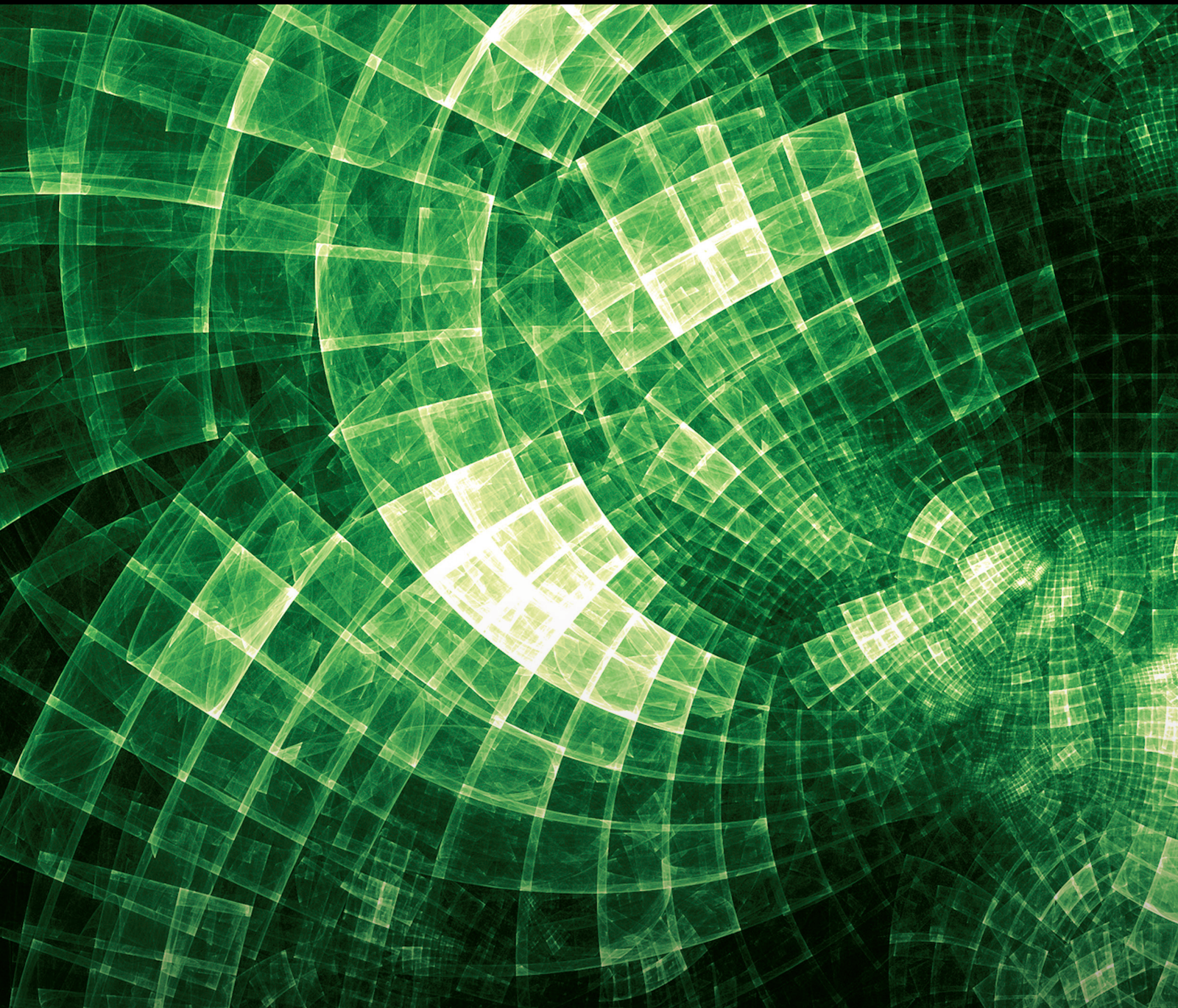


# Applications of Exponential Sums and Character Sums

Lead Guest Editor: Wenpeng Zhang

Guest Editors: Tianping Zhang, Tingting Wang, and Jie Wu







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# **Applications of Exponential Sums and Character Sums**

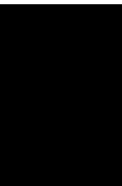


## **Applications of Exponential Sums and Character Sums**

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## Research Article

# On the Hybrid Power Mean of Two-Term Exponential Sums and Cubic Gauss Sums

Shaofan Cao  and Tingting Wang 

College of Science, Northwest A&F University, Yangling, Shaanxi, China

Correspondence should be addressed to Tingting Wang; [ttwang@nwsuaf.edu.cn](mailto:ttwang@nwsuaf.edu.cn)

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In this paper, an interesting third-order linear recurrence formula is presented by using elementary and analytic methods. This formula is concerned with the calculating problem of the hybrid power mean of a certain two-term exponential sums and the cubic Gauss sums. As an application of this result, some exact computational formulas for one kind hybrid power mean of trigonometric sums are obtained.

## 1. Introduction

As usual, let  $p$  be an odd prime. For any integer  $m$ , we define the cubic Gauss sums  $A(m, p) = A(m)$  as follows:

$$A(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right), \quad (1)$$

where  $e(y) = e^{2\pi iy}$  and  $i^2 = -1$ .

In recent years, several scholars have studied the hybrid power mean problems of various trigonometric sums and proved many interesting results. For example, Chen and Hu [1] studied the computational problem of

$$S_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^k \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{mc + \bar{c}}{p}\right) \right|^2, \quad (2)$$

where  $\bar{c}$  denotes the multiplicative inverse of  $c \bmod p$ . That is,  $c \cdot \bar{c} \equiv 1 \bmod p$ . For  $p \equiv 1 \bmod 3$ , they obtained a third-order linear recurrence formula for  $S_k(p)$ .

Li and Hu [2] studied another hybrid power mean

$$\sum_{b=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right) \right|^2 \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right) \right|^2, \quad (3)$$

and gave an exact computational formula for (3).

Some other related papers can also be found in references [3–11]. We would not have to repeat them here.

Very recently, Chen and Chen [12] studied the recursive properties of the hybrid power mean

$$H_k(c, p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{cma^3}{p}\right) \right)^k \cdot \left( \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right)^3, \quad (4)$$

and obtained a third-order linear recurrence formula for it with  $c = 1$  and  $p \equiv 1 \bmod 3$ . That is, they proved the following result:

Let  $p$  be an odd prime with  $p \equiv 1 \bmod 3$ . If 3 is a cubic residue mod  $p$ , then for any integer  $k \geq 3$ , one has the third-order linear recurrence formula:

$$H_k(1, p) = 3pH_{k-2}(1, p) + dpH_{k-3}(1, p), \quad (5)$$



where the first three terms are  $H_0(1, p) = 2p^2 - pd$ ,  $H_1(1, p) = p^2(d - 6)$ , and  $H_2(1, p) = p^2(6p - 5d)$ .

Note that Zhang and Zhang [13] proved the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2 & \text{if } 3 \mid p-1. \end{cases} \quad (6)$$

Perhaps this is the best result for which there is no conditional requirement on the prime  $p$ . The interesting results of the above work motivate us to ask such a problem of whether there exists a similar recursive formula for the hybrid power mean

$$U_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{cma^3}{p}\right) \right)^k \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4, \quad (7)$$

where  $p$  is an odd prime with  $p \equiv 1 \pmod{3}$ .

Obviously, the problem in (7) is much harder than the problem in [12] because we are dealing with the fourth power mean of the two-term exponential sums in (7). Our main contribution is to obtain an identity for the fourth power mean of the two-term exponential sums weighted by third-order character modulo  $p$ , i.e., the following Lemma 3. Then, we use this lemma to derive several interesting recursion formulas for  $U_k(p)$ . In this way, the continuities and the value distribution properties of this kind of trigonometric sums can be described from different views. Of course, the reason why we focus on the calculation of (7) is that the problem is closely related to the number of the solutions of some congruence equation. These contents play a very important role in study of some famous analytic number theory problems, such as Waring problem and Goldbach conjecture.

Through the study, it is found that the problem we studied is closely related to integer 3. If 3 is a cubic residue modulo  $p$ , then there exists a beautiful third-order linear recurrence formula for  $U_k(p)$ , and the first three terms  $U_0(p)$ ,  $U_1(p)$ , and  $U_2(p)$  are integers. If 3 is not a cubic residue mod  $p$ , then we can get the exact value of  $U_2(p)$ . For any other positive integer  $k$ , we can only give a more complex mathematical representation for  $U_k(p)$ . That is, we have the following three results:

**Theorem 1.** Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . If 3 is a cubic residue modulo  $p$ , then for any integer  $k \geq 3$ , we have the third-order linear recurrence formula

$$U_k(p) = 3pU_{k-2}(p) + dpU_{k-3}(p), \quad (8)$$

where the first three terms are  $U_0(p) = 2p^3 - 7p^2$ ,  $U_1(p) = -dp(6p - 1)$ , and  $U_2(p) = p^2(4p^2 - 22p - d^2 + 2)$  and  $d$  is uniquely determined by  $4p = d^2 + 27b^2$  and  $d \equiv 1 \pmod{3}$ .

**Theorem 2.** Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . If 3 is a cubic residue modulo  $p$ , then for any integer  $k \geq 3$ , we also have the third-order linear recurrence formula

$$U_{-k}(p) = \frac{3}{d} \cdot U_{-(k-1)}(p) + \frac{1}{dp} \cdot U_{-(k-3)}(p), \quad (9)$$

where the first three terms are  $U_0(p) = 2p^3 - 7p^2$ ,  $U_{-1}(p) = -p \cdot (2p^2 + p + d^2 - 2)/d$ , and  $U_{-2}(p) = 1 - 3p + 3p(2p^2 + p - 2)/d^2$ .

**Theorem 3.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{3}$ . If 3 is not a cubic residue modulo  $p$ , then we have the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 = p^2(4p^2 - 19p - d^2 - 1). \quad (10)$$

From our theorems, we may immediately deduce the following three corollaries:

**Corollary 1.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{3}$ , then we have

$$\begin{aligned} & \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ &= \begin{cases} p^2(4p^2 - 22p - d^2 + 2) & \text{if 3 is a cubic residue modulo } p, \\ p^2(4p^2 - 19p - d^2 - 1) & \text{if 3 is not a cubic residue modulo } p. \end{cases} \end{aligned} \quad (11)$$

**Corollary 2.** Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . If 3 is a cubic residue modulo  $p$ , then we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^4 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 = p^2(12p^3 - 66p^2 - 9d^2p + 6p + d^2). \quad (12)$$

**Corollary 3.** Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . If 3 is a cubic residue modulo  $p$ , then we have

$$\sum_{m=1}^{p-1} \left| \frac{\sum_{a=0}^{p-1} e((ma^3 + a)/p)}{\sum_{a=0}^{p-1} e(ma^3/p)} \right|^4 = \frac{27}{d^4} \cdot (2p^3 + p^2 - 2p) - \frac{1}{d^2} \cdot (8p^2 + 7p - 11) - 1. \quad (13)$$

Some notes: first in Theorem 2, if  $(3, p-1) = 1$ , then the question we are discussing is trivial. Because in this case, we have

$$\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) = 0. \quad (14)$$

Second, if  $p \equiv 1 \pmod{3}$  and 3 is not a cubic residue modulo  $p$ , then we can only get the exact value of  $U_2(p)$ .

Third, the advantage of our work is that we completely solve the calculation problem of  $U_k(p)$  with  $p \equiv 1 \pmod{3}$ .

Fourth, the mean value estimation of the exponential sums is closely related to the upper and lower bounds of the individual exponential sums. So, by studying the mean value of the positive exponential sums, we can obtain a better upper bound estimation of the exponential sums. If we want to get its lower bound estimation of the exponential sums, we should study the negative power of the exponential sums. Our Theorems 1 and 2 address both types of problems.

Finally, for any fixed positive integer  $h \geq 5$ , whether there is a third-order linear recurrence formula for the hybrid power mean

$$W_k(h, p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^k \cdot \left( \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right)^h, \quad (15)$$

is an open problem, which is the limitation of our work. The other drawback, of course, is that we cannot compute all  $U_k(p)$  when 3 is not a cubic residue modulo  $p$ . In fact, our ultimate goal is to obtain a precise calculation formula for

$W_k(h, p)$  for all positive integers  $h \geq 5$ . In the future, we will continue to improve the research in this aspect. It also requires us to continue to study.

## 2. Several Lemmas

To complete the proofs of our theorems, several simple lemmas are necessary. Hereafter, we will use many properties of the classical Gauss sums and the third-order character modulo  $p$ , all of which can be found in books concerning about Elementary Number Theory or Analytic Number Theory, such as references [14–16], so the related contents will not be repeated here. First we have the following:

**Lemma 1.** If  $p$  is a prime with  $p \equiv 1 \pmod{3}$ , then for any third-order character  $\psi \pmod{p}$ , we have the identity

$$\sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3 = \bar{\psi}(3) \tau^2(\bar{\psi}) - 3p \tau(\psi). \quad (16)$$

*Proof.* First applying trigonometric identity

$$\sum_{m=1}^q e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q|n, \\ 0 & \text{if } q \nmid n, \end{cases} \quad (17)$$

and noting that  $\psi^3 = \chi_0$ , the principal character modulo  $p$ , we have

$$\begin{aligned} \sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3 &= \sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \\ &\quad + \sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \left( \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right) \\ &= 2 \sum_{m=1}^{p-1} \psi(m) \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right) + \sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \\ &\quad + \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=1}^{p-1} \bar{\psi}(a^3 + b^3 + c^3) e\left(\frac{a + b + c}{p}\right) \end{aligned}$$

$$\begin{aligned}
&= -2\tau(\psi) + \tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a^3 + 1) \sum_{b=1}^{p-1} e\left(\frac{b(a+1)}{p}\right) \\
&\quad + \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\psi}(a^3 + b^3 + 1) \sum_{c=1}^{p-1} e\left(\frac{c(a+b+1)}{p}\right) \\
&= -2\tau(\psi) - \tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a^3 + 1) + p\tau(\psi) \sum_{\substack{a=0 \\ a+b+1 \equiv 0 \pmod{p}}}^{p-1} \sum_{b=0}^{p-1} \bar{\psi}(a^3 + b^3 + 1) \\
&\quad - \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\psi}(a^3 + b^3 + 1) \\
&= -2\tau(\psi) - \tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a^3 + 1) + p\tau(\psi) \sum_{a=0}^{p-1} \bar{\psi}(a^3 - (a+1)^3 + 1) \\
&\quad - \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\psi}(a^3 + b^3 + 1).
\end{aligned} \tag{18}$$

Note that  $\psi^2 = \bar{\psi}$  and  $\tau(\psi)\tau(\bar{\psi}) = p$ , and from the properties of Gauss sums and the characteristic function of the third-order character modulo  $p$

$$1 + \psi(a) + \bar{\psi}(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue modulo } p, \\ 0, & \text{if } a \text{ is not a cubic residue modulo } p, \end{cases} \tag{19}$$

we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \bar{\psi}(a^3 + 1) &= \sum_{a=1}^{p-1} \bar{\psi}(a+1)(1 + \psi(a) + \bar{\psi}(a)) \\
&= \sum_{a=1}^{p-1} \bar{\psi}(a+1) + \sum_{a=1}^{p-1} \bar{\psi}(1 + \bar{a}) + \sum_{a=1}^{p-1} \bar{\psi}(a^2 + a) \\
&= -2 + \frac{1}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \sum_{a=1}^{p-1} \bar{\psi}(a) e\left(\frac{b(a+1)}{p}\right) \\
&= -2 + \frac{\tau^2(\bar{\psi})}{\tau(\psi)} = -2 + \frac{\tau^3(\bar{\psi})}{p},
\end{aligned} \tag{20}$$

$$\begin{aligned}
\sum_{a=0}^{p-1} \bar{\psi}(a^3 - (a+1)^3 + 1) &= \sum_{a=0}^{p-1} \bar{\psi}(-3a(a+1)) \\
&= \bar{\psi}(3) \sum_{a=1}^{p-1} \bar{\psi}(a(a+1)) \\
&= \frac{\bar{\psi}(3)\tau^3(\bar{\psi})}{p}.
\end{aligned} \tag{21}$$

Since  $\psi$  is a third-order character modulo  $p$ , for any integer  $c$  with  $(c, p) = 1$ , from the properties of the classical Gauss sums, we have

$$\begin{aligned}
\sum_{a=0}^{p-1} e\left(\frac{ca^3}{p}\right) &= 1 + \sum_{a=1}^{p-1} (1 + \psi(a) + \bar{\psi}(a)) e\left(\frac{ca}{p}\right) \\
&= \bar{\psi}(c)\tau(\psi) + \psi(c)\tau(\bar{\psi}).
\end{aligned} \tag{22}$$

From (22) and the properties of Gauss sums, then we can get

$$\begin{aligned}
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\psi}(a^3 + b^3 + 1) &= \frac{1}{\tau(\psi)} \sum_{c=1}^{p-1} \psi(c) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{ca^3 + cb^3 + c}{p}\right) \\
&= \frac{1}{\tau(\psi)} \sum_{c=1}^{p-1} \psi(c) e\left(\frac{c}{p}\right) \left( \sum_{a=0}^{p-1} e\left(\frac{ca^3}{p}\right) \right)^2 \\
&= \frac{1}{\tau(\psi)} \sum_{c=1}^{p-1} \psi(c) e\left(\frac{c}{p}\right) (\psi(c)\tau^2(\psi) \\
&\quad + 2p + \bar{\psi}(c)\tau^2(\bar{\psi})) \\
&= \tau(\psi)\tau(\bar{\psi}) + 2p - \frac{\tau^3(\bar{\psi})}{p} = 3p - \frac{\tau^3(\bar{\psi})}{p}.
\end{aligned} \tag{23}$$

Combining (18), (20), (21), and (23), we have the identity

$$\sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3 = \bar{\psi}(3)\tau^2(\bar{\psi}) - 3p\tau(\psi). \tag{24}$$

This proves Lemma 1.  $\square$

**Lemma 2.** If  $p$  is a prime with  $p \equiv 1 \pmod{3}$  and  $\psi$  is any third-order character modulo  $p$ , then we have the identity

$$\tau^3(\psi) + \tau^3(\bar{\psi}) = dp, \tag{25}$$

where  $\tau(\psi)$  denotes the classical Gauss sums,  $d$  is uniquely determined by  $4p = d^2 + 27b^2$ , and  $d \equiv 1 \pmod{3}$ .

*Proof.* See [3] or [11].  $\square$

**Lemma 3.** If  $p$  is a prime with  $p \equiv 1 \pmod{3}$ , then for any third-order character  $\psi \pmod{p}$ , we have the identity

$$\sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^4 = (\overline{\psi}(3) - 3p - \overline{\psi}(3)p)\tau^2(\overline{\psi}) - dp\tau(\psi). \quad (26)$$

*Proof.* Note that the two-term exponential sums satisfies

$$\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{-ma^3 - a}{p}\right). \quad (27)$$

So, from the properties of Gauss sums and Lemma 1, we have

$$\begin{aligned} & \sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^4 \\ &= \sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3 \left( \sum_{d=1}^{p-1} e\left(\frac{md^3 + d}{p}\right) \right) \\ & \quad + \sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3 \\ &= \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \overline{\psi}(a^3 + b^3 - c^3 - 1) \sum_{d=1}^{p-1} e\left(\frac{d(a+b-c-1)}{p}\right) \\ & \quad + \overline{\psi}(3)\tau^2(\overline{\psi}) - 3p\tau(\psi) \\ &= p\tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b \equiv c+1 \pmod{p}}}^{p-1} \overline{\psi}(a^3 + b^3 - c^3 - 1) \\ & \quad + \overline{\psi}(3)\tau^2(\overline{\psi}) - 3p\tau(\psi) \\ & \quad - \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \overline{\psi}(a^3 + b^3 - c^3 - 1). \end{aligned} \quad (28)$$

From (22) and Lemma 2, we have

$$\begin{aligned} A^3(m) &= \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^3 \\ &= (\overline{\psi}(c)\tau(\psi) + \psi(c)\tau(\overline{\psi}))^3 \\ &= \tau^3(\psi) + \tau^3(\overline{\psi}) + 3pA(m) \\ &= dp + 3pA(m). \end{aligned} \quad (29)$$

It is clear that  $A(m)$  is a real number, so from the properties of Gauss sums and (29), we have

$$\begin{aligned} & \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \overline{\psi}(a^3 + b^3 - c^3 - 1) \\ &= \sum_{m=1}^{p-1} \psi(m) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{m(a^3 + b^3 - c^3 - 1)}{p}\right) \\ &= \sum_{m=1}^{p-1} \psi(m) e\left(\frac{-m}{p}\right) A^3(m) = \sum_{m=1}^{p-1} \psi(m) e\left(\frac{-m}{p}\right) (dp + 3pA(m)) \\ &= dp\tau(\psi) + 3p \sum_{m=1}^{p-1} \psi(m) (\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})) e\left(\frac{-m}{p}\right) \\ &= dp\tau(\psi) - 3p\tau(\psi) + 3p\tau^2(\overline{\psi}) = p(d-3)\tau(\psi) + 3p\tau^2(\overline{\psi}). \end{aligned} \quad (30)$$

Note that the congruence  $a + b \equiv c + 1 \pmod{p}$  implies the congruence

$$a^3 + b^3 - c^3 - 1 \equiv -3(a+b)(a-1)(b-1) \pmod{p}. \quad (31)$$

So, we have

$$\begin{aligned} & \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b \equiv c+1 \pmod{p}}}^{p-1} \overline{\psi}(a^3 + b^3 - c^3 - 1) \\ &= \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\psi}(-3(a+b)(a-1)(b-1)) \\ &= \overline{\psi}(3) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \overline{\psi}((a+b+2)ab) \\ &= \overline{\psi}(3) \sum_{a=1}^{p-1} \overline{\psi}(a) \sum_{b=1}^{p-1} \overline{\psi}(b(a+b+2)). \end{aligned} \quad (32)$$

It is clear that

$$\begin{aligned} \sum_{b=1}^{p-1} \overline{\psi}(b(a+b+2)) &= \frac{1}{\tau(\psi)} \sum_{c=1}^{p-1} \psi(c) \sum_{b=1}^{p-1} \overline{\psi}(b) e\left(\frac{c(b+a+2)}{p}\right) \\ &= \frac{\tau(\overline{\psi})}{\tau(\psi)} \sum_{c=1}^{p-1} \psi^2(c) e\left(\frac{c(a+2)}{p}\right) = \frac{\tau^2(\overline{\psi})}{\tau(\psi)} \cdot \psi(a+2). \end{aligned} \quad (33)$$



From (32) and (33), we have

$$\begin{aligned}
 \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \bar{\psi}((a+b+2)ab) &= \frac{\tau^2(\bar{\psi})}{\tau(\psi)} \sum_{a=1}^{p-1} \bar{\psi}(a) \psi(a+2) \\
 &= \frac{\tau^2(\bar{\psi})}{\tau(\psi)} \sum_{a=1}^{p-1} \psi(1+2a) \\
 &= \frac{\tau^2(\bar{\psi})}{\tau(\psi)} \sum_{a=1}^{p-1} \psi(1+2a) = -\frac{\tau^2(\bar{\psi})}{\tau(\psi)}.
 \end{aligned} \tag{34}$$

Combining (28), (30), (32), and (34), we may immediately deduce that

$$\sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 = (\bar{\psi}(3) - 3p - \bar{\psi}(3)p)\tau^2(\bar{\psi}) - dp\tau(\psi). \tag{35}$$

This proves Lemma 3.  $\square$

*Proof.* See Zhang and Zhang [13].  $\square$

**Lemma 4.** *If  $p$  is a prime with  $p \equiv 1 \pmod{3}$ , then we have the identity*

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right|^4 = 2p^3 - 7p^2. \tag{36}$$

### 3. Proofs of the Theorems

Now, we prove our main results. First, we prove Theorem 1. If 3 is a cubic residue mod  $p$ , then  $\psi(3) = 1$ . From (22), Lemmas 2 and 3, we have

$$\begin{aligned}
 U_1(p) &= \sum_{m=1}^{p-1} A(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= \sum_{m=1}^{p-1} (\bar{\psi}(m)\tau(\psi) + \psi(m)\tau(\bar{\psi})) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= \tau(\bar{\psi})((1-4p)\tau^2(\bar{\psi}) - dp\tau(\psi)) + \tau(\psi)((1-4p)\tau^2(\psi) - dp\tau(\bar{\psi})) \\
 &= (1-4p)(\tau^3(\psi) + \tau^3(\bar{\psi})) - 2dp^2 = -dp(6p-1).
 \end{aligned} \tag{37}$$

Applying (22), Lemmas 2–4, we have

$$\begin{aligned}
 U_2(p) &= \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^2 \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= \sum_{m=1}^{p-1} (\psi(m)\tau^2(\psi) + 2p + \bar{\psi}(m)\tau^2(\bar{\psi})) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= 2p(2p^3 - 7p^2) + \tau^2(\psi)((1-4p)\tau^2(\bar{\psi}) - dp\tau(\psi)) \\
 &\quad + \tau^2(\bar{\psi})((1-4p)\tau^2(\psi) - dp\tau(\bar{\psi})) \\
 &= p^2(4p^2 - 22p - d^2 + 2).
 \end{aligned} \tag{38}$$

Applying (29), (37), Lemmas 3 and 4, we also have

$$\begin{aligned}
 U_3(p) &= \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^3 \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= \sum_{m=1}^{p-1} (dp + 3pA(m)) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \quad (39) \\
 &= dp(2p^3 - 7p^2) - 3dp^2(6p - 1) \\
 &= dp^2(2p^2 - 25p + 3).
 \end{aligned}$$

If  $k \geq 3$ , then from (29) and the definition of  $U_k(p)$ , we have

$$\begin{aligned}
 U_k(p) &= \sum_{m=1}^{p-1} A^k(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= \sum_{m=1}^{p-1} A^{k-3}(m) (dp + 3pA(m)) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= 3pU_{k-2}(p) + dpU_{k-3}(p). \quad (40)
 \end{aligned}$$

Combining Lemma 4, (37)–(40), we complete the proof of Theorem 1.

