)\phi_j(s)ds\right]\phi_i(\tau)\Theta(\tau)d\tau.
\tag{2.12}
$$

In general $a_{ij} \neq a_{ji}$.

The solution of the set of n_ϕ coupled differential equations (2.9) for coefficients $c_j(x)$, $j = 1, 2, \dots, n_\phi$ provides Galerkin approximation (2.3) to (1.1).

3. Discretization

Equations can be solved using discretization in a space variable. In one-dimesional case, let us introduce a grid of points $(x_1, x_2, \ldots, x_{n_h})$, where $x_l - x_{l-1} = h$. The grid approximation of a second derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} + O\left(h^3\right). \tag{3.1}$$

Then the set of equations (2.9) takes the following form:

$$\begin{aligned} g_i(x_l) &= c_i(x_l) + \frac{1}{h^2}\sum_{j=1}^{n_\phi} a_{ij}\left[-c_j(x_{l-1}) + 2c_j(x_l) - c_j(x_{l+1})\right] - \sum_{j=1}^{n_\phi} b_{ij}c_j(x_l) \\ &= c_i(x_l) + \frac{1}{h^2}\sum_{j=1}^{n_\phi}\left[-a_{ij}c_j(x_{l-1}) + \left(2a_{ij} - h^2 b_{ij}\right)c_j(x_l) - a_{ij}c_j(x_{l+1})\right], \end{aligned} \tag{3.2}$$

where $i = 1, 2, \ldots, n_\phi$ and $l = 1, 2, \ldots, n_h$.

In two-dimensional case, with the grid $(x_1, x_2, \ldots, x_{n_h}) \times (y_1, y_2, \ldots, y_{n_h})$, where $x_l - x_{l-1} = y_m - y_{m-1} = h$ for $l, m = 2, 3, \ldots, n_h$, the set of equations (2.9) takes the form

$$\begin{aligned} g_i(x_l, y_m) = c_i(x_l, y_m) &+ \frac{1}{h^2}\sum_{j=1}^{n_\phi} a_{ij}\left[-c_j(x_{l-1}, y_m) - c_j(x_l, y_{m-1})\right. \\ &\qquad\qquad \left. + 4c_j(x_l, y_m) - c_j(x_{l+1}, y_m) - c_j(x_l, y_{m+1})\right] \\ &- \sum_{j=1}^{n_\phi} b_{ij}c_j(x_l, y_m) \\ = c_i(x_l, y_m) &+ \frac{1}{h^2}\sum_{j=1}^{n_\phi}\left[-a_{ij}c_j(x_{l-1}, y_m) - a_{ij}c_j(x_l, y_{m-1})\right. \\ &\qquad \left. + \left(4a_{ij} - h^2 b_{ij}\right)c_j(x_l, y_m) - a_{ij}c_j(x_{l+1}, y_m) - a_{ij}c_j(x_l, y_{m+1})\right]. \end{aligned} \tag{3.3}$$

Both sets of linear equations (3.2) and (3.3) can be written in a matrix form

$$g = \mathcal{A}c, \tag{3.4}$$

where vectors g, c and matrix $\mathcal{A}$ have block forms

$$g = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_{n_\phi} \end{pmatrix}, \qquad c = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n_\phi} \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} [A_{11}] & \cdots & [A_{1n_\phi}] \\ [A_{21}] & \cdots & [A_{2n_\phi}] \\ \vdots & \ddots & \vdots \\ [A_{n_\phi 1}] & \cdots & [A_{n_\phi n_\phi}] \end{pmatrix}. \tag{3.5}$$

Detailed structure of blocks occurring in (3.5) is given below.

3.1. One-Dimensional Case

Blocks G_i and G_i are n_ϕ-dimensional column vectors. For the sake of space, we present their transpositions

$$\begin{aligned} G_i^T &= (g_i(x_1), g_i(x_2), \ldots, g_i(x_{n_h})), \\ C_i^T &= (c_i(x_1), c_i(x_2), \ldots, c_i(x_{n_h})). \end{aligned} \tag{3.6}$$

Blocks $[A_{ij}]$ have the form

$$[A_{ij}] = \begin{pmatrix} \mu_{ij} & \eta_{ij} & 0 & \cdots & 0 & \theta_{ij} \\ \eta_{ij} & \mu_{ij} & \eta_{ij} & \cdots & 0 & 0 \\ 0 & \eta_{ij} & \mu_{ij} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{ij} & \eta_{ij} \\ \theta_{ij} & 0 & 0 & \cdots & \eta_{ij} & \mu_{ij} \end{pmatrix}_{n_h \times n_h}, \tag{3.7}$$

where $\mu_{ij} = \delta_{ij} + (2/h^2)a_{ij} - b_{ij}$, $\eta_{ij} = -(1/h^2)a_{ij}$, whereas

$$\theta_{ij} = \begin{cases} \eta_{ij} & \text{for periodic boundary conditions,} \\ 0 & \text{for closed boundary conditions.} \end{cases} \tag{3.8}$$

Vectors g and c are $n_\phi n_h$-dimensional, whereas the matrix $\mathcal{A}$ has dimension $n_\phi n_h \times n_\phi n_h$. The matrix $\mathcal{A}$ is a sparse one. The number of nonzero elements of the matrix $\mathcal{A}$ is at most $n_\phi^2(3n_h - 2)$ (with closed boundary conditions) or $3n_\phi^2 n_h$ (with periodic boundary conditions).

3.2. Two-Dimensional Case

In two-dimensional case, $n_\phi n_h^2$-dimensional vectors G_i^T and C_i^T read as

$$\begin{aligned} G_i^T &= (g_i(x_1,y_1), g_j(x_1,y_2),\dots, g_i(x_1,y_{n_h}), g_i(x_2,y_1), g_i(x_2,y_2),\dots, g_i(x_{n_h},y_{n_h})),\\ C_i^T &= (c_i(x_1,y_1), c_i(x_1,y_2),\dots, c_i(x_1,y_{n_h}), c_i(x_2,y_1), c_i(x_2,y_2),\dots, c_i(x_{n_h},y_{n_h})). \end{aligned} \tag{3.9}$$

Blocks $[A_{ij}]$ have the form of embedded blocks

$$[A_{ij}] = \begin{pmatrix} (\alpha_{ij}) & (\beta_{ij}) & (0) & \cdots & (0) & (0) & (\gamma_{ij}) \\ (\beta_{ij}) & (\alpha_{ij}) & (\beta_{ij}) & \cdots & (0) & (0) & (0) \\ (0) & (\beta_{ij}) & (\alpha_{ij}) & \cdots & (0) & (0) & (0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (0) & (0) & (0) & \cdots & (\alpha_{ij}) & (\beta_{ij}) & (0) \\ (0) & (0) & (0) & \cdots & (\beta_{ij}) & (\alpha_{ij}) & (\beta_{ij}) \\ (\gamma_{ij}) & (0) & (0) & \cdots & (0) & (\beta_{ij}) & (\alpha_{ij}) \end{pmatrix}_{n_h\times n_h}, \tag{3.10}$$

where each term $(\cdot)$ is an embedded block of the size $n_h \times n_h$. In particular

$$(\gamma_{ij}) = \begin{cases} (\beta_{ij}) & \text{for periodic boundary conditions,} \\ (0) & \text{for closed boundary conditions,} \end{cases} \tag{3.11}$$

block (0) is a matrix $n_h \times n_h$ with all null elements, block (α_{ij}) is again a sparse matrix of the form

$$(\alpha_{ij}) = \begin{pmatrix} \mu_{ij} & \eta_{ij} & 0 & \cdots & 0 & \theta_{ij} \\ \eta_{ij} & \mu_{ij} & \eta_{ij} & \cdots & 0 & 0 \\ 0 & \eta_{ij} & \mu_{ij} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{ij} & \eta_{ij} \\ \theta_{ij} & 0 & 0 & \cdots & \eta_{ij} & \mu_{ij} \end{pmatrix}_{n_h\times n_h}, \tag{3.12}$$

where $\mu_{ij} = \delta_{ij} + (4/h^2)a_{ij} - b_{ij}$, $\eta_{ij} = -(1/h^2)a_{ij}$,

$$\theta_{ij} = \begin{cases} \eta_{ij} & \text{for periodic boundary conditions,} \\ 0 & \text{for closed boundary conditions} \end{cases} \tag{3.13}$$

and block (β_{ij}) is diagonal

$$(\beta_{ij}) = \begin{pmatrix} \frac{-1}{h^2}a_{ij} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{h^2}a_{ij} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{h^2}a_{ij} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{h^2}a_{ij} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{h^2}a_{ij} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{h^2}a_{ij} \end{pmatrix}_{n_h \times n_h}. \tag{3.14}$$

The matrix $\mathcal{A}$ is the sparse matrix of $n_\phi n_h^2 \times n_\phi n_h^2$ elements. However, only at most $n_\phi^2(5n_h - 4)n_h$ (with closed boundary conditions) or $5n_\phi^2 n_h^2$ (with periodic boundary conditions) elements are nonzero.

3.3. *Basis Functions*

The basis functions $\{\phi_j : j = 1, 2, 3, \ldots\}$ have to be orthogonal on the interval $[0, t]$ with respect to a weight function Θ. We use the set of Legendre polynomials P_l which are solutions of the Legendre differential equation

$$\frac{d}{dx}\left[\left(1 - x^2\right)\frac{d}{dx}P_l(x)\right] + l(l+1)P_l(x) = 0 \tag{3.15}$$

for $l = 0, 1, 2, \ldots$. The Legendre polynomials are orthogonal on the interval $[-1, 1]$ with the weight function $\Theta \equiv 1$

$$\int_{-1}^{1} P_l(x)P_m(x)dx = \frac{2}{2l+1}\delta_{lm}. \tag{3.16}$$

Taking the basis function in the form

$$\widetilde{P}_l(x) = \sqrt{\frac{2j-1}{t}}P_{l-1}\left(\frac{2x}{t} - 1\right), \quad l = 1, 2, 3, \ldots, \tag{3.17}$$

ensures that the functions $\overline{P}_l(x)$ fulfill the orthonormality relations (2.2) on the interval $[0, t]$. Therfeore they can be used as a basis in the Galerkin method. In principle, any set of functions orthonormal on the interval $[0, t]$ can be used. For our purposes, however, the Lagrange polynomials appeared more efficient in practical applications than for instance Chebyshev polynomials.

3.4. Methods for Solving Large Linear Systems

The matrices $\mathcal{A}$, both in one- and two-dimensional case, are sparse matrices. In order to obtain a reasonable approximation of the solution to (1.1), their sizes have to be large. Those facts suggest an application of iterative methods for solving the linear systems (3.4).

In general, the matrix $\mathcal{A}$ is nonsymmetric. We have tested on our examples two iterative methods developed for solving large-scale linear systems of non-symmetric matrices. One of those metods is so called BiCGSTAB (BiConjugate Gradient Stabilized method) [29, 30]. The other one is the GMRES (General Minimal Residual method) [29, 31]. In both methods, a suitable preconditioning is necessary.

For cases discused in the paper, the GMRES method appeared to be more efficient. Usually, after a proper choice of auxiliary parameters of calculations, the GMRES was requiring less number of iterations and converging faster than the BiCGSTAB method.

4. Examples of Numerical Solutions

In this section, several examples of approximate numerical solutions to (1.1) are presented. The function

$$u(x,0) = \frac{1}{1+\exp((|x|-r_1)/r_2)}, \tag{4.1}$$

where $|x| = \sqrt{\sum_{i=1}^{d} x_i^2}$ and d is the space dimension has been taken as the initial condition. Such function, which is substantially different than zero only in a finite region, may represent a distribution of the temperature in a rod (or plane) heated locally or a distribution of gas (or liquid) particles which may diffuse in a nonhomogeneous medium. The values of constants $r_1 = 3$ and $r_2 = 0.3$ in (4.1) were chosen for a clear graphical presentation of the results.

4.1. One-Dimensional Case

For presentation of approximate numerical results we chose an interval $x \in [-10, 10]$ with $n_h = 201$ equidistant grid points. The Hilbert space H_{n_ϕ} was spanned on the basis of $n_\phi = 20$ functions described in Section 3.3.

In our previous study [28], we have shown that when α increases from $\alpha = 1$ to $\alpha = 2$ the solution of unperturbed evolution governed by (1.1) with $k = b = 0$ changes from pure heat (diffusive) behaviour to pure wave motion. Below we present results for fractional cases $\alpha = 1.5$ (an intermediate case) pointing out the effects of perturbations. For the sake of space, we show a few examples only, explaining the general influences of perturbations on the character of solutions.

Figure 1 illustrates the effect of perturbation in the form $k(t) = c\cdot\cos(t)$, where $c \in [0, 1]$ represents an intensity of the perturbation, whereas the function $b \equiv 0$. It is clearly seen that this perturbation, periodic in time variable, produces a wavy formations in space with amlitudes strongly depending on the intensity of perturbation. Results obtained with the opposite sign of the perturbation term, $c \to -c$, (not presented here for the sake of space), show that such perturbation decreases diffusive behaviour of the system and enforces a wavelike evolution.

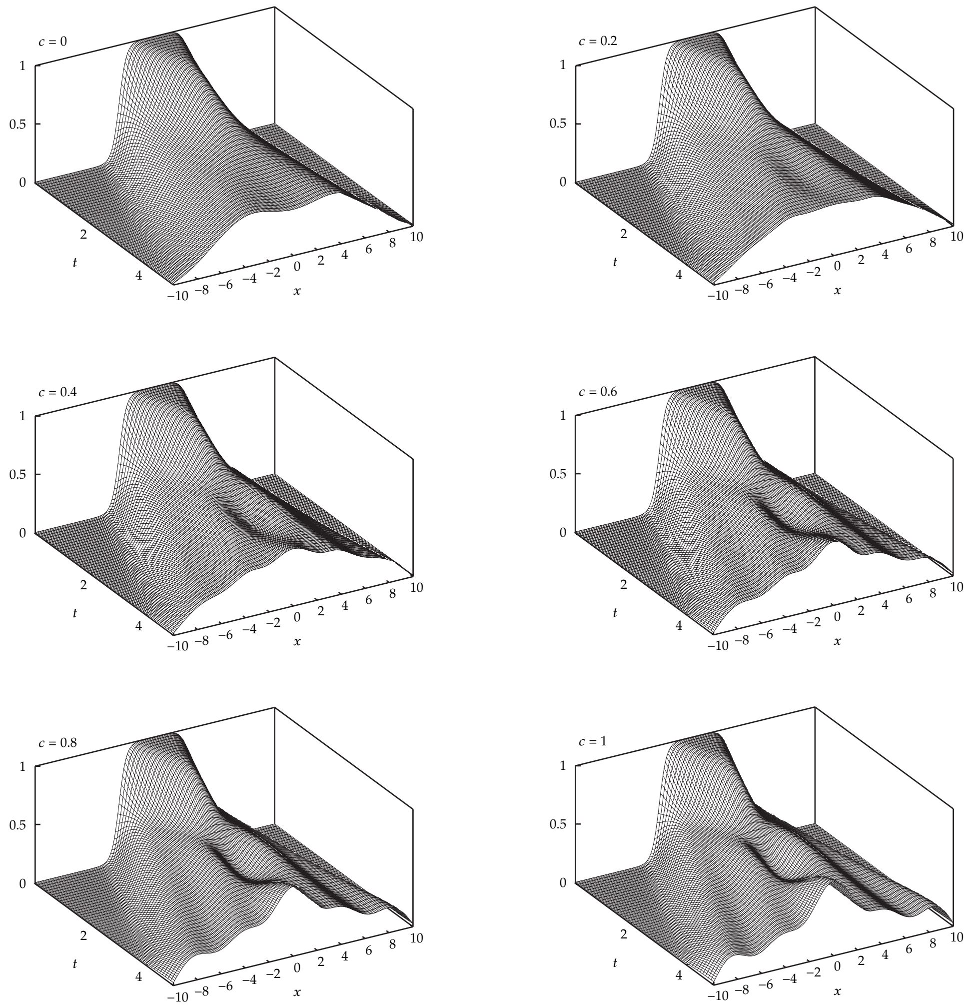

Figure 1: Numerical solution to (1.1) for $\alpha = 1.5$, $t \in [0,5]$, $k(t) = c \cdot \cos(t)$, $b \equiv 0$.

The effect of the perturbation of the form $b = c \cdot e^{-t}$, when $k \equiv 0$, is presented in Figure 2. As in the former case $c \in [0,1]$ represents the intensity of perturbation. It is clear that the perturbation in such form generally increases the amplitude of the solution. The change of sign of the perturbation, $c \to -c$, produces the opposite effect, the amplitude of $u(x,t)$ decreases, like in the case of dumping.

4.2. Two-Dimensional Case

In this subsection, we show some results obtained for two-dimensional case. The calculations have been performed on the grid of $n_h \times n_h$ points, where $n_h = 101$, $x, y \in [-5,5]$, with $n_\phi = 20$ basic functions spanning the Hilbert space H_{n_ϕ}.

Figures 3, 4, and 5 illustrate several examples of the numerical solutions to (1.1) for $\alpha = 1.5$ and different perturbation functions k and b.

The case $k \equiv b \equiv 0$, corresponds to the unperturbed equation. Its solution, as seen from Figure 3, evolves in the wave manner with a significant influence of diffusion due to

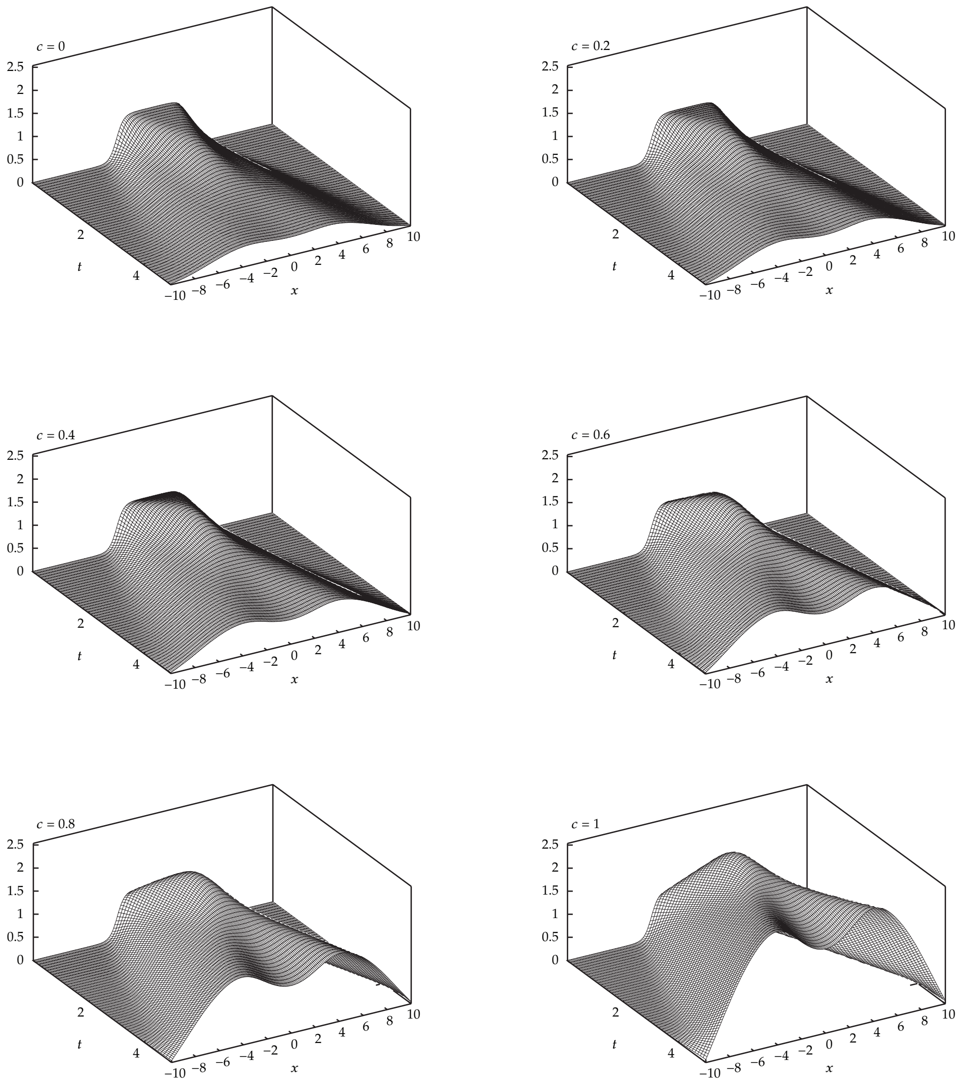

Figure 2: Numerical solution to (1.1) for $\alpha = 1.5$, $t \in [0, 5]$, $k \equiv 0$, $b(t) = c \cdot e^{-t}$.

fractional value of $\alpha = 1.5$, between the pure diffusion case ($\alpha = 1$) and the pure wave case ($\alpha = 2$). For more examples of unperturbed evolution of solutions to (1.1) in two-dimensional case, see [28].

Results obtained for two-dimensional cases with nonzero perturbations generally exhibit the properties similar to those in one-dimensional case. Again, like in the one-dimensional case, the presence of perturbation in the form of $k(t) = \cos(t)$ results in an increase of a wave frequency (see, e.g., Figure 4). In other words the perturbation of such form produces additional wavy behaviour of the solution. The change of sign of the perturbation term changes the phase of that behaviour.

The presence of perturbation term with $b \neq 0$ influences mainly the amplitude of the solution. Comparing Figure 5 to Figure 4, one notices that the amplitude increases with $b = e^{-t}$. Perturbation with the opposite sign $b(t) = -e^{-t}$ results in decrease of amplitude, like in the case of dumping.

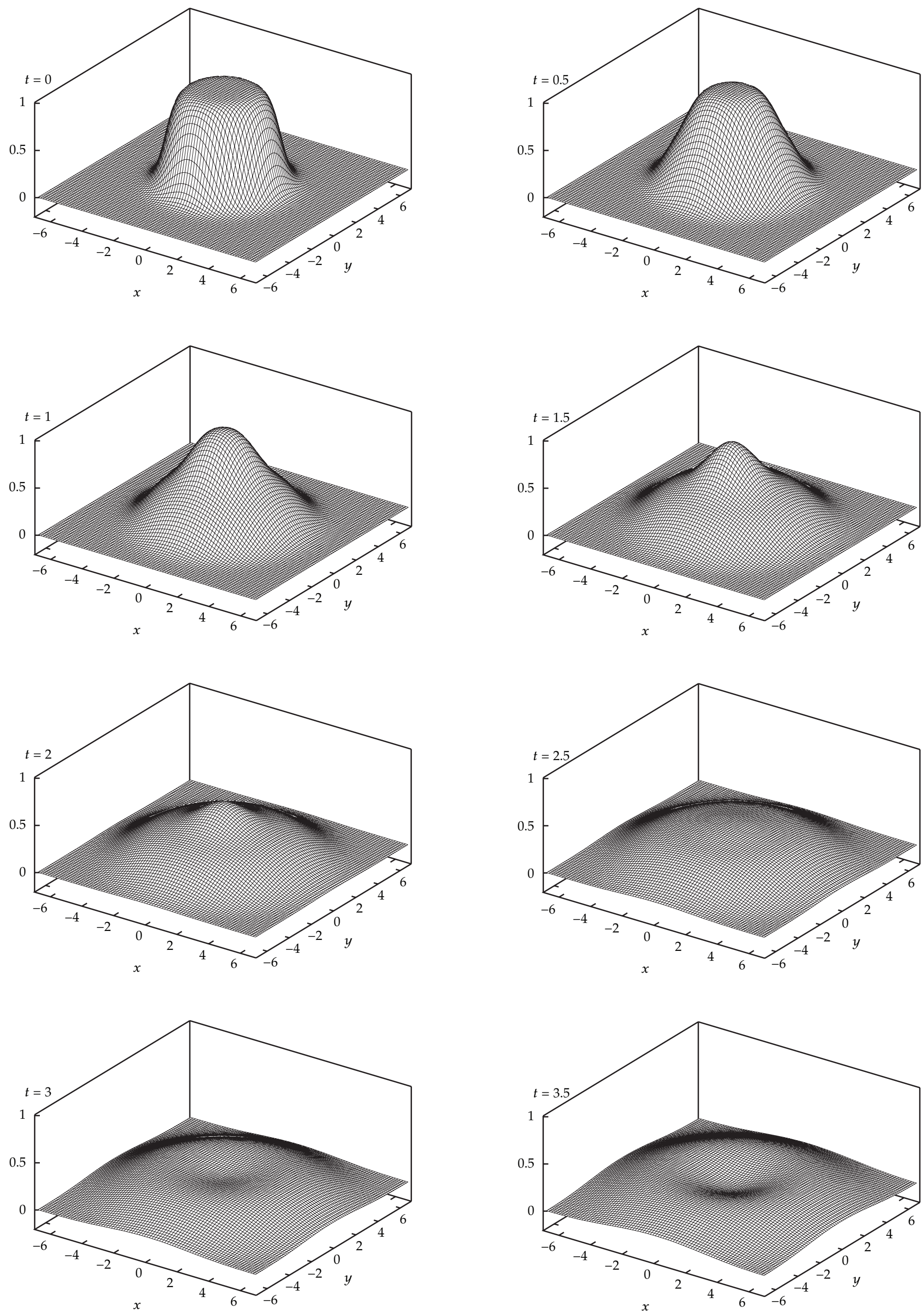

Figure 3: Numerical solution to (1.1) for $\alpha = 1.5$, $t \in [0, 3.5]$, $k \equiv 0$, $b \equiv 0$. Closed boundary conditions.

5. Precision of Numerical Results

A comparison between the analytic and numerical solutions to (1.1) is possible only for one-dimensional case when there is no perturbation and $\alpha = 1$ or $\alpha = 2$. Despite the existence of the analytic solutions for this case for an arbitrary $\alpha \in [1, 2]$, given (for $d = 1$ case) in terms of Mittag-Leffler functions [15, 16], their computation is not practical.

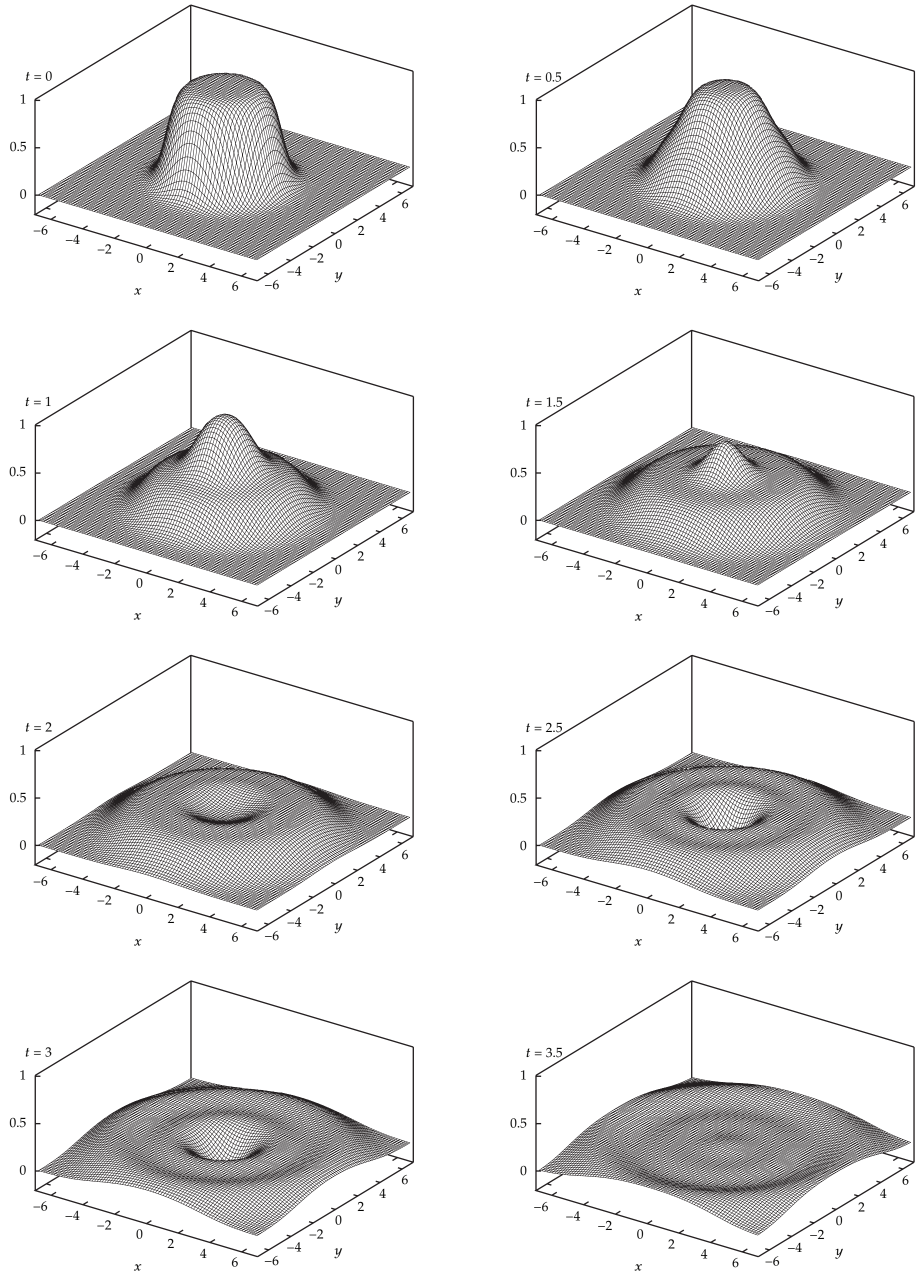

Figure 4: Numerical solution to (1.1) for $\alpha = 1.5$, $t \in [0, 3.5]$, $k(t) = \cos(t)$, $b \equiv 0$. Periodic boundary conditions.

For non-perturbed case, we defined in [28] an error estimate as the maximum of the absolute value of the difference between the exact analytical solution and approximate numerical one

$$\Delta u_{n_\phi, n_h}(t) = \max \left| u^{\text{anal}}_{n_\phi, n_h}(x_i, t) - u^{\text{num}}_{n_\phi, n_h}(x_i, t) \right|_{i=1}^{n_h}, \tag{5.1}$$

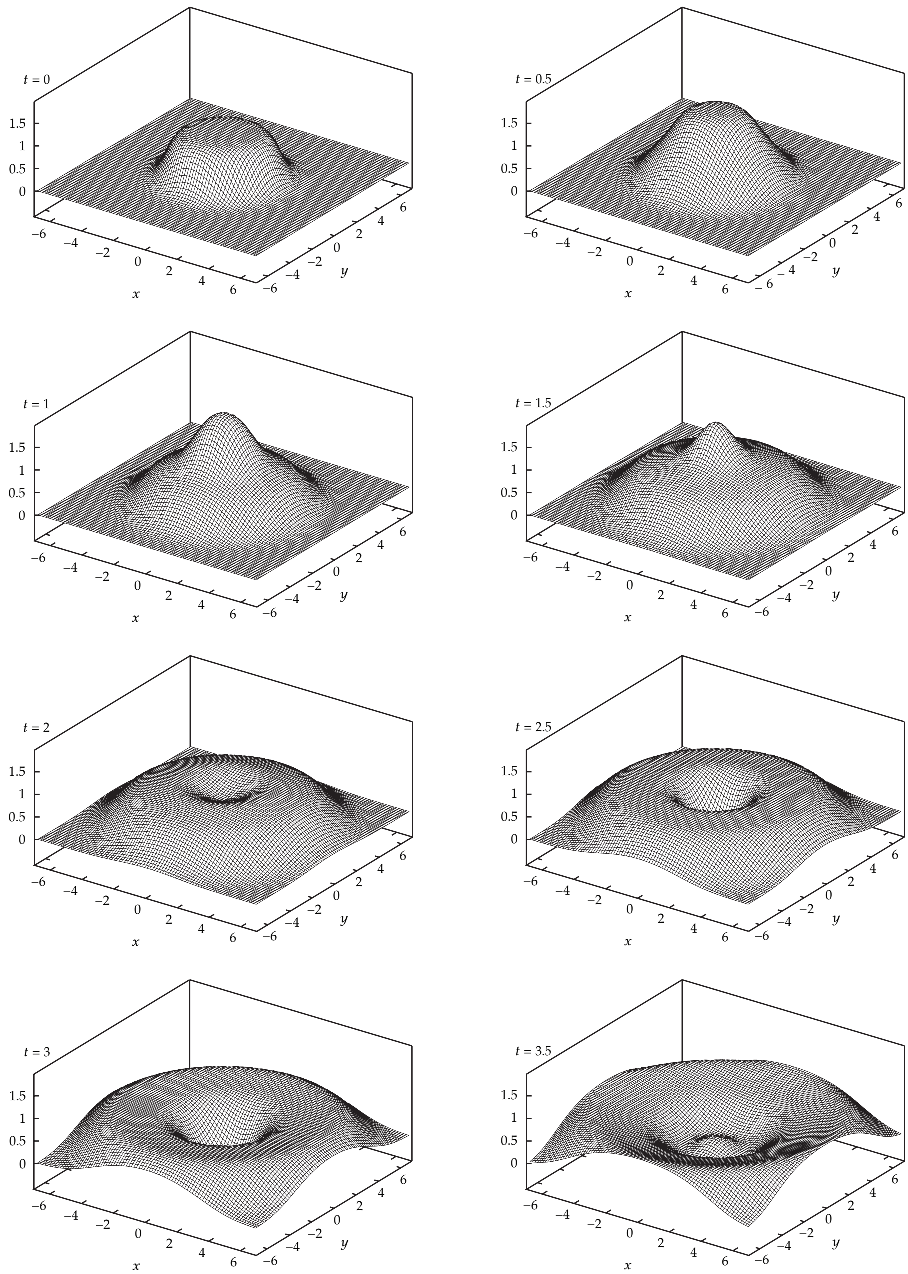

Figure 5: Numerical solution to (1.1) for $\alpha = 1.5$, $t \in [0, 3.5]$, $k(t) = \cos(t)$, $b(t) = e^{-t}$. Periodic boundary conditions.

where maximum is taken over all grid points x_i. For $d = 1$ and $\alpha = 1$ and 2, $n_\phi > 20$, $n_h > 100$, $t \leq 6$ the error estimate $\Delta u_{n_\phi, n_h}(t)$ was always less than 10^{-5}.

When we consider presented method for obtaining numerical solution to fractional perturbed Volterra equation (1.1), there are three levels of numerical errors.

The first level corresponds to the error of Galerkin approximation (2.3) which depends on basic functions number n_ϕ. In a special case, when $k \equiv 0$, $b \equiv 0$ and Δ operator is replaced by identity operator, we can estimate the approximation error using the following result from [27].

Remark 5.1 (see [27, Remark 5.2]). If $u_{n_\phi}(x,t)$ (see (2.3)) is the Legendre-Gauss-Lobatto interpolation of $u(x,t)$, $u \in H^r(I)$, $I = [0,t]$, $t > 0$, then

$$\left\| u - u_{n_\phi} \right\|_{L^\infty(I)} \leq C n_\phi^{3/4-r} \|u\|_{H^r(I)}, \quad r \in \mathbb{N}, \tag{5.2}$$

where C is a positive constant (see [26]). Therefore, we can get the following error bounds:

$$\left\| \int_0^t g_\alpha(t-s)\left(u(x,s) - u_{n_\phi}(x,s)\right) ds \right\|_{L^\infty(I)} \leq C n_\phi^{3/4-r} \|u\|_{H^r(I)}, \quad \alpha > 0, \ r \geq 1. \tag{5.3}$$

Also at the first level, the integrals (2.11) and (2.12) are calculated. In our method, we use a Gauss-Legendre quadrature for numerical integration. The exact error of such quadrature can be found, for example, in Theorem 7.3.5 in [32].

The second level is the Laplacian discretization (Section 3). In this case, the numerical error can be estimated by $O(h^3)$, where h is the spatial grid step.

The last one is is the residual error of GMRES method for solving large linear systems. In our computations, the residual error threshold was set to 10^{-9}, which was small enough to obtain reliable solution.

The joint error estimate from all three levels is not obvious. Moreover, in the considered perturbed case, the analytic solution to (1.1) is not known. Therefore, in order to estimate the accuracy of numerical solutions, we proceed in the following manner which is also applicable for two- and higher-dimensional cases.

When we are not able to confront the numerical solutions with analytic ones, we can investigate how does approximate solution change with increasing numbers of grid points and increasing number of basic functions. One can expect that increasing number of grid points and increasing number of basic functions should result in a better approximation of the true (unknown) solution. Taking appropriate sequences of those numbers and estimating the largest differences between consecutive solutions, one can show convergence of approximation errors. To do it let us define the following quantities.

Let $\Delta_\phi u_{n_\phi,n_h}(t)$ denote the maximum difference between two solutions obtained for the same t and the same grid (defined by n_h) but with different numbers of basic functions n_ϕ and $n_{\phi-2}$.

$$\Delta_\phi u_{n_\phi,n_h}(t) := \max_i \left| u_{n_\phi,n_h}(x_i,t) - u_{n_\phi-2,n_h}(x_i,t) \right|. \tag{5.4}$$

Then let $\Delta_h u_{n_\phi,n_h}(t)$ denote the maximum difference between two solutions obtained for the same t and the same Hilbert space (the same n_ϕ) but with different numbers of grid points (in one direction) n_h and n_{h-10}

$$\Delta_h u_{n_\phi,n_h}(t) := \max_i \left| u_{n_\phi,n_h}(x_i,t) - \tilde{u}_{n_\phi,n_h-10}(x_i,t) \right| , \tag{5.5}$$

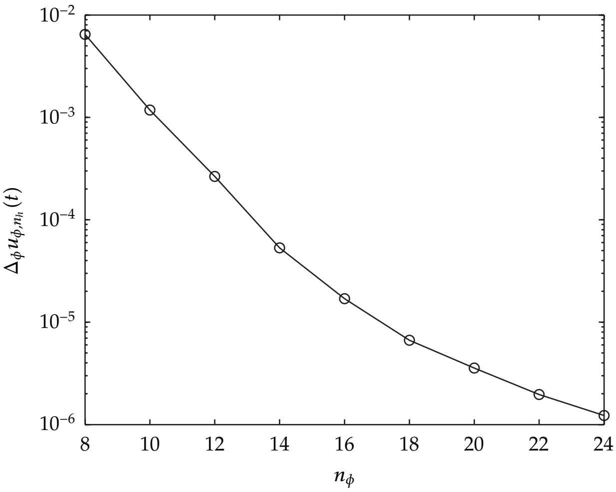

Figure 6: Error estimate (5.4) for one-dimensional case, for $t = 1.8$, $n_h = 121$, $\alpha = 1,5$, $k(t) = \cos(t)$, and $b(t) = e^{-t}$. Periodic boundary conditions.

where $\tilde{u}_{n_\phi,n_h-10}(x_i,t)$ means the value in the node of bigger size obtained from values calculated for grid of smaller size by cubic-spline interpolation.

In two-dimensional cases, the appropriate error estimates read

$$\Delta_\phi u_{n_\phi,n_h}(t) := \max_{i,j}\left|u_{n_\phi,n_h}(x_i,y_j,t) - u_{n_\phi-2,n_h}(x_i,y_j,t)\right|, \tag{5.6}$$

$$\Delta_h u_{n_\phi,n_h}(t) := \max_{i,j}\left|u_{n_\phi,n_h}(x_i,y_j,t) - \tilde{u}_{n_\phi,n_h-10}(x_i,y_j,t)\right|. \tag{5.7}$$

Figures 6, 7, and 8 present some examples of the dependence of the above defined error estimates on the numbers of grid points and the size of the basis. Presented examples contain both one- and two-dimensional cases and two cases of boundary conditions.

The general conclusions of that investigation are the following. In all cases the error estimates decrease fast with increasing number of grid points or with increasing size of the basis. That decrease is in log plots seen as close to a straight line, that is, error estimates decrease almost exponentially. Then taking large enough grid and large enough set of basic functions, one can obtain, in principle, an error less then arbitrary small number. In practice increasing the sizes of basis and grid produces a sharp increase of numerical operations causing accumultion of rounding errors. That property can be compensated by increasing the precision of representation of real numbers (using double or quadruple precision) and so on. All those actions require higher and higher computer power to obtain results in a reasonable computing time.

Our estimates show, however, that a reasonable approximations can be obtained with relatively low values of n_ϕ and n_h.

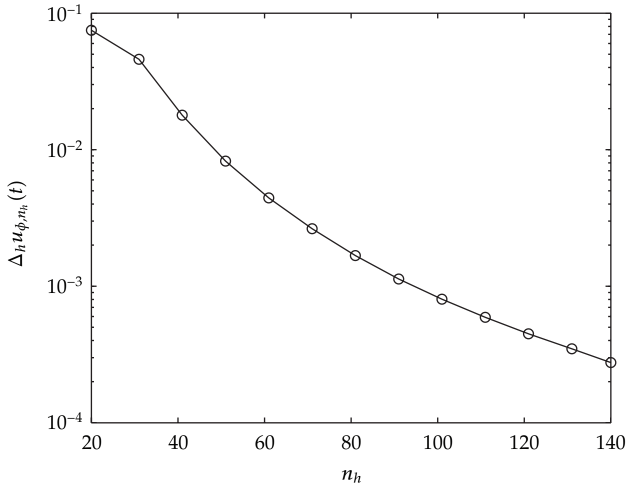

Figure 7: Error estimate (5.5) for one-dimensional case, for $t = 1.8$, $n_\phi = 20$, $\alpha = 1,5$, $k(t) = \cos(t)$, and $b(t) = e^{-t}$. Closed boundary conditions.

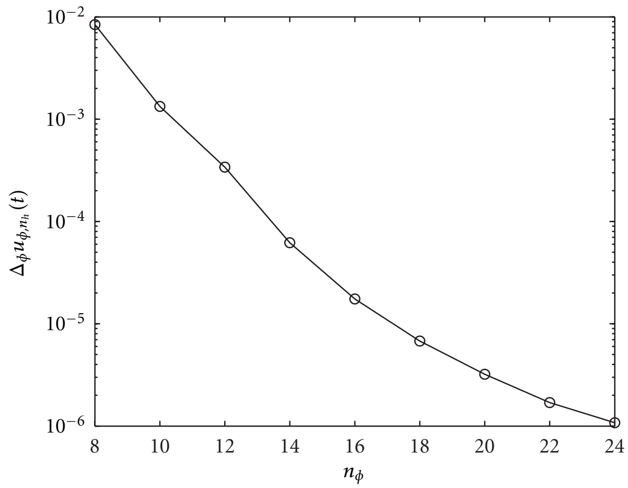

Figure 8: Error estimate (5.6) for two-dimensional case, for $t = 1.8$, $n_h = 101$, $\alpha = 1,5$, $k(t) = \cos(t)$, and $b(t) = e^{-t}$. Closed boundary conditions.

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Research Article

Robust Position Control of PMSM Using Fractional-Order Sliding Mode Controller

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A new robust fractional-order sliding mode controller (FOSMC) is proposed for the position control of a permanent magnet synchronous motor (PMSM). The sliding mode controller (SMC), which is insensitive to uncertainties and load disturbances, is studied widely in the application of PMSM drive. In the existing SMC method, the sliding surface is usually designed based on the integer-order integration or differentiation of the state variables, while in this proposed robust FOSMC algorithm, the sliding surface is designed based on the fractional-order calculus of the state variables. In fact, the conventional SMC method can be seen as a special case of the proposed FOSMC method. The performance and robustness of the proposed method are analyzed and tested for nonlinear load torque disturbances, and simulation results show that the proposed algorithm is more robust and effective than the conventional SMC method.

1. Introduction

Permanent magnet synchronous motor (PMSM) has many applications in industries due to its superior features such as compact structure, high efficiency, high torque to inertia ratio, and high power density [1]. To get fast four-quadrant operation, good acceleration, and smooth starting, the field-oriented control or vector control is used in the design of PMSM drives [1–4]. However, the PMSM is a typical high nonlinear, multivariable coupled system, and its performance is sensitive to external load disturbances, parameter changes in plant, and unmodeled and nonlinear dynamics. To achieve good dynamic response, some robust control strategies such as nonlinear control [5, 6], adaptive control [7, 8], H_∞ control [9–11], and sliding mode control (SMC) [12–18] have been developed.

The SMC is a powerful nonlinear control technique and has been widely used for speed and position control of PMSM system, because it provides a fast dynamic response and is insensitive to external load disturbances and parameter variations. In [14], a fuzzy sliding mode controller was proposed for the speed and position control of PMSM. In [15], a hybrid controller (HC) which consists of a parallel connected sliding mode controller and a neurofuzzy controller was proposed for the speed control of PMSM. In [16], a robust wavelet-neural-network sliding mode controller was proposed which can achieve favourable decoupling control and high-precision speed tracking performance of PMSM. The design of the SMC mainly contains two steps: the first step is to select the sliding surface, which is usually the linear manifold of the state variables and can guarantee the asymptotic stability; the second step is to determine the control output, which drives the system state to the designed sliding surface and constrains the state to the surface subsequently. Usually, the design of sliding surface for a PMSM is limited to integer order, which means that the sliding surface is constructed by the integer-order integration or differentiation of the state variables.

Fractional calculus has a 300-year-old history, and for a long time, it was considered as a pure theoretical subject with nearly no applications. In recent decades, not only the theory of fractional-order calculus is developed greatly, but also the application of fractional controller attracts increasing attention due to the higher degree of freedom provided [19–36]. In [33] the fractional-order adaptation law for integer-order sliding mode control is studied and applied in the 2DOF robot. In [34], the synchronization of chaotic and uncertain Duffing-Holmes system has been done using the sliding mode control strategy and fractional order mathematics. In [35], a robust fractional-order proportion-plus-differential (FOPD) controller for the control of PMSM was proposed. In control practice, it is useful to consider the fractional-order controller design for an integer-order plant. This is due to the fact that the plant model may have already been obtained as an integer-order model in classical sense. In most cases, our objective of using fractional calculus is to apply the fractional-order control (FOC) to enhance the system control performance.

This paper applies the fractional calculus into the sliding surface design and proposes a robust fractional-order sliding mode controller (FOSMC) for the position control of a PMSM. The rest of this paper is organized as follows. In Section 2, the fractional-order calculus operation is introduced. In Section 3, the mathematical model of PMSM is given. In Section 4, the conventional integer-order SMC for PMSM is reviewed. In Section 5, the FOSMC method for position control of PMSM is derived. In Section 6, the robustness of the proposed FOSMC method is analyzed. In Section 7, the effectiveness of the proposed algorithm is illustrated through numerical examples and compared with the conventional integer-order SMC. In Section 8, the guidance for parameters selection and design is given. Finally, conclusions are presented in Section 9.

2. Fractional Calculus

Fractional calculus has been known since the development of the integer-order calculus, but for a long time, it has been considered as a sole mathematical problem. In recent decades, fractional calculus has become an interesting topic among system analyses and control fields.

Fractional calculus is a generalization of integer-order integration and differentiation to non-integer-order ones. Let symbol ${}_aD_t^{\lambda}$ denote the fractional-order fundamental operator, defined as follows [20, 21]:

$$D^{\lambda} \triangleq {}_aD_t^{\lambda} = \begin{cases} \dfrac{d^{\lambda}}{dt^{\lambda}} & R(\lambda) > 0, \\ 1 & R(\lambda) = 0, \\ \int_a^t (d\tau)^{-\lambda} & R(\lambda) < 0, \end{cases} \tag{2.1}$$

where a and t are the limits of the operation, λ is the order of the operation, and generally $\lambda \in R$ and λ can be a complex number.

The two most used definitions for the general fractional differentiation and integration are the Grunwald-Letnikov (GL) definition [22] and the Riemann-Liouville (RL) definition [23]. The GL is given by

$$ {}_aD_t^{\lambda} f(t) = \lim_{h \to 0} h^{-\lambda} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\lambda}{j} f(t - jh), \tag{2.2}$$

where $[\cdot]$ means the integer part.

The RL definition is given as

$$ {}_aD_t^{\lambda} f(t) = \frac{1}{\Gamma(n-\lambda)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\lambda-n+1}} d\tau, \tag{2.3}$$

where $n - 1 < \lambda < n$, and $\Gamma(\cdot)$ is the *Gamma* function.

Having zero initial conditions, the Laplace transformation of the RL definition for a fractional-order λ is given by

$$L\left\{ {}_aD_t^{\lambda} f(t) \right\} = s^{\lambda} F(s), \tag{2.4}$$

where $F(s)$ is the Laplace transformation of $f(t)$.

Distinctly, the fractional-order operator has more degrees of freedom than that with integer order. It is likely that a better performance can be obtained with the proper choice of order.

3. Mathematical Model of PMSM

The PMSM is composed of a stator and a rotor; the rotor is made by a permanent magnet, and the stator has 3-phase windings which are distributed sinusoidally. To get the model of the PMSM, some assumptions are made: *(a) the eddy current and hysteresis losses are ignored; (b) magnetic saturation is neglected; (c) no damp winding is on the rotor; (d) the induced EMF is* *sinusoidal*. Under the above assumptions, the mathematics model of a PMSM can be described in the rotor rotating reference frame as follows [2]:

$$u_d = Ri_d - \omega_e L_q i_q + L_d \frac{di_d}{dt},$$
$$u_q = Ri_q + \omega_e L_d i_d + \omega_e \psi_f + L_q \frac{di_q}{dt}. \tag{3.1}$$

In the above equations, u_d and u_q are voltages in the d- and q-axes, i_d and i_q are currents in the d- and q-axes, L_d and L_q are inductances in the d- and q-axes, R is the stator resistance, ω_e is the electrical angular velocity, and ψ_f is the flux linkage of the permanent magnet.

The corresponding electromagnetic torque is as follows:

$$T_e = P\left[\psi_f i_q + \left(L_d - L_q\right) i_d i_q\right], \tag{3.2}$$

where T_e is the electromagnetic torque, and P is the pole number of the rotor.

For surface PMSM, we have $L_d = L_q$; thus, the electromagnetic torque equation is rewritten as follows:

$$T_e = P\psi_f i_q. \tag{3.3}$$

The associated mechanical equation is as follows:

$$T_e - T_L = J\frac{d\omega_m}{dt} + B\omega_m, \tag{3.4}$$

where J is the motor moment inertia constant, T_L is the external load torque, B is the viscous friction coefficient, and ω_m is the rotor angular speed, and it satisfies

$$\omega_e = P\omega_m. \tag{3.5}$$

In this paper, the $i_d = 0$ decoupled control method is applied, which means that there is no demagnetization effect, and the electromagnetic torque and the armature current are the linear relationship.

4. Review of Conventional SMC

4.1. State Equations of PMSM System

The object of the designed controller is to make the position θ_m strictly follow its desired signal θ_{ref}. Let

$$x_1 = \theta_{\text{ref}} - \theta_m,$$
$$x_2 = \dot{x}_1 = \dot{\theta}_{\text{ref}} - \dot{\theta}_m, \tag{4.1}$$

where x_1 and x_2 are the state error variables of the PMSM system,

$$\begin{aligned}\dot{\theta}_m &= \omega_m,\\ \ddot{\theta}_m &= \dot{\omega}_m.\end{aligned} \tag{4.2}$$

From (4.1) and (4.2), it is obvious that

$$\begin{aligned}\dot{x}_1 &= x_2 = \dot{\theta}_{\text{ref}} - \dot{\theta}_m,\\ \dot{x}_2 &= \ddot{\theta}_{\text{ref}} - \ddot{\theta}_m = \ddot{\theta}_{\text{ref}} - \dot{\omega}_m.\end{aligned} \tag{4.3}$$

Substituting (3.3) and (3.4) into (4.3), we have

$$\dot{x}_2 = \ddot{\theta}_{\text{ref}} - \frac{1}{J}\left[P\psi_f i_q - T_L - B\omega_m\right]. \tag{4.4}$$

Then the state-space equation of the PMSM control system can be written as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ E\end{bmatrix}U + \begin{bmatrix}0\\ F\end{bmatrix}, \tag{4.5}$$

where

$$E = -\frac{P\psi_f}{J}, \qquad F = \ddot{\theta}_{\text{ref}} + \frac{T_L + B\omega_m}{J}, \qquad U = i_q. \tag{4.6}$$

4.2. The Conventional Integer-Order SMC

The design of the SMC usually consists of two steps. Firstly, the sliding surface is designed such that the system motion on the sliding mode can satisfy the design specifications; secondly, a control law is designed to drive the system state to the designed sliding surface and constrains the state to the surface subsequently.

The conventional integer-order sliding surface S is designed as follows [4]:

$$S = cx_1 + x_2, \tag{4.7}$$

where c is set as a positive constant, and the derivative of (4.7) is as follows:

$$\dot{S} = c\dot{x}_1 + \dot{x}_2. \tag{4.8}$$

Substituting (4.3) and (4.4) into (4.8), we have

$$\dot{S} = cx_2 + \ddot{\theta}_{\text{ref}} - \frac{1}{J}\left[P\psi_f i_q - T_L - B\omega_m\right]. \tag{4.9}$$

When $T_L = 0$, and forcing $\dot{S} = 0$, then the control output is obtained as follows:

$$U_{\text{eq}} = i_q = \frac{J}{P\psi_f}\left(cx_2 + \ddot{\theta}_{\text{ref}} + \frac{1}{J}B\omega_m\right). \tag{4.10}$$

Here, U_{eq} is the equivalent control, which keeps the state variables on the sliding surface.

When the system has immeasurable disturbances with upper limit $T_{L\text{-max}}$, then the final control output can be given as

$$U = i_q = U_{\text{eq}} + k\,\text{sgn}(S) = \frac{J}{P\psi_f}\left(cx_2 + \ddot{\theta}_{\text{ref}} + \frac{1}{J}B\omega_m\right) + k\,\text{sgn}(S), \tag{4.11}$$

where k is a positive switch gain, and $\text{sgn}(\cdot)$ denotes the *sign* function defined as

$$\text{sgn}(S) = \begin{cases} 1 & S > 0, \\ 0 & S = 0, \\ -1 & S < 0. \end{cases} \tag{4.12}$$

4.3. Stability Analysis

The *Lyapunov* function is defined as

$$V = \frac{1}{2}S^2. \tag{4.13}$$

According to the *Lyapunov* stability theorem, the sliding surface reaching condition is $S\dot{S} < 0$. Taking the derivative of (4.13) and substituting (4.11) into (4.9), we have

$$\dot{V} = S\dot{S} = S\left[\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\,\text{sgn}(S)\right]. \tag{4.14}$$

From (4.14), it is obvious that when

$$k > \frac{T_{L\text{-max}}}{P\psi_f}, \tag{4.15}$$

then $S\dot{S} < 0$, and the system is globally and asymptotically stable; S and $\dot{S}$ will approach zero in a finite time duration.

5. Proposed Fractional-Order SMC (FOSMC)

In this section, the fractional-order sliding mode controller (FOSMC) for the position control of PMSM will be proposed.

5.1. Design of Fractional-Order Sliding Surface

First, the fractional-order sliding surface is designed as follows:

$$S = k_p x_1 + k_d D^{\mu} x_1 = k_p x_1 + k_d D^{\mu-1} x_2, \tag{5.1}$$

where k_p and k_d are set as positive constants, the function D^{μ} is defined as (2.1), and $0 < \mu < 1$. From (5.1), it can be seen that the fractional-order differentiation of x_1 is used to construct the sliding surface. Meanwhile, as $-1 < \mu - 1 < 0$, the operator $D^{\mu-1}x_2$ in (5.1), which means the $(\mu-1)$th-order integration of x_2, can be seen as a low-pass filter and can reduce the amplitude of high-frequency fluctuations of x_2. In this sense, the fractional-order sliding surface defined by (5.1) is more smooth compared with the conventional sliding surface shown as (4.7).

5.2. Design of FOSMC

Taking the time derivative on both sides of (5.1) yields

$$\dot{S} = k_p \dot{x}_1 + k_d D^{\mu+1} x_1 = k_p x_2 + k_d D^{\mu-1} \dot{x}_2. \tag{5.2}$$

Substituting (4.4) into (5.2), we have

$$\dot{S} = k_p x_2 + k_d D^{\mu-1} \dot{x}_2 = k_p x_2 + k_d D^{\mu-1} \left\{ \ddot{\theta}_{\text{ref}} - \frac{1}{J} \left[P\psi_f i_q - T_L - B\omega_m \right] \right\}, \tag{5.3}$$

when $T_L = 0$, and forcing $\dot{S} = 0$, then the control output can be obtained as follows:

$$D^{\mu-1} \left\{ \ddot{\theta}_{\text{ref}} - \frac{1}{J} \left[P\psi_f i_q - B\omega_m \right] \right\} = -\frac{k_p}{k_d} x_2. \tag{5.4}$$

Taking the $(1-\mu)$th-order derivative on both sides of (5.4) will result in

$$\ddot{\theta}_{\text{ref}} - \frac{1}{J} \left[P\psi_f i_q - B\omega_m \right] = D^{1-\mu} \left(-\frac{k_p}{k_d} x_2 \right). \tag{5.5}$$

From (5.5), the equivalent control can be obtained as

$$U_{\text{eq}} = i_q = \frac{J}{P\psi_f} \left(\frac{k_p}{k_d} D^{1-\mu} x_2 + \ddot{\theta}_{\text{ref}} + \frac{1}{J} B\omega_m \right). \tag{5.6}$$

Similar to (4.11), when the system has load disturbances with upper limit $T_{L\text{-max}}$, then the control output of FOSMC method can be given as

$$U = i_q = U_{\text{eq}} + k\,\text{sgn}(S) = \frac{J}{P\psi_f} \left(\frac{k_p}{k_d} D^{1-\mu} x_2 + \ddot{\theta}_{\text{ref}} + \frac{1}{J} B\omega_m \right) + k\,\text{sgn}(S), \tag{5.7}$$

where μ is called as the order of FOSMC method. If we set $k_p = c$, $k_d = 1$, and let $A = P\psi_f / J$, then the block diagram of the proposed FOSMC method can be shown in Figure 1.

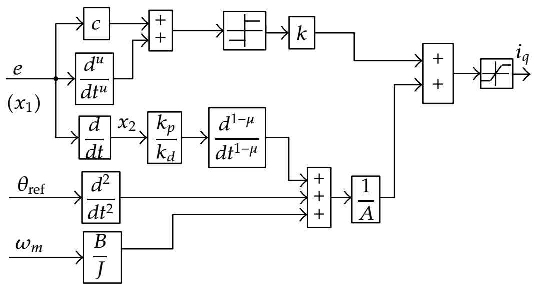


Figure 1: Block diagram of the proposed FOSMC method.

5.3. Stability Analysis of FOSMC with Sign Function

When the *sign* function is used in the control output, then substituting (5.7) into (5.3), we have

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\,\mathrm{sgn}(S)\right). \tag{5.8}$$

From (5.8), we can get the following.

(a) When $S < 0$, then $\mathrm{sgn}(S) = -1$, and we have

$$\delta_1 \triangleq \left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\,\mathrm{sgn}(S)\right) = \left(\frac{T_{L\text{-max}}}{J} + \frac{P\psi_f}{J}k\right) > 0. \tag{5.9}$$

So the $(\mu - 1)$th-order fractional integration of δ_1 is higher than zero, that is,

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\,\mathrm{sgn}(S)\right) > 0, \tag{5.10}$$

which implies that the derivative of the *Lyapunov* function $\dot{V} = S\dot{S} < 0$.

(b) When $S > 0$, then $\mathrm{sgn}(S) = 1$, and we have

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\right). \tag{5.11}$$

From (5.11), it is clear that when

$$\delta_2 \triangleq \frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k < 0, \tag{5.12}$$

that is,

$$k > \frac{T_{L\text{-max}}}{P\psi_f}, \tag{5.13}$$

then the $(\mu - 1)$th-order fractional integration of δ_2 is lower than zero, that is, $\dot{S} < 0$, which means that $\dot{V} = S\dot{S} < 0$.

From (5.8) to (5.13), it is obvious that when

$$k > \frac{T_{L\text{-max}}}{P\psi_f}, \tag{5.14}$$

then the system is globally stable; S and $\dot{S}$ will approach zero in a finite time duration.

Moreover, from (5.8), it can be seen that because of the integration effect by the operator $D^{\mu-1}(\cdot)$, the variation amplitude of $\dot{S}$ in (5.8) is smaller than that of $\dot{S}$ in (4.14), which means that when the *sign* function is used, the sliding surface of the proposed FOSMC method has smaller chattering amplitude than the sliding surface of the conventional SMC method.

5.4. Stability Analysis of FOSMC with Saturation Function

From (5.7), it can be seen that the *sign* function is involved in the output, so the chattering phenomenon will be caused. In this paper, a *saturation* function is adopted to reduce the chattering problem, described as follows:

$$\operatorname{sat}(S) = \begin{cases} 1 & S > \varepsilon, \\ \dfrac{S}{\varepsilon} & -\varepsilon \le S \le \varepsilon, \\ -1 & S < -\varepsilon, \end{cases} \tag{5.15}$$

where $\varepsilon > 0$ denotes the thickness of the boundary layer.

When the *saturation* function is used, the control output can be rewritten as

$$U = i_q = U_{\text{eq}} + k\operatorname{sat}(S) = \frac{J}{P\psi_f}\left(\frac{k_p}{k_d}D^{1-\mu}x_2 + \ddot{\theta}_{\text{ref}} + \frac{1}{J}B\omega_m\right) + k\operatorname{sat}(S), \tag{5.16}$$

then, similar to (5.8), substituting (5.16) into (5.3), we have

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\operatorname{sat}(\mathrm{S})\right). \tag{5.17}$$

From (5.17), the following is clear.

(a) When $S < 0$, then $\operatorname{sat}(S) < 0$,

$$\delta_3 \triangleq \left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\operatorname{sat}(S)\right) > 0. \tag{5.18}$$

So the $(\mu-1)$th-order fractional integration of δ_3 is higher than zero, that is,

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J} k\,\mathrm{sat}(S)\right) > 0, \tag{5.19}$$

which means that the derivative of the *Lyapunov* function $\dot{V} = S\dot{S} < 0$.

(b) When $S > \varepsilon$, then $\mathrm{sat}(S) = 1$, and we have

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J} k\right). \tag{5.20}$$

Similar with (5.11)–(5.13), when

$$k > \frac{T_{L\text{-max}}}{P\psi_f}, \tag{5.21}$$

then $\dot{S} < 0$, which means that $\dot{V} = S\dot{S} < 0$.

(c) When $0 < S \le \varepsilon$, then $\mathrm{sat}(S) = S/\varepsilon$, and we have

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J} k\frac{S}{\varepsilon}\right). \tag{5.22}$$

From (5.22), it can be seen that when

$$\delta_4 \triangleq \left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J} k\frac{S}{\varepsilon}\right) < 0, \tag{5.23}$$

that is,

$$k > \frac{T_{L\text{-max}}}{P\psi_f}\frac{\varepsilon}{S} \ge \frac{T_{L\text{-max}}}{P\psi_f}, \tag{5.24}$$

then $\dot{S} < 0$. Here, it is assumed that a load disturbance with magnitude $T_{L\text{-max}}$ is exerted on the system. From (5.24), it can be seen that when the value of S is very small, then $\varepsilon/S \gg 1$, so the condition for $\dot{S} < 0$ is that the value of k is much higher than $T_{L\text{-max}}/P\psi_f$, but in fact the parameter k will not be given a so high value. Here, it is assumed that k is assigned a minimum value which meets condition (5.21). Then the sliding surface S will undergo the following stages.

(i) In the period $0 < S \ll \varepsilon$, we have $\dot{S} \gg 0$, so the system is unstable, meanwhile S will rapidly arrive at the peak value S^* (where $S^* > \varepsilon$) in a finite time with large initial positive velocity.

(ii) As $S = S^* > \varepsilon$, then from (5.24), it can be seen that the value of k satisfies condition (5.21), so $\dot{S} < 0$ and $\dot{V} = S\dot{S} < 0$, that is, the system is globally stable again. In this moment, because $S > 0$ and $\dot{S} < 0$, then S will decrease with negative velocity.

(iii) When S decreases until $S < \varepsilon$, then from (5.24), it can be seen that the value of k does not satisfy condition (5.21) any longer, which means that $\dot{S} > 0$, then S starts to increase.

(iv) When S increases until $S > \varepsilon$, then similar to (ii), we have $S > 0$ and $\dot{S} < 0$, then S will decrease with negative velocity.

(v) After several oscillations and adjustments between stages (iii) and (iv), the sliding surface function S will finally maintain on the point of $S = \varepsilon$, and the system is in a stable state with $\dot{S} = 0$.

When the system is in the stable state described by (v), then from (4.7) or (5.1), it can be seen that $x_2 = 0$ or $D^{\mu}x_1 = 0$, and the stable position error x_1 can be estimated as follows:

$$S = cx_1 = \varepsilon \Longrightarrow x_1 = \frac{\varepsilon}{c} \tag{5.25}$$

or

$$S = k_p x_1 = \varepsilon \Longrightarrow x_1 = \frac{\varepsilon}{k_p}. \tag{5.26}$$

Generally, when the load disturbance is T_L ($T_L < T_{L\text{-max}}$), then similar to the above analysis, the stable position error x_1 can be estimated as follows:

$$S = cx_1 = \frac{\varepsilon T_L}{kP\psi_f} \Longrightarrow x_1 = \frac{\varepsilon T_L}{ckP\psi_f} \tag{5.27}$$

or

$$S = k_p x_1 = \frac{\varepsilon T_L}{kP\psi_f} \Longrightarrow x_1 = \frac{\varepsilon T_L}{k_p kP\psi_f}. \tag{5.28}$$

With the maximum permissible position error x_1 of the PMSM system, (5.26) or (5.28) will be the constraint in designing the parameter ε and c or k_p.

Remark 5.1. In the above analysis of parts (b) and (c), the integration effect of the operator $D^{\mu-1}(\cdot)$ is ignored temporarily. If the integration effect is considered, then the fractional-order μ will decide the phase delay and variation magnitude of $\dot{S}$. When μ is too small, especially when $\mu = 0$, then the operator $D^{\mu-1}(\cdot)$ becomes a first-order integer integrator, and the long time integration effect will lead to the largest phase delay and smallest variation magnitude of $\dot{S}$, and the stable condition $\dot{V} = S\dot{S} < 0$ may not be satisfied promptly, and so the system will become unstable. When μ is too large, especially when $\mu = 1$, then the operator $D^{\mu-1}(\cdot)$ does not have integration action, and $\dot{S}$ has zero-phase delay and the largest variation magnitude, which are the same as the convention SMC method. When μ is selected as a proper value in the range (0, 1), then the suitable phase delay of $\dot{S}$ will satisfy the stable condition $\dot{V} = S\dot{S} < 0$, and meanwhile, the appropriate variation magnitude of $\dot{S}$ will make the sliding surface S change with small fluctuation, so a better control performance can be obtained.

6. Robustness and Effectiveness Analysis of FOSMC

The robustness and effectiveness of the proposed FOSMC method will be analyzed in the following two aspects.

6.1. Analysis of the Control Output

From the control output of the FOSMC method shown as (5.7) or (5.16), it can be seen that two important terms are included.

(a) The term $D^{1-\mu}x_2$ denotes the $(1-\mu)$th-order differentiation of x_2, so the fractional dimension accelerating change rate of position error is contained in the output, which means that the output of the FOSMC method is more sensitive to the change rate of position error and can provide a prompt output.

(b) The other term is the $\mathrm{sgn}(S)$ in (5.7) or the $\mathrm{sat}(S)$ in (5.16), the former is a high-frequency switching signal, and the latter is a relative smooth switch signal. According to the sliding surface S defined by (5.1), it is clear that an $(\mu-1)$th-order integrator for x_2 is contained, that is, the proposed sliding surface S is more smooth than the conventional sliding surface. In other words, by using the FOSMC method, the chattering of $\mathrm{sgn}(S)$ in (5.7) is eliminated to some degree, and the term $\mathrm{sat}(S)$ in (5.16) is more smooth.

6.2. Analysis of Stable Condition

With (5.8) and (5.17), it can be seen that when substituting the control output into the derivative of fractional-order sliding surface S, we have

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\,\mathrm{sgn}(S)\right) = k_d D^{\mu-1}\delta_1 \tag{6.1}$$

or

$$\dot{S} = k_d D^{\mu-1}\left(\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J}k\,\mathrm{sat}(S)\right) = k_d D^{\mu-1}\delta_3. \tag{6.2}$$

Here, the operator $D^{\mu-1}(\cdot)$ means the fractional-order integration since $0 < \mu < 1$.

The following is assumed:

(i) the value of k is set as a constant which is satisfied with condition (5.14) or (5.21);

(ii) the system is in a reaching state (i.e., $\dot{V} = S\dot{S} < 0$) or in a stable state (i.e., $S = 0$ or constant, and $\dot{S} = 0$).

Then the following three cases will be discussed.

(a) When the system is in a reaching state and $S > 0$, $\dot{S} < 0$, then

$$\dot{S} = k_d D^{\mu-1}\delta_1 < 0 \Longrightarrow \delta_1 < 0 \tag{6.3}$$

or

$$\dot{S} = k_d D^{\mu-1}\delta_3 < 0 \Longrightarrow \delta_3 < 0. \tag{6.4}$$

If an instant load disturbance T_{instant} which is greater than $T_{L\text{-max}}$ is applied on the system, then from (6.1) or (6.2), it can be seen that in this moment $\delta_1 > 0$ or $\delta_3 > 0$, but because of the integration effect by the fractional-order integration operator $D^{\mu-1}(\cdot)$, the integration value that is, $D^{\mu-1}(\delta_1)$ or $D^{\mu-1}(\delta_3)$, will not be greater than zero instantaneous, in other words the system will remain stable for an extra short time.

While for the conventional SMC method, from (4.14), it can be seen that the derivative of sliding surface S is

$$\dot{S} = \left[\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J} k\,\text{sgn}(S)\right] \tag{6.5}$$

or

$$\dot{S} = \left[\frac{T_{L\text{-max}}}{J} - \frac{P\psi_f}{J} k\,\text{sat}(S)\right]. \tag{6.6}$$

It is clear that when an instant load disturbance T_{instant} ($T_{\text{instant}} > T_{L\text{-max}}$) is applied on the system, then $\dot{S} < 0$ immediately, and the system is also unstable at once.

(b) When the system is in a reaching state and $S < 0$, $\dot{S} > 0$, then

$$\dot{S} = k_d D^{\mu-1}\delta_1 > 0 \Longrightarrow \delta_1 > 0 \tag{6.7}$$

or

$$\dot{S} = k_d D^{\mu-1}\delta_3 > 0 \Longrightarrow \delta_3 > 0. \tag{6.8}$$

Similar to the above analysis, when an instant negative load disturbance (i.e., an opposite direction load disturbance) T_{instant} which is smaller than $(-T_{L\text{-max}})$ is applied on the system, then from (6.1) or (6.2), it can be seen that in this moment $\delta_1 > 0$ or $\delta_3 > 0$, but because of the integration effect by the operator $D^{\mu-1}(\cdot)$, the integration value, that is, $D^{\mu-1}(\delta_1)$ or $D^{\mu-1}(\delta_3)$, will not be smaller than zero instantaneously in other words, the system will continue to be stable for an extra short time.

While for the conventional SMC method, it is clear that when an instant negative load disturbance T_{instant} ($T_{\text{instant}} < -T_{L\text{-max}}$) is exerted on the system, then according to (6.5) and (6.6), it can be seen that $\dot{S}$ will be smaller than zero (i.e., $\dot{S} < 0$) immediately, and thus, the system is also unstable at once.

(c) When the system is in a stable state, that is, $S = 0$ or constant, and $\dot{S} = 0$, then

$$\dot{S} = k_d D^{\mu-1}\delta_1 = 0 \Longrightarrow \delta_1 = 0 \tag{6.9}$$

or

$$\dot{S} = k_d D^{\mu-1}\delta_3 = 0 \Longrightarrow \delta_3 = 0. \tag{6.10}$$

If an instant positive or negative load disturbance T_{instant} is applied on the system, then from (6.1) or (6.2), it is obvious that in this moment there is a step change for δ_1 or δ_3, but because of the integration effect by the fractional-order integration operator $D^{\mu-1}(\cdot)$, the integration value, that is, $D^{\mu-1}(\delta_1)$ or $\mathrm{D}^{\mu-1}(\delta_3)$, will not change greatly in a short time, which means that the sliding surface S will change with smaller fluctuation comparing with the conventional SMC method, so a better control performance is obtained.

In addition, when the load disturbance T_{instant} is greater than $T_{L\text{-max}}$, then the same conclusions as those made from the above analysis of (a) and (b) can be obtained.

From the above analysis, it is obvious that the proposed FOSMC method is more robust than the conventional SMC method.

7. Numerical Computation Examples and Simulation

7.1. Approximation of Fractional-Order Operator

The Matlab/Simulink is used to simulate the FOSMC control system. In the simulation, a discrete-time finite-dimensional (z) transfer function is computed to approximate the continuous-time fractional-order operator $D^{\mu}(\cdot)$ by the IRID method [37], that is, dfod = irid_fod(u, T_s, N). In the simulation, the sampling frequency of FOSMC controller is 2 KHz; thus, in the IRID method, $T_s = 0.0005$ sec, and the approximation order is $N = 5$.

7.2. System Block and Configuration

The block diagram of the PMSM drive system using FOSMC method is shown in Figure 2, in which the block “SMC” means the conventional integer-order SMC method, and the block “FOSMC” is the proposed method, which is shown in Figure 1. The performance of the proposed FOSMC is compared with that of the conventional SMC. The rotor of the PMSM is the permanent magnet, and the flux linkage is constant. The specifications of the PMSM are shown in Table 1.

As shown in Figure 2, the drive system has an outer loop of position controller based on FOSMC method and an inner loop including two current controllers, that is, the q-axis and d-axis stator current regulators, both of which are based on PI control algorithm with sampling frequency of 10 KHz, and the d-axis stator current command is set to zero. In the block, ω_{ref} is the reference rotor speed in mechanical revolutions per minute, ω is the rotor speed in mechanical revolutions per minute measured by encoder, and the space vector PWM was used for the PWM generation.

For comparison, we first determine the optimal parameters of the conventional SMC method, and then the corresponding parameters of the new proposed FOSMC method are set similarely, that is, in Figure 2, the following parameters of SMC and FOSMC are set to be the same, that is,

$$k = 3, \qquad k_p = c = 100, \qquad k_d = 1, \qquad \varepsilon = 1. \tag{7.1}$$

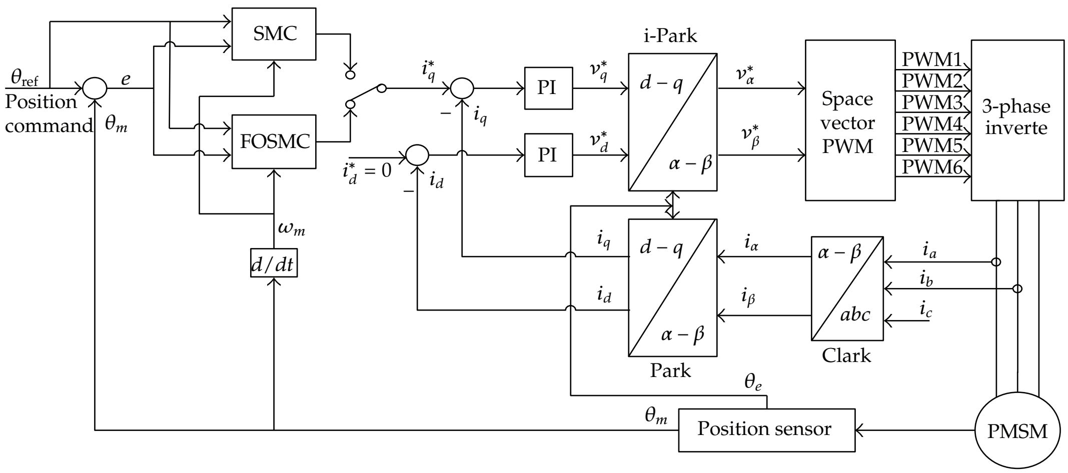
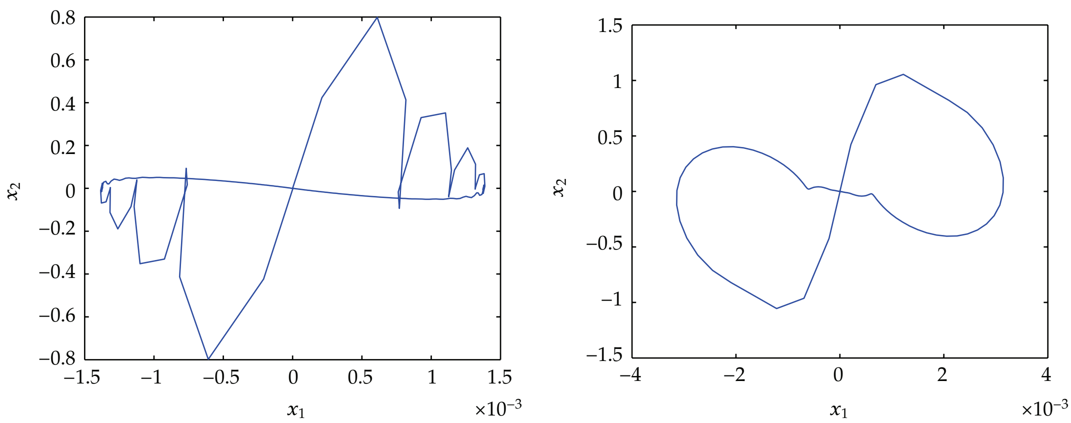


Figure 2: Block diagram of the PMSM position control system.



(a) Phase trace of the conventional SMC method

(b) Phase trace of the proposed FOSMC method ($\mu = 0.6$)

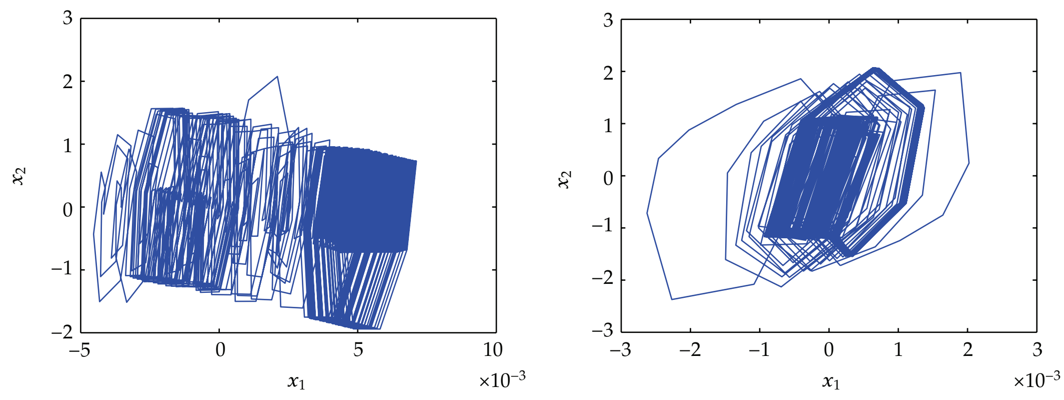
Figure 3: Phase traces of the conventional SMC method and the proposed FOSMC method with *saturation* function.

7.3. Simulation and Comparison

7.3.1. Simulation of Phase Trace

In this simulation, the phase traces by the conventional SMC method and the proposed FOSMC method are simulated and compared. The given position reference is $\theta_{\mathrm{ref}} = \pi$ rad, which is a step input with soft-start mode, and the order of the proposed FOSMC method is $\mu = 0.6$.

Figure 3 shows the simulation results of the phase traces by the conventional SMC method and the proposed FOSMC method with *saturation* function. Figure 4 is similar to Figure 3, and the only difference is that the *saturation* function is replaced by the *sign* function in the two methods.



(a) Phase trace of conventional SMC method (b) Phase trace of the proposed FOSMC method ($\mu = 0.6$)

Figure 4: Phase traces of the conventional SMC method and the proposed FOSMC method with *sign* function.

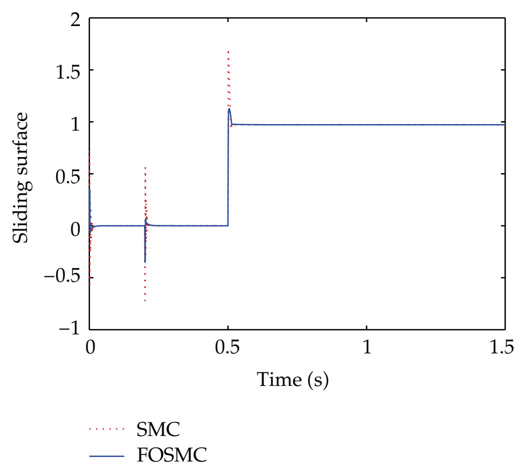


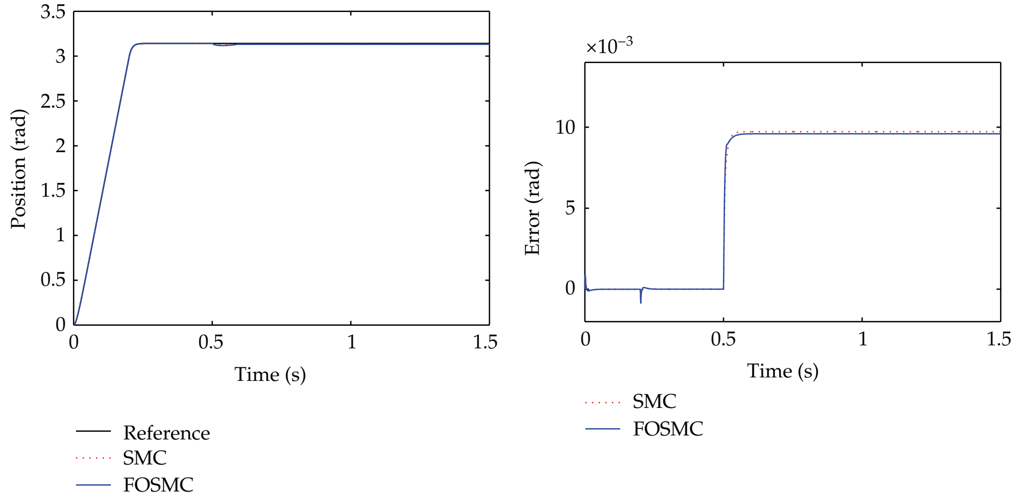
Figure 5: Sliding surfaces with *saturation* function and load disturbance of 2.5 Nm at $t = 0.5$ s ($\mu = 0.6$).

From Figure 3, it can be seen that the phase traces of both methods can reach the sliding surface ($S = 0$) and arrive at the origin finally, but because of the fractional-order integration effect (i.e., the term $D^{\mu-1}x_2$ in S), the phase trace of the proposed FOSMC method is more smooth than that of the conventional SMC method; this also means that the proposed FOSMC has smaller speed vibration, which is consistent with the analysis of Section 5.1.

From Figure 4, it is obvious that the phase trace of the proposed FOSMC method is more focused on the origin than that of the conventional SMC method, which means that the proposed FOSMC has smaller speed error.

7.3.2. Simulation of Stability Condition

In this simulation, the stability condition will be tested. The position reference is step input $\theta_{\text{ref}} = \pi$ rad, the order of FOSMC is $\mu = 0.6$, and other parameters are set as (7.1). From



(a) Position responses

(b) Position error

Figure 6: Position responses and error with *saturation* function and load disturbance of 2.5 Nm at $t = 0.5$ s ($\mu = 0.6$).

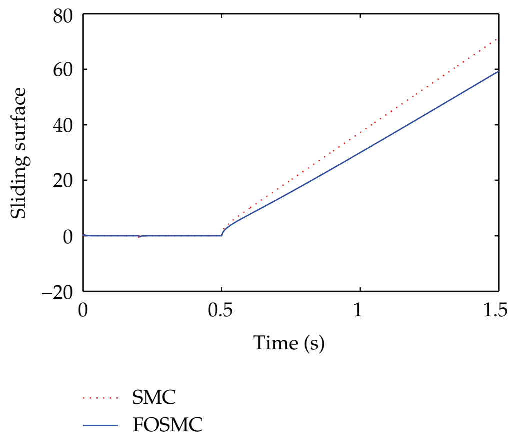
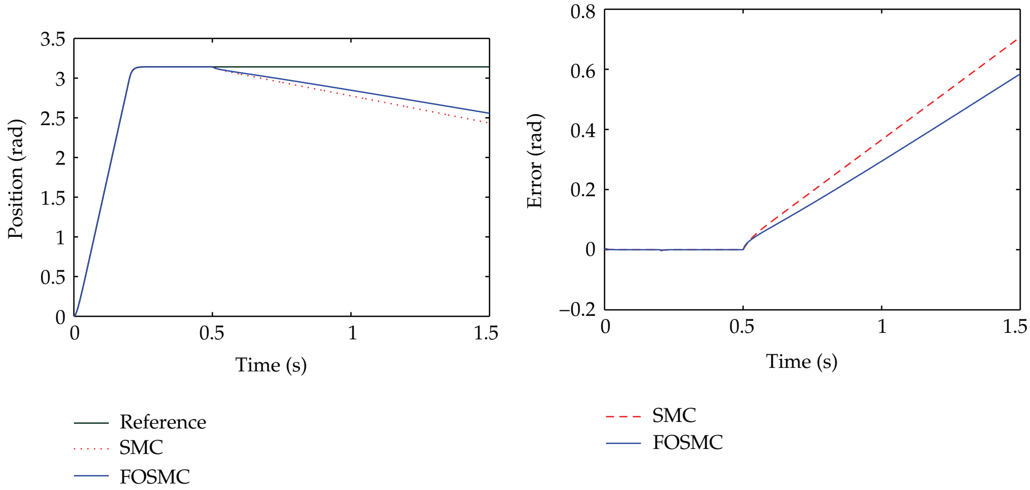


Figure 7: Sliding surfaces with *saturation* function and load disturbance of 2.6 Nm at $t = 0.5$ s ($\mu = 0.6$).

(5.13), (5.21), and Table 1, it can be calculated that the maximum load disturbance is $T_{L\text{-max}} = 2.568$ Nm. In each of the following cases, the conventional SMC method and the proposed FOSMC method are executed.

Figures 5–8 are the time curves of sliding surface function S, position responses, and position error, respectively. The *saturation* function is adopted, and different load disturbance is applied at time $t = 0.5$ s.

In Figures 5 and 6, the load disturbance is 2.5 Nm, and we can see that the system controlled by SMC or FOSMC is stable, because the load disturbance is less than $T_{L\text{-max}}$. Meanwhile, from Figures 5 and 6(b), it can be seen that the stable value of sliding surface function is $S \approx \varepsilon = 1$, and the stable position error is $x_1 \approx \varepsilon / c = 0.01$, which are consistent with the analysis of Section 5.4 and (5.26).



(a) Position responses (b) Position error

Figure 8: Position responses and error with *saturation* function and load disturbance of 2.6 Nm at $t = 0.5\,\mathrm{s}$ ($\mu = 0.6$).

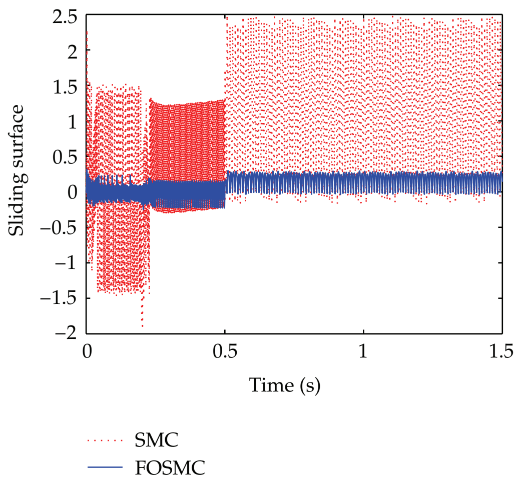
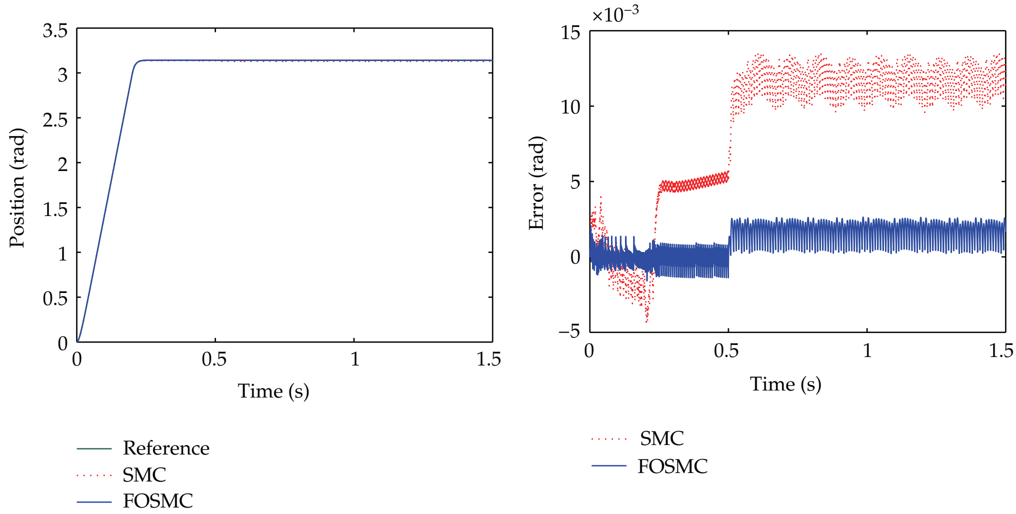


Figure 9: Sliding surfaces with *sign* function and load disturbance of 2.3 Nm at $t = 0.5\,\mathrm{s}$ ($\mu = 0.6$).

From Figure 5, one can see that when the external load is exerted on the system at $t = 0.5\,\mathrm{s}$, the variation amplitude of the sliding surface by the FOSMC method is smaller than that of the conventional SMC method, and consequently, the position error by the FOSMC method is smaller than that by the conventional SMC method, just as shown by Figure 6(b). The above two simulation results meet the analysis of Section 6.2(c).

In Figures 7 and 8, the load disturbance is 2.6 Nm, and it is obvious that the system controlled by SMC or FOSMC method is unstable, just because the load disturbance is greater than $T_{L\text{-max}}$. Moreover, an important result can be obtained from Figures 7 and 8, that is, when the load disturbance is greater than $T_{L\text{-max}}$, although the system is unstable any longer, the



(a) Position responses

(b) Position error

Figure 10: Position responses and error with *sign* function and load disturbance of 2.3 Nm at $t = 0.5$ s ($\mu = 0.6$).

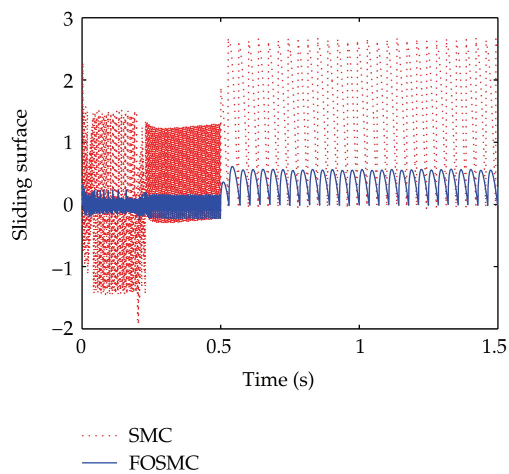
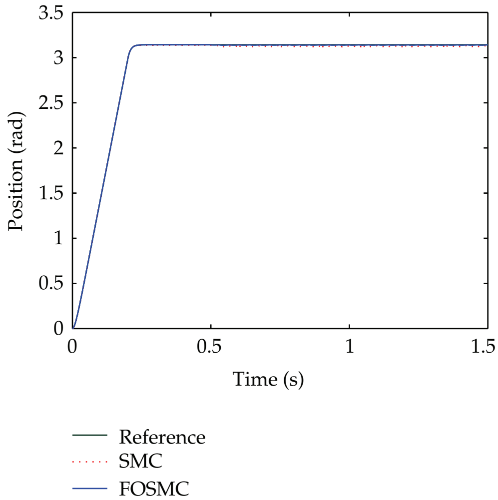


Figure 11: Sliding surfaces with *sign* function and load disturbance of 2.5 Nm at $t = 0.5$ s ($\mu = 0.6$).

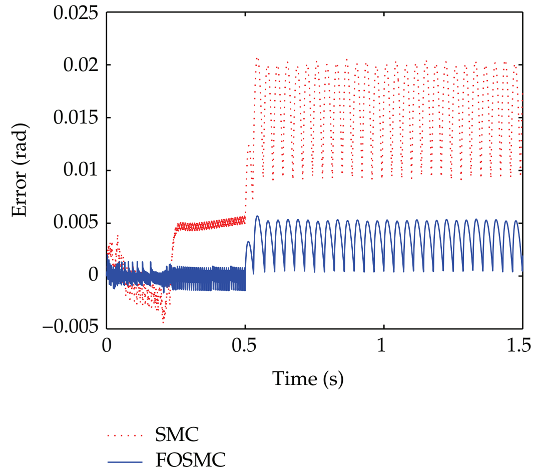
position error by the proposed FOSMC method is smaller than that by the conventional SMC method, which is keeping with the analysis of Section 6.2.

Figures 9–14 are the time curves of sliding surface function S, position responses, and position error, respectively, in which the *sign* function is adopted, and different load disturbance is applied at $t = 0.5$ s.

In Figures 9 and 10, the load disturbance is 2.3 Nm, and we can see that the system is stable under the load disturbance, because the load disturbance is less than $T_{L\text{-max}}$. Because of the use of *sign* function, the chattering phenomenon exists in the sliding surface S, just as shown in Figure 9. Meanwhile from Figures 9 and 10, two important results can be seen, that is, (a) the chattering amplitude of the sliding surface S by the FOSMC method is smaller



(a) Position responses



(b) Position error

Figure 12: Position responses and error with *sign* function and load disturbance of 2.5 Nm at $t = 0.5$ s ($\mu = 0.6$).

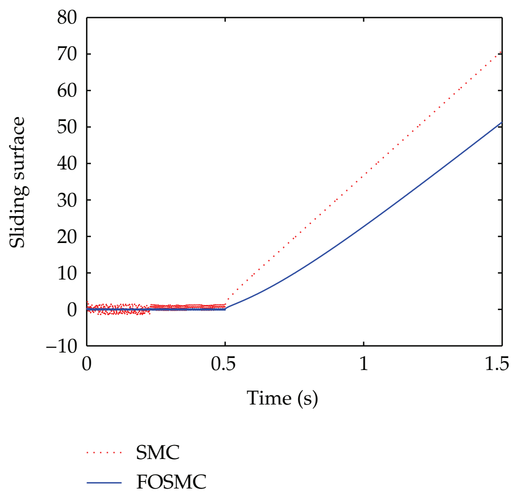
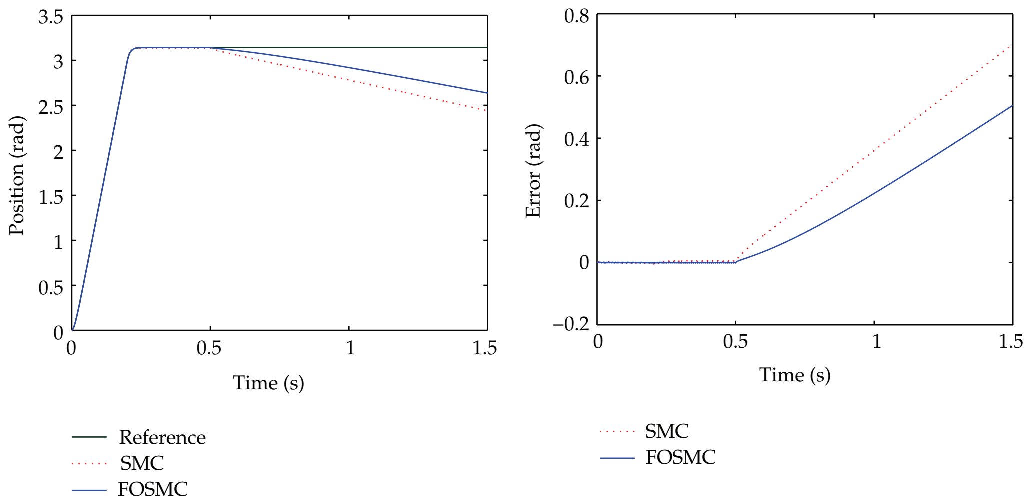


Figure 13: Sliding surfaces with *sign* function and load disturbance of 2.6 Nm at $t = 0.5$ s ($\mu = 0.6$).

than that by the conventional SMC method; (b) the position error by the proposed FOSMC method is also distinctly smaller than that by the conventional SMC method. The above two results meet the analysis of Sections 5.3 and 6.1.

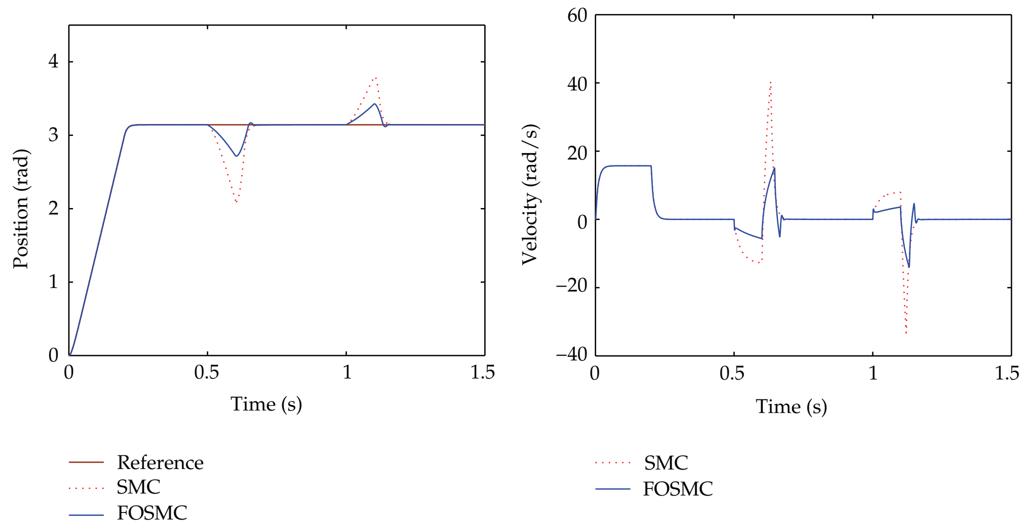
In Figures 11 and 12, the load disturbance is 2.5 Nm, and it can be seen that the system is critically stable after the load disturbance is applied, just because the load disturbance is close to $T_{L\text{-max}}$. And we also can see that the chattering amplitude of the sliding surface S and the position error, by the FOSMC method, are also distinctly smaller than those by the conventional SMC method, which meet the analysis of Sections 5.3 and 6.1.



(a) Position responses

(b) Position error

Figure 14: Position responses with *sign* function and load disturbance of 2.6 Nm at $t = 0.5$ s ($\mu = 0.6$).



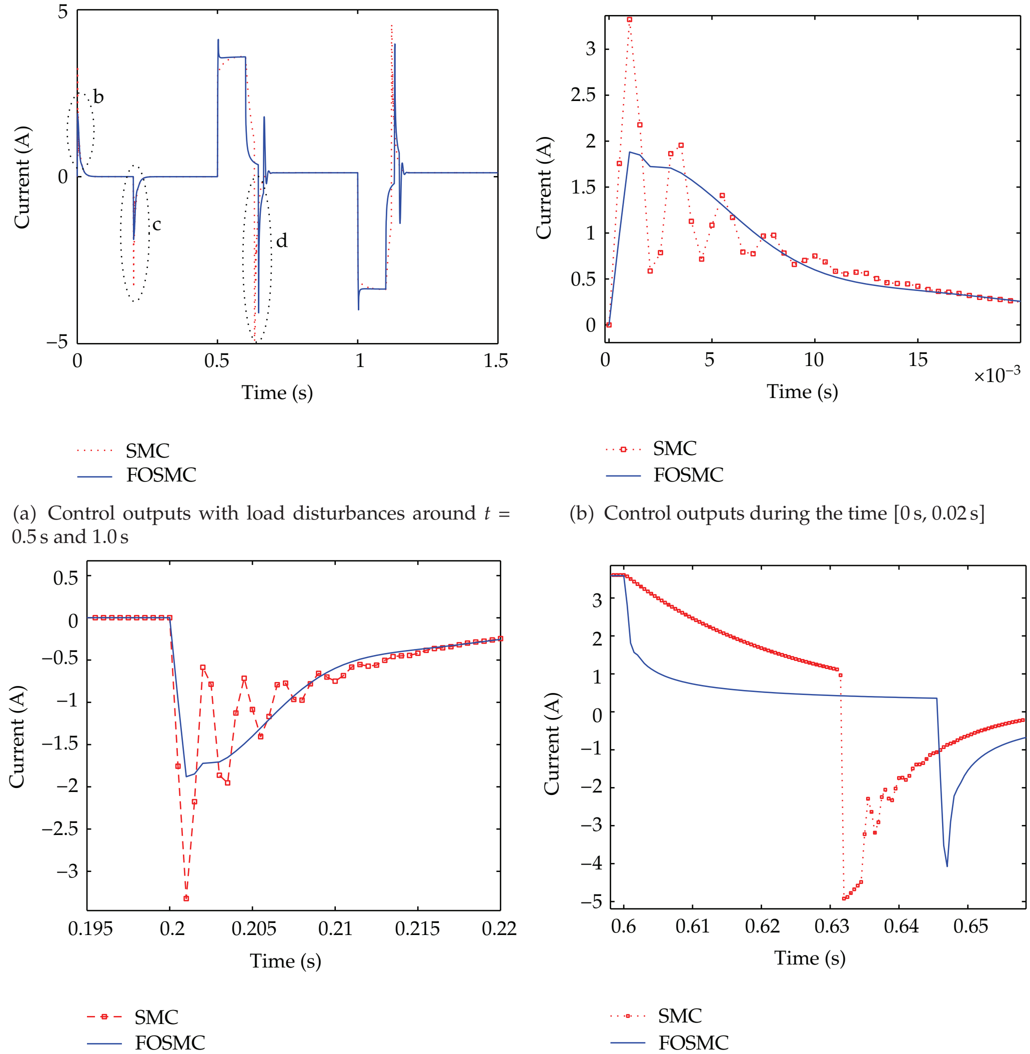
(a) Position responses

(b) Velocity responses

Figure 15: Position responses and velocity responses with load disturbances around $t = 0.5$ s and 1.0 s.

In Figures 13 and 14, the load disturbance is 2.6 Nm, and it is clear that the system driven by FOSMC or SMC method is unstable after the time 0.5 s, just because the load disturbance is greater than $T_{L\text{-max}}$. Meanwhile, although the system is unstable any longer, the position error by the proposed FOSMC method is smaller than that by the conventional SMC method, which is keeping with the analysis of Section 6.2.

All of the above simulation results show the correctness of the stability condition shown by (5.13) or (5.21); meanwhile, the robustness analyses of Sections 5.3, 5.4, and 6 are also verified.



(a) Control outputs with load disturbances around t = 0.5 s and 1.0 s

(b) Control outputs during the time [0 s, 0.02 s]

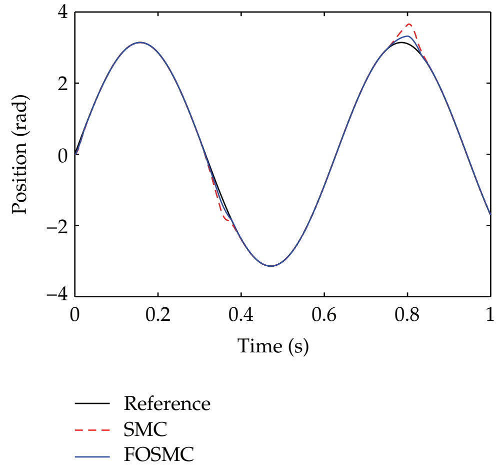
(c) Control outputs during the time [0.195 s, 0.22 s]

(d) Control outputs during the time [0.60 s, 0.66 s]

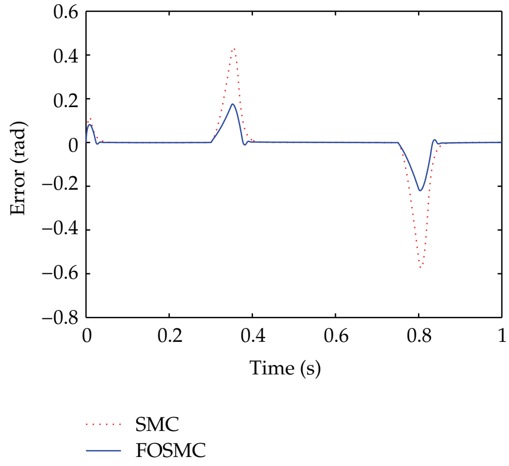
Figure 16: Results of the control outputs.

7.3.3. Simulation of Dynamic Position Response with Step Input Signal

In this simulation, the position reference is $\theta_{\text{ref}} = \pi$ rad, the order of FOSMC is $\mu = 0.6$, and the *saturation* function is adopted. A step disturbance load of 3.1 Nm is applied at $t = 0.5$ s and withdrawn at $t = 0.6$ s, another step disturbance load of -3.1 Nm is applied at $t = 1.0$ s and withdrawn at $t = 1.1$ s. Figures 15(a) and 15(b) show the dynamic position and velocity responses, respectively, of the conventional SMC method and the proposed FOSMC method in the presence of the above disturbances load. Obviously, the position error by the proposed FOSMC method is significantly smaller than that by the conventional SMC method; in other words, the FOSMC method is of more robustness than the conventional SMC method, which is in agreement with the analysis of Section 6.



(a) Position responses



(b) Position error

Figure 17: Position responses and error to sinusoidal input signal with load disturbances at $t = 0.3$ s and 0.75 s.

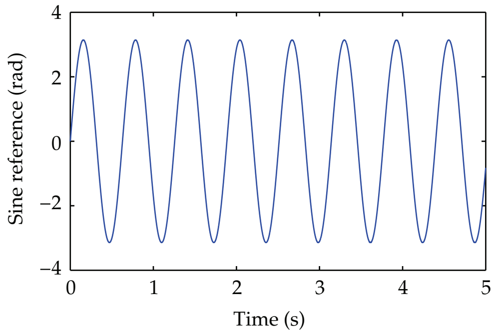


Figure 18: *Sine* reference input.

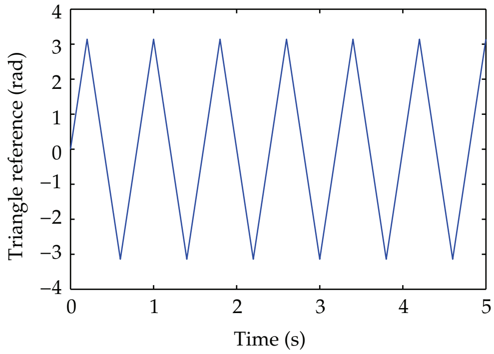


Figure 19: *Triangle* reference input.

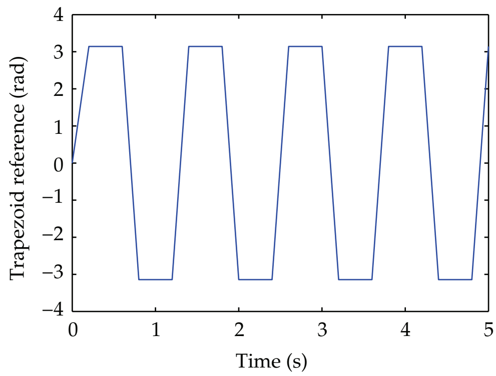


Figure 20: *Trapezoid* reference input.

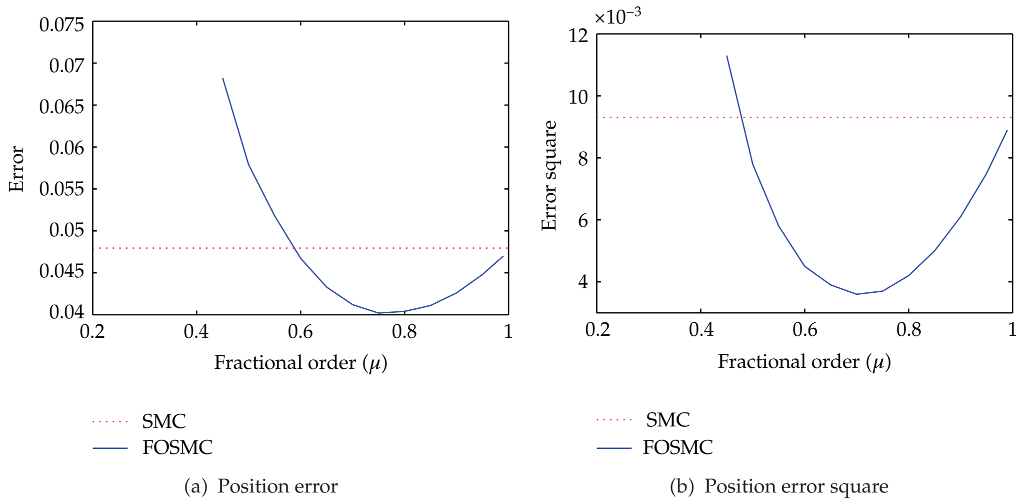


Figure 21: Position error and error square with *sine* reference input (2.3 Nm pulse disturbance, with *saturation* function).

Figure 16(a) shows the control output i_q between $[0\,\mathrm{s}, 1.5\,\mathrm{s}]$, and the particular time period b, c, and d in Figure 16(a) is zoomed in as shown by Figures 16(b), 16(c), and 16(d), respectively. Figure 16(b) shows the control output i_q between $[0\,\mathrm{s}, 0.02\,\mathrm{s}]$, during which the PMSM motor just started; Figure 16(c) shows the control output i_q between $[0.195\,\mathrm{s}, 0.22\,\mathrm{s}]$ in this time period, the position output reaches the desired reference value; Figure 16(d) shows control output i_q between $[0.60\,\mathrm{s}, 0.66\,\mathrm{s}]$, in which the external disturbance load is withdrawn. From Figures 16(b), 16(c), and 16(d), it can be seen that the control output i_q of the proposed FOSMC method is more smooth than that of the conventional SMC method; in other words, the system chattering is eliminated to some degree by the FOSMC method.

7.3.4. Simulation of Dynamic Position Response with Sinusoidal Input Signal

In this simulation, the position reference is a sinusoidal trajectory with $\theta_{\mathrm{ref}}(t) = \pi \sin(10t)$rad, the order of FOSMC is $\mu = 0.6$, and the *saturation* function is used. A step disturbance load

Table 1: PMSM specifications.

Features	Values
Rated Voltage	300 V
Maximum Speed (ω_m)	2400 rpm
Number of Poles (P)	4
Phase Resistance (R_s)	2.46 Ω
Winding Inductance (L_s)	4.233 mh
Motor Inertia (J)	1.02×10^{-3} Kg·m^2
Friction Coefficient (B)	1.0×10^{-4} N·m·s·rad^{-1}
Rotor Flux Linkages ψ_f	0.214 Wb
Torque (Te)	5.25 Nm

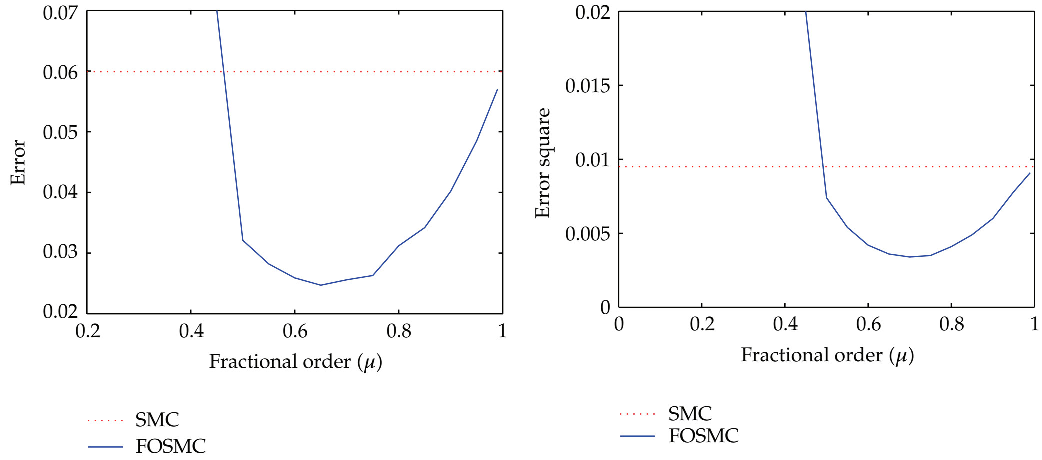
Table 2: Controller Performance.

Controller Type	Δe	$\lvert\Delta e\rvert^2$
SMC	0.8766	0.2899
FOSMC		
$\mu = 0.35$	575.6	90180
$\mu = 0.4$	351.9	34470
$\mu = 0.45$	0.4186	0.0669
$\mu = 0.5$	0.2695	0.0413
$\mu = 0.55$	0.3014	0.0409
$\mu = 0.6$	0.328	0.0463
$\mu = 0.65$	0.363	0.0549
$\mu = 0.7$	0.4057	0.0675
$\mu = 0.75$	0.4575	0.085
$\mu = 0.8$	0.517	0.1078
$\mu = 0.85$	0.5895	0.1384
$\mu = 0.9$	0.6738	0.1781
$\mu = 0.95$	0.7716	0.2292
$\mu = 0.99$	0.8492	0.2745

of 3.1 Nm is applied at $t = 0.3$ s and vanished at $t = 0.35$ s, and another step disturbance load of −3.1 Nm is applied at $t = 0.75$ s and vanished at $t = 0.8$ s. Figures 17(a) and 17(b) show the position responses and position error, respectively. From the results, it is clear that the dynamic tracking error of the proposed FOSMC method is smaller than that of the conventional SMC method.

7.3.5. Controller Performance with Different Fractional Orders

We let the motor angle to track a sinusoidal trajectory $\theta_{\mathrm{ref}}(t) = \pi \sin(10t)$rad, and a pulse load disturbance with 3.1 Nm amplitude, 50% pulse width, and 100 ms period is applied to the PMSM the total running time is 5 seconds. Table 2 shows the controller performance of the conventional SMC method and the proposed FOSMC method with different fractional-order



(a) Position error

(b) Position error square

Figure 22: Position error and error square with *sine* reference input (2.3 Nm pulse disturbance, with *sign* function).

μ, and the *saturation* function is adopted in both methods. In Table 2, the error Δe and error square $|\Delta e|^2$ are defined as follows:

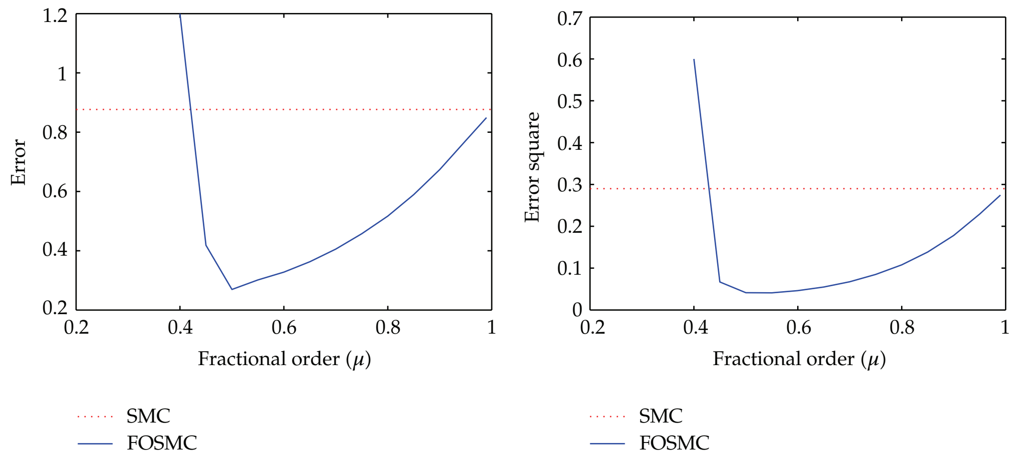
$$\Delta e = \int |\theta_{\text{ref}}(t) - \theta_m(t)| dt,$$
$$|\Delta e|^2 = \int |\theta_{\text{ref}}(t) - \theta_m(t)|^2 dt. \quad (7.2)$$

From Table 2, it can be seen that when the fractional order of the proposed FOSMC method is set to small value, the performance is poor, but once the order is got value between $\mu \in [0.45, 0.95]$, then the position error and error square of the proposed FOSMC method are significantly smaller than those of conventional SMC method, especially when $\mu \in [0.5, 0.6]$. This also means that the control performance of the FOSMC method can be improved by selecting a proper fractional-order μ and designing a corresponding fractional-order sliding surface.

7.3.6. Controller Performance with Different Fractional-Order and Different Reference Input

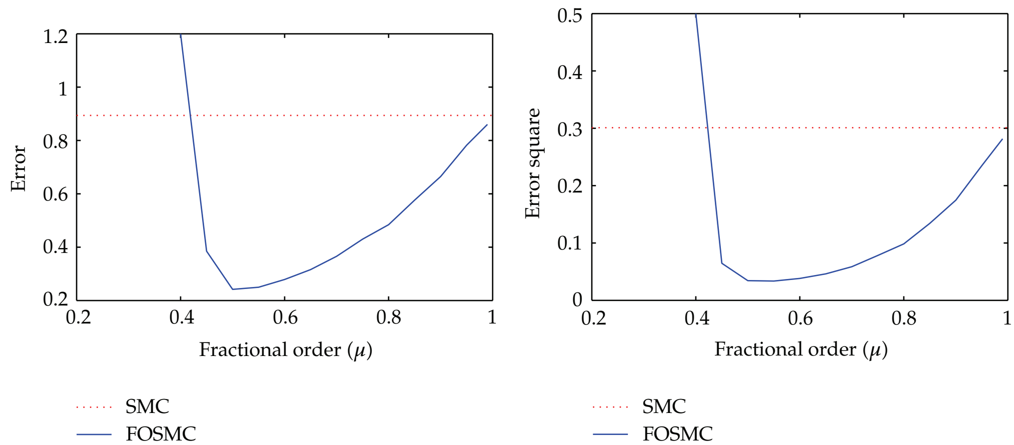
In this simulation, we check the effectiveness of the proposed FOSMC method to another position reference input and find the general regularity between the control performance and the different fractional-order μ. Three position reference inputs, that is, sine wave, triangle wave, and trapezoid wave, are considered and, respectively, shown in Figures 18, 19, and 20. A pulse load disturbance with 50% pulse width, 100 ms period, and alternative amplitude of 3.1 Nm and 2.3 Nm is applied to the PMSM. The total running time is 5 seconds.

In the simulation, for each position reference input, the position error and error square of the proposed FOSMC method and the convention SMC method are regarded as the control



(a) Position error (b) Position error square

Figure 23: Position error and error square with *sine* reference input (3.1 Nm pulse disturbance, with *saturation* function).

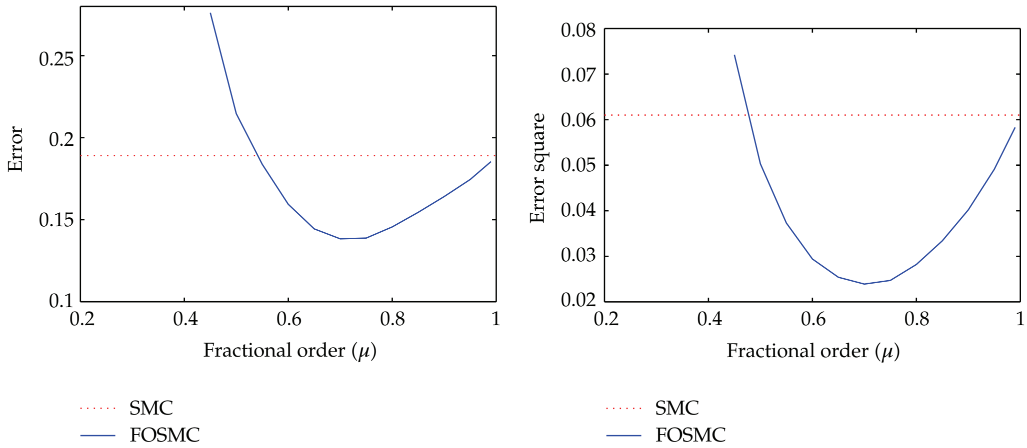


(a) Position error (b) Position error square

Figure 24: Position error and error square with *sine* reference input (3.1 Nm pulse disturbance, with *sign* function).

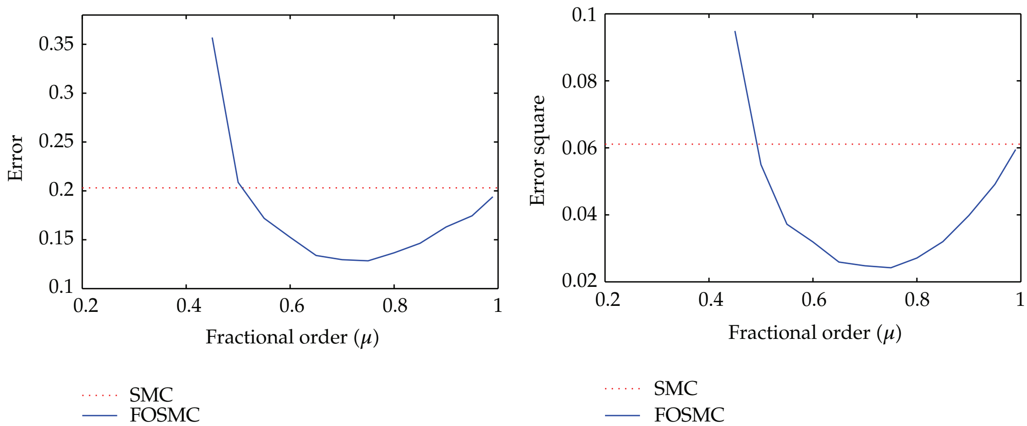
performance. The amplitude of the pulse load disturbance is set as 2.3 Nm and 3.1 Nm, respectively, which means that the system is stable under the load disturbance of 2.3 Nm and unstable under the load disturbance of 3.1 Nm. Moreover, under the two kinds of load amplitude, the *saturation* function and *sign* function are considered, respectively.

For comparison convenience, in Figures 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, and 32, the red dot line represents the error or error square obtained by the conventional SMC method, and it has no relationship with the fractional-order μ, while the green solid line is the error and error square got by the proposed FOSMC method with different fractional-order μ.



(a) Position error (b) Position error square

Figure 25: Position error and error square with *triangle* reference input (2.3 Nm pulse disturbance, with *saturation* function).

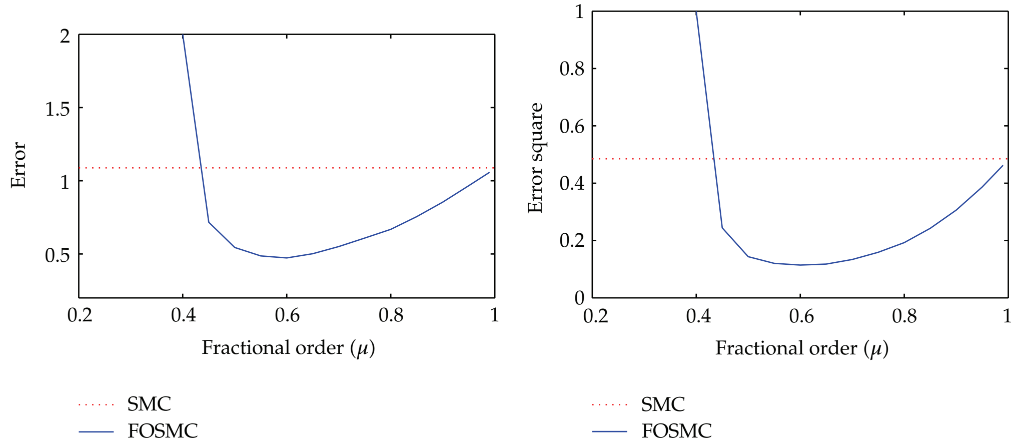


(a) Position error (b) Position error square

Figure 26: Position error and error square with *triangle* reference input (2.3 Nm pulse disturbance, with *sign* function).

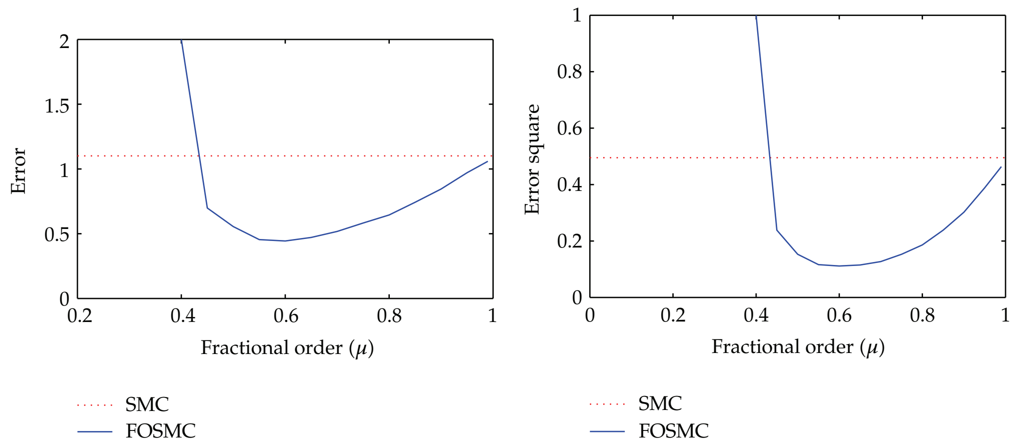
From Figures 21–32, it is clear that when $\mu \in (0,\ \mu^*)$, then the error or error square of the FOSMC method is bigger than that of the conventional SMC method, but when $\mu \in [\mu^*,\ 1)$, the error or error square of the FOSMC method is significantly smaller than that of the conventional SMC method. Moreover, from Figures 21–32, one can see that the value of μ^* is around 0.5; this also means that when the fractional-order $\mu \in [0.5,\ 1)$, then the proposed FOSMC method outperforms the conventional SMC method.

In fact, from the simulation results shown from Figures 21–32, it can be seen that the best selection range for μ is $[0.5,\ 0.6]$.



(a) Position error (b) Position error square

Figure 27: Position error and error square with *triangle* reference input (3.1 Nm pulse disturbance, with *saturation* function).



(a) Position error (b) Position error square

Figure 28: Position error and error square with *triangle* reference input (3.1 Nm pulse disturbance, with *sign* function).

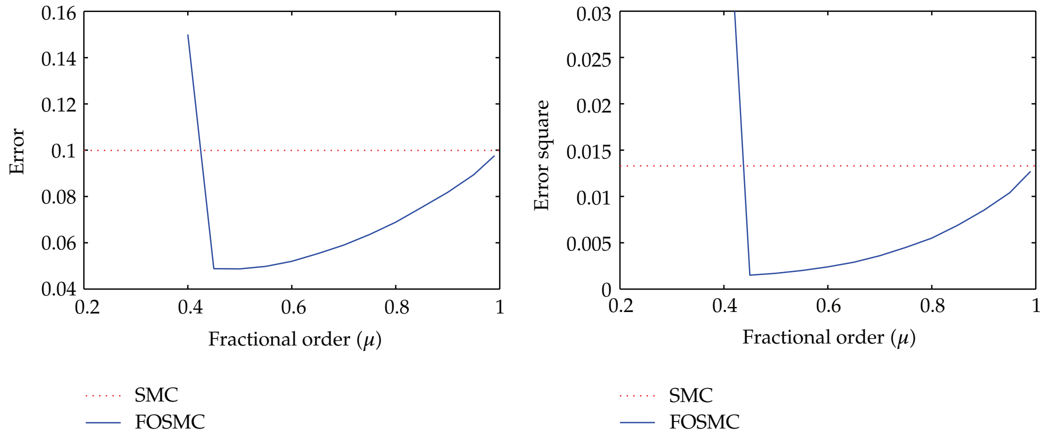
8. Guidance for Parameters Selection and Design of FOSMC

In the proposed FOSMC method, there are five parameters, that is, $k_p, k_d, k, \varepsilon,$ and μ, which need to be designed. From the above analyses and numerical simulation results, one can select and design the five parameters through the following procedures:

(i) select a value in the range $[0.5, \ 0.6]$ or $[0.5, 1)$ for the fractional-order μ;

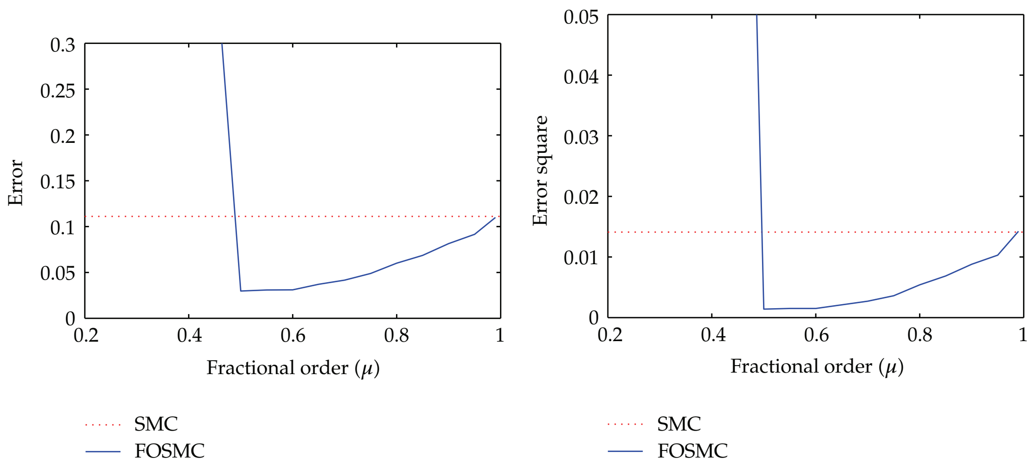
(ii) estimate the maximum load disturbance $T_{L\text{-max}}$;

(iii) according to the $T_{L\text{-max}}$ and (5.21), compute the value range of parameter k and then select a suitable value for k;



(a) Position error (b) Position error square

Figure 29: Position error and error square with *trapezoid* reference input (2.3 Nm pulse disturbance, with *saturation* function).

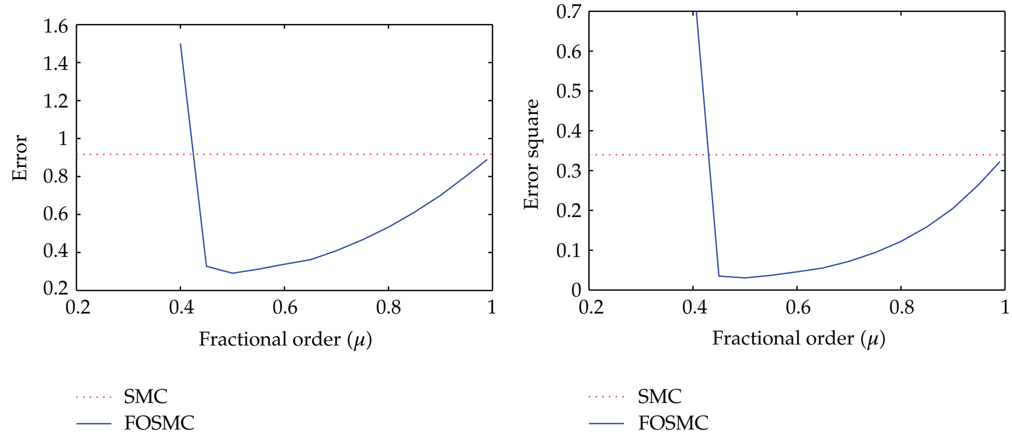


(a) Position error (b) Position error square

Figure 30: Position error and error square with *trapezoid* reference input (2.3 Nm pulse disturbance, with *sign* function).

(iv) because k_d is the coefficient of fractional-order differentiation of position error x_1, if the value of k_d is big, it will be too sensitive to the variation of position error x_1 and cause oscillation. In general, k_d can be set as a suitable small value, for example, in this paper $k_d = 1$;

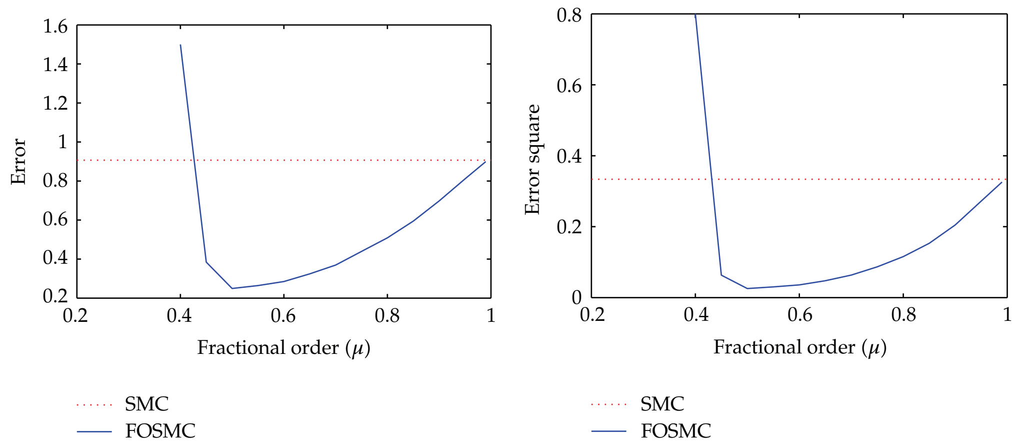
(v) with the maximum permissible position error x_1 of the PMSM system and the parameter k designed by the procedure (iii), then parameters ε and k_p can be designed and selected according to (5.28).



(a) Position error

(b) Position error square

Figure 31: Position error and error square with *trapezoid* reference input (3.1 Nm pulse disturbance, with *saturation* function).



(a) Position error

(b) Position error square

Figure 32: Position error and error square with *trapezoid* reference input (3.1 Nm pulse disturbance, with *sign* function).

9. Conclusions

A new and systematic design of the fractional-order sliding mode controller (FOSMC) for PMSM position control system is presented. By selecting a proper fractional-order μ and designing a fractional-order sliding surface, the control performance such as control precision and system robustness of the proposed FOSMC method is distinctly more excellent than that of the conventional SMC method, because an extra fractional order, the real parameters μ, is involved. The robustness of the proposed FOSMC method is analyzed in detail, and the guidance for parameters selection and design is given. The numerical simulation results demonstrate the effectiveness and robustness of the proposed FOSMC method.

Acknowledgments

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Research Article

On a Class of Abstract Time-Fractional Equations on Locally Convex Spaces

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This paper is devoted to the study of abstract time-fractional equations of the following form: $\mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha_i}u(t)=A\mathbf{D}_t^{\alpha}u(t)+f(t)$, $t>0$, $u^{(k)}(0)=u_k$, $k=0,\ldots,\lceil\alpha_n\rceil-1$, where $n\in\mathbb{N}\setminus\{1\}$, A and $A_1,\ldots,A_{n-1}$ are closed linear operators on a sequentially complete locally convex space E, $0\leq\alpha_1<\cdots<\alpha_n$, $0\leq\alpha<\alpha_n$, $f(t)$ is an E-valued function, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α (Bazhlekova (2001)). We introduce and systematically analyze various classes of k-regularized (C_1,C_2)-existence and uniqueness (propagation) families, continuing in such a way the researches raised in (de Laubenfels (1999, 1991), Kostić (Preprint), and Xiao and Liang (2003, 2002). The obtained results are illustrated with several examples.

1. Introduction and Preliminaries

A great number of abstract time-fractional equations appearing in engineering, mathematical physics, and chemistry can be modeled through the abstract Cauchy problem

$$\begin{gathered}\mathbf{D}_t^{\alpha n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha i}u(t)=A\mathbf{D}_t^{\alpha}u(t)+f(t),\quad t>0,\\ u^{(k)}(0)=u_k,\quad k=0,\ldots,\lceil\alpha_n\rceil-1.\end{gathered}\tag{1.1}$$

For further information about the applications of fractional calculus, the interested reader may consult the monographs by Baleanu et al. [1], Klafter et al. (Eds.) [2], Kilbas et al. [3], Mainardi [4], Podlubny [5], and Samko et al. [6]; we also refer to the references [7–19].

The aim of this paper is to develop some operator theoretical methods for solving the abstract time-fractional equations of the form (1.1). We start by quoting some special cases. The study of qualitative properties of the abstract Basset-Boussinesq-Oseen equation:

$$u'(t) - A\mathbf{D}_t^{\alpha}u(t) + u(t) = f(t), \quad t \geq 0, \ u(0) = 0 \ \ (\alpha \in (0,1)), \tag{1.2}$$

describing the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, has been initiated by Lizama and Prado in [17]. For further results concerning the C-wellposedness of (1.2), [20, 21] are of importance. In [12], Karczewska and Lizama have recently analyzed the following stochastic fractional oscillation equation:

$$u(t) + \int_0^t (t-s)[A\mathbf{D}_s^{\alpha}u(s) + u(s)]ds = W(t), \quad t > 0, \tag{1.3}$$

where $1 < \alpha < 2$, A is the generator of a bounded analytic C_0-semigroup on a Hilbert space H and $W(t)$ denotes an H-valued Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, P)$. The theory of (a,k)-regularized resolvent families (cf. [12, Theorems 3.1 and 3.2]) can be applied in the study of deterministic counterpart of (1.3) in integrated form:

$$u(t) + \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}Au(s)ds + \int_0^t (t-s)u(s)ds = \int_0^t (t-s)f(s)ds, \quad t > 0, \tag{1.4}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $f \in L^1_{\mathrm{loc}}([0,\infty) : E)$. Equation (1.4) generalizes the so-called Bagley-Torvik equation, which can be obtained by plugging $\alpha = 3/2$ in (1.4), and models an oscillation process with fractional damping term (cf. [21] for the analysis of C-wellposedness and perturbation properties of (1.4)). After differentiation, (1.4) becomes, in some sense,

$$u''(t) + A\mathbf{D}_t^{\alpha}u(t) + u(t) = f(t), \quad t \geq 0; \ u(0) = u'(0) = 0. \tag{1.5}$$

Notice also that the periodic solutions for the equation

$$D^{\alpha}u(t) + BD^{\beta}u(t) + Au(t) = f(t), \quad t \in [0,2\pi], \tag{1.6}$$

where A and B are closed linear operators defined on a complex Banach space X, $0 \leq \beta < \alpha \leq 2$, $f \in C([0,2\pi] : X)$ and D^{α} denotes the Liouville-Grünwald fractional derivative of order α, have been studied by Keyantuo and Lizama in [13]. Observe also that Diethelm analyzed in [22, Chapter 8] scalar-valued multiterm Caputo fractional differential equations. Consider, for illustration purposes, the following abstract time-fractional equation:

$$\mathbf{D}_t^{\alpha}u(t) + \mathbf{D}_t^{\beta}u(t) = au(t), \quad t > 0; \ u(0) = u_0, \ u'(0) = 0, \tag{1.7}$$

where $1 < \alpha < 2$, $0 < \beta < \alpha$ and $A = a$ is a certain complex constant. Applying the Laplace transform (see, e.g., [10, (1.23)]), we get:

$$\left(\lambda^{\alpha} + \lambda^{\beta}\right)\tilde{u}(\lambda) - \left(\lambda^{\alpha-1} + \lambda^{\beta-1}\right)u_0 = a\tilde{u}(\lambda). \tag{1.8}$$

Therefore,

$$\tilde{u}(\lambda) = \frac{\lambda^{\alpha-1} + \lambda^{\beta-1}}{\lambda^{\alpha} + \lambda^{\beta} - a}u_0. \tag{1.9}$$

By (24) and (26) in [19], it readily follows that:

$$u(t) = \sum_{n=0}^{\infty}(-1)^n t^{(\alpha-\beta)n}\left[E^{n+1}_{\alpha,(\alpha-\beta)n+1}(at^{\alpha}) + t^{\alpha-\beta}E^{n+1}_{\alpha,(\alpha-\beta)(n+1)+1}(at^{\alpha})\right]u_0, \tag{1.10}$$

where

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(n\alpha+\beta)n!} \tag{1.11}$$

is the generalized Mittag-Leffler function. Here $(\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1)$ $(n \in \mathbb{N})$ and $(\gamma)_0 = 1$. The formula (1.10) shows that it is quite complicated to apply Fourier multiplier theorems to the abstract time-fractional equations of the form (1.1); for some basic references in this direction, the reader may consult [16, 23]. Before going any further, we would also like to observe that Atanacković et al. considered in [8], among many other authors, the following fractional generalization of the telegraph equation:

$$\tau\mathbf{D}^{\alpha}_t u(t) + \mathbf{D}^{\beta}_t u(t) = Du_{xx}, \quad x \in (0,l),\ t > 0, \tag{1.12}$$

where $0 < \beta \leq \alpha \leq 2$, $\tau > 0$ and $D > 0$. In that paper, solutions to signalling and Cauchy problems in terms of a series and integral representation are given.

In the second section, we continue the analysis from our recent paper [15], where it has been assumed that $A_j = c_j I$ for some complex constants $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$; here, and in the sequel of the second section, I denotes the identity operator on E. We introduce and clarify the basic structural properties of various types of k-regularized (C_1, C_2)-existence and uniqueness propagation families. This is probably the best concept for the investigation of integral solutions of the abstract time-fractional equation (1.1) with $A_j \in L(E)$, $1 \leq j \leq n-1$. If there exists an index $j \in \mathbb{N}_{n-1}$ such that $A_j \notin L(E)$, then the vector-valued Laplace transform cannot be so easily applied (cf. Theorems 2.10–2.11), which implies, however, that there exist some limitations to the introduced classes of propagation families. The notion of a strong solution of (1.1) is introduced in Definition 2.1, and the notions of strong and mild solutions of inhomogeneous equations of the form (2.15) below are introduced in Definition 2.7. The generalized variation of parameters formula is proved in Theorem 2.8.

On the other hand, the notions of C_1-existence families and C_2-uniqueness families for the higher order abstract Cauchy problem (ACP_n) were introduced by Xiao and Liang in

[24, Definition 2.1]. In the third section, we will introduce more general classes of (local) k-regularized C_1-existence families for (1.1), k-regularized C_2-uniqueness families for (1.1), and k-regularized C-resolvent families for (1.1). Our intention in this section is to transfer results of [24] to abstract time-fractional equations. In addition, various adjoint type theorems for k-regularized C-resolvent families are considered in Theorem 3.6.

Throughout this paper, we will always assume that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short, and that the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E; in this place, we would like to mention in passing that the locally convex spaces are very important to describe a set of mixed states in quantum theory [2]. The completeness of E, if needed, will be explicitly emphasized. By $L(E)$ is denoted the space of all continuous linear mappings from E into E. Let $\mathcal{B}$ be the family of bounded subsets of E and let $p_B(T) := \sup_{x\in B} p(Tx)$, $p \in \circledast$, $B \in \mathcal{B}$, $T \in L(E)$. Then $p_B(\cdot)$ is a seminorm on $L(E)$ and the system $(p_B)_{(p,B)\in\circledast\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E)$. Recall that $L(E)$ is sequentially complete provided that E is barreled. Henceforth A is a closed linear operator acting on E, $L(E) \ni C$ is an injective operator, and the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)ds$. The domain, resolvent set and range of A are denoted by $D(A)$, $\rho(A)$ and $R(A)$, respectively. Since it makes no misunderstanding, we will identify A with its graph. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \left\{\lambda \in \mathbb{C};\ \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\right\}. \tag{1.13}$$

Suppose F is a linear subspace of E. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax,\ x \in D(A_{|F})$.

Define $E_p := E/p^{-1}(0)$ $(p \in \circledast)$. Then the norm of a class $x + p^{-1}(0)$ is defined by $\|x + p^{-1}(0)\|_{E_p} := p(x)$ $(x \in E)$. The canonical mapping $\Psi_p : E \to E_p$ is continuous and the completion of E_p under the norm $\|\cdot\|_{E_p}$ is denoted by $\overline{E_p}$. Since no confusion seems likely, we will also denote the norms on E_p and $L(E_p)$ ($\overline{E_p}$ and $L(\overline{E_p})$) by $\|\cdot\|$; $L_{\circledast}(E)$ denotes the subspace of $L(E)$ which consists of those bounded linear operators T on E such that, for every $p \in \circledast$, there exists $c_p > 0$ satisfying $p(Tx) \le c_p p(x)$, $x \in E$. If $T \in L_{\circledast}(E)$ and $p \in \circledast$, then the operator $T_p : E_p \to E_p$, defined by $T_p(\Psi_p(x)) := \Psi_p(Tx)$, $x \in E$, belongs to $L(E_p)$. This operator is uniquely extensible to a bounded linear operator $\overline{T_p}$ on $\overline{E_p}$, and the following holds: $\|T_p\| = \|\overline{T_p}\|$. The function $\pi_{qp} : E_p \to E_q$, defined by $\pi_{qp}(\Psi_p(x)) := \Psi_q(x)$, $x \in E$, is a continuous homomorphism of E_p onto E_q, and extends therefore, to a continuous linear homomorphism π_{qp} of $\overline{E_p}$ onto $\overline{E_q}$. The reader may consult [25] for the basic facts about projective limits of Banach spaces (closed linear operators acting on Banach spaces) and their projective limits. Recall, a closed linear operator A acting on E is said to be compartmentalized (w.r.t. $\circledast$) if, for every $p \in \circledast$, $A_p := \{(\Psi_p(x), \Psi_p(Ax)) : x \in D(A)\}$ is a function. Therefore, $T \in L_{\circledast}(E)$ is a compartmentalized operator.

Given $s \in \mathbb{R}$ in advance, set $\lfloor s\rfloor := \sup\{l \in \mathbb{Z} : s \ge l\}$ and $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \le l\}$. The principal branch is always used to take the powers. Set $\mathbb{N}_l := \{1, \ldots, l\}$, $\mathbb{N}_l^0 := \{0, 1, \ldots, l\}$, $0^{\zeta} := 0$, $g_{\zeta}(t) := t^{\zeta-1}/\Gamma(\zeta)$ $(\zeta > 0, t > 0)$ and $g_0 :=$ the Dirac δ-distribution. If $\gamma \in (0, \pi]$, then we define $\Sigma_{\gamma} := \{\lambda \in \mathbb{C} : \lambda \ne 0, |\arg(\lambda)| < \gamma\}$. We refer the reader to [26] and references cited there for the basic material concerning integration in sequentially complete locally convex spaces and vector-valued analytic functions.

Let $\alpha > 0$, let $\beta \in \mathbb{R}$, and let the Mittag-Leffler function $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n+\beta)$, $z \in \mathbb{C}$. In this place, we assume that $1/\Gamma(\alpha n+\beta) = 0$ if $\alpha n+\beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. The Wright function $\Phi_\gamma(t)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geq 0$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. For further information concerning Mittag-Leffler and Wright functions, we refer the reader to [10, Section 1.3].

The following definition has been recently introduced in [27].

Definition 1.1. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$ and A is a closed linear operator on E.

(i) Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) (a,k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)} \subseteq L(E) \times L(E)$ if and only if the mapping $t \mapsto (R_1(t)x, R_2(t)x)$, $t \in [0,\tau)$ is continuous for every fixed $x \in E$ and if the following conditions hold:

(a) $R_i(0) = k(0)C_i$, $i = 1, 2$,

(b) C_2 is injective,

(c)

$$A\int_0^t a(t-s)R_1(s)x\,ds = R_1(t)x - k(t)C_1x, \quad t \in [0,\tau),\ x \in E, \tag{1.14}$$

$$\int_0^t a(t-s)R_2(s)Ax\,ds = R_2(t)x - k(t)C_2x, \quad t \in [0,\tau),\ x \in D(A). \tag{1.15}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(E)$ be strongly continuous. Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ if and only if $R_1(0) = k(0)C_1$ and (1.14) holds.

(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(E)$ be strongly continuous. Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ if and only if $R_2(0) = k(0)C_2$, C_2 is injective and (1.15) holds.

It will be convenient to remind us of the following definitions from [14, 20, 26].

Definition 1.2. (i) Let $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$ and let $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$. A strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) (a,k)-regularized C-resolvent family having A as a subgenerator if and only if the following holds:

(a) $R(t)A \subseteq AR(t)$, $t \in [0,\tau)$, $R(0) = k(0)C$ and $CA \subseteq AC$,

(b) $R(t)C = CR(t)$, $t \in [0,\tau)$,

(c) $R(t)x = k(t)Cx + \int_0^t a(t-s)AR(s)x\,ds$, $t \in [0,\tau)$, $x \in D(A)$,

$(R(t))_{t\in[0,\tau)}$ is said to be nondegenerate if the condition $R(t)x = 0$, $t \in [0,\tau)$ implies $x = 0$, and $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous if, for every $t \in (0,\tau)$, the family $\{R(s) : s \in [0,t]\}$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous.

(ii) Let $\beta \in (0, \pi]$ and let $(R(t))_{t\geq 0}$ be an (a,k)-regularized C-resolvent family. Then it is said that $(R(t))_{t\geq 0}$ is an analytic (a,k)-regularized C-resolvent family of angle β, if there exists a function $\mathbf{R} : \Sigma_\beta \to L(E)$ satisfying that, for every $x \in E$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_\beta$ is analytic as well as that

(a) $\mathbf{R}(t) = R(t)$, $t > 0$ and

(b) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{R}(z)x = k(0)Cx$ for all $\gamma \in (0,\beta)$ and $x \in E$,

$(R(t))_{t\geq 0}$ is said to be an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family, respectively, equicontinuous analytic (a,k)-regularized C-resolvent family of angle β, if for every $\gamma \in (0,\beta)$, there exists $\omega_\gamma \geq 0$, respectively, $\omega_\gamma = 0$, such that the set $\{e^{-\omega_\gamma |z|}\mathbf{R}(z) : z \in \Sigma_\gamma\}$ is equicontinuous. Since there is no risk for confusion, we will identify in the sequel $R(\cdot)$ and $\mathbf{R}(\cdot)$.

Definition 1.3. (i) Let $k \in C([0,\infty))$ and $a \in L^1_{\text{loc}}([0,\infty))$. Suppose that $(R(t))_{t\geq 0}$ is a global (a,k)-regularized C-resolvent family having A as a subgenerator. Then it is said that $(R(t))_{t\geq 0}$ is a quasi-exponentially equicontinuous (q-exponentially equicontinuous, for short) (a,k)-regularized C-resolvent family having A as subgenerator if and only if, for every $p \in \circledast$, there exist $M_p \geq 1$, $\omega_p \geq 0$ and $q_p \in \circledast$ such that:

$$p(R(t)x) \leq M_p e^{\omega_p t} q_p(x), \quad t \geq 0, \ x \in E. \tag{1.16}$$

(ii) Let $\beta \in (0,\pi]$, and let A be a subgenerator of an analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle β. Then it is said that $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family of angle β, if for every $p \in \circledast$ and $\varepsilon \in (0,\beta)$, there exist $M_{p,\varepsilon} \geq 1$, $\omega_{p,\varepsilon} \geq 0$ and $q_{p,\varepsilon} \in \circledast$ such that

$$p(R(z)x) \leq M_{p,\varepsilon} e^{\omega_{p,\varepsilon}|z|} q_{p,\varepsilon}(x), \quad z \in \Sigma_{\beta-\varepsilon}, \ x \in E. \tag{1.17}$$

For a global (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\geq 0}$ having A as subgenerator, it is said that is locally equicontinuous (exponentially equicontinuous, (q-)exponentially equicontinuous, analytic, (q-)exponentially analytic,...) if and only if both $(R_1(t))_{t\geq 0}$ and $(R_2(t))_{t\geq 0}$ are.

The reader may consult [26, Theorems 2.7 and 2.8] for the basic Hille-Yosida type theorems for exponentially equicontinuous (a,k)-regularized C-resolvent families. The characterizations of exponentially equicontinuous, analytic (a,k)-regularized C-resolvent families in terms of spectral properties of their subgenerators are given in [26, Theorems 3.6 and 3.7]. For further information concerning q-exponentially equicontinuous (a,k)-regularized C-resolvent families, we refer the reader to [20, 25].

Henceforth, we assume that $k, k_1, k_2, \ldots$ are scalar-valued kernels and that $a \neq 0$ in $L^1_{\text{loc}}([0,\tau))$. All considered operator families will be nondegenerate.

The following conditions will be used in the sequel:

(H1) A is densely defined and $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous.

(H2) $\rho(A) \neq \emptyset$.

(H3) $\rho_C(A) \neq \emptyset$, $\overline{R(C)} = E$ and $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous.