Note that identity  $A^3(m) = dp + 3pA(m)$ , so we have  $A^{-1}(m) = (A^2(m) - 3p)/dp$ ,  $A^{-2}(m) = (A(m) - 3pA^{-1}(m))/dp$ , and  $A^{-3}(m) = (1/dp) - (3/d) \cdot A^{-2}(m)$ .

From Lemma 4 and Theorem 1, we have the third-order linear recurrence formula:

$$\begin{aligned}
 U_0(p) &= 2p^3 - 7p^2, \\
 U_{-1}(p) &= \frac{1}{dp} (U_2(p) - 3pU_0(p)) = \frac{-p \cdot (2p^2 + p + d^2 - 2)}{d}, \\
 U_{-2}(p) &= \frac{1}{dp} (U_1(p) - 3pU_{-1}(p)) \\
 &= \frac{(1 - 3p)d^2 + 3p(2p^2 + p - 2)}{d^2}. \quad (41)
 \end{aligned}$$

If  $k \geq 3$ , then we have the third-order linear recurrence formula

$$U_{-k}(p) = -\frac{3}{d} \cdot U_{-(k-1)}(p) + \frac{1}{dp} \cdot U_{-(k-3)}(p). \quad (42)$$

This proves Theorem 2.

Now we prove Theorem 3. If  $p \equiv 1 \pmod{3}$  and 3 is not a cubic residue modulo  $p$ , then we have  $\psi(3) + \bar{\psi}(3) + 1 = 0$  or  $\psi(3) + \bar{\psi}(3) = -1$ . From (22), Lemmas 2–4, we have

$$\begin{aligned}
 &\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right|^4 \\
 &= \sum_{m=1}^{p-1} (\psi(m)\tau^2(\psi) + 2p + \bar{\psi}(m)\tau^2(\bar{\psi})) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right)^4 \\
 &= \tau^2(\psi)((\bar{\psi}(3) - 3p - \bar{\psi}(3)p)\tau^2(\bar{\psi}) - dp\tau(\psi)) + 2p(2p^3 - 7p^2) \\
 &\quad + \tau^2(\bar{\psi})(\psi(3) - 3p - \psi(3)p)\tau^2(\psi) - dp\tau(\bar{\psi}) \\
 &= (\psi(3) + \bar{\psi}(3) - 6p - (\psi(3) + \bar{\psi}(3))p)p^2 - dp(\tau^3(\psi) + \tau^3(\bar{\psi})) + 2p^3(2p - 7) \\
 &= -(5p + 1)p^2 - d^2p^2 + 2p^3(2p - 7) = p^2(4p^2 - 19p - d^2 - 1). \quad (43)
 \end{aligned}$$

This completes the proofs of all our results.

## 4. Conclusion

This paper mainly proposed three theorems, which are all closely related to the hybrid power mean of the two-term exponential sums and the classical cubic Gauss sums. Theorems 1 and 2 obtained two interesting third-order linear recurrence formulas of  $U_k(p)$  for  $k \geq 0$  and  $k \leq 0$ , respectively, providing 3 is a cubic residue modulo  $p$ .

However, when 3 is not a cubic residue modulo  $p$ , we gave an exact calculating formula for  $U_2(p)$  in Theorem 3. In general, this work not only generalized the results in reference [12] but also provided some new ideas and methods for the further study of such problems.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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## Research Article

# The Legendre's Symbol Modulo $p$ and Its Some Elementary Results

Jianghua Li  and Yuan Zhang 

College of Sciences, Xi'an University of Technology, Xi'an, Shaanxi, China

Correspondence should be addressed to Jianghua Li; [jianghuali@xaut.edu.cn](mailto:jianghuali@xaut.edu.cn)

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The main purpose of this article is using the elementary methods and the properties of the character sums to study the calculating problem of the number of the solutions for one kind congruence equation modulo  $p$  (an odd prime) and give some interesting identities and asymptotic formulas for it.

## 1. Introduction

Let  $p \geq 3$  be a prime. For any integer  $n$ , the Legendre's symbol  $(n/p)$  modulo  $p$  is defined as follows:

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if } (n, p) = 1 \text{ and } n \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } (n, p) = 1 \text{ and } n \text{ is a quadratic nonresidue modulo } p, \\ 0, & \text{if } p \mid n. \end{cases} \quad (1)$$

This arithmetical function occupies a very important position in the elementary number theory and analytic number theory, and many classical number theory problems are closely related to it, for example, the least quadratic nonresidue problem, the class number formula of the quadratic field, and the prime number structure. In particular, if  $p$  is a prime with  $p \equiv 1 \pmod{4}$ , then we have the identity (see Theorems 4–11 in [1])

$$p = \left( \sum_{a=1}^{p-1/2} \left( \frac{a+\bar{a}}{p} \right) \right)^2 + \left( \sum_{a=1}^{p-1/2} \left( \frac{a+r \cdot \bar{a}}{p} \right) \right)^2 \quad (2)$$

$$\equiv \alpha^2(p) + \beta^2(p),$$

where  $r$  is any quadratic nonresidue modulo  $p$ , and  $a \cdot \bar{a} \equiv 1 \pmod{p}$ .

In addition to its wide range of applications, the Legendre's symbol has a number of interesting and important

properties of its own. The quadratic reciprocity law is one of them. That is, for any two odd primes  $p$  and  $q$  with  $p \neq q$ , we have the identity (see Theorem 9.8 in [2])

$$\left(\frac{q}{p}\right) \cdot \left(\frac{p}{q}\right) = (-1)^{(p-1)(q-1)/4}. \quad (3)$$

Some other papers related to character sums can be referred to [3–9]. There is no need to give detailed examples of their contents here, the interested readers may refer to them.

In this paper, we will consider the following elementary number theory problem related to  $k$ -th residue modulo  $p$ , in particular, the quadratic residue modulo  $p$ .

For any integer  $n$  with  $(n, p) = 1$ , let  $i, j = \pm 1$ , and  $N(n, i, j; p)$  denote the number of the solutions of the congruence equation  $n \equiv a + b \pmod{p}$  ( $1 \leq a, b \leq p-1$ ) such that  $(a/p) = i$  and  $(b/p) = j$ .

**Theorem 1.** Let  $p$  be an odd prime. Then, for any integer  $n$  with  $(n, p) = 1$ , we have the identity

$$N(n, i, j; p) = \frac{p-2}{4} - \frac{i+j}{4} \cdot \left(\frac{n}{p}\right) - \frac{i \cdot j}{4} \cdot \left(\frac{-1}{p}\right). \quad (4)$$

For any fixed positive integer  $k \geq 3$  and any prime  $p$  with  $p \equiv 1 \pmod{k}$ , if we let  $M(n, k; p)$  denote the number of the solutions of the congruence equation  $n \equiv a + b \pmod{p}$  with



$1 \leq a, b \leq p-1$ , both  $a$  and  $b$  are the  $k$ -th residues modulo  $p$ . Then, we also have the following.

**Theorem 2.** Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . Then, for any integer  $n$  with  $(n, p) = 1$ , we have the identity

$$M(n, 3; p) = \frac{p-4}{9} - \frac{2(\lambda(n)\tau^2(\lambda) + \bar{\lambda}(n)\tau^2(\bar{\lambda})) - \bar{\lambda}(n)\tau^3(\lambda) - \lambda(n)\tau^3(\bar{\lambda})}{9p}, \quad (5)$$

if  $n$  is a cubic residue modulo  $p$ , then we have

$$M(n, 3; p) = \frac{p+d-4}{9} - \frac{2(\tau^2(\lambda) + \tau^2(\bar{\lambda}))}{9p}, \quad (6)$$

where  $\lambda$  is any third-order character modulo  $p$ ,  $\tau(\lambda) = \sum_{a=1}^{p-1} \lambda(a)e(a/p)$  denotes the classical Gauss sums,  $4p = d^2 + 27 \cdot b^2$ , and  $d$  is uniquely determined by  $d \equiv 1 \pmod{3}$ .

**Theorem 3.** Let  $p$  be a prime with  $p \equiv 1 \pmod{4}$ . If integer  $n$  is a quadratic residue modulo  $p$ , then have the identity

$$M(n, 4; p) = \begin{cases} 1/16 \cdot (p - 2\alpha(p) - 7), & \text{if } p \equiv 1 \pmod{8}, \\ 1/16 \cdot (p + 6\alpha(p) - 11), & \text{if } p \equiv 5 \pmod{8}, \end{cases} \quad (7)$$

where  $\alpha(p)$  is defined as the same as in (2).

It is clear that from Theorem 1, we can deduce the following several corollaries.

**Corollary 1.** Let  $p$  be an odd prime. Then, for any integer  $n$  with  $(n, p) = 1$ , we have the identity

$$N(n, 1, 1; p) = \begin{cases} p - 5/4, & \text{if } p \equiv 1 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = 1, \\ p - 1/4, & \text{if } p \equiv 1 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = -1, \\ p - 3/4, & \text{if } p \equiv 3 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = 1, \\ p + 1/4, & \text{if } p \equiv 3 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = -1. \end{cases} \quad (8)$$

**Corollary 2.** Let  $p \equiv 3 \pmod{4}$  be an odd prime. Then, for any integer  $n$  with  $(n, p) = 1$ , we have the identity

$$N(n, -1, -1; p) = \begin{cases} \frac{p-1}{4}, & \text{if } p \equiv 1 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = 1, \\ \frac{p-5}{4}, & \text{if } p \equiv 1 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = -1, \\ \frac{p+1}{4}, & \text{if } p \equiv 3 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = 1, \\ \frac{p-3}{4}, & \text{if } p \equiv 3 \pmod{4} \text{ and } \left(\frac{n}{p}\right) = -1. \end{cases} \quad (9)$$

**Corollary 3.** Let  $p \equiv 3 \pmod{4}$  be an odd prime. Then, for any integer  $n$  with  $(n, p) = 1$ , we have the identity

$$N(n, 1, -1; p) = N(n, -1, 1; p) = \begin{cases} \frac{p-1}{4}, & \text{if } p \equiv 1 \pmod{4}, \\ \frac{p-3}{4}, & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (10)$$

Note that the estimate  $|\tau^2(\lambda) + \tau^2(\bar{\lambda})| \leq 2p$  and  $|\alpha(p)| \leq \sqrt{p}$ , from Theorem 2 and Theorem 3; we can also deduce the following corollaries.

**Corollary 4.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{3}$ . Then, for any integer  $n$  with  $(n, p) = 1$ , we have the asymptotic formula

$$M(n, 3; p) = \frac{1}{9} \cdot p + O(\sqrt{p}). \quad (11)$$

**Corollary 5.** Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . Then, for any cubic residue  $n$  modulo  $p$ , we have the identity

$$M(n, 3; p) = \left\lfloor \frac{p+d}{9} \right\rfloor \text{ or } \left\lceil \frac{p+d}{9} \right\rceil - 1. \quad (12)$$

**Corollary 6.** Let  $p$  be a prime with  $p \equiv 1 \pmod{4}$ . Then, for any quadratic residue  $n \pmod{p}$ , we have the asymptotic formula

$$M(n, 4; p) = \frac{1}{16} \cdot p + O(\sqrt{p}). \quad (13)$$

Some notes: it is very interesting that the right hand side of the identity in Corollary 3 does not depend on constant  $n$ . Of course, the research content and methods in this paper can be further promoted.

## 2. Several Lemmas

In this section, we will give several necessary lemmas. Of course, the proofs of the theorems and these lemmas need the knowledge of elementary and analytic number theory, in particular, the properties of the quadratic residues modulo  $p$  and the classical Gauss sums. All these can be found in references [1, 2, 10], and we do not repeat them. First, we have the following.

**Lemma 1.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{3}$ . Then, for any three-order character  $\lambda \pmod{p}$ , we have the identity

$$\tau^3(\lambda) + \tau^3(\bar{\lambda}) = dp, \quad (14)$$

where  $4p = d^2 + 27 \cdot b^2$ , and  $d$  is uniquely determined by  $d \equiv 1 \pmod{3}$ ,

*Proof.* See the study by Zhang and Hu [4] or Berndt and Evans [5].  $\square$

**Lemma 2.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{3}$ . Then, we have

$$\begin{aligned} & \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) (-1 + \bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda}))^2 \\ &= -(2p+1) - 2(\lambda(n)\tau^2(\lambda) + \bar{\lambda}(n)\tau^2(\bar{\lambda})) \\ &+ \bar{\lambda}(n)\tau^3(\lambda) + \lambda(n)\tau^3(\bar{\lambda}). \end{aligned} \quad (15)$$

If  $n$  is a cubic residue modulo  $p$ , then we have

$$\begin{aligned} & \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) (-1 + \bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda}))^2 \\ &= dp - 2p - 1 - 2(\tau^2(\lambda) + \tau^2(\bar{\lambda})), \end{aligned} \quad (16)$$

where  $d$  is defined as the same as in Lemma 1.

*Proof.* For any integer  $1 \leq a \leq p-1$ , note that  $\bar{\lambda} = \lambda^2$  and  $\tau(\lambda) \cdot \tau(\bar{\lambda}) = p$ ; from the properties of the three-order

character  $\lambda \pmod{p}$  and the classical Gauss sums  $\tau(\lambda)$  modulo  $p$ , we have

$$\begin{aligned} & \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) (-1 + \bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda}))^2 \\ &= -1 - 2 \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) (\bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda})) \\ &+ \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) (2p + \lambda(m)\tau^2(\lambda) + \bar{\lambda}(m)\tau^2(\bar{\lambda})) \\ &= -(2p+1) - 2(\lambda(n)\tau^2(\lambda) + \bar{\lambda}(n)\tau^2(\bar{\lambda})) \\ &+ \bar{\lambda}(n)\tau^3(\lambda) + \lambda(n)\tau^3(\bar{\lambda}). \end{aligned} \quad (17)$$

If  $n$  is a cubic residue modulo  $p$ , then we have  $\lambda(-1) = 1$  and  $\lambda(n) = \bar{\lambda}(n) = 1$ ; from (17) and Lemma 1, we have

$$\begin{aligned} & \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) (-1 + \bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda}))^2 \\ &= -(2p+1) - 2(\tau^2(\lambda) + \tau^2(\bar{\lambda})) + \tau^3(\lambda) + \tau^3(\bar{\lambda}) \\ &= dp - 2p - 1 - 2(\tau^2(\lambda) + \tau^2(\bar{\lambda})). \end{aligned} \quad (18)$$

Now, Lemma 2 follows from (18).  $\square$

**Lemma 3.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, for any four-order character  $\psi$  modulo  $p$ , we have the identity

$$\tau^2(\psi) + \tau^2(\bar{\psi}) = 2\sqrt{p} \cdot \alpha(p), \quad (19)$$

where  $\alpha(p)$  is defined as in (2).

*Proof.* See Lemma 2.2 in the study by Chen and Zhang [7].  $\square$

## 3. Proofs of the Theorems

Applying three simple lemmas in Section 2, we can easily complete the proofs of our theorems. First, we prove Theorem 1. Note that the trigonometric identity

$$\sum_{m=0}^{p-1} e\left(\frac{nm}{p}\right) = \begin{cases} p, & \text{if } (n, p) = p, \\ 0, & \text{if } (n, p) = 1, \end{cases} \quad (20)$$

$\tau^2(\chi_2) = (-1/p) \cdot p$ , so from the definition of  $N(n, i, j; p)$  and the properties of the Legendre's symbol, we have

$$\begin{aligned}
N(n, i, j; p) &= \frac{1}{p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{4} \left(1 + i \left(\frac{a}{p}\right)\right) \left(1 + j \left(\frac{b}{p}\right)\right) \sum_{m=0}^{p-1} e\left(\frac{m(a+b-n)}{p}\right) \\
&= \frac{1}{4p} \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(1 + i \left(\frac{a}{p}\right)\right) \left(1 + j \left(\frac{b}{p}\right)\right) e\left(\frac{m(a+b)}{p}\right) + \frac{1}{4p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(1 + i \left(\frac{a}{p}\right)\right) \left(1 + j \left(\frac{b}{p}\right)\right) \\
&= \frac{1}{4p} \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) \left(-1 + i \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e\left(\frac{ma}{p}\right)\right) \cdot \left(-1 + j \sum_{b=1}^{p-1} \left(\frac{b}{p}\right) e\left(\frac{mb}{p}\right)\right) + \frac{1}{4p} \left(p-1 + i \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)\right) \\
&\quad \cdot \left(p-1 + j \sum_{b=1}^{p-1} \left(\frac{b}{p}\right)\right) \\
&= \frac{1}{4p} \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) \left(-1 + i \left(\frac{m}{p}\right) \tau(\chi_2)\right) \left(-1 + j \left(\frac{m}{p}\right) \tau(\chi_2)\right) + \frac{(p-1)^2}{4p} \\
&= \frac{1}{4p} \left(-1 - i \left(\frac{-n}{p}\right) \tau^2(\chi_2) - j \left(\frac{-n}{p}\right) \tau^2(\chi_2) - i \cdot j \cdot \tau^2(\chi_2)\right) + \frac{(p-1)^2}{4p} \\
&= \frac{p-2}{4} - \frac{i+j}{4} \cdot \left(\frac{n}{p}\right) - \frac{i \cdot j}{4} \cdot \left(\frac{-1}{p}\right),
\end{aligned} \tag{21}$$

where  $\chi_2 = (*/p)$  is the Legendre's symbol modulo  $p$ . This proves Theorem 1.

Now, we prove Theorem 2. For any integer  $1 \leq a \leq p-1$ , from the properties of the third-order character  $\lambda$  modulo  $p$ , we have

$$1 + \lambda(a) + \bar{\lambda}(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue modulo } p, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Applying formulas (20) and (22) and Lemma 2, the definition of  $M(n, 3; p)$ , and the properties of the classical Gauss sums, we have

$$\begin{aligned}
M(n, 3; p) &= \frac{1}{9p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} (1 + \lambda(a) + \bar{\lambda}(a)) (1 + \lambda(b) + \bar{\lambda}(b)) \sum_{m=0}^{p-1} e\left(\frac{m(a+b-n)}{p}\right) \\
&= \frac{1}{9p} \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) \left(\sum_{a=1}^{p-1} (1 + \lambda(a) + \bar{\lambda}(a)) e\left(\frac{ma}{p}\right)\right)^2 + \frac{1}{9p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} (1 + \lambda(a) + \bar{\lambda}(a)) (1 + \lambda(b) + \bar{\lambda}(b)) \\
&= \frac{(p-1)^2}{9p} + \frac{1}{9p} \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) (-1 + \bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}))^2 \\
&= \frac{p-4}{9} - \frac{2(\lambda(n) \tau^2(\lambda) + \bar{\lambda}(n) \tau^2(\bar{\lambda})) - \bar{\lambda}(n) \tau^3(\lambda) - \lambda(n) \tau^3(\bar{\lambda})}{9p}.
\end{aligned} \tag{23}$$

If  $n$  is a cubic residue modulo  $p$ , then we have  $\lambda(n) = \bar{\lambda}(n) = 1$ . From (23) and Lemma 1, we may immediately deduce

$$M(n, 3; p) = \frac{p+d-4}{9} - \frac{2}{9p} \cdot (\tau^2(\lambda) + \tau^2(\bar{\lambda})). \tag{24}$$

Now, Theorem 2 follows from (23) and (24).

Similarly, using the methods of proving Theorem 2, we can also deduce Theorem 3. In fact, let  $\psi$  be a four-order character modulo  $p$ , then  $\psi^2 = \chi_2 = (*/p)$ . From the properties of the four-order character, we have

$$1 + \psi(a) + \psi^2(a) + \bar{\psi}(a) = \begin{cases} 4, & \text{if } a \text{ is a quartic residue modulo } p, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

Note that  $\chi_2(-1) = 1$ ; from (25) and the methods of proving Theorem 2, we have

$$\begin{aligned} M(n, 4; p) &= \frac{1}{16p} \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) \left( \sum_{a=1}^{p-1} (1 + \psi(a) + \chi_2(a) + \bar{\psi}(a)) e\left(\frac{ma}{p}\right) \right)^2 \\ &\quad + \frac{1}{16p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} (1 + \psi(a) + \chi_2(a) + \bar{\psi}(a)) (1 + \psi(b) + \chi_2(b) + \bar{\psi}(b)) \\ &= \frac{1}{16p} \sum_{m=1}^{p-1} e\left(\frac{-nm}{p}\right) \left( -1 + \bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}) \right)^2 + \frac{(p-1)^2}{16p} \\ &= \frac{p-3}{16} - \frac{1}{8} \left( \frac{n}{p} \right) - \frac{1}{8p} (1 + \psi(-n) + \bar{\psi}(-n))\tau(\psi)\tau(\bar{\psi}) + \frac{1}{8\sqrt{p}} (\bar{\psi}(-n)\tau^2(\psi) + \psi(-n)\tau^2(\bar{\psi})) \\ &\quad + \frac{\chi_2(n)}{16\sqrt{p}} (\tau^2(\psi) + \tau^2(\bar{\psi})). \end{aligned} \quad (26)$$

If  $p = 8k + 5$ ,  $n$  is a quadratic residue modulo  $p$ , then we have  $\psi(-1) = \psi(n) = 1$  and  $\tau(\psi)\tau(\bar{\psi}) = p$ . Thus, from (26) and Lemma 3, we have

$$\begin{aligned} M(n, 4; p) &= \frac{p-3}{16} - \frac{1}{8} - \frac{3}{8} + \frac{2 \cdot \alpha(p)}{8} + \frac{2\alpha(p)}{16} \\ &= \frac{1}{16} \cdot (p + 6\alpha(p) - 11). \end{aligned} \quad (27)$$

If  $p = 8k + 1$ ,  $n$  is a quadratic residue modulo  $p$ , then we have  $\psi(-1) = -1$ ,  $\psi(n) = 1$ , and  $\tau(\psi)\tau(\bar{\psi}) = -p$ . Thus, from (26) and Lemma 3, we have

$$\begin{aligned} M(n, 4; p) &= \frac{p-3}{16} - \frac{1}{8} - \frac{1}{8} - \frac{2 \cdot \alpha(p)}{8} + \frac{2\alpha(p)}{16} \\ &= \frac{1}{16} \cdot (p - 2\alpha(p) - 7). \end{aligned} \quad (28)$$

Now, Theorem 3 follows from (27) and (28). This completes the proofs of all our results.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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## Research Article

# Some Weighted Sum Formulas for Multiple Zeta, Hurwitz Zeta, and Alternating Multiple Zeta Values

Yuan He <sup>1</sup> and Zhuoyu Chen <sup>2</sup>

<sup>1</sup>School of Mathematics and Information Science, Neijiang Normal University, Neijiang, Sichuan 641100, China

<sup>2</sup>School of Mathematics, Northwest University, Xi'an, Shaanxi 710127, China

Correspondence should be addressed to Zhuoyu Chen; [chenzymath@163.com](mailto:chenzymath@163.com)

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We perform a further investigation for the multiple zeta values and their variations and generalizations in this paper. By making use of the method of the generating functions and some connections between the higher-order trigonometric functions and the Lerch zeta function, we explicitly evaluate some weighted sums of the multiple zeta, Hurwitz zeta, and alternating multiple zeta values in terms of the Bernoulli and Euler polynomials and numbers. It turns out that various known results are deduced as special cases.

## 1. Introduction

Let  $\alpha_1, \alpha_2, \dots, \alpha_k$  be all positive integers with  $\alpha_k \geq 2$ . The multiple zeta values  $\zeta(\alpha_1, \dots, \alpha_k)$  of the depth  $k$  (sometimes called Euler sums) are usually defined by the following series:

$$\zeta(\alpha_1, \dots, \alpha_k) = \sum_{1 \leq n_1 < n_2 < \dots < n_k} \frac{1}{n_1^{\alpha_1} n_2^{\alpha_2} \dots n_k^{\alpha_k}}. \quad (1)$$

These series appear in a variety of fields in mathematics, such as number theory, combinatorics, knot theory, the theory of mixed Tate motives, and quantum field theory; see, for example [1–4].

It is well known that one of the central problems on the multiple zeta values is to determine all the possible  $\mathbb{Q}$ -linear relations among them. Goncharov's ([5], Conjecture 4.2) conjecture implies that it suffices to study the relations among the multiple zeta values of the same weight  $|\mathbf{a}| = \alpha_1 + \alpha_2 + \dots + \alpha_k$ . Perhaps the earliest result in this direction is Euler's [6] sum formula, namely,

$$\sum_{\alpha_1 + \alpha_2 = n} \zeta(\alpha_1, \alpha_2 + 1) = \zeta(n + 1), \quad n \geq 2, \quad (2)$$

where  $\zeta(\cdot)$  is the Riemann zeta function. In fact, there exists a general form of (2), as follows:

$$\sum_{|\mathbf{a}|=n} \zeta(\alpha_1, \alpha_2, \dots, \alpha_k + 1) = \zeta(n + 1), \quad n \geq k, \quad (3)$$

which is referred to as the “sum conjecture” in [7], and was proved by Granville [8] and Zagier [9] independently around the year 1995. Some new relations for the multiple zeta values and their different variations and generalizations have been found in recent years. For example, Ohno and Zudilin [10] in 2008 proved a weighted form of Euler's sum formula:

$$\sum_{\alpha_1 + \alpha_2 = n} 2^{\alpha_2 + 1} \zeta(\alpha_1, \alpha_2 + 1) = (n + 2) \zeta(n + 1), \quad n \geq 2. \quad (4)$$

It becomes obvious from (2) that Ohno and Zudilin's weighted sum formula can be rewritten as

$$\sum_{\alpha_1 + \alpha_2 = n} (2^{\alpha_2 + 1} - 1) \zeta(\alpha_1, \alpha_2 + 1) = (n + 1) \zeta(n + 1), \quad n \geq 2. \quad (5)$$



Based on the equivalence of (4) and (5), Guo and Xie [11] in 2009 extended the weighted sum formula of Ohno and

Zudilin to arbitrary depth and discovered that for positive integers  $n, k$  with  $n \geq k$ ,

$$\sum_{|a|=n} \left[ 2^{\alpha_k} + (2^{\alpha_k} - 1) \left( \sum_{j=2}^{k-1} 2^{S_j - \alpha_k - j} + 2^{S_{k-1} - \alpha_k - (k-1)} \right) \right] \zeta(\alpha_1, \alpha_2, \dots, \alpha_k + 1) = (n+1)\zeta(n+1), \quad (6)$$

where  $S_j = \alpha_{k-j+1} + \dots + \alpha_k + 1$  for  $j = 1, \dots, k-1$ . For another weighted sum formulas of the multiple zeta values, one is referred to [12], where the weight coefficients are given by (symmetric) polynomials of the arguments. On the contrary, let  $m, n, k$  be positive integers with  $m \geq 2$  and  $n \geq k$ , and let  $E(mn, k)$  be the sums of all multiple zeta values of the depth  $k$  and the weight  $mn$  given by

$$E(mn, k) = \sum_{|a|=n} \zeta(m\alpha_1, \dots, m\alpha_k). \quad (7)$$

Gangl et al. [13] in 2004 proved that

$$E(2n, 2) = \frac{3}{4} \zeta(2n). \quad (8)$$

Shen and Cai [14] in 2012 obtained the formulas

$$\begin{aligned} E(2n, 3) &= \frac{5}{8} \zeta(2n) - \frac{1}{4} \zeta(2) \zeta(2n-2), \\ E(2n, 4) &= \frac{35}{64} \zeta(2n) - \frac{5}{16} \zeta(2) \zeta(2n-2). \end{aligned} \quad (9)$$

After that, Hoffman [15] used the theory of symmetric functions to establish the general sum formula

$$E(2n, k) = \frac{(-1)^{n-k} \pi^{2n}}{(2n+1)!} \sum_{l=0}^{n-k} \binom{n-l}{k} \binom{2n+1}{2l} (2-2^{2l}) B_{2l}, \quad (10)$$

where  $B_n$  is the  $n$ -th Bernoulli number. Furthermore, Zhao [16] used the ideas developed in [15] to evaluate the sums of all multiple Hurwitz zeta values of the depth  $k$  and the weight  $2n$  in terms of the Euler numbers. Moreover, Zhao [17] used the theory of symmetric functions to consider the more complicated alternating multiple zeta values and depicted that the sums of all alternating multiple zeta values of the depth  $k$  and the weight  $2n$  can be evaluated in terms of the Riemann zeta function and the Euler numbers. More recently, Chen et al. [18] used the method of the generating functions to express  $E(mn, k)$  by constructing a combinatorial identity of products of the multiple zeta values and the so-called multiple zeta-star values at the repetitions of  $m$ , and then used Muneta's [19] and Nakamura's [20] results to reobtain Hoffman's sum formula (10) and confirm Genčev's ([21], Conjecture 4.1) conjecture on the evaluation of  $E(4n, k)$ .

$$\begin{aligned} E(4n, k) &= \frac{(-1)^{n-k} 2^{2n+1} \pi^{4n}}{(4n+2)!} \sum_{l=0}^{n-k} \binom{n-l}{k} \binom{4n+2}{4l} \frac{1}{(-4)^l} \\ &\quad \times \sum_{r=0}^{2l} \binom{4l}{2r} (-1)^r (2-2^{2r}) B_{2r} (2-2^{4l-2r}) B_{4l-2r}. \end{aligned} \quad (11)$$

Subsequently, Shen and Jia [22] extended the sums of the multiple Hurwitz zeta values previously considered in [23] and showed that the sums of all multiple Hurwitz zeta values of the depth  $k$  and the weight  $mn$  can be expressed by a combinatorial identity of products of the multiple Hurwitz zeta values and the so-called multiple Hurwitz zeta-star values at the repetitions of  $m$ . In particular, Shen and Jia [22] obtained Zhao's [16] sum formula with a slight different notation and evaluated the sums of all multiple Hurwitz zeta values of the depth  $k$  and the weight  $4n$  in terms of the Euler numbers.

Motivated and inspired by the work of the above authors, we explicitly evaluate some weighted sums of the multiple zeta, Hurwitz zeta, and alternating multiple zeta values in terms of the Bernoulli and Euler polynomials and numbers by using the method of the generating functions and some connections between the higher-order trigonometric functions and the Lerch zeta function established by the first author [24]. The results presented here are the corresponding extensions of various known sum formulas.

This paper is organized as follows: In Section 2, we give several weighted sum formulas for the multiple zeta values, some of which generalize the sum formulas (10) and (11), and improve Eie and Ong's [25] weighted sum formulas. In Section 3, we present some similar weighted sum formulas for the multiple Hurwitz zeta values and deduce Shen and Jia's [22] sum formulas as special cases. Section 4 concentrates on the features that have contributed to the weighted sum formulas for the alternating multiple zeta values, and it then turns out that Zhao's [17] sum formula is obtained in a rather simple way.

## 2. Sum Formulas for Multiple Zeta Values

For convenience, in the following, we always denote by  $i$  the square root of  $-1$  such that  $i^2 = -1$ ,  $s(n, k)$  the Stirling numbers of the first kind,  $B_n(x)$  the Bernoulli polynomials, and  $E_n(x)$  the Euler polynomials. It is clear that taking  $x = 0$  and  $x = 1/2$  in the Bernoulli and Euler polynomials gives the Bernoulli numbers  $B_n = B_n(0)$  and the Euler numbers  $E_n = 2^n E_n(1/2)$ , respectively. We refer the reader to two standard books [26, 27] on basic properties for these special

sequences and polynomials. We also write  $[t^n]f(t)$  as the coefficients of  $t^n$  in  $f(t)$  for nonnegative integer  $n$ . We now state our first result as follows.

**Theorem 1.** *Let  $m, n$  be positive integers. Then*

$$\begin{aligned} & \sum_{k=1}^n \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k) \\ &= \frac{(-1)^n 2^{2n+1} \pi^{2n}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2n-m, k), \end{aligned} \quad (12)$$

where, and in what follows,  $\tilde{B}(m, k)$  is the linear combination of the Stirling numbers of the first kind satisfying that for positive integer  $m$  and nonnegative integer  $k$ ,

$$\tilde{B}(m, k) = \sum_{j=k+1}^m \binom{j-1}{k} \frac{s(m, j)}{2^{j-k}}, \quad (13)$$

and  $U(m, n, k)$  is the linear combination of the Bernoulli polynomials given for positive integer  $m$  and nonnegative integers  $n, k$  by

$$U(m, n, k) = \frac{1}{n!} \sum_{l=0}^n \binom{n}{l} \left( \frac{m-1}{2} \right)^{n-l} \frac{B_{l+k+1}(1/2)}{l+k+1}. \quad (14)$$

*Proof.* Recall that Euler's infinite product formula of the sine function is

$$\sin \pi x = \pi x \prod_{j=1}^{\infty} \left( 1 - \frac{x^2}{j^2} \right), \quad (15)$$

which holds true for arbitrary complex number  $x$  (see [26], p. 75 or [28], pp. 12–18). The binomial series asserts that for complex number  $\alpha$  (see [27], p. 37),

$$(1+t)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n, \quad |t| < 1, \quad (16)$$

where  $\binom{\alpha}{n}$  are the binomial coefficients given for non-negative integer  $n$  by

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!}, \quad n \geq 1. \quad (17)$$

So from (15) and (16), we discover that for complex number  $x$  with  $0 < |x| < 1$ ,

$$\begin{aligned} \left( \frac{\sin \pi x}{\pi x} \right)^{-m} &= \prod_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} (-1)^k \binom{-m}{k} \frac{x^{2k}}{j^{2k}} \right) \\ &= \prod_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^{2k}}{j^{2k}} \right). \end{aligned} \quad (18)$$

Comparing the coefficients of  $x^{2n}$  on both sides of (18), it then follows that for complex number  $x$  with  $0 < |x| < 1$ ,

$$\sum_{k=1}^n \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k) = [x^{2n}] \left( \frac{\sin \pi x}{\pi x} \right)^{-m}. \quad (19)$$

We now evaluate the right-hand side of (19) from another view. Let  $q, n$  be positive integers, and let  $\theta_r$  be a real function defined on positive integer  $r$ . If  $\theta_r \neq 0, \pm q, \pm 2q, \dots$ , then (see [24], Theorem 3.2)

$$\csc^n \left( \frac{\pi \theta_r}{q} \right) = -i^n \frac{2^{n+1} e^{(n\pi i \theta_r/q)}}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \tilde{B}(n, k) \phi \left( \frac{\theta_r}{q}, \frac{1}{2}, -k \right), \quad (20)$$

where  $\phi(a, x, s)$  is the Lerch zeta function given for real number  $a$ ,  $x \neq$  negative integer or zero, and complex number  $s$  by

$$\phi(a, x, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n+x)^s}, \quad \Re(s) > 1. \quad (21)$$

Note that the series is an entire function of  $s$  when  $a$  is not an integer. Obviously, replacing  $n$  by  $m$  and  $\theta_r/q$  by  $x$  in (20) gives that for real number  $x \neq 0, \pm 1, \pm 2, \dots$ ,

$$\csc^m(\pi x) = -i^m \frac{2^{m+1} e^{m\pi i x}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \phi \left( x, \frac{1}{2}, -k \right). \quad (22)$$

It follows from (19) and (22) that

$$\begin{aligned} & \sum_{k=1}^n \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k) \\ &= [x^{2n}] \pi^m x^m \left\{ -i^m \frac{2^{m+1} e^{m\pi i x}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \phi \left( x, \frac{1}{2}, -k \right) \right\} \\ &= -i^m \frac{2^{m+1} \pi^m}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) [x^{2n-m}] e^{m\pi i x} \phi \left( x, \frac{1}{2}, -k \right). \end{aligned} \quad (23)$$

It is easily seen from the Taylor series expansion for the complex exponential function and the familiar binomial theorem that

$$\begin{aligned} & [x^{2n-m}] e^{m\pi i x} \phi \left( x, \frac{1}{2}, -k \right) \\ &= [x^{2n-m}] e^{m\pi i x} \sum_{j=0}^{\infty} \frac{e^{2\pi i j x}}{(j + (1/2))^{-k}} \\ &= [x^{2n-m}] \sum_{j=0}^{\infty} \frac{1}{(j + (1/2))^{-k}} \sum_{l=0}^{\infty} \frac{(j + (m/2))^l (2\pi i)^l x^l}{l!} \\ &= \frac{(2\pi i)^{2n-m}}{(2n-m)!} \sum_{j=0}^{\infty} \frac{(j + (m/2))^{2n-m}}{(j + (1/2))^{-k}} \\ &= \frac{(2\pi i)^{2n-m}}{(2n-m)!} \sum_{l=0}^{2n-m} \binom{2n-m}{l} \left( \frac{m-1}{2} \right)^{2n-m-l} \sum_{j=0}^{\infty} \frac{1}{(j + (1/2))^{-(l+k)}}. \end{aligned} \quad (24)$$

Since for nonnegative integer  $n$  (see [29], Theorem 12.13),

$$\zeta(-n, x) = -\frac{B_{n+1}(x)}{n+1}, \quad (25)$$

where  $\zeta(s, x)$  is the Hurwitz zeta function given for real number  $x > 0$  and complex number  $s$  by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \Re(s) > 1, \quad (26)$$

so by applying (25) to (24), we arrive at

$$\left[ x^{2n-m} \right] e^{m\pi i x} \phi\left(x, \frac{1}{2}, -k\right) = -(2\pi i)^{2n-m} U(m, 2n-m, k). \quad (27)$$

Now (12) follows from (23) and (27). This completes the proof of Theorem 1.  $\square$

**Corollary 1.** *Let  $n$  be a positive integer. Then*

$$\sum_{k=1}^n E(2n, k) = \frac{(-1)^n \pi^{2n} (2 - 2^{2n}) B_{2n}}{(2n)!}. \quad (28)$$

*Proof.* Since for nonnegative integer  $n$  (see [26], p. 805),

$$B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1)B_n, \quad (29)$$

so by taking  $m = 1$  in Theorem 1, in view of  $s(1, 1) = 1$ , we get the desired result.

Corollary 1 is usually attributed to Hoffman ([15], Corollary 2) and was previously obtained by Aoki, Kombu, and Ohno ([30], Equation (4.6)), who stated it in the language of the multiple zeta-star values. We are in a good position to use Theorem 1 to yield the following result.  $\square$

**Theorem 2.** *Let  $m, n, k$  be positive integers with  $n \geq k$ . Then*

$$\begin{aligned} & \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k) \\ &= \frac{(-1)^{n-k} 2\pi^{2n}}{(m-1)!} \sum_{l=0}^{n-k} \binom{n-l}{k} 4^l (-1)^{(m-1)(n-l)} V_1(m, n-l) \\ & \quad \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2l-m, k), \end{aligned} \quad (30)$$

where  $V_1(m, n)$  is a rational number given for positive integer  $m$  and nonnegative integer  $n$  by

$$V_1(m, n) = \sum_{\substack{j_1 + \dots + j_m = n \\ j_1, \dots, j_m \geq 0}} \frac{1}{(2j_1 + 1)!} \cdots \frac{1}{(2j_m + 1)!}. \quad (31)$$

*Proof.* Clearly, for real or complex parameter  $\lambda$ ,

$$\begin{aligned} & \sum_{k=1}^n (1+\lambda)^k \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k) \\ &= [x^{2n}] \prod_{j=1}^{\infty} \left( 1 + (1+\lambda) \binom{m}{1} \frac{x^2}{j^2} + \cdots + (1+\lambda) \binom{m+k-1}{k} \frac{x^{2k}}{j^{2k}} + \cdots \right). \end{aligned} \quad (32)$$

Just as a polynomial function of  $z$  in the order of ascending power is divided by another polynomial function of  $z$  in the order of ascending power, we discover that

$$\begin{aligned} & \frac{\lambda \binom{m}{1} z^2 + \cdots + \lambda \binom{m+k-1}{k} z^{2k} + \cdots}{1 + \binom{m}{1} z^2 + \cdots + \binom{m+k-1}{k} z^{2k} + \cdots} \\ &= \lambda \binom{m}{1} z^2 - \lambda \binom{m}{2} z^4 + \cdots + (-1)^{m-1} \lambda \binom{m}{m} z^{2m} \\ &= -\lambda(1-z^2)^m + \lambda. \end{aligned} \quad (33)$$

Applying (33) to the right-hand side of (32), it then follows from (18) that

$$\begin{aligned} & \sum_{k=1}^n (1+\lambda)^k \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k) \\ &= [x^{2n}] \prod_{j=1}^{\infty} \left( 1 + \binom{m}{1} \frac{x^2}{j^2} + \cdots + \binom{m+k-1}{k} \frac{x^{2k}}{j^{2k}} + \cdots \right) \\ & \quad \times \left( 1 - \lambda \left( 1 - \frac{x^2}{j^2} \right)^m + \lambda \right) \\ &= \sum_{l=0}^n [x^{2l}] \left( \frac{\sin \pi x}{\pi x} \right)^{-m} [x^{2n-2l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - \frac{x^2}{j^2} \right)^m + \lambda \right). \end{aligned} \quad (34)$$

Noticing that from (19) and Theorem 1, we have

$$\begin{aligned} [x^{2l}] \left( \frac{\sin \pi x}{\pi x} \right)^{-m} &= \frac{(-1)^l 2^{2l+1} \pi^{2l}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \\ &U(m, 2l - m, k). \end{aligned} \quad (35)$$

We now evaluate the coefficients of  $x^{2n-2l}$  in the infinite product of the right-hand side of (34). Let  $\mu_1(\lambda), \dots, \mu_m(\lambda)$  be all complex numbers determined by the factorization of the polynomial function  $1 - \lambda(1 - z)^m + \lambda$  over the complex number field satisfying that

$$1 - \lambda(1 - z)^m + \lambda = \prod_{l=1}^m (1 + \mu_l(\lambda)z). \quad (36)$$

The famous Vieta's theorem implies that

$$\mu_1(\lambda) \cdots \mu_m(\lambda) = (-1)^{m-1} \lambda. \quad (37)$$

So from (36), (37), and the remarkable formula see [31], Equation (36), or [7], Corollary 2.3,

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}, \quad n \geq 0, \quad (38)$$

where  $\{a\}^n$  denotes the  $n$  repetitions of  $a$ , we get that

$$\begin{aligned} &[x^{2n-2l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - \frac{x^2}{j^2} \right)^m + \lambda \right) \\ &= [x^{2n-2l}] \prod_{l=1}^m \prod_{j=1}^{\infty} \left( 1 + \mu_l(\lambda) \frac{x^2}{j^2} \right) \\ &= [x^{2n-2l}] \prod_{l=1}^m \sum_{j=0}^{\infty} (\mu_l(\lambda))^j \zeta(\{2\}^j) x^{2j} \\ &= \sum_{j_1 + \dots + j_m = n-l, j_1, \dots, j_m \geq 0} (\mu_1(\lambda))^{j_1} \zeta(\{2\}^{j_1}) \cdots (\mu_m(\lambda))^{j_m} \zeta(\{2\}^{j_m}) \\ &= ((-1)^{m-1} \lambda)^{n-l} \pi^{2n-2l} V_1(m, n-l). \end{aligned} \quad (39)$$

Inserting (35) and (39) into (34), it follows that

$$\begin{aligned} &\sum_{k=1}^n (1+\lambda)^k \sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k) \\ &= \frac{2\pi^{2n}}{(m-1)!} \sum_{l=0}^n (-4)^l ((-1)^{m-1} \lambda)^{n-l} V_1(m, n-l) \\ &\quad \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2l - m, k). \end{aligned} \quad (40)$$

Thus (30) follows immediately after making  $k$ -times derivative with respect to  $\lambda$  and then taking  $\lambda = -1$  on both sides of (40). This concludes the proof of Theorem 2.