(H3)′ $\rho_C(A) \neq \emptyset$ and $C^{-1}AC = A$.

(H4) A is densely defined and $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous, or $\rho_C(A) \neq \emptyset$.

(H5) (H1) $\vee$ (H2) $\vee$ (H3) $\vee$ (H3)$'$.

(P1) $k(t)$ is Laplace transformable, that is, it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{k}(\lambda) = \mathcal{L}(k)(\lambda) := \lim_{b\to\infty}\int_0^b e^{-\lambda t}k(t)dt := \int_0^\infty e^{-\lambda t}k(t)dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $\operatorname{abs}(k) := \inf\{\Re\lambda : \tilde{k}(\lambda) \text{ exists}\}$.

2. The Main Structural Properties of k-Regularized (C_1, C_2)-Existence and Uniqueness Propagation Families

In this section, we will always assume that E is a SCLCS, A and $A_1, \ldots, A_{n-1}$ are closed linear operators acting on E, $n \in \mathbb{N} \setminus \{1\}$, $0 \leq \alpha_1 < \cdots < \alpha_n$ and $0 \leq \alpha < \alpha_n$. Our intention is to clarify the most important results concerning the C-wellposedness of (1.1). Set $m_j := \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $m := m_0 := \lceil \alpha \rceil$, $A_0 := A$ and $\alpha_0 := \alpha$.

Definition 2.1. A function $u \in C^{m_n-1}([0,\infty) : E)$ is called a (strong) solution of (1.1) if and only if $A_i \mathbf{D}_t^{\alpha_i} u \in C([0,\infty) : E)$ for $0 \leq i \leq n-1$, $g_{m_n-\alpha_n} * (u - \sum_{k=0}^{m_n-1} u_k g_{k+1}) \in C^{m_n}([0,\infty) : E)$ and (1.1) holds. The abstract Cauchy problem (1.1) is said to be (strongly) C-wellposed if:

(i) for every $u_0, \ldots, u_{m_n-1} \in \bigcap_{0\leq j\leq n-1} C(D(A_j))$, there exists a unique solution $u(t; u_0, \ldots, u_{m_n-1})$ of (1.1);

(ii) for every $T > 0$ and $q \in \circledast$, there exist $c > 0$ and $r \in \circledast$ such that, for every $u_0, \ldots, u_{m_n-1} \in \bigcap_{0\leq j\leq n-1} C(D(A_j))$, the following holds:

$$q(u(t; u_0, \ldots, u_{m_n-1})) \leq c \sum_{k=0}^{m_n-1} r\left(C^{-1}u_k\right), \quad t \in [0,T]. \tag{2.1}$$

In the case of abstract Cauchy problem (ACP$_n$), the definition of C-wellposedness introduced above is slightly different from the corresponding definition introduced by Xiao and Liang [28, Definition 5.2, page 116] in the Banach space setting (cf. also [28, Definition 1.2, page 46] for the case $C = I$). Recall that the notion of a strong C-propagation family is important in the study of existence and uniqueness of strong solutions of the abstract Cauchy problem (ACP$_n$); compare [28, Section 3.5, pages 115–130] for further information in this direction. Suppose now that $u(t) \equiv u(t; u_0, \ldots, u_{m_n-1})$, $t \geq 0$ is a strong solution of (1.1), with $f(t) \equiv 0$ and initial values $u_0, \ldots, u_{m_n-1} \in R(C)$. Convoluting both sides of (1.1) with $g_{\alpha_n}(t)$, and making use of the equality [10, (1.21)], it readily follows that $u(t)$, $t \geq 0$ satisfies the following:

$$\begin{aligned} u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j \left[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot)\right] \\ = g_{\alpha_n-\alpha} * A\left[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot)\right]. \end{aligned} \tag{2.2}$$

In the sequel of this section, we will primarily consider various types of solutions of the integral equation (2.2).

Given $i \in \mathbb{N}^0_{m_n-1}$ in advance, set $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$. Then it is clear that $D_{m_n-1} \subseteq \cdots \subseteq D_0$. Plugging $u_j = 0$, $0 \leq j \leq m_n - 1$, $j \neq i$, in (2.2), one gets:

$$\begin{aligned}
&\left[u(\cdot;\ 0,\dots,u_i,\dots,0) - u_i g_{i+1}(\cdot)\right] \\
&\quad + \sum_{j \in D_i} g_{\alpha_n-\alpha_j} * A_j\left[u(\cdot;\ 0,\dots,u_i,\dots,0) - u_i g_{i+1}(\cdot)\right] \\
&\quad + \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left[g_{\alpha_n-\alpha_j} * A_j u(\cdot;\ 0,\dots,u_i,\dots,0)\right] \\
&= \begin{cases} g_{\alpha_n-\alpha} * Au(\cdot;\ 0,\dots,u_i,\dots,0), & m-1 < i, \\ g_{\alpha_n-\alpha} * A\left[u(\cdot;\ 0,\dots,u_i,\dots,0) - u_i g_{i+1}(\cdot)\right], & m-1 \geq i, \end{cases}
\end{aligned} \tag{2.3}$$

where u_i appears in the ith place ($0 \leq i \leq m_n - 1$) starting from 0. Suppose now $0 < \tau \leq \infty$, $0 \neq K \in L^1_{\mathrm{loc}}([0,\tau))$ and $k(t) = \int_0^t K(s)ds$, $t \in [0,\tau)$. Denote $R_i(t)C^{-1}u_i = (K * u(\cdot;\ 0,\dots, u_i,\dots,0))(t)$, $t \in [0,\tau)$, $0 \leq i \leq m_n - 1$. Convoluting formally both sides of (2.3) with $K(t)$, $t \in [0,\tau)$, one obtains that, for $0 \leq i \leq m_n - 1$:

$$\begin{aligned}
&\left[R_i(\cdot)C^{-1}u_i - (k * g_i)(\cdot)u_i\right] + \sum_{j \in D_i} g_{\alpha_n-\alpha_j} * A_j\left[R_i(\cdot)C^{-1}u_i - (k * g_i)(\cdot)u_i\right] \\
&\quad + \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left[g_{\alpha_n-\alpha_j} * A_j R_i(\cdot)C^{-1}u_i\right] \\
&= \begin{cases} \left(g_{\alpha_n-\alpha} * AR_i\right)(\cdot)C^{-1}u_i, & m-1 < i, \\ g_{\alpha_n-\alpha} * A\left[R_i(\cdot)C^{-1}u_i - (k * g_i)(\cdot)u_i\right], & m-1 \geq i. \end{cases}
\end{aligned} \tag{2.4}$$

Motivated by the above analysis, we introduce the following definition.

Definition 2.2. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $C, C_1, C_2 \in L(E)$, C and C_2 are injective. A sequence $((R_0(t))_{t \in [0,\tau)},\dots,(R_{m_n-1}(t))_{t \in [0,\tau)})$ of strongly continuous operator families in $L(E)$ is called a (local, if $\tau < \infty$):

(i) k-regularized C_1-existence propagation family for (1.1) if and only if $R_i(0) = (k * g_i)(0)C_1$ and the following holds:

$$\begin{aligned}
&\left[R_i(\cdot)x - (k * g_i)(\cdot)C_1x\right] + \sum_{j \in D_i} A_j\left[g_{\alpha_n-\alpha_j} * \left(R_i(\cdot)x - (k * g_i)(\cdot)C_1x\right)\right] \\
&\quad + \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} A_j\left(g_{\alpha_n-\alpha_j} * R_i\right)(\cdot)x \\
&= \begin{cases} A\left(g_{\alpha_n-\alpha} * R_i\right)(\cdot)x, & m-1 < i,\ x \in E, \\ A\left[g_{\alpha_n-\alpha} * \left(R_i(\cdot)x - (k * g_i)(\cdot)C_1x\right)\right](\cdot), & m-1 \geq i,\ x \in E, \end{cases}
\end{aligned} \tag{2.5}$$

for any $i = 0,\dots,m_n - 1$.

(ii) k-regularized C_2-uniqueness propagation family for (1.1) if and only if $R_i(0) = (k * g_i)(0)C_2$ and

$$
\begin{aligned}
&\left[R_i(\cdot)x - (k * g_i)(\cdot)C_2x\right] + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \left[R_i(\cdot)A_jx - (k * g_i)(\cdot)C_2A_jx\right] \\
&\quad + \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} \left(g_{\alpha_n-\alpha_j} * R_i(\cdot)A_jx\right)(\cdot) \\
&= \begin{cases} \left(g_{\alpha_n-\alpha} * R_i(\cdot)Ax\right)(\cdot), & m-1 < i, \\ g_{\alpha_n-\alpha} * \left[R_i(\cdot)Ax - (k * g_i)(\cdot)C_2Ax\right](\cdot), & m-1 \geq i, \end{cases}
\end{aligned}
\tag{2.6}
$$

for any $x \in \bigcap_{0\leq j\leq n-1} D(A_j)$ and $i \in \mathbb{N}^0_{m_n-1}$.

(iii) k-regularized C-resolvent propagation family for (1.1), in short k-regularized C-propagation family for (1.1), if $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C-uniqueness propagation family for (1.1), and if for every $t \in [0,\tau)$, $i \in \mathbb{N}^0_{m_n-1}$ and $j \in \mathbb{N}^0_{n-1}$, one has $R_i(t)A_j \subseteq A_jR_i(t)$, $R_i(t)C = CR_i(t)$ and $CA_j \subseteq A_jC$.

The above classes of propagation families can be defined by purely algebraic equations (cf. [11, 15, 27]). We will not go into further details about this topic here.

As indicated before, we will consider only nondegenerate k-regularized C-resolvent propagation families for (1.1). In case $k(t) = g_{\zeta+1}(t)$, where $\zeta \geq 0$, it is also said that $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a ζ-times integrated C-resolvent propagation family for (1.1); 0-times integrated C-resolvent propagation family for (1.1) is simply called C-resolvent propagation family for (1.1). For a k-regularized (C_1, C_2)-existence and uniqueness family $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$, it is said that is locally equicontinuous (exponentially equicontinuous, (q-)exponentially equicontinuous, analytic, (q-)exponentially analytic,...) if and only if all single operator families $(R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)}$ are. The above terminological agreements and abbreviations can be simply understood for the classes of k-regularized C_1-existence propagation families and k-regularized C_2-uniqueness propagation families. The class of k-regularized (C_1, C_2)-existence and uniqueness propagation families for (1.1) can be also introduced (cf. Definitions 1.1 and 3.1 below).

In case that $A_j = c_jI$, where $c_j \in \mathbb{C}$ for $1 \leq j \leq n-1$, it is also said that the operator A is a subgenerator of $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$. Now we would like to notice the following: if A is a subgenerator of a k-regularized C-resolvent propagation family $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ for (1.1), then, in general, there do not exist $a_i \in L^1_{\mathrm{loc}}([0,\tau))$, $i \in \mathbb{N}^0_{m_n-1}$ and $k_i \in C([0,\tau))$ such that $(R_i(t))_{t\in[0,\tau)}$ is an (a_i, k_i)-regularized C-resolvent family with subgenerator A; the same observation holds for the classes of k-regularized C_1-existence propagation families and k-regularized C_2-uniqueness propagation families. Despite this fact, the structural results for k-regularized C-resolvent propagation families can be derived by using appropriate modifications of the proofs of corresponding results for (a,k)-regularized C-resolvent families. Furthermore, these results can be clarified for any single operator family $(R_i(t))_{t\in[0,\tau)}$ of the tuple $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$.

Let $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ be a k-regularized C-resolvent propagation family with subgenerator A. Then one can simply prove that the validity of condition (H5) implies the following functional equation:

$$\begin{aligned}
&\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]+\sum_{j=1}^{n-1}c_j g_{\alpha_n-\alpha_j}*\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]\\
&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}c_j\left[g_{\alpha_n-\alpha_j+i}*k\right](\cdot)Cx \\
&\quad=\begin{cases}A\left[g_{\alpha_n-\alpha}*R_i\right](\cdot)x, & m-1<i,\quad x\in E,\\ A\left[g_{\alpha_n-\alpha}*\left(R_i(\cdot)x-(k*g_i)(\cdot)Cx\right)\right], & m-1\geq i,\quad x\in E,\end{cases}
\end{aligned}\tag{2.7}$$

for any $i=0,\ldots,m_n-1$. The set consisted of all subgenerators of $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$, denoted by $\chi(R)$, need not to be finite. Notice that the supposition $A\in\chi(R)$ obviously implies $C^{-1}AC\in\chi(R)$. The integral generator $\widehat{A}$ of $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ is defined as the set of all pairs $(x,y)\in E\times E$ such that, for every $i=0,\ldots,m_n-1$ and $t\in[0,\tau)$, the following holds:

$$\begin{aligned}
&\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]+\sum_{j=1}^{n-1}c_j g_{\alpha_n-\alpha_j}*\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]\\
&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}c_j\left[g_{\alpha_n-\alpha_j+i}*k\right](\cdot)Cx \\
&\quad=\begin{cases}\left[g_{\alpha_n-\alpha}*R_i\right](\cdot)y, & m-1<i,\\ g_{\alpha_n-\alpha}*\left[R_i(\cdot)y-(k*g_i)(\cdot)Cy\right], & m-1\geq i.\end{cases}
\end{aligned}\tag{2.8}$$

It is a linear operator on E which extends any subgenerator $A\in\chi(R)$ and satisfies $\widehat{A}=C^{-1}\widehat{A}C$. We have the following.

(i) $R_i(t)(\lambda-A)^{-1}C=(\lambda-A)^{-1}CR_i(t)$, $t\in[0,\tau)$, provided $A\in\chi(R)$, $\lambda\in\rho_C(A)$ and $0\leq i\leq m_n-1$.

(ii) Let $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ be locally equicontinuous. Then:

(a) $\widehat{A}$ is a closed linear operator.

(b) $\widehat{A}\in\chi(R)$, if $R_i(t)R_i(s)=R_i(s)R_i(t)$, $0\leq t,s<\tau$, $i\in\mathbb{N}^0_{m_n-1}$.

(c) $\widehat{A}=C^{-1}AC$, if $A\in\chi(R)$ and (H5) holds. Furthermore, the condition (H5) can be replaced by (2.7).

(iii) Let $\{A,B\}\subseteq\chi(R)$. Then $Ax=Bx$, $x\in D(A)\cap D(B)$, and $A\subseteq B\Leftrightarrow D(A)\subseteq D(B)$. Assume that (2.7) holds for A, and that (2.7) holds for A replaced by B. Then we have the following:

(a) $C^{-1}AC=C^{-1}BC$ and $C(D(A))\subseteq D(B)$.

(b) A and B have the same eigenvalues.

(c) $A\subseteq B\Rightarrow\rho_C(A)\subseteq\rho_C(B)$.

Albeit the similar assertions can be considered in general case, we will omit the corresponding discussion even in the case that $A_j \in L(E)$ for $1 \leq j \leq n-1$.

Proposition 2.3. *Let $i \in \mathbb{N}^0_{m_n-1}$, and let $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ be a locally equicontinuous k-regularized C-resolvent propagation family for* (1.1). *If* (2.5) *holds with $C_1 = C$, then the following holds:*

(i) *the equality*

$$R_i(t)R_i(s) = R_i(s)R_i(t), \quad 0 \leq t,\ s < \tau \tag{2.9}$$

holds provided $m-1 < i$ and the following condition:

($\diamond$) *any of the assumptions $f(t) + \sum_{j\in D_i} A_j(g_{\alpha_n-\alpha_j} * f)(t) = 0$, $t \in [0,\tau)$, or $A(g_{\alpha_n-\alpha} * f)(t) = 0$, for some $f \in C([0,\tau) : E)$, implies $f(t) = 0$, $t \in [0,\tau)$;*

(ii) *the equality* (2.9) *holds provided $m-1 \geq i$, $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$, and the following condition:*

($\diamond\diamond$) *if $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} A_j(g_{\alpha_n-\alpha_j} * f)(t) = 0$, $t \in [0,\tau)$, for some $f \in C([0,\tau) : E)$, then $f(t) = 0$, $t \in [0,\tau)$.*

Proof. Let $x \in E$ and $s \in [0,\tau)$ be fixed. Define $u_i(t) := R_i(t)R_i(s)x - R_i(s)R_i(t)x$, $t \in [0,\tau)$. Using (2.5), it is not difficult to prove that

$$A\int_0^t g_{\alpha_n-\alpha}(t-r)u(r)dr = u(t) + \sum_{j=1}^{n-1}\int_0^t A_j\Big(g_{\alpha_n-\alpha_j} * u\Big)(r)dr = 0, \quad t \in [0,\tau). \tag{2.10}$$

Let $m-1 < i$. Convoluting both sides of (2.10) with $R_i(\cdot)$, we easily infer that $u(t) + \sum_{j=1}^{n-1} A_j(g_{\alpha_n-\alpha_j} * u)(t) = 0$, $t \in [0,\tau)$ and $A(g_{\alpha_n-\alpha} * u)(t) = 0$, $t \in [0,\tau)$. Now the equality (2.9) follows from ($\diamond$). The proof is quite similar in the case $m-1 \geq i$. □

Remark 2.4. The equations (1.1) with $\alpha = 0$ are much easier to deal with, since in this case, $m = 0$ and $m-1 < i$ for all $i \in \mathbb{N}^0_{m_n-1}$. In general, (1.1) with $\alpha > 0$ cannot be reduced to an equivalent equation of the previously considered form.

Proposition 2.5. *Suppose $((R_{j,0}(t))_{t\in[0,\tau)}, \dots, (R_{j,m_n-1}(t))_{t\in[0,\tau)})$ is a locally equicontinuous k_j-regularized C-resolvent propagation family for* (1.1), *$j = 1, 2$, and $0 \leq i \leq m_n - 1$. Then we have the following.*

(i) *If $m-1 < i$ and ($\diamond$) holds, then*

$$(k_1 * R_{2,i})(t)x = (k_2 * R_{1,i})(t)x, \quad x \in \bigcap_{j=0}^{n-1} D(A_j),\ t \in [0,\tau). \tag{2.11}$$

If, additionally,

$$\bigcap_{j=0}^{n-1} D(A_j) \text{ is dense in } E, \tag{2.12}$$

then (2.11) *holds for all* $x \in E$.

(ii) *The equality* (2.11) *holds provided* $m-1 \geq i$, $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$ *and* ($\diamond\diamond$)*; assuming additionally* (2.12), *we have the validity of* (2.11) *for all* $x \in E$.

Proof. We will only prove the second part of proposition. Let $x \in \bigcap_{j=0}^{n-1} D(A_j)$. Then the functional equation of $(R_{j,i}(t))_{t\in[0,\tau)}$ $(j = 1, 2)$ implies:

$$\begin{aligned}
&\left[(k_2 * g_i) * (R_{1,i}(\cdot)x - (k_1 * g_i)(\cdot)Cx)\right](\cdot) \\
&= \Bigg\{ R_{2,i}(\cdot) + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \left[R_{2,i}(\cdot)A_j - (k * g_i)(\cdot)CA_j\right] \\
&\quad + \sum_{j\notin D_i} g_{\alpha_n-\alpha_j} * R_{2,i}(\cdot)A_j - g_{\alpha_n-\alpha} * \left[R_{2,i}(\cdot)A - (k * g_i)(\cdot)CA\right] \Bigg\} \\
&* \left[R_{1,i}(\cdot)x - (k * g_i)(\cdot)Cx\right](\cdot) \\
&= \Bigg\{ R_{2,i}(\cdot) + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \left[R_{2,i}(\cdot)A_j - (k * g_i)(\cdot)CA_j\right] + \sum_{j\notin D_i} g_{\alpha_n-\alpha_j} * R_{2,i}(\cdot)A_j \Bigg\} \\
&* \left[R_{1,i}(\cdot) - (k_1 * g_i)(\cdot)Cx\right](\cdot) \\
&\quad - \left[R_{2,i}(\cdot)x - (k_2 * g_i)(\cdot)C\right] * A\left(g_{\alpha_n-\alpha} * \left[R_{1,i}(\cdot)x - (k_1 * g_i)(\cdot)Cx\right]\right)(\cdot),
\end{aligned} \tag{2.13}$$

which yields after a tedious computation:

$$\sum_{j\notin D_i} g_{\alpha_n-\alpha_j} * A_j[(k_2 * R_{1,i})(\cdot) - (k_1 * R_{2,i})(\cdot)] \equiv 0. \tag{2.14}$$

In view of ($\diamond\diamond$), the above equality shows that $(k_2 * R_{1,i})(t)x = (k_1 * R_{2,i})(t)x$, $t \in [0, \tau)$. It can be simply verified that the condition (2.12) implies that (2.9) holds for all $x \in E$. □

Proposition 2.6. *Let* $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ *be a locally equicontinuous k-regularized* C_1*-existence propagation family (k-regularized* C_2*-unique-ness propagation family, k-regularized C-resolvent propagation family) for* (1.1), *and let* $b \in L^1_{\text{loc}}([0,\tau))$ *be a kernel. Then the tuple*$(((b * R_0)(t))_{t\in[0,\tau)}, \ldots, ((b * R_{m_n-1})(t))_{t\in[0,\tau)})$ *is a locally equicontinuous* $(k * b)$*-regularized* C_1*-existence propagation family* ($(k * b)$*-regularized* C_2*-uniqueness propagation family,* $(k * b)$*-regularized C-resolvent propagation family) for* (1.1).

Suppose now E is complete, (1.1) is C-wellposed, $\bigcap_{j=0}^{n-1} D(A_j)$ is dense in E and $0 \leq i \leq m_n - 1$. Set $R_i(t)x := u(t;\ 0, \ldots, Cx, \ldots, 0)(t)$, $t \geq 0$, $x \in \bigcap_{j=0}^{n-1} D(A_j)$, where $0 \leq i \leq m_n - 1$ and Cx appears in the ith place in the preceding expression. Since we have assumed that E is complete, the operator $R_i(t)$ $(t \geq 0)$ can be uniquely extended (cf. also (ii) of Definition 2.1) to a bounded linear operator on E. It can be easily proved that $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a locally equicontinuous C-uniqueness propagation family for (1.1), and that the assumption $CA_j \subseteq A_jC$, $j \in \mathbb{N}_{n-1}^0$ implies $R_i(t)C = CR_i(t)$, $t \geq 0$. In case that $A_j = c_jI$, where $c_j \in \mathbb{C}$ for $1 \leq j \leq n-1$, one can apply the arguments given in the proof of [29, Proposition 1.1, page 32] in order to see that $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a locally equicontinuous C-resolvent propagation family for (1.1). Regrettably, it is not clear how one can prove in general case that $R_i(t)A_j \subseteq A_jR_i(t)$, $j \in \mathbb{N}_{n-1}^0$, $t \geq 0$.

The following definition also appears in [15].

Definition 2.7. Let $T > 0$ and $f \in C([0,T] : E)$. Consider the following inhomogeneous equation:

$$u(t) + \sum_{j=1}^{n-1}\left(g_{\alpha_n-\alpha_j} * A_ju\right)(t) = f(t) + \left(g_{\alpha_n-\alpha} * Au\right)(t), \quad t \in [0,T]. \tag{2.15}$$

A function $u \in C([0,T] : E)$ is said to be

(i) a strong solution of (2.15) if and only if $A_ju \in C([0,T] : E)$, $j \in \mathbb{N}_{n-1}^0$ and (2.15) holds for every $t \in [0,T]$;

(ii) a mild solution of (2.15) if and only if $(g_{\alpha_n-\alpha_j} * u)(t) \in D(A_j)$, $t \in [0,T]$, $j \in \mathbb{N}_{n-1}^0$ and

$$u(t) + \sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} * u\right)(t) = f(t) + A\left(g_{\alpha_n-\alpha} * u\right)(t), \quad t \in [0,T]. \tag{2.16}$$

It is clear that every strong solution of (2.15) is also a mild solution of the same problem. The converse statement is not true, in general. One can similarly define the notion of a strong (mild) solution of the problem (2.2).

Let $0 < \tau \leq \infty$, and let $T \in (0,\tau)$. Then the following holds:

(a) if $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a C_1-existence propagation family for (1.1), then the function $u(t) = \sum_{i=0}^{m_n-1} R_i(t)x_i$, $t \in [0,T]$, is a mild solution of (2.2) with $u_i = C_1x_i$ for $0 \leq i \leq m_n - 1$;

(b) if $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a C_2-uniqueness propagation family for (1.1), and $A_jR_i(t)x = R_i(t)A_jx$, $t \in [0,T]$, $x \in \bigcap_{j=0}^{n-1} D(A_j)$, $i \in \mathbb{N}_{m_n-1}^0$, $j \in \mathbb{N}_{n-1}^0$, then the function $u(t) = \sum_{i=0}^{m_n-1} R_i(t)C_2^{-1}u_i$, $t \in [0,T]$, is a strong solution of (2.2), provided $u_i \in C_2(\bigcap_{j=0}^{n-1} D(A_j))$ for $0 \leq i \leq m_n - 1$.

Theorem 2.8. *Suppose* $((R_0(t))_{t\in[0,\tau)},\dots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ *is a locally equicontinuous k-regularized C_2-uniqueness propagation family for* (1.1), (2.5) *holds,* $T\in(0,\tau)$ *and* $f\in C([0,T]:E)$. *Then the following holds:*

(i) *if* $m-1<i$, *then any strong solution* $u(t)$ *of* (2.15) *satisfies the equality:*

$$(R_i * f)(t) = (k * g_i * C_2u)(t) + \sum_{j\in D_i}\Big(g_{\alpha_n-\alpha_j+i} * k * C_2A_ju\Big)(t), \tag{2.17}$$

for any $t\in[0,T]$. *Therefore, there is at most one strong (mild) solution for* (2.15), *provided that* $(\diamond)$ *holds,*

(ii) *if* $m-1\geq i$, *then any strong solution* $u(t)$ *of* (2.15) *satisfies the equality:*

$$(R_i * f)(t) = -\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\Big(g_{\alpha_n-\alpha_j+i} * k * C_2A_ju\Big)(t), \quad t\in[0,T]. \tag{2.18}$$

Therefore, there is at most one strong (mild) solution for (2.15), *provided that* $\mathbb{N}_{n-1}\setminus D_i\neq\emptyset$ *and that* $(\diamond\diamond)$ *holds.*

Proof. We will only prove the second part of theorem. Let $m-1\geq i$. Taking into account (2.6), we get:

$$\begin{aligned}
[R_i-(k*g_iC)]*f &= [R_i-(k*g_iC)]*\left\{u+\sum_{j=1}^{n-1}\Big(g_{\alpha_n-\alpha_j}*A_ju\Big)-(g_{\alpha_n-\alpha}*Au)\right\}\\
&= [R_i-(k*g_iC)]*\left(u+\sum_{j=1}^{n-1}\Big(g_{\alpha_n-\alpha_j}*A_ju\Big)\right)\\
&\quad -\left\{[R_i-(k*g_iC)]+\sum_{j\in D_i}\Big[g_{\alpha_n-\alpha_j}*(R_i(\cdot)A_jx-(k*g_i)(\cdot)C_2A_jx)\Big]\right.\\
&\quad \left.+\sum_{j\notin D_i}\Big(g_{\alpha_n-\alpha_j}*R_i(\cdot)A_jx\Big)\right\}*u\\
&= -\sum_{\mathbb{N}_{n-1}\setminus D_i}\Big(g_{\alpha_n-\alpha_j+i}*k*C_2A_ju\Big)(t), \quad t\in[0,T].
\end{aligned} \tag{2.19}$$

This implies the uniqueness of strong solutions to (2.15), provided that $\mathbb{N}_{n-1}\setminus D_i\neq\emptyset$ and that $(\diamond\diamond)$ holds. The uniqueness of mild solutions in the above case follows from the fact that, for every such a solution $u(t)$, there exists a sufficiently large $\zeta>0$ such that the function $(g_\zeta * u)(\cdot)$ is a strong solution of (2.15), with $f(\cdot)$ replaced by $(g_\zeta * f)(\cdot)$ therein. □

If $((R_0(t))_{t\geq0},\dots,(R_{m_n-1}(t))_{t\geq0})$ is a (local) k-regularized C-resolvent propagation family for (1.1), then Theorem 2.8 shows that there exist certain relations between single operator families $(R_0(t))_{t\geq 0},\ldots,$ and $(R_{m_n-1}(t))_{t\geq 0}$ (cf. also [15] and [28, page 116]). It would take too long to analyze such relations in detail.

The subsequent theorems can be shown by modifying the arguments given in the proof of [30, Theorem 2.2.1].

Theorem 2.9. *Suppose $k(t)$ satisfies (P1), $\omega \geq \max(0, \mathrm{abs}(k))$, $(R_i(t))_{t\geq 0}$ is strongly continuous, and the family $\{e^{-\omega t}R_i(t) : t \geq 0\}$ is equicontinuous, provided $0 \leq i \leq m_n - 1$. Let A be a closed linear operator on E, let $C_1, C_2 \in L(E)$, and let C_2 be injective. Set $P_\lambda := \lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j - A$, $\lambda \in \mathbb{C} \setminus \{0\}$.*

(i) *Suppose $A_j \in L(E)$, $j \in \mathbb{N}_{n-1}$. Then $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C_1-existence propagation family for* (1.1) *if and only if the following conditions hold.*

(a) *The equality*

$$P_\lambda \int_0^\infty e^{-\lambda t}R_i(t)x\,dt = \lambda^{\alpha_n-\alpha-i}\tilde{k}(\lambda)C_1x + \sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}\tilde{k}(\lambda)A_jC_1x, \tag{2.20}$$

holds provided $x \in E$, $i \in \mathbb{N}^0_{m_n-1}$, $m-1 < i$ and $\Re\lambda > \omega$.

(b) *The equality*

$$P_\lambda \int_0^\infty e^{-\lambda t}\left[R_i(t)x - (k * g_i)(t)C_1x\right]dt = -\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}\tilde{k}(\lambda)A_jC_1x, \tag{2.21}$$

holds provided $x \in E$, $i \in \mathbb{N}^0_{m_n-1}$, $m-1 \geq i$ and $\Re\lambda > \omega$.

(ii) *Suppose $R_i(0) = (k*g_i)(0)C_2x$, $x \in E\setminus\overline{\bigcap_{0\leq j\leq n-1}D(A_j)}$, $i \in \mathbb{N}^0_{m_n-1}$. Then $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C_2-uniqueness propagation family for* (1.1) *if and only if, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, and for every $x \in \bigcap_{0\leq j\leq n-1}D(A_j)$, the following equality holds:*

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\left[R_i(t)x - (k * g_i)(t)C_2x\right]dt \\
&\quad + \sum_{j\in D_i}\lambda^{\alpha_j-\alpha_n}\int_0^\infty e^{-\lambda t}\left[R_i(t)x - (k * g_i)(t)C_2A_jx\right]dt \\
&\quad + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha_n}\int_0^\infty e^{-\lambda t}R_i(t)A_jx\,dt \\
&= \begin{cases}\lambda^{\alpha-\alpha_n}\int_0^\infty e^{-\lambda t}R_i(t)Ax\,dt, & m-1 < i,\\ \lambda^{\alpha-\alpha_n}\int_0^\infty e^{-\lambda t}\left[R_i(t)Ax - (k * g_i)(t)C_2Ax\right]dt, & m-1 \geq i.\end{cases}
\end{aligned} \tag{2.22}$$

Theorem 2.10. *Suppose $k(t)$ satisfies (P1), $\omega \geq \max(0, \mathrm{abs}(k))$, $(R_i(t))_{t\geq 0}$ is strongly continuous, and the family $\{e^{-\omega t}R_i(t) : t \geq 0\}$ is equicontinuous, provided $0 \leq i \leq m_n - 1$. Let $CA_j \subseteq A_jC$,*

$j \in \mathbb{N}_{n-1}^0$, $A_j \in L(E)$, $j \in \mathbb{N}_{n-1}$, $A_iA_j = A_jA_i$, $i,\ j \in \mathbb{N}_{n-1}$ and $A_jA \subseteq AA_j$, $j \in \mathbb{N}_{n-1}$. Assume, additionally, that the operator $\lambda^{\alpha_n - i} + \sum_{j \in D_i} \lambda^{\alpha_j - i} A_j$ is injective for every $i \in \mathbb{N}_{m_n-1}^0$ with $m-1 < i$ and for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$, and that the operator $\sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \lambda^{\alpha_j - i} A_j$ is injective for every $i \in \mathbb{N}_{m_n-1}^0$ with $m-1 \geq i$ and for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$. Then $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C-resolvent propagation family for (1.1), *and* (2.5) *holds, if and only if the equalities* (2.20)-(2.21) *are fulfilled.*

Keeping in mind Theorem 2.10, one can simply clarify the most important Hille-Yosida type theorems for exponentially equicontinuous k-regularized C-resolvent propagation families (cf. also [15] and [26, Theorem 2.8] for further information in this direction). Notice also that the preceding theorem can be slightly reformulated for k-regularized (C_1, C_2)-existence and uniqueness resolvent propagation families.

The analytical properties of k-regularized C-resolvent propagation families are stated in the following two theorems whose proofs are omitted (cf. [14, Theorems 2.16-2.17] and [26, Lemma 3.3, Theorems 3.4, 3.6, and 3.7]).

Theorem 2.11. *Suppose $\beta \in (0, \pi/2]$, $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is an analytic k-regularized C-resolvent propagation family for* (1.1), *$k(t)$ satisfies (P1),* (2.5) *holds, and $\widetilde{k}(\lambda)$ can be analytically continued to a function $\widehat{k} : \omega + \Sigma_{(\pi/2)+\beta} \to \mathbb{C}$, where $\omega \geq \max(0, \operatorname{abs}(k))$. Suppose $CA_j \subseteq A_jC$, $j \in \mathbb{N}_{n-1}^0$, $A_j \in L(E)$, $j \in \mathbb{N}_{n-1}$, $A_iA_j = A_jA_i$, $i, j \in \mathbb{N}_{n-1}$ and $A_jA \subseteq AA_j$, $j \in \mathbb{N}_{n-1}$. Let the family*

$$\left\{e^{-\omega z} R_i(z) : z \in \Sigma_\gamma\right\} \text{ be equicontinuous, provided } i \in \mathbb{N}_{m_n-1}^0 \text{ and } \gamma \in (0, \beta), \tag{2.23}$$

and let the set

$$\left\{(\lambda - \omega)\widehat{k}(\lambda)\lambda^{-i} : \lambda \in \omega + \Sigma_{(\pi/2)+\gamma}\right\} \tag{2.24}$$

be bounded provided $\gamma \in (0, \beta)$ and $m-1 \geq i$. Set

$$N_i := \left\{\lambda \in \omega + \Sigma_{(\pi/2)+\beta} : \widehat{k}(\lambda)\left(\lambda^{\alpha_n} + \sum_{j \in D_i} \lambda^{\alpha_j} A_j\right) \text{ is injective}\right\}, \tag{2.25}$$

provided $m-1 < i$, and

$$N_i := \left\{\lambda \in \omega + \Sigma_{(\pi/2)+\beta} : \widehat{k}(\lambda)\left(\lambda^{\alpha_n} + \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \lambda^{\alpha_j} A_j\right) \text{ is injective}\right\}, \tag{2.26}$$

provided $m-1 \geq i$. Suppose N_i is an open connected subset of $\mathbb{C}$, and the set $N_i \cap \{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$ has a limit point in $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$, for any $i \in \mathbb{N}_{m_n-1}^0$. Then the operator P_λ is injective for every $\lambda \in N_i$ and $i \in \mathbb{N}_{m_n-1}^0$,

$$\lim_{\lambda \to +\infty, \lambda \in N_i} \lambda \widetilde{k}(\lambda) P_\lambda^{-1}\left(\lambda^{\alpha_n - \alpha - i} + \sum_{j \in D_i} \lambda^{\alpha_j - \alpha - i} A_j\right) Cx = (k * g_i)(0)Cx, \tag{2.27}$$

provided $m-1<i$ *and* $x\in E$, *and*

$$\lim_{\lambda\to+\infty,\lambda\in N_i}\lambda\widetilde{k}(\lambda)P_\lambda^{-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}A_jCx=0, \tag{2.28}$$

provided $m-1\geq i$ *and* $x\in E$. *Suppose, additionally, that there exists* $\mu\in\mathbb{C}$ *such that* $P_\mu^{-1}C\in L(E)$. *Then the family*

$$\begin{aligned}\Bigg\{&(\lambda-\omega)\widehat{k}(\lambda)\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j-C^{-1}AC\right)^{-1}\\&\times\left(\lambda^{\alpha_n-\alpha-i}C+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}A_jC\right):\lambda\in N_i\cap\left(\omega+\Sigma_{(\pi/2)+\gamma}\right)\Bigg\}\text{ is equicontinuous,}\end{aligned} \tag{2.29}$$

provided $m-1<i$ *and* $\gamma\in(0,\beta)$, *respectively, the family*

$$\begin{aligned}\Bigg\{&(\lambda-\omega)\widehat{k}(\lambda)\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j-C^{-1}AC\right)^{-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}A_jC\\&:\lambda\in\ N_i\cap\left(\omega+\Sigma_{(\pi/2)+\gamma}\right)\Bigg\}\ \text{ is equicontinuous,}\end{aligned} \tag{2.30}$$

provided $m-1\geq i$ *and* $\gamma\in(0,\beta)$, *the mapping*

$$\lambda\longmapsto\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j-C^{-1}AC\right)^{-1}\left(\lambda^{\alpha_n-\alpha-i}C+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}A_jC\right)x, \tag{2.31}$$

defined for $\lambda\in N_i$, *is analytic, provided* $m-1<i$ *and* $x\in E$, *and the mapping*

$$\lambda\longmapsto\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j-C^{-1}AC\right)^{-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}A_jCx,\quad\lambda\in N_i, \tag{2.32}$$

is analytic, provided $m-1\geq i$ *and* $x\in E$.

Theorem 2.12. *Assume* $k(t)$ *satisfies (P1),* $\omega\geq\max(0,\mathrm{abs}(k))$, $\beta\in(0,\pi/2]$ *and, for every* $i\in\mathbb{N}^0_{m_n-1}$ *with* $m-1\geq i$, *the function* $(k*g_i)(t)$ *can be analytically extended to a function* $k_i:\Sigma_\beta\to\mathbb{C}$ *satisfying that, for every* $\gamma\in(0,\beta)$, *the set* $\{e^{-\omega z}k_i(z):z\in\Sigma_\gamma\}$ *is bounded. Let* $CA_j\subseteq A_jC$, $j\in\mathbb{N}^0_{n-1}$, $A_j\in L(E)$, $j\in\mathbb{N}_{n-1}$, $A_iA_j=A_jA_i$, $i,j\in\mathbb{N}_{n-1}$ *and* $A_jA\subseteq AA_j$, $j\in\mathbb{N}_{n-1}$. *Assume, additionally, that for each* $i\in\mathbb{N}^0_{m_n-1}$ *the set* $V_i:=N_i\cap\{\lambda\in\mathbb{C}:\Re\lambda>\omega\}$ *contains the set* $\{\lambda\in\mathbb{C}:\Re\lambda>\omega,\widetilde{k}(\lambda)\neq0\}$, *and that* $R(\lambda^{\alpha_n}C+\sum_{j\in D_i}\lambda^{\alpha_j}A_jC)\subseteq R(P_\lambda)$, *provided* $m-1<i$ *and* $\lambda\in V_i$,

respectively, $R(\lambda^{\alpha_n}C+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j}A_jC)\subseteq R(P_\lambda)$, *provided* $m-1\geq i$ *and* $\lambda\in V_i$ *(cf. the formulation of preceding theorem). Suppose also that the operator* $\lambda^{\alpha_n}I+\sum_{j\in D_i}\lambda^{\alpha_j}A_j$ *is injective, provided* $m-1<i$ *and* $\lambda\in V_i$*, and that the operator* $\lambda^{\alpha_n}I+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j}A_j$ *is injective, provided* $m-1\geq i$ *and* $\lambda\in V_i$*. Let* $q_i:\omega+\Sigma_{(\pi/2)+\beta}\to L(E)$ $(0\leq i\leq m_n-1)$ *satisfy that, for every* $x\in E$*, the mapping* $\lambda\mapsto q_i(\lambda)x$*,* $\lambda\in\omega+\Sigma_{(\pi/2)+\beta}$ *is analytic as well as that:*

$$q_i(\lambda)x=\widetilde{k}(\lambda)P_\lambda^{-1}\left(\lambda^{\alpha_n-\alpha-i}C+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}A_jC\right)x,\quad x\in E,\ \lambda\in V_i, \tag{2.33}$$

provided $m-1<i$*,*

$$q_i(\lambda)x=-\widetilde{k}(\lambda)P_\lambda^{-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}A_jCx,\quad x\in E,\ \lambda\in V_i, \tag{2.34}$$

provided $m-1\geq i$*,*

$$\text{the family }\left\{(\lambda-\omega)q_i(\lambda):\lambda\in\omega+\Sigma_{(\pi/2)+\gamma}\right\}\text{ is equicontinuous}\quad\forall\gamma\in(0,\beta), \tag{2.35}$$

and, in the case $\overline{D(A)}\neq E$*,*

$$\lim_{\lambda\to+\infty}\lambda q_i(\lambda)x=\begin{cases}(k*g_i)(0)Cx, & x\notin\overline{D(A)},\ m-1<i,\\ 0, & x\notin\overline{D(A)},\ m-1\geq i.\end{cases} \tag{2.36}$$

Then there exists an exponentially equicontinuous, analytic k*-regularized* C*-resolvent propagation family* $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$ *for* (1.1)*. Furthermore, the family* $\{e^{-\omega z}R_i(z):z\in\Sigma_\gamma\}$ *is equicontinuous for all* $i\in\mathbb{N}^0_{m_n-1}$ *and* $\gamma\in(0,\beta)$*,* (2.5) *holds, and* $R_i(z)A_j\subseteq A_jR_i(z)$*,* $z\in\Sigma_\beta$*,* $j\in\mathbb{N}^0_{n-1}$*.*

In this paper, we will not consider differential properties of k-regularized C-resolvent (propagation) families. For more details, the interested reader may consult [30], and especially, [26, Theorems 3.18–3.20]. Notice also that the assertion of [26, Proposition 3.12] can be reformulated for k-regularized C-resolvent (propagation) families.

In the following theorem, which possesses several obvious consequences, we consider q-exponentially equicontinuous k-regularized I-resolvent propagation families in complete locally convex spaces.

Theorem 2.13. *(i) Suppose* $k(0)\neq0$*,* $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$ *is a* q*-exponentially equicontinuous* k*-regularized* I*-resolvent propagation family for* (1.1)*,* $A_j\in L_{\circledast}(E)$*,* $j\in\mathbb{N}_{n-1}$*, and for every* $p\in\circledast$*, there exist* $M_p\geq1$ *and* $\omega_p\geq0$ *such that*

$$p(R_i(t)x)\leq M_pe^{\omega_pt}p(x),\quad t\geq0,\ x\in E,\ 0\leq i\leq m_n-1. \tag{2.37}$$

Then A *is a compartmentalized operator and, for every seminorm* $p\in\circledast$*,* $((\overline{R_{0,p}(t)})_{t\geq0},\ldots,(\overline{R_{m_n-1,p}(t)})_{t\geq0})$ *is an exponentially bounded* k*-regularized* $\overline{I_p}$*-resolvent propagation family for* (1.1)*,* *in* $\overline{E_p}$, *with* A_j *replaced by* $\overline{A_{j,p}}$ $(0 \leq j \leq n-1)$. *Furthermore,*

$$\left\|\overline{R_{i,p}(t)}\right\| \leq M_p e^{\omega_p t}, \quad t \geq 0,\ 0 \leq i \leq m_n - 1, \tag{2.38}$$

and $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ *is a q-exponentially equicontinuous, analytic k-regularized* $\overline{I_p}$*-resolvent propagation family of angle* $\beta \in (0, \pi]$, *provided that* $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ *is. Assume additionally that* (2.5) *holds. Then, for every* $p \in ⊛$, (2.5) *holds with* A_j *and* $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ *replaced by* $\overline{A_{j,p}}$ *and* $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$.

(ii) Suppose $k(t)$ *satisfies (P1), E is complete, A is a compartmentalized operator in E,* $A_j = c_j I$ *for some* $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$ *and, for every* $p \in ⊛$, $\overline{A_p}$ *is a subgenerator (the integral generator, in fact) of an exponentially bounded k-regularized* $\overline{I_p}$*-resolvent propaga-tion family* $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ *in* $\overline{E_p}$ *satisfying* (2.38), *and* (2.5) *with A and* $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ *replaced, respectively, by* $\overline{A_p}$ *and* $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$. *Suppose, additionally, that* $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$ *and* $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} |c_j|^2 > 0$, *provided* $m - 1 \geq i$. *Then, for every* $p \in ⊛$, (2.37) *holds* $(0 \leq i \leq m_n - 1)$ *and A is a subgenerator (the integral generator, in fact) of a q-exponentially equicontinuous k-regularized I-resolvent propagation family* $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ *satisfying* (2.5). *Furthermore,* $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ *is a q-exponentially equicontinuous, analytic k-regularized I-resolvent propagation family of angle* $\beta \in (0, \pi]$ *provided that, for every* $p \in ⊛$, $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ *is a q-exponentially bounded, analytic k-regularized* $\overline{I_p}$*-resolvent propagation family of angle* β.

Proof. The proof is almost completely similar to that of [20, Theorem 3.1], and we will only outline a few relevant facts needed for the proof of (i). Suppose $x, y \in D(A)$ and $p(x) = p(y)$ for some $p \in ⊛$. Then (2.6) in combination with (2.37) implies that $\Psi_p(R_i(t)A(x-y)) = 0$, $t \geq 0$, provided $m - 1 < i$, and $\Psi_p(R_i(t)A(x-y) - (k * g_i)(t)(x-y)) = 0$, $t \geq 0$, provided $m - 1 \geq i$. In any case, $\Psi_p(R_i(t)A(x-y)) = 0, t \geq 0$, which implies $p(R_i(t)A(x-y)) = 0$, $t \geq 0$, and in particular $p(k(0)A(x-y)) = 0$. Since $k(0) \neq 0$, we obtain $p(Ax - Ay) = 0$ and $p(Ax) = p(Ay)$. Therefore, A is a compartmentalized operator. It is clear that (2.38) holds and that the mapping $t \mapsto \overline{R_{i,p}(t)}x_p$, $t \geq 0$ is continuous for any $x_p \in E_p$. This implies by the standard limit procedure that the mapping $t \mapsto \overline{R_{i,p}(t)}\overline{x_p}$, $t \geq 0$ is continuous for any $\overline{x_p} \in \overline{E_p}$. Now we will prove that, for every $p \in ⊛$, the operator A_p is closable for the topology of $\overline{E_p}$. In order to do that, suppose (x_n) is a sequence in $D(A)$ with $\lim_{n\to\infty}\Psi_p(x_n) = 0$ and $\lim_{n\to\infty}\Psi_p(Ax_n) = y$, in $\overline{E_p}$. Using the dominated convergence theorem, (2.6) and (2.37), we get that $\int_0^t g_{\alpha_n-\alpha}(t-s)\overline{R_{i,p}(s)}y\,ds = \lim_{n\to\infty}\int_0^t g_{\alpha_n-\alpha}(t-s)\overline{R_{i,p}(s)}\Psi_p(Ax_n)ds = 0$, for any $t \geq 0$. Taking the Laplace transform, one obtains $\overline{R_{i,p}(t)}y = 0$, $t \geq 0$. Since $\overline{R_{i,p}(0)} = k(0)\overline{I_p}$, we get that $y = 0$ and that A_p is closable, as claimed. Suppose $0 \leq i \leq m_n - 1$. It is checked at once that $\overline{R_{i,p}(t)}\,\overline{A_{j,p}} \subseteq \overline{A_{j,p}}\,\overline{R_{i,p}(t)}$, $t \geq 0$, $i \in \mathbb{N}^0_{m_n-1}$, $j \in \mathbb{N}_{n-1}$. The functional equation (2.6) for the operators $\overline{A_{j,p}}$, $0 \leq j \leq n-1$ and $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ can be trivially verified, which also holds for the functional equation (2.6) in case of its validity for the operators A_j, $0 \leq j \leq n-1$, and $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$. The remaining part of the proof can be obtained by copying the final part of the proof of [20, Theorem 3.1(i)]. □

Remark 2.14. In the second part of Theorem 2.13, we must restrict ourselves to the case in which $A_j = c_j I$ for some $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$. As a matter of fact, it is not clear how one can prove that the operator $\lambda^{\alpha_n}\overline{I_p} + \sum_{j\in D_i}\lambda^{\alpha_j}\overline{A_{j,p}}$ is injective, provided $m-1 < i$, $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$, as well as that the operator $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j}\overline{A_{j,p}}$ is injective, provided $m-1 \geq i$, $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$. Then Theorem 2.10 is inapplicable, which implies that the argumentation used in the proof of [20, Theorem 3.1(ii)] does not work for the proof of fact that, for every $i \in \mathbb{N}^0_{m_n-1}$ and $t > 0$, $\{\overline{R_{i,p}(t)} : p \in \circledast\}$ is a projective family of operators.

3. k-Regularized (C_1, C_2)-Existence and Uniqueness Families for (1.1)

Throughout this section, we will always assume that X and Y are sequentially complete locally convex spaces. By $L(Y, X)$ is denoted the space which consists of all bounded linear operators from Y into X. The fundamental system of seminorms which defines the topology on X, respectively, Y, is denoted by $\circledast_X$, respectively, $\circledast_Y$. The symbol I designates the identity operator on X.

Let $0 < \tau \leq \infty$. A strongly continuous operator family $(W(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be locally equicontinuous if and only if, for every $T \in (0, \tau)$ and for every $p \in \circledast_X$, there exist $q_p \in \circledast_Y$ and $c_p > 0$ such that $p(W(t)y) \leq c_p q_p(y)$, $y \in Y$, $t \in [0, T]$; the notion of equicontinuity of $(W(t))_{t\in[0,\tau)}$ is defined similarly. Notice that $(W(t))_{t\in[0,\tau)}$ is automatically locally equicontinuous in case that the space Y is barreled.

Following Xiao and Liang [24], we introduce the following definition.

Definition 3.1. Suppose $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (1.1) if and only if, for every $y \in Y$, the following holds: $E(\cdot)y \in C^{m_n-1}([0, \tau) : X)$, $E^{(i)}(0)y = 0$ for every $i \in \mathbb{N}_0$ with $i < m_n - 1$, $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0, \tau) : X)$ for $0 \leq j \leq n-1$, and

$$E^{(m_n-1)}(t)y + \sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} * E^{(m_n-1)}\right)(t)y - A\left(g_{\alpha_n-\alpha} * E^{(m_n-1)}\right)(t)y = k(t)C_1 y, \tag{3.1}$$

for any $t \in [0, \tau)$.

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (1.1) if and only if, for every $\tau \in [0, \tau)$ and $x \in \bigcap_{0\leq j\leq n-1} D(A_j)$, the following holds:

$$U(t)x + \sum_{j=1}^{n-1}\left(g_{\alpha_n-\alpha_j} * U(\cdot)A_j x\right)(t) - \left(g_{\alpha_n-\alpha} * U(\cdot)Ax\right)(t)y = \left(k * g_{m_n-1}\right)(t)C_2 x. \tag{3.2}$$

(iii) A strongly continuous family $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)}) \subseteq L(Y, X) \times L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized (C_1, C_2)-existence and uniqueness family for (1.1) if and only if $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (1.1), and $(U(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness family for (1.1).

(iv) Suppose $Y = X$ and $C = C_1 = C_2$. Then a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C-resolvent family for (1.1) if and only if $(R(t))_{t\in[0,\tau)}$ is a k-regularized C-uniqueness family for (1.1), $R(t)A_j \subseteq A_j R(t)$, for $0 \le j \le n-1$ and $t \in [0,\tau)$, as well as $R(t)C = CR(t)$, $t \in [0,\tau)$, and $CA_j \subseteq A_j C$, for $0 \le j \le n-1$.

In case $k(t) = g_{\zeta+1}(t)$, where $\zeta \ge 0$, it is also said that $(E(t))_{t\in[0,\tau)}$ is a ζ-times integrated C_1-existence family for (1.1); 0-times integrated C_1-existence family for (1.1) is also said to be a C_1-existence family for (1.1). The notion of (exponential) analyticity of C_1-existence families for (1.1) is taken in the sense of Definition 1.2(ii); the above terminological agreement can be simply understood for all other classes of uniqueness and resolvent families introduced in Definition 3.1.

Integrating both sides of (3.1) sufficiently many times, we easily infer that (cf. [24, Definition 2.1, page 151; and (2.8), page 153]):

$$E^{(l)}(t)y + \sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} * E^{(l)}\right)(t)y - A\left(g_{\alpha_n-\alpha} * E^{(l)}\right)(t)y = \left(k * g_{m_n-1-l}\right)(t)C_1 y, \tag{3.3}$$

for any $t \in [0,\tau)$, $y \in Y$ and $l \in \mathbb{N}^0_{m_n-1}$. In this place, it is worth noting that the identity (3.3), with $k(t) = 1$, $l = 0$, $\tau = \infty$ and $\alpha_j = j$ $(0 \le j \le n-1)$, has been used in [24] for the definition of a C_1-existence family for (ACP$_n$). It can be simply proved that this definition is equivalent with the corresponding one given by Definition 3.1.

Proposition 3.2. *Let* $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)})$ *be a k-regularized (C_1, C_2)-existence and uniqueness family for* (1.1), *and let* $(U(t))_{t\in[0,\tau)}$ *be locally equicontinuous. If* $A_j \in L(X)$, $j \in \mathbb{N}_{n-1}$ *or* $\alpha \le \min(\alpha_1, \ldots, \alpha_{n-1})$, *then* $C_2E(t)y = U(t)C_1 y$, $t \in [0,\tau)$, $y \in Y$.

Proof. Let $y \in Y$ be fixed. Using the local equicontinuity of $(U(t))_{t\in[0,\tau)}$, we easily infer that the mappings $t \mapsto ((g_{\alpha_n-\alpha} * U) * E(\cdot)y)(t)$, $t \in [0,\tau)$ and $t \mapsto (U * (g_{\alpha_n-\alpha} * E(\cdot)y))(t)$, $t \in [0,\tau)$ are continuous and coincide. The prescribed assumptions also imply that, for every $j \in \mathbb{N}_{n-1}$, $t \in [0,\tau)$ and $y \in Y$,

$$\left(g_{\alpha_n-\alpha} * U * A_j\left(g_{\alpha_n-\alpha_j} * E(\cdot)y\right)\right)(t)y = \left(g_{\alpha_n-\alpha} * UA_j * g_{\alpha_n-\alpha} * E(\cdot)y\right)(t)y. \tag{3.4}$$

Keeping in mind (3.2)-(3.3) and the foregoing arguments, we get that

$$\begin{aligned} &g_{\alpha_n-\alpha} * U * \left[E(\cdot)y + \sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} * E\right)(\cdot)y - k(\cdot)C_1 y\right] \\ &\quad = g_{\alpha_n-\alpha} * UA * \left[g_{\alpha_n-\alpha} * E\right](\cdot)y \\ &\quad = \left[U(\cdot) + \sum_{j=1}^{n-1}\left(g_{\alpha_n-\alpha_j} * U(\cdot)A_j\right) - k(\cdot)C_2\right] * g_{\alpha_n-\alpha} * E(\cdot)y. \end{aligned} \tag{3.5}$$

This, in turn, implies the required equality $C_2E(t)y = U(t)C_1 y$, $t \in [0,\tau)$. □

Definition 3.3. Suppose $0 \leq i \leq m_n - 1$. Then we define $D_i' := \{j \in \mathbb{N}_{n-1}^0 : m_j - 1 \geq i\}$, $D_i'' := \mathbb{N}_{n-1}^0 \setminus D_i'$ and

$$\mathbf{D}_i := \left\{ x \in \bigcap_{j \in D_i''} D(A_j) : A_j u_i \in R(C_1), j \in D_i'' \right\}. \tag{3.6}$$

In the first part of subsequent theorem (cf. also [24, Remark 2.2, Example 2.5, Remark 2.6]), we will consider the most important case $k(t) = 1$. The analysis is similar if $k(t) = g_{n+1}(t)$ for some $n \in \mathbb{N}$.

Theorem 3.4. *(i) Suppose $(E(t))_{t\in[0,\tau)}$ is a C_1-existence family for (1.1), $T \in (0,\tau)$, and $u_i \in \mathbf{D}_i$ for $0 \leq i \leq m_n - 1$. Then the function*

$$\begin{aligned} u(t) = \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) &- \sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n - \alpha_j} * E^{(m_n-1-i)} \right)(t) v_{i,j} \\ &+ \sum_{i=m}^{m_n-1} \left(g_{\alpha_n - \alpha} * E^{(m_n-1-i)} \right)(t) v_{i,0}, \quad 0 \leq t \leq T, \end{aligned} \tag{3.7}$$

is a strong solution of the problem (2.2) on $[0,T]$, where $v_{i,j} \in Y$ satisfy $A_j u_i = C_1 v_{i,j}$ for $0 \leq j \leq n-1$.

(ii) Suppose $(U(t))_{t\in[0,\tau)}$ is a locally equicontinuous k-regularized C_2-uniqueness family for (1.1), and $T \in (0,\tau)$. Then there exists at most one strong (mild) solution of (2.2) on $[0,T]$, with $u_i = 0$, $i \in \mathbb{N}_{m_n-1}^0$.

Proof. A straightforward computation involving (3.3) shows that

$$\begin{aligned} &u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot) + \sum_{j=1}^{n-1} A_j \left(g_{\alpha_n - \alpha_j} * \left[u(\cdot) - \sum_{i=0}^{m_j-1} u_i g_{i+1}(\cdot) \right] \right) \\ &= -\sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n - \alpha_j} * R^{(m_n-1-i)} \right)(\cdot) v_{i,j} + \sum_{i=m}^{m_n-1} \left(g_{\alpha_n - \alpha} * R^{(m_n-1-i)} \right)(\cdot) v_{i,0} \\ &\quad + \sum_{j=1}^{n-1} A_j \left(g_{\alpha_n - \alpha_j} * \left\{ \sum_{i=m_j}^{m_n-1} g_{i+1}(\cdot) u_i - \sum_{i=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n - \alpha_l} * R^{(m_n-1-i)} \right)(\cdot) v_{i,l} \right.\right. \\ &\quad \left.\left. + \sum_{i=m}^{m_n-1} \left(g_{\alpha_n - \alpha} * R^{(m_n-1-i)} \right)(\cdot) v_{i,0} \right\} \right) \\ &= -\sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n - \alpha_j} * R^{(m_n-1-i)} \right)(\cdot) v_{i,j} + \sum_{i=m}^{m_n-1} \left(g_{\alpha_n - \alpha} * R^{(m_n-1-i)} \right)(\cdot) v_{i,0} \\ &\quad + \sum_{j=1}^{n-1} \sum_{i=m_j}^{m_n-1} C_1 v_{i,j} g_{\alpha_n - \alpha_j + i + 1}(\cdot) - \sum_{i=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_i} g_{\alpha_n - \alpha_l} \end{aligned}$$

$$
\begin{aligned}
&* \left[-R^{(m_n-1-i)}(\cdot)v_{i,l} + A\left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)}\right)(\cdot)v_{i,l} + g_{i+1}(\cdot)C_1 v_{i,l}\right] \\
&+ \sum_{i=m}^{m_n-1} g_{\alpha_n-\alpha} * \left[-R^{(m_n-1-i)}(\cdot)v_{i,0} + A\left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)}\right)(\cdot)v_{i,0} + g_{i+1}(\cdot)C_1 v_{i,0}\right] \\
&= g_{\alpha_n-\alpha} * A\left[u(\cdot) - \sum_{i=0}^{m-1} u_i g_{i+1}(\cdot)\right],
\end{aligned}
\tag{3.8}
$$

since

$$
\sum_{j=1}^{n-1}\sum_{i=m_j}^{m_n-1} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot) = \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot). \tag{3.9}
$$

This implies that $u(t)$ is a mild solution of (2.2) on $[0,T]$. In order to complete the proof of (i), it suffices to show that $\mathbf{D}_t^{\alpha_n}u(t) \in C([0,T]:X)$ and $A_i\mathbf{D}_t^{\alpha_i}u \in C([0,T]:X)$ for all $i \in \mathbb{N}_{n-1}^0$. Towards this end, notice that the partial integration implies that, for every $t \in [0,T]$,

$$
\begin{aligned}
g_{m_n-\alpha_n} * \left[u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\right](t) &= \sum_{i=m}^{m_n-1}\left(g_{m_n-\alpha+i} * E^{(m_n-1)}\right)(t)v_{i,0} \\
&\quad - \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\left(g_{m_n-\alpha_j+i} * E^{(m_n-1)}\right)(t)v_{i,j}.
\end{aligned}
\tag{3.10}
$$

Therefore, $\mathbf{D}_t^{\alpha_n}u \in C([0,T]:X)$ and

$$
\begin{aligned}
\mathbf{D}_t^{\alpha_n}u(t) &= \frac{d^{m_n}}{dt^{m_n}}\left\{g_{m_n-\alpha_n} * \left[u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\right](t)\right\} \\
&= \sum_{i=m}^{m_n-1}\left(g_{i-\alpha} * E^{(m_n-1)}\right)(t)v_{i,0} - \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\left(g_{i-\alpha_j} * E^{(m_n-1)}\right)(t)v_{i,j}.
\end{aligned}
\tag{3.11}
$$

Suppose, for the time being, $i \in \mathbb{N}_{n-1}^0$. Then $A_i u_j \in R(C_1)$ for $j \geq m_i$. Moreover, the inequality $l \geq \alpha_j$ holds provided $0 \leq l \leq m_n-1$ and $j \in \mathbb{N}_{n-1}\setminus D_l$, and $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0,T]:X)$ for $0 \leq j \leq n-1$ and $y \in Y$. Now it is not difficult to prove that

$$
\begin{aligned}
A_i\mathbf{D}_t^{\alpha_i}u(\cdot) &= \sum_{j=m_i}^{m_n-1} g_{j+1-\alpha_i}(\cdot)A_i u_j - \sum_{l=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_l}\left[g_{l-\alpha_j} * A_i\left(g_{\alpha_n-\alpha_i} * E^{(m_n-1)}\right)\right](\cdot)v_{l,j} \\
&\quad + \sum_{l=m}^{m_n-1}\left[g_{l-\alpha} * A_i\left(g_{\alpha_n-\alpha_i} * E^{(m_n-1)}\right)\right](\cdot)v_{l,0} \in C([0,T]:X),
\end{aligned}
\tag{3.12}
$$

finishing the proof of (i). The second part of theorem can be proved as follows. Suppose $u(t)$ is a strong solution of (2.2) on $[0,T]$, with $u_i = 0$, $i \in \mathbb{N}^0_{m_n-1}$. Using this fact and the equality

$$\int_0^t \int_0^{t-s} g_{\alpha_n-\alpha_j}(r)U(t-s-r)A_j u(s)\,dr\,ds = \int_0^t \int_0^s g_{\alpha_n-\alpha_j}(r)U(t-s)A_j u(s-r)\,dr\,ds, \tag{3.13}$$

for any $t \in [0,T]$ and $j \in \mathbb{N}^0_{n-1}$, we easily infer that (for more general results, see [31, Proposition 2.4(i)], and [29, page 155]):

$$\begin{aligned}(U * u)(t) &= \left(k * g_{m_n-1}C_2 * u\right)(t) \\ &\quad + \int_0^t \int_0^{t-s} \left[g_{\alpha_n-\alpha_j}(r)U(t-s-r)A_j u(s) - g_{\alpha_n-\alpha}(r)U(t-s-r)Au(s)\right] dr\,ds \\ &= \left(k * g_{m_n-1}C_2 * u\right)(t) + (U * u)(t), \quad t \in [0,T].\end{aligned} \tag{3.14}$$

Therefore, $(k * g_{m_n-1}C_2 * u)(t) = 0$, $t \in [0,T]$ and $u(t) = 0$, $t \in [0,T]$. □

Before proceeding further, we would like to notice that the solution $u(t)$, given by (3.7), need not to be of class $C^1([0,T] : X)$, in general. Using integration by parts, it is checked at once that (3.7) is an extension of the formula [24, (2.5); Theorem 2.4, page 152]. Notice, finally, that the proof of Theorem 3.4(ii) is much simpler than that of [24, Theorem 2.4(ii)].

The standard proof of following theorem is omitted (cf. also [24, Theorem 2.7, Remark 2.8, Theorem 2.9] and [28, Chapter 1]).

Theorem 3.5. *Suppose $k(t)$ satisfies (P1), $(E(t))_{t\geq 0} \subseteq L(Y,X)$, $(U(t))_{t\geq 0} \subseteq L(X)$, $\omega \geq \max(0, \operatorname{abs}(k))$, $C_1 \in L(Y,X)$ and $C_2 \in L(X)$ is injective. Set $\mathbf{P}_\lambda := I + \sum_{j=1}^{n-1} \lambda^{\alpha_j-\alpha_n} A_j - \lambda^{\alpha-\alpha_n} A$, $\Re\lambda > 0$.*

(i) *(a) Let $(E(t))_{t\geq 0}$ be a k-regularized C_1-existence family for (1.1), let the family $\{e^{-\omega t}E(t) : t \geq 0\}$ be equicontinuous, and let the family $\{e^{-\omega t}A_j(g_{\alpha_n-\alpha_j} * E)(t) : t \geq 0\}$ be equicontinuous ($0 \leq j \leq n-1$). Then the following holds:*

$$\mathbf{P}_\lambda \int_0^\infty e^{-\lambda t}E(t)y\,dt = \tilde{k}(\lambda)\lambda^{1-m_n}C_1 y, \quad y \in Y,\ \Re\lambda > \omega. \tag{3.15}$$

(b) Let the operator $\mathbf{P}_\lambda$ be injective for every $\lambda > \omega$ with $\tilde{k}(\lambda) \neq 0$. Suppose, additionally, that there exist strongly continuous operator families $(W(t))_{t\geq 0} \subseteq L(Y,X)$ and $(W_j(t))_{t\geq 0} \subseteq L(Y,X)$ such that $\{e^{-\omega t}W(t) : t \geq 0\}$ and $\{e^{-\omega t}W_j(t) : t \geq 0\}$ are equicontinuous ($0 \leq j \leq n-1$) as well as that

$$\begin{aligned}\int_0^\infty e^{-\lambda t}W(t)y\,dt &= \tilde{k}(\lambda)\mathbf{P}_\lambda^{-1}C_1 y, \\ \int_0^\infty e^{-\lambda t}W_j(t)y\,dt &= \tilde{k}(\lambda)\lambda^{\alpha_j-\alpha_n}A_j\mathbf{P}_\lambda^{-1}C_1 y,\end{aligned} \tag{3.16}$$

for every $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$ *and* $\tilde{k}(\lambda) \neq 0$, $y \in Y$ *and* $j \in \mathbb{N}^0_{n-1}$. *Then there exists a* k-*regularized* C_1-*existence family for* (1.1), *denoted by* $(E(t))_{t\geq 0}$. *Furthermore,* $E^{(m_n-1)}(t)y = W(t)y$, $t \geq 0$, $y \in Y$ *and* $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(t)y = W_j(t)y$, $t \geq 0$, $y \in Y$, $j \in \mathbb{N}^0_{n-1}$.

(ii) *Let the assumptions of* (i) *hold with* $k(t) = 1$. *If* $m_n > 1$, *then one suppose additionally that, for every* $j \in \mathbb{N}^0_{n-1}$, *there exists a strongly continuous operator family* $(V_j(t))_{t\geq 0} \subseteq L(Y, X)$ *such that* $\{e^{-\omega t}V_j(t) : t \geq 0\}$ *is equicontinuous as well as that*

$$\int_0^\infty e^{-\lambda t}V_j(t)y\,dt = \lambda^{\alpha_j-\alpha_n-1}\mathbf{P}_\lambda^{-1}A_jC_1y, \tag{3.17}$$

for every $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$, *and* $y \in D(A_jC_1)$. *Let* $u_i \in \mathbf{D}_i$, *and let* $C_1v_i = u_i$ *for some* $v_i \in Y$ $(0 \leq i \leq m_n - 1)$. *Then, for every* $p \in \circledast_X$, *there exist* $c_p > 0$ *and* $q_p \in \circledast_Y$ *such that the corresponding solution* $u(t)$ *satisfies the following estimate:*

$$p(u(t)) \leq c_p e^{\omega t}\sum_{i=0}^{m_n-1} q_p(v_i), \quad t \geq 0, \text{ if } \omega > 0, \tag{3.18}$$

$$p(u(t)) \leq c_p g_{m_n}(t)\sum_{i=0}^{m_n-1} q_p(v_i), \quad t \geq 0, \text{ if } \omega = 0. \tag{3.19}$$

(iii) *Suppose* $(U(t))_{t\geq 0}$ *is strongly continuous and the operator family* $\{e^{-\omega t}U(t) : t \geq 0\}$ *is equicontinuous. Then* $(U(t))_{t\geq 0}$ *is a* k-*regularized* C_2-*uniqueness family for* (1.1) *if and only if, for every* $x \in \bigcap_{j=0}^{n-1} D(A_j)$, *the following holds:*

$$\int_0^\infty e^{-\lambda t}U(t)\mathbf{P}_\lambda x\,dt = \tilde{k}(\lambda)\lambda^{1-m_n}C_2x, \quad \Re\lambda > \omega. \tag{3.20}$$

The Hausdorff locally convex topology on E^* defines the system $(|\cdot|_B)_{B\in\mathcal{B}}$ of seminorms on E^*, where $|x^*|_B := \sup_{x\in B}|\langle x^*, x\rangle|$, $x^* \in E^*$, $B \in \mathcal{B}$. Let us recall that E^* is sequentially complete provided that E is barreled. Following Wu and Zhang [32], we also define on E^* the topology of uniform convergence on compacts of E, denoted by $\mathcal{C}(E^*, E)$; more precisely, given a functional $x_0^* \in E^*$, the basis of open neighborhoods of x_0^* with respect to $\mathcal{C}(E^*, E)$ is given by $N(x_0^* : \mathbf{K}, \varepsilon) := \{x^* \in E^* : \sup_{x\in\mathbf{K}}|\langle x^* - x_0^*, x\rangle| < \varepsilon\}$, where $\mathbf{K}$ runs over all compacts of E and $\varepsilon > 0$. Then $(E^*, \mathcal{C}(E^*, E))$ is locally convex, complete and the topology $\mathcal{C}(E^*, E)$ is finer than the topology induced by the calibration $(|\cdot|_B)_{B\in\mathcal{B}}$.

Now we focus our attention to the adjoint type theorems for (local) k-regularized C-resolvent families. The proof of following theorem follows from the arguments given in the proofs of [26, Theorems 2.14 and 2.15]; because of that, we will omit it.

Theorem 3.6. *(i) Suppose* X *is barreled,* $\zeta > 0$, $(R(t))_{t\in[0,\tau)}$ *is a* k-*regularized* C-*resolvent family for* (1.1), *and* $\overline{\bigcap_{j=0}^{n-1} D(A_j)} = \overline{R(C)} = X$. *Then* $((g_\zeta * R(\cdot)^*)(t))_{t\in[0,\tau)}$ *is a* k-*regularized* C^*-*resolvent family for* (1.1), *with* A_j *replaced by* A_j^* $(0 \leq j \leq n-1)$.

(ii) Suppose X is barreled, $(R(t))_{t\in[0,\tau)}$ is a (local, global exponentially equicontinuous) k-regularized C-resolvent family for (1.1), *and* $\overline{\bigcap_{j=0}^{n-1} D(A_j)} = \overline{R(C)} = X$. *Put* $Z := \overline{\bigcap_{j=0}^{n-1} D(A_j^*)}$. *Then* $(R(t)^*_{|Z})_{t\in[0,\tau)}$, *is a (local, global exponentially equicontinuous) k-regularized* $C^*_{|Z}$*-resolvent family for* (1.1), *in Z.*

(iii) Suppose $(R(t))_{t\in[0,\tau)}$ *is a locally equicontinuous k-regularized C-resolvent family for* (1.1), *and* $\overline{\bigcap_{j=0}^{n-1} D(A_j)} = \overline{R(C)} = X$. *Then* $(R(t)^*)_{t\in[0,\tau)}$ *is a locally equicontinuous k-regularized* C^**-resolvent family for* (1.1), *in* $(X^*, \mathcal{C}(X^*, X))$, *with* A_j *replaced by* A_j^* $(0 \le j \le n-1)$. *Furthermore, if* $(R(t))_{t\ge 0}$ *is exponentially equicontinuous, then* $(R(t)^*)_{t\ge 0}$ *is also exponentially equicontinuous.*

Notice here that a similar theorem can be proved for the class of k-regularized C-resolvent propagation families.

Let $f \in C([0,T] : X)$. Convoluting both sides of (1.1) with $g_{\alpha_n}(t)$, we get that

$$\begin{aligned} u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j \left[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot) \right] \\ = g_{\alpha_n-\alpha} * A \left[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot) \right] + \left(g_{\alpha_n} * f \right)(\cdot), \quad t \in [0,T]. \end{aligned} \tag{3.21}$$

In the subsequent theorem, whose proof follows from a slight modification of the proof of [24, Theorem 3.1(i)], we will analyze inhomogeneous Cauchy problem (3.21) in more detail.

Theorem 3.7. *Suppose* $(E(t))_{t\in[0,\tau)}$ *is a locally equicontinuous* C_1*-existence family for* (1.1), $T \in (0,\tau)$, *and* $u_i \in \mathbf{D}_i$ *for* $0 \le i \le m_n - 1$. *Let* $f \in C([0,T] : X)$, *let* $g \in C([0,T] : Y)$ *satisfy* $C_1 g(t) = f(t)$, $t \in [0,T]$, *and let* $G \in C([0,T] : Y)$ *satisfy* $(g_{\alpha_n-m_n+1} * g)(t) = (g_1 * G)(t)$, $t \in [0,T]$. *Then the function*

$$\begin{aligned} u(t) = \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \left(g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)} \right)(t) v_{i,j} \\ + \sum_{i=m}^{m_n-1} \left(g_{\alpha_n-\alpha} * E^{(m_n-1-i)} \right)(t) v_{i,0} + \int_0^t E(t-s)G(s)\,ds, \quad 0 \le t \le T \end{aligned} \tag{3.22}$$

is a mild solution of the problem (3.21) *on* $[0,T]$, *where* $v_{i,j} \in Y$ *satisfy* $A_j u_i = C_1 v_{i,j}$ *for* $0 \le j \le n-1$. *If, additionally,* $g \in C^1([0,T] : Y)$ *and* $(E^{(m_n-1)}(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ *is locally equicontinuous, then the solution* $u(t)$, *given by* (3.22), *is a strong solution of* (1.1) *on* $[0,T]$.

Remark 3.8. Suppose that all conditions quoted in the first part of the above theorem hold, and the family $(E^{(m_n-1)}(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is locally equicontinuous. We assume, instead of condition $g \in C^1([0,T] : Y)$, that there exists a locally equicontinuous C_2-uniqueness family for (1.1) on $[0,\tau)$, as well as that there exist functions $h_j \in L^1([0,T] : Y)$ such that $A_j f(t) = C_1 h_j(t)$, $t \in [0,T]$, $0 \le j \le n-1$ (cf. also the formulation of [24, Theorem 3.1(ii)]). Using the functional equation for $(E(t))_{t\in[0,\tau)}$, one can simply prove that, for every $\sigma \in [0,T]$, the function

$$r_\sigma(\cdot) = E(\cdot)g(\sigma) - g_{m_n}(\cdot)f(\sigma) + \sum_{j=1}^{n-1}\Big(g_{\alpha_n-\alpha_j} * E(\cdot)h_j(\sigma)\Big)(\cdot) - \big(g_{\alpha_n-\alpha} * E(\cdot)h_0(\sigma)\big)(\cdot) \tag{3.23}$$

is a mild solution of the problem

$$u(t) + \sum_{j=1}^{n-1}A_j\Big(g_{\alpha_n-\alpha_j} * u\Big)(t) - A\big(g_{\alpha_n-\alpha} * u\big)(t) = 0, \quad t \in [0,T]. \tag{3.24}$$

By the uniqueness of solutions, we have that the following holds:

$$E(t-\sigma)g(\sigma) - g_{m_n}(t-\sigma)f(\sigma) + \sum_{l=1}^{n-1}\big(g_{\alpha_n-\alpha_l} * E(\cdot)h_l(\sigma)\big)(t-\sigma)\big(g_{\alpha_n-\alpha} * E(\cdot)h_0(\sigma)\big)(t-\sigma) = 0, \tag{3.25}$$

provided $0 \le t, \sigma \le T$ and $\sigma \le t$. Fix $i \in \mathbb{N}_{n-1}^0$. Then the above equality implies that, for every $j \in \mathbb{N}_{m_n-1}^0$ with $j \le \min(\lfloor \alpha_i + m_n - \alpha_{n-1} - 1\rfloor, \lfloor \alpha_i + m_n - \alpha - 1\rfloor)$, one has:

$$\begin{aligned} &A_iE^{(j)}(t-\sigma)g(\sigma) - g_{m_n-j}(t-\sigma)C_1h_i(\sigma) + \sum_{l=1}^{n-1}A_i\Big(g_{\alpha_n-\alpha_l} * E^{(j)}(\cdot)h_l(\sigma)\Big)(t-\sigma) \\ &\quad - A_i\Big(g_{\alpha_n-\alpha} * E^{(j)}(\cdot)h_0(\sigma)\Big)(t-\sigma) = 0, \end{aligned} \tag{3.26}$$

provided $0 \le t, \sigma \le T$ and $\sigma \le t$. For such an index j, we conclude from (3.26) that the mapping $t \mapsto \int_0^t A_iE^{(j)}(t-\sigma)g(\sigma)d\sigma$, $t \in [0,T]$ is continuous. Observe now that the condition

$$\alpha_n - \alpha_i - m_n + \min(\lfloor \alpha_i + m_n - \alpha_{n-1} - 1\rfloor, \lfloor \alpha_i + m_n - \alpha - 1\rfloor) \ge 0, \quad i \in \mathbb{N}_{n-1}^0, \tag{3.27}$$

which holds in the case of abstract Cauchy problem (ACP_n), shows that the mapping $t \mapsto A_i[g_{\alpha_n-\alpha_i-m_n+j} * E^{(j)} * g](t)$, $t \in [0,T]$ is continuous as well as that the mapping $t \mapsto (d/dt)[E^{(m_n-1)} * g](t)$, $t \in [0,T]$ is continuous. Hence, the validity of condition (3.27) implies that the function $u(t)$, given by (3.22), is a strong solution of (1.1) on $[0,T]$.

4. Subordination Principles

The proof of following theorem can be derived by using Theorem 3.5 and the argumentation given in [10, Section 3].

Theorem 4.1. *Suppose $C_1 \in L(Y, X)$, $C_2 \in L(X)$ is injective and $\gamma \in (0,1)$.*

(i) *Let $\omega \geq \max(0, \mathrm{abs}(k))$, and let the assumptions of Theorem 3.5*(i)-(b) *hold. Put*

$$W_\gamma(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(t^{-\gamma}s)W(s)y\,ds, \quad t > 0,\ y \in Y,\ W_\gamma(0) := W(0). \tag{4.1}$$

Define, for every $j \in \mathbb{N}_{n-1}^0$ and $t \geq 0$, $W_{j,\gamma}(t)$ by replacing $W(t)$ in (4.1) *with $W_j(t)$. Suppose that there exist a number $\nu > 0$ and a continuous kernel $k_\gamma(t)$ satisfying (P1) and $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma-1}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. Then there exists an exponentially bounded k_γ-regularized C_1-existence family $(E_\gamma(t))_{t\geq 0}$ for* (1.1), *with α_j replaced by $\alpha_j\gamma$ therein ($0 \leq j \leq n-1$). Furthermore, the family $\{(1 + t^{\lceil \alpha_n\gamma\rceil - 2})^{-1}e^{-\omega^{1/\gamma}t}E_\gamma(t) : t \geq 0\}$ is equicontinuous.*

(ii) *Let $\omega \geq 0$, let the assumptions of Theorem 3.5(ii) hold, and let $k(t) = k_\gamma(t) = 1$. Define, for every $j \in \mathbb{N}_{n-1}^0$ and $t \geq 0$, $V_{j,\gamma}(t)$ by replacing $W(t)$ in* (4.1) *with $V_j(t)$. Then, for every $j \in \mathbb{N}_{n-1}^0$, the family $\{e^{-\omega^{1/\gamma}t}V_{j,\gamma}(t) : t \geq 0\}$ is equicontinuous,*

$$\int_0^\infty e^{-\lambda t}V_{j,\gamma}(t)y\,dt = \lambda^{\alpha_j\gamma - \alpha_n\gamma - 1}\mathbf{P}_{\lambda^\gamma}^{-1}A_jC_1y, \tag{4.2}$$

for every $\lambda \in \mathbb{C}$ with $\Re(\lambda^\gamma) > \omega$, and $y \in D(A_jC_1)$. Let $u_i \in \mathbf{D}_{i,\gamma}$ (defined in the obvious way), and let $C_1v_i = u_i$ for some $v_i \in Y$ ($0 \leq i \leq \lceil \alpha_n\gamma\rceil - 1$). Then, for every $p \in ⊛_X$, there exist $c_p > 0$ and $q_p \in ⊛_Y$ such that the corresponding solution $u(t)$ satisfies the following estimate:

$$\begin{aligned} p(u(t)) &\leq c_p e^{\omega^{1/\gamma}t}\sum_{i=0}^{\lceil \alpha_n\gamma\rceil - 1} q_p(v_i), \quad t \geq 0,\ \text{if } \omega > 0, \\ p(u(t)) &\leq c_p g_{\lceil \alpha_n\gamma\rceil}(t)\sum_{i=0}^{\lceil \alpha_n\gamma\rceil - 1} q_p(v_i), \quad t \geq 0,\ \text{if } \omega = 0. \end{aligned} \tag{4.3}$$

(iii) *Suppose $(U(t))_{t\geq 0}$ is a k-regularized C_2-uniqueness family for* (1.1), *and the family $\{e^{-\omega t}U(t) : t \geq 0\}$ is equicontinuous. Define, for every $t \geq 0$, $U_\gamma(t)$ by replacing $W(t)$ in* (4.1) *with $U(t)$. Suppose that there exist a number $\nu > 0$ and a continuous kernel $k_\gamma(t)$ satisfying (P1) and $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma(2-m_n)-2+\lceil \alpha_n\gamma\rceil}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. Then there exists a k_γ-regularized C_2-uniqueness family for* (1.1), *with α_j replaced by $\alpha_j\gamma$ therein ($0 \leq j \leq n-1$). Furthermore, the family $\{e^{-\omega^{1/\gamma}t}U_\gamma(t) : t \geq 0\}$ is equicontinuous.*

Remark 4.2. (i) Consider the situation of Theorem 4.1(iii). Then we have the obvious equality $(k * g_{m_n - 1})(0) = (k_\gamma * g_{\lceil \alpha_n\gamma\rceil - 1})(0)$. If $\sigma \geq 1$, $k(t) = g_\sigma(t)$ and $(\sigma - 1 + m_n - 1)\gamma + 1 - \lceil \alpha_n\gamma\rceil \geq 0$ (this inequality holds provided $\sigma \geq 2$), then $k_\gamma(t) = g_{(\sigma-1+m_n-1)\gamma+2-\lceil \alpha_n\gamma\rceil}(t)$.

(ii) Let $b \in L_{\mathrm{loc}}^1([0,\infty))$ be a kernel, and let $(U(t))_{t\in[0,\tau)}$ be a (local) k-regularized C_2-uniqueness family for (1.1). Then $((b * U)(t))_{t\in[0,\tau)}$ is a $(b * k)$-regularized C_2-uniqueness family for (1.1).

(iii) Concerning the analytical properties of k_γ-regularized C_1-existence families in Theorem 4.1(i), the following facts should be stated.

(a) The mapping $t \mapsto E_\gamma(t)$, $t > 0$ admits an extension to $\Sigma_{\min(((1/\gamma)-1)(\pi/2),\pi)}$ and, for every $y \in Y$, the mapping $z \mapsto E_\gamma(z)y$, $z \in \Sigma_{\min(((1/\gamma)-1)(\pi/2),\pi)}$ is analytic.

(b) Let $\varepsilon \in (0, \min(((1/\gamma) - 1)(\pi/2), \pi))$, and let $(W(t))_{t\geq 0}$ be equicontinuous. Then $(E_\gamma(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic k_γ-regularized C_1-existence family of angle $\min(((1/\gamma)-1)(\pi/2), \pi)$, and for every $p \in \circledast_X$, there exist $M_{p,\varepsilon} > 0$ and $q_{p,\varepsilon} \in \circledast_Y$ such that

$$p(E_\gamma(z)y) \leq M_{p,\varepsilon} q_{p,\varepsilon}(y)\left(1 + |z|^{\lceil \alpha_n \gamma \rceil - 1}\right), \quad z \in \Sigma_{\min(((1/\gamma)-1)(\pi/2),\pi)-\varepsilon}. \tag{4.4}$$

(c) $(E_\gamma(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic k_γ-regularized C_1-exis-tence family of angle $\min(((1/\gamma) - 1)(\pi/2), \pi/2)$.

The similar statements hold for the k_γ-regularized C_2-uniqueness family $(U_\gamma(t))_{t\geq 0}$ in Theorem 4.1(iii).

The results on k-regularized (C_1, C_2)-existence and uniqueness families can be applied in the study of following abstract Volterra equation:

$$u(t) = f(t) + \sum_{j=0}^{n-1} (a_j * A_j u)(t), \quad t \in [0, \tau), \tag{4.5}$$

where $0 < \tau \leq \infty$, $f \in C([0, \tau) : X)$, $a_0, \ldots, a_{n-1} \in L^1_{\text{loc}}([0, \tau))$, and $A_0, \ldots, A_{n-1}$ are closed linear operators on X. As in Definition 2.7, by a mild solution, respectively, strong solution, of (4.5), we mean any function $u \in C([0, \tau) : X)$ such that $A_j(a_j * u)(t) \in C([0, \tau) : X)$, $j \in \mathbb{N}^0_{n-1}$ and that

$$u(t) = f(t) + \sum_{j=0}^{n-1} A_j (a_j * u)(t), \quad t \in [0, \tau), \tag{4.6}$$

respectively, any function $u \in C([0, \tau) : X)$ such that $u(t) \in \bigcap_{j=0}^{n-1} D(A_j)$, $t \in [0, \tau)$ and that (4.5) holds.

We need the following definition.

Definition 4.3. Suppose $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (4.5) if and only if

$$E(t)y = k(t)C_1 y + \sum_{j=0}^{n-1} A_j (a_j * E)(t)y, \quad t \in [0, \tau), \; y \in Y. \tag{4.7}$$

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (4.5) if and only if

$$U(t)x = k(t)C_2 x + \sum_{j=0}^{n-1} (a_j * A_j U)(t)x, \quad t \in [0, \tau), \; x \in \bigcap_{j=0}^{n-1} D(A_j). \tag{4.8}$$

Notice also that one can introduce the classes of k-regularized (C_1, C_2)-existence and uniqueness families as well as k-regularized C-resolvent families for (4.5); compare Definition 3.1. The full analysis of k-regularized (C_1, C_2)-existence and uniqueness families for (4.5) falls out from the framework of this paper.

The following facts are clear.

(i) Suppose $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (4.5). Then, for every $y \in Y$, the function $u(t) = E(t)y$, $t \in [0,\tau)$, is a mild solution of (4.5) with $f(t) = k(t)C_1 y$, $t \in [0,\tau)$.

(ii) Let $(U(t))_{t\in[0,\tau)}$ be a locally equicontinuous k-regularized C_2-uniqueness family for (4.5). Then there exists at most one mild (strong) solution of (4.5).

The proof of following subordination principle is standard and therefore omitted (cf. the proofs of [29, Theorem 4.1, page 101] and [24, Theorem 2.7]).

Theorem 4.4. *(i) Suppose there is an exponentially equicontinuous k-regularized C_1-existence family for* (1.1)*. Let $c(t)$ be completely positive, let $c(t)$, $k(t)$ and $k_1(t)$ satisfy (P1), and let $\omega_0 > 0$ be such that, for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, the following holds:*

$$\begin{gathered}\tilde{a}_j(\lambda) = -\widetilde{k_1}(\lambda)\tilde{c}(\lambda)^{1+\alpha_n-\alpha_j}\frac{\lambda}{\tilde{k}(1/\tilde{c}(\lambda))}, \quad j \in \mathbb{N}_{n-1}, \\ \tilde{a}_0(\lambda) = -\widetilde{k_1}(\lambda)\tilde{c}(\lambda)^{1+\alpha_n-\alpha}\frac{\lambda}{\tilde{k}(1/\tilde{c}(\lambda))}.\end{gathered} \tag{4.9}$$

Assume, additionally, that there exist a number $z \in \mathbb{C}$ and a function $k_2(t)$ satisfying (P1) so that, for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, one has:

$$\frac{\tilde{k}_1(\lambda)}{\tilde{k}(1/\tilde{c}(\lambda))} = z + \tilde{k}_2(\lambda). \tag{4.10}$$

Then there exists an exponentially equicontinuous k_1-regularized C_1-existence family for (4.5).

(ii) Suppose there is an exponentially equicontinuous k-regularized C_2-uniqueness family for (1.1)*. Let $c(t)$ be completely positive, let $c(t)$, $k(t)$ and $k_1(t)$ satisfy (P1), and let $\omega_0 > 0$ be such that, for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, the following holds:*

$$\tilde{a}_j(\lambda) = \tilde{c}(\lambda)^{\alpha_n-\alpha_j}, j \in \mathbb{N}^0_{n-1}, \qquad \tilde{k}_1(\lambda) = \lambda^{-1}\tilde{c}(\lambda)^{m_n-2}\tilde{k}\left(\frac{1}{\tilde{c}(\lambda)}\right). \tag{4.11}$$

Then there exists an exponentially equicontinuous k_1-regularized C_2-uniqueness family for (4.5).

It is not difficult to reformulate Theorem 4.4 for the class of strong C-propagation families (cf. also Example 5.3 below).

Although our analysis tends to be exhaustive, we cannot cover, in this limited space, many interested subjects. For example, the characterizations of some special classes of q-exponentially equicontinuous k-regularized (C_1, C_2)-existence and uniqueness families in complete locally convex spaces. We also leave to the interested reader the problem of clarifying the Trotter-Kato type theorems for introduced classes.

5. Examples and Applications

We start this section with the following example.

Example 5.1. Suppose $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$ and, for every $i \in \mathbb{N}^0_{m_n-1}$ with $m-1 \geq i$, one has $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$ and $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} |c_j|^2 > 0$. Let $A_j = c_j I$ for $1 \leq j \leq n-1$.

(i) (a) Suppose $0 < \delta \leq 2$, $\sigma \geq 1$, $(\pi\delta/2(\alpha_n - \alpha)) - (\pi/2) > 0$, and A is a subgenerator of an exponentially equicontinuous (g_δ, g_σ)-regularized C-resolvent family $(R_\delta(t))_{t\geq 0}$ which satisfies the following equality:

$$A\int_0^t g_\delta(t-s) R_\delta(s)x\, ds = R_\delta(t)x - g_\sigma(t)Cx, \quad x \in E,\ t \geq 0. \tag{5.1}$$

Put $\sigma' := \max(1, 1+(\alpha_n-\alpha)(\sigma-1)\delta^{-1})$ and $\theta := \min(\pi/2, \pi\delta/2(\alpha_n-\alpha)-(\pi/2))$. By [26, Theorem 2.7], we have that, for every sufficiently small $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that $\omega_\varepsilon + \Sigma_{(\pi/2)\delta-\varepsilon} \subseteq \rho_C(A)$ and the family $\{|\lambda|^{(\delta-\sigma)/\delta}(1 + |\lambda|^{1/\delta})(\lambda - A)^{-1}C : \lambda \in \omega_\varepsilon + \Sigma_{(\pi/2)\alpha-\varepsilon}\}$ is equicontinuous. Notice also that

$$\begin{aligned}
&\arg\left(\lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} c_j \lambda^{\alpha_j-\alpha}\right)\\
&\quad = \arg\left(\lambda^{\alpha_n-\alpha}|\lambda|^{\alpha-((\alpha_{n-1}+\alpha_n)/2)} + \sum_{j=1}^{n-1} c_j \lambda^{\alpha_j-\alpha}|\lambda|^{\alpha-((\alpha_{n-1}+\alpha_n)/2)}\right)\\
&\quad \approx \arg\left(\lambda^{\alpha_n-\alpha}|\lambda|^{\alpha-((\alpha_{n-1}+\alpha_n)/2)}\right)\\
&\quad = (\alpha_n - \alpha)\arg(\lambda), \quad \lambda \longrightarrow \infty,\ \arg(\lambda) < \frac{\pi}{\alpha_n - \alpha}.
\end{aligned} \tag{5.2}$$

Due to the choice of θ, we have that, for every sufficiently small $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that, for every $\lambda \in \omega_\varepsilon + \Sigma_{(\pi/2)+\theta-\varepsilon}$, one has:

$$\arg\left(\lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} c_j \lambda^{\alpha_j-\alpha}\right) < \frac{\pi}{2}\delta - \varepsilon. \tag{5.3}$$

Therefore, we have the following: if the operator A is densely defined, then the above inequality in combination with Theorem 2.12 indicates that A is a subgenerator of an exponentially equicontinuous, analytic $(\sigma'-1)$-times integrated C-resolvent propagation family $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ being the angle of analyticity; if the operator A is not densely defined, then the above conclusion continues to hold with σ' replaced by any number $\sigma''>\sigma'$.

(a′) Suppose $0<\delta\leq 2$, $\sigma\geq 1$, $(\delta((\pi/2)+\gamma)/(\alpha_n-\alpha))-(\pi/2)>0$, A is a subgenerator of an exponentially equicontinuous, analytic (g_δ,g_σ)-regularized C-resolvent family $(R_\delta(t))_{t\geq 0}$ of angle $\gamma\in(0,\pi/2]$, and (5.1) holds. Put $\sigma_1:=\sigma'$ and $\theta_1:=\min(\pi/2,(\delta((\pi/2)+\gamma)/(\alpha_n-\alpha))-(\pi/2))$. If the operator A is densely defined, then it follows from [26, Theorem 3.6] and the above analysis that the operator $C^{-1}AC$ is the integral generator of an exponentially equicontinuous, analytic (σ_1-1)-times integrated C-resolvent propagation family $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ_1 being the angle of analyticity; if the operator A is not densely defined, then the above conclusion continues to hold with σ_1 replaced by any number $\sigma_2>\sigma_1$. Now we will apply this result to the following fractional analogue of the telegraph equation:

$$\mathbf{D}_t^{\alpha_2}u(t,x)+c_1\mathbf{D}_t^{\alpha_1}u(t,x)=D\Delta_x u(t,x),\quad t>0,\ x\in\mathbb{R}^n, \tag{5.4}$$

where $c_1>0$, $D>0$ and $0<\alpha_1\leq\alpha_2<2$. Let E be one of the spaces $L^p(\mathbb{R}^n)$ $(1\leq p\leq\infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ and $0\leq l\leq n$. Put $\mathbb{N}_0^l:=\{\alpha\in\mathbb{N}_0^n:\alpha_{l+1}=\cdots=\alpha_n=0\}$ and recall that the space E_l $(0\leq l\leq n)$ is defined by $E_l:=\{f\in E: f^{(\alpha)}\in E \text{ for all } \alpha\in\mathbb{N}_0^l\}$. The totality of seminorms $(q_\alpha(f):=\|f^{(\alpha)}\|_E, f\in E_l;\ \alpha\in\mathbb{N}_0^l)$ induces a Fréchet topology on E_l. Let T_l possess the same meaning as in [33], and let $A:=D\Delta$ act with its maximal distributional domain. Suppose first $E\neq L^\infty(\mathbb{R}^n)$ and $E\neq C_b(\mathbb{R}^n)$. Then the operator A is the integral generator of an exponentially equicontinuous, analytic C_0-semigroup of angle $\pi/2$, which implies that A is the integral generator of an exponentially equicontinuous, analytic I-regularized resolvent propagation family $(R_0(t))_{t\geq 0}$, if $\alpha_2\leq 1$, respectively, $((R_0(t))_{t\geq 0},(R_1(t))_{t\geq 0})$ if $\alpha_2>1$, of angle $\zeta=\min(\pi/2,(\pi/\alpha_2)-(\pi/2))$; the established conclusion also holds in the Fréchet nuclear space Ξ which consists of those smooth functions on $\mathbb{R}^n$ with period 1 along each coordinate axis [26]. In this place, we would like to observe that it is not clear whether the angle of analyticity of constructed I-regularized resolvent propagation families, in the case that $\alpha_1<\alpha_2<1$, can be improved by allowing that ζ takes the value $\min(\pi,(\pi/\alpha_2)-(\pi/2))$. Suppose now $E=L^\infty(\mathbb{R}^n)$ or $E=C_b(\mathbb{R}^n)$. Then, for every $\sigma'>1$, the operator A is the integral generator of an exponentially equicontinuous, analytic $(\sigma'-1)$-times integrated I-regularized resolvent propagation family $(R_0(t))_{t\geq 0}$, if $\alpha_2\leq 1$, respectively, $((R_0(t))_{t\geq 0},(R_1(t))_{t\geq 0})$ if $\alpha_2>1$, of angle $\min(\pi/2,(\pi/\alpha_2)-(\pi/2))$.

(b) Suppose $0<\delta\leq 2$, $\sigma\geq 1$, $(\pi\delta/2(\alpha_n-\alpha))-(\pi/2)>0$, $a>0$, $b\in(0,1)$, $k_{a,b}(t):=\mathcal{L}^{-1}(\exp(-a\lambda^b))(t)$, $t\geq 0$ and A is a subgenerator of an exponentially equicontinuous $(g_\delta, k_{a,b})$-regularized C-resolvent family $(R_{a,b}(t))_{t\geq 0}$ which satisfies the following equality:

$$A\int_0^t g_\delta(t-s)R_{a,b}(s)x\,ds = R_{a,k}(t)x - k_{a,b}(t)Cx, \quad x \in E,\ t \geq 0. \tag{5.5}$$

Let θ be defined as in (a). Then it is checked at once that $(\alpha_n - \alpha)b\delta^{-1} < 1$ and $(\alpha_n - \alpha)b\delta^{-1}((\pi/2)+\theta) < \pi/2$. Put $k_1(t) := k_{a_1,b_1}(t)$, $t \geq 0$, where $b_1 := (\alpha_n - \alpha)b\delta^{-1}$ and $a_1 > a(\cos((\alpha_n-\alpha)b\delta^{-1}((\pi/2)+\theta)))^{-1}$. It is clear that, for every $\theta' \in (0,\theta)$, there exists a sufficiently large $\omega_{\theta'} > 0$ such that, for every $\lambda \in \omega_{\theta'} + \Sigma_{(\pi/2)+\theta'}$,

$$\begin{aligned} &\frac{\left|\widetilde{k_1}(\lambda)\right|}{\left|\widetilde{k}\left(\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}c_j\lambda^{\alpha_j-\alpha}\right)^{1/\delta}\right)\right|} \\ &\quad \leq \left|\widetilde{k_1}(\lambda)\right|\exp\left(a|\lambda|^{b_1}+\sum_{j=1}^{n-1}\left|c_j\right||\lambda|^{(\alpha_j-\alpha)b/\delta}\right). \end{aligned} \tag{5.6}$$

Arguing as in (a), we reveal that A is a subgenerator of an exponentially equicontinuous, analytic k_1-regularized C-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ being the angle of analyticity.

(b′) Suppose $0 < \delta \leq 2$, $\sigma \geq 1$, $\delta(((\pi/2)+\gamma)/(\alpha_n-\alpha)) - (\pi/2) > 0$, $A_j = c_jI$ $(1 \leq j \leq n-1)$, $a > 0$, $b \in (0,1)$, A is a subgenerator of an exponentially equicontinuous, analytic $(g_\delta, k_{a,b})$-regularized C-resolvent family $(R_{a,b}(t))_{t\geq 0}$ of angle $\gamma \in (0,\pi/2]$, and (5.5) holds. Assume, additionally, that $b(1+(2\gamma/\pi)) \leq 1$. Define θ_1 as in $(a)'$, and $k_2(t) := k_{a_2,b_2}(t)$, $t \geq 0$, where $b_2 := (\alpha_n-\alpha)b\delta^{-1}$ and $a_2 > a(\cos((\alpha_n-\alpha)b\delta^{-1}((\pi/2)+\theta_1)))^{-1}$. Then one can simply verify that $(\alpha_n-\alpha)b < \delta$ and $(\alpha_n-\alpha)b\delta^{-1}((\pi/2)+\gamma) \leq \pi/2$. Making use of [26, Theorem 3.6] and the foregoing arguments, we obtain that the operator $C^{-1}AC$ is the integral generator of an exponentially equicontinuous, analytic k_2-regularized C-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ being the angle of analyticity. Before proceeding further, we would like to recommend for the reader [14, 20, 21, 26, 30, 34] for some examples of (nondensely defined, in general) differential operators generating various types of $(g_\sigma, k_{a,b})$-regularized C-resolvent families.

(ii) Suppose E is complete, $0 < \delta \leq 2$, $(\pi\delta/2(\alpha_n-\alpha)) - (\pi/2) > 0$, and A is the densely defined generator of a q-exponentially equicontinuous (g_δ, g_1)-regularized I-resolvent family $(R_\delta(t))_{t\geq 0}$ which satisfies that, for every $p \in \circledast$, there exist $M_p \geq 1$ and $\omega_p \geq 0$ such that $p(R_\delta(t)x) \leq M_p e^{\omega_p t}p(x)$, $t \geq 0$, $x \in E$. By [20, Theorem 3.1], we infer that A is a compartmentalized operator and that, for every $p \in \circledast$, the operator $\overline{A_p}$ is the integral generator of an exponentially bounded (g_δ, g_1)-regularized $\overline{I_p}$-resolvent family in $\overline{E_p}$. Then the first part of this example shows that $\overline{A_p}$ is the integral generator of an exponentially bounded, analytic $\overline{I_p}$-resolvent propagation family, with $\min(\pi/2, (\pi\delta/2(\alpha_n-\alpha)) - (\pi/2))$ being the angle of analyticity. By Theorem 2.13(ii), we obtain that A is the integral generator of a q-exponentially equicontinuous, analytic I-resolvent propagation family $((R_0(t))_{t\in[0,\tau)},\dots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ for (1.1), and that the corresponding angle of analyticity is $\min(\pi/2,(\pi\delta/2(\alpha_n-\alpha))-(\pi/2))$. It can be simply shown that, for every $p\in\circledast$ and $i\in\mathbb{N}^0_{m_n-1}$, there exist $M_{p,i}\geq 1$ and $\omega_{p,i}\geq 0$ such that $p(R_i(t)x)\leq M_{p,i}e^{\omega_{p,i}t}p(x)$, $t\geq 0$, $x\in E$. In the continuation, we will also present some other applications of (a,k)-regularized C-resolvent families in the analysis of some special cases of (1.1); as already mentioned, this theory is inapplicable if some of initial values $u_0,\dots,u_{m_n-1}$ is a non-zero element of E. Consider the abstract Basset-Boussinesq-Oseen equation (1.2) and assume that E is complete. Set $a_\alpha(t):=\mathcal{L}^{-1}(\lambda^\alpha/(\lambda+1))(t)$, $t\geq 0$, $k_\alpha(t):=e^{-t}$, $t\geq 0$ and $\delta_\alpha:=\min(\pi/2,(\pi\alpha/2(1-\alpha)))$. Suppose A is the integral generator of a q-exponentially equicontinuous (g_1,g_1)-regularized I-resolvent family $(R(t))_{t\geq 0}$ satisfying (2.37); cf. [20, 25] for important examples of differential operators generating q-exponentially equicontinuous (g_δ,g_1)-regularized I-resolvent families. Then it has been proved in [20] that A is the integral generator of a q-exponentially equicontinuous, analytic (a_α,k_α)-regularized resolvent family of angle δ_α. Notice, finally, that the choice of function $a_\alpha(t)$ instead of $g_1(t)$ has some advantages.

Example 5.2. Suppose $1\leq p\leq\infty$, $E:=L^p(\mathbb{R})$, $m:\mathbb{R}\to\mathbb{C}$ is measurable, $a_j\in L^\infty(\mathbb{R})$, $(A_jf)(x):=a_j(x)f(x)$, $x\in\mathbb{R}$, $f\in E$ $(1\leq j\leq n-1)$ and $(Af)(x):=m(x)f(x)$, $x\in\mathbb{R}$, with maximal domain. Assume $s\in(1,2)$, $\delta=1/s$, $M_p=p!^s$ and $k_\delta(t)=\mathcal{L}^{-1}(e^{-\lambda^\delta})(t)$, $t\geq 0$. Denote by $M(t)$ the associated function of the sequence (M_p) [30] and put $\Lambda_{\alpha',\beta',\gamma'}:=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geq{\gamma'}^{-1}M(\alpha'\lambda)+\beta'\}$, $\alpha'>0$, $\beta'>0$, $\gamma'>0$. Clearly, there exists a constant $C_s>0$ such that $M(\lambda)\leq C_s|\lambda|^{1/s}$, $\lambda\in\mathbb{C}$. Hereafter we assume that the following condition holds:

(H) for every $\tau>0$, there exist $\alpha'>0$, $\beta'>0$ and $d>0$ such that $\tau\leq\cos(\delta\pi/2)/C_s{\alpha'}^{1/s}$ and

$$\left|\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)\right|\geq d,\quad x\in\mathbb{R},\ \lambda\in\Lambda_{\alpha',\beta',1}. \tag{5.7}$$

Notice that the above condition holds provided $n=2$, $\alpha_2-\alpha=2$, $\alpha_2-\alpha_1=1$ and $m(x)=(1/4)a_1^2(x)-(1/16)a_1^4(x)-1$, $x\in\mathbb{R}$ (cf. [31]), and that the validity of condition (H) does not imply, in general, the essential boundedness of the function $m(\cdot)$. We will prove that A is the integral generator of a global (not exponentially bounded, in general) k_δ-regularized I-resolvent propagation family $((R_0(t))_{t\geq 0},\dots,(R_{m_n-1}(t))_{t\geq 0})$ for (1.1). Clearly, it suffices to show that, for every $\tau\in(0,\infty)$, A is the integral generator of a local k_δ-regularized I-resolvent propagation family for (1.1) on $[0,\tau)$. Suppose that $\tau>0$ is given in advance, and that $\alpha'>0$, $\beta'>0$ and $d>0$ satisfy (H), for this τ. Let Γ denote the upwards oriented boundary of ultralogarithmic region $\Lambda_{\alpha',\beta',1}$. Put, for every $t\in[0,\tau)$, $f\in E$ and $x\in\mathbb{R}$,

$$(R_i(t)f)(x):=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}\,d\lambda, \tag{5.8}$$

if $m-1<i$, and

$$\left(R_i(t)f\right)(x) := \frac{(-1)}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\lambda^{\alpha_j-\alpha-i}a_j(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}\,d\lambda + \left(k_\delta * g_i\right)(t)f(x), \quad (5.9)$$

if $m-1\geq i$. It is clear that, for every $i\in\mathbb{N}^0_{m_n-1}$, $R_i(t)A_j\subseteq A_jR_i(t)$, $t\in[0,\tau)$, $j\in\mathbb{N}^0_{n-1}$ and that $(R_i(t))_{t\in[0,\tau)}\subseteq L(E)$ is strongly continuous. Furthermore, the Cauchy theorem implies that $R_i(0)=0=k_\delta(0)$, $i\in\mathbb{N}^0_{m_n-1}$. Now we will prove that the identity (2.6) holds provided $m-1<i$ and $C_2=I$. Let $f\in D(A)$. Then a straightforward computation involving Cauchy theorem shows that (2.6) holds, with x replaced by $f(\cdot)$ therein, if and only if:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&\quad+\sum_{j=1}^{n-1}\frac{1}{2\pi i}\int_\Gamma\left(\int_0^t g_{\alpha_n-\alpha_j}(t-s)e^{\lambda s}\,ds\right)e^{-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{l\in D_i}\lambda^{\alpha_l-\alpha-i}g_l(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{l=1}^{n-1}\lambda^{\alpha_l-\alpha}a_l(x)-m(x)}d\lambda\\
&\quad-\frac{1}{2\pi i}\int_\Gamma\left(\int_0^t g_{\alpha_n-\alpha}(t-s)e^{\lambda s}\,ds\right)e^{-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]m(x)f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\left[\lambda^{-i}f(x)+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha_n-i}a_j(x)f(x)\right]d\lambda.
\end{aligned} \quad (5.10)$$

Using [28, Lemma 5.5, page 23] and the Cauchy theorem, the above equality is equivalent with:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&\quad+\sum_{j=1}^{n-1}\frac{1}{2\pi i}\int_\Gamma\frac{e^{\lambda t-\lambda^\delta}}{\lambda^{\alpha_n-\alpha_j}}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{l\in D_i}\lambda^{\alpha_l-\alpha-i}g_l(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{l=1}^{n-1}\lambda^{\alpha_l-\alpha}a_l(x)-m(x)}d\lambda\\
&\quad-\frac{1}{2\pi i}\int_\Gamma\frac{e^{\lambda t-\lambda^\delta}}{\lambda^{\alpha_n-\alpha}}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]m(x)f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\left[\lambda^{-i}f(x)+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha_n-i}a_j(x)f(x)\right]d\lambda,
\end{aligned} \quad (5.11)$$

which is true because the integrands appearing on both sides of this equality are equal identically. One can similarly prove that the identity (2.6) holds provided $m-1\geq i$ and $C_2=I$, so that $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$, defined in the obvious way, is a k_δ-regularized I-resolvent propagation family for (1.1), with subgenerator A. Notice that the condition (H) implies $m(\cdot)/(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(\cdot)-m(\cdot))\in L^{\infty}(\mathbb{R})$ for all $\lambda\in\Lambda_{\alpha',\beta',1}$, which has as a further consequence that $R(R_i(t))\subseteq D(A)$, provided $t\geq 0$ and $m-1<i$, and that $R(R_i(t)-(k_\delta * g_i)(t))\subseteq D(A)$, provided $t\geq 0$ and $m-1\geq i$. The equality (2.5) holds for $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$, the integral generator of $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$, defined similarly as in the second section, coincides with the operator A, which is the unique subgenerator of $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$. Notice that, for every compact set $K\subseteq[0,\infty)$, there exists $h_K>0$ such that

$$\sup_{t\in K,p\in\mathbb{N}_0,i\in\mathbb{N}^0_{m_n-1}}\frac{\left\|h_k^p(d^p/dt^p)R_i(t)\right\|}{p!^s}<\infty, \tag{5.12}$$

and that one can similarly consider the generation of local $k_{1/2}$-regularized I-resolvent propagation families which oblige a modification of the property stated above with $s=2$. Now we would like to give an example of k_δ-regularized I-resolvent propagation family for (1.1) in which $A_j\notin L(E)$ for some $j\in\mathbb{N}_{n-1}$. Assume $n=2$, $\alpha_2-\alpha=2$, $\alpha_2-\alpha_1=1$, $a_1(x)=-2x$, $x\in\mathbb{R}$ and $m(x)=x^2-x^4-1$, $x\in\mathbb{R}$. Define A_1, A and $R_i(\cdot)$ as before ($i=0,1$). Then the established conclusions continue to hold since, for every $\tau>0$, there exist $\alpha'>0$, $\beta'>0$ and $d>0$ such that (H) holds as well as that:

$$\frac{x^2+(x^4-x^2+1)|\lambda|^{-2}}{\left|\lambda^2-2x\lambda+(x^4-x^2+1)\right|}\leq d,\quad x\in\mathbb{R},\quad \lambda\in\Lambda_{\alpha',\beta',1}. \tag{5.13}$$

Notice, finally, that it is not so difficult to construct examples of local k-regularized C-resolvent propagation families which cannot be extended beyond its maximal interval of existence.

Example 5.3. Suppose $1\leq p\leq\infty$, $X:=L^p(\mathbb{R})$, $a\in\mathbb{R}$, $r>0$, $\vartheta(\cdot)\in W^{1,\infty}(\mathbb{R})$, $1/2<\gamma\leq 1$, $T>0$, $f\in C([0,T]:X)$, and $(d/dt)(g_{2\gamma-1}*(d/dx)f(t,\cdot))\in C([0,T]:X)$. Put $A_1:=ad/dx$ and $Au:=r\Delta u-\vartheta(\cdot)u$ with maximal distributional domain. Now we will focus our attention to the following fractional analogue of damped Klein-Gordon equation:

$$\begin{gathered}\mathbf{D}_t^{2\gamma}u(t,x)+a\frac{\partial}{\partial x}\mathbf{D}_t^{\gamma}u(t,x)-r\Delta_x u(t,x)+\vartheta(x)u(t,x)=f(t,x),\quad t>0,\quad x\in\mathbb{R},\\ u(0,x)=\phi(x),\qquad u_t(0,x)=\psi(x),\quad x\in\mathbb{R}.\end{gathered} \tag{5.14}$$

The case $\gamma=1$ has been analyzed in [24, Example 4.1], showing that there exists an exponentially bounded I-uniqueness family for (5.14) and that, for every $\mu_0\in\rho(A_1)$, there exists an exponentially bounded $(\mu_0-A_1)^{-1}$-existence family for (5.14) with $Y=X$. It is worth noting that the estimates obtained in cited example enables one to simply verify that the conditions of Theorem 4.1(i)-(ii) hold with $k(t)=1$ and $C_1=(\mu_0-A_1)^{-1}$, and that the conditions of Theorem 4.1(iii) hold with $k(t)=t$ and $C_2=I$. This implies that there exists an exponentially bounded $g_{2\gamma}$-regularized I-uniqueness family $(U_\gamma(t))_{t\geq 0}$ for (5.14) with $\alpha_j=j\gamma$, $j=0,1,2$, and that there exists an exponentially bounded $(\mu_0-A_1)^{-1}$-existence family $(E_\gamma(t))_{t\geq 0}$ for (5.14) with $\alpha_j=j\gamma$, $j=0,1,2$. Applying Theorem 3.7, we obtain that, for every $\phi\in W^{3,p}(\mathbb{R})$ and $\psi\in W^{3,p}(\mathbb{R})$, there exists a unique mild solution $u(t,x)$ of the corresponding problem (3.21) as well as that there exist $M \geq 1$ and $\omega \geq 0$ such that the following estimate holds for each $t \geq 0$:

$$\|u(t,x)\|_{L^p(\mathbb{R})} \leq Me^{\omega t}\Bigg[\|\phi\|_{W^{1,p}(\mathbb{R})} + \|\psi\|_{W^{1,p}(\mathbb{R})} + \int_0^t (t-s)^{2\gamma-2}\|f(s,\cdot)\|_{L^p(\mathbb{R})}ds \\ + \int_0^t \left\|\frac{d}{ds}\left(g_{2\gamma-1} * \frac{d}{dx}f(s,\cdot)\right)\right\|_{L^p(\mathbb{R})} ds\Bigg]. \tag{5.15}$$

It is checked at once that the solution $u(t,x)$ is analytically extensible to the sector $\Sigma_{((1/\gamma)-1)(\pi/2)}$, provided that $f(t,x) \equiv 0$. Suppose now $\vartheta(x) \equiv \vartheta > 0$, $\kappa \geq |1/2 - 1/p|$, provided $1 < p < \infty$, respectively, $\kappa > 1/2$, provided $p \in \{1, \infty\}$, $C := (1-\Delta)^{-(1/2)\kappa}$ and $f(t,x) \equiv 0$. Then there exists a strong C-propagation family $\{(S_0(t))_{t\geq 0}, (S_1(t))_{t\geq 0}\}$ for the problem (5.14) with $\gamma = 1$ (cf. [28, Example 5.8, page 130]). Using [10, (1.23), page 12; Theorems 3.1–3.3, pages 40–42] and [28, Proposition 5.3(iii), page 116], it readily follows that, for every $\phi \in W^{p,2}(\mathbb{R})$ and $\psi \in W^{p,2}(\mathbb{R})$, the function $u_\gamma(t,\cdot)$, $t > 0$, given by

$$u_\gamma(t,\cdot) := \int_0^\infty t^{-\gamma}\Phi(st^{-\gamma})\left[S_1(s)\phi + S_1'(s)\phi\right]ds \\ + \int_0^t g_{1-\gamma}(t-s)\int_0^\infty s^{-\gamma}\Phi(rs^{-\gamma})S_1(r)\psi dr ds, \tag{5.16}$$

is a unique strong solution of the corresponding integral equation (3.21) with $u_0 = C\phi$ and $u_1 = C\psi$; obviously, this solution is analytically extensible to the sector $\Sigma_{((1/\gamma)-1)(\pi/2)}$. Notice also that one can similarly consider (cf. [24, Example 4.2] for more details) the results concerning the existence and uniqueness of mild solutions of the following time-fractional equation:

$$\mathbf{D}_t^{2\gamma}u(t,x) + \left(\rho_1\frac{\partial^3}{\partial x^3} - \rho_2\frac{\partial^2}{\partial x^2}\right)\mathbf{D}_t^{\gamma}u(t,x) + \left(c\frac{\partial^2}{\partial x^2} + a(x)\right)u(t,x) = f(t,x), \tag{5.17}$$

$$u(0,x) = \phi(x), \qquad u_t(0,x) = \psi(x), \tag{5.18}$$

and that Theorem 4.4 can be applied in the analysis of the following integral equation:

$$u(t,x) = a\int_0^t a_1(t-s)\frac{\partial}{\partial x}u(s,x)ds + \int_0^t a_2(t-s)\left[r\Delta_x u(s,x) - \vartheta(x)u(s,x)\right]ds + f(t,x), \tag{5.19}$$

for certain kernels $a_1(t)$ and $a_2(t)$. We leave details to the interested reader.

Consider now the following slight modification of (5.14):

$$\mathbf{D}_t^{2\gamma}u(t,x)+a\frac{\partial}{\partial x}\mathbf{D}_t^{\gamma}u(t,x)-re^{i(2-2\gamma)(\pi/2)}\Delta_x u(t,x)+\vartheta(x)u(t,x)=f(t,x),\quad t>0,\ x\in\mathbb{R},$$
$$u(0,x)=\phi(x),\qquad \left(\mathbf{D}_t^{\gamma}u(t,x)\right)_{|t=0}=\psi(x),\quad x\in\mathbb{R}. \tag{5.20}$$

Suppose now that $a\neq 0$ (for further information concerning the case $a=0$, [21, 23] may be of some importance). Although the equality $\mathbf{D}_t^{2\gamma}u(t,x)=\mathbf{D}_t^{\gamma}u(t,x)\mathbf{D}_t^{\gamma}u(t,x)$ does not hold in general, we would like to point out that the existence and uniqueness of mild solutions to the homogeneous counterpart of (5.20) cannot be so easily proved for initial values belonging to the Sobolev space $W^{k,p}(\mathbb{R})$, for some $k\in\mathbb{N}$. In order to better explain this, we will introduce the new function $v(t,x)$ by $v(t,x):=\mathbf{D}_t^{\gamma}u(t,x)$. Then (5.20) can be rewritten in the following equivalent matricial form:

$$\mathbf{D}_t^{\gamma}\left[u(t,x)\ \ v(t,x)\right]^T=\begin{bmatrix}0 & 1\\ -re^{i(2-2\gamma)\pi/2} & -aix\end{bmatrix}(A)\left[u(t,x)\ \ v(t,x)\right]^T,\quad t\geq 0, \tag{5.21}$$

where $A=-id/dx$; see, for example, [35, 36]. The characteristic values of associated polynomial matrix $P(x):=\left[\begin{smallmatrix}0 & 1\\ -re^{i(2-2\gamma)(\pi/2)} & -aix\end{smallmatrix}\right]$ are $\lambda_{1,2}(x)=(1/2)(-aix\pm\sqrt{a^2+4re^{i(2-2\gamma)(\pi/2)}})$, $x\in\mathbb{R}$, which implies that the condition of Petrovskii for systems of abstract time-fractional equations, that is, $\sup_{x\in\mathbb{R}}\Re((\lambda_{1,2}(x))^{1/\gamma})<\infty$, is not satisfied [36]. Notice, finally, that (1.1) cannot be converted to an equivalent matrix form, except for some very special values of $\alpha_0,\ldots,\alpha_n$.

Before proceeding further, we would like to observe that several examples of k-times integrated (C_1,C_2)-existence and uniqueness families, acting on products of possibly different Banach spaces ($k\in\mathbb{N}$), can be constructed following the consideration given in [37, Section 7].

Example 5.4. Let $s'>1$,

$$E:=\left\{f\in C^{\infty}[0,1];\ \|f\|:=\sup_{p\geq 0}\frac{\|f^{(p)}\|_{\infty}}{p!^{s'}}<\infty\right\},$$
$$A:=-\frac{d}{ds},\qquad D(A):=\left\{f\in E;\ f'\in E, f(0)=0\right\}. \tag{5.22}$$

Then $\rho(A)=\mathbb{C}$, and for every $\eta>1$, $\|R(\lambda:A)\|=O(e^{\eta|\lambda|})$, $\lambda\in\mathbb{C}$ [21]. Consider now the complex non-zero polynomials $P_j(z)=\sum_{l=0}^{n_j}a_{j,l}z^l$, $z\in\mathbb{C}$, $a_{j,n_j}\neq 0$ $(0\leq j\leq n-1)$, and define, for every $\lambda\in\mathbb{C}$ and $j\in\mathbb{N}_{n-1}^0$, the operator $P_j(A)$ by $D(P_j(A)):=D(A^{n_j})$ and $P_j(A)f:=\sum_{l=0}^{n_j}a_{j,l}A^lf$, $f\in D(P_j(A))$. Our intention is to analyze the smoothing properties of solutions of the equation (3.21) with $A_j:=p_j(A)$, $j\in\mathbb{N}_{n-1}^0$, $u_k=0$, $k\in\mathbb{N}_{m_n-1}^0$, and a suitable chosen function $f(t)$. In order to do that, set $N:=\max(dg\ (P_0),\ldots,dg\ (P_{n-1}))$,

$\mathcal{P}_\lambda(z) := 1+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha_n}P_j(z)-\lambda^{\alpha-\alpha_n}P_0(z)$ $(\lambda\in\mathbb{C}\setminus\{0\}, z\in\mathbb{C})$, and after that, $\Phi := \{\lambda\in\mathbb{C}\setminus\{0\} : dg\ (\mathcal{P}_\lambda(\cdot)) = N, \mathcal{P}_\lambda(0)\neq 0\}$. Then it is not difficult to prove (cf. [21, Example 2.10]) that, for every $\lambda\in\mathbb{C}\setminus\{0\}$, $\mathbf{P}_\lambda^{-1} = (I+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha_n}A_j-\lambda^{\alpha-\alpha_n}A)^{-1}\in L(E)$ and that

$$\mathbf{P}_\lambda^{-1} = (-1)^N g(\lambda)^{-1}R(z_{1,\lambda} : A)\cdots R(z_{N,\lambda} : A), \quad \lambda\in\Phi, \tag{5.23}$$

where $z_{1,\lambda},\ldots,z_{N,\lambda}$ denote the zeroes of $\mathcal{P}_\lambda(z)$ and $g(\lambda) := N!^{-1}\mathcal{P}_\lambda^{(N)}(0)$, $\lambda\in\Phi$. Suppose now that the following condition holds:

(H) there exist $\sigma\in(0,1)$, $\omega>0$ and $m>0$ such that, for every $j\in\mathbb{N}_{n-1}^0$, one has: $|z_{j,\lambda}|\leq m|\lambda|^\sigma$, $\lambda\in\Phi$, $\Re\lambda>\omega$.

It is well known from the elementary courses of numerical analysis [38] that the condition:

(H_1) there exist $\sigma\in(0,1)$, $\omega>0$ and $m>0$ such that, for every $j\in\mathbb{N}_{n-1}^0$, one has:

$$\left|\frac{N!\mathcal{P}_\lambda^{(j)}(0)}{j!\mathcal{P}_\lambda^{(N)}(0)}\right|^{1/(N-j)} \leq \frac{1}{2}m|\lambda|^\sigma, \quad \lambda\in\Phi,\ \Re\lambda>\omega, \tag{5.24}$$

implies (H). The validity of last condition can be simply verified in many concrete situations, and it seems that slightly better estimates can be obtained only in the case of very special equations of the form (1.1). We would also like to point out that the condition (H) need not to be satisfied, in general. Using (5.23), the inequality $\|A^lR(\mu_1 : A)\cdots R(\mu_l : A)\| \leq (1+|\mu_1|\|R(\mu_1 : A)\|)\cdots(1+|\mu_l|\|R(\mu_l : A)\|)$ $(l\in\mathbb{N}, \mu_1,\ldots,\mu_l\in\mathbb{C})$, as well as the continuity of mappings $\lambda\mapsto\mathbf{P}_\lambda^{-1}$, $\Re\lambda>\omega$ and $\lambda\mapsto A_j\mathbf{P}_\lambda^{-1}$, $\Re\lambda>\omega$, for $0\leq j\leq n-1$, we obtain the existence of a positive polynomial $p(\cdot)$ such that

$$\left\|\mathbf{P}_\lambda^{-1}\right\| + \sum_{j=0}^{n-1}\left\|A_j\mathbf{P}_\lambda^{-1}\right\| \leq p(|\lambda|)e^{mN|\lambda|^\sigma}, \quad \Re\lambda>\omega. \tag{5.25}$$

In what follows, we will use the following family of kernels. Define, for every $l>0$, the entire function $\omega_l(\cdot)$ by $\omega_l(\lambda) := \prod_{p=1}^\infty(1+(l\lambda/p^s))$, $\lambda\in\mathbb{C}$, where $s := \sigma^{-1}$. Then it is clear that $|\omega_l(\lambda)| \geq \sup_{k\in\mathbb{N}}\prod_{p=1}^k|1+(l\lambda/p^s)| \geq \sup_{k\in\mathbb{N}}\prod_{p=1}^k l|\lambda|/p^s \geq \sup_{k\in\mathbb{N}}(l|\lambda|)^k/p!^s$, $\lambda\in\mathbb{C}$, $\Re\lambda\geq 0$. Hence, $|\omega_l(\lambda)|\geq e^{M(l|\lambda|)}$, $\lambda\in\mathbb{C}$, $\Re\lambda\geq 0$, where $M(\lambda) := \sup_{p\in\mathbb{N}_0}\ln|\lambda|^p/p!^s$, $\lambda\in\mathbb{C}\setminus\{0\}$ and $M(0) := 0$. It is also worth noting that, for every $\zeta\in(0,\pi/2)$, $p\in\mathbb{N}_0$ and $\lambda\in\Sigma_{(\pi/2)+\zeta}$, we have $|1+(l\lambda/p^s)|\geq l|\Im\lambda|/p^s\geq l(1+\tan\zeta)^{-1}|\lambda|/p^s$, and

$$|\omega_l(\lambda)|\geq e^{M(l(1+\tan\zeta)^{-1}|\lambda|)}, \quad \zeta\in\left(0,\frac{\pi}{2}\right),\ l>0,\ \lambda\in\Sigma_{(\pi/2)+\zeta}. \tag{5.26}$$

Put now $K_l(t) := \mathcal{L}^{-1}(1/\omega_l(\lambda))(t)$, $t\geq 0$, $l>0$. Then, for every $l>0$, $0\in\operatorname{supp}K_l$, $K_l(0)=0$ and $K_l(t)$ is infinitely differentiable for $t\geq 0$. By Theorem 3.5(i)-(b) and (iii), we easily infer from (5.25) that there exists $k>0$ such that, for every $l>k$, there exists an exponentially bounded K_l-regularized I-resolvent family $(E_l(t))_{t\geq 0}$ for (1.1), with $Y=X=E$. Furthermore, the mapping $t \mapsto E_l(t)$, $t \geq 0$ is infinitely differentiable in the uniform operator topology of $L(E)$ and, for every compact set $K \subseteq [0,\infty)$ and for every $l > k$, there exists $h_{K,l} > 0$ such that

$$\sup_{p\geq 0, t\in K} \frac{h_{K,l}^{p} \left\| E_l^{(p)}(t) \right\|}{p!^{s}} < \infty. \tag{5.27}$$

One can similarly construct examples of exponentially bounded, analytic K_l-regularized I-resolvent families.

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Research Article

Existence of Mild Solutions for a Class of Fractional Evolution Equations with Compact Analytic Semigroup

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This paper deals with the existence of mild solutions for a class of fractional evolution equations with compact analytic semigroup. We prove the existence of mild solutions, assuming that the nonlinear part satisfies some local growth conditions in fractional power spaces. An example is also given to illustrate the applicability of abstract results.

1. Introduction

The differential equations involving fractional derivatives in time have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics, and science. Numerous applications can be found in electrochemistry, control, porous media, electromagnetic, see for example, [1–5] and references therein. Hence the study of such equations has become an object of extensive study during recent years, see [6–23] and references therein.

In this paper, we consider the existence of the following fractional evolution equation:

$$\begin{gathered} D^q\, u(t) + Au(t) = f(t, u(t), Gu(t)), \quad t \in J = [0,\ T], \\ u(0) = x_0, \end{gathered} \tag{1.1}$$

where D^q is the Caputo fractional derivative of order $q \in (0,1)$, $-A$ is the infinitesimal generator of a compact analytic semigroup $S(\cdot)$ of uniformly bounded linear operators, f is the nonlinear term and will be specified later, and

$$Gu(t) = \int_0^t K(t,s)u(s)ds \tag{1.2}$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, R^+), \Delta = \{(t,s) : 0 \leq s \leq t \leq T\}, R^+ = [0, +\infty)$. Throughout this paper, we denote by $K^* := \max_{(t,s)\in\Delta} K(t,s)$

In some existing articles, the fractional differential equations were treated under the hypothesis that nonlinear term satisfies Lipschitz conditions or linear growth conditions. It is obvious that these conditions are not easy to be verified sometimes. To make the things more applicable, in this work, we will prove the existence of mild solutions for (1.1) under some new conditions. More precisely, the nonlinear term only satisfies some local growth conditions (see conditions (H_1) and (H_2)). These conditions are much weaker than Lipschitz conditions and linear growth conditions. The main techniques used here are fractional calculus, theory of analytic semigroup, and Schauder fixed point theorem.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given on the fractional power of the generator of a compact analytic semigroup and the definition of mild solutions of (1.1). In Section 3, we study the existence of mild solutions for (1.1). In Section 4, an example is given to illustrate the applicability of abstract results obtained in Section 3.

2. Preliminaries

In this section, we introduce some basic facts about the fractional power of the generator of a compact analytic semigroup and the fractional calculus that are used throughout this paper.

Let X be a Banach space with norm $\|\cdot\|$. Throughout this paper, we assume that $-A : D(A) \subset X \to X$ is the infinitesimal generator of a compact analytic semigroup $S(t)$ $(t \geq 0)$ of uniformly bounded linear operator in X, that is, there exists $M \geq 1$ such that $\|S(t)\| \leq M$ for all $t \geq 0$. Without loss of generality, let $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$. Then for any $\alpha > 0$, we can define $A^{-\alpha}$ by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{(\alpha-1)} S(t) dt. \tag{2.1}$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of X. Hence we can define A^α by $A^\alpha := (A^{-\alpha})^{-1}$, which is a closed bijective linear operator in X. It can be shown that each A^α has dense domain and that $D(A^\beta) \subset D(A^\alpha)$ for $0 \leq \alpha \leq \beta$. Moreover, $A^{\alpha+\beta}x = A^\alpha A^\beta x = A^\beta A^\alpha x$ for every $\alpha, \beta \in \mathbb{R}$ and $x \in D(A^\mu)$ with $\mu := \max\{\alpha, \beta, \alpha+\beta\}$, where $A^0 = I$, I is the identity in X. (For proofs of these facts, we refer to the literature [24–26]).

We denote by X_α the Banach space of $D(A^\alpha)$ equipped with norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in D(A^\alpha)$, which is equivalent to the graph norm of A^α. Then we have $X_\beta \hookrightarrow X_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$ (with $X_0 = X$), and the embedding is continuous. Moreover, A^α has the following basic properties.

Lemma 2.1 (see [24]). *A^α has the following properties.*

(i) *$S(t) : X \to X_\alpha$ for each $t > 0$ and $\alpha \geq 0$.*

(ii) *$A^\alpha S(t)x = S(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \geq 0$.*

(iii) *For every* $t > 0$, $A^{\alpha}S(t)$ *is bounded in* X *and there exists* $M_{\alpha} > 0$ *such that*

$$\|A^{\alpha}S(t)\| \leq M_{\alpha}t^{-\alpha}. \tag{2.2}$$

(iv) $A^{-\alpha}$ *is a bounded linear operator for* $0 \leq \alpha \leq 1$ *in* X.

In the following, we denote by $C(J, X_{\alpha})$ the Banach space of all continuous functions from J into X_{α} with supnorm given by $\|u\|_C = \sup_{t\in J}\|u(t)\|_{\alpha}$ for $u \in C(J, X_{\alpha})$. From Lemma 2.1(iv), since $A^{-\alpha}$ is a bounded linear operator for $0 \leq \alpha \leq 1$, there exists a constant C_{α} such that $\|A^{-\alpha}\| \leq C_{\alpha}$ for $0 \leq \alpha \leq 1$.

For any $t \geq 0$, denote by $S_{\alpha}(t)$ the restriction of $S(t)$ to X_{α}. From Lemma 2.1(i) and (ii), for any $x \in X_{\alpha}$, we have

$$\begin{gathered}\|S(t)x\|_{\alpha} = \|A^{\alpha} \cdot S(t)x\| = \|S(t) \cdot A^{\alpha}x\| \leq \|S(t)\| \cdot \|A^{\alpha}x\| = \|S(t)\| \cdot \|x\|_{\alpha}, \\ \|S(t)x - x_{\alpha}\| = \|A^{\alpha} \cdot S(t)x - A^{\alpha}x\| = \|S(t) \cdot A^{\alpha}x - A^{\alpha}x\| \longrightarrow 0\end{gathered} \tag{2.3}$$

as $t \to 0$. Therefore, $S(t)$ $(t \geq 0)$ is a strongly continuous semigroup in X_{α}, and $\|S_{\alpha}(t)\|_{\alpha} \leq \|S(t)\|$ for all $t \geq 0$. To prove our main results, the following lemma is also needed.

Lemma 2.2 (see [27]). $S_{\alpha}(t)$ $(t \geq 0)$ *is an immediately compact semigroup in* X_{α}*, and hence it is immediately norm-continuous.*

Let us recall the following known definitions in fractional calculus. For more details, see [16–20, 23].

Definition 2.3. The fractional integral of order $\sigma > 0$ with the lower limits zero for a function f is defined by

$$I^{\sigma}f(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}f(s)ds, \quad t > 0, \tag{2.4}$$

where Γ is the gamma function.

The Riemann-Liouville fractional derivative of order $n-1 < \sigma < n$ with the lower limits zero for a function f can be written as

$${}^{L}D^{\sigma}f(t) = \frac{1}{\Gamma(n-\sigma)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\sigma-1}f(s)ds, \quad t > 0,\ n \in \mathbb{N}. \tag{2.5}$$

Also the Caputo fractional derivative of order $n-1 < \sigma < n$ with the lower limits zero for a function $f \in C^n[0,\infty)$ can be written as

$$D^{\sigma}f(t) = \frac{1}{\Gamma(n-\sigma)}\int_0^t (t-s)^{n-\sigma-1}f^{(n)}(s)ds, \quad t > 0,\ n \in \mathbb{N}. \tag{2.6}$$

Remark 2.4. (1) The Caputo derivative of a constant is equal to zero.

(2) If f is an abstract function with values in X, then integrals which appear in Definition 2.3 are taken in Bochner's sense.

Lemma 2.5 (see [12]). *A measurable function $h : J \to X$ is Bochner integrable if $\|h\|$ is Lebesgue integrable.*

For $x \in X$, we define two families $\{U(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\geq 0}$ of operators by

$$U(t)x = \int_0^\infty \eta_q(\theta) S(t^q\theta)x d\theta, \qquad V(t)x = q\int_0^\infty \theta\eta_q(\theta)S(t^q\theta)x d\theta, \quad 0 < q < 1, \tag{2.7}$$

where

$$\eta_q(\theta) = \frac{1}{q}\theta^{-1-(1/q)}\rho_q\left(\theta^{-(1/q)}\right), \qquad \rho_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty), \tag{2.8}$$

η_q is a probability density function defined on $(0,\infty)$, which has properties $\eta_q(\theta) \geq 0$ for all $\theta \in (0,\infty)$ and

$$\int_0^\infty \eta_q(\theta)d\theta = 1, \qquad \int_0^\infty \theta\eta_q(\theta)d\theta = \frac{1}{\Gamma(q+1)}. \tag{2.9}$$

The following lemma follows from the results in [7, 11–13].

Lemma 2.6. *The operators U and V have the following properties.*

(i) *For fixed $t \geq 0$ and any $x \in X_\alpha$, we have*

$$\|U(t)x\|_\alpha \leq M\|x\|_\alpha, \qquad \|V(t)x\|_\alpha \leq \frac{qM}{\Gamma(1+q)}\|x\|_\alpha = \frac{M}{\Gamma(q)}\|x\|_\alpha. \tag{2.10}$$

(ii) *The operators $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.*

(iii) *$U(t)$ and $V(t)$ are norm-continuous in X for $t > 0$.*

(iv) *$U(t)$ and $V(t)$ are compact operators in X for $t > 0$.*

(v) *For every $t > 0$, the restriction of $U(t)$ to X_α and the restriction of $V(t)$ to X_α are norm-continuous.*

(vi) *For every $t > 0$, the restriction of $U(t)$ to X_α and the restriction of $V(t)$ to X_α are compact operators in X_α.*

Based on an overall observation of the previous related literature, in this paper, we adopt the following definition of mild solution of (1.1).

Definition 2.7. By a mild solution of (1.1), we mean a function $u \in C(J, X_\alpha)$ satisfying

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s), Gu(s)) ds \tag{2.11}$$

for all $t \in J$.

3. Existence of Mild Solutions

In this section, we give the existence theorems of mild solutions of (1.1). The discussions are based on fractional calculus and Schauder fixed point theorem. Our main results are as follows.

Theorem 3.1. *Assume that the following condition on f is satisfied.*

(H_1) *There exists a constant $\beta \in [\alpha, 1]$ such that $f : J \times X_\alpha \times X_\alpha \longrightarrow X_\beta$ satisfies:*

(i) *for each $(x, y) \in X_\alpha \times X_\alpha$, the function $f(\cdot, x, y) : J \longrightarrow X_\beta$ is measurable;*

(ii) *for each $t \in J$, the function $f(t, \cdot, \cdot) : X_\alpha \times X_\alpha \longrightarrow X_\beta$ is continuous;*

(iii) *for any $r > 0$, there exists a function $g_r \in L^\infty(J, \mathbb{R}^+)$ such that*

$$\sup_{\|x\|_\alpha \le r,\ \|y\|_\alpha \le K^* Tr} \|f(t, x, y)\|_\beta \le g_r(t), \quad t \in J, \tag{3.1}$$

and there is a constant $\gamma > 0$ such that

$$\liminf_{r \longrightarrow +\infty} \frac{1}{r} \int_0^t \frac{g_r(s)}{(t-s)^{1-q}} ds \le \gamma < +\infty. \tag{3.2}$$

If $x_0 \in X_\alpha$ and $MC_{\beta-\alpha}\gamma < \Gamma(q)$, then (1.1) has at least one mild solution.

Proof. Define an operator Q by

$$(Qu)(t) = U(t)x_0 + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s), Gu(s)) ds, \quad t \in J. \tag{3.3}$$

It is not difficult to verify that $Q : C(J, X_\alpha) \longrightarrow C(J, X_\alpha)$. We will use Schauder fixed point theorem to prove that Q has fixed points in $C(J, X_\alpha)$.

For any $r > 0$, let $B_r := \{u \in C(J, X_\alpha) : \|u(t)\|_\alpha \le r,\ t \in J\}$. We first show that there is a positive number r such that $Q(B_r) \subset B_r$. If this were not the case, then for each $r > 0$, there would exist $u_r \in B_r$ and $t_r \in J$ such that $\|(Qu_r)(t_r)\|_\alpha > r$. Thus, from Lemma 2.6(i) and (H_1)(iii), we see that

$$\begin{aligned} r < \|(Qu_r)(t_r)\|_\alpha &\le \|U(t_r)x_0\|_\alpha + \int_0^{t_r} (t_r - s)^{q-1} \left\| V(t_r - s) f(s, u_r(s), Gu_r(s)) \right\|_\alpha ds \\ &\le M\|x_0\|_\alpha + \int_0^{t_r} (t_r - s)^{q-1} \left\| A^{\alpha-\beta} V(t_r - s) \cdot A^\beta f(s, u_r(s), Gu_r(s)) \right\| ds \\ &\le M\|x_0\|_\alpha + \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^{t_r} (t_r - s)^{q-1} g_r(s) ds. \end{aligned} \tag{3.4}$$

Dividing on both sides by r and taking the lower limit as $r \to +\infty$, we have

$$MC_{\beta-\alpha}\gamma \ge \Gamma(q), \tag{3.5}$$

which is a contradiction. Hence $Q(B_r) \subset B_r$ for some $r > 0$.

To complete the proof, we separate the rest of proof into the following three steps.

Step 1. $Q : B_r \to B_r$ is continuous.

Let $\{u_n\} \subset B_r$ with $u_n \to u \in B_r$ as $n \to \infty$. From the assumption (H_1)(ii), for each $s \in J$, we have

$$f(s, u_n(s), Gu_n(s)) \longrightarrow f(s, u(s), Gu(s)) \tag{3.6}$$

as $n \to \infty$. Since $\|f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s))\|_\beta \le 2g_r(s)$, by the Lebesgue dominated convergence theorem, for each $t \in J$, we have

$$\begin{aligned} \|(Qu_n)(t) - (Qu)(t)\|_\alpha &\le \int_0^t (t-s)^{q-1} \left\| V(t-s) \left[f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s)) \right] \right\|_\alpha ds \\ &\le \int_0^t (t-s)^{q-1} \Big\| A^{\alpha-\beta} V(t-s) \\ &\qquad \cdot A^\beta \left[f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s)) \right] \Big\| ds \\ &\le \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big\| f(s, u_n(s), Gu_n(s)) \\ &\qquad - f(s, u(s), Gu(s)) \big\|_\beta ds \longrightarrow 0 \end{aligned} \tag{3.7}$$

as $n \to \infty$, which implies that $Q : B_r \to B_r$ is continuous.

Step 2. $(QB_r)(t) := \{(Qu)(t) : u \in B_r\}$ is relatively compact in X_α for all $t \in J$.

It follows from (2.9) and (3.3) that $(QB_r)(0) = \{(Qu)(0) : u \in B_r\} = \{x_0\}$ is compact in X_α. Hence it is only necessary to consider the case of $t > 0$. For each $t \in (0,T]$, $\epsilon \in (0,t)$, and any $\delta > 0$, we define a set $(Q_{\epsilon,\delta}B_r)(t)$ by

$$(Q_{\epsilon,\delta}B_r)(t) := \{(Q_{\epsilon,\delta}u)(t) : u \in B_r\}, \tag{3.8}$$

where

$$\begin{aligned}(Q_{\epsilon,\delta}u)(t) &= \int_\delta^\infty \eta_q(\theta)S(t^q\theta)d\theta x_0 \\ &\quad + q\int_0^{t-\epsilon}(t-s)^{q-1}\int_\delta^\infty \theta\eta_q(\theta)S\big((t-s)^q\theta\big)d\theta\cdot f(s,u(s),Gu(s))ds \\ &= S(\epsilon^q\delta)\bigg[\int_\delta^\infty \eta_q(\theta)S(t^q\theta-\epsilon^q\delta)d\theta x_0 \\ &\quad + q\int_0^{t-\epsilon}(t-s)^{q-1}\int_\delta^\infty \theta\eta_q(\theta)S\big((t-s)^q\theta-\epsilon^q\delta\big)d\theta\cdot f(s,u(s),Gu(s))ds\bigg].\end{aligned} \tag{3.9}$$

Then the set $(Q_{\epsilon,\delta}B_r)(t)$ is relatively compact in X_α since by Lemma 2.2, the operator $S_\alpha(\epsilon^q\delta)$ is compact in X_α. For any $u \in B_r$ and $t \in (0,T]$, from the following inequality:

$$\begin{aligned}\|(Qu)(t)-(Q_{\epsilon,\delta}u)(t)\|_\alpha &\le \left\|\int_0^\delta \eta_q(\theta)S(t^q\theta)d\theta x_0\right\|_\alpha \\ &\quad + \bigg\|q\int_0^t (t-s)^{q-1}\int_0^\delta \theta\eta_q(\theta)S\big((t-s)^q\theta\big)d\theta \\ &\qquad \cdot f(s,u(s),Gu(s))ds\bigg\|_\alpha \\ &\quad + \bigg\|q\int_0^t (t-s)^{q-1}\int_\delta^\infty \theta\eta_q(\theta)S\big((t-s)^q\theta\big)d\theta\cdot f(s,u(s),Gu(s))ds \\ &\qquad - q\int_0^{t-\epsilon}(t-s)^{q-1}\int_\delta^\infty \theta\eta_q(\theta)S\big((t-s)^q\theta\big)d\theta \\ &\qquad \cdot f(s,u(s),Gu(s))ds\bigg\|_\alpha \\ &\le M\|x_0\|_\alpha\int_0^\delta \eta_q(\theta)d\theta + qMC_{\beta-\alpha}\|g_r\|_{L^\infty}\int_0^t (t-s)^{q-1}ds\int_0^\delta \theta\eta_q(\theta)d\theta \\ &\quad + qMC_{\beta-\alpha}\|g_r\|_{L^\infty}\int_{t-\epsilon}^t (t-s)^{q-1}ds\int_0^\infty \theta\eta_q(\theta)d\theta\end{aligned}$$

$$\leq M\|x_0\|_\alpha \int_0^\delta \eta_q(\theta)d\theta + MC_{\beta-\alpha}T^q\|g_r\|_{L^\infty} \times \int_0^\delta \theta\eta_q(\theta)d\theta + \frac{MC_{\beta-\alpha}\|g_r\|_{L^\infty}}{\Gamma(q+1)}\epsilon^q. \tag{3.10}$$

One can obtain that the set $(QB_r)(t)$ is relatively compact in X_α for all $t \in (0,T)$. And since it is compact at $t = 0$, we have the relatively compactness of $(QB_r)(t)$ in X_α for all $t \in J$.

Step 3. $QB_r := \{Qu \in C(J, X_\alpha) : u \in B_r\}$ is equicontinuous.
For $\tau \in [0,T)$, by (3.3), we have

$$\begin{aligned} \|(Qu)(\tau) - (Qu)(0)\|_\alpha &\leq \|U(\tau)x_0 - x_0\|_\alpha \\ &\quad + \left\|\int_0^\tau (\tau-s)^{q-1}V(\tau-s)f(s,u(s),Gu(s))ds\right\|_\alpha \\ &\leq \|U(\tau) - I\| \cdot \|x_0\|_\alpha + \frac{MC_{\beta-\alpha}\|g_r\|_{L^\infty}}{\Gamma(q+1)}\tau^q. \end{aligned} \tag{3.11}$$

Hence it is only necessary to consider the case of $t > 0$. For $0 < t_1 < t_2 \leq T$, by Lemma 2.1 and Lemma 2.6(i), we have

$$\begin{aligned} \|(Qu)(t_2) - (Qu)(t_1)\|_\alpha &\leq \|U(t_2)x_0 - U(t_1)x_0\|_\alpha \\ &\quad + \left\|\int_0^{t_2} (t_2-s)^{q-1}V(t_2-s)f(s,u(s),Gu(s))ds \right. \\ &\quad \left. - \int_0^{t_1} (t_1-s)^{q-1}V(t_1-s)f(s,u(s),Gu(s))ds\right\|_\alpha \\ &\leq \|U(t_2) - U(t_1)\| \cdot \|x_0\|_\alpha \\ &\quad + \left\|\int_{t_1}^{t_2} (t_2-s)^{q-1}V(t_2-s)f(s,u(s),Gu(s))ds\right\|_\alpha \\ &\quad + \left\|\int_0^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right]V(t_2-s)f(s,u(s),Gu(s))ds\right\|_\alpha \\ &\quad + \left\|\int_0^{t_1} (t_1-s)^{q-1}f(s,u(s),Gu(s))[V(t_2-s) - V(t_1-s)]ds\right\|_\alpha \\ &\leq \|U(t_2) - U(t_1)\| \cdot \|x_0\|_\alpha + \frac{MC_{\beta-\alpha}\|g_r\|_{L^\infty}}{\Gamma(q+1)}(t_2-t_1)^q \\ &\quad + \frac{MC_{\beta-\alpha}\|g_r\|_{L^\infty}}{\Gamma(q+1)}\left[t_2^q - t_1^q - (t_2-t_1)^q\right] \end{aligned}$$

$$+\left\|\int_{0}^{t_1}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]\cdot f(s,u(s),Gu(s))ds\right\|_{\alpha}$$
$$\triangleq I_1+I_2+I_3+I_4. \tag{3.12}$$

From Lemma 2.6(v), we see that $I_1 \to 0$ as $t_2 \to t_1$ independently of $u \in B_r$. From the expressions of I_2 and I_3, it is clear that $I_2 \to 0$ and $I_3 \to 0$as $t_2 \to t_1$ independently of $u \in B_r$. For any $\epsilon \in (0, t_1)$, we have

$$\begin{aligned}
I_4 &\le \left\|\int_{0}^{t_1-\epsilon}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]\cdot f(s,u(s),Gu(s))ds\right\|_{\alpha} \\
&\quad+\left\|\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]\cdot f(s,u(s),Gu(s))ds\right\|_{\alpha} \\
&\le \frac{1}{q}C_{\beta-\alpha}\|g_r\|_{L^\infty}(T^q+\epsilon^q)\sup_{0\le s\le t_1-\epsilon}\|V(t_2-s)-V(t_1-s)\|+\frac{2MC_{\beta-\alpha}\|g_r\|_{L^\infty}}{\Gamma(q+1)}\epsilon^q.
\end{aligned} \tag{3.13}$$

It follows from Lemma 2.6(v) that $I_4 \to 0$ as $t_2 \to t_1$ and $\epsilon \to 0$ independently of $u \in B_r$. Therefore, we prove that QB_r is equicontinuous.

Thus, the Arzela-Ascoli theorem guarantees that Q is a compact operator. By the Schauder fixed point theorem, the operator Q has at least one fixed point u^* in B_r, which is a mild solution of (1.1). This completes the proof. □

Remark 3.2. In assumption (H_1)(iii), if the function $g_r(t)$ is independent of t, then we can easily obtain a constant $\gamma > 0$ satisfying (3.2). For example, if there is a constant $a_f > 0$ such that

$$\|f(t,x,y)\|_{\beta} \le a_f(1+\|x\|_{\alpha}+\|y\|_{\alpha}) \tag{3.14}$$

for all $x, y \in X_\alpha$ and $t \in J$, then for any $r > 0$, $x, y \in X_\alpha$ with $\|x\|_\alpha \le r$, $\|y\|_\alpha \le K^*Tr$, we have $\|f(t,x,y)\|_\beta \le a_f + a_f(1+K^*T)r \triangleq g_r(t)$, where $g_r(t)$ is independent of t. Thus, $\gamma := (1/q)a_f T^q(1+K^*T) > 0$ is the constant in (3.2).

More generally, if f satisfies the following condition:

(H2) there is a constant $\beta \in [\alpha, 1]$ such that $f : J \times X_\alpha \times X_\alpha \to X_\beta$ satisfies:

  (i) for each $(x, y) \in X_\alpha \times X_\alpha$, the function $f(\cdot, x, y) : J \to X_\beta$ is measurable,

  (ii) for any $r > 0$, there exists a function $\ell \in L^\infty(J, \mathbb{R}^+)$ such that

$$\|f(t,x_1,y_1)-f(t,x_2,y_2)\|_{\beta} \le \ell(t)(\|x_1-x_2\|_{\alpha}+\|y_1-y_2\|_{\alpha}) \tag{3.15}$$

for any $x_i, y_i \in X_\alpha$ with $\|x_i\|_\alpha \le r$, $\|y_i\|_\alpha \le K^*Tr\,(i = 1, 2)$ and $t \in J$, then we have the following existence and uniqueness theorem.