It is easy to check that taking  $m = 1$  in Theorem 2 and then applying (29) and  $s(1, 1) = 1$  leads to Hoffman's formula (10). It is worth noticing that the formula (40) can also be regarded as an extension of Theorem 1. In a similar consideration to Theorem 2, we have the following result.  $\square$

**Theorem 3.** Let  $m, n, k$  be the positive integers with  $n \geq k$ . Then

$$\begin{aligned} &\sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(4\alpha_1, \dots, 4\alpha_k) \\ &= \frac{(-1)^{n-k} 2^{2n+m+1} \pi^{4n}}{(m-1)!} \sum_{l=0}^{n-k} \binom{n-l}{k} (-4)^l (-1)^{(m-1)(n-l)} V_2(m, n-l) \\ &\quad \times \sum_{r=0}^{2l} \frac{(-1)^r}{(4l-2r)!} B_{4l-2r}^{(m)} \left( \frac{m}{2} \right) \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r - m, k), \end{aligned} \quad (41)$$

where  $V_2(m, n)$  is a rational number given for positive integer  $m$  and nonnegative integer  $n$  by

$$V_2(m, n) = \sum_{\substack{j_1 + \dots + j_m = n \\ j_1, \dots, j_m \geq 0}} \frac{1}{(4j_1 + 2)!} \cdots \frac{1}{(4j_m + 2)!} \quad (42)$$

and  $B_n^{(m)}(x)$  are the higher-order Bernoulli polynomials defined by the generating function (see [32]):

$$\left(\frac{t}{e^t - 1}\right)^m e^{xt} = \sum_{n=0}^{\infty} B_n^{(m)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (43)$$

*Proof.* We know from (16) and (33) that

$$\begin{aligned} & \sum_{k=1}^n (1+\lambda)^k \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(4\alpha_1, \dots, 4\alpha_k) \\ &= [x^{4n}] \prod_{j=1}^{\infty} \left( 1 + (1+\lambda) \binom{m}{1} \frac{x^4}{j^4} + \dots + (1+\lambda) \binom{m+k-1}{k} \frac{x^{4k}}{j^{4k}} + \dots \right) \\ &= [x^{4n}] \prod_{j=1}^{\infty} \left( 1 + \binom{m}{1} \frac{x^4}{j^4} + \dots + \binom{m+k-1}{k} \frac{x^{4k}}{j^{4k}} + \dots \right) \\ & \quad \times \left( 1 - \lambda \left( 1 - \frac{x^4}{j^4} \right)^m + \lambda \right) \\ &= \sum_{l=0}^n [x^{4l}] \prod_{j=1}^{\infty} \left( 1 - \frac{x^4}{j^4} \right)^{-m} [x^{4n-4l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - \frac{x^4}{j^4} \right)^m + \lambda \right). \end{aligned} \quad (44)$$

If we replace  $x$  by  $ix$  in (15), then we have

$$\prod_{j=1}^{\infty} \left( 1 + \frac{x^2}{j^2} \right)^{-m} = \left( \frac{\sinh \pi x}{\pi x} \right)^{-m} = \sum_{n=0}^{\infty} B_n^{(m)} \left( \frac{m}{2} \right) \frac{(2\pi x)^n}{n!}. \quad (45)$$

It follows from (15), (35), and (45) that

$$\begin{aligned} [x^{4l}] \prod_{j=1}^{\infty} \left( 1 - \frac{x^4}{j^4} \right)^{-m} &= \sum_{r=0}^{2l} [x^{4l-2r}] \prod_{j=1}^{\infty} \left( 1 + \frac{x^2}{j^2} \right)^{-m} [x^{2r}] \prod_{j=1}^{\infty} \left( 1 - \frac{x^2}{j^2} \right)^{-m} \\ &= \sum_{r=0}^{2l} \frac{(2\pi)^{4l-2r}}{(4l-2r)!} B_{4l-2r}^{(m)} \left( \frac{m}{2} \right) [x^{2r}] \left( \frac{\sin \pi x}{\pi x} \right)^{-m} \\ &= \frac{2^{4l+1} \pi^{4l}}{(m-1)!} \sum_{r=0}^{2l} \frac{(-1)^r}{(4l-2r)!} B_{4l-2r}^{(m)} \left( \frac{m}{2} \right) \\ & \quad \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r-m, k). \end{aligned} \quad (46)$$

On the contrary, from (36), (37), and the well known formula (see [31], Equation (37)),

$$\zeta(\{4\}^n) = \frac{2^{2n+1} \pi^{4n}}{(4n+2)!}, \quad n \geq 0, \quad (47)$$

we obtain that

$$\begin{aligned}
& [x^{4n-4l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - \frac{x^4}{j^4} \right)^m + \lambda \right) \\
&= [x^{4n-4l}] \prod_{l=1}^m \prod_{j=1}^{\infty} \left( 1 + \mu_l(\lambda) \frac{x^4}{j^4} \right) \\
&= [x^{4n-4l}] \prod_{l=1}^m \sum_{j=0}^{\infty} (\mu_l(\lambda))^j \zeta(\{4\}^j) x^{4j} \\
&= \sum_{\substack{j_1+\dots+j_m=n-l \\ j_1, \dots, j_m \geq 0}} (\mu_1(\lambda))^{j_1} \zeta(\{4\}^{j_1}) \dots (\mu_m(\lambda))^{j_m} \zeta(\{4\}^{j_m}) \\
&= \left( (-1)^{m-1} \lambda \right)^{n-l} 2^{2n-2l+m} \pi^{4n-4l} V_2(m, n-l).
\end{aligned} \tag{48}$$

Inserting (46) and (48) into (44), we have

$$\begin{aligned}
& \sum_{k=1}^n (1+\lambda)^k \sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(4\alpha_1, \dots, 4\alpha_k) \\
&= \frac{2^{2n+m+1} \pi^{4n}}{(m-1)!} \sum_{l=0}^n 4^l \left( (-1)^{m-1} \lambda \right)^{n-l} V_2(m, n-l) \\
&\quad \times \sum_{r=0}^{2l} \frac{(-1)^r}{(4l-2r)!} B_{4l-2r}^{(m)} \left( \frac{m}{2} \right) \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r-m, k).
\end{aligned} \tag{49}$$

Therefore, we get (41) and finish the proof of Theorem 3 when making  $k$ -times derivative with respect to  $\lambda$  and then taking  $\lambda = -1$  on both sides of (49).

It is trivial to check that Genčev's conjecture (11) holds true when taking  $m = 1$  in Theorem 3. We here remark that Theorems 2 and 3, as well as the generalizations of the sum formulas (10) and (11), are the improvements of the results recently obtained by Eie and Ong ([25], Theorems 2.1 and 2.2), where they expressed the left-hand side of (30) by a combinatorial identity involving the higher derivative of one function and the sums of products of  $m$  Bernoulli polynomials, and the left-hand side of (41) by a combinatorial identity involving the higher derivative of another function and the sums of products of  $2m$  Bernoulli polynomials.  $\square$

### 3. Sum Formulas for Multiple Hurwitz Zeta Values

In this section, we shall study the multiple Hurwitz zeta values defined by the series (see [23, 33])

$$\begin{aligned}
& \zeta\left(\alpha_1, \dots, \alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2}\right) \\
&= \sum_{1 \leq n_1 < n_2 < \dots < n_k} \frac{1}{(n_1 - (1/2))^{\alpha_1} (n_2 - (1/2))^{\alpha_2} \dots (n_k - (1/2))^{\alpha_k}},
\end{aligned} \tag{50}$$

where  $\alpha_1, \alpha_2, \dots, \alpha_k$  are all positive integers with  $\alpha_k \geq 2$  and present some weighted sum formulas for them. The results showed here are very analogous to the ones in Section 2.

**Theorem 4.** Let  $m, n$  be positive integers. Then

$$\begin{aligned}
& \sum_{k=1}^n \sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta\left(2\alpha_1, \dots, 2\alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2}\right) \\
&= \frac{(-1)^{m+n-1} 2^{m+2n} \pi^{2n}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \hat{U}(m, 2n, k),
\end{aligned} \tag{51}$$

where  $\hat{U}(m, n, k)$  is the linear combination of the Euler polynomials given for positive integer  $m$  and nonnegative integers  $n, k$  by

$$\hat{U}(m, n, k) = \frac{1}{n!} \sum_{l=0}^n \binom{n}{l} \left( \frac{m-1}{2} \right)^{n-l} E_{l+k} \left( \frac{1}{2} \right). \tag{52}$$

*Proof.* It is well known that Euler's infinite product formula of the cosine function is

$$\cos \pi x = \prod_{j=1}^{\infty} \left( 1 - \frac{4x^2}{(2j-1)^2} \right), \tag{53}$$

which holds true for arbitrary complex number  $x$  (see [26], p. 75 or [28], pp. 12–18). Hence, we obtain from (16) and (53) that for complex number  $x$  with  $|x| < 1/2$ ,

$$\begin{aligned}
& [x^{2n}] (\cos \pi x)^{-m} \\
&= [x^{2n}] \prod_{j=1}^{\infty} \left( 1 - \frac{x^2}{(j - (1/2))^2} \right)^{-m} \\
&= [x^{2n}] \prod_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} (-1)^k \binom{-m}{k} \frac{x^{2k}}{(j - (1/2))^{2k}} \right) \\
&= [x^{2n}] \prod_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^{2k}}{(j - (1/2))^{2k}} \right) \\
&= \sum_{k=1}^n \sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta\left(2\alpha_1, \dots, 2\alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2}\right).
\end{aligned} \tag{54}$$

If we replace  $x$  by  $1/2 - x$  in (22), we get that for real number  $x \neq \pm(1/2), \pm(3/2), \dots$ ,

$$(\cos \pi x)^{-m} = -i^m \frac{2^{m+1} e^{m\pi i((1/2)-x)}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \phi\left(\frac{1}{2} - x, \frac{1}{2}, -k\right), \tag{55}$$



which can be converted by the expression of the higher-order secant function stated in ([24], Theorem 3.2). It then follows that

$$\begin{aligned} [x^{2n}] (\cos \pi x)^{-m} &= \frac{(-1)^{m-1} 2^{m+1}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \\ &\quad \times [x^{2n}] e^{-m\pi i x} \phi\left(\frac{1}{2} - x, \frac{1}{2}, -k\right). \end{aligned} \quad (56)$$

Observe that

$$\begin{aligned} &[x^{2n}] e^{-m\pi i x} \phi\left(\frac{1}{2} - x, \frac{1}{2}, -k\right) \\ &= [x^{2n}] e^{-m\pi i x} \sum_{j=0}^{\infty} (-1)^j \frac{e^{-2\pi i j x}}{(j + (1/2))^{-k}} \\ &= [x^{2n}] \sum_{j=0}^{\infty} (-1)^j \frac{1}{(j + (1/2))^{-k}} \sum_{l=0}^{\infty} \frac{(j + (m/2))^l (-2\pi i)^l x^l}{l!} \\ &= \frac{(2\pi i)^{2n}}{(2n)!} \sum_{j=0}^{\infty} (-1)^j \frac{(j + (m/2))^{2n}}{(j + (1/2))^{-k}} \\ &= \frac{(2\pi i)^{2n}}{(2n)!} \sum_{l=0}^{2n} \binom{2n}{l} \left(\frac{m-1}{2}\right)^{2n-l} \sum_{j=0}^{\infty} (-1)^j \frac{1}{(j + (1/2))^{-(l+k)}}. \end{aligned} \quad (57)$$

Since for nonnegative integer  $n$  (see [34], Corollary 3),

$$\eta(-n, x) = \frac{1}{2} E_n(x), \quad (58)$$

where  $\eta(s, x)$  is the alternating Hurwitz zeta function (also called Hurwitz Euler-eta function) given for real number  $x > 0$  and complex number  $s$  by

$$\eta(s, x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+x)^s}, \quad \Re(s) > 0, \quad (59)$$

so by applying (58) to (57), we find that

$$[x^{2n}] e^{-m\pi i x} \phi\left(\frac{1}{2} - x, \frac{1}{2}, -k\right) = \frac{(2\pi i)^{2n}}{2} \hat{U}(m, 2n, k). \quad (60)$$

Inserting (60) into (56), we have

$$[x^{2n}] (\cos \pi x)^{-m} = \frac{(-1)^{m+n-1} 2^{m+2n} \pi^{2n}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \hat{U}(m, 2n, k). \quad (61)$$

Thus, we complete the proof of Theorem 4 by equating (54) and (61).

In what follows, we denote by  $T(mn, k)$  the sums of all multiple Hurwitz zeta values of the depth  $k$  and the weight  $mn$  given for positive integers  $m, n, k$  with  $m \geq 2$  and  $n \geq k$  by

$$T(mn, k) = \sum_{|\mathbf{a}|=n} \zeta\left(m\alpha_1, \dots, m\alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2}\right). \quad (62)$$

It follows that we state the following result.  $\square$

**Corollary 2.** *Let  $n$  be a positive integer. Then*

$$\sum_{k=1}^n T(2n, k) = \frac{(-1)^n \pi^{2n} E_{2n}}{(2n)!}. \quad (63)$$

*Proof.* By taking  $m = 1$  in Theorem 4, in view of  $s(1, 1) = 1$  and  $E_n = 2^n E_n(1/2)$  for nonnegative integer  $n$ , the desired result follows immediately.

We mention that Corollary 2 was obtained by Shen and Jia ([22], p. 265) in the language of the multiple Hurwitz zeta-star values. We now give another weighted sum formula for the multiple Hurwitz zeta values as follows.  $\square$

**Theorem 5.** *Let  $m, n, k$  be positive integers with  $n \geq k$ . Then*

$$\begin{aligned} &\sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta\left(2\alpha_1, \dots, 2\alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2}\right) \\ &= \frac{(-1)^{m+n-k-1} 2^m \pi^{2n}}{(m-1)!} \sum_{l=0}^{n-k} \binom{n-l}{k} 4^l (-1)^{(m-1)(n-l)} V_3(m, n-l) \\ &\quad \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \hat{U}(m, 2l, k), \end{aligned} \quad (64)$$

where  $V_3(m, n)$  is a rational number given for positive integer  $m$  and nonnegative integer  $n$  by

$$V_3(m, n) = \sum_{\substack{j_1 + \dots + j_m = n \\ j_1, \dots, j_m \geq 0}} \frac{1}{(2j_1)!} \cdots \frac{1}{(2j_m)!}. \quad (65)$$

*Proof.* We obtain from (33) and (54) that

$$\begin{aligned}
 & \sum_{k=1}^n (1+\lambda)^k \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta\left(2\alpha_1, \dots, 2\alpha_k; \frac{1}{2}, \dots, \frac{1}{2}\right) \\
 &= [x^{2n}] \prod_{j=1}^{\infty} \left( 1 + (1+\lambda) \binom{m}{1} \frac{x^2}{(j - (1/2))^2} + \dots + (1+\lambda) \binom{m+k-1}{k} \frac{x^{2k}}{(j - (1/2))^{2k}} + \dots \right) \\
 &= [x^{2n}] \prod_{j=1}^{\infty} \left( 1 + \binom{m}{1} \frac{x^2}{(j - (1/2))^2} + \dots + \binom{m+k-1}{k} \frac{x^{2k}}{(j - (1/2))^{2k}} + \dots \right) \\
 &\quad \times \left( 1 - \lambda \left( 1 - \frac{x^2}{(j - (1/2))^2} \right)^m + \lambda \right) \\
 &= \sum_{l=0}^n [x^{2l}] (\cos \pi x)^{-m} [x^{2n-2l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - \frac{x^2}{(j - (1/2))^2} \right)^m + \lambda \right).
 \end{aligned} \tag{66}$$

Noticing that from (61), we have

$$[x^{2l}] (\cos \pi x)^{-m} = \frac{(-1)^{m+l-1} 2^{m+2l} \pi^{2l}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \bar{B}(m, k) \hat{U}(m, 2l, k), \tag{67}$$

and from (36), (37), and the formula (see [33] or [22], p. 265),

$$\zeta\left(\left\{2; -\frac{1}{2}\right\}^n\right) = \frac{\pi^{2n}}{(2n)!}, \quad n \geq 0, \tag{68}$$

we deduce that

$$\begin{aligned}
 & [x^{2n-2l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - \frac{x^2}{(j - (1/2))^2} \right)^m + \lambda \right) \\
 &= [x^{2n-2l}] \prod_{l=1}^m \prod_{j=1}^{\infty} \left( 1 + \mu_l(\lambda) \frac{x^2}{(j - (1/2))^2} \right) \\
 &= [x^{2n-2l}] \prod_{l=1}^m \sum_{j=0}^{\infty} (\mu_l(\lambda))^j \zeta(\{2; -(1/2)\}^j) x^{2j} \\
 &= \sum_{\substack{j_1 + \dots + j_m = n-l \\ j_1, \dots, j_m \geq 0}} (\mu_1(\lambda))^{j_1} \zeta(\{2; -(1/2)\}^{j_1}) \dots \\
 &\quad \cdot (\mu_m(\lambda))^{j_m} \zeta(\{2; -(1/2)\}^{j_m}) \\
 &= (-1)^{m-1} \lambda^{n-l} \pi^{2n-2l} V_3(m, n-l).
 \end{aligned} \tag{69}$$

Inserting (67) and (69) into (66), we get that

$$\begin{aligned}
 & \sum_{k=1}^n (1+\lambda)^k \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta\left(2\alpha_1, \dots, 2\alpha_k; \frac{1}{2}, \dots, \frac{1}{2}\right) \\
 &= \frac{(-1)^{m-1} 2^m \pi^{2n}}{(m-1)!} \sum_{l=0}^n (-4)^l ((-1)^{m-1} \lambda)^{n-l} V_3(m, n-l) \\
 &\quad \times \sum_{k=0}^{m-1} (-1)^k \bar{B}(m, k) \hat{U}(m, 2l, k).
 \end{aligned} \tag{70}$$

Thus the desired result follows by making  $k$ -times derivative with respect to  $\lambda$  and then taking  $\lambda = -1$  on both sides of (70). This completes the proof of Theorem 5.  $\square$

**Corollary 3.** Let  $n, k$  be positive integers with  $n \geq k$ . Then

$$T(2n, k) = \frac{(-1)^{n-k} \pi^{2n}}{(2n)!} \sum_{l=0}^{n-k} \binom{n-l}{k} \binom{2n}{2l} E_{2l}. \tag{71}$$

*Proof.* Taking  $m = 1$  in Theorem 5, in light of  $s(1, 1) = 1$  and  $E_n = 2^n E_n(1/2)$  for nonnegative integer  $n$ , we get the desired result.

Corollary 3 is a general form of Shen and Cai's [23] results for the cases  $2 \leq k \leq 5$  in  $T(2n, k)$  and was also found by Shen and Jia ([22], p. 265). For an equivalent version of Corollary 3, see ([16], Theorem 1.3) for details. Similarly, we state the following result.  $\square$

**Theorem 6.** Let  $m, n, k$  be positive integers with  $n \geq k$ . Then

$$\begin{aligned} & \sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta\left(4\alpha_1, \dots, 4\alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2}\right) \\ &= \frac{(-1)^{m+n-k-1} 2^{m+2n} \pi^{4n}}{(m-1)!} \sum_{l=0}^{n-k} \binom{n-l}{k} (-4)^l (-1)^{(m-1)(n-l)} V_4(m, n-l) \\ & \quad \times \sum_{r=0}^{2l} \frac{(-1)^r}{(4l-2r)!} E_{4l-2r}^{(m)}\left(\frac{m}{2}\right) \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \hat{U}(m, 2r, k), \end{aligned} \quad (72)$$

where  $V_4(m, n)$  is a rational number given for positive integer  $m$  and nonnegative integer  $n$  by

$$V_4(m, n) = \sum_{\substack{j_1 + \dots + j_m = n \\ j_1, \dots, j_m \geq 0}} \frac{1}{(4j_1)!} \cdots \frac{1}{(4j_m)!}, \quad (73)$$

and  $E_n^{(m)}(x)$  are the higher-order Euler polynomials defined by the generating function (see [35])

$$\left(\frac{2}{e^t + 1}\right)^m e^{xt} = \sum_{n=0}^{\infty} E_n^{(m)}(x) \frac{t^n}{n!}, \quad |t| < \pi. \quad (74)$$

*Proof.* With the help of (16) and (33), we discover that

$$\begin{aligned} & \sum_{k=1}^n (1+\lambda)^k \sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta\left(4\alpha_1, \dots, 4\alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2}\right) \\ &= [x^{4n}] \prod_{j=1}^{\infty} \left(1 + (1+\lambda) \binom{m}{1} \frac{x^4}{(j - (1/2))^4} + \dots + (1+\lambda) \binom{m+k-1}{k} \frac{x^{4k}}{(j - (1/2))^{4k}} + \dots\right) \\ &= [x^{4n}] \prod_{j=1}^{\infty} \left(1 + \binom{m}{1} \frac{x^4}{(j - (1/2))^4} + \dots + \binom{m+k-1}{k} \frac{x^{4k}}{(j - (1/2))^{4k}} + \dots\right) \\ & \quad \times \left(1 - \lambda \left(1 - \frac{x^4}{(j - (1/2))^4}\right)^m + \lambda\right) \\ &= \sum_{l=0}^n [x^{4l}] \prod_{j=1}^{\infty} \left(1 - \frac{x^4}{(j - (1/2))^4}\right)^{-m} \\ & \quad \times [x^{4n-4l}] \prod_{j=1}^{\infty} \left(1 - \lambda \left(1 - \frac{x^4}{(j - (1/2))^4}\right)^m + \lambda\right). \end{aligned} \quad (75)$$

By replacing  $x$  by  $ix$  in (53), we find that

$$\prod_{j=1}^{\infty} \left(1 + \frac{x^2}{(j - (1/2))^2}\right)^{-m} = (\cosh \pi x)^{-m} = \sum_{n=0}^{\infty} E_n^{(m)}\left(\frac{m}{2}\right) \frac{(2\pi x)^n}{n!}. \quad (76)$$

It follows from (53), (67), and (76) that

$$\begin{aligned}
& [x^{4l}] \prod_{j=1}^{\infty} \left( 1 - \frac{x^4}{(j - (1/2))^4} \right)^{-m} \\
&= \sum_{r=0}^{2l} [x^{4l-2r}] \prod_{j=1}^{\infty} \left( 1 + \frac{x^2}{(j - (1/2))^2} \right)^{-m} [x^{2r}] \prod_{j=1}^{\infty} \left( 1 - \frac{x^2}{(j - (1/2))^2} \right)^{-m} \\
&= \sum_{r=0}^{2l} \frac{(2\pi)^{4l-2r}}{(4l-2r)!} E_{4l-2r}^{(m)} \left( \frac{m}{2} \right) [x^{2r}] (\cos \pi x)^{-m} \\
&= \frac{(-1)^{m-1} 2^{m+4l} \pi^{4l}}{(m-1)!} \sum_{r=0}^{2l} \frac{(-1)^r}{(4l-2r)!} E_{4l-2r}^{(m)} \left( \frac{m}{2} \right) \\
&\quad \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \hat{U}(m, 2r, k).
\end{aligned} \tag{77}$$

If we apply (36), (37), and the formula (see [22], p. 265),

$$\zeta \left( \left\{ 4; -\frac{1}{2} \right\}^n \right) = \frac{2^{2n} \pi^{4n}}{(4n)!} \quad n \geq 0, \tag{78}$$

to the second infinite product in the right-hand side of (75), we have

$$\begin{aligned}
& [x^{4n-4l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - \frac{x^4}{(j - (1/2))^4} \right)^m + \lambda \right) \\
&= [x^{4n-4l}] \prod_{l=1}^m \prod_{j=1}^{\infty} \left( 1 + \mu_l(\lambda) \frac{x^4}{(j - (1/2))^4} \right) \\
&= [x^{4n-4l}] \prod_{l=1}^m \sum_{j=0}^{\infty} (\mu_l(\lambda))^j \zeta \left( \left\{ 4; -\frac{1}{2} \right\}^j \right) x^{4j} \\
&= \sum_{\substack{j_1 + \dots + j_m = n-l \\ j_1, \dots, j_m \geq 0}} (\mu_1(\lambda))^{j_1} \zeta \left( \left\{ 4; -\frac{1}{2} \right\}^{j_1} \right) \dots (\mu_m(\lambda))^{j_m} \zeta \left( \left\{ 4; -\frac{1}{2} \right\}^{j_m} \right) \\
&= \left( (-1)^{m-1} \lambda \right)^{n-l} 2^{2n-2l} \pi^{4n-4l} V_4(m, n-l).
\end{aligned} \tag{79}$$

Inserting (77) and (79) into (75), it then follows that

$$\begin{aligned}
& \sum_{k=1}^n (1+\lambda)^k \sum_{|a|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta \left( 4\alpha_1, \dots, 4\alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2} \right) \\
&= \frac{(-1)^{m-1} 2^{m+2n} \pi^{4n}}{(m-1)!} \sum_{l=0}^n 4^l \left( (-1)^{m-1} \lambda \right)^{n-l} V_4(m, n-l) \\
&\quad \times \sum_{r=0}^{2l} \frac{(-1)^r}{(4l-2r)!} E_{4l-2r}^{(m)} \left( \frac{m}{2} \right) \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \hat{U}(m, 2r, k).
\end{aligned} \tag{80}$$

Thus we prove (72) immediately by making  $k$ -times derivative with respect to  $\lambda$  and then taking  $\lambda = -1$  on both sides of (80). This concludes the proof of Theorem 6.