Theorem 3.3. *Assume that the condition* (H_2) *is satisfied. If* $x_0 \in X_\alpha$ *and* $MC_{\beta-\alpha}T^q(1 + K^*T)\|\ell\|_{L^\infty} < \Gamma(q+1)$, *then* (1.1) *has a unique mild solution.*

Proof. For any $r > 0$, if $x, y \in X_\alpha$ with $\|x\|_\alpha \le r$, $\|y\|_\alpha \le K^*Tr$, then from (H_2)(ii), we have

$$\|f(t,x,y)\|_\beta \le \ell(t)(1+K^*T)r + b(t) \triangleq g_r(t), \tag{3.16}$$

where $b(t) = \|f(t,0,0)\|_\beta$. Therefore, the condition (H_1)(iii) is satisfied with $\gamma = ((1 + K^*T)T^q\|\ell\|_{L^\infty})/q$. By Theorem 3.1, (1.1) has at least one mild solution $u^* \in B_r$.

Let $u_1, u_2 \in B_r$ be the solutions of (1.1). We show that $u_1 \equiv u_2$. Since $u_1(t) = (Qu_1)(t)$ and $u_2(t) = (Qu_2)(t)$ for all $t \in J$, we have

$$\begin{aligned}
\|u_1(t) - u_2(t)\|_\alpha &= \|(Qu_1)(t) - (Qu_2)(t)\|_\alpha \\
&\le \int_0^t (t-s)^{q-1}\|V(t-s)[f(s,u_1(s),Gu_1(s)) - f(s,u_2(s),Gu_2(s))]\|_\alpha ds \\
&= \int_0^t (t-s)^{q-1}\left\|A^{\alpha-\beta}V(t-s)\cdot A^\beta[f(s,u_1(s),Gu_1(s)) - f(s,u_2(s),Gu_2(s))]\right\|ds \\
&\le \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\cdot\|f(s,u_1(s),Gu_1(s)) - f(s,u_2(s),Gu_2(s))\|_\beta ds \\
&\le \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\cdot\ell(s)(\|u_1(s) - u_2(s)\|_\alpha + \|Gu_1(s) - Gu_2(s)\|_\alpha)ds \\
&\le \frac{MC_{\beta-\alpha}\|\ell\|_{L^\infty}}{\Gamma(q)}(1+K^*T)\int_0^t (t-s)^{q-1}\cdot\|u_1(s) - u_2(s)\|_\alpha ds.
\end{aligned} \tag{3.17}$$

By using the Gronwall-Bellman inequality (see [14, Theorem 1]), we can deduce that $\|u_1(t) - u_2(t)\|_\alpha = 0$ for all $t \in J$, which implies that $u_1 \equiv u_2$. Hence (1.1) has a unique mild solution $u^* \in B_r$. This completes the proof. □

Remark 3.4. In Theorem 3.3, we only assume that f satisfies a local Lioschitz condition (see condition (H_2)), and an existence and uniqueness result is obtained. If $f(t,u,v) \equiv f(t,u) : J \times X_\alpha \to X$, then the assumption (H_2) deletes the linear growth condition (3) of assumption (Hf) in [12]. Therefore, the Theorem 3.3 extends and improves the main result in [12].

4. An Example

Assume that $X = L^2[0,\pi]$ equipped with its natural norm and inner product defined, respectively, for all $u, v \in L^2[0,\pi]$, by

$$\|u\|_X = \left(\int_0^\pi |u(x)|^2 dx\right)^{1/2}, \qquad \langle u,v\rangle = \int_0^\pi u(x)\overline{v(x)}dx. \tag{4.1}$$

Consider the following fractional partial differential equation:

$$\begin{aligned}
&\frac{\partial^{1/2}}{\partial t^{1/2}}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)=g\Big(x,t,u(x,t),\textstyle\int_0^t K(t,s)u(x,s)ds\Big), \quad t\in[0,T],\ x\in[0,\pi]\\
&u(0,t)=u(\pi,t)=0, \quad t\in[0,T],\\
&u(x,0)=u_0(x), \quad x\in[0,\pi],
\end{aligned}\tag{4.2}$$

where $T>0$ is a constant.

Let the operator $A: D(A)\subset X\to X$ be defined by

$$D(A):=\{v\in X: v''\in X,\ v(0)=v(\pi)=0\}, \qquad Au=-\frac{\partial^2 u}{\partial x^2}.\tag{4.3}$$

It is well known that A has a discrete spectrum with eigenvalues of the form $n^2, n\in\mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n=\sqrt{(2/\pi)}\sin(nx)$. Moreover, $-A$ generates a compact analytic semigroup $S(t)$ $(t\ge 0)$ in X, and

$$S(t)u=\sum_{n=1}^{\infty}e^{-n^2t}\langle u,z_n\rangle z_n.\tag{4.4}$$

It is not difficult to verify that $\|S(t)\|_{\mathcal{L}(X)}\le e^{-t}$ for all $t\ge 0$. Hence, we take $M=1$.

The following results are also well known.

(I) The operator A can be written as

$$Au=\sum_{n=1}^{\infty}n^2\langle u,z_n\rangle z_n\tag{4.5}$$

for every $u\in D(A)$.

(II) The operator $A^{1/2}$ is given by

$$A^{1/2}u=\sum_{n=1}^{\infty}n\langle u,z_n\rangle z_n\tag{4.6}$$

for each $u\in D(A^{1/2}):=\{v\in X:\sum_{n=1}^{\infty}n<v,\ z_n>z_n\in X\}$ and $\|A^{-(1/2)}\|_{\mathcal{L}(X)}=1$.

Lemma 4.1 (see [28]). *If $m\in D(A^{1/2})$, then m is absolutely continuous, $m'\in X$ and $\|m'\|_X=\|A^{1/2}m\|_X$.*

Let $X_{1/2}=(D(A^{1/2}),\|\cdot\|_{1/2})$, where $\|x\|_{1/2}:=\|A^{1/2}x\|_X$ for all $x\in D(A^{1/2})$. Assume that $g:[0,\pi]\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions.

(i) For each $(x,t)\in[0,\pi]\times[0,T]$, the function $g(x,t,\cdot,\cdot)$ is continuous.

(ii) For each $(\xi,\eta)\in\mathbb{R}^2$, the function $g(\cdot,\cdot,\xi,\eta)$ is measurable.

(iii) For each $t \in [0,T]$ and $\xi, \eta \in \mathbb{R}$, $g(\cdot, t, \xi, \eta)$ is differentiable, and $(\partial/\partial x)g(x,t,\xi,\eta) \in X$.

(iv) $g(0,\cdot,\cdot,\cdot) = g(\pi,\cdot,\cdot,\cdot) = 0$.

(v) There exist the functions $\ell_1, \ell_0 \in L^\infty([0,T],\mathbb{R}^+)$ such that

$$\left|\frac{\partial}{\partial x} g(x,t,\xi,\eta)\right| \le \ell_1(t)\left(|\xi| + |\eta|\right) + \ell_0(t) \tag{4.7}$$

for all $(x,t,\xi,\eta) \in [0,\pi] \times [0,T] \times \mathbb{R} \times \mathbb{R}$.

Define $f(t, u(t), Gu(t))(x) = g(x,t,u(x,t), \int_0^t K(t,s)u(x,s)ds)$. Then, for each $\phi \in X_{1/2}$, from assumptions (iii) and (iv), we have

$$\begin{aligned}\langle f(t,\phi,G\phi), z_n\rangle &= \int_0^\pi g\left(x,t,\phi(x,t), \int_0^t K(t,s)\phi(x,s)ds\right) \cdot \sqrt{\frac{2}{\pi}}\sin(nx)dx \\ &= \frac{1}{n}\int_0^\pi \left(\frac{\partial}{\partial x} g\left(x,t,\phi(x,t), \int_0^t K(t,s)\phi(x,s)ds\right)\right) \cdot \sqrt{\frac{2}{\pi}}\cos(nx)dx.\end{aligned} \tag{4.8}$$

This implies from (II) that $f : [0,T] \times X_{1/2} \times X_{1/2} \to X_{1/2}$. Moreover, for any $r > 0$, by Minkowski inequality, assumption (v) and Lemma 4.1, we have

$$\begin{aligned}\sup_{\|\phi\|_{1/2}\le r} \|f(t,\phi,G\phi)\|_{1/2} &= \sup_{\|\phi\|_{1/2}\le r} \left\|\frac{\partial}{\partial x} g\left(x,t,\phi(x,t), \int_0^t K(t,s)\phi(x,s)ds\right)\right\|_X \\ &= \sup_{\|\phi\|_{1/2}\le r} \left(\int_0^\pi \left|\frac{\partial}{\partial x} g\left(x,t,\phi(x,t), \int_0^t K(t,s)\phi(x,s)ds\right)\right|^2 dx\right)^{1/2} \\ &\le \sup_{\|\phi\|_{1/2}\le r} \left(\int_0^\pi \left[\ell_1(t)\left(|\phi(x,t)| + \left|\int_0^t K(t,s)\phi(x,s)ds\right|\right)\right.\right. \\ &\qquad \left.\left. +\ell_0(t)\right]^2 dx\right)^{1/2} \\ &\le \sup_{\|\phi\|_{1/2}\le r} \left[\ell_1(t)\left(\|\phi\|_X + K^*T\|\phi\|_X\right) + \ell_0(t)\right] \\ &= \sup_{\|\phi\|_{1/2}\le r} \left[(1+K^*T)\ell_1(t)\left\|A^{-(1/2)} \cdot A^{1/2}\phi\right\|_X + \ell_0(t)\right] \\ &\le (1+K^*T)r\ell_1(t) + \ell_0(t) \triangleq g_r(t).\end{aligned} \tag{4.9}$$

Therefore, f satisfies the condition (H_1) with $\gamma = 2T^{1/2}(1+K^*T)\|\ell_1\|_{(L^\infty)}$. Thus, (4.2) has at least one mild solution provided that $\gamma < \sqrt{\pi}$ due to Theorem 3.1.

Assume furthermore that the function g satisfies the following:

(vi) for any $r > 0$, there exists a function $\ell_2 \in L^\infty([0,T],\mathbb{R}^+)$ such that

$$\left|\frac{\partial}{\partial x}g(x,t,\xi_1,\eta_1) - \frac{\partial}{\partial x}g(x,t,\xi_2,\eta_2)\right| \le \ell_2(t)(|\xi_1-\xi_2| + |\eta_1-\eta_2|) \tag{4.10}$$

for (x,t,ξ_1,η_1), $(x,t,\xi_2,\eta_2) \in [0,\pi]\times[0,T]\times\mathbb{R}\times\mathbb{R}$ with $|\xi_i| \le r$ and $|\eta_i| \le K^*Tr$, $i = 1, 2$.

Then for each $\phi_1, \phi_2 \in X_{1/2}$, by Lemma 4.1, we have

$$\begin{aligned}
\|f(t,\phi_1,G\phi_1) - f(t,\phi_2,G\phi_2)\|_{1/2} &= \left\|A^{1/2}[f(t,\phi_1,G\phi_1) - f(t,\phi_2,G\phi_2)]\right\|_X \\
&= \left\|\frac{\partial}{\partial x}g\left(x,t,\phi_1(x,t),\int_0^t K(t,s)\phi_1(x,s)ds\right)\right. \\
&\quad \left. -\frac{\partial}{\partial x}g\left(x,t,\phi_2(x,t),\int_0^t K(t,s)\phi_2(x,s)ds\right)\right\|_X \\
&= \left(\int_0^\pi \left|\frac{\partial}{\partial x}g\left(x,t,\phi_1(x,t),\int_0^t K(t,s)\phi_1(x,s)ds\right)\right.\right. \\
&\quad \left.\left. -\frac{\partial}{\partial x}g\left(x,t,\phi_2(x,t),\int_0^t K(t,s)\phi_2(x,s)ds\right)\right|^2 dx\right)^{1/2} \\
&\le \left(\int_0^\pi \left|\ell_2(t)\left(|\phi_1(x,t)-\phi_2(x,t)|\right.\right.\right. \\
&\quad + \left|\int_0^t K(t,s)\phi_1(x,s)ds\right. \\
&\quad \left.\left.\left. -\int_0^t K(t,s)\phi_2(x,s)ds\right|\right)\right|^2 dx\right)^{1/2} \\
&\le \ell_2(t)\left[\left(\int_0^\pi |\phi_1(x,t)-\phi_2(x,t)|^2 dx\right)^{1/2}\right. \\
&\quad + \left(\int_0^\pi\left|\int_0^t K(t,s)\phi_1(x,s)ds\right.\right. \\
&\quad \left.\left.\left. -\int_0^t K(t,s)\phi_2(x,s)ds\right|^2 dx\right)^{1/2}\right]
\end{aligned}$$

$$\begin{aligned}
&= \ell_2(t)\left(\|\phi_1-\phi_2\|_X + \|G\phi_1 - G\phi_2\|_X\right)\\
&= \ell_2(t)\Big(\left\|A^{-(1/2)}\cdot A^{1/2}(\phi_1-\phi_2)\right\|_X\\
&\qquad + \left\|A^{-(1/2)}\cdot A^{1/2}(G\phi_1-G\phi_2)\right\|_X\Big)\\
&\le \ell_2(t)\left(\left\|A^{1/2}(\phi_1-\phi_2)\right\|_X + \left\|A^{1/2}(G\phi_1-G\phi_2)\right\|_X\right)\\
&= \ell_2(t)\left(\|\phi_1-\phi_2\|_{1/2} + \|G\phi_1-G\phi_2\|_{1/2}\right).
\end{aligned} \tag{4.11}$$

This shows that f satisfies the condition (H_2). Hence by Theorem 3.3, the mild solution of (4.2) is unique.

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Research Article

A Characteristic Difference Scheme for Time-Fractional Heat Equations Based on the Crank-Nicholson Difference Schemes

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We consider the numerical solution of a time-fractional heat equation, which is obtained from the standard diffusion equation by replacing the first-order time derivative with *Riemann-Liouville* fractional derivative of order α, where $0 < \alpha < 1$. The main purpose of this work is to extend the idea on Crank-Nicholson method to the time-fractional heat equations. We prove that the proposed method is unconditionally stable, and the numerical solution converges to the exact one with the order $O(\tau^2 + h^2)$. Numerical experiments are carried out to support the theoretical claims.

1. Introduction

Fractional calculus is one of the most popular subjects in many scientific areas for decades. Many problems in applied science, physics and engineering are modeled mathematically by the fractional partial differential equations (FPDEs). We can see these models adoption in viscoelasticity [1, 2], finance [3, 4], hydrology [5, 6], engineering [7, 8], and control systems [9–11]. FPDEs may be investigated into two fundamental types: time-fractional differential equations and space-fractional differential equations.

Several different methods have been used for solving FPDEs. For the analytical solutions to problems, some methods have been proposed: the variational iteration method [12, 13], the Adomian decomposition method [13–16], as well as the Laplace transform and Fourier transform methods [17, 18].

On the other hand, numerical methods which based on a finite-difference approximation to the fractional derivative, for solving FDPEs [19–24], have been proposed. A practical numerical method for solving multidimensional fractional partial differential equations, using a variation on the classical alternating-directions implicit (ADI) Euler method, is presented in [25]. Many finite-difference approximations for the FPDEs are only first-order accurate. Some second-order accurate numerical approximations for the *space*-fractional differential equations were presented in [26–28]. Here, we propose a Crank-Nicholson-type method for *time*-fractional differential heat equations with the accuracy of order $O(\tau^2 + h^2)$.

In this work, we consider the following time-fractional heat equation:

$$\begin{gathered}
\frac{\partial_M^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \quad (0 < x < 1, 0 < t < 1), \\
u(0,x) = r(x), \quad 0 \le x \le 1, \\
u(t,0) = 0, \quad u(t,1) = 0, \quad 0 \le t \le 1.
\end{gathered} \tag{1.1}$$

Here, the term $\partial_M^\alpha u(t,x)/\partial t^\alpha$ denotes α-order-modified Riemann-Liouville fractional derivative [29] given with the formula:

$$\frac{\partial_M^\alpha u(t,x)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\partial}{\partial t} \displaystyle\int_0^t \dfrac{u(s,x) - u(0,x)}{(t-s)^\alpha} ds, & \text{if } 0 < \alpha < 1, \\ \dfrac{\partial}{\partial t} u(t,x), & \text{if } \alpha = 1, \end{cases} \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function.

Remark 1.1. If $r(x) = 0$, then the Riemann-Liouville and the modified Riemann-Liouville fractional derivatives are identical, since the Riemann-Liouville derivative is given by the following formula:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\partial}{\partial t} \displaystyle\int_0^t \dfrac{u(s,x)}{(t-s)^\alpha} ds, & \text{if } 0 < \alpha < 1, \\ \dfrac{\partial}{\partial t} u(t,x), & \text{if } \alpha = 1. \end{cases} \tag{1.3}$$

If $r(x)$ is nonzero, then there are some problems about the existence of the solutions for the heat equation (1.1). To rectify the situation, two main approaches can be used: the modified Riemann-Liouville fractional derivative can be used [29] or the initial condition should be modified [30]. We chose the first approach in our work.

2. Discretization of the Problem

In this section, we introduce the basic ideas for the numerical solution of the time-fractional heat equation (1.1) by Crank-Nicholson difference scheme.

For some positive integers M and N, the grid sizes in space and time for the finite-difference algorithm are defined by $h = 1/M$ and $\tau = 1/N$, respectively. The grid points in the space interval $[0,1]$ are the numbers $x_i = ih$, $i = 0,1,2,\ldots,M$, and the grid points in the time interval $[0,1]$ are labeled $t_n = n\tau$, $n = 0,1,2,\ldots,N$. The values of the functions U and f at the grid points are denoted $U_i^n = U(t_n, x_i)$ and $f_i^n = f(t_n, x_i)$, respectively.

As in the classical Crank-Nicholson difference scheme, we will obtain a discrete approximation to the fractional derivative $\partial^\alpha U(t,x)/\partial t^\alpha$ at $(t_{n+(1/2)}, x_i)$. Let

$$H(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(s,x)-u(0,x)}{(t-s)^\alpha} ds. \tag{2.1}$$

Then, we have

$$\frac{\partial^\alpha U(t_{n+1/2}, x_i)}{\partial t^\alpha} = \frac{\partial}{\partial t} H(t_{n+1/2}, x_i) = \frac{H(t_{n+1}, x_i) - H(t_n, x_i)}{\tau} + O\left(\tau^2\right). \tag{2.2}$$

Now, we will find the approximations for $H(t_{n+1}, x_i)$ and $H(t_n, x_i)$:

$$\begin{aligned} H(t_{n+1}, x_i) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} \frac{u(s,x_i)-u(0,x_i)}{(t_{n+1}-s)^\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n+1} \int_{(j-1)\tau}^{j\tau} \frac{u(s,x_i)}{(t_{n+1}-s)^\alpha} ds - u(0,x_i) \frac{((n+1)\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n+1} \int_{(j-1)\tau}^{j\tau} \left[\frac{(s-t_j)}{-\tau} U_i^{j-1} + \frac{(s-t_{j-1})}{\tau} U_i^j + O\left(\tau^2\right) \right] \frac{1}{(t_{n+1}-s)^\alpha} ds \\ &\qquad - U_i^0 \frac{((n+1)\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \\ &= \tau \sum_{j=0}^{n} (a_j - j b_j) U_i^{n-j} - \tau \sum_{j=0}^{n} (a_j - (j+1) b_j) U_i^{n-j+1} - U_i^0 \frac{((n+1)\tau)^{1-\alpha}}{\Gamma(2-\alpha)} + R_{n+1}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R_{n+1} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n+1} \int_{(j-1)\tau}^{j\tau} O\left(\tau^2\right) \frac{ds}{(t_{n+1}-s)^\alpha} \\ &= \frac{1}{(1-\alpha)\Gamma(1-\alpha)} O\left(\tau^2\right) \sum_{j=1}^{n+1} \left[(n-j+2)^{1-\alpha} - (n-j+1)^{1-\alpha} \right] \tau^{1-\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} (n+1)^{1-\alpha} O\left(\tau^{3-\alpha}\right). \end{aligned} \tag{2.4}$$

Similarly, we can obtain

$$H(t_n, x_i) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{u(s, x_i) - u(0, x_i)}{(t_n - s)^{\alpha}} ds$$
$$= \tau \sum_{j=1}^{n} (a_{j-1} - (j-1) b_{j-1}) U_i^{n-j} - \tau \sum_{j=1}^{n} (a_{j-1} - j b_{j-1}) U_i^{n-j+1} - U_i^0 \frac{(n\tau)^{1-\alpha}}{\Gamma(2-\alpha)} + R_n, \tag{2.5}$$

where $R_n = (1/\Gamma(2-\alpha)) n^{1-\alpha} O(\tau^{3-\alpha})$ and

$$a_j = \frac{\tau^{-\alpha}}{(2-\alpha)\Gamma(1-\alpha)} \left[(j+1)^{2-\alpha} - j^{2-\alpha} \right], \qquad b_j = \frac{\tau^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \left[(j+1)^{1-\alpha} - j^{1-\alpha} \right]. \tag{2.6}$$

Then, we can write the following approximation:

$$\frac{\partial^{\alpha} U(t_{n+1/2}, x_i)}{\partial t^{\alpha}} = \frac{H(t_{n+1}, x_i) - H(t_n, x_i)}{\tau} + O\left(\tau^2\right)$$
$$= q_n U_i^0 + \sum_{j=0}^{n} p_j U_i^{n+1-j} + \frac{R^{n+1} - R^n}{\tau} + O\left(\tau^2\right)$$
$$= q_n U_i^0 + \sum_{j=0}^{n} p_j U_i^{n+1-j} + \frac{1}{\Gamma(2-\alpha)} \left[(n+1)^{1-\alpha} - n^{1-\alpha} \right] O\left(\tau^{2-\alpha}\right) + O\left(\tau^2\right)$$
$$= q_n U_i^0 + \sum_{j=0}^{n} p_j U_i^{n+1-j} + \frac{1}{\Gamma(2-\alpha)} \left[\frac{(n+1)^{1-\alpha} - n^{1-\alpha}}{\tau} \right] O\left(\tau^{3-\alpha}\right) + O\left(\tau^2\right)$$
$$= q_n U_i^0 + \sum_{j=0}^{n} p_j U_i^{n+1-j} + \frac{1}{\Gamma(2-\alpha)} \left[\frac{(\tau(n+1))^{1-\alpha} - (\tau n)^{1-\alpha}}{\tau} \right] O\left(\tau^2\right) + O\left(\tau^2\right), \tag{2.7}$$

where

$$\begin{aligned}
q_0 &= 3a_0 - a_1 + 2b_1 - 2b_0, \\
q_n &= a_n - a_{n-1} + (n-1) b_{n-1} - (n+1) b_n, \quad \text{for } 1 \le n \le N-1, \\
p_0 &= b_0 - a_0, \\
p_1 &= 2a_0 - a_1 + 2b_1 - b_0, \\
p_j &= \left(-a_{j-2} + 2a_{j-1} - a_j\right) + \left(j-2\right) b_{j-2} - (2j-1) b_{j-1} + \left(j+1\right) b_j, \quad \text{for } j \ge 2.
\end{aligned} \tag{2.8}$$

On the other hand, using the mean-value theorem, we get

$$\frac{(\tau(n+1))^{1-\alpha} - (\tau n)^{1-\alpha}}{\tau} = f'(c) = \text{constant}, \tag{2.9}$$

where $f(x) = x^{1-\alpha}$ and $t_n < c < t_{n+1}$. So, we obtain the following second-order approximation for the modified Riemann-Liouville derivative:

$$\begin{aligned}\frac{\partial^\alpha U(t_{n+(1/2)}, x_i)}{\partial t^\alpha} &= \frac{H(t_{n+1}, x_i) - H(t_n, x_i)}{\tau} + O\left(\tau^2\right) \\ &= q_n U_i^0 + \sum_{j=0}^{n} p_j U_i^{n+1-j} + O\left(\tau^2\right).\end{aligned} \tag{2.10}$$

3. Crank-Nicholson Difference Scheme

Using the approximation above, we obtain the following difference scheme which is accurate of order $O(\tau^2 + h^2)$:

$$\begin{gathered} q_n U_i^0 + \sum_{j=0}^{n} p_j U_i^{n+1-j} - \left[\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{2h^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{2h^2}\right] \\ = f\left(t_n + \frac{\tau}{2}, x_i\right), \quad 0 \le n \le N-1,\ 1 \le i \le M-1, \\ U_i^0 = r(x_i), \quad 1 \le i \le M-1, \\ U_0^n = 0, \quad U_M^n = 0, \quad 0 \le n \le N. \end{gathered} \tag{3.1}$$

We can arrange the system above to obtain

$$\begin{gathered} \left(-\frac{1}{2h^2}\right)\left(U_{i+1}^{n+1} + U_{i+1}^{n}\right) + q_n U_i^0 + \sum_{j=0}^{n} p_j U_i^{n+1-j} + \left(-\frac{1}{2h^2}\right)\left(U_{i-1}^{n+1} + U_{i-1}^{n}\right) \\ = f\left(t_n + \frac{\tau}{2}, x_i\right), \quad 0 \le n \le N-1,\ 1 \le i \le M-1, \\ U_i^0 = r(x_i), \quad 1 \le i \le M-1, \\ U_0^n = 0, \quad U_M^n = 0, \quad 0 \le n \le N. \end{gathered} \tag{3.2}$$

The difference scheme above can be written in matrix form:

$$AU_{i+1} + BU_i + AU_{i-1} = \varphi_i, \tag{3.3}$$

where $\varphi_i = [\varphi_i^0, \varphi_i^1, \varphi_i^2, \ldots, \varphi_i^N]^T$, $\varphi_i^0 = r(x_i)$, $\varphi_i^n = f(t_{n+1/2}, x_i)$, $1 \le n \le N$, $1 \le i \le M$, and $U_i = [U_i^0, U_i^1, U_i^2, \ldots, U_i^N]^T$.

Here, $A_{(N+1)\times(N+1)}$ and $B_{(N+1)\times(N+1)}$ are the matrices of the form

$$A = \left(-\frac{1}{2h^2}\right)\begin{bmatrix} 0 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & & & & & \\ q_0 + \frac{1}{h^2} & p_0 + \frac{1}{h^2} & & & & \\ q_1 & p_1 + \frac{1}{h^2} & p_0 + \frac{1}{h^2} & & & \\ q_2 & p_2 & p_1 + \frac{1}{h^2} & p_0 + \frac{1}{h^2} & & \\ \vdots & & \ddots & \ddots & \ddots & \\ q_{N-1} & p_{N-1} & \cdots & p_2 & p_1 + \frac{1}{h^2} & p_0 + \frac{1}{h^2} \end{bmatrix}. \tag{3.4}$$

We note that the unspecified entries are zero at the matrices above.

Using the idea on the modified Gauss-Elimination method, we can convert (3.3) into the following form:

$$U_i = \alpha_{i+1} U_{i+1} + \beta_{i+1}, \quad i = M-1, \dots, 2, 1, 0. \tag{3.5}$$

This way, the two-step form of difference schemes in (3.3) is transformed to one-step method as in (3.5).

Now, we need to determine the matrices α_{i+1} and β_{i+1} satisfying the last equality. Since $U_0 = \alpha_1 U_1 + \beta_1 = 0$, we can select $\alpha_1 = O_{(N+1)\times(N+1)}$ and $\beta_1 = O_{(N+1)\times 1}$. Combining the equalities $U_i = \alpha_{i+1} U_{i+1} + \beta_{i+1}$ and $U_{i-1} = \alpha_i U_i + \beta_i$ and the matrix equation (3.3), we have

$$(A + B\alpha_{i+1} + A\alpha_i\alpha_{i+1})U_{i+1} + \left(B\beta_{i+1} + A\alpha_i\beta_{i+1} + A\beta_i\right) = \varphi_i. \tag{3.6}$$

Then, we write

$$\begin{gathered} A + B\alpha_{i+1} + A\alpha_i\alpha_{i+1} = 0, \\ B\beta_{i+1} + A\alpha_i\beta_{i+1} + A\beta_i = \varphi_i, \end{gathered} \tag{3.7}$$

where $1 \le i \le M-1$.

So, we obtain the following pair of formulas:

$$\begin{aligned} \alpha_{i+1} &= -(B + A\alpha_i)^{-1} A, \\ \beta_{i+1} &= (B + A\alpha_i)^{-1} (\varphi_i - A\beta_i), \end{aligned} \tag{3.8}$$

where $1 \le i \le M - 1$.

4. Stability of the Method

The stability analysis is done by using the analysis of the eigenvalues of the iteration matrix α_i $(1 \le i \le M)$ of the scheme (3.5).

Let $\rho(A)$ denote the spectral radius of a matrix A, that is, the maximum of the absolute value of the eigenvalues of the matrix A.

We will prove that $\rho(\alpha_i) < 1$, $(1 \le i \le M)$, by induction.

Since α_1 is a zero matrix $\rho(\alpha_1) = 0 < 1$.

Moreover, $\alpha_2 = -B^{-1}A, \rho(\alpha_2) = \rho(-B^{-1}A) = \dfrac{-1}{1/h^2 + p_0} \cdot \dfrac{-1}{2h^2} = \dfrac{1/h^2}{2(1/h^2 + p_0)}$, since α_2 is of the form

$$\alpha_2 = \begin{bmatrix} 0 & & & & \\ * & \dfrac{1/h^2}{2(1/h^2 + p_0)} & & & \\ * & * & \dfrac{1/h^2}{2(1/h^2 + p_0)} & & \\ & & & \ddots & \\ * & * & * & & \dfrac{1/h^2}{2(1/h^2 + p_0)} \end{bmatrix}_{(N+1)\times(N+1)}, \tag{4.1}$$

$$p_0 = b_0 - a_0 = \frac{\tau^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} - \frac{\tau^{-\alpha}}{(2-\alpha)\Gamma(1-\alpha)} = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} > 0,$$

therefore, $\rho(\alpha_2) < 1$.

Now, assume $\rho(\alpha_i) < 1$. After some calculations, we find that

$$\begin{aligned} \alpha_{i+1} &= -(B + A\alpha_i)^{-1} A \\ &= \left(\frac{1}{2h^2}\right) \begin{bmatrix} 0 & & & & \\ * & \dfrac{1}{B_{2,2} - (1/2h^2)\alpha_{i_{2,2}}} & & & \\ * & * & \dfrac{1}{B_{3,3} - (1/2h^2)\alpha_{i_{3,3}}} & & \\ * & * & * & \ddots & \\ * & * & * & & \dfrac{1}{B_{N+1,N+1} - (1/2h^2)\alpha_{i_{N+1,N+1}}} \end{bmatrix} \end{aligned} \tag{4.2}$$

and we already know that $B_{j,j} = 1/h^2 + w_0$ and $\alpha_{i_{j,j}} = \rho(\alpha_i)$ for $2 \le j \le N+1$:

$$\rho(\alpha_{i+1}) = \left| \frac{1/2h^2}{1/h^2 + p_0 - (1/2h^2)\rho(\alpha_i)} \right| = \frac{M^2}{2\left[M^2(1-\rho(\alpha_i)/2) + p_0\right]}. \tag{4.3}$$

Since $0 \le \rho(\alpha_i) < 1$, it follows that $\rho(\alpha_{i+1}) < 1$. So, $\rho(\alpha_i) < 1$ for any i, where $1 \le i \le M$.

Remark 4.1. The convergence of the method follows from the Lax equivalence theorem [31] because of the stability and consistency of the proposed scheme.

5. Numerical Analysis

Example 5.1. Consider

$$\begin{gathered}\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{2t^{(2-\alpha)}}{\Gamma(3-\alpha)} \sin\left(x - x^2\right) \\ + t^2\left[\sin\left(x-x^2\right)(1-2x)^2 + 2\cos\left(x-x^2\right)\right], \quad (0 < x < 1, 0 < t < 1), \\ u(0,x) = 0, \quad 0 \le x \le 1, \\ u(t,0) = 0, \quad u(t,1) = 0, \quad 0 \le t \le 1.\end{gathered} \tag{5.1}$$

Exact solution of this problem is $U(t,x) = t^2 \sin(1-x)x$. The solution by the Crank-Nicholson scheme is given in Figure 1. The errors when solving this problem are listed in the Table 1 for various values of time and space nodes.

The errors in the table are calculated by the formula $\max_{0\le n\le M,\, 0\le k\le N}|u(t_k,x_n) - U_n^k|$ and the *error rate* formula is $|E_k|/|E_{k+1}|$.

Example 5.2. Consider

$$\begin{gathered}\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{24t^{(2-\alpha)}}{\Gamma(5-\alpha)}\left(x^2 - x\right) - 2t^4, \quad (0 < x < 1, 0 < t < 1), \\ u(0,x) = 0, \quad 0 \le x \le 1, \\ u(t,0) = 0, \quad u(t,1) = 0, \quad 0 \le t \le 1.\end{gathered} \tag{5.2}$$

Exact solution of this problem is $U(t,x) = t^4 x(x-1)$. The solution by the Crank-Nicholson scheme is given in Figure 2. The errors when solving this problem are listed in Table 2 for various values of time and space nodes and several values of α.

It can be concluded from the tables and the figures that when the step size is reduced by a factor of 1/2, the error decreases by about 1/4. The numerical results support the claim about the order of the convergence.

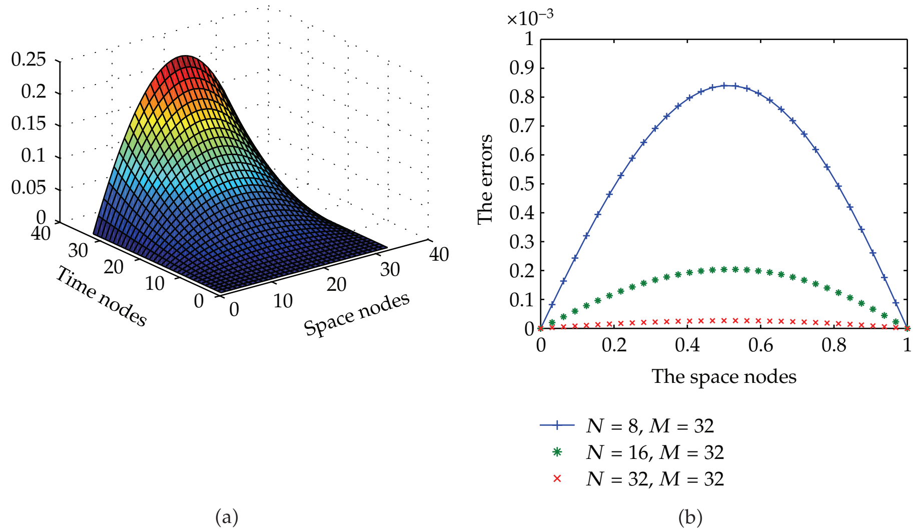

Figure 1: (a) The approximate solutions of Example 5.1 by the proposed method when $N = 32$, $M = 32$, and $\alpha = 0.5$. (b) The errors for some values of M and N when $t = 1$ and $\alpha = 0.5$.

Table 1: Error table for Example 5.1.

		$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.9$	
M	N	Error	Rate	Error	Rate	Error	Rate
32	8	0.0018870311	—	0.0016846217	—	0.0009754809	—
32	16	0.0004703510	4.01	0.0004052354	4.16	0.0002461078	3.97
32	32	0.0001172029	4.01	0.0000969929	4.18	0.0000650942	3.78
32	64	0.0000291961	4.01	0.00002314510	4.19	0.0000198362	3.28

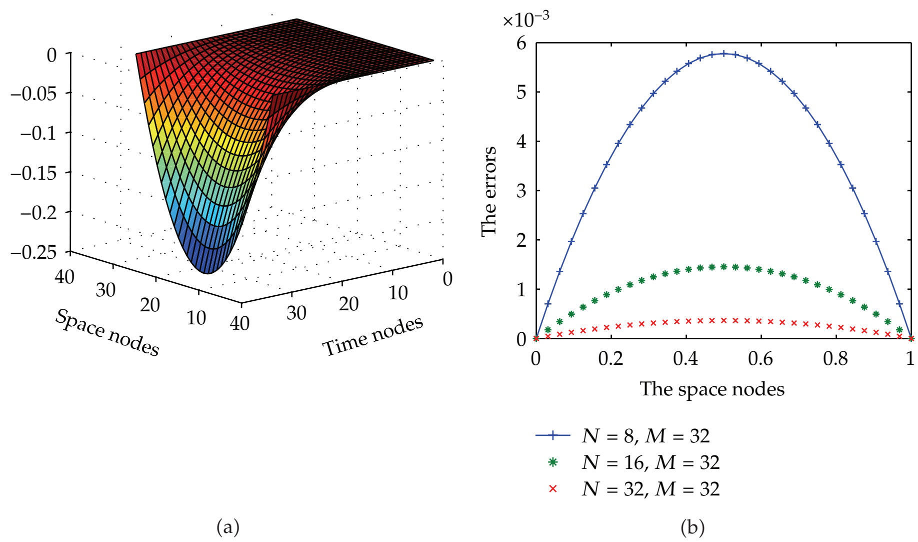

Figure 2: (a) The approximate solutions of Example 5.2 by the proposed method when $N = 32$, $M = 32$, and $\alpha = 0.5$. (b) The errors for some values of M and N when $t = 1$ and $\alpha = 0.5$.

Table 2: The errors for some values of M, N, and α.

		$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.8$	
M	N	Error	Rate	Error	Rate	Error	Rate
4	4	0.02321328680	—	0.02286737567	—	0.02173420667	—
8	8	0.00583004420	3.98	0.00577931685	3.96	0.00554721754	3.92
16	16	0.00146112785	3.99	0.00145293106	3.98	0.00140076083	3.96
32	32	0.00036572715	3.995	0.00036424786	3.99	0.00035252421	3.97
64	64	0.00009148685	3.998	0.00009122231	3.99	0.00008860379	3.98

6. Conclusion

In this work, the Crank-Nicholson difference scheme was successfully extended to solve the time-fractional heat equations. A second-order approximation for the Riemann-Liouville fractional derivative is obtained. It is proven that the time-fractional Crank-Nicholson difference scheme is unconditionally stable and convergent. Numerical results are in good agreement with the theoretical results.

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Research Article

Alternative Forms of Compound Fractional Poisson Processes

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We study here different fractional versions of the compound Poisson process. The fractionality is introduced in the counting process representing the number of jumps as well as in the density of the jumps themselves. The corresponding distributions are obtained explicitly and proved to be solution of fractional equations of order less than one. Only in the final case treated in this paper, where the number of jumps is given by the fractional-difference Poisson process defined in Orsingher and Polito (2012), we have a fractional driving equation, with respect to the time argument, with order greater than one. Moreover, in this case, the compound Poisson process is Markovian and this is also true for the corresponding limiting process. All the processes considered here are proved to be compositions of continuous time random walks with stable processes (or inverse stable subordinators). These subordinating relationships hold, not only in the limit, but also in the finite domain. In some cases the densities satisfy master equations which are the fractional analogues of the well-known Kolmogorov one.

1. Introduction and Preliminary Results

The fractional Poisson process (FPP), which we will denote by $\mathcal{N}_\beta(t)$, $t > 0$, $\beta \in (0,1]$, has been introduced in [1], by replacing, in the differential equation governing the Poisson process, the time derivative with a fractional one. Later, in [2, 3], it was proved to be a renewal process with Mittag-Leffler distributed waiting times (and therefore with infinite mean). In [4] it has been expressed as the composition $N(\mathcal{T}_\beta(t))$ of a standard Poisson process N with the fractional diffusion $\mathcal{T}_\beta$, independent of N. A full characterization of $\mathcal{N}_\beta$ in terms of its finite multidimensional distributions can be found in [5]. In [6] the coincidence between $\mathcal{N}_\beta$ and the fractal time Poisson process (FTPP) defined as $N(\mathcal{L}_\beta(t))$ has been proved, where

$\mathcal{L}_\beta(t)$, $t \geq 0$ is the inverse of the stable subordinator $\mathcal{A}_\beta(t)$ of index β (with parameters $\mu = 0$, $\theta = 1$, $\sigma = (t\cos \pi\beta/2)^{1/\beta}$, in the notation of [7], that we will adopt hereafter). Thus, the process $\mathcal{A}_\beta$ is characterized by the following Laplace pairs:

$$\mathbb{E}e^{-k\mathcal{A}_\beta(t)} = e^{-k^\beta t}, \quad k,t > 0,$$
$$\int_0^{+\infty} e^{-st} h_\beta(x,t)dt = x^{\beta-1}E_{\beta,\beta}\left(-sx^\beta\right), \quad s,x > 0, \tag{1.1}$$

where $E_{\beta,\delta}$ is the Mittag-Leffler function of parameters β, δ and $h_\beta(x,t)$ is the density of $\mathcal{A}_\beta(t)$. The inverse stable subordinator $\mathcal{L}_\beta$ is defined by the following relation:

$$\mathcal{L}_\beta(t) := \inf\{s : \mathcal{A}_\beta(s) = t\}, \quad z,t > 0, \tag{1.2}$$

and therefore we get

$$\mathbb{E}e^{-k\mathcal{L}_\beta(t)} = E_{\beta,1}\left(-kt^\beta\right), \quad k,t > 0,$$
$$\int_0^{+\infty} e^{-st} l_\beta(x,t)dt = s^{\beta-1}e^{-xs^\beta}, \quad s,x > 0, \tag{1.3}$$

where $l_\beta(x,t)$ is the density of $\mathcal{L}_\beta(t)$.

We will make use also of different forms of FPP such as the alternative fractional Poisson process in [8] and the fractional-difference Poisson process in [9].

In this paper we study several fractional compound Poisson processes and, to help the reader, we list the acronyms used throughout the paper by the end of the paper.

The first form of fractional compound Poisson process has been introduced in [10], in the form of a continuous time random walk with infinite-mean waiting times (see also [11]). This corresponds to the following random walk time changed via the FTPP, that is,

$$Y_\beta(t) = \sum_{j=1}^{N(\mathcal{L}_\beta(t))} X_j, \quad t \geq 0, \tag{1.4}$$

with X_j, $j = 1,2,\ldots$ are i.i.d. random variables, independent from N and $\mathcal{L}_\beta$. The last assumption (that we will adopt throughout the paper) corresponds to the so-called uncoupled case.

In [6] it is proved that subordinating random walk to the fractional Poisson process $\mathcal{N}_\beta(t)$, $t \geq 0$, produces the same one-dimensional distribution. The (generalized) density function of $Y_\beta(t)$ can be expressed as

$$g_{Y_\beta}(y,t) := E_{\beta,1}\left(-\lambda t^\beta\right)\delta(y) + f_{Y_\beta}(y,t), \quad y,t \geq 0, \tag{1.5}$$

where the first term refers to the probability mass concentrated in the origin, $\delta(y)$ denotes the Dirac delta function, and f_{Y_β} denotes the density of the absolutely continuous component. The function g_{Y_β} given in (1.5) satisfies the following fractional master equation, that is,

$$\frac{\partial^\beta}{\partial t^\beta} g_{Y_\beta}(y,t) = -\lambda g_{Y_\beta}(y,t) + \lambda \int_{-\infty}^{+\infty} g_{Y_\beta}(y-x,t) f_X(x)dx, \tag{1.6}$$

where $\partial^\beta/\partial t^\beta$ is the Caputo fractional derivative of order $\beta \in (0,1]$ (see, for example, [12]) and the random variables X_j, $j = 1,2,\dots$ have continuous density f_X.

We also recall the following result proved in [13] for the rescaled version of the time-fractional compound Poisson process (hereafter TFCPP): if the random variables $X_j, j = 1,2,\dots$ are centered and have finite variance, then

$$c^{-\beta/2} Y_\beta(ct) \Longrightarrow W(\mathcal{L}_\beta(t)), \quad c \longrightarrow \infty, \tag{1.7}$$

where W is a standard Brownian motion and $\Rightarrow$ denotes weak convergence.

A detailed exposition of the theory of TFCPP and continuous time random walks can be found in [14, 15], where the density f_{Y_β} is expressed in terms of successive derivatives of the Mittag-Leffler function as follows:

$$f_{Y_\beta}(y,t) = \sum_{n=1}^{\infty} f_X^{*n}(y) \Pr\{\mathcal{N}_\beta(t) = n\} = \sum_{n=1}^{\infty} f_X^{*n}(y) \frac{(\lambda t^\beta)^n}{n!} \frac{\partial^n}{\partial x^n} E_{\beta,1}(x)\bigg|_{x=-\lambda t^\beta}, \quad t,y \ge 0, \tag{1.8}$$

where f_X^{*n} is the nth convolution of the density f_X of the r.v.'s X_j.

A further asymptotic result has been proved in [15], under the assumption that the density of the jump variables (which we will denote, in this special case, as X_j^*) behaves asymptotically as

$$\widehat{f}_{X^*}(h\kappa) := \int_{-\infty}^{+\infty} e^{i\kappa h x} \widehat{f}_{X^*}(x)dx \simeq 1 - h^\alpha |\kappa|^\alpha, \quad h \longrightarrow 0, \; \alpha \in (0,1], \tag{1.9}$$

where $\widehat{\cdot}$ denotes the Fourier transform. In this case the TFCPP is defined as $Y_\beta^*(t) = \sum_{j=1}^{\mathcal{N}_\beta(t)} X_j^*$ and the rescaled version displays the following weak convergence:

$$hY_\beta^*\left(\frac{t}{r}\right) \Longrightarrow Z(t), \tag{1.10}$$

for $h, r \to 0$, s.t. $h^\alpha / r^\beta \to 1$. The characteristic function of the limiting process $Z(t)$ is given by

$$E_{\beta,1}\left(-\lambda t^\beta |\kappa|^\alpha\right) \tag{1.11}$$

and thus it can be represented as $\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$, where $\mathcal{S}_\alpha$ is a symmetric α-stable process with parameters $\mu = 0$, $\theta = 0$, $\sigma = (t \cos \pi\alpha/2)^{1/\alpha}$. For $\beta < 1$, the inverse stable subordinator

$\mathcal{L}_\beta(t)$ is not Markovian as well as not Lévy (see [10]) and the same is true for $\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$, as remarked in [15]; moreover, the density $u = u(y,t)$ of the latter is the solution to the space-time fractional equation:

$$\frac{\partial^\beta u}{\partial t^\beta} = \lambda \frac{\partial^\alpha u}{\partial |y|^\alpha}, \qquad u(y,0) = \delta(y), \quad y \in \mathbb{R},\ t > 0, \tag{1.12}$$

where $\partial^\alpha / \partial|y|^\alpha$ denotes the Riesz-Feller derivative of order $\alpha \in (0,1]$ (see [16]). Thus, in the special case $\alpha = 1$, it reduces to the composition of a Cauchy process with $\mathcal{L}_\beta$.

Finally, we recall the following result proved in [17]: under the assumption of heavy tailed r.v.'s representing the jumps, that is,

$$\Pr\{|X_j| > x\} \sim x^{-\alpha}, \quad x \longrightarrow \infty, \tag{1.13}$$

the following convergence holds, as $c \to \infty$,

$$c^{-\beta/\alpha} Y_\beta(ct) \Longrightarrow \mathcal{O}(\mathcal{L}_\beta(t)). \tag{1.14}$$

In (1.14) $\mathcal{O}$ is a α-stable Lévy process with density $p_\alpha(x,t)$ and characteristic function

$$\widehat{p}_\alpha(\kappa, t) = e^{tb[q(-i\kappa)^\alpha + (1-q)(i\kappa)^\alpha]}, \quad \text{for } \begin{cases} b < 0,\ 0 < \alpha < 1 \\ \text{or } b > 0,\ 1 < \alpha < 2, \end{cases} \tag{1.15}$$

under the assumption that $\lim_{x\to\infty} \Pr\{X_j < -x\}/\Pr\{|X_j| > x\} = q \in 0,1]$. The density of the limiting process is proved to satisfy the following time and space fractional equation:

$$D^\beta_{0+,t} u = qbD^\alpha_{-,x} u + (1-q)bD^\alpha_{0+,x} u, \tag{1.16}$$

where the fractional derivatives are intended in the Riemann-Liouville sense (see [12], formulae (2.2.3) and (2.2.4), page 80).

We present, in this paper, different versions of the compound Poisson process (CPP), fractional (under different acceptions) with respect to time and space; we provide for them analytic expressions of the distributions and some composition relationships with stable and inverse-stable processes, holding not only in the scaling limit, but also in the finite domain.

Tables 1 and 2 provide a summary of these results in the finite and asymptotic domains, respectively.

We assume here exponential jumps (generalized later to Mittag-Leffler), since this allows to obtain explicit equations (fractional in most cases) driving these fractional CPP's for any finite value of the time and space arguments. This kind of explicit formulae, together with the knowledge of the related governing differential equations, is of great importance in many actuarial applications (see, for example, [18], Section 4.2). In risk theory it is related to the Tweedie's compound Poisson model (see [19]). The hypothesis of exponential jumps has been widely applied also in other fields: in natural sciences it leads to the so-called compound Poisson-Gamma model, which is used for rainfall prediction (see, for example, [20]).

2. Time-Fractional Compound Poisson Processes

We consider different forms of TFCPP, starting with the more familiar one given in (1.4) and then comparing the results with those obtained for an alternative definition of FPP.

2.1. The Standard Case

In order to get a form of the density of the TFCPP more explicit than (1.8), we assume that the X_j's are exponentially distributed: in this case it can be expressed in terms of the generalized Mittag-Leffler function:

$$E^{\gamma}_{\alpha,\delta}(x) = \sum_{j=0}^{\infty} \frac{(\gamma)^{(j)}}{j!} \frac{x^j}{\Gamma(\alpha j + \delta)}, \quad \alpha, \delta, \gamma \in \mathbb{C}, \ \mathcal{R}(\alpha), \mathcal{R}(\delta) > 0, \tag{2.1}$$

where $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ is the rising factorial (or Pochhammer symbol). Moreover, we can obtain the fractional partial-differential equation satisfied by the density of its absolutely continuous component.

Theorem 2.1. *The process*

$$Y_\beta(t) = \sum_{j=1}^{\mathcal{N}_\beta(t)} X_j, \quad t \geq 0, \tag{2.2}$$

with X_j, $j = 1, 2\ldots$, independent and exponentially distributed with parameter ξ, has the following distribution:

$$\Pr\{Y_\beta(t) \leq y\} = E_{\beta,1}\left(-\lambda t^\beta\right) 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{Y_\beta}(z,t)\,dz, \quad t \geq 0, \ y \in \mathbb{R}, \tag{2.3}$$

where

$$f_{Y_\beta}(y,t) = \frac{e^{-\xi y}}{y} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)!} E^{n+1}_{\beta,\beta n+1}\left(-\lambda t^\beta\right) 1_{[0,+\infty)}(y), \quad t \geq 0. \tag{2.4}$$

The function $f_{Y_\beta}(y,t)$ given in (2.4) *satisfies the following partial differential equation:*

$$\xi \frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta} = -\left[\lambda + \frac{\partial^\beta}{\partial t^\beta}\right] \frac{\partial}{\partial y} f_{Y_\beta}, \quad t, y \geq 0, \tag{2.5}$$

where $\partial^\beta / \partial t^\beta$ denotes the Caputo fractional derivative with the conditions

$$\begin{gathered} f_{Y_\beta}(y,0) = 0, \\ \int_0^{+\infty} f_{Y_\beta}(y,t)\,dy = 1 - E_{\beta,1}\left(-\lambda t^\beta\right). \end{gathered} \tag{2.6}$$

Proof . Formula (1.8) can be rewritten by considering that $f_X^{*n}(y) = \xi^n y^{n-1} e^{-\xi y}/(n-1)!$ and using the expression of $\Pr\{\mathcal{N}_\beta(t) = n\}$ in terms of generalized Mittag-Leffler functions (see [21]), that is,

$$\Pr\{\mathcal{N}_\beta(t) = n\} = \lambda^n t^{n\beta} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right), \quad n \geq 0. \tag{2.7}$$

In order to derive (2.5), we evaluate the following partial derivatives of (2.4):

$$\begin{aligned}
\frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) &= \frac{e^{-\xi y}}{y t^\beta} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)!n!} \sum_{j=0}^{\infty} \frac{(n+j)!(-\lambda t^\beta)^j}{j!\Gamma(\beta j + \beta n - \beta + 1)}, \\
\frac{\partial}{\partial y} f_{Y_\beta}(y,t) &= -\frac{\xi e^{-\xi y}}{y} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)!n!} \sum_{j=0}^{\infty} \frac{(n+j)!(-\lambda t^\beta)^j}{j!\Gamma(\beta j + \beta n + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2} \sum_{n=2}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)!n!} \sum_{j=0}^{\infty} \frac{(n+j)!(-\lambda t^\beta)^j}{j!\Gamma(\beta j + \beta n + 1)}, \\
\frac{\partial}{\partial y}\frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) &= -\frac{\xi e^{-\xi y}}{y t^\beta} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{n!(n-1)!} \sum_{j=0}^{\infty} \frac{(n+j)!(-\lambda t^\beta)^j}{j!\Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2 t^\beta} \sum_{n=2}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)!n!} \sum_{j=0}^{\infty} \frac{(n+j)!(-\lambda t^\beta)^j}{j!\Gamma(\beta j + \beta n - \beta + 1)} \\
&= -\frac{\xi e^{-\xi y}}{y t^\beta} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{((n-1)!)^2} \sum_{j=0}^{\infty} \frac{(n+j-1)!(-\lambda t^\beta)^j}{j!\Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad - \frac{\xi e^{-\xi y}}{y t^\beta} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{n!(n-1)!} \sum_{j=1}^{\infty} \frac{(n+j-1)!(-\lambda t^\beta)^j}{(j-1)!\Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2 t^\beta} \sum_{n=2}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)!(n-1)!} \sum_{j=0}^{\infty} \frac{(n+j-1)!(-\lambda t^\beta)^j}{j!\Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2 t^\beta} \sum_{n=2}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)!n!} \sum_{j=0}^{\infty} \frac{(n+j-1)!(-\lambda t^\beta)^j}{(j-1)!\Gamma(\beta j + \beta n - \beta + 1)}.
\end{aligned} \tag{2.8}$$

By inserting (2.8) in (2.5), the equation is satisfied. Finally, it can be easily verified that the initial condition holds. In order to check the second condition in (2.6), we integrate f_{Y_β} with respect to y:

$$\int_0^\infty \frac{e^{-\xi y}}{y} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)!} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right) dy = \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta\right)^n}{(n-1)!} \frac{(n-1)!}{\xi^n} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\lambda t^{\beta}\right)^{n} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^{\beta}\right) - E_{\beta,1}\left(-\lambda t^{\beta}\right) \\
&= 1 - E_{\beta,1}\left(-\lambda t^{\beta}\right),
\end{aligned}
\tag{2.9}
$$

where, in the last step, we have applied formula (2.30) of [21], for $u = 1$. □

2.1.1. *The Nonfractional Case* $\beta = 1$

From (2.4), we obtain the distribution of the standard CPP, defined as $Y(t) = \sum_{n=1}^{N(t)} X_j$, under the assumption of exponential jumps X_j, which reads

$$
\Pr\{Y(t) \le y\} = e^{-\lambda t} 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_Y(z,t)dz, \qquad t \ge 0,\ y \in \mathbb{R}, \tag{2.10}
$$

where

$$
\begin{aligned}
f_Y(y,t) &= \frac{e^{-\xi y - \lambda t}}{y} \sum_{n=1}^{\infty} \frac{(\lambda \xi t y)^{n}}{n!(n-1)!} 1_{[0,+\infty)}(y) \\
&= \lambda \xi t e^{-\xi y - \lambda t} W_{1,2}(\lambda \xi t y) 1_{[0,+\infty)}(y), \quad t \ge 0,
\end{aligned}
\tag{2.11}
$$

$$
W_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^{j}}{j!\Gamma(\alpha j + \beta)}, \quad \alpha > -1,\ \beta, z \in \mathbb{C}, \tag{2.12}
$$

is the Wright function. Equation (4.2.8) in [18] provides another expression of f_Y in terms of the modified Bessel function. The density (2.11) satisfies the following equation:

$$
\xi \frac{\partial}{\partial t} f_Y = -\left[\lambda + \frac{\partial}{\partial t}\right] \frac{\partial}{\partial y} f_Y, \tag{2.13}
$$

with conditions

$$
\begin{aligned}
&f_Y(y,0) = 0, \\
&\int_{0}^{+\infty} f_Y(y,t)dy = 1 - e^{-\lambda t},
\end{aligned}
\tag{2.14}
$$

as can be easily verified directly.

Now we recall the following subordination law presented in [6] in a more general setting:

$$
Y_{\beta}(t) \overset{d}{=} Y(\mathcal{L}_{\beta}(t)), \tag{2.15}
$$

where $\mathcal{L}_\beta(t)$, $t \geq 0$ is the inverse stable subordinator defined by (1.2). We give an explicit proof of (2.15), which will be useful to prove analogous results in the next sections. We start with the evaluation of the Laplace transform (hereafter denoted by $\widetilde{\cdot}$) of $Y_\beta(t)$ with respect to y: by considering the probability generating function of $\mathcal{N}_\beta$, that is,

$$\mathbb{E}u^{\mathcal{N}_\beta(t)} = E_{\beta,1}\left(-\lambda t^\beta(1-u)\right), \quad |u| \leq 1, \tag{2.16}$$

we get

$$\widetilde{g}_{Y_\beta}(k,t) := \mathbb{E}e^{-kY_\beta(t)} = E_{\beta,1}\left(-\frac{\lambda k}{k+\xi}t^\beta\right). \tag{2.17}$$

Formula (2.17), Laplace transformed with respect to t, gives

$$\widetilde{\widetilde{g}}_{Y_\beta}(k,s) := \int_0^{+\infty} e^{-st}\widetilde{g}_{Y_\beta}(k,t)dt = \frac{s^{\beta-1}(k+\xi)}{s^\beta(k+\xi)+k\lambda}, \tag{2.18}$$

which can be rewritten as

$$\begin{aligned} \widetilde{\widetilde{g}}_{Y_\beta}(k,s) &= s^{\beta-1}\int_0^{+\infty} e^{-s^\beta t}\mathbb{E}e^{-kY(t)}dt \\ &= [\text{by } (1.3)] \\ &= \int_0^{+\infty} \mathbb{E}e^{-kY(z)}\widetilde{l}_\beta(z;s)dz, \end{aligned} \tag{2.19}$$

where $\widetilde{l}_\beta(z;s) := \int_0^{+\infty} e^{-st}l_\beta(z,t)dt$. Thus, by inverting the double Laplace transform, we get

$$\Pr\{Y_\beta(t) \in dy\} = \int_0^{+\infty} \Pr\{Y(z) \in dy\}l_\beta(z,t)dz. \tag{2.20}$$

Now it is also easy to derive (2.5), since we can write in particular from (2.20) that

$$f_{Y_\beta}(y,t) = \int_0^{+\infty} f_Y(y,z)l_\beta(z,t)dz \tag{2.21}$$

and thus we get

$$\begin{aligned} \frac{\partial^\beta}{\partial t^\beta}f_{Y_\beta}(y,t) &= \int_0^{+\infty} f_Y(y,z)\frac{\partial^\beta}{\partial t^\beta}l_\beta(z,t)dz \\ &= -\int_0^{+\infty} f_Y(y,z)\frac{\partial}{\partial z}l_\beta(z,t)dz. \end{aligned} \tag{2.22}$$

Indeed, it is well known that $\mathcal{L}_\beta(t)$ is governed by the following equation:

$$\frac{\partial^\beta}{\partial t^\beta} l_\beta(z,t) = -\frac{\partial}{\partial z} l_\beta(z,t), \qquad l_\beta(z,0) = \delta(z), \quad z,t \geq 0. \tag{2.23}$$

By integrating by parts and applying the initial condition, (2.22) becomes

$$\begin{aligned} \frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) &= \int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) l_\beta(z,t) dz \\ &= [\text{by } (2.13)] \\ &= -\frac{1}{\xi}\frac{\partial}{\partial y}\int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) l_\beta(z,t) dz - \frac{\lambda}{\xi}\frac{\partial}{\partial y}\int_0^{+\infty} f_Y(y,z) l_\beta(z,t) dz \\ &= -\frac{1}{\xi}\frac{\partial}{\partial y}\frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) - \frac{\lambda}{\xi}\frac{\partial}{\partial y} f_{Y_\beta}(y,t). \end{aligned} \tag{2.24}$$

2.2. An Alternative Case

We consider now a different model of TFCPP, based on the alternative definition of FPP given in [4], that is,

$$\Pr\left\{\overline{N}_\beta(t) = k\right\} = \frac{\left(\lambda t^\beta\right)^k}{\Gamma(\beta k + 1)} \frac{1}{E_{\beta,1}\left(\lambda t^\beta\right)}, \quad t,k \geq 0. \tag{2.25}$$

The process with the above state probabilities plays a crucial role in the evolution of some random motions (see [22]) and can be considered as a fractional version of the Poisson process because its probability generating function (displayed below) satisfies a fractional equation (see formula (4.5) of [4]). The distribution (2.25) can be interpreted as a weighted Poisson distribution (for the general concept of discrete weighted distribution see, e.g., [23], page 90, and the references cited therein) and, as explained in [8], the weights that do not depend on t; actually we have

$$\Pr\left\{\overline{N}_\beta(t) = k\right\} = \frac{w_k p_k\left(t^\beta\right)}{\sum_{j\geq 0} w_j p_j\left(t^\beta\right)}, \quad t,k \geq 0, \tag{2.26}$$

where $w_j = j!/\Gamma(\beta j + 1)$, $j = 0,1,\ldots$ (for all t) and $p_j(t) = ((\lambda t)^j/j!)e^{-\lambda t}$, $j = 0,1,\ldots$ are the distribution of the standard Poisson process N with intensity λ. We also recall [24] where one can find a sample path version of the weighted Poisson process.

We remark that the corresponding process is not Markovian, as $\mathcal{N}_\beta$, and moreover is not a renewal. Nevertheless, it is, for some aspects, more similar to the standard Poisson process N than $\mathcal{N}_\beta$. For example, the rate of the asymptotic behavior of its moments is the same as for N.

The moment generating function is given by

$$\mathbb{E}e^{\theta \overline{N}_\beta(t)} = \frac{E_{\beta,1}(\lambda t^\beta e^\theta)}{E_{\beta,1}(\lambda t^\beta)}, \tag{2.27}$$

so that we get

$$\mathbb{E}\overline{N}_\beta(t) = \frac{\lambda t^\beta}{\beta}\frac{E_{\beta,\beta}(\lambda t^\beta)}{E_{\beta,1}(\lambda t^\beta)}. \tag{2.28}$$

By applying the following asymptotic formula of the Mittag-Leffler function

$$E_{\beta,\nu}(z) \simeq \frac{1}{\beta} z^{(1-\nu)/\beta} \exp\left\{z^{1/\beta}\right\}, \quad \text{as } z \longrightarrow \infty, \tag{2.29}$$

(see, for example, [25] or [26]) we get

$$\mathbb{E}\overline{N}_\beta(t) \simeq \frac{1}{\beta}\lambda^{1/\beta} t, \quad \text{as } t \longrightarrow \infty, \tag{2.30}$$

while for $\mathcal{N}_\beta$ the mean value behaves asymptotically as t^β.

We define the alternative TFCPP as

$$\overline{Y}_\beta(t) = \sum_{j=1}^{\overline{N}_\beta(t)} X_j, \quad t \geq 0,\ \beta \in (0,1], \tag{2.31}$$

where again X_j's are i.i.d. with exponential distribution, independent from $\overline{N}_\beta$. Under this assumption we obtain the following result on the distribution of $\overline{Y}_\beta$.

Theorem 2.2. *The process $\overline{Y}_\beta$ defined in* (2.31)*, with X_j, $j = 1, 2, \ldots$, independent and exponentially distributed with parameter ξ, has the following distribution:*

$$\Pr\left\{\overline{Y}_\beta(t) \leq y\right\} = \frac{1}{E_{\beta,1}(\lambda t^\beta)} 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{\overline{Y}_\beta}(z,t)\,dz, \quad t \geq 0,\ y \in \mathbb{R}, \tag{2.32}$$

where

$$f_{\overline{Y}_\beta}(y,t) = \frac{\lambda \xi t^\beta e^{-\xi y}}{E_{\beta,1}(\lambda t^\beta)} W_{\beta,\beta+1}\left(\lambda \xi t^\beta y\right) 1_{[0,+\infty)}(y), \quad t \geq 0. \tag{2.33}$$

Proof. The density (2.33) can be obtained as follows:

$$f_{\overline{Y}_\beta}(y,t) = \frac{e^{-\xi y}}{E_{\beta,1}(\lambda t^\beta)} \sum_{n=1}^{\infty} \frac{(\lambda t^\beta)^n}{\Gamma(\beta n+1)} \frac{\xi^n y^{n-1}}{(n-1)!}$$
$$= \frac{\lambda \xi t^\beta e^{-\xi y}}{E_{\beta,1}(\lambda t^\beta)} \sum_{l=0}^{\infty} \frac{(\lambda \xi t^\beta)^l}{l!\Gamma(\beta l+\beta+1)}. \tag{2.34}$$

Moreover, one can check that

$$\int_0^\infty f_{\overline{Y}_\beta}(y,t)\,dy = 1 - \frac{1}{E_{\beta,1}(\lambda t^\beta)}, \tag{2.35}$$

and this completes the proof. □

Remark 2.3. For $\beta = 1$, formula (2.33) reduces to (2.11). We note that, as happens for the standard case, the density in (2.33) is expressed in terms of a single Wright function instead of an infinite sum of generalized Mittag-Leffler functions (as for the process Y_β). Nevertheless, the presence of a Mittag-Leffler in the denominator does not allow to evaluate the equation satisfied by $f_{\overline{Y}_\beta}$.

2.2.1. Asymptotic Results

The analogy with the standard case is even more evident in the asymptotic behavior of the rescaled version of (2.31). Under the assumption (1.9) for the r.v.'s X_j^*, we can prove that, as $h, r \to 0$, s.t. $h^\alpha / r \to 1$ (not depending on β),

$$h\overline{Y}_\beta^*\left(\frac{t}{r}\right) = \sum_{j=1}^{\overline{N}_\beta(t/r)} hX_j^* \Longrightarrow \mathcal{S}_\alpha^\beta(t), \tag{2.36}$$

where $\mathcal{S}_\alpha^\beta$ is a symmetric α-stable Lévy process with $\mu = \theta = 0$ and $\sigma = ((1/\beta)\lambda^{1/\beta} t\cos(\pi\alpha/2))^{1/\alpha}$. Indeed, the characteristic function of (2.36) can be written as

$$\begin{aligned}\widehat{g}_{h\overline{Y}_\beta^*}\left(\kappa, \frac{t}{r}\right) &= \mathbb{E}e^{i\kappa h\overline{Y}_\beta^*(t/r)} = \frac{1}{E_{\beta,1}(\lambda(t^\beta/r^\beta))} \sum_{n=0}^{\infty} \frac{\left(\lambda(t^\beta/r^\beta)\widehat{f}_{hX}(\kappa)\right)^n}{\Gamma(\beta n+1)} \\ &= \frac{E_{\beta,1}\left(\lambda \widehat{f}_{hX}(\kappa)(t^\beta/r^\beta)\right)}{E_{\beta,1}(\lambda(t^\beta/r^\beta))} \\ &= [\text{by the assumption (1.9)}]\end{aligned}$$

$$\simeq \frac{E_{\beta,1}\left(\lambda\left(t^\beta/r^\beta\right)-\lambda\left(t^\beta h^\alpha|\kappa|^\alpha/r^\beta\right)\right)}{E_{\beta,1}\left(\lambda\left(t^\beta/r^\beta\right)\right)}$$
$$= [\text{for } (2.29)]$$
$$\simeq \exp\left\{\frac{\lambda^{1/\beta}t}{r}\left[\left(1-h^\alpha|\kappa|^\alpha\right)^{1/\beta}-1\right]\right\}. \tag{2.37}$$

By considering the generalized binomial theorem, we get from (2.37) that

$$\begin{aligned}\widehat{g}_{h\overline{Y}^*_\beta}\left(\kappa,\frac{t}{r}\right) &\simeq \exp\left\{\frac{\lambda^{1/\beta}t}{r}\sum_{j=0}^{\infty}\binom{\frac{1}{\beta}}{j}\left(-h^\alpha|\kappa|^\alpha\right)^j\right\}\\ &= \exp\left\{\frac{\lambda^{1/\beta}t}{r}\left[1-\frac{h^\alpha|\kappa|^\alpha}{\beta}+o(h^\alpha)\right]\right\}.\end{aligned} \tag{2.38}$$

Therefore, the limiting process is represented by the α-stable process $\mathcal{S}_\alpha^\beta$ with characteristic function $e^{-(1/\beta)\lambda^{1/\beta}t|\kappa|^\alpha}$, instead of the subordinated process $\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$ obtained in the limit when considering the FPP $\mathcal{N}_\beta$; note that $\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$ coincides with $Z(t)$ in (1.10). It is clear that the dependence on β is limited to the scale parameter; the space-fractional equation satisfied by its density is therefore given by

$$\frac{\partial u}{\partial t}=\frac{\lambda^{1/\beta}}{\beta}\frac{\partial^\alpha u}{\partial|y|^\alpha}, \qquad u(y,0)=\delta(y), \quad y\in\mathbb{R},\ t\geq 0, \tag{2.39}$$

instead of (1.12). For $\alpha=1$ the density of the limiting process reduces to a Cauchy with scale parameter $\lambda^{1/\beta}t/\beta$.

3. Space-Fractional Compound Poisson Process

We define now a space-fractional version of the compound Poisson process (which we will indicate hereafter by SFCPP): indeed, its distribution satisfies (2.5), but with integer time derivative and fractional space derivative. We consider the standard CPP

$$Y^{(\alpha)}(t)=\sum_{j=1}^{N(t)}X_j^{(\alpha)}, \quad \alpha\in(0,1], \tag{3.1}$$

where, as usual, $N(t),\ t>0$ is a standard Poisson process with parameter λ and the random variables $X_j^{(\alpha)}$ have the following heavy tail distribution:

$$f_{X^{(\alpha)}}(x)=\xi x^{\alpha-1}E_{\alpha,\alpha}(-\xi x^\alpha), \quad x>0,\ \alpha\in(0,1] \tag{3.2}$$

for $\xi > 0$. The Laplace transform of (3.2) is $\widetilde{f}_{X^{(\alpha)}}(k) = \xi/(k^\alpha + \xi)$. The distribution of $X_j^{(\alpha)}$ given in (3.2) is usually called Mittag-Leffler and coincides with the geometric-stable law of index α (hereafter $\mathcal{GS}_\alpha$) with parameters $\mu = 0$, $\theta = 1$, and $\sigma = [\cos(\pi\alpha/2)/\xi]^{1/\alpha}$ (see [27]). The density of $\sum_{j=1}^n X_j^{(\alpha)}$ is given by

$$f_{X^{(\alpha)}}^{*n}(y) = \xi^n y^{\alpha n-1} E_{\alpha,\alpha n}^n(-\xi y^\alpha) \tag{3.3}$$

with Laplace transform

$$\widetilde{f}_{X^{(\alpha)}}^{*n}(k) = \frac{\xi^n}{(k^\alpha + \xi)^n}. \tag{3.4}$$

Note that (3.3) coincides with the density of the nth event waiting time for the fractional Poisson process $\mathcal{N}_\alpha$ (see [21]). It is easy to check that the variable $X_j^{(\alpha)}$ displays the asymptotic behavior (1.13).

Theorem 3.1. *The process $Y^{(\alpha)}$ defined in* (3.1)*, with $X_j^{(\alpha)}$, $j = 1, 2, \ldots$, independent and distributed according to* (3.2)*, has the following distribution:*

$$\Pr\left\{Y^{(\alpha)}(t) \le y\right\} = e^{-\lambda t} 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{Y^{(\alpha)}}(z,t)\,dz, \quad t \ge 0,\ y \in \mathbb{R}, \tag{3.5}$$

where

$$f_{Y^{(\alpha)}}(y,t) = \frac{e^{-\lambda t}}{y} \sum_{n=1}^{\infty} \frac{(\xi\lambda t y^\alpha)^n}{n!} E_{\alpha,\alpha n}^n(-\xi y^\alpha) 1_{(0,+\infty)}(y), \quad t \ge 0. \tag{3.6}$$

The density (3.6) *satisfies the following equation:*

$$\xi \frac{\partial}{\partial t} f_{Y^{(\alpha)}} = -\left[\lambda + \frac{\partial}{\partial t}\right] \frac{\partial^\alpha}{\partial y^\alpha} f_{Y^{(\alpha)}}, \quad t \ge 0,\ y > 0, \tag{3.7}$$

with conditions

$$\begin{gathered} f_{Y^{(\alpha)}}(y,0) = 0, \\ \int_0^{+\infty} f_{Y^{(\alpha)}}(y,t)\,dy = 1 - e^{-\lambda t}. \end{gathered} \tag{3.8}$$

The following composition rule holds for the one-dimensional distribution of (3.1)*:*

$$Y^{(\alpha)}(t) \stackrel{d}{=} \mathcal{A}_\alpha(Y(t)), \tag{3.9}$$

where $\mathcal{A}_\alpha(t)$ is the stable subordinator defined in (1.1) *and Y is the standard CPP.*

Proof. We start by noting that the absolutely continuous part of the distribution is defined in $(0,\infty)$, with the exclusion of $y = 0$, where only the discrete component gives some contribution.

In order to check (3.7) we evaluate the following fractional derivatives, arguing as in the proof of Theorem 2.1:

$$\begin{aligned}
\frac{\partial}{\partial t} f_{Y^{(\alpha)}}(y,t) &= -\frac{\lambda e^{-\lambda t}}{y}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{(n-1)!n!}\sum_{j=0}^{\infty}\frac{(n+j-1)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n)}+\\
&\quad+\frac{e^{-\lambda t}}{yt}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{((n-1)!)^2}\sum_{j=0}^{\infty}\frac{(n+j-1)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n)},\\
\frac{\partial^{\alpha}}{\partial y^{\alpha}} f_{Y^{(\alpha)}}(y,t) &= \frac{e^{-\lambda t}}{y^{1+\alpha}}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{n!(n-1)!}\sum_{j=0}^{\infty}\frac{(n+j-1)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n-\alpha)},\\
\frac{\partial^{\alpha}}{\partial y^{\alpha}}\frac{\partial}{\partial t} f_{Y^{(\alpha)}}(y,t) &= -\frac{e^{-\lambda t}}{y^{1+\alpha}}\sum_{n=2}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{n!(n-2)!}\sum_{j=0}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n-\alpha)}+\\
&\quad+\frac{e^{-\lambda t}}{y^{1+\alpha}t}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{((n-1)!)^2}\sum_{j=1}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{(j-1)!\Gamma(\alpha j+\alpha n-\alpha)}+\\
&\quad+\frac{e^{-\lambda t}}{y^{1+\alpha}t}\sum_{n=2}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{(n-1)!(n-2)!}\sum_{j=0}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n-\alpha)}+\\
&\quad+\frac{e^{-\lambda t}}{y^{1+\alpha}}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{n!(n-1)!}\sum_{j=1}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{(j-1)!\Gamma(\alpha j+\alpha n-\alpha)}.
\end{aligned}\tag{3.10}$$

The initial condition is immediately satisfied by (3.6), while the second condition in (3.8) can be verified as follows:

$$\begin{aligned}
\int_0^{\infty} e^{-ky} f_{Y^{(\alpha)}}(y,t)dy &= e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\xi\lambda t)^n}{n!k^{\alpha n}}\sum_{j=0}^{\infty}\frac{(n+j-1)!}{j!k^{\alpha j}}\\
&= e^{-\lambda t}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\xi\lambda t}{k^{\alpha}+\xi}\right)^n\\
&= e^{-\lambda t}\left(e^{\lambda t\xi/(k^{\alpha}+\xi)}-1\right),
\end{aligned}\tag{3.11}$$

which, for $k = 0$, becomes $1 - e^{-\lambda t}$. The composition rule given in (3.9) can be verified by taking the Laplace transform of $Y^{(\alpha)}$,

$$\widetilde{g}_{Y^{(\alpha)}}(k,t) := \mathbb{E}e^{-kY^{(\alpha)}(t)} = e^{-(\lambda k^{\alpha}/(k^{\alpha}+\xi))t},\tag{3.12}$$

which Laplace transformed with respect to t gets

$$\widetilde{\widetilde{g}}_{Y^{(\alpha)}}(k,s) := \int_0^{\infty} e^{-st}\widetilde{g}_{Y^{(\alpha)}}(k,t)dt = \frac{k^{\alpha}+\xi}{k^{\alpha}(\lambda+s)+s\xi} = \int_0^{+\infty} \mathbb{E}e^{-k^{\alpha}Y(z)}e^{-sz}dz. \tag{3.13}$$

Thus,

$$\widetilde{g}_{Y^{(\alpha)}}(k,t) = \mathbb{E}e^{-k^{\alpha}Y(t)} = \int_0^{+\infty} e^{-k^{\alpha}v}\Pr\{Y(t)\in dv\}, \tag{3.14}$$

so that, by (1.1), we get

$$\Pr\left\{Y^{(\alpha)}(t)\in dy\right\} = \int_0^{+\infty} h_{\alpha}(y,v)\Pr\{Y(t)\in dv\}dy, \tag{3.15}$$

and formula (3.9) follows. □

Remark 3.2. Equation (3.15) yields an alternative proof of (3.7) noting that the density of $\mathcal{A}_{\alpha}^{\lambda,\xi}$ satisfies the following equation (where the space-fractional derivative is defined now in the Caputo sense):

$$\frac{\partial u}{\partial t} = -\frac{\partial^{\alpha}u}{\partial y^{\alpha}}, \qquad u(y,0)=\delta(y), \quad y,t\geq 0. \tag{3.16}$$

Indeed, we get

$$\begin{aligned}
\frac{\partial^{\alpha}}{\partial y^{\alpha}}f_{Y^{(\alpha)}}(y,t) &= \int_0^{+\infty}\frac{\partial^{\alpha}}{\partial y^{\alpha}}h_{\alpha}(y,v)f_Y(v,t)dv \\
&= -\int_0^{+\infty}\frac{\partial}{\partial v}h_{\alpha}(y,v)f_Y(v,t)dv \\
&= \int_0^{+\infty}h_{\alpha}(y,v)\frac{\partial}{\partial v}f_Y(v,t)dv \\
&= [\text{by (2.5)}] \\
&= -\frac{\xi}{\lambda}\frac{\partial}{\partial t}\int_0^{+\infty}h_{\alpha}(y,v)f_Y(v,t)dv - \frac{1}{\lambda}\frac{\partial}{\partial t}\int_0^{+\infty}h_{\alpha}(y,v)\frac{\partial}{\partial v}f_Y(v,t)dv \\
&= -\frac{\xi}{\lambda}\frac{\partial}{\partial t}f_{Y^{(\alpha)}}(y,t) - \frac{1}{\lambda}\frac{\partial}{\partial t}\frac{\partial^{\alpha}}{\partial y^{\alpha}}f_{Y^{(\alpha)}}(y,t).
\end{aligned} \tag{3.17}$$

By considering (3.9) together with (1.2), we can write the following relationship:

$$F_{Y^{(\alpha)}(t)}(z) = \Pr\left\{Y^{(\alpha)}(t)\leq z\right\} = \Pr\{\mathcal{A}_{\alpha}(Y(t))\leq z\} = \Pr\{Y(t)\geq \mathcal{L}_{\alpha}(z)\}, \tag{3.18}$$

while for the first version of TFCPP we had, from (2.15), that $F_{Y_{\beta}(t)}(z) = \Pr\{Y(\mathcal{L}_{\beta}(t))\leq z\}$.

We finally note that the process $Y^{(\alpha)}$ is still a Markovian and Lévy process, since it is substantially a special case of CPP.

3.1. Special Cases

For $\alpha = 1$, since the X_j's reduce to exponential r.v.'s, from (3.6) and (3.7) we retrieve the results (2.11) and (2.13) valid for the standard CPP, under the exponential assumption for X_j's. As a direct check of (3.9), we can consider the special case $\alpha = 1/2$, so that the law $h_{1/2}(\cdot, z)$ can be written explicitly as the density of the first passage time of a standard Brownian motion through the level $z > 0$. Then by considering (3.15) we can write

$$\begin{aligned}
\Pr\{Y_{1/2}(t) \in dy\} &= \int_0^{+\infty} h_{1/2}(y, v) f_Y(v, t) dv\, dy \\
&= \int_0^{+\infty} \frac{z e^{-z^2/2y}}{\sqrt{2\pi y^3}} \frac{e^{-\xi z - \lambda t}}{z} \sum_{n=1}^{\infty} \frac{(\lambda \xi t z)^n}{n!(n-1)!} dz\, dy \\
&= \frac{e^{-\lambda t}}{y} \sum_{n=1}^{\infty} \frac{(\lambda \xi t)^n}{n!(n-1)!} (-1)^n \frac{d^n}{d\xi^n} \int_0^{+\infty} \frac{e^{-z^2/2y}}{\sqrt{2\pi y}} e^{-\xi z} dz\, dy \\
&= \frac{e^{-\lambda t}}{2y} \sum_{n=1}^{\infty} \frac{(\lambda \xi t)^n}{n!(n-1)!} (-1)^n \frac{d^n}{d\xi^n} E_{1/2,1}\left(-\xi y^{1/2}\right) dy,
\end{aligned} \tag{3.19}$$

where the last equality holds by (2.11)-(2.12) in [28]; then, by (1.10.3) in [12], we get

$$\begin{aligned}
\Pr\{Y_{1/2}(t) \in dy\} &= \frac{e^{-\lambda t}}{2y} \sum_{n=1}^{\infty} \frac{(\lambda \xi t)^n}{(n-1)!} \frac{y^{n/2}}{n!} \sum_{j=0}^{\infty} \frac{(n+j)!(-\xi y^{1/2})^j}{j!\Gamma(j/2 + n/2 + 1)} dy \\
&= \frac{e^{-\lambda t}}{y} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t y^{1/2}\right)^n}{n!} E^n_{1/2, n/2}\left(-\xi y^{1/2}\right) dy.
\end{aligned} \tag{3.20}$$

3.2. Asymptotic Results

We study now the asymptotic behavior of the rescaled version of $Y^{(\alpha)}$ defined as

$$hY^{(\alpha)}\left(\frac{t}{r}\right) = \sum_{j=1}^{N(t/r)} hX_j^{(\alpha)}, \tag{3.21}$$

for $h, r \to 0$. The Fourier transform of the r.v.'s, $X_j^{(\alpha)}$, for any $\alpha \in (0, 1)$, is given by

$$\widehat{f}_{X^{(\alpha)}}(\kappa) = \frac{1}{1 + (1/\xi)\cos(\pi\alpha/2)|\kappa|^{\alpha}(1 - i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))} \tag{3.22}$$

(see [27], formula (2.4.1)), which, in the limit, behaves as

$$\widehat{f}_{X^{(\alpha)}}(h\kappa) \simeq 1 - Ah^{\alpha}|\kappa|^{\alpha}, \quad h \longrightarrow 0, \tag{3.23}$$

where $A = (1/\xi)\cos(\pi\alpha/2)(1-i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))$. Thus, the characteristic function of (3.21) can be written as

$$\begin{aligned}\widehat{g}_{hY^{(\alpha)}}\left(\kappa, \frac{t}{r}\right) &= e^{\lambda(t/r)[\widehat{f}_{X^{(\alpha)}}(h\kappa)-1]} \\ &\simeq e^{-(\lambda t/\xi)\cos(\pi\alpha/2)|\kappa|^{\alpha}(1-i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))}, \quad \alpha \in (0,1),\end{aligned} \tag{3.24}$$

for $h, r \to 0$, s.t. $h^{\alpha}/r \to 1$. We can conclude that

$$hY^{(\alpha)}\left(\frac{t}{r}\right) \Longrightarrow \mathcal{A}_{\alpha}^{\lambda,\xi}(t), \tag{3.25}$$

where the limiting process is represented, in this case, by an α-stable subordinator $\mathcal{A}_{\alpha}^{\lambda,\xi}(t)$ with parameters $\mu = 0$, $\theta = 1$, $\sigma = ((\lambda t/\xi)\cos\pi\alpha/2)^{1/\alpha}$, whose density satisfies

$$\frac{\partial u}{\partial t} = -\frac{\lambda}{\xi}\frac{\partial^{\alpha} u}{\partial y^{\alpha}}, \qquad u(y,0) = \delta(y), \quad y > 0, \ t > 0. \tag{3.26}$$

4. Compound Poisson Processes Fractional in Time and Space

We consider now together the results obtained in the previous sections, by defining a CPP fractional both in space and time (STFCPP), that is,

$$Y_{\beta}^{(\alpha)}(t) = \sum_{j=1}^{\mathcal{N}_{\beta}(t)} X_j^{(\alpha)}, \quad t > 0, \tag{4.1}$$

where $X_j^{(\alpha)}$'s are i.i.d. with density (3.2) and $\mathcal{N}_{\beta}(t)$, $t > 0$ is again the FPP.

Theorem 4.1. *The process $Y_{\beta}^{(\alpha)}(t), t > 0$, defined in* (4.1) *has the following distribution:*

$$\Pr\left\{Y_{\beta}^{(\alpha)}(t) \le y\right\} = E_{\beta,1}\left(-\lambda t^{\beta}\right)1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{Y_{\beta}^{(\alpha)}}(z,t)dz, \quad t \ge 0, \ y \in \mathbb{R}, \tag{4.2}$$

where

$$f_{Y_{\beta}^{(\alpha)}}(y,t) = \frac{1}{y}\sum_{n=1}^{\infty}\left(\lambda\xi t^{\beta}y^{\alpha}\right)^{n} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^{\beta}\right)E_{\alpha,\alpha n}^{n}(-\xi y^{\alpha})1_{(0,+\infty)}(y), \quad t \ge 0. \tag{4.3}$$

The density $f_{Y_\beta^{(\alpha)}}$ solves the following equation:

$$\xi \frac{\partial^\beta}{\partial t^\beta} f_{Y_{\alpha\beta}} = -\left[\lambda + \frac{\partial^\beta}{\partial t^\beta}\right] \frac{\partial^\alpha}{\partial y^\alpha} f_{Y_\beta^{(\alpha)}}, \quad t \geq 0, \; y > 0, \tag{4.4}$$

with conditions

$$\begin{gathered} f_{Y_\beta^{(\alpha)}}(y, 0) = 0, \\ \int_0^{+\infty} f_{Y_\beta^{(\alpha)}}(y, t) dy = 1 - E_{\beta,1}\left(-\lambda t^\beta\right). \end{gathered} \tag{4.5}$$

The following equality of the one-dimensional distributions holds:

$$Y_\beta^{(\alpha)}(t) \stackrel{d}{=} \mathcal{S}_\alpha\left(Y_\beta(t)\right). \tag{4.6}$$

Proof. In order to check (4.4) we evaluate the following fractional derivatives:

$$\begin{aligned} \frac{\partial}{\partial t^\beta} f_{Y_\beta^{(\alpha)}}(y, t) &= \frac{1}{y t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y^\alpha\right)^n}{(n-1)! n!} \left(\sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n - \beta + 1)}\right) \left(\sum_{r=0}^\infty \frac{(n+r-1)!\left(-\xi y^\alpha\right)^r}{r! \Gamma(\alpha r + \alpha n)}\right), \\ \frac{\partial^\alpha}{\partial y^\alpha} f_{Y_\beta^{(\alpha)}}(y, t) &= \frac{1}{y^{1+\alpha}} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y^\alpha\right)^n}{n!(n-1)!} \left(\sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n + 1)}\right) \left(\sum_{r=0}^\infty \frac{(n+r-1)!\left(-\xi y^\alpha\right)^r}{r! \Gamma(\alpha r + \alpha n - \alpha)}\right), \\ \frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial}{\partial t^\beta} f_{Y_\beta^{(\alpha)}}(y, t) &= \frac{1}{y^{1+\alpha} t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y^\alpha\right)^n}{n!(n-1)!} \left(\sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n - \beta + 1)}\right) \\ &\quad \times \left(\sum_{r=0}^\infty \frac{(n+r-1)!\left(-\xi y^\alpha\right)^r}{r! \Gamma(\alpha r + \alpha n - \alpha)}\right). \end{aligned} \tag{4.7}$$

By some algebraic manipulations we finally get (4.4). While the initial condition is trivially satisfied, the second condition in (4.5) can be checked as follows:

$$\begin{aligned} \int_0^\infty e^{-ky} f_{Y_\beta^{(\alpha)}}(y, t) dy &= \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta\right)^n}{k^{\alpha n}} E_{\beta, \beta n+1}^{n+1}\left(-\lambda t^\beta\right) \sum_{r=0}^\infty \binom{n+r-1}{r} \left(-\frac{\xi}{k^\alpha}\right)^r \\ &= \sum_{n=1}^\infty \left(\frac{\lambda \xi t^\beta}{k^\alpha + \xi}\right)^n E_{\beta, \beta n+1}^{n+1}\left(-\lambda t^\beta\right) \\ &= \left[\text{by (2.30) of [21]}\right] \\ &= E_{\beta,1}\left(-\frac{\lambda \xi t^\beta k^\alpha}{k^\alpha + \xi}\right) - E_{\beta,1}\left(-\lambda t^\beta\right), \end{aligned} \tag{4.8}$$

which, for $k = 0$, becomes $1 - E_{\beta,1}(-\lambda t^\beta)$.

The relationship (4.6) can be checked by evaluating the double Laplace transform of $\mathcal{Y}_\beta^{(\alpha)}$ as follows:

$$\tilde{\tilde{g}}_{\mathcal{Y}_\beta^{(\alpha)}}(k,s) = \int_0^{+\infty} \mathbb{E} e^{-k\mathcal{Y}_\beta^{(\alpha)}(t)} e^{-st} dt = \frac{s^{\beta-1}(k^\alpha+\xi)}{s^\beta(k^\alpha+\xi)+\lambda k^\alpha}. \tag{4.9}$$

We then rewrite formula (4.9) as

$$\tilde{\tilde{g}}_{\mathcal{Y}_\beta^{(\alpha)}}(k,s) = s^{\beta-1}\int_0^{+\infty} e^{-s^\beta z} \mathbb{E} e^{-k^\alpha \mathcal{Y}(z)} dz \tag{4.10}$$

and we follow the same lines which lead to (2.15) to get the conclusion. □

Remark 4.2. For $\alpha = 1$ formulae (4.3) and (4.4) coincide with (2.4) and (2.5), while for $\alpha = \beta = 1$ we get (3.6) and (3.7).

From (4.6), by considering (1.2), we get the following relation:

$$F_{\mathcal{Y}_\beta^{(\alpha)}(t)}(z) = \Pr\left\{\mathcal{Y}_\beta^{(\alpha)}(t) \le z\right\} = \Pr\{\mathcal{S}_\alpha(\mathcal{Y}_\beta(t)) \le z\} = \Pr\{\mathcal{Y}_\beta(t) \ge \mathcal{L}_\alpha(z)\}, \tag{4.11}$$

where $\mathcal{L}_\alpha$ is the inverse stable subordinator.

4.1. Asymptotic Results

For the rescaled version of $\mathcal{Y}_\beta^{(\alpha)}$ we obtain the following asymptotic result, which agrees with (1.14) and (1.15) proved in [17]: the characteristic function of the process

$$h\mathcal{Y}_\beta^{(\alpha)}\left(\frac{t}{r}\right) = \sum_{j=1}^{\mathcal{N}_\beta(t/r)} hX_j^{(\alpha)} \tag{4.12}$$

can be written as

$$\widehat{g}_{h\mathcal{Y}_\beta^{(\alpha)}}\left(\kappa,\frac{t}{r}\right) = E_{\beta,1}\left(\lambda\frac{t^\beta}{r^\beta}\left[\widehat{f}_{hX^{(\alpha)}}(\kappa)-1\right]\right). \tag{4.13}$$

By applying formula (3.23) we conclude that (4.13) converges, for $h, r \to 0$ s.t. $h^\alpha/r^\beta \to 1$, to

$$E_{\beta,1}\left(-\frac{\lambda t^\beta}{\xi}|\kappa|^\alpha\cos\frac{\pi\alpha}{2}\left(1-i\,\mathrm{sgn}(\kappa)\tan\frac{\pi\alpha}{2}\right)\right) \tag{4.14}$$

so that the process $hY_{\beta}^{(\alpha)}(t/r)$ converges weakly to the α-stable subordinator $\mathcal{A}_{\alpha}^{\lambda,\xi}(t)$, composed with the inverse β-stable subordinator $\mathcal{L}_{\beta}$. Indeed, the characteristic function of $\mathcal{A}_{\alpha}^{\lambda,\xi}(\mathcal{L}_{\beta}(t))$ can be evaluated as follows:

$$\begin{aligned}\int_0^{+\infty} e^{-st}\widehat{f}_{\mathcal{A}_{\alpha}^{\lambda,\xi}(\mathcal{L}_{\beta})}(\kappa,t)dt &= \int_0^{+\infty} e^{-st}dt\int_0^{+\infty} e^{i\kappa y}dy\int_0^{+\infty} p_{\alpha}^{\lambda,\xi}(y;z)l_{\beta}(z,t)dz \\ &= s^{\beta-1}\int_0^{+\infty} e^{-z\lambda|\kappa|^{\alpha}A}e^{-zs^{\beta}}dz \\ &= \frac{s^{\beta-1}}{\lambda|\kappa|^{\alpha}A+s^{\beta}},\end{aligned} \tag{4.15}$$

where $h_{\alpha}^{\lambda,\xi}(y,z)$ is the law of $\mathcal{A}_{\alpha}^{\lambda,\xi}(z)$ and $A = (1/\xi)\cos(\pi\alpha/2)(1 - i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))$. By inverting the Laplace transform in (4.15) we get (4.14). The density of $\mathcal{A}_{\alpha}^{\lambda,\xi}(\mathcal{L}_{\beta}(t))$ satisfies the following equation:

$$\frac{\partial^{\beta}u}{\partial t^{\beta}} = -\frac{\lambda}{\xi}\frac{\partial^{\alpha}u}{\partial y^{\alpha}}, \quad y,t>0, \tag{4.16}$$

as can be easily seen from (4.15) (see also [29]). A relevant special case of this result can be obtained by taking $\alpha = \beta = \nu$, so that the composition $\mathcal{A}_{\nu}^{\lambda,\xi}(\mathcal{L}_{\nu}(t))$ is proved to display a Lamperti-type law (see on this topic [30, 31]); therefore, the latter can be seen as the weak limit of the STFCPP.

We note that in the particular case $\beta = 1$, the Fourier transform (4.14) reduces to (3.24) and correspondingly (4.16) coincides with (3.26).

Finally, we consider the case where we have $\overline{N}_{\beta}(t)$ in place of $N_{\beta}(t)$. If the jumps are Mittag-Leffler distributed, we get the following space-time fractional CPP:

$$\overline{Y}_{\alpha,\beta}(t) = \sum_{j=1}^{\overline{N}_{\beta}(t)} X_j^{(\alpha)}, \tag{4.17}$$

whose distribution is given by

$$\Pr\left\{\overline{Y}_{\alpha,\beta}\le y\right\} = \frac{1}{E_{\beta,1}(-\lambda t^{\beta})}1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{\overline{Y}_{\alpha,\beta}}(z,t)dz, \quad t\ge 0,\ y\in\mathbb{R}, \tag{4.18}$$

where

$$f_{\overline{Y}_{\alpha,\beta}}(y,t) = \frac{1}{E_{\beta,1}(\lambda t^{\beta})}\sum_{n=1}^{\infty}\frac{\left(\lambda\xi t^{\beta}y^{\alpha}\right)^n}{\Gamma(\beta n+1)}E_{\alpha,\alpha n}^{n}(-\xi y^{\alpha})1_{[0,+\infty)}(y), \quad t\ge 0. \tag{4.19}$$

The rescaled version of (4.17) is defined as

$$h\overline{Y}_{\alpha,\beta}\left(\frac{t}{r}\right) = \sum_{j=1}^{\overline{N}_\beta(t/r)} hX_j^{(\alpha)} \Longrightarrow \mathcal{A}_\alpha^{\lambda,\xi}(t) \tag{4.20}$$

for $h, r \to 0$, s.t. $h^\alpha/r \to 1$, where again $\mathcal{A}_\alpha^{\lambda,\xi}$ denotes the α-stable subordinator with characteristic function given in (3.24) (last line). Thus, in the limit, the fractional nature of the counting process $\overline{N}_\beta$ does not exert any influence, in analogy with the result given in (3.25).

5. Fractional-Difference Compound Poisson Process

We present now a final version of the fractional CPP, where the fractionality of the counting process is referred to the difference operator involved in the recursive equation governing its distribution. Let B denote the standard backward shift operator, $\Delta = 1 - B$, and let γ be a fractional parameter in $(0,1]$, then the fractional recursive differential equation

$$\frac{d}{dt}p_k^\Delta(t) = -\lambda^\gamma \Delta^\gamma p_k^\Delta(t), \qquad p_k^\Delta(0) = 1_{[k=0]}, \tag{5.1}$$

has been introduced in [9]. In (5.1) the following definition of the fractional difference operator Δ^γ of a function $f(n)$ has been used (see [12], formula (2.8.2), page121):

$$\Delta^\gamma f(n) = \sum_{j=0}^{\infty} \frac{(-1)^j (\gamma)_j}{j!} f(n-j), \tag{5.2}$$

where $(x)_n = x(x-1)\cdots(x-(n-1))$ is the falling factorial. We use the notation $p_k^\Delta(t) := \Pr\{N_\Delta(t) = k\}$, $k \geq 0$, $t > 0$, and we have

$$p_k^\Delta(t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\gamma t)^r}{r!} (\gamma r)_k, \quad \gamma \in (0,1]. \tag{5.3}$$

It can be proved that N_Δ is not a renewal process, by verifying that the density of the kth event waiting time cannot be expressed as kth convolution of i.i.d. random variables. Nevertheless, $N_\Delta(t)$ is a Lévy process, with infinite expected value for any t. Moreover, by (5.3), one can check that (as $h \to 0$)

$$\Pr\{N_\Delta(h) = k\} = (-1)^{k+1} \frac{\lambda^\gamma (\gamma)_k}{k!} h + o(h), \quad \forall k \geq 1 \tag{5.4}$$

instead of $o(h)$ for $k \geq 2$, as for the standard or the time-fractional Poisson process. We can obtain (5.1) from (5.4) by taking into account that the increments are independent and stationary.

Let us define the corresponding fractional-difference compound Poisson process (hereafter ΔFCPP) as

$$Y_\Delta(t) = \sum_{j=1}^{N_\Delta(t)} X_j, \quad t \geq 0,\ \gamma \in (0,1] \tag{5.5}$$

so that we can obtain, under the assumption of i.i.d. exponential X_j's, the distribution of Y_Δ together with the differential equation which is satisfied by its absolutely continuous component.

Theorem 5.1. *For $\gamma \in (0,1]$, the distribution of the process Y_Δ defined in* (5.5)*, with X_j, $j = 1,2,\ldots$, independent and exponentially distributed with parameter ξ, is given by*

$$\Pr\{Y_\Delta(t) < y\} = e^{-\lambda^\gamma t} 1_{[0,+\infty)}(y) + \int_0^y f_{Y_\Delta}(z,t)dz, \quad t,y \geq 0, \tag{5.6}$$

where

$$f_{Y_\Delta}(y,t) = \frac{e^{-\xi y}}{y} \sum_{n=1}^{\infty} \frac{(-\xi y)^n}{n!(n-1)!} \sum_{r=0}^{\infty} \frac{(-\lambda^\gamma t)^r}{r!} (\gamma r)_n 1_{[0,+\infty)}(y), \quad t \geq 0. \tag{5.7}$$

The density f_{Y_Δ} solves the differential equation:

$$\xi D^{1/\gamma}_{-,t} f_{Y_\Delta} = \left[\lambda - D^{1/\gamma}_{-,t}\right] \frac{\partial}{\partial y} f_{Y_\Delta}, \tag{5.8}$$

where $D^{1/\gamma}_{0-;\,t}$ is the right-sided fractional Riemann-Liouville derivative on the half-axis $\mathbb{R}^+$, with conditions

$$\begin{gathered} f_{Y_\Delta}(y,0) = 0, \\ D^r_{-,t} f_{Y_\Delta}(y,t)\Big|_{t=0} = \Phi_r(y), \\ \int_0^{+\infty} f_{Y_\Delta}(y,t)dy = 1 - e^{-\lambda^\gamma t}, \end{gathered} \tag{5.9}$$

where $\Phi_r(y) = (\lambda^{\gamma r} e^{-\xi y}/y) \sum_{n=1}^{\infty}((-\xi y)^n (\gamma r)_n / n!(n-1)!)$.

The following subordinating relationship holds for (5.5)*:*

$$Y_\Delta(t) \overset{d}{=} Y(\mathcal{A}_\gamma(t)), \tag{5.10}$$

where, as usual, $\mathcal{A}_\gamma$ denotes the γ-stable subordinator.

Proof. Formula (5.7) can be easily derived by (5.3) and can be checked by verifying that, for $\gamma = 1$, it reduces to (2.11):

$$
\begin{aligned}
f_{Y_\Delta}(y,t)\big|_{\gamma=1} &= \frac{e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(-\xi y)^n}{n!(n-1)!}\sum_{r=n}^{\infty}\frac{(-\lambda t)^r}{r!}(r)_n \\
&= \frac{e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(-\xi y)^n}{n!(n-1)!}(-\lambda t)^n e^{-\lambda t}.
\end{aligned}
\tag{5.11}
$$

We now prove the relationship (5.10) as follows. The Laplace transform of $Y_\Delta(t)$ is given by

$$
\begin{aligned}
\widetilde{g}_{Y_\Delta}(k,t) = \mathbb{E}e^{-kY_\Delta(t)} &= \sum_{n=0}^{\infty}\frac{(-\xi)^n}{n!}\frac{1}{(k+\xi)^n}\sum_{r=0}^{\infty}\frac{(-\lambda^\gamma t)^r}{r!}(\gamma r)_n \\
&= \sum_{r=0}^{\infty}\frac{(-\lambda^\gamma t)^r}{r!}\sum_{n=0}^{\infty}\frac{(\gamma r)_n}{n!}\left(-\frac{\xi}{k+\xi}\right)^n \\
&= \sum_{r=0}^{\infty}\frac{(-\lambda^\gamma t)^r}{r!}\left(1-\frac{\xi}{k+\xi}\right)^{\gamma r} = e^{-\lambda^\gamma k^\gamma t/(k+\xi)^\gamma};
\end{aligned}
\tag{5.12}
$$

moreover, (5.12) can be rewritten as

$$
\begin{aligned}
\widetilde{g}_{Y_\Delta}(k,t) &= \int_0^{+\infty} e^{-\lambda z}e^{\lambda\xi z/(k+\xi)}h_\gamma(z,t)dz \\
&= \sum_{n=0}^{\infty}\frac{(\lambda\xi)^n}{n!(\xi+k)^n}\int_0^{+\infty} z^n e^{-\lambda z}h_\gamma(z,t)dz \\
&= \int_0^{+\infty}\mathbb{E}e^{-kY(z)}h_\gamma(z,t)dz,
\end{aligned}
\tag{5.13}
$$

which gives (5.10). Thus, we also have that

$$
f_{Y_\Delta}(y,t) = \int_0^{+\infty} f_Y(y,z)h_\gamma(z,t)dz. \tag{5.14}
$$

In order to prove that (5.7) satisfies (5.8), we recall the following result proved in [32]: the density $h_{\nu/n}(y,t)$ of the stable subordinator $\mathcal{A}_{\nu/n}$ is governed by the following equation (as well as by (3.16) for $\alpha = \nu/n$):

$$
D^n_{-,t}h_{\nu/n} = \frac{\partial^\nu}{\partial y^\nu}h_{\nu/n}, \quad y,t \ge 0,\ \nu \in (0,1], \tag{5.15}
$$

for $n \in \mathbb{N}$, with conditions

$$\begin{aligned} h_{\nu/n}(0,t) &= 0, \\ h_{\nu/n}(y,0) &= \delta(y), \\ D^r_{-,t} h_{\nu/n}(y,t)\Big|_{t=0} &= \frac{y^{-(\nu r/n)-1}}{\Gamma(-r\nu/n)}, \quad r = 1,\dots,n-1. \end{aligned} \tag{5.16}$$

We can prove that the slightly different result holds:

$$D^{1/\gamma}_{-,t} h_\gamma = \frac{\partial}{\partial y} h_\gamma, \quad y,t \geq 0,\ \gamma \in (0,1],\ n = \left\lfloor \frac{1}{\gamma} \right\rfloor + 1 \tag{5.17}$$

with the following conditions

$$\begin{aligned} h_\gamma(0,t) &= 0, \\ h_\gamma(y,0) &= \delta(y), \\ D^r_{-,t} h_\gamma(y,t)\Big|_{t=0} &= \frac{y^{-\gamma r-1}}{\Gamma(-\gamma r)}, \quad r = 1,\dots,n-1. \end{aligned} \tag{5.18}$$

Equation (5.17) can be checked by resorting to the Laplace transform with respect to y as follows:

$$\begin{aligned} D^{1/\gamma}_{-,t} \widetilde{h}_\gamma(k,t) &= D^{1/\gamma}_{-,t} e^{-k^\gamma t} \\ &= [\text{by (2.2.15) of [12]}] \\ &= k e^{-k^\gamma t} \\ &= \int_0^{+\infty} e^{-ky} \frac{\partial}{\partial y} h_\gamma(y,t) dy. \end{aligned} \tag{5.19}$$

Analogously, we can check (5.18): in particular we get

$$\begin{aligned} D^r_{-,t} \widetilde{h}_\gamma(k,t)\Big|_{t=0} &= (-1)^r \frac{\partial^r}{\partial t^r} \widetilde{h}_\gamma(k,t)\Big|_{t=0} \\ &= k^{\gamma r} \\ &= \int_0^{+\infty} e^{-ky} \frac{y^{-\gamma r-1}}{\Gamma(-\gamma r)} dy. \end{aligned} \tag{5.20}$$

We now take the derivative of (5.14) of order $1/\gamma$ with respect to t:

$$
\begin{aligned}
D_{-,t}^{1/\gamma} f_{Y_\Delta}(y,t) &= \int_0^{+\infty} f_Y(y,z) D_{-,t}^{1/\gamma} h_\gamma(z,t)\,dz \\
&= \int_0^{+\infty} f_Y(y,z) \frac{\partial}{\partial z} h_\gamma(z,t)\,dz \\
&= [\text{by considering (2.11)}] \\
&= -\int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) h_\gamma(z,t)\,dz \\
&= \frac{1}{\xi}\frac{\partial}{\partial y}\int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) h_\gamma(z,t)\,dz + \frac{\lambda}{\xi}\frac{\partial}{\partial y} f_{Y_\Delta}(y,t) \\
&= -\frac{1}{\xi}\frac{\partial}{\partial y} D_{-,t}^{1/\gamma} f_{Y_\Delta}(y,t) + \frac{\lambda}{\xi}\frac{\partial}{\partial y} f_{Y_\Delta}(y,t).
\end{aligned}
\tag{5.21}
$$

We remark that, for $\gamma = 1$, $D_{-,t}^{1/\gamma} f_{Y_\Delta} = -\partial f_{Y_\Delta}/\partial t$, and therefore the previous equation reduces to (2.13). Finally, we have to check (5.9): the first initial condition is trivially satisfied, while the second condition can be checked either directly by taking the derivatives of (5.7) or by noting that

$$
\begin{aligned}
D_{-,t}^{r} f_{Y_\Delta}(y,t) &= \int_0^{+\infty} f_Y(y,z) D_{-,t}^{r} h_\gamma(z,t)\,dz \\
&= [\text{by (5.18)}] \\
&= \frac{e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(\lambda\xi y)^n}{n!(n-1)!}\int_0^{+\infty} e^{-\lambda z}\frac{z^{n-r\gamma-1}}{\Gamma(-r\gamma)}\,dz \\
&= \frac{\lambda^{\gamma n} e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(\xi y)^n}{n!(n-1)!}\frac{\Gamma(n-r\gamma)}{\Gamma(-r\gamma)} \\
&= \frac{\lambda^{\gamma n} e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(\xi y)^n}{n!(n-1)!}(-\gamma r)^{(n)} \\
&= \frac{\lambda^{\gamma n} e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(-\xi y)^n}{n!(n-1)!}(\gamma r)_n,
\end{aligned}
\tag{5.22}
$$

where, in the last step, we have applied the following relationship between falling and rising factorial $(x)_n = (-1)^n (x)^{(n)}$. The last condition in (5.9) holds, since $\widetilde{g}_{Y_\Delta}(0,t) = 1$ by (5.12), so that $\widetilde{f}_{Y_\Delta}(0,t) = \widetilde{g}_{Y_\Delta}(0,t) - e^{-\lambda^\gamma t} = 1 - e^{-\lambda^\gamma t}$. □

Remark 5.2. We show that the distribution of the ΔFCPP satisfies a fractional master equation of order $1/\gamma$ greater than one, when the jumps have an arbitrary continuous density f_X. If we consider the (generalized) density function of $Y_\Delta(t)$,

$$g_{Y_\Delta}(y,t) := e^{-\lambda^\gamma t}\delta(y) + f_{Y_\Delta}(y,t), \quad y,t \geq 0, \tag{5.23}$$

then we get

$$D_{-,t}^{1/\gamma} g_{Y_\Delta}(y,t) = \lambda g_{Y_\Delta}(y,t) - \lambda \int_{-\infty}^{+\infty} g_{Y_\Delta}(y-x,t) f_X(x)dx, \tag{5.24}$$

which is analogue to (1.6) for the TFCPP Y_β. Indeed, by (5.10), we can write (5.23) as

$$g_{Y_\Delta}(y,t) = \int_0^{+\infty} g_Y(y,z) h_\gamma(z,t)dz, \tag{5.25}$$

where $g_Y(y,t) = e^{-\lambda t}\delta(y) + f_Y(y,t)$ and $f_Y(y,t)$ are the density of the standard CPP. By taking the fractional time-derivative of (5.25) we get

$$\begin{aligned}
D_{-,t}^{1/\gamma} f_{Y_\Delta}(y,t) &= \int_0^{+\infty} g_Y(y,z) D_{-,t}^{1/\gamma} h_\gamma(z,t)dz \\
&= [\text{by } (5.17)] \\
&= \int_0^{+\infty} g_Y(y,z) \frac{\partial}{\partial z} h_\gamma(z,t)dz \\
&= [\text{by } (5.18)] \\
&= -\int_0^{+\infty} \frac{\partial}{\partial z} g_Y(y,z) h_\gamma(z,t)dz \\
&= [\text{by the Kolmogorov master equation}] \\
&= \lambda \int_0^{+\infty} g_Y(y,z) h_\gamma(z,t)dz - \lambda \int_{-\infty}^{+\infty} f_X(x) \int_0^{+\infty} g_Y(y-x,z) h_\gamma(z,t)dz\, dx,
\end{aligned} \tag{5.26}$$

which coincides with (5.24). For $\gamma = 1$ (5.24) reduces to the well-known master equation of the standard CPP, by considering again that $D_{-,t}^{1/\gamma} f_{Y_\Delta} = -\partial f_{Y_\Delta}/\partial t$.

5.1. Asymptotic Results

We study the asymptotic behavior of the rescaled version of (5.5) under the two alternative assumptions on the r.v.'s representing the jumps: for X_j^* distributed according to (1.9) and for $X_j^{(\alpha)}$ with density (3.2). In the first case, we have that

$$hY_\Delta\left(\frac{t}{r}\right) = \sum_{j=1}^{N_\Delta(t/r)} hX_j^* \Longrightarrow \mathcal{S}_{\alpha\gamma}(t) \tag{5.27}$$

for $h, r \to 0$, s.t. $h^{\alpha\gamma}/r \to 1$, where $\mathcal{S}_{\alpha\gamma}(t)$ is a symmetric stable process of index $\alpha\gamma$ (which is strictly less than one) and parameters $\mu = 0$, $\theta = 0$, and $\sigma = (t\cos\pi\alpha\gamma/2)^{1/\alpha\gamma}$. Indeed, the characteristic function of (5.27) can be evaluated, by considering that the probability generating function of N_Δ is $G(u,t) = e^{-\lambda^\gamma t(1-u)^\gamma}$ (see [9]) as follows:

$$\begin{aligned} \widehat{f}_{hY_\Delta}\left(\kappa, \frac{t}{r}\right) &= e^{-(\lambda^\gamma t/r)(1-\widehat{f}_{hX^*}(\kappa))^\gamma} \\ &\simeq e^{-\lambda^\gamma t|\kappa|^{\alpha\gamma}}. \end{aligned} \tag{5.28}$$

Under the assumption of Mittag-Leffler distributed $X_j^{(\alpha)}$'s, we get instead the following result: the rescaled process

$$hY_\Delta^{(\alpha)}\left(\frac{t}{r}\right) = \sum_{j=1}^{N_\Delta(t/r)} hX_j^{(\alpha)} \Longrightarrow \mathcal{A}_{\alpha\gamma}^{\lambda,\xi}(t) \tag{5.29}$$

can be written as

$$\begin{aligned} \widehat{f}_{hY_\Delta^{(\alpha)}}\left(\kappa, \frac{t}{r}\right) &\simeq \exp\left\{-\frac{\lambda^\gamma t}{\xi^\gamma r}\left(\cos\frac{\pi\alpha}{2}\right)^\gamma h^{\alpha\gamma}|\kappa|^{\alpha\gamma}\left(1 - i\,\mathrm{sgn}(\kappa)\tan\frac{\pi\alpha}{2}\right)^\gamma\right\} \\ &\simeq \exp\left\{-\frac{\lambda^\gamma t}{\xi^\gamma}|\kappa|^{\alpha\gamma}\exp\left\{-i\,\mathrm{sgn}(\kappa)\frac{\pi\alpha\gamma}{2}\right\}\right\}, \end{aligned} \tag{5.30}$$

for $h, r \to 0$, s.t. $h^{\alpha\gamma}/r \to 1$. The last line of (5.30) corresponds to the Fourier transform of a stable subordinator $\mathcal{A}_{\alpha\gamma}^{\lambda,\xi}$ of index $\alpha\gamma$ and with parameters $\mu = 0$, $\theta = 1$, and $\sigma = ((\lambda^\gamma t/\xi^\gamma)\cos\pi\alpha\gamma/2)^{1/\alpha\gamma}$. Therefore, in both cases, the limiting processes are simply the stable symmetric process and the stable subordinator of index $\alpha\gamma$, respectively, instead of their compositions with the inverse stable subordinator as happened when $\mathcal{N}_\beta$ was used as counting process.

Table 1: Main results in finite domain.

Process		Equation
CPP	$Y(t)$	$\xi\dfrac{\partial}{\partial t}=-\left[\lambda+\dfrac{\partial}{\partial t}\right]\dfrac{\partial}{\partial y}$
TFCPP	$Y_\beta(t)\overset{d}{=}Y(\mathcal{L}_\beta(t))$	$\xi\dfrac{\partial^\beta}{\partial t^\beta}=-\left[\lambda+\dfrac{\partial^\beta}{\partial t^\beta}\right]\dfrac{\partial}{\partial y}$
SFCPP	$Y^{(\alpha)}(t)\overset{d}{=}\mathcal{S}_\alpha(Y(t))$	$\xi\dfrac{\partial}{\partial t}=-\left[\lambda+\dfrac{\partial}{\partial t}\right]\dfrac{\partial^\alpha}{\partial y^\alpha}$
STFCPP	$Y_\beta^{(\alpha)}(t)\overset{d}{=}\mathcal{S}_\alpha(Y_\beta(t))$	$\xi\dfrac{\partial^\beta}{\partial t^\beta}=-\left[\lambda+\dfrac{\partial^\beta}{\partial t^\beta}\right]\dfrac{\partial^\alpha}{\partial y^\alpha}$
ΔFCPP	$Y_\Delta(t)\overset{d}{=}Y(\mathcal{A}_\gamma(t))$	$\xi D_{-,t}^{1/\gamma}=\left[\lambda-D_{-,t}^{1/\gamma}\right]\dfrac{\partial}{\partial y}$

Table 2: Main results in asymptotic domain.

Process	Hypothesis on jumps	Limiting process	Limiting equation
TFCPP Y_β	X_j^*	$\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$	$\dfrac{\partial^\beta u}{\partial t^\beta}=\lambda\dfrac{\partial^\alpha u}{\partial\lvert y\rvert^\alpha}$
″	$X_j^{(\alpha)}$	$\mathcal{A}_\alpha(\mathcal{L}_\beta(t))$	$\dfrac{\partial^\beta u}{\partial t^\beta}=-\lambda\dfrac{\partial^\alpha u}{\partial y^\alpha}$
Altern. TFCPP $\overline{Y}_\beta$	X_j^*	$\mathcal{S}_\alpha^\beta(t)$	$\dfrac{\partial u}{\partial t}=\dfrac{\lambda^{1/\beta}}{\beta}\dfrac{\partial^\alpha u}{\partial\lvert y\rvert^\alpha}$
″	$X_j^{(\alpha)}$	$\mathcal{A}_\alpha^{\lambda,\xi}(t)$	$\dfrac{\partial u}{\partial t}=-\dfrac{\lambda}{\xi}\dfrac{\partial^\alpha u}{\partial y^\alpha}$
ΔFCPP Y_Δ	X_j^*	$\mathcal{S}_{\alpha\gamma}(t)$	$\dfrac{\partial u}{\partial t}=\lambda^\gamma\dfrac{\partial^{\alpha\gamma}u}{\partial\lvert y\rvert^{\alpha\gamma}}$
″	$X_j^{(\alpha)}$	$\mathcal{A}_{\alpha\gamma}^{\lambda,\xi}(t)$	$\dfrac{\partial u}{\partial t}=-\dfrac{\lambda^\gamma}{\xi}\dfrac{\partial^{\alpha\gamma}u}{\partial y^{\alpha\gamma}}$

Acronym

CPP: Compound Poisson process
TFCPP: Time-fractional compound Poisson process
SFCPP: Space-fractional compound Poisson process
STFCPP: Space-time fractional compound Poisson process
ΔFCPP: Fractional-difference compound Poisson process.

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Research Article

Dynamic Properties of a Forest Fire Model

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The reaction-diffusion equations have been widely used in physics, chemistry, and other areas. Forest fire can also be described by such equations. We here propose a fighting forest fire model. By using the normal form approach theory and center manifold theory, we analyze the stability of the trivial solution and Hopf bifurcation of this model. Finally, we give the numerical simulations to illustrate the effectiveness of our results.

1. Introduction

The forest fire is an important issue in the world. It has brought us huge losses. It not only burns our forests but also destroys the local ecological environment. Many factors lead to forest fires. Several authors have studied them in depth [1–6]. Some important organizations, especially the USDA Forest Service, have also researched them in their themes [7].

Reaction-diffusion equations have been applied in forest fire model for several years. Some authors analyzed the dynamical behavior of the fire front propagations using hyperbolic reaction-diffusion equations [8]. Lots of articles related to percolation theory [9] and self-organized criticality [10] are trying to provide a different dynamical model for the spread of the fire. In this paper, the model describes the condition that people are putting out the fire when the fire is spreading. We analyze dynamic properties of the reaction-diffusion equations. Kolmogorov et al. proposed the famous KPP model [11] in the 1930s. From then on, it had been applied in various fields including forest fire:

$$u_t = d_1 u_{xx} + u + f(u), \quad x \in R, \ t \geq 0, \tag{1.1}$$

where $u = u(x,t)$ can be seen as the area of the burned forest. u_{xx} is a diffusion term of u in space, and d_1 is the diffusion coefficient. $f(u)$ is a nonlinear function. The equation can describe the speed of fire spreading. Zeldovich et al. gave the famous theory of combustion and explosions [12]. We can get inspiration from it:

The people will go to put out the fire as soon as they realize the forest fire. We can use a reaction-diffusion equation to describe it.

$$v_t = d_2 v_{xx} - cv + g(v). \tag{1.2}$$

In this equation, $v = v(x,t)$ is the area where the fire has been put out. v_{xx} is a diffusion term of v in space, and d_2 is the diffusion coefficient. c is the resurgence probability of v. $g(v)$ is a nonlinear function which represents the ability of people to put out the fire.

Now, let us consider the two reaction-diffusion equations together. As we know, u and v influence each other. Thus, f and g must be functions of u, v. We define $g(u,v)$ by referring to the combustion model [13]:

$$g(u,v) = \frac{uv}{b(u+1)}, \quad b > 0. \tag{1.3}$$

Since $g(u,v)$ has opposite effect on the fire area (or u), we can also define $f(u,v)$ by taking into account KPP model [8]:

$$f(u,v) = -au^2 - \frac{uv}{b(u+1)}. \tag{1.4}$$

Then we get a new model:

$$\begin{gathered} u_t - d_1 u_{xx} = u - au^2 - \frac{uv}{b(u+1)}, \\ v_t - d_2 v_{xx} = -cv + \frac{uv}{b(u+1)}, \\ u_x(0,t) = v_x(0,t) = 0, \qquad u_x(l\pi,t) = v_x(l\pi,t) = 0, \\ u(x,0) \geq 0, \quad v(x,0) \geq 0, \quad x \in (0, l\pi). \end{gathered} \tag{1.5}$$

Define

$$H = L^2[0,\pi] \times L^2[0,\pi] = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f, g \in L^2[0,\pi] \right\}, \tag{1.6}$$

and an inner product is given by

$$\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle = \langle f_1, f_2 \rangle_{L^2} + \langle g_1, g_2 \rangle_{L^2} = \frac{2}{\pi} \int_0^{\pi} \overline{f_1} f_2 \, dx + \frac{2}{\pi} \int_0^{\pi} \overline{g_1} g_2 \, dx, \tag{1.7}$$

where $(f_1, g_1)^T \in H, (f_2, g_2)^T \in H$. From the standpoint of biology, we are only interested in the dynamics of model (1.5) in the region:

$$R_+^2 = \{(u, v) \mid u > 0, v > 0\}. \tag{1.8}$$

2. Stability Analysis

Firstly, we consider the location stability [14, 15] and the number of the equilibria of model (1.5) in R_+^2. We can also study autowave solutions [16] of the model. The interior equilibrium point is a root of the following equation:

$$\begin{aligned} u - au^2 - \frac{uv}{b(u+1)} &= 0, \\ -cv + \frac{uv}{b(u+1)} &= 0. \end{aligned} \tag{2.1}$$

It is obvious that (2.1) has an only real solution $Y_0 = (u_0, v_0)$, where

$$u_0 = \frac{bc}{1-bc}, \qquad v_0 = ab\left(\frac{1}{a} - u_0\right)(1 + u_0), \tag{2.2}$$

and $b < 1/c(a+1)$.

Now, we analyze the asymptotic stability of (u_0, v_0) by Lyapunov function.

Lemma 2.1. *For the model* (1.5),

(1) *if* $a \geq 1$, (u_0, v_0) *is global asymptotic stability.*

(2) *if* $a < 1$ *and* $(1-a)/c \leq b \leq 1/(ac+c)$, (u_0, v_0) *also has global asymptotic stability.*

Proof. Defining

$$\omega(u, v) = \int_0^{l\pi} \int_{u_0}^{u} \frac{r/b(r+1) - c}{r/(r+1)} dr\, dx + \frac{1}{b} \int_0^{l\pi} \int_{v_0}^{v} \frac{s - v_0}{s} ds\, dx, \tag{2.3}$$

we can get

$$\frac{\partial \omega}{\partial t} = \frac{1}{b} \int_0^{l\pi} (h(u) - h(u_0))(p(u) - p(u_0)) dx + Y(t), \tag{2.4}$$

where

$$\begin{aligned} h(u) &= \frac{u}{u+1}, \qquad p(u) = (1 - au)(u+1), \\ Y(t) &= -d_1 c \int_0^{l\pi} \frac{h'(u)}{h^2(u)} u_x^2 dx + \frac{d_2 v_0}{b} \int_0^{l\pi} \frac{v_x^2}{v^2} dx. \end{aligned} \tag{2.5}$$

In what follows, we split it into two cases to prove. If $a \geq 1$, for all $u > 0$, $p'(u) < 0$, since $h'(u) = 1/(u+1)^2 > 0$, we can get

$$(h(u) - h(u_0))(p(u) - p(u_0)) \leq 0. \tag{2.6}$$

If $a < 1$ and $(1-a)/c \leq b \leq 1/(ac+c)$ (equal to $v_0 \leq b$), we can still get (2.6). That is to say

$$w_t(u,v) < 0. \tag{2.7}$$

We prove the conclusion. □

Because of the conclusion of Lemma 2.1, we always assume $a < 1$ and $0 < b < (1-a)/(2ac-c)$. Introducing perturbations $u^* = u - u_0, v^* = v - v_0$, and replace (u^*, v^*) with (u, v), for which model (1.5) yields

$$\begin{gathered}
u_t - d_1 u_{xx} = u + u_0 - a(u+u_0)^2 - \frac{(u+u_0)(v+v_0)}{b(u+u_0+1)}, \\
v_t - d_2 v_{xx} = -c(v+v_0) + \frac{(u+u_0)(v+v_0)}{b(u+u_0+1)}, \\
u_x(0,t) = v_x(0,t) = 0, \qquad u_x(l\pi,t) = v_x(l\pi,t) = 0, \\
u(x,0) \geq 0, \quad v(x,0) \geq 0, \quad x \in (0, l\pi).
\end{gathered} \tag{2.8}$$

Now we can get the linearized system of parametric model (2.8) at $(0,0)$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \left(\tilde{\Delta} + L(b)\right)\begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.9}$$

where

$$\tilde{\Delta} = \begin{pmatrix} 0 & d_1 \dfrac{\partial^2}{\partial x^2} \\ d_2 \dfrac{\partial^2}{\partial x^2} & 0 \end{pmatrix}, \qquad L(b) = \begin{pmatrix} \dfrac{u_0(1-a-au_0)}{1+u_0} & -c \\ \dfrac{1-au_0}{1+u_0} & 0 \end{pmatrix}. \tag{2.10}$$

The eigenvalues of $\tilde{\Delta}$ are as follows:

$$\left\{-d_2\frac{n^2}{l^2}, -d_1\frac{n^2}{l^2}\right\}_{n=0}^{+\infty}, \tag{2.11}$$

and the corresponding eigenvectors as follows:

$$\left\{\beta_n^1, \beta_{nk}^2\right\}_{n=0}^{+\infty}, \tag{2.12}$$

where

$$\beta_n^1 = \begin{pmatrix} 0 \\ \cos\left(\frac{n}{l}x\right) \end{pmatrix}, \qquad \beta_n^2 = \begin{pmatrix} \cos\left(\frac{n}{l}x\right) \\ 0 \end{pmatrix}. \tag{2.13}$$

Define for all $y \in H$

$$y = \sum_{k=1}^{n} Y_k^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \qquad Y_k = \begin{pmatrix} \langle y, \beta_k^1 \rangle \\ \langle y, \beta_k^2 \rangle \end{pmatrix}. \tag{2.14}$$

It is easy to get, $\lambda \in (\widetilde{\Delta} + L(b))$, if and only if the equation

$$\sum_{k=1}^{n} Y_k^T \left(E\lambda - L(b) - \begin{pmatrix} -d_1 \dfrac{k^2}{l^2} & 0 \\ 0 & -d_2 \dfrac{k^2}{l^2} \end{pmatrix} \right) \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} = 0 \tag{2.15}$$

is held.

We obtain

$$\left| E\lambda - L(b) - \begin{pmatrix} -d_1 \dfrac{n^2}{l^2} & 0 \\ 0 & -d_2 \dfrac{n^2}{l^2} \end{pmatrix} \right| = 0. \tag{2.16}$$

Rewrite it as

$$\lambda^2 - T_n(b)\lambda + D_n(b) = 0, \tag{2.17}$$

where

$$\begin{aligned} T_n(b) &= bc\left(1 - a - 2a\frac{bc}{1 - bc}\right) - \frac{(d_1 + d_2)n^2}{l^2}, \\ D_n(b) &= c(1 - bc - abc) - d_2 bc\left(1 - a - \frac{2abc}{1 - bc}\right)\frac{n^2}{l^2} + \frac{d_1 d_2 n^4}{l^4}. \end{aligned} \tag{2.18}$$

From (2.18), when $(1-a)/(ac+c) < b < (1-a)/c$ is held, we can get $T_n(b) < 0$, $D_n(b) > 0$. So the system's eigenvalues have negative real part, and (u_0, v_0) has local asymptotic stability. Then, we can conclude that the system has Hopf bifurcation [14] in $b \in (0, (1-a)/(ac+c))$.

Define

$$B = \left\{ b_0 \mid T_n(b_0) = 0, D_n(b_0) > 0, T_j(b_0) \neq 0, D_j(b_0) \neq 0, \forall j \neq n \right\}. \tag{2.19}$$

$b_0 \in B \cup (0, (1-a)/(ac+c))$, $\alpha(b) \pm i\omega(b)$ are characteristic roots of $\widetilde{\Delta} + L(b)$, where

$$\begin{gathered}\alpha(b) = \frac{1}{2}\left(A(b) - \frac{d_1+d_2}{l^2}n^2\right), \qquad \omega(b) = \sqrt{D_n(b) - \alpha^2(b)}, \\ A(b) = bc\left(1 - a - 2a\frac{bc}{1-bc}\right).\end{gathered} \tag{2.20}$$

Now we compute transversality condition:

$$\begin{aligned}\alpha'(b_0) &= \frac{1}{2}a(1-b_0c)^2\left(\frac{1}{a} - 1 - \frac{4b_0c}{1-b_0c} - \left(\frac{b_0c}{1-b_0c}\right)^2\right) \\ &\quad\times\begin{cases} > 0, & 0 < b_0 < \dfrac{1}{c}\left(1 - \sqrt{\dfrac{2a}{1+a}}\right), \\ < 0, & \dfrac{1}{c}\left(1 - \sqrt{\dfrac{2a}{1+a}}\right) < b_0 < \dfrac{1-a}{ac+c}.\end{cases}\end{aligned} \tag{2.21}$$

Now we consider $A(0) = A(b_0^B) = 0$ and $A(b)$ is positive in $(0, b_0^B)$. So we can get the maximum value of $A(b)$ (*defined as* $A(b^*)$):

Define

$$l_n = n\sqrt{\frac{d_1+d_2}{A(b^*)}}, \quad n \in \mathbb{N},\ A(b^*) = a\left(\sqrt{\frac{1}{a}+1} - \sqrt{2}\right)^2, \tag{2.22}$$

for all $l \in (l_n, l_{n+1}]$, $0 \le j \le n$, $b_{j,-}^B$ and $b_{j,+}^B$ are two roots of the equation

$$A(b) = \frac{d_1+d_2}{l^2}j^2. \tag{2.23}$$

It is easy to get

$$0 < b_{1,-}^B < \cdots < b_{n,-}^B < \frac{1}{c}\left(1 - \sqrt{\frac{2a}{1+a}}\right) < b_{n,+}^B < \cdots < b_{1,+}^B < b_0^B. \tag{2.24}$$

Then we give the condition of $D_n(b_{j,\pm}^B) \neq 0$ especially $D_n(b_{j,\pm}^B) > 0$.

As we know

$$D_n(b) \ge ac - d_2A(b^*)\frac{n^2}{l^2} + d_1d_2\frac{n^4}{l^4}. \tag{2.25}$$

Then $D_n(b) > 0$ is held if and only if

$$\begin{gathered} d_1 d_2 > 0, \\ (d_2 A(b^*))^2 - 4d_1 d_2 ac > 0. \end{gathered} \tag{2.26}$$

Theorem 2.2. *Assume that $d_1, d_2, c > 0,\ 0 < a < 1$, and the equation is held:*

$$d_2 A^2(b^*) - 4d_1 ac > 0, \tag{2.27}$$

where

$$l_n = n\sqrt{\frac{d_1 + d_2}{A(b^*)}}, \quad n \in \mathbb{N},\ A(b^*) = a\left(\sqrt{\frac{1}{a} + 1} - \sqrt{2}\right)^2, \tag{2.28}$$

then for all $l \in (l_n, l_{n+1}]$, existing $b = b^B_{j,\pm}$ or $b = b^B_0$; there are Hopf bifurcations at the real solution of model (1.5).

Furthermore

$$0 < b^B_{1,-} < \cdots < b^B_{n,-} < \frac{1}{c}\left(1 - \sqrt{\frac{2a}{1+a}}\right) < b^B_{n,+} < \cdots < b^B_{1,+} < b^B_0. \tag{2.29}$$

3. Hopf Bifurcation

In the above section, we have already obtained the conditions which ensure that model (2.8) undergoes the Hopf bifurcation at the critical values b_0 or $b_{j,\pm}$ $(j = 1, \cdots)$. In the following part, we will study the direction and stability of the Hopf bifurcation based on the normal form approach theory and center manifold theory introduced by Hassard at al. [14].

Firstly, by the transformation $u^* = u - u_0,\ v^* = v - v_0$, and replacing (u^*, v^*) with (u, v), the parametric system (1.5) is equivalent to the following functional differential equation (FDE) system:

$$\frac{\partial U}{\partial t} = \left(\tilde{\Delta} + L(b_0)\right) + F(b_0, U), \tag{3.1}$$

where

$$U = (u, v)^T, \qquad F(b_0, U) = \begin{pmatrix} f - f_u u - f_v v \\ g - g_u u - g_v v \end{pmatrix}. \tag{3.2}$$

The adjoint operator of $L_n(b)$ is defined as

$$L^*(b_0) = \begin{pmatrix} d_1 \dfrac{\partial^2}{\partial x^2} + f_u(b_0) & f_v(b_0) \\ g_u(b_0) & d_2 \dfrac{\partial^2}{\partial x^2} + g_v(b_0) \end{pmatrix}. \tag{3.3}$$

It is easy to get

$$\langle u, L(b_0)v \rangle = \langle L^*(b_0)u, v \rangle. \tag{3.4}$$

From the discussions in Section 2, define $q^* = (a_n^*, b_n^*)^T \cos(n/l)x$. We have

$$L^*(b_0)q^* = -i\omega q^*, \qquad \langle q^*, q \rangle = 1, \qquad \langle q^*, \overline{q} \rangle = 0. \tag{3.5}$$

Decompose X as $X = X^c \oplus X^s$, where $X^c = \{zq + \overline{zq} \mid z \in C\}$ and $X^s = \{u \in x \mid \langle q^*, u \rangle = 0\}$. For all $(u, v) \in X$, existing $z \in \mathbb{C}$ and $\omega = (\omega_1, \omega_2) \in X^s$, we can obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = zq + \overline{zq} + \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{3.6}$$

Rewrite (3.1) as

$$\begin{aligned} \dot{z} &= i\omega_0 z + \langle q^*, F_0^* \rangle, \\ \dot{\omega} &= L(b_0)\omega + H(z, \overline{z}, \omega), \end{aligned} \tag{3.7}$$

where

$$F_0^* = zq + \overline{zq} + \omega, \qquad H(z, \overline{z}, \omega) = F_0^* - \langle q^*, F_0^* \rangle q - \left\langle \overline{q^*}, F_0^* \right\rangle \overline{q}. \tag{3.8}$$

Using the same notations as in [11],

$$F_0^*(U) = \frac{1}{2}Q(U, U) + \frac{1}{6}C(U, U, U) + O\left|U^4\right|, \tag{3.9}$$

where $U = (u, v)$ and Q, C are symmetrical multilinear functions. We can compute

$$Q(q, q) = \begin{pmatrix} A_n^1 \\ A_n^2 \end{pmatrix} \cos^2 \frac{n}{l}x, \qquad Q(q, \overline{q}) = \begin{pmatrix} B_n^1 \\ B_n^2 \end{pmatrix} \cos^2 \frac{n}{l}x, \qquad C(q, q, \overline{q}) = \begin{pmatrix} C_n^1 \\ C_n^2 \end{pmatrix} \cos^3 \frac{n}{l}x, \tag{3.10}$$

where

$$\begin{aligned}
A_n^1 &= f_{uu}a_n^2 + 2f_{uv}a_nb_n + f_{vv}b_n^2,\\
A_n^2 &= g_{uu}a_n^2 + 2g_{uv}a_nb_n + g_{vv}b_n^2,\\
B_n^1 &= f_{uu}|a_n|^2 + f_{uv}\left(a_n\overline{b_n} + \overline{a_n}b_n\right) + f_{vv}|b_n|^2,\\
B_n^2 &= g_{uu}|a_n|^2 + g_{uv}\left(a_n\overline{b_n} + \overline{a_n}b_n\right) + g_{vv}|b_n|^2,\\
C_n^1 &= f_{uuu}|a_n|^2a_n + f_{uuv}\left(2|a_n|^2b_n + a_n^2\overline{b_n}\right) + f_{uvv}\left(2b_n^2a_n + b_n^2\overline{a_n}\right) + f_{vvv}|b_n|^2b_n,\\
C_n^2 &= g_{uuu}|a_n|^2a_n + g_{uuv}\left(2|a_n|^2b_n + a_n^2\overline{b_n}\right) + g_{uvv}\left(2b_n^2a_n + b_n^2\overline{a_n}\right) + g_{vvv}|b_n|^2b_n.
\end{aligned} \tag{3.11}$$

Define

$$H(z,\overline{z},\omega) = \frac{1}{2}H_{20}z^2 + H_{11}z\overline{z} + \frac{1}{2}H_{02}\overline{z}^2 + o(|z||\omega|), \tag{3.12}$$

where

$$\begin{aligned}
H_{20} &= Q(q,q) - \langle q^*, Q(q,q)\rangle q - \left\langle \overline{q^*}, Q(q,q)\right\rangle \overline{q},\\
H_{11} &= Q(q,\overline{q}) - \langle q^*, Q(q,\overline{q})\rangle q - \left\langle \overline{q^*}, Q(q,\overline{q})\right\rangle \overline{q}.
\end{aligned} \tag{3.13}$$

On the center manifold, we have

$$\omega = \frac{1}{2}\omega_{20}z^2 + \omega_{11}z\overline{z} + \frac{1}{2}\omega_{02}\overline{z}^2 + o\left(|z|^3\right). \tag{3.14}$$

We can obtain

$$w_{20} = [2i\omega_0 I - L_n(b_0)]^{-1}H_{20}, \qquad \omega_{11} = [L_n(b_0)]^{-1}H_{11}. \tag{3.15}$$

Comparing (3.9) and (3.13), we can get

$$H_{20} = \begin{cases} Q(q,q) - \begin{pmatrix} A_n^1 \\ A_n^2 \end{pmatrix}\cos^2\dfrac{n}{l}x - \begin{pmatrix} A_n^1 \\ A_n^2 \end{pmatrix}\left(\dfrac{1}{2}\cos^2\dfrac{n}{2}x + \dfrac{1}{2}\right), & n \in \mathbb{N}^*,\\ \begin{pmatrix} A_0^1 \\ A_0^2 \end{pmatrix} - \langle q^*, Q(q,q)\rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle q^*, Q(q,q)\rangle \begin{pmatrix} \overline{a_0} \\ \overline{b_0} \end{pmatrix}, & n = 0. \end{cases} \tag{3.16}$$

Similarly

$$\omega_{11} = \begin{cases} -\dfrac{1}{2}[L(b_o)]^{-1}\begin{pmatrix} C_n^1 \\ C_n^2 \end{pmatrix}\left(\cos^2\dfrac{n}{2}+1\right), & n \in \mathbb{N}^*, \\ -[L(b_0)]^{-1}\left[\begin{pmatrix} c_0^1 \\ c_0^2 \end{pmatrix} - \langle q^*, Q(q,\overline{q})\rangle \begin{pmatrix} a_0^1 \\ b_0^2 \end{pmatrix} - \left\langle \overline{q^*}, Q(q,\overline{q})\right\rangle \begin{pmatrix} \overline{a_0} \\ \overline{b_0} \end{pmatrix}\right], & n = 0. \end{cases} \tag{3.17}$$

Then on the center manifold rewrite (dU) as

$$\frac{dz}{dt} = i\omega_0 z + \langle q^*, F_0^* \rangle = i\omega_0 z + \sum_{2\le i+j\le 3} \frac{g_{ij}}{i!j!} z^i \overline{z}^j + o\left(|z|^4\right), \tag{3.18}$$

where

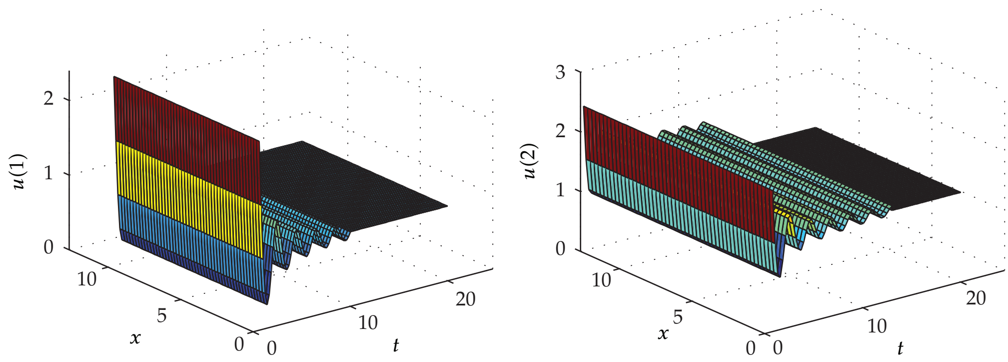
$$\begin{aligned} g_{20} &= \langle q^*, Q(q,q)\rangle = \frac{4c\omega_0 a^2 - a(1-a)^2\omega_0 - 2ca^2(3a-1)i}{(1-a^2)\omega_0}, \\ g_{11} &= \langle q^*, Q(q,\overline{q})\rangle = \frac{a(a-q)\omega_0 - 2ca^2 i}{(1+a)\omega_0}, \\ g_{02} &= \langle q^*, Q(\overline{q},\overline{q})\rangle = \frac{a(1-a)^2 + 2c\omega_0 a^2 - 4ca^2 i}{(1-a^2)\omega_0}, \\ g_{21} &= 2\langle q^*, Q(w_{11},q)\rangle + \langle q^*, Q(w_{20},\overline{q})\rangle + \langle q^*, C(q,q,\overline{q})\rangle \\ &= \frac{-12a^3(1-a)\omega_0 - 8ca^3\omega_0 + 4ca^{3(3-5a)i}}{(1-a)(1+a)^2\omega_0}. \end{aligned} \tag{3.19}$$

Using conclusions in [14] we can get

$$C_1(b) = \frac{g_{20}g_{11}(3\alpha(b) + i\omega(b))}{2(\alpha^2(b) + \omega^2(b))} + \frac{|g_{11}|^2}{\alpha(b) + i\omega(b)} + \frac{|g_{02}|^2}{2(\alpha(b) + 3i\omega(b))} + \frac{g_{21}}{2}, \tag{3.20}$$

then

$$\begin{gathered} C_1(b_0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{b'(\tau_n)\}}, \qquad \beta_2 = 2\operatorname{Re}\{c_1(0)\}, \\ T_2 = \frac{2\pi}{\omega_0}\left(1 + \tau_2 s^2\right) + o\left(s^4\right), \end{gathered} \tag{3.21}$$



(a)

(b)

Figure 1: When $b = 0.1$, the positive equilibrium point Y_0 is asymptotically stable.

where

$$\tau_2 = -\frac{1}{\omega_0}\left[\mathrm{Im}(c_1(b_0)) - \frac{\mathrm{Re}(c_1(b_0))}{\alpha'(b_0)}\omega'(b_0)\right]. \tag{3.22}$$

Now we give a conclusion.

Conclusion. (1) The sign of μ_2 determines the direction of Hopf bifurcation. When $\mu_2 > 0$, the Hopf bifurcation is supercritical; when $\mu_2 < 0$, the Hopf bifurcation is subcritical.

(2) β_2 determines the stability of bifurcated periodic solutions. When $\beta_2 < 0$, the periodic solutions are stable; when $\beta_2 > 0$, the periodic solutions are unstable.

(3) T_2 determines the period of bifurcated periodic solutions. When $T_2 > 0$, the period increases; when $T_2 < 0$, the period decreases.

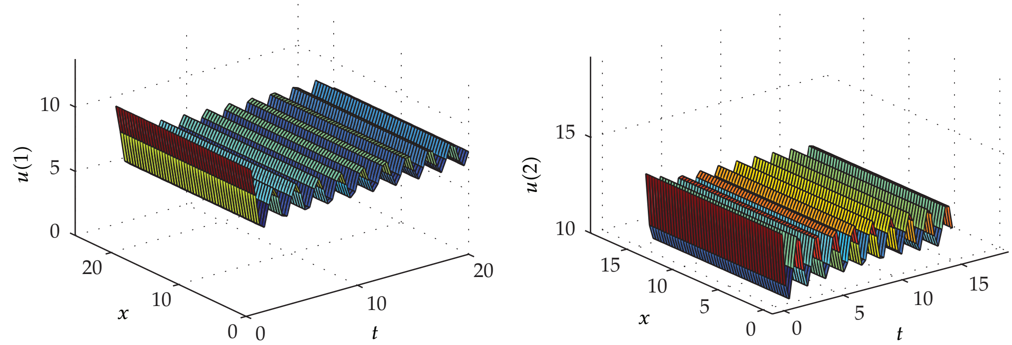
4. Example

In this section, we use a numerical simulation to illustrate the analytical results we obtained in previous sections.

Let $x \in (0, l\pi)$, $d_1 = 1$, $d_2 = 3$, $c = 4$, $a = 0.0588$. The system (1.5) is

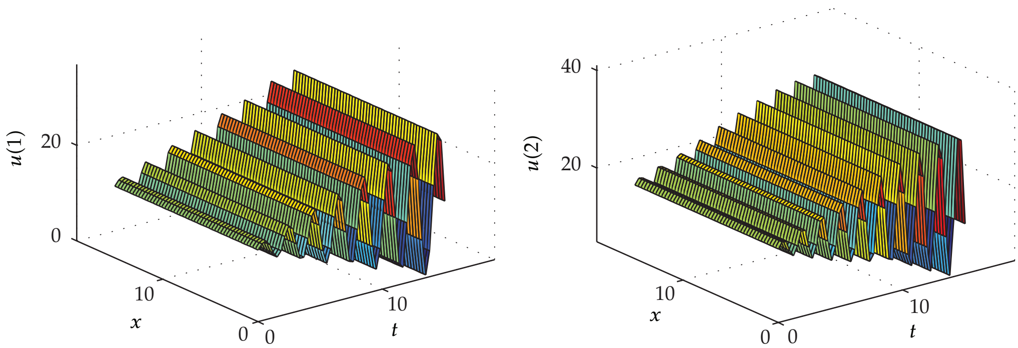
$$\begin{gathered}
u_t - u_{xx} = u - 0.0588u^2 - \frac{uv}{b(u+1)}, \\
v_t - 3v_{xx} = -4v + \frac{uv}{b(u+1)}, \\
u_x(0,t) = v_x(0,t) = 0, \qquad u_x(l\pi,t) = v_x(l\pi,t) = 0, \\
u(x,0) \geq 0, \quad v(x,0) \geq 0, \quad x \in (0, l\pi).
\end{gathered} \tag{4.1}$$

Now we determine the direction of a Hopf bifurcation with $b \in B$ and the other properties of bifurcating periodic solutions based on the theory of Hassard et al. [14], as discussed before. By means of software MATLAB 7.0, we can get some figures to illustrate the effectiveness of



(a) (b)

Figure 2: When $b = 0.1253$, periodic solutions occur from Y_0.



(a) (b)

Figure 3: When $b = 1.3$, the positive equilibrium point Y_0 is unstable.

our results. $b_0^B = 0.2222$, $b^* = 0.1667$, $A(b^*) = 0.4706$, $l_n = 2.9155n$, and (2.27) is held. When $n = 2$, $l = 3.1162$, $(l_n, l_{n+1}] = (2.9155, 5.8302]$, $l \in (l_n, l_{n+1}]$. We can get $B = (0.1253, 0.1944, 0.2222)$. The only positive equilibrium point of (4.1) is $Y_0 = (4b/(1-4b), b(0.0588-u_0)(1+u_0)/0.0588)$. When $b = 0.1253$, we can compute

$$\operatorname{Re} c_1(0.1253) = 0.2856 > 0, \qquad \mu_2 = -0.3423 < 0, \qquad T_2 = 1.2346 > 0. \tag{4.2}$$

The positive equilibrium point of (4.1) is unstable and the Hopf bifurcation is supercritical. The positive equilibrium point Y_0 of system (4.1) is locally asymptotically stable when $b = 0.1$ as is illustrated by computer simulations in Figure 1. And periodic solutions occur from Y_0 when $b = 0.1253$ as is illustrated by computer simulations in Figure 2. When $b = 1.3$, we can easily show that the positive equilibrium point Y_0 is unstable as is illustrated in Figure 3. From the above results, we can conclude that the stability properties of the system could switch with parameter b.

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Research Article

On the Definitions of Nabla Fractional Operators

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We show that two recent definitions of discrete nabla fractional sum operators are related. Obtaining such a relation between two operators allows one to prove basic properties of the one operator by using the known properties of the other. We illustrate this idea with proving power rule and commutative property of discrete fractional sum operators. We also introduce and prove summation by parts formulas for the right and left fractional sum and difference operators, where we employ the Riemann-Liouville definition of the fractional difference. We formalize initial value problems for nonlinear fractional difference equations as an application of our findings. An alternative definition for the nabla right fractional difference operator is also introduced.

1. Introduction

The following definitions of the backward (nabla) discrete fractional sum operators were given in [1, 2], respectively. For any given positive real number α, we have

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t - \rho(s)\right)^{\overline{\alpha-1}} f(s), \tag{1.1}$$

where $t \in \{a+1, a+2, \ldots\}$, and

$$\triangledown_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} \left(t - \rho(s)\right)^{\overline{\alpha-1}} f(s), \tag{1.2}$$

where $t \in \{a, a+1, a+2, \ldots\}$.

One significant difference between these two operators is that the sum in (1.1) starts at $a+1$ and the sum in (1.2) starts at a. In this paper we aim to answer the following question:

Do these two definitions lead the development of the theory of the nabla fractional difference equations in two directions?

In order to answer this question, we first obtain a relation between the operators in (1.1) and (1.2). Then we illustrate how such a relation helps one to prove basic properties of the one operator if similar properties of the other are already known.

In recent years, discrete fractional calculus gains a great deal of interest by several mathematicians. First Miller and Ross [3] and then Gray and Zhang [1] introduced discrete versions of the Riemann-Liouville left fractional integrals and derivatives, called the fractional sums and differences with the delta and nabla operators, respectively. For recent developments of the theory, we refer the reader to the papers [2, 4–19]. For further reading in this area, we refer the reader to the books on fractional differential equations [20–23].

The paper is organized as follows. In Section 2, we summarize some of basic notations and definitions in discrete nabla calculus. We employ the Riemann-Liouville definition of the fractional difference. In Section 3, we obtain two relations between the operators $\nabla_a^{-\alpha}$ and $\triangledown_a^{-\alpha}$. So by the use of these relations we prove some properties for $\nabla_a^{-\alpha}$-operator. Section 4 is devoted to summation by parts formulas. In Section 5, we formalize initial value problems and obtain corresponding summation equation with $\nabla_a^{-\alpha}$-operator. This section can be considered as an application of the results in Section 3. Finally, in Section 6, a definition of the nabla right fractional difference resembling the nabla right fractional sum is formulated. This definition can be used to prove continuity of the nabla right fractional differences with respect to the order α.

2. Notations and Basic Definitions

Definition 2.1. (i) For a natural number m, the m rising (ascending) factorial of t is defined by

$$t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \quad t^{\overline{0}} = 1. \tag{2.1}$$

(ii) For any real number α, the rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{\dots,-2,-1,0\}, \ 0^{\overline{\alpha}} = 0. \tag{2.2}$$

Throughout this paper, we will use the following notations.

(i) For real numbers a and b, we denote $\mathbb{N}_a = \{a, a+1, \dots\}$ and ${}_b\mathbb{N} = \{\dots, b-1,\ b\}$.

(ii) For $n \in \mathbb{N}$, we define

$$_{\ominus}\Delta^n f(t) := (-1)^n \Delta^n f(t). \tag{2.3}$$

Definition 2.2. Let $\rho(t) = t - 1$ be the backward jump operator. Then

(i) the (nabla) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t - \rho(s)\right)^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1} \tag{2.4}$$

(ii) the (nabla) right fractional sum of order $\alpha > 0$ (ending at b) is defined by

$${}_b\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \left(s - \rho(t)\right)^{\overline{\alpha-1}} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s) - t)^{\overline{\alpha-1}} f(s), \quad t \in_{b-1} \mathbb{N}. \tag{2.5}$$

We want to point out that the nabla left fractional sum operator has the following characteristics.

(i) $\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$.

(ii) $\nabla_a^{-n} f(t)$ satisfies the nth order discrete initial value problem

$$\nabla^n y(t) = f(t), \qquad \nabla^i y(a) = 0, \quad i = 0, 1, \ldots, n-1. \tag{2.6}$$

(iii) The Cauchy function $(t - \rho(s))^{\overline{n-1}}/\Gamma(n)$ satisfies $\nabla^n y(t) = 0$.

In the same manner, it is worth noting that the nabla right fractional sum operator has the following characteristics.

(i) ${}_b\nabla^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

(ii) ${}_b\nabla^{-n} f(t)$ satisfies the nth order discrete initial value problem

$${}_\ominus\Delta^n y(t) = f(t), \qquad {}_\ominus\Delta^i y(b) = 0, \quad i = 0, 1, \ldots, n-1. \tag{2.7}$$

(iii) The Cauchy function $(s - \rho(t))^{\overline{n-1}}/\Gamma(n)$ satisfies ${}_\ominus\Delta^n y(t) = 0$.

Definition 2.3. (i) The (nabla) left fractional difference of order $\alpha > 0$ is defined by

$$\nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \frac{\nabla^n}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t} \left(t - \rho(s)\right)^{\overline{n-\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1}. \tag{2.8}$$

(ii) The (nabla) right fractional difference of order $\alpha > 0$ is defined by

$${}_b\nabla^{\alpha} f(t) = {}_\ominus\Delta^n {}_b\nabla^{-(n-\alpha)} f(t) = \frac{{}_\ominus\Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} \left(s - \rho(t)\right)^{\overline{n-\alpha-1}} f(s), \quad t \in_{b-1} \mathbb{N}. \tag{2.9}$$

Here and throughout the paper $n = [\alpha] + 1$, where $[\alpha]$ is the greatest integer less than or equal α.

Regarding the domains of the fractional difference operators we observe the following.

(i) The nabla left fractional difference ∇_a^{α} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n}$ (on $\mathbb{N}_a$ if we think $f = 0$ before a).

(ii) The nabla right fractional difference ${}_b\nabla^{\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-n}\mathbb{N}$ (on ${}_b\mathbb{N}$ if we think $f = 0$ after b).

3. A Relation between the Operators $\nabla_a^{-\alpha}$ and $\triangledown_a^{-\alpha}$

In this section we illustrate how two operators, $\nabla_a^{-\alpha}$ and $\triangledown_a^{-\alpha}$ are related.

Lemma 3.1. *The following holds:*

(i) $\triangledown_{a+1}^{-\alpha} f(t) = \nabla_a^{-\alpha} f(t)$,

(ii) $\triangledown_a^{-\alpha} f(t) = (1/\Gamma(\alpha))(t-a+1)^{\overline{\alpha-1}} f(a) + \nabla_a^{-\alpha} f(t)$.

Proof. The proof of (i) follows immediately from the above definitions (1.1) and (1.2). For the proof of (ii), we have

$$\begin{aligned}
\triangledown_a^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s) \\
&= \frac{1}{\Gamma(\alpha)} (t-a+1)^{\overline{\alpha-1}} f(a) + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s) \qquad (3.1) \\
&= \frac{1}{\Gamma(\alpha)} (t-a+1)^{\overline{\alpha-1}} f(a) + \nabla_a^{-\alpha} f(t).
\end{aligned}$$

□

Next three lemmas show that the above relations on the operators (1.1) and (1.2) help us to prove some identities and properties for the operator $\nabla_a^{-\alpha}$ by the use of known identities for the operator $\triangledown_a^{-\alpha}$.

Lemma 3.2. *The following holds:*

$$\nabla_a^{-\alpha} \nabla f(t) = \nabla \nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a). \qquad (3.2)$$

Proof. It follows from Lemma 3.1 and Theorem 2.1 in [13]

$$\begin{aligned}
\nabla_a^{-\alpha} \nabla f(t) &= \triangledown_{a+1}^{-\alpha} \nabla f(t) = \nabla \triangledown_a^{-\alpha} f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) \\
&= \nabla \left\{ \frac{1}{\Gamma(\alpha)} (t-a+1)^{\overline{\alpha-1}} f(a) + \nabla_a^{-\alpha} f(t) \right\} - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\alpha-1)}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-2}}f(a) + \nabla\nabla_a^{-\alpha}f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a) \\
&= \nabla\nabla_a^{-\alpha}f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a).
\end{aligned}
\tag{3.3}
$$

□

Lemma 3.3. *Let $\alpha > 0$ and $\beta > -1$. Then for $t \in \mathbb{N}_a$, the following equality holds*

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\alpha+\mu}}. \tag{3.4}$$

Proof. It follows from Theorem 2.1 in [13]

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \triangledown_{a+1}^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\alpha+\mu}}. \tag{3.5}$$

□

Lemma 3.4. *Let f be a real-valued function defined on $\mathbb{N}_a$, and let $\alpha, \beta > 0$. Then*

$$\nabla_a^{-\alpha}\nabla_a^{-\beta}f(t) = \nabla_a^{-(\alpha+\beta)}f(t) = \nabla_a^{-\beta}\nabla_a^{-\alpha}f(t). \tag{3.6}$$

Proof. It follows from Lemma 3.1 and Theorem 2.1 in [2]

$$\nabla_a^{-\alpha}\nabla_a^{-\beta}f(t) = \triangledown_{a+1}^{-\alpha}\triangledown_{a+1}^{-\beta}f(t) = \triangledown_{a+1}^{-(\alpha+\beta)}f(t) = \nabla_a^{-(\alpha+\beta)}f(t). \tag{3.7}$$

□

Remark 3.5. Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, by Lemma 3.2 we have

$$\nabla\nabla_a^{\alpha}f(t) = \nabla\nabla^n\left(\nabla_a^{-(n-\alpha)}f(t)\right) = \nabla^n\left(\nabla\nabla_a^{-(n-\alpha)}f(t)\right) \tag{3.8}$$

or

$$\nabla\nabla_a^{\alpha}f(t) = \nabla^n\left[\nabla_a^{-(n-\alpha)}\nabla f(t) + \frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}f(a)\right]. \tag{3.9}$$

Then, using the identity

$$\nabla^n\frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} \tag{3.10}$$

we verified that (3.2) is valid for any real α.

By using Lemma 3.1, Remark 3.5, and the identity $\nabla(t-a)^{\overline{\alpha-1}} = (\alpha-1)(t-a)^{\overline{\alpha-2}}$, we arrive inductively at the following generalization.

Theorem 3.6. *For any real number α and any positive integer p, the following equality holds:*

$$\nabla_{a+p-1}^{-\alpha}\nabla^{p}f(t)=\nabla^{p}\nabla_{a+p-1}^{-\alpha}f(t)-\sum_{k=0}^{p-1}\frac{\left(t-(a+p-1)\right)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\nabla^{k}f(a+p-1),\tag{3.11}$$

where f is defined on $\mathbb{N}_a$.

Lemma 3.7. *For any $\alpha>0$, the following equality holds:*

$${}_{b}\nabla_{\ominus}^{-\alpha}\Delta f(t)={}_{\ominus}\Delta\,{}_{b}\nabla^{-\alpha}f(t)-\frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(b).\tag{3.12}$$

Proof. By using of the following summation by parts formula

$$\begin{aligned}&\Delta_{s}\left[\left(\rho(s)-\rho(t)\right)^{\overline{\alpha-1}}f(s)\right]\\&\quad=(\alpha-1)\left(s-\rho(t)\right)^{\overline{\alpha-2}}f(s)+\left(s-\rho(t)\right)^{\overline{\alpha-1}}\Delta f(s)\end{aligned}\tag{3.13}$$

we have

$$\begin{aligned}{}_{b}\nabla_{\ominus}^{-\alpha}\Delta f(t)&=-\frac{1}{\Gamma(\alpha)}\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-1}}\Delta f(s)\\&=\frac{1}{\Gamma(\alpha)}\left[-\sum_{s=t}^{b-1}\Delta_{s}\left(\left(\rho(s)-\rho(t)\right)^{\overline{\alpha-1}}f(s)\right)+(\alpha-1)\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-2}}f(s)\right]\\&=\frac{1}{\Gamma(\alpha-1)}\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-2}}f(s)-\frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(b).\end{aligned}\tag{3.14}$$

On the other hand,

$$\begin{aligned}&{}_{\ominus}\Delta\,{}_{b}\nabla^{-\alpha}f(t)\\&\quad=-\frac{1}{\Gamma(\alpha)}\sum_{s=t}^{b-1}\Delta_{t}\left(s-\rho(t)\right)^{\overline{\alpha-1}}f(s)=\frac{1}{\Gamma(\alpha-1)}\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-2}}f(s),\end{aligned}\tag{3.15}$$

where the identity

$$\Delta_{t}\left(s-\rho(t)\right)^{\overline{\alpha-1}}=-(\alpha-1)\left(s-\rho(t)\right)^{\overline{\alpha-2}}\tag{3.16}$$

and the convention that $(0)^{\overline{\alpha-1}}=0$ are used. □

Remark 3.8. Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, by the help of Lemma 3.7 we have

$$\begin{aligned} {}_{\ominus}\Delta\, {}_{b}\nabla^{\alpha} f(t) &= {}_{\ominus}\Delta_{\ominus}\Delta^{n}\left({}_{b}\nabla^{-(n-\alpha)} f(t)\right) \\ &= {}_{\ominus}\Delta^{n}\left({}_{\ominus}\Delta\, {}_{b}\nabla^{-(n-\alpha)} f(t)\right) \end{aligned} \tag{3.17}$$

or

$${}_{\ominus}\Delta\, {}_{b}\nabla^{\alpha} f(t) = {}_{\ominus}\Delta^{n}\left[{}_{b}\nabla_{\ominus}^{-(n-\alpha)} \Delta f(t) + \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} f(b)\right]. \tag{3.18}$$

Then, using the identity

$${}_{\ominus}\Delta^{n} \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(b-t)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} \tag{3.19}$$

we verified that (3.12) is valid for any real α.

By using Lemma 3.7, Remark 3.8, and the identity $\Delta(b-t)^{\overline{\alpha-1}} = -(\alpha-1)(b-t)^{\overline{\alpha-2}}$, we arrive inductively at the following generalization.

Theorem 3.9. *For any real number α and any positive integer p, the following equality holds:*

$${}_{b-p+1}\nabla_{\ominus}^{-\alpha} \Delta^{p} f(t) =_{\ominus} \Delta^{p}\, {}_{b-p+1}\nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-p+1-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\, {}_{\ominus}\Delta^{k} f(b-p+1), \tag{3.20}$$

where f is defined on ${}_{b}N$.

We finish this section by stating the commutative property for the right fractional sum operators without giving its proof.

Lemma 3.10. *Let f be a real valued function defined on ${}_{b}\mathbb{N}$, and let $\alpha, \beta > 0$. Then*

$${}_{b}\nabla^{-\alpha}\left[{}_{b}\nabla^{-\beta} f(t)\right] =_{b} \nabla^{-(\alpha+\beta)} f(t) = {}_{b}\nabla^{-\beta}\left[{}_{b}\nabla^{-\alpha} f(t)\right]. \tag{3.21}$$

4. Summation by Parts Formulas for Fractional Sums and Differences

We first state summation by parts formula for nabla fractional sum operators.

Theorem 4.1. *For $\alpha > 0$, $a, b \in \mathbb{R}$, f defined on $\mathbb{N}_a$ and g defined on ${}_{b}\mathbb{N}$, the following equality holds*

$$\sum_{s=a+1}^{b-1} g(s) \nabla_{a}^{-\alpha} f(s) = \sum_{s=a+1}^{b-1} f(s)_{b}\nabla^{-\alpha} g(s). \tag{4.1}$$

Proof. By the definition of the nabla left fractional sum we have

$$\sum_{s=a+1}^{b-1} g(s)\nabla_a^{-\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{b-1} g(s) \sum_{r=a+1}^{s} (s-\rho(r))^{\overline{\alpha-1}} f(r). \tag{4.2}$$

If we interchange the order of summation we reach (4.1). □

By using Theorem 3.6, Lemma 3.4, and $\nabla_a^{-(n-\alpha)} f(a) = 0$, we prove the following result.

Theorem 4.2. *For $\alpha > 0$, and f defined in a suitable domain $\mathbb{N}_a$, the following are valid*

$$\nabla_a^{\alpha} \nabla_a^{-\alpha} f(t) = f(t), \tag{4.3}$$

$$\nabla_a^{-\alpha} \nabla_a^{\alpha} f(t) = f(t), \quad \text{when } \alpha \notin \mathbb{N}, \tag{4.4}$$

$$\nabla_a^{-\alpha} \nabla_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a), \quad \text{when } \alpha = n \in \mathbb{N}. \tag{4.5}$$

We recall that $D^{-\alpha} D^{\alpha} f(t) = f(t)$, where $D^{-\alpha}$ is the Riemann-Liouville fractional integral, is valid for sufficiently smooth functions such as continuous functions. As a result of this it is possible to obtain integration by parts formula for a certain class of functions (see [23] page 76, and for more details see [22]). Since discrete functions are continuous we see that the term $\nabla_a^{-(1-\alpha)} f(t)|_{t=a}$, for $0 < \alpha < 1$ disappears in (4.4), with the application of the convention that $\sum_{s=a+1}^{a} f(s) = 0$.

By using Theorem 3.9, Lemma 3.10, and ${}_b\nabla^{-(n-\alpha)} f(b) = 0$, we obtain the following.

Theorem 4.3. *For $\alpha > 0$, and f defined in a suitable domain ${}_b\mathbb{N}$, we have*

$${}_b\nabla^{-\alpha}\ {}_b\nabla^{-\alpha} f(t) = f(t), \tag{4.6}$$

$${}_b\nabla_b^{-\alpha} \nabla^{\alpha} f(t) = f(t), \quad \text{when } \alpha \notin \mathbb{N}, \tag{4.7}$$

$${}_b\nabla^{-\alpha}{}_b\nabla^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{k}}}{k!}\, {}_{\ominus}\Delta^k f(b), \quad \text{when } \alpha = n \in \mathbb{N}. \tag{4.8}$$

Theorem 4.4. *Let $\alpha > 0$ be noninteger. If f is defined on ${}_bN$ and g is defined on N_a, then*

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha} g(s) = \sum_{s=a+1}^{b-1} g(s)_b\nabla^{\alpha} f(s). \tag{4.9}$$

Proof. Equation (4.7) implies that

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha} g(s) = \sum_{s=a+1_b}^{b-1} \nabla^{-\alpha}\left({}_b\nabla^{\alpha} f(s)\right)\nabla_a^{\alpha} g(s). \tag{4.10}$$

And by Theorem 4.1 we have

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha} g(s) = \sum_{s=a+1}^{b-1} {}_b\nabla^{\alpha} f(s)\nabla_a^{-\alpha}\nabla_a^{\alpha} g(s). \tag{4.11}$$

Then the result follows by (4.4). □

5. Initial Value Problems

Let us consider the following initial value problem for a nonlinear fractional difference equation

$$\nabla_{a-1}^{\alpha} y(t) = f(t, y(t)) \quad \text{for} \quad t = a+1, a+2, \ldots, \tag{5.1}$$

$$\nabla_{a-1}^{-(1-\alpha)} y(t)|_{t=a} = y(a) = c, \tag{5.2}$$

where $0 < \alpha < 1$ and a is any real number.

Apply the operator $\nabla_a^{-\alpha}$ to each side of (5.1) to obtain

$$\nabla_a^{-\alpha}\nabla_{a-1}^{\alpha} y(t) = \nabla_a^{-\alpha} f(t, y(t)). \tag{5.3}$$

Then using the definition of the fractional difference and sum operators we obtain

$$\begin{gathered} \nabla_a^{-\alpha}\left\{\nabla\nabla_{a-1}^{-(1-\alpha)} y(t)\right\} = \nabla_a^{-\alpha} f(t, y(t)) \\ \nabla_a^{-\alpha}\left\{\nabla\nabla_a^{-(1-\alpha)} y(t) + \nabla\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\}\right\} = \nabla_a^{-\alpha} f(t, y(t)), \end{gathered} \tag{5.4}$$

$$\nabla_a^{-\alpha}\nabla_a^{\alpha} y(t) + \nabla_a^{-\alpha}\nabla\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} = \nabla_a^{-\alpha} f(t, y(t)). \tag{5.5}$$

It follows from Lemma 3.2 that

$$\nabla_a^{-\alpha}\nabla\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} = \nabla\nabla_a^{-\alpha}\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(a). \tag{5.6}$$

Note that

$$\nabla_a^{-\alpha}\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} = \nabla_{a-1}^{-\alpha}\frac{(t-(a-1))^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(1-\alpha)} y(a). \tag{5.7}$$

Hence we obtain

$$
\begin{aligned}
\nabla_a^{-\alpha}\nabla&\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)\right\} \\
&=\nabla\left\{\nabla_{a-1}^{-\alpha}\frac{(t-(a-1))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)-\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(1-\alpha)}y(a)\right\}-\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a) \\
&=y(a)\nabla\left\{\Gamma(1-\alpha)(t-(a-1))^{\overline{0}}\right\}-(\alpha-1)y(a)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha)}-\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a) \\
&=-\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a),
\end{aligned}
\tag{5.8}
$$

which follows from the power rule in Lemma 3.3.

Let us put this expression back in (5.5) and use (4.4), we have

$$y(t)=\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)+\nabla_a^{-\alpha}f(t,y(t)). \tag{5.9}$$

Thus, we have proved the following lemma.

Lemma 5.1. *y is a solution of the initial value problem,* (5.1), (5.2), *if, and only if, y has the representation* (5.9).

Remark 5.2. A similar result has been obtained in the paper [13] with the operator $\triangledown_a^\alpha$. And the initial value problem has been defined in the following form

$$\triangledown_a^\alpha y(t)=f(t,y(t)) \quad \text{for } t=a+1,\ a+2,\dots, \tag{5.10}$$

$$\triangledown_a^{-(1-\alpha)}y(t)|_{t=a}=y(a)=c, \tag{5.11}$$

where $0<\alpha\le 1$ and a is any real number. The subscript a of the term $\triangledown_a^\alpha y(t)$ on the left hand side of (5.10) indicates directly that the solution has a domain starts at a. The nature of this notation helps us to use the nabla transform easily as one can see in the papers [2, 12]. On the other hand, the subscript $a-1$ in (5.1) indicates that the solution has a domain starts at a.

6. An Alternative Definition of Nabla Fractional Differences

Recently, the authors in [18], by the help of a nabla Leibniz's Rule, have rewritten the nabla left fractional difference in a form similar to the definition of the nabla left fractional sum. In this section, we do this for the nabla right fractional differences.

The following delta Leibniz's Rule will be used:

$$\Delta_t\sum_{s=t}^{b-1}g(s,t)=\sum_{s=t}^{b-1}\Delta_t g(s,t)-g(t,\ t+1). \tag{6.1}$$

Using the following identity

$$\Delta_t \big(s - \rho(t)\big)^{\overline{\alpha}} = -\alpha \big(s - \rho(t)\big)^{\overline{\alpha-1}}, \tag{6.2}$$

and the definition of the nabla right fractional difference (ii) of Definition 2.3, for $\alpha > 0$, $\alpha \notin \mathbb{N}$ we have

$$\begin{aligned}
{}_b\nabla^{\alpha} f(t) &= (-1)^n \Delta^n {}_b\nabla^{-(n-\alpha)} f(t) \\
&= \frac{(-1)^n \Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} \big(s - \rho(t)\big)^{\overline{n-\alpha-1}} f(s) \\
&= \frac{(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha)} \Delta_t \sum_{s=t}^{b-1} \big(s - \rho(t)\big)^{\overline{n-\alpha-1}} f(s) \\
&= \frac{(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha)} \left[-(n-\alpha-1) \sum_{s=t}^{b-1} \big(s - \rho(t)\big)^{\overline{n-\alpha-2}} f(s) - (t-t)^{\overline{n-\alpha-1}} \right] \\
&= \frac{-(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha-1)} \sum_{s=t}^{b-1} \big(s - \rho(t)\big)^{\overline{n-\alpha-2}} f(s).
\end{aligned} \tag{6.3}$$

By applying the Leibniz's Rule (6.1), $n-1$ number of times we get

$${}_b\nabla^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=t}^{b-1} \big(s - \rho(t)\big)^{\overline{-\alpha-1}} f(s). \tag{6.4}$$

In the above, it is to be insisted that $\alpha \notin \mathbb{N}$ is required due the fact that the term $1/\Gamma(-\alpha)$ is undefined for negative integers. Therefore we can proceed and unify the definitions of nabla right fractional sums and differences similar to Definition 5.3 in [18]. Also, the alternative formula (6.4) can be employed, similar to Theorem 5.4 in [18], to show that the nabla right fractional difference ${}_b\nabla^{\alpha} f$ is continuous with respect to $\alpha \geq 0$.

7. Conclusions

In fractional calculus there are two approaches to obtain fractional derivatives. The first approach is by iterating the integral and then defining a fractional order by using Cauchy formula to obtain Riemann fractional integrals and derivatives. The second approach is by iterating the derivative and then defining a fractional order by making use of the binomial theorem to obtain Grünwald-Letnikov fractional derivatives. In this paper we followed the discrete form of the first approach via the nabla difference operator. However, we noticed that in the right fractional difference case we used both the nabla and delta difference operators. This setting enables us to obtain reasonable summation by parts formulas for nabla fractional sums and differences in Section 4 and to obtain an alternative definition for nabla right fractional differences through the delta Leibniz's Rule in Section 6.

While following the discrete form of the first approach, two types of fractional sums and hence fractional differences appeared; one type by starting from a and the other type, which obeys the general theory of nabla time scale calculus, by starting from $a+1$ in the left case and ending at $b-1$ in the right case. Section 3 discussed the relation between the two types of operators, where certain properties of one operator are obtained by using the second operator.

An initial value problem discussed in Section 5 is an important application exposing the derived properties of the two types of operators discussed throughout the paper, where the solution representation was obtained explicitly for order $0 < \alpha < 1$. Regarding this example we remark the following. In fractional calculus, Initial value problems usually make sense for functions not necessarily continuous at a (left case) so that the initial conditions are given by means of a^+. Since sequences are nice continuous functions then in Theorem 4.2, which is the tool in solving our example, the identity (4.4) appears without any initial condition. To create an initial condition in our example we shifted the fractional difference operator so that it started at $a-1$.

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Research Article

Application of Homotopy Perturbation and Variational Iteration Methods for Fredholm Integrodifferential Equation of Fractional Order

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This paper presents the application of homotopy perturbation and variational iteration methods as numerical methods for Fredholm integrodifferential equation of fractional order with initial-boundary conditions. The fractional derivatives are described in Caputo sense. Some illustrative examples are presented.

1. Introduction

Fractional differential equations have attracted much attention, recently, see for instance [1–4]. This is mostly due to the fact that fractional calculus provides an efficient and excellent instrument for the description of many practical dynamical phenomena arising in engineering and scientific disciplines such as, physics, chemistry, biology, economy, viscoelasticity, electrochemistry, electromagnetic, control, porous media and many more, see for example, [5, 6].

During the past decades, the topic of fractional calculus has attracted many scientists and researchers due to its applications in many areas, see [4, 7–9]. Thus several researchers have investigated existence results for solutions to fractional differential equations, see [10, 11]. Further, many mathematical formulation of physical phenomena lead to integrodifferential equations, for example, mostly these type of equations arise in fluid dynamics, biological models and chemical kinetics, and continuum and statistical mechanics, for more details see [12–16]. Integrodifferential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. The homotopy perturbation method and variational iteration method which are proposed by He [17–26] are of the methods which have received much concern. These methods have been successfully applied by many authors, such as the works in [19, 27, 28].

In this work, we study the Integrodifferential equations which are combination of differential and Fredholm-Volterra equations that have the fractional order. In particular, we applied the HPM and VIM for fractional Fredholm Integrodifferential equations with constant coefficients

$$\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(t) = g(t) + \lambda \int_0^a H(x,t) y(t) dt, \quad a \leq x,\ t \leq b, \tag{1.1}$$

under the initial-boundary conditions

$$D_*^{\alpha} y(a) = y(0), \tag{1.2}$$

$$D_*^{\alpha} y(0) = y'(a), \tag{1.3}$$

where a is constant, and $1 < \alpha < 2$, and D_*^{α} is the fractional derivative in the Caputo sense.

For the geometrical applications and physical understanding of the fractional Integrodifferential equations, see [14, 26]. Further, we also note that fractional integro-differential equations were associated with a certain class of phase angles and suggested a new way for understanding of Riemann's conjecture, see [29].

In present paper, we apply the HPM and VIM to solve the linear and nonlinear fractional Fredholm Integrodifferential equations of the form (1.1). The paper is organized as follows. In Section 2, some basic definitions and properties of fractional calculus theory are given. In Section 3, the basic idea of HPM exists. In Section 4, also is the basic idea of VIM. In Sections 5 and 6, analysis of HPM and VIM exsists, respectively. some examples are given in Section 7. Concluding remarks are listed in Section 8.

2. Preliminaries

In order to modeling the real world application the fractional differential equations are considered by using the fractional derivatives. Thus, in this section, we give some basic definitions and properties of fractional calculus theory which is used in this paper. There are many different starting points for the discussion of classical fractional calculus, see for example, [30]. One can begin with a generalization of repeated integration. If $f(t)$ is absolutely integrable on $[0, b)$, as in [31] then

$$\int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t2} f(t_1) dt_1 = \frac{1}{(n+1)!} \int_0^t (t - t_1)^{n-1} f(t_1) dt_1 = \frac{1}{(n+1)!} t^{n-1} * f(t), \tag{2.1}$$

where $n = 1, 2, \ldots$, and $0 \leq t \leq b$. On writing $\Gamma(n) = (n-1)!$, an immediate generalization in the form of the operation I^{α} defined for $\alpha > 0$ is

$$(I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - t_1)^{\alpha - 1} f(t_1) dt_1 = \frac{1}{\Gamma(\alpha)} t^{\alpha - 1} * f(t), \quad 0 \leq t < b, \tag{2.2}$$

where $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1} * f(t) = \int_0^t f(t - t_1)^{\alpha - 1}(t_1) dt_1$ is called the convolution product of $t^{\alpha-1}$ and $f(t)$. Equation (2.2) is called the Riemann-Liouville fractional integral of order α for the function $f(t)$. Then, we have the following definitions.

Definition 2.1. A real function $f(x)$, $x > 0$ is said to be in space $C\mu$, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_{μ}^{n} if $f^n \in \mathbb{R}_{\mu}$, $n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C\mu$, $\mu \geq -1$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t) dt, \quad \alpha > 0, \ t > 0. \tag{2.3}$$

In particular, $J^0 f(x) = f(x)$.

For $\beta \geq 0$ and $\gamma \geq -1$, some properties of the operator J^{α}:

(1) $J^{\alpha} J^{\beta} f(x) = J^{\alpha + \beta} f(x)$,

(2) $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$,

(3) $J^{\alpha} x^{\gamma} = (\Gamma(\gamma + 1)/\Gamma(\alpha + \gamma + 1)) x^{\alpha + \gamma}$.

Definition 2.3. The Caputo fractional derivative of $f \in C_{-1}^{m}$, $m \in \mathbb{N}$ is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x f^m(t) dt, \quad m - 1 < \alpha \leq m. \tag{2.4}$$

Lemma 2.4. *If $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $f \in C_{\mu}^{m}$, $\mu > -1$ then the following two properties hold*

(1) $D^{\alpha}[J^{\alpha} f(x)] = f(x)$,

(2) $J^{\alpha}[D^{\alpha} f(x)] = f(x) - \sum_{k=1}^{m-1} f^k(0)(x^k/k!)$.

Now, if $f(x)$ is expanded to the block pulse functions, then the Riemann-Liouville fractional integral becomes

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha - 1} * f(x) \simeq \xi^T \frac{1}{\Gamma(\alpha)} \left\{ x^{\alpha - 1} * \phi_m(x) \right\}. \tag{2.5}$$

Thus, if $x^{\alpha-1} * \phi_m(x)$ can be integrated, then expanded in block pulse functions, the Riemann-Liouville fractional integral is solved via the block pulse functions. Thus, one notes on that

Kronecker convolution product can be expanded in order to define the Riemann-Liouville fractional integrals for matrices by using the Block Pulse operational matrix as follows:

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1}\phi_m(t_1)dt_1 \simeq F_\alpha \phi_m(t), \tag{2.6}$$

where

$$F_\alpha = \left(\frac{b}{m}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ 0 & 1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & 1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.7}$$

see [32].

3. Homotopy Perturbation Method

To illustrate the basic idea of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3.1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{3.2}$$

where A is a general differential operator; B is a boundary operator; $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Equation (3.1) therefor, can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3.3}$$

By the homotopy technique [33–35], we construct a homotopy $v(r,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \ r \in \Omega \tag{3.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{3.5}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation of (3.1) which satisfies the boundary conditions. From (3.2) and (3.3) we have

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, \\ H(v,1) &= A(v) - f(r) = 0 \end{aligned} \tag{3.6}$$

the changing in the process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Now, assume that the solution of (3.2) and (3.3) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{3.7}$$

Setting $p = 1$ results in the approximate solution of (3.1).

Therefore,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{3.8}$$

4. The Variational Iteration Method

To illustrate the basic concepts of VIM, we consider the following differential equation

$$L(u) + N(u) = g(x), \tag{4.1}$$

where L is a linear operator; N is nonlinear operator, and $g(x)$ is an nonhomogeneous term. According to VIM, one constructs a correction functional as follows:

$$y_{n+1} = y_n + \int_0^x \lambda \left[L y_n(s) - N \widetilde{y_n}(s) \right] ds, \tag{4.2}$$

where λ is a general Lagrange multiplier, and $\widetilde{y_n}$ denotes restricted variation that is $\delta \widetilde{y_n} = 0$.

5. Analysis of Homotopy Perturbation Method

To illustrate the basic concepts of HPM for Fredholm Integrodifferential equation (1.1) with boundary conditions (1.2) and (1.3). We use the view of He in [19, 20], where the following homotopy was constructed for (1.1) as the following:

$$(1-p) \sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) + p \left[\sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) - g(t) - \lambda \int_a^b H(x,t) y(x) dx \right] = 0 \tag{5.1}$$

or

$$\sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) = p \left[g(t) + \lambda \int_a^b H(x,t) y(x) dx \right], \tag{5.2}$$

where $p \in [0,1]$ is an embedding parameter. If $p = 0$, (5.2) becomes linear fractional differential equation

$$\sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) = 0, \tag{5.3}$$

and when $p = 1$, the (5.2) turn out to be the original equation. In view of basic assumption of HPM, solution of (1.1) can be expressed as a power series in p

$$y(x) = y_0(x) + p_1 y_1(x) + p_2 y_2(x) + \cdots, \tag{5.4}$$

when $p = 1$, we get the approximate solution of (5.4)

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots. \tag{5.5}$$

The convergence of series (5.5) has been proved in [21]. Substitution (5.4) into (5.2), and equating the terms with having identical power of p, we obtain the following series of equations:

$$\begin{aligned}
p^0 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_0 = 0, \\
p^1 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_1 = g(t) - \lambda \int_a^b H(x,t) y_0(x) dx, \\
p^2 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_2 = -\lambda \int_a^b H(x,t) y_1(x) dx, \\
p^3 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_3 = -\lambda \int_a^b H(x,t) y_2(x) dx, \\
&\vdots
\end{aligned} \tag{5.6}$$

with the initial-boundary conditions

$$D_*^{\alpha} y(a) = y(0), \qquad D_*^{\alpha} y(0) = y'(a). \tag{5.7}$$

The initial approximation can be chosen in the following manner.

$$y_0 = \sum_{j=0}^{1} \gamma_j \frac{x^j}{j!} = \gamma_0 + \gamma_1 x, \quad \text{where } \gamma_0 = D_*^{\alpha} y(a) \gamma_1 = D_*^{\alpha} y(0). \tag{5.8}$$

Note that the (5.6) can be solved by applying the operator J_*^α and by some computation, we approximate the series solution of HPM by the following N-term truncated series

$$\chi_n(x) = y_0(x) + y_1(x) + \cdots + y_{N-1}(x), \tag{5.9}$$

which is the approximate solution of (1.1)–(1.3).

6. Analysis of VIM

To solve the fractional Integrodifferential equation by using the variational iteration method, with boundary conditions (1.2) and (1.3) we construct the following correction functional:

$$y_{k+1}(x) = y_k(x) + J^\alpha\left[\mu\left(\sum_{k=0}^{\infty} P_k D_*^\alpha y(x) - \tilde{g}(x) - \lambda \int_0^a H(x,s)\tilde{y}_k(s)ds\right)\right] \tag{6.1}$$

or

$$y_{k+1}(x) = y_k(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\mu(s)\left[\sum_{k=0}^{\infty} P_k D_*^\alpha y(s) - \tilde{g}_k(s) - \lambda \int_0^a H(x,p)\tilde{y}_k(p)dp\right], \tag{6.2}$$

where μ is a general Lagrange multiplier, and $\tilde{g}_k(x)$ and $\tilde{y}_k(x)$ are considered as restricted variation, that is, $\delta\tilde{g}_k(x) = 0$ and $\delta\tilde{y}_k(x) = 0$.

Making the above correction functional stationary, the following condition can be obtained

$$\delta y_{k+1}(x) = \delta y_k(x) + \int_0^x (x-s)^{\alpha-1}\mu(s)\left[\sum_{k=0}^{\infty} P_k \delta D_*^\alpha y(s) - \delta\tilde{g}_k(s) - \lambda \int_0^a H(x,p)\delta\tilde{y}_k(p)dp\right]. \tag{6.3}$$

It's boundary condition can be obtained as follows:

$$1 - \mu'(s)\big|_{x=s} = 0, \qquad \mu(s)\big|_{x=s} = 1. \tag{6.4}$$

The Lagrange multipliers can be identified as follows:

$$\mu(s) = \frac{1}{2}(x-s). \tag{6.5}$$

We obtain the following iteration formula by substitution of (6.5) in (6.2):

$$y_{k+1}(x) = y_k(x) + \frac{1}{2\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}(s-x)\left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s) - \tilde{g}_k(s) - \lambda\int_0^a H(x,p)\tilde{y}_k(p)dp\right]ds. \tag{6.6}$$

That is,

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s) - \tilde{g}_k(s) - \lambda\int_0^a H(x,p)\tilde{y}_k(p)dp\right]ds. \tag{6.7}$$

This yields the following iteration formula:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2}J^{\alpha}\left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) - g_k(x) - \lambda\int_0^a H(x,s)\tilde{y}_k(s)ds\right]. \tag{6.8}$$

The initial approximation y_0 can be chosen by the following manner which satisfies initial-boundary conditions (1.2)-(1.3)

$$y_0 = \gamma_0 + \gamma_1 x, \quad \text{where } \gamma_0 = D_*^{\alpha} y(a)\ \gamma_1 = D_*^{\alpha} y(0). \tag{6.9}$$

We can obtain the following first-order approximation by substitution of (6.9) in (6.8)

$$y_1(x) = y_0(x) - \frac{(\alpha-1)}{2}J^{\alpha}\left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) - g_0(x) - \lambda\int_0^a H(x,s)\tilde{y}_0(s)ds\right]. \tag{6.10}$$

Finally, by substituting the constant values of γ_0 and γ_1 in (6.10) we have the results as the approximate solutions of (1.1)–(1.3), see the further details in [36–40].

7. Applications

In this section, we have applied homotopy perturbation method and variational iteration method to fractional Fredholm Integrodifferential equations with known exact solution.

Example 7.1. Consider the following linear Fredholm Integrodifferential equation:

$$D^{\alpha} y(x) = \left(\frac{3}{2} + \frac{e^{2x}}{2}\right) + \int_0^x e^t y(t)dt \quad 0 \le x \le 1,\ 1 < \alpha \le 2, \tag{7.1}$$

with initial boundary conditions

$$y(0) = 1, \qquad y'(1) = e \tag{7.2}$$

the exact solution is $y(x) = e^x$. Now we construct

$$D^\alpha y(x) = p\left(\left(\frac{3}{2} + \frac{e^{2x}}{2}\right) + \int_0^x e^t y(t)dt\right). \tag{7.3}$$

Substitution of (5.4) in (7.3) and then equating the terms with same powers of p, we get the series

$$\begin{aligned}
&p^0 : D^\alpha y_0(x) = 0,\\
&p^1 : D^\alpha y_1(x) = \left(\frac{3}{2} + \frac{2}{3}e^{2x}\right) + \int_0^x e^t y_0(t)dt,\\
&p^2 : D^\alpha y_2(x) = -\int_0^x e^t y_1(t)dt.\\
&\ \ \vdots
\end{aligned} \tag{7.4}$$

Now applying the operator J_α to the equations (7.4) and using initial-boundary conditions yields

$$y_0(x) = 1, \tag{7.5}$$

$$y_1(x) = 1 + Ax + J^\alpha\left(\left(\frac{3}{2} + \frac{e^{2x}}{2}\right) + \int_0^x e^t y_0 dt\right), \tag{7.6}$$

$$y_2(x) = J^\alpha\left(\int_0^x e^t y_1 dt\right), \tag{7.7}$$

$$y_n(x) = J^\alpha\left(\int_0^x e^t y_{n-1} dt\right), \quad n = 2, 3, 4, \ldots. \tag{7.8}$$

Then by solving (7.5)–(7.8), we obtain $y_1, y_2, \ldots$ as

$$y_1(x) = 1 + Ax + \frac{5x^\alpha}{2\Gamma(\alpha+1)} + \frac{2x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{3x^{\alpha+2}}{2\Gamma(\alpha+3)} + \frac{5x^{\alpha+3}}{6\Gamma(\alpha+4)}, \tag{7.9}$$

$$\begin{aligned}
y_2(x) = {}& \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (A+1)\frac{x^{\alpha+2}}{2\Gamma(\alpha+3)} + \left(\frac{A}{3} + \frac{1}{2}\right)\frac{x^{\alpha+3}}{\Gamma(\alpha+4)}\\
&+ \left(\frac{A}{8} + \frac{1}{12}\right)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{Ax^{\alpha+5}}{15\Gamma(\alpha+6)} + \frac{5x^{2\alpha+1}}{2\Gamma(2\alpha+2)} + \cdots.
\end{aligned} \tag{7.10}$$

Table 1: Values of A for different values of α.

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	−2.33209843875457	−1.906444021198994	−0.88898224618462	−0.098915873901025

Table 2: Value of A for different values of α using (7.14).

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	1.23429062479478	0.73267858113358	0.66218167845861	0.54744784230252

Now, we can form the 2 term approximation as follows:

$$\begin{aligned}\phi_2(x) &= 1 + Ax + \frac{5x^{\alpha}}{2\Gamma(\alpha+1)} + \frac{3x^{\alpha+1}}{\Gamma(\alpha+2)} + (A+2)\frac{x^{\alpha+2}}{2\Gamma(\alpha+3)} \\ &\quad + (A+4)\frac{x^{\alpha+3}}{3\Gamma(\alpha+4)} + \left(\frac{A}{8}+\frac{1}{12}\right)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{Ax^{\alpha+5}}{15\Gamma(\alpha+6)} + \frac{5x^{2\alpha+1}}{2\Gamma(2\alpha+2)} + \cdots,\end{aligned} \tag{7.11}$$

where A can be determined by imposing initial-boundary conditions (7.2) on ϕ_2. Table 1 shows the values of A for different values of α.

Now, we solve (7.1)-(7.2) by variational iteration method. According to variational iteration method, the formula (6.8) for (7.1) can be expressed in the following form:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2}J^{\alpha}\left[D^{\alpha}y(x) - \left(\frac{3}{2}+\frac{e^{2x}}{2}\right) - \int_0^x e^t y(t)dt\right]. \tag{7.12}$$

Then, in order to avoid the complex and difficult fractional integration, we can consider the truncated Taylor expansions for exponential term in (7.6)–(7.8) for example, $e^x \sim 1 + x + x^2/2 + x^3/6$ and further, suppose that an initial approximation has the following form which satisfies the inial-boundary conditions

$$y_0(x) = 1 + Ax. \tag{7.13}$$

Now by iteration formula (7.12), the first approximation takes the following form:

$$\begin{aligned}y_1(x) &= y_0(x) - \frac{(\alpha-1)}{2}J^{\alpha}\left[D^{\alpha}y_0(x) - \left(\frac{3}{2}+\frac{e^{2x}}{2}\right) - \int_0^x e^t y_0(t)dt\right] \\ &= 1 + Ax + \frac{(\alpha-1)}{2}x^{\alpha}\left[\frac{5}{\Gamma(\alpha+1)} + \frac{2x}{\Gamma(\alpha+2)} + \frac{(A+3)x^2}{2\Gamma(\alpha+3)}\right. \\ &\qquad \left. + \frac{(5/2+A)x^3}{3\Gamma(\alpha+4)} + \frac{(A/2+1/6)x^4}{6\Gamma(\alpha+5)} - \frac{Ax^5}{30\Gamma(\alpha+6)}\right].\end{aligned} \tag{7.14}$$

By imposing initial-boundary conditions (7.2) on y_1, we can obtain the values of A for different α which we show in Table 2.

Example 7.2. Consider the following linear Fredholm Integrodifferential equation:

$$D^{\alpha}y(x) = \left(1-\frac{x}{4}\right) + \int_0^x xty^2(t)dt \quad 0 \le x \le 1,\ 1 < \alpha \le 2 \tag{7.15}$$

with initial boundary conditions

$$y(0) = 0, \qquad y'(1) = 1. \tag{7.16}$$

then the exact solution is $y(x) = x$. By applying the HPM, we have

$$D^{\alpha}y(x) = p\left(\left(1-\frac{x}{4}\right) + \int_0^x xty^2(t)dt\right). \tag{7.17}$$

Substitution of (5.4) in (7.15) and then equating the terms with same powers of p, we get the following series expressions:

$$\begin{aligned}
&p^0 : D^{\alpha}y_0(x) = 0,\\
&p^1 : D^{\alpha}y_1(x) = \left(1-\frac{x}{4}\right) + \int_0^x xt{y_0}^2(t)dt,\\
&p^2 : D^{\alpha}y_2(x) = 2\int_0^x xty_0(t)y_1(t)dt,\\
&p^3 : D^{\alpha}y_3(x) = \int_0^x xt\left(y_0(t)y_2(t) + {y_1}^2(t)\right)dt,\\
&p^4 : D^{\alpha}y_4(x) = \int_0^x xt\left(2y_0(t)y_4(t) + 2y_1y_3 + {y_2}^2(t)\right)dt,\\
&\quad\vdots
\end{aligned} \tag{7.18}$$

Applying the operator J_{α} to (7.18) and using initial-boundary conditions, then we get

$$\begin{aligned}
&y_0(x) = 0,\\
&y_1(x) = Ax + J^{\alpha}\left(\left(1-\frac{x}{4}\right) + \int_0^x xt{y_0}^2(t)dt\right),\\
&y_2(x) = 0,\\
&y_3(x) = J^{\alpha}\left(\int_0^x xt\left(y_0(t)y_2(t) + {y_1}^2(t)\right)dt\right),\\
&y_4(x) = J^{\alpha}\left(\int_0^x xt\left(2y_0(t)y_4(t) + 2y_1y_3 + {y_2}^2(t)\right)dt\right),\\
&\quad\vdots
\end{aligned} \tag{7.19}$$

Table 3: Value of A for different values of α.

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	0.179304	0.153796	0.0673477	0.124989

Table 4: Value of A for different values of α.

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	0.88967375	0.76492075	0.650263	0.5625

Thus, by solving (7.19), we obtain $y_1, y_2, y_3, \ldots$

$$y_1(x) = Ax + \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{\alpha+1}}{4\Gamma(\alpha+2)},$$
$$y_2(x) = 0,$$
$$y_3(x) = \frac{A^2 x^{\alpha+5}}{4\Gamma(\alpha+6)} + \frac{2Ax^{2\alpha+4}}{(\alpha+3)\Gamma(\alpha+1)\Gamma(2\alpha+5)} + \frac{x^{3\alpha+3}}{(2\alpha+2)\Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(3\alpha+4)} + \cdots . \tag{7.20}$$

Now, we can form the 3 term approximation

$$\begin{aligned}\phi_2(x) = Ax &+ \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{\alpha+1}}{4\Gamma(\alpha+2)} + \frac{A^2 x^{\alpha+5}}{4\Gamma(\alpha+6)} + \frac{2Ax^{2\alpha+4}}{(\alpha+3)\Gamma(\alpha+1)\Gamma(2\alpha+5)} \\ &+ \frac{x^{3\alpha+3}}{(2\alpha+2)\Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(3\alpha+4)} + \cdots ,\end{aligned} \tag{7.21}$$

where A can be determined by imposing initial-boundary conditions (7.16) on ϕ_2. Thus, we have Table 3.

Similarly, by variational iteration method we have the following form:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2} J^\alpha \left[D^\alpha y(x) - \left(1 - \frac{x}{4}\right) + \int_0^x xty^2(t)dt \right], \tag{7.22}$$

where we suppose that an initial approximation has the following form which satisfies the initial-boundary conditions $y_0(x) = Ax$. Now by using the iteration formula, the first approximation takes the following form:

$$\begin{aligned} y_1(x) &= y_0(x) - \frac{(\alpha-1)}{2} J^\alpha \left[D^\alpha y_0(x) - \left(1 - \frac{x}{4}\right) + \int_0^x xt{y_0}^2(t)dt \right] \\ &= Ax + \frac{(\alpha-1)}{2} J^\alpha \left[\frac{-x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{4\Gamma(\alpha+2)} \right]. \end{aligned} \tag{7.23}$$

By imposing initial-boundary conditions, we can obtain the following Table 4.

8. Conclusion

In this work, homotopy perturbation method (HPM) and variational iteration method (VIM) have been applied to linear and nonlinear initial-boundary value problems for fractional Fredholm Integrodifferential equations. Two examples are presented in order to illustrate the accuracy of the present methods. Comparisons of HPM and VIM with exact solution have been given in the Tables 1–4.

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Research Article

Shannon Information and Power Law Analysis of the Chromosome Code

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This paper studies the information content of the chromosomes of twenty-three species. Several statistics considering different number of bases for alphabet character encoding are derived. Based on the resulting histograms, word delimiters and character relative frequencies are identified. The knowledge of this data allows moving along each chromosome while evaluating the flow of characters and words. The resulting flux of information is captured by means of Shannon entropy. The results are explored in the perspective of power law relationships allowing a quantitative evaluation of the DNA of the species.

1. Introduction

During the last years the genome sequencing project produced a large volume of data that is presently available for computational processing [1–14]. Researchers have been tackling the information content of the deoxyribonucleic acid (DNA), but interesting questions remain still open [15–21].

This paper addresses the information flow along each DNA strand. For this purpose several statistics are developed, and the relative frequencies of distinct types of symbol associations are evaluated. The concepts of character, word, word delimiter, and phrase are defined, and the information content of each chromosome message is quantified. Power law (PL) relationships emerge in the information locus. PL distributions, often known as heavy tail distributions, Pareto laws, Zipf laws, or others, have been largely reported in the modeling of distinct real phenomena [22–31]. It was recognized [11, 32–34] that DNA has an information structure that reveals long range behavior, somehow in the line of thought of systems with dynamics described by the tools of Fractional Calculus (FC) [35–37]. It is

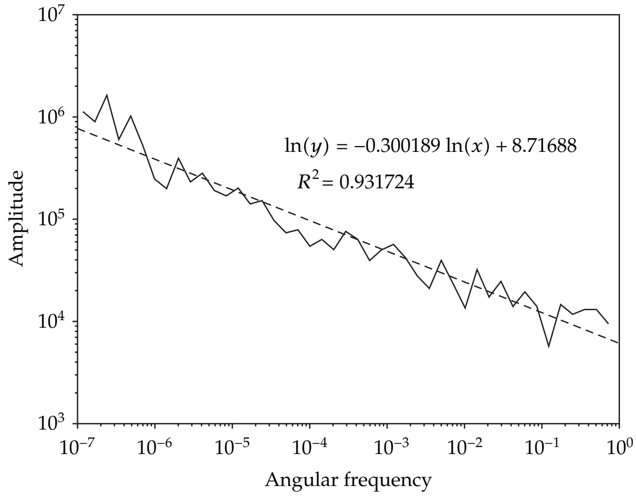

Figure 1: Amplitude of the Fourier transform versus frequency ω for chromosome 1 of the human being (solid line) and PL approximation (dashed line).

known the existence of a strong relationship between FC and PL; nevertheless, up to the present state of knowledge, no formal demonstration supported that observation based on empirical and experimental measurements. Therefore, it is not a surprise that both FC and PL descriptions emerge when analyzing DNA with distinct mathematical tools. In the present study PL descriptions are applied for condensing the charts characterizing the chromosomes of twenty-three species.

Having these ideas in mind, this paper is organized as follows. Section 2 presents the DNA sequence decoding concepts, the mathematical tools and formulates the algorithm that computes the information for each chromosome and species. Section 3 analyzes the DNA information dynamical content of 463 chromosomes corresponding to a set of twenty-three species. Finally, Section 4 outlines the main conclusions.

2. Preliminary Notes on the DNA Information

In the DNA double helix there are four distinct nitrogenous bases, namely, thymine, cytosine, adenine, and guanine, denoted by the symbols $\{T, C, A, G\}$. Each type of base on one strand connects with only one type of base on the other strand, forming the base pairing $A - T$ and $G - C$. Besides the four symbols $\{T, C, A, G\}$, the available chromosome data includes a fifth symbol "N" which is believed to have no practical meaning for the DNA decoding.

For processing the DNA information a possible technique is to convert the symbols into a numerical value. In previous papers was adopted the direct symbol translation $= 1 + i0$, $C = -1+i0$, $T = 0+i$, $G = 0-i$, $N = 0+i0$, where $i = \sqrt{-1}$. We can move along the DNA strip, one symbol (base) at a time. The resulting values form a "signal" $x(t)$ where "t" can be interpreted as a pseudotime. The signal can be treated by the Fourier transform $F\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-i\omega}dt$, where ω represents the angular frequency.

Figure 1 shows one example with the amplitude of the Fourier transform for chromosome 1 of the human being. The frequency interval $10^{-7} \leq \omega \leq 10^0$ is adopted and a PL approximation is superimposed revealing a strong correlation.

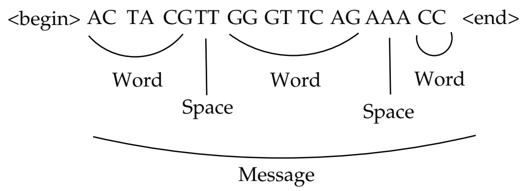

Figure 2: Example of a message when considering $n = 2$, {TT, AA} ≡ "spaces", {TC, TA, TG, CT, CC, CA, CG, AT, AC, AG, GT, GC, GA, GG} ≡ "word characters." Multiple consecutive spaces are considered as a single space.

This technique has, however, one drawback which is the initial assignment of numerical values to the DNA symbols. Therefore, it is important to design an alternative method of analysis avoiding that problem, but, on the other hand, capable of revealing fractional order phenomena. Bearing this strategy in mind, in this paper is adopted an approach based on the histograms of symbol alignment, information theory, and PL approximations.

This study focuses over twenty-three species yielding a space of 463 chromosomes. Therefore, denoting by N_j the number of chromosomes of species $j = 1, \ldots, 23$, we consider the $\{\text{Species}, \text{Tag}, N_j\}_j$ given by {Mosquito (*Anopheles gambiae*), Ag, 5}$_1$, {Honeybee, (*Apis mellifera*), Am, 16}$_2$, {*Caenorhabditis briggsae*, Cb, 6}$_3$, {*Caenorhabditis elegans*, Ce, 6}$_4$, {Chimpanzee, Ch, 25}$_5$, {Dog, Dg, 39}$_6$, {*Drosophila simulans*, Ds, 6}$_7$, {*Drosophila yakuba*, Dy, 10}$_8$, {Horse, Eq, 32}$_9$, {Chicken, Ga, 31}$_{10}$, {Human, Ho, 24}$_{11}$, {Medaka, Me, 24}$_{12}$, {Mouse, Mm, 21}$_{13}$, {Opossum, Op, 9}$_{14}$, {Orangutan, Or, 24}$_{15}$, {Cow, Ox, 30}$_{16}$, {Pig, Po, 19}$_{17}$, {Rat, Rn, 21}$_{18}$, {Yeast (*Saccharomyces cerevisiae*), Sc, 16}$_{19}$, {Stickleback, St, 21}$_{20}$, {Zebra Finch, Tg, 32}$_{21}$, {Tetraodon, Tn, 21}$_{22}$ and {Zebrafish, Zf, 25}$_{23}$.

The DNA information decoding is addressed in this paper, and we start by defining the underlying concepts. The fundamental unit is the "symbol" that, in our case, consists in one of the four possibilities $\{T, C, A, G\}$, while "N" is simply disregarded. Each "character" is represented by an n-tuple association ($n = 1, 2, \ldots$) of the 4 symbols, resulting in a total of 4^n possible symbols per character. For example, with $n = 2$ we get a maximum of 4^2 characters represented by the 16 two-symbol sequences {TT, TC, TA, TG, CT, CC, CA, CG, AT, AC, AA, AG, GT, GC, GA, GG}. The sequences are obtained when moving sequentially along the DNA. The characters may have different significance and are divided into two classes, namely, characters with relevant information, to be denoted in the sequel as "word characters," and delimiters denoted as "spaces." Therefore, joining consecutive "word characters" yields a "word," that ends in the presence of one or more consecutive "spaces" (i.e., multiple spaces are considered as a single space). When the complete association of consecutive words is fulfilled, we obtain a "message."

Figure 2 depicts a simple example of a message with 21 symbols and 3 words. The message {ACTACGTTGGGTTCAGAAACC} is processed according to the proposed scheme for $n = 2$ and considering the 2 sequences {TT, AA} as spaces, and the 14 sequences {TC, TA, TG, CT, CC, CA, CG, AT, AC, AG, GT, GC, GA, GG} as characters. Therefore, the resulting words are {AC TA CG}, {GG GT TC AG} and {CC}.

We verify that we may have words with different lengths and that it is considered as a single space any repetition of spaces. The message finishes when the end of the DNA strand is attained, and, therefore, it is not considered the case of multiple messages for each chromosome.

After defining the concepts for symbol, character (with the categories of word character and space), and message, we need to establish the numerical value to be adopted by n and the method for measuring the information. In what concerns n no *a priori* optimal value is considered. Therefore, in the experiments is analyzed the influence when going from $n = 1$ up to $n = 12$, or, correspondingly, when going from 4^1 up to 4^{12} symbols per character. This evaluation is performed for one chromosome. Based on this first assessment, given the huge computational load required by high values of n, the set of twenty-three species, totalizing 463 chromosomes, is analyzed for $n = \{1, \ldots, 8\}$. In what concerns the information measurement it is adopted the Shannon information [38–49] $I_i = -\ln(p_i)$ where I_i represents the quantity of information of event i that has a probability p_i. In this topic we can refer to [50] calculating also the Shannon information for short DNA words of differing lengths, where the authors find that genomes share universal statistical properties. It is also worth mentioning that other entropies, such as the Rényi, Tsallis, and Ubriaco definitions [51, 52] were tested. Nevertheless, experiments with these expressions and distinct numerical values of the parameters did not reveal any significant conceptual difference. Therefore, for simplicity in the sequel it is adopted merely the Shannon definition.

In our case, for a n-tuple symbol encoding, the occurrence of the ith character within the 4^n set has probability ${p_i}^{\text{char},n}$ leading to information $-\ln({p_i}^{\text{char},n})$, and, therefore, the total information content of a word $I^{\text{word},n}$ yields

$$I^{\text{word},n} = -\sum_{i=1}^{m} \ln\left({p_i}^{\text{char},n}\right), \tag{2.1}$$

where m represents the total number of word characters including the first space. In fact, it was numerically evaluated the effect of including, or not, the space information but, due to its low importance, the final effect is negligible. Therefore, it is considered the inclusion of one space as the information for delimiting the word, while further consecutive repetitions of spaces are disregarded.

The message information is the sum of all word information:

$$I^{\text{mes},n} = \sum_{i=1}^{r} {I_i}^{\text{word},n}, \tag{2.2}$$

where r denotes the total number of words included in the message (i.e., the chromosome).

The information measurement requires the knowledge of ${p_i}^{\text{char},n}$. While we can expect an equilibrium of probabilities for $n = 1$, that may be not true for larger values of n. Therefore, in the sequel it is adopted a numerical procedure that starts by reading the chromosome message based on the n-tuple character setup leading to the construction of one histogram per chromosome. In the set of 4^n bins are chosen, by inspection, those that are more frequent (and have smaller information content) for the role of spaces. In a second phase, the relative frequencies, which are adopted as approximants to the probabilities, and the information values (2.1) and (2.2) are calculated numerically while traveling along the DNA strand.

This strategy does not consider some *a priori* optimal value of n. Therefore, as mentioned previously, several distinct values of n will be studied before establishing any conclusions.

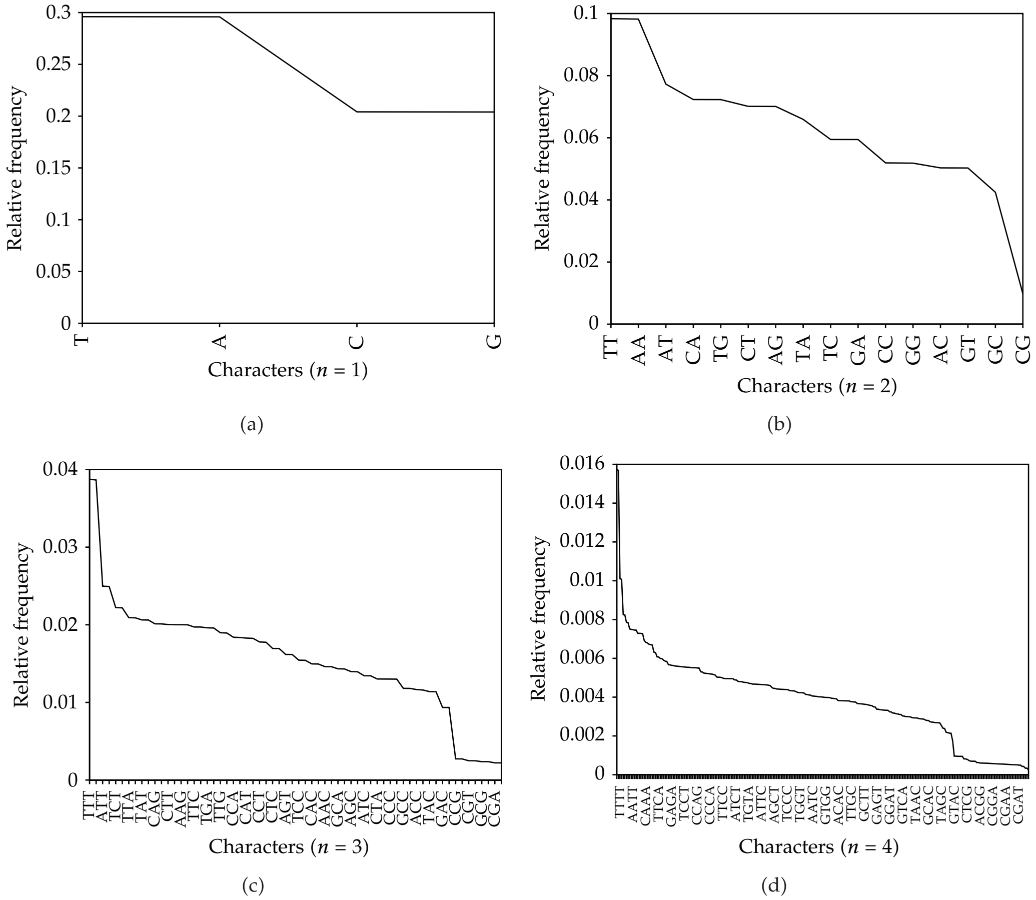

Figure 3: Histograms for Ho12 and $n = \{1, 2, 3, 4\}$.

3. Capturing the DNA Information

We start by considering Human chromosome 12 (Ho12) and $n = \{1, \ldots, 12\}$. This chromosome is represented by a medium size file (130 Mbytes) and may be considered a good compromise between length and computational load.

Figure 3 depicts the histograms for $n = \{1, 2, 3, 4\}$ where, for simplifying the visualization, the characters are ordered by decreasing magnitude of relative frequency. For the histograms construction two counting methods were envisaged: (i) counting with disjoint set of n symbols and (ii) counting the sets while sliding one symbol at a time. At first sight it seems that (i) is the most straightforward, but if we consider that we do not have reliable information for starting and synchronizing the counting, then method (ii) is more robust and, therefore, is adopted in the sequel.

Figure 4 shows the word information dynamics when travelling along the Ho12 strand for $n = \{1, 2, 3, 4\}$. We observe the existence of quantum information levels that somehow vanish when n increases. This is due to finite number of quantifying levels of information that occur before a space terminates a word. The number of quantum levels increases with n while

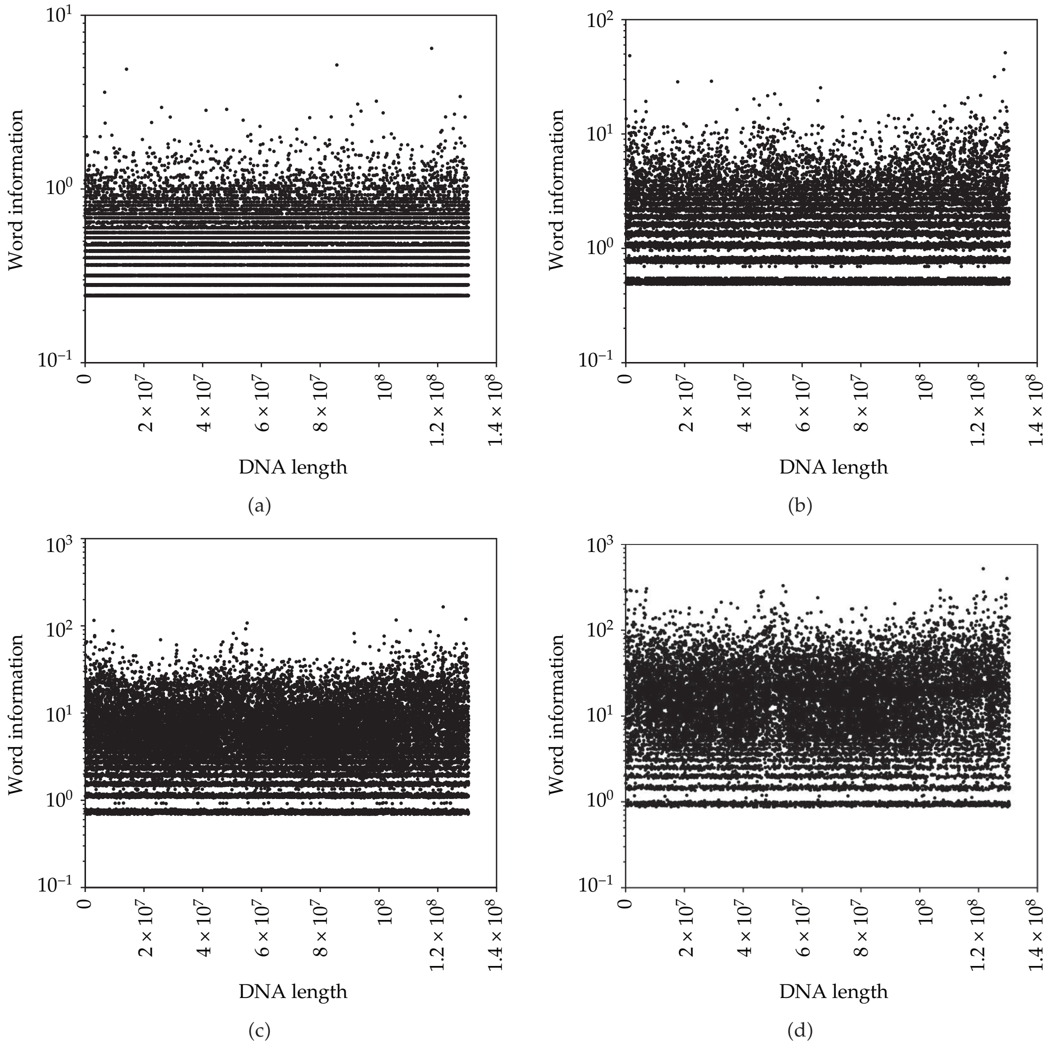

Figure 4: Word information *versus* length for the Ho12 and $n = \{1, 2, 3, 4\}$.

the length of each word increases. Besides this interesting effect, we also note a considerable randomness and a uniform behavior along all length of the strand.

The total chromosome information, the number of words N_w, and the average word information I_{av} versus n are depicted in Figures 5(a) and 5(b). We verify a maximum of the total chromosome information for $n = 3$. For larger values of n the information decreases slightly due to the effect of dropping out repeated consecutive spaces. Therefore, we can say that large values of n seem to lead to a slightly better estimate of the total information content, while the cases of $n = 1$ or $n = 2$ lead to an inferior measurement process. We also observe that the number of words decreases with n but its average information varies in the opposite way. Therefore, it is relevant to plot one variable against the other, with n as parameter (Figure 5(c)). A PL trendline approximation demonstrates that the two quantities are inversely proportional. In fact, we get numerically $I_{\text{av}} = aN_w{}^b$ with $a = 2.07\,10^8$,

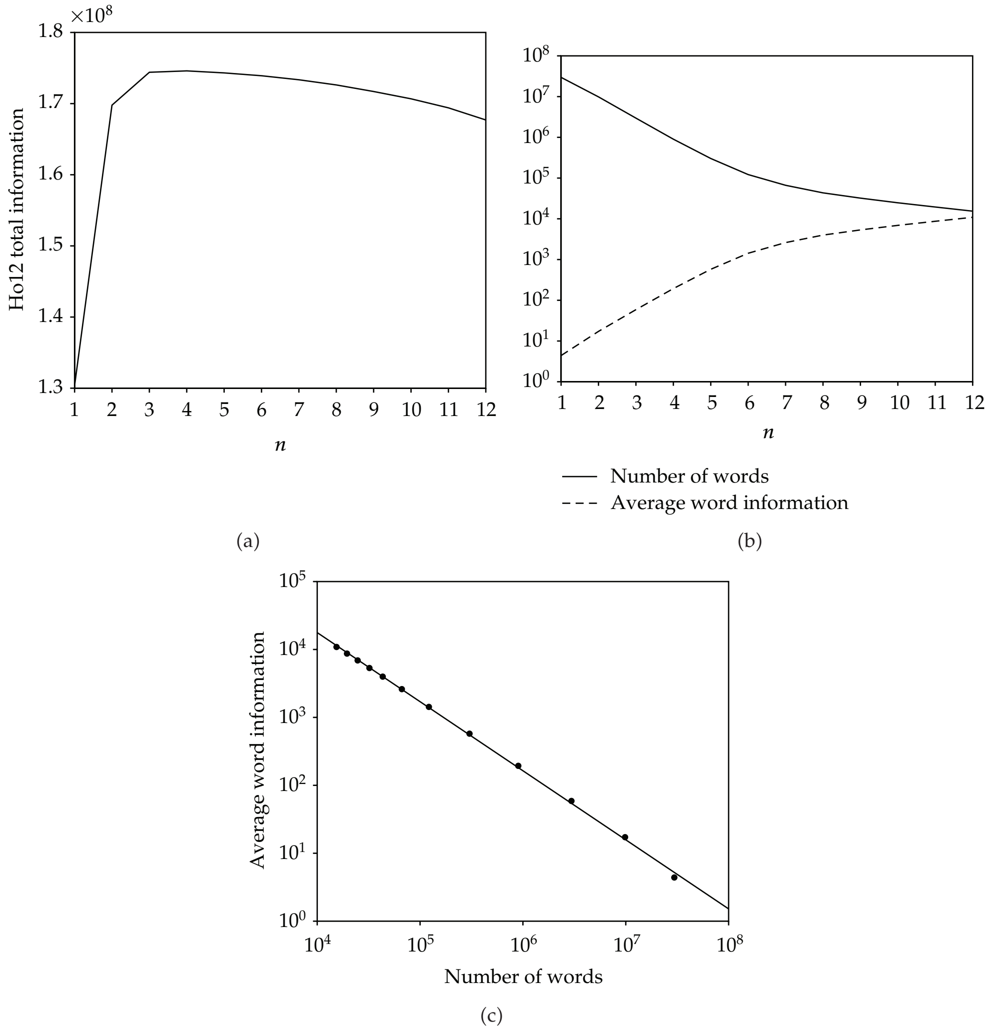

Figure 5: Chromosome Ho12: (a) total information *versus* n, (b) average word information and number of words *versus* n, (c) average word information *versus* number of words.

$b = -1.02$. For the rest of the chromosomes it was observed a similar type of behavior, but with different numerical values for the parameters.

For other values of n the resulting histograms reveal identical characteristics, namely, two characters with a very large relative frequency (depicted at the left part of the histograms of Figure 3). Furthermore, experiments with other chromosomes lead to similar results. The two characters are simply a succession of symbols A or T and the corresponding n-tuples (i.e., $A \cdots A$ and $T \cdots T$) are adopted in the sequel as "spaces."

Figure 6 shows the total information, that is, the information resulting from summing the information of all the chromosomes of each species *versus* the corresponding number of chromosomes, for character encoding with $n = 8$. We observe a weak correlation between both variables.

Figure 7 shows the length of each chromosome L_i^{crom} versus its information content $I_i^{\text{crom},n}$, $i = 1, \ldots, 463$, estimated by the proposed method with $n = 8$. In this case we observe

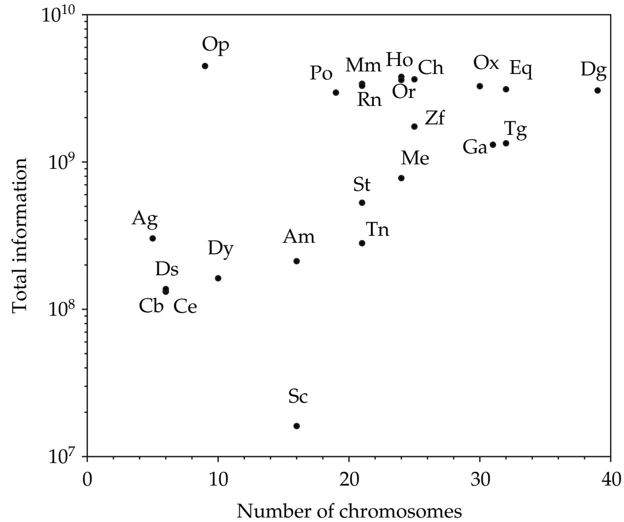

Figure 6: Total information for each species versus the number of chromosomes with $n = 8$.

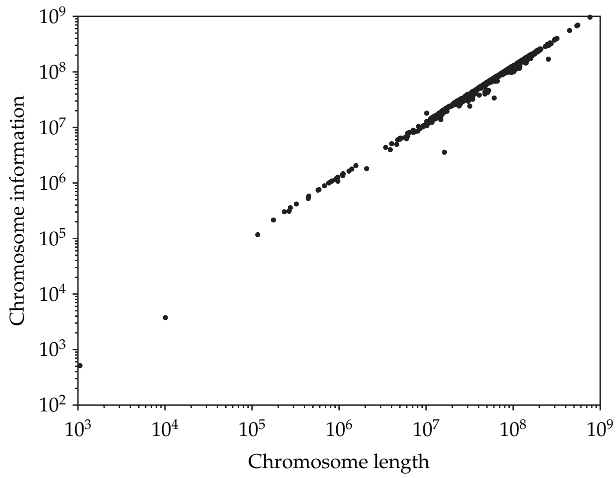

Figure 7: Chromosome length L_i^{crom} versus its information content $I_i^{\text{crom},8}$, $i = 1, \ldots, 463$, $n = 8$.

a strong correlation between both variables, meaning that the implementation of the DNA code has a large similarity between all species. In fact, we can calculate a PL trendline over the 463 chromosomes yielding the relationship $I_i^{\text{crom},8} = 0.79(L_i^{\text{crom}})^{1.03}$.

Bearing these ideas in mind it was decided to explore the PL behavior, that is, the relation $I_{\text{av}} = aN_w{}^b$, $a > 0$, $b < 0$, of the average word information I_{av} versus the number of words N_w (with n as parameter) per chromosome. The extensive evaluation of the 463 chromosomes for $n = \{1, \ldots, 8\}$ leads to the locus (a, b) of the PL trendline depicted in Figure 8. The point for chromosome DyYh is not included to allow a better visualization of the remaining set of points. Moreover, the individual chromosome labels are not included to make the plot more readable.

We verify that the map produces clear patterns, not only by grouping the chromosomes of each species but also by the relative positioning of the different species.

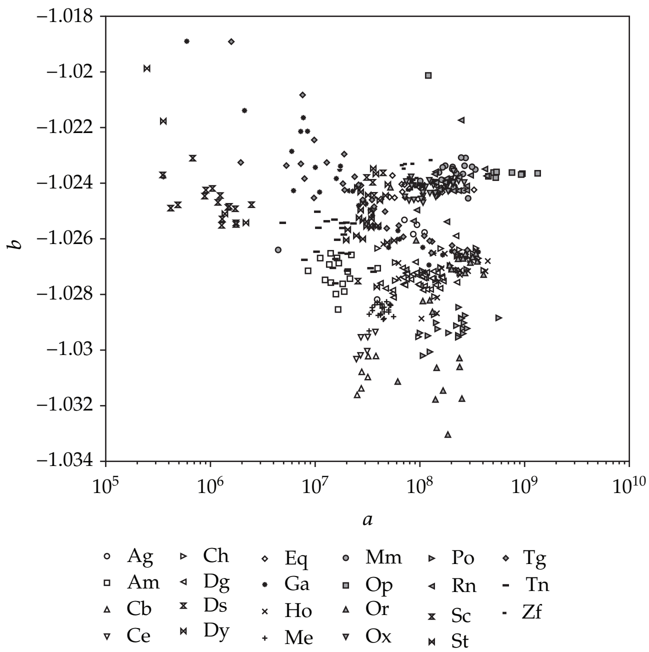

Figure 8: Locus (a, b) of the power law parameters for the 463 chromosomes and $n = \{1, \ldots, 8\}$.

Nevertheless, the large number of points complicates the visualization. Therefore, it was decided to represent each species by a single point having for coordinates the geometric and arithmetic averages of parameters a and b, respectively. Figure 9 depicts the resulting locus where is now easier to analyze the previously mentioned relations. The microchromosomes Ga32 and Tg16, which have a very small base pair counting, were not included in the calculations because they significantly disturb the results.

We verify the emergence of clusters that are in reasonable accordance with phylogenetics, going from the less "complex" species at left up to the most "complex" species at the right. The cluster of mammals is at the right and includes the subcluster of primates {Ho, Ch, Or}, with Ch closer to Hu than Or. In the rest of mammals it is interesting to see Po close to the primates and the position of the marsupial Op relatively distant from the placental mammals. In what concerns the rest of the points we notice Cb close to Ce and, in a middle position, the clusters of birds {Ga, Tg}, fishes {Tn, St, Me, Zf}, and insects {Dy, Ds, Am, Ag}.

In conclusion, the proposed information measure leads to an assertive and quantitative classification of chromosomes and species. Furthermore, it can be further explored for decoding in more detail other aspects of the DNA code in association with the FC tools.

4. Conclusions

Chromosomes have a code based on a four-symbol alphabet, and it can be analyzed with methods usually adopted in information processing. The information structure has resemblances to those occurring in systems characterized by fractional dynamics. Nevertheless, schemes based on assigning numerical values to the DNA symbols may deform the information, and alternative methods that avoid such problem need to be implemented. In this paper it was proposed a scheme based on the Shannon information theory. Bearing these ideas in

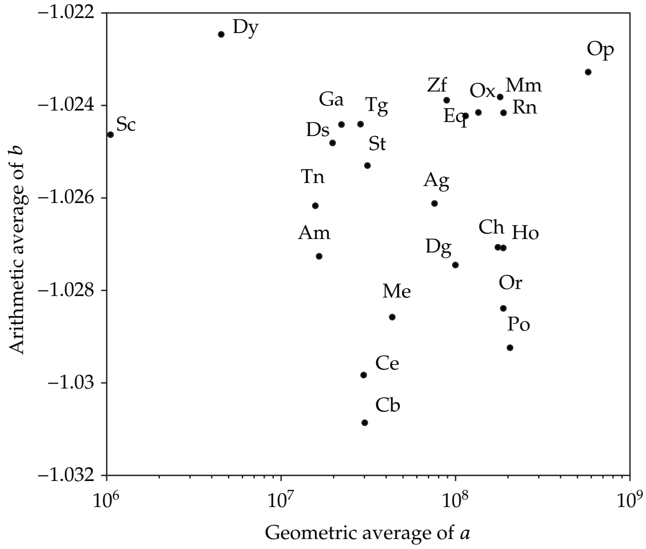

Figure 9: Locus of geometric average of a versus the arithmetic average of b for the twenty-three species.

mind, the chromosomes were processed in the perspective of a PL relationship between the average information and the total number of words, for distinct values of character encoding. For condensing the information an averaging of the PL parameters was also adopted. The resulting locus revealed the emergence of clearly interpretable patterns in accordance with current knowledge in phylogenetics. The proposed methodology opens new directions of research for DNA information processing and supports the recent discoveries that fractional phenomena are present in this biological structure.

Acknowledgments

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Research Article

Observer-Type Consensus Protocol for a Class of Fractional-Order Uncertain Multiagent Systems

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This paper investigates the consensus problem for a class of fractional-order uncertain multiagent systems with general linear node dynamics. Firstly, an observer-type consensus protocol is proposed based on the relative observer states of neighboring agents. Secondly, based on property of the Kronecker product and stability theory of fractional-order system, some sufficient conditions are presented for robust asymptotical stability of the observer-based fractional-order control systems. Thirdly, robust stabilizing controllers are derived by using linear matrix inequality approach and matrix's singular value decomposition. Our results are in the form of linear matrix inequalities which can easily be solved by LMI toolbox in MATLAB. Finally, a numerical simulation is performed to show the effectiveness of the theoretical results.

1. Introduction

The coordination problem of multiagent dynamical systems have attracted an increasing attention in recent years due to its applications in sensor networks, robotic teams, satellites formation [1–3]. Particularly, the consensus of multiagent systems, which has been extensively studied in the past few years [4–8].

Note that for most of aforementioned results, the agent dynamics is assumed to be first-order, second-order, or high-order integrators, which may be restrictive in many cases. Recently, consensus of multiagent systems with general linear node dynamics has received dramatic attention [9–11], the output feedback consensus problem is studies in [9] for high-order linear system, the consensus is reached if there exists a stable compensator. In [10], the consensus problem of multiagent systems with general form of linear dynamics is investigated under a time-invariant communication topology, an observer-type consensus protocol based on relative output measurements between neighboring agents has been proposed. Li et al. [11] investigate the consensus problems for multiagent systems with continuous-time and discrete-time linear dynamics, distributed reduce-order consensus protocols have been proposed based on the information of relative outputs of neighboring agents. In the above-mentioned works on consensus of multiagent systems, the systems are described by an integer-order dynamics, but many phenomena in nature cannot be explained in the framework of integer-order dynamics, for example, the synchronized motion of agents in fractional circumstances, such as molecule fluids and porous media, the stress-strain relationship demonstrates non-integer-order dynamics rather than integer-order dynamics [12]. In addition, fractional-order systems provide an excellent instrument for the description of memory and hereditary properties of various materials and processes, such as dielectric polarization, electrode-electrolyte polarization, and visco-elastic systems [13–16]. For the fractional-order dynamical systems, it is very difficult and inconvenient to construct Lyapunov functions, because there exist substantial differences between fractional-order differential systems and integer-order differential ones. As a way of efficiently solving the robust stability and stabilization problem, the linear matrix inequality approach is presented [17–20], which provides the designing method of state feedback controllers for fractional-order systems. A recent study [21] investigates the distributed formation control problem for multiple fractional-order systems under dynamic interaction, sufficient conditions on the network topology are given to ensure the formation control. In [21], the coordination algorithms for networked fractional-order systems are studied when the fixed interaction graph is directed, the coordination algorithms for integer-order system can be considered a special case of those for fractional-order systems. Only two of the above studies considered fractional-order multiagent systems. In addition, in previous results, static consensus protocols based on relative states of neighboring agents are used, which require the absolute output measurement of each agent to be available, which is impractical in many cases, the agent states in relatively large scale networks are not often completely available. Thus, in many application, one often needs to estimate the agent states through available measurements and then utilizes the estimated states to achieve certain design objectives such as consensus of multiagent systems, synchronization of complex networks [22–25]. To date, very little research effort has been done about the consensus problem for fractional-order uncertain multiagent systems. The purpose of our study is to fill this gap.

Motivated by the above discussion, the consensus problem is investigated for a class of fractional-order uncertain multiagent systems with general linear node dynamics. An observer-type consensus protocol is proposed based on the relative observer states of neighboring agents. Some sufficient conditions are presented for robust asymptotical stability of the observer-based fractional-order control systems, and robust stabilizing controllers are derived by using linear matrix inequality approach and matrix's singular value decomposition. Finally, an illustrative example is provided to demonstrate the effectiveness of the proposed approach.

The main novelties of this study are summarized as follows: (1) a novel observer-type consensus protocol is proposed based on the relative observer states of neighboring agents. Different from [26], the observer state is used instead of the agent's state in consensus protocol, and the dynamic behavior of multiagent is described by fractional-order system; (2) the uncertainty is considered in multiagent systems due to external disturbing factors such as environment temperature, voltage fluctuation, and mutual interfere among components; (3) the feedback gain matrices can be derived by matrix's singular value decomposition, and the consensus criteria are in the form of linear matrix inequalities which can be solved by applying the LMI toolbox.

The rest of this paper is organized as follows. In Section 2, preliminaries and problem statement are given. In Section 3, the consensus conditions are derived by using linear matrix inequality approach and matrix's singular value decomposition. In Section 4, a simulation example is provided to show the advantages of the obtained results. Conclusions are presented in Section 5.

2. Preliminaries and Problem Statement

2.1. Graph Theory Notions

Let $g = \{v, \varepsilon, A\}$ be a weighted directed graph of order N, with the set of nodes $v = \{v_1, v_2, \ldots, v_N\}$, an edge set $\varepsilon \subseteq v \times v$, and a weighted adjacency matrix $A = (a_{ij})_{N\times N}$ with $a_{ij} > 0$ if $(v_j, v_i) \in \varepsilon$ and $a_{ij} = 0$, otherwise. The neighbor set of node i is defined by $N_i = \{j \in v \mid (v_j, v_i) \in \varepsilon\}$, and the in-degree and out-degree of node i are defined as

$$\deg_{\text{in}}(i) = \sum_{j=1, j\neq i}^{N} a_{ij}, \qquad \deg_{\text{out}}(i) = \sum_{j=1, j\neq i}^{N} a_{ji}. \tag{2.1}$$

A diagraph is called balanced if $\deg_{\text{in}}(i) = \deg_{\text{out}}(i)$ for all $i \in v$.

The Laplacian matrix $L = (l_{ij})_{N\times N}$ associated with the adjacency matrix A is defined as

$$\begin{aligned} l_{ij} &= -a_{ij} \qquad (i \neq j), \\ l_{ii} &= -\sum_{j=1, j\neq i}^{N} a_{ij}, \qquad (i = 1, 2, \ldots, N). \end{aligned} \tag{2.2}$$

It is straightforward to verify that L has at least one zero eigenvalue with a corresponding eigenvalue with a corresponding eigenvector **1**, where **1** is an all-one column vector with a compatible size.

2.2. Caputo Fractional Operator

With the development of fractional calculus, it has been found that many physical systems show fractional dynamical behavior because of special materials and chemical properties, which can be described more accurately using fractional-order calculus than traditional integer-order calculus [27, 28]. Therefore, fractional-order calculus has become a hot research issue in recent years. There are many definitions of fractional derivatives [29–31], such as the Riemann-Liouville derivative and the Caputo derivative which are used in fractional systems. In physical systems, Caputo fractional derivative is more appropriate for describing the initial value problem of fractional differential equations, the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with clear physical interpretations [17]. Therefore, the following Caputo fractional operator is adopted in this paper for fractional derivatives of order α:

$$D^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)d\tau \quad (m-1<\alpha<m), \tag{2.3}$$

where $m \in \mathbb{Z}_{+}$, $\Gamma(\cdot)$ is a gamma function given by $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$.

In order to simulate the fractional-order multiagent systems, a predictor corrector algorithm is introduced as follows.

The fractional-order differential equation is given by

$$\begin{aligned} &D^{\alpha}x(t) = f(t, x(t)) \quad (0 \le t \le T, 0 < \alpha < 1), \\ &x^{(i)}(0) = x_0^{(i)} \quad (i = 0, 1, 2, \ldots, n-1) \end{aligned} \tag{2.4}$$

which is equivalent to the following Volterra integral equation:

$$x(t) = \sum_{i=0}^{\lceil a\rceil-1}\frac{t^i}{i!}x_0^{(i)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau, x(\tau))d\tau. \tag{2.5}$$

Set $h = T/N \ (N \in \mathbb{Z}_{+})$ and $t_n = nh(n = 1, 2, \ldots, N)$, where h is the step size, T is simulation time, and N is the number of sample points, (2.4) can be discretized as follows:

$$\begin{aligned} x_h(t_{n+1}) = &\sum_{i=0}^{\lceil a\rceil-1}\frac{t_{n+1}^i}{i!}x_0^{(i)} + \frac{h^{\alpha}}{\Gamma(\alpha+2)}f\left(t_{n+1}, x_h^p(t_{n+1})\right) \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n}a_{j,n+1}f(t_j, x_h(t_j)), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} x_h^p(t_{n+1}) &= \sum_{i=0}^{n-1}\frac{t_{n+1}^i}{i!}x_0^{(i)} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}b_{j,n+1}f(t_j, x_h(t_j)), \\ a_{j,n+1} &= \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha+1} & (j=0) \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & (1 \le j \le n) \\ 1 & (j = n+1), \end{cases} \\ b_{j,n+1} &= \frac{h^{\alpha}}{\alpha}\left((n-j+1)^{\alpha} - (n-j)^{\alpha}\right), \end{aligned} \tag{2.7}$$

the estimation error of this approximation is $e = \max_{j=0,1,\ldots,N}|x(t_j) - x_h(t_j)| = O(h^p)$, where $p = \min(2, 1+\alpha)$.

2.3. Problem Formulation

Consider multiagent systems of N identical linear dynamical systems, the dynamics of agent i is described by

$$\begin{gathered} D^{\alpha} x_i(t) = A x_i(t) + B u_i(t) \\ y_i(t) = C x_i(t) \quad (i = 1, 2, \ldots, N), \end{gathered} \tag{2.8}$$

where $0 < \alpha < 1$ is the fractional order, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)) \in \mathbb{R}^n$ is position state vector of ith agent, $u_i(t) = (u_{i1}(t), u_{i2}(t), \ldots, u_{im}(t)) \in \mathbb{R}^m$ is the control input, and $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{ip}(t)) \in \mathbb{R}^p$ is the measured output. A, B, and C are some real matrices with compatible dimensions, and

$$A = A_0 + \Delta A(t), \qquad B = B_0 + \Delta B(t), \tag{2.9}$$

where $\Delta A(t)$ and $\Delta B(t)$ represent the parameter uncertainties satisfying the following conditions:

$$\Delta A(t) = D_A F_A(t) E_A, \qquad \Delta B(t) = D_B F_B(t) E_B, \tag{2.10}$$

and D_A, E_A, D_B, and E_B are some constant matrices with appropriate dimensions, $F_A(t)$ and $F_B(t)$ are the uncertainties satisfying $F_A^T(t) F_A(t) \le I$, $F_B^T(t) F_B(t) \le I$, in which $I \in \mathbb{R}^{n \times n}$ denotes the identity matrix.

Remark 2.1. In [9–11], observer-based protocols are proposed for consensus of linear multiagent systems with reduced order, where the multiagent systems are described by integer-order systems, uncertainty is not considered. But in (2.1), every agent is required with the same dynamics and uncertainties, which have certain conservation. In order to reduce this conservation, we can consider a more general multiagent systems:

$$\begin{gathered} D^{\alpha} x_i(t) = A_i x_i(t) + B_i u_i(t) \\ y_i(t) = C_i x_i(t) \quad (i = 1, 2, \ldots, N), \\ A_i = A_{i0} + \Delta A_i(t), \qquad B_i = B_{i0} + \Delta B_i(t), \end{gathered} \tag{2.11}$$

where $\Delta A_i(t)$ and $\Delta B_i(t)$ represent the parameter uncertainties of ith agent satisfying the following conditions:

$$\Delta A_i(t) = D_{Ai} F_{Ai}(t) E_{Ai}, \qquad \Delta B_i(t) = D_{Bi} F_{Bi}(t) E_{Bi}, \tag{2.12}$$

for the above model, we can get similar results, but the mathematical treatment becomes more complicated, the important and meaningful research topics will be considered for our future research.

Based on the design idea of the observer, a Luenberger-type fractional-order linear observer is constructed as follows:

$$D^{\alpha}\hat{x}_i(t) = A\hat{x}_i(t) + Bu_i(t) + K_2\big(y_i(t) - \hat{y}_i(t)\big)\hat{y}_i(t) = C\hat{x}_i(t) \quad (i = 1, 2, \ldots, N), \tag{2.13}$$

where $\hat{x}_i(t)$ and $\hat{y}_i(t)$ are the state and the output of observer. $K_2 \in \mathbb{R}^{n\times p}$ is a feedback gain matrix to be determined later.

Similar to [26], observer-type consensus protocol based on the relative observer states between neighboring agents is given as

$$u_i(t) = K_1 \sum_{j\in N_i} a_{ij}\big(\hat{x}_i(t) - \hat{x}_j(t)\big) = -\sum_{j\in N_i} l_{ij} K_1 \hat{x}_j(t) \quad (i = 1, 2, \ldots, N), \tag{2.14}$$

where $K_1 \in \mathbb{R}^{p\times n}$ is the feedback gain matrix to be designed.

Remark 2.2. It should be noticed that the consensus protocol (2.14) is based on the relative states of the neighboring observers. To our best knowledge, it is novel for fractional-order multiagent systems.

Substituting (2.14) into (2.8), and (2.13) respectively, and using Kronecker product, (2.8) and (2.13) can be rewritten in the following compact form:

$$\begin{gathered}
D^{\alpha}x(t) = (I_N \otimes A)x(t) - (L \otimes BK_1)\hat{x}(t), \\
y(t) = (I_N \otimes C)x(t), \\
D^{\alpha}\hat{x}(t) = (I_N \otimes A - L \otimes BK_1)\hat{x}(t) + (I_N \otimes K_2C)(x(t) - \hat{x}(t)), \\
\hat{y}(t) = (I_N \otimes C)\hat{x}(t),
\end{gathered} \tag{2.15}$$

Let $e(t) = x(t) - \hat{x}(t)$, the closed-loop system is given by

$$\begin{gathered}
D^{\alpha}\hat{x}(t) = (I_N \otimes A - L \otimes BK_1)\hat{x}(t) + (I_N \otimes K_2C)e(t) \\
D^{\alpha}e(t) = (I_N \otimes A - I_N \otimes K_2C)e(t)
\end{gathered} \tag{2.16}$$

which can be rewritten as

$$D^{\alpha}X(t) = A_K X(t), \tag{2.17}$$

where

$$X(t) = \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix}, \qquad A_K = \begin{bmatrix} I_N \otimes A - L \otimes BK_1 & I_N \otimes K_2C \\ 0 & I_N \otimes A - I_N \otimes K_2C \end{bmatrix}. \tag{2.18}$$

In the following, an LMI-based design method is developed for the consensus of a class of linear fractional-order uncertain multiagents. Before giving the main results, the following definition and lemmas are introduced.

Definition 2.3 (see [26]). The consensus problem of multiagent systems (2.8) is solved by protocol (2.14) if the states of (2.8) satisfy

$$\lim_{t\to\infty}\left\|x_i(t)-x_j(t)\right\|=0 \quad (i=1,2,\ldots,N). \tag{2.19}$$

Lemma 2.4 (see [32]). *Let $a \in \mathbb{R}$ and A, B, C, D be matrices with appropriate dimensions. The following properties can be proved by the definition of Kronecker product:*

$$\begin{aligned}
&(1)\ a(A\otimes B)=(aA)\otimes B=A\otimes(aB),\\
&(2)\ (A\otimes B)^T=A^T\otimes B^T,\\
&(3)\ (A\otimes B)(C\otimes D)=(AC)\otimes(BD),\\
&(4)\ A\otimes B\otimes C=(A\otimes B)\otimes C=A\otimes(B\otimes C).
\end{aligned} \tag{2.20}$$

Lemma 2.5 (see [33]). *Let $A \in \mathbb{R}^{n\times n}$ be a real matrix, a necessary and sufficient condition for the asymptotical stability of $D^\alpha x(t) = Ax(t)$ is*

$$\left|\arg(\operatorname{spec}(A))\right| > \frac{\alpha\pi}{2}, \tag{2.21}$$

where $\operatorname{spec}(A)$ *is the spectrum of all eigenvalues of A, fractional order satisfying* $0 < \alpha < 2$.

Lemma 2.6 (see [18]). *Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha < 1$, then the fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable if and only if there exist two real symmetric positive definite matrices $P_{k1} \in \mathbb{R}^{n\times n} (k = 1, 2)$ and two skew-symmetric matrices $P_{k2} \in \mathbb{R}^{n\times n} (k = 1, 2)$, such that*

$$\begin{gathered}
\sum_{i=1}^{2}\sum_{j=1}^{2}\operatorname{sym}\left\{\Theta_{ij}\otimes\left(AP_{ij}\right)\right\}<0,\\
\begin{bmatrix} P_{11} & P_{12}\\ -P_{12} & P_{11}\end{bmatrix}>0 \qquad \begin{bmatrix} P_{21} & P_{22}\\ -P_{22} & P_{21}\end{bmatrix}>0,
\end{gathered} \tag{2.22}$$

where

$$\begin{aligned}
\Theta_{11}&=\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}, & \Theta_{12}&=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix},\\
\Theta_{21}&=\begin{bmatrix}\sin\theta & \cos\theta\\ -\cos\theta & \sin\theta\end{bmatrix}, & \Theta_{22}&=\begin{bmatrix}-\cos\theta & \sin\theta\\ -\sin\theta & -\cos\theta\end{bmatrix},
\end{aligned} \tag{2.23}$$

and $\theta = \alpha\pi/2$.

Lemma 2.7 (see[34]). *Let H and E be real matrices of appropriate dimensions with $F(t)$ satisfying $F^T(t)F(t) < I$, and there exists positive scalar $\varepsilon > 0$, such that*

$$HF(t)E+E^TF^T(t)H^T<\varepsilon HH^T+\varepsilon^{-1}E^TE. \tag{2.24}$$

Recall that for any matrix $\Pi \in \mathbb{R}^{m\times n}$ with full row rank, there exists a singular value decomposition of Π as follows:

$$\Pi = U[S \quad 0]V^T, \tag{2.25}$$

where $S \in \mathbb{R}^{m\times p}$ is a diagonal matrix with positive elements in decreasing order, $p = \min\{m, n\}$, $U \in \mathbb{R}^{p\times p}$ and $V \in \mathbb{R}^{n\times n}$ are unitary matrices. Then the following, lemma holds.

Lemma 2.8 (see [19]). *Given matrix $\Pi \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(\Pi) = p$, assume that $X \in \mathbb{R}^{n\times n}$ is a symmetric matrix, there exists a matrix $\overline{X} \in \mathbb{R}^{m\times n}$ satisfying $\Pi X = \overline{X}\Pi$ if and only if X can be expressed as*

$$X = V\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix} V^T, \tag{2.26}$$

where $X_{11} \in \mathbb{R}^{m\times m}$ and $X_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$.

Remark 2.9. As we known, the existing results about stability of fractional-order deterministic systems are based on Lemma 2.5, which cannot be applied directing to fractional-order uncertain systems, because this method needs to compute all eigenvalues of system (2.17), it is hard to yield all eigenvalues for fractional-order systems with uncertain parameters, the paper can effectively avoid this difficulty by using Lemma 2.6.

3. Main Results

In this section, a sufficient condition is first derived for robust asymptotic stability of uncertain fractional-order linear systems (2.17), based on the stability criterion, an LMI-based approach is proposed for designing observer-type consensus protocol for the uncertain fractional-order multiagent systems (2.8).

Theorem 3.1. *For given two feedback gain matrix $K_1 \in \mathbb{R}^{p\times n}$ and $K_2 \in \mathbb{R}^{n\times p}$, the uncertain fractional-order multiagent systems* (2.17) *are asymptotically stable if there exist two positive matrices $Q_1 \in \mathbb{R}^{n\times n}$, $Q_2 \in \mathbb{R}^{n\times n}$ and four real positive scalars a_1, a_2, b_1, and b_2 such that the following linear matrix inequality holds:*

$$\begin{bmatrix} \Pi_1 & * & * & * & * \\ \Pi_2 & -a_1 \otimes I_{4Nn} & * & * & * \\ \Pi_2 & 0 & -a_2 \otimes I_{4Nn} & * & * \\ \Pi_3 & 0 & 0 & -b_1 \otimes I_{4Nn} & * \\ \Pi_3 & 0 & 0 & 0 & -b_2 \otimes I_{4Nn} \end{bmatrix} < 0, \tag{3.1}$$

where

$$\Pi_1 = \begin{bmatrix} \Pi_{11} & * & * & * \\ \Pi_{21} & \Pi_{22} & * & * \\ 0 & 0 & \Pi_{11} & * \\ 0 & 0 & \Pi_{21} & \Pi_{22} \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} I_N \otimes E_A Q_1 & 0 & 0 & 0 \\ 0 & I_N \otimes E_A Q_2 & 0 & 0 \\ 0 & 0 & I_N \otimes E_A Q_1 & 0 \\ 0 & 0 & 0 & I_N \otimes E_A Q_2 \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} I_N \otimes E_B K_1 Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_N \otimes E_B K_1 Q_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \Pi_{11} &= 2\left(I_N \otimes A_0 Q_1 + I_N \otimes Q_1 A_0^T - L \otimes B_0 K_1 Q_1 - L^T \otimes Q_1 K_1^T B_0^T\right)\sin\theta \\ &\quad + (a_1 + a_2) I_N \otimes D_A D_A^T + (b_1 + b_2) L L^T \otimes D_B D_B^T, \\ \Pi_{21} &= 2\left(I_N \otimes Q_2 C^T K_2^T\right)\sin\theta, \\ \Pi_{22} &= 2\left(I_N \otimes A_0 Q_2 + I_N \otimes Q_2 A_0^T - I_N \otimes K_2 C Q_2 - I_N \otimes Q_2 C^T K_2^T\right)\sin\theta + (a_1 + a_2) \otimes D_A D_A^T. \end{aligned} \tag{3.2}$$

Proof. It follows from Lemma 2.6 that the system (2.17) is asymptotically stable if there exist two real symmetric positive definite matrices $P_{k1} \in \mathbb{R}^{2n\times 2n}$ $(k = 1,2)$ and two skew-symmetric matrices $P_{k2} \in \mathbb{R}^{2n\times 2n}$ $(k = 1,2)$ such that the following linear matrix inequality holds:

$$\Pi = \sum_{i=1}^{2}\sum_{j=1}^{2} \mathrm{sym}\left\{\theta_{ij} \otimes \left(A_K P_{ij}\right)\right\} < 0. \tag{3.3}$$

Setting $P_{11} = P_{21} = Q = \mathrm{diag}\{I_N \otimes Q_1, I_N \otimes Q_2\}$ and $P_{12} = P_{22} = 0$, then (3.3) will degenerate to the following inequality:

$$\Pi = \sum_{i=1}^{2} \mathrm{sym}\{\theta_{i1} \otimes (A_K Q)\} < 0, \tag{3.4}$$

then system (2.17) is asymptotically stable. By simple calculation, the following equality holds:

$$\begin{aligned} A_K Q &= \begin{bmatrix} I_N \otimes A_0 Q_1 - L \otimes B_0 K_1 Q_1 & I_N \otimes K_2 C Q_2 \\ 0 & I_N \otimes A_0 Q_2 - I_N \otimes K_2 C Q_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} I_N \otimes D_A F_A(t) E_A Q_1 & 0 \\ 0 & I_N \otimes D_A F_A(t) E_A Q_2 \end{bmatrix} + \begin{bmatrix} -L \otimes D_B F_B(t) E_B K_1 Q_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.5}$$

For the convenience of later analysis, we denote

$$\Delta_1 = \begin{bmatrix} I_N \otimes A_0 Q_1 - L \otimes B_0 K_1 Q_1 & I_N \otimes K_2 C Q_2 \\ 0 & I_N \otimes A_0 Q_2 - I_N \otimes K_2 C Q_2 \end{bmatrix}, \tag{3.6}$$

$$\Delta_2 = \begin{bmatrix} I_N \otimes D_A F_A(t) E_A Q_1 & 0 \\ 0 & I_N \otimes D_A F_A(t) E_A Q_2 \end{bmatrix}, \tag{3.7}$$

$$\Delta_3 = \begin{bmatrix} -L \otimes D_B F_B(t) E_B K_1 Q_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.8}$$

We rewrite (3.5) as follows:

$$\begin{aligned} \Pi &= \sum_{i=1}^{2} \mathrm{sym}\{\theta_{i1} \otimes (A_K Q)\} \\ &= \sum_{i=1}^{2} \mathrm{sym}\{\theta_{i1} \otimes \Delta_1\} + \sum_{i=1}^{2} \mathrm{sym}\{\theta_{i1} \otimes \Delta_2\} + \sum_{i=1}^{2} \mathrm{sym}\{\theta_{i1} \otimes \Delta_3\}. \end{aligned} \tag{3.9}$$

Based on Lemma 2.6, it follows from (3.5) that

$$\begin{aligned} \sum_{i=1}^{2} \mathrm{sym}\{\theta_{i1} \otimes \Delta_1\} &= \mathrm{sym}\left\{ \begin{bmatrix} 2\sin\theta & 0 \\ 0 & 2\sin\theta \end{bmatrix} \otimes \Delta_1 \right\} \\ &= \begin{bmatrix} \Pi_{11} - \Phi_1 & * & * & * \\ \Pi_{21} & \Pi_{22} - \Phi_2 & * & * \\ 0 & 0 & \Pi_{11} - \Phi_1 & * \\ 0 & 0 & \Pi_{21} & \Pi_{22} - \Phi_2 \end{bmatrix}, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \Phi_1 &= (a_1 + a_2) \otimes D_A D_A^T + (b_1 + b_2) L L^T \otimes D_B D_B^T, \\ \Phi_2 &= (a_1 + a_2) \otimes D_A D_A^T. \end{aligned} \tag{3.11}$$

Note that $\theta_{i1}\theta_{i1}^T = I_2$ $(i = 1, 2)$, for any real positive scalars a_1 and a_2, it follows from (3.9) and Lemma 2.7 that

$$\begin{aligned} &\sum_{i=1}^{2} \mathrm{sym}\{\theta_{i1} \otimes \Delta_2\} \\ &\quad = \sum_{i=1}^{2} \mathrm{sym}\left\{ \theta_{i1} \otimes \left\{ I_N \otimes \begin{bmatrix} D_A & 0 \\ 0 & D_A \end{bmatrix} \begin{bmatrix} F_A(t) & 0 \\ 0 & F_A(t) \end{bmatrix} \begin{bmatrix} E_A Q_1 & 0 \\ 0 & E_A Q_2 \end{bmatrix} \right\} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \operatorname{sym}\left\{ \theta_{i1} \otimes \left\{ \begin{bmatrix} I_N \otimes D_A & 0 \\ 0 & I_N \otimes D_A \end{bmatrix} \begin{bmatrix} I_N \otimes F_A(t) & 0 \\ 0 & I_N \otimes F_A(t) \end{bmatrix} \begin{bmatrix} I_N \otimes E_A Q_1 & 0 \\ 0 & I_N \otimes E_A Q_2 \end{bmatrix} \right\} \right\} \\
&= \sum_{i=1}^{2} \operatorname{sym}\left\{ \left(\theta_{i1} \otimes \begin{bmatrix} I_N \otimes D_A & 0 \\ 0 & I_N \otimes D_A \end{bmatrix} \right) \left(I_2 \otimes \begin{bmatrix} I_N \otimes F_A(t) & 0 \\ 0 & I_N \otimes F_A(t) \end{bmatrix} \right) \right. \\
&\qquad \left. \times \left(I_2 \otimes \begin{bmatrix} I_N \otimes E_A Q_1 & 0 \\ 0 & I_N \otimes E_A Q_2 \end{bmatrix} \right) \right\} \\
&\le \sum_{i=1}^{2} \left\{ a_i \left(\theta_{i1} \otimes \begin{bmatrix} I_N \otimes D_A & 0 \\ 0 & I_N \otimes D_A \end{bmatrix} \right) \left(\theta_{i1} \otimes \begin{bmatrix} I_N \otimes D_A & 0 \\ 0 & I_N \otimes D_A \end{bmatrix} \right)^T \right. \\
&\qquad \left. + a_i^{-1} \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A Q_1 & 0 \\ 0 & I_N \otimes D_A Q_2 \end{bmatrix} \right)^T \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A Q_1 & 0 \\ 0 & I_N \otimes D_A Q_2 \end{bmatrix} \right) \right\} \\
&= \sum_{i=1}^{2} \left\{ a_i \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A D_A^T & 0 \\ 0 & I_N \otimes D_A D_A^T \end{bmatrix} \right) \right. \\
&\qquad \left. + a_i^{-1} \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A Q_1 & 0 \\ 0 & I_N \otimes D_A Q_2 \end{bmatrix} \right)^T \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A Q_1 & 0 \\ 0 & I_N \otimes D_A Q_2 \end{bmatrix} \right) \right\}.
\end{aligned}
\tag{3.12}
$$

By the same argument, for any real positive scalars b_1 and b_2, one has

$$
\begin{aligned}
&\sum_{i=1}^{2} \operatorname{sym}\{\theta_{i1} \otimes \Delta_3\} \\
&\le \sum_{i=1}^{2} \left\{ b_i \left(I_2 \otimes \begin{bmatrix} LL^T \otimes D_B D_B^T & 0 \\ 0 & 0 \end{bmatrix} \right) \right. \\
&\qquad \left. + b_i^{-1} \left(I_2 \otimes \begin{bmatrix} I_N \otimes E_B K_1 Q_1 & 0 \\ 0 & 0 \end{bmatrix} \right)^T \left(I_2 \otimes \begin{bmatrix} I_N \otimes E_B K_1 Q_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right\}.
\end{aligned}
\tag{3.13}
$$

Combined with (3.10)–(3.13), we can obtain

$$
\begin{aligned}
\Pi \le{} & \operatorname{sym}\left\{ \begin{bmatrix} 2\sin(\theta) & 0 \\ 0 & 2\sin(\theta) \end{bmatrix} \otimes \Delta_1 \right\} + \sum_{i=1}^{2} \left\{ a_i \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A D_A^T & 0 \\ 0 & I_N \otimes D_A D_A^T \end{bmatrix} \right) \right\} \\
& + \sum_{i=1}^{2} \left\{ b_i \left(I_2 \otimes \begin{bmatrix} LL^T \otimes D_B D_B^T & 0 \\ 0 & 0 \end{bmatrix} \right) \right\} \\
& + \sum_{i=1}^{2} a_i^{-1} \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A Q_1 & 0 \\ 0 & I_N \otimes D_A Q_2 \end{bmatrix} \right)^T \left(I_2 \otimes \begin{bmatrix} I_N \otimes D_A Q_1 & 0 \\ 0 & I_N \otimes D_A Q_2 \end{bmatrix} \right) \\
& + \sum_{i=1}^{2} b_i^{-1} \left(I_2 \otimes \begin{bmatrix} I_N \otimes E_B K_1 Q_1 & 0 \\ 0 & 0 \end{bmatrix} \right)^T \left(I_2 \otimes \begin{bmatrix} I_N \otimes E_B K_1 Q_1 & 0 \\ 0 & 0 \end{bmatrix} \right),
\end{aligned}
\tag{3.14}
$$

we can immediately obtain (3.14) from (3.1) by using the Schur complement. This completes the proof. □

Remark 3.2. If K_1 and K_2 are not given beforehand, owing to the existence of the nonlinear terms such as K_1Q_1, K_2CQ_2, the matrix inequality (3.1) is not an LMI. However, applying Lemma 2.8, it can be transformed into an LMI, and the main results are given in the following theorem.

Theorem 3.3. *Assume that singular value decomposition of output matrix C with full-row rank is $C = U[S\ \ 0]V^T$, then the closed-loop control system* (2.17) *under the observer-type consensus protocol* (2.14) *is robust asymptotically stable, if there exist symmetric positive matrices $Q_1 \in \mathbb{R}^{n\times n}$, $Q_{11} \in \mathbb{R}^{p\times p}$, $Q_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$ and two matrices $X_1 \in \mathbb{R}^{p\times n}$, $X_2 \in \mathbb{R}^{n\times p}$, and four real positive scalars a_1, a_2, b_1, and b_2 such that the following linear matrix inequality holds:*

$$\begin{bmatrix} \overline{\Pi}_1 & * & * & * & * \\ \Pi_2 & -a_1 \otimes I_{4Nn} & * & * & * \\ \Pi_2 & 0 & -a_2 \otimes I_{4Nn} & * & * \\ \overline{\Pi}_3 & 0 & 0 & -b_1 \otimes I_{4Nn} & * \\ \overline{\Pi}_3 & 0 & 0 & 0 & -b_2 \otimes I_{4Nn} \end{bmatrix} < 0, \tag{3.15}$$

where

$$\begin{gathered} \overline{\Pi}_1 = \begin{bmatrix} \overline{\Pi}_{11} & * & * & * \\ \overline{\Pi}_{21} & \overline{\Pi}_{22} & * & * \\ 0 & 0 & \overline{\Pi}_{11} & * \\ 0 & 0 & \overline{\Pi}_{21} & \overline{\Pi}_{22} \end{bmatrix}, \\ \overline{\Pi}_3 = \begin{bmatrix} I_N \otimes E_B X_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_N \otimes E_B X_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \overline{\Pi}_{11} = 2\left(I_N \otimes A_0 Q_1 + I_N \otimes Q_1 A_0^T - L \otimes B_0 X_1 - L^T \otimes X_1^T B_0^T\right)\sin\theta \\ + (a_1 + a_2) I_N \otimes D_A D_A^T + (b_1 + b_2) L L^T \otimes D_B D_B^T, \\ \overline{\Pi}_{21} = 2\left(I_N \otimes C^T X_2^T\right)\sin\theta, \\ \overline{\Pi}_{22} = 2\left(I_N \otimes A_0 Q_2 + I_N \otimes Q_2 A_0^T - I_N \otimes X_2 C - I_N \otimes C^T X_2^T\right)\sin\theta + (a_1 + a_2) \otimes D_A D_A^T, \\ Q_2 = V\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} V^T, \end{gathered} \tag{3.16}$$

then the feedback gain matrices are given by

$$K_1 = X_1 Q_1^{-1}, \qquad K_2 = X_2 U S Q_{11}^{-1} S^{-1} U^{-1}. \tag{3.17}$$

Proof. Since $C = U[S\ \ 0]V^T$ and $Q_2 = V\left[\begin{smallmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{smallmatrix}\right]V^T$, from Lemma 2.5, there exists $\overline{Q}_2$, such that $CQ_2 = \overline{Q}_2 C$, where $\overline{Q}_2 = USQ_{11}S^{-1}U^{-1}$, it is easy derived that $\overline{Q}_2^{-1} = USQ_{11}^{-1}S^{-1}U^{-1}$. Setting $K_1Q_1 = X_1$ and $K_2\overline{Q}_2 = X_2$, and (3.1) is inequivalent to (3.15). Moreover, the feedback gain matrices are obtained by

$$\begin{aligned} K_1 &= X_1 Q_1^{-1}, \\ K_2 &= X_2 \overline{Q}_2^{-1} = X_2 USQ_{11}^{-1}S^{-1}U^{-1}. \end{aligned} \tag{3.18}$$

□

Remark 3.4. Linear matrix inequality technique has attracted much more attention and has wide applications because of its high performance in analysis and design in the control systems, which in particular has a better advantage in dealing with uncertain systems. Noting that Theorem 3.3 provides an LMI-based method of designing feed-back gain matrices, that is, K_1 and K_2 can easily be obtained by LMI toolbox.

In particular, if the position state of multiagent can be obtained, then we can select the following consensus protocol:

$$u_i(t) = -\sum_{j\in N_i} l_{ij} K x_j(t) \quad (i = 1, 2 \ldots, N), \tag{3.19}$$

substituting (3.19) into (2.8), we have

$$\begin{aligned} D^\alpha x(t) &= (I_N \otimes A - L \otimes BK)x(t), \\ y(t) &= (I_N \otimes C)x(t). \end{aligned} \tag{3.20}$$

Based on (3.20), the following consensus criteria can derived without proof.

Corollary 3.5. *The fractional-order uncertain multiagent systems* (3.20) *can achieve consensus by protocol* (3.19) *if there exist symmetric positive matrix $Q \in \mathbb{R}^{n\times n}$ and a matrix $X \in \mathbb{R}^{p\times n}$, and four real-positive scalars a_1, a_2, b_1, and b_2, such that*

$$\begin{bmatrix} \widehat{\Pi}_1 & * & * & * & * \\ \widehat{\Pi}_2 & -a_1 \otimes I_{2Nn} & * & * & * \\ \widehat{\Pi}_2 & 0 & -a_2 \otimes I_{2Nn} & * & * \\ \widehat{\Pi}_3 & 0 & 0 & -b_1 \otimes I_{2Nn} & * \\ \widehat{\Pi}_3 & 0 & 0 & 0 & -b_2 \otimes I_{2Nn} \end{bmatrix} < 0, \tag{3.21}$$

where

$$\widehat{\Pi}_1 = \begin{bmatrix} \widehat{\Pi}_{11} & * \\ 0 & \widehat{\Pi}_{11} \end{bmatrix},$$

$$\widehat{\Pi}_2 = \begin{bmatrix} I_N \otimes E_A Q & 0 \\ 0 & I_N \otimes E_A Q \end{bmatrix},$$

$$\widehat{\Pi}_3 = \begin{bmatrix} I_N \otimes E_B X & 0 \\ 0 & I_N \otimes E_A X \end{bmatrix},$$

$$\widehat{\Pi}_{11} = 2\left(I_N \otimes A_0 Q + I_N \otimes Q A_0^T - L \otimes B_0 X - L^T \otimes X^T B_0^T\right) \sin\theta, \tag{3.22}$$

then the feedback gain matrix is given as follows:

$$K = XQ^{-1}. \tag{3.23}$$

4. A Numerical Example

In this section, a numerical example is given to verify the effectiveness of proposed observer-type consensus protocol in the preceding section.

Example 4.1. Consider fractional-order uncertain multiagent systems consisting of four agents, the interaction diagraph is shown in Figure 1 with the weights on the connections. The related parameters are given as follows:

$$A_0 = \begin{bmatrix} -0.4 & 0.35 & -0.65 \\ -0.9 & -2.7 & 1.1 \\ -0.5 & -1.35 & -2.25 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.45 \\ 0.75 \\ 0.75 \end{bmatrix} \quad C = \begin{bmatrix} 1.5 \\ 2 \\ 1 \end{bmatrix}^T,$$
$$D_A = \begin{bmatrix} -0.1 & 0.05 & 0.1 \\ -0.1 & -0.3 & 0.1 \\ -0.15 & -0.08 & -0.4 \end{bmatrix}, \quad E_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_B = \begin{bmatrix} -0.01 & 0.05 & 0.01 \\ -0.15 & -0.08 & -0.04 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.1}$$
$$E_B = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.15 \end{bmatrix}, \quad F_A(t) = F_B(t) = \begin{bmatrix} \sin(0.1\pi t) & 0 & 0 \\ 0 & \cos(0.1\pi t) & 0 \\ 0 & 0 & \sin(0.1\pi t) \end{bmatrix}.$$

For given fractional-order $\alpha = 0.9$, by using the Matlab LMI toolbox in Theorem 3.3, the feasible solution (3.15) is given as follows:

$$Q_1 = \begin{bmatrix} 6.8316 & -2.7412 & -0.5239 \\ -2.7412 & 5.3247 & 0.6660 \\ -0.5237 & 0.6660 & 5.2857 \end{bmatrix}, \quad Q_{11} = 11.7744, \quad Q_{22} = \begin{bmatrix} 8.4600 & 1.9039 \\ 1.9039 & 6.1330 \end{bmatrix}$$
$$X_1 = \begin{bmatrix} 0.0701 & -0.0483 & -0.1461 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.8652 & -2.9990 & -2.7827 \end{bmatrix}^T \tag{4.2}$$

the gain matrices are given as follows:

$$K_1 = \begin{bmatrix} 0.0075 & -0.0019 & -0.0267 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0735 & -0.2547 & -0.2363 \end{bmatrix}^T. \tag{4.3}$$

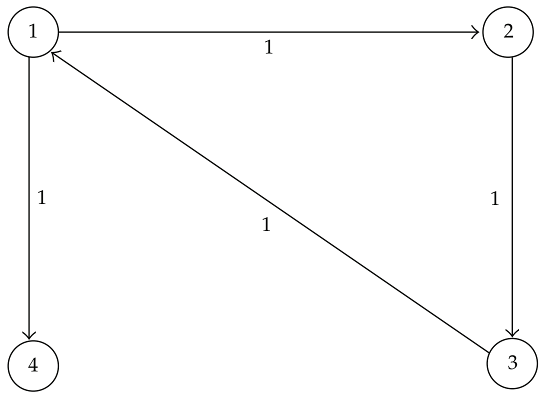

Figure 1: The topology structure of the agents.

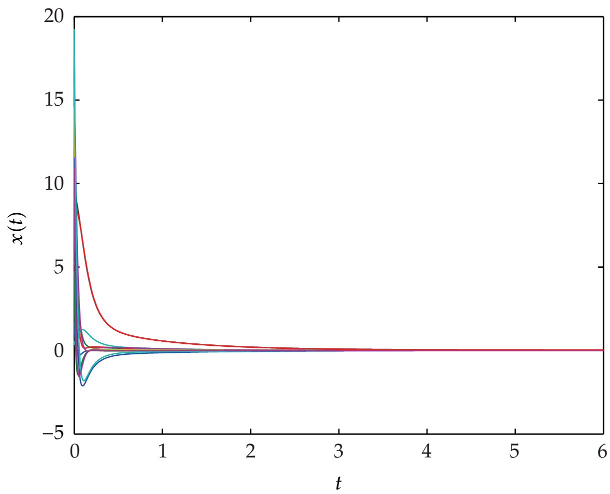

Figure 2: The position curves $x_i(t)\quad(i = 1, 2, 3, 4)$ for fractional-order $\alpha = 0.9$.

In the numerical simulation, the position state curves of four multiagents are shown in Figure 2 with random initial state values, it can be seen that the consensus is achieved, and the response of the error $e(t)$ between the position state $x(t)$ and its estimate $\hat{x}(t)$ are shown in Figure 3, which converge to zero. Therefore, the numerical simulation perfectly supports our theoretical results. To provided relatively complete information, the gain matrices are also listed in Table 1 for different fractional orders.

Remark 4.2. If fractional order $\alpha = 1$, multiagent systems (2.8) reduce to integer-order multiagent systems as follows:

$$\begin{gathered} Dx_i(t) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t) \quad (i = 1, 2, \ldots, N), \end{gathered} \tag{4.4}$$

similar to the process of Section 2, we can obtain the following augment system:

$$DX(t) = A_K X(t). \tag{4.5}$$

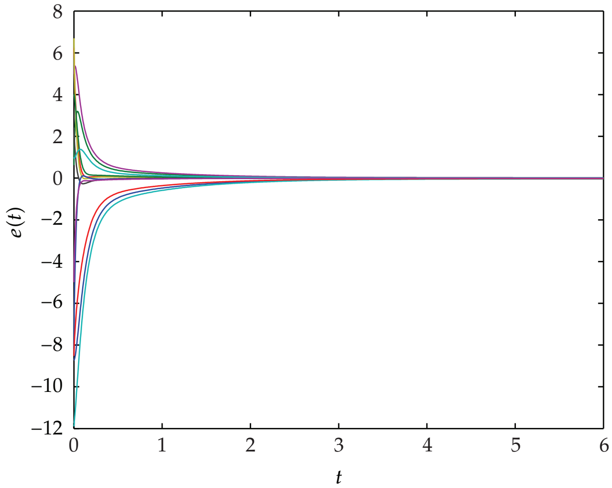

Figure 3: the error response $e(t)$ between the position $x(t)$ and its estimate $\hat{x}(t)$ for fractional-order $\alpha = 0.9$.

Table 1: Controller gain matrices K_1 and K_2 for different fractional order α.

Fractional orders α	Feedback gain matrices K_1 and K_2
0.4	$K_1 = [0.0031\ \ -0.0136\ \ -0.0953]$, $K_2 = [0.0137\ \ -0.2220\ \ -0.1835]^T$
0.5	$K_1 = [0.0173\ \ -0.0001\ \ -0.0160]$, $K_2 = [0.0728\ \ -0.2379\ \ -0.2279]^T$
0.6	$K_1 = [0.0127\ \ -0.0006\ \ -0.0255]$, $K_2 = [0.0686\ \ -0.2462\ \ -0.2304]^T$
0.7	$K_1 = [0.0097\ \ -0.0012\ \ -0.0281]$, $K_2 = [0.0691\ \ -0.2507\ \ -0.2326]^T$
0.8	$K_1 = [0.0082\ \ -0.0017\ \ -0.0275]$, $K_2 = [0.0716\ \ -0.2533\ \ -0.2348]^T$
0.9	$K_1 = [0.0075\ \ -0.0019\ \ -0.0267]$, $K_2 = [0.0735\ \ -0.2547\ \ -0.2363]^T$
0.99	$K_1 = [-0.0098\ \ -0.0118\ 0.0852]$, $K_2 = [0.2423\ \ -0.1882\ \ -0.2746]^T$
0.999	$K_1 = [-0.0098\ \ -0.0118\ 0.0852]$, $K_2 = [0.2423\ \ -0.1882\ \ -0.2746]^T$
0.9999	$K_1 = [-0.0098\ \ -0.0118\ 0.0852]$, $K_2 = [0.2423\ \ -0.1882\ \ -0.2746]^T$
$\alpha \to 1^-$	$K_1 = [-0.0098\ \ -0.0118\ 0.0852]$, $K_2 = [0.2423\ \ -0.1882\ \ -0.2746]^T$

Construct the following Lyapunov function:

$$V(t) = X^T(t)PX(t), \tag{4.6}$$

where $P = \operatorname{diag}\{I_N \otimes P_1, I_N \otimes P_2\}$.

We can easily obtain similar results with Theorem 3.1 and Theorem 3.3, by using LMI toolbox in MATLAB, two unknown matrices can be obtained as follows:

$$K_1 = \begin{bmatrix}-0.0098 & -0.0118 & 0.0852\end{bmatrix}, \qquad K_2 = \begin{bmatrix}0.2423 & -0.1882 & -0.2746\end{bmatrix}^T, \tag{4.7}$$

the results are consistent with corresponding fractional-order case in Table 1, when $\alpha \to 1^-$.

5. Conclusions

The study investigates the consensus for a class of fractional-order uncertain multiagent systems. The consensus criteria are derived by applying the observer-type consensus protocol and stability theory of the fractional-order system, and these criteria are in the form of linear matrix inequalities which can be readily solved by applying the LMI toolbox. A numerical example is provided to demonstrate the validity of the presented consensus protocol.

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Research Article

Positive Solutions for Nonlinear Fractional Differential Equations with Boundary Conditions Involving Riemann-Stieltjes Integrals

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We consider the existence of positive solutions for a class of nonlinear integral boundary value problems for fractional differential equations. By using some fixed point theorems, the existence and multiplicity results of positive solutions are obtained. The results obtained in this paper improve and generalize some well-known results.

1. Introduction

This paper is concerned with the existence of positive solutions to the following boundary value problem (BVP) for fractional differential equation:

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f\left(t,u(t),u'(t),\dots,u^{(n-2)}(t)\right)=0,\quad 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ u^{(n-2)}(1)=\beta\left[u^{(n-2)}\right], \end{gathered}\tag{1.1}$$

where $\beta[v] = \int_0^1 v(t)dA(t)$ is a linear functional on $C[0,1]$ given by a Riemann-Stieltjes integral with A representing a suitable function of bounded variation, D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $n-1<\alpha\le n$, $n\ge 2$, $f:[0,1]\times\mathbb{R}^{n-1}\longrightarrow\mathbb{R}^{+}$ satisfies the Carathéodory type conditions, $\mathbb{R}=(-\infty,+\infty)$ and $\mathbb{R}^{+}=[0,+\infty)$.

Fractional differential equations arise in the modeling and control of many real-world systems and processes particularly in the fields of physics, chemistry, aerodynamics, electrodynamics of complex media, and polymer rheology. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Hence, intensive research has been carried out worldwide to study the existence of solutions of nonlinear fractional differential equations (see [1–25]). For example, by means of a mixed monotone method, Zhang [11] studied a unique positive solution for the singular boundary value problem

$$\begin{gathered} D_{0+}^{\alpha}u(t)+q(t)f\left(t,u(t),u'(t),\dots,u^{(n-2)}(t)\right)=0, \quad 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=u^{(n-2)}(1)=0, \end{gathered} \tag{1.2}$$

where $\alpha\in(n-1,n]$, $n\geq 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f=g+h$ is nonlinear, and g and h have different monotone properties.

Recently, nonlocal boundary value problems for fractional differential equations were investigated intensively [13–23]. In [14], Bai concerned the existence and uniqueness of a positive solution for the following nonlocal problem:

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \\ u(0)=0, \qquad \beta u(\eta)=u(1), \end{gathered} \tag{1.3}$$

where $1<\alpha\leq 2$, $0<\beta\eta^{\alpha-1}<1$, $0<\eta<1$, D_{0+}^{α} is the standard Riemann-Liouville differentiation. The function f is continuous on $[0,1]\times\mathbb{R}^{+}$.

In [20], El-Shahed and Nieto investigated the existence of nontrivial solutions for the following nonlinear m-point boundary value problem of fractional type:

$$\begin{gathered} {}_{R}D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad t\in[0,1],\ \alpha\in(n-1,n],\ n\in\mathbb{N}, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \qquad u(1)=\sum_{i=1}^{m-2}a_iu(\eta_i), \end{gathered} \tag{1.4}$$

where $n\geq 2$, $a_i>0$ $(i=1,2,\dots,m-2)$, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$. Also the authors considered the analogous problem using the Caputo fractional derivative:

$$\begin{gathered} {}_{C}D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad t\in[0,1],\ \alpha\in(n-1,n],\ n\in\mathbb{N}, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \qquad u(1)=\sum_{i=1}^{m-2}a_iu(\eta_i). \end{gathered} \tag{1.5}$$

Under certain growth conditions on the nonlinearity, several sufficient conditions for the existence of nontrivial solution are obtained by using the Leray-Schauder nonlinear alternative.

Inspired by the work of the above papers, the aim of this paper is to establish the existence and multiplicity of positive solutions of the BVP (1.1). We discuss the boundary value problem with the Riemann-Stieltjes integral boundary conditions, that is, the BVP (1.1), which includes fractional order two-point, three-point, multipoint, and nonlocal boundary value problems as special cases. Moreover, the $\beta[\cdot]$ in (1.1) is a linear function on $C[0,1]$ denoting the Riemann-Stieltjes integral; the A in the Riemann-Stieltjes integral is of bounded variation, namely, dA can be a signed measure. By using the Krasnosel'skii fixed point theorem, the Leray-Schauder nonlinear alternative and the Leggett-Williams fixed point theorem, some existence and multiplicity results of positive solutions are obtained.

The rest of this paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence and multiplicity of positive solutions of the BVP (1.1) are established by using some fixed point theorems. In Section 4, we give four examples to demonstrate the application of our theoretical results.

2. Basic Definitions and Preliminaries

We begin this section with some preliminaries of fractional calculus. Let $\alpha > 0$ and $n = [\alpha]+1$, where $[\alpha]$ is the largest integer smaller than or equal to α. For a function $f : (0,+\infty) \to R$, we define the fractional integral of order α of f as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \tag{2.1}$$

provided the integral exists. The fractional derivative of order $\alpha > 0$ of a continuous function f is defined by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds, \tag{2.2}$$

provided the right-hand side is pointwise defined on $(0,+\infty)$. We recall the following properties [26, 27] which are useful for the sequel. For $\alpha > 0$, $\beta > 0$, we have

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t), \qquad D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t). \tag{2.3}$$

As an example, we can choose a function f such that f, $D_{0+}^{\alpha} f \in C(0,+\infty) \cap L_{\mathrm{loc}}^1(0,+\infty)$.

For $\alpha > 0$, the general solution of the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ with $u \in C(0,1) \cap L(0,1)$ is given by

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \cdots + c_n t^{\alpha-n}, \tag{2.4}$$

where $c_i \in R$ $(i = 1,2,\ldots,n)$. Hence for $u \in C(0,1) \cap L(0,1)$, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \cdots + c_n t^{\alpha-n}. \tag{2.5}$$

Set

$$G_0(t,s)=\frac{1}{\Gamma(\alpha-n+2)}\begin{cases}[t(1-s)]^{\alpha-n+1}-(t-s)^{\alpha-n+1}, & 0\le s\le t\le 1,\\ [t(1-s)]^{\alpha-n+1}, & 0\le t\le s\le 1.\end{cases} \tag{2.6}$$

Lemma 2.1 (see [11]). *Let* $y\in C_r[0,1]$ $(C_r[0,1]=\{y\in C[0,1],\ t^r y\in C[0,1],\ 0\le r<1\})$. *Then the boundary value problem,*

$$\begin{gathered}D_{0+}^{\alpha-n+2}v(t)=y(t),\quad 0<t<1,\ n-1<\alpha\le n,\ n\ge 2,\\ v(0)=0,\qquad v(1)=0,\end{gathered} \tag{2.7}$$

has a unique solution

$$v(t)=\int_0^1 G_0(t,s)y(s)\,ds. \tag{2.8}$$

Lemma 2.2 (see [11]). *The function* $G_0(t,s)$ *defined by* (2.6) *satisfies the following properties:*

(i) $G_0(t,s)\ge 0$, $G_0(t,s)\le G_0(s,s)$ *for all* $t,s\in[0,1]$;

(ii) *there exist a positive function* $\rho\in C(0,1)$ *and* $0<\xi<\eta<1$ *such that*

$$\min_{t\in[\xi,\eta]}G_0(t,s)\ge\rho(s)G_0(s,s),\quad s\in(0,1), \tag{2.9}$$

where

$$\rho(s)=\begin{cases}\dfrac{[\eta(1-s)]^{\alpha-n+1}-(\eta-s)^{\alpha-n+1}}{[s(1-s)]^{\alpha-n+1}}, & s\in(0,r],\\ \left(\dfrac{\xi}{s}\right)^{\alpha-n+1}, & s\in[r,1).\end{cases} \tag{2.10}$$

By (2.4), the unique solution of the problem

$$D_{0+}^{\alpha-n+2}v(t) = 0, \quad 0 < t < 1, \ n-1 < \alpha \leq n, \ n \geq 2, \\ v(0) = 0, \qquad v(1) = \beta[v] \tag{2.11}$$

is $\gamma(t) = t^{\alpha-n+1}$, with $\beta[v]$ replaced by 1. As in [21], the Green's function for boundary value problem (2.11) is given by

$$G(t,s) = \frac{\gamma(t)}{1-\beta[\gamma]}\mathcal{G}(s) + G_0(t,s), \tag{2.12}$$

where $\mathcal{G}(s) := \int_0^1 G_0(t,s)dA(t)$.

Lemma 2.3. *Let $0 \leq \beta[\gamma] < 1$ and $\mathcal{G}(s) \geq 0$ for $s \in [0,1]$, the Green function $G(t,s)$ defined by* (2.12) *has the following properties:*

(i) $G(t,s) \geq 0$, $G(t,s) \leq (1+\beta[1]/(1-\beta[\gamma]))G_0(s,s)$ *for all* $t,s \in [0,1]$;

(ii) $\min_{t\in[\xi,\eta]} G(t,s) \geq \min_{t\in[\xi,\eta]} G_0(t,s) \geq \rho(s)G_0(s,s)$, $s \in (0,1)$.

Proof. By Lemma 2.2, it is easy to prove this lemma, so we omit it. □

Let $X = C[0,1]$. It follows that $(X, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined by the supernorm $\|x\| = \sup_{t\in[0,1]}|x(t)|$. $P = \{x \in X : x(t) \geq 0, t \in [0,1]\}$. Clearly P is a cone of X. Now, in the following, we give the assumptions to be used throughout the rest of this paper.

(H_1) A is a function of bounded variation, $\mathcal{G}(s) \geq 0$ for $s \in [0,1]$ and $0 \leq \beta[\gamma] < 1$.

(H_2) $f : [0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}^+$ satisfies the following conditions of Carathéodory type:

(i) $f(\cdot, x)$ is Lebesgue measurable for each fixed $x \in \mathbb{R}^{n-1}$;

(ii) $f(t,\cdot)$ is continuous for a.e. $t \in [0,1]$.

In order to overcome the difficulty due to the dependance of f on derivatives, we consider the following modified problem:

$$D_{0+}^{\alpha-n+2}v(t) + f\left(t, I_{0+}^{n-2}v(t), I_{0+}^{n-3}v(t), \ldots, I_{0+}^{1}v(t), v(t)\right) = 0, \quad 0 < t < 1, \\ v(0) = 0, \qquad v(1) = \beta[v], \tag{2.13}$$

where $n-1 < \alpha \leq n$, $n \geq 2$.

Lemma 2.4. *The nonlocal fractional order boundary value problem* (1.1) *has a positive solution if and only if the nonlinear fractional integrodifferential equation* (2.13) *has a positive solution.*

Proof. If u is a positive solution of the fractional order boundary value problem (1.1), let $v(t) = D_{0+}^{n-2}u(t)$. Then from the boundary value conditions of (1.1) and the definition of the Riemann-Liouville fractional integral and derivative, we have

$$\begin{aligned}
v(t) &= D_{0+}^{n-2}u(t) = u^{(n-2)}(t),\\
I_{0+}^{1}v(t) &= I_{0+}^{1}u^{(n-2)}(t) = \frac{1}{\Gamma(1)}\int_0^t u^{(n-2)}(s)\,ds = u^{(n-3)}(t),\\
I_{0+}^{2}v(t) &= I_{0+}^{2}u^{(n-2)}(t) = \frac{1}{\Gamma(2)}\int_0^t (t-s)u^{(n-2)}(s)\,ds = u^{(n-4)}(t),\\
&\vdots\\
I_{0+}^{n-2}v(t) &= I_{0+}^{n-2}u^{(n-2)}(t) = \frac{1}{\Gamma(n-2)}\int_0^t (t-s)^{n-3}u^{(n-2)}(s)\,ds = u(t),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
D_{0+}^{\alpha-n+2}v(t) &= \frac{1}{\Gamma(2n-\alpha-2)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{2n-\alpha-3}u^{(n-2)}(s)\,ds\\
&= \frac{1}{\Gamma(2n-\alpha-3)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{2n-\alpha-4}u^{(n-3)}(s)\,ds\\
&= \cdots\\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}u(s)\,ds\\
&= D_{0+}^{\alpha}u(t),
\end{aligned} \tag{2.15}$$

which imply that $v(0) = u^{(n-2)}(0) = 0$, $v(1) = u^{(n-2)}(1) = \beta[u^{(n-2)}] = \beta[v]$. Thus $v(t)$ is a positive solution of the nonlinear fractional integrodifferential equation (2.13).

On the other hand, if v is a positive solution of the nonlinear fractional integrodifferential equation (2.13), let $u(t) = I_{0+}^{n-2}v(t)$, then by (2.3) and the definition of the Riemann-Liouville fractional derivative, we have

$$\begin{aligned}
u'(t) &= D_{0+}^{1}u(t) = D_{0+}^{1}I_{0+}^{n-2}v(t) = D_{0+}^{1}I_{0+}^{1}I_{0+}^{n-3}v(t) = I_{0+}^{n-3}v(t),\\
u''(t) &= D_{0+}^{2}u(t) = D_{0+}^{2}I_{0+}^{n-2}v(t) = D_{0+}^{2}I_{0+}^{2}I_{0+}^{n-4}v(t) = I_{0+}^{n-4}v(t),\\
&\vdots\\
u^{(n-3)}(t) &= D_{0+}^{n-3}u(t) = D_{0+}^{n-3}I_{0+}^{n-2}v(t) = D_{0+}^{n-3}I_{0+}^{n-3}I_{0+}^{1}v(t) = I_{0+}^{1}v(t),\\
u^{(n-2)}(t) &= D_{0+}^{n-2}u(t) = D_{0+}^{n-2}I_{0+}^{n-2}v(t) = v(t),
\end{aligned}$$

$$\begin{aligned}
D_{0+}^{\alpha}u(t) &= \frac{d^n}{dt^n}I_{0+}^{n-\alpha}u(t) = \frac{d^n}{dt^n}I_{0+}^{n-\alpha}I_{0+}^{n-2}v(t) = \frac{d^n}{dt^n}I_{0+}^{2n-\alpha-2}v(t) = D_{0+}^{\alpha-n+2}v(t) \\
&= -f\left(t, I_{0+}^{n-2}v(t), I_{0+}^{n-3}v(t), \dots, I_{0+}^{1}v(t), v(t)\right) \\
&= -f\left(t, u(t), u'(t), \dots, u^{(n-3)}(t), u^{(n-2)}(t)\right), \quad 0 < t < 1,
\end{aligned} \tag{2.16}$$

which imply that $u(0) = u'(0) = \cdots = u^{(n-3)}(0) = 0$, $u^{(n-2)}(0) = v(0) = 0$, $u^{(n-2)}(1) = v(1) = \beta[v] = \beta[u^{(n-2)}]$. Moreover, it follows from the monotonicity and property of I_{0+}^{n-2} that $I_{0+}^{n-2}v(t) \in C([0,1], \mathbb{R}^+)$. Consequently, $u(t) = I_{0+}^{n-2}v(t)$ is a positive solution of the fractional order boundary value problem (1.1). □

By Lemma 2.4, we will concentrate our study on (2.13). We here define an operator $T : P \to P$ by

$$Tv(t) = \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) ds, \quad t \in [0,1]. \tag{2.17}$$

Clearly, v is a fixed point of T in P, and so v is a positive solution of BVP (2.13).

In order to prove our main results, we need the following lemmas.

Lemma 2.5 (see [28]). *Let X be a real Banach space, Ω be a bounded open subset of X, where $\theta \in \Omega$, $T : \overline{\Omega} \to X$ is a completely continuous operator. Then, either there exist $x \in \partial\Omega$, $\mu \in (0,1)$ such that $\mu T(x) = x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Lemma 2.6 (see [29]). *Let X be a real Banach space, P be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open sets of X with $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tx\| \le \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Tx\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \le \|x\|$, $x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.7 (see [30, 31]). *Let P be a cone in a real Banach space X, $P_c = \{x \in P : \|x\| < c\}$, φ be a nonnegative continuous concave functional on P such that $\varphi(x) \le \|x\|$ for all $x \in \overline{P}_c$, and $P(\varphi, b, d) = \{x \in P : b \le \varphi(x), \|x\| \le d\}$. Suppose that $T : \overline{P}_c \to \overline{P}_c$ is completely continuous and there exist positive constants $0 < a < b < d \le c$ such that*

(C_1) $\{x \in P(\varphi, b, d) : \varphi(x) > b\} \neq \phi$ *and* $\varphi(Tx) > b$ *for* $x \in P(\varphi, b, d)$,

(C_2) $\|Tx\| < a$ *for* $x \in \overline{P}_a$,

(C_3) $\varphi(Tx) > b$ *for* $x \in P(\varphi, b, c)$ *with* $\|Tx\| > d$.

Then T has at least three fixed points x_1, x_2, and x_3 satisfying

$$\|x_1\| < a, \quad b < \varphi(x_2), \quad a < \|x_3\| \quad \text{with} \quad \varphi(x_3) < b. \tag{2.18}$$

Remark 2.8. If $d = c$, then condition (C_1) of Lemma 2.7 implies condition (C_3) of Lemma 2.7.

For notational convenience, we introduce the following constants:

$$L_1 = \int_\xi^\eta \rho(s) G_0(s,s)\, ds, \qquad L_2 = \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) \int_0^1 G_0(s,s)\, ds, \tag{2.19}$$

and a nonnegative continuous concave functional φ on the cone P defined by

$$\varphi(v) = \min_{\xi \le t \le \eta} |v(t)|. \tag{2.20}$$

3. Main Results

In this section, we present and prove our main results.

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold and there exist nonnegative functions* $h_1, h_2, \ldots, h_{n-1} \in L[0,1]$ *such that*

$$\left| f(t, x_1, x_2, \ldots, x_{n-1}) - f(t, y_1, y_2, \ldots, y_{n-1}) \right| \le \sum_{i=1}^{n-1} h_i(t) \left| x_i - y_i \right|, \tag{3.1}$$

for almost every $t \in [0,1]$ *and all* $(x_1, x_2, \ldots, x_{n-1})$, $(y_1, y_2, \ldots, y_{n-1}) \in R^{n-1}$.

If

$$0 < \int_0^1 G_0(s,s) \sum_{i=1}^{n-1} h_i(s)\, ds < \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}, \tag{3.2}$$

then BVP (1.1) *has a unique positive solution.*

Proof. We will show that T is a contraction mapping. For any $v_1, v_2 \in P$ and $1 \le i \le n-2$, by the definition of fractional integral, we obtain

$$\begin{aligned} \left| I_{0+}^i v_1(t) - I_{0+}^i v_2(t) \right| &= \left| \frac{1}{(i-1)!} \int_0^t (t-s)^{i-1} (v_1(s) - v_2(s))\, ds \right| \\ &\le \frac{1}{(i-1)!} \int_0^t (t-s)^{i-1} |v_1(s) - v_2(s)|\, ds \\ &\le \frac{1}{(i-1)!} \int_0^t (t-s)^{i-1}\, ds \|v_1 - v_2\| \\ &= \frac{1}{i!} t^i \|v_1 - v_2\| \le \|v_1 - v_2\|. \end{aligned} \tag{3.3}$$

So, for any $v_1, v_2 \in P$, by (3.3) and Lemma 2.3, we have

$$\begin{aligned}
|Tv_1(t) - Tv_2(t)| &= \left| \int_0^1 G(t,s)\Big[f\Big(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\Big)\right. \\
&\qquad \left. -f\Big(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\Big)\Big]ds\right| \\
&\leq \int_0^1 G(t,s)\Big|f\Big(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\Big) \\
&\qquad -f\Big(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\Big)\Big|ds \\
&\leq \int_0^1 \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) G_0(s,s) \sum_{i=1}^{n-1} h_i(s) \left| I_{0+}^{n-1-i}v_1(s) - I_{0+}^{n-1-i}v_2(s)\right| ds \\
&\leq \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) \int_0^1 G_0(s,s) \sum_{i=1}^{n-1} h_i(s)\, ds \|v_1 - v_2\|.
\end{aligned} \tag{3.4}$$

This implies that

$$\|Tv_1 - Tv_2\| \leq \kappa \|v_1 - v_2\|, \tag{3.5}$$

where $\kappa = (1 + \beta[1]/(1-\beta[\gamma])) \int_0^1 G_0(s,s) \sum_{i=1}^{n-1} h_i(s)\, ds \in (0,1)$. By the Banach contraction mapping principle, we deduce that T has a unique fixed point v^*. Thus, by Lemma 2.4, $u^*(t) = I_{0+}^{n-2}v^*(t)$ is a unique positive solution of BVP (1.1). □

Lemma 3.2. *Assume that* (H_1) *and* (H_2) *hold and the following conditions are satisfied.*

(H_3) *There exist nonnegative real-valued functions* $q, p_1, p_2, \dots, p_{n-1} \in L[0,1]$ *such that*

$$f(t, x_1, x_2, \dots, x_{n-1}) \leq q(t) + \sum_{i=1}^{n-1} p_i(t)|x_i|, \tag{3.6}$$

for almost every $t \in [0,1]$ *and all* $(x_1, x_2, \dots, x_{n-1}) \in R^{n-1}$.

Then $T : P \to P$ *is a completely continuous operator.*

Proof. For any $v \in P$, as $G(t,s) \geq 0$ for all $t, s \in [0,1]$, we have $Tv(t) \geq 0$, so $T(P) \subset P$. Let $D \subset P$ be any bounded set. Then there exists a constant $L > 0$ such that $\|v\| \leq L$ for any $v \in D$. Moreover for any $v \in D$, $s \in [0,1]$, $v(s) \leq \|v\| \leq L$. Proceeding as for (3.3), we obtain

$$\left| I_{0+}^{n-1-i}v(s)\right| = I_{0+}^{n-1-i}v(s) \leq \|v\| \leq L, \quad i = 1, 2, \dots, n-1. \tag{3.7}$$

Thus,

$$\begin{aligned}
|Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\
&\le \int_0^1 \left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right) G_0(s,s) \left[q(s) + \sum_{i=1}^{n-1} p_i(s) I_{0+}^{n-1-i} v(s)\right] ds \\
&\le \left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right) \int_0^1 G_0(s,s) \left[q(s) + \sum_{i=1}^{n-1} p_i(s) \|v\|\right] ds \\
&\le \left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right) (L+1) \int_0^1 G_0(s,s) \left[q(s) + \sum_{i=1}^{n-1} p_i(s)\right] ds \\
&< +\infty.
\end{aligned} \tag{3.8}$$

Therefore, $T(D)$ is uniformly bounded.

Now we show that $T(D)$ is equicontinuous on $[0,1]$. Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$. Hence, for any $\varepsilon > 0$, there exists a constant $\delta_0 > 0$ such that for any $s \in [0,1]$, $t, t' \in [0,1]$, when $|t-t'| < \delta_0$, it holds

$$\left|G(t,s) - G(t',s)\right| < \left[1 + (L+1)\int_0^1 \left(q(s) + \sum_{i=1}^{n-1} p_i(s)\right) ds\right]^{-1} \varepsilon. \tag{3.9}$$

Consequently, for any $t, t' \in [0,1]$ and $|t-t'| < \delta_0$, we have

$$\begin{aligned}
\left|Tv(t) - Tv(t')\right| &\le \int_0^1 \left|G(t,s) - G(t',s)\right| f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\
&\le \int_0^1 \left|G(t,s) - G(t',s)\right| \left[q(s) + \sum_{i=1}^{n-1} p_i(s)\|v\|\right] ds \\
&\le (L+1) \int_0^1 \left|G(t,s) - G(t',s)\right| \left[q(s) + \sum_{i=1}^{n-1} p_i(s)\right] ds \\
&< \varepsilon.
\end{aligned} \tag{3.10}$$

This implies that $T(D)$ is equicontinuous. Thus according to the Ascoli-Arzela Theorem, $T(D)$ is a relatively compact set.

In the end, we show that $T : P \to P$ is continuous. Assume that $v_m, v_0 \in P$ $(m = 1,2,\ldots)$, $v_m \to v_0$ $(m \to +\infty)$, then

$$|v_m(t) - v_0(t)| \le \|v_m - v_0\| \longrightarrow 0, \tag{3.11}$$

and $\|v_m\| \le L$ ($m = 0, 1, 2, \ldots$), where L is a positive constant. Keeping in mind that f satisfies Carathéodory conditions on $[0,1] \times \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
&\lim_{m \to +\infty} f\left(t, I_{0+}^{n-2} v_m(t), I_{0+}^{n-3} v_m(t), \ldots, I_{0+}^{1} v_m(t), v_m(t)\right) \\
&\quad = f\left(t, I_{0+}^{n-2} v_0(t), I_{0+}^{n-3} v_0(t), \ldots, I_{0+}^{1} v_0(t), v_0(t)\right), \quad \text{for a.e. } t \in [0,1].
\end{aligned} \tag{3.12}$$

Proceeding as for (3.3), for $m \in \mathbb{N}$ we obtain

$$\left| I_{0+}^{n-1-i} v_m(s) \right| = I_{0+}^{n-1-i} v_m(s) \le \|v_m\| \le L \quad i = 1, 2, \ldots, n-1. \tag{3.13}$$

This together with (3.6),

$$0 \le f\left(t, I_{0+}^{n-2} v_m(t), I_{0+}^{n-3} v_m(t), \ldots, I_{0+}^{1} v_m(t), v_m(t)\right) \le q(t) + L \sum_{i=1}^{n-1} p_i(t). \tag{3.14}$$

The Lebesgue dominated convergence theorem gives

$$\begin{aligned}
&\lim_{m \to +\infty} \int_0^1 \left| f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \\
&\qquad \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right| ds = 0.
\end{aligned} \tag{3.15}$$

Now we deduce from (3.15), Lemma 2.3

$$\begin{aligned}
&|Tv_m(t) - Tv_0(t)| \\
&\quad = \left| \int_0^1 G(t,s) \left[f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \right. \\
&\qquad \left. \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right] ds \right| \\
&\quad \le \left(1 + \frac{\beta[1]}{1 - \beta[\gamma]}\right) \int_0^1 G_0(s,s) \left| f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \\
&\qquad \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right| ds \\
&\quad \le \left(1 + \frac{\beta[1]}{1 - \beta[\gamma]}\right) \max_{s \in [0,1]} G_0(s,s) \int_0^1 \left| f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \\
&\qquad \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right| ds
\end{aligned} \tag{3.16}$$

that $\|Tv_m - Tv_0\| \to 0$, as $m \to +\infty$. So $T : P \to P$ is continuous. Therefore $T : P \to P$ is completely continuous. □

Remark 3.3. If $f : [0,1] \times \mathbb{R}^{n-1}$ is continuous, by similar argument as above, we can show that T is completely continuous.