In particular, we discover Shen and Jia's ([22], p. 266) result as follows.  $\square$

**Corollary 4.** Let  $n, k$  be positive integers with  $n \geq k$ . Then

$$T(4n, k) = \frac{(-1)^{n-k} \pi^{4n}}{(4n)!} \sum_{l=0}^{n-k} \binom{n-l}{k} \binom{4n}{4l} (-1)^l 4^{n-l} \\ \times \sum_{r=0}^{2l} \binom{4l}{2r} (-1)^r E_{2r} E_{4l-2r}. \quad (81)$$

*Proof.* Since  $E_n(x) = E_n^{(1)}(x)$  for nonnegative integer  $n$ , so by taking  $m = 1$  in Theorem 6, the desired result follows from  $s(1, 1) = 1$  and  $E_n = 2^n E_n(1/2)$  for nonnegative integer  $n$ .  $\square$

#### 4. Sum Formulas for Alternating Multiple Zeta Values

As shown in [36, 37], the alternating multiple zeta values  $\zeta(\alpha_1, \dots, \alpha_k; \varepsilon_1, \dots, \varepsilon_k)$  of depth  $k$  (also called alternating Euler sums) are defined by the series

$$\zeta(\alpha_1, \dots, \alpha_k; \varepsilon_1, \dots, \varepsilon_k) = \sum_{1 \leq n_1 < n_2 < \dots < n_k} \frac{\varepsilon_1^{n_1} \varepsilon_2^{n_2} \dots \varepsilon_k^{n_k}}{n_1^{\alpha_1} n_2^{\alpha_2} \dots n_k^{\alpha_k}}, \quad (82)$$

where  $\alpha_1, \alpha_2, \dots, \alpha_k$  are all positive integers and  $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \dots, \varepsilon_k = \pm 1$  with  $(\alpha_k, \varepsilon_k) \neq (1, 1)$ . We next use the methods and results described in the second and third sections to obtain some weighted sum formulas for the alternating multiple zeta values.

**Theorem 7.** Let  $m, n$  be positive integers. Then

$$\sum_{k=1}^n \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k; (-1)^{\alpha_1}, \dots, (-1)^{\alpha_k}) \\ = \frac{2\pi^{2n}}{(m-1)!} \sum_{r=0}^n \frac{(-1)^r}{(2n-2r)!} E_{2n-2r}^{(m)} \left( \frac{m}{2} \right) \\ \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r-m, k). \quad (83)$$

*Proof.* By substituting  $x/2$  for  $x$  in (15) and  $xi/2$  for  $x$  in (53), we find that

$$\frac{\sin(\pi x/2)}{(\pi x/2)} \cosh \frac{\pi x}{2} = \prod_{j=1}^{\infty} \left( 1 - \frac{x^2}{(2j)^2} \right) \prod_{j=1}^{\infty} \left( 1 + \frac{x^2}{(2j-1)^2} \right) \\ = \prod_{j=1}^{\infty} \left( 1 - (-1)^j \frac{x^2}{j^2} \right). \quad (84)$$

It follows from (16) and (84) that

$$[x^{2n}] \left( \frac{\sin(\pi x/2)}{(\pi x/2)} \cosh \frac{\pi x}{2} \right)^{-m} \\ = [x^{2n}] \prod_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} \binom{-m}{k} (-1)^j \frac{x^{2k}}{j^{2k}} \right) \\ = [x^{2n}] \prod_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} \binom{m+k-1}{k} (-1)^j \frac{x^{2k}}{j^{2k}} \right) \\ = \sum_{k=1}^n \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k; (-1)^{\alpha_1}, \dots, (-1)^{\alpha_k}). \quad (85)$$

Trivially, the left-hand side of (85) can be written as

$$[x^{2n}] \left( \frac{\sin(\pi x/2)}{(\pi x/2)} \cosh \frac{\pi x}{2} \right)^{-m} = \sum_{r=0}^n [x^{2n-2r}] \left( \cosh \frac{\pi x}{2} \right)^{-m} [x^{2r}] \left( \frac{\sin(\pi x/2)}{(\pi x/2)} \right)^{-m}. \quad (86)$$

It is easy to see that from the generating function of the higher-order Euler polynomials, we have

$$[x^{2n-2r}] \left( \cosh \frac{\pi x}{2} \right)^{-m} = \frac{\pi^{2n-2r}}{(2n-2r)!} E_{2n-2r}^{(m)} \left( \frac{m}{2} \right), \quad (87)$$

and from (22) and (27), we can write

$$\begin{aligned} [x^{2r}] \left( \frac{\sin(\pi x/2)}{(\pi x/2)} \right)^{-m} &= -i^m \frac{2\pi^m}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) \\ &\quad \times [x^{2r-m}] e^{(m\pi i x/2)} \phi\left(\frac{x}{2}, \frac{1}{2}, -k\right) \\ &= \frac{(-1)^r 2\pi^{2r}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r-m, k). \end{aligned} \quad (88)$$

Inserting (87) and (88) into (86), in view of (85), we get (83) immediately and thus finish the proof of Theorem 7.

If we denote by  $\Xi(mn, k)$  the sums of all alternating multiple zeta values of the depth  $k$  and the weight  $mn$  given for positive integers  $m, n, k$  with  $m \geq 2$  and  $n \geq k$  by

$$\Xi(mn, k) = \sum_{|\mathbf{a}|=n} \zeta(m\alpha_1, \dots, m\alpha_k; (-1)^{\alpha_1}, \dots, (-1)^{\alpha_k}), \quad (89)$$

then we obtain Zhao's ([17], Equation (7)) result as follows.  $\square$

**Corollary 5.** *Let  $n$  be a positive integer. Then*

$$\sum_{k=1}^n \Xi(2n, k) = \frac{\pi^{2n}}{4^n (2n)!} \sum_{r=0}^n \binom{2n}{2r} (-1)^r (2-2^{2r}) B_{2r} E_{2n-2r}. \quad (90)$$

*Proof.* By taking  $m = 1$  in Theorem 7 and then using  $s(1, 1) = 1$ , the desired result follows from (29) and  $E_n = 2^n E_n(1/2)$  for nonnegative integer  $n$ .  $\square$

**Theorem 8.** *Let  $m, n, k$  be positive integers with  $n \geq k$ . Then*

$$\begin{aligned} &\sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k; (-1)^{\alpha_1}, \dots, (-1)^{\alpha_k}) \\ &= \frac{(-1)^{n-k} 2\pi^{2n}}{(m-1)!} \sum_{l=0}^{n-k} \binom{n-l}{k} \frac{(-1)^{(m-1)(n-l)+l}}{2^{n-l}} V_5(m, n-l) \\ &\quad \times \sum_{r=0}^l \frac{(-1)^r}{(2l-2r)!} E_{2l-2r}^{(m)} \left( \frac{m}{2} \right) \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r-m, k), \end{aligned} \quad (91)$$

where  $V_5(m, n)$  is a rational number given for positive integer  $m$  and nonnegative integer  $n$  by

$$V_5(m, n) = \sum_{\substack{j_1 + \dots + j_m = n \\ j_1, \dots, j_m \geq 0}} \frac{(-1)^{\lfloor j_1 + 1/2 \rfloor}}{(2j_1 + 1)!} \cdots \frac{(-1)^{\lfloor j_m + 1/2 \rfloor}}{(2j_m + 1)!}, \quad (92)$$

and  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to real number  $x$ .

*Proof.* It is easily seen from (33) and (85) that the following identities are complete:

$$\begin{aligned} & \sum_{k=1}^n (1+\lambda)^k \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k; (-1)^{\alpha_1}, \dots, (-1)^{\alpha_k}) \\ &= [x^{2n}] \prod_{j=1}^{\infty} \left\{ 1 + (1+\lambda) \binom{m}{1} (-1)^j \frac{x^2}{j^2} + \dots + (1+\lambda) \binom{m+k-1}{k} (-1)^k \frac{x^{2k}}{j^{2k}} + \dots \right\} \\ &= [x^{2n}] \prod_{j=1}^{\infty} \left( 1 + \binom{m}{1} (-1)^j \frac{x^2}{j^2} + \dots + \binom{m+k-1}{k} (-1)^k \frac{x^{2k}}{j^{2k}} + \dots \right) \\ & \quad \times \left( 1 - \lambda \left( 1 - (-1)^j \frac{x^2}{j^2} \right)^m + \lambda \right) \\ &= \sum_{l=0}^n [x^{2l}] \left( \frac{\sin(\pi x/2)}{(\pi x/2)} \cosh \frac{\pi x}{2} \right)^{-m} \\ & \quad \times [x^{2n-2l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - (-1)^j \frac{x^2}{j^2} \right)^m + \lambda \right). \end{aligned} \quad (93)$$

Noticing that from (85) and Theorem 7, we have

$$\begin{aligned} & [x^{2l}] \left( \frac{\sin(\pi x/2)}{(\pi x/2)} \cosh \frac{\pi x}{2} \right)^{-m} \\ &= \frac{2\pi^{2l}}{(m-1)!} \sum_{r=0}^l \frac{(-1)^r}{(2l-2r)!} E_{2l-2r}^{(m)} \left( \frac{m}{2} \right) \\ & \quad \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r-m, k), \end{aligned} \quad (94)$$

and from (36), (37), and the formula (see [17], Lemma 2.1),

$$\zeta(\{2; -1\}^n) = \frac{(-1)^{\lfloor n+1/2 \rfloor} \pi^{2n}}{2^n (2n+1)!}, \quad (95)$$

we obtain

$$\begin{aligned} & [x^{2n-2l}] \prod_{j=1}^{\infty} \left( 1 - \lambda \left( 1 - (-1)^j \frac{x^2}{j^2} \right)^m + \lambda \right) \\ &= [x^{2n-2l}] \prod_{l=1}^m \prod_{j=1}^{\infty} \left( 1 + \mu_l(\lambda) (-1)^j \frac{x^2}{j^2} \right) \\ &= [x^{2n-2l}] \prod_{l=1}^m \sum_{j=0}^{\infty} (\mu_l(\lambda))^j \zeta(\{2; -1\}^j) x^{2j} \\ &= \sum_{\substack{j_1 + \dots + j_m = n-l \\ j_1, \dots, j_m \geq 0}} (\mu_1(\lambda))^{j_1} \zeta(\{2; -1\}^{j_1}) \cdots (\mu_m(\lambda))^{j_m} \zeta(\{2; -1\}^{j_m}) \\ &= \frac{((-1)^{m-1} \lambda)^{n-l} \pi^{2n-2l}}{2^{n-l}} V_5(m, n-l). \end{aligned} \quad (96)$$



Inserting (94) and (96) into (93), it then follows that

$$\begin{aligned} & \sum_{k=1}^n (1+\lambda)^k \sum_{|\mathbf{a}|=n} \prod_{j=1}^k \binom{\alpha_j + m - 1}{\alpha_j} \zeta(2\alpha_1, \dots, 2\alpha_k; (-1)^{\alpha_1}, \dots, (-1)^{\alpha_k}) \\ &= \frac{2\pi^{2n}}{(m-1)!} \sum_{l=0}^n \frac{((-1)^{m-1}\lambda)^{n-l}}{2^{n-l}} V_5(m, n-l) \sum_{r=0}^l \frac{(-1)^r}{(2l-2r)!} E_{2l-2r}^{(m)}\left(\frac{m}{2}\right) \\ & \quad \times \sum_{k=0}^{m-1} (-1)^k \tilde{B}(m, k) U(m, 2r-m, k). \end{aligned} \quad (97)$$

Therefore (91) follows immediately when making  $k$ -times derivative with respect to  $\lambda$  and then taking  $\lambda = -1$  on both sides of (97). This concludes the proof of Theorem 8.  $\square$

**Corollary 6.** Let  $n, k$  be positive integers with  $n \geq k$ . Then

$$\begin{aligned} \Xi(2n, k) &= \frac{(-1)^{n-k} \pi^{2n}}{(2n+1)!} \sum_{l=0}^{n-k} \binom{n-l}{k} \binom{2n+1}{2l} \frac{(-1)^{\lfloor n-l+1/2 \rfloor + l}}{2^{n+l}} \\ & \quad \times \sum_{r=0}^l \binom{2l}{2r} (-1)^r (2-2^{2r}) B_{2r} E_{2l-2r}. \end{aligned} \quad (98)$$

*Proof.* By taking  $m = 1$  in Theorem 8, in light of  $s(1, 1) = 1$ , we get the desired result after using (29) and  $E_n = 2^n E_n(1/2)$  for nonnegative integer  $n$ .

It is well known that the values of the Riemann zeta function at even positive integers can be computed by the Bernoulli numbers (see [29], Theorem 12.17):

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}, \quad n \geq 1. \quad (99)$$

Hence, by applying (99) to Corollary 6, one can easily get Zhao's ([17], Theorem 1.3) sum formula:

$$\Xi(2n, k) = \sum_{l=0}^{n-k} \binom{n-l}{k} \frac{(-1)^{\lfloor l-n-1/2 \rfloor + k}}{2^{n+l-1} (2n-2l+1)!} \sum_{r=0}^l \zeta(2l-2r) \frac{\pi^{2n-2l+2r} E_{2r}}{(2r)!}, \quad (100)$$

where  $\zeta(\bar{n}) = (2^{1-n} - 1)\zeta(n)$  for nonnegative integer  $n$ .  $\square$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# One-Kind Hybrid Power Means of the Two-Term Exponential Sums and Quartic Gauss Sums

Xiaoxue Li<sup>1</sup> and Li Chen<sup>2</sup> 

<sup>1</sup>School of Science, Xi'an Aeronautical University, Xi'an, Shaanxi, China

<sup>2</sup>School of Mathematics, Northwest University, Xi'an, Shaanxi, China

Correspondence should be addressed to Li Chen; chenli\_0928@163.com

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The main purpose of this article is using the analytic methods and the properties of the classical Gauss sums to study the calculating problem of the hybrid power mean of the two-term exponential sums and quartic Gauss sums and then prove two interesting linear recurrence formulas. As applications, some asymptotic formulas are obtained.

## 1. Introduction

Let  $q \geq 3$  be a fixed integer. For any integers  $k \geq 2$  and  $m$  with  $(m, q) = 1$ , the two-term exponential sums  $G(m, k; q)$  and quartic Gauss sums  $G(m, q)$  are defined by

$$G(m, k; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + a}{q}\right), \quad (1)$$

$$G(m, q) = \sum_{a=0}^{q-1} e\left(\frac{ma^4}{q}\right),$$

where, as usual,  $e(y) = e^{2\pi i y}$  and  $i^2 = -1$ .

These sums play a significant role in the research of analytic number theory, and many number theory problems are closely related to them. Therefore, for the sake of promoting the development of research work in related fields, it is necessary to study the various properties of  $G(m, k; q)$  and  $G(m, q)$ . Some research results in these fields can be found in references [1–12]. We will not list all of them. For example, Zhang and Zhang [1] proved the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases} \quad (2)$$

where  $p$  be an odd prime and  $n$  denotes any integer with  $(n, p) = 1$ .

Zhang and Han [2] obtained the identity:

$$\sum_{a=1}^{p-1} \left| \sum_{n=0}^{p-1} e\left(\frac{n^3 + an}{p}\right) \right|^6 = 5p^4 - 8p^3 - p^2, \quad (3)$$

where  $p$  denotes an odd prime with  $3 \nmid (p-1)$ .

Zhang and Zhang [3] derived that, for any prime  $p$ , one has the identity:

$$\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 = \begin{cases} p^2(\delta - 3), & \text{if } p \equiv 1 \pmod{6}, \\ p^2(\delta + 3), & \text{if } p \equiv -1 \pmod{6}, \end{cases} \quad (4)$$

where, as usual,  $(\cdot/p) = \chi_2$  denotes the Legendre's symbol modulo  $p$ ,  $d \cdot \bar{d} \equiv 1 \pmod{p}$ , and  $\delta = \sum_{d=1}^{p-1} ((d-1+\bar{d})/p)$  is an integer which satisfies the estimate  $|\delta| \leq 2\sqrt{p}$ .

The author [4] studied the following hybrid power mean:

$$M_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right)^k \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2, \quad (5)$$

and obtained two interesting fourth-order linear recurrence formula as follows:

$$(1) M_k(p) = -2pM_{k-2}(p) + 8p\alpha M_{k-3}(p) - p(9p - 4\alpha^2)M_{k-4}(p), \text{ where the first four item in sequence } \{M_k(p)\} \text{ is } M_0(p) = p(p-3); M_1(p) = 2p\alpha; M_2(p) = -p(p^2 - 3p - 4\alpha^2); \text{ and } M_3(p) = 2p^2\alpha(3p - 14)$$

$$(2) M_k(p) = 6pM_{k-2}(p) + 8p\alpha M_{k-3}(p) - p(p - 4\alpha^2)M_{k-4}(p)$$

The first four items in the sequence  $\{M_k(p)\}$  is  $M_0(p) = p(p-3)$ ;  $M_1(p) = -6p\alpha$ ;  $M_2(p) = p(3p^2 - 17p - 4\alpha^2)$ ; and  $M_3(p) = 6p^2\alpha(p-8)$ .

Inspired by references [1–4], in this paper, we will consider the following  $2k$ -th hybrid power mean:

$$A_k(p) = \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^{2k} \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4, \quad (6)$$

and  $k$ -th hybrid power mean

$$B_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right)^k \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4. \quad (7)$$

Of course, the work in this paper looks a little imaginative with that in [4], but they have different essence, and the main difference lies in the power of the two-term exponential sums. In fact, it is a lot easier that we are dealing with the quadratic power mean of the two-term exponential sum in [4]. In this paper, we are dealing with the fourth power of the two-term exponential sums, and it is very difficult.

In this paper, we will give a second-order linear recurrence formula for  $A_k(p)$  and a fourth-order linear recurrence formula for  $B_k(p)$  by using the properties of Legendre's symbol and the classical Gauss sums. That is, we will prove the following results.

**Theorem 1.** *If  $p > 3$  is an odd prime with  $p \equiv 1 \pmod{4}$ , for any four-order character  $\chi_4 \pmod{p}$ , we have*

$$\sum_{m=1}^{p-1} \chi_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 = \tau^2(\chi_4) + \chi_4(4) \cdot p \cdot \tau^2(\chi_4) \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(u^2 + u + \bar{u} + \bar{u}^2) + 2\chi_4(-36) \cdot p^{5/2}, \quad (8)$$

where  $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e(a/p)$  denotes the classical Gauss sums.

$$A_k(p) = 6p \cdot A_{k-1}(p) - (9p^2 - 4p\alpha^2) \cdot A_{k-2}(p), \quad k \geq 2, \quad (9)$$

**Theorem 2.** *If  $p > 3$  is an odd prime with  $p \equiv 5 \pmod{8}$ , then we have the second-order linear recurrence formula:*

with the initial values

$$A_0(p) = \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases}$$

$$A_1(p) = 3p \cdot A_0(p) - 2\sqrt{p} \cdot \alpha \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4, \quad (10)$$

$$\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 = \begin{cases} p^2(\delta - 3), & \text{if } p \equiv 1 \pmod{6}, \\ p^2(\delta + 3), & \text{if } p \equiv -1 \pmod{6}, \end{cases}$$

where  $(*/p) = \chi_2$  denotes Legendre's symbol modulo  $p$ .  $\alpha = \alpha(p) = \sum_{a=1}^{(p-1)/2} ((a+\bar{a})/p)$  and  $\delta = \delta(p) = \sum_{d=1}^{p-1} ((d-1+\bar{d})/p)$  are two integers, and they satisfy the estimates  $|\alpha| \leq \sqrt{p}$  and  $|\delta| \leq 2\sqrt{p}$ ,  $d \cdot \bar{d} \equiv 1 \pmod{p}$ .

**Theorem 3.** *If  $p > 3$  is an odd prime with  $p \equiv 1 \pmod{8}$ , then we have the fourth-order linear recurrence formula:*

$$B_k(p) = 6p \cdot B_{k-2}(p) + 8p\alpha \cdot B_{k-3}(p) + (4p\alpha^2 - p^2) \cdot B_{k-4}(p), \quad (11)$$

with the initial values

$$B_0(p) = \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases}$$

$$B_1(p) = \sqrt{p} \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + \tau(\bar{\chi}_4) \cdot \sum_{m=1}^{p-1} \chi_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + \tau(\chi_4) \cdot \sum_{m=1}^{p-1} \bar{\chi}_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4;$$

$$B_2(p) = 3p \cdot B_0(p) + 2\sqrt{p}\alpha \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + 2\sqrt{p} \cdot \tau(\bar{\chi}_4) \cdot \sum_{m=1}^{p-1} \bar{\chi}_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + 2\sqrt{p} \cdot \tau(\chi_4) \cdot \sum_{m=1}^{p-1} \chi_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4,$$

$$B_3(p) = \alpha \cdot B_2(p) + 5p \cdot B(p) + 3p\alpha \cdot B_0(p) \\ + 2\sqrt{p} \cdot (p - \alpha^2) \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4. \quad (12)$$

From (4), Theorem 1, and Weil's works [13,14], we have the estimates:

$$\left| \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \right| = O(p^{5/2}), \\ \left| \sum_{m=1}^{p-1} \chi_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \right| = O(p^{5/2}), \\ \left| \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right)^{2k-1} \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \right| = O_k(p^{(2k+3)/2}). \quad (13)$$

Applying (2), Theorems 2 and 3, the properties of the linear recursive sequences, and these three estimates, we can deduce the following three corollaries.