Theorem 3.4. *Assume that* (H_1)–(H_3) *hold. If*

$$\int_0^1 G_0(s,s)\sum_{i=1}^{n-1} p_i(s)\,ds < \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}, \tag{3.17}$$

then BVP (1.1) *has at least one positive solution.*

Proof. Let

$$\Omega = \{v \in P : \|v\| < r\}, \quad \text{where } r = \frac{(1+\beta[1]/(1-\beta[\gamma]))\int_0^1 G_0(s,s)q(s)ds}{1-(1+\beta[1]/(1-\beta[\gamma]))\int_0^1 G_0(s,s)\sum_{i=1}^{n-1} p_i(s)ds}, \tag{3.18}$$

we have $\Omega \subset P$. From Lemma 3.2, we know that $T : \Omega \to P$ is completely continuous. If there exists $v \in \partial\Omega$, $\mu \in (0,1)$ such that

$$v = \mu T v, \tag{3.19}$$

then by (H_3) and (3.19), we have

$$\begin{aligned} v(t) = \mu T v(t) &= \mu \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\ &\le \mu \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\ &\le \mu\left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)\left[\int_0^1 G_0(s,s)q(s)\,ds + \int_0^1 G_0(s,s)\sum_{i=1}^{n-1}p_i(s)ds\|v\|\right], \end{aligned} \tag{3.20}$$

which implies that

$$\begin{aligned} \|v\| &\le \mu\left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)\left[\int_0^1 G_0(s,s)q(s)\,ds + r\int_0^1 G_0(s,s)\sum_{i=1}^{n-1}p_i(s)ds\right] \\ &< \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)\left[\int_0^1 G_0(s,s)q(s)ds + r\int_0^1 G_0(s,s)\sum_{i=1}^{n-1}p_i(s)ds\right] = r. \end{aligned} \tag{3.21}$$

This means that $v \notin \partial\Omega$. By Lemma 2.5, T has a fixed point $\widehat{v} \in \overline{\Omega}$. By Lemma 2.4, BVP (1.1) has at least one positive solution $\widehat{u}(t) = I_{0+}^{n-2}\widehat{v}(t)$. □

Theorem 3.5. *Assume that* (H_1)*–*(H_3) *hold. If there exist two positive constants* $r_1 < r_2$ *such that*

(i) $f(t, x_1, x_2, \ldots, x_{n-1}) \le L_2^{-1} r_2$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0, r_2] \times \cdots \times [0, r_2]$,

(ii) $f(t, x_1, x_2, \ldots, x_{n-1}) \ge L_1^{-1} r_1$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0, r_1] \times \cdots \times [0, r_1]$,

where L_1, L_2 *are defined by* (2.19)*, then BVP* (1.1) *has at least one positive solution.*

Proof. Let $\Omega_2 = \{v \in P : \|v\| < r_2\}$. For any $v \in \partial\Omega_2$, we have $\|v\| = r_2$ and $0 \le v(t) \le r_2$ for every $t \in [0,1]$. Similar to (3.7), for $0 \le v(s) \le r_2$, we have

$$0 \le \left| I_{0+}^{n-1-i} v(s) \right| = I_{0+}^{n-1-i} v(s) \le \|v\| \le r_2, \quad i = 1, 2, \ldots, n-1. \tag{3.22}$$

It follows from condition (i) that

$$f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) \le L_2^{-1} r_2, \quad \text{for } (s, v) \in [0,1] \times [0, r_2]. \tag{3.23}$$

Thus, for any $v \in \partial\Omega_2$, by (3.23) and Lemma 2.3, we have

$$\begin{aligned} |Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) ds \\ &\le \int_0^1 G(t,s) L_2^{-1} r_2 \, ds \le \left(1 + \beta[1]/(1 - \beta[\gamma])\right) L_2^{-1} r_2 \int_0^1 G_0(s,s) ds \\ &= r_2 = \|v\|, \quad t \in [0,1], \end{aligned} \tag{3.24}$$

which means that

$$\|Tv\| \le \|v\|, \quad v \in \partial\Omega_2. \tag{3.25}$$

On the other hand, let $\Omega_1 = \{v \in P : \|v\| < r_1\}$. For any $v \in \partial\Omega_1$, we have $\|v\| = r_1$ and $0 \le v(t) \le r_1$ for every $t \in [0,1]$. Similar to (3.23), from condition (ii), we can get

$$f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) \ge L_1^{-1} r_1, \quad \text{for } (s, v) \in [0,1] \times [0, r_1]. \tag{3.26}$$

Hence for any $t \in [\xi, \eta]$, $v \in \partial\Omega_1$, by (3.26) and Lemma 2.3 we have

$$\begin{aligned} |Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) ds \\ &\ge \int_0^1 G(t,s) L_1^{-1} r_1 \, ds \ge L_1^{-1} r_1 \int_\xi^\eta G(t,s) ds \\ &\ge L_1^{-1} r_1 \int_\xi^\eta \rho(s) G_0(s,s) ds \\ &= r_1 = \|v\|. \end{aligned} \tag{3.27}$$

Thus we get

$$\|Tv\| \geq \|v\|, \quad v \in \partial\Omega_1. \tag{3.28}$$

By (3.25), (3.28), and Lemma 2.6, T has a fixed point $\tilde{v} \in \overline{\Omega}_2 \setminus \Omega_1$ such that $r_1 \leq \|\tilde{v}\| \leq r_2$. By Lemma 2.4, BVP (1.1) has at least one positive solution $\tilde{u}(t) = I_{0+}^{n-2}\tilde{v}(t)$. □

Theorem 3.6. *Assume that (H_1)–(H_3) hold. If there exist constants $0 < a < b < c$ such that*

(I) $f(t, x_1, x_2, \ldots, x_{n-1}) < L_2^{-1}a$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0,a] \times \cdots \times [0,a]$,

(II) $f(t, x_1, x_2, \ldots, x_{n-1}) \leq L_2^{-1}c$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0,c] \times \cdots \times [0,c]$,

(III) $f(t, x_1, x_2, \ldots, x_{n-1}) \geq L_1^{-1}b$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [\xi, \eta] \times [(b/(n-2)!)\xi^{n-2}, c] \times [(b/(n-3)!)\xi^{n-3}, c] \times \cdots \times [b\xi, c] \times [b, c]$,

where L_1, L_2 are defined by (2.19)*, then BVP* (1.1) *has at least three positive solutions u_1, u_2, and u_3 satisfying*

$$\begin{gathered}
\left\|D_{0+}^{n-2}u_1\right\| < a, \qquad b < \varphi\left(D_{0+}^{n-2}u_2\right) < \left\|D_{0+}^{n-2}u_2\right\| \leq c, \\
a < \left\|D_{0+}^{n-2}u_3\right\|, \qquad \varphi\left(D_{0+}^{n-2}u_3\right) < b.
\end{gathered} \tag{3.29}$$

Proof. We will show that all conditions of Lemma 2.7 are satisfied.

First, if $v \in \overline{P}_c$, then $\|v\| \leq c$. So we have $0 \leq v(t) \leq c$, $t \in [0,1]$. Similar to (3.23), it follows from condition (II) that

$$f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) \leq L_2^{-1}c, \quad \text{for } (s,v) \in [0,1] \times [0,c]. \tag{3.30}$$

Thus, for any $v \in \overline{P}_c$, by (3.30), we have

$$\begin{aligned}
|Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\
&\leq \int_0^1 G(t,s) L_2^{-1}c\, ds \leq \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) L_2^{-1}c \int_0^1 G_0(s,s) ds \\
&= c,
\end{aligned} \tag{3.31}$$

which means that $\|Tv\| \leq c$, $v \in \overline{P}_c$. Therefore, $T : \overline{P}_c \to \overline{P}_c$. By Lemma 3.2, we know that $T : \overline{P}_c \to \overline{P}_c$ is completely continuous.

Next, similar to (3.30) and (3.31), it follows from condition (I) that if $v \in \overline{P}_a$ then $\|Tv\| < a$. So the condition (C_2) of Lemma 2.7 holds.

Now, we take $v(t) = (b+c)/2$, $t \in [0,1]$. It is easy to see that $v(t) = (b+c)/2 \in P(\varphi, b, c)$, and so

$$\varphi(v) = \min_{\xi \le t \le \eta} |v(t)| = \frac{b+c}{2} > b, \tag{3.32}$$

where $\varphi(v)$ is defined by (2.20). This proves that $\{v \in P(\varphi, b, c) : \varphi(v) > b\} \ne \phi$.

On the other hand, if $v \in P(\varphi, b, c)$, then $b \le v(t) \le c$, $t \in [\xi, \eta]$. By the definition of fractional integral, for any $t \in [\xi, \eta]$, $1 \le i \le n-2$, we obtain

$$\begin{aligned} \frac{b}{i!}\xi^i &\le \frac{b}{(i-1)!}\int_0^t (t-s)^{i-1}\,ds \le I_{0+}^i v(t) = \frac{1}{(i-1)!}\int_0^t (t-s)^{i-1} v(s)ds \\ &\le \frac{c}{(i-1)!}\int_0^t (t-s)^{i-1}\,ds \\ &\le \frac{c}{i!}\eta^i \le c. \end{aligned} \tag{3.33}$$

It follows from (3.33) and condition (III) that

$$f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) \ge L_1^{-1}b, \quad \text{for } s \in [\xi, \eta],\ v \in P(\varphi, b, c). \tag{3.34}$$

Hence we have

$$\begin{aligned} \varphi(Tv) &= \min_{\xi \le t \le \eta} |Tv(t)| \\ &= \min_{\xi \le t \le \eta} \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right)ds \\ &\ge L_1^{-1} b \min_{\xi \le t \le \eta} \int_0^1 G(t,s)ds > L_1^{-1} b \int_\xi^\eta \rho(s) G_0(s,s)ds = b, \end{aligned} \tag{3.35}$$

which implies that $\varphi(Tv) > b$, for $v \in P(\varphi, b, c)$. This shows that condition (C_1) of Lemma 2.7 is also satisfied.

By Lemma 2.7 and Remark 2.8, BVP (2.13) has at least three positive solutions v_1, v_2, and v_3 such that $\|v_1\| < a$, $b < \varphi(v_2) < \|v_2\| \le c$, and $a < \|v_3\|$, $\varphi(v_3) < b$. By Lemma 2.4, BVP (1.1) has at least three positive solutions $u_i(t) = I_{0+}^{n-2}v_i(t)$, $(i = 1, 2, 3)$. By (2.3), we have $D_{0+}^{n-2}u_i(t) = D_{0+}^{n-2}I_{0+}^{n-2}v_i(t) = v_i$, $i = 1, 2, 3$. So u_1, u_2, u_3 are three positive solutions of BVP (1.1) satisfying

$$\begin{aligned} \left\|D_{0+}^{n-2}u_1\right\| < a, \qquad b < \varphi\left(D_{0+}^{n-2}u_2\right) < \left\|D_{0+}^{n-2}u_2\right\| \le c, \\ a < \left\|D_{0+}^{n-2}u_3\right\|, \qquad \varphi\left(D_{0+}^{n-2}u_3\right) < b. \end{aligned} \tag{3.36}$$

The proof of Theorem 3.6 is completed. □

4. Examples

Example 4.1. Consider the following problem:

$$D_{0+}^{7/2}u(t)+\frac{(1-t)^3e^tu}{(1+e^t)(1+u)}+\frac{1}{2}t^2\sin^2 u'+\frac{1}{4}tu''=0,\quad 0<t<1,$$
$$u(0)=u'(0)=u''(0)=0,\qquad u''(1)=\beta[u''].\tag{4.1}$$

Let $\beta[u'']=(1/2)\ u''(1/2)$. Then

$$G_0(t,s)=\frac{1}{\Gamma(3/2)}\begin{cases}[t(1-s)]^{1/2}-(t-s)^{1/2}, & 0\le s\le t\le 1,\\ [t(1-s)]^{1/2}, & 0\le t\le s\le 1.\end{cases}\tag{4.2}$$

$$\mathcal{G}(s)=\frac{1}{2}G_0\left(\frac{1}{2},s\right)\ge 0,\qquad \beta[1]=\int_0^1 dA(t)=\frac{1}{2},\qquad \beta[\gamma]=\int_0^1 t^{1/2}dA(t)=\frac{\sqrt{2}}{4}<1.\tag{4.3}$$

Let

$$f(t,x,y,z)=\frac{(1-t)^3e^tx}{(1+e^t)(1+x)}+\frac{1}{2}t^2\sin^2 y+\frac{1}{4}tz,$$
$$h_1(t)=\frac{(1-t)^3e^t}{1+e^t},\qquad h_2(t)=\frac{1}{2}t^2,\qquad h_3(t)=\frac{1}{4}t.\tag{4.4}$$

Then f is a nonnegative continuous function on $[0,1]\times(\mathbb{R}^+)^3$ and, for any (t,x_1,y_1,z_1) and $(t,x_2,y_2,z_2)\in[0,1]\times(\mathbb{R}^+)^3$, satisfies

$$\left|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\right|\le h_1(t)|x_1-x_2|+h_2(t)\left|y_1-y_2\right|+h_3(t)|z_1-z_2|.\tag{4.5}$$

So we have

$$\begin{aligned}\int_0^1 G_0(s,s)\sum_{i=1}^3 h_i(s)\,ds &\le \frac{1}{\Gamma(3/2)}\int_0^1 (s(1-s))^{1/2}\left((1-s)^3+s^2+s\right)ds\\ &=\frac{B(3/2,9/2)+B(7/2,3/2)+B(5/2,3/2)}{\Gamma(3/2)}=\frac{33}{128}\sqrt{\pi}\approx 0.4569608\\ &<\left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}=\frac{4-\sqrt{2}}{6-\sqrt{2}}\approx 0.5638901,\end{aligned}\tag{4.6}$$

where $B(\cdot,\cdot)$ denotes a Beta function. So all conditions of Theorem 3.1 are satisfied. Thus, by Theorem 3.1, BVP (4.1) has at least one positive solution.

Example 4.2. Consider the following problem:

$$D_{0+}^{5/2}u(t)+\frac{1}{2}\left(t-t^2\right)\ln(1+u)+\frac{1}{2}t^2u'+t^3+\sin t=0,\quad 0<t<1,$$
$$u(0)=u'(0)=0,\qquad u'(1)=\beta\left[u'\right], \tag{4.7}$$

where $\beta[u']=\int_0^1 u'(s)dA(s)$ with

$$A(s)=\begin{cases}0, & s\in\left[0,\dfrac{1}{4}\right),\\[2mm] 2, & s\in\left[\dfrac{1}{4},\dfrac{9}{16}\right),\\[2mm] 1, & s\in\left[\dfrac{9}{16},1\right].\end{cases} \tag{4.8}$$

Set

$$f(t,x,y)=\frac{1}{2}\left(t-t^2\right)\ln(1+x)+\frac{1}{2}t^2y+t^3+\sin t,\qquad p_1(t)=\frac{1}{2}\left(t-t^2\right),$$
$$p_2(t)=\frac{1}{2}t^2,\qquad q(t)=t^3+1. \tag{4.9}$$

Then $f:[0,1]\times[0,+\infty)\times[0,+\infty)\longrightarrow[0,+\infty)$ is continuous and

$$f(t,x,y)\leq p_1(t)x+p_2(t)y+q(t). \tag{4.10}$$

As in [21], $\beta[\gamma]=\int_0^1\gamma(t)dA(t)=2\sqrt{1/4}+(-1)\times\sqrt{9/16}=1/4<1$, $\mathcal{G}(s)=\int_0^1 G_0(t,s)dA(t)\geq 0$,

$$\begin{aligned}\int_0^1 G_0(s,s)\left(p_1(s)+p_2(s)\right)ds &\leq \frac{1}{\Gamma(3/2)}\int_0^1(s(1-s))^{1/2}\left(\left(s-s^2\right)+s^2\right)ds\\ &=\frac{B(5/2,5/2)+B(7/2,3/2)}{\Gamma(3/2)}=\frac{1}{8}\sqrt{\pi}\approx 0.22155673\\ &<\left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}=\frac{3}{7}\approx 0.42857143,\end{aligned} \tag{4.11}$$

where $B(\cdot,\cdot)$ denotes a Beta function and $G_0(t,s)$ is defined by (4.2). So all conditions of Theorem 3.4 are satisfied. Thus, by Theorem 3.4, BVP (4.7) has at least one positive solution.

Example 4.3. Consider the following problem:

$$D_{0+}^{5/2}u(t) + \frac{t^2}{5}\ln(1+u) + \frac{te^t u'}{10+10e^t} + \frac{\sin t}{20} + \frac{1}{2} = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u'(1) = \beta[u'], \tag{4.12}$$

where $\beta[u'] = \int_0^1 u'(s)dA(s)$ with $A(s)$ as given by (4.8). Set

$$f(t,x,y) = \frac{t^2}{5}\ln(1+x) + \frac{te^t y}{10+10e^t} + \frac{\sin t}{20} + \frac{1}{2}, \qquad p_1(t) = \frac{t^2}{5},$$
$$p_2(t) = \frac{te^t}{10+10e^t}, \qquad q(t) = \frac{\sin t}{20} + \frac{1}{2}. \tag{4.13}$$

Then $f : [0,1] \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous and

$$f(t,x,y) \le p_1(t)x + p_2(t)y + q(t). \tag{4.14}$$

By Example 4.2, $\beta[\gamma] = \int_0^1 \gamma(t)dA(t) = 1/4 < 1$, $\mathcal{G}(s) = \int_0^1 G_0(t,s)dA(t) \ge 0$, where $G_0(t,s)$ is defined by (4.2). As in [1, 3], we also take $\xi = 1/4$, $\eta = 3/4$, then

$$L_1^{-1} = \left(\int_{1/4}^{3/4} \rho(s)G_0(s,s)\,ds\right)^{-1} \approx 13.6649,$$
$$L_2^{-1} = \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}\left(\int_0^1 G_0(s,s)\,ds\right)^{-1} = \frac{12}{7\sqrt{\pi}} \approx 0.967182. \tag{4.15}$$

Choosing $r_1 = 1/30$, $r_2 = 1$, we have

$$f(t,x,y) \le 0.85 \le L_2^{-1}r_2, \quad \text{for } (t,x,y) \in [0,1]\times[0,1]\times[0,1],$$
$$f(t,x,y) \ge 0.5 \ge L_1^{-1}r_1, \quad \text{for } (t,x,y) \in [0,1]\times\left[0,\frac{1}{30}\right]\times\left[0,\frac{1}{30}\right]. \tag{4.16}$$

So all conditions of Theorem 3.5 are satisfied. Thus, by Theorem 3.5, BVP (4.12) has at least one positive solution.

Example 4.4. Consider the following problem:

$$D_{0+}^{5/2}u(t) + f(t,u(t),u'(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u'(1) = \beta[u'], \tag{4.17}$$

where $\beta[u'] = \int_0^1 u'(s)dA(s)$ with $A(s)$ as given by (4.8). Set

$$f(t,x,y) = \begin{cases} \dfrac{t^2}{100}\ln(1+x) + 15y^2 + \dfrac{t}{1000}, & (t,x,y) \in [0,1]\times[0,1]\times[0,1], \\ \dfrac{t^2}{100}\ln(1+x) + \dfrac{29}{2} + \dfrac{1}{2}y + \dfrac{t}{1000}, & (t,x,y) \in [0,1]\times\left((\mathbb{R}^+)^2 \setminus [0,1]^2\right), \end{cases} \quad (4.18)$$

$$p_1(t) = \frac{t^2}{100}, \qquad p_2(t) = 15, \qquad q(t) = \frac{t}{1000} + \frac{29}{2}.$$

Then

$$f(t,x,y) \le p_1(t)x + p_2(t)y + q(t). \quad (4.19)$$

By Example 4.3, $\beta[\gamma] = \int_0^1 \gamma(t)dA(t) = 1/4 < 1$, $\mathcal{G}(s) = \int_0^1 G_0(t,s)dA(t) \ge 0$,

$$L_1^{-1} = \left(\int_{1/4}^{3/4} \rho(s)G_0(s,s)ds\right)^{-1} \approx 13.6649,$$
$$L_2^{-1} = \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}\left(\int_0^1 G_0(s,s)ds\right)^{-1} = \frac{12}{7\sqrt{\pi}} \approx 0.967182. \quad (4.20)$$

Choosing $a = 1/20$, $b = 1$, $c = 100$, we have

$$f(t,x,y) \le 0.044845 < L_2^{-1}a \approx 0.048359, \quad \text{for } (t,x,y) \in [0,1]\times\left[0,\frac{1}{20}\right]\times\left[0,\frac{1}{20}\right],$$
$$f(t,x,y) \ge 14.5025 \ge L_1^{-1}b \approx 13.6649, \quad \text{for } (t,x,y) \in \left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4},100\right]\times[1,100], \quad (4.21)$$
$$f(t,x,y) \le 65.501 \le L_2^{-1}c \approx 96.7182, \quad \text{for } (t,x,y) \in [0,1]\times[0,100]\times[0,100].$$

So all conditions of Theorem 3.6 are satisfied. Thus, by Theorem 3.6, BVP (4.17) has at least three positive solutions u_1, u_2, and u_3, satisfying

$$\max_{0\le t\le 1}\left|u_1'(t)\right| < \frac{1}{20}, \qquad 1 < \min_{1/4\le t\le 3/4}\left|u_2'(t)\right| < \max_{0\le t\le 1}\left|u_2'(t)\right| \le 100,$$
$$\frac{1}{20} \le \max_{0\le t\le 1}\left|u_3'(t)\right| \le 100 \qquad \text{with } \min_{1/4\le t\le 3/4}\left|u_3'(t)\right| < 1. \quad (4.22)$$

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Research Article

A Weighted Variant of Riemann-Liouville Fractional Integrals on $\mathbb{R}^n$

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We introduce certain type of weighted variant of Riemann-Liouville fractional integral on $\mathbb{R}^n$ and obtain its sharp bounds on the central Morrey and λ-central BMO spaces. Moreover, we establish a sufficient and necessary condition of the weight functions so that commutators of weighted Hardy operators (with symbols in λ-central BMO space) are bounded on the central Morrey spaces. These results are further used to prove sharp estimates of some inequalities due to Weyl and Cesàro.

1. Introduction

Let $0 < \alpha < 1$. The well-known *Riemann-Liouville fractional integral* I_α is defined by

$$I_\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > 0, \tag{1.1}$$

for all locally integrable functions f on $(0,\infty)$. The study of Riemann-Liouville fractional integral has a very long history and number of papers involved its generalizations, variants, and applications. For the earlier development of this kind of integrals and many important applications in fractional calculus, we refer the interested reader to the book [1]. Among numerous material dealing with applications of fractional calculus to (ordinary or partial) differential equations, we choose to refer to [2] and references therein.

As the classical n-dimensional generalization of I_α, the well-known Riesz potential (the solution of Laplace equation) $\mathcal{I}_\alpha$ with $0 < \alpha < n$ is defined by setting, for all locally integrable functions f on $\mathbb{R}^n$,

$$\mathcal{I}_\alpha f(x) := C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(t)}{|x-t|^{n-\alpha}} dt, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $C_{n,\alpha} := \pi^{n/2} 2^\alpha (\Gamma(\alpha/2))/(\Gamma((n-\alpha)/2))$. The importance of Riesz potentials lies in the fact that they are indeed smoothing operators and have been extensively used in many different areas such as potential analysis, harmonic analysis, and partial differential equations. Here we refer to the paper [3], which is devoted to the sharp constant in the Hardy-Littlewood-Sobolev inequality related to $\mathcal{I}_\alpha$.

This paper focused on another generalization, the weighted variants of Riemann-Liouville fractional integrals on $\mathbb{R}^n$. We investigate the boundedness of these weighted variants on the type of central Morrey and central Campanato spaces and also give the sharp estimates. This development begins with an equivalent definition of I_α as

$$x^\alpha I_\alpha f(x) = \int_0^1 f(tx) \frac{1}{\Gamma(\alpha)(1-t)^{1-\alpha}} dt, \quad x > 0. \tag{1.3}$$

More generally, we use a positive function (weight function) $\omega(t)$ to replace $1/(\Gamma(\alpha)(1-t)^{1-\alpha})$ in (1.3) and generalize the parameter x from the positive axle to the Euclidean space $\mathbb{R}^n$ therein. We then derive a weighted generalization of $|x|^\alpha I_\alpha$ on $\mathbb{R}^n$, which is called the weighted Hardy operator (originally named weighted Hardy-Littlewood avarage) H_ω.

More precise, let ω be a positive function on $[0,1]$. The *weighted Hardy operator* H_ω is defined by setting, for all complex-valued measurable functions f on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$H_\omega f(x) := \int_0^1 f(tx)\omega(t) dt. \tag{1.4}$$

Under certain conditions on ω, Carton-Lebrun and Fosset [4] proved that H_ω maps $L^p(\mathbb{R}^n)$, $1 < p < \infty$, into itself; moreover, the operator H_ω commutes with the Hilbert transform when $n = 1$, and with certain Calderón-Zygmund singular integrals including the Riesz transform when $n \geq 2$. Obviously, for $n = 1$ and $0 < \alpha < 1$, if we take $\omega(t) := 1/(\Gamma(\alpha)(1-t)^{1-\alpha})$, then as mentioned above, for all $x > 0$,

$$H_\omega f(x) = x^{-\alpha} I_\alpha f(x). \tag{1.5}$$

A further extension of [4] was due to Xiao [5] as follows.

Theorem A. *Let $1 < p < \infty$. Then, H_ω is bounded on $L^p(\mathbb{R}^n)$ if and only if*

$$\mathbb{A} := \int_0^1 t^{-n/p}\omega(t) dt < \infty. \tag{1.6}$$

Moreover,

$$\left\| H_\omega f \right\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} = \mathbb{A}. \tag{1.7}$$

Remark 1.1. Notice that the condition (1.6) implies that ω is integrable on $[0,1]$ since $\int_0^1 \omega(t)dt \le \int_0^1 t^{-n/p}\omega(t)dt$. We naturally assume ω is integrable on $[0,1]$ throughout this paper.

Obviously, Theorem A implies the celebrated result of Hardy et al. [6, Theorem 329], namely, for all $0 < \alpha < 1$ and $1 < p < \infty$,

$$\|I_\alpha\|_{L^p(dx) \to L^p(x^{-p\alpha}dx)} = \frac{\Gamma(1-1/p)}{\Gamma(1+\alpha-1/p)}. \tag{1.8}$$

The constant $\mathbb{A}$ in (1.6) also seems to be of interest as it equals to $p/(p-1)$ if $\omega \equiv 1$ and $n = 1$. In this case, H_ω is precisely reduced to the *classical Hardy operator* H defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0, \tag{1.9}$$

which is the most fundamental integral averaging operator in analysis. Also, a celebrated operator norm estimate due to Hardy et al. [6], that is,

$$\|H\|_{L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)} = \frac{p}{p-1} \tag{1.10}$$

with $1 < p < \infty$, can be deduced from Theorem A immediately.

Recall that $\mathrm{BMO}(\mathbb{R}^n)$ is defined to be the space of all $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|b\|_{\mathrm{BMO}} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|}\int_B |b(x) - b_B| \, dx < \infty, \tag{1.11}$$

where $b_B = (1/|B|)\int_B b$ and the supremum is taken over all balls B in $\mathbb{R}^n$ with sides parallel to the axes. It is well known that $L^\infty(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$, since $\mathrm{BMO}(\mathbb{R}^n)$ contains unbounded functions such as $\log|x|$. Another interesting result of Xiao in [5] is that the weighted Hardy operator H_ω is bounded on $\mathrm{BMO}(\mathbb{R}^n)$, if and only if

$$\int_0^1 \omega(t)dt < \infty. \tag{1.12}$$

Moreover,

$$\|H_\omega\|_{\mathrm{BMO}(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)} = \int_0^1 \varphi(t)dt. \tag{1.13}$$

In recent years, several authors have extended and considered the action of weighted Hardy operators on various spaces. We mention here, the work of Rim and Lee [7], Kuang [8], Krulić et al. [9], Tang and Zhai [10], Tang and Zhou [11].

The main purpose of this paper is to make precise the mapping properties of weighted Hardy operators on the central Morrey and λ-central BMO spaces. The study of the central Morrey and λ-central BMO spaces are traced to the work of Wiener [12, 13] on describing the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L^q $(1 < q < \infty)$ spaces. Beurling [14] extended this idea and defined a pair of dual Banach spaces A^q and $B^{q'}$, where $1/q + 1/q' = 1$. To be precise, A^q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L^q spaces. The space $B^{q'}$ is expressed as the intersection of the corresponding weighted $L^{q'}$ spaces. Later, Feichtinger [15] observed that the space $B^{q'}$ can be equivalently described by the set of all locally q'-integrable functions f satisfying that

$$\|f\|_{B^{q'}} = \sup_{k\geq 0}\left(2^{-kn/q'}\|f\chi_k\|_{q'}\right) < \infty, \tag{1.14}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$, $k = 1, 2, 3, \ldots$, and $\|\cdot\|_{q'}$ is the norm in $L^{q'}$. By duality, the space A^q, called Beurling algebra now, can be equivalently described by the set of all locally q-integrable functions f satisfying that

$$\|f\|_{A^q} = \sum_{k=0}^{\infty} 2^{kn/q'}\|f\chi_k\|_q < \infty. \tag{1.15}$$

Based on these, Chen and Lau [16] and García-Cuerva [17] introduced an atomic space HA^q associated with the Beurling algebra A^q and identified its dual as the space CMO^q, which is defined to be the space of all locally q-integrable functions f satisfying that

$$\sup_{R\geq 1}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\left|f(x) - f_{B(0,R)}\right|^q dx\right)^{1/q} < \infty. \tag{1.16}$$

By replacing $k \in \mathbb{N} \cup \{0\}$ with $k \in \mathbb{Z}$ in (1.3) and (1.6), we obtain the spaces $\dot{A}^q$ and $\dot{B}^{q'}$, which are the homogeneous version of the spaces A^q and $B^{q'}$, and the dual space of $\dot{A}^q$ is just $\dot{B}^{q'}$. Related to these homogeneous spaces, in [18, 19], Lu and Yang introduced the homogeneous counterparts of HA^q and CMO^q, denoted by $\mathrm{H\dot{A}}^q$ and $\mathrm{C\dot{M}O}^q$, respectively. These spaces were originally denoted by HK^q and CBMO_q in [18, 19]. Recall that the space $\mathrm{C\dot{M}O}^q$ is defined to be the space of all locally q-integrable functions f satisfying that

$$\sup_{R>0}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\left|f(x) - f_{B(0,R)}\right|^q dx\right)^{1/q} < \infty. \tag{1.17}$$

It was also proved by Lu and Yang that the dual space of $\mathrm{H\dot{A}}^q$ is just $\mathrm{C\dot{M}O}^q$.

In 2000, Alvarez et al. [20] introduced the following λ-central bounded mean oscillation spaces and the central Morrey spaces, respectively.

Definition 1.2. Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined to be the space of all locally q-integrable functions f satisfying that

$$\|f\|_{\dot{B}^{q,\lambda}} = \sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)|^q dx\right)^{1/q} < \infty. \tag{1.18}$$

Definition 1.3. Let $\lambda < 1/n$ and $1 < q < \infty$. A function $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ is said to belong to the λ-central bounded mean oscillation space $\mathrm{C\dot{M}O}^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{C\dot{M}O}^{q,\lambda}} = \sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)-f_{B(0,R)}|^q dx\right)^{1/q} < \infty. \tag{1.19}$$

We remark that if two functions which differ by a constant are regarded as a function in the space $\mathrm{C\dot{M}O}^{q,\lambda}$, then $\mathrm{C\dot{M}O}^{q,\lambda}$ becomes a Banach space. Apparently, (1.19) is equivalent to the following condition:

$$\sup_{R>0}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)-c|^q dx\right)^{1/q} < \infty. \tag{1.20}$$

Remark 1.4. $\dot{B}^{q,\lambda}$ is a Banach space which is continuously included in $\mathrm{C\dot{M}O}^{q,\lambda}$. One can easily check $\dot{B}^{q,\lambda}(\mathbb{R}^n) = \{0\}$ if $\lambda < -1/q$, $\dot{B}^{q,0}(\mathbb{R}^n) = \dot{B}^q(\mathbb{R}^n)$, $\dot{B}^{q,-1/q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, and $\dot{B}^{q,\lambda}(\mathbb{R}^n) \supsetneq L^q(\mathbb{R}^n)$ if $\lambda > -1/q$. Similar to the classical Morrey space, we only consider the case $-1/q < \lambda \le 0$ in this paper.

Remark 1.5. The space $\mathrm{C\dot{M}O}^{q,\lambda}$ when $\lambda = 0$ is just the space $\mathrm{C\dot{M}O}^q$. It is easy to see that $\mathrm{BMO} \subset \mathrm{C\dot{M}O}^q$ for all $1 < q < \infty$. When $\lambda \in (0, 1/n)$, then the space $\mathrm{C\dot{M}O}^{q,\lambda}$ is just the central version of the Lipschitz space $\mathrm{Lip}_\lambda(\mathbb{R}^n)$.

Remark 1.6. If $1 < q_1 < q_2 < \infty$, then by Hölder's inequality, we know that $\dot{B}^{q_2,\lambda} \subset \dot{B}^{q_1,\lambda}$ for $\lambda \in \mathbb{R}$, and $\mathrm{C\dot{M}O}^{q_2,\lambda} \subset \mathrm{C\dot{M}O}^{q_1,\lambda}$ for $\lambda < 1/n$.

For more recent generalization about central Morrey and Campanato space, we refer to [21]. We also remark that in recent years, there exists an increasing interest in the study of Morrey-type spaces and the related theory of operators; see, for example, [22].

In this paper, we give sufficient and necessary conditions on the weight ω which ensure that the corresponding weighted Hardy operator H_ω is bounded on $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and $\mathrm{C\dot{M}O}^{q,\lambda}(\mathbb{R}^n)$. Meanwhile, we can work out the corresponding operator norms. Moreover, we establish a sufficient and necessary condition of the weight functions so that commutators of weighted Hardy operators (with symbols in central Campanato-type space) are bounded on the central Morrey-type spaces. These results are further used to prove sharp estimates of some inequalities due to Weyl and Cesàro.

2. Sharp Estimates of H_ω

Let us state our main results.

Theorem 2.1. *Let $1 < q < \infty$ and $-1/q < \lambda \le 0$. Then H_ω is a bounded operator on $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ if and only if*

$$\mathbb{B} := \int_0^1 t^{n\lambda}\omega(t)dt < \infty. \tag{2.1}$$

Moreover, when (2.1) *holds, the operator norm of H_ω on $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is given by*

$$\|H_\omega\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)\to\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \mathbb{B}. \tag{2.2}$$

Proof. Suppose (2.1) holds. For any $R > 0$, using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|(H_\omega f)(x)\right|^q dx\right)^{1/q} \\
&\quad\le \int_0^1\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|f(tx)\right|^q dx\right)^{1/q}\omega(t)dt \\
&\quad= \int_0^1\left(\frac{1}{|B(0,tR)|^{1+\lambda q}}\int_{B(0,tR)}\left|f(x)\right|^q dx\right)^{1/q} t^{n\lambda}\omega(t)dt \\
&\quad\le \|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)}\int_0^1 t^{n\lambda}\omega(t)dt.
\end{aligned} \tag{2.3}$$

It implies that

$$\|H_\omega\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)\to\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le \int_0^1 t^{n\lambda}\omega(t)dt. \tag{2.4}$$

Thus H_ω maps $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ into itself.

The proof of the converse comes from a standard calculation. If H_ω is a bounded operator on $\dot{B}^{q,\lambda}(\mathbb{R}^n)$, take

$$f_0(x) = |x|^{n\lambda}, \quad x \in \mathbb{R}^n. \tag{2.5}$$

Then

$$\|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \Omega_n^{-\lambda}\frac{1}{(nq\lambda+n)^{1/q}}, \tag{2.6}$$

where $\Omega_n = \pi^{n/2}/(\Gamma(1+n/2))$ is the volume of the unit ball in $\mathbb{R}^n$.

We have

$$H_\omega f_0 = f_0 \int_0^1 t^{n\lambda}\omega(t)\,dt, \tag{2.7}$$

$$\|H_\omega\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)\to\dot{B}^{q,\lambda}(\mathbb{R}^n)} \geq \int_0^1 t^{n\lambda}\omega(t)\,dt. \tag{2.8}$$

(2.8) together with (2.4) yields the desired result. □

Corollary 2.2. (i) *For* $0<\alpha<1$, $1<q<\infty$, *and* $-1/q<\lambda\leq 0$,

$$\|I_\alpha\|_{\dot{B}^{q,\lambda}(dx)\to\dot{B}^{q,\lambda}(x^{-q\alpha}dx)} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\alpha+\lambda)}. \tag{2.9}$$

(ii) *For* $1<q<\infty$ *and* $-1/q<\lambda\leq 0$,

$$\|H\|_{\dot{B}^{q,\lambda}\to\dot{B}^{q,\lambda}} = \frac{1}{1+\lambda}. \tag{2.10}$$

Next, we state the corresponding conclusion for the space $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$.

Theorem 2.3. *Let* $1<q<\infty$ *and* $0\leq\lambda<1/n$. *Then* H_ω *is a bounded operator on* $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ *if and only if* (2.1) *holds. Moreover, when* (2.1) *holds, the operator norm of* H_ω *on* $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ *is given by*

$$\|H_\omega\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)\to C\dot{M}O^{q,\lambda}(\mathbb{R}^n)} = \mathbb{B}. \tag{2.11}$$

Proof. Suppose (2.1) holds. If $f\in C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$, then for any $R>0$ and ball $B(0,R)$, using Fubini's theorem, we see that

$$(H_\omega f)_{B(0,R)} = \int_0^1\left(\frac{1}{|B(0,R)|}\int_{B(0,R)} f(tx)\,dx\right)\omega(t)\,dt = \int_0^1 f_{B(0,tR)}\omega(t)\,dt. \tag{2.12}$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|(H_\omega f)(x)-(H_\omega f)_{B(0,R)}\right|^q dx\right)^{1/q}\\
&\quad=\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|\int_0^1\left(f(tx)-f_{B(0,tR)}\right)dt\right|^q dx\right)^{1/q}\\
&\quad\leq\int_0^1\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|f(tx)-f_{B(0,tR)}\right|^q dx\right)^{1/q}\omega(t)dt\\
&\quad=\int_0^1\left(\frac{1}{|B(0,tR)|^{1+\lambda q}}\int_{B(0,tR)}\left|f(x)-f_{B(0,tR)}\right|^q dx\right)^{1/q}t^{n\lambda}\omega(t)dt\\
&\quad\leq\|f\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}\int_0^1 t^{n\lambda}\omega(t)dt,
\end{aligned}\tag{2.13}$$

which implies H_ω is bounded on $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ and

$$\|H_\omega\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)\to C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}\leq\mathbb{B}.\tag{2.14}$$

Conversely, if H_ω is a bounded operator on $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$, take

$$f_0(x)=\begin{cases}|x|^{n\lambda}, & x\in\mathbb{R}^n_r,\\ -|x|^{n\lambda}, & x\in\mathbb{R}^n_l,\end{cases}\tag{2.15}$$

where $\mathbb{R}^n_r$ and $\mathbb{R}^n_l$ denote the right and the left halves of $\mathbb{R}^n$, separated by the hyperplane $x_1=0$, and x_1 is the first coordinate of $x\in\mathbb{R}^n$.

Thus, by a standard calculation, we see that $(f_0)_{B(0,R)}=0$ and

$$\begin{gathered}
\|f_0\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}=\Omega_n^{-\lambda}\frac{1}{(nq\lambda+n)^{1/q}},\\
H_\omega f_0=f_0\int_0^1 t^{n\lambda}\omega(t)dt.
\end{gathered}\tag{2.16}$$

From this formula we have

$$\|H_\omega\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)\to C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}\geq\mathbb{B}.\tag{2.17}$$

The proof is complete. □

Corollary 2.4. (i) *For $1 < q < \infty$ and $0 \le \lambda < 1$, we have*

$$\|H\|_{C\dot{M}O^{q,\lambda} \to C\dot{M}O^{q,\lambda}} = \frac{1}{1+\lambda}. \tag{2.18}$$

(ii) *For $1 < q < \infty$, we have $\|H\|_{C\dot{M}O^{q} \to C\dot{M}O^{q}} = 1$.*

3. A Characterization of Weight Functions via Commutators

A well-known result of Coifman et al. [23] states that the commutator generated by Calderón-Zygmund singular integrals and BMO functions is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Recently, we introduced the commutators of weighted Hardy operators and BMO functions introduced in [24]. For any locally integrable function b on $\mathbb{R}^n$ and integrable function $\omega : [0,1] \to [0,\infty)$, the *commutator of the weighted Hardy operator* H^b_ω is defined by

$$H^b_\omega f := bH_\omega f - H_\omega(bf). \tag{3.1}$$

It is easy to see that when $b \in L^\infty(\mathbb{R}^n)$ and ω satisfies the condition (1.6), then the commutator H^b_ω is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. An interesting choice of b is that it belongs to the class of $\mathrm{BMO}(\mathbb{R}^n)$. When symbols $b \in \mathrm{BMO}(\mathbb{R}^n)$, the condition (1.6) on weight functions ω can not ensure the boundedness of H^b_ω on $L^p(\mathbb{R}^n)$. Via controlling H^b_ω by the Hardy-Littlewood maximal operators instead of sharp maximal functions, we [24] established a sufficient and necessary (more stronger) condition on weight functions ω which ensures that H^b_ω is bounded on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$. More recently, Fu and Lu [25] studied the boundedness of H^b_ω on the classical Morrey spaces. Tang et al. [26] and Tang and Zhou [11] obtained the corresponding result on some Herz-type and Triebel-Lizorkin-type spaces. We also refer to the work [27] for more general m-linear Hardy operators.

Similar to [24], we are devoted to the construction of a sufficient and necessary condition (which is stronger than $\mathbb{B} = \infty$ in Theorem 2.1) on the weight functions so that commutators of weighted Hardy operators (with symbols in λ-central BMO space) are bounded on the central Morrey spaces. For the boundedness of commutators with symbols in central BMO spaces, we refer the interested reader to [28, 29] and Mo [30].

Theorem 3.1. *Let $1 < q_1 < q < \infty, 1/q_1 = 1/q + 1/q_2$, $-1/q < \lambda < 0$. Assume further that ω is a positive integrable function on $[0,1]$. Then, the commutator H^b_ω is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda}(\mathbb{R}^n)$, for any $b \in C\dot{M}O^{q_2}(\mathbb{R}^n)$, if and only if*

$$\mathbb{C} := \int_0^1 t^{n\lambda}\omega(t)\log\frac{2}{t}\,dt < \infty. \tag{3.2}$$

Remark 3.2. The condition (2.1), that is, $\mathbb{B} < \infty$, is weaker than $\mathbb{C} < \infty$. In fact, let

$$\mathbb{D} := \int_0^1 t^{n\lambda}\omega(t)\log\frac{1}{t}\,dt < \infty. \tag{3.3}$$

By $\mathbb{C} = \mathbb{B}\log 2 + \mathbb{D}$, we know that $\mathbb{C} < \infty$ implies $\mathbb{B} < \infty$. But the following example shows that $\mathbb{B} < \infty$ does not imply $\mathbb{C} < \infty$. For $0 < \beta < 1$, if we take

$$e^{s(-n\lambda-1)}\widetilde{\omega}(s) = \begin{cases} s^{-1+\beta}, & 0 < s \leq 1, \\ s^{-1-\beta}, & 1 < s < \infty, \\ 0, & s = 0, \infty, \end{cases} \tag{3.4}$$

and $\omega(t) = \widetilde{\omega}(\log(1/t))$, where $0 \leq t \leq 1$, then $\mathbb{B} < \infty$ and $\mathbb{C} = \infty$.

Proof. (i) Let $R \in (0, \infty)$. Denote $B(0, R)$ by B and $B(0, tR)$ by tB. Assume $\mathbb{C} < \infty$. We get

$$\begin{aligned} \left(\frac{1}{|B|}\int_B \left|H_\omega^b f(x)\right|^{q_1} dx\right)^{1/q_1} &\leq \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b(x) - b(tx))f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &\leq \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b(x) - b_B)f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &\quad + \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b_B - b_{tB})f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &\quad + \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b(tx) - b_{tB})f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{3.5}$$

By the Minkowski inequality and the Hölder inequality (with $1/q_1 = 1/q + 1/q_2$), we have

$$\begin{aligned} I_1 &\leq \int_0^1 \left(\frac{1}{|B|}\int_B \left|(b(x) - b_B)f(tx)\right|^{q_1} dx\right)^{1/q_1} \omega(t)dt \\ &\leq \int_0^1 \left(\frac{1}{|B|}\int_B |b(x) - b_B|^{q_2} dx\right)^{1/q_2} \left(\frac{1}{|B|}\int_B \left|f(tx)\right|^q dx\right)^{1/q} \omega(t)dt \\ &\leq |B|^\lambda \|b\|_{\dot{\mathrm{CMO}}^{q_2}} \int_0^1 \left(\frac{1}{|tB|^{1+q\lambda}}\int_{tB} \left|f(x)\right|^q dx\right)^{1/q} t^{n\lambda}\omega(t)dt \\ &\leq |B|^\lambda \|b\|_{\dot{\mathrm{CMO}}^{q_2}} \left\|f\right\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda}\omega(t)dt. \end{aligned} \tag{3.6}$$

Similarly, we have

$$
\begin{aligned}
I_3 &\leq \int_0^1 \left(\frac{1}{|B|} \int_B \left| (b(tx) - b_{tB}) f(tx) \right|^{q_1} dx \right)^{1/q_1} \omega(t) dt \\
&\leq \int_0^1 \left(\frac{1}{|tB|} \int_{tB} |b(x) - b_{tB}|^{q_2} dx \right)^{1/q_2} \left(\frac{1}{|tB|} \int_{tB} \left| f(x) \right|^{q} dx \right)^{1/q} \omega(t) dt \\
&\leq |B|^{\lambda} \|b\|_{\dot{CMO}^{q_2}} \int_0^1 \left(\frac{1}{|tB|^{1+q\lambda}} \int_{tB} \left| f(x) \right|^{q} dx \right)^{1/q} t^{n\lambda} \omega(t) dt \\
&\leq C|B|^{\lambda} \|b\|_{\dot{CMO}^{q_2}} \left\| f \right\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda} \omega(t) dt.
\end{aligned}
\tag{3.7}
$$

Now we estimate I_2,

$$
\begin{aligned}
I_2 &\leq \int_0^1 \left(\frac{1}{|B|} \int_B \left| f(tx) \right|^{q_1} dx \right)^{1/q_1} |b_B - b_{tB}| \omega(t) dt \\
&\leq \left\| f \right\|_{\dot{B}^{q,\lambda}} \int_0^1 |tB|^{\lambda} |b_B - b_{tB}| \omega(t) dt \\
&= \left\| f \right\|_{\dot{B}^{q,\lambda}} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |tB|^{\lambda} |b_B - b_{tB}| \omega(t) dt \\
&\leq \left\| f \right\|_{\dot{B}^{q,\lambda}} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |tB|^{\lambda} \left\{ \left(\sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| \right) + |b_{2^{-k-1}B} - b_{tB}| \right\} \omega(t) dt.
\end{aligned}
\tag{3.8}
$$

We see that

$$
\begin{aligned}
\sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| &\leq C \sum_{i=0}^{k} \left(\frac{1}{|2^{-i}B|} \int_{2^{-i}B} \left| b(y) - b_{2^{-i}B} \right|^{q_2} dy \right)^{1/q_2} \\
&\leq C \|b\|_{\dot{CMO}^{q_2}} (k+1).
\end{aligned}
\tag{3.9}
$$

Therefore,

$$
I_2 \leq C|B|^{\lambda} \|b\|_{\dot{CMO}^{q_2}} \left\| f \right\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \tag{3.10}
$$

Combining the estimates of I_1, I_2, and I_3, we conclude that H_{ω}^{b} is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1}(\mathbb{R}^n)$.

Conversely, assume that for any $b \in \dot{CMO}^{q_2}$, H_{ω}^{b} is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_2,\lambda}(\mathbb{R}^n)$. We need to show that $\mathbb{C} < \infty$. Since $\mathbb{C} = \mathbb{B} \log 2 + \mathbb{D}$, we will prove that $\mathbb{B} < \infty$ and $\mathbb{D} < \infty$, respectively. To this end, let

$$
b_0(x) = \log|x| \tag{3.11}
$$

for all $x \in \mathbb{R}^n$. Then it follows from Remark 1.5 that $b_0 \in \mathrm{BMO} \subset C\dot{M}O^{q_2}$, and

$$\left\| H_\omega^{b_0} \right\|_{\dot{B}^{q,\lambda} \to \dot{B}^{q_1,\lambda}} < \infty. \tag{3.12}$$

Let $f_0(x) = |x|^{n\lambda}$, $x \in \mathbb{R}^n$. Then

$$\begin{gathered} \| f_0 \|_{\dot{B}^{q,\lambda}} = \Omega_n^{-\lambda} \frac{1}{(nq\lambda + n)^{1/q}}, \\ H_\omega^{b_0} f_0(x) = |x|^{n\lambda} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \end{gathered} \tag{3.13}$$

For $\lambda > -1/q > -1/q_1$, we obtain

$$\left\| H_\omega^{b_0} f_0 \right\|_{\dot{B}^{q_1,\lambda}} = \Omega_n^{-\lambda} \frac{1}{(nq_1\lambda + n)^{1/q_1}} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \tag{3.14}$$

So,

$$\left\| H_\omega^{b_0} \right\|_{\dot{B}^{q_1,\lambda} \to \dot{B}^{q,\lambda}} \geq C_{n,\lambda,q,q_1} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \tag{3.15}$$

Therefore, we have

$$\mathbb{D} < \infty. \tag{3.16}$$

On the other hand,

$$\begin{gathered} \int_0^{1/2} t^{n\lambda} \omega(t) dt \leq C \int_0^{1/2} t^{n\lambda} \omega(t) \log \frac{1}{t} dt < \infty, \\ \int_{1/2}^1 t^{n\lambda} \omega(t) dt < \infty, \end{gathered} \tag{3.17}$$

since $t^{n\lambda}$ and $\omega(t)$ are integrable functions on $[1/2, 1]$. Combining the above estimates, we get

$$\mathbb{B} < \infty. \tag{3.18}$$

Combining (3.18) and (3.16), we then obtain the desired result. □

Notice that comparing with Theorems 2.1 and 2.3, we need a priori assumption in Theorem 3.1 that ω is integrable on $[0, 1]$. However, by Remark 1.1, this assumption is reasonable in some sense.

When $b \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n)$ with $\lambda_2 > 0$, namely, b is a central λ-Lipschitz function, we have the following conclusion. The proof is similar to that of Theorem 3.1. We give some details here.

Theorem 3.3. *Let $1 < q_1 < q < \infty$, $1/q_1 = 1/q + 1/q_2$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 < \lambda_2 < 1/n$, and $\lambda_1 = \lambda + \lambda_2$. If (2.1) holds true, then for all $b \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n)$, the corresponding commutator H_ω^b is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$.*

Proof. Let I_1, I_2, and I_3 be as in the proof of Theorem 3.1. Then, following the estimates of I_1 and I_3 in the proof of Theorem 3.1, we see that

$$\begin{aligned} I_1 &\le |B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda}\omega(t)dt, \\ I_3 &\le |B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda_1}\omega(t)dt \\ &\le |B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda}\omega(t)dt. \end{aligned} \tag{3.19}$$

For I_2, we also have

$$I_2 \le \|f\|_{\dot{B}^{q,\lambda}} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |tB|^{\lambda} \left\{ \left(\sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| \right) + |b_{2^{-k-1}B} - b_{tB}| \right\} \omega(t)dt. \tag{3.20}$$

Since now $0 < \lambda_2 < 1/n$, we see that

$$\begin{aligned} \sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| &\le C \sum_{i=0}^{k} \left(\frac{1}{|2^{-i}B|} \int_{2^{-i}B} |b(y) - b_{2^{-i}B}|^{q_2} dy \right)^{1/q_2} \\ &\le C \|b\|_{C\dot{M}O^{q_2,\lambda_2}} |B|^{\lambda_2} \sum_{i=0}^{k} 2^{-in\lambda_2} \\ &\le C \|b\|_{C\dot{M}O^{q_2,\lambda_2}} |B|^{\lambda_2}. \end{aligned} \tag{3.21}$$

Therefore,

$$I_2 \le C|B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda}\omega(t)dt. \tag{3.22}$$

Combining the estimates of I_1, I_2, and I_3, we conclude that H_ω^b is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$. □

Different from Theorem 3.1, it is still unknown whether the condition (2.1) in Theorem 3.3 is sharp. That is, whether the fact that H_ω^b is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$ for all $b \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n)$ induces (2.1)?

More general, we may extend the previous results to the kth order commutator of the weighted Hardy operator. Given $k \geq 1$ and a vector $\vec{b} = (b_1, \ldots, b_k)$, we define the higher order commutator of the weighted Hardy operator as

$$H_\omega^{\vec{b}} f(x) = \int_0^1 \left(\prod_{j=1}^{k} (b_j(x) - b_j(tx)) \right) f(tx)\omega(t)dt, \quad x \in \mathbb{R}^n. \tag{3.23}$$

When $k = 0$, we understand that $H_\omega^{\vec{b}} = H_\omega$. Notice that if $k = 1$, then $H_\omega^{\vec{b}} = H_\omega^{b}$.

Using the method in the proof of Theorems 3.1 and 3.3, we can also get the following Theorem 3.4. For the sake of convenience, we give the sketch of the proof of Theorem 3.4(i) here.

Theorem 3.4. *Let $k \geq 2$, $1 < q_1 < q, q_2, \ldots, q_k < \infty$, $1/q_1 = 1/q + \sum_{i=2}^{k} 1/q_i$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 \leq \lambda_2, \ldots, \lambda_k < 1/n$, and $\lambda_1 = \lambda + \sum_{i=2}^{k} \lambda_i$.*

(i) *Assume further that ω is a positive integrable function on $[0,1]$. The commutator $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda}(\mathbb{R}^n)$, for any $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$, if and only if*

$$\int_0^1 t^{n\lambda}\omega(t) \left(\log \frac{2}{t} \right)^{k-1} dt < \infty. \tag{3.24}$$

(ii) *Let $\lambda_2, \ldots, \lambda_k > 0$ and $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k,\lambda_k}(\mathbb{R}^n)$. If* (2.1) *holds true, then the corresponding commutator $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$.*

Proof. Let $R \in (0, \infty)$. Denote $B(0,R)$ by B and $B(0,tR)$ by tB. Assume $\mathbb{C} < \infty$. We get

$$\begin{aligned}
&\left(\frac{1}{|B|} \int_B \left| H_\omega^{\vec{b}} f(x) \right|^{q_1} dx \right)^{1/q_1} \\
&\quad \leq \left\{ \frac{1}{|B|} \int_B \left[\int_0^1 \left| \left(\prod_{j=2}^{k} (b_j(x) - b_j(tx)) \right) f(tx) \right| \omega(t)dt \right]^{q_1} dx \right\}^{1/q_1} \\
&\quad \leq C \sum_{I \subset \{2,\ldots,k\}} \sum_{J \subset \{2,\ldots,k\}, J \cap I = \emptyset} \left\{ \frac{1}{|B|} \int_B \left[\int_0^1 \left| \left(\prod_{i \in I} \prod_{j \in J} \prod_{m \in \{2,\ldots,k\} \setminus (I \cup J)} (b_i(x) - b_i(tx)) \right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. \times \left(b_j(x) - (b_j)_B \right) (b_m(tx) - (b_m)_{tB}) \right) f(tx) \right| \omega(t)dt \right]^{q_1} dx \right\}^{1/q_1}.
\end{aligned} \tag{3.25}$$

Then, applying the Minkowski inequality and the Hölder inequality (with $1/q_1 = 1/q + \sum_{i=2}^{k} 1/q_i$), and repeating the arguments in the proof of Theorem 3.1, $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1}(\mathbb{R}^n)$ for any $\vec{b} = (b_2, \dots, b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$, provided

$$\int_0^1 t^{n\lambda}\omega(t)\left(\log\frac{2}{t}\right)^{k-1} dt < \infty. \tag{3.26}$$

Conversely, assume that $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1}(\mathbb{R}^n)$ for any $\vec{b} = (b_2, \dots, b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$. We choose $\vec{b} = (b_2, \dots, b_k)$ with $b_j(x) = \log|x|$ for all $x \in \mathbb{R}^n$ and $j \in \{2, \dots, k\}$. Then $\vec{b} \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$. Repeating the argument in the proof of Theorem 3.1 then yields the desired conclusion. □

We point out that, it is still unknown whether the condition (2.1) in Theorem 3.4(ii) is sharp.

4. Adjoint Operators and Related Results

In this section, we focus on the corresponding results for the adjoint operators of weighted Hardy operators.

Recall that the *weighted Cesàro operator* G_ω is defined by

$$G_\omega f(x) = \int_0^1 f\left(\frac{x}{t}\right) t^{-n}\omega(t)dt, \quad x \in \mathbb{R}^n. \tag{4.1}$$

If $0 < \alpha < 1$, $n = 1$, and $\omega(t) = 1/(\Gamma(\alpha)((1/t) - 1)^{1-\alpha})$, then $G_\omega f(\cdot)$ is reduced to $(\cdot)^{1-\alpha} J_\alpha f(\cdot)$, where J_α is a variant of Weyl integral operator and defined by

$$J_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} \frac{dt}{t} \tag{4.2}$$

for all $x \in (0, \infty)$. When $\omega \equiv 1$ and $n = 1$, G_ω is the classical *Cesàro operator*:

$$Gf(x) = \begin{cases} \displaystyle\int_x^\infty \frac{f(y)}{y} dy, & x > 0, \\ \displaystyle -\int_{-\infty}^x \frac{f(y)}{y} dy, & x < 0. \end{cases} \tag{4.3}$$

It was pointed out in [5] that the weighted Hardy operator H_ω and the weighted Cesàro operator G_ω are adjoint mutually, namely,

$$\int_{\mathbb{R}^n} g(x) H_\omega f(x) dx = \int_{\mathbb{R}^n} f(x) G_\omega g(x) dx \tag{4.4}$$

for all admissible pairs f and g.

Since $\dot{A}^q$ and $\dot{B}^{q'}$ are a pair of dual Banach spaces, it follows from Theorem 2.1 the following.

Theorem 4.1. *Let $1 < q < \infty$. Then G_ω is bounded on $\dot{A}^q(\mathbb{R}^n)$ if and only if*

$$\mathbb{E} := \int_0^1 \omega(t)dt < \infty. \tag{4.5}$$

Moreover, when (4.5) holds, the operator norm of G_ω on $\dot{A}^q(\mathbb{R}^n)$ is given by

$$\|G_\omega\|_{\dot{A}^q(\mathbb{R}^n)\to\dot{A}^q(\mathbb{R}^n)} = \mathbb{E}. \tag{4.6}$$

Corollary 4.2. (i) *For $0 < \alpha < 1$ and $1 < q < \infty$,*

$$\|J_\alpha\|_{\dot{A}^q(dx)\to\dot{A}^q(x^{q(1-\alpha)}dx)} = \frac{\Gamma(1)}{\Gamma(1+\alpha)}. \tag{4.7}$$

(ii) *For $1 < q < \infty$, we have*

$$\|G\|_{\dot{A}^q(\mathbb{R}^n)\to\dot{A}^q(\mathbb{R}^n)} = 1. \tag{4.8}$$

Since the dual space of $H\dot{A}^q(1 < q < \infty)$ is isomorphic to $\mathrm{C\dot{M}O}^{q'}$ (see [18, 19]), Theorem 2.3 implies the following result.

Theorem 4.3. *Let $1 < q < \infty$. Then G_ω is a bounded operator on $H\dot{A}^q(\mathbb{R}^n)$ if and only if (4.5) holds. Moreover, when (4.5) holds, the operator norm of G_ω on $H\dot{A}^q(\mathbb{R}^n)$ is given by*

$$\|G_\omega\|_{H\dot{A}^q(\mathbb{R}^n)\to H\dot{A}^q(\mathbb{R}^n)} = \mathbb{E}. \tag{4.9}$$

Corollary 4.4. *For $1 < q < \infty$, we have*

$$\|G\|_{H\dot{A}^q\to H\dot{A}^q} = 1. \tag{4.10}$$

Following the idea in Section 3, we define the higher order commutator of the weighted Cesàro operator as

$$G_\omega^{\vec{b}} f(x) = \int_0^1 \left(\prod_{j=1}^{k}\left(b_j\left(\frac{x}{t}\right) - b_j(x)\right)\right) f\left(\frac{x}{t}\right) t^{-n}\omega(t)dt, \quad x \in \mathbb{R}^n. \tag{4.11}$$

When $k = 0$, $G_\omega^{\vec{b}}$ is understood as G_ω. Notice that if $k = 1$, then $G_\omega^{\vec{b}} = G_\omega^{b}$. Similar to the proofs of Theorems 3.1 and 3.3, we have the following result.

Theorem 4.5. *Let $k \geq 2$, $1 < q_1 < q, q_2, \ldots, q_k < \infty$, $1/q_1 = 1/q + \sum_{i=2}^{k} 1/q_i$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 \leq \lambda_2, \ldots, \lambda_k < 1/n$, and $\lambda_1 = \lambda + \sum_{i=2}^{k} \lambda_i$.*

(i) *Assume further that* ω *is a positive integrable function on* $[0,1]$. *The commutator* $G_\omega^{\vec{b}}$ *is bounded from* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *to* $\dot{B}^{q_1,\lambda}(\mathbb{R}^n)$, *for any* $\vec{b} = (b_2,\ldots,b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$, *if and only if*

$$\int_0^1 t^{-n(\lambda+1)}\omega(t)\left(\log\frac{2}{t}\right)^{k-1} dt < \infty. \tag{4.12}$$

(ii) *Let* $\lambda_2,\ldots,\lambda_k > 0$ *and* $\vec{b} = (b_2,\ldots,b_k) \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k,\lambda_k}(\mathbb{R}^n)$. *Then the corresponding commutator* $G_\omega^{\vec{b}}$ *is bounded from* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *to* $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$, *provided that*

$$\int_0^1 t^{-n(\lambda+1)}\omega(t)dt < \infty. \tag{4.13}$$

We conclude this paper with some comments on the discrete version of the weighted Hardy and Cesàro operators.

Let $\mathbb{N}_0$ be the set of all nonnegative integers and $2^{-\mathbb{N}_0}$ denote the set $\{2^{-j} : j \in \mathbb{N}_0\}$. Let now φ be a nonnegative function defined on $2^{-\mathbb{N}_0}$ and f be a complex-valued measurable function on $\mathbb{R}^n$. The *discrete weighted Hardy operator* $\widetilde{H}_\omega$ is defined by

$$\left(\widetilde{H}_\omega f\right)(x) = \sum_{k=0}^{\infty} 2^{-k} f\left(2^{-k}x\right)\omega\left(2^{-k}\right), \quad x \in \mathbb{R}^n, \tag{4.14}$$

and the corresponding *discrete weighted Cesàro operator* is defined by setting, for all $x \in \mathbb{R}^n$,

$$\left(\widetilde{G}_\omega f\right)(x) = \sum_{k=0}^{\infty} f\left(2^k x\right)2^{k(n-1)}\omega\left(2^{-k}\right). \tag{4.15}$$

We remark that, by the same argument as above with slight modifications, all the results related to the operators H_ω and G_ω in Sections 1–4 are also true for their discrete versions $\widetilde{H}_\omega$ and $\widetilde{G}_\omega$.

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Research Article

Existence of Three Solutions for a Nonlinear Fractional Boundary Value Problem via a Critical Points Theorem

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This paper is concerned with the existence of three solutions to a nonlinear fractional boundary value problem as follows: $(d/dt)((1/2)_0D_t^{\alpha-1}({}_0^CD_t^\alpha u(t)) - (1/2)_tD_T^{\alpha-1}({}_t^CD_T^\alpha u(t))) + \lambda a(t)f(u(t)) = 0$, a.e. $t \in [0,T], u(0) = u(T) = 0$, where $\alpha \in (1/2, 1]$, and λ is a positive real parameter. The approach is based on a critical-points theorem established by G. Bonanno.

1. Introduction

Differential equations with fractional order have recently proved to be strong tools in the modeling of many physical phenomena in various fields of physical, chemical, biology, engineering, and economics. There has been significant development in fractional differential equations, one can see the monographs [1–5] and the papers [6–20] and the references therein.

Critical-point theory, which proved to be very useful in determining the existence of solution for integer-order differential equation with some boundary conditions, for example, one can refer to [21–25]. But till now, there are few results on the solution to fractional boundary value problem which were established by the critical-point theory, since it is often very difficult to establish a suitable space and variational functional for fractional boundary value problem. Recently, Jiao and Zhou [26] investigated the following fractional boundary value problem:

$$\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}\,(u'(t))\right) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T], \qquad (1.1)$$
$$u(0) = u(T) = 0$$

by using the critical point theory, where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$, respectively, $F : [0,T] \times \mathbf{R}^N \to \mathbf{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x.

In this paper, by using the critical-points theorem established by Bonanno in [27], a new approach is provided to investigate the existence of three solutions to the following fractional boundary value problems:

$$\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{\alpha-1}\left({}_0^C D_t^{a} u(t)\right) - \frac{1}{2}\,{}_tD_T^{\alpha-1}\left({}_t^C D_T^{a} u(t)\right)\right) + \lambda a(t) f(u(t)) = 0, \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0, \tag{1.2}$$

where $\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order α respectively, λ is a positive real parameter, $f : \mathbf{R} \to \mathbf{R}$ is a continuous function, and $a : \mathbf{R} \to \mathbf{R}$ is a nonnegative continuous function with $a(t) \not\equiv 0$.

2. Preliminaries

In this section, we first introduce some necessary definitions and properties of the fractional calculus which are used in this paper.

Definition 2.1 (see [5]). Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order α for function f denoted by ${}_aD_t^{-\alpha} f(t)$ and ${}_tD_b^{-\alpha} f(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds, \quad t \in [a,b],\ \alpha > 0, \\ {}_tD_b^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\,ds, \quad t \in [a,b],\ \alpha > 0, \end{aligned} \tag{2.1}$$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2 (see [5]). Let $\gamma \geq 0$ and $n \in \mathbf{N}$.

(i) If $\gamma \in (n-1, n)$ and $f \in AC^n([a,b], \mathbf{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^CD_t^{\gamma} f(t)$ and ${}_t^CD_b^{\gamma} f(t)$, respectively, exist almost everywhere on $[a,b]$, ${}_a^CD_t^{\gamma} f(t)$ and ${}_t^CD_b^{\gamma} f(t)$ are represented by

$$\begin{aligned} {}_a^CD_t^{\gamma} f(t) &= \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-s)^{n-\gamma-1} f^{(n)}(s)\,ds, \quad t \in [a,b], \\ {}_t^CD_b^{\gamma} f(t) &= \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^b (s-t)^{n-\gamma-1} f^{(n)}(s)\,ds, \quad t \in [a,b], \end{aligned} \tag{2.2}$$

respectively.

(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a,b],\mathbf{R}^N)$, then ${}_a^C D_t^{n-1} f(t)$ and ${}_t^C D_b^{n-1} f(t)$ are represented by

$$ {}_a^C D_t^{n-1} f(t) = f^{(n-1)}(t), \qquad {}_t^C D_b^{n-1} f(t) = (-1)^{(n-1)} f^{(n-1)}(t), \quad t \in [a,b]. \tag{2.3}$$

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [2, 5].

Property 1 (see [2, 5]). we have the following property of fractional integration:

$$\int_a^b \left[{}_a D_t^{-\gamma} f(t) \right] g(t) dt = \int_a^b \left[{}_t D_b^{-\gamma} g(t) \right] f(t) dt, \quad \gamma > 0 \tag{2.4}$$

provided that $f \in L^p([a,b],\mathbf{R}^N)$, $g \in L^q([a,b],\mathbf{R}^N)$, and $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1+\gamma$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1+\gamma$.

Property 2 (see [5]). Let $n \in \mathbf{N}$ and $n-1 < \gamma \leq n$. If $f \in AC^n([a,b],\mathbf{R}^N)$ or $f \in C^n([a,b],\mathbf{R}^N)$, then

$$\begin{gathered} {}_a D_t^{-\gamma} \left({}_a^C D_t^{\gamma} f(t) \right) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (t-a)^j, \\ {}_t D_b^{-\gamma} \left({}_t^C D_b^{\gamma} f(t) \right) = f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!} (b-t)^j, \end{gathered} \tag{2.5}$$

for $t \in [a,b]$. In particular, if $0 < \gamma \leq 1$ and $f \in AC([a,b],\mathbf{R}^N)$ or $f \in C^1([a,b],\mathbf{R}^N)$, then

$$ {}_a D_t^{-\gamma} \left({}_a^C D_t^{\gamma} f(t) \right) = f(t) - f(a), \qquad {}_t D_b^{-\gamma} \left({}_t^C D_b^{\gamma} f(t) \right) = f(t) - f(b). \tag{2.6}$$

Remark 2.3. In view of Property 1 and Definition 2.2, it is obvious that $u \in AC([0,T])$ is a solution of BVP (1.2) if and only if u is a solution of the following problem:

$$\begin{gathered} \frac{d}{dt}\left(\frac{1}{2} {}_0 D_t^{-\beta}(u'(t)) + \frac{1}{2} {}_t D_T^{-\beta}(u'(t)) \right) + \lambda a(t) f(u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{2.7}$$

where $\beta = 2(1-\alpha) \in [0,1)$.

In order to establish a variational structure for BVP (1.2), it is necessary to construct appropriate function spaces.

Denote by $C_0^\infty[0,T]$ the set of all functions $g \in C^\infty[0,T]$ with $g(0) = g(T) = 0$.

Definition 2.4 (see [26]). Let $0 < \alpha \leq 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty[0,T]$ with respect to the norm

$$\|u\|_\alpha = \left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^2 dt + \int_0^T |u(t)|^2 dt\right)^{1/2}, \quad \forall u \in E_0^\alpha. \tag{2.8}$$

Remark 2.5. It is obvious that the fractional derivative space E_0^α is the space of functions $u \in L^2[0,T]$ having an α-order Caputo fractional derivative ${}_0^C D_t^\alpha u \in L^2[0,T]$ and $u(0) = u(T) = 0$.

Proposition 2.6 (see [26]). *Let $0 < \alpha \leq 1$. The fractional derivative space E_0^α is reflexive and separable Banach space.*

Lemma 2.7 (see [26]). *Let $1/2 < \alpha \leq 1$. For all $u \in E_0^\alpha$, one has the following:*

(i)

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\|{}_0^C D_t^\alpha u\right\|_{L^2}. \tag{2.9}$$

(ii)

$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}} \left\|{}_0^C D_t^\alpha u\right\|_{L^2}. \tag{2.10}$$

By (2.9), we can consider E_0^α with respect to the norm

$$\|u\|_\alpha = \left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^2 dt\right)^{1/2} = \left\|{}_0^C D_t^\alpha u\right\|_{L^2}, \quad \forall u \in E_0^\alpha \tag{2.11}$$

in the following analysis.

Lemma 2.8 (see [26]). *Let $1/2 < \alpha \leq 1$, then for all any $u \in E_0^\alpha$, one has*

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq -\int_0^T {}_0^C D_t^\alpha u(t) \cdot {}_t^C D_T^\alpha u(t) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2. \tag{2.12}$$

Our main tool is the critical-points theorem [27] which is recalled below.

Theorem 2.9 (see [27]). *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbf{R}$ be a nonnegative continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbf{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = \Psi(x_0) = 0$, and that*

(i) $\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda\Psi(x)) = +\infty, forall \lambda \in [0, +\infty]$. *Further, assume that there are $r > 0$, $x_1 \in X$ such that*

(ii) $r < \Phi(x_1)$;

(iii) $\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x) < (r/(r + \Phi(x_1)))\Psi(x_1)$.

Then, for each

$$\lambda \in \Lambda_1 = \left] \frac{\Phi(x_1)}{\Psi(x_1) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x)}, \frac{r}{\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x)} \right[, \tag{2.13}$$

the equation

$$\Phi'(x) - \lambda \Psi'(x) = 0 \tag{2.14}$$

has at least three solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subset \left[0, \frac{hr}{(r(\Psi(x_1)/\Phi(x_1))) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(x)} \right] \tag{2.15}$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, (2.14) *has at least three solutions in X whose norms are less than σ.*

3. Main Result

For given $u \in E_0^\alpha$, we define functionals $\Phi, \Psi : E^\alpha \to \mathbf{R}$ as follows:

$$\begin{gathered} \Phi(u) := -\frac{1}{2} \int_0^T {}_0^C D_t^\alpha u(t) \cdot {}_t^C D_T^\alpha u(t) dt, \\ \Psi(u) := \int_0^T a(t) F(u(t)) dt, \end{gathered} \tag{3.1}$$

where $F(u) = \int_0^u f(s) ds$. Clearly, Φ and Ψ are Gateaux differentiable functional whose Gateaux derivative at the point $u \in E_0^\alpha$ are given by

$$\begin{gathered} \Phi'(u)v = -\frac{1}{2} \int_0^T \left({}_0^C D_t^\alpha u(t) \cdot {}_t^C D_T^\alpha v(t) + {}_t^C D_T^\alpha u(t) \cdot {}_0^C D_t^\alpha v(t) \right) dt, \\ \Psi'(u)v = \int_0^T a(t) f(u(t)) v(t) dt = -\int_0^T \int_0^t a(s) f(u(s)) ds \cdot v'(t) dt, \end{gathered} \tag{3.2}$$

for every $v \in E_0^\alpha$. By Definition 2.2 and Property 2, we have

$$\Phi'(u)v = \int_0^T \left(\frac{1}{2} {}_0D_t^{\alpha-1}\left({}_0^C D_t^\alpha u(t) \right) - \frac{1}{2} {}_tD_T^{\alpha-1}\left({}_t^C D_T^\alpha u(t) \right) \right) \cdot v'(t)dt. \tag{3.3}$$

Hence, $I_\lambda = \Phi - \lambda\Psi \in C^1(E_0^\alpha, \mathbf{R})$. If $u_* \in E_0^\alpha$ is a critical point of I_λ, then

$$\begin{aligned} 0 &= I_\lambda'(u_*)v \\ &= \int_0^T \left(\frac{1}{2} {}_0D_t^{\alpha-1}\left({}_0^C D_t^\alpha u_*(t) \right) - \frac{1}{2} {}_tD_T^{\alpha-1}\left({}_t^C D_T^\alpha u_*(t) \right) \right. \\ &\qquad \left. +\lambda \int_0^t a(s) f(u_*(s)) ds \right) \cdot v'(t)dt, \end{aligned} \tag{3.4}$$

for $v \in E_0^\alpha$. We can choose $v \in E_0^\alpha$ such that

$$v(t) = \sin \frac{2k\pi t}{T} \quad \text{or} \quad v(t) = 1 - \cos \frac{2k\pi t}{T}, \quad k = 1, 2, \ldots. \tag{3.5}$$

The theory of Fourier series and (3.4) imply that

$$\frac{1}{2} {}_0D_t^{\alpha-1}\left({}_0^C D_t^\alpha u_*(t) \right) - \frac{1}{2} {}_tD_T^{\alpha-1}\left({}_t^C D_T^\alpha u_*(t) \right) + \lambda \int_0^t a(s) f(u_*(s)) ds = C \tag{3.6}$$

a.e. on $[0, T]$ for some $C \in \mathbf{R}$. By (3.6), it is easy to know that $u_* \in E_0^\alpha$ is a solution of BVP (1.2).

By Lemma 2.7, if $\alpha > 1/2$, we have for each $u \in E_0^\alpha$ that

$$\|u\|_\infty \le \Omega \left(\int_0^T \left| {}_0^C D_t^\alpha u(t) \right|^2 dt \right)^{1/2} = \Omega \|u\|_\alpha, \tag{3.7}$$

where

$$\Omega = \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}. \tag{3.8}$$

Given two constants $c \ge 0$ and $d \ne 0$, with $c \ne \sqrt{(2A(\alpha)/|\cos(\pi\alpha)|)}\Omega \cdot d$, where Ω as in (3.8).

For convenience, set

$$A(\alpha) := \frac{8\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)} T^{1-2\alpha} \left(\left(1 + 3^{3-2\alpha} \right) 2^{4\alpha-5} - 2^{2\alpha-3} - 1 \right). \tag{3.9}$$

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $a : \mathbf{R} \to \mathbf{R}$ be a nonnegative continuous function with $a(t) \not\equiv 0$, and $1/2 < \alpha \leq 1$. Put $F(x) = \int_0^x f(s)ds$ for every $x \in \mathbb{R}$, and assume that there exist four positive constants c, d, μ, and p, with $c < \sqrt{(2A(\alpha)/|\cos(\pi\alpha)|)}\Omega \cdot d$ and $p < 2$, such that*

(H1) $F(x) \leq \mu(1 + |x|^p)$, *for all* $x \in \mathbb{R}$;

(H2) $F(x) \geq 0$*for all* $x \in [0, \Gamma(2-\alpha)d]$, *and*

$$F(x) < \frac{|\cos(\pi\alpha)|c^2}{(|\cos(\pi\alpha)|c^2 + 2\Omega^2 A(\alpha)d^2)\int_0^T a(t)dt} \times \left[F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(s)F(s)ds\right], \quad \forall x \in [-c, c], \tag{3.10}$$

where $b(s) = a((T/4\Gamma(2-\alpha)d)s) + a(T - (T/4\Gamma(2-\alpha)d)s)$. *Then, for each*

$$\lambda \in \Lambda_1 = \left]\frac{A(\alpha)d^2}{\Re_a + \Re\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx - \int_0^T a(t)dt \cdot \max_{|x|\leq c} F(x)}, \frac{c^2|\cos(\pi\alpha)|}{2\Omega^2\int_0^T a(t)dt \cdot \max_{|x|\leq c} F(x)}\right[, \tag{3.11}$$

where $\Re_a$ and $\Re$ denote $F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt$ and $T/(4\Gamma(2-\alpha)d)$ respectively, the problem (1.2) *admits at least three solutions in E_0^α and, moreover, for each $h > 1$, there exists an open interval*

$$\Lambda_2 \subset \left[0, \frac{hA(\alpha)d^2}{\Re_a + \Re\int_0^{2\Gamma(2-\alpha)} b(x)F(x)dx - (2\Omega^2 A(\alpha)d^2/c^2|\cos(\pi\alpha)|)\int_0^T a(t)dt \cdot \max_{|x|\leq c} F(x)}\right] \tag{3.12}$$

such that, for each $\lambda \in \Lambda_2$, the problem (1.2) *admits at least three solutions in E_0^α whose norms are less that σ.*

Proof. Let Φ, Ψ be the functionals defined in the above. By the Lemma 5.1 in [26], Φ is continuous and convex, hence it is weakly sequentially lower semicontinuous. Moreover, Φ is coercive, continuously Gateaux differentiable functional whose Gateaux derivative admits a continuous inverse on E_0^α. The functional Ψ is well defined, continuously Gateaux differentiable and with compact derivative. It is well known that the critical point of the functional $\Phi - \lambda\Psi$ in E_0^α is exactly the solution of BVP (1.2). □

From (H1) and (2.12), we get

$$\lim_{\|u\|_\alpha \to +\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty, \tag{3.13}$$

for all $\lambda \in [0, +\infty[$. Put

$$u_1(t) = \begin{cases} \dfrac{4\Gamma(2-\alpha)d}{T} t, & t \in \left[0, \dfrac{T}{4}\right[, \\ \Gamma(2-\alpha)d, & t \in \left[\dfrac{T}{4}, \dfrac{3T}{4}\right], \\ \dfrac{4\Gamma(2-\alpha)d}{T}(T-t), & t \in \left]\dfrac{T}{4}, T\right]. \end{cases} \tag{3.14}$$

It is easy to check that $u_1(0) = u_1(T) = 0$ and $u_1 \in L^2[0,T]$. The direct calculation shows

$$\begin{gathered} {}_0^C D_t^\alpha u_1(t) = \begin{cases} \dfrac{4d}{T} t^{1-\alpha}, & t \in \left[0, \dfrac{T}{4}\right[, \\ \dfrac{4d}{T}\left(t^{1-\alpha} - \left(t - \dfrac{T}{4}\right)^{1-\alpha}\right), & t \in \left[\dfrac{T}{4}, \dfrac{3T}{4}\right], \\ \dfrac{4d}{T}\left(t^{1-\alpha} - \left(t - \dfrac{T}{4}\right)^{1-\alpha} - \left(t - \dfrac{3T}{4}\right)^{1-\alpha}\right), & t \in \left]\dfrac{3T}{4}, T\right], \end{cases} \\ \begin{aligned} \|u_1\|_\alpha^2 &= \int_0^T \left({}_0^C D_t^\alpha u_1(t)\right)^2 dt = \int_0^{T/4} + \int_{T/4}^{3T/4} + \int_{3T/4}^T \left({}_0^C D_t^\alpha u_1(t)\right)^2 dt \\ &= \frac{16d^2}{T^2}\Bigg[\int_0^T t^{2(1-\alpha)} dt + \int_{T/4}^T \left(t - \frac{T}{4}\right)^{2(1-\alpha)} dt + \int_{3T/4}^T \left(t - \frac{3T}{4}\right)^{2(1-\alpha)} dt \\ &\qquad - 2\int_{T/4}^T t^{1-\alpha}\left(t - \frac{T}{4}\right)^{1-\alpha} dt - 2\int_{3T/4}^T t^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt \\ &\qquad + 2\int_{3T/4}^T \left(t - \frac{T}{4}\right)^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt\Bigg] \\ &= \frac{16d^2}{T^2}\Bigg[\left(1 + \left(\frac{3}{4}\right)^{3-2\alpha} + \left(\frac{1}{4}\right)^{3-2\alpha}\right)\frac{T^{3-2\alpha}}{3-2\alpha} - 2\int_{T/4}^T t^{1-\alpha}\left(t - \frac{T}{4}\right)^{1-\alpha} dt \\ &\qquad - 2\int_{3T/4}^T t^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt + 2\int_{3T/4}^T \left(t - \frac{T}{4}\right)^{1-\alpha}\left(t - \frac{3T}{4}\right)^{1-\alpha} dt\Bigg] < \infty \quad . \end{aligned} \end{gathered} \tag{3.15}$$

That is, ${}_0^C D_t^\alpha u_1 \in L^2[0,T]$. Thus, $u_1 \in E_0^\alpha$. Moreover, the direct calculation shows

$$
\begin{gathered}
{}_t^C D_T^\alpha u_1(t) = \begin{cases} \dfrac{4d}{T}\left((T-t)^{1-\alpha} - \left(\dfrac{3T}{4}-t\right)^{1-\alpha} - \left(\dfrac{T}{4}-t\right)^{1-\alpha}\right), & t \in \left[0, \dfrac{T}{4}\right[, \\ \dfrac{4d}{T}\left((T-t)^{1-\alpha} - \left(\dfrac{3T}{4}-t\right)^{1-\alpha}\right), & t \in \left[\dfrac{T}{4}, \dfrac{3T}{4}\right], \\ \dfrac{4d}{T}(T-t)^{1-\alpha}, & t \in \left]\dfrac{3T}{4}, T\right], \end{cases} \\
\begin{aligned}
\Phi(u_1) &= -\frac{1}{2}\int_0^T {}_0^C D_t^\alpha u_1(t) \cdot {}_t^C D_T^\alpha u_1(t)\,dt \\
&= -\frac{8d^2}{T^2}\Bigg[\int_0^{T/4} t^{1-\alpha}\left((T-t)^{1-\alpha} - \left(\frac{3T}{4}-t\right)^{1-\alpha} - \left(\frac{T}{4}-t\right)^{1-\alpha}\right)dt \\
&\quad + \int_{T/4}^{3T/4}\left(t^{1-\alpha} - \left(t-\frac{T}{4}\right)^{1-\alpha}\right)\left((T-t)^{1-\alpha} - \left(\frac{3T}{4}-t\right)^{1-\alpha}\right)dt \\
&\quad + \int_{3T/4}^{T}\left(t^{1-\alpha} - \left(t-\frac{T}{4}\right)^{1-\alpha} - \left(t-\frac{3T}{4}\right)^{1-\alpha}\right)(T-t)^{1-\alpha}dt\Bigg] \\
&= -\frac{8d^2}{T^2}\Bigg[\int_0^T t^{1-\alpha}(T-t)^{1-\alpha}dt - \int_0^{T/4} t^{1-\alpha}\left(\frac{T}{4}-t\right)^{1-\alpha}dt \\
&\quad + \int_{T/4}^{3T/4}\left(t-\frac{T}{4}\right)^{1-\alpha}\left(\frac{3T}{4}-t\right)^{1-\alpha}dt - \int_{3T/4}^{T}\left(t-\frac{3T}{4}\right)^{1-\alpha}(T-t)^{1-\alpha}dt \\
&\quad - \int_0^{3T/4} t^{1-\alpha}\left(\frac{3T}{4}-t\right)^{1-\alpha} - \int_{T/4}^{T}\left(t-\frac{T}{4}\right)^{1-\alpha}(T-t)^{1-\alpha}dt\Bigg] \\
&= \frac{8\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{1-2\alpha}d^2\left(\left(1+3^{3-2\alpha}\right)2^{4\alpha-5} - 2^{2\alpha-3} - 1\right) = A(\alpha)d^2, \\
\Psi(u_1) &= \int_0^T a(t)F(u_1(t))\,dt \\
&= \int_0^{T/4} a(t)F\left(\frac{4\Gamma(2-\alpha)d}{T}t\right)dt + \int_{T/4}^{3T/4} a(t)F(\Gamma(2-\alpha)d)\,dt \\
&\quad + \int_{3T/4}^{T} a(t)F\left(\frac{4\Gamma(2-\alpha)d}{T}(T-t)\right)dt \\
&= F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)\,dt + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(x)F(x)\,dx.
\end{aligned}
\end{gathered}
\tag{3.16}
$$

Let $r = (|\cos(\pi\alpha)|/2\Omega^2)c^2$. Since $c < \sqrt{(2A(\alpha)/|\cos(\pi\alpha)|)}\Omega \cdot d$, we obtain $r < \Phi(u_1)$.

By (2.12) and (3.7), one has $\Phi(u) \leq r \Rightarrow \|u\|_\infty \leq c$. Thus,

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) \leq \max_{|x|\leq c} F(x) \int_0^T a(t)dt. \tag{3.17}$$

Moreover, we have

$$\begin{aligned}
&\frac{r}{r+\Phi(u_1)}\Psi(u_1)\\
&\quad= \frac{\left(|\cos(\pi\alpha)|/2\Omega^2\right)c^2}{(|\cos(\pi\alpha)|/2\Omega^2)c^2 + A(\alpha)d^2}\\
&\qquad\times\left[F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx\right]\\
&\quad= \frac{|\cos(\pi\alpha)|c^2}{|\cos(\pi\alpha)|c^2 + 2\Omega^2 A(\alpha)d^2}\\
&\qquad\times\left[F(\Gamma(2-\alpha)d)\int_{T/4}^{3T/4} a(t)dt + \frac{T}{4\Gamma(2-\alpha)d}\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx\right].
\end{aligned} \tag{3.18}$$

Hence, from (H2) one has

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u) < \frac{r}{r+\Phi(u_1)}\Psi(u_1). \tag{3.19}$$

Now, taking into account that

$$\begin{aligned}
&\frac{\Phi(u_1)}{\Psi(u_1) - \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)}\\
&\quad\leq \frac{A(\alpha)d^2}{\Re_a + \Re\int_0^{\Gamma(2-\alpha)d} b(x)F(x)dx - \int_0^T a(t)dt\cdot\max_{|x|\leq c} F(x)},\\
&\frac{r}{\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)} \geq \frac{c^2|\cos(\pi\alpha)|}{2\Omega^2\int_0^T a(t)dt\cdot\max_{|x|\leq c} F(x)},\\
&\frac{hr}{r(\Psi(u_1)/\Phi(u_1)) - \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}}\Psi(u)}\\
&\quad\leq \frac{hA(\alpha)d^2}{\Re_a + \Re\int_0^{2\Gamma(2-\alpha)} b(x)F(x)dx - (2\Omega^2 A(\alpha)d^2/c^2|\cos(\pi\alpha)|)\int_0^T a(t)dt\cdot\max_{|x|\leq c} F(x)}\\
&\quad= m.
\end{aligned} \tag{3.20}$$

Thus, by Theorem 2.9 it follows that, for each $\lambda \in \Lambda_1$, BVP (1.2) admits at least three solutions, and there exists an open interval $\Lambda_2 \subset [0, m]$ and a real positive number σ such that, for each $\lambda \in \Lambda_2$, BVP (1.2) admits at least three solutions in E_0^{α} whose norms are less than σ.

Finally, we give an example to show the effectiveness of the results obtained here.

Let $\alpha = 0.8$, $T = 1$, $a(t) \equiv 1$, and $f(u) = e^{-u}u^8(9-u) + \sqrt{u}$. Then BVP (1.2) reduces to the following boundary value problem:

$$\begin{gathered}\frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-0.2}\left({}_0^CD_t^{0.8}u(t)\right) - \frac{1}{2}{}_tD_1^{-0.2}\left({}_t^CD_1^{0.8}u(t)\right)\right) + \lambda\left(e^{-u}u^8(9-u) + \sqrt{u}\right) \\ = 0, \quad \text{a.e. } t \in [0,1], \\ u(0) = u(1) = 0.\end{gathered} \tag{3.21}$$

Example 3.2. Owing to Theorem 3.1, for each $\lambda \in]0.291, 0.318[$, BVP (3.21) admits at least three solutions. In fact, put $c = 1$ and $d = 2$, it is easy to calculate that $\Omega = 1.1089$, $A(0.8) = 1.3313$, and

$$\sqrt{\frac{2A(0.8)}{|\cos(0.8\pi)|}}\Omega \cdot d = 4.0235 > 1 = c. \tag{3.22}$$

Since

$$F(x) = \int_0^x f(s)ds = e^{-x}x^9 + \frac{2}{3}x^{3/2}, \tag{3.23}$$

we have that condition (H1) holds. Moreover, $F(x) \geq 0$ for each $x \in [0, 2\Gamma(1.2)]$, and

$$\begin{aligned}&\frac{|\cos(0.8\pi)|}{|\cos(0.8\pi)| + 2\Omega^2 A(0.8)\cdot 2^2}\left[\frac{1}{2}F(2\Gamma(1.2)) + \frac{1}{4\Gamma(1.2)}\int_0^{2\Gamma(1.2)} F(s)ds\right] \\ &\quad > 1.064 > 1.0345 = e^{-1} + \frac{2}{3} \geq F(x), \quad |x| \leq 1,\end{aligned} \tag{3.24}$$

which implies that condition (H2) holds. Thus, by Theorem 3.1, for each $\lambda \in]0.291, 0.318[$, the problem (3.21) admits at least three nontrivial solutions in $E_0^{0.8}$. Moreover, for each $h > 1$, there exists an open interval $\Lambda \subset]0, 3.4674h[$ and a real positive number σ such that, for each $\lambda \in \Lambda$, the problem (3.21) admits at least three solutions in $E_0^{0.8}$ whose norms are less than σ.

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Research Article

Finite Element Solutions for the Space Fractional Diffusion Equation with a Nonlinear Source Term

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We consider finite element Galerkin solutions for the space fractional diffusion equation with a nonlinear source term. Existence, stability, and order of convergence of approximate solutions for the backward Euler fully discrete scheme have been discussed as well as for the semidiscrete scheme. The analytical convergent orders are obtained as $O(k + h^{\tilde{\gamma}})$, where $\tilde{\gamma}$ is a constant depending on the order of fractional derivative. Numerical computations are presented, which confirm the theoretical results when the equation has a linear source term. When the equation has a nonlinear source term, numerical results show that the diffusivity depends on the order of fractional derivative as we expect.

1. Introduction

Fractional calculus is an old mathematical topic but it has not been attracted enough for almost three hundred years. However, it has been recently proven that fractional calculus is a significant tool in the modeling of many phenomena in various fields such as engineering, physics, porous media, economics, and biological sciences. One can see related references in [1–7].

In the classical diffusion model, it is assumed that particles are distributed in a normal bell-shaped pattern based on the Brownian motion. In general, the nature of diffusion is characterized by the mean squared displacement

$$\left\langle (\triangle r)^2 \right\rangle = 2d\kappa_\mu t^\mu, \tag{1.1}$$

where d is the spatial dimension and κ_μ is the diffusion constant. The classical normal diffusion case arises when the exponent $\mu = 1$. When $\mu \neq 1$, anomalous diffusions arise.

The anomalous diffusion is classified as the process is subdiffusive (diffusive slowly) when $\mu < 1$ or superdiffusive (diffusive fast) when $\mu > 1$.

As mentioned before, in many real problems, it is more adequate to use anomalous diffusion described by fractional derivatives than the classical normal diffusion [4, 5, 8–12]. One typical model for anomalous diffusion is the fractional superdiffusion equation arising in chaotic and turbulent processes, where the usual second derivative in space is replaced by a fractional derivative of order $1 < \mu < 2$.

In this paper we discuss Galerkin approximate solutions for the space fractional diffusion equation with a nonlinear source term. The equation is described as

$$\frac{\partial u(x,t)}{\partial t} = \kappa_\mu \nabla^\mu u(x,t) + f(x,t,u) \tag{1.2}$$

with an initial condition

$$u(x,0) = u_0, \quad x \in \Omega \subset \mathbf{R} \tag{1.3}$$

and boundary conditions

$$u(x,t) = 0, \quad x \in \partial\Omega,\ 0 \le t \le T, \tag{1.4}$$

where κ_μ denotes the anomalous diffusion coefficient and $\partial\Omega$ is the boundary of the domain Ω. And the differential operator ∇^μ is

$$\nabla^\mu = \frac{1}{2}\,{}_aD_x^\mu + \frac{1}{2}\,{}_xD_b^\mu, \tag{1.5}$$

where ${}_aD_x^\mu$ and ${}_xD_b^\mu$ are called the left and the right Riemann-Liouville space fractional derivatives of order μ, respectively, defined by

$$\begin{aligned}\mathbf{D}^\mu u &:= {}_aD_x^\mu u(x) = D^n\,{}_aD_x^{\mu-n} u(x) = \frac{1}{\Gamma(n-\mu)}\frac{d^n}{dx^n}\int_a^x (x-\xi)^{n-\mu-1}u(\xi)d\xi,\\ \mathbf{D}^{\mu *} u &:= {}_xD_b^\mu u(x) = (-D)^n\,{}_xD_b^{\mu-n} u(x) = \frac{(-1)^n}{\Gamma(n-\mu)}\frac{d^n}{dx^n}\int_x^b (\xi-x)^{n-\mu-1}u(\xi)d\xi.\end{aligned} \tag{1.6}$$

Here n is the smallest integer such that $n-1 \le \mu < n$.

Throughout this paper, we will assume that the nonlinear source term $f(x,t,u)$ is locally Lipschitz continuous with constants C_l and C_f such that

$$\|f(u) - f(v)\|_{L^2(\Omega)} \le C_l \|u - v\|_{L^2(\Omega)}, \tag{1.7}$$

$$\|f(u)\|_{L^2(\Omega)} \le C_f \|u\|_{L^2(\Omega)} \tag{1.8}$$

for $u, v \in \{w \in H_0^{\mu/2}(\Omega) \mid \|w\|_{L^2(\Omega)} \le l\}$.

Baeumer et al. [8, 13] have proved existence and uniqueness of a strong solution for (1.2) using the semigroup theory when $f(x,t,u)$ is globally Lipschitz continuous. Furthermore, when $f(x,t,u)$ is locally Lipschitz continuous, existence of a unique strong solution has also been shown by introducing the cut-off function.

Finite difference methods have been studied in [14–16] for linear space fractional diffusion problems. They used the right-shifted Grüwald-Letnikov approximate for the fractional derivative since the standard Grüwald-Letnikov approximate gives the unconditional instability even for the implicit method. Using the right-shifted Grüwald-Letnikov approximation, the method of lines has been applied in [12] for numerical approximate solutions.

For the space fractional diffusion problems with a nonlinear source term, Lynch et al. [17] used the so-called L2 and L2C methods in [6] and compared computational accuracy of them. Baeumer et al. [8] give existence of the solution and computational results using finite difference methods. Choi et al. [18] have shown existence and stability of numerical solutions of an implicit finite difference equation obtained by using the right-shifted Grüwald-Letnikov approximation. For the time fractional diffusion equations, explicit and implicit finite difference methods have been used in [11, 19–23].

Compared to finite difference methods on the fractional diffusion equation, finite element methods have been rarely discussed. Ervin and Roop [24] have considered finite element analysis for stationary linear advection dispersion equations, and Roop [25] has studied finite element analysis for nonstationary linear advection dispersion equations. The finite element numerical approximations have been discussed for the time and space fractional Fokker-Planck equation in Deng [9] and for the space general fractional diffusion equations with a nonlocal quadratic nonlinearity but a linear source term in Ervin et al. [26].

As far as we know, finite element methods have not been considered for the space fractional diffusion equation with nonlinear source terms. In this paper, we will discuss finite element solutions for the problem (1.2)–(1.4) under the assumption of existence of a sufficiently regular solution u of the equation. Finite element numerical analysis of the semidiscrete and fully discrete methods for (1.2)–(1.4) will be considered using the backward Euler method in time and Galerkin finite element method in space as well as the semidiscrete method. We will discuss existence, uniqueness, and stability of the numerical solutions for the problem (1.2)–(1.4). Also, L^2-error estimate will be considered for the problem (1.2)–(1.4).

The outline of the paper is as follows. We introduce some properties of the space fractional derivatives in Section 2, which will be used in later discussion. In Section 3, the semidiscrete variational formulation for (1.2) based on Galerkin method is given. Existence, stability and L^2-error estimate of the semidiscrete solution are analyzed. In Section 4, existence and unconditional stability of approximate solutions for the fully discrete backward Euler method are shown following the idea of the semidiscrete method. Further, L^2-error estimates are obtained, whose convergence is of $O(k + h^{\tilde{\gamma}})$, where $\tilde{\gamma} = \mu$ if $\mu \neq 3/2$ and $\tilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < 1/2$, if $\mu = 3/2$. Finally, numerical examples are given in order to see the theoretical convergence order discussed in Section 5. We will see that numerical solutions of fractional diffusion equations diffuse more slowly than that of the classical diffusion problem and diffusivity depends on the order of fractional derivatives.

2. The Variational Form

In this section we will consider the variational form of problem (1.2)–(1.4) and show existence and stability of the weak solution. We first recall some basic properties of Riemann-Liouville fractional calculus [9, 24].

For any given positive number $\mu > 0$, define the seminorm

$$|u|_{J_L^\mu(\mathbf{R})} = \|\mathbf{D}^\mu u\|_{L^2(\mathbf{R})} \tag{2.1}$$

and the norm

$$\|u\|_{J_L^\mu(\mathbf{R})} = \left(\|u\|^2_{L^2(\mathbf{R})} + |u|^2_{J_L^\mu(\mathbf{R})}\right)^{1/2}, \tag{2.2}$$

where the left fractional derivative space $J_L^\mu(\mathbf{R})$ denotes the closure of $C_0^\infty(\mathbf{R})$ with respect to the norm $\|\cdot\|_{J_L^\mu(\mathbf{R})}$.

Similarly, we may define the right fractional derivative space $J_R^\mu(\mathbf{R})$ as the closure of $C_0^\infty(\mathbf{R})$ with respect to the norm $\|\cdot\|_{J_R^\mu(\mathbf{R})}$, where

$$\|u\|_{J_R^\mu(\mathbf{R})} = \left(\|u\|^2_{L^2(\mathbf{R})} + |u|^2_{J_R^\mu(\mathbf{R})}\right)^{1/2} \tag{2.3}$$

and the seminorm

$$|u|_{J_R^\mu(\mathbf{R})} = \|\mathbf{D}^{\mu *} u\|_{L^2(\mathbf{R})}. \tag{2.4}$$

Furthermore, with the help of Fourier transform we define a seminorm

$$|u|_{H^\mu(\mathbf{R})} = \left\| |\omega|^\mu \widehat{u} \right\|_{L^2(\mathbf{R})} \tag{2.5}$$

and the norm

$$\|u\|_{H^\mu(\mathbf{R})} = \left(\|u\|^2_{L^2(\mathbf{R})} + |u|^2_{H^\mu(\mathbf{R})}\right)^{1/2}. \tag{2.6}$$

Here $H^\mu(\mathbf{R})$ denotes the closure of $C_0^\infty(\mathbf{R})$ with respect to $\|\cdot\|_{H^\mu(\mathbf{R})}$. It is known in [24] that the spaces $J_L^\mu(\mathbf{R})$, $J_R^\mu(\mathbf{R})$, and $H^\mu(\mathbf{R})$ are all equal with equivalent seminorms and norms. Analogously, when the domain Ω is a bounded interval, the spaces $J_{L,0}^\mu(\Omega)$, $J_{R,0}^\mu(\Omega)$, and $H_0^\mu(\Omega)$ are equal with equivalent seminorms and norms [24, 27].

The following lemma on the Riemann-Liouville fractional integral operators will be used in our analysis, which can be proved by using the property of Fourier transform [24].

Lemma 2.1. *For a given $\mu > 0$ and a real valued function u*

$$(\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u) = \cos(\pi\mu)\|\mathbf{D}^\mu u\|^2_{L^2(\mathbf{R})}. \tag{2.7}$$

Remark 2.2. It follows from (2.7) that we may use the following norm:

$$\|u\|^2_{H_0^{\mu/2}(\mathbf{R})} = \|u\|^2_{L_2(\mathbf{R})} + \kappa_\mu \left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right| |u|^2_{H_0^{\mu/2}(\mathbf{R})} \tag{2.8}$$

instead of the norm $\|u\|_{H^\mu(\mathbf{R})}$.

For the seminorm on $H_0^{\mu}(\Omega)$ with $\Omega = (a,b)$, the following fractional Poincaré-Friedrich's inequality holds. For the proof, we refer to [9, 24].

Lemma 2.3. *For $u \in H_0^{\mu}(\Omega)$, there is a positive constant C such that*

$$\|u\|_{L^2(\Omega)} \le C|u|_{H_0^{\mu}(\Omega)} \tag{2.9}$$

and for $0 < s < \mu,\ s \ne n - 1/2,\ n - 1 \le \mu < n,\ n \in \mathbb{N}$,

$$|u|_{H_0^{s}(\Omega)} \le C|u|_{H_0^{\mu}(\Omega)}. \tag{2.10}$$

Hereafter, a positive number C will denote a generic constant. Also the semigroup property and the adjoint property hold for the Riemann-Liouville fractional integral operators [9, 24]: for all $\mu, \nu > 0$, if $u \in L^p(\Omega)$, $p \ge 1$, then

$$\begin{aligned} {}_aD_x^{-\mu}\,{}_aD_x^{-\nu}u(x) &= {}_aD_x^{-\mu-\nu}u(x), \quad \forall x \in \Omega, \\ {}_xD_b^{-\mu}\,{}_xD_b^{-\nu}u(x) &= {}_xD_b^{-\mu-\nu}u(x), \quad \forall x \in \Omega, \end{aligned} \tag{2.11}$$

and specially

$$\left({}_aD_x^{-\mu}u, v\right)_{L^2(\Omega)} = \left(u, {}_xD_b^{-\mu}v\right)_{L^2(\Omega)}, \quad \forall u, v \in L^2(\Omega). \tag{2.12}$$

In the rest of this section, we will consider a weak problem for (1.2)–(1.4) with $1 < \mu < 2$: find a function $u \in H_0^{\mu/2}(\Omega)$ such that

$$(u_t, v) = (\kappa_\mu \nabla^\mu u, v) + (f(u), v), \quad \forall v \in H_0^{\mu/2}(\Omega). \tag{2.13}$$

Since there is a weak solution of (2.13) when f is locally Lipschitz continuous as in [8, 13], we here only discuss the stability of the weak solution, to show that we need the following lemma.

Lemma 2.4. *For all $v \in H_0^{\mu/2}(\Omega)$, the following inequality holds:*

$$-(\kappa_\mu \nabla^\mu v, v) \ge \kappa_\mu \left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right| |v|^2_{H_0^{\mu/2}(\Omega)}. \tag{2.14}$$

Proof. Following the ideas in [9, 26], we obtain the following inequality by using the properties (2.11)-(2.12) and Lemmas 2.1–2.3:

$$\begin{aligned}
-(\kappa_\mu \nabla^\mu v, v) &= -\frac{\kappa_\mu}{2}\left\{\left({}_aD_x^\mu v, v\right) + \left({}_xD_b^\mu v, v\right)\right\} \\
&= -\frac{\kappa_\mu}{2}\left\{\int_a^b \left(D^2{}_aD_x^{-(2-\mu)} v\right) v\, dx + \int_a^b \left((-D)^2{}_xD_b^{-(2-\mu)} v\right) v\, dx\right\} \\
&= \frac{\kappa_\mu}{2}\left\{\int_a^b \left(D_a D_x^{-(2-\mu)} v\right) Dv\, dx + \int_a^b \left(D_x D_b^{-(2-\mu)} v\right) Dv\, dx\right\} \\
&= \frac{\kappa_\mu}{2}\left\{\int_a^b \left({}_aD_x^{-(2-\mu)} Dv\right) Dv\, dx + \int_a^b \left({}_xD_b^{-(2-\mu)} Dv\right) Dv\, dx\right\} \\
&= \frac{\kappa_\mu}{2}\left\{\int_a^b \left({}_aD_x^{-(2-\mu)/2}{}_aD_x^{-(2-\mu)/2} Dv\right) Dv\, dx\right. \\
&\qquad \left. + \int_a^b \left({}_xD_b^{-((2-\mu)/2)}{}_xD_b^{-(2-\mu)/2} Dv\right) Dv\, dx\right\} \\
&= \frac{\kappa_\mu}{2}\left\{\int_a^b \left({}_aD_x^{-(2-\mu)/2} Dv\right)\left({}_xD_b^{-(2-\mu)/2} Dv\right) dx\right. \\
&\qquad \left. + \int_a^b \left({}_xD_b^{-(2-\mu)/2} Dv\right)\left({}_aD_x^{-(2-\mu)/2} Dv\right) dx\right\} \\
&= -\kappa_\mu \left(\mathbf{D}^{\mu/2} v, \mathbf{D}^{(\mu/2)*} v\right) \\
&= -\kappa_\mu \cos\left(\pi \cdot \frac{\mu}{2}\right) \left\|\mathbf{D}^{\mu/2} v\right\|_{L^2(\Omega)}^2 \\
&\geq \kappa_\mu \left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right| |v|_{H_0^{\mu/2}(\Omega)}^2 .
\end{aligned} \tag{2.15}$$

This completes the proof. □

We consider the stability of a weak solution u for (2.13).

Theorem 2.5. *Let u be a solution of* (2.13). *Then there is a constant C such that*

$$\|u(t)\|_{L^2(\Omega)} \leq C\|u(0)\|_{L^2(\Omega)}. \tag{2.16}$$

Proof. Taking $v = u(t)$ in (2.13), we obtain

$$(u_t, u) - (\kappa_\mu \nabla^\mu u, u) = (f(u), u). \tag{2.17}$$

Since the second term on the left hand side is nonnegative from Lemma 2.4, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} &\le \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \kappa_\mu\left|\cos\left(\pi\cdot\frac{\mu}{2}\right)\right||u|^2_{H_0^{\mu/2}(\Omega)} \\ &\le \|f(u)\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\le C_f\|u\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.18}$$

Integrating both sides with respect to t, we obtain

$$\|u(t)\|^2_{L^2(\Omega)} \le \|u(0)\|^2_{L^2(\Omega)} + C\int_0^t \|u(s)\|^2_{L^2(\Omega)}ds. \tag{2.19}$$

An application of Gronwall's inequality gives that there is a constant C such that

$$\|u(t)\|^2_{L^2(\Omega)} \le C\|u(0)\|^2_{L^2(\Omega)}. \tag{2.20}$$

This completes the proof. □

3. The Semidiscrete Variational Form

In this section, we will analyze the stability and error estimates of Galerkin finite element solutions for the semidiscrete variational formulation for (1.2).

Let S_h be a partition of Ω with a grid parameter h such that $\overline{\Omega} = \{\cup K \mid K \in S_h\}$ and $h = \max_{K\in S_h} h_K$, where h_K is the width of the subinterval K. Associated with the partition S_h, we may define a finite-dimensional subspace $V_h \subset H_0^{\mu/2}(\Omega)$ with a basis $\{\varphi_i\}_{i=1}^N$ of piecewise polynomials. Then the semidiscrete variational problem is to find $u_h \in V_h$ such that

$$(u_{h,t}, v) = \left(\kappa_\mu \nabla^\mu u_h, v\right) + \left(f(u_h), v\right), \quad \forall v \in V_h, \tag{3.1}$$

$$u_h(x,0) = u_0, \tag{3.2}$$

$$u_h(a,t) = u_h(b,t) = 0. \tag{3.3}$$

Since u_h can be represented as

$$u_h(x,t) = \sum_{i=1}^N \alpha_i(t)\varphi_i(x), \tag{3.4}$$

we may rewrite (3.1) in a matrix form:

$$\mathbf{A}\dot{\mathbf{u}}(t) + \mathbf{B}\mathbf{u} = \mathbf{F}(\mathbf{u}), \tag{3.5}$$

where $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ and vectors $\mathbf{u}$ and $\mathbf{F}$ are

$$\begin{gathered}
\mathbf{A} = (a_{ij}), \quad a_{ij} = (\varphi_i, \varphi_j), \\
\mathbf{B} = (b_{ij}), \quad b_{ij} = -\frac{\kappa_\mu}{2}\left[\left(\mathbf{D}^{\mu/2}\varphi_i, \mathbf{D}^{(\mu/2)*}\varphi_j\right) + \left(\mathbf{D}^{\mu/2}\varphi_j, \mathbf{D}^{(\mu/2)*}\varphi_i\right)\right], \\
\mathbf{F}(\mathbf{u}) = (F_j), \quad F_j = \left(f\left(\sum_{l=1}^{N}\alpha_l\varphi_l\right), \varphi_j\right), \\
\mathbf{u} = (\alpha_1(t), \alpha_2(t), \dots, \alpha_N(t))^T.
\end{gathered} \tag{3.6}$$

It follows from $\sum_{i,j=1}^{N} \alpha_i\alpha_j(\varphi_i, \varphi_j) = (\sum_{i=1}^{N} \alpha_i\varphi_i, \sum_{j=1}^{N} \alpha_j\varphi_j) \geq 0$ and Lemma 2.4 that matrices $\mathbf{A}$ and $\mathbf{B}$ are nonnegative definite and nonsingular. Thus this system (3.5) of ordinary differential equations has a unique solution since f is locally Lipschitz continuous.

The stability for the semidiscrete variational problem (3.1) can be obtained by following the proof of Theorem 2.5, which is

$$\|u_h\|_{L^2(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}. \tag{3.7}$$

Now we will consider estimates of error between the weak solution of (2.13) and the one of semidiscrete form (3.1). The finite dimensional subspace $V_h \subset H_0^{\mu/2}(\Omega)$ is chosen so that the interpolation $I^h u$ of u satisfies an approximation property [9, 28]: for $u \in H^\gamma(\Omega)$, $0 < \gamma \leq n$, and $0 \leq s \leq \gamma$, there exists a constant C depending only on Ω such that

$$\left\|u - I^h u\right\|_{H^s(\Omega)} \leq Ch^{\gamma - s}\|u\|_{H^\gamma(\Omega)}. \tag{3.8}$$

Since the norm $\|\cdot\|_{H^s(\Omega)}$ is equivalent to the seminorm $|\cdot|_{H^s(\Omega)}$, we may replace (3.8) by the relation

$$\left\|u - I^h u\right\|_{H^s(\Omega)} \leq Ch^{\gamma - s}|u|_{H^\gamma(\Omega)}. \tag{3.9}$$

Further we need an adjoint problem to find $w \in H^\mu(\Omega) \cap H_0^{\mu/2}(\Omega)$ satisfying

$$\begin{gathered}
-\kappa_\mu \nabla^\mu w = g, \quad \text{in } \Omega, \\
w = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.10}$$

Bai and Lü [29] have proved existence of a solution to the problem (3.10). We assume as in Ervin and Roop [24] that the solution w satisfies the regularity

$$\|w\|_{H^\mu(\Omega)} \leq C\|g\|_{L^2(\Omega)}, \quad \mu \neq \frac{3}{2}, \tag{3.11}$$

$$\|w\|_{H^{\mu-\epsilon}(\Omega)} \leq C\|g\|_{L^2(\Omega)}, \quad \mu = \frac{3}{2}, \ 0 < \epsilon < \frac{1}{2}. \tag{3.12}$$

Let $\widetilde{u}_h = P_h u$ be the elliptic projection $P_h : H_0^{\mu/2}(\Omega) \rightarrow V_h$ of the exact solution u, which is defined by

$$-\kappa_\mu(\nabla^\mu(u-\widetilde{u}_h), v) = 0, \quad \forall v \in V_h. \tag{3.13}$$

Let $\theta = u_h - \widetilde{u}_h$ and $\rho = \widetilde{u}_h - u$. Then the error is expressed as

$$e_h = u_h - u = (u_h - \widetilde{u}_h) + (\widetilde{u}_h - u) = \theta + \rho. \tag{3.14}$$

First, we consider the following estimates on ρ.

Lemma 3.1. *Let $\widetilde{u}_h$ be a solution of* (3.13) *and let $u \in H^\mu(\Omega) \cap H_0^{\mu/2}(\Omega)$ be the solution of* (2.13). *Let $\rho(t) = \widetilde{u}_h(t) - u(t)$. Then there is a constant C such that*

$$\begin{aligned} \left\|\rho(t)\right\|_{L^2(\Omega)} &\leq C h^{\widetilde{\gamma}} \|u(t)\|_{H^\gamma(\Omega)}, \\ \left\|\rho_t(t)\right\|_{L^2(\Omega)} &\leq C h^{\widetilde{\gamma}} \|u(t)\|_{H^\gamma(\Omega)}, \end{aligned} \tag{3.15}$$

where $\widetilde{\gamma} = \mu$ if $\mu \neq 3/2$ and $\widetilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < 1/2$ if $\mu = 3/2$.

Proof. It follows from the fractional Poincaré-Friedrich's inequality and the adjoint property (2.12) that for $\psi, \chi \in V_h \subset H_0^{\mu/2}(\Omega)$

$$\begin{aligned} \left(\mathbf{D}^\mu \psi, \chi\right) &= \int_a^b \left(\mathbf{D}^{\mu/2}\psi\right) \mathbf{D}^{(\mu/2)*} \chi \, dx \\ &\leq |\psi|_{J_{L,0}^{\mu/2}(\Omega)} |\chi|_{J_{R,0}^{\mu/2}(\Omega)} \\ &\leq C\|\psi\|_{H_0^{\mu/2}(\Omega)} \|\chi\|_{H_0^{\mu/2}(\Omega)}. \end{aligned} \tag{3.16}$$

Similarly we obtain

$$\left(\mathbf{D}^{\mu*}\psi, \chi\right) = \int_a^b \left(\mathbf{D}^{(\mu/2)*}\psi\right) \mathbf{D}^{\mu/2} \chi \, dx \leq C\|\psi\|_{H_0^{\mu/2}(\Omega)} \|\chi\|_{H_0^{\mu/2}(\Omega)}. \tag{3.17}$$

It follows from Lemma 2.4 that for $v \in V_h$

$$\begin{aligned} \kappa_\mu \left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right| |u - \widetilde{u}_h|^2_{H_0^{\mu/2}(\Omega)} &\leq -\kappa_\mu(\nabla^\mu(u-\widetilde{u}_h), u - \widetilde{u}_h) \\ &\leq -\kappa_\mu(\nabla^\mu(u-\widetilde{u}_h), u - v) - \kappa_\mu(\nabla^\mu(u-\widetilde{u}_h), v - \widetilde{u}_h) \\ &\leq C\|u - \widetilde{u}_h\|_{H_0^{\mu/2}(\Omega)} \|u - v\|_{H_0^{\mu/2}(\Omega)}. \end{aligned} \tag{3.18}$$

Using the equivalence of seminorms and norms, we obtain

$$\|u - \widetilde{u}_h\|_{H_0^{\mu/2}(\Omega)} \le C \inf_{v \in V_h} \|u - v\|_{H_0^{\mu/2}(\Omega)} \le C \left\| u - I^h u \right\|_{H_0^{\mu/2}(\Omega)}. \tag{3.19}$$

In case of $\mu \neq 3/2$ and $v \in V_h$, by taking $g = \rho$ in (3.10) and using (3.13), (3.16)-(3.17) and the adjoint property (2.12), we have

$$\begin{aligned}(\rho, \rho) &= -\kappa_\mu (\nabla^\mu w, \rho) \\ &= -\kappa_\mu (\nabla^\mu (w - v), \rho) - \kappa_\mu (\nabla^\mu \rho, v) \\ &= -\kappa_\mu (\nabla^\mu (w - v), \rho) \\ &\le C \|w - v\|_{H_0^{\mu/2}(\Omega)} \left\|\rho\right\|_{H_0^{\mu/2}(\Omega)}.\end{aligned} \tag{3.20}$$

Taking $v = I^h w$ in the previously mentioned inequalities, we have

$$\begin{aligned}\left\|\rho\right\|_{L^2(\Omega)}^2 &\le C \left\| w - I^h w \right\|_{H_0^{\mu/2}(\Omega)} \left\|\rho\right\|_{H_0^{\mu/2}(\Omega)} \\ &\le C h^{\mu/2} \|w\|_{H^\mu(\Omega)} \left\| u - I^h u \right\|_{H_0^{\mu/2}(\Omega)} \\ &\le C h^{\mu/2} \left\|\rho\right\|_{L^2(\Omega)} h^{\mu/2} \|u\|_{H^\mu(\Omega)}.\end{aligned} \tag{3.21}$$

Thus we obtain

$$\left\|\rho\right\|_{L^2(\Omega)} \le C h^\mu \|u\|_{H^\mu(\Omega)}. \tag{3.22}$$

We now differentiate (3.13). Then we obtain $-\kappa_\mu (\nabla^\mu \rho_t, v) = 0$ for all $v \in V_h$. Using the previous duality arguments again, we have

$$\left\|\rho_t\right\|_{L^2(\Omega)} \le C h^\mu \|u\|_{H^\mu(\Omega)}. \tag{3.23}$$

In case of $\mu = 3/2$, we can similarly prove (3.15) by applying the assumption (3.12). This completes the proof. □

We now consider the estimates on θ.

Lemma 3.2. *Let u_h and $\widetilde{u}_h$ be the solutions of* (3.1)–(3.3) *and* (3.13)*, respectively. Let $\theta(t) = u_h(t) - \widetilde{u}_h(t)$. Then there is a constant C such that*

$$\|\theta(t)\|_{L^2(\Omega)} \le C h^{\widetilde{\gamma}}, \tag{3.24}$$

where $\widetilde{\gamma} = \mu$ if $\mu \neq 3/2$ and $\widetilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < 1/2$ if $\mu = 3/2$.

Proof. It follows from (3.1) and (3.13) that for $v \in V_h$,

$$(\theta_t, v) - \kappa_\mu(\nabla^\mu \theta, v) = \big(f(u_h) - f(u), v\big) - \big(\rho_t, v\big). \tag{3.25}$$

Replacing $v = \theta$ in (3.25), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\Omega)} \le C_l\|u_h - u\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)} + \big\|\rho_t\big\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)}. \tag{3.26}$$

Using Young's inequality

$$\begin{aligned}\frac{d}{dt}\|\theta\|^2_{L^2(\Omega)} &\le C\Big(\|u_h - \tilde{u}_h\|_{L^2(\Omega)} + \|\tilde{u}_h - u\|_{L^2(\Omega)}\Big)\|\theta\|_{L^2(\Omega)} + \big\|\rho_t\big\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)} \\ &\le C\Big(\|\theta\|_{L^2(\Omega)} + \big\|\rho\big\|_{L^2(\Omega)} + \big\|\rho_t\big\|_{L^2(\Omega)}\Big)\|\theta\|_{L^2(\Omega)} \\ &\le C_1\|\theta\|^2_{L^2(\Omega)} + C_2\big\|\rho\big\|^2_{L^2(\Omega)} + C_3\big\|\rho_t\big\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.27}$$

Integration on time t gives

$$\|\theta(t)\|^2_{L^2(\Omega)} \le \|\theta(0)\|^2_{L^2(\Omega)} + C\int_0^t \|\theta\|^2_{L^2(\Omega)} ds + C\int_0^t \Big(\big\|\rho\big\|^2_{L^2(\Omega)} + \big\|\rho_t\big\|^2_{L^2(\Omega)}\Big) ds. \tag{3.28}$$

Applying Gronwall's inequality, we obtain

$$\|\theta(t)\|^2_{L^2(\Omega)} \le C_1\|\theta(0)\|^2_{L^2(\Omega)} + C_2\int_0^t \Big(\big\|\rho\big\|^2_{L^2(\Omega)} + \big\|\rho_t\big\|^2_{L^2(\Omega)}\Big) ds. \tag{3.29}$$

Since

$$\begin{aligned}\|\theta(0)\|_{L^2(\Omega)} &\le \|u_h(0) - u(0)\|_{L^2(\Omega)} + \|\tilde{u}_h(0) - u(0)\|_{L^2(\Omega)} \\ &\le Ch^{\tilde{\gamma}}\|u_0\|_{H^\gamma(\Omega)},\end{aligned} \tag{3.30}$$

we obtain the desired inequality

$$\|\theta(t)\|_{L^2(\Omega)} \le Ch^{\tilde{\gamma}}, \tag{3.31}$$

where $\tilde{\gamma} = \mu$ if $\mu \neq 3/2$ and $\tilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < 1/2$, if $\mu = 3/2$. □

Combining Lemmas 3.1 and 3.2, we obtain the following error estimates.

Theorem 3.3. *Let u_h and u be the solutions of* (3.1)–(3.3) *and* (1.2)–(1.4)*, respectively. Then there is a constant* $C(u)$ *such that*

$$\begin{aligned} \|u(t)-u_h(t)\|_{L^2(\Omega)} &\le C(u)h^{\mu}, \quad \mu \ne \frac{3}{2}, \\ \|u(t)-u_h(t)\|_{L^2(\Omega)} &\le C(u)h^{\mu-\epsilon}, \quad \mu = \frac{3}{2},\ 0<\epsilon<\frac{1}{2}. \end{aligned} \tag{3.32}$$

4. The Fully Discrete Variational Form

In this section, we consider a fully discrete variational formulation of (1.2). Existence and uniqueness of numerical solutions for the fully discrete variational formulation are discussed. The corresponding error estimates are also analyzed.

For the temporal discretization let $k = T/M$ for a positive integer M and $t_m = mk$. Let u^m be the solution of the backward Euler method defined by

$$\frac{u^{m+1}-u^m}{k} = \kappa_\mu \nabla^\mu u^{m+1} + f\left(u^{m+1}\right) \tag{4.1}$$

with an initial condition

$$u^0(x) = u_0, \quad x \in \Omega = (a,b) \tag{4.2}$$

and boundary conditions

$$u^{m+1}(a) = u^{m+1}(b) = 0, \quad m = 0,1,\dots,M-1. \tag{4.3}$$

Then we get the fully discrete variational formulation of (1.2) to find $u^{m+1} \in H_0^{\mu/2}(\Omega)$ such that for all $v \in H_0^{\mu/2}(\Omega)$

$$\left(u^{m+1},v\right) - k\left(\kappa_\mu \nabla^\mu u^{m+1},v\right) = \left(kf\left(u^{m+1}\right),v\right) + \left(u^m,v\right). \tag{4.4}$$

Thus a finite Galerkin solution $u_h^{m+1} \in V_h \subset H_0^{\mu/2}(\Omega)$ is a solution of the equation

$$\left(u_h^{m+1},v_h\right) - k\kappa_\mu\left(\nabla^\mu u_h^{m+1},v_h\right) = k\left(f\left(u_h^{m+1}\right),v_h\right) + \left(u_h^m,v_h\right), \quad \forall v_h \in V_h \tag{4.5}$$

with an initial condition

$$u_h^0 = u_0 \tag{4.6}$$

and boundary conditions

$$u_h^{m+1}(a) = u_h^{m+1}(b) = 0, \quad m = 0, 1, \ldots, M-1. \tag{4.7}$$

Now we prove the existence and uniqueness of solutions for (4.5) using the Brouwer fixed-point theorem.

Theorem 4.1. *There exists a unique solution $u_h^{m+1} \in V_h \subset H_0^{\mu/2}(\Omega)$ of* (4.5)–(4.7).

Proof. Let

$$G\left(u_h^{m+1}\right) = u_h^{m+1} - k\kappa_\mu \nabla^\mu u_h^{m+1} - kf\left(u_h^{m+1}\right) - u_h^m. \tag{4.8}$$

Then $G(v)$ is obviously a continuous function from V_h to V_h. In order to show the existence of solution for $G(v) = 0$, we adopt the mathematical induction. Assume that $u_h^0, u_h^1, \ldots, u_h^m$ exist for $m < M$. It follows from (1.8), Lemma 2.4, and Young's inequality that

$$\begin{aligned}
(G(v), v) &= (v, v) - \left(u_h^m, v\right) - k\left(\kappa_\mu \nabla^\mu v, v\right) - k\left(f(v), v\right) \\
&\geq \|v\|_{L^2(\Omega)}^2 - \left\|u_h^m\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + k\kappa_\mu \left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right| |v|_{H_0^{\mu/2}(\Omega)}^2 - C_f k \|v\|_{L^2(\Omega)}^2 \\
&\geq \|v\|_{L^2(\Omega)}^2 - \left\|u_h^m\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} - C_f k \|v\|_{L^2(\Omega)}^2 \\
&\geq \|v\|_{L^2(\Omega)}^2 - \frac{1}{2}\left(\left\|u_h^m\right\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2\right) - C_f k \|v\|_{L^2(\Omega)}^2 \\
&= \left(\frac{1}{2} - C_f k\right) \|v\|_{L^2(\Omega)}^2 - \frac{1}{2}\left\|u_h^m\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.9}$$

If we take sufficiently small k so that $k < 1/2C_f$ and $\|v\|_{L^2(\Omega)} > \|u_h^m\|_{L^2(\Omega)} / (1 - 2C_f k)$, then the Brouwer's fixed-point theorem implies the existence of a solution.

For the proof of the uniqueness of solutions, we assume that u and v are two solutions of (4.5). Then we obtain

$$(u - v, \psi) = k\kappa_\mu \left(\nabla^\mu (u - v), \psi\right) + k\left(f(u) - f(v), \psi\right), \quad \forall \psi \in V_h \subset H_0^{\mu/2}(\Omega). \tag{4.10}$$

Replacing $\psi = u - v$ in the above equation and applying Lemma 2.4, we obtain

$$\begin{aligned}
\|u - v\|_{L^2(\Omega)}^2 &\leq -k\kappa_\mu \left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right| |u - v|_{H_0^{\mu/2}(\Omega)} + k\left\|f(u) - f(v)\right\|_{L^2(\Omega)} \|u - v\|_{L^2(\Omega)} \\
&\leq k\left\|f(u) - f(v)\right\|_{L^2(\Omega)} \|u - v\|_{L^2(\Omega)} \\
&\leq kC_l \|u - v\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.11}$$

This implies $u - v = 0$ since $u(0) = v(0)$. □

The following theorem presents the unconditional stability for (4.4).

Theorem 4.2. *The fully discrete scheme* (4.4) *is unconditionally stable. In fact, for any* m

$$\left\|u^{m+1}\right\|_{L^2(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}. \tag{4.12}$$

Proof. It follows from (1.8), Lemma 2.4, and Young's inequality that by taking $v = u^{m+1}$ in (4.4), we obtain

$$\begin{aligned}
0 &= \left(u^{m+1}, u^{m+1}\right) - k\left(\kappa_\mu \nabla^\mu u^{m+1}, u^{m+1}\right) - k\left(f\left(u^{m+1}\right), u^{m+1}\right) - \left(u^m, u^{m+1}\right) \\
&\geq \left\|u^{m+1}\right\|^2_{L^2(\Omega)} + k\kappa_\mu\left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right|\left|u^{m+1}\right|^2_{H_0^{\mu/2}(\Omega)} \\
&\quad - C_f k\left\|u^{m+1}\right\|^2_{L^2(\Omega)} - \|u^m\|_{L^2(\Omega)}\left\|u^{m+1}\right\|_{L^2(\Omega)} \\
&\geq \frac{1}{2}\left\|u^{m+1}\right\|^2_{L^2(\Omega)} + k\kappa_\mu\left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right|\left|u^{m+1}\right|^2_{H_0^{\mu/2}(\Omega)} \\
&\quad - C_f k\left\|u^{m+1}\right\|^2_{L^2(\Omega)} - \frac{1}{2}\|u^m\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.13}$$

Then

$$\begin{aligned}
\frac{1}{2}\left\|u^{m+1}\right\|^2_{L^2(\Omega)} &\leq \frac{1}{2}\left\|u^{m+1}\right\|^2_{L^2(\Omega)} + k\kappa_\mu\left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right|\left|u^{m+1}\right|^2_{H_0^{\mu/2}(\Omega)} \\
&\leq C_f k\left\|u^{m+1}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\|u^m\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.14}$$

Adding the above inequality from $m = 0$ to m, we obtain

$$\left(1 - 2C_f k\right)\left\|u^{m+1}\right\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} + 2C_f k\sum_{j=1}^{m}\left\|u^j\right\|^2_{L^2(\Omega)}. \tag{4.15}$$

Applying the discrete Gronwall's inequality with sufficiently small k such that $k < 1/2C_f$, we obtain the desired result. □

The following theorem is an error estimate for the fully discrete problem (4.4).

Theorem 4.3. *Let* u *be the exact solution of* (1.2) *and let* u^m *be the solution of* (4.4)*. Then there is a constant* C *such that*

$$\|u(t_m) - u^m\|_{L^2(\Omega)} \leq Ck. \tag{4.16}$$

Proof. Let $e^m = u(t_m) - u^m$ be the error at t_m. It follows from (1.2) and (4.4) that for any $v \in H_0^{\mu/2}(\Omega)$

$$\left(e^{m+1}, v\right) - k\left(\kappa_\mu \nabla^\mu e^{m+1}, v\right) = k\left(f(u(t_{m+1})) - f\left(u^{m+1}\right), v\right) + (e^m, v) + \left(kr^{m+1}, v\right), \tag{4.17}$$

where $r = O(k)$. Taking $v = e^{m+1}$,

$$\begin{aligned} \left\|e^{m+1}\right\|_{L^2(\Omega)}^2 &\le \left\|e^{m+1}\right\|_{L^2(\Omega)}^2 + k\kappa_\mu\left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right|\left|e^{m+1}\right|_{H_0^{\mu/2}(\Omega)}^2 \\ &\le k\left\|f(u(t_{m+1})) - f(u^{m+1})\right\|_{L^2(\Omega)}\left\|e^{m+1}\right\|_{L^2(\Omega)} \\ &\quad + \|e^m\|_{L^2(\Omega)}\left\|e^{m+1}\right\|_{L^2(\Omega)} + \left\|kr^{m+1}\right\|_{L^2(\Omega)}\left\|e^{m+1}\right\|_{L^2(\Omega)}. \end{aligned} \tag{4.18}$$

Applying the locally Lipschitz continuity of f and Young's inequality, we obtain

$$\begin{aligned} \left\|e^{m+1}\right\|_{L^2(\Omega)}^2 &\le kC_l\left\|e^{m+1}\right\|_{L^2(\Omega)}^2 + \|e^m\|_{L^2(\Omega)}\left\|e^{m+1}\right\|_{L^2(\Omega)} + \left\|kr^{m+1}\right\|_{L^2(\Omega)}\left\|e^{m+1}\right\|_{L^2(\Omega)} \\ &\le kC_l\left\|e^{m+1}\right\|_{L^2(\Omega)}^2 + \varepsilon_1\|e^m\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon_1}\left\|e^{m+1}\right\|_{L^2(\Omega)}^2 \\ &\quad + \varepsilon_2\left\|kr^{m+1}\right\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon_2}\left\|e^{m+1}\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.19}$$

That is,

$$\left(1 - \frac{1}{4\varepsilon_1} - \frac{1}{4\varepsilon_2}\right)\left\|e^{m+1}\right\|_{L^2(\Omega)}^2 \le kC_l\left\|e^{m+1}\right\|_{L^2(\Omega)}^2 + \varepsilon_1\|e^m\|_{L^2(\Omega)}^2 + \varepsilon_2\left\|kr^{m+1}\right\|_{L^2(\Omega)}^2. \tag{4.20}$$

Denoting $\varepsilon_0 = 1 - 1/4\varepsilon_1 - 1/4\varepsilon_2$ and adding the above equation from $m = 0$ to m, we obtain

$$(\varepsilon_0 - kC_l)\left\|e^{m+1}\right\|_{L^2(\Omega)}^2 \le \varepsilon_1\left\|e^0\right\|_{L^2(\Omega)}^2 + (kC_l + \varepsilon_1 - \varepsilon_0)\sum_{i=1}^{m}\left\|e^i\right\|_{L^2(\Omega)}^2 + \varepsilon_2\sum_{i=1}^{m+1}\left\|kr^i\right\|_{L^2(\Omega)}^2. \tag{4.21}$$

Applying the discrete Gronwall's inequality with sufficiently small k such that $(\varepsilon_0 - \varepsilon_1)/C_l < k < \varepsilon_0/C_l$, we obtain the desired result since $\sum_{i=1}^{m+1}\|kr^i\|_{L^2(\Omega)} \le Ck$ and $\|e^0\|_{L^2(\Omega)} = \|u(0) - u^0\|_{L^2(\Omega)} = 0$. □

As in the previous section, denote $\theta^{m+1} = u_h^{m+1} - \tilde{u}_h^{m+1}$ and $\rho^{m+1} = \tilde{u}_h^{m+1} - u(t_{m+1})$. Here $\tilde{u}_h^{m+1}$ is the elliptic projection of $u(t_{m+1})$ defined in (3.13). Then

$$e_h^{m+1} = \theta^{m+1} + \rho^{m+1}. \tag{4.22}$$

Theorem 4.4. *Let u be the exact solution of* (1.2)–(1.4) *and let* $\{u_h^m\}_{m=0}^M$ *be the solution of* (4.5)–(4.7). *Then when* $\mu \neq 3/2$

$$\left\| u(t_{m+1}) - u_h^{m+1} \right\|_{L^2(\Omega)} \leq Ck + Ch^{\mu} \| u(t_{m+1}) \|_{H^{\mu}(\Omega)} \tag{4.23}$$

and when $\mu = 3/2,\ 0 < \epsilon < 1/2$,

$$\left\| u(t_{m+1}) - u_h^{m+1} \right\|_{L^2(\Omega)} \leq Ck + Ch^{\mu-\epsilon} \| u(t_{m+1}) \|_{H^{\mu-\epsilon}(\Omega)}. \tag{4.24}$$

Proof. Since we know the estimates on ρ from Lemma 3.1, we have only to show boundedness of θ^{m+1}. Using the property (3.13), we obtain for $v \in V_h$

$$\begin{aligned} \left(\theta^{m+1}, v\right) - k\left(\kappa_\mu \nabla^\mu \theta^{m+1}, v\right) = k\left(f\left(u_h^{m+1}\right) - f(u(t_{m+1})), v\right) + \left(u_h^m - u(t_m), v\right) \\ - \left(kr^{m+1}, v\right) - \left(\rho^{m+1}, v\right), \end{aligned} \tag{4.25}$$

where $r = O(k)$.

Taking $v = \theta^{m+1}$ and applying Lemma 2.4, the locally Lipschitz continuity of f, Young's inequality, and the triangle inequality, we obtain

$$\begin{aligned} \left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2 &\leq \left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2 + k\kappa_\mu \left|\cos\left(\pi \cdot \frac{\mu}{2}\right)\right| \left|\theta^{m+1}\right|_{H_0^{\mu/2}(\Omega)}^2 \\ &\leq k\left\| f(u_h^{m+1}) - f(u(t_{m+1})) \right\|_{L^2(\Omega)} \left\|\theta^{m+1}\right\|_{L^2(\Omega)} + \left\| e_h^m \right\|_{L^2(\Omega)} \left\|\theta^{m+1}\right\|_{L^2(\Omega)} \\ &\quad + \left\| kr^{m+1} \right\|_{L^2(\Omega)} \left\|\theta^{m+1}\right\|_{L^2(\Omega)} + \left\|\rho^{m+1}\right\|_{L^2(\Omega)} \left\|\theta^{m+1}\right\|_{L^2(\Omega)} \\ &\leq kC_l \left\| e_h^{m+1} \right\|_{L^2(\Omega)} \left\|\theta^{m+1}\right\|_{L^2(\Omega)} \\ &\quad + \left(\|\theta^m\|_{L^2(\Omega)} + \|\rho^m\|_{L^2(\Omega)} \right) \left\|\theta^{m+1}\right\|_{L^2(\Omega)} \\ &\quad + \left\| kr^{m+1} \right\|_{L^2(\Omega)} \left\|\theta^{m+1}\right\|_{L^2(\Omega)} + \left\|\rho^{m+1}\right\|_{L^2(\Omega)} \left\|\theta^{m+1}\right\|_{L^2(\Omega)} \\ &\leq kC_l \left(1 + \frac{1}{4\varepsilon_6}\right) \left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2 + \left(\frac{1}{4\varepsilon_3} + \frac{1}{4\varepsilon_4} + \frac{1}{4\varepsilon_5} + \frac{1}{4\varepsilon_6}\right) \left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2 \\ &\quad + \varepsilon_3 \|\theta^m\|_{L^2(\Omega)}^2 + \varepsilon_4 \left\| kr^{m+1} \right\|_{L^2(\Omega)}^2 + \varepsilon_5 \|\rho^m\|_{L^2(\Omega)}^2 + (1 + kC_l)\varepsilon_6 \left\|\rho^{m+1}\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.26}$$

This implies that

$$\begin{aligned}
&\left(1-\frac{1}{4\varepsilon_3}-\frac{1}{4\varepsilon_4}-\frac{1}{4\varepsilon_5}-\frac{1}{4\varepsilon_6}\right)\left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2 \\
&\quad\le kC_l\left(1+\frac{1}{4\varepsilon_6}\right)\left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2+\varepsilon_3\|\theta^m\|_{L^2(\Omega)}^2+\varepsilon_4\left\|kr^{m+1}\right\|_{L^2(\Omega)}^2 \\
&\qquad+\varepsilon_5\left\|\rho^m\right\|_{L^2(\Omega)}^2+(1+kC_l)\varepsilon_6\left\|\rho^{m+1}\right\|_{L^2(\Omega)}^2.
\end{aligned}\tag{4.27}$$

Denote $\varepsilon_7 = 1-1/4\varepsilon_3-1/4\varepsilon_4-1/4\varepsilon_5-1/4\varepsilon_6$ and $\varepsilon_8 = 1+1/4\varepsilon_6$. Then adding the above inequality from $m=0$ to m, we obtain

$$\begin{aligned}
(\varepsilon_7-kC_l\varepsilon_8)\left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2 &\le \varepsilon_3\left\|\theta^0\right\|_{L^2(\Omega)}^2+(kC_l\varepsilon_8+\varepsilon_3-\varepsilon_7)\sum_{i=1}^{m}\left\|\theta^i\right\|_{L^2(\Omega)}^2 \\
&\quad+\varepsilon_4\sum_{i=1}^{m+1}\left\|kr^i\right\|_{L^2(\Omega)}^2+\varepsilon_5\sum_{i=0}^{m}\left\|\rho^i\right\|_{L^2(\Omega)}^2 \\
&\quad+(1+kC_l)\varepsilon_6\sum_{i=1}^{m+1}\left\|\rho^i\right\|_{L^2(\Omega)}^2.
\end{aligned}\tag{4.28}$$

Applying the discrete Gronwall's inequality with sufficiently small k such that $(\varepsilon_7-\varepsilon_3)/\varepsilon_8C_l < k < \varepsilon_7/C_l\varepsilon_8$,

$$\left\|\theta^{m+1}\right\|_{L^2(\Omega)}^2 \le C_1\left\|\theta^0\right\|_{L^2(\Omega)}^2+C_2\sum_{i=1}^{m+1}\left\|kr^i\right\|_{L^2(\Omega)}^2+C_3\sum_{i=0}^{m+1}\left\|\rho^i\right\|_{L^2(\Omega)}^2.\tag{4.29}$$

Also, using Lemma 3.1 and the initial conditions (1.3) and (4.6), we obtain

$$\begin{aligned}
\left\|\theta^0\right\|_{L^2(\Omega)} &\le \left\|u_h^0-u(0)\right\|_{L^2(\Omega)}+\left\|\tilde{u}_h^0-u(0)\right\|_{L^2(\Omega)} \\
&\le Ch^{\tilde{\gamma}}\|u_0\|_{H^\gamma(\Omega)}.
\end{aligned}\tag{4.30}$$

Since $\sum_{i=1}^{m+1}\|kr^i\|_{L^2(\Omega)} \le Ck$, we get

$$\left\|\theta^{m+1}\right\|_{L^2(\Omega)} \le Ck+C(u)h^{\tilde{\gamma}},\tag{4.31}$$

where $\tilde{\gamma} = \mu$ if $\mu \ne 3/2$ and $\tilde{\gamma} = \mu-\epsilon$, $0<\epsilon<1/2$, if $\mu = 3/2$. Thus we obtain the desired result. □

Table 1: L^2-error and order of convergence in x when $\mu = 1.6$.

h	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ Error	Order
1/4	$8.37811e-03$	—
1/8	$2.73537e-03$	1.615
1/16	$8.75752e-04$	1.643
1/32	$2.83167e-04$	1.629

5. Numerical Experiments

In this section, we present numerical results for the Galerkin approximations which supports the theoretical analysis discussed in the previous section.

Let S_h denote a uniform partition of Ω and let V_h denote the space of continuous piecewise linear functions defined on S_h. In order to implement the Galerkin finite element approximation, we adapt finite element discretization on the spatial axis and the backward Euler finite difference scheme along the temporal axis. We associate shape functions of space V_h with the standard basis of the functions on the uniform interval with length h.

Example 5.1. We first consider a space fractional linear diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \nabla^{\mu} u(x,t) + \frac{2t}{t^2+1} u(x,t) - \left(t^2+1\right) \times \left(\frac{\left\{x^{2-\mu} + (1-x)^{2-\mu}\right\}}{\Gamma(3-\mu)} - \frac{6\left\{x^{3-\mu} + (1-x)^{3-\mu}\right\}}{\Gamma(4-\mu)} + \frac{12\left\{x^{4-\mu} + (1-x)^{4-\mu}\right\}}{\Gamma(5-\mu)} \right) \tag{5.1}$$

with an initial condition

$$u(x,0) = x^2(1-x)^2, \quad x \in [0,1] \tag{5.2}$$

and boundary conditions

$$u(0,t) = u(1,t) = 0. \tag{5.3}$$

In this case, the exact solution is

$$u(x,t) = \left(t^2+1\right) x^2 (1-x)^2. \tag{5.4}$$

Tables 1, 2, and 4 show the order of convergence and L^2-error between the exact solution and the Galerkin approximate solution of the fully discrete backward Euler method for (5.1) when $\mu = 1.6$, $\mu = 1.8$ and $\mu = 1.5$, respectively. For numerical computation, the temporal step size $k = 0.001$ is used in all three cases. Table 3 shows L^2-errors and orders of convergence for the Galerkin approximate solution when $\mu = 1.8$ and the spatial step size $h = 0.0625$.

Table 2: L^2-error and order of convergence in x when $\mu = 1.8$.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$ Error	Order
1/4	$8.03045e-03$	—
1/8	$2.28959e-03$	1.810
1/16	$6.32962e-04$	1.855
1/32	$1.76406e-04$	1.843

Table 3: L^2-error and order of convergence in t when $\mu = 1.8$.

k	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$ Error	Ratio
1/20	$4.20420e-03$	—
1/30	$2.94873e-03$	0.951
1/40	$2.31793e-03$	0.954
1/50	$1.93046e-03$	0.961

Table 4: L^2-error and order of convergence in x when $\mu = 1.5$.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$ Error	Order
1/4	$5.47750e-03$	—
1/8	$2.20129e-03$	1.315
1/16	$8.86858e-04$	1.312
1/32	$3.57629e-04$	1.310

According to Tables 1–3, we may find the order of convergence of $O(k+h^{\mu})$ for this linear fractional diffusion problem (5.1)–(5.3) when $\mu \neq 3/2$. Furthermore, Table 4 shows orders of numerical convergence for the problem when $\mu = 3/2$, where we may see that the order of convergence is of $O(k+h^{\mu-\epsilon})$, $0<\epsilon<1/2$. It follows from Tables 1–4 that numerical computations confirm the theoretical results.

We plot the exact solution and approximate solutions obtained by the backward Euler Galerkin method using $h = 1/32$ and $k = 1/1000$ for (5.1) with $\mu = 1.6$ and $\mu = 1.8$. Figure 1 shows the contour plots of an exact solution and numerical solutions at $t = 1$, and Figure 2 shows log-log graph for the order of convergence.

Example 5.2. We consider a space fractional diffusion equation with a nonlinear Fisher type source term which is described as

$$\frac{\partial u(x,t)}{\partial t} = \kappa_{\mu}\nabla^{\mu}u(x,t) + \lambda u(x,t)\big(1-\beta u(x,t)\big) \tag{5.5}$$

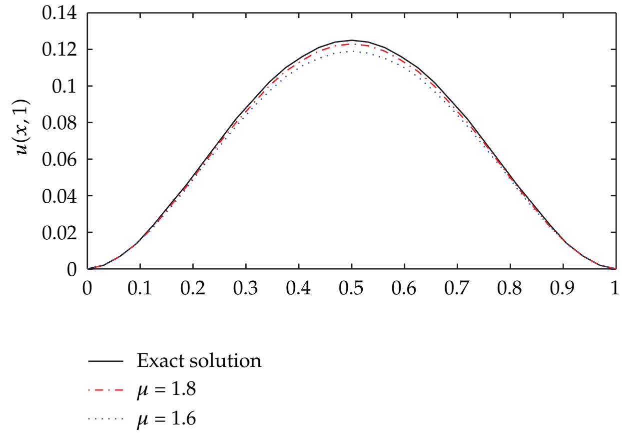

Figure 1: Exact and numerical solutions with $\mu = 1.6$ and $\mu = 1.8$.

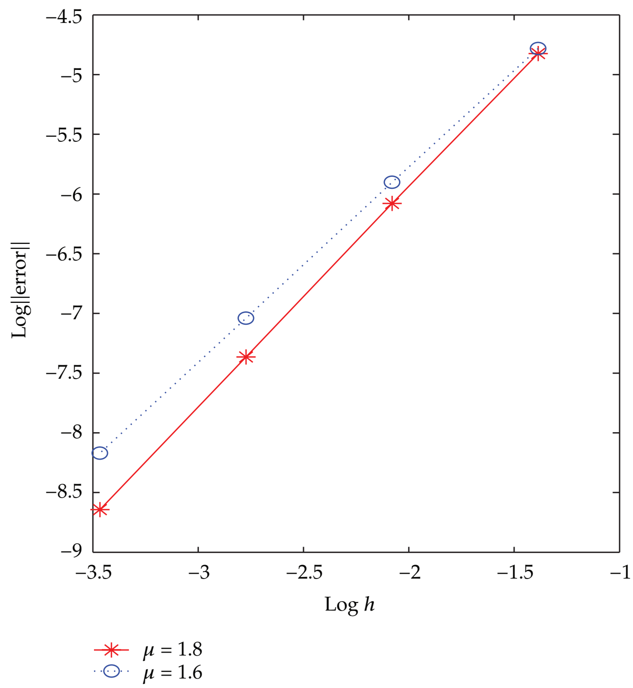

Figure 2: Log-log plots of the error for the rate of convergence.

with an initial condition

$$u(x, 0) = u_0(x) \tag{5.6}$$

and boundary conditions

$$u(-1, t) = u(1, t) = 0. \tag{5.7}$$

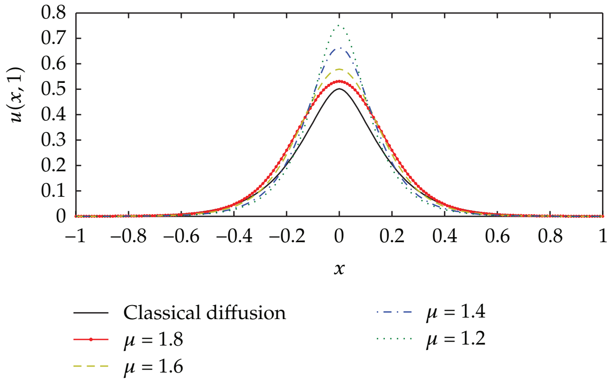

Figure 3: Numerical solutions for (5.5) with (5.8).

In fact, we will consider the case of $\kappa_\mu = 0.1$, $\beta = 1$ in (5.5) with an initial condition

$$u_0(x) = \begin{cases} e^{-10x}, & x \geq 0, \\ e^{10x}, & x < 0. \end{cases} \tag{5.8}$$

For numerical computations, we have to take care of the nonlinear term $f(u) = \lambda u(1 - \beta u)$. This gives a complicated nonlinear matrix. In order to avoid the difficulty of solving nonlinear system, we adopted a linearized method replacing $\lambda u^{n+1}(1 - \beta u^{n+1})$ by $\lambda u^{n+1}(1 - \beta u^{n})$. Figure 3 shows contour plots of numerical solutions at $t = 1$ for (5.5)–(5.8) with $\lambda = 0.25$. For numerical computations, step sizes $h = 0.01$ and $k = 0.005$ are used. From the numerical results we may find that numerical solutions converge to the solution of classical diffusion equation as μ approaches to 2.

Example 5.3. We now consider (5.5) with $\kappa_\mu = 0.1$, $\beta = 1$ and boundary conditions

$$\lim_{|x| \to \infty} u(x, t) = 0. \tag{5.9}$$

We will consider an initial condition with a sharp peak in the middle as

$$u_0(x) = \operatorname{sech}^2(10x) \tag{5.10}$$

and an initial condition with a flat roof in the middle as

$$u_0(x) = \begin{cases} e^{-10(x-1)}, & x > 1, \\ 1, & -1 < x \leq 1, \\ e^{10(x+1)}, & x \leq -1. \end{cases} \tag{5.11}$$

Tang and Weber [30] have obtained computational solutions for (5.5) with initial conditions (5.10) and (5.11) using a Petrov-Galerkin method when (5.5) is a classical diffusion

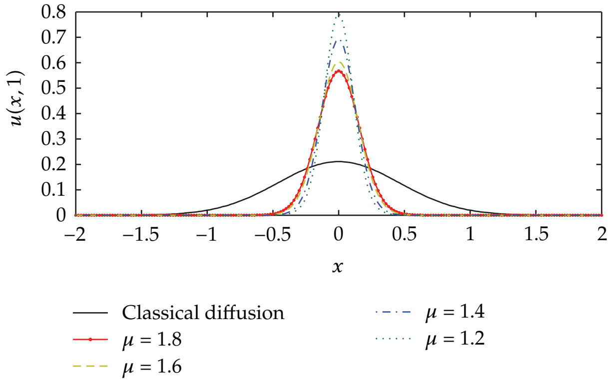

Figure 4: Numerical solutions at $t = 1$ for (5.5) and (5.10) with $\lambda = 0.25$.

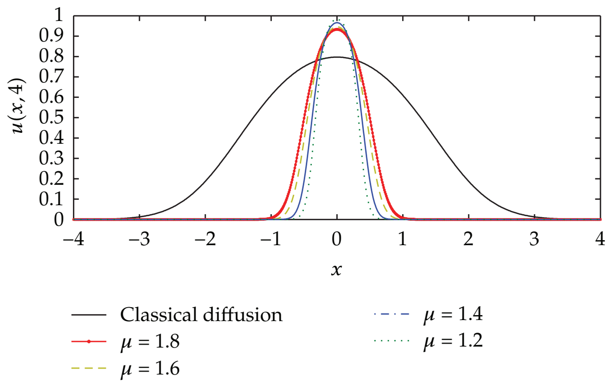

Figure 5: Numerical solutions at $t = 4$ for (5.5) and (5.10) with $\lambda = 1$.

problem. We obtain computational results using the method as in Example 5.2. Figure 4 shows contour plots of numerical solutions at $t = 1$ for (5.5) with an initial condition (5.10) when $\Omega = (-2, 2)$ and $\lambda = 0.25$. Figure 5 shows also contour plots of numerical solutions at $t = 4$ for (5.5) and (5.10) when $\Omega = (-4, 4)$ and $\lambda = 1$. In both cases, step sizes $h = 0.01$ and $k = 0.005$ are used for computation. According to Figures 4 and 5, we may see that the diffusivity depends on μ but it is far less than that of the classical solution. That is, the fractional diffusion problem keeps the peak in the middle for longer time than the classical one does.

Figure 6 shows contour plots of numerical solutions for (5.5) with an initial condition (5.10) when $\mu = 1.8$, $\Omega = (-2, 2)$ and $\lambda = 1$. In this case, step sizes $h = 0.01$ and $k = 0.005$ are also used for computation. But the period of time is from $t = 0$ to $t = 5$. According to Figure 6, we may see that the peak goes down rapidly for a short time, and it begins to go up after the contour arrives at the lowest level.

Figure 7 shows contour plots of numerical solutions at $t = 1$ for (5.5) with an initial condition (5.11) when $\Omega = (-4, 4)$ and $\lambda = 0.25$. In this case, step sizes $h = 0.01$ and $k = 0.005$ are also used for computation. According to Figure 7, we may find that the fractional diffusion problem keeps the flat roof in the middle for longer time than the classical one does.

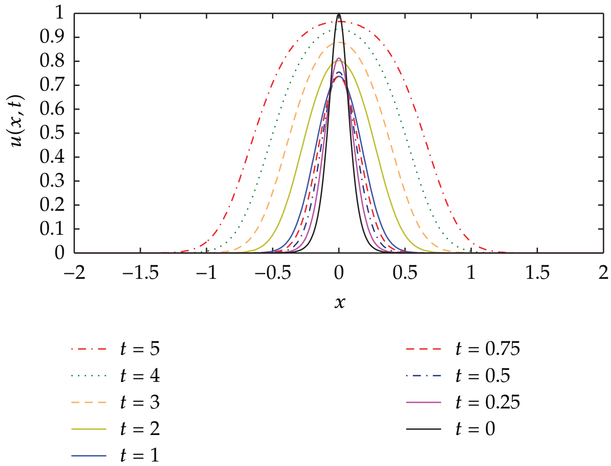

Figure 6: Numerical solutions for (5.5) and (5.10) with $\lambda = 1$.

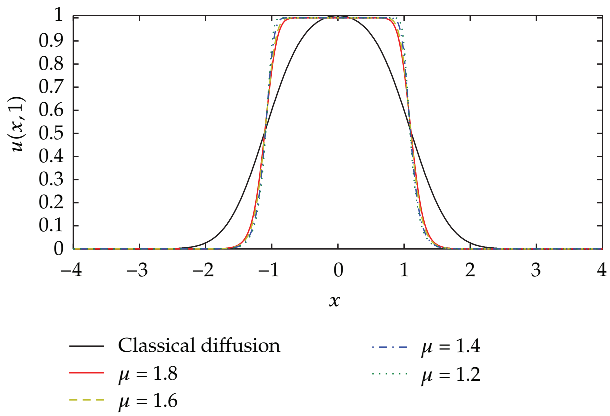

Figure 7: Numerical solutions for (5.5) and (5.11) with $\lambda = 0.25$.

6. Concluding Remarks

Galerkin finite element methods are considered for the space fractional diffusion equation with a nonlinear source term. We have derived the variational formula of the semidiscrete scheme by using the Galerkin finite element method in space. We showed existence and stability of solutions for the semidiscrete scheme. Furthermore, we derived the fully time-space discrete variational formulation using the backward Euler method. Existence and uniqueness of solutions for the fully discrete Galerkin method have been discussed. Also we proved that the scheme is unconditionally stable, and it has the order of convergence of $O(k + h^{\tilde{\gamma}})$, where $\tilde{\gamma}$ is a constant depending on the order of fractional derivative. Numerical computations confirm the theoretical results discussed in the previous section for the problem with a linear source term. For the fractional diffusion problem with a nonlinear source term, we may find that the diffusivity depends on the order of fractional derivative, and numerical solutions of fractional order problems are less diffusive than the solution of a classical diffusion problem.

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Research Article

Dynamic Properties of the Fractional-Order Logistic Equation of Complex Variables

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We study the dynamic properties (equilibrium points, local and global stability, chaos and bifurcation) of the continuous dynamical system of the logistic equation of complex variables. The existence and uniqueness of uniformly Lyapunov stable solution will be proved.

1. Introduction

Dynamical properties and chaos synchronization of deterministic nonlinear systems have been intensively studied over the last two decades on a large number of real dynamical systems of physical nature (i.e., those that involve real variables). However, there are also many interesting cases involving complex variables. As an example, we mention here the complex Lorenz equations, complex Chen and Lü chaotic systems, and some others (see [1–8] and the references therein).

The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only among mathematicians, but also among physicists and engineers (see [9–16] and references therein).

Consider the following fractional-order Logistic equation of complex variables:

$$D^{\alpha} z(t) = \rho z(t)(1 - z(t)) = \rho z(t) - \rho z^2(t), \quad t > 0, \tag{1.1}$$

$$z(0) = z_o = x_o + i y_o, \tag{1.2}$$

where

$$z(t) = x(t) + iy(t), \quad |z(t)| \le 1,$$
$$\rho = a + ib, \quad a, b > 0. \tag{1.3}$$

Here we study the dynamic properties (equilibrium points, local and global stability, chaos and bifurcation) of the continuous dynamical system of complex variables (1.1)-(1.2). The the existence of a unique uniformly stable solution and the continuous dependence of the solution on the initial data (1.2) are also proved.

Now we give the definition of fractional-order integration and fractional-order differentiation.

Definition 1.1. The fractional integral of order $\beta \in R^+$ of the function $f(t), t \in I$ is

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \tag{1.4}$$

and the Caputo's definition for the fractional order derivative of order $\alpha \in (0,1]$ of $f(t)$ is given by

$$D^{\alpha} f(t) = I^{1-\alpha} \frac{d}{dt} f(t). \tag{1.5}$$

2. Existence and Uniqueness

The following lemma (formulation of the problem) can be easily proved.

Lemma 2.1. *The discontinuous dynamical system* (1.1)-(1.2) *can be transformed to the system*

$$D^{\alpha} x(t) = ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t), \quad t > 0, \tag{2.1}$$

$$D^{\alpha} y(t) = bx(t) + ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t), \quad t > 0, \tag{2.2}$$

with the initial values

$$x(0) = x_o, \quad y(0) = y_o, \tag{2.3}$$

where $|x(t)| \le 1$ *and* $|y(t)| \le 1$.

Let $C[0,T]$ *be the class of continuous functions defined on* $[0,T]$.

Let Y *be the class of columns vectors* $(x(t), y(t))^{\tau}$, $x, y \in C[0,T]$ *with the norm*

$$\left\| (x,y)^{\tau} \right\|_Y = \|x\| + \|y\| = \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |y(t)|. \tag{2.4}$$

Let X be the class of columns vectors $(x(t), y(t))^{\tau}$, $x, y \in C[0,T]$ *with the equivalent norm*

$$\|(x,y)^{\tau}\|_X = \|x\|^* + \|y\|^* = \sup_{t\in[0,T]} e^{-Nt}|x(t)| + \sup_{t\in[0,T]} e^{-Nt}|y(t)|, \quad N > 0. \tag{2.5}$$

Write the problem (2.1)-(2.3) *in the following matrix form:*

$$\begin{aligned} D^{\alpha}(x,y)^{\tau} = \Big(&ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t),\ bx(t) \\ &+ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t)\Big)^{\tau}, \end{aligned} \tag{2.6}$$

and

$$(x(0), y(0))^{\tau} = (x_o, y_o)^{\tau}, \tag{2.7}$$

where τ *is the transpose of the matrix.*

Now we have the following theorem.

Theorem 2.2. *The problem* (2.6)-(2.7) *has a unique solution* $(x,y) \in X$.

Proof. Integrating (2.6) α-times we obtain

$$\begin{aligned} (x(t), y(t))^{\tau} = (x(0), y(0))^{\tau} + I^{\alpha}\Big(&ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t),\ bx(t) \\ &+ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t)\Big)^{\tau}. \end{aligned} \tag{2.8}$$

Define the operator $F : X \rightarrow X$ by

$$\begin{aligned} F(x(t), y(t))^{\tau} = (x(0), y(0))^{\tau} + I^{\alpha}\Big(&ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t),\ bx(t) \\ &+ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t)\Big)^{\tau}, \end{aligned} \tag{2.9}$$

then by direct calculations, we can get

$$\|F(x,y) - F(u,v)^{\tau}\|_X \le K\|(x,y) - (u,v)^{\tau}\|_X, \tag{2.10}$$

where

$$K = 5(a+b)\frac{1}{N^{\alpha}}. \tag{2.11}$$

Choose N large enough we find that $K < 1$ and by the contraction fixed theorem [17] the problem (2.6)-(2.7) has a unique solution $(x,y) \in X$.

From the continuity of the solution we deduce that (see [10])

$$I^{\alpha}\Big(ax(t) - by(t) - a\Big(x^2(t) - y^2(t)\Big) + 2bx(t)y(t),\ bx(t) \\ +ay(t) - b\Big(x^2(t) - y^2(t)\Big) - 2ax(t)y(t)\Big)^{\tau}\Big|_{t=0} = 0, \tag{2.12}$$

then the solution satisfies the initial condition. Differentiating (2.8), then by the same way as in ([18, 19]), we deduce that the integral equation (2.8) satisfies the problem (2.6)-(2.7) which completes the proof. □

3. Uniform Stability

Theorem 3.1. *The solution of the problem* (2.6)–(2.7) *is uniformly stable in the sense that*

$$|x_o - x_o^*| + |y_o - y_o^*| \le \delta \Longrightarrow \|(x,y) - (x^*,y^*)\|_X \le \epsilon, \tag{3.1}$$

where $(x^*(t), y^*(t))$ *is the solution of the differential equation* (2.6) *with the initial data*

$$(x(0), y(0))^{\tau} = (x_o^*, y_o^*)^{\tau}. \tag{3.2}$$

Proof. Direct calculations give

$$\|(x,y) - (x^*,y^*)^{\tau}\|_X \le |x_o - x_o^*| + |y_o - y_o^*| + K\|(x,y) - (x^*,y^*)^{\tau}\|_X, \tag{3.3}$$

which implies that

$$\|(x,y) - ((x^*,y^*)^{\tau}\|_X \le (1-K)^{-1}(|x_o - x_o^*| + |y_o - y_o^*|) \le \epsilon, \tag{3.4}$$

$$\epsilon = (1-K)^{-1}\delta. \tag{3.5}$$

□

4. Equilibrium Points and Their Asymptotic Stability

Let $\alpha \in (0,1]$ and consider the system ([9, 20–22])

$$D^{\alpha}y_1(t) = f_1(y_1, y_2), \\ D^{\alpha}y_2(t) = f_2(y_1, y_2), \tag{4.1}$$

with the initial values

$$y_1(0) = y_{o1},\ y_2(0) = y_{o2}. \tag{4.2}$$

To evaluate the equilibrium points, let

$$D^{\alpha} y_j(t) = 0 \Longrightarrow f_j\left(y_1^{\mathrm{eq}}, y_2^{\mathrm{eq}}\right) = 0, \quad j = 1, 2, \tag{4.3}$$

from which we can get the equilibrium points $y_1^{\mathrm{eq}}, y_2^{\mathrm{eq}}$.

To evaluate the asymptotic stability, let

$$y_j(t) = y_j^{\mathrm{eq}} + \varepsilon_j(t). \tag{4.4}$$

So the the equilibrium point $(y_1^{\mathrm{eq}}, y_2^{\mathrm{eq}})$ is locally asymptotically stable if both the eigenvalues of the Jacobian matrix A

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{bmatrix} \tag{4.5}$$

evaluated at the equilibrium point satisfies $(|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2)$ ([9, 20–23]).

For the fractional-order Logistic equation of complex variables consider the following:

$$\begin{aligned} D^{\alpha} x(t) &= a x(t) - b y(t) - a\left(x^2(t) - y^2(t)\right) + 2b x(t) y(t), \quad t > 0, \\ D^{\alpha} y(t) &= b x(t) + a y(t) - b\left(x^2(t) - y^2(t)\right) - 2a x(t) y(t), \quad t > 0. \end{aligned} \tag{4.6}$$

To evaluate the equilibrium points, let

$$\begin{aligned} D^{\alpha} x &= 0, \\ D^{\alpha} y &= 0, \end{aligned} \tag{4.7}$$

then $(x_{\mathrm{eq}}, y_{\mathrm{eq}}) = (0,0), (1,0)$, are the equilibrium points.

For $(x_{\mathrm{eq}}, y_{\mathrm{eq}}) = (0,0)$ we find that

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{4.8}$$

its eigenvalues are

$$\lambda = a \mp bi. \tag{4.9}$$

A sufficient condition for the local asymptotic stability of the equilibrium point $(0,0)$ is

$$\left|\arg(\lambda_1)\right| > \frac{\alpha\pi}{2}, \quad \left|\arg(\lambda_2)\right| > \frac{\alpha\pi}{2}, \quad 0 < \alpha < 1, \tag{4.10}$$

that is,

$$\frac{b}{a} > \tan\left(\frac{\alpha\pi}{2}\right) \tag{4.11}$$

and x_0 is small.

For $(x_{eq}, y_{eq}) = (1, 0)$ we find that

$$A = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} \tag{4.12}$$

its eigenvalues are

$$\lambda = -a \pm bi. \tag{4.13}$$

A sufficient condition for the local asymptotic stability of the equilibrium point $(1, 0)$ is $a > 0$ and x_0 is not close to zero.

5. Numerical Methods and Results

An Adams-type predictor-corrector method has been introduced and investigated further in ([24–26]). In this paper we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equation.

The key to the derivation of the method is to replace the original problem (2.1)-(2.2) by an equivalent fractional integral equations

$$\begin{aligned} x(t) &= x(0) + I^{\alpha}\left[ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t)\right], \\ y(t) &= y(0) + I^{\alpha}\left[bx(t) + ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t)\right], \end{aligned} \tag{5.1}$$

and then apply the PECE (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions displayed in Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 for different $0 < \alpha \leq 1$. In Figures 1–4 we take $x(0) = 0.1$, $y(0) = 0.9, a = 0.1, b = 0.9$ and found that the equilibrium point $(0, 0)$ is local asymptotic stable for $\alpha = 0.8, 0.9$ because the condition $b/a > \tan(\alpha\pi/2)$ is satisfied and the equilibrium point $(1, 0)$ is local asymptotic stable for $\alpha = 1.0$. In Figures 5–8 we take $x(0) = 0.2$, $y(0) = 0.7, a = 0.1, b = 0.5$ and found that the equilibrium point $(0, 0)$ is local asymptotic stable for $\alpha = 0.8$ because the condition $b/a > \tan(\alpha\pi/2)$ is satisfied and the equilibrium point $(1, 0)$ is local asymptotic stable for $\alpha = 0.9, 1.0$. In Figures 9–12 we take $x(0) = 0.5$, $y(0) = 0.5, a = 0.1, b = 0.4$ and found that the equilibrium point $(1, 0)$ is local asymptotic stable for $\alpha = 0.8, 0.9, 1.0$.

6. Conclusions

In this paper we considered the fractional-order Logistic equations of complex variables. Here we studied the dynamic properties (equilibrium points, local and global stability, chaos

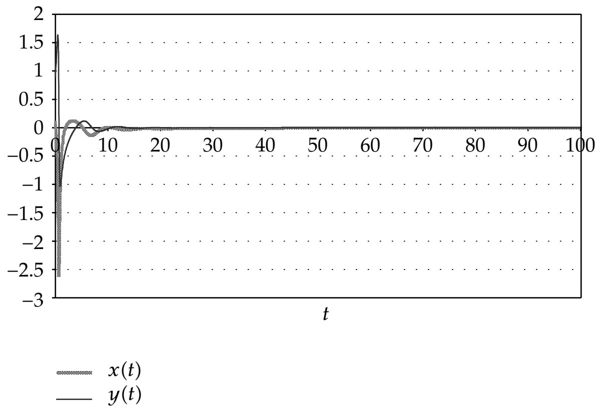


Figure 1: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$, alpha = 0.8.

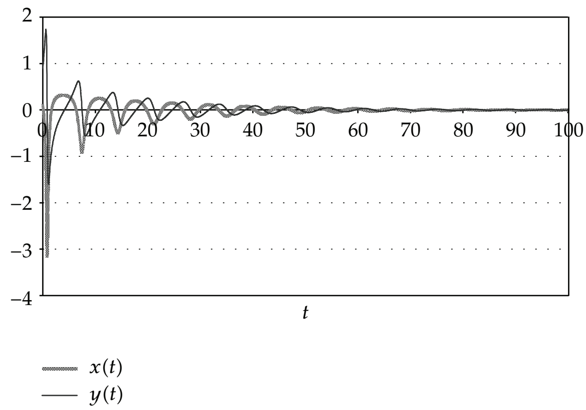


Figure 2: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$, alpha = 0.9.

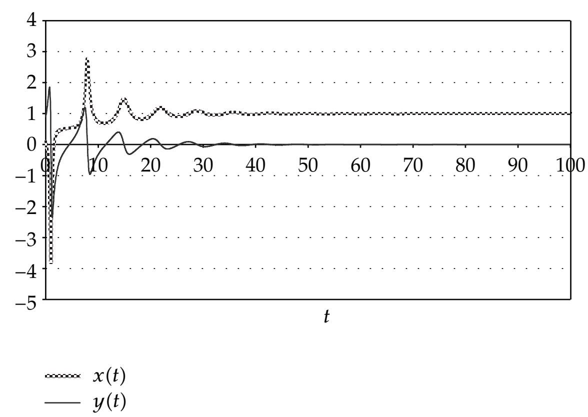
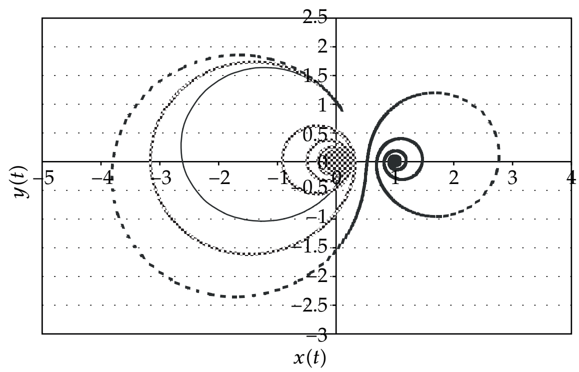
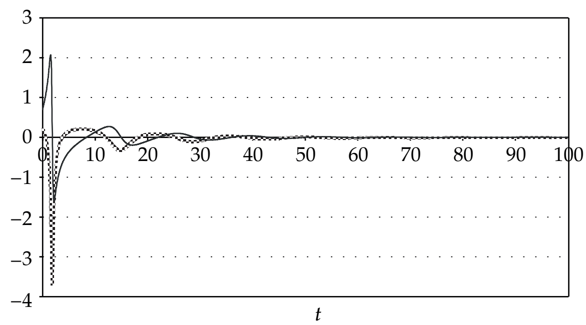


Figure 3: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$, alpha = 1.0.



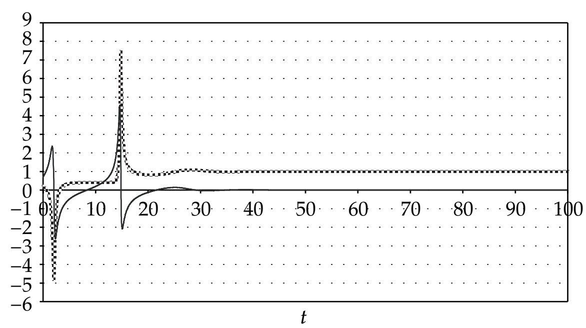
$\alpha = 0.8$
$\alpha = 0.9$
$\alpha = 1$

Figure 4: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$.



$x(t)$
$y(t)$

Figure 5: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$, alpha = 0.8.



$x(t)$
$y(t)$

Figure 6: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$, alpha = 0.9.

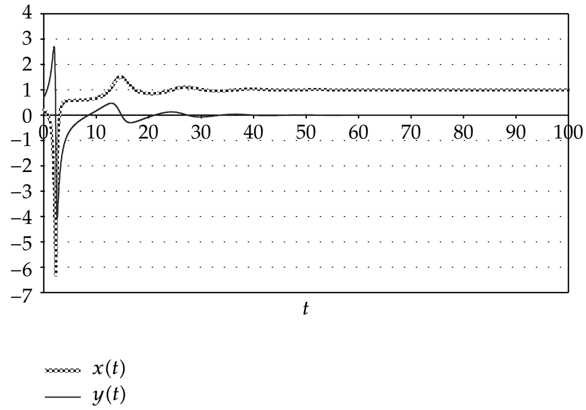


Figure 7: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$, alpha = 1.0.

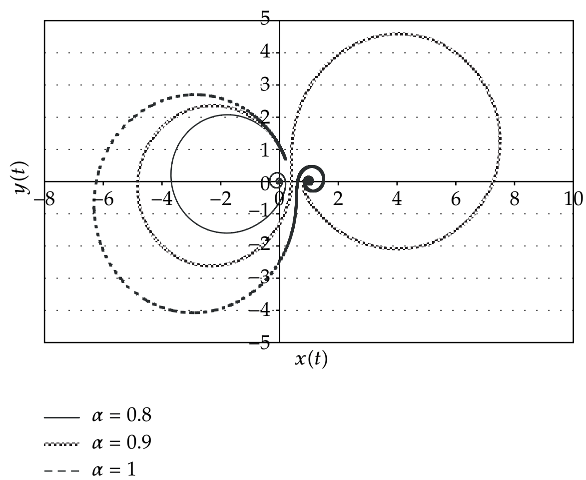


Figure 8: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$.

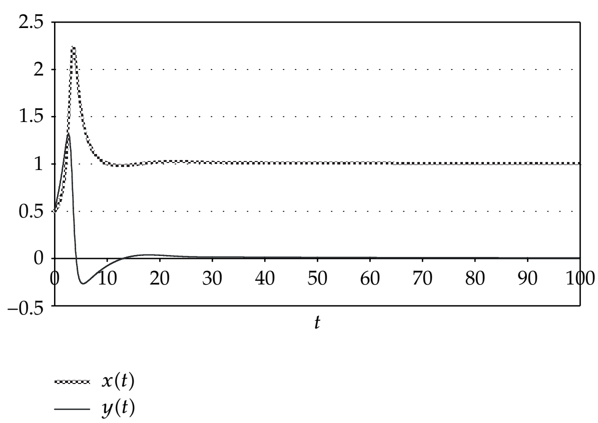


Figure 9: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$, alpha = 0.8.

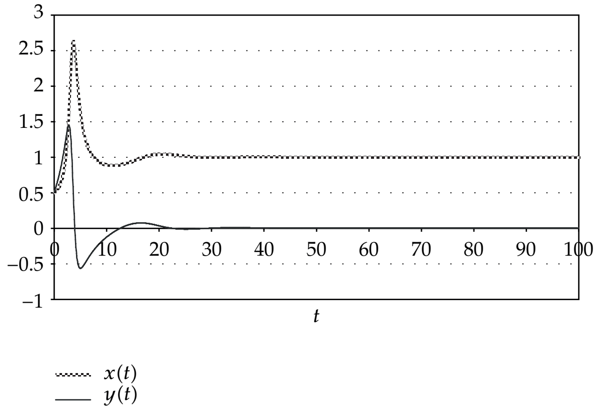


Figure 10: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$, alpha = 0.9.

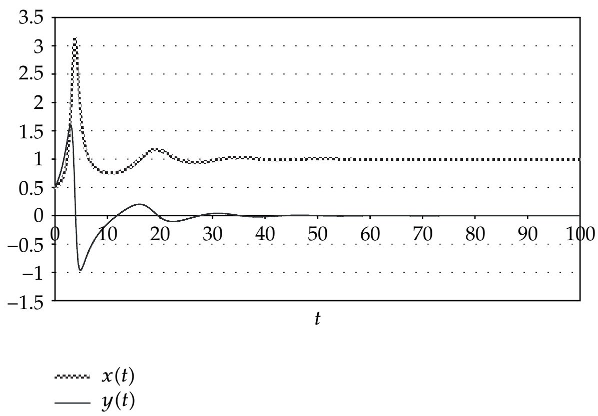


Figure 11: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$, alpha = 1.0.

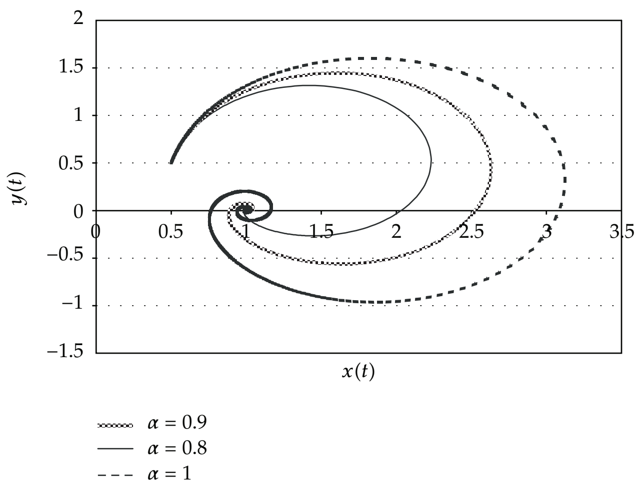


Figure 12: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$.

and bifurcation). The existence of a unique uniformly stable solution and the continuous dependence of the solution on the initial data (1.2) are also proved. Also we studied the numerical solution of the system (1.1)-(1.2).

We like to argue that fractional-order equations are more suitable than integer-order ones in modeling biological, economic, and social systems (generally complex adaptive systems) where memory effects are important.

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Research Article

A Note on Impulsive Fractional Evolution Equations with Nondense Domain

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This paper is concerned with the existence of integral solutions for nondensely defined fractional functional differential equations with impulse effects. Some errors in the existing paper concerned with nondensely defined fractional differential equations are pointed out, and correct formula of integral solutions is established by using integrated semigroup and some probability densities. Sufficient conditions for the existence are obtained by applying the Banach contraction mapping principle. An example is also given to illustrate our results.

1. Introduction

The aim in this paper is to study the existence of the integral solutions for the fractional semilinear differential equations of the form

$$\begin{gathered} D^q y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0,b], \quad t \neq t_k, \quad k = 1,\dots,m, \\ \Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{1.1}$$

where $0 < q < 1$, D^q is the Caputo fractional derivative. $f : J \times \mathfrak{D} \to E$ is a given function, $\mathfrak{D} = \{\psi : [-\tau, 0] \to E, \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, and E is a real Banach space with the norm $|\cdot|$. Denoting the domain of A by $D(A)$, $A : D(A) \subset E \to E$ is nondensely closed linear operator on E, $\phi \in \mathfrak{D}$. $I_k : E \to E$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits at t_k of $y(t)$ as usual; we assume $y(t_k^-) = y(t_k)$. $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ represents the jump in the state y at time t_k. Moreover, for any $t \in J$, the histories y_t belong to $\mathfrak{D}$ defined by $y_t(\varsigma) = y(t+\varsigma), \varsigma \in [-\tau, 0]$.

In the past decades, the theory of fractional differential equations has become an important area of investigation because of its wide applicability in many branches of physics, economics, and technical sciences [1–10]. In recent years, many authors were devoted to mild solutions to fractional evolution equations, and there have been a lot of interesting works. For instance, in [11], El-Borai discussed the following equation in Banach space X:

$$\begin{gathered} D^q u(t) = Au(t) + B(t)u(t), \\ u(0) = u_0, \end{gathered} \tag{1.2}$$

where A generates an analytic semigroup, and the solution was given in terms of some probability densities. In [12], Zhou and Jiao concerned the existence and uniqueness of mild solutions for fractional evolution equations by some fixed point theorems. Cao et al. [13] studied the α-mild solutions for a class of fractional evolution equations and optimal controls in fractional powder space. For more information on this subject, the readers may refer to [14–16] and the references therein.

Research on integer order differential evolution equations including a nondensely defined operator was initialed by Da Prato and Sinestrari [17] and has been extensively investigated by many authors [18–25]. The main methods used in their work are based on integrated semigroup theory. Recently, existence results for integral solutions of nondensely defined fractional evolution equations were established in some papers [9, 26]. But there are some errors in transforming integral solution into an available form. For example, definition of integral solution [9] is given by

$$x(t) = S(t)\left(x_0 - g(x)\right) + \lim_{\lambda \to \infty} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) B(\lambda, A) f(s, x(s)) ds, \quad t \geq 0. \tag{1.3}$$

Here $D(A) \subset E$ and $\overline{D(A)} \neq E$. $B(\lambda, A) := \lambda(\lambda I - A)^{-1}$ will be introduced in next section. If we let f take values in $\overline{D(A)}$, then (1.3) becomes

$$x(t) = S(t)\left(x_0 - g(x)\right) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) f(s, x(s)) ds. \tag{1.4}$$

According to [19], integral solution should be mild solution in this case. But as pointed in [14], (1.4) is not the mild solution.

Motivated by these papers and the fact that impulse effects exist widely in the realistic situations, we give the definition of integral solution and prove the existence results for impulsive semilinear fractional differential equations with nondensely defined operators. The rest of the paper will be organized as follows. In Section 2, we will recall some basic definitions and preliminary facts from integrated semigroups and fractional derivation and integration which would be used later. Section 3 is devoted to the existence of integral solutions of problem (1.1). We present an example to illustrate our results in Section 4. At last, we end the paper with a conclusion.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary results which would be used in the rest of the paper.

We denote by $C([0,b];E)$ the Banach space of all continuous functions from $[0,b]$ into E with the norm

$$\|x\|_\infty = \sup\{|y(t)| : t \in [0,b]\}. \tag{2.1}$$

For $\phi \in \mathfrak{D}$ the norm of ϕ is defined by

$$\|\phi\|_{\mathfrak{D}} = \sup\{|\phi(\varsigma)| : \varsigma \in [-\tau,0]\}. \tag{2.2}$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E, with the norm

$$\|N\| = \sup\{|N(y)| : |y| = 1\}, \tag{2.3}$$

where $N \in B(E)$ and $y \in E$. Let $L^p([0,b];E)$ be the space of E-valued Bochner function on $[0,b]$ with the norm

$$\|x\|_{L^p} = \left(\int_0^b |y(s)|^p ds\right)^{1/p}, \quad 1 \leq p < \infty. \tag{2.4}$$

In order to define an integral solution of problem (1.1), we will introduce the set of functions

$$\begin{gathered}\mathrm{PC} = \Big\{y : J \longrightarrow \overline{D(A)}, \text{ is continuous except for } t = t_k,\ k = 1,2,\ldots,m, \\ \text{there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ such that } y(t_k^-) = y(t_k)\Big\}.\end{gathered} \tag{2.5}$$

Endowed with the norm $\|y\|_{\mathrm{PC}} = \sup_{t\in J}|y(t)|$, $(\mathrm{PC}, \|\cdot\|_{\mathrm{PC}})$ is a Banach space.

Seting

$$\Omega = \Big\{y : [-\tau,b] \longrightarrow \overline{D(A)} : y \in \mathfrak{D} \cap \mathrm{PC}\Big\}, \tag{2.6}$$

then Ω is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y\|_{\mathfrak{D}}, \|y\|_{\mathrm{PC}}\}. \tag{2.7}$$

Definition 2.1 (see [27]). Letting E be a Banach space, an integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:

(i) $S(0) = 0$;

(ii) $t \to S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$ for all $t, s \geq 0$.

Definition 2.2 (see [28]). An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbf{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ for all $\lambda > \omega$.

Proposition 2.3 (see [27]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A), \qquad S(t)x = A\int_0^t S(s)xds + tx. \tag{2.8}$$

Definition 2.4 (see [29]). We say that a linear operator A satisfies the Hille-Yosida condition if there exists $M \geq 0$ and $\omega \in \mathbf{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda - \omega)^n \left\|(\lambda I - A)^{-n}\right\|,\ n \in \mathbf{N}, \lambda > \omega\right\} \leq M. \tag{2.9}$$

Here and hereafter, we assume that A satisfies the Hille-Yosida condition. Let us introduce the part A_0 of A in $\overline{D(A)}$: $A_0 = A$ on $D(A_0) = \{x \in D(A); Ax \in \overline{D(A)}\}$. Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. We note that $(S'(t))_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by A_0 and $\|S'(t)\| \leq Me^{\omega t}$, $t \geq 0$, where M and ω are the constants considered in the Hille-Yosida condition [28, 30].

Let $B(\lambda, A) := \lambda(\lambda I - A)^{-1}$; then for all $x \in \overline{D(A)}$, $B(\lambda, A)x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition it is easy to see that $\lim_{\lambda\to\infty}|B(\lambda, A)x| \leq M|x|$.

For more properties on integral semigroup theory the interested reader may refer to [18, 30].

Definition 2.5 (see [3]). The Riemann-Liouville fractional integral of order $q \in \mathbf{R}^+$ of a function $h : (0, b] \to E$ is defined by

$$I_0^q h(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds, \tag{2.10}$$

provided the right-hand side is pointwise defined on $(0, b]$ and where Γ is the gamma function.

Remark 2.6. According to [10], $I_0^q I_0^\beta = I_0^{q+\beta}$ holds for all $q, \beta \geq 0$.

Definition 2.7 (see [3]). The Caputo fractional derivative of order $0 < q < 1$ of a function $f \in C^1([0, \infty), E)$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q} f'(s)ds, \quad t > 0. \tag{2.11}$$

3. Main Results

In this section we will establish the existence and uniqueness of integral solution for problem (1.1).

Definition 3.1. A function $y \in \Omega$ is said to be an integral solutions of (1.1) if

(i) $\int_{t_k}^{t}(t-s)^{q-1}y(s)ds \in D(A)$ for $t \in (t_k, t_{k+1}],\ k = 0,1,\dots,m$,

(ii) $y(t) = \phi(t),\ t \in [-\tau, 0]$,

(iii)

$$y(t) = \begin{cases} \phi(0) + \dfrac{1}{\Gamma(q)}A\displaystyle\int_0^t (t-s)^{q-1}y(s)ds + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}f(s,y_s)ds, & t \in (0,t_1], \\ y(t_1^-) + I_1(y(t_1^-)) + \dfrac{1}{\Gamma(q)}A\displaystyle\int_{t_1}^t (t-s)^{q-1}y(s)ds & \\ \quad + \dfrac{1}{\Gamma(q)}\displaystyle\int_{t_1}^t (t-s)^{q-1}f(s,y_s)ds, & t \in (t_1,t_2], \\ \vdots & \\ y(t_m^-) + I_m(y(t_m^-)) + \dfrac{1}{\Gamma(q)}A\displaystyle\int_{t_m}^t (t-s)^{q-1}y(s)ds & \\ \quad + \dfrac{1}{\Gamma(q)}\displaystyle\int_{t_m}^t (t-s)^{q-1}f(s,y_s)ds, & t \in (t_m,b]. \end{cases} \tag{3.1}$$

Lemma 3.2. *If y is an integral solution of* (1.1)*, then for all $t \in [0,b]$, $y(t) \in \overline{D(A)}$. In particular, $\phi(0)$, $y(t_1^-) + I_1(y(t_1^-))$,...,$y(t_m^-) + I_1(y(t_m^-))$ belong to $\overline{D(A)}$.*

Proof. Using Remark 2.6, for each $t \in (t_k, t_{k+1}]$, $I_{t_k}^1 y(t) = I_{t_k}^{1-q} I_{t_k}^q y(t) \in D(A)$ since $I_{t_k}^q y(t) \in D(A)$. Consequently, for $h > 0$ such that $t + h \in (t_k, t_{k+1}]$, $(1/h)\int_t^{t+h} y(s)ds \in D(A)$ because $I_{t_k}^1 y(t) = \int_{t_k}^t y(s)ds \in D(A)$. Hence, we deduce that $y(t) = \lim_{h\to 0}(1/h)\int_t^{t+h} y(s)ds \in \overline{D(A)}$. The proof is completed. □

Lemma 3.3 (see [31]). *Let $\Psi_q(\theta) = (1/\pi)\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}(\Gamma(nq+1)/n!)\sin(n\pi q)$, $\theta \in \mathbf{R}^+$; then $\Psi_q(\theta)$ is a one-sided stable probability density function, and its Laplace transform is given by*

$$\int_0^{\infty} e^{-p\theta}\Psi_q(\theta)d\theta = e^{-p^q}, \quad q \in (0,1),\ p > 0. \tag{3.2}$$

Lemma 3.4. *For $t \in (0,b]$, the integral solution in Definition 3.1 is given by*

$$y(t) = \begin{cases} \mathfrak{S}(t)\phi(0) + \lim\limits_{\lambda\to\infty}\displaystyle\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)f(s,y_s)ds, & t \in (0,t_1], \\ \mathfrak{S}(t-t_1)\left(y(t_1^-) + I_1(y(t_1^-))\right) & \\ \quad + \lim\limits_{\lambda\to\infty}\displaystyle\int_{t_1}^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)f(s,y_s)ds, & t \in (t_1,t_2], \\ \vdots & \\ \mathfrak{S}(t-t_m)\left(y(t_m^-) + I_m(y(t_m^-))\right) & \\ \quad + \lim\limits_{\lambda\to\infty}\displaystyle\int_{t_m}^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)f(s,y_s)ds, & t \in (t_m,b], \end{cases} \tag{3.3}$$

where

$$\mathfrak{S}(t) = \int_0^\infty h_q(\theta) S'(t^q\theta) d\theta, \quad \mathfrak{T}(t) = q\int_0^\infty \theta h_q(\theta) S'(t^q\theta) d\theta, \tag{3.4}$$

where $h_q(\theta) = (1/q)\theta^{-1-1/q}\Psi_q(\theta^{-1/q})$ *is the probability density function defined on* $\mathbf{R}^+$.

Proof. From the definition, for $t \in (0, t_1]$ we have

$$y(t) = \phi(0) + \frac{1}{\Gamma(q)} A\int_0^t (t-s)^{q-1} y(s) ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y_s) ds, \quad t \in [0, b]. \tag{3.5}$$

Consider the Laplace transform

$$v(p) = \int_0^\infty e^{-pt} B(\lambda, A) y(t) dt, \qquad w(p) = \int_0^\infty e^{-pt} B(\lambda, A) f(t, y_t) dt, \quad p > 0. \tag{3.6}$$

Note that for each $0 < t \le t_1$, $B_\lambda y(t)$, $B(\lambda, A) f(t, y_t) \in D(A)$, then we have $v(p), w(p) \in \overline{D(A)}$. Applying (3.6) to (3.5) yields

$$\begin{aligned} v(p) &= \frac{1}{p} B(\lambda, A)\phi(0) + \frac{1}{p^q} A v(p) + \frac{1}{p^q} w(p) \\ &= p^{q-1}(p^q I - A)^{-1} B(\lambda, A)\phi(0) + (p^q I - A)^{-1} w(p) \\ &= p^{q-1}\int_0^\infty e^{-p^q s} S'(s) B(\lambda, A)\phi(0) ds + \int_0^\infty e^{-p^q s} S'(s) w(p) ds, \end{aligned} \tag{3.7}$$

where I is the identity operator defined on E.

From (3.2), we get

$$\begin{aligned} &p^{q-1}\int_0^\infty e^{-p^q s} S'(s) B(\lambda, A)\phi(0) ds \\ &= \int_0^\infty p^{q-1} e^{-(pt)^q} S'(t^q) B(\lambda, A)\phi(0) q t^{q-1} dt \\ &= \int_0^\infty -\frac{1}{p}\frac{d}{dt}\left(e^{-(pt)^q}\right) S'(t^q) B(\lambda, A)\phi(0) dt \\ &= \iint_0^\infty \left(\theta\Psi_q(\theta) e^{-pt\theta} S'(t^q) B(\lambda, A)\phi(0)\right) d\theta\, dt \\ &= \iint_0^\infty \left(\Psi_q(\theta) e^{-ps} S'\left(\left(\frac{s}{\theta}\right)^q\right) B(\lambda, A)\phi(0)\right) d\theta\, ds \\ &= \int_0^\infty e^{-pt}\left(\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B(\lambda, A)\phi(0) d\theta\right) dt, \end{aligned}$$

$$
\begin{aligned}
&\int_0^\infty e^{-p^q s} S'(s) w(p)\, ds \\
&\quad = \iint_0^\infty e^{-p^q s} e^{-pt} S'(s) B(\lambda, A) f(t, y_t)\, dt\, ds \\
&\quad = \iint_0^\infty q s^{q-1} e^{-(ps)^q} e^{-pt} S'(s^q) B(\lambda, A) f(t, y_t)\, dt\, ds \\
&\quad = \iiint_0^\infty q \Psi_q(\theta) e^{-ps\theta} e^{-pt} S'(s^q) B(\lambda, A) f(t, y_t)\, d\theta\, dt\, ds \\
&\quad = \iiint_0^\infty q \Psi_q(\theta) e^{-p(s+t)} \frac{s^{q-1}}{\theta^q} S'\left(\left(\frac{s}{\theta}\right)^q\right) B(\lambda, A) f(t, y_t)\, d\theta\, dt\, ds \\
&\quad = \int_0^\infty e^{-ps} q \int_0^s \int_0^\infty \Psi_q(\theta) \frac{(s-t)^{q-1}}{\theta^q} S'\left(\frac{(s-t)^q}{\theta^q}\right) B(\lambda, A) f(t, y_t)\, d\theta\, dt\, ds \\
&\quad = \int_0^\infty e^{-pt} q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B(\lambda, A) f(s, y_s)\, d\theta\, ds\, dt.
\end{aligned}
\tag{3.8}
$$

According to (3.7) and (3.8), we have

$$
\begin{aligned}
v(p) = &\int_0^\infty e^{-pt} \left(\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B(\lambda, A) \phi(0)\, d\theta \right) dt \\
&+ \int_0^\infty e^{-pt} q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B(\lambda, A) f(s, y_s)\, d\theta\, ds\, dt.
\end{aligned}
\tag{3.9}
$$

Inverting the last Laplace transform, we obtain

$$
\begin{aligned}
B(\lambda, A) y(t) = &\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B(\lambda, A) \phi(0)\, d\theta \\
&+ q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B(\lambda, A) f(s, y_s)\, d\theta\, ds \\
= &\int_0^\infty h_q(\theta) S'(t^q \theta) B(\lambda, A) \phi(0)\, d\theta \\
&+ q \int_0^t \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'((t-s)^q \theta) B(\lambda, A) f(s, y_s)\, d\theta\, ds.
\end{aligned}
\tag{3.10}
$$

In view of $\lim_{\lambda\to\infty}B(\lambda,A)x = x$ for $x \in \overline{D(A)}$ and Lemma 3.2, we have

$$\begin{aligned} y(t) &= \int_0^\infty h_q(\theta)S'(t^q\theta)\phi(0)d\theta + \lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\left((t-s)^q\theta\right)B(\lambda,A) \\ &\quad\times f\left(s,y_s\right)d\theta\, ds \\ &= \mathfrak{S}(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)f\left(s,y_s\right)ds. \end{aligned} \tag{3.11}$$

For $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we can prove the results by the similar methods used previously. The proof is completed. □

Remark 3.5. According to [31], one can easily check that

$$\int_0^\infty \theta h_q(\theta)d\theta = \int_0^\infty \frac{1}{\theta^q}\Psi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}. \tag{3.12}$$

We are now in a position to state and prove our main results for the existence and uniqueness of solutions of problem (1.1).

Let us list the following hypotheses.

(H1) A satisfies the Hille-Yosida condition, and assume that $\overline{M} := \sup\{\|S'(t)\| : t \in [0,+\infty]\} < \infty$.

(H2) For $u \in \mathfrak{D}$, $f(\cdot,u) : [0,b] \to E$ is strongly measurable.

(H3) There exists a constant $q_1 \in (0,q)$ and $l \in L^{1/q_1}([0,b];\mathbf{R}^+)$ such that

$$\left|f(t,u)\right| \le l(t), \text{ a.e. } t \in J, \text{ and each } u \in \mathfrak{D}. \tag{3.13}$$

(H4) There exists $\rho > 0$ such that

$$|I_k(u) - I_k(v)| \le \rho|u - v| \quad \forall u,v \in E,\ k = 1,\ldots,m. \tag{3.14}$$

(H5) There exists a constant κ such that

$$\left|f(t,u) - f(t,v)\right| \le \kappa\|u - v\|_{\mathfrak{D}}, \quad \text{for } t \in J \text{ and every } u,v \in \mathfrak{D}. \tag{3.15}$$

Theorem 3.6. *Assuming that hypotheses* (H1)–(H5) *hold, then problem* (1.1) *has a unique integral solution $y \in \Omega$ provided that* $\overline{M}(1+\rho) + (\overline{M}M\kappa b^q/\Gamma(1+q)) < 1$.

Proof. Define $Q : \Omega \to \Omega$ by

$$(Qy)(t) = \phi(t), \quad t \in [-\tau, 0], \tag{3.16}$$

and for $t \in J$,

$$(Qy)(t) = \begin{cases} \mathfrak{S}(t)\phi(0) + \lim\limits_{\lambda \to \infty} \displaystyle\int_0^t (t-s)^{q-1} \mathfrak{T}(t-s) \times B(\lambda, A) f(s, y_s) ds, & t \in (0, t_1], \\ \mathfrak{S}(t-t_1)\left(y(t_1^-) + I_1(y(t_1^-))\right) & \\ \qquad + \lim\limits_{\lambda \to \infty} \displaystyle\int_{t_1}^t (t-s)^{q-1} \mathfrak{T}(t-s) B(\lambda, A) f(s, y_s) ds, & t \in (t_1, t_2], \\ \vdots & \\ \mathfrak{S}(t-t_m)\left(y(t_m^-) + I_m(y(t_m^-))\right) & \\ \qquad + \lim\limits_{\lambda \to \infty} \displaystyle\int_{t_m}^t (t-s)^{q-1} \mathfrak{T}(t-s) B(\lambda, A) f(s, y_s) ds, & t \in (t_m, b]. \end{cases} \tag{3.17}$$

Firstly we check that Q is well defined on Ω.

For each $y \in \Omega$, take $t \in (0, t_1]$. It is obvious that $\mathfrak{S}(t)\phi(0)$ is well defined. Direct calculation shows that $(t-s)^{q-1} \in L^{(1/(1-q_1))}[0, t]$, for $t \in [0, t_1]$ and $q_1 \in (0, q)$. Let

$$a = \frac{q-1}{1-q_1} \in (-1, 0), \qquad M_1 = \|l\|_{L^{1/q_1}[0,b]}. \tag{3.18}$$

Then for $t \in [0, t_1]$, we have

$$\begin{aligned} \int_0^t \left| (t-s)^{q-1} f(s, y_s) \right| ds &\leq \left(\int_0^t (t-s)^{((q-1)/(1-q_1))} ds \right)^{1-q_1} \|l\|_{L^{1/q_1}[0,t]} \\ &\leq \frac{M_1}{(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}. \end{aligned} \tag{3.19}$$

From (H1), (3.12), (3.19), and the fact that $\|B(\lambda, A)\| \leq M$, we get

$$\begin{aligned} &\int_0^t \left| \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'((t-s)^q \theta) B(\lambda, A) f(s, y_s) d\theta \right| ds \\ &\quad \leq M\overline{M} \int_0^t \int_0^\infty \theta h_q(\theta) \left| (t-s)^{q-1} f(s, y_s) \right| d\theta \, ds \\ &\quad \leq \frac{MM_0}{\Gamma(1+q)} \int_0^t \left| (t-s)^{q-1} f(s, y_s) \right| ds \\ &\quad \leq \frac{MM_0 M_1}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}, \quad \text{for } t \in [0, t_1]. \end{aligned} \tag{3.20}$$

It means that $| \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'((t-s)^q \theta) B(\lambda, A) f(s, y_s) d\theta |$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in [0, t_1]$. Therefore $\int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'((t-s)^q \theta) B(\lambda, A) f(s, y_s) d\theta$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in [0, t_1]$.

From [19], we know $\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}S'((t-s)^q\theta)B(\lambda,A)f(s,y_s)ds$ exists; then

$$\begin{aligned}
&\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)f(s,y_s)ds\\
&\quad=\lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'((t-s)^q\theta)B(\lambda,A)f(s,y_s)d\theta\, ds\\
&\quad=\lim_{\lambda\to\infty} q\int_0^\infty \theta h_q(\theta)\int_0^t (t-s)^{q-1}S'((t-s)^q\theta)B(\lambda,A)f(s,y_s)ds\, d\theta\\
&\quad= q\int_0^\infty \theta h_q(\theta)\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}S'((t-s)^q\theta)B(\lambda,A)f(s,y_s)ds\, d\theta
\end{aligned}\tag{3.21}$$

exists. Therefore we get $(Qy)(\cdot)$ which is well defined on $[0,t_1]$.

For $t\in(t_k,t_{k+1}]$, $k=1,2,\ldots,m$, similar discussion could obtain $(Qy)(\cdot)$ is well defined. Hence, Q is well defined on Ω.

Secondly, we will prove operator Q is a contraction.

For $t\in(0,t_1]$ and $y,z\in\Omega$, by the hypotheses and $\|B(\lambda,A)\|\le M$, we get

$$\begin{aligned}
&|(Qy)(t)-(Qz)(t)|\\
&\quad=\left|\lim_{\lambda\to\infty}\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)(f(s,y_s)-f(s,z_s))ds\right|\\
&\quad=\left|\lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'((t-s)^q\theta)B(\lambda,A)(f(s,y_s)-f(s,z_s))d\theta\, ds\right|\\
&\quad\le\frac{\overline{M}M}{\Gamma(1+q)}\int_0^t q(t-s)^{q-1}|f(s,y_s)-f(s,z_s)|ds\\
&\quad\le\frac{\overline{M}M\kappa}{\Gamma(1+q)}\int_0^t q(t-s)^{q-1}\|y_s-z_s\|_{\mathfrak{D}}ds\\
&\quad\le\frac{\overline{M}M\kappa b^q}{\Gamma(1+q)}\|y-z\|_\Omega.
\end{aligned}\tag{3.22}$$

Now take $t\in(t_k,t_{k+1}]$, $k=1,2,\ldots,m$ and $y,z\in\Omega$:

$$\begin{aligned}
&|(Qy)(t)-(Qz)(t)|\\
&\quad\le\left|\mathfrak{S}(t-t_k)\left[y(t_k^-)+I_k(y(t_k^-))-z(t_k^-)-I_k(z(t_k^-))\right]\right|\\
&\qquad+\left|\lim_{\lambda\to\infty}\int_{t_k}^t (t-s)^{q-1}\mathfrak{T}(t-s)B(\lambda,A)(f(s,y_s)-f(s,z_s))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \overline{M}(1+\rho)\|y-z\|_{\Omega}+\frac{\overline{M}M\kappa b^{q}}{\Gamma(1+q)}\|y-z\|_{\Omega} \\
&\leq \left(\overline{M}(1+\rho)+\frac{\overline{M}M\kappa b^{q}}{\Gamma(1+q)}\right)\|y-z\|_{\Omega}.
\end{aligned}
\tag{3.23}
$$

In view of $\overline{M}(1+\rho)+(\overline{M}M\kappa b^{q}/\Gamma(1+q))<1$, we have that the operator Q is a contraction. By the Banach contraction principle we have that Q has a unique fixed point $y\in\Omega$, which gives rise to a unique integral solution to the problem (1.1). The proof is finished. □

Remark 3.7. For impulsive Caputo fractional differential equations, its integral solutions (or mild solutions; see [14]) can be expressed only by using piecewise functions. Thus Definition 2.3 given in [15] is unsuitable.

4. An Example

As an application of our results we consider the following fractional differential equations of the form

$$
\begin{gathered}
D^{q}u(t,z)=\frac{\partial^{2}}{\partial z^{2}}u(t,z)+F(t,u_{t}(\varsigma,z)),\quad z\in[0,\pi],\quad t\in[0,1]\setminus\left\{\frac{1}{2}\right\}, \\
u\left(\frac{1}{2}^{+},z\right)-u\left(\frac{1}{2}^{-},z\right)=\rho u\left(\frac{1}{2}^{-},z\right),\quad z\in[0,\pi], \\
u(t,0)=u(t,\pi)=0,\quad t\in[0,1], \\
u(\varsigma,z)=\phi(\varsigma,z),\quad \varsigma\in[-1,0],\quad z\in[0,\pi].
\end{gathered}
\tag{4.1}
$$

Consider $E=C([0,\pi];\mathbf{R})$ endowed with the supnorm and the operator $A:D(A)\subset E\to E$ defined by

$$
D(A)=\left\{u\in C^{2}([0,\pi];\mathbf{R}):u(t,0)=u(t,\pi)=0\right\},\qquad Au=\frac{\partial^{2}}{\partial z^{2}}u(t,z). \tag{4.2}
$$

Now, we have $\overline{D(A)}=\{u\in E:u(t,0)=u(t,\pi)=0\}\neq E$. As we know from [17] that A satisfies the Hille-Yosida condition with $(0,+\infty)\subseteq\rho(A)$ and $\lambda>0$, $|R(\lambda,A)|\leq 1/\lambda$. Hence, operator A satisfies (H1) and $M=\overline{M}=1/2$.

Then the system (4.1) can be reformulated as

$$
\begin{gathered}
D^{q}y(t)=Ay(t)+f(t,y_{t}),\quad t\in J:=[0,b],\quad t\neq\frac{1}{2}, \\
\Delta y\big|_{t=1/2}=I\left(y\left(\frac{1}{2}^{-}\right)\right),\quad k=1,\ldots,m, \\
y(t)=\phi(t),\quad t\in[-\tau,0],
\end{gathered}
\tag{4.3}
$$

where $y(t)(z)=u(t,z)$, $f(t,y_{t})(z)=F(t,u_{t}(\varsigma,z))$, $I(x)=\rho x$, $\phi(t)(z)=\phi(t,z)$.

If we take $q = 1/3$, $\rho = 1/10$, $f(t, y_t) = (1/(t+1)(t+2)) \sin y_t$. We easily get that

$$\left| f(t,u) - t(t,v) \right| \leq \frac{1}{3} \|u - v\|_{\mathfrak{D}}, \quad \text{for } t \in J \text{ and every } u, v \in \mathfrak{D}. \tag{4.4}$$

Then all conditions of Theorem 3.6 are satisfied and we deduce (4.1) has a unique integral solution.

5. Conclusions

An essence error of the formula of solutions which appeared in the recent work on the nondensely defined fractional evolution differential equations is reported in this work. A correct formula of integral solutions for nondensely defined fractional evolution equations could be obtained from the results in this paper.

In view of the complicated definitions for integral or mild solutions for impulsive fractional evolution equations, many fixed point theorems related to completely continuous operators are hard to be used to establish the existence results. As far as we know, only [14] applied Leray Schauder Alternative theorem to the existence of mild solutions of impulsive fractional differential equations. But there is a mistake in proving that $\Gamma(B_r)$ is equicontinuous (page 2009, Step 3). How to overcome this difficulty is our next work.

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Research Article

Existence and Uniqueness of Positive Solutions for a Singular Fractional Three-Point Boundary Value Problem

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We investigate the existence and uniqueness of positive solutions for the following singular fractional three-point boundary value problem $D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, 0 < t < 1$, $u(0) = u'(0) = u''(0) = 0, u''(1) = \beta u''(\eta)$, where $3 < \alpha \leq 4$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative and $f : (0,1] \times [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} f(t,\cdot) = \infty$ (i.e., f is singular at $t = 0$). Our analysis relies on a fixed point theorem in partially ordered metric spaces.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications (see, e.g., [1–5]).

Recently, many papers have appeared dealing with the existence of solutions of nonlinear fractional boundary value problems.

In [6], the authors studied the existence and multiplicity of positive solutions for the boundary value problem:

$$\begin{gathered} D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, by using some fixed point theorem on cones.

In [7], the authors considered the following nonlinear fractional boundary value problem:

$$\begin{aligned}
&D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\
&u(0) = u'(0) = u''(0) = u''(1) = 0,
\end{aligned} \tag{1.2}$$

where $3 < \alpha \leq 4$ and $f : [0,1] \times [0,\infty) \rightarrow [0,\infty)$ is continuous. They obtained their results by using lower and upper solution method and fixed point theorems.

In [8] the authors investigated the existence and uniqueness of positive and nondecreasing solutions for a class of singular fractional boundary value problems by using a fixed point theorem in partially ordered metric spaces.

Very recently, in [9] the authors studied the existence of solutions of the following three-point boundary value problem:

$$\begin{aligned}
&D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\
&u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \beta u''(\eta),
\end{aligned} \tag{1.3}$$

where $3 < \alpha \leq 4$, $0 < \eta < 1$, $0 < \beta\eta^{\alpha-3} < 1$ and $f : [0,1] \times [0,\infty) \rightarrow [0,\infty)$ is continuous.

Motivated by [8, 9], in this paper we discuss the existence and uniqueness of positive solutions for Problem (1.3) assuming that $f : (0,1] \times [0,\infty) \rightarrow [0,\infty)$ is such that $\lim_{t\to 0^+} f(t,\cdot) = \infty$ (i.e., f is singular at $t = 0$). Our main tool is a fixed point theorem in partially ordered metric spaces which appears in [10].

2. Preliminaries and Basic Facts

For the convenience of the reader, we present some notations and lemmas which will be used in the proof of our results.

Definition 2.1 (see [5]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \rightarrow \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$ and where $\Gamma(\alpha)$ denotes the classical gamma function.

Definition 2.2 (see [2]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \rightarrow \mathbb{R}$ is defined as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

The following two lemmas can be found in [2] and they are crucial in finding an integral representation of the boundary value problem (1.3).

Lemma 2.3 (see [2]). *Assume that $u \in C(0,1) \cap L^1(0,1)$ and $\alpha > 0$.*

Then the fractional differential equation

$$D_{0^+}^{\alpha} u(t) = 0 \tag{2.3}$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ and $n = [\alpha] + 1$, as unique solution.

Lemma 2.4 (see [2]). *Assume that $u \in C(0,1) \cap L^1(0,1)$ with fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ and $n = [\alpha] + 1$.

By using Lemma 2.4, in [9] the authors proved the following result.

Lemma 2.5 (see [9]). *Let $0 < \eta < 1$ and $\beta \neq 1/\eta^{\alpha-3}$ and $h \in C[0,1]$.*

Then the boundary value problem

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \beta u''(\eta), \end{gathered} \tag{2.6}$$

where $3 < \alpha \leq 4$, has as unique solution

$$u(t) = \int_0^1 G(t,s) h(s) ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})} \int_0^1 H(\eta, s) h(s) ds, \tag{2.7}$$

where

$$\begin{gathered} G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \\ H(t,s) = \frac{\partial^2 G(t,s)}{\partial t^2} = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \left[t^{\alpha-3}(1-s)^{\alpha-3} - (t-s)^{\alpha-3} \right], & 0 \leq s \leq t \leq 1, \\ \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} t^{\alpha-3}(1-s)^{\alpha-3}, & 0 \leq t \leq s \leq 1. \end{cases} \end{gathered} \tag{2.8}$$

Remark 2.6. In [9] it is proved that G is a continuous function on $[0,1] \times [0,1]$, $G(t,s) \geq 0$, $G(t,1) = 0$ and

$$\sup_{0\leq t\leq 1} \int_0^1 G(t,s)ds = \frac{2}{(\alpha-2)\Gamma(\alpha+1)}, \qquad \int_0^1 H(\eta,s)ds = \frac{\eta^{\alpha-3}(\alpha-1)(1-\eta)}{\Gamma(\alpha)}. \tag{2.9}$$

In the sequel, we present the fixed point theorem which we will use later. Previously, we present the following class of functions.

By $\mathcal{S}$ we denote the class of functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \longrightarrow 1 \text{ implies } t_n \longrightarrow 0. \tag{2.10}$$

Examples of functions belonging to $\mathcal{S}$ are $\beta(t) = kt$ with $0 \leq k < 1$ and $\beta(t) = 1/(1+t)$.

The fixed point theorem which we will use later appears in [10].

Theorem 2.7 (see [10]). *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq Tx_0$. Suppose that there exists $\beta \in \mathcal{S}$ such that*

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y) \quad \text{for } x,y \in X \text{ with } x \geq y. \tag{2.11}$$

Assume that either T is continuous or X is such that

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \longrightarrow x \text{ then } x_n \leq x \ \forall n \in \mathbb{N}. \tag{2.12}$$

Besides, if

$$\text{for all } x,y \in X \text{ there exists } z \in X \text{ which is comparable to } x,\ y, \tag{2.13}$$

then T has a unique fixed point.

In our considerations, we will work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbb{R}, x \text{ is continuous}\}$ with the classical metric given by $d(x,y) = \sup_{0\leq t\leq 1}|x(t) - y(t)|$.

Notice that this space can be equipped with a partial order given by

$$x,y \in C[0,1], \quad x \leq y \Longleftrightarrow x(t) \leq y(t) \quad \text{for any } t \in [0,1]. \tag{2.14}$$

In [11] it is proved that $(C[0,1],\leq)$ satisfies condition (2.12) of Theorem 2.7. Moreover, for $x,y \in C[0,1]$, as the function $\max\{x,y\} \in C[0,1]$, $(C[0,1],\leq)$ satisfies condition (2.13).

3. Main Result

Our starting point of this section is the following lemma.

Lemma 3.1. *Suppose that $0 < \sigma < 1$, $3 < \alpha \leq 4$, and $F : (0,1] \to \mathbb{R}$ is a continuous function with $\lim_{t\to 0^+} F(t) = \infty$. If $t^\sigma F(t)$ is a continuous function on $[0,1]$ then the function defined by*

$$L(t) = \int_0^1 G(t,s)F(s)ds \tag{3.1}$$

is continuous on $[0,1]$, where $G(t,s)$ is the Green's function appearing in Lemma 2.5.

Proof. We divide the proof into three cases.

Case 1 ($t_0 = 0$). It is clear that $L(0) = 0$.

Since $t^\sigma F(t)$ is a continuous function on $[0,1]$, we can find a constant $M > 0$ such that $|t^\sigma F(t)| \leq M$ for any $t \in [0,1]$.

Then, we get

$$\begin{aligned}
|L(t) - L(0)| = |L(t)| &= \left|\int_0^1 G(t,s)F(s)ds\right| = \left|\int_0^1 G(t,s)s^{-\sigma}s^\sigma F(s)ds\right| \\
&= \left|\int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}s^\sigma F(s)ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma}s^\sigma F(s)ds\right| \\
&= \left|\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma}s^\sigma F(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}s^\sigma F(s)ds\right| \\
&\leq \left|\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma}s^\sigma F(s)ds\right| + \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}s^\sigma F(s)ds\right| \\
&\leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3}s^{-\alpha}ds + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}s^{-\alpha}ds \\
&= \frac{Mt^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3}s^{-\alpha}ds + \frac{Mt^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \left(1-\frac{s}{t}\right)^{\alpha-1} s^{-\alpha}ds \\
&= \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha-2) + \frac{Mt^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \left(1-\frac{s}{t}\right)^{\alpha-1} s^{-\alpha}ds.
\end{aligned} \tag{3.2}$$

If in the integral $\int_0^t (1-(s/t))^{\alpha-1}s^{-\sigma}ds$ we use the change of variables $u = s/t$ then we have

$$\int_0^t \left(1-\frac{s}{t}\right)^{\alpha-1} s^{-\sigma}ds = t^{1-\sigma}\int_0^1 (1-u)^{\alpha-1}u^{-\sigma}du = t^{1-\sigma}\beta(1-\sigma,\alpha). \tag{3.3}$$

This and (3.2) give us

$$|L(t)| \leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha-2) + \frac{Mt^{\alpha-\sigma}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha) \tag{3.4}$$

and letting $t \to 0$, we see that $|L(t)| \to 0$.

This proves the continuity of L at $t_0 = 0$.

Case 2 ($t_0 \in (0,1)$). We take $t_n \to t_0$ and we have to prove that $L(t_n) \to L(t_0)$.

Without loss of generality, we can take $t_n > t_0$ (the same argument works for $t_n < t_0$).

In fact,

$$\begin{aligned}
|L(t_n) - L(t_0)| = &\left| \int_0^{t_n} \frac{t_n^{\alpha-1}(1-s)^{\alpha-3} - (t_n - s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds \right.\\
&+ \int_{t_n}^1 \frac{t_n^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds\\
&- \int_0^{t_0} \frac{t_0^{\alpha-1}(1-s)^{\alpha-3} - (t_0 - s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds\\
&\left. - \int_{t_0}^1 \frac{t_0^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds \right|\\
= &\left| \int_0^1 \frac{t_n^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds - \int_0^{t_n} \frac{(t_n - s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds \right.\\
&\left. - \int_0^1 \frac{t_0^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds + \int_0^{t_0} \frac{(t_0 - s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds \right|\\
= &\left| \int_0^1 \frac{\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds \right.\\
&- \int_0^{t_0} \frac{(t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds\\
&\left. - \int_{t_0}^{t_n} \frac{(t_n - s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, ds \right|\\
\leq &\frac{M\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3} s^{-\sigma}\, ds\\
&+ \frac{M}{\Gamma(\alpha)} \int_0^{t_0} \left((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1}\right) s^{-\sigma}\, ds\\
&+ \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t_n} (t_n - s)^{\alpha-1} s^{-\sigma}\, ds\\
\leq &\frac{M\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)}{\Gamma(\alpha)} \beta(1-\sigma, \alpha-2) + \frac{M}{\Gamma(\alpha)} I_n^1 + \frac{M}{\Gamma(\alpha)} I_n^2,
\end{aligned} \tag{3.5}$$

where

$$I_n^1 = \int_0^{t_0} \left((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right) s^{-\sigma} ds,$$
$$I_n^2 = \int_{t_0}^{t_n} (t_n - s)^{\alpha-1} s^{-\sigma} ds. \tag{3.6}$$

In the sequel, we will prove that $I_n^1 \to 0$ when $n \to \infty$.

In fact, as $t_n \to t_0$, then

$$\left((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right) s^{-\sigma} \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.7}$$

Moreover,

$$\left| \left((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right) s^{-\sigma} \right| \leq \left(|t_n - s|^{\alpha-1} + |t_0 - s|^{\alpha-1} \right) s^{-\sigma} \leq 2 s^{-\sigma} \tag{3.8}$$

and, as

$$\int_0^1 2 s^{-\sigma} ds = 2 \left[\frac{s^{-\sigma+1}}{-\sigma+1} \right]_0^1 = \frac{2}{1-\sigma} < \infty, \tag{3.9}$$

we have that the sequence $((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1}) s^{-\sigma}$ converges pointwise to the zero function and $|((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1}) s^{-\sigma}|$ is bounded by a function belonging to $L^1[0,1]$, then by Lebesgue's dominated convergence theorem

$$I_n^1 \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.10}$$

Now, we will prove that $I_n^2 \to 0$ when $n \to \infty$.

In fact, as

$$I_n^2 = \int_{t_0}^{t_n} (t_n - s)^{\alpha-1} s^{-\sigma} ds \leq \int_{t_0}^{t_n} s^{-\sigma} ds = \frac{1}{1-\sigma} \left(t_n^{1-\sigma} - t_0^{1-\sigma} \right) \tag{3.11}$$

and letting $n \to \infty$ and, taking into account that $t_n \to t_0$, from the last expression we get

$$I_n^2 \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.12}$$

Finally, from (3.5), (3.10), and (3.12) we get

$$|L(t_n) - L(t_0)| \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.13}$$

This proves the continuity of L at t_0.

Case 3 ($t_0 = 1$). It is easily checked that $L(1) = 0$.

Now, following the same lines that in the proof of Case 1, we can demonstrate the continuity of L at $t_0 = 1$.

This finishes the proof. □

Lemma 3.2. *Suppose that* $0 < \sigma < 1$, $3 < \alpha \leq 4$, $0 < \beta\eta^{\alpha-3} < 1$, *and* $F : (0,1] \to \mathbb{R}$ *is a continuous function with* $\lim_{t\to 0^+} F(t) = \infty$.

If $t^{\sigma}F(t)$ *is a continuous function on* $[0,1]$ *then the function defined by*

$$N(t) = \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})} \int_0^1 H(\eta,s)F(s)ds \tag{3.14}$$

is continuous on $[0,1]$, *where* $H(t,s)$ *is the function appearing in Lemma 2.5.*

Proof. Since $t^{\sigma}F(t)$ is continuous on $[0,1]$, there exists a constant $M > 0$ such that $|t^{\sigma}F(t)| \leq M$ for any $t \in [0,1]$.

Taking into account that

$$|H(t,s)| \leq \frac{2(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \tag{3.15}$$

we have

$$\begin{aligned} \left|\int_0^1 H(\eta,s)F(s)ds\right| &= \left|\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma}F(s)ds\right| \\ &\leq \frac{2M(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_0^1 s^{-\sigma}\, ds = \frac{2M(\alpha-1)(\alpha-2)}{\Gamma(\alpha)(1-\alpha)} < \infty, \end{aligned} \tag{3.16}$$

and, consequently, the function N is continuous at any point $t \in [0,1]$. □

Remark 3.3. Notice that the function $H(t,s)$ appearing in Lemma 2.5 which is defined as

$$H(t,s) = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[t^{\alpha-3}(1-s)^{\alpha-3} - (t-s)^{\alpha-3}\right], & 0 \leq s \leq t \leq 1, \\ \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} t^{\alpha-3}(1-s)^{\alpha-3}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.17}$$

is continuous function on $[0,1] \times [0,1]$ and, moreover, $H(t,s) \geq 0$.

In fact, for $0 \leq t \leq s \leq 1$ it is clear that $H(t,s) \geq 0$.

In the case, $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned} H(t,s) &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[t^{\alpha-3}(1-s)^{\alpha-3} - (t-s)^{\alpha-3}\right] \\ &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[(t-ts)^{\alpha-3} - (t-s)^{\alpha-3}\right] \geq 0. \end{aligned} \tag{3.18}$$

This proves the nonnegative character of the function H on $[0,1] \times [0,1]$.

Lemma 3.4. *Suppose that* $0 < \sigma < 1$. *Then*

$$\sup_{0\le t\le 1}\int_0^1 G(t,s)s^{-\sigma}ds = \frac{1}{\Gamma(\alpha)}\left(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha)\right), \tag{3.19}$$

where $G(t,s)$ *is the function appearing in Lemma 2.5.*

Proof. By definition of $G(t,s)$, we have

$$\begin{aligned}\int_0^1 G(t,s)s^{-\sigma}ds &= \int_0^t G(t,s)s^{-\sigma}ds + \int_t^1 G(t,s)s^{-\sigma}ds \\ &= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}s^{-\sigma}ds \\ &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}s^{-\sigma}ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\alpha}ds \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-3}s^{-\sigma}ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\sigma}ds.\end{aligned} \tag{3.20}$$

As we saw in Case 1 of Lemma 3.1.

$$\int_0^t (t-s)^{\alpha-1}s^{-\sigma}ds = \frac{t^{\alpha-\sigma}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha) \tag{3.21}$$

and, therefore,

$$\int_0^t G(t,s)s^{-\sigma}ds = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha-2) - \frac{t^{\alpha-\sigma}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha). \tag{3.22}$$

Now, using elemental calculus it is easily seen that the function

$$\varphi(t) = \frac{\beta(1-\sigma,\alpha-2)}{\Gamma(\alpha)}t^{\alpha-1} - \frac{\beta(1-\sigma,\alpha)}{\Gamma(\alpha)}t^{\alpha-\sigma} \tag{3.23}$$

is increasing on the interval $[0,1]$ and, therefore,

$$\sup_{0\le t\le 1}\int_0^1 G(t,s)s^{-\sigma}ds = \sup_{0\le t\le 1}\varphi(t) = \varphi(1) = \frac{1}{\Gamma(\alpha)}\left(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha)\right). \tag{3.24}$$

□

Lemma 3.5. *Suppose that* $0 < \sigma < 1$ *then*

$$\int_0^1 H(\eta,s)s^{-\sigma}ds = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2}\right)\beta(1-\sigma,\alpha-2), \tag{3.25}$$

where $H(t,s)$ is the function appearing in Lemma 2.5.

Proof. By definition of $H(t,s)$, we have

$$\begin{aligned}
\int_0^1 H(\eta,s)s^{-\sigma}ds &= \int_0^\eta H(\eta,s)s^{-\sigma}ds + \int_\eta^1 H(\eta,s)s^{-\sigma}ds \\
&= \int_0^\eta \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[\eta^{\alpha-3}(1-s)^{\alpha-3} - (\eta-s)^{\alpha-3}\right]s^{-\sigma}ds \\
&\quad + \int_\eta^1 \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}(1-s)^{\alpha-3}s^{-\sigma}ds \\
&= \int_0^1 \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}(1-s)^{\alpha-3}s^{-\sigma}ds - \int_0^\eta \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}(\eta-s)^{\alpha-3}s^{-\sigma}ds \\
&= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}\int_0^1 (1-s)^{\alpha-3}s^{-\sigma}ds - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-3}s^{-\sigma}ds \\
&= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}\beta(1-\sigma,\alpha-2) - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-3}s^{-\sigma}ds.
\end{aligned} \tag{3.26}$$

By a similar argument that the one used in the Case 1 of Lemma 3.1, we have

$$\begin{aligned}
\int_0^1 H(\eta,s)s^{-\sigma}ds &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}\beta(1-\sigma,\alpha-2) - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-\sigma-2}\beta(1-\sigma,\alpha-2) \\
&= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\eta^{\alpha-3} - \eta^{\alpha-\sigma-2}\right)\beta(1-\sigma,\alpha-2).
\end{aligned} \tag{3.27}$$

This finishes the proof. □

By commodity, we denote by K the constant given by

$$K = \frac{1}{\Gamma(\alpha)}\left[\left(1 + \frac{\beta(\eta^{\alpha-3} - \eta^{\alpha-\sigma-2})}{1-\beta\eta^{\alpha-3}}\right)\beta(1-\sigma,\alpha-2) - \beta(1-\sigma,\alpha)\right]. \tag{3.28}$$

Moreover, we introduce the following class of functions which will be used in the main result of the paper. By $\mathcal{A}$ we denote the class of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following:

(i) ϕ is nondecreasing.

(ii) $\phi(x) < x$ for any $x > 0$.

(iii) $\beta(x) = \phi(x)/x \in \mathcal{S}$, where $\mathcal{S}$ is the class of functions introduced in Remark 2.6.

Theorem 3.6. *Let $0 < \sigma < 1$, $3 < \alpha \leq 4$, $0 < \eta < 1$, $0 < \beta\eta^{\alpha-3} < 1$, and $f : (0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function with $\lim_{t\to 0^+} f(t,\cdot) = \infty$ and such that $t^{\sigma} f(t,y)$ is a continuous function on $[0,1] \times [0,\infty)$. Assume that there exists $0 < \lambda \leq 1/K$ such that for $x, y \in [0,\infty)$ with $y \geq x$ and $t \in [0,1]$*

$$0 \leq t^{\sigma}\left(f(t,y) - f(t,x)\right) \leq \lambda\phi(y-x), \tag{3.29}$$

where $\phi \in \mathcal{A}$.

Then Problem (1.3) *has a unique positive solution (this means that $x(t) > 0$ for $t \in (0,1)$).*

Proof. Consider the cone:

$$P = \{u \in C[0,1] : u(t) \geq 0\}. \tag{3.30}$$

Since P is a closed set of $C[0,1]$, P is a complete metric space with the distance given by $d(u,v) = \sup_{0\leq t\leq 1}|u(t) - v(t)|$, for $u, v \in P$.

It is easily checked that P satisfies conditions (2.12) and (2.13) of Theorem 2.7.

Now, for $u \in P$ we define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s) f(s,u(s))ds. \tag{3.31}$$

By Lemmas 3.1 and 3.2, for $u \in P$ we have $Tu \in C[0,1]$.

Moreover, in view of the nonnegative character of $G(t,s)$, $H(\eta,s)$, and $f(s,x)$, we have that $Tu \in P$ for $u \in P$.

In what follows, we check that assumptions in Theorem 2.7 are satisfied.

Firstly, we will prove that T is nondecreasing.

In fact, by (3.29), for $u \geq v$ we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)f(s,u(s))ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)f(s,u(s))ds \\ &= \int_0^1 G(t,s)s^{-\sigma}s^{\sigma}f(s,u(s))ds \\ &\quad + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma}f(s,u(s))ds \\ &\geq \int_0^1 G(t,s)s^{-\sigma}s^{\sigma}f(s,v(s))ds \\ &\quad + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma}f(s,v(s))ds \\ &= \int_0^1 G(t,s)f(s,v(s))ds \\ &\quad + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)f(s,v(s))ds = (Tv)(t).\end{aligned} \tag{3.32}$$

This proves that T is a nondecreasing operator.

On the other hand, for $u \geq v$ and $u \neq v$, we have

$$\begin{aligned}d(Tu,Tv) &= \sup_{0\leq t\leq 1}|(Tu)(t)-(Tv)(t)| = \sup_{0\leq t\leq 1}((Tu)(t)-(Tv)(t)) \\ &= \sup_{0\leq t\leq 1}\left[\int_0^1 G(t,s)(f(s,u(s))-f(s,v(s)))ds\right. \\ &\qquad \left. + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)(f(s,u(s))-f(s,v(s)))ds\right] \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-\sigma}s^{\sigma}(f(s,u(s))-f(s,v(s)))ds \\ &\quad + \frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma}(f(s,u(s))-f(s,v(s)))ds \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-\sigma}\lambda(\phi(u(s)-v(s)))ds \\ &\quad + \frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}\lambda(\phi(u(s)-v(s)))ds.\end{aligned} \tag{3.33}$$

Since ϕ is nondecreasing, the last inequality implies

$$\begin{aligned} d(Tu,Tv) &\le \lambda\phi(d(u,v))\sup_{0\le t\le 0}\int_0^1 G(t,s)s^{-\sigma}ds \\ &\quad + \frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\lambda\phi(d(u,v))\int_0^1 H(\eta,s)s^{-\sigma}ds \\ &= \lambda\phi(d(u,v))\left[\sup_{0\le t\le 0}\int_0^1 G(t,s)s^{-\sigma}\,ds + \frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}ds\right]. \end{aligned} \tag{3.34}$$

Now, from Lemmas 3.4 and 3.5 it follows:

$$\begin{aligned} d(Tu,Tv) &\le \lambda\phi(d(u,v))\left[\frac{1}{\Gamma(\alpha)}(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha)) + \frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\right. \\ &\qquad \left.\times\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2}\right)\beta(1-\sigma,\alpha-2)\right] \\ &= \lambda\phi(d(u,v))\left[\frac{1}{\Gamma(\alpha)}(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha))\right. \\ &\qquad \left. + \frac{\beta(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2})}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\beta(1-\sigma,\alpha-2)\right] \\ &= \lambda\phi(d(u,v))\left[\frac{1}{\Gamma(\alpha)}\left[\left(1+\frac{\beta(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2})}{1-\beta\eta^{\alpha-3}}\right)\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha)\right]\right] \\ &= \lambda\phi(d(u,v))K. \end{aligned} \tag{3.35}$$

Since $0<\lambda\le 1/K$, from the last inequality we obtain

$$d(Tu,Tv) \le \lambda\phi(d(u,v))K \le \phi(d(u,v)), \tag{3.36}$$

and, since $u\ne v$,

$$d(Tu,Tv) \le \frac{\phi(d(u,v))}{d(u,v)}d(u,v) = \beta(d(u,v))d(u,v). \tag{3.37}$$

Since this inequality is obviously satisfied for $u=v$, we have

$$d(Tu,Tv) \le \beta(d(u,v))d(u,v) \quad \text{for any } u,v\in P \text{ with } u\ge v. \tag{3.38}$$

Finally, since the zero function satisfies $0 \leq T0$, Theorem 2.7 says us that the operator T has a unique fixed point in P, or, equivalently, Problem (1.3) has a unique nonnegative solution x in $C[0,1]$.

Now, we will prove that x is a positive solution.

In contrary case, we can find $0 < t^* < 1$ such that $x(t^*) = 0$.

Taking into account that the nonnegative solution x of Problem (1.3) is a fixed point of the operator, we have

$$x(t) = \int_0^1 G(t,s)f(s,x(s))ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)f(s,x(s))ds \tag{3.39}$$

and, particularly,

$$x(t^*) = \int_0^1 G(t^*,s)f(s,x(s))ds + \frac{\beta t^{*\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)f(s,x(s))ds = 0. \tag{3.40}$$

Since both summands in the right hand are nonnegative (see Remarks 2.6 and 3.3) we have

$$\begin{gathered}\int_0^1 G(t^*,s)f(s,x(s))ds = 0,\\ \int_0^1 H(\eta,s)f(s,x(s))ds = 0.\end{gathered} \tag{3.41}$$

Given the nonnegative character of $G(t,s)$, $H(\eta,s)$, and $f(s,u)$, we have

$$\begin{gathered}G(t^*,s)f(s,x(s)) = 0 \quad \text{a.e. } (s),\\ H(\eta,s)f(s,x(s)) = 0 \quad \text{a.e. } (s).\end{gathered} \tag{3.42}$$

Taking into account that $\lim_{t\to 0^+} f(t,0) = \infty$, this means that for $M > 0$ we can find $\delta > 0$ such that for $s \in [0,1] \cap (0,\delta)$ we have $f(s,0) > M$. Notice that $[0,1] \cap (0,\delta) \subset \{s \in [0,1] : f(s,x(s)) > M\}$ and $\mu([0,1]\cap(0,\delta)) > 0$, where μ is the Lebesgue measure on $[0,1]$.

This and (3.42) give us that

$$\begin{gathered}G(t^*,s) = 0 \quad \text{a.e. } (s),\\ H(\eta,s) = 0 \quad \text{a.e. } (s),\end{gathered} \tag{3.43}$$

and this is a contradiction since $G(t^*,s)$ and $H(\eta,s)$ are rational functions in the variable s.

Therefore, $x(t) > 0$ for $t \in (0,1)$.

This finishes the proof. □

In order to present an example which illustrates our results, we need to prove some properties about the hyperbolic tangent function.

Previously, we recalled some definitions.

Definition 3.7. A function $f : [0, \infty) \to [0, \infty)$ is said to be subadditive if it satisfies

$$f(x + y) \le f(x) + f(y) \quad \text{for any } x, y \in [0, \infty). \tag{3.44}$$

An example of subadditive function is the square root function, that is, $f(x) = \sqrt{x}$.

Remark 3.8. Suppose that $f : [0, \infty) \to [0, \infty)$ is subadditive and $y \le x$ then

$$f(x) - f(y) \le f(x - y). \tag{3.45}$$

In fact, since

$$f(x) = f(x - y + y) \le f(x - y) + f(y) \tag{3.46}$$

this inequality implies that

$$f(x) - f(y) \le f(x - y). \tag{3.47}$$

Recall that a function $f : [0, \infty) \to [0, \infty)$ is concave if for any $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$.

$$f(\lambda x + (1 - \lambda) y) \ge \lambda f(x) + (1 - \lambda) f(y). \tag{3.48}$$

Lemma 3.9. *Let $f : [0, \infty) \to [0, \infty)$ be a concave function with $f(0) = 0$. Then f is subadditive.*

Proof. We take $x, y \in [0, \infty)$.

Since f is concave and $f(0) = 0$, we get

$$\begin{aligned} f(x) &= f\left(\frac{y}{x+y} 0 + \frac{x}{x+y}(x+y)\right) \ge \frac{y}{x+y} f(0) + \frac{x}{x+y} f(x+y) = \frac{x}{x+y} f(x+y), \\ f(y) &= f\left(\frac{x}{x+y} 0 + \frac{y}{x+y}(x+y)\right) \ge \frac{x}{x+y} f(0) + \frac{y}{x+y} f(x+y) = \frac{y}{x+y} f(x+y). \end{aligned} \tag{3.49}$$

Adding these inequalities, we have

$$f(x) + f(y) \ge \frac{x}{x+y} f(x+y) + \frac{y}{x+y} f(x+y) = f(x+y). \tag{3.50}$$

This proves the lemma. $\square$

In what follows, we will prove that the function

$$f(x) = \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \tag{3.51}$$

belongs to the class $\mathcal{A}$ previously defined.

Lemma 3.10. *The function* $f : [0,\infty) \to [0,\infty)$ *defined as*

$$f(x) = \tanh x = \frac{e^{2x}-1}{e^{2x}+1} \tag{3.52}$$

satisfies:

(a) $f \in \mathcal{A}$.

(b) f *is subadditive.*

Proof. (a) Since $f'(x) = 4e^{2x}/(e^{2x}+1)^2 > 0$ for $x > 0$, f is nondecreasing.

Moreover, the function

$$g(x) = x - \tanh x = x - \frac{e^{2x}-1}{e^{2x}+1} \tag{3.53}$$

has as derivative

$$g'(x) = \frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2} > 0 \quad \text{for } x > 0 \tag{3.54}$$

and, consequently, g is strictly nondecreasing on $(0,\infty)$.

Since $g(0) = 0$, we have $0 = g(0) < g(x)$ for $x > 0$ or, equivalently, $f(x) = \tanh x < x$ for $x > 0$.

In order to prove that $\beta(x) = \tanh x/x \in \mathcal{S}$, notice that if $\beta(t_n) \to 1$ then the sequence (t_n) is a bounded sequence.

In fact, in contrary case $t_n \to \infty$ and we have

$$\beta(t_n) = \frac{\tanh t_n}{t_n} \longrightarrow 0 \tag{3.55}$$

which contradicts the fact that $\beta(t_n) \to 1$.

Now, we suppose that $\beta(t_n) \to 1$ and $t_n \nrightarrow 0$.

Then, we can find $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $\varrho_n \geq n$ with $t_{\varrho_n} \geq \varepsilon$.

Since (t_n) is a bounded sequence (because $\beta(t_n) \to 1$) we can find a subsequence of (t_{ϱ_n}), which we will denote of the same way, such that $t_{\varrho_n} \to a$.

As $\beta(t_n) \to 1$, it follows that

$$\beta\left(t_{\varrho_n}\right) = \frac{\tanh t_{\varrho_n}}{t_{\varrho_n}} \longrightarrow \frac{\tanh a}{a} = 1 \tag{3.56}$$

and, as the unique solution of the equation $\tanh x = x$ on $[0,\infty)$ is $x_0 = 0$, we deduce that $a = 0$.

Therefore, $t_{\varrho_n} \to 0$ and this implies that there exists $n_0 \in \mathbb{N}$ such that $t_{\varrho_n} < \varepsilon$ for $n \geq n_0$.

This contradicts the fact that $t_{\varrho_n} \geq \varepsilon$ for any $n \in \mathbb{N}$.

Therefore, $t_n \to 0$.

This proves that $\beta(x) = \tanh x/x \in \mathcal{S}$.

Therefore, $f \in \mathcal{A}$.

(b) Since $\tanh 0 = 0$ and

$$(\tanh x)'' = \frac{8e^{2x}(1-e^{2x})}{(e^{2x}+1)^3} < 0 \quad \text{for } x > 0, \tag{3.57}$$

this means that $f(x) = \tanh x$ is a concave function with $\tanh 0 = 0$ and, by Lemma 3.9, $f(x) = \tanh x$ is subadditive. □

Remark 3.11. By Remark 3.8 and by (b) of Lemma 3.9, for $x, y \in [0,\infty)$ with $y \le x$

$$\tanh x - \tanh y \le \tanh(x-y). \tag{3.58}$$

Now, we present an example which illustrates our result.

Example 3.12. Consider the following singular fractional boundary value problem

$$\begin{aligned} &D_{0^+}^{7/2}u(t) + \frac{\lambda(t^2+1)\tanh u(t)}{t^{1/2}} = 0, \quad 0<t<1, \\ &u(0) = u'(0) = u''(0) = 0, \quad u''(1) = u''\left(\frac{1}{4}\right). \end{aligned} \tag{3.59}$$

In this case, $\sigma = 1/2$, $\eta = 1/4$, $\beta = 1$ and $\alpha = 7/2$.

Moreover, in this case $f(t,u) = \lambda(t^2+1)\tanh u/t^{1/2}$ for $(t,u) \in (0,1] \times [0,\infty)$.

Notice that f is continuous in $(0,1] \times [0,\infty)$ and $\lim_{t\to 0^+} f(t,\cdot) = \infty$.

Now, we check that $f(t,u)$ satisfies assumptions appearing in Theorem 3.6.

It is clear that $t^{1/2}f(t,u) = \lambda(t^2+1)\tanh u$ is a continuous function on $[0,1] \times [0,\infty)$.

Moreover, by Lemma 3.10 and Remark 3.11, for $u \ge v$ and $t \in [0,1]$ we have

$$\begin{aligned} 0 &\le t^{1/2}(f(t,u) - f(t,v)) \\ &= \lambda\left(t^2+1\right)(\tanh u - \tanh v) \\ &\le \lambda\left(t^2+1\right)\tanh(u-v) \le 2\lambda\tanh(u-v), \end{aligned} \tag{3.60}$$

where $f(x) = \tanh x$ is a function belonging to $\mathcal{A}$ (see, Lemma 3.10).

Finally, Theorem 3.6 says us that Problem (3.59) has a unique positive solution for

$$2\lambda \le \frac{1}{K} = \frac{\Gamma(7/2)}{\left(1+\left((1/4)^{1/2}-1/4\right)/\left(1-(1/4)^{1/2}\right)\right)\beta(1/2,3/2) \ -\beta(1/2,7/2)} = \frac{30}{7\sqrt{\pi}}. \tag{3.61}$$

Or, equivalently, for $\lambda \le 15/7\sqrt{\pi}$.

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Research Article

On Antiperiodic Boundary Value Problems for Higher-Order Fractional Differential Equations

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We study an antiperiodic boundary value problem of nonlinear fractional differential equations of order $q \in (4, 5]$. Some existence results are obtained by applying some standard tools of fixed-point theory. We show that solutions for lower-order anti-periodic fractional boundary value problems follow from the solution of the problem at hand. Our results are new and generalize the existing results on anti-periodic fractional boundary value problems. The paper concludes with some illustrating examples.

1. Introduction

In the preceding years, there has been a great advancement in the study of fractional calculus. A variety of results on initial and boundary value problems of fractional order, ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions, have appeared in the literature. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and fitting of experimental data [1–5]. For an updated account of mathematical tools for fractional models and methods of solutions for fractional differential equations, we refer the reader to a recent text [6] by Baleanu et al. Fractional derivatives are also regarded as an excellent tool for the description of memory and hereditary properties of various materials and processes [7]. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. For more details and examples, see [8–20].

Antiperiodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have received a considerable attention. Examples include antiperiodic trigonometric polynomials in the study of interpolation problems, antiperiodic wavelets, antiperiodic boundary conditions in physics, and so forth (for details, see [21] and the references therein). Some recent work on antiperiodic boundary value problems of fractional-order can be found in [21–27] and references therein.

In this paper, we consider an antiperiodic boundary value problems of fractional differential equations of order $q \in (4,5]$ given by

$$\begin{gathered} {}^cD^q x(t) = f(t,x(t)), \quad t \in [0,T],\ T>0,\ 4<q\le 5, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T), \\ x'''(0) = -x'''(T), \qquad x^{(iv)}(0) = -x^{(iv)}(T), \end{gathered} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q and f is a given continuous function.

The main objective of the present work is to develop the existence theory for problem (1.1) and relate problem (1.1) with lower-order fractional antiperiodic boundary value problems. Our results are new and give further insight into the characteristics of fractional-order antiperiodic boundary value problems.

2. Preliminaries

Definition 2.1 (see [4]). The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q>0, \tag{2.1}$$

provided the integral exists.

Definition 2.2 (see [4]). For at least n-times continuously differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1<q<n,\ n=[q]+1, \tag{2.2}$$

where $[q]$ denotes the integer part of the real number q.

To study the nonlinear problem (1.1), we need the following lemma, which deals with a linear variant of problem (1.1).

Lemma 2.3. *For any $y \in C[0,T]$, the unique solution of the boundary value problem:*

$$\begin{aligned}
&{}^{c}D^{q}x(t) = y(t), \quad t \in [0,T],\ 4 < q \leq 5,\\
&x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T),\\
&x'''(0) = -x'''(T), \qquad x^{(iv)}(0) = -x^{(iv)}(T)
\end{aligned} \tag{2.3}$$

is

$$x(t) = \int_0^T G(t,s)y(s)ds, \tag{2.4}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \dfrac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ \quad + \dfrac{(6t^2T - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \dfrac{(2Tt^3 - t^4 - tT^3)(T-s)^{q-5}}{48\Gamma(q-4)}, & 0 < s < t < T, \\ -\dfrac{(T-s)^{q-1}}{2\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ \quad + \dfrac{(6t^2T - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \dfrac{(2Tt^3 - t^4 - tT^3)(T-s)^{q-5}}{48\Gamma(q-4)}, & 0 < t < s < T. \end{cases} \tag{2.5}$$

Proof. It is well known [4] that the solution of ${}^{c}D^{q}x(t) = y(t)$ can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds - b_o - b_1 t - b_2 t^2 - b_3 t^3 - b_4 t^4, \tag{2.6}$$

where $b_o,\ b_1,\ b_2,\ b_3,$ and $b_4 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions for problem (2.3) in (2.6), we find that

$$\begin{aligned}
b_o = {}& \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)ds - \frac{T}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)ds \\
&+ \frac{T^3}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)ds,
\end{aligned}$$

$$
\begin{aligned}
b_1 &= \frac{1}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)ds - \frac{T}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)ds + \frac{T^3}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)ds, \\
b_2 &= \frac{1}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)ds - \frac{T}{8}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)ds, \\
b_3 &= \frac{1}{12}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)ds - \frac{T}{24}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)ds, \\
b_4 &= \frac{1}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)ds.
\end{aligned}
\tag{2.7}
$$

Substituting the values of $b_o,\ b_1,\ b_2,\ b_3$, and b_4 in (2.6), we obtain

$$
\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)ds \\
&\quad + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)ds \\
&\quad + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)ds + \frac{(2Tt^3-t^4-tT^3)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)ds \\
&= \int_0^T G(t,s)y(s)ds,
\end{aligned}
\tag{2.8}
$$

where $G(t,s)$ is given by (2.5). This completes the proof. □

2.1. Relationship with Lower-Order Problems

We observe that the first term in expressions for $G(t,s)$ given by (2.5) corresponds to the Green's function for the problem:

$$
\begin{gathered}
{}^cD^q x(t) = f(t,x(t)), \quad t \in [0,T],\ T > 0,\ 0 < q \le 1, \\
x(0) = -x(T);
\end{gathered}
\tag{2.9}
$$

the first two terms in (2.5) form Green's function for the problem [21]:

$$
\begin{gathered}
{}^cD^q x(t) = f(t,x(t)), \quad t \in [0,T],\ T > 0,\ 1 < q \le 2, \\
x(0) = -x(T), \qquad x'(0) = -x'(T);
\end{gathered}
\tag{2.10}
$$

the first three terms in (2.5) give the Green's function for the problem [22]:

$$\begin{gathered} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0, T],\ T > 0,\ 2 < q \leq 3, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T); \end{gathered} \tag{2.11}$$

while the first four terms in (2.5) yield the Green's function for the antiperiodic problem [23]:

$$\begin{gathered} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0, T],\ T > 0,\ 3 < q \leq 4, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T), \qquad x'''(0) = -x'''(T). \end{gathered} \tag{2.12}$$

From the above deductions, it can easily be concluded that Green's function (2.5) for an antiperiodic boundary value problem of fractional order $q \in (4, 5]$ contains Green's function (or solution) for lower-order fractional antiperiodic problems. We can further interpret that the last term in expressions for Green's function (2.5) arises due to consideration of the order $q \in (4, 5]$, whereas the remaining terms correspond to the lower-order problems. This observation gives a useful insight into the study of antiperiodic fractional boundary value problems that a unit-increase in the fractional order of the problem gives rise to a new term in expressions for Green's function, preserving the terms corresponding to lower-order antiperiodic problems. In other words, one can say that Green's function (or solution) for a higher-order antiperiodic fractional boundary value problem inherits all the characteristics of lower-order fractional antiperiodic problems. Hence, our results generalize the existing results on antiperiodic fractional boundary value problems ([21–23]).

3. Existence Results

Let $\mathcal{E} := C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions defined on $[0, T] \times \mathbb{R}$ endowed with a topology of uniform convergence with the norm $\|x\| = \sup_{t \in [0,T]} |x(t)|$.

To prove the existence results for problem (1.1), we need the following known results [28].

Theorem 3.1. *Let X be a Banach space. Assume that $T : X \to X$ is completely continuous operator and the set*

$$V = \{u \in X \mid u = \mu T u, 0 < \mu < 1\} \tag{3.1}$$

is bounded. Then T has a fixed point in X.

Theorem 3.2. *Let X be a Banach space. Assume that Ω is an open-bounded subset of X with $\theta \in \Omega$ and let $T : \overline{\Omega} \to X$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega. \tag{3.2}$$

Then T has a fixed point in $\overline{\Omega}$.

By Lemma 2.3, we define an operator $\mathcal{U} : \mathcal{E} \to \mathcal{E}$ as

$$\begin{aligned}
(\mathcal{U}x)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
& + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s,x(s))ds \\
& + \frac{(6t^2T - 4t^3 - T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s,x(s))ds \\
& + \frac{(2Tt^3 - t^4 - tT^3)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s,x(s))ds, \quad t \in [0,T].
\end{aligned} \tag{3.3}$$

Observe that the problem (1.1) has a solution if and only if the operator equation $\mathcal{U}x = x$ has a fixed point.

Theorem 3.3. *Assume that there exists a positive constant L_1 such that $|f(t,x)| \le L_1$ for $t \in [0,T], x \in \mathbb{R}$. Then the problem* (1.1) *has at least one solution.*

Proof. First of all, we show that the operator $\mathcal{U}$ is completely continuous. Note that the operator $\mathcal{U}$ is continuous in view of the continuity of f. Let $\mathcal{B} \subset \mathcal{E}$ be a bounded set. By the assumption that $|f(t,x)| \le L_1$, for $x \in \mathcal{B}$, we have

$$\begin{aligned}
|(\mathcal{U}x)(t)| \le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left|f(s,x(s))\right|ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \left|f(s,x(s))\right|ds \\
& + \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \left|f(s,x(s))\right|ds \\
& + \frac{1}{4}|t(T-t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} \left|f(s,x(s))\right|ds \\
& + \frac{\left|6t^2T - 4t^3 - T^3\right|}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} \left|f(s,x(s))\right|ds \\
& + \frac{\left|2Tt^3 - t^4 - tT^3\right|}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} \left|f(s,x(s))\right|ds \\
\le & L_1\Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}ds \\
& \quad + \frac{|T-2t|}{4\Gamma(q-1)}\int_0^T (T-s)^{q-2}ds + \frac{|t(T-t)|}{4\Gamma(q-2)}\int_0^T (T-s)^{q-3}ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left|6t^2T-4t^3-T^3\right|}{48\Gamma(q-3)}\int_0^T(T-s)^{q-4}ds+\frac{\left|2Tt^3-t^4-tT^3\right|}{48\Gamma(q-4)}\int_0^T(T-s)^{q-5}ds\Bigg]\\
&\leq L_1\left[\frac{T^q}{2\Gamma(q+1)}\left(3+\frac{q}{2}+\frac{q(q-1)(5q^2-9q+46)}{384}\right)\right]=L_2,
\end{aligned}
\tag{3.4}
$$

which implies that $\|(\mathcal{U}x)\|\leq L_2$. Further, we find that

$$
\begin{aligned}
\left|(\mathcal{U}x)'(t)\right| = {} & \int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}\left|f(s,x(s))\right|ds+\frac{1}{2}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}\left|f(s,x(s))\right|ds\\
&+\frac{|T-2t|}{4}\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}\left|f(s,x(s))\right|ds+\frac{|t(T-t)|}{4}\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}\left|f(s,x(s))\right|ds\\
&+\frac{\left|6Tt^2-4t^3-T^3\right|}{48}\int_0^T\frac{(T-s)^{q-5}}{\Gamma(q-4)}\left|f(s,x(s))\right|ds\\
\leq {} & L_1\left[\int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}ds+\frac{1}{2}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}ds\right.\\
&+\frac{|T-2t|}{4}\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}ds+\frac{|t(T-t)|}{4}\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}ds\\
&\left.+\frac{\left|6Tt^2-4t^3-T^3\right|}{48}\int_0^T\frac{(T-s)^{q-5}}{\Gamma(q-4)}\right]\\
\leq {} & L_1\left[\frac{T^{q-1}}{2\Gamma(q)}\left(3+\frac{(q-1)(q^2-2q+12)}{24}\right)\right]=L_3.
\end{aligned}
\tag{3.5}
$$

Hence, for $t_1,t_2\in[0,T]$, we have

$$
|(\mathcal{U}x)(t_2)-(\mathcal{U}x)(t_1)|\leq\int_{t_1}^{t_2}\left|(\mathcal{U}x)'(s)\right|ds\leq L_3(t_2-t_1). \tag{3.6}
$$

This implies that $\mathcal{U}$ is equicontinuous on $[0,T]$. Thus, by the Arzela-Ascoli theorem, the operator $\mathcal{U}:\mathcal{E}\to\mathcal{E}$ is completely continuous.

Next, we consider the set

$$
V=\left\{x\in\mathcal{E}\mid x=\mu\mathcal{U}x,\ 0<\mu<1\right\}, \tag{3.7}
$$

and show that the set V is bounded. Let $x \in V$, then $x = \mu \mathcal{U} x,\ 0 < \mu < 1$. For any $t \in [0,T]$, we have

$$\begin{aligned}
x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
& + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s,x(s))ds \\
& + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s,x(s))ds \\
& + \frac{(2Tt^3-t^4-tT^3)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s,x(s))ds,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
|x(t)| = \mu|(\mathcal{U}x)(t)| \le {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds \\
& + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))|ds \\
& + \frac{|t(T-t)|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s,x(s))|ds \\
& + \frac{|6t^2T-4t^3-T^3|}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s,x(s))|ds \\
& + \frac{|2Tt^3-t^4-tT^3|}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} |f(s,x(s))|ds \\
\le {} & L_1\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}ds \right.\\
& + \frac{|T-2t|}{4\Gamma(q-1)}\int_0^T (T-s)^{q-2}ds + \frac{|t(T-t)|}{4\Gamma(q-2)}\int_0^T (T-s)^{q-3}ds \\
& + \frac{|6t^2T-4t^3-T^3|}{48\Gamma(q-3)}\int_0^T (T-s)^{q-4}\Gamma(q-3)ds \\
& \left. + \frac{|2Tt^3-t^4-tT^3|}{48\Gamma(q-4)}\int_0^T (T-s)^{q-5}ds\right] \\
\le {} & \max_{t\in[0,T]}\left\{\frac{2|t^q|+T^q}{2\Gamma(q+1)} + \frac{|T-2t|T^{q-1}}{4\Gamma(q)} + \frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)} + \frac{|6t^2T-4t^3-T^3|T^{q-3}}{48\Gamma(q-2)} \right.\\
& \left. + \frac{|2Tt^3-t^4-tT^3|T^{q-4}}{48\Gamma(q-3)}\right\}L_1 \\
= {} & M_1.
\end{aligned} \tag{3.9}$$

Thus, $\|x\| \le M_1$ for any $t \in [0,T]$. So, the set V is bounded. Thus, by the conclusion of Theorem 3.1, the operator $\mathcal{U}$ has at least one fixed point, which implies that (1.1) has at least one solution. □

Theorem 3.4. *Let there exist a small positive number τ such that $|f(t,x)| \le \delta|x|$ for $0 < |x| < \tau$, where $\delta > 0$ satisfies the condition*

$$\max_{t\in[0,T]}\left\{\frac{2|t^q|+T^q}{2\Gamma(q+1)}+\frac{|T-2t|T^{q-1}}{4\Gamma(q)}+\frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)}+\frac{\left|6t^2T-4t^3-T^3\right|T^{q-3}}{48\Gamma(q-2)}\right.$$
$$\left.+\frac{\left|2Tt^3-t^4-tT^3\right|T^{q-4}}{48\Gamma(q-3)}\right\}\delta \le 1. \tag{3.10}$$

Then the problem (1.1) *has at least one solution.*

Proof. Let us define $\mathcal{B}_\tau = \{x \in \mathcal{E} \mid \|x\| < \tau\}$ and take $x \in \mathcal{E}$ such that $\|x\| = \tau$, that is, $x \in \partial\mathcal{B}_\tau$. As before, it can be shown that $\mathcal{U}$ is completely continuous and

$$\|\mathcal{U}x\| \le \max_{t\in[0,T]}\left\{\frac{2|t^q|+T^q}{2\Gamma(q+1)}+\frac{|T-2t|T^{q-1}}{4\Gamma(q)}+\frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)}+\frac{\left|6t^2T-4t^3-T^3\right|T^{q-3}}{48\Gamma(q-2)}\right.$$
$$\left.+\frac{\left|2Tt^3-t^4-tT^3\right|T^{q-4}}{48\Gamma(q-3)}\right\}\delta\|x\|, \tag{3.11}$$

which in view of (3.10) yields $\|\mathcal{U}x\| \le \|x\|, x \in \partial\mathcal{B}_\tau$. Therefore, by Theorem 3.2, the operator $\mathcal{U}$ has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution. □

Our next existence result is based on Krasnoselskii's fixed point theorem [29].

Theorem 3.5. *Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be the operators such that (i) $Ax + By \in M$ whenever $x,\ y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.6. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function. Further, we assume that*

(A_1) $|f(t,x) - f(t,y)| \le L|x-y|$, *for all* $t \in [0,T],\ x,y \in \mathbb{R}$;

(A_2) $|f(t,x)| \le \mu(t)$, *for all* $(t,x) \in [0,T] \times \mathbb{R}$, *and* $\mu \in \mathbb{C}([0,T], R^+)$.

Then the problem (1.1) *has at least one solution on* $[0,T]$ *if*

$$\frac{LT^q}{2\Gamma(q+1)}\left(1+\frac{q}{2}+\frac{q(q-1)}{8}+\frac{q(q-1)(q-2)}{24}+\frac{5q(q-1)(q-2)(q-3)}{384}\right) < 1. \tag{3.12}$$

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$\bar{r} \geq \frac{\|\mu\| T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right) \tag{3.13}$$

and consider $\mathcal{B}_{\bar{r}} = \{x \in \mathcal{E} : \|x\| \leq \bar{r}\}$. We define the operators $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathcal{B}_{\bar{r}}$ as

$$\begin{aligned}
(\mathcal{U}_1 x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds, \\
(\mathcal{U}_2 x)(t) &= -\frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, x(s)) ds \\
&\quad + \frac{1}{4}(t(T-t)) \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s, x(s)) ds \\
&\quad + \frac{(6t^2T - 4t^3 - T^3)}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s, x(s)) ds \\
&\quad + \frac{(2Tt^3 - t^4 - tT^3)}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s, x(s)) ds.
\end{aligned} \tag{3.14}$$

For $x, y \in \mathcal{B}_{\bar{r}}$, we find that

$$\|\mathcal{U}_1 x + \mathcal{U}_2 y\| \leq \frac{\|\mu\| T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right) \leq r. \tag{3.15}$$

Thus, $\mathcal{U}_1 x + \mathcal{U}_2 y \in \mathcal{B}_{\bar{r}}$. It follows from the assumption (A_1) that $\mathcal{U}_2$ is a contraction mapping for

$$\frac{LT^q}{2\Gamma(q+1)} \left(1 + \frac{q}{2} + \frac{q(q-1)}{8} + \frac{q(q-1)(q-2)}{24} + \frac{5q(q-1)(q-2)(q-3)}{384} \right) < 1. \tag{3.16}$$

Continuity of f implies that the operator $\mathcal{U}_1$ is continuous. Also, $\mathcal{U}_1$ is uniformly bounded on $\mathcal{B}_{\bar{r}}$ as

$$\|\mathcal{U}_1 x\| \leq \frac{\|\mu\| T^q}{\Gamma(q+1)}. \tag{3.17}$$

Now we prove the compactness of the operator $\mathcal{U}_1$. In view of (A_1), we define

$$\sup_{(t,x)\in[0,T]\times\mathcal{B}_{\bar{r}}} |f(t,x)| = f_m < \infty, \tag{3.18}$$

and consequently, for $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$\begin{aligned}|(\mathcal{U}_1 x)(t_2) - (\mathcal{U}_1 x)(t_1)| \le \frac{f_m}{\Gamma(q)} \Bigg| \int_0^{t_1} \Big[(t_2 - s)^{q-1} - (t_1 - s)^{q-1} \Big] ds \\ + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \Bigg|,\end{aligned} \tag{3.19}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. So $\mathcal{U}_1$ is relatively compact on $\mathcal{B}_{\overline{r}}$. Hence, By the Arzela-Ascoli theorem, $\mathcal{U}_1$ is compact on $\mathcal{B}_{\overline{r}}$. Thus all the assumptions of Theorem 3.5 are satisfied. Therefore, the conclusion of Theorem 3.5 applies and the antiperiodic fractional boundary value problem (1.1) has at least one solution on $[0,T]$. This completes the proof. □

Theorem 3.7. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function satisfying the condition*

$$\big| f(t,x) - f(t,y) \big| \le L \big| x - y \big|, \quad \forall t \in [0,T], \ x, y \in \mathbb{R} \tag{3.20}$$

with

$$L\Delta < 1, \tag{3.21}$$

$$\Delta = \frac{T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right). \tag{3.22}$$

Then the antiperiodic boundary value problem (1.1) *has a unique solution.*

Proof. Let us define $\sup_{t\in[0,T]} |f(t,0)| = M$ and select $r_\kappa \ge M\Delta/(1-\kappa)$ where $L\Delta \le \kappa < 1$. Then we show that $\mathcal{U} B_{r_\kappa} \subset B_{r_\kappa}$, where $B_{r_\kappa} = \{x \in \mathcal{E} : \|x\| \le r_\kappa\}$. For $x \in B_{r_\kappa}$, we have

$$\begin{aligned}\|(\mathcal{U}x)\| \le \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big(\big| f(s,x(s)) - f(s,0) \big| + \big| f(s,0) \big| \big) ds \\ + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \big(\big| f(s,x(s)) - f(s,0) \big| + \big| f(s,0) \big| \big) ds \\ + \frac{1}{4} |T - 2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \big(\big| f(s,x(s)) - f(s,0) \big| + \big| f(s,0) \big| \big) ds\end{aligned}$$

$$
\begin{aligned}
&\qquad + \frac{1}{4}|t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} \left(\left| f(s,x(s)) - f(s,0) \right| + \left| f(s,0) \right| \right) ds \\
&\qquad + \frac{1}{48}\left| 6t^2T - 4t^3 - T^3 \right| \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} \left(\left| f(s,x(s)) - f(s,0) \right| + \left| f(s,0) \right| \right) ds \\
&\qquad + \frac{\left| 2Tt^3 - t^4 - tT^3 \right|}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} \left(\left| f(s,x(s)) - f(s,0) \right| + \left| f(s,0) \right| \right) ds \Bigg\} \\
&\le (Lr_\kappa + M) \max_{t\in[0,T]} \Bigg\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} ds \\
&\qquad + \frac{|T-2t|}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds + \frac{|t(T-t)|}{4\Gamma(q-2)} \int_0^T (T-s)^{q-3} ds \\
&\qquad + \frac{1}{48\Gamma(q-3)} \left| 6t^2T - 4t^3 - T^3 \right| \int_0^T (T-s)^{q-4} ds \\
&\qquad + \frac{\left| 2Tt^3 - t^4 - tT^3 \right|}{48\Gamma(q-4)} \int_0^T (T-s)^{q-5} ds \Bigg\} \\
&\le (Lr_\kappa + M) \left[\frac{T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2-9q+46)}{384} \right) \right] \\
&= (Lr_\kappa + M)\Delta \le \kappa r_\kappa + M\Delta \le r_\kappa,
\end{aligned}
\tag{3.23}
$$

where (3.22) is used. Now, for $x, y \in \mathcal{E}$, we obtain

$$
\begin{aligned}
&\| (\mathcal{U}x) - (\mathcal{U}y) \| \\
&\le \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \| f(s,x(s)) - f(s,y(s)) \| ds \\
&\qquad + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \| f(s,x(s)) - f(s,y(s)) \| ds \\
&\qquad + \frac{1}{4}|T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \| f(s,x(s)) - f(s,y(s)) \| ds \\
&\qquad + \frac{1}{4}|t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} \| f(s,x(s)) - f(s,y(s)) \| ds \\
&\qquad + \frac{1}{48}\left| 6t^2T - 4t^3 - T^3 \right| \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} \| f(s,x(s)) - f(s,y(s)) \| ds \\
&\qquad + \frac{1}{48}\left| 2Tt^3 - t^4 - tT^3 \right| \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} \| f(s,x(s)) - f(s,y(s)) \| ds \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\le L\|x-y\| \max_{t\in[0,T]}\left\{ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}ds \right.\\
&\qquad + \frac{|T-2t|}{4\Gamma(q-1)}\int_0^T (T-s)^{q-2}ds + \frac{|t(T-t)|}{4\Gamma(q-2)}\int_0^T (T-s)^{q-3}ds\\
&\qquad + \frac{1}{48\Gamma(q-3)}\left|6t^2T-4t^3-T^3\right|\int_0^T (T-s)^{q-4}\Gamma(q-3)ds\\
&\qquad \left. + \frac{1}{48}\left|2Tt^3-t^4-tT^3\right|\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}\right\}\\
&\le \frac{LT^q}{2\Gamma(q+1)}\left(3+\frac{q}{2}+\frac{q(q-1)(5q^2-9q+46)}{384}\right)\|x-y\| = \Delta L\|x-y\|,
\end{aligned}
\tag{3.24}
$$

where we have used (3.22). It follows by the condition (3.21) that $\mathcal{U}$ is a contraction. So, by Banach's contraction mapping principle, problem (1.1) has a unique solution. □

Example 3.8. Consider the following antiperiodic fractional boundary value problem:

$$
\begin{gathered}
{}^{C}D^q x(t) = \frac{e^{(1-\cos^2 x(t))^2}\left[4\sin 2t + 8\ln\left(17+5\cos^2 x(t)\right)\right]}{\sqrt{(17+\sin x(t))}}, \quad 0<t<1,\\
x(0)=-x(1), \qquad x'(0)=-x'(1), \qquad x''(0)=-x''(1),\\
x'''(0)=-x'''(1), \qquad x^{iv}(0)=-x^{iv}(1),
\end{gathered}
\tag{3.25}
$$

where $4<q\le 5$ and $T=1$.

Clearly, $|f(t,x)| \le L_1 = e(1+2\ln 22)$, and the hypothesis of Theorem 3.3 holds. Therefore, the conclusion of Theorem 3.3 applies to problem (3.25).

Example 3.9. Consider the following problem:

$$
\begin{gathered}
{}^{C}D^q x(t) = x\left(a^2+x^3(t)\right)^{1/2} + 2\left(1+t^4\right)^3(1-\cos x(t)), \quad x\neq 0,\ a>0,\ 0<t<1,\\
x(0)=-x(1), \qquad x'(0)=-x'(1), \qquad x''(0)=-x''(1),
\end{gathered}
\tag{3.26}
$$

$$
x'''(0)=-x'''(1), \qquad x^{iv}(0)=-x^{iv}(1),
\tag{3.27}
$$

where $4<q\le 5$, and $T=1$.

For sufficiently small x (ignoring x^2 and higher powers of x), we have

$$
\left| x\left(a^2+x^3(t)\right)^{1/2} + 2\left(1+t^4\right)^3(1-\cos x(t))\right| \le a|x|,
\tag{3.28}
$$

where $a \leq \delta$, and (3.10) takes the form

$$\left(\frac{3}{2\Gamma(q+1)} + \frac{1}{4\Gamma(q)} + \frac{1}{16\Gamma(q-1)} + \frac{1}{48\Gamma(q-2)} + \frac{5}{768\Gamma(q-3)}\right)\delta \leq 1 \tag{3.29}$$

(in particular, for $q = 9/2$, $\delta \leq 1920\sqrt{\pi}/313$). Thus all the assumptions of Theorem 3.4 hold. Consequently, the conclusion of Theorem 3.4 implies that the problem (3.26) has at least one solution

Example 3.10. Consider the following antiperiodic fractional boundary value problem:

$$\begin{gathered}
{}^{c}D^{9/2}x(t) = \frac{1}{\sqrt{(t+2025)}}\left(\frac{|x|}{1+|x|} + \tan^{-1}x\right) + \sin t, \quad t \in [0,\pi], \\
x(0) = -x(\pi), \qquad x'(0) = -x'(\pi), \qquad x''(0) = -x''(\pi), \\
x'''(0) = -x'''(\pi), \qquad x^{(iv)}(0) = -x^{(iv)}(\pi),
\end{gathered} \tag{3.30}$$

where $q = 9/2$, and $T = \pi$. Clearly, $L = 2/45$ as $|f(t,x) - f(t,y)| \leq 2/45|x-y|$. Further,

$$L\Delta = \frac{LT^q}{2\Gamma(q+1)}\left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2-9q+46)}{384}\right) = \frac{313\pi^4}{43200} < 1. \tag{3.31}$$

Thus, all the assumptions of by Theorem 3.7 are satisfied. Hence, the fractional boundary value problem (3.30) has a unique solution on $[0,\pi]$.

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Research Article

Positive Solutions for a Fractional Boundary Value Problem with Changing Sign Nonlinearity

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We discuss the existence of positive solutions to the following fractional m-point boundary value problem with changing sign nonlinearity $D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, 0<t<1, u(0)=0, D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i)$, where λ is a positive parameter, $1<\alpha\le 2$, $0<\beta<\alpha-1$, $0<\xi_1<\cdots<\xi_{m-2}<1$ with $\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-\beta-1}<1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, f and may be singular at $t=0$ and/or $t=1$ and also may change sign. The work improves and generalizes some previous results.

1. Introduction

In this paper, we consider the following fractional differential equation with m-point boundary conditions:

$$\begin{aligned} &D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, \quad 0<t<1, \\ &u(0)=0, \quad D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i), \end{aligned} \tag{1.1}$$

where $1<\alpha\le 2$, $\lambda>0$ is a parameter, $0<\beta<\alpha-1$, $0<\xi_1<\cdots<\xi_{m-2}<1$ with $\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-\beta-1}<1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, and $f\in C((0,1)\times[0,+\infty)\to(-\infty,+\infty))$ may be singular at $t=0$ and/or $t=1$ and also may change sign. In this paper, by a positive solution to (1.1), we mean a function $u\in C[0,1]$ which is positive on $(0,1]$ and satisfies (1.1).

In recent years, great efforts have been made worldwide to study the existence of solutions for nonlinear fractional differential equations by using nonlinear analysis methods [1–24]. Fractional-order multipoint boundary value problems (BVP) have particularly attracted a great deal of attention (see, e.g., [13–19]). In [10], the authors discussed some properties of the Green function for the Direchlet-type BVP of nonlinear fractional differential equations

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \\ u(0)=0, \quad u(1)=0, \end{gathered} \tag{1.2}$$

where $1<\alpha<2$, D_{0+}^{α} is the standard Riemann-Liouville derivative and $f: C([0,1]\times[0,+\infty) \to [0,+\infty))$ is continuous. By using the Krasnosel'skii fixed-point theorem, the existence of positive solutions was obtained under some suitable conditions on f.

In [14], the authors investigated the existence and multiplicity of positive solutions by using some fixed-point theorems for the fractional differential equation given by

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \\ u(0)=0, \quad D_{0+}^{\beta}u(1)=aD_{0+}^{\beta}u(\xi), \end{gathered} \tag{1.3}$$

where $1<\alpha\leq 2$, $0\leq\beta\leq 1$, $0<\xi<1$, $0\leq a\leq 1$ with $a\xi^{\alpha-\beta-2}<1-\beta$, $0\leq\alpha-\beta-1$, f is nonnegative.

It should be noted that in most of the works in literature the nonlinearity needs to be nonnegative in order to establish positive solutions. As far as we know, semipositone fractional nonlocal boundary value problems with $1<\alpha\leq 2$ have been seldom studied due to the difficulties in finding and analyzing the corresponding Green function.

In [23], the authors investigated the following fractional differential equation with three-point boundary conditions:

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t))+e(t)=0, \quad 0<t<1, \\ u(0)=0, \quad D_{0+}^{\beta}u(1)=aD_{0+}^{\beta}u(\xi), \end{gathered} \tag{1.4}$$

where $1<\alpha\leq 2$, $0<\beta\leq 1$, $0<\xi<1$, $0\leq a\leq 1$,$0\leq\alpha-\beta-1$, $e(t)\in L[0,1]$, and f satisfies the Caratheodory conditions. The authors obtained the properties of the Green function for (1.4) as follows:

$$\frac{\beta t^{\alpha-1}s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\leq G(t,s)\leq\frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)\left(1-a\xi^{\alpha-\beta-1}\right)}. \tag{1.5}$$

By using the Schauder fixed-point theorem, the authors obtained the existence of positive solution of (1.4) with the following assumptions:

(A_1) for each $L>0$, there exists a function $\phi_L\succ 0$ such that $f(t,t^{\alpha-1}x)\geq\phi_L(t)$ for a.e. $t\in(0,1)$, for all $x\in(0,L]$;

(A_2) there exist $g(x), h(x)$, and $k(t) \succ 0$, such that

$$0 \le f(t,x) \le k(t)\{g(x)+h(x)\}, \quad \forall x \in (0,\infty), \text{ a.e. } t \in (0,1), \tag{1.6}$$

here $g : (0,\infty) \to [0,\infty)$ is continuous and nonincreasing, $h : [0,\infty) \to [0,\infty)$ is continuous, and h/g is nondecreasing;

(A_3) There exist two positive constants $R > r > 0$ such that

$$\begin{gathered} R > \Phi_{R1} + \gamma_* \ge r > 0, \\ \int_0^1 k(s) g\left(r s^{\alpha-1}\right) ds < +\infty, \\ R \ge \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)\left(1 - a\xi^{\alpha-\beta-1}\right)} k(s) g\left(r s^{\alpha-1}\right) ds + \gamma_*. \end{gathered} \tag{1.7}$$

Here

$$\Phi_{R1} = \int_0^1 \frac{\beta s (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \phi_R(s) ds. \tag{1.8}$$

The assumptions on nonlinearity are not suitable for frequently used conditions, such as superlinear or some sublinear. For instance, $f(t,x) = x^{\mu}$, $\mu > 0$, obviously, f does not satisfy (A_1).

Inspired by the previous work, the aim of this paper is to establish conditions for the existence of positive solutions of the more general BVP (1.1). Our work presented in this paper has the following new features. Firstly, we consider few cases of $1 < \alpha \le 2$ which has been studied before, and in dealing with the difficulties related to the Green function for this case, some new properties of the Green function have been discovered. Secondly, the BVP (1.1) possesses singularity; that is, f may be singular at $t = 0$ and/or $t = 1$. Thirdly, the nonlinearity f may change sign and may be unbounded from below. Finally, we impose weaker positivity conditions on the nonlocal boundary term; that is, some of the coefficients η_i may be negative.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are to be used to prove our main results. We also discover some new positive properties of the corresponding Green function. In Section 3, we discuss the existence of positive solutions of the semipositone BVP (1.1). In Section 4, we give an example to demonstrate the application of our theoretical results.

2. Basic Definitions and Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can also be found in the recent literature.

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \tag{2.1}$$

provided that the right-hand side is pointwisely defined on $(0, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwisely defined on $(0, +\infty)$.

Lemma 2.3 (see [3]). *Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, $D_{0+}^{\alpha} u \in L(0,1)$;*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.3}$$

where $c_i \in R$, $i = 1, 2, \ldots, n$, $n - 1 < \alpha \leq n$.

Set

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.4}$$

$$p(s) = 1 - \sum_{s \leq \xi_i} \eta_i \left(\frac{\xi_i - s}{1-s}\right)^{\alpha-\beta-1}, \tag{2.5}$$

$$G(t,s) = G_0(t,s) + q(s) t^{\alpha-1}, \tag{2.6}$$

where

$$q(s) = \frac{p(s) - p(0)}{\Gamma(\alpha) p(0)} (1-s)^{\alpha-\beta-1}, \quad p(0) = 1 - \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1}. \tag{2.7}$$

For convenience in presentation, we here list the assumption to be used throughout the paper.

(H_1) $p(0) > 0$, $q(s) \geq 0$ on $[0,1]$.

Remark 2.4. If $\eta_i = 0$ $(i = 1, \ldots, m-2)$, we have $p(0) = 1$ and $q(s) \equiv 0$. If $\eta_i \geq 0$ $(i = 1, \ldots, m-2)$ and $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$, we have $q(s) \geq 0$ on $[0,1]$.

Lemma 2.5 (see [14]). *Assume that $g(t) \in L[0,1]$ and $\alpha > \beta > 0$. Then*

$$D_{0+}^{\beta} \int_0^t (t-s)^{\alpha-1} g(s)ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)ds. \tag{2.8}$$

Lemma 2.6. *Assume (H_1) holds, and $y(t) \in L[0,1]$. Then the unique solution of the problem*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\beta} u(\xi_i) \end{gathered} \tag{2.9}$$

is

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{2.10}$$

where $G(t,s)$ is the Green function of the boundary value problem (2.9).

Proof. From Lemma 2.3, the solution of (2.9) is

$$u(t) = -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{2.11}$$

Consequently,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{2.12}$$

From $u(0) = 0$, we have $c_2 = 0$.

By Lemma 2.5, we have

$$D_{0+}^{\beta} u(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \tag{2.13}$$

Therefore,

$$\begin{gathered} D_{0+}^{\beta} u(1) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)}, \\ D_{0+}^{\beta} u(\xi_i) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} y(s) ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} \xi_i^{\alpha-\beta-1}. \end{gathered} \tag{2.14}$$

By $D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i)$, we have

$$\begin{aligned} c_1 &= \frac{\int_0^1 (1-s)^{\alpha-1}y(s)ds-\sum_{i=1}^{m-2}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}y(s)ds}{\Gamma(\alpha)p(0)} \\ &= \frac{\int_0^1 (1-s)^{\alpha-\beta-1}p(s)y(s)ds}{\Gamma(\alpha)p(0)}. \end{aligned} \tag{2.15}$$

Therefore, the solution of (2.9) is

$$\begin{aligned} u(t) &= c_1t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned} \tag{2.16}$$

□

Lemma 2.7. *The function $G_0(t,s)$ has the following properties:*

(1) $G_0(t,s)>0$, *for* $t,s\in(0,1)$;

(2) $\Gamma(\alpha)G_0(t,s)\le t^{\alpha-1}$, *for* $t,s\in[0,1]$;

(3) $\beta t^{\alpha-1}h(s)\le\Gamma(\alpha)G_0(t,s)\le h(s)t^{\alpha-2}$, *for* $t,s\in(0,1)$,

where

$$h(s)=s(1-s)^{\alpha-\beta-1}. \tag{2.17}$$

Proof. (1) When $0<t\le s<1$, it is clear that

$$G_0(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1}>0. \tag{2.18}$$

When $0<s\le t<1$, we have

$$\begin{aligned} t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1} &\ge t^{\alpha-1}(1-s)^{\alpha-\beta-1}-t^{\alpha-1}(1-s)^{\alpha-1} \\ &= t^{\alpha-1}(1-s)^{\alpha-\beta-1}\left[1-(1-s)^{\beta}\right]>0. \end{aligned} \tag{2.19}$$

(2) By (2.4), for any $t,s\in[0,1]$, we have

$$\Gamma(\alpha)G_0(t,s)\le t^{\alpha-1}(1-s)^{\alpha-\beta-1}\le t^{\alpha-1}. \tag{2.20}$$

In the following, we will prove (3).

(i) When $0 < s \leq t < 1$, noticing that $0 < \beta < \alpha - 1 \leq 1$, we have

$$\frac{\partial}{\partial \beta}\left\{t^{\alpha-2}s(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-s)^{\alpha-\beta-1}\right\} = t^{\alpha-2}(1-s)^{\alpha-\beta-1}(t-s)\ln(1-s) \leq 0. \tag{2.21}$$

Therefore,

$$\begin{aligned} t^{\alpha-2}s(1-s)^{\alpha-\beta-1} - \left(t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\right) &\geq t^{\alpha-2}s - t^{\alpha-1} + (t-s)^{\alpha-1} \\ &= -t^{\alpha-2}(t-s) + (t-s)^{\alpha-1} \geq 0, \end{aligned} \tag{2.22}$$

which implies

$$\Gamma(\alpha)G_0(t,s) \leq h(s)t^{\alpha-2}. \tag{2.23}$$

On the other hand, we have

$$\frac{d}{ds}\left\{\beta s + (1-s)^{\beta}\right\} \leq 0, \quad s \in [0,1). \tag{2.24}$$

Therefore, $\beta s + (1-s)^{\beta} \leq 1$, which implies

$$\left[1 - (1-s)^{\beta}\right] \geq \beta s. \tag{2.25}$$

Then

$$\begin{aligned} \Gamma(\alpha)G_0(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ &\geq t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\beta}(t-ts)^{\alpha-\beta-1} \\ &= \left[1 - \left(1 - \frac{s}{t}\right)^{\beta}\right] t^{\alpha-1}(1-s)^{\alpha-\beta-1} \\ &\geq \left[1 - (1-s)^{\beta}\right] t^{\alpha-1}(1-s)^{\alpha-\beta-1} \\ &\geq \beta t^{\alpha-1}h(s). \end{aligned} \tag{2.26}$$

(ii) When $0 < t \leq s < 1$, we have

$$\begin{aligned} \Gamma(\alpha)G_0(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} = t^{\alpha-2}t(1-s)^{\alpha-\beta-1} \\ &\leq t^{\alpha-2}s(1-s)^{\alpha-\beta-1} = h(s)t^{\alpha-2}, \end{aligned} \tag{2.27}$$

On the other hand, clearly we have

$$\Gamma(\alpha)G_0(t,s) = t^{\alpha-1}(1-s)^{\alpha-\beta-1} \geq \beta t^{\alpha-1}h(s). \tag{2.28}$$

The inequalities (2.23)–(2.28) imply that (3) holds. □

By Lemma 2.7, we have the following results.

Lemma 2.8. *Assume (H_1) holds, then the Green function defined by (2.6) satisfies*

(1) $G(t,s) > 0$, *for all* $t,s \in (0,1)$;

(2) $G(t,s) \leq t^{\alpha-1}((1/(\Gamma(\alpha))) + q(s))$, *for all* $t,s \in [0,1]$;

(3) $\beta t^{\alpha-1}\Phi(s) \leq G(t,s) \leq t^{\alpha-2}\Phi(s)$, *for all* $t,s \in (0,1)$,

where

$$\Phi(s) = \left(\frac{h(s)}{\Gamma(\alpha)} + q(s)\right). \tag{2.29}$$

Lemma 2.9. *Assume (H_1) holds, then the function $G^*(t,s) =: t^{2-\alpha}G(t,s)$ satisfies*

(1) $G^*(t,s) > 0$, *for all* $t,s \in (0,1)$;

(2) $G^*(t,s) \leq t((1/(\Gamma(\alpha))) + q(s))$, *for all* $t,s \in [0,1]$;

(3) $\beta t\Phi(s) \leq G^*(t,s) \leq \Phi(s)$, *for all* $t,s \in [0,1]$.

For convenience, we list here four more assumptions to be used later:

(H_2) $f \in C((0,1)\times[0,+\infty),(-\infty,+\infty))$ satisfies

$$f(t,x) \geq -r(t), \quad f\left(t,t^{\alpha-2}x\right) \leq z(t)g(x), \quad t \in (0,1),\ x \in [0,+\infty), \tag{2.30}$$

where $r,z \in C((0,1),[0,+\infty))$, $g \in C([0,+\infty),[0,+\infty))$.

(H_3) $\int_0^1 r(s)ds < +\infty$, $0 < \int_0^1 z(s)ds < +\infty$.

(H_4) There exists $[a,b] \subset (0,1)$ such that

$$\liminf_{x\to+\infty}\min_{t\in[a,b]}\frac{f(t,x)}{x} = +\infty. \tag{2.31}$$

(H_5) There exists $[c,d] \subset (0,1)$ such that

$$\begin{gathered}\liminf_{x\to+\infty}\min_{t\in[c,d]} f(t,x) = +\infty,\\ \lim_{x\to+\infty}\frac{g(x)}{x} = 0.\end{gathered} \tag{2.32}$$

Remark 2.10. The second limit of (H_5) implies

$$\lim_{u\to+\infty}\frac{g^*(u)}{u}=0, \tag{2.33}$$

where

$$g^*(u)=\max_{x\in[0,u]}g(x). \tag{2.34}$$

Proof. By $\lim_{u\to+\infty}(g(u)/u)=0$, for any $\epsilon>0$, there exists $N_1>0$, such that for any $u>N_1$ we have

$$0\le g(u)<\epsilon u. \tag{2.35}$$

Let $N=\max\{N_1,((g^*(N_1))/\epsilon)\}$, for any $u>N$ we have

$$0\le g^*(u)<\epsilon u+g^*(N_1)<2\epsilon u. \tag{2.36}$$

Therefore, $\lim_{u\to+\infty}((g^*(u))/u)=0$. □

Lemma 2.11. *Assume* (H_1) *holds and* $r(t)\in C(0,1)\cap L[0,1]$ *is nonnegative, then the BVP*

$$\begin{gathered}D^{\alpha}_{0+}u(t)+r(t)=0,\quad 0<t<1,\\ u(0)=0,\quad D^{\beta}_{0+}u(1)=\sum_{i=1}^{m-2}\eta_i D^{\beta}_{0+}u(\xi_i)\end{gathered} \tag{2.37}$$

has a unique solution $\omega(t)=\int_0^1 G(t,s)r(s)ds$ *with* $\omega(t)\le kt^{\alpha-1}$*, where*

$$k=\int_0^1\left(\frac{1}{\Gamma(\alpha)}+q(s)\right)r(s)ds,\quad t\in[0,1]. \tag{2.38}$$

Proof. By Lemma 2.6, $\omega(t)=\int_0^1 G(t,s)r(s)ds$ is the unique solution of (2.37). By (2) of Lemma 2.8, we have

$$\omega(t)=\int_0^1 G(t,s)r(s)ds\le t^{\alpha-1}\int_0^1\left(\frac{1}{\Gamma(\alpha)}+q(s)\right)r(s)ds. \tag{2.39}$$

□

Let $E=C[0,1]$ be endowed with the maximum norm $\|u\|=\max_{0\le t\le 1}|u(t)|$ and define a cone P by

$$P=\{u(t)\in E:\ \text{there exists } l_u>0\ \text{ such that }\ \beta t\|u\|\le u(t)\le l_u t\}, \tag{2.40}$$

and then set $B_r=\{u(t)\in E:\|u\|<r\}$, $P_r=P\cap B_r$, $\partial P_r=P\cap\partial B_r$.

Next we consider the following singular nonlinear BVP:

$$D_{0+}^{\alpha}u(t)+\lambda\left[f\left(t,[u(t)-\lambda\omega(t)]^{+}\right)+r(t)\right]=0,\quad 0<t<1,$$
$$u(0)=0,\quad D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i),\tag{2.41}$$

where $\lambda>0$, $[v(t)]^{+}=\max\{v(t),0\}$, $\omega(t)$ is defined in Lemma 2.11.

Let

$$Tu(t)=\lambda\int_0^1 G^*(t,s)\left[f\left(s,\left[s^{\alpha-2}u(s)-\lambda\omega(s)\right]^{+}\right)+r(s)\right]ds.\tag{2.42}$$

Clearly, if $u(t)\in P$ is a fixed point of T, then $y(t)=t^{\alpha-2}u(t)$ is a positive solution of (2.41).

Lemma 2.12. *Suppose that (H_1)–(H_3) hold. Then $T:P\to P$ is a completely continuous operator.*

Proof. It is clear that T is well defined on P. For any $u\in P$, Lemma 2.9 implies

$$Tu(t)\geq \beta t\lambda\int_0^1\Phi(s)\left[f\left(s,\left[s^{\alpha-2}u(s)-\lambda\omega(s)\right]^{+}\right)+r(s)\right]ds.\tag{2.43}$$

On the other hand,

$$Tu(t)\leq \lambda\int_0^1\Phi(s)\left[f\left(s,\left[s^{\alpha-2}u(s)-\lambda\omega(s)\right]^{+}\right)+r(s)\right]ds.\tag{2.44}$$

Therefore, $Tu(t)\geq \beta t\|Tu\|$. Noticing that

$$Tu(t)\leq \lambda t\int_0^1\left(\frac{1}{\Gamma(\alpha)}+q(s)\right)\left[f\left(s,\left[s^{\alpha-2}u(s)-\lambda\omega(s)\right]^{+}\right)+r(s)\right]ds,\tag{2.45}$$

we have $T:P\to P$.

Using the Ascoli-Arzela theorem, we can then get that $T:P\to P$ is a completely continuous operator. □

Lemma 2.13 (see [25]). *Let E be a real Banach space and let $P\subset E$ be a cone. Assume that Ω_1 and Ω_2 are two-bounded open subsets of E with $\theta\in\Omega_1,\overline{\Omega}_1\subset\Omega_2$, $T:P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ a completely continuous operator such that either*

(1) *$\|Tu\|\leq\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Tu\|\geq\|u\|$, $u\in P\cap\partial\Omega_2$, or*

(2) *$\|Tu\|\geq\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Tu\|\leq\|u\|$, $u\in P\cap\partial\Omega_2$.*

Then T has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. Existence of Positive Solutions

Theorem 3.1. *Suppose that* (H_1)–(H_4) *hold. Then there exists* $\lambda^* > 0$ *such that the BVP* (1.1) *has at least one positive solution for any* $\lambda \in (0, \lambda^*)$.

Proof. Choose $r_1 > k\beta^{-1}$. Let

$$\lambda^* = \min\left\{1, \frac{r_1}{(g^*(r_1)+1)\int_0^1 \Phi(s)(z(s)+r(s))ds}\right\}, \tag{3.1}$$

where

$$g^*(r) = \max_{x\in[0,r]} g(x). \tag{3.2}$$

In the following of the proof, we suppose $\lambda \in (0, \lambda^*)$.

For any $u \in \partial P_{r_1}$, noticing $u(t) \geq \beta t r_1$ and Lemma 2.11, we have

$$t^{\alpha-2}u(t) - \lambda\omega(t) \geq (\beta r_1 - \lambda k)t^{\alpha-1} \geq (\beta r_1 - k)t^{\alpha-1} \geq 0, \tag{3.3}$$

$$r_1 \geq u(t) - \lambda t^{2-\alpha}\omega(t) \geq (\beta r_1 - k)t \geq 0. \tag{3.4}$$

Therefore,

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G^*(t,s)\Big(f\Big(s, \big[s^{\alpha-2}u(s) - \lambda\omega(s)\big]^+\Big) + r(s)\Big)ds \\ &\leq \lambda\int_0^1 \Phi(s)\Big(z(s)g\Big(\big[u(s) - \lambda s^{2-\alpha}\omega(s)\big]^+\Big) + r(s)\Big)ds \\ &\leq \lambda(g^*(r_1)+1)\int_0^1 \Phi(s)(z(s)+r(s))ds \\ &< \lambda^*(g^*(r_1)+1)\int_0^1 \Phi(s)(z(s)+r(s))ds \leq r_1. \end{aligned} \tag{3.5}$$

Thus,

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial P_{r_1}. \tag{3.6}$$

Now choose a real number

$$L > \frac{2}{\lambda\beta^2\int_a^b \Phi(s)s^{\alpha-1}ds}. \tag{3.7}$$

By (H_4), there exists a constant $N > 0$ such that

$$f(t,x) > Lx, \quad \text{for any } t \in [a,b],\ x \geq N. \tag{3.8}$$

Let

$$r_2 = r_1 + \frac{2k}{\beta} + \frac{2N}{\beta a^{\alpha-1}}. \tag{3.9}$$

Then for any $u \in \partial P_{r_2}$, we have

$$t^{\alpha-2}u(t) - \lambda\omega(t) \geq (\beta r_2 - k)t^{\alpha-1} \geq \frac{\beta r_2}{2}t^{\alpha-1}, \quad \forall t \in (0,1]. \tag{3.10}$$

Thus, for any $t \in [a,b]$, we have $t^{\alpha-2}u(t) - \lambda\omega(t) > N$. Hence, we get

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\lambda\int_0^1 G^*(t,s)\left[f\left(s,\left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^+\right) + r(s)\right]ds \\
&\geq \max_{t\in[0,1]}\lambda\int_a^b G^*(t,s)f\left(s,\left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]\right)ds \\
&\geq \max_{t\in[0,1]}\lambda L\int_a^b G^*(t,s)\left(s^{\alpha-2}u(s) - \lambda\omega(s)\right)ds \\
&\geq \max_{t\in[0,1]}\lambda L\int_a^b G^*(t,s)\frac{\beta r_2}{2}s^{\alpha-1}ds \\
&\geq \max_{t\in[0,1]}\frac{\lambda L\beta^2 r_2}{2}t\int_a^b \Phi(s)s^{\alpha-1}ds \\
&= \frac{\lambda L\beta^2 r_2}{2}\int_a^b \Phi(s)s^{\alpha-1}ds \geq r_2.
\end{aligned} \tag{3.11}$$

Therefore,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial P_{r_2}. \tag{3.12}$$

By Lemma 2.13, T has a fixed point $u \in P$ such that $r_1 \leq \|u\| \leq r_2$. Let $\overline{u}(t) = t^{\alpha-2}u(t) - \lambda\omega(t)$. Since $\|u\| \geq r_1$, by (3.3) we have $\overline{u}(t) \geq 0$ on $(0,1]$ and $\lim_{t\to 0^+}t^{\alpha-2}u(t) = 0$. Notice that $\omega(t)$ is the solution of (2.37) and $t^{\alpha-2}u(t)$ is the solution of (2.41). Thus, $\overline{u}(t)$ is a positive solution of the BVP (1.1). □

Theorem 3.2. *Suppose that* (H_1)*–*(H_3) *and* (H_5) *hold. Then there exists* $\lambda^* > 0$ *such that the BVP* (1.1) *has at least one positive solution for any* $\lambda \in (\lambda^*, +\infty)$.

Proof. By the first limit of (H_5), there exists $N > 0$ such that

$$f(t,x) \geq \frac{2k}{\beta^2 \int_c^d \Phi(s)ds}, \quad \text{for any } t \in [c,d],\ x \geq N. \tag{3.13}$$

Let

$$\lambda^* = \frac{N}{kc^{\alpha-1}}. \tag{3.14}$$

In the following part of the proof, we suppose $\lambda > \lambda^*$.

Let

$$R_1 = \frac{2\lambda k}{\beta}. \tag{3.15}$$

Then for any $u \in \partial P_{R_1}$, we have

$$t^{\alpha-2}u(t) - \lambda\omega(t) \geq (\beta R_1 - \lambda k)t^{\alpha-1} = \lambda k t^{\alpha-1} \geq \lambda^* k t^{\alpha-1}, \quad \forall t \in (0,1]. \tag{3.16}$$

Therefore, $t^{\alpha-2}u(t) - \lambda\omega(t) \geq N$, for any $t \in [c,d]$ and $u \in \partial P_{R_1}$. Then

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G^*(t,s)\left[f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^+\right) + r(s)\right]ds \\ &\geq \lambda \int_c^d G^*(t,s) f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^+\right)ds \\ &\geq \frac{2\lambda k}{\beta^2 \int_c^d \Phi(s)ds} \int_c^d G^*(t,s)ds \\ &\geq \frac{2\lambda k t}{\beta \int_c^d \Phi(s)ds} \int_c^d \Phi(s)ds = R_1 t. \end{aligned} \tag{3.17}$$

This implies

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial P_{R_1}. \tag{3.18}$$

On the other hand, $g(x)$ is continuous on $[0,+\infty)$, and thus from the second limit of (H_5), we have

$$\lim_{x \to +\infty} \frac{g^*(x)}{x} = 0, \tag{3.19}$$

where $g^*(x)$ is defined by (3.2). For

$$\epsilon = \left(2\lambda \int_0^1 \Phi(s) z(s) ds \right)^{-1}, \tag{3.20}$$

there exists $X_0 > 0$ such that $g(u) \leq \varepsilon x$ for any $x \geq X_0$ and $u \in [0, x]$.

Let

$$R_2 = X_0 + R_1 + 2\lambda \int_0^1 \Phi(s) r(s) ds. \tag{3.21}$$

For any $u \in \partial P_{R_2}$, by (3.16) we can get $R_2 \geq u(t) - \lambda t^{2-\alpha}\omega(t) \geq 0$, for all $t \in [0, 1]$. Therefore,

$$\begin{aligned} \|Tu\| &\leq \lambda \int_0^1 \Phi(s) \left[z(s) g\left(\left[u(s) - \lambda s^{2-\alpha}\omega(s) \right]^+ \right) + r(s) \right] ds \\ &\leq \lambda \varepsilon R_2 \int_0^1 \Phi(s) z(s) ds + \lambda \int_0^1 \Phi(s) r(s) ds \\ &\leq \frac{R_2}{2} + \frac{R_2}{2} = R_2. \end{aligned} \tag{3.22}$$

Thus,

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial P_{R_2}. \tag{3.23}$$

By Lemma 2.13, T has a fixed point $u \in P$ such that $R_1 \leq \|u\| \leq R_2$. Let $\overline{u}(t) = t^{\alpha-2}u(t) - \lambda\omega(t)$. Since $\|u\| \geq R_1$, by (3.16) we have $\overline{u}(t) \geq 0$ on $(0, 1]$ and $\lim_{t\to 0^+} t^{\alpha-2}u(t) = 0$. Notice that $\omega(t)$ is a solution of (2.37) and $t^{\alpha-2}u(t)$ is a solution of (2.41). Thus, $\overline{u}(t)$ is a positive solution of the BVP (1.1). □

By the proof of Theorem 3.2, we have the following corollary.

Corollary 3.3. *The conclusion of Theorem 3.2 is valid if* (H_5) *is replaced by* (H_5^*)*. There exist* $[c, d] \subset (0, 1)$ *and* $N > 0$ *such that for any* $t \in [c, d]$ *and* $x \geq N$*,*

$$\begin{gathered} f(t, x) \geq \frac{2k}{\beta^2 \int_c^d \Phi(s) ds}, \\ \lim_{x\to+\infty} \frac{g(x)}{x} = 0. \end{gathered} \tag{3.24}$$

4. Example

Example 4.1 (a 4-point BVP with coefficients of both signs). Consider the following problem:

$$\begin{gathered} D_{0+}^{7/4}u(t) + \lambda f(t,u(t)) = 0, \quad t \in (0,1), \ u(0) = 0, \\ D_{0+}^{1/4}u(1) = D_{0+}^{1/4}u\left(\frac{1}{4}\right) - \frac{1}{2}D_{0+}^{1/4}u\left(\frac{4}{9}\right), \end{gathered} \tag{4.1}$$

where

$$f(t,x) = x^2 + \ln t. \tag{4.2}$$

We have

$$G_0(t,s) = \frac{1}{\Gamma(7/4)}\begin{cases} t^{3/4}(1-s)^{1/2}, & 0 \le t \le s \le 1, \\ t^{3/4}(1-s)^{1/2} - (t-s)^{3/4}, & 0 \le s \le t \le 1, \end{cases} \tag{4.3}$$

$$p(s) = \begin{cases} 1 - \left(\dfrac{(1/4)-s}{1-s}\right)^{1/2} - \dfrac{1}{2}\left(\dfrac{(4/9)-s}{1-s}\right)^{1/2}, & 0 \le s \le \dfrac{1}{4}, \\ 1 - \dfrac{1}{2}\left(\dfrac{(4/9)-s}{1-s}\right)^{1/2}, & \dfrac{1}{4} < s \le \dfrac{4}{9}, \\ 1, & \dfrac{4}{9} < s \le 1. \end{cases} \tag{4.4}$$

By direct calculations, we have $p(0) = (1/6)$ and $q(s) \ge 0$, which implies that (H_1) holds.

Let $r(t) = -\ln t$, $z(t) = t^{-1/2}$, $g(x) = x^2$. It is easy to see that (H_2) and (H_3) hold. Moreover,

$$\liminf_{x \to +\infty} \min_{t \in [(1/4),(3/4)]} \frac{f(t,x)}{x} = +\infty. \tag{4.5}$$

Therefore, the assumptions of Theorem 3.1 are satisfied. Thus, Theorem 3.1 ensures that there exists $\lambda^* > 0$ such that the BVP (4.1) has at least one positive solution for any $\lambda \in (0,\lambda^*)$.

Remark 4.2. Noticing that λx^2 does not satisfy (A_1), therefore, the work in the present paper improves and generalizes the main results of [23].

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Research Article

Coupled Coincidence Point and Coupled Fixed Point Theorems via Generalized Meir-Keeler Type Contractions

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We prove coupled coincidence point and coupled fixed point results of $F : X \times X \to X$ and $g : X \to X$ involving Meir-Keeler type contractions on the class of partially ordered metric spaces. Our results generalize some recent results in the literature. Also, we give some illustrative examples and application.

1. Introduction and Preliminaries

Fixed point theory has wide applications in many areas. In economics it has applications in the study of market stability, in dynamic systems it is used to deterministic timed systems on feedback semantics, and in the theory of differential and integral equations to demonstrate the existence and uniqueness of solutions; see, for example, [1–5]. On the other hand, fixed point theory, in particular fixed point iteration, has also numerous applications in engineering. For example, use of the fixed point iteration in image retrieval provides much better accuracy [6]. Fixed point algorithms proved to be very successful in practical optimization of the contrast functions in independent component analysis in neural-network research, as well as in statistics and signal processing [7]. These algorithms optimize the contrast functions very fast and reliably. Relaxation in linear systems and relaxation and stability in neural networks are also analyzed by means of fixed point iteration [8].

The problem of existence and uniqueness of fixed points in partially ordered sets has been studied thoroughly because of its interesting nature. The first result in this direction was given by Turinici [9], where he extended the Banach contraction principle in partially ordered sets. Ran and Reurings [10] presented some applications of Turinici's theorem to matrix equations. The result of Turinici was further extended and refined in [11–25]. In particular, Gnana Bhaskar and Lakshmikantham in [12] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. They also discussed an application of their result by investigating the existence and uniqueness of solution of the periodic boundary value problem:

$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in [0, T], \\ u(0) &= u(T), \end{aligned} \tag{1.1}$$

where the function f satisfies certain conditions. Following this trend, Harjani et al. [4] studied the existence and uniqueness of solutions of a nonlinear integral equation as an application of coupled fixed points. Very recently, motivated by [5], Jleli and Samet [13] discussed the existence and uniqueness of a positive solution for the singular nonlinear fractional differential equation boundary value problem:

$$\begin{aligned} D^{\alpha}_{0^+} u(t) &= f(t, u(t), u(t)), \quad 0 < t < 1, \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \tag{1.2}$$

where $3 < \alpha \leq 4$ is a real number, $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative and $f : (0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, $\lim_{t \to 0^+} f(t, \cdot, \cdot) = +\infty$ (f is singular at $t = 0$) for all $t \in (0, 1]$, $f(t, \cdot, \cdot)$ is nondecreasing with respect to first component and decreasing with respect to its second and third components.

On the other hand, Lakshmikantham and Ćirić [19] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces which extend the coupled fixed point theorem given in [12]. Recently, Samet [23] proved some coupled fixed point theorems under a generalized Meir-Keeler contractive condition.

In this paper, we introduce the definition of weak generalized g-Meir-Keeler type contractions and prove some coupled coincidence point theorems for such contractions. The theorems presented here generalize, enrich, and improve the previous results. Moreover, they have application potential in the theory of existence and uniqueness of solutions of boundary value problems.

Hereafter, we assume that $X \neq \emptyset$ and we use the notation

$$X^k = \underbrace{X \times X \times \cdots \times X}_{k\text{-many}}. \tag{1.3}$$

Let $\mathbb{R}$ be the set of real numbers.

Definition 1.1 (see [12]). Let $(X, \leq)$ be a partially ordered set and $F : X^2 \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} x_1 \leq x_2 &\Longrightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \\ y_1 \leq y_2 &\Longrightarrow F(x, y_2) \leq F(x, y_1), \quad \text{for } y_1, y_2 \in X. \end{aligned} \tag{1.4}$$

Definition 1.2 (see [12]). An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \to X$ if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{1.5}$$

The following result of Gnana Bhaskar and Lakshmikantham [12] was also proved in the context of cone metric spaces in [16].

Theorem 1.3 (see [12]). *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a given mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall u \leq x, y \leq v. \tag{1.6}$$

Assume either F is continuous, or X satisfies the following property:

(i) *if a nondecreasing sequence $\{x_n\} \in X$ converges to x, then $x_n \leq x$, for all n;*

(ii) *if a nonincreasing sequence $\{y_n\} \in X$ converges to y, then $y \leq y_n$, for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Inspired by Definition 1.1, Lakshmikantham and Ćirić [19] introduced the concept of the mixed g-monotone property.

Definition 1.4 (see [19]). Let $(X, \leq)$ be a partially ordered set. Let $F : X^2 \to X$ and let $g : X \to X$. The mapping F is said to have the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in x and is monotone g-nonincreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} g(x_1) \leq g(x_2) &\Longrightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \\ g(y_1) \leq g(y_2) &\Longrightarrow F(x, y_2) \leq F(x, y_1), \quad \text{for } y_1, y_2 \in X. \end{aligned} \tag{1.7}$$

It is clear that Definition 1.4 reduces to Definition 1.1 when g is the identity map.

Definition 1.5 (see [19]). An element $(x, y) \in X^2$ is called a coupled coincidence point of the mappings $F : X^2 \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), \qquad F(y, x) = g(y). \tag{1.8}$$

Moreover, $(x, y) \in X^2$ is called a common coupled fixed point of F and g if

$$F(x, y) = g(x) = x, \qquad F(y, x) = g(y) = y. \tag{1.9}$$

Definition 1.6 (see [19]). Let $F : X^2 \to X$ and let $g : X \to X$. The mappings F and g are said to commute if

$$g(F(x, y)) = F(g(x), g(y)), \quad \forall x, y \in X. \tag{1.10}$$

In 2009, Lakshmikantham and Ćirić [19] also proved a common coupled fixed point on partially ordered complete metric spaces.