**Corollary 1.** If  $p$  is an odd prime with  $p \equiv 5 \pmod{8}$ , for any positive integer  $k$ , we have the asymptotic formula:

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^{2k} \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4 \\ = p^3 \cdot [(3p + 2\sqrt{p}|\alpha|)^k + (3p - 2\sqrt{p}|\alpha|)^k] \\ + O_k(p^{(2k+5)/2}), \quad (14)$$

where  $O_k$  denotes the big- $O$  constant depending only on the positive integer  $k$ .

Especially for  $k = 2$ , we have the asymptotic formula:

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^4 \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4 = 2p^4 \cdot (9p + 4\alpha^2) + O(p^{9/2}). \quad (15)$$

**Corollary 2.** If  $p$  is an odd prime with  $p \equiv 1 \pmod{8}$ , for any positive integer  $k$ , we have the asymptotic formula:

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^{2k} \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4 \\ = p^3 \cdot \left[ (3p + 2\sqrt{2p^2 + p\alpha^2})^k + (3p - 2\sqrt{2p^2 + p\alpha^2})^k \right] + O_k(p^{(2k+5)/2}), \quad (16)$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^4 \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4 = 2p^4 \cdot (17p + 4\alpha^2) + O(p^{9/2}).$$

**Corollary 3.** *If  $p$  is an odd prime with  $p \equiv 1 \pmod{4}$ , then we have the asymptotic formula:*

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^2 \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4 = 6p^4 + O(p^{7/2}). \quad (17)$$

## 2. Several Lemmas

In this section, we will give four basic lemmas that they are all necessary in the proofs of the theorems. Certainly, the proofs of these lemmas need some theoretical knowledge of elementary and analytic number theory. They can be found in references [15–17]. Firstly, we have the following:

**Lemma 1.** *If  $p > 3$  is an odd prime with  $p \equiv 1 \pmod{4}$ , for any four-order character  $\chi_4 \pmod{p}$ , we have*

$$\sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 = \chi_4(-36) \cdot p^{3/2}. \quad (18)$$

*Proof.* Let  $\chi_2$  denotes the Legendre's symbol modulo  $p$ . Then, for any integer  $b$ , from the properties of the Legendre's symbol modulo  $p$ , we have

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = \begin{cases} \chi_2(b) \cdot \tau(\chi_2), & \text{if } p \nmid b, \\ p, & \text{if } p \mid b. \end{cases} \quad (19)$$

Note that  $\chi_4^3 = \bar{\chi}_4$  and  $\tau(\chi_2) = \sqrt{p}$ , and using the definition and properties of Gauss sums, reduced residue system modulo  $p$ , and formula (19), we have

$$\begin{aligned} & \sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \\ &= \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=1}^{p-1} \chi_4(m) e\left(\frac{m(a^3 + b^3) + a + b}{p}\right) \\ &= \tau(\chi_4) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\chi}_4(a^3 + b^3) e\left(\frac{a+b}{p}\right) \\ &= \tau(\chi_4) \sum_{a=0}^{p-1} \bar{\chi}_4(a^3) e\left(\frac{a}{p}\right) + \tau(\chi_4) \sum_{a=0}^{p-1} \sum_{b=1}^{p-1} \bar{\chi}_4(a^3 + b^3) e\left(\frac{a+b}{p}\right) \\ &= \tau^2(\chi_4) + \tau^2(\chi_4) \sum_{a=0}^{p-1} \bar{\chi}_4(a^3 + 1) \bar{\chi}_4(a + 1) \\ &= \tau^2(\chi_4) + \tau^2(\chi_4) \sum_{b=0}^{p-1} \bar{\chi}_4(b^3 - 3b^2 + 3b) \bar{\chi}_4(b) \\ &= \tau^2(\chi_4) + \tau^2(\chi_4) \sum_{b=1}^{p-1} \bar{\chi}_4(1 - 3\bar{b} + 3\bar{b}^2) \\ &= \tau^2(\chi_4) \sum_{b=0}^{p-1} \bar{\chi}_4(3b^2 - 3b + 1) \\ &= \chi_2(2) \tau^2(\chi_4) \sum_{b=0}^{p-1} \bar{\chi}_4(3(2b-1)^2 + 1) \\ &= \bar{\chi}_4(12) \tau^2(\chi_4) \sum_{b=0}^{p-1} \bar{\chi}_4(b^2 + 3). \end{aligned} \quad (20)$$

Note that  $\tau(\chi_4)\tau(\bar{\chi}_4) = \chi_4(-1) \cdot p$ , and from the properties of Gauss sums, we have

$$\begin{aligned} \sum_{a=0}^{p-1} \chi_4(a^2 + r) &= \frac{1}{\tau(\bar{\chi}_4)} \sum_{b=1}^{p-1} \bar{\chi}_4(b) \sum_{a=0}^{p-1} e\left(\frac{b(a^2 + r)}{p}\right) \\ &= \begin{cases} \frac{\bar{\chi}_4(-r) \cdot \tau^2(\chi_4)}{\sqrt{p}}, & \text{if } p \nmid r, \\ 0, & \text{if } p \mid r. \end{cases} \end{aligned} \quad (21)$$

From (20) and (21), we have

$$\sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 = \chi_4(-36) \cdot p^{3/2}. \quad (22)$$

This proves Lemma 1.  $\square$

**Lemma 2.** If  $p > 3$  is an odd prime with  $p \equiv 1 \pmod{4}$ , for any four-order character  $\chi_4 \pmod{p}$ , we have

$$\begin{aligned} &\sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \left( \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \right) \\ &= (p+1) \cdot \tau^2(\chi_4) + \chi_4(-36) \cdot p^{3/2} \\ &\quad \cdot \sum_{a=0}^{p-1} \bar{\chi}_4(a+2) \chi_4(a^3 + 2). \end{aligned} \quad (23)$$

*Proof.* From Lemma 1, we have

$$\begin{aligned} &\sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \left( \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \right) \\ &= \sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \left( \sum_{b=1}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \right) \\ &\quad + \sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \\ &= \tau^2(\chi_4) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\chi}_4(a^3 + b^3 - 1) \bar{\chi}_4(a + b - 1) \\ &\quad + \chi_4(-36) \cdot p^{3/2}. \end{aligned} \quad (24)$$

Let  $c = b - 1$ , then from the properties of the complete residue system modulo  $p$ , we have

$$\begin{aligned} &\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\chi}_4(a^3 + b^3 - 1) \bar{\chi}_4(a + b - 1) \\ &= \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(a^3 + (c+1)^3 - 1) \bar{\chi}_4(a + c) \\ &= \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(a^3 + c^3 + 3c^2 + 3c) \bar{\chi}_4(a + c) \\ &= p + \sum_{a=0}^{p-1} \sum_{c=1}^{p-1} \bar{\chi}_4(a^3 c^3 + c^3 + 3c^2 + 3c) \bar{\chi}_4(ac + c) \\ &= p + \sum_{a=0}^{p-1} \sum_{c=1}^{p-1} \bar{\chi}_4(a^3 + 1 + 3c + 3c^2) \bar{\chi}_4(a + 1) \\ &= p + \chi_4(4) \sum_{a=0}^{p-1} \sum_{c=1}^{p-1} \bar{\chi}_4(4a^3 + 1 + 3(2c+1)^2) \bar{\chi}_4(a + 1) \\ &= p + \chi_4(4) \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(4a^3 + 1 + 3c^2) \bar{\chi}_4(a + 1) \\ &\quad - \sum_{a=0}^{p-1} \bar{\chi}_4(a^3 + 1) \bar{\chi}_4(a + 1). \end{aligned} \quad (25)$$

From (19) and the methods of proving (20), we have



$$\begin{aligned}
\sum_{a=0}^{p-1} \bar{\chi}_4(a^3 + 1) \bar{\chi}_4(a + 1) &= \sum_{b=1}^{p-1} \bar{\chi}_4(3b^2 - 3b + 1) \\
&= \chi_4(4) \sum_{c=0}^{p-1} \bar{\chi}_4(3c^2 + 1) - 1 = \frac{\chi_4(4)}{\tau(\chi_4)} \sum_{b=1}^{p-1} \chi_4(b) \sum_{c=0}^{p-1} e\left(\frac{b(3c^2 + 1)}{p}\right) - 1 \\
&= \frac{\chi_4(-36) \cdot \tau^2(\bar{\chi}_4)}{\sqrt{p}} - 1.
\end{aligned} \tag{26}$$

From (19), we have

$$\begin{aligned}
&\sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(4a^3 + 1 + 3c^2) \bar{\chi}_4(a + 1) \\
&= \frac{1}{\tau(\chi_4)} \sum_{a=0}^{p-1} \bar{\chi}_4(a + 1) \sum_{b=1}^{p-1} \chi_4(b) \sum_{c=0}^{p-1} e\left(\frac{b(4a^3 + 1 + 3c^2)}{p}\right) \\
&= \frac{\chi_2(3) \cdot \sqrt{p} \cdot \tau(\bar{\chi}_4)}{\tau(\chi_4)} \sum_{a=0}^{p-1} \bar{\chi}_4(a + 1) \chi_4(4a^3 + 1) \\
&= \frac{\chi_4(-9) \cdot \tau^2(\bar{\chi}_4)}{\sqrt{p}} \cdot \sum_{a=0}^{p-1} \bar{\chi}_4(a + 2) \chi_4(a^3 + 2).
\end{aligned} \tag{27}$$

Combining (24)–(27), we have the identity

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \left( \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \right) \\
&= (p + 1) \cdot \tau^2(\chi_4) + \chi_4(-36) \cdot p^{3/2} \\
&\quad \cdot \sum_{a=0}^{p-1} \bar{\chi}_4(a + 2) \chi_4(a^3 + 2).
\end{aligned} \tag{28}$$

This proves Lemma 2.  $\square$

**Lemma 3.** If  $p > 3$  is an odd prime with  $p \equiv 1 \pmod{4}$ , then we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \sum_{c=1}^{p-1} e\left(\frac{-mc^3 - c}{p}\right) \\
&= \chi_4(4) \cdot p \cdot \tau^2(\chi_4) \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(u^2 + u + \bar{u} + \bar{u}^2) + 2\chi_4(-36) \cdot p^{5/2} \\
&\quad - p \cdot \tau^2(\chi_4) - \chi_4(-36) \cdot p^{3/2} \cdot \sum_{a=0}^{p-1} \bar{\chi}_4(a + 2) \chi_4(a^3 + 2).
\end{aligned} \tag{29}$$

*Proof.* From the properties of the classical Gauss sums and the methods of proving Lemmas 1 and 2, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \sum_{c=1}^{p-1} e\left(\frac{-mc^3 - c}{p}\right) \\
&= \tau^2(\chi_4) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(a^3 + b^3 - c^3 - 1) \bar{\chi}_4(a + b - c - 1).
\end{aligned} \tag{30}$$

Let  $u = a - 1$  and  $v = b - c$ , then from the properties of the complete residue system modulo  $p$ , we have

$$\begin{aligned}
& \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(a^3 + b^3 - c^3 - 1) \bar{\chi}_4(a + b - c - 1) \\
&= \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4((u+1)^3 + (v+c)^3 - c^3 - 1) \bar{\chi}_4(u+v) \\
&= \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(u^3 + 3u^2 + 3u + v^3 + 3v^2c + 3vc^2) \bar{\chi}_4(u+v) \\
&= \sum_{u=0}^{p-1} \sum_{v=1}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(u^3 + 3u^2v + 3uv^2 + 1 + 3c + 3c^2) \bar{\chi}_4(u+1) + p \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(3u^2 + 3u + 1) \\
&= \chi_4(4) \sum_{u=0}^{p-1} \sum_{v=1}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(u^3 + 3u(2v+u)^2 + 1 + 3(2c+1)^2) \bar{\chi}_4(u+1) \\
&\quad + \chi_4(4) \cdot p \cdot \sum_{u=0}^{p-1} \bar{\chi}_4(3(2u+1)^2 + 1) - p \\
&= \chi_4(4) \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(u^3 + 3uv^2 + 1 + 3c^2) \bar{\chi}_4(u+1) \\
&\quad - \chi_4(4) \sum_{u=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(4u^3 + 1 + 3c^2) \bar{\chi}_4(u+1) + \chi_4(4) \cdot p \cdot \sum_{u=0}^{p-1} \bar{\chi}_4(3u^2 + 1) - p.
\end{aligned} \tag{31}$$

From (19), we have

$$\begin{aligned}
\sum_{u=0}^{p-1} \bar{\chi}_4(3u^2 + 1) &= \frac{1}{\tau(\chi_4)} \cdot \sum_{b=1}^{p-1} \chi_4(b) \sum_{u=0}^{p-1} e\left(\frac{b(3u^2 + 1)}{p}\right) \\
&= \frac{\chi_2(3) \cdot \sqrt{p}}{\tau(\chi_4)} \cdot \sum_{b=1}^{p-1} \bar{\chi}_4(b) e\left(\frac{b}{p}\right) = \frac{\chi_4(-9) \cdot \tau^2(\bar{\chi}_4)}{\sqrt{p}}.
\end{aligned} \tag{32}$$

From (19), we also have

$$\begin{aligned}
& \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \sum_{c=0}^{p-1} \bar{\chi}_4(u^3 + 3uv^2 + 1 + 3c^2) \bar{\chi}_4(u+1) \\
&= \frac{1}{\tau(\chi_4)} \sum_{u=0}^{p-1} \bar{\chi}_4(u+1) \sum_{b=1}^{p-1} \chi_4(b) \sum_{v=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{b(u^3 + 3uv^2 + 1 + 3c^2)}{p}\right) \\
&= \frac{p}{\tau(\chi_4)} \sum_{u=1}^{p-1} \bar{\chi}_4(u^3 + u^2) \sum_{b=1}^{p-1} \chi_4(b) e\left(\frac{b(u^3 + 1)}{p}\right) + \chi_4(-9) \cdot \sqrt{p} \cdot \tau^2(\bar{\chi}_4) \\
&= p \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(u^3 + u^2) \bar{\chi}_4(u^3 + 1) + \chi_4(-9) \cdot \sqrt{p} \cdot \tau^2(\bar{\chi}_4) \\
&= p \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(u^2 + u + \bar{u} + \bar{u}^2) + \chi_4(-9) \cdot \sqrt{p} \cdot \tau^2(\bar{\chi}_4).
\end{aligned} \tag{33}$$

Now, combining (27)–(33), we have the identity:

$$\begin{aligned}
& \sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \sum_{c=1}^{p-1} e\left(\frac{-mc^3 - c}{p}\right) \\
&= \chi_4(4) \cdot p \cdot \tau^2(\chi_4) \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(u^2 + u + \bar{u} + \bar{u}^2) + 2\chi_4(-36) \cdot p^{5/2} \\
&\quad - p \cdot \tau^2(\chi_4) - \chi_4(-36) \cdot p^{3/2} \cdot \sum_{a=0}^{p-1} \bar{\chi}_4(a+2) \chi_4(a^3 + 2).
\end{aligned} \tag{34}$$

This proves Lemma 3.  $\square$

*Proof.* See Lemma 2.2 in Han [5].  $\square$

**Lemma 4.** Let  $p > 3$  is an odd prime with  $p \equiv 1 \pmod{4}$ , then for any four-order character  $\chi_4 \pmod{p}$ , we have the identity:

$$\tau^2(\chi_4) + \tau^2(\bar{\chi}_4) = 2\sqrt{p} \cdot \alpha, \tag{35}$$

where  $\alpha = \alpha(p) = \sum_{a=1}^{(p-1)/2} ((a + \bar{a})/p)$  is an integer.

### 3. Proofs of the Theorems

In this section, we will complete the proofs of our theorems. In fact, for any prime  $p$  with  $p \equiv 1 \pmod{4}$ , from Lemmas 2 and 3, we have

$$\begin{aligned}
& \sum_{m=1}^{p-1} \chi_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\
&= \sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \left( \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \right) \\
&\quad + \sum_{m=1}^{p-1} \chi_4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 \sum_{b=0}^{p-1} e\left(\frac{-mb^3 - b}{p}\right) \sum_{c=1}^{p-1} e\left(\frac{-mc^3 - c}{p}\right) \\
&= (p+1) \cdot \tau^2(\chi_4) + \chi_4(-36) \cdot p^{3/2} \cdot \sum_{a=0}^{p-1} \bar{\chi}_4(a+2) \chi_4(a^3 + 2) \\
&\quad + \chi_4(4) \cdot p \cdot \tau^2(\chi_4) \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(u^2 + u + \bar{u} + \bar{u}^2) + 2\chi_4(-36) \cdot p^{5/2} \\
&\quad - p \cdot \tau^2(\chi_4) - \chi_4(-36) \cdot p^{3/2} \cdot \sum_{a=0}^{p-1} \bar{\chi}_4(a+2) \chi_4(a^3 + 2) \\
&= \tau^2(\chi_4) + \chi_4(4) \cdot p \cdot \tau^2(\chi_4) \cdot \sum_{u=1}^{p-1} \bar{\chi}_4(u^2 + u + \bar{u} + \bar{u}^2) + 2\chi_4(-36) \cdot p^{5/2}.
\end{aligned} \tag{36}$$

This proves Theorem 1.

Now, we prove Theorem 2. If  $p \equiv 5 \pmod{8}$ , then for any four-order character  $\chi_4 \pmod{p}$ , we have  $\chi_4(-1) = -1$ . From

the properties of the Gauss sums and the four-order character modulo  $p$ , we have

$$\begin{aligned} G(m, p) &= \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1} (1 + \chi_4(a) + \chi_2(a) + \bar{\chi}_4(a)) \cdot e\left(\frac{ma}{p}\right) \\ &= \bar{\chi}_4(m) \cdot \tau(\chi_4) + \chi_2(m) \cdot \tau(\chi_2) + \chi_4(m) \cdot \tau(\bar{\chi}_4) \\ &= \bar{\chi}_4(m) \cdot \tau(\chi_4) + \chi_2(m) \cdot \sqrt{p} + \chi_4(m) \cdot \tau(\bar{\chi}_4), \end{aligned} \quad (37)$$

$$\overline{G(m, p)} = -\chi_4(m) \cdot \tau(\bar{\chi}_4) - \bar{\chi}_4(m) \cdot \tau(\chi_4) + \chi_2(m) \cdot \sqrt{p}. \quad (38)$$

So, from (37), (38), and Lemma 4, we have

So, for any integer  $k \geq 2$ , from (2), (4), (39), and (40), we have

$$\begin{aligned} |G(m, p)|^2 &= p + |\chi_4(m) \cdot \tau(\bar{\chi}_4) + \bar{\chi}_4(m) \cdot \tau(\chi_4)|^2 \\ &= 3p - \chi_2(m)(\tau^2(\chi_4) + \tau^2(\bar{\chi}_4)) \\ &= 3p - 2\chi_2(m)\sqrt{p}\alpha, \end{aligned} \quad (39)$$

$$|G(m, p)|^4 = 6p|G(m, p)|^2 + 4p\alpha^2 - 9p^2. \quad (40)$$

$$\begin{aligned} A_k(p) &= \sum_{m=1}^{p-1} |G(m, p)|^{2k-4} \cdot |G(m, p)|^4 \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4 \\ &= \sum_{m=1}^{p-1} |G(m, p)|^{2k-4} \cdot (6p|G(m, p)|^2 + 4p\alpha^2 - 9p^2) \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4 \\ &= 6p \cdot A_{k-1}(p) - (9p^2 - 4p\alpha^2) \cdot A_{k-2}(p), \end{aligned} \quad (41)$$

with the initial values

$$\begin{aligned} A_0(p) &= \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases} \\ A_1(p) &= 3p \cdot A_0(p) - 2\sqrt{p} \cdot \alpha \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4, \\ \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 &= \begin{cases} p^2(\delta - 3), & \text{if } p \equiv 1 \pmod{6}, \\ p^2(\delta + 3), & \text{if } p \equiv -1 \pmod{6}, \end{cases} \end{aligned} \quad (42)$$

where  $\delta$  is the same as the definition in (4). This proves Theorem 2.

Now, we prove Theorem 3. If  $p \equiv 1 \pmod{8}$ , then note that  $\chi_4(-1) = -1$ , so from (37) and Lemma 4, we have

$$G^2(m, p) = 3p + \chi_2(m) \cdot 2\sqrt{p}\alpha + 2\sqrt{p}(\bar{\chi}_4(m)\tau(\bar{\chi}_4) + \chi_4(m) \cdot \tau(\chi_4)), \quad (43)$$

$$G^3(m, p) = \alpha \cdot G^2(m, p) + 5p \cdot G(m, p) + \chi_2(m) \cdot 2\sqrt{p} \cdot (p - \alpha^2) + 3p\alpha, \quad (44)$$

$$G^4(m, p) = 6p \cdot G^2(m, p) + 8p\alpha \cdot G(m, p) + (4p\alpha^2 - p^2). \quad (45)$$

So, for any integer  $k \geq 4$ , from (43) and (44), we have

$$B_k(p) = 6p \cdot B_{k-2}(p) + 8p\alpha \cdot B_{k-3}(p) + (4p\alpha^2 - p^2) \cdot B_{k-4}(p), \quad (46)$$

with the initial values

$$B_0(p) = \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases}$$

$$B_1(p) = \sqrt{p} \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + \tau(\bar{\chi}_4) \cdot \sum_{m=1}^{p-1} \chi_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + \tau(\chi_4) \cdot \sum_{m=1}^{p-1} \bar{\chi}_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4,$$

$$B_2(p) = 3p \cdot B_0(p) + 2\sqrt{p}\alpha \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + 2\sqrt{p} \cdot \tau(\bar{\chi}_4) \cdot \sum_{m=1}^{p-1} \bar{\chi}_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \\ + 2\sqrt{p} \cdot \tau(\chi_4) \cdot \sum_{m=1}^{p-1} \chi_4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4,$$

$$B_3(p) = \alpha \cdot B_2(p) + 5p \cdot B_1(p) + 3p\alpha \cdot B_0(p)$$

$$+ 2\sqrt{p} \cdot (p - \alpha^2) \cdot \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4. \quad (47)$$

This proves Theorem 3.

Now, we will give a simple proof for Corollary 2. Note the estimate

$$\left| \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right)^{2k-1} \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 \right| = O_k(p^{(2k+3)/2}), \quad (48)$$

and the identity

$$B_{2k}(p) = 6p \cdot B_{2k-2}(p) + 8p\alpha \cdot B_{2k-3}(p) \\ + (4p\alpha^2 - p^2) \cdot B_{2k-4}(p). \quad (49)$$

From (48) and (49), we have

$$B_{2k}(p) = 6p \cdot B_{2(k-1)}(p) + (4p\alpha^2 - p^2) \cdot B_{2(k-2)}(p) \\ + O_k(p^{k+2}). \quad (50)$$

Note that the linear recurrence sequence  $U_n = 6p \cdot U_{n-1} + (4p\alpha^2 - p^2) \cdot U_{n-2}$  has the general terms:

$$U_n = C_1 \cdot \left( 3p + 2\sqrt{2p^2 + p\alpha^2} \right)^n \\ + C_2 \cdot \left( 3p - 2\sqrt{2p^2 + p\alpha^2} \right)^n, \quad n \geq 0. \quad (51)$$

If  $U_0 = 2p^3$  and  $U_1 = 6p^4$ , then we have  $C_1 = C_2 = p^3$  and

$$U_n = p^3 \cdot \left[ \left( 3p + 2\sqrt{2p^2 + p\alpha^2} \right)^n \right. \\ \left. + \left( 3p - 2\sqrt{2p^2 + p\alpha^2} \right)^n \right], \quad n \geq 0. \quad (52)$$

Combining (50) and (52), we have the asymptotic formula:

$$B_{2k}(p) = p^3 \cdot \left[ \left( 3p + 2\sqrt{2p^2 + p\alpha^2} \right)^k \right. \\ \left. + \left( 3p - 2\sqrt{2p^2 + p\alpha^2} \right)^k \right] + O_k(p^{(2k+5)/2}). \quad (53)$$

This completes the proofs of our all results.

## 4. Conclusion

The main results of this paper are two linear recurrence formulas for the hybrid power means of the two-term exponential sums  $G(m, k; q)$  and the quartic Gauss sums  $G(m, q)$ . As some applications of these results, we obtained a sharp asymptotic formula for the hybrid power means:

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^{2k} \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^3 + b}{p}\right) \right|^4, \quad (54)$$

for all positive integer  $k$  and odd prime  $p$  with  $p \equiv 1 \pmod{4}$ . All these results are new contributions to the related fields.

## Data Availability

Our field is theoretical mathematics, and we do not use any data.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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## Research Article

# Autocorrelation and Linear Complexity of Binary Generalized Cyclotomic Sequences with Period $pq$

Yan Wang, Liantao Yan , Qing Tian, and Liping Ding

School of Science, Xi'an University of Architecture and Technology, Xi'an 710055, China

Correspondence should be addressed to Liantao Yan; [yanliantao@xauat.edu.cn](mailto:yanliantao@xauat.edu.cn)

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Ding constructed a new cyclotomic class  $(V_0, V_1)$ . Based on it, a construction of generalized cyclotomic binary sequences with period  $pq$  is described, and their autocorrelation value, linear complexity, and minimal polynomial are confirmed. The autocorrelation function  $C_S(w)$  is 3-level if  $p \equiv 3 \pmod{4}$ , and  $C_S(w)$  is 5-level if  $p \equiv 1 \pmod{4}$ . The linear complexity  $LC(S) > (pq/2)$  if  $p \equiv 1 \pmod{8}$ ,  $p > q + 1$ , or  $p \equiv 3 \pmod{4}$  or  $p \equiv -3 \pmod{8}$ . The results show that these sequences have quite good cryptographic properties in the aspect of autocorrelation and linear complexity.

## 1. Introduction

Pseudorandom sequences with good cryptography properties have wide applications in CDMA, global positioning systems, and stream ciphers. The security of stream ciphers depends on the randomness of the key stream, which makes the construction of pseudorandom sequences to be an important research direction. Many researchers focused on cyclotomic sequences, which have good balance property. Linear complexity and autocorrelation are important criteria for measuring unpredictability of cyclotomic sequences.

Let  $F_l$  denote a finite field with  $l$  elements, where  $l$  is a prime power. A sequence  $S = \{s_i\}$  is periodic if there exists a positive integer  $N$  such that  $s_{j+N} = s_j$  for all  $j \geq 0$ .

Let  $S = \{s_i\}$  be a periodic sequence over  $F_l$  with period  $N$ . The periodic autocorrelation function of binary sequence  $S$  is defined by

$$C_S(w) = \sum_{i=0}^{N-1} (-1)^{s_{i+w} + s_i}, \quad (1)$$

where  $0 \leq w \leq N - 1$ .

Autocorrelation function measures the amount of similarity between sequence  $S$  and a shift of  $S$  by  $w$  shifts. Only when the values of  $C_S(w)$  distribute flat and low,

sequence  $S$  is easy to distinguish from each time shifted version of itself. The autocorrelation function with the ideal distribution of values is two-valued, which is given as

$$C_S(w) = \begin{cases} N & \text{if } w = 0, \\ 1, & \text{otherwise.} \end{cases} \quad (2)$$

Sequences with ideal autocorrelation functions have many applications in cryptography, coding, and other communication engineering.

Linear complexity of  $S$ , denoted by  $LC(S)$ , is the least integer  $L$  of a linear recurrence relation over  $F_l$  satisfied by  $S$ :

$$-c_0 s_{i+L} = c_1 s_{i+L-1} + \cdots + c_L s_i, \quad i \geq 0, \quad (3)$$

where  $c_0, c_1, \dots, c_L \in F_l$ . The linear complexity of a sequence is also defined to be the length of the shortest linear feedback shift register which can generate the sequence. It is an important criterion of randomness of sequences in stream ciphers. To resist the attack from Berlekamp-Massey algorithm, the sequences used in cipher systems should have large linear complexity. If  $LC(S) \geq (N/2)$ , where  $N$  is the least period of  $S$ , then  $S$  is considered to be good from the viewpoint of linear complexity.

The minimal polynomial  $m(x)$  of  $S$  is

$$m(x) = \frac{x^N - 1}{\gcd(x^N - 1, S(x))}, \quad (4)$$

and the linear complexity  $LC(S)$  of  $S$  is given by  $N - \deg(\gcd(x^N - 1, S(x)))$ , where  $S(x)$  is the generating polynomial of  $S$ , that is,

$$S(x) = s_0 + s_1x + s_2x^2 + \cdots + s_{N-1}x^{N-1}. \quad (5)$$

Sequences from cyclotomic and generalized cyclotomic are important families of pseudorandom sequences.

For an integer  $N \geq 2$ , let  $Z_N = \{0, 1, \dots, N-1\}$  denote the residue class ring of integers modulo  $N$  and  $Z_N^*$  be the multiplicative group consisting of all invertible elements in  $Z_N$ . A partition  $\{D_0^{(N)}, D_1^{(N)}, \dots, D_{d-1}^{(N)}\}$  of  $Z_N^*$  is a family of sets satisfying  $D_i^{(N)} \cap D_j^{(N)} = \emptyset$  for all  $i \neq j$  and  $Z_N^* = \bigcup_{i=0}^{d-1} D_i^{(N)}$ . Suppose  $D_0^{(N)}$  is a multiplicative subgroup of  $Z_N^*$  and there exist elements  $g \in Z_N^*$  such that  $D_i^{(N)} = g^i D_0^{(N)}$  for all  $i = 1, \dots, d-1$ ; then,  $D_i^{(N)}$  are called classical cyclotomic classes of order  $d$  with respect to  $N$  when  $N$  is prime and generalized cyclotomic classes of order  $d$  with respect to  $N$  when  $N$  is composite. The sequences constructed by them are called classical cyclotomic sequences and generalized cyclotomic sequences, respectively. Gauss [1] first proposed the concept of cyclotomic, divided the multiplicative group  $Z_p^*$ , and then divided the residual class ring  $Z_p$  to construct Gauss classical cyclotomic. Whiteman [2] divided the multiplicative group  $Z_{pq}^*$  and then divided the residual class ring  $Z_{pq}$  to construct Whiteman generalized cyclotomic. Ding and Hellesteth [3] divided the multiplicative group  $Z_{p_1^{e_1} p_2^{e_2} \dots p_t^{e_t}}^*$  and then divided the residual class ring  $Z_{p_1^{e_1} p_2^{e_2} \dots p_t^{e_t}}$  to construct the Ding generalized cyclotomic. The above three kinds of cyclotomic theories are the most representative and the most widely used cyclotomic theories.

Classical cyclotomic sequences include Legendre sequences,  $d$ -degree residual sequences, and Hall sextic residue sequences. Damgaard [4] determined the autocorrelation value of Legendre sequences, and then Ding et al. [5] determined their linear complexity. Kim and Song [6] determined the linear complexity of Hall sextic residue sequences.

Ding [7] constructed Whiteman generalized cyclotomic sequences of order 2 and confirmed their linear complexity. And Ding [8] determined the autocorrelation value of Whiteman generalized cyclotomic sequences of order 2. Bai [9] constructed Whiteman generalized cyclotomic sequences of order 4 and determined their linear complexity. Yan et al. [10] extended Whiteman generalized cyclotomic sequences to the case of order  $2^k$ .

Bai [9] determined the autocorrelation value of Ding generalized cyclotomic sequences with period  $pq$  of order 2. And Bai et al. [11] confirmed they had high linear complexity. Yan et al. [12] constructed Ding generalized cyclotomic sequences with period  $p^m$  and confirmed they had high linear complexity. Edemskiy [13] constructed a kind of balanced binary generalized cyclotomic sequences with

period  $p^{n+1}$ . Zhang et al. [14] determined the linear complexity of generalized cyclotomic sequences with period  $2p^m$ . Hu et al. [15] constructed generalized cyclotomic sequences with period  $p^{m+1}q^{n+1}$  and determined their linear complexity. Ke et al. [16] determined the linear complexity and the autocorrelation value of Ding generalized cyclotomic sequences with period  $2p^m$ . Chang et al. [17] constructed binary generalized cyclotomic sequences with period  $pqr$  and determined their linear complexity and minimal polynomial.

Ding [18] constructed a new cyclotomic class  $(V_0, V_1)$  and obtained a kind of cyclic code from it. Liu and Chen [19] determined binary generalized cyclotomic sequences with period  $pq$  based on the new cyclotomic class  $(V_0, V_1)$  and determined their autocorrelation value, linear complexity, and minimal polynomial.

In this paper, based on Ding's new cyclotomic class  $(V_0, V_1)$ , a simple construction of binary generalized cyclotomic sequences with period  $pq$  is constructed, and their autocorrelation value, linear complexity, and minimal polynomial are confirmed. The remainder of this paper is organized as follows. Section 2 proposes a construction of generalized cyclotomic binary sequences. Section 3 calculates the autocorrelation value, linear complexity, and minimal polynomial of the new sequences and compares our results with [19]. Section 4 concludes this paper.

## 2. Preliminaries

**Lemma 1** (see [20]). *Let  $m_1 > 0, m_2 > 0, a_1$ , and  $a_2$  be integers. The system of congruences,*

$$\begin{cases} x \equiv a_1 \pmod{m_1}, \\ x \equiv a_2 \pmod{m_2}, \end{cases} \quad (6)$$

*has solutions if and only if  $\gcd(m_1, m_2) | a_1 - a_2$ .*

*If the above condition is satisfied, the solution is unique modulo  $\text{lcm}(m_1, m_2)$ .*

Let  $N = pq$ , where  $p$  and  $q$  are two distinct odd primes. Let  $g$  be the unique common primitive root of  $p$  and  $q$ . The existence and uniqueness of  $g$  are guaranteed by Lemma 1. Similarly, there exists a unique integer  $x$  which satisfies the following system of congruences:

$$\begin{cases} x \equiv g \pmod{p}, \\ x \equiv 1 \pmod{q}. \end{cases} \quad (7)$$

Let  $d = \gcd(p-1, q-1)$  and  $e = (\gcd(p-1)(q-1)/d)$ . According to Whiteman [2], Whiteman generalized cyclotomic class of order  $d$  is

$$D_i = \{g^s x^i : s = 0, 1, \dots, e-1\}, \quad i = 0, 1, \dots, d-1. \quad (8)$$

It can be easily seen that  $D_i \cap D_j^{(N)} = \emptyset$  for all  $i \neq j$  and  $Z_N^* = \bigcup_{i=0}^{d-1} D_i$ .

Define two sets



$$\begin{aligned} P &= \{p, 2p, \dots, (q-1)p\}, \\ Q &= \{q, 2q, \dots, (p-1)q\}. \end{aligned} \quad (9)$$

Then,

$$Z_{pq} = \bigcup_{i=0}^{d-1} D_i \cup P \cup Q \cup \{0\}. \quad (10)$$

**Lemma 2** (see [7]). *Let*

$$\begin{aligned} d &= \gcd(p-1, q-1) = 2, \\ C_0 &= \{0\} \cup Q \cup D_0, \\ C_1 &= P \cup D_1. \end{aligned} \quad (11)$$

And a new binary generalized cyclotomic sequence  $S$  of order 2 is

$$s_i = \begin{cases} 0, & \text{if } i \in C_0, \\ 1, & \text{if } i \in C_1. \end{cases} \quad (12)$$

Let  $m = \text{ord}_{pq}(2)$  and  $\alpha$  be a primitive  $pq$ th root of unity in finite field  $F_{2^m}$ ,  $d_i(x) = \prod_{j \in D_i} (x - \alpha^j)$ ,  $i = 0, 1$ . Then,

(1) When  $p \equiv 1 \pmod{8}$ ,  $q \equiv 3 \pmod{8}$  or  $p \equiv -3 \pmod{8}$ ,  $q \equiv -1 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= pq - 1, \\ m(x) &= \frac{x^{pq} - 1}{x - 1}. \end{aligned} \quad (13)$$

(2) When  $p \equiv -1 \pmod{8}$ ,  $q \equiv 3 \pmod{8}$  or  $p \equiv 3 \pmod{8}$ ,  $q \equiv -1 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= (p-1)q, \\ m(x) &= \frac{x^{pq} - 1}{x^q - 1}. \end{aligned} \quad (14)$$

(3) When  $p \equiv -1 \pmod{8}$ ,  $q \equiv -3 \pmod{8}$  or  $p \equiv 3 \pmod{8}$ ,  $q \equiv 1 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= pq - p - q + 1, \\ m(x) &= \frac{(x^{pq} - 1)(x - 1)}{(x^p - 1)(x^q - 1)}. \end{aligned} \quad (15)$$

(4) When  $p \equiv 1 \pmod{8}$ ,  $q \equiv -1 \pmod{8}$  or  $p \equiv -3 \pmod{8}$ ,  $q \equiv 3 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= \frac{pq + p + q - 3}{2}, \\ m(x) &= \frac{x^{pq} - 1}{(x-1)d_0(x)}. \end{aligned} \quad (16)$$

(5) When  $p \equiv -1 \pmod{8}$ ,  $q \equiv 1 \pmod{8}$  or  $p \equiv 3 \pmod{8}$ ,  $q \equiv -3 \pmod{8}$ ,

$$\text{LC}(S) = \frac{(p-1)(q-1)}{2}, \quad (17)$$

$$m(x) = d_1(x).$$

(6) When  $p \equiv -1 \pmod{8}$ ,  $q \equiv 1 \pmod{8}$  or  $p \equiv 3 \pmod{8}$ ,  $q \equiv 3 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= \frac{(p-1)(q+1)}{2}, \\ m(x) &= \frac{(x^p - 1)d_1(x)}{x - 1}. \end{aligned} \quad (18)$$

**Lemma 3** (see [8]). *Let*

$$\begin{aligned} d &= \gcd(p-1, q-1) = 2, \\ C_0 &= \{0\} \cup Q \cup D_0, \\ C_1 &= P \cup D_1. \end{aligned} \quad (19)$$

And a binary generalized cyclotomic sequence  $S$  of order 2 is defined by

$$s_i = \begin{cases} 0, & \text{if } i \in C_0, \\ 1, & \text{if } i \in C_1. \end{cases} \quad (20)$$

Then,

(1) When  $((p-1)(q-1)/4)$  is even,

$$C_S(w) = \begin{cases} q - p - 3, & w \in P, \\ 1 + p - q, & w \in Q, \\ -1, & w \in Z_N^*. \end{cases} \quad (21)$$

(2) When  $((p-1)(q-1)/4)$  is odd,

$$C_S(w) = \begin{cases} q - p - 3, & w \in P, \\ 1 + p - q, & w \in Q, \\ -3, & w \in D_0, \\ 1, & w \in D_1. \end{cases} \quad (22)$$

**Lemma 4** (see [9]). *Let*

$$\begin{aligned} d &= \gcd(p-1, q-1) = 4, \\ C_0 &= \{0\} \cup Q \cup D_0 \cup D_2, \\ C_1 &= P \cup D_1 \cup D_3. \end{aligned} \quad (23)$$

And a binary generalized cyclotomic sequence  $S$  of order 4 is defined by

$$s_i = \begin{cases} 0 & \text{if } i \in C_0, \\ 1 & \text{if } i \in C_1. \end{cases} \quad (24)$$

(1) If  $2 \in D_0$  or  $2 \in D_2$ , then

$$\text{LC}(S) = \frac{(p+1)(q-1)}{2}. \quad (25)$$

If  $2 \in D_1$  or  $2 \in D_3$ , then

$$\text{LC}(S) = p(q-1). \quad (26)$$

(2) When  $p \equiv -1 \pmod{8}$ ,  $q \equiv 3 \pmod{8}$  or  $p \equiv 3 \pmod{8}$ ,  $q \equiv -1 \pmod{8}$ ,

$$C_S(w) = \begin{cases} pq, & w \in 0, \\ q-p-3, & w \in P, \\ 1+p-q, & w \in Q, \\ 1, & w \in D_0 \cup D_2, \\ -3, & w \in D_1 \cup D_3. \end{cases} \quad (27)$$

**Lemma 5** (see [10]). Let

$$\begin{aligned} d &= \gcd(p-1, q-1) = 2^k, \\ C_0 &= \{0\} \cup Q \cup \left( \bigcup_{i=0}^{2^{k-1}-1} D_i \right), \\ C_1 &= P \cup \left( \bigcup_{i=2^{k-1}}^{2^k-1} D_i \right). \end{aligned} \quad (28)$$

And a binary generalized cyclotomic sequence  $S$  of order  $2^k$  is defined by

$$s_i = \begin{cases} 0, & \text{if } i \in C_0, \\ 1, & \text{if } i \in C_1. \end{cases} \quad (29)$$

Let  $m = \text{ord}_{pq}(2)$  and  $\alpha$  be a  $pq$ th primitive root of unity in finite field  $F_{2^m}$ ,

$$S_i(x) = \sum_{j \in P \cup \left( \bigcap_{t=2^{k-1}+i}^{2^k-1+i} D_t \right)} x^j, \quad i = 0, 1, \dots, 2^k - 1, \quad (30)$$

$$\Lambda = \left\{ i_j : S_{i_j}(\alpha) = 0, \quad j = 0, 1, \dots, 2^{k-1} - 1 \right\}.$$

Then,

(1) If for all  $s$ ,  $g^s \equiv 2 \pmod{pq}$  is true,

$$\begin{aligned} \text{LC}(S) &= (p-1)q, \\ m(x) &= \frac{x^{pq} - 1}{x^q - 1}. \end{aligned} \quad (31)$$

(2) When there exists  $s$  such that  $g^s \equiv 2 \pmod{pq}$ ,

$$\begin{aligned} \text{LC}(S) &= \frac{(p+1)(q-1)}{2}, \\ m(x) &= \frac{x^{pq} - 1}{(x^p - 1)(\prod_{i \in \Lambda} d_i(x))}, \end{aligned} \quad (32)$$

where  $d_i(x) = \prod_{j \in D_i} (x - \alpha^j)$ .

The following are Ding's new cyclotomic class  $(V_0, V_1)$ .

Assume  $d = \gcd(p-1, q-1) = 2$ ; let

$$\begin{aligned} V_0 &= \{g^s x^l : 0 \leq s \leq e-1, \quad 0 \leq l \leq d-1, 2|s+l\}, \\ V_1 &= \{g^s x^l : 0 \leq s \leq e-1, \quad 0 \leq l \leq d-1, 2 \nmid s+l\}. \end{aligned} \quad (33)$$

With the above preparations, a partition of  $Z_N^*$  is

$$Z_N^* = V_0 \cup V_1, \quad (34)$$

Then,

$$Z_{pq} = V_0 \cup V_1 \cup P \cup Q \cup \{0\}. \quad (35)$$

Let

$$\begin{aligned} C_0 &= \{0\} \cup Q \cup V_0, \\ C_1 &= P \cup V_1. \end{aligned} \quad (36)$$

And a binary generalized cyclotomic sequence  $S$  with period  $pq$  constructed in [19] is

$$s_i = \begin{cases} 0, & \text{if } i \in C_0, \\ 1, & \text{if } i \in C_1. \end{cases} \quad (37)$$

**Lemma 6** (see [19]). Let  $S = \{s_i\}$  be the binary sequences defined. Then, the autocorrelation of  $S$  is

$$C_S(w) = \begin{cases} pq - 2p - 2, & w \in P, \\ 1 + p - q - (q-1)\eta_w((-1)^{(p-1/2)} + 1), & w \in Q, \\ -q - (q-2)\eta_w((-1)^{(p-1/2)} + 1), & w \in Z_N^*. \end{cases} \quad (38)$$

**Lemma 7** (see [19]). Let  $d_i(x) = \prod_{j \in V_i} (x - \alpha^j)$ ,  $i = 0, 1$ . Then,

(1) When  $p \equiv 1 \pmod{8}$ ,

$$\text{LC}(S) = \frac{pq + q - p - 1}{2}. \quad (39)$$

(2) When  $p \equiv -1 \pmod{8}$ ,

$$\text{LC}(S) = \frac{pq - p - q + 1}{2}. \quad (40)$$

(3) When  $p \equiv 3 \pmod{8}$ ,

$$\text{LC}(S) = pq - p - q + 1. \quad (41)$$

(4) When  $p \equiv -3 \pmod{8}$ ,

$$\text{LC}(S) = pq - p. \quad (42)$$

Now, let

$$\begin{aligned} C_0 &= \{0\} \cup P \cup V_0, \\ C_1 &= Q \cup V_1. \end{aligned} \quad (43)$$

A new binary cyclotomic sequence  $S = \{s_i\}$  with period  $pq$  is defined by

$$s_i = \begin{cases} 0 & \text{if } (i \bmod pq) \in C_0, \\ 1 & \text{if } (i \bmod pq) \in C_1. \end{cases} \quad (44)$$

### 3. Main Results

**3.1. Autocorrelation of Our New Sequences.** Let

$$\eta_i = \begin{cases} 1 & \text{if } i \text{ is the quadratic residue of module } p, \\ -1 & \text{if } i \text{ is not the quadratic residue of module } p. \end{cases} \quad (45)$$

**Lemma 8** (see [19]). *Let  $V_0$  and  $V_1$  be the sets defined above; then,*

- (1)  $2 \in V_0$  if and only if  $p \equiv \pm 1 \pmod{8}$
- (2)  $2 \in V_1$  if and only if  $p \equiv \pm 3 \pmod{8}$

**Lemma 9** (see [19]). *Let  $V_0$  be the sets defined above; then,  $n \in V_0$  if and only if  $(1/2)(1 + \eta_n) = 1$ ,  $0 \leq n \leq N - 1$ .*

**Lemma 10.** *Let  $1 \leq w \leq N - 1$ ; then,*

$$\begin{aligned} (1) \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ p|i+w}}^{N-1} \eta_{i(i+w)} &= \begin{cases} (q-2)(p-1), & w \in P, \\ 1-q, & w \in Q, \\ 2-q, & w \in Z_N^*. \end{cases} \\ (2) \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ p|i+w}}^{N-1} \eta_i &= \begin{cases} 0, & w \in P, \\ (q-1)\eta_{-w}, & w \notin P. \end{cases} \\ (3) \sum_{\substack{i=0 \\ p|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_{i+w} &= \begin{cases} 0, & w \in P, \\ (q-1)\eta_w, & w \notin P. \end{cases} \\ (4) \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ q|i+w, pq|i+w}}^{N-1} \eta_i &= \begin{cases} -\eta_{-w} & w \in Z_N^*, \\ 0 & w \notin Z_N^*, \end{cases} \\ (5) \sum_{\substack{i=1 \\ q|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_{i+w} &= \begin{cases} -\eta_w & w \in Z_N^*, \\ 0 & w \notin Z_N^*. \end{cases} \\ (6) \sum_{\substack{i=0 \\ p|i \\ p|i+w}}^{N-1} 1 &= \begin{cases} q, & w \in P, \\ 0, & w \notin P. \end{cases} \\ (7) \sum_{\substack{i=1 \\ q|i \\ p|i+w}}^{N-1} 1 &= \begin{cases} 0, & w \in P, \\ 1, & w \notin P. \end{cases} \\ (8) \sum_{\substack{i=0 \\ p|i, q|i+w \\ pq|i+w}}^{N-1} 1 &= \begin{cases} 0, & w \in P, \\ 1, & w \notin P. \end{cases} \\ (9) \sum_{\substack{i=1 \\ q|i, q|i+w \\ pq|i+w}}^{N-1} 1 &= \begin{cases} p-2, & w \in Q, \\ 0, & w \notin Q. \end{cases} \end{aligned}$$