Theorem 1.7 (see [19]). *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ and let $g : X \to X$ such that F has the mixed g-monotone property. Suppose that there exists $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(g(x), g(u)) + d(g(y), g(v))], \tag{1.11}$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(v) \leq g(y)$. Suppose $F(X^2) \subseteq g(X)$, g is continuous and commutes with F. Also suppose that either F is continuous or X has the following property:

$$\begin{aligned} &\text{if a nondecreasing sequence } \{x_n\} \longrightarrow x, \quad \text{then } x_n \leq x, \ \forall n, \\ &\text{if a nonincreasing sequence } \{y_n\} \longrightarrow y, \quad \text{then } y \leq y_n, \ \forall n. \end{aligned} \tag{1.12}$$

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$; that is, F and g have a coupled coincidence point.

In 2010, Samet [23] introduced the mixed strict monotone property.

Definition 1.8 (see [23]). Let $(X, \leq)$ be a partially ordered set and let $F : X^2 \to X$. F is said to have mixed strict monotone property if $F(x, y)$ is monotone increasing in x and is monotone decreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} x_1 < x_2 &\Longrightarrow F(x_1, y) < F(x_2, y), \quad \text{for } x_1, x_2 \in X, \\ y_1 < y_2 &\Longrightarrow F(x, y_2) < F(x, y_1), \quad \text{for } y_1, y_2 \in X. \end{aligned} \tag{1.13}$$

Also, Samet [23] defined generalized Meir-Keeler contractions as follows.

Definition 1.9 (see [23]). Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X. Let $F : X \times X \to X$. The mapping F is said to be a generalized Meir-Keeler type contraction if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{1}{2}[d(x, u) + d(y, v)] < \varepsilon + \delta(\varepsilon) \Longrightarrow d(F(x, y), F(u, v)) < \varepsilon, \tag{1.14}$$

for all $x, y, u, v \in X$ with $x \leq u$, $y \geq v$.

The existence and uniqueness of common coupled coincidence points via generalized Meir-Keeler type contractions was investigated by Samet [23].

Theorem 1.10 (see [23]). *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ be a map satisfying the following conditions:*

(i) *F has the mixed strict monotone property,*

(ii) *F is a generalized Meir-Keeler type contraction,*

(iii) *there exist $x_0, y_0 \in X$ such that*

$$x_0 < F(x_0, y_0), \qquad y_0 \geq F(y_0, x_0). \tag{1.15}$$

Assume either F is continuous or X satisfies the following property:

(i) *if a nondecreasing sequence $\{x_n\} \in X$ converges to x, then $x_n \leq x$, for all n,*

(ii) *if a nonincreasing sequence $\{y_n\} \in X$ converges to y, then $y \leq y_n$, for all n.*

Then F has a coupled fixed point in X^2; that is, there exist $x, y \in X$ such that

$$F(x, y) = x, \qquad F(y, x) = y. \tag{1.16}$$

Very recently, Gordji et al. [26] replaced the *mixed g-monotone property* by the *mixed strict g-monotone property.*

Definition 1.11 (see [26]). Let $(X, \leq)$ be a partially ordered set. Let $F : X^2 \to X$ and let $g : X \to X$. F is said to have the mixed strict g-monotone property if $F(x, y)$ is monotone g-increasing in x and is monotone g-decreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} g(x_1) < g(x_2) &\Longrightarrow F(x_1, y) < F(x_2, y), \quad \text{for } x_1, x_2 \in X, \\ g(y_1) < g(y_2) &\Longrightarrow F(x, y_2) < F(x, y_1), \quad \text{for } y_1, y_2 \in X. \end{aligned} \tag{1.17}$$

If we replace g with identity map in (1.17), we get Definition 1.8 of the mixed strict monotone property of F.

Gordji et al. [26] gave also the following definition.

Definition 1.12 (see [26]). Let $(X, d, \leq)$ be a partially ordered metric space and $F : X \times X \to X$, $g : X \to X$. The operator F is said to be a generalized g-Meir-Keeler type contraction if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right] < \varepsilon + \delta(\varepsilon) \Longrightarrow d(F(x, y), F(u, v)) < \varepsilon, \tag{1.18}$$

for all $x, y, u, v \in X$ with $g(x) \leq g(u)$, $g(y) \geq g(v)$.

Note that if we replace g with the identity in (1.18), we get Definition 1.9 of generalized Meir-Keeler type contraction.

Gordji et al. [26] proved the following theorem.

Theorem 1.13 (see [26]). *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ and let $g : X \to X$ be mappings such that $F(X^2) \subseteq g(X)$, g is continuous and commutes with F. Suppose also that F satisfies the following conditions:*

(i) *F is continuous,*

(ii) *F has the mixed strict g-monotone property,*

(iii) *F is a generalized g-Meir-Keeler type contraction,*

(iv) *there exist $x_0, y_0 \in X$ such that*

$$g(x_0) < F(x_0, y_0), \qquad g(y_0) \geq F(y_0, x_0). \tag{1.19}$$

Then F and g have a coupled coincidence point in X^2; that is, there exist $x, y \in X$ such that

$$F(x, y) = g(x), \qquad F(y, x) = g(y). \tag{1.20}$$

In this paper, we proved coupled coincidence point results in the setting of partially ordered metric spaces. Also, the existence and uniqueness of a common coupled fixed point of $F : X \times X \to X$ and $g : X \to X$ is studied. Our results improve the results of Berinde [15] and Gordji et al. [26]. We give two examples and an application that illustrate our results.

2. Existence of Coupled Fixed Point

We start this section with the following definition which is modification of Definition 1.12.

Definition 2.1. Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a metric space. Let $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to be a weak generalized g-Meir-Keeler type contraction if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned} &\varepsilon \leq \frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right] < \varepsilon + \delta(\varepsilon) \\ &\quad \Longrightarrow \frac{1}{2}\left[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))\right] < \varepsilon, \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Remark 2.2. If we replace g with the identity in (2.1), we get the definition of a weak Meir-Keeler type contraction; that is, for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{1}{2}\left[d(x, u) + d(y, v)\right] < \varepsilon + \delta(\varepsilon) \Longrightarrow \frac{1}{2}\left[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))\right] < \varepsilon, \tag{2.2}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$.

Note that (2.2) corresponds to a Meir-Keeler contraction type studied very recently by Berinde [15].

The following fact can be derived easily from Definition 2.1.

Lemma 2.3. *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a metric space. Let $F : X \times X \to X$ and $g : X \to X$. If F is a weak generalized g-Meir-Keeler type contraction, then we have*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) < [d(g(x), g(u)) + d(g(y), g(v))], \tag{2.3}$$

for all $x, y, u, v \in X$ with $g(x) < g(u), g(y) \geq g(v)$ or $g(x) \leq (u), g(y) > g(v)$.

Proof. Without loss of generality, suppose that $g(x) < g(u),\ g(y) \geq g(v)$ where $x, y, u, v \in X$. It is clear that $d(g(x), g(u)) + d(g(y), g(v)) > 0$. Set $\varepsilon = (1/2)[d(g(x), g(u)) + d(g(y), g(v))] > 0$. Since F is a weak generalized g-Meir-Keeler type contraction, then for this ε, there exits $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{aligned} &\varepsilon \leq \frac{1}{2}[d(g(x_0), g(u_0)) + d(g(y_0), g(v_0))] < \varepsilon + \delta \\ &\Longrightarrow \frac{1}{2}[d(F(x_0, y_0), F(u_0, v_0)) + d(F(y_0, x_0), F(v_0, u_0))] < \varepsilon, \end{aligned} \tag{2.4}$$

for all $x_0, y_0, u_0, v_0 \in X$ with $g(x_0) < g(u_0),\ g(y_0) \geq g(v_0)$. The result follows by choosing $x = x_0,\ y = y_0,\ u = u_0$ and $z = z_0$, that is:

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) < d(g(x), g(u)) + d(g(y), g(v)). \tag{2.5}$$

□

Next, we state an existence theorem of a coupled coincidence point for $F : X^2 \to X$ and $g : X \to X$.

Theorem 2.4. *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ and $g : X \to X$ be mappings such that $F(X^2) \subseteq g(X)$. Moreover, assume that g is continuous and commutes with F. Suppose also that the following conditions hold:*

(i) *F is continuous,*

(ii) *F has the mixed strict g-monotone property,*

(iii) *F is a weak generalized g-Meir-Keeler type contraction,*

(iv) *there exist $x_0, y_0 \in X$ such that*

$$g(x_0) < F(x_0, y_0), \qquad g(y_0) \geq F(y_0, x_0). \tag{2.6}$$

Then F and g have a coupled coincidence point; that is, there exist $x, y \in X$ such that

$$F(x, y) = g(x), \qquad F(y, x) = g(y). \tag{2.7}$$

Proof. Let $(x_0, y_0) \in X^2$ be a point satisfying (iv); that is, $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. We define the sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Because of the assumption $F(X^2) \subseteq g(X)$, we can choose $(x_1, y_1) \in X^2$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. By the same argument, we can take $(x_2, y_2) \in X^2$ in such a way that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Inductively, we define

$$g(x_{n+1}) = F(x_n, y_n), \qquad g(y_{n+1}) = F(y_n, x_n) \quad \forall n = 0, 1, 2, \ldots. \tag{2.8}$$

We claim that the the sequence $\{g(x_n)\}$ is increasing and the sequence $\{g(y_n)\}$ is decreasing, that is:

$$\begin{aligned} &\cdots > g(x_n) > g(x_{n-1}) > \cdots > g(x_1) > g(x_0), \\ &\cdots < g(y_n) < g(y_{n-1}) < \cdots < g(y_1) \leq g(y_0). \end{aligned} \tag{2.9}$$

We will use mathematical induction to show (2.9). By assumption (iv), we have

$$g(x_0) < F(x_0, y_0) = g(x_1), \qquad g(y_0) \geq F(y_0, x_0) = g(y_1). \tag{2.10}$$

Assume that (2.9) holds for some $n \geq 1$. Regarding the mixed strict g-monotone property of F, we have

$$g(x_{n-1}) < g(x_n) \Longrightarrow \begin{cases} F(x_{n-1}, y_{n-1}) < F(x_n, y_{n-1}), \\ F(y_{n-1}, x_{n-1}) > F(y_{n-1}, x_n). \end{cases} \tag{2.11}$$

By repeating the same arguments, we observe that

$$g(y_{n-1}) > g(y_n) \Longrightarrow \begin{cases} F(x_n, y_{n-1}) < F(x_n, y_n), \\ F(y_{n-1}, x_n) > F(y_n, x_n). \end{cases} \tag{2.12}$$

Combining the previous inequalities, together with (2.8), we get

$$\begin{aligned} g(x_n) &= F(x_{n-1}, y_{n-1}) < F(x_n, y_n) = g(x_{n+1}), \\ g(y_n) &= F(y_{n-1}, x_{n-1}) > F(y_n, x_n) = g(y_{n+1}). \end{aligned} \tag{2.13}$$

We conclude that (2.9) holds for all $n \geq 1$. Set

$$\Delta_n = d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})). \tag{2.14}$$

Making use of Lemma 2.3 and (2.8), we obtain

$$\begin{aligned} & d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \\ & \quad = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ & \quad < d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)). \end{aligned} \tag{2.15}$$

Thus, we have $\Delta_n < \Delta_{n-1}$. Hence, the sequence $\{\Delta_n\}$ is monotone decreasing and clearly bounded below by 0. Therefore, $\lim_{n\to\infty}\Delta_n = L$ for some $L \geq 0$.

We show that $L = 0$. Suppose the contrary; that is, $L \neq 0$. Then, for some positive integer k, we have for all $n \geq k$

$$\varepsilon \leq \frac{\Delta_n}{2} = \frac{1}{2}\left[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))\right] < \varepsilon + \delta(\varepsilon), \tag{2.16}$$

where we choose $\varepsilon = L/2$. In particular, for $n = k$

$$\varepsilon \leq \frac{\Delta_k}{2} = \frac{1}{2}\left[d(g(x_k), g(x_{k+1})) + d(g(y_k), g(y_{k+1}))\right] < \varepsilon + \delta(\varepsilon). \tag{2.17}$$

Regarding the assumption (iii) and (2.17), we have

$$\frac{1}{2}\left[d(F(x_k, y_k), F(x_{k+1}, y_{k+1})) + d(F(y_k, x_k), F(y_{k+1}, x_{k+1}))\right] < \varepsilon, \tag{2.18}$$

which by (2.8) is equivalent to

$$\frac{1}{2}\left[d(g(x_{k+1}), g(x_{k+2})) + d(g(y_{k+1}), g(y_{k+2}))\right] < \varepsilon. \tag{2.19}$$

Hence, we obtain

$$\frac{\Delta_{k+1}}{2} < \varepsilon, \tag{2.20}$$

which contradicts (2.16) for $n = k + 1$. Thus, we deduce that $L = 0$, that is:

$$\lim_{n\to\infty}\Delta_n = \lim_{n\to\infty}\left[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))\right] = 0. \tag{2.21}$$

This implies that

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0 = \lim_{n\to\infty} d(g(y_n), g(y_{n+1})). \tag{2.22}$$

We claim that the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Take an arbitrary $\varepsilon > 0$. It follows from (2.21) that there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2}\left[d\left(g(x_k), g(x_{k+1})\right) + d\left(g(y_k), g(y_{k+1})\right)\right] < \delta(\varepsilon). \tag{2.23}$$

Without loss of the generality, assume that $\delta(\varepsilon) \leq \varepsilon$ and define the following set:

$$\Pi := \left\{(x, y) \in X^2 : d\left(x, g(x_k)\right) + d\left(y, g(y_k)\right) < 2(\varepsilon + \delta(\varepsilon)),\ x > g(x_k),\ y \leq g(y_k)\right\}. \tag{2.24}$$

Take $\wedge = (g(X), g(X)) \cap \Pi$. We claim that

$$(F(p, q), F(q, p)) \in \wedge \quad \forall (x, y) = (g(p), g(q)) \in \wedge \text{ where } p, q \in X. \tag{2.25}$$

Take $(x, y) = (g(p), g(q)) \in \Pi$. Then, by (2.23) and the triangle inequality we have

$$\begin{aligned}
&\frac{1}{2}\left[d\left(g(x_k), F(p, q)\right) + d\left(g(y_k), F(q, p)\right)\right] \\
&\quad\leq \frac{1}{2}\left[d\left(g(x_k), g(x_{k+1})\right) + d\left(g(x_{k+1}), F(p, q)\right)\right] \\
&\quad\quad+ \frac{1}{2}\left[d\left(g(y_k), g(y_{k+1})\right) + d\left(g(y_{k+1}), F(q, p)\right)\right] \\
&\quad= \frac{1}{2}\left[d\left(g(x_k), g(x_{k+1})\right) + d\left(g(y_k), g(y_{k+1})\right)\right] + \frac{1}{2} d\left(F(p, q), F(x_k, y_k)\right) \\
&\quad\quad+ \frac{1}{2} d\left(F(y_k, x_k), F(q, p)\right) \\
&\quad< \delta(\varepsilon) + \frac{1}{2} d\left(F(p, q), F(x_k, y_k)\right) + \frac{1}{2} d\left(F(y_k, x_k), F(q, p)\right).
\end{aligned} \tag{2.26}$$

We distinguish two cases.

First Case. $(1/2)[d(x, g(x_k)) + d(y, g(y_k))] = (1/2)[d(g(p), g(x_k)) + d(g(q), g(y_k))] \leq \varepsilon$.

By Lemma 2.3 and the definition of Π, (2.26) turns into

$$\begin{aligned}
&\frac{1}{2}\left[d\left(g(x_k), F(p, q)\right) + d\left(g(y_k), F(q, p)\right)\right] \\
&\quad< \delta(\varepsilon) + \frac{1}{2} d\left(F(p, q), F(x_k, y_k)\right) + \frac{1}{2} d\left(F(y_k, x_k), F(q, p)\right) \\
&\quad< \delta(\varepsilon) + \frac{1}{2}\left[d\left(g(p), g(x_k)\right) + d\left(g(q), g(y_k)\right)\right] \leq \delta(\varepsilon) + \varepsilon.
\end{aligned} \tag{2.27}$$

Second Case. $\varepsilon < (1/2)[d(x, g(x_k)) + d(y, g(y_k))] = (1/2)[d(g(p), g(x_k)) + d(g(q), g(y_k))] < \varepsilon + \delta(\varepsilon)$.

In this case, we have

$$\varepsilon < \frac{1}{2}\left[d\left(g(p), g(x_k)\right) + d\left(g(q), g(y_k)\right)\right] < \varepsilon + \delta(\varepsilon). \tag{2.28}$$

Since $x = g(p) > g(x_k)$ and $y = g(q) \leq g(y_k)$, by (ii), we get

$$\frac{1}{2}\left[d\left(F(p,q), F(x_k, y_k)\right) + d\left(F(y_k, x_k), F(q,p)\right)\right] < \varepsilon. \tag{2.29}$$

Thus, combining (2.26) and (2.29), we obtain

$$\frac{1}{2}\left[d\left(g(x_k), F(p,q)\right) + d\left(g(y_k), F(q,p)\right)\right] < \varepsilon + \delta(\varepsilon). \tag{2.30}$$

On the other hand, using (i), it is obvious that

$$F(p,q) > g(x_k), \qquad F(q,p) \leq g(y_k). \tag{2.31}$$

We conclude that $(F(p,q), F(q,p)) \in \Pi$. Since $F(X^2) \subset g(X)$, so

$$\left(F(p,q), F(q,p)\right) \in \wedge, \tag{2.32}$$

that is, (2.25) holds. By (2.23), we have $(g(x_{k+1}), g(y_{k+1})) \in \wedge$. This implies with (2.25) that

$$\begin{aligned}\left(g(x_{k+1}), g(y_{k+1})\right) \in \wedge \implies & \left(F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1})\right) = \left(g(x_{k+2}), g(y_{k+2})\right) \in \wedge \\ \implies & \left(F(x_{k+2}, y_{k+2}), F(y_{k+2}, x_{k+2})\right) = \left(g(x_{k+3}), g(y_{k+3})\right) \in \wedge \\ \implies & \cdots \implies \left(g(x_n), g(y_n)\right) \in \wedge \implies \cdots .\end{aligned} \tag{2.33}$$

Then, for all $n > k$, we have $(g(x_n), g(y_n)) \in \wedge$. This implies that for all $n, m > k$, we have

$$\begin{aligned}d\left(g(x_n), d(x_m)\right) + d\left(g(y_n), g(y_m)\right) \leq\ & d\left(g(x_n), g(x_k)\right) + d\left(g(y_n), g(y_k)\right) \\ & + d\left(g(x_k), g(x_m)\right) + d\left(g(y_k), g(y_m)\right) \\ <\ & 4(\varepsilon + \delta(\varepsilon)) \leq 8\varepsilon.\end{aligned} \tag{2.34}$$

Thus, the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy in (X, d).

Since (X, d) is complete, so there exist $x, y \in X$ such that

$$\begin{aligned}\lim_{n \to \infty} d\left(x, g(x_n)\right) = 0, \\ \lim_{n \to \infty} d\left(y, g(y_n)\right) = 0.\end{aligned} \tag{2.35}$$

Finally, by continuity of F and g, the commutativity of F and g, and using exactly the same argument of Lakshmikantham and Ćirić [19], we get that $F(x,y) = g(x)$ and $F(y,x) = g(y)$, which completes the proof. □

Remark 2.5. Theorem 2.4 holds if we replace (iv) by the following: there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0), \qquad g(y_0) > F(y_0, x_0). \tag{2.36}$$

Theorem 2.6. *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a metric space. Let $F : X^2 \to X$ and let $g : X \to X$ be mappings such that $F(X^2) \subseteq g(X)$. Assume that X satisfies the following property:*

(a) *if $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1, 2, \ldots$ and $x_n \to x$, then $x_n < x$ for each $n = 1, 2, \ldots$,*

(b) *if $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1, 2, \ldots$ and $y_n \to y$, then $y_n > y$ for each $n = 1, 2, \ldots$.*

Suppose the following conditions hold:

(i) *F has the mixed strict g-monotone property,*

(ii) *F is a weak generalized g-Meir-Keeler type contraction,*

(iii) *$g(X)$ is a complete subspace of (X, d),*

(iv) *there exist $x_0, y_0 \in X$ such that*

$$g(x_0) < F(x_0, y_0), \qquad g(y_0) \geq F(y_0, x_0). \tag{2.37}$$

Then F and g have a coupled coincidence point; that is, there exist $x, y \in X$ such that

$$F(x,y) = g(x), \qquad F(y,x) = g(y). \tag{2.38}$$

Proof. Proceeding exactly as in Theorem 2.4, we have that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the complete metric space $(g(X), d)$. Then, there exist $x, y \in X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$. Since $\{g(x_n)\}$ is increasing and $\{g(y_n)\}$ is decreasing, using the assumptions (a) and (b), we have

$$g(x_n) < g(x), \qquad g(y_n) > g(y), \tag{2.39}$$

for each $n \geq 0$. Using triangle inequality together with (2.8), we find

$$\begin{aligned} d(F(x,y), g(x)) &\leq d(F(x,y), g(x_n)) + d(g(x_n), g(x)) \\ &= d(F(x,y), F(x_{n-1}, y_{n-1})) + d(g(x_n), g(x)). \end{aligned} \tag{2.40}$$

Similarly,

$$\begin{aligned} d(F(y,x),g(y)) &\le d(F(y,x),g(y_n)) + d(g(y_n),g(y)) \\ &= d(F(y,x),F(y_{n-1},x_{n-1})) + d(g(y_n),g(y)). \end{aligned} \tag{2.41}$$

Taking side-by-side sum of the above mentioned inequalities and having in mind (2.39), the fact that $g(x_n) \to g(x)$, $g(y_n) \to g(y)$ and Lemma 2.3, we get

$$\begin{aligned} & d(F(x,y),g(x)) + d(F(y,x),g(y)) \\ &\quad \le d(F(x,y),F(x_{n-1},y_{n-1})) + d(F(y,x),F(y_{n-1},x_{n-1})) \\ &\qquad + d(g(x_n),g(x)) + d(g(y_n),g(y)) \\ &\quad < [d(g(x),g(x_{n-1})) + d(g(y),g(y_{n-1}))] + d(g(x_n),g(x)) + d(g(y_n),g(y)) \longrightarrow 0, \end{aligned} \tag{2.42}$$

as $n \to \infty$. Hence, we end up with $d(F(x,y),g(x)) = 0 = d(F(y,x),g(y))$, that is, $F(x,y) = g(x)$ and $F(y,x) = g(y)$, which completes the proof. □

As a particular case of Theorems 2.4 and 2.6, we state the following corollary where the function g is taken as the identity function.

Corollary 2.7. *Let $(X,\le)$ be a partially ordered set, and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $F : X^2 \to X$. Suppose that F satisfies the following conditions:*

(i) *F has the mixed strict monotone property,*

(ii) *F is a weak Meir-Keeler type contraction,*

(iii) *there exist $x_0, y_0 \in X$ such that*

$$x_0 < F(x_0,y_0), \qquad y_0 \ge F(y_0,x_0). \tag{2.43}$$

Assume either F is continuous or X satisfies the following property:

(a) *if $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1,2,\ldots$ and $x_n \to x$, then $x_n < x$ for each $n = 1,2,\ldots$,*

(b) *if $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1,2,\ldots$ and $y_n \to y$, then $y_n > y$ for each $n = 1,2,\ldots$.*

Then F has a coupled fixed point; that is, there exist $x, y \in X$ such that

$$F(x,y) = x, \qquad F(y,x) = y. \tag{2.44}$$

We give the following examples.

Example 2.8. Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$. Set $F : X^2 \to X$ and let $g : X \to X$ be defined as $F(x,y) = (3x^3 - 7y^3)/12$ and $g(x) = x^3$. Then, the mapping F has the strict mixed monotone property. We claim that condition (2.1) holds, but the condition (1.18) is not satisfied.

Note that in order to guarantee (1.18), we must have

$$\varepsilon \leq \frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right] < \varepsilon + \delta(\varepsilon) \Longrightarrow d(F(x,y), F(u,v)) < \varepsilon, \tag{2.45}$$

for $x, y, u, v \in X$ with $g(x) \leq g(u)$, $g(y) \geq g(v)$. This means that

$$\varepsilon \leq \frac{1}{2}\left(\left|x^3 - u^3\right| + \left|y^3 - v^3\right|\right) < \varepsilon + \delta(\varepsilon) \Longrightarrow \left|\frac{3x^3 - 7y^3}{12} - \frac{3u^3 - 7v^3}{12}\right| < \varepsilon. \tag{2.46}$$

Choosing $x = u$ for simplicity (so $g(x) = g(u)$), we get

$$\varepsilon \leq \frac{1}{2}\left(\left|y^3 - v^3\right|\right) < \varepsilon + \delta(\varepsilon), \quad g(y) \geq g(v). \tag{2.47}$$

Hence for $g(y) > g(v)$, (2.46) implies that

$$\left|\frac{3x^3 - 7y^3}{12} - \frac{3u^3 - 7v^3}{12}\right| = \left|\frac{7v^3 - 7y^3}{12}\right| = \frac{7}{12}\left|y^3 - v^3\right| < \varepsilon. \tag{2.48}$$

Combining (2.47) and(2.48), we get that

$$2\varepsilon \leq \left|y^3 - v^3\right| < \frac{12}{7}\varepsilon < 2\varepsilon, \tag{2.49}$$

which is a contradiction.

On the other hand, F and g satisfy (2.1). Indeed, if we take the sum of

$$\begin{aligned} \left|\frac{3x^3 - 7y^3}{12} - \frac{3u^3 - 7v^3}{12}\right| &\leq \frac{3}{12}\left|x^3 - u^3\right| + \frac{7}{12}\left|v^3 - y^3\right|, \quad g(x) \leq g(u),\ g(y) \geq g(v), \\ \left|\frac{3y^3 - 7x^3}{12} - \frac{3v^3 - 7y^3}{12}\right| &\leq \frac{3}{12}\left|v^3 - y^3\right| + \frac{7}{12}\left|x^3 - u^3\right|, \quad g(x) \leq g(u),\ g(y) \geq g(v), \end{aligned} \tag{2.50}$$

and divide by 2, we obtain for $g(x) \le g(u)$ and let $g(y) \ge g(v)$

$$\begin{aligned}
&\frac{1}{2}\left[d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))\right] \\
&\quad = \frac{1}{2}\left(\left|\frac{3x^3-7y^3}{12} - \frac{3u^3-7v^3}{12}\right| + \left|\frac{7x^3-3y^3}{12} - \frac{7u^3-3v^3}{12}\right|\right) \\
&\quad \le \frac{5}{12}\left(\left|x^3-u^3\right| + \left|y^3-v^3\right|\right) \\
&\quad = \frac{5}{6}\left\{\frac{1}{2}\left[d(g(x),g(u)) + d(g(y),g(v))\right]\right\}.
\end{aligned} \tag{2.51}$$

Choosing $\delta(\varepsilon) < \varepsilon/5$, we get the desired result. Note also that $x_0 = -1$ and $y_0 = 1$ satisfy (2.6).

So Theorem 2.4 can be applied to F ad g in this example to conclude that F and g have a coupled coincidence point $(0,0)$, while Theorem 1.13 cannot be applied since (1.18) is not satisfied.

Example 2.9. Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$. Set $F : X^2 \to X$ and let $g : X \to X$ be defined as $F(x,y) = (x-2y)/4$ and $g(x) = 2x$. Then, the mapping F has the strict mixed monotone property. We claim that condition (2.1) holds for $g(x) \le g(u)$ and $g(y) \ge g(v)$. Indeed,

$$\begin{aligned}
&\frac{1}{2}\left[d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))\right] \\
&\quad = \frac{1}{2}\left(\left|\frac{x-2y}{4} - \frac{u-2v}{4}\right| + \left|\frac{y-2x}{4} - \frac{v-2u}{4}\right|\right) \\
&\quad = \frac{3}{8}((u-x) + (y-v)) \\
&\quad = \frac{3}{8}\left\{\frac{1}{2}\left[d(g(x),g(u)) + d(g(y),g(v))\right]\right\}.
\end{aligned} \tag{2.52}$$

Choosing $\delta(\varepsilon) < 5\varepsilon/3$, we get the desired result. Note also that $x_0 = 0$ and $y_0 = 1$ satisfy (2.6).

All hypotheses of Theorem 2.4 are satisfied. Here, F and g have a coupled coincidence point $(0,0)$.

3. Uniqueness of Common Coupled Fixed Point

In this section we will prove the uniqueness of a common coupled fixed point. We endow the product space X^2 with the following partial order:

$$(u,v) \le (x,y) \Longleftrightarrow u \le x, y \ge v, \quad \forall (x,y),\ (u,v) \in X^2. \tag{3.1}$$

Note that a pair $(x,y) \in X^2$ is comparable with $(u,v) \in X^2$ if either $(x,y) \le (u,v)$ or $(u,v) \le (x,y)$. We next state the conditions for the existence and uniqueness of a common coupled fixed point of maps F and g.

Theorem 3.1. *In addition to the hypotheses of Theorem 2.4 (resp., Theorem 2.6), assume that for all* (x, y), $(x^*, y^*) \in X^2$, *there exists* $(a, b) \in X^2$ *such that* $(F(a, b), F(b, a))$ *is comparable to both* $(F(x, y), F(y, x))$ *and* $(F(x^*, y^*), F(y^*, x^*))$. *Then,* F *and* g *have a unique common coupled fixed point, that is:*

$$x = g(x) = F(x, y), \qquad y = g(y) = F(y, x). \tag{3.2}$$

Proof. The set of coupled coincidence points of F and g is not empty due to Theorem 2.4 (resp., Theorem 2.6). We suppose that $(x, y), (x^*, y^*) \in X^2$ are two coupled coincidence points of F and g. We distinguish the following two cases.

First Case. $(F(x, y), F(y, x))$ is comparable to $(F(x^*, y^*), F(y^*, x^*)$ with respect to the ordering in X^2, where

$$F(x, y) = g(x), \qquad F(y, x) = g(y), \qquad F(x^*, y^*) = g(x^*), \qquad F(y^*, x^*) = g(y^*). \tag{3.3}$$

Without loss of the generality, we may assume that

$$g(x) = F(x, y) < F(x^*, y^*) = g(x^*), \qquad g(y) = F(y, x) \geq F(y^*, x^*) = g(y^*). \tag{3.4}$$

By Lemma 2.3, we have

$$\begin{aligned} d(g(x), g(x^*)) + d(g(y), g(y^*)) &= d(F(x, y), F(x^*, y^*)) + d(F(y, x), F(y^*, x^*)) \\ &< d(g(x), g(x^*)) + d(g(y), g(y^*)), \end{aligned} \tag{3.5}$$

which is a contradiction. Therefore, we have $g(x) = g(x^*)$ and $g(y) = g(y^*)$.

Second Case. Suppose that $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$ are not comparable. By assumption there exists $(a, b) \in X^2$ such that $(F(a, b), F(b, a))$ is comparable to both $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$.

Setting $a = a_0, b = b_0$, as in the proof of Theorem 2.4, we define the sequences $\{g(a_n)\}$ and $\{g(b_n)\}$ as follows:

$$g(a_{n+1}) = F(a_n, b_n), \quad g(b_{n+1}) = F(b_n, a_n) \quad \forall n = 0, 1, 2, \ldots. \tag{3.6}$$

Since $(F(x, y), F(y, x)) = (g(x), g(y))$ and $(F(a, b), F(b, a)) = (g(a_1), g(b_1))$ are comparable, we may assume without loss of generality that $g(x) < g(a_1)$ and $g(y) \geq g(b_1)$. Inductively, we observe that $g(x) < g(a_n)$ and $g(y) \geq g(b_n)$ for all $n \geq 1$. Thus, by Lemma 2.3, we get that

$$\begin{aligned} d(g(x), g(a_{n+1})) + d(g(y), g(b_{n+1})) &= d(F(x, y), F(a_n, b_n)) + d(F(y, x), F(b_n, a_n)) \\ &< d(g(x), g(a_n)) + d(g(y), g(b_n)). \end{aligned} \tag{3.7}$$

Set $\lambda_n = d(g(x), g(a_n)) + d(g(y), g(b_n))$. Hence, for each $n \geq 0$

$$\lambda_{n+1} < \lambda_n. \tag{3.8}$$

Therefore, the sequence $\{\lambda_n\}$ is decreasing and bounded below by 0. Hence, it converges to some $L \geq 0$. Assume that $L > 0$. Then, for some positive integer k, we have for all $n \geq k$

$$\varepsilon \leq \frac{\lambda_n}{2} = \frac{1}{2}\left[d(g(x), g(a_n)) + d(g(y), g(b_n))\right] < \varepsilon + \delta(\varepsilon), \tag{3.9}$$

where we choose $\varepsilon = L/2$. In particular, for $n = k$

$$\varepsilon \leq \frac{\lambda_k}{2} = \frac{1}{2}\left[d(g(x), g(a_k)) + d(g(y), g(b_k))\right] < \varepsilon + \delta(\varepsilon). \tag{3.10}$$

Having in mind (3.10) and the fact that F is a weak generalized g-Meir-Keeler contraction, we get that

$$\frac{1}{2}\left[d(F(x,y), F(a_k, b_k)) + d(F(y,x), F(b_k, a_k))\right] < \varepsilon, \tag{3.11}$$

which is equivalent to

$$\frac{1}{2}\left[d(g(x), g(a_{k+1})) + d(g(y), g(b_{k+1}))\right] < \varepsilon. \tag{3.12}$$

Hence, we obtain

$$\frac{\lambda_{k+1}}{2} < \varepsilon, \tag{3.13}$$

which contradicts (3.9) for $n = k+1$. Thus, we deduce that $L = 0$, that is:

$$\lim_{n \to \infty} d(g(x), g(a_n)) + d(g(y), g(b_n)) = 0. \tag{3.14}$$

In a similar manner, we can show that

$$\lim_{n \to \infty} d(g(x^*), g(a_n)) + d(g(y^*), g(b_n)) = 0. \tag{3.15}$$

By the triangle inequality, we have

$$\begin{aligned}
0 \leq d(g(x), g(x^*)) &\leq d(g(x), g(a_n)) + d(g(a_n), g(x^*)) \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\
0 \leq d(g(y), g(y^*)) &\leq d(g(y), g(b_n)) + d(g(b_n), g(y^*)) \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.16}$$

Combining all observations mentioned previously, we get $d(g(x^*), g(x)) = 0$ and $d(g(y^*), g(y)) = 0$. Hence, we have

$$g(x) = g(x^*), \qquad g(y) = g(y^*). \tag{3.17}$$

Last, we show that $g(x) = x$ and $g(y) = y$. Let $g(x) = u$ and $g(y) = v$. By the commutativity of F and g and the fact that $g(x) = F(x, y)$ and $F(y, x) = g(y)$, we have

$$\begin{gathered} g(u) = g(g(x)) = g(F(x,y)) = F(g(x), g(y)) = F(u,v), \\ g(v) = g(g(y)) = g(F(y,x)) = F(g(y), g(x)) = F(v,u). \end{gathered} \tag{3.18}$$

Thus, (u, v) is a coupled coincidence point of F and g. However, according to (3.17), we must have

$$g(x) = g(u), \qquad g(y) = g(v). \tag{3.19}$$

Hence, we deduce

$$u = g(u) = F(u, v), \qquad v = g(v) = F(v, u), \tag{3.20}$$

that is, the pair (u, v) is the coupled common fixed point of F and g.

We claim that (u, v) is the unique coupled common fixed point of F and g. Assume that (z, w) is another coupled common fixed point of F and g. But,

$$u = g(u) = g(z) = z, \qquad v = g(v) = g(w) = w \tag{3.21}$$

follows from (3.17). □

The particular case in which g is the identity function can be given as a corollary.

Corollary 3.2. *In addition to the hypotheses of Corollary 2.7, assume that for all $(x, y), (x^*, y^*) \in X^2$, there exists $(a, b) \in X^2$ such that $(F(a, b), F(b, a))$ is comparable to both $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then, F has a unique coupled fixed point.*

4. Application

In this section we give an application of the main theorems relevant to weak generalized g-Meir-Keeler type contractions. For this, we need the following theorem.

Theorem 4.1. *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X. Let $F : X^2 \to X$ and let $g : X \to X$ be two given mappings. Let also $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the following:*

(i) *$\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$,*

(ii) *ϕ is nondecreasing and right continuous,*

(iii) *for any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for all* $x, y, u, v \in X$ *with* $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$

$$\begin{aligned} \varepsilon \leq \phi\left(\frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right]\right) < \varepsilon + \delta(\varepsilon) \\ \Longrightarrow \phi\left(\frac{1}{2}\left[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))\right]\right) < \varepsilon. \end{aligned} \tag{4.1}$$

Then the mapping F is a weak generalized g-Meir-Keeler contraction.

Proof. By the condition (i) $\phi(\varepsilon) > 0$ for any $\varepsilon > 0$. Then according to (iii), for $\phi(\varepsilon) > 0$ there exists $\gamma > 0$ such that, for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$

$$\begin{aligned} \phi(\varepsilon) \leq \phi\left(\frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right]\right) < \phi(\varepsilon) + \gamma \\ \Longrightarrow \phi\left(\frac{1}{2}\left[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))\right]\right) < \phi(\varepsilon). \end{aligned} \tag{4.2}$$

Since ϕ is right continuous, so there exists $\delta > 0$ such that

$$\phi(\varepsilon + \delta) < \phi(\varepsilon) + \gamma. \tag{4.3}$$

Now, fix $x, y, u, v \in X$ satisfying $g(x) \leq g(u)$, $g(y) \geq g(v)$ and

$$\varepsilon \leq \frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right] < \varepsilon + \delta. \tag{4.4}$$

Since ϕ is nondecreasing, so we have

$$\phi(\varepsilon) \leq \phi\left(\frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right]\right) \leq \phi(\varepsilon + \delta) < \phi(\varepsilon) + \gamma. \tag{4.5}$$

From (4.2),

$$\phi\left(\frac{1}{2}\left[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))\right]\right) < \phi(\varepsilon). \tag{4.6}$$

Regarding the nondecreasing behavior of the function ϕ, we get

$$\frac{1}{2}\left[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))\right] < \varepsilon. \tag{4.7}$$

Consequently, F is a weak generalized g-Meir-Keeler type contraction. □

If g is the identity function, we derive the following special case of the Theorem 4.1.

Corollary 4.2. *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X. Let $F : X^2 \to X$. Let also $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the following:*

(i) $\phi(0) = 0$ *and* $\phi(t) > 0$ *for all* $t > 0$,

(ii) ϕ *is nondecreasing and right continuous,*

(iii) *for any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for all* $x, y, u, v \in X$ *with* $x \leq u$ *and* $y \geq v$,

$$\begin{aligned} &\varepsilon \leq \phi\left(\frac{1}{2}\left[d(x,u) + d(y,v)\right]\right) < \varepsilon + \delta(\varepsilon) \\ &\Longrightarrow \phi\left(\frac{1}{2}\left[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))\right]\right) < \varepsilon. \end{aligned} \tag{4.8}$$

Then, the mapping F is a weak Meir-Keeler contraction.

The subsequent results are particular cases of Theorems 2.4 and 4.1.

Corollary 4.3. *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ and let $g : X \to X$ be two given mappings such that $F(X^2) \subseteq g(X)$, g is continuous and commutes with F. Also, suppose the following conditions:*

(i) F *is continuous,*

(ii) F *has the mixed strict g-monotone property,*

(iii) *for any* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that for all* $x, y, u, v \in X$ *satisfying* $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$,

$$\begin{aligned} &\varepsilon \leq \int_0^{(1/2)[d(g(x),g(u))+d(g(y),g(v))]} \phi(s)ds < \varepsilon + \delta(\varepsilon) \\ &\Longrightarrow \int_0^{(1/2)[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \phi(s)ds < \varepsilon, \end{aligned} \tag{4.9}$$

where $\phi : [0, \infty) \to [0, \infty)$ *is locally integrable and for all* $t > 0$

$$\int_0^t \phi(s)ds > 0, \tag{4.10}$$

(iv) *there exist* $x_0, y_0 \in X$ *such that*

$$g(x_0) < F(x_0, y_0), \qquad g(y_0) \geq F(y_0, x_0). \tag{4.11}$$

Then, F and g have a coupled coincidence point; that is, there exist $x, y \in X$ such that

$$F(x, y) = g(x), \qquad F(y, x) = g(y). \tag{4.12}$$

If, in addition, for all (x, y), $(x^*, y^*) \in X^2$, there exists $(a, b) \in X^2$ such that $(F(a, b), F(b, a))$ is comparable to both $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$, then F and g have a unique common coupled fixed point.

Corollary 4.4. *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \to X$ be a mapping satisfying the following conditions:*

(i) *F is continuous,*

(ii) *F has the mixed strict monotone property,*

(iii) *for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $x, y, u, v \in X$ satisfying $x \leq u$ and $y \geq v$,*

$$\begin{aligned} \varepsilon \leq \int_0^{(1/2)[d(x,u)+d(y,v)]} \phi(s)ds < \varepsilon + \delta(\varepsilon) \\ \Longrightarrow \int_0^{(1/2)[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \phi(s)ds < \varepsilon, \end{aligned} \tag{4.13}$$

where $\phi : [0, \infty) \to [0, \infty)$ is locally integrable and for all $t > 0$

$$\textstyle\int_0^t \phi(s)ds > 0, \tag{4.14}$$

(iv) *there exist $x_0, y_0 \in X$ such that*

$$x_0 < F(x_0, y_0), \qquad y_0 \geq F(y_0, x_0). \tag{4.15}$$

Then F has a coupled fixed point; that is, there exist $x, y \in X$ such that

$$F(x, y) = x, \qquad F(y, x) = y. \tag{4.16}$$

If, in addition, for all (x, y), $(x^*, y^*) \in X^2$, there exists $(a, b) \in X^2$ such that $(F(a, b), F(b, a))$ is comparable to both $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$, then F has a unique coupled fixed point.

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Research Article

The Existence of Positive Solutions for Fractional Differential Equations with Sign Changing Nonlinearities

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We investigate the existence of at least two positive solutions to eigenvalue problems of fractional differential equations with sign changing nonlinearities in more generalized boundary conditions. Our analysis relies on the Avery-Peterson fixed point theorem in a cone. Some examples are given for the illustration of main results.

1. Introduction

The theory of fractional differential equations has become an important aspect of differential equations (see [1–8]). Boundary value problems of fractional differential equations have been investigated in many papers (see [9–46]). The existence of positive solutions to boundary value problems of fractional differential equations has been studied by many authors when nonlinearities are positive (see [9–24]). There are a few papers to study the existence of positive solutions of semipositone fractional differential equations. For example, using the Krasnoselskii fixed point theorem, Yuan et al. [9] discussed the existence of positive solutions for the singular positone and semipositone two-point boundary value problems

$$\begin{gathered} D_{0^+}^{\alpha} u(t) = \mu a(t) f(t, u(t)), \\ u(0) = u'(0) = u(1) = u'(1) = 0, \quad 3 < \alpha \leq 4, \end{gathered} \tag{1.1}$$

where $\mu > 0$, a and f are continuous. In [10], Wang et al. studied the existence of positive solutions for the singular semipositone two-point boundary value problems

$$D_{0^+}^{\alpha}u(t) + \lambda f(t, u(t)) = 0, \tag{1.2}$$

$$u(0) = u'(0) = u(1) = 0, \quad 2 < \alpha \le 3, \tag{1.3}$$

where $\lambda > 0$, f is continuous.

In [11], using Krasnoselskii fixed point theorem, Goodrich discussed the existence of at least one positive solutions for the system of fractional boundary value problems

$$\begin{gathered} -D_{0^+}^{\nu_1}y_1(t) = \lambda_1 a_1(t) f(y_1(t), y_2(t)), \qquad -D_{0^+}^{\nu_2}y_2(t) = \lambda_2 a_2(t) g(y_1(t), y_2(t)), \\ y_1^{(i)}(0) = y_2^{(i)}(0) = 0, \quad 0 \le i \le n-2, \qquad D_{0^+}^{\alpha}y_1(t)|_{t=1} = \phi_1(y), \qquad D_{0^+}^{\alpha}y_2(t)|_{t=1} = \phi_2(y), \end{gathered} \tag{1.4}$$

where a_1, a_2, f, and g are nonnegative for $t \in [0,1]$.

Motivated by the excellent results mentioned above, in this paper, we investigate the existence of at least two positive solutions for the problem

$$\begin{gathered} D_{0^+}^{\alpha}y(t) + \lambda f(t, y(t)) = 0, \quad t \in [0,1], \\ y^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ D_{0^+}^{\beta}y(t)|_{t=1} = h(y), \end{gathered} \tag{1.5}$$

where $\lambda > 0$, $\alpha \in (n-1, n]$, $n \ge 3$, $1 \le \beta \le n-2 < \alpha - 1$, $f \in C([0,1] \times \mathbb{R}^+, [-M, \infty))$, $M > 0$, $h \in C\ (C(\mathbb{R}^+), \mathbb{R}^+)$, $\mathbb{R}^+ = [0, \infty)$. The main tool is the Avery-Peterson theorem. To the best of our knowledge, this is the first paper dealing with eigenvalue problems of fractional differential equations with sign changing nonlinearities involving more general boundary conditions. Our results improve some of the earlier work presented in [10, 17, 46].

2. Preliminaries

For the convenience of the readers, we present here some necessary definitions and lemmas from fractional calculus theory. For more details see [1, 2].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractiontal derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s) ds, \tag{2.2}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$, where $n = [\alpha] + 1$.

Lemma 2.3. *Assume $f \in C[0,1]$, $q \geq p \geq 0$, then*

$$D_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{q-p} f(t). \tag{2.3}$$

Lemma 2.4. *Assume $\alpha > 0$, then*

(1) *If $\lambda > -1$, $\lambda \neq \alpha - i$, $i = 1,2,\ldots,[\alpha]+1$, $t > 0$, then*

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}. \tag{2.4}$$

(2) $D_{0^+}^{\alpha} t^{\alpha-i} = 0$, *$i = 1,2,\ldots,n$.*

(3) $D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$, *for a.e. $t \in [0,1]$, where $u \in L^1[0,1]$.*

(4) $D_{0^+}^{\alpha} u(t) = 0$ *if and only if*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, where n is the least integer greater than or equal to α.

Lemma 2.5 (see[11]). *Given $g \in C[0,1]$, y is a solution of the problem*

$$\begin{gathered} D_{0^+}^{\alpha} y(t) + g(t) = 0, \quad t \in [0,1], \\ y^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ D_{0^+}^{\beta} y(t)|_{t=1} = h(y), \end{gathered} \tag{2.6}$$

if and only if it satisfies

$$y(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} h(y) + \int_0^1 G(t,s) g(s) ds, \tag{2.7}$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.8}$$

Lemma 2.6 (see[11]). *$G(t,s)$ is continuous on $[0,1] \times [0,1]$ and*

$$0 \le G(t,s) \le G(1,s), \quad t,s \in [0,1]. \tag{2.9}$$

Lemma 2.7. *$G(t,s) \ge t^{\alpha-1}G(1,s)$, $t,s \in [0,1]$.*

Proof. For $s \le t$,

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= t^{\alpha-1}\frac{(1-s)^{\alpha-\beta-1} - (1-s/t)^{\alpha-1}}{\Gamma(\alpha)} \\ &\ge t^{\alpha-1}\frac{(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}}{\Gamma(\alpha)} = t^{\alpha-1}G(1,s). \end{aligned} \tag{2.10}$$

For $s > t$,

$$G(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \ge t^{\alpha-1}\frac{(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} = t^{\alpha-1}G(1,s). \tag{2.11}$$

□

By simple calculation, we can get

$$\int_0^1 G(1,s)ds = \frac{\beta}{(\alpha-\beta)\Gamma(\alpha+1)}, \qquad \int_{1/2}^1 G(1,s)ds = \frac{2^\beta\alpha - \alpha + \beta}{2^\alpha(\alpha-\beta)\Gamma(\alpha+1)}. \tag{2.12}$$

By Lemma 2.5, we can easily get the following lemma.

Lemma 2.8. *The boundary value problem*

$$\begin{aligned} &D_{0^+}^\alpha u(t) + 1 = 0, \quad 0 < t < 1, \\ &u^{(i)}(0) = D_{0^+}^\beta u(t)|_{t=1} = 0, \quad 0 \le i \le n-2, \end{aligned} \tag{2.13}$$

has a unique solution

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\alpha-\beta} - \frac{t}{\alpha}\right). \tag{2.14}$$

Obviously, u satisfies

$$\frac{\beta t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)} \le u(t) \le \frac{t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}, \quad t \in [0,1]. \tag{2.15}$$

Lemma 2.9. *$\widetilde{y} \ge \lambda Mu$ is a solution of the following problem:*

$$\begin{gathered} D_{0^+}^{\alpha}\widetilde{y} + \lambda\left[f\left(t, \widetilde{y} - \lambda Mu\right) + M\right] = 0, \quad t \in [0,1], \\ \widetilde{y}^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ D_{0^+}^{\beta}\widetilde{y}(t)|_{t=1} = h(\widetilde{y} - \lambda Mu), \end{gathered} \tag{2.16}$$

if and only if $y = \widetilde{y} - \lambda Mu$ is a positive solution of (1.5).

Proof. In fact, if y is a positive solution of the problem (1.5), by Lemma 2.8, we get that y satisfies

$$\begin{gathered} D_{0^+}^{\alpha}(y + \lambda Mu) + \lambda\left[f(t, y) + M\right] = 0, \quad t \in [0,1], \\ (y + \lambda Mu)^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ D_{0^+}^{\beta}(y + \lambda Mu)|_{t=1} = h(y). \end{gathered} \tag{2.17}$$

Take $\widetilde{y} = y + \lambda Mu$. Then $\widetilde{y}$ satisfies (2.16) and $\widetilde{y} \ge \lambda Mu$.

On the other hand, if $\widetilde{y}$ is a solution of (2.16) and $\widetilde{y} \ge \lambda Mu$. Take $y = \widetilde{y} - \lambda Mu$. By Lemma 2.8, we can easily get that y satisfies (1.5). Clearly, $y \ge 0$. □

Define functions $\widetilde{h}$, $\widetilde{f}$ and an operator $T : C[0,1] \to C[0,1]$ by

$$\begin{gathered} \widetilde{h}(y) = h(\max\{y - \lambda Mu, 0\}), \qquad \widetilde{f}(t, y) = f(t, \max\{y - \lambda Mu, 0\}) + M. \\ Ty(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1}\widetilde{h}(y) + \lambda\int_0^1 G(t,s)\widetilde{f}(s, y(s))ds. \end{gathered} \tag{2.18}$$

Obviously, $y \ge \lambda Mu$ is a fixed point of the operator T if and only if $y - \lambda Mu$ is a positive solution of the problem (1.5).

Take $X = C[0,1]$ with norm $\|x\| = \max_{t\in[0,1]}|x(t)|$. Define a cone P by

$$P = \left\{y \in C[0,1] \mid y(t) \ge t^{\alpha-1}\|y\|,\ t \in [0,1]\right\}. \tag{2.19}$$

Lemma 2.10. *$T : P \to P$ is a completely continuous operator.*

Proof. Take $y \in P$. By Lemmas 2.6 and 2.7, we get

$$Ty(t) \geq t^{\alpha-1}\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\widetilde{h}(y) + \lambda\int_0^1 G(1,s)\widetilde{f}(s,y(s))ds\right] \geq t^{\alpha-1}\|Ty\|. \tag{2.20}$$

So, $T : P \to P$. Let $\Omega \subset P$ be bounded. It follows from the continuity of h, f that there exist constants M_1 and M_2 such that $\widetilde{h}(y) \leq M_1$ and $\widetilde{f}(t,y) \leq M_2$ for $t \in [0,1]$, $y \in \Omega$. Thus,

$$\|Ty\| \leq \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}M_1 + \lambda M_2\int_0^1 G(1,s)ds. \tag{2.21}$$

That is $T(\Omega)$ is bounded. For $y \in \Omega$, $t_1, t_2 \in [0,1]$,

$$\left|Ty(t_1) - Ty(t_2)\right| \leq M_1\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\left|t_1^{\alpha-1} - t_2^{\alpha-1}\right| + \lambda M_2\int_0^1 |G(t_1,s) - G(t_2,s)|ds. \tag{2.22}$$

By the uniform continuity of $t^{\alpha-1}$ and $G(t,s)$, we get that $T(\Omega)$ is equicontinuous. Obviously, $T : P \to P$ is continuous. By the Arzela-Ascoli theorem, we get that $T : P \to P$ is completely continuous. □

Definition 2.11. A map ϕ is said to be a nonnegative, continuous, and concave functional on a cone P of a real Banach space E if and only if $\phi : P \to \mathbb{R}^+$ is continuous and

$$\phi(tx + (1-t)y) \geq t\phi(x) + (1-t)\phi(y), \tag{2.23}$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.12. A map Φ is said to be a nonnegative, continuous, and convex functional on a cone P of a real Banach space E iff $\Phi : P \to \mathbb{R}^+$ is continuous and

$$\Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y), \tag{2.24}$$

for all $x, y \in P$ and $t \in [0,1]$.

Let φ and Θ be nonnegative, continuous, and convex functional on P, Φ a nonnegative, continuous, and concave functional on P, and Ψ a nonnegative continuous functional on P. Then, for positive numbers a, b, c, and d, we define the following sets:

$$\begin{gathered}
P(\varphi, d) = \{x \in P : \varphi(x) < d\}, \\
P(\varphi, \Phi, b, d) = \{x \in P : b \leq \Phi(x),\ \varphi(x) \leq d\}, \\
P(\varphi, \Theta, \Phi, b, c, d) = \{x \in P : b \leq \Phi(x),\ \Theta(x) \leq c,\ \varphi(x) \leq d\}, \\
R(\varphi, \Psi, a, d) = \{x \in P : a \leq \Psi(x),\ \varphi(x) \leq d\}.
\end{gathered} \tag{2.25}$$

We will use the following fixed point theorem of Avery and Peterson to study the problem (1.5).

Theorem 2.13 (see [47]). *Let P be a cone in a real Banach space E. Let φ and Θ be nonnegative, continuous, and convex functionals on P, Φ a nonnegative, continuous, and concave functional on P, and Ψ a nonnegative continuous functional on P satisfying $\Psi(kx) \leq k\Psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers M and d,*

$$\Phi(x) \leq \Psi(x), \quad \|x\| \leq M\varphi(x) \tag{2.26}$$

for all $x \in \overline{P(\varphi, d)}$. Suppose that

$$T : \overline{P(\varphi, d)} \longrightarrow \overline{P(\varphi, d)} \tag{2.27}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that the following conditions are satisfied:

(S1) $\{x \in P(\varphi, \Theta, \Phi, b, c, d) : \Phi(x) > b\} \neq \emptyset$ *and* $\Phi(Tx) > b$ *for* $x \in P(\varphi, \Theta, \Phi, b, c, d)$;

(S2) $\Phi(Tx) > b$ *for* $x \in P(\varphi, \Phi, b, d)$ *with* $\Theta(Tx) > c$;

(S3) $0 \notin R(\varphi, \Psi, a, d)$ *and* $\Psi(Tx) < a$ *for* $x \in R(\varphi, \Psi, a, d)$ *with* $\Psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, such that

$$\begin{gathered} \varphi(x_i) \leq d, \quad \text{for } i = 1, 2, 3, \\ b < \Phi(x_1), \quad a < \Psi(x_2), \quad \Phi(x_2) < b, \quad \Psi(x_3) < a. \end{gathered} \tag{2.28}$$

3. Main Results

We define a concave function $\Phi(x) = \min_{t\in[1/2,1]}|x(t)|$ and convex functions $\Psi(x) = \Theta(x) = \varphi(x) = \|x\|$.

Theorem 3.1. *Assume that there exists a constant $0 < l < \Gamma(\alpha)/\Gamma(\alpha - \beta)$, such that $h(y) \leq l\|y\|$ for $y \in P$. In addition, suppose that there exist constants k, a, b, c, d with $k > 2^{2\alpha-1}\beta\Gamma(\alpha)/(\Gamma(\alpha) - \Gamma(\alpha - \beta)l)$, $[\Gamma(\alpha) - \Gamma(\alpha - \beta)l]\alpha M/\beta\Gamma(\alpha) < a < b - [\Gamma(\alpha) - \Gamma(\alpha - \beta)l]\alpha M/\beta\Gamma(\alpha) < 2^{\alpha-1}b < c < d$, such that the following conditions hold:*

(C1) $f(t, y) \leq d - M$, *for* $(t, y) \in [0, 1] \times [0, d]$;

(C2) $f(t, y) \geq kb - M$, *for* $(t, y) \in [1/2, 1] \times [b - [\Gamma(\alpha) - \Gamma(\alpha - \beta)l]\alpha M/\beta\Gamma(\alpha), c]$;

(C3) $f(t, y) \leq a - M$, *for* $(t, y) \in [0, 1] \times [0, a]$.

Then the problem (1.5) *has at least two positive solutions for*

$$\frac{2^{2\alpha-1}(\alpha - \beta)\Gamma(\alpha + 1)}{k\left[\alpha(2^{\beta} - 1) + \beta\right]} < \lambda < \frac{\alpha(\alpha - \beta)\left[\Gamma(\alpha) - \Gamma(\alpha - \beta)l\right]}{\beta}. \tag{3.1}$$

Proof. Take $y \in \overline{P(\varphi, d)}$. By $\|\max\{y - \lambda Mu, 0\}\| \le d$, (C1), Lemma 2.6, (2.12), and (3.1), we have

$$\begin{aligned}\|Ty\| &\le \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} ld + \lambda d \int_0^1 G(1,s)ds \\ &= \left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} l + \lambda \int_0^1 G(1,s)ds\right] d \le d.\end{aligned} \tag{3.2}$$

This means that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

It is easy to see that $\{y \in P(\varphi, \Theta, \Phi, b, c, d) : \Phi(y) > b\} \neq \emptyset$. $y \in P(\varphi, \Theta, \Phi, b, c, d)$ implies $\min_{t\in[1/2,1]} y(t) \ge b$, $\|y\| \le c$. It follows from (2.15) and (3.1) that $\min_{t\in[1/2,1]}(y - \lambda Mu) \ge b - \alpha[\Gamma(\alpha) - \Gamma(\alpha-\beta)l]M/\beta\Gamma(\alpha)$. By (C2), (2.12), (3.1), and Lemma 2.7, we get

$$\Phi(Ty) = \min_{t\in[1/2,1]} Ty(t) \ge \lambda \min_{t\in[1/2,1]} \int_0^1 G(t,s)\tilde{f}(s, y(s))ds \ge \left(\frac{1}{2}\right)^{\alpha-1} \lambda kb \int_{1/2}^1 G(1,s)ds > b. \tag{3.3}$$

So, the condition (S1) of Theorem 2.13 holds.

Take $y \in P(\varphi, \Phi, b, d)$ with $\Theta(Ty) > c$. By $Ty \in P$, we get

$$\min_{t\in[1/2,1]} Ty(t) \ge \min_{t\in[1/2,1]} t^{\alpha-1}\|Ty\| \ge \frac{1}{2^{\alpha-1}}\|Ty\| > \frac{1}{2^{\alpha-1}} c > b. \tag{3.4}$$

Thus, (S2) holds.

By $a > 0$, we have $0 \notin R(\varphi, \Psi, a, d)$. Take $y \in R(\varphi, \Psi, a, d)$ with $\Psi(y) = a$. By (C3), we get

$$\begin{aligned}\Psi(Ty) = \|Ty\| &\le \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} la + \lambda a \int_0^1 G(1,s)ds \\ &= \left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} l + \lambda \int_0^1 G(1,s)ds\right] a \le a.\end{aligned} \tag{3.5}$$

By Theorem 2.13, we get that T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\varphi, d)}$ such that $\|y_i\| \le d$, $i = 1, 2, 3$, and

$$b < \Phi(y_1), \quad a < \Psi(y_2), \quad \Phi(y_2) < b, \quad \Psi(y_3) < a. \tag{3.6}$$

If $y \in P$ and $\|y\| > a$, by (2.15) and (3.1), we have

$$y(t) \ge t^{\alpha-1}\|y\| > at^{\alpha-1} > \lambda Mu(t). \tag{3.7}$$

Obviously, $\|y_1\| > b > a$ and $\|y_2\| > a$. So, $y_1 - \lambda Mu$, $y_2 - \lambda Mu$ are two positive solutions of (1.5). The proof is completed. □

4. Example

For convenience, we define the following notations:

$$[a,b] := \{x : x \in \mathbb{R}, a \le x \le b\}, \quad (a,b] := \{x : x \in \mathbb{R}, a < x \le b\}, \quad \text{for } a,b \in \mathbb{R},\ a < b. \tag{4.1}$$

Example 4.1. Consider the following boundary value problem:

$$\begin{gathered} D_{0^+}^{5/2} y(t) + \lambda f(t, y(t)) = 0, \quad t \in [0,1], \\ y(0) = y'(0) = 0, \\ y'(1) = h(y), \end{gathered} \tag{4.2}$$

where $h(y) = \int_0^1 y(s)dg(s)$, $g(s)$ is a bounded variation function on $[0,1]$ with $0 < \bigvee_0^1(g) \le 1 < 3/2$,

$$f(t,y) = \begin{cases} \sin\left(t - \dfrac{1}{2}\right)\pi - 1 - \sqrt{y}, & (t,y) \in [0,1] \times [0,6], \\ \sin\left(t - \dfrac{1}{2}\right)\pi - 1 + 601(y-6) - \sqrt{y}, & (t,y) \in [0,1] \times (6,7], \\ \sin\left(t - \dfrac{1}{2}\right)\pi + 600 - \sqrt{y}, & (t,y) \in [0,1] \times (7,900], \\ \sin\left(t - \dfrac{1}{2}\right)\pi + 570, & (t,y) \in [0,1] \times (900,+\infty). \end{cases} \tag{4.3}$$

Corresponding to the problem (1.5), we get that $\alpha = 5/2$, $\beta = 1$, $n = 3$, $h(y) \le \|y\| \bigvee_0^1(g)$. Take $l = 1$, $k = 50$, $M = a = 6$, $b = 12$, $c = 36$, $d = 620$.

By simple calculation, we can get that the conditions of Theorem 3.1 are satisfied. So, when $(9/35)\sqrt{\pi} < \lambda < (15/16)\sqrt{\pi}$, the problem (4.2) has at least two positive solutions.

Example 4.2. Consider the following boundary value problem:

$$\begin{gathered} D_{0^+}^{7/2} y(t) + \lambda f(t, y(t)) = 0, \quad t \in [0,1], \\ y(0) = y'(0) = y''(0) = 0, \\ D_{0^+}^{3/2} y(t)|_{t=1} = h(y), \end{gathered} \tag{4.4}$$

where $h(y) = \int_0^1 y(s)dg(s)$, $g(s)$ is a bounded variation function on $[0,1]$ with $0 < \bigvee_0^1(g) \le (15/8)\sqrt{\pi} - 1$,

$$f(t,y) = \begin{cases} -\dfrac{3}{4}\cos\dfrac{\pi}{2}t - \dfrac{1}{4}\sqrt{y}, & (t,y) \in [0,1] \times [0,1], \\ -\dfrac{3}{4}\cos\dfrac{\pi}{2}t + 3205(y-1) - \dfrac{1}{4}\sqrt{y}, & (t,y) \in [0,1] \times (1,1.2], \\ -\dfrac{3}{4}\cos\dfrac{\pi}{2}t + 641 - \dfrac{1}{4}\sqrt{y}, & (t,y) \in [0,1] \times (1.2,12], \\ -\dfrac{3}{4}\cos\dfrac{\pi}{2}t + 641 - \dfrac{1}{2}\sqrt{3}, & (t,y) \in [0,1] \times (12,+\infty). \end{cases} \tag{4.5}$$

Corresponding to the problem (1.5), we get that $\alpha = 7/2$, $\beta = 3/2$, $n = 4$, $h(y) \le \|y\|\bigvee_0^1(g)$. Take $l = (15/8)\sqrt{\pi} - 1$, $k = 320$, $M = a = 1$, $b = 2$, $c = 12$, $d = 642$.

Obviously, $f \in C\ ([0,1] \times \mathbb{R}^+, [-1,\infty))$, $h(y) \le l\|y\|$. By simple calculation, we can get that $k, l, a, b, c, d, \alpha, \beta, M$ satisfy $0 < l < \Gamma(\alpha)/\Gamma(\alpha-\beta)$, $k > 2^{2\alpha-1}\beta\Gamma(\alpha)/(\Gamma(\alpha) - \Gamma(\alpha-\beta)l)$, and

$$\frac{[\Gamma(\alpha) - \Gamma(\alpha-\beta)l]\alpha M}{\beta\Gamma(\alpha)} < a < b - \frac{[\Gamma(\alpha) - \Gamma(\alpha-\beta)l]\alpha M}{\beta\Gamma(\alpha)} < 2^{\alpha-1}b < c < d. \tag{4.6}$$

It is easy to see that $f(t,y) + 1 \le 642$, for $(t,y) \in [0,1] \times [0,642]$, and $f(t,y) + 1 \le 1$, for $(t,y) \in [0,1] \times [0,1]$. So, conditions (C1) and (C3) of Theorem 3.1 hold.

For $(t,y) \in [1/2,1] \times [2 - 56/45\sqrt{\pi}, 12]$, $f(t,y) + 1 \ge kb = 640$. Therefore, condition (C2) of Theorem 3.1 holds. So, for $(21/8(7\sqrt{2}-2))\sqrt{\pi} < \lambda < 14/3$, the problem (4.4) has at least two positive solutions.

Specially, in Example 4.2, we take

$$g(t) = \begin{cases} 0, & t < \xi, \\ 1, & t \ge \xi, \end{cases} \tag{4.7}$$

where $0 < \xi < 1$ and all other conditions remain unchanged. Then $h(y) = y(\xi)$. Clearly, $\vee_0^1(g) = 1 < (15/8)\sqrt{\pi} - 1$. The problem (4.4) has at least two positive solutions for $(21/8(7\sqrt{2}-2))\sqrt{\pi} < \lambda < 14/3$.

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Research Article

Existence and Uniqueness of Solutions for Initial Value Problem of Nonlinear Fractional Differential Equations

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We discuss the initial value problem for the nonlinear fractional differential equation $L(D)u = f(t,u)$, $t \in (0,1]$, $u(0) = 0$, where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}$, $0 < s_1 < s_2 < \cdots < s_n < 1$, and $a_j < 0$, $j = 1,2,\ldots,n-1$, D^{s_j} is the standard Riemann-Liouville fractional derivative and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. We extend the basic theory of differential equation, the method of upper and lower solutions, and monotone iterative technique to the initial value problem. Some existence and uniqueness results are established.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order, so fractional differential equations have wider application. Fractional differential equations have gained considerable importance; it can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetic.

In the recent years, there has been a significant development in fractional calculus and fractional differential equations; see Kilbas et al. [1], Miller and Ross [2], Podlubny [3], Baleanu et al. [4], and so forth. Research on the solutions of fractional differential equations is very extensive, such as numerical solutions, see El-Mesiry et al. [5] and Hashim et al. [6], mild solutions, see Chang et al. [7] and Chen et al. [8], the existence and uniqueness of solutions for initial and boundary value problem, see [9–30], and so on.

With the deep study, many papers that studied the fractional equations contained more than one fractional differential operator; see [16–20].

Babakhani and Daftardar-Gejji in [16] considered the initial value problem of nonlinear fractional differential equation

$$L(D)u = f(t,u), \quad u(0) = 0, \quad 0 < t < 1. \tag{1.1}$$

By using Banach fixed point theorem and fixed point theorem on a cone some results of existence and uniqueness of solutions are established.

Zhang in [17] studied the singular initial value problem for fractional differential equation by nonlinear alternative of Leray-Schauder theorem:

$$L(D)u = f(t,u), \quad t^{1-s_n}u(t)|_{t=0} = 0, \quad 0 < t \leq 1. \tag{1.2}$$

In above two equations, $L(D)$ is defined $L(D) := D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}$, where $0 < s_1 < s_2 < \cdots < s_n < 1$, and $a_j > 0$, $j = 1, 2, \ldots, n-1$, D^{s_j} is the standard Riemann-Liouville fractional derivative.

McRae in [14] studied the initial value problem by the method of upper and lower solutions and monotone iterative technique:

$$\begin{gathered} D^q u = f(t,u), \quad t \in (t_0, T], \ 0 < q < 1, \\ u(t_0) = u^0 = u(t)(t-t_0)^{1-q}|_{t=t_0}. \end{gathered} \tag{1.3}$$

In this paper, we use similar method as in [16] to consider the initial value problem:

$$\begin{gathered} L(D)u = f(t,u), \quad t \in (0,1], \\ u(0) = 0, \end{gathered} \tag{1.4}$$

where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}$, $0 < s_1 < s_2 < \cdots < s_n < 1$, and $a_j < 0$, $j = 1, 2, \ldots, n-1$, D^{s_j} is the standard Riemann-Liouville fractional derivative and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Since f is assumed continuous, the IVP (1.4) is equivalent to the following Volterra fractional integral equation:

$$u(t) = \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_{n-1} - 1} u(s)ds + \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n - 1} f(s, u(s))ds. \tag{1.5}$$

In Section 2, we give some definitions and lemmas that will be useful to our main results. In Section 3, we will use the basic theory of differential equation, the method of upper and lower solutions, and monotone iterative technique to investigate the initial value problem (1.4), and some existence and uniqueness results are established. In Section 4, an example is presented to illustrate the main results.

2. Preliminaries

In this section, we need the following definitions and lemmas that will be useful to our main results. These materials can be found in the recent literatures; see [1, 11, 16].

Definition 2.1 (see [1]). Let $\Omega = [a,b](-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\begin{aligned} I_{a+}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a, \\ I_{b-}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t)dt, \quad x < b, \end{aligned} \tag{2.1}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We denote $I_{0+}^{\alpha} f(x)$ by $I^{\alpha} f(x)$ in the following paper.

Definition 2.2 (see [1]). Let $\Omega = [a,b]$ $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional derivatives $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\begin{aligned} D_{a+}^{\alpha} f(x) &= \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha}\right) f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)dt, \quad x > a, \\ D_{b-}^{\alpha} f(x) &= \left(-\frac{d}{dx}\right)^n \left(I_{b-}^{n-\alpha}\right) f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b (x-t)^{n-\alpha-1} f(t)dt, \quad x < b, \end{aligned} \tag{2.2}$$

respectively, where $n = [\alpha] + 1$, $[\alpha]$ means the integral part of α. These derivatives are called the left-sided and the right-sided fractional derivatives. We denote $D_{0+}^{\alpha} f(x)$ by $D^{\alpha} f(x)$ in the following paper.

Definition 2.3. Letting $v, w \in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $s_n < \lambda < 1$, we say that w is an upper solution of (1.4) if

$$\begin{aligned} L(D)w &\geq f(t,w), \\ w(0) &\geq 0, \end{aligned} \tag{2.3}$$

and v is a lower solution of (1.4) if

$$\begin{aligned} L(D)v &\leq f(t,v), \\ v(0) &\leq 0. \end{aligned} \tag{2.4}$$

Next, we will list the following lemma from [11] that is useful for our main results.

Lemma 2.4 (see [11, Lemma 2.1]). *Let $m \in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $q < \lambda < 1$ such that for any $t_1 \in (0,1]$, we have*

$$m(t_1) = 0, \quad m(t) \leq 0 \quad \text{for } 0 \leq \mathrm{t} \leq \mathrm{t}_1. \tag{2.5}$$

Then it follows that $D^q m(t_1) \geq 0$.

Corollary 2.5. *Let $m \in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $s_n < \lambda < 1$ such that for any $t_1 \in (0,1]$, we have*

$$m(t_1) = 0, \quad m(t) \leq 0 \quad \text{for } 0 \leq \mathrm{t} \leq \mathrm{t}_1. \tag{2.6}$$

Then it follows that $L(D)m(t_1) \geq 0$ provided $a_j < 0$, $j = 1,2,\ldots,n-1$.

Lemma 2.6. *Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[0,1]$, for each $\epsilon > 0$, where $L(D)u_\epsilon(t) = f(t,u_\epsilon(t))$, $u_\epsilon(0) = 0$ and $|f(t,u_\epsilon(t))| \leq M$ for $0 \leq t \leq 1$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous on $[0,1]$.*

Proof. Since $\{u_\epsilon(t)\}$ is a family of continuous functions on $[0,1]$, there exists $l > 0$ such that $|u_\epsilon(t)| \leq l$ for $0 \leq t \leq 1$.

Let $\delta < \min\{(\sum_{j=1}^{n-1} \epsilon\Gamma(s_n - s_j + 1)/(4l|a_j|))^{1/(s_n - s_{n-1})}, (\epsilon\Gamma(s_n+1)/(4M))^{1/s_n}\}$. For $0 \leq t_1 < t_2 \leq 1$, $t_2 - t_1 < \delta$, we get

$$\begin{aligned} |u_\epsilon(t_2) - u_\epsilon(t_1)| &= \left| \sum_{j=1}^{n-1} I^{s_n - s_j} a_j u(t_2) - \sum_{j=1}^{n-1} I^{s_n - s_j} a_j u(t_1) + I^{s_n} f(t_2, u(t_2)) - I^{s_n} f(t_1, u(t_1)) \right| \\ &= \left| \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^{t_1} \left[(t_2 - s)^{s_n - s_j - 1} - (t_1 - s)^{s_n - s_j - 1} \right] u(s)ds \right. \\ &\quad \left. + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_{t_1}^{t_2} (t_2 - s)^{s_n - s_j - 1} u(s)ds + \frac{1}{\Gamma(s_n)} \int_{t_1}^{t_2} (t_2 - s)^{s_n - 1} f(s,u(s))ds \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(s_n)}\int_0^{t_1}\left[(t_2-s)^{s_n-1}-(t_1-s)^{s_n-1}\right]f(s,u(s))ds\Bigg| \\
&\le \sum_{j=1}^{n-1}\frac{l|a_j|}{\Gamma(s_n-s_j)}\int_0^{t_1}\left[(t_1-s)^{s_n-s_j-1}-(t_2-s)^{s_n-s_j-1}\right]ds \\
&\quad+\sum_{j=1}^{n-1}\frac{l|a_j|}{\Gamma(s_n-s_j)}\int_{t_1}^{t_2}(t_2-s)^{s_n-s_j-1}ds+\frac{M}{\Gamma(s_n)}\int_{t_1}^{t_2}(t_2-s)^{s_n-1}ds \\
&\quad+\frac{M}{\Gamma(s_n)}\int_0^{t_1}\left[(t_1-s)^{s_n-1}-(t_2-s)^{s_n-1}\right]ds \\
&=\sum_{j=1}^{n-1}\frac{l|a_j|}{\Gamma(s_n-s_j+1)}\left(t_1^{s_n-s_j}-t_2^{s_n-s_j}\right)+\sum_{j=1}^{n-1}\frac{2l|a_j|}{\Gamma(s_n-s_j+1)}(t_2-t_1)^{s_n-s_j} \\
&\quad+\frac{M}{\Gamma(s_n+1)}\left(t_1^{s_n}-t_2^{s_n}\right)+\frac{2M}{\Gamma(s_n+1)}(t_2-t_1)^{s_n} \\
&\le \sum_{j=1}^{n-1}\frac{2l|a_j|}{\Gamma(s_n-s_j+1)}(t_2-t_1)^{s_n-s_{n-1}}+\frac{2M}{\Gamma(s_n+1)}(t_2-t_1)^{s_n} \\
&\le \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}
\tag{2.7}
$$

Thus, $\{u_\epsilon(t)\}$ is equicontinuous on $[0,1]$. □

Lemma 2.7 (see [16, Theorem 4.2]). *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous and Lipschitz with respect to second variable with Lipschitz constant L. Let a_j satisfy*

$$0<\frac{L}{\Gamma(s_n+1)}+\sum_{j=1}^{n-1}\frac{|a_j|}{\Gamma(s_n-s_j+1)}<1. \tag{2.8}$$

Then IVP (1.4) *has a unique solution.*

Lemma 2.8. *Let $v,w\in C([0,1],\mathbb{R})$ be locally Hölder continuous with exponent $q<\lambda<1$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$ and*

$$L(D)w\ge f(t,w),\quad L(D)v\le f(t,v),\quad 0<t\le 1 \tag{2.9}$$

one of the nonstrict inequalities being strict. Then $v(0)<w(0)$ implies $v(t)<w(t)$, $0\le t\le 1$.

Proof. Suppose that $v(t)<w(t)$, $0\le t\le 1$ is not true. We suppose the inequality $L(D)>f(t,w(t))$. Letting $m(t)=v(t)-w(t)$, there exists $0<t_1\le 1$ such that $m(t)\le 0$, $0\le t\le t_1$, and

$m(t_1) = 0$. Then by Corollary 2.5, we can obtain $L(D)m(t_1) \geq 0$. From the conditions and the definition of $m(t)$, we have

$$f(t_1, v(t_1)) \geq L(D)v(t_1) \geq L(D)w(t_1) > f(t_1, w(t_1)). \tag{2.10}$$

This is a contradiction to $v(t_1) = w(t_1)$. The proof is complete. □

Lemma 2.9. *Assume that the conditions of Lemma 2.8 hold with nonstrict inequalities* (2.3) *and* (2.4). *Furthermore, suppose that*

$$f(t,x) - f(t,y) \leq N(x-y), \quad \text{where } x \geq y,\ N > 0. \tag{2.11}$$

Then $v(0) \leq w(0)$ *implies* $v(t) \leq w(t)$, $0 \leq t \leq 1$ *provided* $N < 1/\Gamma(1-s_n) - \sum_{j=1}^{n-1} a_j/\Gamma(1-s_j)$.

Proof. Let $w_\epsilon(t) = w(t) + \epsilon$. For small $\epsilon > 0$, we have

$$w_\epsilon(0) > w(0), \qquad w_\epsilon(t) > w(t), \quad 0 \leq t \leq 1. \tag{2.12}$$

Then, from (2.11) and (2.12) we get

$$\begin{aligned}
L(D)w_\epsilon(t) &= L(D)w(t) + L(D)\epsilon \\
&= f(t,w(t)) + \epsilon\left[\frac{t^{-s_n}}{\Gamma(1-s_n)} - \sum_{j=1}^{n-1}\frac{a_j t^{-s_j}}{\Gamma(1-s_j)}\right] \\
&\geq f(t,w_\epsilon(t)) - N\epsilon + \epsilon\left[\frac{t^{-s_n}}{\Gamma(1-s_n)} - \sum_{j=1}^{n-1}\frac{a_j t^{-s_j}}{\Gamma(1-s_j)}\right] \\
&\geq f(t,w_\epsilon(t)) - N\epsilon + \epsilon\left[\frac{1}{\Gamma(1-s_n)} - \sum_{j=1}^{n-1}\frac{a_j}{\Gamma(1-s_j)}\right] \\
&> f(t,w_\epsilon(t)), \quad 0 < t \leq 1.
\end{aligned} \tag{2.13}$$

Applying Lemma 2.8, we obtain $v(t) < w_\epsilon(t)$, $0 \leq t \leq 1$. By the arbitrariness of $\epsilon > 0$, we can conclude that $v(t) \leq w(t)$. The proof is complete. □

Corollary 2.10. *The function* $f(t,u) = \sigma(t)u$, *where* $\sigma(t) \leq N$, *is admissible in Lemma 2.9 to yield* $v(t) \leq 0$ *on* $0 \leq t \leq 1$.

3. Main Results

In this section, we establish the existence and uniqueness criteria of solutions for initial value problem (1.4).

Theorem 3.1. *Assume that $f \in C(R_0, \mathbb{R})$, where $R_0 = \{(t,u) : 0 \le t \le 1, |u(t)| \le b\}$ and $|f(t,u)| \le M$. Then IVP (1.4) possesses at least one solution $u(t)$ on $0 \le t \le \alpha$, where $\alpha = \min\{1, (b\Gamma(1+s_n)/(2M))^{1/s_n}, (\sum_{j=1}^{n-1} \Gamma(s_n - s_j + 1)/(2|a_j|))^{1/(s_n - s_{n-1})}\}$.*

Proof. Let $u_0(t)$ be a continuous function on $[-\delta, 0]$, $\delta > 0$, such that $u_0(0) = 0$, $|u_0(t)| \le b$ and $|L(D)u_0(t)| \le M$, where $D_{0-}^{s_j} u_0(t)$, $j = 1, 2, \ldots, n-1$ are the continuous fractional derivatives. For $0 < \epsilon \le \delta$, we define the function $u_\epsilon(t) = u_0(t)$ on $[-\delta, 0]$ and

$$u_\epsilon(t) = \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n - 1} f(s, u_\epsilon(s-\epsilon))ds + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} u_\epsilon(s-\epsilon)ds \tag{3.1}$$

on $[0, \alpha_1]$, where $\alpha_1 = \min\{\epsilon, \alpha\}$. We observe that $D^{s_j} u_\epsilon(t)$, $j = 1, 2, \ldots, n$ exist for $t \in [0, \alpha_1]$ and

$$\begin{aligned} |u_\epsilon(t)| &\le \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n-1} \left| f(s, u_\epsilon(s-\epsilon)) \right| ds + \sum_{j=1}^{n-1} \frac{|a_j|}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} |u_\epsilon(s-\epsilon)| ds \\ &\le \frac{M}{\Gamma(s_n)} \int_0^t (t-s)^{s_n-1} ds + \sum_{j=1}^{n-1} \frac{b|a_j|}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} ds \\ &= \frac{M}{\Gamma(s_n+1)} t^{s_n} + \sum_{j=1}^{n-1} \frac{b|a_j|}{\Gamma(s_n - s_j + 1)} t^{s_n - s_j} \\ &\le \frac{M}{\Gamma(s_n+1)} \alpha^{s_n} + \sum_{j=1}^{n-1} \frac{b|a_j|}{\Gamma(s_n - s_j + 1)} \alpha^{s_n - s_{n-1}} \\ &\le \frac{b}{2} + \frac{b}{2} = b. \end{aligned} \tag{3.2}$$

If $\alpha_1 < \alpha$, we can employ (3.1) to extend $u_\epsilon(t)$ as a continuously fractional differentiable function on $[-\delta, \alpha_2]$, $\alpha_2 = \min\{\alpha, 2\epsilon\}$ such that $u_\epsilon(t) \le b$ holds. Continuing this process, we can define $u_\epsilon(t)$ over $[-\delta, \alpha]$ so that $u_\epsilon(t) \le b$; it has a continuous fractional derivative and satisfies (3.1) on the same interval $[-\delta, \alpha]$. Furthermore, $|L(D)u_\epsilon(t)| \le M$, since $|f(t, u_\epsilon(t-\epsilon))| \le M$ on R_0. Therefore, from Lemma 2.6, the family $\{u_\epsilon(t)\}$ is an equicontinuous and uniformly bounded function. An application of Ascoli-Arzela Theorem shows the existence of a sequence $\{\epsilon_n\}$ such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n \to 0$ as $n \to \infty$, and $u(t) = \lim_{n\to\infty} u_{\epsilon_n}(t)$ exists uniformly on $[-\delta, \alpha]$. Due to f being uniformly continuous, we can obtain $f(t, u_{\epsilon_n}(t - \epsilon_n))$ which uniformly tends to $f(t, u(t))$, and $u_{\epsilon_n}(t - \epsilon_n)$ uniformly tends to $u(t)$ as $n \to \infty$. Therefore, term by term, integration of (3.1) with $\epsilon = \epsilon_n$, $\alpha_1 = \alpha$ yields

$$u(t) = \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n-1} f(s, u(s))ds + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \int_0^t (t-s)^{s_n - s_j - 1} u(s)ds. \tag{3.3}$$

This proves that $u(t)$ is a solution of IVP (1.4) and the proof is complete. □

Theorem 3.2. *Let $v, w \in C([0,1], \mathbb{R})$ be lower and upper solutions of the IVP (1.4) which are locally Hölder continuous with exponent $s_n < \lambda < 1$ such that $v(t) \le w(t)$, $t \in [0,1]$ and $f \in C(\Omega, \mathbb{R})$, where $\Omega = \{(t,u) : v(t) \le u(t) \le w(t), t \in [0,1]\}$. Furthermore, suppose that*

$$\left(\sum_{j=1}^{n-1} \frac{\Gamma(s_n - s_j + 1)}{2|a_j|}\right)^{1/(s_n - s_{n-1})} \ge 1. \tag{3.4}$$

Then there exists a solution $u(t)$ of IVP (1.4) satisfying $v(t) \le u(t) \le w(t)$ on $[0,1]$.

Proof. For the need of proof, we define function $p(t,u) : [0,1] \times \mathbb{R} \to \mathbb{R}$ as

$$p(t,u) = \max\{v(t), \min\{u, w(t)\}\}. \tag{3.5}$$

Therefore, $f(t, p(t,u))$ defines a continuous extension of f to $[0,1] \times R$ which is also bounded because f is bounded on Ω. Then by Theorems 3.1 and 3.2, we can obtain that the initial value problem

$$\begin{gathered} L(D)u = f(t, p(t,u)), \quad t \in (0,1], \\ u(0) = 0, \end{gathered} \tag{3.6}$$

has a solution on $[0,1]$.

Clearly, from the definition of function $p(t,u)$, we know that if IVP (3.6) exits a solution $u(t)$ satisfying $v(t) \le u(t) \le w(t)$ on $[0,1]$, then $u(t)$ is also a solution of IVP (1.4). In the following, we will prove that the solution $u(t)$ of IVP (3.6) satisfies $v(t) \le u(t) \le w(t)$ on $[0,1]$.

For any $\epsilon > 0$, we consider

$$w_\epsilon(t) = w(t) + \epsilon, \qquad v_\epsilon(t) = v(t) - \epsilon. \tag{3.7}$$

Then, we get

$$w_\epsilon(0) = w(0) + \epsilon, \qquad v_\epsilon(0) = v(0) - \epsilon. \tag{3.8}$$

Therefore, it follows that $v_\epsilon(0) < u(0) < w_\epsilon(0)$. Next, we will show that $v_\epsilon(t) < u(t) < w_\epsilon(t)$, $t \in [0,1]$. Suppose that it is not true. Then there exists $t_1 \in (0,1]$ such that

$$u(t_1) = w_\epsilon(t_1), \quad v_\epsilon(t) < u(t) < w_\epsilon(t), \quad 0 \le t < t_1. \tag{3.9}$$

Therefore, $u(t_1) > w(t_1)$, $p(t_1, u(t_1)) = w(t_1)$ and $v(t_1) \leq p(t_1, u(t_1)) \leq w(t_1)$. Letting $m(t) = u(t) - w_\epsilon(t)$, we have $m(t_1) = 0$ and $m(t) \leq 0$, $0 \leq t \leq t_1$. Then from Corollary 2.5, we can obtain $L(D)m(t_1) \geq 0$ and

$$\begin{aligned} f(t_1, w(t_1)) &= f(t_1, p(t_1, w(t_1))) \\ &= L(D)u(t_1) \geq L(D)w_\epsilon(t_1) \\ &= L(D)w(t_1) + L(D)\epsilon(t_1) \\ &= L(D)w(t_1) + \epsilon\left[\frac{t_1^{-s_n}}{\Gamma(1-s_n)} - \sum_{j=1}^{n-1}\frac{a_j t_1^{-s_j}}{\Gamma(1-s_j)}\right] \\ &> L(D)w(t_1) = f(t_1, w(t_1)), \end{aligned} \tag{3.10}$$

which is a contradiction. The other case $v_\epsilon(t) < u(t)$ can be proved similarly.

Hence, we get $v_\epsilon(t) < u(t) < w_\epsilon(t)$ on $[0,1]$. Letting $\epsilon \to 0$, we obtain $v(t) \leq u(t) \leq w(t)$ on $[0,1]$. The proof is complete. □

Now, we will give the existence of maximal and minimal solutions of initial value problem (1.4).

Theorem 3.3. *Let $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, v_0, w_0 be lower and upper solutions of* (1.4) *such that $v_0 \leq w_0$ on* $[0,1]$*. Furthermore, suppose that*

$$f(t,x) - f(t,y) \geq -N(x-y), \quad \text{for } \mathrm{v}_0 \leq \mathrm{y} \leq \mathrm{x} \leq \mathrm{w}_0,\ \mathrm{N} \geq 0, \tag{3.11}$$

and a_j satisfy

$$0 < \frac{N}{\Gamma(s_n+1)} + \sum_{j=1}^{n-1}\frac{|a_j|}{\Gamma(s_n - s_j + 1)} < 1. \tag{3.12}$$

Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that $v_n \to \rho$, $w_n \to r$ as $n \to \infty$ uniformly on $[0,1]$*, where ρ and r are minimal and maximal solutions of IVP* (1.4)*, respectively.*

Proof. For any $\eta \in C([0,1], \mathbb{R})$ satisfying $v_0 \leq \eta \leq w_0$, we consider the following linear fractional differential equation:

$$\begin{gathered} L(D)u = f(t,\eta) - N(u-\eta), \quad t \in (0,1], \\ u(0) = 0. \end{gathered} \tag{3.13}$$

Obviously, the right hand side of (3.13) satisfies the Lipschitz condition. From (3.11) and Lemma 2.7, it is clear that for every η, there exists a unique solution u of (3.13) on $[0,1]$.

Define the operator T by $T\eta = u$ and use it to construct the sequences $\{v_n\}$, $\{w_n\}$. We need to prove the following propositions hold:

(i) $v_0 \leq Tv_0$, $w_0 \geq Tw_0$;

(ii) T is a monotone operator on the segment

$$\langle v_0, w_0 \rangle = \{u \in C([0,1], \mathbb{R}) : v_0 \leq u \leq w_0\}. \tag{3.14}$$

To prove (i), let $Tv_0 = v_1$, where v_1 is the unique solution of (3.13) with $\eta = v_0$. Letting $p = v_0 - v_1$, we have

$$\begin{gathered} L(D)p = L(D)v_0 - L(D)v_1 \leq f(t, v_0) - \left[f(t, v_0) - N(v_1 - v_0)\right] = -Np, \\ p(0) = v_0(0) - v_1(0) \leq 0. \end{gathered} \tag{3.15}$$

By Corollary 2.10, we can obtain that $p(t) \leq 0$ on $[0,1]$, that is, $v_0 \leq Tv_0$.

Similarly, we can get $w_0 \geq Tw_0$.

To prove (ii), let $\eta_1, \eta_2 \in \langle v_0, w_0 \rangle$ such that $\eta_1 \leq \eta_2$. Assume that $u_1 = T\eta_1$ and $u_2 = T\eta_2$. Setting $p = u_1 - u_2$, then using the condition (3.11), we have

$$\begin{aligned} L(D)p &= L(D)u_1 - L(D)u_2 = f(t, \eta_1) - N(u_1 - \eta_1) - \left[f(t, \eta_2) - N(u_2 - \eta_2)\right] \\ &\leq -N(\eta_1 - \eta_2) - N(u_1 - \eta_1) + N(u_2 - \eta_2) = -Np, \\ &\qquad p(0) = u_1(0) - u_2(0) = 0. \end{aligned} \tag{3.16}$$

From Corollary 2.10, we can obtain that $p(t) \leq 0$ on $[0,1]$, which implies $T\eta_1 \leq T\eta_2$. And (ii) is proved.

Therefore, we can define the sequences $v_n = Tv_{n-1}$, $w_n = Tw_{n-1}$. From the previous discussion, we can get

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_1 \leq w_0 \quad \text{on } [0,1]. \tag{3.17}$$

Clearly, the sequences $\{v_n\}$, $\{w_n\}$ are uniformly bounded on $[0,1]$. From (3.13), we have $|L(D)v_n|$, $|L(D)w_n|$ which are also uniformly bounded. By Lemma 2.6, we know that $\{v_n\}$, $\{w_n\}$ are equicontinuous on $[0,1]$. Then applying Ascoli-Arzela Theorem, there exist subsequences $\{v_{n_k}\}$, $\{w_{n_k}\}$ that converge uniformly on $[0,1]$. From (3.17), we can see that the entire sequences $\{v_n\}$, $\{w_n\}$ converge uniformly and monotonically to ρ, r, respectively, as $n \to \infty$. It is now easy to show that ρ, r are solutions of IVP (1.4) by the corresponding Volterra fractional integral equation for (3.13).

In the following, we will prove that ρ and r are the minimal and maximal solutions of IVP (1.4), respectively. We need to show that if u is any solution of IVP (1.4) satisfying $v_0 \leq u \leq w_0$ on $[0,1]$, then we have $v_0 \leq \rho \leq u \leq r \leq w_0$ on $[0,1]$.

We assume that for some n, $v_n \le u \le w_n$ on $[0,1]$ and letting $p = v_{n+1} - u$, we have

$$\begin{aligned} L(D)p &= L(D)v_{n+1} - L(D)u = f(t,v_n) - N(v_{n+1} - v_n) - f(t,u) \\ &\le -N(v_n - u) - N(v_{n+1} - v_n) = -Np, \\ &p(0) = v_{n+1}(0) - u(0) = 0, \end{aligned} \tag{3.18}$$

which implies $v_{n+1} \le u$. Similarly, we have $u \le w_{n+1}$ on $[0,1]$. Since $v_0 \le u \le w_0$ on $[0,1]$, this proves $v_n \le u \le w_n$ for all n by induction. Letting $n \to \infty$, we conclude that $\rho \le u \le r$ on $[0,1]$ and the proof is complete. □

Theorem 3.4. *Suppose that the conditions of Theorem 3.3 hold. In addition, we assume*

$$f(t,x) - f(t,y) \le N(x-y), \quad v_0 \le y \le x \le w_0, \quad N > 0. \tag{3.19}$$

Then $\rho = r = u$ is the unique solution of IVP (1.4) *provided* $N < 1/\Gamma(1-s_n) - \sum_{j=1}^{n-1} a_j/\Gamma(1-s_j)$.

Proof. We have proved $\rho \le r$ in Theorem 3.3, so we just need to prove $\rho \ge r$. Letting $p = r - \rho$, we get

$$\begin{gathered} L(D)p = f(t,r) - f(t,\rho) \le Np, \\ p(0) = r(0) - \rho(0) = 0. \end{gathered} \tag{3.20}$$

From Corollary 2.10, we obtain $p \le 0$ on $[0,1]$, which implies $\rho \ge r$. Hence, $\rho = r = u$ is the unique solution of IVP (1.4). □

4. Examples

In this paper, we will present an example to illustrate the main results.

Example 4.1. Consider the initial value problem of fractional differential equation

$$\begin{gathered} D^{0.8}u + 0.4D^{0.6}u = \frac{u^2 t^{0.2}}{10\Gamma(0.2)} - \frac{ut^{0.4}}{2\Gamma(0.4)}, \quad t \in (0,1], \\ u(0) = 0. \end{gathered} \tag{4.1}$$

Choose $w = 5$, $v = -5$; then we can obtain

$$\begin{gathered} D^{0.8}w + 0.4D^{0.6}w \ge \frac{w^2 t^{0.2}}{10\Gamma(0.2)} - \frac{wt^{0.4}}{2\Gamma(0.4)}, \\ D^{0.8}v + 0.4D^{0.6}v \le \frac{v^2 t^{0.2}}{10\Gamma(0.2)} - \frac{vt^{0.4}}{2\Gamma(0.4)}. \end{gathered} \tag{4.2}$$

That is, v and w are the lower and upper solutions of initial value problem (4.1). Furthermore, v and w are locally continuous with exponent $1 > \lambda > 0.8$.

Since

$$\left(\frac{\Gamma(0.8-0.6+1)}{2|-0.4|}\right)^{1/(0.8-0.6)} = 1.9914 > 1, \tag{4.3}$$

then by Theorem 3.2, there exists a solution $u(t)$ of initial value problem (4.1) satisfying $-5 \le u(t) \le 5$.

Next, we will prove the existence of maximal and minimal solutions for initial value problem (4.1) by using Theorem 3.3.

Let $v_0 = -5$ and $w_0 = 5$ be lower and upper solutions of (4.1). Furthermore, for any $-5 \le y \le x \le 5$, we have

$$\begin{aligned} f(t,x) - f(t,y) &= \frac{x^2 t^{0.2}}{10\Gamma(0.2)} - \frac{x t^{0.4}}{2\Gamma(0.4)} - \frac{y^2 t^{0.2}}{10\Gamma(0.2)} + \frac{y t^{0.4}}{2\Gamma(0.4)} \\ &= \frac{t^{0.2}}{10\Gamma(0.2)}(x-y)(x+y) - \frac{t^{0.4}}{2\Gamma(0.4)}(x-y) \\ &\ge -\frac{1}{2\Gamma(0.4)}(x-y). \end{aligned} \tag{4.4}$$

Then let $N = 1/2\Gamma(0.4) \approx 0.2254$. We get

$$0 < \frac{N}{\Gamma(0.8+1)} + \frac{|-0.4|}{\Gamma(0.8-0.6+1)} \approx 0.6777 < 1. \tag{4.5}$$

Thus, from Theorem 3.3, there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that $v_n \to \rho$, $w_n \to r$ as $n \to \infty$ uniformly on $[0,1]$, where ρ and r are minimal and maximal solutions of initial value problem (4.1), respectively.

In addition,

$$\begin{aligned} f(t,x) - f(t,y) &= \frac{x^2 t^{0.2}}{10\Gamma(0.2)} - \frac{x t^{0.4}}{2\Gamma(0.4)} - \frac{y^2 t^{0.2}}{10\Gamma(0.2)} + \frac{y t^{0.4}}{2\Gamma(0.4)} \\ &= \frac{t^{0.2}}{10\Gamma(0.2)}(x-y)(x+y) - \frac{t^{0.4}}{2\Gamma(0.4)}(x-y) \\ &\le \frac{10}{10\Gamma(0.2)}(x-y) \\ &\le N(x-y), \\ 0 < N &< \frac{1}{\Gamma(1-0.8)} - \frac{-0.4}{\Gamma(1-0.6)} \approx 0.3982. \end{aligned} \tag{4.6}$$

Hence, by Theorem 3.4, initial value problem (4.1) has a unique solution.

5. Conclusion

In this paper, we considered the initial value problem of nonlinear fractional differential equation

$$\begin{gathered} L(D)u = f(t,u), \quad t \in (0,1], \\ u(0) = 0. \end{gathered} \tag{5.1}$$

The basic theory of differential equation, the method of upper and lower solutions, and monotone iterative technique have been applied for the existence and uniqueness of solutions of the initial value problem. And several results were obtained. Besides, we studied the existence of minimal and maximal solutions. In Section 4, we also give an example to illustrate our results.

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Research Article

A New Fractional Integral Inequality with Singularity and Its Application

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We prove an integral inequality with singularity, which complements some known results. This inequality enables us to study the dependence of the solution on the initial condition to a fractional differential equation in the weighted space.

1. Introduction

Integral inequalities provide an excellent tool for the properties of solutions to differential equations, such as boundedness, existence, uniqueness, and stability (e.g., see [1–10]). For this reason, the study of integral inequalities has been emphasized by many authors. For example, in 1919, Gronwall in [11] proved a remarkable inequality which can be described by the following.

Suppose that $x(t)$ satisfies the relation

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds, \quad t_0 \leq t < T, \tag{1.1}$$

where all the functions involved are continuous on the interval $[t_0, T)$, $T \leq \infty$, and $k(t) \geq 0$. Consider

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)\exp\left(\int_{s}^{t} k(\tau)d\tau\right)ds, \quad t_0 \leq t < T. \tag{1.2}$$

The inequality has attracted and continues to attract considerable attention in the literature. In 2007, Ye et al. [12] reported an integral inequality with singular kernel. The inequality can be stated as follows.

If $\beta > 0$, $a(t)$ is a nonnegative and locally integrable on $0 \le t < T$, $g(t)$ is a nonnegative, nondecreasing continuous function on $0 \le t < T$, and $g(t) \le M$, where $T \le \infty$, M is a positive constant. Further suppose that $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with

$$u(t) \le a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s)\,ds, \quad 0 \le t < T. \tag{1.3}$$

Then

$$u(t) \le a(t) + \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} \int_0^t (t-s)^{n\beta-1} a(s)\,ds, \quad 0 \le t < T. \tag{1.4}$$

Besides the above-mentioned inequalities, there are still many inequalities (e.g., see [13–15]).

But in the analysis of the dependence of the solution on the initial condition of a fractional differential equation in the weighted space, the bounds provided by the existing inequalities are not adequate. So it is natural and necessary to seek new inequality in order to obtain our desired results. In this paper, we present a new integral inequality, and then apply our inequality to investigate the dependence of the solution on the initial condition of a fractional differential equations in the weighted space.

2. An Integral Inequality

In this section, our main aim is to establish an integral inequality with singularity. Before proceeding, we give some useful definitions and lemmas.

Definition 2.1 (see [14, 16]). The gamma function is defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,\ z > 0$.

Definition 2.2 (see [14, 16]). The beta function is defined by $B(z,w) = \int_0^1 (1-t)^{z-1} t^{w-1} dt,\ z,w > 0$.

The beta function is connected with gamma function by the following relation [3, 14]:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad z,w > 0. \tag{2.1}$$

Lemma 2.3 (see [14]). *Let $z > 0$, $a,b \in R$. Then the quotient expansion of two gamma functions at infinity can be represented as follows:*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1 + O\left(\frac{1}{z}\right)\right), \quad z \to \infty. \tag{2.2}$$

Lemma 2.4. *Let $z > 0$, $a, b \in R$. Then one has*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = O\left(z^{a-b}\right), \quad z \longrightarrow \infty. \tag{2.3}$$

Proof. By Lemma 2.3, we have $\lim_{z\to\infty}(\Gamma(z+a)/\Gamma(z+b))/z^{a-b} = \lim_{z\to\infty}(1+O(1/z)) = 1$, which proves that $\Gamma(z+a)/\Gamma(z+b) = O(z^{a-b})$ as $z \to \infty$. The proof of this lemma is completed. □

Based on Lemma 2.4, we can define a function.

Definition 2.5. Let $b > a > 0$, $\rho > 0$. Then the following definition:

$$F_{\rho,a,b}(z) := \sum_{k=0}^{\infty} c_k z^k, \quad z \in R \tag{2.4}$$

is well defined, where c_0 is a positive constant, and $c_{k+1} = (\Gamma(k\rho+a)/\Gamma(k\rho+b))c_k$.

Proof. We only need to show that the series in (2.4) is uniformly convergent for $z \in R$. By Lemma 2.4, we know that $c_{k+1}/c_k = \Gamma(k\rho+a)/\Gamma(k\rho+b) = O((k\rho)^{a-b})$ as $k \to \infty$. Since $b > a > 0$, $c_{k+1}/c_k \to 0$ as $k \to \infty$. This implies that the series in (2.4) is uniformly convergent for $z \in R$. It follows that the definition is well defined. □

Lemma 2.6. *Let $z, w > 0$, $t, s \in R$ and $t \neq s$. Then one has*

$$\int_s^t (t-\tau)^{z-1}(\tau-s)^{w-1}d\tau = (t-s)^{z+w-1}\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{2.5}$$

Proof. Making the substitution $\tau = s + \xi(t-s)$ and combining the relation (2.1), we obtain

$$\begin{aligned}\int_s^t (t-\tau)^{z-1}(\tau-s)^{w-1}d\tau &= (t-s)^{z+w-1}\int_0^1 (1-\xi)^{z-1}\xi^{w-1}d\xi \\ &= (t-s)^{z+w-1}B(z,w) = (t-s)^{z+w-1}\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.\end{aligned} \tag{2.6}$$

The proof of this lemma is completed. □

Now we can state the integral inequality.

Theorem 2.7. *Let $\alpha, \beta, \gamma > 0$, $\delta = \alpha+\gamma-1 > 0$, $\nu = \beta+\gamma-1 > 0$, $a > 0$, and let $b(t)$ be a nonnegative, nondecreasing continuous function on $0 \le t < T$, $b(t) \le M$, where $T \le \infty$, M is a positive constant. Further suppose that $u(t)$ is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \le t < T$ with*

$$u(t) \le at^{\alpha-1} + b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds, \quad 0 \le t < T. \tag{2.7}$$

Then one has

$$u(t) \leq a t^{\alpha-1} F_{\nu,\delta,\delta+\beta}\left(\Gamma(\beta) b(t) t^{\beta}\right), \quad 0 \leq t < T. \tag{2.8}$$

Proof. For convenience, we define an operator

$$(\mathcal{R}u)(t) = b(t) \int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds. \tag{2.9}$$

Then (2.7) can be rewritten in the form

$$u(t) \leq a t^{\alpha-1} + (\mathcal{R}u)(t). \tag{2.10}$$

Since $b(t)$ and $u(t)$ are nonnegative, it is easy to induce that

$$u(t) \leq \sum_{k=0}^{n} \left(\mathcal{R}^k a t^{\alpha-1}\right)(t) + \left(\mathcal{R}^{n+1} u\right)(t), \quad n \in N. \tag{2.11}$$

Let us prove that the following relation

$$(\mathcal{R}^n u)(t) \leq \begin{cases} b(t)(\Gamma(\beta) b(t))^{n-1} \displaystyle\prod_{i=1}^{n-1} \frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)} \int_0^t (t-s)^{n\nu-\gamma} s^{\gamma-1} u(s) ds, & 0 < \gamma < 1, \\ \dfrac{(\Gamma(\beta) b(t))^n t^{(n-1)(\gamma-1)}}{\Gamma(n\beta)} \displaystyle\int_0^t (t-s)^{n\beta-1} s^{\gamma-1} u(s) ds, & \gamma \geq 1, \end{cases} \tag{2.12}$$

holds for any $n \in N^+$, where $\prod_{i=1}^{0} 1 = 1$, and $(\mathcal{R}^n u)(t) \to 0$ as $n \to \infty$ for each t in $0 \leq t < T$.

Obviously, inequality (2.12) is valid for $n = 1$, due to $\prod_{i=1}^{0} 1 = 1$. Suppose that the inequality is satisfied for any fixed $n \in N^+$. Let us verify that it is also satisfied for $n+1$. We first prove the case $0 < \gamma < 1$. According to the induction hypothesis and Lemma 2.6, we have

$$\begin{aligned}
\left(\mathcal{R}^{n+1}u\right)(t) &= b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}(\mathcal{R}^n u)(s)ds \\
&\le b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\int_0^s (s-\tau)^{n\nu-\gamma}\tau^{\gamma-1}u(\tau)d\tau ds \\
&= b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}s^{\gamma-1}(s-\tau)^{n\nu-\gamma}ds \\
&\le b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{n\nu-1}ds \\
&= b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t \tau^{\gamma-1}u(\tau)(t-\tau)^{n\nu+\beta-1}\frac{\Gamma(\beta)\Gamma(n\nu)}{\Gamma(n\nu+\beta)}d\tau \\
&= b(t)\left(\Gamma(\beta)b(t)\right)^{n}\prod_{i=1}^{n}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t (t-\tau)^{(n+1)\nu-\gamma}\tau^{\gamma-1}u(\tau)d\tau,
\end{aligned} \tag{2.13}$$

which is estimated with the help of

$$s^{\gamma-1} \le (s-\tau)^{\gamma-1}, \quad 0 \le \tau \le s,\ 0 < \gamma < 1. \tag{2.14}$$

So, for the case $0 < \gamma < 1$, inequality (2.12) is true for any $n \in N^+$. Now we prove the case $\gamma \ge 1$. Similarly, according to the induction hypothesis and Lemma 2.6, we get

$$\begin{aligned}
\left(\mathcal{R}^{n+1}u\right)(t) &= b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}(\mathcal{R}^n u)(s)ds \\
&\le b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\frac{\left(\Gamma(\beta)b(s)\right)^n s^{(n-1)(\gamma-1)}}{\Gamma(n\beta)}\int_0^s (s-\tau)^{n\beta-1}\tau^{\gamma-1}u(\tau)d\tau ds \\
&= b(t)\frac{\left(\Gamma(\beta)b(t)\right)^n}{\Gamma(n\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}s^{n(\gamma-1)}(s-\tau)^{n\beta-1}ds \\
&\le b(t)\frac{\left(\Gamma(\beta)b(t)\right)^n t^{n(\gamma-1)}}{\Gamma(n\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{n\beta-1}ds \\
&= b(t)\frac{\left(\Gamma(\beta)b(t)\right)^n t^{n(\gamma-1)}}{\Gamma(n\beta)}\int_0^t \tau^{\gamma-1}u(\tau)(t-\tau)^{n\beta+\beta-1}\frac{\Gamma(\beta)\Gamma(n\beta)}{\Gamma(n\beta+\beta)}d\tau \\
&= \frac{\left(\Gamma(\beta)b(t)\right)^{n+1}t^{n(\gamma-1)}}{\Gamma((n+1)\beta)}\int_0^t \tau^{\gamma-1}u(\tau)(t-\tau)^{(n+1)\beta-1}d\tau,
\end{aligned} \tag{2.15}$$

which is calculated with the help of

$$s^{n(\gamma-1)} \leq t^{n(\gamma-1)}, \quad 0 \leq s \leq t, \ \gamma \geq 1, \ n \in N^+. \tag{2.16}$$

So, for the case $\gamma \geq 1$, inequality (2.12) is true for any $n \in N^+$. Based on this analysis, we conclude that inequality (2.12) holds for any $n \in N^+$.

Next, we show that $(\mathcal{R}^n u)(t) \to 0$ as $n \to \infty$. Now, we go back to inequality (2.12). For the case $0 < \gamma < 1$, we denote $K_n(t,s) = B_n(t-s)^{n\nu-\gamma}$, where $B_n = b(t)(\Gamma(\beta)b(t))^{n-1}\prod_{i=1}^{n-1}(\Gamma(i\nu)/\Gamma(i\nu+\beta))$. Note that

$$B_1 = b(t), \quad \frac{B_{n+1}}{B_n} = \Gamma(\beta)b(t)\frac{\Gamma(n\nu)}{\Gamma(n\nu+\beta)}. \tag{2.17}$$

Since $b(t) \leq M$, by Lemma 2.4, we obtain $B_{n+1}/B_n \to 0$ as $n \to \infty$. This implies that $K_n(t,s) \to 0$ as $n \to \infty$. It follows that $(R^n u)(t) \to 0$ as $n \to \infty$ for the case $0 < \gamma < 1$. For the case $\gamma \geq 1$, we denote $\overline{K}_n(t,s) = \overline{B}_n(t-s)^{n\beta-1}$, where $\overline{B}_n = (\Gamma(\beta)b(t))^n t^{(n-1)(\gamma-1)}/\Gamma(n\beta)$. Note that

$$\overline{B}_1 = b(t), \quad \frac{\overline{B}_{n+1}}{\overline{B}_n} = \Gamma(\beta)b(t)t^{\gamma-1}\frac{\Gamma(n\beta)}{\Gamma(n\beta+\beta)}. \tag{2.18}$$

Using the same arguments as above, we know that $\overline{K}_n(t,s) \to 0$ as $n \to \infty$. It follows that $(\mathcal{R}^n u)(t) \to 0$ as $n \to \infty$ for the case $\gamma \geq 1$. So, it has $(\mathcal{R}^n u)(t) \to 0$ as $n \to \infty$ for the two cases $0 < \gamma < 1$ and $\gamma \geq 1$. This, together with (2.11), leads to $u(t) \leq \sum_{k=0}^{\infty}(\mathcal{R}^k a t^{\alpha-1})(t)$.

Finally, we show that

$$\left(\mathcal{R}^k a t^{\alpha-1}\right)(t) \leq a\left(\Gamma(\beta)b(t)\right)^k c_k t^{\alpha-1}t^{k\nu}, \quad k \in N, \tag{2.19}$$

where $c_0 = 1$, $c_k = \prod_{i=0}^{k-1}(\Gamma(i\nu+\delta)/\Gamma(i\nu+\delta+\beta))$, $k \in N^+$.

Obviously, inequality (2.19) is true for $k = 0$. Suppose that the inequality is satisfied for any fixed $k \in N$. Let us verify that it is also satisfied for $k+1$. According to the induction hypothesis and Lemma 2.6, we obtain

$$\begin{aligned}
\left(\mathcal{R}^{k+1} a t^{\alpha-1}\right)(t) &\leq b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\left(\mathcal{R}^k a s^{\alpha-1}\right)(s)ds \\
&\leq a\left(\Gamma(\beta)b(t)\right)^k c_k b(t)\int_0^t (t-s)^{\beta-1}s^{k\nu+\delta-1}ds \\
&= a\left(\Gamma(\beta)b(t)\right)^k c_k b(t)t^{k\nu+\delta+\beta-1}\frac{\Gamma(\beta)\Gamma(k\nu+\delta)}{\Gamma(k\nu+\delta+\beta)} \\
&= a\left(\Gamma(\beta)b(t)\right)^{k+1} c_{k+1}t^{\alpha-1}t^{(k+1)\nu}.
\end{aligned} \tag{2.20}$$

This proves that inequality (2.19) is satisfied for any $k \in N$. In other words, we have proved that

$$u(t) \le \sum_{k=0}^{\infty} a\left(\Gamma(\beta)b(t)\right)^k c_k t^{\alpha-1} t^{k\nu}, \tag{2.21}$$

where $c_0 = 1$, $c_k = \prod_{i=0}^{k-1}(\Gamma(i\nu + \delta)/\Gamma(i\nu + \delta + \beta))$, $k \in N^+$. By virtue of Definition 2.5, we can arrive at inequality (2.8) and the proof of this theorem is completed. □

For the case $b(t) \equiv b > 0$ in Theorem 2.7, we can obtain the following corollary, which can be found in [17].

Corollary 2.8. *Let $\alpha, \beta, \gamma > 0$, $\delta = \alpha + \gamma - 1 > 0$, $\nu = \beta + \gamma - 1 > 0$, $a, b > 0$. And suppose that $u(t)$ is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \le t < T$ $(T \le \infty)$ with*

$$u(t) \le a t^{\alpha-1} + b\int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s)ds, \quad 0 \le t < T. \tag{2.22}$$

Then one has

$$u(t) \le a t^{\alpha-1} F_{\nu,\delta,\delta+\beta}\left(\Gamma(\beta) b t^{\beta}\right), \quad 0 \le t < T. \tag{2.23}$$

For $\alpha = \gamma = 1$ in Theorem 2.7, we can arrive at the following corollary, which can be found in [12].

Corollary 2.9. *Let β, $a > 0$, $b(t)$ be a nonnegative, nondecreasing continuous function on $0 \le t < T$, $b(t) \le M$, where $T \le \infty$, M is a positive constant. And suppose that $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a + b(t)\int_0^t (t-s)^{\beta-1} u(s)ds, \quad 0 \le t < T. \tag{2.24}$$

Then one has

$$u(t) \le a E_{\beta}\left(\Gamma(\beta) b(t) t^{\beta}\right), \quad 0 \le t < T. \tag{2.25}$$

3. Application

In this section, we will apply our established result to study the dependence of the solution on the initial condition of a fractional differential equation with the Riemann-Liouville derivative.

For the reader's convenience, we first recall several definitions of the Reimann-Liouville integral and derivative. From now on, we assume that T is a finite positive constant, that is, $T \neq \infty$.

Definition 3.1 (see [14, 16]). Let $0 < p < 1$. The Riemann-Liouville integral of order p is defined by

$$\left(I_{0^+}^{p} x\right)(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} x(s) ds, \quad 0 \le t \le T. \tag{3.1}$$

Definition 3.2 (see [14, 16]). Let $0 < p < 1$. The Riemann-Liouville derivative of order p is defined by

$$\left(D_{0^+}^{p} x\right)(t) = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t (t-s)^{-p} x(s) ds, \quad 0 \le t \le T. \tag{3.2}$$

Now we consider the following initial value problem of the form

$$\left(D_{0^+}^{p} x\right)(t) = f(t, x(t)), \quad \lim_{t \to 0^+} \left(I_{0^+}^{1-p} x\right)(t) = x_0, \quad 0 < p < 1, \ 0 < t \le T, \ x_0 \in R. \tag{3.3}$$

With regard to problem (3.3), the existence and uniqueness of the solution can be found in the book by Kilbas et al. [14]. For the completeness of this paper, we give the existence and uniqueness of the solution to (3.3) in the weighted space $C_{1-p}([0,T])$. The space $C_{1-p}([0,T])$ consists of all functions $x \in C((0,T])$ such that $t^{1-p}x(t) \in C([0,T])$, which turns out to be a Banach space when endowed with the norm $|x|_{1-p} = \max_{0 \le t \le T} |t^{1-p}x(t)|$.

Theorem 3.3 (see [14]). *Let $0 < p < 1$, and $f(t,x) : (0,T] \times R \to R$ be a function such that for any $x \in R$, $f(t,x) \in C_{1-p}([0,T])$. Further assume that for any $t \in (0,T]$, $x, y \in R$, the following inequality*

$$\left| f(t,x) - f(t,y) \right| \le L \left| x - y \right| \tag{3.4}$$

holds, where $L > 0$ is a constant. Then there exists a unique solution $x(t)$ to problem (3.3) *in the space $C_{1-p}([0,T])$.*

Theorem 3.4. *Let $0 < p < 1$, and $f(t,x) : (0,T] \times R \to R$ be a function such that for any $x \in R$, $f(t,x) \in C_{1-p}([0,T])$. Further assume that for any $t \in (0,T]$, $x, y \in R$, the following inequality*

$$\left| f(t,x) - f(t,y) \right| \le L \left| x - y \right| \tag{3.5}$$

holds, where $L > 0$ is a constant. Assume that x and y are the solutions of problem (3.3) *and*

$$\left(D_{0^+}^{p} y\right)(t) = f(t, y(t)), \quad \lim_{t \to 0^+} \left(I_{0^+}^{1-p} y\right)(t) = y_0, \quad 0 < t \le T, \ y_0 \in R, \tag{3.6}$$

respectively. Then, for $0 \le t \le T$, one has

$$t^{1-p}\left|x(t)-y(t)\right| \le \begin{cases} \dfrac{\left|x_0-y_0\right|}{\Gamma(p)} F_{2p-1,p,2p}(Lt), & \dfrac{1}{2}<p<1,\ 0\le t\le T, \\[2ex] \dfrac{\left|x_0-y_0\right|}{\Gamma(p)} F_{2p+q-1,p+q,2p+q}(L^{*}t), & 0<p\le\dfrac{1}{2},\ 0\le t<1, \\[2ex] \dfrac{\left|x_0-y_0\right|}{\Gamma(p)} F_{2p+q'-1,p+q',2p+q'}(L't), & 0<p\le\dfrac{1}{2},\ 1\le t\le T, \end{cases} \tag{3.7}$$

where q, q', L^{*}, L' are positive constants such that

$$\begin{gathered} 1-2p<q<\log_t^{L/L^{*}}, \quad 0<p\le\frac{1}{2},\ 0<t<1, \\ q'>\max\left\{1-2p,\log_t^{L/L'}\right\}, \quad 0<p\le\frac{1}{2},\ 1\le t\le T. \end{gathered} \tag{3.8}$$

Proof. The proof is rather technical. We first prove the case $1/2<p<1$ and $0\le t\le T$. Since $x(t)$ and $y(t)$ are the solutions of (3.3) and (3.6), we have

$$x(t)=\frac{x_0 t^{p-1}}{\Gamma(p)}+\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s,x(s))ds, \tag{3.9}$$

$$y(t)=\frac{y_0 t^{p-1}}{\Gamma(p)}+\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s,y(s))ds. \tag{3.10}$$

Subtracting (3.10) from (3.9) and using the Lipschitz condition (3.5), we obtain

$$\left|x(t)-y(t)\right| \le \frac{\left|x_0-y_0\right|t^{p-1}}{\Gamma(p)}+\frac{L}{\Gamma(p)}\int_0^t (t-s)^{p-1}\left|x(s)-y(s)\right|ds. \tag{3.11}$$

Taking into account that $x(t), y(t) \in C_{1-p}([0,T])$, we multiply at both sides of inequality (3.11) by t^{1-p} to get

$$t^{1-p}\left|x(t)-y(t)\right| \le \frac{\left|x_0-y_0\right|}{\Gamma(p)}+\frac{Lt^{1-p}}{\Gamma(p)}\int_0^t (t-s)^{p-1}s^{p-1}s^{1-p}\left|x(s)-y(s)\right|ds. \tag{3.12}$$

Denote $u(t)=t^{1-p}|x(t)-y(t)|$. Then, (3.12) can be written as

$$u(t) \le \frac{\left|x_0-y_0\right|}{\Gamma(p)}+\frac{Lt^{1-p}}{\Gamma(p)}\int_0^t (t-s)^{p-1}s^{p-1}u(s)ds. \tag{3.13}$$

Putting $a = |x_0 - y_0|/\Gamma(p)$, $b(t) = Lt^{1-p}/\Gamma(p)$, $\alpha = 1$, $\beta = p$, $\gamma = p$, we see that $\alpha, \beta, \gamma > 0$, $\delta = \alpha + \gamma - 1 = p > 0$, $\nu = \beta + \gamma - 1 = 2p - 1 > 0$, $a > 0$, and $b(t)$ is a nondecreasing continuous function due to $p, L > 0$. So, applying Theorem 2.7 to (3.13), we obtain

$$u(t) \leq \frac{\left|x_0 - y_0\right|}{\Gamma(p)} F_{2p-1,p,2p}(Lt), \quad \frac{1}{2} < p < 1,\ 0 \leq t \leq T. \tag{3.14}$$

Next, we prove the case $0 < p \leq 1/2$ and $0 \leq t < 1$. Notice that the Lipschitz condition (3.5) holds for each t in $t \in (0, T]$. Since $t > 0$ and $L > 0$, we can always choose two positive constants q, L^* such that

$$1 - 2p < q < \log_t^{L/L^*}, \quad 0 < t < 1. \tag{3.15}$$

Condition (3.15) means that $0 \leq 1 - 2p < q$ and $L < L^* t^q$. That is to say, if the Lipschitz condition (3.5) holds for each t in $t \in (0, T]$, then we can always choose two positive constants q, L^* such that the following condition

$$0 \leq 1 - 2p < q, \quad \left|f(t,u) - f(t,v)\right| \leq L^* t^q |u - v| \tag{3.16}$$

holds for each t in $t \in (0, T]$.

Subtracting (3.10) from (3.9) and using condition (3.16), we obtain

$$\left|x(t) - y(t)\right| \leq \frac{\left|x_0 - y_0\right| t^{p-1}}{\Gamma(p)} + \frac{L^*}{\Gamma(p)} \int_0^t (t-s)^{p-1} s^q \left|x(s) - y(s)\right| ds. \tag{3.17}$$

Multiplying t^{1-p} on both sides of (3.17), we get

$$u(t) \leq \frac{\left|x_0 - y_0\right|}{\Gamma(p)} + \frac{L^* t^{1-p}}{\Gamma(p)} \int_0^t (t-s)^{p-1} s^{p+q-1} u(s) ds, \tag{3.18}$$

where $u(t)$ is defined as before. Now, putting $a = |x_0 - y_0|/\Gamma(p)$, $b(t) = L^* t^{1-p}/\Gamma(p)$, $\alpha = 1$, $\beta = p$, $\gamma = p + q$, we see that $\alpha, \beta, \gamma > 0$, $\delta = \alpha + \gamma - 1 = p + q > 0$, $\nu = \beta + \gamma - 1 = 2p + q - 1 = q - (1 - 2p) > 0$, $a > 0$, $b(t)$ is a nondecreasing continuous function due to $p, L^* > 0$. So, applying Theorem 2.7 to (3.18), we have

$$u(t) \leq \frac{\left|x_0 - y_0\right|}{\Gamma(p)} F_{2p+q-1,p+q,2p+q}(L^* t), \quad 0 < p \leq \frac{1}{2},\ 0 \leq t < 1. \tag{3.19}$$

Finally, we prove the case $0 < p \leq 1/2$ and $1 \leq t \leq T$. Since $t > 0$ and $L > 0$, we can always choose two positive constants q', L' such that

$$q' > \max\left\{1 - 2p, \log_t^{L/L'}\right\}, \quad 1 \leq t \leq T. \tag{3.20}$$

Condition (3.20) means that $0 \leq 1 - 2p < q'$ and $L < L't^q$. Using the same arguments as above, we can obtain that

$$u(t) \leq \frac{|x_0 - y_0|}{\Gamma(p)} F_{2p+q'-1,p+q',2p+q'}(L't), \quad 0 < p \leq \frac{1}{2}, \ 1 \leq t \leq T, \tag{3.21}$$

where $u(t)$ is defined as before. So the conclusion of this theorem is true. □

From the proof of Theorem 3.4, we can see that the integral inequality in Theorem 2.7 is very useful. This demonstrates that our investigation is meaningful.

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Research Article

A Multidimensional Scaling Analysis of Musical Sounds Based on Pseudo Phase Plane

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This paper studies musical opus from the point of view of three mathematical tools: entropy, pseudo phase plane (PPP), and multidimensional scaling (MDS). The experiments analyze ten sets of different musical styles. First, for each musical composition, the PPP is produced using the time series lags captured by the average mutual information. Second, to unravel hidden relationships between the musical styles the MDS technique is used. The MDS is calculated based on two alternative metrics obtained from the PPP, namely, the average mutual information and the fractal dimension. The results reveal significant differences in the musical styles, demonstrating the feasibility of the proposed strategy and motivating further developments towards a dynamical analysis of musical sounds.

1. Introduction

For many centuries, philosophers, music composers, and mathematicians worked intensively to find mathematical formulae that could explain the process of music creation. As a matter of fact, music and mathematics are intricately related: strings vibrate at certain frequencies and sound waves can be described by mathematical equations. Although it seems not possible to find an expression that models the musical works, it is recognized that there are certain inherent mathematical structures in all types of music. Through the history of music, we have been faced with the proposal of formal techniques for melody composition, claiming that musical pieces can be created as a result of applying certain rules to some given initial material [1–12]. More recently, the growth of computing power made it possible to generate music automatically.

The concept of entropy was introduced in the field of thermodynamics by Clausius (1862) and Boltzmann (1896) and was later applied by Shannon (1948) and Jaynes (1957) in information theory [13–15]. However, recently more general entropy measures were proposed, allowing the relaxation of the additivity axiom for application in several types of complex systems [16–24]. The novel ideas are presently under a large development and open up promising perspectives.

The pseudo phase space (PPS) is used to analyze signals with nonlinear behavior. For the two-dimensional case it is called pseudo phase plane (PPP) [25–27]. To reconstruct the PPS it is necessary to find the adequate time lag between the signal and one delayed image of the original signal. To determine the proper lag (or time delay) often the mutual information concept is used.

The Multidimensional Scaling (MDS) has its origins in psychometrics and psychophysics, where it is used as a tool for perceptual and cognitive modeling. From the beginning MDS has been applied in many fields, such as psychology, sociology, anthropology, economy, and educational research. In the last decades this technique has been applied also in others areas, including computational chemistry [28], machine learning [29], concept maps [30], and wireless network sensors [31].

Bearing these facts in mind, the present study combines the referred concepts and is organized as follows. Section 2 introduces a brief description of the fundamental concepts. Section 3 formulates and develops the musical study through several entropy measures and MDS analysis. Finally, Section 4 outlines the main conclusions.

2. Fundamental Concepts

This section presents the main tools adopted in this study, namely, the musical signals, the PPP, the fractional dimension, and the MDS.

2.1. Musical Sounds

In the context of this study, a musical work is a set of one or more time-sequenced digital data streams, representing a certain time sampling of the original musical source. For all musical objects, the original data streams result from sampling at 44 kHz, subsequently converted to a single (mono-) digital data series, each sample being a 32-bit signed floating value.

These sounds have a strong variability, making difficult their direct comparison in the time domain. In this line of thought, several tests were developed to obtain methods that establish a compromise between smoothing the high signal variability and handling the rhythm and style time evolution that are the essence of each composition. The Shannon entropy S of the signals is shown to be an appropriate method:

$$S = -\sum_{x \in X} p(x) \ln[p(x)], \tag{2.1}$$

where X is the set of all possible events and $p(x)$ is the probability that event i occurs so that $\sum_{x \in X} p(x) = 1$.

For a bidimensional random variable the join entropy becomes

$$S = -\sum_{x \in X} \sum_{y \in Y} p(x, y) \ln[p(x, y)]. \tag{2.2}$$

2.2. Pseudo Phase Plane

The PPS is used to analyze signals with nonlinear behavior. The proper time lagT_d, for the delay measurements, and the adequate dimension $d \in N$ of the space must be determined in order to achieve the phase space. In the PPS the measurement $s(t)$ forms the *pseudo* vector $y(t)$ according to

$$y(t) = [s(t), s(t+T_d), \dots, s(t+(d-1)T_d)]. \tag{2.3}$$

The vector $y(t)$ can be plotted in a d-dimensional space forming a curve in the PPS. If $d = 2$ we have a two-dimensional space, and, therefore, the PPP is a particular case of the PPS technique.

The procedure of choosing a sufficiently large d is formally known as embedding and any dimension that works is called an embedding dimension d_E. The number of measurements d_E should provide a phase space dimension, in which the geometrical structure of the plotted PPS is completely unfold and where there are no hidden points in the resulting plot.

Among others [26], the method of delays is the most common method for reconstructing the phase space. Several techniques have been proposed to choose an appropriate time delay [27]. One line of thought is to choose T_d based on the correlation of the time series with its delayed image. The difficulty of correlation to deal with nonlinear relations leads to the use of the mutual information. This concept, from the information theory [32], recognizes the nonlinear properties of the series and measures their dependence. The average mutual information for the two series of variables t and $t + T_d$ is given by

$$I(t, t+T_d) = \int_t \int_{t+T_d} F_1\{s(t), s(t+T_d)\} \log_2 \frac{F_1\{s(t), s(t+T_d)\}}{F_2\{s(t)\}F_3\{s(t+T_d)\}} dt d(t+T_d), \tag{2.4}$$

where $F_1\{s(t), s(t+T_d)\}$ is a bidimensional probability density function and $F_2\{s(t)\}$ and $F_3\{s(t+T_d)\}$ are the marginal probability distributions of the two series $s(t)$ and $s(t+T_d)$, respectively.

The index I allows us to obtain the time lag required to construct the pseudo phase space. For finding the best value T_d of the delay, I is computed for a range of delays and the first minimum is chosen. Usually I is referred [25–27] as the preferred alternative to select the proper time delay T_d.

2.3. Fractal Dimension

The fractal dimension is a quantity that gives an indication of how completely a spatial representation appears to fill space. There are many specific methods to compute the fractal dimension [33, 34]. The most popular methods are the Hausdorff and box-counting dimensions. Here the box-counting dimension method is used due to its simplicity of implementation and is defined as

$$fd = \lim_{\varepsilon \to 0} \frac{\ln[N(\varepsilon)]}{\ln(1/\varepsilon)}, \tag{2.5}$$

where $N(\varepsilon)$ represents the minimal number of covering cells (e.g., boxes) of size ε required to cover the set analyzed. The slope on a plot of $\ln[N(\varepsilon)]$ versus $\ln(1/\varepsilon)$ provides an estimate of the fractal dimension.

2.4. Multidimensional Scaling

MDS is a generic name for a family of algorithms that construct a configuration of points in a low-dimensional space from information about interpoint distances measured in high-dimensional space. The new geometrical configuration of points, preserving the proximities of the high dimensional space, facilitates the perception underlying structure of the data and often makes it much easier to analyze. The problem addressed by MDS can be stated as follows: given n items in an m-dimensional space and an $n\times n$ matrix $\mathbf{C}$ of proximity measures among the items, MDS produces a p-dimensional configuration $\Phi, p \leq m$, representing the items such that the distances among the points in the new space reflect, with some degree of fidelity, the proximities in the data. The proximity measures the closeness (in MDS terms usually referred as similarities) among the items and, in general, it is a distance measure: the more similar two items are, the smaller their distance is.

The Minkowski distance metric provides a general way to specify distance for quantitative data in a multidimensional space:

$$d_{ij} = \left(\sum_{k=1}^{m} w_k \left|x_{ik} - x_{jk}\right|^r\right)^{1/r}, \tag{2.6}$$

where m is the number of dimensions, x_{ik} is the value of the kth component of object i, and w_k is a weight factor.

For $w_k = 1$, if $r = 2$ then (2.6) yields the Euclidean distance, and if $r = 1$ then it leads to the city-block (or Manhattan) distance. In practice, the Euclidean distance is generally used, but there are several other definitions that can be applied, including for binary data [35].

Typically MDS is used to transform the data into two or three dimensions for visualizing the result to uncover the data hidden structure, but any $p < m$ is possible. Some authors use a rule of thumb to determine the maximum number of m, which is to ensure that there are at least twice as many pairs of items than the number of parameters to be estimated, resulting in $m \geq 4p + 1$ [36]. The geometrical representation obtained with MDS is indeterminate with respect to translation, rotation, and reflection [37].

There are two forms of MDS, namely, the metric MDS and the nonmetric MDS. The metric MDS uses the actual values of dissimilarities, while nonmetric MDS effectively uses only their ranks [38, 39]. Metric MDS assumes that the dissimilarities δ_{ij} calculated in the original m-dimensional data and distances d_{ij} in the p-dimensional space are related as follows:

$$d_{ij} \approx f(\delta_{ij}), \tag{2.7}$$

where f is a continuous monotonic function. Metric (scaling) refers to the type of transformation f of the dissimilarities and its form determines the MDS model. If $d_{ij} = \delta_{ij}$ (it means $f = 1$) and a Euclidean distance is used then we obtain the classical (metric) MDS.

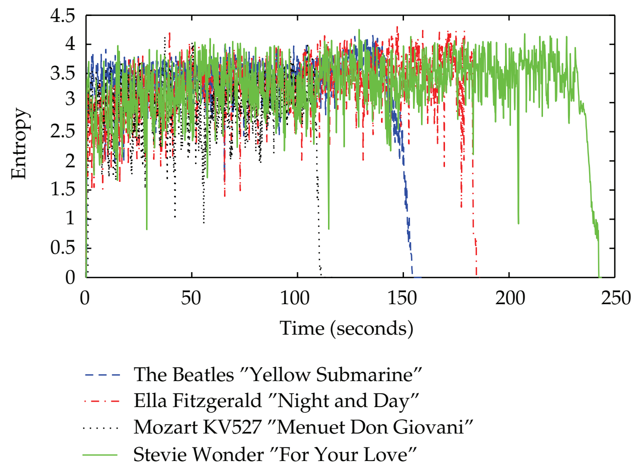


Figure 1: Entropy S versus time t of four musical compositions using a sliding window of $T = 1$ second. Musical compositions—The Beatles: "*Yellow Submarine*," Ella Fitzgerald: "*Night and Day*," Mozart: "*KV527 Menuet Don Giovani*," and Stevie Wonder: "*For Your Love*."

In metric MDS the dissimilarities between all objects are known numbers and they are approximated by distances. Therefore, objects are mapped into a low-dimensional space, distances are calculated and compared with the dissimilarities. Then objects are moved in such way that the fit becomes better, until an objective function (called stress function in the context of MDS) is minimized.

In nonmetric MDS, the metric properties of f are relaxed, but the rank order of the dissimilarities must be preserved. The transformation function f obeys the monotonicity constraint $\delta_{ij} < \delta_{rs} \Rightarrow f(\delta_{ij}) \leq f(\delta_{rs})$ for all objects. The advantage of nonmetric MDS is that no assumptions need to be made about the underlying transformation function f. Therefore, it can be used in situations that only the rank order of dissimilarities is known (ordinal data). Additionally, it can be used in cases which there are incomplete information. In such cases, the configuration Φ is constructed from a subset of the distances, and, at the same time, the other (missing) distances are estimated by monotonic regression. In nonmetric MDS it is assumed that $d_{ij} \approx f(\delta_{ij})$ and, therefore, $f(\delta_{ij})$ are often referred as the disparities [40–42] in contrast to the original dissimilarities δ_{ij}, on one hand, and the distances d_{ij} of the configuration space, on the other hand. In this context, the disparity is a measure of how well the distance d_{ij} matches the dissimilarity δ_{ij}.

With further developments over the years, MDS techniques are commonly classified according to the type of data to analyze. From this point of view, the techniques are embedded into the following MDS categories [35, 42]: (i) one-way *versus* multiway: in K-way MDS each pair of objects has K dissimilarity measures from different replications (e.g., repeated measures); (ii) one-mode *versus* multimode: similar to (i) but the K dissimilarities are qualitatively different (e.g., distinct experimental conditions).

There is no rigorous statistical method to evaluate the quality and the reliability of the results obtained by an MDS analysis. However, there are two methods often used for that purpose: the Shepard plot and the stress. The Shepard plot is a scatter plot of the dissimilarities and disparities against the distances, usually overlaid with a line having unitary slope. The plot provides a qualitative evaluation of the goodness of fit. On the other hand, the stress value gives a quantitative evaluation. Additionally, the stress plotted as a

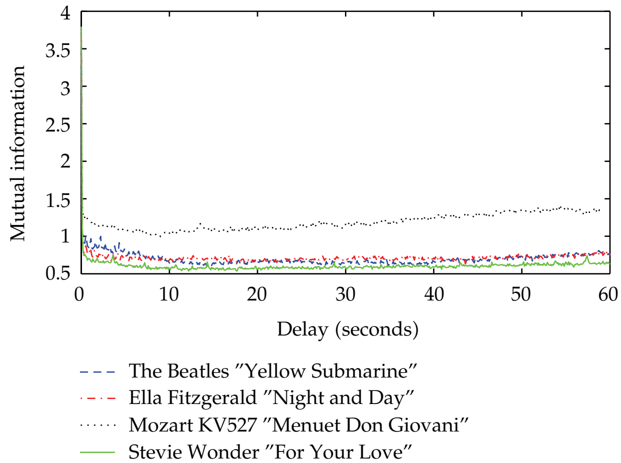


Figure 2: Average mutual information I versus lag T_d of four musical compositions—The Beatles: “*Yellow Submarine*,” Ella Fitzgerald: “*Night and Day*,” Mozart: “*KV527 Menuet Don Giovani*,” and Stevie Wonder: “*For Your Love*.”

function of dimensionality can be used to estimate the adequate p-dimension (known as scree plot). When the curve ceases to decrease significantly the resulting “elbow” may correspond to a substantial improvement in fit.

Beyond the aspects referred before, there are other developments of MDS that include Procrustean methods, individual differences models (also known as three-way models), and constrained config uration.

In the Procrustean methods the data is analyzed by scaling each replication separately and then comparing or aggregating the different MDS solutions. The individual differences models scale a set of K dissimilarity matrices into only one MDS solution. The procedure of constraints on the configuration (which Borg and Groenen called “confirmatory MDS” [43]) is used when the researcher has some substantive underlying theory regarding a decomposition of the dissimilarities and, consequently, tries to restrain the configuration space.

3. Study of Musical Sounds

This section develops the musical study using entropy applied to a large sample of representative musical works. Once having the entropy measurements, the corresponding time lags and the PPP are calculated. Finally, an MDS analysis is performed using two alternative criteria, namely, based on mutual information and fractal dimension.

3.1. Entropy Analysis of Musical Compositions

For the calculation of the entropy S is considered a rectangular window of duration T that slides over time t capturing a limited part of the signal evolution. Each new window overlaps 50% with the previous one. For the signal captured in the window a histogram of relative frequency of amplitudes is obtained and $S(t)$ calculated. Several experiments demonstrated that a sampling window with width $T = 1$ represented a good compromise between

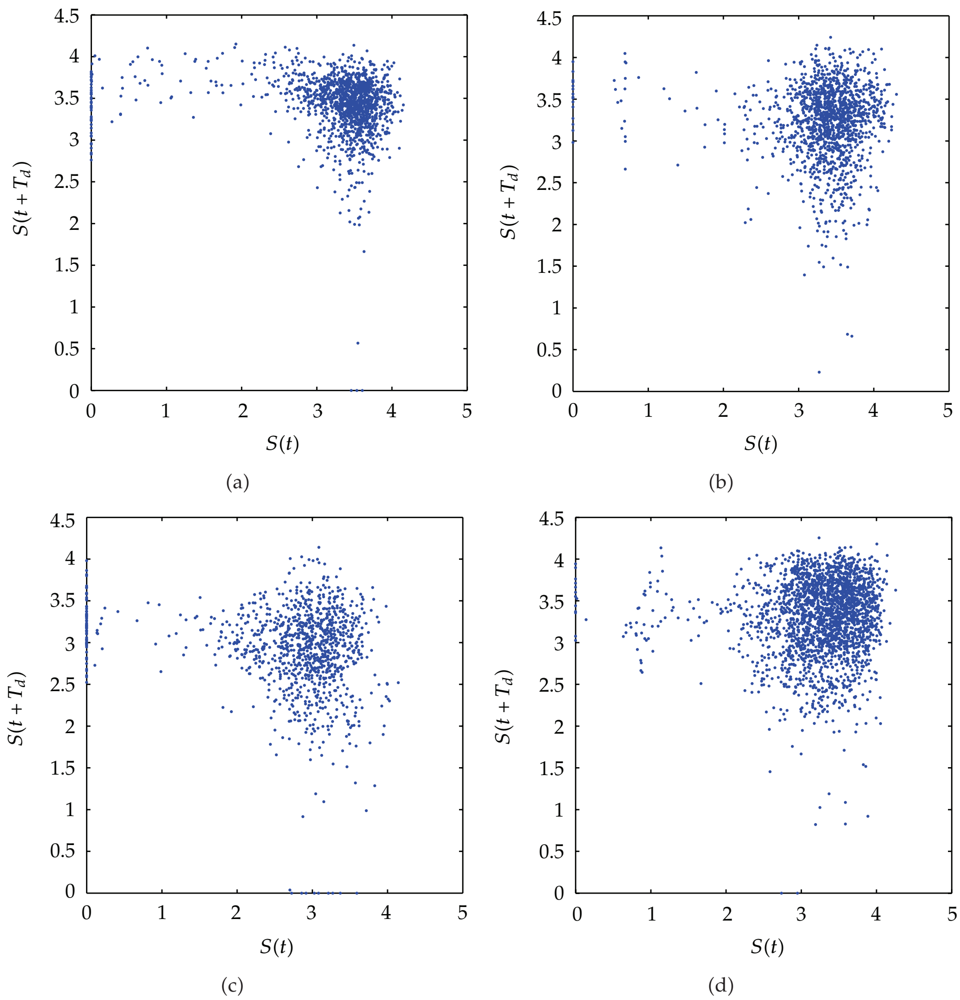


Figure 3: PPP of four musical compositions: (a) The Beatles: "*Yellow Submarine*"; (b) Ella Fitzgerald: "*Night and Day*"; (c) Mozart: "*KV527 Menuet Don Giovani*"; (d) Stevie Wonder: "*For Your Love*."

the original signal's frequency (tenths of microseconds) and the musical piece's duration (hundreds of seconds).

Figure 1 shows the evolution of several musical sounds viewed through the entropy versus time for a sliding window of $T = 1$. The entropy curves represent four different compositions, namely, The Beatles: "*Yellow Submarine*," Ella Fitzgerald: "*Night and Day*," Mozart: "*KV527 Minuet Don Giovanni*," and Stevie Wonder: "*For Your Love*."

3.2. Pseudo Phase Plane of Entropy Curves from Musical Compositions

Having established the concept of time evolution of the entropy measure for musical compositions, the question of how the entropies of compositions with different "types" are interrelated was investigated. Several music titles from different "types" were selected: "Classical" (49 titles), "Easy" (31), "Electro" (16), "Jazz" (50), "Brazilian Music" (18), "Portuguese

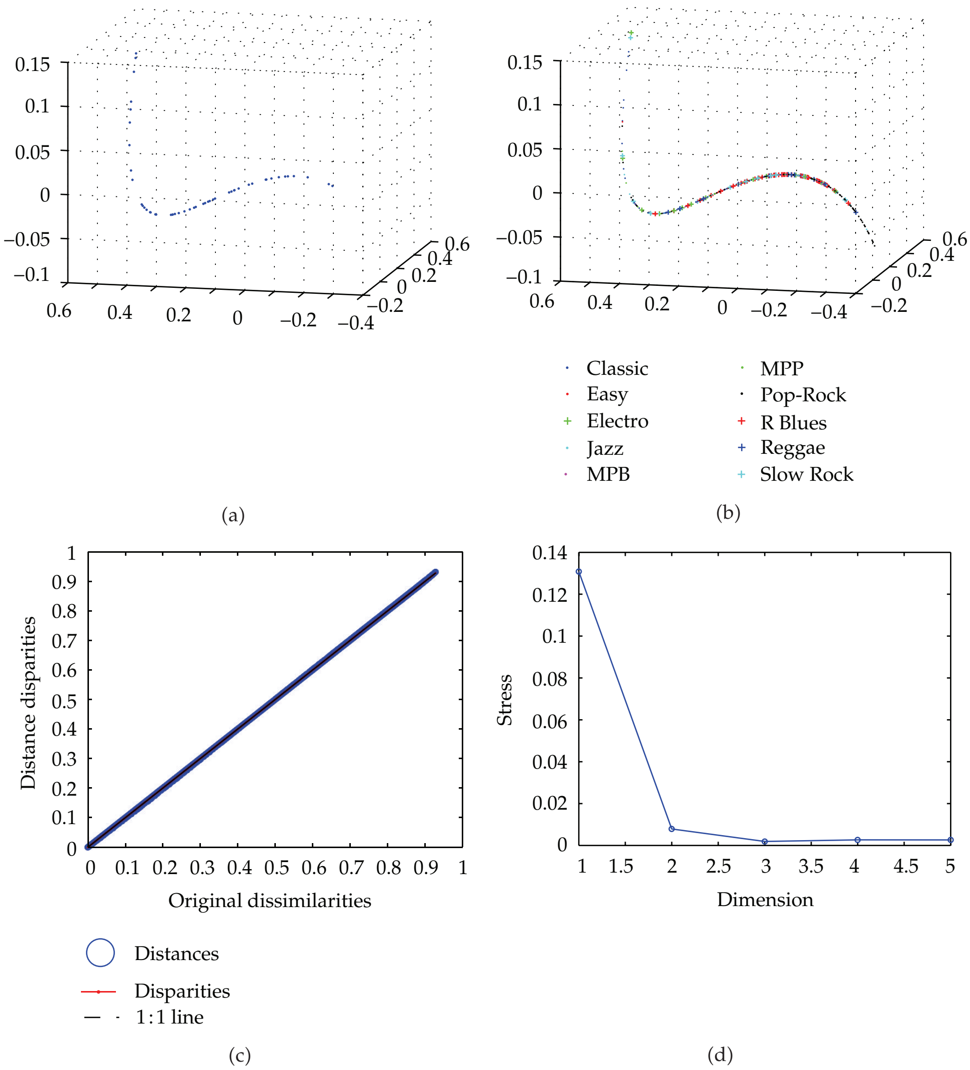


Figure 4: MDS using c_{ij}^{I} for (a) classic compositions; (b) all musical compositions tested; (c) Shepard plot for 3D; (d) scree plot.

Music" (17), "Pop and Rock" (167), "Rhythm Blues" (44), "Reggae" (15), and "Slow Rock" (19). These samples lead to a population of $N = 426$ music titles.

For each signal $S(t)$ derived from the 426 compositions, the average mutual information I was calculated. For example, Figure 2 shows the average mutual information I versus lag T_d of four musical compositions—The Beatles: "*Yellow Submarine*," Ella Fitzgerald: "*Night and Day*," Mozart: "*KV527 Menuet Don Giovani*," and Stevie Wonder: "*For Your Love*." The minimum of the average mutual information $I_{\min}$ and the corresponding delay yield $(T_d, I_{\min}) = \{(14.3, 0.6), (43.6, 0.6), (9, 1), (12.2, 0.5)\}$, respectively. To reconstruct the PPP, the first minimum of I was considered. The corresponding PPPs are represented in Figure 3.

Usually T_d is just calculated for the PPP reconstruction. However, the time lag represents a "memory" of previous parts of the time series and, therefore, this information

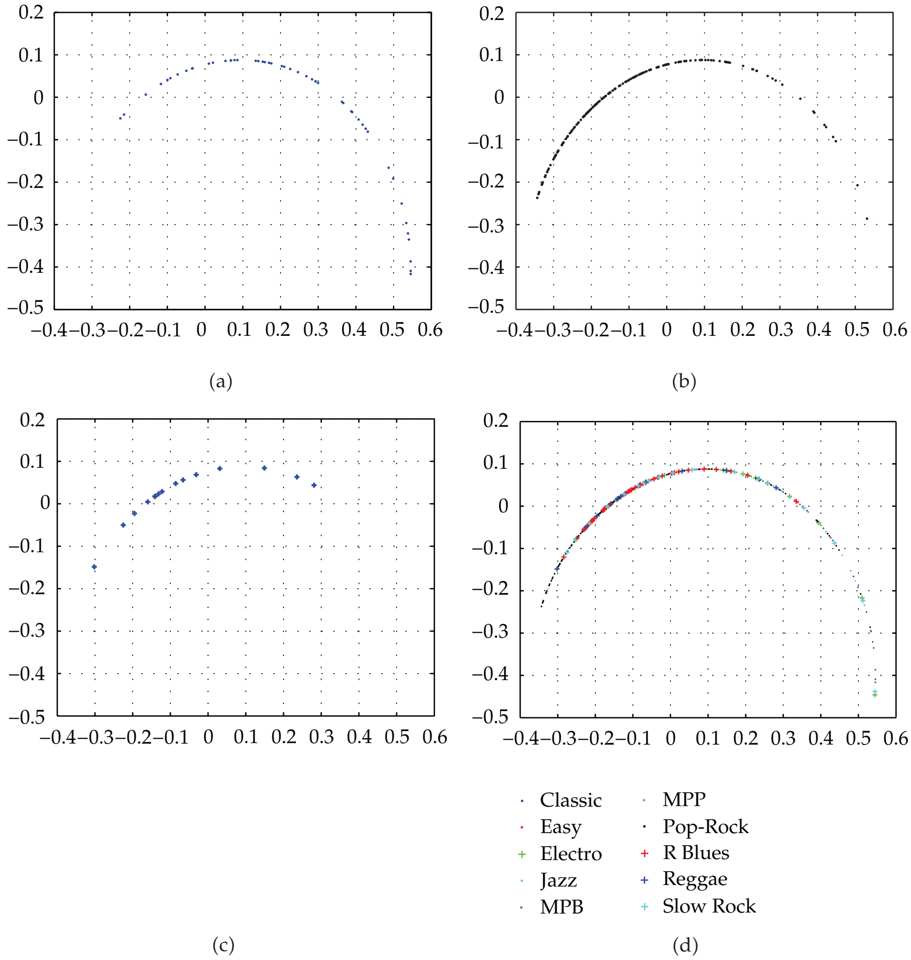


Figure 5: 2D locus generated by MDS using c_{ij}^{I} for: (a) Classic; (b) Pop-Rock; (c) Reggae; all musical compositions tested.

is related with the fractional dynamics embedded in the music [44–46]. Consequently, the value of $I_{\min}$ and the characteristics of the PPP chart obtained for T_d are important details to be included in the MDS maps to be formulated in the next subsection.

3.3. Multidimensional Scaling Analysis of Musical Compositions

In order to reveal hypothetical relationships between the musical compositions the MDS technique is used. Two alternative metrics to compare objects i and j were adopted, namely,

$$c_{ij}^{I} = e^{-(I_{\min_i} - I_{\min_j})^2}, \quad i, j = 1, \ldots, N, \tag{3.1}$$

$$c_{ij}^{fd} = e^{-(fd_i - fd_j)^2}, \quad i, j = 1, \ldots, N, \tag{3.2}$$

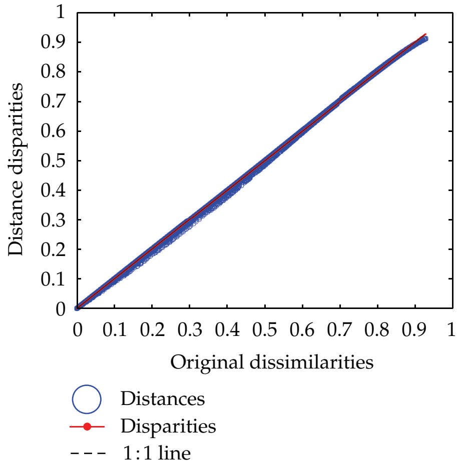


Figure 6: Shepard plot for 2D MDS using c_{ij}^{I}.

where N is the total number of music, c_{ij}^{I} defined in (3.1) is based on the minimal of the average mutual information $I_{\min}$, and c_{ij}^{fd} defined in (3.2) is based on the fractal dimension fd of the reconstructed PPP.

For each of the two indices a 426×426 symmetrical matrix $\mathbf{C}$ with 1's in the main diagonal was calculated and the MDS maps obtained.

Figure 4(a) shows the locus of the classic compositions obtained by MDS using c_{ij}^{I} for the dimension $p = 3$. The locus obtained with this exponential type of metric forms a curve. Due to space limitations we are only depicting the locus obtained for some individual types of music. The tests developed show that each type of music occupies a certain segment in the curve obtained for all the musical compositions (Figure 4(b)). Figures 4(c) and 4(d) depict two tests computed to evaluate the consistency of the results obtained by MDS analysis. The Shepard plot (Figure 4(c)) shows the fitting of the 3D configuration distances to the dissimilarities. The value of the stress function versus the dimension is shown in Figure 4(d), that allows the estimation of the adequate p-dimension. An "elbow" occurs at dimension two for a low value of stress, which corresponds to a significant improvement in fit. From the scree plot can be concluded that the improvement obtained for the increasing of the p-dimension from $p = 2$ to $p = 3$ is very low. Therefore, the 2D MDS configuration is appropriate.

In this line of thought, Figures 5(a)–5(c) show the 2D locus for the Classic, Pop and Rock, and Reggae types of music, respectively. The Classic music compositions (Figure 5(a)) occupy a segment of approximately 80% of the curve obtained for all the musical compositions tested (Figure 5(d)). This segment begins near one end of the curve. The Pop and Rock music is located over a segment of approximately 80% of the curve beginning near the other end (Figure 5(b)). Therefore, approximately 60% of the positions for these two types of music are superimposed in the center of the curve. For the Pop and Rock most of the positions are concentrated in the half of the segment positioned at the opposite side of the classic music. The Reggae music compositions are located over a limited zone near the center of the curve (Figure 5(c)). Figure 5(d) shows the curve obtained for the 426 musical titles tested. The Jazz zone is centered approximately in the middle of the curve and corresponds to the superimposed zone of the Classic and the Pop and Rock. The Rhythm Blues titles are

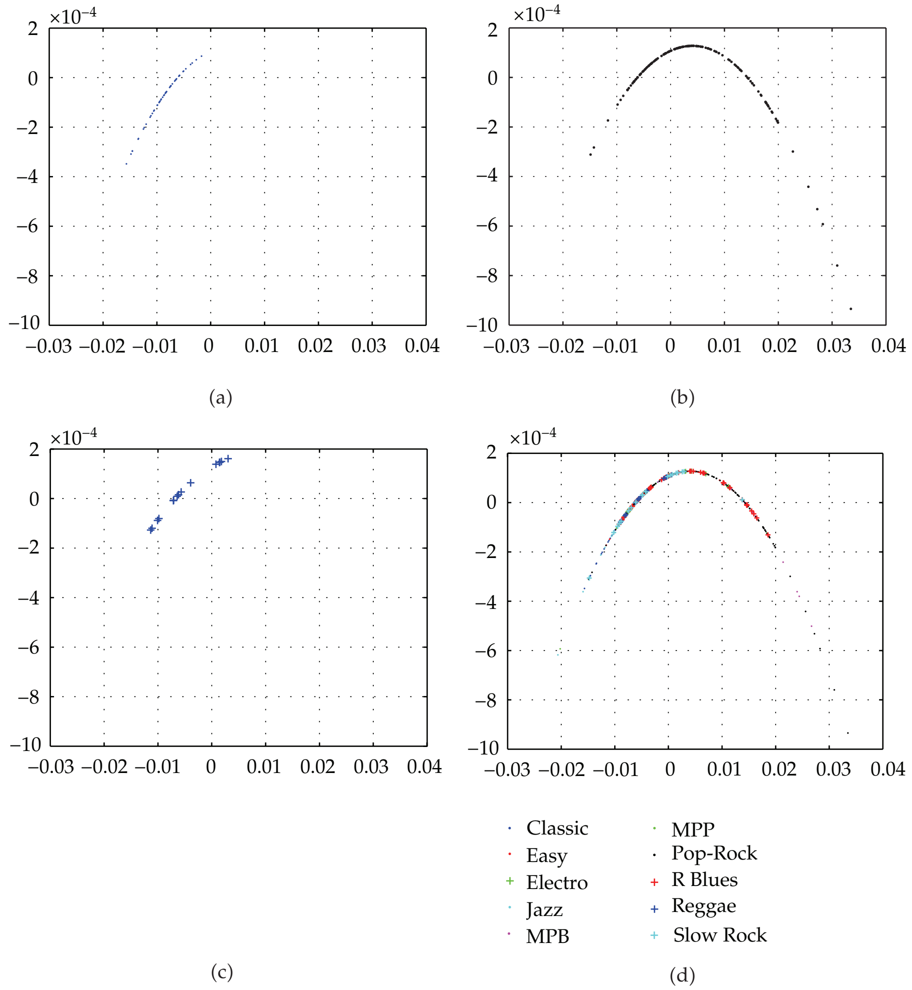
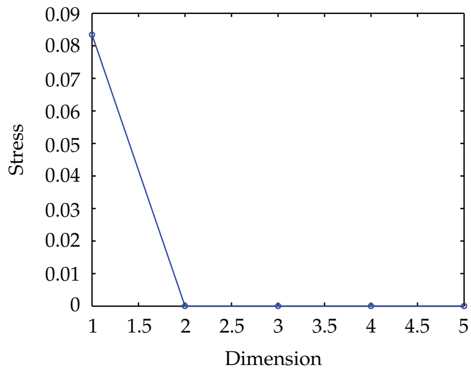


Figure 7: 2D locus generated by MDS using c_{ij}^{fd} for (a) Classic; (b) Pop-Rock; (c) Reggae; (d) all musical compositions tested.

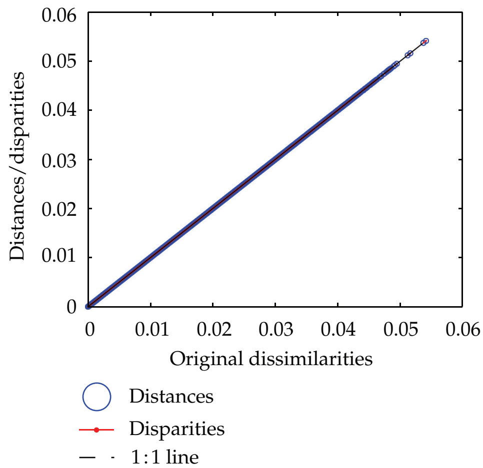
located approximately in the same zone of that corresponding to the Reggae. The Slow Rock and the Electro types occupy approximately the same segment that corresponds to the Classic music, nevertheless in a scattered way near the end of the curve. The Easy type occupies a shorter segment than the one occupied by the Slow Rock and the Electro. Finally, the Brazilian and the Portuguese compositions occupy a segment that corresponds approximately to the Reggae one, but with a slightly shift to the side of the Classic music. The shift is more pronounced for the case of the Portuguese music.

Figure 6 depicts the Shepard plot that confirms the good fitting of the 2D configuration distances to the dissimilarities.

Figure 7 shows the locus of the musical compositions obtained by MDS using the metric c_{ij}^{fd}. Figures 7(a)–7(c) show the locus for the Classic, Pop and Rock, and Reggae types of music, respectively. The Classic music compositions form a segment located in one end of the curve (Figure 7(a)). The Pop and Rock musical opus occupies the most part of the curve



(a)



(b)

Figure 8: Evaluation of MDS results using c_{ij}^{fd}: scree plot (a); Shepard plot for 2D (b).

in a scattered way, but with a slightly superimposition over the Classic (Figure 7(b)). The Reggae music compositions are located on a limited zone superimposed over the Classic and the Pop and Rock compositions (Figure 7(c)).

Figure 7(d) shows the locus of the 426 musical titles. In general the relative positions for the others types of music are similar to those obtained for c_{ij}^{I}. Nevertheless the positions achieved with the metric c_{ij}^{fd} are represented in a curve shorter than the one obtained with c_{ij}^{I} that occasionally can make the analysis difficult.

Figure 8 shows the scree and Shepard plots to evaluate the results obtained by MDS using c_{ij}^{fd}. Again, an "elbow" occurs at dimension two for a low value of stress (Figure 8(a)), which corresponds to a significant improvement in fit. Additionally, the Shepard plot (Figure 8(b)) shows the fitting of the 2D configuration distances to the dissimilarities.

The results obtained with the proposed tools, namely, the MDS and the PPP, together with the tested metrics proved to be assertive methods to analyze the musical compositions.

4. Conclusions

Through the history of music many authors tried to find mathematical formulae that could explain the process of music creation. In this perspective, the study analyzes the musical compositions from a mathematical view point. The representation in the time domain of the music compositions presents characteristics which makes difficult their direct comparison. To overcome this limitation the Shannon entropy was used together with other tools, namely, the pseudo phase plane and multidimensional scaling. These tools were applied to an aggregate of different type sets of music compositions. The proposed tools proved to be assertive methods to analyze music. In future work, we plan to pursue several research directions to help us understand the behavior of the musical signals. These include other techniques to measure the similarities of the signals.

Acknowledgments

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Research Article

Existence and Uniqueness of Solution for a Class of Nonlinear Fractional Order Differential Equations

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We discuss the existence and uniqueness of solution to nonlinear fractional order ordinary differential equations $(\mathfrak{D}^{\alpha} - \rho t \mathfrak{D}^{\beta})x(t) = f(t, x(t), \mathfrak{D}^{\gamma} x(t))$, $t \in (0,1)$ with boundary conditions $x(0) = x_0$, $x(1) = x_1$ or satisfying the initial conditions $x(0) = 0$, $x'(0) = 1$, where $\mathfrak{D}^{\alpha}$ denotes Caputo fractional derivative, ρ is constant, $1 < \alpha < 2$, and $0 < \beta + \gamma \leq \alpha$. Schauder's fixed-point theorem was used to establish the existence of the solution. Banach contraction principle was used to show the uniqueness of the solution under certain conditions on f.

1. Introduction

Fractional calculus deals with generalization of differentiation and integration to the fractional order [1, 2]. In the last few decades the fractional calculus and fractional differential equations have found applications in various disciplines [2–6]. Owing to the increasing applications, a considerable attention has been given to exact and numerical solutions of fractional differential equations [2, 6–11]. Many papers were dedicated to the existence and the uniqueness of the fractional differential equations, to the analytic methods for solving fractional differential equations, e.g., Greens function method, the Mellin transform method, and the power series (see for example references [2, 6–26] and the references therein). On this line of taught in this manuscript we proved the existence and uniqueness of a specific nonlinear fractional order ordinary differential equations within Caputo derivatives. Very recently in [27–31], the authors and other researchers studied the existence and uniqueness of solutions of some classes of fractional differential equations with delay. The paper is organized as follows: In Section 2 we introduce some necessary definitions and mathematical preliminaries of fractional calculus. In Section 3 sufficient conditions are established for the existence and uniqueness of solutions for a class fractional order differential equations satisfying the boundary conditions or satisfying the initial conditions. In order to illustrate our results several examples are presented in Section 3.

2. Fractional Integral and Derivatives

In this section, we present some notations, definitions, and preliminary facts that will be used further in this work. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations. Therefore, in this work we will use the Caputo fractional derivative $\mathfrak{D}$ proposed by Caputo in his work on the theory of viscoelasticity [32].

Let $\alpha \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$ and $x \in C((0,\infty),\mathbb{R})$; then the Caputo fractional derivative of order α defined by

$$\mathfrak{D}^{\alpha}x(t) = \mathcal{I}^{n-\alpha}\left(\frac{d^n x(t)}{dt^n}\right), \tag{2.1}$$

where

$$\mathcal{I}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, \tag{2.2}$$

is the Riemann-Liouville fractional integral operator of order α and Γ is the gamma function.

The fractional integral of $x(t) = (t-a)^{\beta}$, $a \geq 0$, $\beta > -1$ is given as

$$\mathcal{I}^{\alpha}x(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{2.3}$$

For $\alpha,\beta \geq 0$, we have the following properties of fractional integrals and derivative [33].

The fractional order integral satisfies the semigroup property

$$\mathcal{I}^{\alpha}\left(\mathcal{I}^{\beta}x(t)\right) = \mathcal{I}^{\beta}(\mathcal{I}^{\alpha}x(t)) = \mathcal{I}^{\alpha+\beta}x(t). \tag{2.4}$$

The integer order derivative operator $\mathfrak{D}^m$ commutes with fractional order $\mathfrak{D}^{\alpha}$, that is:

$$\mathfrak{D}^m(\mathfrak{D}^{\alpha}x(t)) = \mathfrak{D}^{m+\alpha}x(t) = \mathfrak{D}^{\alpha}(\mathfrak{D}^m x(t)). \tag{2.5}$$

The fractional operator and fractional derivative operator do not commute in general. Then the following result can be found in [33, 34].

Lemma 2.1 (see [33, 34]). *For $\alpha > 0$, the general solution of the fractional differential equation $\mathfrak{D}^{\alpha} x(t) = 0$ is given by*

$$x(t) = \sum_{i=0}^{r-1} c_i t^i, \quad c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, r-1, \ r = [\alpha] + 1, \tag{2.6}$$

where $[\alpha]$ denotes the integer part of the real number α.

In view of Lemma 2.1 it follows that

$$\mathcal{I}^{\alpha}\left(\mathfrak{D}^{\alpha} x(t)\right) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{r-1} t^{r-1} \quad \text{for some } c_i \in \mathbb{R}, \ i = 0, 1, \ldots, r-1. \tag{2.7}$$

But in the opposite way we have

$$\mathfrak{D}^{\alpha}\left(\mathcal{I}^{\beta}(t)\right) = \mathfrak{D}^{\alpha-\beta} x(t). \tag{2.8}$$

Proposition 2.2. *Assume that $x : [0, \infty) \to \mathbb{R}$ is continuous and $0 < \beta \le \alpha$. Then*

(i) $\mathcal{I}^{\alpha}(t x(t)) = t \mathcal{I}^{\alpha} x(t) - \alpha \mathcal{I}^{\alpha+1} x(t)$,

(ii) $\mathcal{I}^{\alpha}\{t \mathfrak{D}^{\beta} x(t)\} = t \mathcal{I}^{\alpha-\beta} x(t) - \alpha \mathcal{I}^{\alpha-\beta+1} x(t)$.

The proof of the above proposition can be found in [9, page 53].

As a pursuit of this in this paper, we discuss the existence and uniqueness of solution for nonlinear fractional order differential equations

$$\left(\mathfrak{D}^{\alpha} - \rho t \mathfrak{D}^{\beta}\right) x(t) = f\left(t, x(t), \mathfrak{D}^{\gamma} x(t)\right), \quad t \in (0, 1), \tag{2.9}$$

satisfying the boundary conditions

$$x(0) = x_0, \qquad x(1) = x_1, \tag{2.10}$$

or satisfying the initial conditions

$$x(0) = x_0, \qquad x'(0) = 1, \tag{2.11}$$

where $1 < \alpha \le 2$ and $0 < \beta + \gamma \le \alpha$.

In the following, we present the existence and the uniqueness results for fractional differential equation (2.9) with boundary conditions (2.10).

3. Existence and Uniqueness of Solutions

Lemma 3.1. *Assume that* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Then* $x \in C[0,1]$ *is a solution of the boundary value problem* (2.9) *and* (2.10) *if and only if* $x(t)$ *is the solution of the integral equation*

$$\begin{aligned} x(t) &= -c_0 - c_1 t + \rho t I^{\alpha-\beta} x(t) - \rho\alpha I^{\alpha-\beta+1} x(t) + I^{\alpha} f(t, x(t), \mathfrak{D}^{\gamma} x(t)) \\ &= x_0 + (x_1 - x_0)t + \int_0^1 \mathcal{G}(t,s)\,ds, \end{aligned} \tag{3.1}$$

for some constants c_0, c_1 *where* $\mathcal{G}(t,s)$ *given by*

$$\mathcal{G}(t,s) = \begin{cases} \mathcal{G}_1(t,s) & 0 \le s < t, \\ \mathcal{G}_2(t,s), & t \le s \le 1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \mathcal{G}_1(t,s) &= \rho\left\{ \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s) \\ &\quad + \left\{ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \right\} f(s, x(s), \mathfrak{D}^{\gamma} x(s)), \\ \mathcal{G}_2(t,s) &= \rho t \left\{ \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right\} x(s) \\ &\quad - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s)). \end{aligned} \tag{3.3}$$

Proof. Assume that $x \in C[0,1]$ is a solution of the fractional differential equation (2.9) satisfying boundary conditions (2.10). Then in view of Lemma 2.1 and Proposition 2.2, we have

$$\begin{aligned} x(t) &= \rho t I^{\alpha-\beta} x(t) - \rho\alpha I^{\alpha-\beta+1} x(t) + I^{\alpha} f(t, x(t), \mathfrak{D}^{\gamma} x(t)) - c_0 - c_1 t \\ &= \rho \int_0^t \left\{ \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s)\,ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s))\,ds - c_0 - c_1 t, \end{aligned} \tag{3.4}$$

for some constants c_0 and c_1. Hence using the boundary conditions (2.10) we obtain $c_0 = -x_0$ and

$$c_1 = x_0 - x_1 + \rho \int_0^1 \left\{ \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s))ds. \tag{3.5}$$

Substituting $c_0 = -x_0$ and c_1 into (3.4) we get

$$x(t) = x_0 + (x_1 - x_0)t - \rho t \int_0^1 \left\{ \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s)ds \tag{3.6}$$

$$- t \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s))ds \tag{3.7}$$

$$+ \int_0^t \left\{ \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s)ds \tag{3.8}$$

$$\begin{aligned}
&+ \rho \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s))ds, \\
&= x_0 + (x_1 - x_0)t \\
&\quad + \rho \int_0^t \left\{ \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} x(s)ds \\
&\quad + \int_0^t \left\{ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \right\} f(s, x(s), \mathfrak{D}^{\gamma} x(s))ds \\
&\quad + \rho \int_t^1 \left\{ \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right\} x(s)ds \\
&\quad - \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma} x(s))ds \\
&= x_0 + (x_1 - x_0)t + \int_0^1 \mathcal{G}(t,s)ds.
\end{aligned} \tag{3.9}$$

We consider the space

$$\mathcal{B} = \{x(t) : x(t) \in C[0,1], \mathfrak{D}^{\gamma} x(t) \in C[0,1]\}, \tag{3.10}$$

furnished with the norm

$$\|x(t)\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|\mathfrak{D}^{\gamma}x(t)|. \tag{3.11}$$

The space $\mathcal{B}$ is a Banach space [35]. □

Theorem 3.2. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous, and there exists a function $\eta : [0,1] \to [0,\infty]$, such that $f(t,x,y) \le \eta(t) + a|x| + b|y|$, $a,b \ge 0$, $2a + 2b + \alpha|\rho| \le 2\delta$ where $\delta = \min\{\Gamma(\alpha-\beta-\gamma+2), \Gamma(\alpha-\beta-\gamma+1), \Gamma(\alpha-\gamma+1), \Gamma(\alpha+1)\}$. Then, the boundary value problem* (2.9), (2.10) *has a solution.*

Proof. Define an operator $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{F}x(t) = x_0 + (x_1 - x_0)t - t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),\mathfrak{D}^{\gamma}x(s))ds \tag{3.12}$$

$$+ \rho t\int_0^1 \left\{\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\}x(s)ds \tag{3.13}$$

$$+ \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t) + I^{\alpha}f(t,x(t),\mathfrak{D}^{\gamma}x(t)) \tag{3.14}$$

$$= x_0 + (x_1 - x_0)t + \int_0^1 \mathcal{G}(t,s)ds. \tag{3.15}$$

In order to show that the boundary value problem (2.9), (2.10) has a solution, it is sufficient to prove that the operator $\mathcal{F}$ has a fixed point. For $s \le t$, from (3.2), we have

$$\begin{aligned}
|\mathcal{G}(t,s)| &\le |\rho|\left\{\frac{2\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{2(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\}|x(s)| \\
&\quad + \left\{\frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}\right\}\left|f(s,x(s),\mathfrak{D}^{\gamma}x(s))\right| \\
&\le m_1\left\{\alpha(1-s)^{\alpha-\beta} + (1-s)^{\alpha-\beta-1}\right\}|x(s)| \\
&\quad + m_1(1-s)^{\alpha-1}\left|f(s,x(s),\mathfrak{D}^{\gamma}x(s))\right| \\
&\le m_1\left\{(\alpha+1)(1-s)^{\alpha-\beta-1}|x(s)| + (1-s)^{\alpha-1}\left|f(s,x(s),\mathfrak{D}^{\gamma}x(s))\right|\right\} \\
&\le m_1(1-s)^{\alpha-\beta-1}\left\{3|x(s)| + \left|f(s,x(s),\mathfrak{D}^{\gamma}x(s))\right|\right\} \\
&\le m_1(1-s)^{\alpha-\beta-1}\left\{(3+a)|x(s)| + \eta(s) + b|\mathfrak{D}^{\gamma}x(s)|\right\} \\
&\le m_1m_2(1-s)^{\alpha-\beta-1},
\end{aligned} \tag{3.16}$$

where

$$m_1 = \max\left\{\frac{2|\rho|}{\Gamma(\alpha-\beta+1)}, \frac{2|\rho|}{\Gamma(\alpha-\beta)}, \frac{2}{\Gamma(\alpha)}\right\},$$
$$m_2 = \max\left\{(3+a)|x(s)|, \eta(s), b|\mathfrak{D}^{\gamma}x(s)|,\ 0 \le s \le 1\right\}. \tag{3.17}$$

On the other hand, for $s > t$, we arrive at same conclusion. Therefore,

$$\int_0^1 |\mathcal{G}(t,s)|ds \le m_1 m_2 \int_0^1 (1-s)^{\alpha-\beta-1} ds = \frac{m_1 m_2}{\alpha-\beta}. \tag{3.18}$$

Choose $\mathcal{R} \ge \max\{\mathcal{R}_1, \mathcal{R}_2\}$, where $\mathcal{R}_1 = \max\{m_1 m_2/2(\alpha-\beta), (1/2)(2|x_0| + |x_1|)\}$ and

$$\mathcal{R}_2 = \max\left\{\frac{5|x_1 - x_0|}{2\Gamma(1-\gamma)}, \frac{5\|\eta\|}{2\Gamma(\alpha-\gamma+1)}, \frac{5\|\eta\|}{2\Gamma(\alpha+1)}, \frac{5|\rho|\alpha}{2\Gamma(\alpha-\beta+2)}, \frac{5|\rho|}{2\Gamma(\alpha-\beta+1)}\right\}. \tag{3.19}$$

Define the set $\Omega = \{x \in \mathcal{B} : \|x\| \le 8\mathcal{R}\}$. For $x \in \Omega$, using (3.15) and (3.18), we obtain

$$|\mathcal{F}x(t)| \le |x_0| + |x_1 - x_0|t + \int_0^1 |\mathcal{G}(t,s)|ds \le 2|x_0| + |x_1| + \frac{m_1 m_2}{\alpha-\beta} \le 2\mathcal{R} + 2\mathcal{R} = 4\mathcal{R}. \tag{3.20}$$

From the Caputo derivative and with using (3.12)–(3.14), we have

$$\begin{aligned}
\mathfrak{D}^{\gamma}(\mathcal{F}x(t)) &= I^{1-\gamma}\left\{\frac{d\mathcal{F}x(t)}{dt}\right\} \\
&= -I^{1-\gamma}\frac{d}{dt}\left\{t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \mathfrak{D}^{\gamma}x(s))ds\right\} \\
&\quad + I^{1-\gamma}\frac{d}{dt}\left\{\rho t\int_0^1 \left\{\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\} x(s)ds\right\} \\
&\quad + I^{1-\gamma}\left\{\frac{d}{dt}\left[x_0 + (x_1 - x_0)t + \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t)\right]\right\} \\
&\quad + I^{1-\gamma}\left\{\frac{d}{dt} I^{\alpha} f(t, x(t), \mathfrak{D}^{\gamma}x(t))\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -I^{1-\gamma}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),\mathfrak{D}^{\gamma}x(s))ds \\
&\quad + \rho I^{1-\gamma}\int_0^1 \left\{\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\}x(s)ds \\
&\quad + I^{1-\gamma}\left\{x_1 - x_0 + \rho(1-\alpha)I^{\alpha-\beta}x(t) + \rho t I^{\alpha-\beta-1}x(t) + I^{\alpha-1}f(t,x(t),\mathfrak{D}^{\gamma}x(t))\right\}.
\end{aligned}
\tag{3.21}
$$

Then, (2.3) yields

$$
\begin{aligned}
\mathfrak{D}^{\gamma}(\mathcal{F}x(t)) = &-\frac{t^{1-\gamma}}{\Gamma(1-\gamma)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),\mathfrak{D}^{\gamma}x(s))ds \\
&+ \frac{\rho t^{1-\gamma}}{\Gamma(1-\gamma)}\int_0^1 \left\{\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right\}x(s)ds \\
&+ \frac{(x_1-x_0)t^{1-\gamma}}{\Gamma(1-\gamma)} + \rho(1-\alpha)I^{\alpha-\beta-\gamma+1}x(t) \\
&+ \rho t I^{\alpha-\beta-\gamma}x(t) + I^{\alpha-\gamma}f(t,x(t),\mathfrak{D}^{\gamma}x(t)).
\end{aligned}
\tag{3.22}
$$

Hence,

$$
\begin{aligned}
|\mathfrak{D}^{\gamma}(\mathcal{F}x(t))| \le &\frac{t^{1-\gamma}}{\Gamma(\alpha+1)}\{\eta(t) + a|x(t)| + b|\mathfrak{D}^{\gamma}x(t)|\} \\
&+ |\rho|\left\{\frac{\alpha}{\Gamma(\alpha-\beta+2)} + \frac{1}{\Gamma(\alpha-\beta+1)}\right\}t^{1-\gamma} \\
&+ \frac{x_1-x_0}{\Gamma(1-\gamma)} + \frac{|\rho(1-\alpha)|\|x\|}{\Gamma(\alpha-\beta-\gamma+1)}\int_0^t (t-s)^{\alpha-\beta-\gamma}ds \\
&+ \frac{|\rho|\|x\|}{\Gamma(\alpha-\beta-\gamma)}\int_0^t (t-s)^{\alpha-\beta-\gamma-1}ds + I^{\alpha-\gamma}\{\eta(t) + a|x(t)| + b|\mathfrak{D}^{\gamma}x(t)|\} \\
\le &\frac{t^{1-\gamma}}{\Gamma(\alpha+1)}\{\eta(t) + a|x(t)| + b|\mathfrak{D}^{\gamma}x(t)|\} \\
&+ |\rho|\left\{\frac{\alpha}{\Gamma(\alpha-\beta+2)} + \frac{1}{\Gamma(\alpha-\beta+1)}\right\}t^{1-\gamma} \\
&+ \frac{|x_1-x_0|}{\Gamma(1-\gamma)} + \frac{|\rho(1-\alpha)|\|x\|t^{\alpha-\beta-\gamma+1}}{\Gamma(\alpha-\beta-\gamma+2)} + \frac{|\rho|\|x\|t^{\alpha-\beta-\gamma}}{\Gamma(\alpha-\beta-\gamma+1)} + \frac{\|\eta\| + (a+b)\|x\|}{\Gamma(\alpha-\gamma+1)}t^{\alpha-\gamma}.
\end{aligned}
\tag{3.23}
$$

Thus,

$$\begin{aligned}
|\mathfrak{D}^{\gamma}(\mathcal{F}x(t))| \le{}& \frac{|x_1 - x_0|}{\Gamma(1-\gamma)} + \frac{\|\eta\|}{\Gamma(\alpha-\gamma+1)} + \frac{\|\eta\|}{\Gamma(\alpha+1)} \\
&+ \mathcal{R}\left(\frac{|\rho(1-\alpha)|}{\Gamma(\alpha-\beta-\gamma+2)}, \frac{|\rho|}{\Gamma(\alpha-\beta-\gamma+1)} + \frac{a+b}{\Gamma(\alpha-\gamma+1)} + \frac{a+b}{\Gamma(\alpha+1)}\right) \\
&+ \frac{\rho\alpha}{\Gamma(\alpha-\beta+2)} + \frac{|\rho|}{\Gamma(\alpha-\beta+1)} \\
\le{}& 2\mathcal{R} + \mathcal{R}\left(\frac{|\rho|(\alpha-1)}{\delta} + \frac{|\rho|}{\delta} + \frac{2a+2b}{\delta}\right) = 2\mathcal{R} + \frac{2a+2b+\alpha|\rho|}{\delta}\mathcal{R} \le 2\mathcal{R} + 2\mathcal{R} = 4\mathcal{R}.
\end{aligned} \tag{3.24}$$

Therefore, $\|\mathcal{F}x(t)\| \le 4\mathcal{R} + 4\mathcal{R} = 8\mathcal{R}$. Thus, $\mathcal{F} : \Omega \to \Omega$. Finally, it remains to show that $\mathcal{F}$ is completely continuous. For any $x \in \Omega$, let $\ell = \max_{t\in[0,1]}|f(t, x(t), \mathfrak{D}^{\gamma}x(t))|$; then for $0 \le t_1 \le t_2 \le 1$ and using (3.12)–(3.14), we have

$$\begin{aligned}
|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)| \le{}& |x_1 - x_0||t_2 - t_1| + \ell|t_2 - t_1|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&+ |\rho||t_2 - t_1|\int_0^1 \left\{\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right\}x(s)ds \\
&+ \ell\left|\int_0^{t_2}\left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)ds\right| \\
&+ |\rho|\|x\|\left|\int_0^{t_2}\left(\frac{t_2(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t_2-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)ds\right. \\
&\qquad \left. -\left(\frac{t_1(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t_1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)ds\right| \\
\le{}& |x_1 - x_0||t_2 - t_1| + \frac{\ell|t_2 - t_1|}{\Gamma(\alpha+1)} \\
&+ |\rho|\|x\||t_2 - t_1|\left(\frac{1}{\Gamma(\alpha-\beta+1)} - \frac{\alpha}{\Gamma(\alpha-\beta+2)}\right) + \ell\frac{|t_1 - t_2|^{\alpha} + |t_1^{\alpha} - t_2^{\alpha}|}{\Gamma(\alpha+1)} \\
&+ \alpha\|x\|\frac{|t_1 - t_2|^{\alpha-\beta+1} + \left|t_1^{\alpha-\beta+1} - t_2^{\alpha-\beta+1}\right|}{\Gamma(\alpha-\beta+2)} + \|x\||t_1 - t_2|\frac{t_1^{\alpha-\beta} + |t_1 - t_2|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.
\end{aligned} \tag{3.25}$$

Hence, it follows that $\|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)\| \to 0$, as $t_2 \to t_1$. By the Arzela-Ascoli theorem, $\mathcal{F} : \Omega \to \Omega$ is completely continuous. Thus by using the Schauder fixed-point theorem, it was proved that the boundary value problem (2.9), (2.10) has a solution. □

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If there exists a constant μ such that $|f(t,x,y) - f(t,\tilde{x},\tilde{y})| \le \mu(|x-\tilde{x}| + |y-\tilde{y}|)$ for each $t \in [0,1]$ and all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$ and $4\mathcal{M} + 3\mu \le 1$, where*

$$\mathcal{M} = \max\left\{ \frac{2|\rho|}{\Gamma(\alpha-\beta+1)}, \frac{|\rho|(1+\alpha)}{\Gamma(\alpha-\beta+2)}, \frac{|\rho(1-\alpha)|}{\Gamma(\alpha-\beta-\gamma+2)}, \frac{|\rho|}{\Gamma(\alpha-\beta-\gamma+1)} \right\}. \tag{3.26}$$

Then the boundary value problem (2.9) *with boundary conditions* (2.10) *has a unique solution.*

Proof. Under condition on f, we have

$$\begin{aligned}
|\mathcal{F}x(t) - \mathcal{F}\tilde{x}(t)| &\le t\left|\rho\int_0^1 \left\{ \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} [\tilde{x}(s) - x(s)]\,ds\right| \\
&\quad + t\left|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f(s,\tilde{x}(s),\mathfrak{D}^\gamma\tilde{x}(s)) - f(s,x(s),\mathfrak{D}^\gamma x(s))\right] ds\right| \\
&\quad + \left|\rho\int_0^t \left\{ \frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\} [x(s) - \tilde{x}(s)]\,ds\right| \\
&\quad + \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f(s,x(s),\mathfrak{D}^\gamma x(s)) - f(s,\tilde{x}(s),\mathfrak{D}^\gamma\tilde{x}(s))\right] ds\right| \\
&\le \left|\frac{\rho}{\Gamma(\alpha-\beta+1)} - \frac{\alpha\rho}{\Gamma(\alpha-\beta+2)}\right| \|x-\tilde{x}\| + \frac{2\mu\|\tilde{x}-x\|}{\Gamma(\alpha+1)} \\
&\quad + \left|\frac{\rho t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{\rho\alpha t}{\Gamma(\alpha-\beta+2)}\right| \|x-\tilde{x}\| + \frac{2\mu\|x-\tilde{x}\|t^\alpha}{\Gamma(\alpha+1)} \\
&\le \left(\frac{|\rho|(1+t^{\alpha-\beta})}{\Gamma(\alpha-\beta+1)} + \frac{|\rho|(1+\alpha)t}{\Gamma(\alpha-\beta+2)} + \frac{2\mu(1+t^\alpha)}{\Gamma(\alpha+1)} \right) \|x-\tilde{x}\|.
\end{aligned} \tag{3.27}$$

Using (3.22) we conclude

$$\begin{aligned}
|\mathfrak{D}^\gamma(\mathcal{F}x)(t) - \mathfrak{D}^\gamma(\mathcal{F}\tilde{x})(t)| &\le |\rho(1-\alpha)|\left|I^{\alpha-\beta-\gamma+1}(x(t)-\tilde{x}(t))\right| + t\left|\rho I^{\alpha-\beta-\gamma}(x(t)-\tilde{x}(t))\right| \\
&\quad + \left|I^{\alpha-\gamma}\left(f(t,x(t),\mathfrak{D}^\gamma x(t)) - f(t,\tilde{x}(t),\mathfrak{D}^\gamma\tilde{x}(t))\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|\rho(1-\alpha)|\|x-\tilde{x}\|}{\Gamma(\alpha-\beta-\gamma+1)} \int_0^t (t-s)^{\alpha-\beta-\gamma} ds \\
&\quad + \frac{|\rho|\|x-\tilde{x}\|}{\Gamma(\alpha-\beta-\gamma)} \int_0^t (t-s)^{\alpha-\beta-\gamma-1} ds \\
&\quad + \frac{2\mu\|x-\tilde{x}\|}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} ds \\
&\leq \left(\frac{\rho|1-\alpha|t^{\alpha-\beta-\gamma+1}}{\Gamma(\alpha-\beta-\gamma+2)} + \frac{\rho t^{\alpha-\beta-\gamma}}{\Gamma(\alpha-\beta-\gamma+1)} + \frac{2\mu t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) \|x-\tilde{x}\|.
\end{aligned} \tag{3.28}$$

Thus, we have

$$\|\mathcal{F}x(t)-\mathcal{F}\tilde{x}(t)\| \leq \left(4\mathcal{M}+\frac{6\mu}{\Gamma(\alpha+1)}\right) < (4\mathcal{M}+3\mu)\|x-\tilde{x}\|. \tag{3.29}$$

Therefore, by the contraction mapping theorem, the boundary value problem (2.9), (2.10) has a unique solution. □

Theorem 3.4. *Let $f : [0,1] \to [0,\infty]$, such that $f(t,x,y) \leq \eta(t) + a|x| + b|y|$, $a,b \geq 0$ with $a+b+\alpha|\rho \leq \delta|$ where $\delta = \min\{\Gamma(\alpha-\beta-\gamma+1), \Gamma(\alpha-\beta-\gamma+2), \Gamma(\alpha-\beta-\gamma+3)\}$. Then the initial value problem* (2.9), (2.10) *has a solution.*

Proof. In view of Lemma 2.1 and Proposition 2.2, we have

$$x(t) = \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t) + I^{\alpha} f(t,x(t),\mathfrak{D}^{\gamma}x(t)) - c_0 - c_1 t. \tag{3.30}$$

Then,

$$x'(t) = \rho(1-\alpha) I^{\alpha-\beta}x(t) + \rho t I^{\alpha-\beta-1}x(t) + I^{\alpha=1} f(t,x(t),\mathfrak{D}^{\gamma}x(t)) - c_0 - c_1 t. \tag{3.31}$$

By initial conditions we have $c_0 = -x_0$ and $c_1 = -1$. Define an operator $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}x(t) = x_0 + t + \rho t I^{\alpha-\beta}x(t) - \rho\alpha I^{\alpha-\beta+1}x(t) + I^{\alpha} f(t,x(t),\mathfrak{D}^{\gamma}x(t)). \tag{3.32}$$

Can be easily to prove that $\mathcal{T} : \Omega \to \Omega$ is completely continuous as operator $\mathcal{F}$. □

Theorem 3.5. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If there exists a constant μ such that $|f(t,x,y)-f(t,\tilde{x},\tilde{y})| \leq \mu(|x-\tilde{x}|+|y-\tilde{y}|)$ for each $t \in [0,1]$ and all $x,\tilde{x},y,\tilde{y} \in \mathbb{R}$ and $3(\mathcal{M}+\mu) \leq 1$, where*

$$\mathcal{M} = \max\left\{ \frac{|\rho|}{\Gamma(\alpha-\beta+1)}, \frac{|\rho|\alpha}{\Gamma(\alpha-\beta+2)}, \frac{|\rho|}{\Gamma(\alpha-\beta-\gamma+1)} \right\}, \tag{3.33}$$

then the initial value value problem (2.9), (2.11) *has a unique solution.*

The proof of the Theorem 3.5 is similar to the proof of Theorem 3.3. Note that

$$\frac{d\mathcal{T}x(t)}{dt} = 1 + \rho(1-\alpha)\mathcal{I}^{\alpha-\beta}x(t) + \rho t\mathcal{I}^{\alpha-\beta-1}x(t) + \mathcal{I}^{\alpha-1}f(t, x(t), \mathfrak{D}^{\gamma}x(t)). \tag{3.34}$$

Then using Proposition 2.2 we have,

$$\begin{aligned}
&\mathfrak{D}^{\gamma}(\mathcal{T}x(t)) \\
&\quad = \mathcal{I}^{1-\gamma}\left\{\frac{d\mathcal{T}x(t)}{d\mathcal{T}}\right\} \\
&\quad = \frac{t^{1-\gamma}}{\Gamma(1-\gamma)} + (\alpha(1-\rho) - \rho(1-\alpha))\mathcal{I}^{\alpha-\beta-\gamma+1}x(t) + \rho t\mathcal{I}^{\alpha-\beta-\gamma}x(t) + \mathcal{I}^{\alpha-\gamma}f(t, x(t), \mathfrak{D}^{\gamma}x(t))
\end{aligned} \tag{3.35}$$

Example 3.6. Consider the following boundary value problem for nonlinear fractional order differential equation:

$$\begin{gathered}
\left(\mathfrak{D}^{3/2} - t\mathfrak{D}^{1/2}\right)x(t) = \left(3e^t + \frac{1}{10}x(t) + \frac{1}{10}\mathfrak{D}^{1/2}x(t)\right)^{1/3}, \quad t \in (0,1), \\
x(0) = x_0, \qquad x(1) = x_1.
\end{gathered} \tag{3.36}$$

Then, (3.36) with assumed boundary conditions has a solution in Ω.

In Example 3.6 $f(t, x(t), \mathfrak{D}^{\gamma}x(t)) = \sqrt[3]{3e^t + (1/10)x(t) + (1/10)\mathfrak{D}^{1/2}x(t)}$ satisfies the conditions required in Theorem 3.2, that is

$$f\left(t, x(t), \mathfrak{D}^{1/2}x(t)\right) \le e^t + \frac{1}{30}|x(t)| + \frac{1}{30}\left|\mathfrak{D}^{1/2}x(t)\right| \tag{3.37}$$

and $\delta = \min\{\Gamma(3/2), \Gamma(2), \Gamma(5/2)\} = \Gamma(3/2) = \sqrt{\pi}/2$ and $2a + 2b + \alpha\rho = 47/30 < 2\delta = \sqrt{\pi}$.

Example 3.7. Consider the following boundary value problem for nonlinear fractional order differential equation:

$$\begin{gathered}
\left(\mathfrak{D}^{3/2} - (1/8)t\mathfrak{D}^{1/2}\right)x(t) = \frac{1}{21}x(t) + \frac{1}{21}\mathfrak{D}^{1/2}x(t), \quad t \in (0,1), \\
x(0) = x_0, \qquad x(1) = x_1.
\end{gathered} \tag{3.38}$$

Then, (3.38) with assumed boundary conditions has unique solution in Ω.

In Example 3.7 $f(t, x(t), \mathfrak{D}^{\gamma}x(t)) = (1/21)x(t) + (1/21)\mathfrak{D}^{1/2}x(t)$ satisfies the conditions required in Theorem 3.3. $L = \max\{1/3\sqrt{\pi}, 1/8\sqrt{\pi}, 1/12\sqrt{\pi}, 1/4\sqrt{\pi}\} = 1/3\sqrt{\pi}$ and $4\mathcal{M} + 3\mu = 4/3\sqrt{\pi} + 1/7 < 1$.

4. Conclusion

We considered a class of nonlinear fractional order differential equations involving Caputo fractional derivative with lower terminal at 0 in order to study the existence solution satisfying the boundary conditions or satisfying the initial conditions. The unique solution under Lipschitz condition is also derived. In order to illustrate our results several examples are presented. The presented research work can be generalized to multiterm nonlinear fractional order differential equations with polynomial coefficients.

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Research Article

Positive Solutions of an Initial Value Problem for Nonlinear Fractional Differential Equations

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We investigate the existence and multiplicity of positive solutions for the nonlinear fractional differential equation initial value problem $D^{\alpha}_{0+}u(t) + D^{\beta}_{0+}u(t) = f(t, u(t))$, $u(0) = 0$, $0 < t < 1$, where $0 < \beta < \alpha < 1$, D^{α}_{0+} is the standard Riemann-Liouville differentiation and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous. By using some fixed-point results on cones, some existence and multiplicity results of positive solutions are obtained.

1. Introduction

Fractional differential equations have been subjected to an intense debate during the last few years (see, e.g., [1–5] and the references therein). This trend is due to the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering [5–15]. The fractional differential equations started to be used extensively in studying the dynamical systems possessing memory effect. Comprehensive treatment of the fractional equations techniques such as Laplace and Fourier transform method, method of Green function, Mellin transform, and some numerical techniques are given in [5, 7, 9] and the references therein. In classical approach, linear initial fractional differential equations are solved by special functions [9, 16]. In some papers, for nonlinear problems, techniques of functional analysis such as fixed point theory, the Banach contraction principle, and Leray-Schauder theory are applied for solving such kind of the problems (see, e.g., [17–19] and the references therein). The existence of nonlinear fractional differential equations of one time fractional derivative is considered in [6, 7, 9, 20]. Also, the existence and multiplicity of positive solutions to nonlinear Dirichlet problem

$$D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \; u(0) = u(1) = 0, \; 1 < \alpha \leq 2, \; \alpha \in \mathbb{R}, \tag{1.1}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and D_{0+}^{α} is the Riemann-Liouville differentiation, have been reviewed by some authors (see e.g., [18–21] and the references therein).

In this paper, by using some fixed-point results, we investigate the existence and multiplicity of positive solutions for the nonlinear fractional differential equation initial value problem

$$D_{0+}^{\alpha}u(t) + D_{0+}^{\beta}u(t) = f(t,u(t)), \quad u(0) = 0, \; 0 < t < 1, \tag{1.2}$$

where $0 < \beta < \alpha < 1$, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Now, we present some necessary notions. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by $I^{\alpha}f(t) := (1/\Gamma(\alpha))\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau$ [20]. Also, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by $D^{\alpha}f(t) := (1/\Gamma(n-\alpha))(d/dt)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau$, where $n = [\alpha] + 1$ and the right side is pointwise defined on $(0,\infty)$ ([20]). The formula of Laplace transform for the Riemann-Liouville derivative is defined by

$$L\{D^{\alpha}f(t);s\} = s^{\alpha}\tilde{f}(s)\sum_{k=0}^{m-1}\left[D^k I^{m-\alpha}\right]f(0^+)s^{m-k-1} \tag{1.3}$$

when the limiting values $f^{(k)}(0^+)$ are finite and $m-1 < \alpha < m$. This formula simplifies to $L\{D^{\alpha}f(t);s\} = s^{\alpha}\tilde{f}(s)$ [21]. Also, two-parametric Mittag-Leffler function is defined by $E_{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} z^k/\Gamma(k\alpha+\beta)$ for $\alpha > 0$ and $\beta > 0$ [21]. Analytic properties and asymptotical expansion of this function are given in [9]. For example, if $\alpha < 2$, $\pi\alpha/2 < \mu < \min(\pi, \pi\alpha)$, $\beta \in \mathbb{R}$ and c_3 is a real constant, then $|E_{\alpha,\beta}(z)| \leq c_3/(1+|z|)$, whenever $|z| \geq 0$ and $\mu \leq |\arg z| \leq \pi$. Also, by using the formula for integration of the Mittag-Leffler function term by term, we have (see [9])

$$\int_0^z t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})dt = z^{\beta}E_{\alpha,\beta+1}(\lambda t^{\alpha}). \tag{$*$}$$

Let P be a cone in a Banach space E. The map $\theta : P \to [0,\infty]$ is said to be a nonnegative continuous concave functional whenever θ is continuous and $\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y)$ for all $x, y \in P$ and $0 \leq t \leq 1$ [20]. We need the following fixed point theorems for obtaining our results.

Lemma 1.1 (see [22]). *Let E be a Banach space, P a cone in E, and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\overline{\Omega_1} \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$

holds. Then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 1.2 (see [23]). *Let P be a cone in a real Banach space E, c, b, and d positive real numbers, $P_c = \{x \in P : \|x\| \leq c\}$, θ a nonnegative concave functional on P such that $\theta(x) \leq \|x\|$ for all $x \in \overline{P_c}$ and*

$$P(\theta, b, d) = \{x \in P : b \leq \theta(x),\ \|x\| \leq d\}. \tag{1.4}$$

Suppose that $A : \overline{P_c} \to \overline{P_c}$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that

- (c_1) *$\{x \in P(\theta, b, d) : \theta(x) > b\} \neq \emptyset$, and for some $x \in P(\theta, b, d)$ we have $\theta(Ax) > b$,*
- (c_2) *$\|Ax\| < a$ for all x with $\|x\| \leq a$,*
- (c_3) *$\theta(Ax) > b$ for all $x \in P(\theta, b, c)$ with $\|Ax\| > d$.*

Then A has at least three fixed points x_1, x_2, and x_3 such that $\|x_1\| < a$, $b < \theta(x_2)$, $a < \|x_3\|$ with $\theta(x_3) < b$.

Note that the condition (c_1) implies (c_3) whenever $d = c$.

2. Main Results

As we know, there is an integral form of the solution for the following equation:

$$D^{\alpha}_{0+}u(t) + D^{\beta}_{0+}u(t) = f(t, u(t)), \quad u(0) = 0,\ 0 < t < 1, \tag{2.1}$$

Suppose that the functions u and f are continuous on $[0, 1]$. Then $u(t) = \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau$ is a solution for (2.1), where $G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ and $E_{\alpha,\beta}$ is the two-parameter function of the Mittag-Leffler type (see [9]). Now, we give an equivalent solution for (2.1). In fact, if we apply the Laplace transform to (2.1), then by using a calculation and finding the inverse Laplace transform we get that $u(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}) * f(t, u(t))$ is an equivalent solution for (2.1). In this way, note that

$$D^{\alpha}u(t) + D^{\beta}u(t) = \left(D^{\alpha}G(t) + D^{\beta}G(t)\right) * f(t, u(t)), \tag{2.2}$$

where $G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$. But, we have

$$\begin{aligned} D^{\alpha}G(t) + D^{\beta}G(t) &= t^{-1}E_{\alpha-\beta,0}\left(-t^{\alpha-\beta}\right) + t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha-\beta}\left(-t^{\alpha-\beta}\right) \\ &= E_{\alpha-\beta,0}\left(-t^{\alpha-\beta}\right) - E_{\alpha-\beta,0}\left(-t^{\alpha-\beta}\right) - \frac{1}{t}\frac{1}{\Gamma(\alpha-\beta)}. \end{aligned} \tag{2.3}$$

Since $\lim_{t\to 0}(1/t)(1/\Gamma(\alpha-\beta)) = \delta(t)$, we get $D^{\alpha}G(t) + D^{\beta}G(t) = \delta(t)$ and so

$$D^{\alpha}u(t) + D^{\beta}u(t) = \delta(t) * f(t, u(t)) = f(t, u(t)). \tag{2.4}$$

Now, we establish some results on existence and multiplicity of positive solutions for the problem (2.1). Let $E = (C[0, 1], \|\cdot\|_{\infty})$ be endowed via the order $u \leq v$ if and only if $u(t) \leq v(t)$

for all $t \in [0,1]$. Consider the cone $P = \{u \in E \mid u(t) \geq 0\}$ and the nonnegative continuous concave functional $\theta(u) = \inf_{1/2<t<1}|u(t)|$. Now, we give our first result.

Lemma 2.1. *Define $T : P \to P$ by $Tu(t) := \int_0^t G(t-\tau)f(\tau,u(\tau))d\tau$, where $G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ and $E_{\alpha,\beta}(z)$ is the two-parameter function of the Mittag-Leffler type. Then T is completely continuous.*

Proof. Since the mappings G and f are nonnegative and continuous, it is easy to see that T is continuous. Now, we show that T is a relatively compact operator. This implies that T is completely continuous. Let $\Omega \subset P$ be a bounded subset. Then there exists a positive constant $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$. Put $L = \sup_{0\leq t\leq 1}|f(t,u(t))| + 1$. Then, for each $u \in \Omega$, we have

$$\begin{aligned}|Tu(t)| &= \left|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right)f(\tau,u(\tau))d\tau\right| \\ &\leq L\left|-t^{\alpha}E_{\alpha-\beta,\alpha+1}\left(-t^{\alpha-\beta}\right)\right| \leq L\left|\frac{-t^{\alpha}}{1+\left|-t^{\alpha-\beta}\right|}\right| \leq Lt^{\alpha} \leq L,\end{aligned} \tag{2.5}$$

where $0 < \alpha < 1$ and $t \in [0,1]$. Thus, T is uniformly bounded. Now, we show that T is equicontinuous. Let $t,\tau \in [0,1]$ and $t_1 \leq t_2$. Thus,

$$\begin{aligned}&|Tu(t_1) - Tu(t_2)| \\ &= \left|\int_0^{t_1} G(t_1-\tau)f(\tau,u(\tau))d\tau - \int_0^{t_2} G(t_2-\tau)f(\tau,u(\tau))d\tau\right| \\ &= \left|\int_0^{t_1} \left(G(t_1-\tau)f(\tau,u(\tau)) - G(t_2-\tau)f(\tau,u(\tau))\right)d\tau + \int_{t_2}^{t_1} G(t_2-\tau)f(\tau,u(\tau))d\tau\right| \\ &\leq \left|\int_0^{t_1} \left[G(t_1-\tau)f(\tau,u(\tau)) - G(t_2-\tau)f(\tau,u(\tau))\right]d\tau\right| + \left|\int_{t_2}^{t_1} G(t_2-\tau)f(\tau,u(\tau))d\tau\right|.\end{aligned} \tag{2.6}$$

Now, by using the formula for integration of the Mittag-Leffler function term by term given in $(*)$, we obtain that

$$\begin{aligned}&|Tu(t_1) - Tu(t_2)| \\ &\leq \|f\|\left[\left(\frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|} - \frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|} + \frac{(t_2-t_1)^{\alpha}}{1+\left|-(t_2-t_1)^{\alpha-\beta}\right|}\right)\right. \\ &\qquad \left.+\left(\frac{t_2^{\alpha}}{1+\left|-t_2^{\alpha-\beta}\right|} - \frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|} - \frac{(t_2-t_1)^{\alpha}}{1+\left|-(t_2-t_1)^{\alpha-\beta}\right|}\right)\right] \\ &= \|f\|\left[\frac{t_2^{\alpha}}{1+\left|-t_2^{\alpha-\beta}\right|} - \frac{t_1^{\alpha}}{1+\left|-t_1^{\alpha-\beta}\right|}\right] \leq \|f\|\left[\frac{\left(t_2^{\alpha}-t_1^{\alpha}\right) - t_2^{\alpha}\left(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right) + t_2^{\alpha-\beta}\left(t_2^{\alpha}-t_1^{\alpha}\right)}{\left(1+\left|-t_1^{\alpha-\beta}\right|\right)\left(1+\left|-t_2^{\alpha-\beta}\right|\right)}\right].\end{aligned} \tag{2.7}$$

Thus, by using the formula $t_2^s - t_1^s = (t_2 - t_1)/(t_2^{s-1} + \cdots + t_1^{s-1})$, we obtain a common factor $(t_1 - t_2)$. This implies that small changes of u cause small changes of Tu. that is, T is equicontinuous. Now by using the Arzela-Ascoli theorem, we get that T is a relatively compact operator. □

Theorem 2.2. *Suppose that in the problem* (1.2) *there exists a positive real number $r > 0$ such that*

(A_1) $f(t,u) \leq \alpha r$ *for all* $(t,u) \in [0,1] \times [0,r]$,

(A_2) $f(t,u) \geq 0$ *for all* $t \in [0,1]$ *with* $u(t) = 0$.

Then the problem (1.2) *has a positive solution u such that* $0 \leq |u| \leq r$.

Example 2.3. Consider the nonlinear fractional differential equation initial value problem

$$D^{3/2}u(t) + D^{1/2}u(t) + u(t) + \sin t = 0, \quad u(0) = 0, \ (0 < t < 1). \tag{2.8}$$

Put $r = 2$ and $\alpha = 3/2$. Since $f(t,u) = u(t) + \sin t \leq u + 1 \leq 3 = \alpha r$ for all $(t,u) \in [0,1] \times [0,2]$ and $f(t,u) = u + \sin t \geq 0$ for all $(t,u) \in [0,1] \times \{0\}$, by using Theorem 2.2 we get that this problem has a positive solution we get that this problem has a positive solution u with $0 \leq \|u\| \leq 2$.

Proof. First, let us to consider the operator $(Tu)(t) = \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau$, where $G(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})(0 < \beta < \alpha < 1)$. By using Lemma 2.1, T is completely continuous and note that u is a solution of the problem (1.2) if and only if $u = T(u)$. Let $\Omega_1 = \{u \in P : \|u\| = 0\}$ and $\Omega_2 = \{u \in P : \|u\| u \in \partial\Omega_1\}$ we have $u(t) = 0$ for all $t \in [0,1]$. By using the assumption (A_2), we have

$$(Tu)(t) = \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau \geq 0 = \|u\| \tag{2.9}$$

and so $\|Tu\| \geq \|u\|$. Also, for $u \in \partial\Omega_2$ we have $0 \leq u(t) \leq r$ for all $t \in [0,1]$. By using the assumption (A_1) we have

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^t G(t-\tau) f(\tau, u(\tau)) d\tau \leq \alpha r \int_0^t (t-\tau)^{\alpha-1} d\tau = r t^{\alpha} \leq r = \|u\|. \tag{2.10}$$

This completes the proof. □

Theorem 2.4. *Suppose that in the problem* (2.1) *there exist positive real numbers $0 < a < b < c$ such that*

(A_1) $f(t,u) < \alpha a$ *for all* $(t,u) \in [0,1] \times [0,a]$,

(A_2) $f(t,u) > Nb$ *for all* $(t,u) \in [1/2,1] \times [b,c]$, *where*

$$N^{-1} = \inf_{1/2 < t < 1} \left| \int_0^t G(t-s) ds \right|, \tag{2.11}$$

(A_3) $f(t,u) \leq \alpha c$ *for all* $(t,u) \in [0,1] \times [0,c]$.

Then the problem (2.1) *has at least there positive solutions* u_1, u_2, *and* u_3 *such that* $\sup_{0\le t\le 1}|u_1(t)| < a$, $b < \inf_{1/2\le t\le 1}|u_2(t)| < \sup_{1/2\le t\le 1}|u_2(t)| \le c$, $a < \sup_{0\le t\le 1}|u_3(t)| \le c$ *and* $\inf_{1/2\le t\le 1}|u_3(t)| < b$.

Proof. Define $P_c = \{x \in P : \|x\| \le c\}$. Then, $\|u\| \le c$ for all $u \in \overline{P_c}$. Note that, the assumption (A_3) implies that $f(t, u(t)) \le \alpha c$ for all t. Thus,

$$\|Tu\| = \sup_{0\le t\le 1}\left|\int_0^t G(t-\tau)f(\tau, u(\tau))d\tau\right| \le \alpha c\int_0^t (t-\tau)^{\alpha-1}d\tau = \alpha c\frac{t^\alpha}{\alpha} = ct^\alpha \le c. \tag{2.12}$$

Hence, T is a operator on $\overline{P_c}$. Also, note that the assumption (A_1) implies that $f(t, u(t)) < \alpha a$ for all $0 \le t \le 1$. Thus, the condition (c_2) in Lemma 1.2 holds. It is sufficient that we show that the condition (c_1) in Lemma 1.2 holds. Put $u(t) = (b+c)/2$ for all $0 \le t \le 1$. It is easy to see that $u(t) \in P(\theta, b, c)$ and $\theta(u) = \theta((b+c)/2) > b$. Thus, $\{u \in P(\theta, b, c) : \theta(u) > b\} \neq \emptyset$ and so $b \le u(t) \le c$ for all $u \in P(\theta, b, c)$ and $1/2 \le t \le 1$. But, the assumption (A_2) implies that $f(t, u(t)) \ge Nb$ for all $1/2 \le t \le 1$ and so

$$\theta(Tu) = \inf_{1/2\le t\le 1}|(Tu)(t)| = \inf_{1/2\le t\le 1}\left|\int_0^t G(t-\tau)f(\tau, u(\tau))d\tau\right| > NbN^{-1} = b. \tag{2.13}$$

Thus, $\theta(Tu) > b$ for all $u \in P(\theta, b, c)$. This shows that the condition (c_1) in Lemma 1.2 holds. This completes the proof. □

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Research Article

A Coupled System of Nonlinear Fractional Differential Equations with Multipoint Fractional Boundary Conditions on an Unbounded Domain

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This paper investigates the existence of solutions for a coupled system of nonlinear fractional differential equations with m-point fractional boundary conditions on an unbounded domain. Some standard fixed point theorems are applied to obtain the main results. The paper concludes with two illustrative examples.

1. Introduction

In the last few decades, the subject of fractional calculus has gained considerable popularity and importance as it finds its applications in numerous fields of science and engineering. Some of the areas of recent applications of fractional models include fluid mechanics, solute transport or dynamical processes in porous media, material viscoelastic theory, dynamics of earthquakes, control theory of dynamical systems, and biomathematics. In the aforementioned areas, there are phenomena with estrange kinetics involving microscopic complex dynamical behaviour that cannot be characterized by classical derivative models. It has been learnt through experimentation that most of the processes associated with complex systems have nonlocal dynamics possessing long-memory in time, and the integral and derivative operators of fractional order do have some of these characteristics. Thus, due to the modeling capabilities of fractional integrals and derivatives for complex phenomena, the fractional modelling has emerged as a powerful tool and has accounted for the rapid development of the theory of fractional differential equations. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes [1]. The presence of memory term in such models not only takes into account the history of the process involved but also carries its impact to present and future development of the process. For more details and applications, we refer the reader to the books [2–6]. For some recent work on the topic, see [7–27] and references therein.

The study of coupled systems involving fractional differential equations is also important as such systems occur in various problems of applied nature. For some recent results on systems of fractional differential equations, see [28–35].

Much of the work on fractional differential equations has been considered on finite domain and there are few papers dealing with infinite domain [36–43]. In this paper, we discuss the existence and uniqueness of the solutions of a coupled system of nonlinear fractional differential equations with m-point boundary conditions on an unbounded domain. Precisely, we consider the following problem:

$$\begin{gathered} D^p u(t) + f(t, v(t)) = 0, \quad 2 < p < 3, \\ D^q v(t) + g(t, u(t)) = 0, \quad 2 < q < 3, \\ u(0) = u'(0) = 0, \qquad D^{p-1}u(+\infty) = \sum_{i=1}^{m-2}\beta_i u(\xi_i), \\ v(0) = v'(0) = 0, \qquad D^{q-1}v(+\infty) = \sum_{i=1}^{m-2}\gamma_i v(\xi_i), \end{gathered} \tag{1.1}$$

where $t \in J = [0, +\infty)$, $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, D^p and D^q denote Riemann-Liouville fractional derivatives of order p and q, respectively, and $\beta_i > 0$, and $\gamma_i > 0$ are such that $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{p-1} < \Gamma(p)$ and $0 < \sum_{i=1}^{m-2} \gamma_i \xi_i^{q-1} < \Gamma(q)$.

2. Preliminaries

For the convenience of the readers, in this section we first present some useful definitions and lemmas.

Definition 2.1 (see [5]). The Riemann-Liouville fractional derivative of order δ for a continuous function f is defined by

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\delta-1} f(s)ds, \quad n = [\delta] + 1, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 (see [5]). The Riemann-Liouville fractional integral of order δ for a function f is defined as

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} f(s)ds, \quad \delta > 0, \tag{2.2}$$

provided that such integral exists.

For the forthcoming analysis, we define the spaces

$$X = \left\{ u \in C[0, +\infty) : \sup_{t \in J} \frac{|u(t)|}{1 + t^{p-1}} < +\infty \right\},$$
$$Y = \left\{ v \in C[0, +\infty) : \sup_{t \in J} \frac{|v(t)|}{1 + t^{q-1}} < +\infty \right\} \tag{2.3}$$

equipped with the norms

$$\|u\|_X = \sup_{t \in J} \frac{|u(t)|}{1 + t^{p-1}}, \tag{2.4}$$

$$\|v\|_Y = \sup_{t \in J} \frac{|v(t)|}{1 + t^{q-1}}. \tag{2.5}$$

Obviously X and Y are Banach spaces.

Lemma 2.3 (see [38]). *Let $h \in C([0, +\infty))$. For $2 < \alpha < 3$, the fractional boundary value problem*

$$D^{\alpha} u(t) + h(t) = 0,$$
$$u(0) = u'(0) = 0, \qquad D^{\alpha - 1} u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i) \tag{2.6}$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t, s) h(s) ds, \tag{2.7}$$

where

$$G(t, s) = G^*(t, s) + G^{**}(t, s), \tag{2.8}$$

with

$$G^*(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t - s)^{\alpha - 1}, & 0 \le s \le t < +\infty, \\ t^{\alpha - 1}, & 0 \le t \le s < +\infty. \end{cases} \tag{2.9}$$

$$G^{**}(t, s) = \frac{\sum_{i=1}^{m-2} \beta_i t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}} G^*(\xi_i, s). \tag{2.10}$$

Lemma 2.4 (see [38]). *For $(s, t) \in [0, +\infty) \times [0, +\infty)$, $G(t, s)/1 + t^{\alpha-1} \le L_1$, where*

$$L_1 = \frac{1}{\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} \beta_i \xi_{m-2}^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}\right)}. \tag{2.11}$$

3. Main Results

This section is devoted to some existence and uniqueness results for problem (1.1).

Define the space

$$Z = \{(u, v) \mid u \in X,\ v \in Y\} \tag{3.1}$$

equipped with the norm

$$\|(u, v)\|_Z = \max\{\|u\|_X, \|v\|_Y\}. \tag{3.2}$$

Clearly Z is a Banach space.

Let an operator $Q : Z \to Z$ be defined by

$$\begin{aligned} Q(u, v) &= (Q_1(v), Q_2(u)) \\ &= \left(\int_0^{+\infty} G_1(t, s) f(t, v(s)) ds, \int_0^{+\infty} G_2(t, s) g(t, u(s)) ds\right), \end{aligned} \tag{3.3}$$

where $G_1(t, s) = G_{11}(t, s) + G_{12}(t, s)$, $G_2(t, s) = G_{21}(t, s) + G_{22}(t, s)$, with

$$\begin{aligned} G_{11}(t, s) &= \frac{1}{\Gamma(p)} \begin{cases} t^{p-1} - (t-s)^{p-1}, & 0 \le s \le t < +\infty, \\ t^{p-1}, & 0 \le t \le s < +\infty, \end{cases} \\ G_{12}(t, s) &= \frac{\sum_{i=1}^{m-2} \beta_i t^{p-1}}{\Gamma(p) - \sum_{i=1}^{m-2} \beta_i \xi_i^{p-1}} G_{11}(\xi_i, s), \\ G_{21}(t, s) &= \frac{1}{\Gamma(q)} \begin{cases} t^{q-1} - (t-s)^{q-1}, & 0 \le s \le t < +\infty, \\ t^{q-1}, & 0 \le t \le s < +\infty, \end{cases} \\ G_{22}(t, s) &= \frac{\sum_{i=1}^{m-2} \gamma_i t^{q-1}}{\Gamma(q) - \sum_{i=1}^{m-2} \gamma_i \xi_i^{q-1}} G_{21}(\xi_i, s). \end{aligned} \tag{3.4}$$

Observe that the problem (1.1) has a solution if and only if the operator Q defined by (3.3) has a fixed point.

Lemma 3.1. *For $(s, t) \in [0, +\infty) \times [0, +\infty)$, one has*

$$\frac{G_1(t, s)}{1 + t^{p-1}} \le L, \qquad \frac{G_2(t, s)}{1 + t^{q-1}} \le L, \tag{3.5}$$

where

$$L = \max\left\{ \frac{1}{\Gamma(p)} + \frac{\sum_{i=1}^{m-2} \beta_i \xi_{m-2}^{p-1}}{\Gamma(p)\left(\Gamma(p) - \sum_{i=1}^{m-2} \beta_i \xi_i^{p-1}\right)}, \frac{1}{\Gamma(q)} + \frac{\sum_{i=1}^{m-2} \gamma_i \xi_{m-2}^{q-1}}{\Gamma(q)\left(\Gamma(q) - \sum_{i=1}^{m-2} \gamma_i \xi_i^{q-1}\right)} \right\}. \tag{3.6}$$

Theorem 3.2. *Assume that*

(H_1) *there exist nonnegative functions* $a(t), b(t) \in C[0, +\infty)$ *such that*

$$\begin{gathered}\left|f(t,x)\right| \leq a(t)|x| + b(t), \qquad t \in [0,+\infty), \\ \int_0^{+\infty}\left(1+t^{q-1}\right)a(t)dt < \frac{1}{L}, \qquad \int_0^{+\infty} b(t)dt < +\infty;\end{gathered} \tag{3.7}$$

(H_2) *there exist nonnegative functions* $c(t), d(t) \in C[0, +\infty)$ *such that*

$$\begin{gathered}\left|g(t,y)\right| \leq c(t)\left|y\right| + d(t), \qquad t \in [0,+\infty), \\ \int_0^{+\infty}\left(1+t^{p-1}\right)c(t)dt < \frac{1}{L}, \qquad \int_0^{+\infty} d(t)dt < +\infty.\end{gathered} \tag{3.8}$$

Then the system (1.1) *has a solution.*

Proof. Let us take

$$R > \max\left\{\frac{L\int_0^{+\infty} b(s)ds}{1 - L\int_0^{+\infty}(1+s^{q-1})a(s)ds}, \frac{L\int_0^{+\infty} d(s)ds}{1 - L\int_0^{+\infty}(1+s^{p-1})c(s)ds}\right\}, \tag{3.9}$$

and define

$$B_R = \{(u,v) \in Z \mid \|(u,v)\|_Z \leq R\}. \tag{3.10}$$

Obviously, B_R is a bounded closed and convex set of Z.

As a first step, we show that the operator Q is $B_R \to B_R$.

For any $(u,v) \in B_R$, we have

$$\begin{aligned}\|Q_1 v\|_X &= \sup_{t\in J}\frac{1}{1+t^{p-1}}\left|\int_0^{+\infty} G_1(t,s)f(s,v(s))ds\right| \\ &\leq \sup_{t\in J}\frac{1}{1+t^{p-1}}\int_0^{+\infty} G_1(t,s)(a(s)|v(s)| + b(s))ds \\ &\leq L\int_0^{+\infty}\left(1+s^{q-1}\right)a(s)ds\|v\|_Y + L\int_0^{+\infty} b(s)ds \\ &< \frac{L\int_0^{+\infty} b(s)ds}{1 - L\int_0^{+\infty}(1+t^{q-1})a(s)ds} \\ &< R.\end{aligned} \tag{3.11}$$

Similarly, we can get

$$\begin{aligned}\|Q_2u\|_Y &= \sup_{t\in J}\frac{1}{1+t^{q-1}}\left|\int_0^{+\infty} G_2(t,s)g(s,u(s))ds\right| \\ &\le L\int_0^{+\infty}\left(1+s^{p-1}\right)c(s)ds\|u\|_X + L\int_0^{+\infty} d(s)ds \\ &< \frac{L\int_0^{+\infty} d(s)ds}{1-L\int_0^{+\infty}(1+s^{p-1})c(s)ds} \\ &< R.\end{aligned}\tag{3.12}$$

That is, $\|Q(u,v)\|_Z \le R$. Thus, $QB_R \subset B_R$.

Next, we show that Q is completely continuous. By continuity of f, g, G_1, and G_2, it follows that Q is continuous. On the other hand, by a similar process used in [38], we can easily prove that the operators Q_1 and Q_2 are equicontinuous. Therefore it follows that QB_R is an equicontinuous set. Also, it is uniformly bounded as $QB_R \subset B_R$. Thus, we conclude that Q is a completely continuous operator. Hence, by Schauder fixed point theorem, there exists a solution of (1.1). This completes the proof. □

Theorem 3.3. *Assume that*

(H_3) *there exist* $0 < \rho_1 < 1$ *and nonnegative functions* $a_1(t), b_1(t) \in C[0,+\infty)$ *such that*

$$\begin{gathered}\left|f(t,x)\right| \le a_1(t)|x|^{\rho_1} + b_1(t), \quad t\in[0,+\infty), \\ \int_0^{+\infty}\left(1+t^{q-1}\right)a_1(t)dt < +\infty, \quad \int_0^{+\infty} b_1(t)dt < +\infty.\end{gathered}\tag{3.13}$$

(H_4) *there exist* $0 < \rho_2 < 1$ *and nonnegative functions* $c_1(t), d_1(t) \in C[0,+\infty)$ *such that*

$$\begin{gathered}\left|g(t,y)\right| \le c_1(t)\left|y\right|^{\rho_2} + d_1(t), \quad t\in[0,+\infty), \\ \int_0^{+\infty}\left(1+t^{p-1}\right)c_1(t)dt < +\infty, \quad \int_0^{+\infty} d_1(t)dt < +\infty.\end{gathered}\tag{3.14}$$

Then the system (1.1) *has a solution.*

Proof. In this case, we take

$$\begin{aligned}R > \max\Bigg\{&2L\int_0^{+\infty} b_1(s)ds, \left(2L\int_0^{+\infty}\left(1+s^{q-1}\right)a_1(s)ds\right)^{1/(1-\rho_1)}, \\ &2L\int_0^{+\infty} d_1(s)ds, \left(2L\int_0^{+\infty}\left(1+s^{p-1}\right)c_1(s)ds\right)^{1/(1-\rho_2)}\Bigg\}.\end{aligned}\tag{3.15}$$

The rest of the proof is similar to that of Theorem 3.2. So we omit it. □

Remark 3.4. By taking $\rho_1, \rho_2 > 1$ (instead of $0 < \rho_1 < 1$, $0 < \rho_2 < 1$) in (H_3) and (H_4), one can show that (1.1) has a solution.

Theorem 3.5. *Assume that*

(H_5) the functions f and g satisfy Lipschitz condition; that is, there exist nonnegative functions $K_1(t)$ and $K_2(t)$ such that

$$\begin{aligned} \left|f(t,x)-f(t,y)\right| &\le K_1(t)\left|x-y\right|, \quad t\in[0,+\infty), \\ \left|g(t,x)-g(t,y)\right| &\le K_2(t)\left|x-y\right|, \quad t\in[0,+\infty). \end{aligned} \tag{3.16}$$

Then the problem (1.1) *has a unique solution if*

$$\mu = L\int_0^{+\infty} K_1(s)\left(1+s^{q-1}\right)ds < 1, \qquad \tau = L\int_0^{+\infty} K_2(s)\left(1+s^{p-1}\right)ds < 1. \tag{3.17}$$

Proof. For any $(u_1,v_1), (u_2,v_2) \in Z$, we have

$$\begin{aligned} \|Q_1v_2 - Q_1v_1\|_X &= \sup_{t\in J}\frac{1}{1+t^{p-1}}\left|\int_0^{+\infty} G_1(t,s)\left[f(s,v_2(s)) - f(s,v_1(s))\right]ds\right| \\ &\le \sup_{t\in J}\int_0^{+\infty}\frac{G_1(t,s)}{1+t^{p-1}}K_1(s)|(v_2-v_1)(s)|ds \\ &\le L\int_0^{+\infty}K_1(s)\left(1+s^{q-1}\right)ds\|v_2-v_1\|_Y \\ &= \mu\|v_2-v_1\|_Y. \end{aligned} \tag{3.18}$$

Similarly, it can be shown that

$$\begin{aligned} \|Q_2u_2 - Q_2u_1\|_Y &= \sup_{t\in J}\frac{1}{1+t^{q-1}}\left|\int_0^{+\infty} G_1(t,s)\left(g(s,u_2(s)) - f(s,u_2(s))\right)ds\right| \\ &\le L\int_0^{+\infty}K_2(s)\left(1+s^{p-1}\right)ds\|u_2-u_1\|_X \\ &= \tau\|u_2-u_1\|_X. \end{aligned} \tag{3.19}$$

Thus, we get

$$\|Q(u_2,v_2) - Q(u_1,v_1)\|_Z \le \max\{\mu,\tau\}\|(u_2,v_2)-(u_1,v_1)\|_Z. \tag{3.20}$$

Obviously, Q is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

4. Example

Example 4.1. Consider the following multipoint boundary value problem on an unbounded domain:

$$\begin{aligned}
&D^{9/4}u(t) + \frac{\sin t \ln(1+|v(t)|)}{(1+t^{7/4})(2+t)^2} + (1+\cos 2t)e^{-t} = 0,\\
&D^{11/4}v(t) + \frac{e^{-5t}\sin|u(t)|}{3(1+t^{5/4})(1+t)^2} + \frac{4}{(t+4)^2} = 0,\\
&u(0) = u'(0) = 0, \qquad D^{5/4}u(+\infty) = \frac{2}{5}u\left(\frac{1}{4}\right) + \frac{1}{10}u(1),\\
&v(0) = v'(0) = 0, \qquad D^{7/4}v(+\infty) = \frac{3}{10}u\left(\frac{1}{4}\right) + \frac{1}{5}u(1).
\end{aligned} \tag{4.1}$$

Here $t \in [0,+\infty)$, $p = 9/4$, $q = 11/4$, $\xi_1 = 1/4$, $\xi_2 = 1$, $\beta_1 = 2/5$, $\beta_2 = 1/10$, $\gamma_1 = 3/10$, and $\gamma_2 = 1/5$. One has

$$f(t,v(t)) = \frac{\sin t \ \ln(1+|v(t)|)}{(1+t^{7/4})(2+t)^2} + (1+\cos 2t)e^{-t}, \qquad g(t,u(t)) = \frac{e^{-5t}\sin|u(t)|}{3(1+t^{5/4})(1+t)^2} + \frac{4}{(t+4)^2}. \tag{4.2}$$

For $a(t) = 1/(1+t^{7/4})(2+t)^2$, $b(t) = 2e^{-t}$, $c(t) = 1/3(1+t^{5/4})(1+t)^2$, $d(t) = 4/(t+4)^2$, by direct calculation we find that

$$\begin{aligned}
L &= \max\left\{\frac{1}{\Gamma(p)} + \frac{\sum_{i=1}^{m-2}\beta_i\xi_{m-2}^{p-1}}{\Gamma(p)\left(\Gamma(p) - \sum_{i=1}^{m-2}\beta_i\xi_i^{p-1}\right)}, \frac{1}{\Gamma(q)}\right.\\
&\qquad\left. + \frac{\sum_{i=1}^{m-2}\gamma_i\xi_{m-2}^{q-1}}{\Gamma(q)\left(\Gamma(q) - \sum_{i=1}^{m-2}\gamma_i\xi_i^{q-1}\right)}\right\}\\
&= \max\left\{\frac{1}{\Gamma(9/4)} + \frac{(2/5)+(1/10)}{\Gamma(9/4)\left(\Gamma(9/4) - 2/5(1/4)^{5/4} - (1/10)\right)}, \frac{1}{\Gamma(11/4)}\right.\\
&\qquad\left. + \frac{(3/10)+(1/5)}{\Gamma(11/4)\left(\Gamma(11/4) - 3/10(1/4)^{7/4} - (1/5)\right)}\right\}\\
&= 1.341213,\\
&\left|f(t,x)\right| \le a(t)|x| + b(t), \qquad \left|g(t,y)\right| \le c(t)\left|y\right| + d(t), \qquad t \in [0,+\infty),\\
&\int_0^{+\infty}\left(1+t^{q-1}\right)a(t)dt = \frac{1}{2} < \frac{1}{L} = 0.745594, \qquad \int_0^{+\infty} b(t)dt = 2 < +\infty,\\
&\int_0^{+\infty}\left(1+t^{p-1}\right)c(t)dt = \frac{1}{3} < \frac{1}{L} = 0.745594, \qquad \int_0^{+\infty} d(t)dt = 1 < +\infty.
\end{aligned} \tag{4.3}$$

Thus all conditions of Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, the couple system of nonlinear fractional differential (4.1) has at least one solution.

Example 4.2. Consider the following problem on an unbounded domain:

$$\begin{gathered} D^p u(t) + M_1(t)\sin v(t) + N_1(t) = 0, \\ D^q v(t) + \frac{M_2(t)}{1+u^2(t)} + N_2(t) = 0, \\ u(0) = u'(0) = 0, \qquad D^{p-1}u(+\infty) = \frac{2}{5}u\left(\frac{1}{4}\right) + \frac{1}{10}u(1), \\ v(0) = v'(0) = 0, \qquad D^{q-1}v(+\infty) = \frac{3}{10}u\left(\frac{1}{4}\right) + \frac{1}{5}u(1). \end{gathered} \tag{4.4}$$

Here $t \in [0,+\infty)$, $2 < p, q < 3$, $\xi_1 = 1/4$, $\xi_2 = 1$, $\beta_1 = 2/5$, $\beta_2 = 1/10$, $\gamma_1 = 3/10$, and $\gamma_2 = 1/5$, $M_1(t),\ M_2(t),\ N_1(t),\ N_2(t) \in C([0,+\infty),\mathbb{R})$.

With

$$f(t,v(t)) = M_1(t)\sin v(t) + N_1(t), \qquad g(t,u(t)) = \frac{M_2(t)}{1+u^2(t)} + N_2(t), \tag{4.5}$$

we have

$$\begin{aligned} \left|f(t,x) - f(t,y)\right| &= |M_1(t)|\left|\sin x - \sin y\right| \le |M_1(t)|\left|x-y\right|, \quad t \in [0,+\infty), \\ \left|g(t,x) - g(t,y)\right| &= |M_2(t)|\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| \le |M_2(t)|\left|x-y\right|, \quad t \in [0,+\infty), \end{aligned} \tag{4.6}$$

where $K_1(t) = |M_1(t)|$, $K_2(t) = |M_2(t)|$. So, the condition (H_5) holds. Let us assume that

$$\mu = L\int_0^{+\infty} |M_1(s)|\left(1+s^{q-1}\right)ds < 1, \qquad \tau = L\int_0^{+\infty} |M_2(s)|\left(1+s^{p-1}\right)ds < 1. \tag{4.7}$$

For example, condition (4.7) holds if we take

$$p = \frac{9}{4}, \qquad q = \frac{11}{4}, \qquad M_1(t) = \frac{1}{(1+t^{7/4})(2+t)^2}, \qquad M_2(t) = \frac{1}{3(1+t^{5/4})(1+t)^2}. \tag{4.8}$$

Thus all the conditions of Theorem 3.5 are satisfied. Therefore, by the conclusion of Theorem 3.5, the coupled system (4.4) has a unique solution.

5. Conclusion

We have shown the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with multipoint fractional boundary conditions on a semi-infinite domain. Our existence results are based on Schauder's fixed point theorem, while the uniqueness result is obtained by applying Banach's contraction mapping principle. The existence of solutions for (1.1) has been addressed for different kinds of growth conditions. Our approach is simple and can easily be applied to a variety of problems. This has been demonstrated by solving two examples.

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Research Article

On a Differential Equation Involving Hilfer-Hadamard Fractional Derivative

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This paper studies a fractional differential inequality involving a new fractional derivative (Hilfer-Hadamard type) with a polynomial source term. We obtain an exponent for which there does not exist any global solution for the problem. We also provide an example to show the existence of solutions in a wider space for some exponents.

1. Introduction

Fractional derivatives have proved to be very efficient and adequate to describe many phenomena with memory and hereditary processes. These phenomena are abundant in science, engineering (viscoelasticity, control, porous media, mechanics, electrical engineering, electromagnetism, etc.) as well as in geology, rheology, finance, and biology. Unlike the classical derivatives, fractional derivatives have the ability to characterize adequately processes involving a past history. We are witnessing a huge development of fractional calculus and methods in the theory of differential equations. Indeed, after the appearance of the papers by Bagley and Torvik [1–3], researchers started to deal directly with differential equations containing fractional derivatives instead of ignoring them as it used to be the case. For analytical treatments, we may refer the reader to [4–36], and for some applications, one can consult [1–3, 8, 25, 26, 26, 27, 27–31, 33, 34, 37–49] to cite but a few.

We will consider the problem:

$$\begin{gathered}\left(\mathfrak{D}_{a^+}^{\alpha,\beta}u\right)(t) = f[t,u(t)], \quad 0<\alpha<1,\ 0\le\beta\le1,\ t>a>0,\\ \left(\mathfrak{D}_{a^+}^{(\beta-1)(1-\alpha)}u\right)(a) = u_0 \ge 0,\end{gathered} \tag{1.1}$$

where $\mathfrak{D}_{a^+}^{\alpha,\beta}u$ is a new type of fractional derivative we will define below and u_0 is a given constant. This new fractional derivative interpolates the Hadamard fractional derivative and its Caputo counterpart [26, 34], in the same way the Hilfer fractional derivative interpolates the Riemann-Liouville fractional derivative and the Caputo fractional derivative. That is why we are naming it after Hilfer and Hadamard.

A nonexistence result for global solutions of the problem (1.1) will be proved when $f[t,u(t)] \geq (\log(t/a))^{\mu}|u(t)|^m$ for some $m > 1$ and $\mu \in \mathbf{R}$. That is we consider the Cauchy problem:

$$\begin{gathered}\left(\mathfrak{D}_{a^+}^{\alpha,\beta}u\right)(t) \geq \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m, \quad t > a > 0,\ m > 1,\ \mu \in \mathbf{R}, \\ \left(\mathfrak{D}_{a^+}^{\gamma-1}u\right)(a) = u_0 \geq 0,\end{gathered} \tag{1.2}$$

where $\gamma = \alpha+\beta-\alpha\beta$ and show that no solutions can exist for all time for certain values of μ and m. Clearly, sufficient conditions for nonexistence provide necessary conditions for existence of solutions. In addition, we construct an example for which there exist solutions for some powers m and in some appropriate space.

The existence and uniqueness of solutions for problem (1.1) has been discussed in [50] in the space $C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b]$ defined by

$$C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b] = \left\{y \in C_{1-\gamma,\log}[a,b], \mathfrak{D}_{a^+}^{\alpha,\beta}y \in C_{\mu,\log}[a,b]\right\}, \tag{1.3}$$

where

$$C_{\gamma,\log}[a,b] = \left\{g : (a,b] \longrightarrow \mathbf{R} : \left(\log\frac{x}{a}\right)^{\gamma}g(x) \in C[a,b]\right\} \tag{1.4}$$

for $0 \leq \mu < 1$ and $C_{0,\log}[a,b] = C[a,b]$.

We also point out here that the case where $\mathfrak{D}_{a^+}^{\alpha,\beta}$ is the usual Riemann-Liouville fractional derivative has been studied in [26] (see also references therein). There are very few papers [26, 29] dealing with the pure Hadamard case, that is, when $\beta = 0$.

The rest of the paper is divided into three sections. In Section 2, we present some definitions, notations, and lemmas which will be needed later in our proof. Section 3 is devoted to the nonexistence result and Section 4 contains an example of existence of solutions.

2. Preliminaries

In this section, we present some background material for the forthcoming analysis. For more details, see [25, 26, 33, 42, 51, 52].

Definition 2.1. The space $X_c^p(a,b)$ $(c \in \mathbf{R}, 1 \le p \le \infty)$ consists of those real-valued Lebesgue measurable functions g on (a,b) for which $\|g\|_{X_c^p} < \infty$, where

$$\|g\|_{X_c^p} = \left(\int_a^b \left|t^c g(t)\right|^p \frac{dt}{t}\right)^{1/p}, \quad 1 \le p < \infty, \; c \in \mathbf{R},$$
$$\|g\|_{X_c^\infty} = \operatorname*{ess\,sup}_{a \le x \le b} \left|x^c g(x)\right|, \quad c \in \mathbf{R}. \tag{2.1}$$

In particular, when $c = 1/p$, we see that $X_{1/p}^p(a,b) = L_p(a,b)$.

Definition 2.2. Let $\Omega = [a,b]$ $(0 < a < b < \infty)$ be a finite interval and $0 \le \gamma < 1$, we introduce the weighted space $C_{\gamma,log}[a,b]$ of continuous functions g on $(a,b]$:

$$C_{\gamma,\log}[a,b] = \left\{ g \in C(a,b] : \left(\log \frac{x}{a}\right)^\gamma g(x) \in C[a,b] \right\}. \tag{2.2}$$

In the space $C_{\gamma,\log}[a,b]$, we define the norm:

$$\|g\|_{C_{\gamma,\log}} = \left\| \left(\log \frac{x}{a}\right)^\gamma g(x) \right\|_C, \qquad \|g\|_{C_{0,\log}} = \|g\|_\infty. \tag{2.3}$$

Definition 2.3. Let $\delta = x(d/dx)$ be the δ-derivative, for $n \in \mathbf{N}$, we denote by $C_{\delta,\gamma}^n[a,b]$ $(0 \le \gamma < 1)$ the Banach space of functions g which have continuous δ-derivatives on $[a,b]$ up to order $n-1$ and have the derivative $\delta^n g$ of order n on $(a,b]$ such that $\delta^n g \in C_{\gamma,\log}[a,b]$:

$$C_{\delta,\gamma}^n[a,b] = \left\{ g : (a,b] \longrightarrow \mathbf{R} : \delta^k g \in C[a,b], k = 0,\dots,n-1, \delta^n g \in C_{\gamma,\log}[a,b] \right\} \tag{2.4}$$

with the norm:

$$\|g\|_{C_{\delta,\gamma}^n} = \sum_{k=0}^{n-1} \left\|\delta^k g\right\|_C + \|\delta^n g\|_{C_{\gamma,\log}}. \tag{2.5}$$

When $n = 0$, we set

$$C_{\delta,\gamma}^0[a,b] = C_{\gamma,\log}[a,b]. \tag{2.6}$$

Definition 2.4. Let (a,b) $(0 \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $\mathbf{R}^+$ and let $\alpha > 0$. The Hadamard left-sided fractional integral $\mathcal{J}_{a^+}^\alpha f$ of order $\alpha > 0$ is defined by

$$\left(\mathcal{J}_{a^+}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad a < x < b \tag{2.7}$$

provided that the integral exists. When $\alpha = 0$, we set

$$\mathcal{J}_{a^+}^{0} f = f. \tag{2.8}$$

Definition 2.5. Let (a, b) $(0 \leq a < b \leq \infty)$ be a finite or infinite interval of the half-axis $\mathbf{R}^+$ and let $\alpha > 0$. The Hadamard right-sided fractional integral $\mathcal{J}_{b^-}^{\alpha} f$ of order $\alpha > 0$ is defined by

$$\left(\mathcal{J}_{b^-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad a < x < b, \tag{2.9}$$

provided that the integral exists. When $\alpha = 0$, we set

$$\mathcal{J}_{b^-}^{0} f = f. \tag{2.10}$$

Definition 2.6. The left-sided Hadamard fractional derivative of order $0 \leq \alpha < 1$ on (a, b) is defined by

$$\left(\mathfrak{D}_{a^+}^{\alpha} f\right)(x) := \delta\left(\mathcal{J}_{a^+}^{1-\alpha} f\right)(x), \tag{2.11}$$

that is,

$$\left(\mathfrak{D}_{a^+}^{\alpha} f\right)(x) = \left(x \frac{d}{dx}\right) \frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{-\alpha} \frac{f(t)dt}{t}, \quad a < x < b. \tag{2.12}$$

When $\alpha = 0$, we set

$$\mathfrak{D}_{a^+}^{0} f = f. \tag{2.13}$$

Definition 2.7. The right-sided Hadamard fractional derivative of order α $(0 \leq \alpha < 1)$ on (a, b) is defined by

$$\left(\mathfrak{D}_{b^-}^{\alpha} f\right)(x) := -\delta\left(\mathcal{J}_{b^-}^{1-\alpha} f\right)(x), \tag{2.14}$$

that is,

$$\left(\mathfrak{D}_{b^-}^{\alpha} f\right)(x) = -\left(x \frac{d}{dx}\right) \frac{1}{\Gamma(1-\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{-\alpha} \frac{f(t)dt}{t}, \quad a < x < b. \tag{2.15}$$

When $\alpha = 0$, we set

$$\mathfrak{D}_{b^-}^{0} f = f. \tag{2.16}$$

Lemma 2.8. *If $\alpha > 0,\ \beta > 0$ and $0 < a < b < \infty$, then*

$$\left(\mathcal{J}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1},$$
$$\left(\mathfrak{D}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}. \tag{2.17}$$

In particular, if $\beta = 1$, then the Hadamard fractional derivative of a constant is not equal to zero:

$$\left(\mathfrak{D}_{a^+}^{\alpha}1\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\log\frac{x}{a}\right)^{-\alpha}, \tag{2.18}$$

when $0 < \alpha < 1$.

Lemma 2.9. *Let $0 < a < b < \infty,\ \alpha > 0$, and $0 \le \mu < 1$.*

(a) *If $\mu > \alpha > 0$, then $\mathcal{J}_{a^+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C_{\mu-\alpha,\log}[a,b]$. In particular, $\mathcal{J}_{a^+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.*

(b) *If $\mu \le \alpha$, then $\mathcal{J}_{a^+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C[a,b]$. In particular, $\mathcal{J}_{a^+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.*

This lemma justifies the following one

Lemma 2.10 (the semigroup property of the fractional integration operator $\mathcal{J}_{a^+}^{\alpha}$). *Let $\alpha > 0,\ \beta > 0$, and $0 \le \mu < 1$. If $0 < a < b < \infty$, then, for $f \in C_{\mu,\log}[a,b]$,*

$$\mathcal{J}_{a^+}^{\alpha}\mathcal{J}_{a^+}^{\beta}f = \mathcal{J}_{a^+}^{\alpha+\beta}f \tag{2.19}$$

holds at any point $x \in (a,b]$. When $f \in C[a,b]$, this relation is valid at any point $x \in [a,b]$.

Lemma 2.11. *Let $0 \le \alpha < 1$ and $0 \le \gamma < 1$. If $f \in C_{\gamma,\log}^{1}[a,b]$, then the fractional derivatives $\mathfrak{D}_{a^+}^{\alpha}$ and $\mathfrak{D}_{b^-}^{\alpha}$ exist on $(a,b]$ and $[a,b)$, respectively, $(a > 0)$ and can be represented in the forms:*

$$\left(\mathfrak{D}_{a^+}^{\alpha}f\right)(x) = \frac{f(a)}{\Gamma(1-\alpha)}\left(\log\frac{x}{a}\right)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_a^x\left(\log\frac{x}{t}\right)^{-\alpha}f'(t)dt,$$
$$\left(\mathfrak{D}_{b^-}^{\alpha}f\right)(x) = \frac{f(b)}{\Gamma(1-\alpha)}\left(\log\frac{b}{x}\right)^{-\alpha} - \frac{1}{\Gamma(1-\alpha)}\int_x^b\left(\log\frac{t}{x}\right)^{-\alpha}f'(t)dt, \tag{2.20}$$

respectively.

Lemma 2.12 (fractional integration by Parts). *Let $\alpha > 0$ and $1 \leq p \leq \infty$. If $\varphi \in L_p(\mathbf{R}^+)$ and $\psi \in X^q_{-1/p}$, then*

$$\int_0^\infty \varphi(x)\left(\mathcal{J}_+^\alpha \psi\right)(x)\frac{dx}{x} = \int_0^\infty \psi(x)\left(\mathcal{J}_-^\alpha \varphi\right)(x)\frac{dx}{x}, \tag{2.21}$$

where $1/p + 1/q = 1$.

Definition 2.13. The fractional derivative ${}^c\mathfrak{D}_{a^+}^\alpha f$ of order α $(0 < \alpha < 1)$ on (a, b) defined by

$${}^c\mathfrak{D}_{a^+}^\alpha f = \mathcal{J}_{a^+}^{1-\alpha}\delta f, \tag{2.22}$$

where $\delta = x(d/dx)$, is called the Hadamard-Caputo fractional derivative of order α.

Now, motivated by the Hilfer fractional derivative introduced in [41, 42], we introduce the new fractional derivative which we call Hilfer-Hadamard fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$:

$$\left(\mathfrak{D}_{a^+}^{\alpha,\beta} u\right)(t) = \mathcal{J}_{a^+}^{\beta(1-\alpha)}\left(t\frac{d}{dt}\right)\left(\mathcal{J}_{a^+}^{(1-\beta)(1-\alpha)} u\right)(t). \tag{2.23}$$

The Hilfer fractional derivative interpolates the Riemann-Liouville fractional derivative and the Caputo fractional derivative. This new one interpolates the Hadamard fractional derivative and its caputo counterpart. Indeed, for $\beta = 0$, we find the Hadamard fractional derivative as defined in Definition 2.6 and, for $\beta = 1$, we find its Caputo type counterpart (Definition 2.13).

Theorem 2.14 (Young's inequality). *If a and b are nonnegative real numbers and p and q are positive real numbers such that $1/p + 1/q = 1$, then one has*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.24}$$

Equality holds if and only if $a^p = b^q$.

3. Nonexistence Result

Before we state and prove our main result, we will start with the following lemma.

Lemma 3.1. *If $\alpha > 0$ and $f \in C[a, b]$, then*

$$\begin{aligned}\left(\mathcal{J}_{a+}^\alpha f\right)(a) &= \lim_{t \to a}\left(\mathcal{J}_{a+}^\alpha f\right)(t) = 0,\\ \left(\mathcal{J}_{b-}^\alpha f\right)(b) &= \lim_{t \to b}\left(\mathcal{J}_{b-}^\alpha f\right)(t) = 0.\end{aligned} \tag{3.1}$$

Proof. Since $f \in C[a,b]$, then on $[a,b]$ we have $|f(t)| < M$ for some positive constant M. Therefore,

$$\begin{aligned}\left|\left(\mathcal{J}_{a+}^{\alpha} f\right)(t)\right| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s)| \frac{ds}{s} \leq \frac{M}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{M}{\Gamma(\alpha+1)} \left(\log \frac{t}{a}\right)^{\alpha}.\end{aligned} \tag{3.2}$$

As $\alpha > 0$, we see that

$$\left(\mathcal{J}_{a+}^{\alpha} f\right)(a) = \lim_{t \to a} \left(\mathcal{J}_{a+}^{\alpha} f\right)(t) = 0. \tag{3.3}$$

In a similar manner, we prove the second part of the lemma. □

The proof of the next result is based on the test function method developed by Mitidieri and Pokhozhaev in [52].

Theorem 3.2. *Assume that $\mu \in \mathbf{R}$ and $m < (1+\mu)/(1-\gamma)$. Then, Problem* (1.2) *does not admit global nontrivial solutions in $C_{1-\gamma,\log}^{\gamma}[a,b]$, where*

$$C_{1-\gamma,\log}^{\gamma}[a,b] = \left\{ y \in C_{1-\gamma,\log}[a,b] : \mathfrak{D}_{a^+}^{\gamma} y \in C_{1-\gamma,\log}[a,b] \right\} \tag{3.4}$$

when $u_0 \geq 0$.

Proof. Assume that a nontrivial solution exists for all time $t > a$. Let $\varphi(t) \in C^1([a,\infty))$ be a test function satisfying $\varphi(t) \geq 0$, $\varphi(t)$ is non-increasing and such that

$$\varphi(t) := \begin{cases} 1, & a \leq t \leq \theta T, \\ 0, & t \geq T, \end{cases} \tag{3.5}$$

for some $T > a$ and some θ ($\theta < 1$) such that $a < \theta T < T$. Multiplying the inequality in (1.2) by $\varphi(t)/t$ and integrating over $[a,T]$, we get

$$\int_a^T \varphi(t) \left(\mathfrak{D}_{a^+}^{\alpha,\beta} u\right)(t) \frac{dt}{t} \geq \int_a^T \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t}. \tag{3.6}$$

Observe that the integral in left-hand side exists and the one in the right-hand side exists for $m < (1+\mu)/(1-\gamma)$ when $u \in C_{1-\gamma,\log}^{\gamma}[a,b]$. Moreover, from the definition of $(\mathfrak{D}_{a^+}^{\alpha,\beta} u)(t)$, we can rewrite (3.6) as

$$\int_a^T \varphi(t) \left(\mathcal{J}_{a^+}^{\beta(1-\alpha)} t \frac{d}{dt} \mathcal{J}_{a^+}^{1-\gamma} u\right)(t) \frac{dt}{t} \geq \int_a^T \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t}. \tag{3.7}$$

By virtue of Lemma 2.12 (after extending by zero outside $[a,T]$), we may deduce from (3.7) that

$$\int_a^T \frac{d}{dt}\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi(t)\right)(t)dt \geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}. \tag{3.8}$$

Notice that Lemma 2.12 is valid in our case since $((\log(t/a))^{(1-\gamma)}(\mathfrak{D}_{a^+}^{\gamma}u) \in C[a,T]$ implies that $|(\log(t/a))^{(1-\gamma)}(\mathfrak{D}_{a^+}^{\gamma}u)(t)| \leq M$ on $[a,T]$ for some positive constant M)

$$\begin{aligned}\int_a^T \left|t^{-1/p}\left(\mathfrak{D}_{a^+}^{\gamma}u\right)(t)\right|^{p'}\frac{dt}{t} &\leq M\int_a^T t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t}\\ &\leq M\int_a^{\infty} t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t}.\end{aligned} \tag{3.9}$$

Let $s=(p'-1)(\log(t/a))$, then by the definition of the Gamma function,

$$\begin{aligned}\int_a^T \left|t^{-1/p}\left(\mathfrak{D}_{a^+}^{\gamma}u\right)(t)\right|^{p'}\frac{dt}{t} &\leq \frac{Ma^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\int_0^{\infty} s^{-p'(1-\gamma)}e^{-s}ds\\ &\leq \frac{Ma^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\Gamma(1-p'(1-\gamma)) < \infty.\end{aligned} \tag{3.10}$$

Hence, $t(d/dt)(\mathcal{J}_{a^+}^{1-\gamma}u)t = (\mathfrak{D}_{a^+}^{\gamma}u)(t) \in X_{-1/p}^{p'}$ (and $\varphi \in L_p$) for some $p > 1/\gamma$.

An integration by parts in (3.8) yields

$$\begin{aligned}&\left[\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)\right]_{t=a}^{T} - \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)dt\\ &\quad\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t},\end{aligned} \tag{3.11}$$

or

$$\begin{aligned}&-u_0\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a^+) - \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)dt\\ &\quad\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}\end{aligned} \tag{3.12}$$

because $(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi)(T)=0$ (see Lemma 3.1) and

$$\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) = \left(\mathfrak{D}_{a^+}^{\gamma-1}u\right)(a^+) = u_0. \tag{3.13}$$

Multiplying by t/t inside the integral in the left hand side of (3.12), we see that

$$\begin{aligned} L &:= \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\left(-t\frac{d}{dt}\right)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)\frac{dt}{t} \\ &\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t)\frac{dt}{t}. \end{aligned} \tag{3.14}$$

It appears from Definition 2.7 that

$$L = \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\left(\mathfrak{D}_{T^-}^{1-\beta(1-\alpha)}\varphi\right)(t)\frac{dt}{t}, \tag{3.15}$$

and from Lemma 2.11, we see that

$$L = \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\left[\frac{\varphi(T)}{\Gamma(\beta(1-\alpha))}\left(\log\frac{T}{t}\right)^{\beta(1-\alpha)-1} - \frac{1}{\Gamma(\beta(1-\alpha))}\int_t^T \left(\log\frac{s}{t}\right)^{\beta(1-\alpha)-1}\varphi'(s)ds\right]\frac{dt}{t}. \tag{3.16}$$

Since $\varphi(T) = 0$ and

$$\frac{1}{\Gamma(\beta(1-\alpha))}\int_t^T \left(\log\frac{s}{t}\right)^{\beta(1-\alpha)-1}\varphi'(s)ds = \left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\delta\varphi\right)(t), \tag{3.17}$$

the last equality becomes

$$\begin{aligned} L &= -\int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\delta\varphi\right)(t)\frac{dt}{t} \\ &\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t)\frac{dt}{t}. \end{aligned} \tag{3.18}$$

Note that $\delta\varphi \in L_p$ and by the same argument as the one used at the beginning of the proof we may show that $\mathcal{J}_{a^+}^{1-\gamma} u \in X_{-1/p}^{p'}$ since $\mathcal{J}_{a^+}^{1-\gamma} u \in C_{1-\gamma,\log}[a,T]$.

Therefore, Lemma 2.12 again allows us to write

$$L = -\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\mathcal{J}_{a^+}^{1-\gamma} u\right)(t)\frac{dt}{t}, \tag{3.19}$$

and by the semigroup property Lemma 2.10

$$L = -\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha} u\right)(t)\frac{dt}{t}. \tag{3.20}$$

On the other hand,

$$\begin{aligned}\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha}u\right)(t)\frac{dt}{t} &= \frac{1}{\Gamma(1-\alpha)}\int_a^T \delta\varphi(t)\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{u(s)}{s}ds\frac{dt}{t}\\ &\le \frac{1}{\Gamma(1-\alpha)}\int_a^T \left|\delta\varphi(t)\right|\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{|u(s)|}{s}ds\frac{dt}{t}.\end{aligned}\tag{3.21}$$

As φ is nonincreasing, we have $\varphi(s) \ge \varphi(t)$ for all $t \ge s$ and $1/\varphi^{1/m}(s) \le 1/\varphi^{1/m}(t)$, $m > 1$. Also, it is clear that

$$\varphi'(t) = 0, \quad t \in [a, \theta T].\tag{3.22}$$

Therefore,

$$\begin{aligned}L &\le \frac{1}{\Gamma(1-\alpha)}\int_a^T \left|\delta\varphi(t)\right|\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{|u(s)|\varphi^{1/m}(s)}{s\varphi^{1/m}(s)}ds\frac{dt}{t}\\ &\le \frac{1}{\Gamma(1-\alpha)}\int_{\theta T}^T \frac{\left|\delta\varphi(t)\right|}{\varphi^{1/m}(t)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\frac{|u(s)|\varphi^{1/m}(s)}{s}ds\frac{dt}{t}.\end{aligned}\tag{3.23}$$

Definition 2.4 allows us to write

$$L \le \int_{\theta T}^T \frac{\left|\delta\varphi(t)\right|}{\varphi^{1/m}(t)}\left(\mathcal{J}_{a^+}^{1-\alpha}|u|\varphi^{1/m}\right)(t)\frac{dt}{t}.\tag{3.24}$$

By the same argument as the one used at the beginning of the proof, we may show that $|u(t)|\varphi^{1/m}(t) \in X_{-1/p}^{p'}$ ($|u(t)|\varphi^{1/m}(t) \le |u(t)|$). Moreover, it is easy to see that $\delta\varphi(t)/\varphi^{1/m}(t) \in L_p$ (for, otherwise, we consider $\varphi^{\lambda}(t)$ with some sufficiently large λ). Thus, we can apply Lemma 2.12 to get

$$L \le \int_{\theta T}^T |u(t)|\varphi^{1/m}(t)\left(\mathcal{J}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)\frac{dt}{t}.\tag{3.25}$$

Next, we multiply by $(\log(t/a))^{\mu/m}.(\log(t/a))^{-\mu/m}$ inside the integral in the right-hand side of (3.25):

$$L \le \int_{\theta T}^T \left(\mathcal{J}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)|u(t)|\varphi^{1/m}(t)\frac{(\log(t/a))^{\mu/m}}{(\log(t/a))^{\mu/m}}\frac{dt}{t}.\tag{3.26}$$

For $\mu \ge 0$, we have $(\log(t/a))^{-\mu/m} \le (\log(\theta T/a))^{-\mu/m}$ (because $-\mu/m < 0$ and $t > \theta T$). It follows that

$$L \le \left(\log\frac{\theta T}{a}\right)^{-\mu/m}\int_{\theta T}^T \left(\mathcal{J}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)\left(\log\frac{t}{a}\right)^{\mu/m}|u(t)|\varphi^{1/m}(t)\frac{dt}{t}.\tag{3.27}$$

By using the Young inequality (see Theorem 2.14), with m and m' such that $1/m + 1/m' = 1$, in the right-hand side of (3.27), we find

$$\begin{aligned} L &\leq \frac{1}{m}\int_{\theta T}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} + \frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t} \\ &\leq \frac{1}{m}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} + \frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t}. \end{aligned} \tag{3.28}$$

Clearly, from (3.14) and (3.28), we see that

$$\begin{aligned} &\frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t} \\ &\quad\geq \left(1-\frac{1}{m}\right)\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}, \end{aligned} \tag{3.29}$$

or

$$\frac{1}{m'}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{m'}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t}. \tag{3.30}$$

Therefore, by Definition 2.5, we have

$$\begin{aligned} &\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \\ &\quad\leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\theta T}^{T}\left(\int_{t}^{T}\left(\log\frac{s}{t}\right)^{-\alpha}\frac{|\delta\varphi(s)|}{\varphi^{1/m}(s)}\frac{ds}{s}\right)^{m'}\frac{dt}{t}. \end{aligned} \tag{3.31}$$

The change of variable $\sigma T = t$ yields

$$\begin{aligned} &\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \\ &\quad\leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\theta}^{1}\left(\int_{\sigma T}^{T}\left(\log\frac{s}{\sigma T}\right)^{-\alpha}\frac{|\varphi'(s)|}{\varphi(s)^{1/m}}ds\right)^{m'}\frac{d\sigma}{\sigma}. \end{aligned} \tag{3.32}$$

Another change of variable $r = s/T$ gives

$$\begin{aligned} &\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \\ &\quad\leq \frac{(\log(\theta T/a))^{-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\theta}^{1}\left(\int_{\sigma}^{1}\left(\log\frac{r}{\sigma}\right)^{-\alpha}\frac{|\varphi'(r)|}{\varphi(r)^{1/m}}dr\right)^{m'}\frac{d\sigma}{\sigma}. \end{aligned} \tag{3.33}$$

We may assume that the integral term in the right-hand side of (3.33) is convergent, that is,

$$\frac{1}{\Gamma^{m'}(1-\alpha)}\int_{\theta}^{1}\left(\int_{\sigma}^{1}\left(\ln\frac{r}{\sigma}\right)^{-\alpha}\frac{|\varphi'(r)|}{\varphi(r)^{1/m}}dr\right)^{m'}d\sigma \leq C, \tag{3.34}$$

for some positive constant C, for otherwise we consider $\varphi^{\lambda}(r)$ with some sufficiently large λ. Therefore

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \leq C\left(\log\frac{\theta T}{a}\right)^{-\mu m'/m}. \tag{3.35}$$

If $\mu > 0$, then

$$\left(\log\frac{\theta T}{a}\right)^{-\mu m'/m} \longrightarrow 0, \tag{3.36}$$

as $T \longrightarrow \infty$. Finally, from (3.35), we obtain

$$\lim_{T\to\infty}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} = 0. \tag{3.37}$$

We reach a contradiction since the solution is not supposed to be trivial.

In the case $\mu = 0$ we have $-\mu m'/m = 0$ and the relation (3.35) ensures that

$$\lim_{T\to\infty}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \leq C. \tag{3.38}$$

Moreover, it is clear that

$$\begin{aligned}&\left(\log\frac{\theta T}{a}\right)^{-\mu/m}\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)\left(\log\frac{t}{a}\right)^{\mu/m}|u(t)|\varphi^{1/m}(t)\frac{dt}{t}\\ &\quad\leq\left(\log\frac{\theta T}{a}\right)^{-\mu/m}\left[\int_{\theta T}^{T}\left(\mathcal{D}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m'}(t)\frac{dt}{t}\right]^{1/m'}\left[\int_{\theta T}^{T}\left(\log\frac{t}{a}\right)^{\mu}|u(t)|^{m}\varphi(t)\frac{dt}{t}\right]^{1/m}.\end{aligned} \tag{3.39}$$

This relation, together with (3.27) (relations (3.28) and (3.31) also are used without θ), implies that

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t} \leq K\left[\int_{\theta T}^{t}\left(\log\frac{t}{a}\right)^{\mu}|u(t)|^{m}\varphi(t)\frac{dt}{t}\right]^{1/m} \tag{3.40}$$

for some positive constant K, with

$$\lim_{T\to\infty}\int_{\theta T}^{T}\left(\log\frac{t}{a}\right)^{\mu}|u(t)|^{m}\varphi(t)\frac{dt}{t}=0 \tag{3.41}$$

due to the convergence of the integral in (3.38). This is again a contradiction.

If $\mu<0$, we have $(\log(t/a))^{-\mu/m}\leq(\log(T/a))^{-\mu/m}$ (because $-\mu/m>0$ and $t<T$). Then, the change of variables $t=(T/a)^{\sigma}$ and $s=(T/a)^{r}$ in (3.27) yields

$$\begin{aligned}&\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}\\&\quad\leq\frac{(\log(T/a))^{1-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\ln\theta T/\ln(T/a)}^{\ln T/\ln(T/a)}\left(\int_{\sigma}^{\ln T/\ln(T/a)}\left(\ln\frac{(T/a)^{r}}{(T/a)^{\sigma}}\right)^{-\alpha}\frac{|\varphi'(r)|}{\varphi^{1/m}(r)}dr\right)^{m'}d\sigma,\end{aligned} \tag{3.42}$$

or

$$\begin{aligned}&\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}\\&\quad\leq\frac{(\log(T/a))^{1-\alpha m'-\mu m'/m}}{\Gamma^{m'}(1-\alpha)}\int_{\ln\theta T/\ln(T/a)}^{\ln T/\ln(T/a)}\left(\int_{\sigma}^{\ln T/\ln(T/a)}(r-\sigma)^{-\alpha}\frac{|\varphi'(r)|}{\varphi^{1/m}(r)}dr\right)^{m'}d\sigma.\end{aligned} \tag{3.43}$$

The expression $|\varphi'(r)|/\varphi^{1/m}(r)$ may be assumed bounded (or else we use $\varphi^{\lambda}(r)$ with a large value of λ). Hence,

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)|u(t)|^{m}\frac{dt}{t}\leq C\left(\log\frac{T}{a}\right)^{-m'-\mu m'/m} \tag{3.44}$$

for some positive constant C. □

Although we are concerend here about nonexistence of solutions, using standard techniques, one may show the existence of local solutions of Problem (1.1) with $1<m<(1+\mu)/(1-\gamma)$. However, according to Theorem 3.2, such a solution cannot be continued for all time in $C^{\gamma}_{1-\gamma,\log}[a,b]$. This is a phenomenon which occurs often in parabolic and hyperbolic problems with sources of polynomial type. In the absence of strong dissipations, these sources are the cause of blowup in finite time (of local solutions). For this reason, they are called blowup terms.

4. Example

For our example, we need the following lemma.

Lemma 4.1. *The following result holds for the fractional derivative operator* $\mathfrak{D}_{a^+}^{\alpha,\beta}$*:*

$$\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right]\right)(t)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\left(\log\frac{t}{a}\right)^{\gamma-\alpha-1},\quad t>a;\ \gamma>0, \tag{4.1}$$

where $0<\alpha<1$ *and* $0\leq\beta\leq 1$.

Proof. We observe from Lemma 2.8 that

$$\left(\mathcal{J}_{a^+}^{(1-\alpha)(1-\beta)}\left(\log\frac{s}{a}\right)^{\gamma-1}\right)(t)=\frac{\Gamma(\gamma)}{\Gamma((1-\alpha)(1-\beta)+\gamma)}\left(\log\frac{t}{a}\right)^{\gamma+(1-\alpha)(1-\beta)-1}. \tag{4.2}$$

Therefore,

$$\begin{aligned}&\left(t\frac{d}{dt}\right)\left(\mathcal{J}_{a^+}^{(1-\alpha)(1-\beta)}\left(\log\frac{s}{a}\right)^{\gamma-1}\right)(t)\\&\quad=\frac{[\gamma+(1-\alpha)(1-\beta)-1]\Gamma(\gamma)}{\Gamma((1-\alpha)(1-\beta)+\gamma)}\left(\log\frac{t}{a}\right)^{\gamma+(1-\alpha)(1-\beta)-2},\end{aligned} \tag{4.3}$$

which, in light of the definition of $\mathfrak{D}_{a^+}^{\alpha,\beta}$, yields

$$\begin{aligned}&\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right]\right)(t)\\&\quad=\frac{\Gamma(\gamma)}{\Gamma((1-\alpha)(1-\beta)+\gamma-1)}\left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\left(\log\frac{s}{a}\right)^{\gamma+(1-\alpha)(1-\beta)-2}\right)(t).\end{aligned} \tag{4.4}$$

From Lemma 2.8 again, we have

$$\begin{aligned}&\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right]\right)(t)\\&\quad=\frac{\Gamma(\gamma)}{\Gamma(\beta(1-\alpha)+\gamma+(1-\alpha)(1-\beta)-1)}\left(\log\frac{t}{a}\right)^{\beta(1-\alpha)+\gamma+(1-\alpha)(1-\beta)-2}\\&\quad=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\left(\log\frac{t}{a}\right)^{\gamma-\alpha-1}.\end{aligned} \tag{4.5}$$

The proof is complete. □

Example 4.2. Consider the following differential equation of Hilfer-Hadamard-type fractional derivative of order $0<\alpha<1$ and type $0\leq\beta\leq 1$:

$$\left(\mathfrak{D}_{a^+}^{\alpha,\beta}y\right)(t)=\lambda\left(\log\frac{t}{a}\right)^{\mu}[y(t)]^m\quad(t>a>0;m>1) \tag{4.6}$$

with real $\lambda, \mu \in \mathbf{R}^+$ ($\lambda \neq 0$). Suppose that the solution has the following form:

$$y(t) = c\left(\log \frac{t}{a}\right)^{\nu}. \tag{4.7}$$

Our aim next is to find the values of c and ν. By using Lemma 4.1 we have

$$\left(\mathfrak{D}_{a^+}^{\alpha,\beta}\left[c\left(\log \frac{s}{a}\right)^{\nu}\right]\right)(t) = \frac{c\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}\left(\log \frac{t}{a}\right)^{\nu-\alpha}. \tag{4.8}$$

Therefore,

$$\frac{c\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}\left(\log \frac{t}{a}\right)^{\nu-\alpha} = \lambda\left(\log \frac{t}{a}\right)^{\mu}\left[c\left(\log \frac{t}{a}\right)^{\nu}\right]^{m}. \tag{4.9}$$

It can be directly shown that $\nu = (\alpha+\mu)/(1-m)$ and $c = [\Gamma((\alpha+\mu)/(1-m)+1)/\lambda\Gamma((m\alpha+\mu)/(1-m)+1)]^{1/(m-1)}$. If $(m\alpha+\mu)/(1-m) > -1$, that is, $m > (1+\mu)/(1-\alpha)$, then (4.6) has the exact solution:

$$y(t) = \left[\frac{\Gamma((\alpha+\mu)/(1-m)+1)}{\lambda\Gamma((m\alpha+\mu)/(1-m)+1)}\right]^{1/(m-1)}\left(\log \frac{t}{a}\right)^{(\alpha+\mu)/(1-m)}. \tag{4.10}$$

This solution satisfies the initial condition when $(\alpha+\mu)/(1-m) \geq \gamma - 1 > -1$. Note that there is an overlap of the interval of existence in this example and the interval of nonexistence in the previous theorem. This may be explained by the fact that this solution is in $C_{1-\gamma,\log}[a,b]$ but not in $C_{1-\gamma,\log}^{\gamma}[a,b]$.

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Research Article

The Approximate Solution of Fractional Fredholm Integrodifferential Equations by Variational Iteration and Homotopy Perturbation Methods

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Variational iteration method and homotopy perturbation method are used to solve the fractional Fredholm integrodifferential equations with constant coefficients. The obtained results indicate that the method is efficient and also accurate.

1. Introduction

The topic of fractional calculus has attracted many scientists because of its several applications in many areas, such as physics, chemistry, and engineering. For a detail survey with collections of applications in various fields, see, for example, [1–3].

Further, the fractional derivatives technique has been employed for solving linear fractional differential equations including the fractional integrodifferential equations; in this way, much of the efforts is devoted to searching for methods that generate accurate results, see [4, 5]. In this work, we present two different methods, namely, homotopy perturbation method and variational iteration method [6], for solving a fractional Fredholm integrodifferential equations with constant coefficients. There is a vast literature, and we only mention the works of Liao which treat a homotopy method in [7, 8].

For the nonlinear equations with derivatives of integer order, many methods are used to derive approximation solution [9–14]. However, for the fractional differential equation, there are some limited approaches, such as Laplace transform method [3], the Fourier transform method [15], the iteration method [16], and the operational calculus method [17].

Recently, there has been considerable researches in fractional differential equations due to their numerous applications in the area of physics and engineering [18], such as phenomena in electromagnetic theory, acoustics, electrochemistry, and material science [3, 16, 18, 19]. Similarly, there is also growing interest in the integrodifferential equations which are combination of differential and Fredholm-Volterra equations. In this work, we study these kind of equations that have the fractional order usually difficult to solve analytically, thus a numerical method is required, for example, the successive approximations, Adomian decomposition, Chebyshev and Taylor collocation, Haar Wavelet, Tau and Walsh series methods.

This note is devoted to the application of variational iteration method (VIM) and homotopy perturbation method (HPM) for solving fractional Fredholm integrodifferential equations with constant coefficients:

$$\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(t) = g(x) + \lambda \int_0^a H(x,t) y(t) dt, \quad a \le x,\ t \le b, \tag{1.1}$$

under the initial-boundary conditions

$$D_*^{\alpha} y(a) = y(0), \tag{1.2}$$

$$D_*^{\alpha} y(0) = y'(a), \tag{1.3}$$

where a is constant and $1 < \alpha < 2$ and D_*^{α} is the fractional derivative operator given in the Caputo sense. For the physical understanding of the fractional integrodifferential equations, see [20]. Further, we also note that fractional integrodifferential equations were associated with a certain class of phase angles and suggested a new way for understanding of Riemann's conjecture, see [21].

Outline of this paper is as follows. Section 2 contains preliminaries on fractional calculus. Section 3 is a short review of the homotopy method and Section 4 variational iteration method. Sections 5 and 6 are devoted to VIM and HPM analysis, respectively. Concluding remarks with suggestions for future work are listed in Section 7.

2. Description of the Fractional Calculus

In the following, we give the necessary notations and basic definitions and properties of fractional calculus theory; for more details, see [3, 13, 16, 22].

Definition 2.1. A real function $f(x),\ x > 0$, is said to be in the space C_α, $\alpha \in R$ if there exists a real number $(p > \alpha)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) = C([0,\infty))$. Clearly, $C_\alpha \subset C_\beta$, if $\beta \le \alpha$.

Definition 2.2. A function $f(x),\ x > 0$, is said to be in space C_α^m, $m \in N$, if $f^{(m)} \in C_\alpha$.

Definition 2.3. The Riemann-Liouville fractional integral of order $\mu \geq 0$ for a function $f \in C_\alpha$, $(\alpha \geq 1)$ is defined as

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad \mu > 0,\ t > 0, \tag{2.1}$$

in particular $I^0 f(t) = f(t)$.

Definition 2.4. The Caputo fractional derivative of $f \in C_{-1}^m$, $m \in N$, is defined as

$$D_c^\mu f(t) = \begin{cases} \left[I^{m-\mu} f^{(m)}(t)\right], & m-1 < \mu \leq m,\ m \in N, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \tag{2.2}$$

Note that

(i) $I^\mu t^\gamma = (\Gamma(\gamma+1)/\Gamma(\gamma+\mu+1)) t^{\gamma+\mu}$, $\mu > 0$, $\gamma > -1$, $t > 0$,

(ii) $I^\mu\ {}^C D_{0^+}^\mu f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+)(t^k/k!)$, $m-1 < \mu \leq m$, $m \in N$,

(iii) ${}^C D_{0^+}^\mu f(t) = D^\mu (f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+)(t^k/k!))$, $m-1 < \mu \leq m$, $m \in N$,

(iv)

$$D^\beta I^\alpha f(t) = \begin{cases} I^{\alpha-\beta} f(t), & \text{if } \alpha > \beta, \\ f(t), & \text{if } \alpha = \beta, \\ D^{\beta-\alpha} f(t), & \text{if } \alpha < \beta, \end{cases} \tag{2.3}$$

(v) ${}^C D_{0^+}^\beta D^m f(t) = D^{\beta+m} f(t)$, $m = 0, 1, 2, \ldots, n-1 < \beta < n$.

Definition 2.5 (see [3, 16]). The Riemann-Liouville fractional integral operator of order $\rho \geq 0$ for a function $f \in C_\mu$, $(\mu \geq -1)$ is defined as

$$K^\rho f(x) = \frac{1}{\Gamma(\rho)} \int_0^x (x-t)^{\rho-1} f(t) dt, \quad \rho > 0,\ x > 0, \qquad K^0 f(x) = f(x), \tag{2.4}$$

having the properties

$$\begin{gathered} K^\rho K^\beta f(x) = K^{\rho+\beta} f(x), \\ K^\rho x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\rho+\beta+1)} x^{\alpha+\beta}. \end{gathered} \tag{2.5}$$

According to the Caputo's derivatives, we obtain the following expressions:

$$
\begin{gathered}
{}^{C}D^{\mu}C = 0, \quad C = \text{constant}, \\
{}^{C}D^{\mu}t^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0, & \beta \leq \alpha - 1. \end{cases}
\end{gathered} \tag{2.6}
$$

Lemma 2.6. *If $m-1 < \alpha \leq m$, $m \in N$, $f \in C_{\mu}^{m}$, $\mu \geq -1$, then the following two properties hold:*

$$
(1)\ D^{\alpha}K^{\alpha}f(t) = f(t), \qquad (2)\ (D^{\alpha}K^{\alpha})f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\frac{t^{k}}{k!}. \tag{2.7}
$$

In fact, Kılıçman and Zhour introduced the Kronecker convolution product and expanded to the Riemann-Liouville fractional integrals of matrices by using the Block Pulse operational matrix as follows:

$$
\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-t_1)^{\alpha-1}\phi_m(t_1)dt_1 \simeq F_{\alpha}\phi_m(t), \tag{2.8}
$$

where

$$
F_{\alpha} = \left(\frac{b}{m}\right)^{\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ 0 & 1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & 1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.9}
$$

see [23].

In our work, we consider Caputo fractional derivatives and apply the homotopy method in order to derive an approximate solutions of the fractional integrodifferential equations.

3. Homotopy Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$
A(u) + f(\mathbf{r}) = 0, \quad \mathbf{r} \in \boldsymbol{\Omega}, \tag{3.1}
$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad \mathbf{r} \in \mathbf{\Gamma}, \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytical function, and Γ is the boundary of the domain Ω, see [24].

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Equation (3.1), therefore, can be rewritten as follows:

$$L(u) + N(u) - f(\mathbf{r}) = 0. \tag{3.3}$$

By using the homotopy technique that was proposed by Liao in [7, 8], we construct a homotopy of (3.1) $v(\mathbf{r}, p) : \Omega \times [0,1] \rightarrow \mathcal{R}$ which satisfies

$$\mathscr{H}(v,p) = (1-p)[L(v) - L(u_0)] + p\left[A(v) + f(\mathbf{r})\right] = 0, \quad p \in [0,1], \ \mathbf{r} \in \mathbf{\Omega}, \tag{3.4}$$

or

$$\mathscr{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(\mathbf{r})\right] = 0, \tag{3.5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation which satisfies the boundary conditions. By using (3.4) and (3.5), we have

$$\begin{aligned} \mathscr{H}(v,0) &= L(v) - L(u_0) = 0, \\ \mathscr{H}(v,1) &= A(v) - f(\mathbf{r}) = 0. \end{aligned} \tag{3.6}$$

The changing in the process of p from zero to unity is just that of $v(\mathbf{r}, p)$ from u_0 to $u(\mathbf{r})$. In a topology, this is also

known deformation, further $L(v) - L(u_0)$ and $A(v) - f(\mathbf{r})$ are homotopic.

Now, assume that the solution of (3.4) and (3.5) can be expressed as

$$v = v + pv_1 + p^2 v_2 + \cdots . \tag{3.7}$$

The approximate solution of (3.1), therefore, can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{3.8}$$

The convergence of the series of (3.8) has been proved in the [25, 26].

4. The Variational Iteration Method

To illustrate the basic concepts of the VIM, we consider the following differential equation:

$$Lu + Nu = g(x), \tag{4.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an nonhomogenous term; for more details, see [19].

According to the VIM, one construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left[Lu_n(s) + N\tilde{u}_n(s) - g(s)\right]ds, \tag{4.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and the subscript n denotes the order of approximation, $\tilde{u}_u$ is considered variation [6, 27], that is, $\delta\tilde{u}_u = 0$.

5. Analysis of VIM

To solve the fractional integrodifferential equation (1.1) by using the variational iteration method, with boundary conditions (1.2), one can construct the following correction functional:

$$y_{k+1}(x) = y_k(x) + \int_0^t \mu \sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s)ds - \mu\tilde{g}_k(x) - \lambda \int_a^b \mu H(x,s)\tilde{y}_k(s)ds, \tag{5.1}$$

where μ is a general Lagrange multiplier and $\tilde{g}_k(x)$ and $\tilde{y}_k(x)$ are considered as restricted variations, that is, $\delta\tilde{g}_k(x) = 0$ and $\delta\tilde{y}_k(x) = 0$.

Making the above correction functional stationary, the following conditions can be obtained:

$$\delta y_{k+1}(x) = \delta y_k(x) + \int_0^t \left[\sum_{k=0}^{\infty} P_k \mu(s)\delta D_*^{\alpha} y(s) - \delta\tilde{g}_k(x) - \lambda\int_a^b H(x,s)\mu(s)\delta\tilde{y}_k(s)ds\right], \tag{5.2}$$

having the boundary conditions as follows:

$$1 - \mu'(s)\big|_{x=s} = 0, \qquad \mu(s)\big|_{x=s} = 1. \tag{5.3}$$

The Lagrange multipliers can be identified as follows:

$$\mu(s) = \frac{1}{2}(s - x). \tag{5.4}$$

Substituting the value of μ from (5.4) into correction functional of (5.1) leads to the following iteration formulae:

$$\begin{aligned}
y_{k+1}(x) &= y_k(x) + \frac{\mu}{2\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}(s-x) \\
&\quad\times\left[\int_0^t \sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s)ds - \tilde{g}_k(x) - \lambda\int_a^b H(x,s)\tilde{y}_k(s)\right]ds, \\
y_{k+1}(x) &= y_k(x) - \frac{\mu(\alpha-1)}{2\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\
&\quad\times\left[\int_0^t \sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s)ds - \tilde{g}_k(x) - \lambda\int_a^b H(x,s)\tilde{y}_k(s)\right]ds,
\end{aligned} \tag{5.5}$$

by applying formulae (2.4), we get

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)K^{\alpha}}{2\Gamma(\alpha)}\left[\int_0^t \mu\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(v)dv - \mu\tilde{g}_k(x) - \lambda\int_a^b \mu H(x,v)\tilde{y}_k(v)\right]dv. \tag{5.6}$$

The initial approximation can be chosen in the following manner which satisfies initial boundary conditions (1.2)-(1.3):

$$y_0(x) = v_0 + v_1 x, \quad \text{where } v_1 = D_*^{\alpha} y(0),\ v_0 = D_*^{\alpha} y(a). \tag{5.7}$$

We can obtain the following first-order approximation by substitution of (5.7) into (5.6)

$$y_1(x) = y_0(x) - \frac{(\alpha-1)K^{\alpha}}{2\Gamma(\alpha)}\left[\int_0^t \mu\sum_{k=0}^{N} P_k D_*^{\alpha} y(v)dv - \mu\tilde{g}_0(x) - \lambda\int_a^b \mu H(x,v)\tilde{y}_k(v)\right]dv. \tag{5.8}$$

Substituting the constant value of v_0 and v_1 in the expression (5.8) results in the approximation solution of (1.1)–(1.3).

6. Analysis of HPM

This section illustrates the basic of HPM for fractional Fredholm integrodifferential equations with constant coefficients (1.1) with initial-boundary conditions (1.2).

In view of HPM [25, 26], construct the following homotopy for (1.1):

$$\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) = p\left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) + \left(g(t) - \lambda\int_a^b H(x,t)y(x)dx\right)\right]. \tag{6.1}$$

In view of basic assumption of HPM, solution of (1.1) can be expressed as a power series in p:

$$y(x) = D_*^\alpha y_0(x) + pD_*^\alpha y_1(x) + p^2 D_*^\alpha y_2(x) + p^3 D_*^\alpha y_3(x) + \cdots. \tag{6.2}$$

If we put $p \to 1$ in (6.2), we get the approximate solution of (1.1):

$$y(x) = D_*^\alpha y_0(x) + D_*^\alpha y_1(x) + D_*^\alpha y_2(x) + D_*^\alpha y_3(x) + \cdots. \tag{6.3}$$

The convergence of series (6.3) has been proved in [28].

Now, we substitute (6.2) into (6.1); then equating the terms with identical power of p, we obtain the following series of linear equations:

$$\begin{aligned}
p^0 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_0(t) = 0, \\
p^1 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_1(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_0(t) - \lambda \int_a^b H(x,t) y_0(x) dx, \\
p^2 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_2(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_1(t) + g(x) - \lambda \int_a^b H(x,t) y_1(x) dx, \\
p^3 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_3(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_2(t) - \lambda \int_a^b H(x,t) y_2(x) dx, \\
p^4 &: \sum_{k=0}^{\infty} P_k D_*^\alpha y_4(t) = \sum_{k=0}^{\infty} P_k D_*^\alpha y_3(t) - \lambda \int_a^b H(x,t) y_3(x) dx,
\end{aligned} \tag{6.4}$$

with the initial-boundary conditions

$$D_*^\alpha y(a) = y(0), \quad D_*^\alpha y(0) = y'(a). \tag{6.5}$$

We can also take the initial approximation in the following manner which satisfies initial-boundary conditions (1.2)-(1.3):

$$y_0(x) = v_0 + v_1 x, \quad \text{where } v_1 = D_*^\alpha y(0), \quad v_0 = D_*^\alpha y(a). \tag{6.6}$$

Note that (6.4) can be solved by applying the operator K^β, which is the inverse of operator D^α we approximate the series solution of HPM by the following n-term truncated series [29]:

$$\chi_n(x) = D_*^\alpha y_0(x) + D_*^\alpha y_1(x) + D_*^\alpha y_2(x) + D_*^\alpha y_3(x) + \cdots + D_*^\alpha y_{n-1}(x), \tag{6.7}$$

which results, the approximate solutions of (1.2)-(1.3). For further analysis, the variational iteration method, see [30] and the algorithm by the homotopy perturbation method, see [31].

7. Conclusion

The proposed methods are used to solve fractional Fredholm integrodifferential equations with constant coefficients. Comparison of the results obtained by the present method with that obtained by other method reveals that the present method is very effective and convenient. Unfortunately, the disadvantage of the second method is that the embedding parameter p is quite casual, and often enough the approximations obtained by this method will not be uniform. So, in our future work we expect to study this kind of equation by using a combination of the variational iteration method and the homotopy perturbation method which has shown reliable results in supplying analytical approximation that converges very rapidly. However, we note that the papers [32, 33] suggest alternative ways for similar problems.

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Research Article

A Generalized q-Mittag-Leffler Function by q-Captuo Fractional Linear Equations

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Some Caputo q-fractional difference equations are solved. The solutions are expressed by means of a new introduced generalized type of q-Mittag-Leffler functions. The method of successive approximation is used to obtain the solutions. The obtained q-version of Mittag-Leffler function is thought as the q-analogue of the one introduced previously by Kilbas and Saigo (1995).

1. Introduction and Preliminaries

The concept of fractional calculus is not new. However, it has gained its popularity and importance during the last three decades or so. This is due to its distinguished applications in numerous diverse fields of science and engineering (see, e.g., [1–6] and the references therein). The q-calculus is also not of recent appearance. It was initiated in the twenties of the last century. For the basic concepts in q-calculus we refer the reader to [7]. Discrete and q-fractional difference equations are discrete versions of fractional differential equations. An extensive work has been done in discrete fractional dynamic equations and discrete fractional variational calculus (see [8–12]). Some of the authors applied the delta analysis and some applied nabla analysis. Since the domain of nabla operatos is more stable, the nabla approach could be preferable. In this paper we apply the nabla approach in the quantum case with $0 < q < 1$, but also the delta approach is possible [13]. During the last decade many authors applied diverse methods, such as homotopy perturbation method, to derive approximate analytical solutions of systems of fractional differential equations into Caputo and Riemann (see [14–18]). In this paper, we apply a direct method to express the solution of a certain linear Caputo q-fractional differential equation by means of a new introduced generalized q-Mittag-Leffler function.

Starting from the q-analogue of Cauchy formula [19], Al-Salam started the fitting of the concept of q-fractional calculus. After that he [20, 21] and Agarwal [22] continued on by studying certain q-fractional integrals and derivatives, where they proved the semigroup properties for left and right (Riemann) type fractional integrals but without variable lower limit and variable upper limit, respectively. Recently, the authors in [23] generalized the notion of the (left) fractional q-integral and q-derivative by introducing variable lower limit and proved the semigroup properties.

Very recently and after the appearance of time-scale calculus (see, e.g., [24]), some authors started to pay attention and apply the techniques of time scale to discrete fractional calculus (see [25–28]) benefitting from the results announced before in [29]. All of these results are mainly about fractional calculus on the time scales $\mathbb{T}_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $h\mathbb{Z}$ [30]. As a contribution in this direction and being motivated by what is mentioned before, in this paper we introduce the q-analogue of a generalized type Mittag-Leffler function used before by Kilbas and Saigo in [31]. Such functions are obtained by solving linear q-Caputo initial value problems. The results obtained in this paper generalize also the results of [32]. Indeed, the authors in [32] solved a linear Caputo q-fractional difference equation of the form

$$\left({}_qC_a^\alpha y\right)(x) = \lambda y(x) + f(x), \quad y(a) = b, \quad 0 < \alpha < 1, \tag{1.1}$$

where the solution was expressed by means of discrete q-Mittag-Leffler functions. In this paper, we solve an equation of the form

$$\begin{gathered}\left({}_qC_a^\alpha y\right)(x) = \lambda (x-a)_q^\beta \, y\left(q^{-\beta} x\right), \qquad y(a) = b, \\ 0 < \alpha < 1,\ \beta > -\alpha,\ \lambda \in \mathbb{R},\ b \in \mathbb{R},\end{gathered} \tag{1.2}$$

where the solution is expressed by means of a more general discrete q-Mittag-Leffler functions generalizing the ones obtained by (1.1), as (1.1) is obtained from (1.2) by setting $\beta = 0$. Finally, we generalize to the higher-order case for any $\alpha > 0$, where higher-order q-Mittag-Leffler functions are obtained.

For the theory of q-calculus we refer the reader to the survey of [7], and for the basic definitions and results for the q-fractional calculus we refer to [28]. Here we will summarize some of those basics.

For $0 < q < 1$, let $\mathbb{T}_q$ be the time scale:

$$\mathbb{T}_q = \left\{q^n : n \in \mathbb{Z}\right\} \cup \{0\}, \tag{1.3}$$

where $\mathbb{Z}$ is the set of integers. More generally, if α is a nonnegative real number, then we define the time scale

$$\mathbb{T}_q^\alpha = \left\{q^{n+\alpha} : n \in \mathbb{Z}\right\} \cup \{0\}, \tag{1.4}$$

and we write $\mathbb{T}_q^0 = \mathbb{T}_q$.

For a function $f : T_q \to \mathbb{R}$, the nabla q-derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in \mathbb{T}_q - \{0\}. \tag{1.5}$$

The nabla q-integral of f is given by

$$\int_0^t f(s)\nabla_q s = (1-q)t\sum_{i=0}^{\infty} q^i f\left(tq^i\right), \tag{1.6}$$

and for $0 \leq a \in T_q$,

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\ \nabla_q s - \int_0^a f(s)\ \nabla_q s. \tag{1.7}$$

On the other hand

$$\int_t^{\infty} f(s)\nabla_q s = (1-q)t\sum_{i=1}^{\infty} q^{-i} f\left(tq^{-i}\right), \tag{1.8}$$

and for $0 < b < \infty$ in $\mathbb{T}_q$,

$$\int_t^b f(s)\nabla_q s = \int_t^{\infty} f(s)\nabla_q s - \int_b^{\infty} f(s)\nabla_q s. \tag{1.9}$$

By the fundamental theorem in q-calculus we have

$$\nabla_q \int_0^t f(s)\nabla_q s = f(t), \tag{1.10}$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(s)\nabla_q s = f(t) - f(0). \tag{1.11}$$

Also the following identity will be helpful:

$$\nabla_q \int_a^t f(t,s)\nabla_q s = \int_a^t \nabla_q f(t,s)\ \nabla_q s + f(qt,t). \tag{1.12}$$

Similarly the following identity will be useful as well:

$$\nabla_q \int_t^b f(t,s)\nabla_q s = \int_{qt}^b \nabla_q f(t,s)\ \nabla_q s - f(t,t). \tag{1.13}$$

The q-derivative in (1.12) and (1.13) is applied with respect to t.

From the theory of q-calculus and the theory of time scale more generally, the following product rule is valid:

$$\nabla_q\big(f(t)g(t)\big) = f(qt)\nabla_q g(t) + \nabla_q f(t)g(t). \tag{1.14}$$

The q-factorial function for $n \in \mathbb{N}$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1}\left(t - q^i s\right). \tag{1.15}$$

When α is a nonpositive integer, the q-factorial function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{(1-(s/t))q^i}{(1-(s/t))q^{i+\alpha}}. \tag{1.16}$$

We summarize some of the properties of q-factorial functions, which can be found mainly in [28], in the following lemma.

Lemma 1.1. *One has the following.*

(i) $(t-s)_q^{\beta+\gamma} = (t-s)_q^\beta (t-q^\beta s)_q^\gamma$.

(ii) $(at-as)_q^\beta = a^\beta (t-s)_q^\beta$.

(iii) *The nabla q-derivative of the q-factorial function with respect to t is*

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q}(t-s)_q^{\alpha-1}. \tag{1.17}$$

(iv) *The nabla q-derivative of the q-factorial function with respect to s is*

$$\nabla_q (t-s)_q^\alpha = -\frac{1-q^\alpha}{1-q}\left(t-qs\right)_q^{\alpha-1}, \tag{1.18}$$

where $\alpha, \gamma, \beta \in \mathbb{R}$.

Definition 1.2 (see [32]). Let $\alpha > 0$. *If* $\alpha \notin \mathbb{N}$, then the α-order Caputo (left) q-fractional derivative of a function f is defined by

$$ {}_qC_a^{\alpha} f(t) \triangleq {}_qI_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s, \tag{1.19} $$

where $n = [\alpha] + 1$.

If $\alpha \in \mathbb{N}$, then ${}_qC_a^{\alpha} f(t) \triangleq \nabla_q^n f(t)$.

It is clear that ${}_qC_a^{\alpha}$ maps functions defined on T_q to functions defined on T_q, and that ${}_bC_q^{\alpha}$ maps functions defined on $T_q^{1-\alpha}$ to functions defined on T_q.

The following identity which is useful to transform Caputo q-fractional difference equations into q-fractional integrals, will be our key in solving the q-fractional linear type equation by using successive approximation.

Proposition 1.3 ([32]). *Assume that* $\alpha > 0$ *and* f *is defined in suitable domains. Then*

$$ {}_qI_a^{\alpha}\, {}_qC_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a), \tag{1.20} $$

and if $0 < \alpha \le 1$, *then*

$$ {}_qI_a^{\alpha}\, {}_qC_a^{\alpha} f(t) = f(t) - f(a). \tag{1.21} $$

The following identity [23] is essential to solve linear q-fractional equations:

$$ {}_qI_a^{\alpha} (x-a)_q^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha} \quad (0 < a < x < b), \tag{1.22} $$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$. The q-analogue of Mittag-Leffler function with double index (α, β) is introduced in [32]. It was defined as follows.

Definition 1.4 ([32]). For $z, z_0 \in \mathbf{C}$ and $\Re(\alpha) > 0$, the q-Mittag-Leffler function is defined by

$$ {}_qE_{\alpha,\beta}(\lambda, z-z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z-z_0)_q^{\alpha k}}{\Gamma_q(\alpha k+\beta)}. \tag{1.23} $$

When $\beta = 1$, we simply use ${}_qE_{\alpha}(\lambda, z-z_0) := {}_qE_{\alpha,1}(\lambda, z-z_0)$.

2. Main Results

The following is to be the q-analogue of the generalized Mittag-Leffler function introduced by Kilbas and Saigo [31] (see also [3] page 48).

Definition 2.1. For $\alpha, l, \lambda \in \mathbb{C}$ are complex numbers and $m \in \mathbb{R}$ such that $\Re(\alpha) > 0, m > 0, a \geq 0$, and $\alpha(jm+l) \neq -1, -2, -3, \ldots$, the generalized q-Mittag-Leffler function (of order 0) is defined by

$$ {}_qE_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-(k(k-1)/2)\alpha(m-1)(\alpha l+\alpha)} c_k (x-a)_q^{\alpha k m}, \tag{2.1} $$

where

$$ c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}, \qquad k = 1,2,3,\ldots, \tag{2.2} $$

while the generalized q-Mittag-Leffler function (of order r), $r = 0,1,2,3,\ldots$, is defined by

$$ {}_qE^r_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-k\alpha(m-1)r} q^{-(k(k-1)/2)\alpha(m-1)(\alpha l+\alpha)} c_k (x-q^r a)_q^{\alpha k m}. \tag{2.3} $$

Note that ${}_qE^0_{\alpha,m,l}(\lambda, x-a) = {}_qE_{\alpha,m,l}(\lambda, x-a)$.

Remark 2.2. In particular, if $m = 1$, then the generalized q-Mittag-Leffler function is reduced to the q-Mittag-Leffler function, apart from a constant factor $\Gamma_q(\alpha l+1)$. Namely,

$$ {}_qE_{\alpha,1,l}(\lambda, x-a) = \Gamma_q(\alpha l+1) {}_qE_{\alpha,\alpha l+1}(\lambda, x-a). \tag{2.4} $$

This turns to be the q-analogue of the identity $E_{\alpha,1,l}(z) = \Gamma(\alpha l+1)E_{\alpha,\alpha l+1}(z)$ (see [3] page 48).

Example 2.3. Consider the q-Caputo difference equation:

$$ \begin{gathered} \left({}_qC_a^{\alpha} y\right)(x) = \lambda (x-a)_q^{\beta} y\left(q^{-\beta}x\right), \qquad y(a) = b, \\ 0 < \alpha < 1,\ \beta > -\alpha,\ \lambda \in \mathbb{R},\ b \in \mathbb{R}. \end{gathered} \tag{2.5} $$

Applying Proposition 1.3 we have

$$ y(x) = y(a) + \lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} y\left(q^{-\beta}x\right)\right]. \tag{2.6} $$

The method of successive applications implies that

$$ y_m(x) = y(a) + \lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} y_{m-1}\left(q^{-\beta}x\right)\right], \qquad m = 1,2,3,\ldots, \tag{2.7} $$

where $y_0(x) = b$. Then by the help of (1.22) we have

$$y_1(x) = b + b\lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(x-a)_q^{\beta+\alpha},$$
$$y_2(x) = b + b\lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta}\left\{1+\lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}\left(q^{-\beta}x-a\right)_q^{\beta+\alpha}\right\}\right]. \tag{2.8}$$

Then by (i) and (ii) of Lemma 1.1,

$$y_2(x) = b + b\lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} + \lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+\alpha}\right]. \tag{2.9}$$

Again by (1.22) we conclude that

$$y_2(x) = b + b\lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} + \lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+\alpha}\right]. \tag{2.10}$$

Then (1.22) leads to

$$y_2(x) = b\left[1 + \lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(x-a)_q^{\beta+\alpha} + \lambda^2\frac{\Gamma_q(2\beta+\alpha+1)}{\Gamma_q(2\beta+2\alpha+1)}q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+2\alpha}\right]. \tag{2.11}$$

Proceeding inductively, for each $m = 1, 2, \ldots$ we obtain

$$y_m(x) = b\left[1 + \sum_{k=1}^{m}\lambda^k q^{-\beta(k(k-1)/2)(\alpha+\beta)}c_k(x-a)_q^{k(\alpha+\beta)}\right], \tag{2.12}$$

where

$$c_k = \prod_{j=0}^{k-1}\frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}, \qquad m = 1+\frac{\beta}{\alpha},\ l = \frac{\beta}{\alpha},\ k = 1, 2, 3, \ldots. \tag{2.13}$$

If we let $m \to \infty$, then we obtain the solution

$$y(x) = b\left[1 + \sum_{k=1}^{\infty}\lambda^k q^{-\beta(k(k-1)/2)(\alpha+\beta)}c_k(x-a)_q^{k(\alpha+\beta)}\right]. \tag{2.14}$$

Now, by means of Definition 2.1, we can state the following.

Theorem 2.4. *The solution of the q-Caputo difference equation* (2.5) *is given by*

$$y(x) = b\, {}_qE_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda, x-a). \tag{2.15}$$

Remark 2.5. (1) If in (2.5) $\beta = 0$, then in accordance with (2.4) and Example 9 in [32] we have

$$ {}_qE_{\alpha,1,0}(\lambda, x-a) = {}_qE_{\alpha,1}(\lambda, x-a) = {}_qE_{\alpha}(\lambda, x-a). \tag{2.16}$$

(2) The solution of the q-Cauchy problem

$$\begin{gathered} \left({}_qC_a^{1/2}y\right)(x) = \lambda\ (x-a)_q^{\beta}y\left(q^{-\beta}x\right), \qquad y(a) = b, \\ 0<\alpha<1,\ \beta > -\frac{1}{2},\ \lambda\in\mathbb{R},\ b\in\mathbb{R}, \end{gathered} \tag{2.17}$$

is given by

$$y(x) = b\,{}_qE_{1/2,1+2\beta,2\beta}(\lambda, x-a). \tag{2.18}$$

For the sake of generalization to the higher-order case, we consider the fractional q-initial value problem:

$$\left({}_qC_a^{\alpha}y\right)(x) = \lambda(x-a)_q^{\beta}y\left(q^{-\beta}x\right), \quad y^{(k)}(a) = b_k \quad (b_k\in\mathbb{R}, k=0,1,\ldots,n-1), \tag{2.19}$$

where

$$n-1<\alpha<n, \quad \beta>-\alpha, \quad \lambda\in\mathbb{R}, \quad b\in\mathbb{R}. \tag{2.20}$$

Theorem 2.6. *The solution of the fractional q-initial value problem* (2.19) *is of the following form:*

$$y(x) = \sum_{r=0}^{n-1}\frac{b_r}{\Gamma_q(r+1)}(x-a)_q^r\,{}_qE^r_{\alpha,((1+\beta)/\alpha),((\beta+r)/\alpha)}(\lambda, x-a). \tag{2.21}$$

Proof. The proof follows by the help of (1.20) and let Lemma 1.1 and by applying the successive approximation with

$$y_0(x) = \sum_{k=0}^{n-1}\frac{(t-a)_q^k}{\Gamma_q(k+1)}\nabla_q^k f(a), \tag{2.22}$$

□

Note that when $0<\alpha<1$, that is, $n=1$, the solution of Example 2.3 is recovered.
Next, we solve a nonhomogenous versions of (2.5).

Lemma 2.7. *Let $r\in\mathbb{N}$, $\alpha>0$, and let f be defined on $\mathbb{T}_q$. Then*

$${}_qI_a f\left(q^{-r}t\right) = q^{r\alpha}\left({}_qI_{q^{-r}a}f\right)\left(q^{-r}t\right) \quad \forall t\in\mathbb{T}_q. \tag{2.23}$$

In particular, if $a = 0$, then

$$ {}_qI_0 f\left(q^{-r}t\right) = q^{r\alpha}\left({}_qI_0 f\right)\left(q^{-r}t\right) \quad \forall t \in \mathbb{T}_q. \tag{2.24}$$

Proof. The proof can be achieved by making use of Theorem 1 in [28] for integration by substitution (for details see [24]). Indeed,

$$\begin{aligned} {}_qI_a f\left(q^{-r}t\right) &= \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1} f\left(q^{-r}s\right)\nabla_q s \\ &= \frac{q^r}{\Gamma_q(\alpha)}\int_{q^{-r}a}^{q^{-r}t} \left(t-qq^r s\right)_q^{\alpha-1} f(s)\nabla_q s \\ &= \frac{q^{r\alpha}}{\Gamma_q(\alpha)}\int_{q^{-r}a}^{q^{-r}t} \left(q^{-r}t-qs\right)_q^{\alpha-1} f(s)\nabla_q s \\ &= q^{r\alpha}\left({}_qI_{q^{-r}a} f\right)\left(q^{-r}t\right). \end{aligned} \tag{2.25}$$

□

Consider the q-fractional initial value problem:

$$\left({}_qC_0^{\alpha}y\right)(x) = \lambda x^{\beta} y\left(q^{-\beta}x\right) + f(x), \qquad y(0) = b, \tag{2.26}$$

where

$$0 < \alpha < 1, \quad \beta > -\alpha, \quad \beta \in \mathbb{N}_0, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{2.27}$$

If we apply the successive approximation as in Example 2.3 and use Lemma 2.7, then we can state the following

Theorem 2.8. *The solution of the initial value problem* (2.26) *is expressed by*

$$y(x) = b\, {}_qE_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda, x) + \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma_q(\alpha k+\alpha)} q^{-\alpha\beta(k(k+1)/2)}\int_0^x (x-qt)_q^{\alpha k+\alpha} f\left(q^{-k\beta}t\right)\nabla_q t. \tag{2.28}$$

Remark 2.9. If in (2.26) we set $\beta = 0$, then Example 9 in [32] is recovered for a = 0.

Definition 2.10. A function $f : \mathbb{T}_q \to \mathbb{R}$ is called periodic with period $\beta \in \mathbb{N}_1$ *if* β is the smallest natural number such that $f(q^{\beta}t) = f(t)$, for all $t \in \mathbb{T}_q$.

Consider the nonhomogeneous initial value problem:

$$\left({}_qC_0^{\alpha}y\right)(x) = \lambda(x-a)_q^{\beta} y\left(q^{-\beta}x\right) + f(x), \qquad y(a) = b, \tag{2.29}$$

where

$$0 < \alpha < 1, \quad \beta > -\alpha, \quad \beta \in \mathbb{N}_0, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{2.30}$$

If we apply the successive approximation as in Example 2.3, then we state the following.

Theorem 2.11. *If in* (2.29) *either $\beta = 0$ or f is periodic with period dividing β, then the solution is given by*

$$y(x) = b_q E_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda, x-a) + \int_a^x (x-qt)_q^{\alpha-1} {}_qE_{\alpha,\alpha}(\lambda, x-q^\alpha t) f(t) \nabla_q t. \tag{2.31}$$

Clearly, if $\beta = 0$, then the result in Example 9 in [32] is recovered as well.

For the sake of completeness, it would be interesting if the h-discrete fractional analogue, or more generally the (q, h)-analogue of the general q-Mittag-Leffler functions are obtained, possibly better, by applying nabla calculus (see [33–35]). However, this needs preparations in the Caputo case and it might be very complicated.

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