*Proof.* For the proof of (1) and (6)–(9), one can refer to in [19]; we just prove (2)–(5).

$$\begin{aligned} \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ p|i+w}}^{N-1} \eta_i &= \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ p|i+w}}^{N-1} \eta_{-w} = \eta_{-w} \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ p|i+w}}^{N-1} 1 = \begin{cases} 0, & w \in P, \\ (q-1)\eta_{-w}, & w \notin P, \end{cases} \\ \sum_{\substack{i=0 \\ p|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_{i+w} &= \sum_{\substack{i=0 \\ p|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_w = \eta_w \sum_{\substack{i=0 \\ p|i \\ \gcd(i+w,pq)=1}}^{N-1} 1 = \begin{cases} 0, & w \in P, \\ (q-1)\eta_w, & w \notin P, \end{cases} \\ \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ q|i+w, pq|i+w}}^{N-1} \eta_i &= \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ q|i+w}}^{N-1} \eta_i - \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ q|i+w, pq|i+w}}^{N-1} \eta_i = 0 - \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ q|i+w, pq|i+w}}^{N-1} \eta_{-w} = -\eta_{-w} \sum_{\substack{i=0 \\ \gcd(i,pq)=1 \\ q|i+w, pq|i+w}}^{N-1} 1 = \begin{cases} -\eta_{-w} & w \in Z_N^*, \\ 0 & w \notin Z_N^*, \end{cases} \\ \sum_{\substack{i=1 \\ q|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_{i+w} &= \sum_{\substack{i=0 \\ q|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_{i+w} - \sum_{\substack{i=0 \\ q|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_{i+w} = 0 - \sum_{\substack{i=0 \\ q|i \\ \gcd(i+w,pq)=1}}^{N-1} \eta_w = -\eta_w \sum_{\substack{i=0 \\ q|i \\ \gcd(i+w,pq)=1}}^{N-1} 1 = \begin{cases} -\eta_w & w \in Z_N^*, \\ 0 & w \notin Z_N^*. \end{cases} \end{aligned} \quad (46)$$

□

**Theorem 1.** Let  $S = \{s_i\}$  be the new binary sequences defined in (44); then, the autocorrelation of  $S$  is

$$C_S(w) = \begin{cases} pq - 2p + 2, & w \in P, \\ p - q - 3 + (q - 1)\eta_w((-1)^{(p-1/2)} + 1), & w \in Q, \\ -q + q\eta_w((-1)^{(p-1/2)} + 1), & w \in Z_N^*. \end{cases} \quad (47)$$

*Proof.* By the definition of  $S$ ,

$$(-1)^{S_n} = \begin{cases} \eta_n, & \gcd(n, pq) = 1, \\ 1, & p|n, \\ -1, & q|n, n > 0. \end{cases} \quad (48)$$

Let  $1 \leq w \leq N - 1$ ; then,

$$\begin{aligned} C_S(w) &= \sum_{i=0}^{N-1} (-1)^{s_{i+w} + s_i} = \sum_{\substack{i=0 \\ \gcd(i, pq)=1 \\ \gcd(i+w, pq)=1}}^{N-1} \eta_{i(i+w)} + \sum_{\substack{i=0 \\ \gcd(i, pq)=1 \\ p|i+w}}^{N-1} \eta_i - \sum_{\substack{i=0 \\ \gcd(i, pq)=1 \\ q|i+w, pq \nmid i+w}}^{N-1} \eta_i - \sum_{\substack{i=1 \\ q|i \\ \gcd(i+w, pq)=1}}^{N-1} \eta_{i+w} \\ &\quad - \sum_{\substack{i=1 \\ q|i \\ p|i+w}}^{N-1} 1 + \sum_{\substack{i=1 \\ q|i, q|i+w \\ pq \nmid i+w}}^{N-1} 1 + \sum_{\substack{i=0 \\ p|i \\ \gcd(i+w, pq)=1}}^{N-1} \eta_{i+w} + \sum_{\substack{i=0 \\ p|i \\ p|i+w}}^{N-1} 1 - \sum_{\substack{i=0 \\ p|i, q|i+w \\ pq \nmid i+w}}^{N-1} 1 \\ &= \begin{cases} pq - 2p + 2 & \text{if } w \in P, \\ p - q - 3 + (q - 1)\eta_w((-1)^{(p-1/2)} + 1) & \text{if } w \in Q, \\ -q + q\eta_w((-1)^{(p-1/2)} + 1) & \text{if } w \in Z_N^*. \end{cases} \end{aligned} \quad (49)$$

The autocorrelation function of the new sequences  $C_S(w)$  is 5-level if  $p \equiv 1 \pmod{4}$ .  $C_S(w)$  is 3-level.  $\square$

### 3.2. Linear Complexity and Minimal Polynomial of Our New Sequences

**Lemma 11** (see [19]). Let  $V_0$  and  $V_1$  be the sets defined above; then,

$$\begin{cases} tV_0 \in V_0, tV_1 \in V_1, & \text{if } t \in V_0, \\ tV_0 \in V_1, tV_1 \in V_0, & \text{if } t \in V_1. \end{cases} \quad (50)$$

Denote

$$\begin{aligned} S(x) &= s_0 + s_1x + s_2x^2 + \cdots + s_{N-1}x^{N-1} \\ &= \sum_{i \in C_1} x^i = \sum_{i \in Q} x^i + \sum_{i \in V_1} x^i. \end{aligned} \quad (51)$$

Assume  $m = \text{ord}_{pq}(2)$  and  $\alpha$  is a  $pq$ th primitive root of unity in finite field  $F_{2^m}$ . According to the Blahut theorem, the linear complexity of sequence  $S = \{s_i\}$  is

$$\text{LC}(S) = pq - \left| \{t: S(\alpha^t) = 0, \quad 0 \leq t \leq pq - 1\} \right|. \quad (52)$$

**Lemma 12.** Let  $1 \leq t \leq N - 1$ ; then,

$$S(\alpha^t) = \begin{cases} S(\alpha), & t \in V_0, \\ 1 + S(\alpha), & t \in V_1, \\ \frac{p-1}{2} \pmod{2}, & t \in P, \\ 1, & t \in Q, \\ 0, & t \in \{0\}. \end{cases} \quad (53)$$

*Proof.* Let  $t \in V_0$ ; then, by Lemma 11,

$$\begin{aligned} S(\alpha^t) &= \sum_{i \in Q} \alpha^{ti} + \sum_{i \in V_1} \alpha^{ti} \\ &= \sum_{i \in tQ} \alpha^i + \sum_{i \in tV_1} \alpha^i \\ &= \sum_{i \in Q} \alpha^i + \sum_{i \in V_1} \alpha^i = S(\alpha). \end{aligned} \quad (54)$$

Let  $t \in V_1$ ,

$$\begin{aligned} 0 &= \alpha^{pq} - 1 = (\alpha - 1)(1 + \alpha + \cdots + \alpha^{pq-1}) \\ &= (\alpha - 1) \left( 1 + \sum_{i \in V_0} \alpha^i + \sum_{i \in V_1} \alpha^i + \sum_{i \in P} \alpha^i + \sum_{i \in Q} \alpha^i \right). \end{aligned} \quad (55)$$

Then, by Lemma 11,

$$\begin{aligned}
 S(\alpha^t) &= \sum_{i \in Q} \alpha^{ti} + \sum_{i \in V_1} \alpha^{ti} \\
 &= \sum_{i \in tQ} \alpha^i + \sum_{i \in tV_1} \alpha^i \\
 &= \sum_{i \in Q} \alpha^i + \sum_{i \in V_0} \alpha^i \quad (56) \\
 &= \sum_{i \in Q} \alpha^i - \sum_{i \in V_1} \alpha^i + 1 \\
 &\equiv (S(\alpha) + 1) \pmod{2}.
 \end{aligned}$$

Let  $t \in P$ ; then,

$$\begin{aligned}
 V_1 \bmod q &= \{g^s x^l \bmod q: 0 \leq s \leq e-1, 0 \leq l \leq 1, 2 \nmid s+l\} \\
 &= \left\{g^{2s} x \bmod q: 0 \leq s \leq \frac{e}{2}-1\right\} \cup \left\{g^{2s+1} \bmod q: 0 \leq s \leq \frac{e}{2}-1\right\} \quad (57) \\
 &= \left\{g^{2s} \bmod q: 0 \leq s \leq \frac{e}{2}-1\right\} \cup \left\{g^{2s+1} \bmod q: 0 \leq s \leq \frac{e}{2}-1\right\} = \{1, 2, \dots, q-1\}.
 \end{aligned}$$

When  $s$  runs through  $\{0, 1, \dots, e-1\}$ ,  $l$  runs through  $\{0, 1\}$ , and  $2 \nmid s+l$ , the set  $V_1 \bmod q$  takes on each element in  $\{1, 2, \dots, q-1\}$  exactly  $(p-1/2)$  times. Therefore,

$$\begin{aligned}
 S(\alpha^t) &= \sum_{i \in Q} \alpha^{ti} + \sum_{i \in V_1} \alpha^{ti} = ((p-1) \bmod 2) + \left(\frac{p-1}{2} \bmod 2\right) \\
 &\cdot \sum_{i \in P} \alpha^i = \frac{p-1}{2} \bmod 2. \quad (58)
 \end{aligned}$$

Let  $t \in Q$ ; then,

$$\begin{aligned}
 V_1 \bmod p &= \{g^s x^l \bmod p: 0 \leq s \leq e-1, 0 \leq l \leq 1, 2 \nmid s+l\} \\
 &= \left\{g^{2s} x \bmod p: 0 \leq s \leq \frac{e}{2}-1\right\} \\
 &\cup \left\{g^{2s+1} \bmod p: 0 \leq s \leq \frac{e}{2}-1\right\} \\
 &= \left\{g^{2s+1} \bmod p: 0 \leq s \leq \frac{e}{2}-1\right\} = \{g^1, g^3, \dots, g^{p-2}\}. \quad (59)
 \end{aligned}$$

When  $s$  runs through  $\{0, 1, \dots, e-1\}$ ,  $l$  runs through  $\{0, 1\}$ , and  $2 \nmid s+l$ , the set  $V_1 \bmod p$  takes on each element in  $\{g^1, g^3, \dots, g^{p-2}\}$  exactly  $q-1$  times. Therefore,

$$\begin{aligned}
 S(\alpha^t) &= \sum_{i \in Q} \alpha^{ti} + \sum_{i \in V_1} \alpha^{ti} = \sum_{i \in Q} \alpha^i + ((q-1) \bmod 2) \\
 &\cdot \sum_{i \in \{g^1, g^3, \dots, g^{p-2}\}} \alpha^i = -1 \equiv 1 \bmod 2. \quad (60)
 \end{aligned}$$

Let  $t \in \{0\}$ ; then,

$$\begin{aligned}
 S(\alpha^t) &= S(1) = p-1 + \frac{(p-1)(q-1)}{2} \\
 &= \frac{(p-1)(q+1)}{2} \bmod 2 = 0. \quad (61)
 \end{aligned}$$

□

**Lemma 13.** Let  $V_0$  and  $V_1$  be the sets defined above; then,  $2 \in V_0$  if and only if  $S(\alpha) \in \{0, 1\}$ .

*Proof.* Since the characteristic of finite field  $F_{2^m}$  is 2,  $(S(\alpha))^2 = S(\alpha^2)$ .

Let  $2 \in V_0$ ; then,  $2V_i = V_i$ ,  $i = 0, 1$ .

$$\begin{aligned}
 (S(\alpha))^2 &= S(\alpha^2) = \sum_{i \in Q} \alpha^{2i} + \sum_{i \in V_1} \alpha^{2i} \\
 &= \sum_{i \in Q} \alpha^i + \sum_{i \in V_1} \alpha^i = S(\alpha). \quad (62)
 \end{aligned}$$

Therefore,  $S(\alpha) \in \{0, 1\}$ .

Let  $2 \in V_1$ ; then,  $2V_i = V_{(i+1) \bmod 2}$ ,  $i = 0, 1$ .

$$\begin{aligned}
 (S(\alpha))^2 &= S(\alpha^2) = \sum_{i \in Q} \alpha^{2i} + \sum_{i \in V_1} \alpha^{2i} \\
 &= \sum_{i \in Q} \alpha^i + \sum_{i \in V_0} \alpha^i = S(\alpha) + 1. \quad (63)
 \end{aligned}$$

Therefore,  $S(\alpha) \notin \{0, 1\}$ .

Since  $2 \notin P \cup Q$ ,  $2 \in V_0$  if and only if  $S(\alpha) \in \{0, 1\}$ . □

**Theorem 2.** Let  $d_i(x) = \prod_{j \in V_i} (x - \alpha^j)$ ,  $i = 0, 1$ . Then,

(1) When  $p \equiv 1 \bmod 8$ ,

$$\begin{aligned} \text{LC}(S) &= \frac{pq + p - q - 1}{2}, \\ m(x) &= \frac{x^{pq} - 1}{(x^q - 1)d_i(x)}. \end{aligned} \quad (64)$$

(2) When  $p \equiv -1 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= \frac{pq + p + q - 3}{2}, \\ m(x) &= \frac{x^{pq} - 1}{(x^q - 1)d_i(x)}. \end{aligned} \quad (65)$$

(3) When  $p \equiv 3 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= pq - 1, \\ m(x) &= \frac{x^{pq} - 1}{x - 1}. \end{aligned} \quad (66)$$

(4) When  $p \equiv -3 \pmod{8}$ ,

$$\begin{aligned} \text{LC}(S) &= pq - q, \\ m(x) &= \frac{x^{pq} - 1}{x^q - 1}. \end{aligned} \quad (67)$$

*Proof.* Let  $\alpha$  be a  $pq$ th primitive root of unity in finite field  $F_{2^m}$ . Then,

$$\begin{aligned} x^p - 1 &= \prod_{i \in \{0\} \cup Q} (x - \alpha^i), \\ x^q - 1 &= \prod_{i \in \{0\} \cup P} (x - \alpha^i). \end{aligned} \quad (68)$$

Define  $d_i(x) = \prod_{j \in V_i} (x - \alpha^j)$ ,  $i = 0, 1$ . It can be easily seen that

$$\begin{aligned} x^N - 1 &= (x - 1)(x - \alpha) \dots (x - \alpha^{N-1}) \\ &= \frac{(x^p - 1)(x^q - 1)d_0(x)d_1(x)}{x - 1}. \end{aligned} \quad (69)$$

□

*Case 1.*  $p \equiv 1 \pmod{8}$ : choose  $\alpha$  such that  $S(\alpha) = 0$ . Then,

$$S(\alpha^t) = \begin{cases} 1, & t \in Q \cup V_1, \\ 0, & t \in 0 \cup P \cup V_0. \end{cases} \quad \text{LC}(S) = pq - q - \frac{(p-1)(q-1)}{2} = \frac{pq + p - q - 1}{2}, m(x) = \frac{x^{pq} - 1}{(x^q - 1)d_0(x)}. \quad (70)$$

*Case 2.*  $p \equiv -1 \pmod{8}$ : choose  $\alpha$  such that  $S(\alpha) = 0$ . Then,

$$S(\alpha^t) = \begin{cases} 1, & t \in P \cup Q \cup V_1, \\ 0, & t \in \{0\} \cup V_0, \end{cases} \quad \text{LC}(S) = pq - 1 - \frac{(p-1)(q-1)}{2} = \frac{pq + p + q - 3}{2}, m(x) = \frac{x^{pq} - 1}{(x - 1)d_0(x)}. \quad (71)$$

*Case 3.*  $p \equiv 3 \pmod{8}$ . Then,

$$\begin{aligned} S(\alpha^t) &= \begin{cases} \neq 0, & t \in P \cup Q \cup V_0 \cup V_1, \\ 0, & t \in \{0\}, \end{cases} \\ \text{LC}(S) &= pq - 1, \\ m(x) &= \frac{x^{pq} - 1}{x - 1}. \end{aligned} \quad (72)$$

*Case 4.*  $p \equiv -3 \pmod{8}$ . Then,

$$\begin{aligned} S(\alpha^t) &= \begin{cases} \neq 0, & t \in Q \cup V_0 \cup V_1, \\ 0, & t \in \{0\} \cup P, \end{cases} \\ \text{LC}(S) &= pq - q, \\ m(x) &= \frac{x^{pq} - 1}{x^q - 1}. \end{aligned} \quad (73)$$

The linear complexity of the new sequences is  $\text{LC}(S) > (pq/2)$  if  $p \equiv 1 \pmod{8}$ ,  $p > q + 1$ , or  $p \equiv -1 \pmod{8}$ ;  $\text{LC}(S) = pq - q$  or  $pq - 1$  if  $p \equiv \pm 3 \pmod{8}$ , which is very close to period  $pq$ .

The following are some examples.

*Example 1.* Let  $p = 7$  and  $q = 3$ . Then,

$$\begin{aligned} P &= \{7, 14\}, \\ Q &= \{3, 6, 9, 12, 15, 18\}, \\ V_0 &= \{1, 2, 4, 8, 11, 16\}, \\ V_1 &= \{5, 10, 13, 17, 19, 20\}. \end{aligned} \quad (74)$$

Our corresponding new binary sequence of period 21 is as follows: 0001011001101101111.

By using Magma, the autocorrelation value of the above sequence is 3-level, which is consistent with the case  $p \equiv 3 \pmod{4}$  in Theorem 1. And the linear complexity of the

TABLE 1: Comparison of autocorrelation  $C_S(w)$ .

$C_S(w)$	Sequences of Liu and Chen	New sequences
$w \in P$	$pq - 2p - 2$	$pq - 2p + 2$
$w \in Q$	$1 + p - q - (q - 1)\eta_w((-1)^{(p-1/2)} + 1)$	$p - q - 3 + (q - 1)\eta_w((-1)^{(p-1/2)} + 1)$
$w \in Z_N^*$	$-q - (q - 2)\eta_w((-1)^{(p-1/2)} + 1)$	$-q + q\eta_w((-1)^{(p-1/2)} + 1)$

TABLE 2: Comparison of linear complexity  $LC(S)$ .

$LC(S)$	Sequences of Liu and Chen	New sequences
$p \equiv 1 \pmod{8}$	$(pq - p + q - 1)/2$	$(pq + p - q - 1)/2$
$p \equiv -1 \pmod{8}$	$((p - 1)(q - 1))/2$	$(pq + p + q - 3)/2$
$p \equiv 3 \pmod{8}$	$pq - p - q + 1$	$pq - 1$
$p \equiv -3 \pmod{8}$	$pq - p$	$pq - q$

above sequence is equal to 14, which is consistent with the case  $p \equiv -1 \pmod{8}$  in Theorem 2.

*Example 2.* Let  $p = 11$  and  $q = 3$ . Then,

$$\begin{aligned}
 P &= \{11, 22\}, \\
 Q &= \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}, \\
 V_0 &= \{1, 4, 5, 14, 16, 20, 23, 25, 26, 31\}, \\
 V_1 &= \{2, 7, 8, 10, 13, 17, 19, 28, 29, 32\}.
 \end{aligned} \tag{75}$$

Our corresponding new binary sequence of period 33 is as follows: 00110011110110101110100100111101.

By using Magma, the autocorrelation value of the above sequence is 3-level, which is consistent with the case  $p \equiv 3 \pmod{4}$  in Theorem 1. And the linear complexity of the above sequence is equal to 32, which is consistent with the case  $p \equiv 3 \pmod{8}$  in Theorem 2.

*Example 3.* Let  $p = 13$  and  $q = 3$ . Then,

$$\begin{aligned}
 P &= \{13, 26\}, \\
 Q &= \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}, \\
 V_0 &= \{1, 4, 10, 14, 16, 17, 22, 23, 25, 29, 35, 38\}, \\
 V_1 &= \{2, 5, 7, 8, 11, 19, 20, 28, 31, 32, 34, 37\}.
 \end{aligned} \tag{76}$$

Our corresponding new binary sequence of period 39 is as follows: 00110111110110010011110010011011110110.

By using Magma, the autocorrelation value of the above sequence is 5-level, which is consistent with the case  $p \equiv 1 \pmod{4}$  in Theorem 1. And the linear complexity of the

above sequence is equal to 36, which is consistent with the case  $p \equiv -3 \pmod{8}$  in Theorem 2.

*Example 4.* Let  $p = 17$  and  $q = 3$ . Then,

$$\begin{aligned}
 P &= \{17, 34\}, \\
 Q &= \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48\}, \\
 V_0 &= \{1, 2, 4, 8, 13, 16, 19, 25, 26, 32, 35, 38, 43, 47, 49, 50\}, \\
 V_1 &= \{5, 7, 10, 11, 14, 20, 22, 23, 28, 29, 31, 37, 40, 41, 44, 46\}.
 \end{aligned} \tag{77}$$

Our corresponding new binary sequence of period 51 is as follows: 00010111011110110010111110011111010011011-1101110100.

By using Magma, the linear complexity of the above sequence is equal to 32, which is consistent with the case  $p \equiv 1 \pmod{8}$  in Theorem 1, but the autocorrelation value of the above sequence is 4-level, which is consistent with the case  $p \equiv 1 \pmod{4}$  in Theorem 2.

**3.3. Comparisons of Results.** The comparisons of our results with [19] are listed in Tables 1 and 2.

The comparisons show the following:

- (i) When  $p \equiv 3 \pmod{4}$ , the autocorrelation  $C_S(w)$  of the two sequences is unequal, but they are 3-level. When  $p \equiv 1 \pmod{4}$ , the autocorrelation of the two sequences is unequal, but both of them are 5-level.
- (ii) When  $p \equiv 1 \pmod{4}$ ,  $p > q$ , or  $p \equiv 3 \pmod{4}$ , the linear complexity of our new sequences is larger.

## 4. Conclusion

This paper presents a construction of generalized cyclotomic binary sequences with period  $pq$  based on Ding's new cyclotomic class  $(V_0, V_1)$ . And the autocorrelation value, linear complexity, and minimal polynomial of our new sequences are determined. The autocorrelation function  $C_S(w)$  is 5-level if  $p \equiv 1 \pmod{4}$ .  $C_S(w)$  is 3-level, and  $S$  has almost optimal autocorrelation if  $p \equiv 3 \pmod{4}$ . The linear complexity  $LC(S) > (pq/2)$  if  $p \equiv 1 \pmod{8}$ ,  $p > q + 1$ , or  $p \equiv -1 \pmod{8}$ ;  $LC(S) = pq - q$  or  $pq - 1$  if  $p \equiv \pm 3 \pmod{8}$ , which is very close to the period. The results show that our

new sequences have quite good cryptographic properties in the aspect of autocorrelation and linear complexity.

## Data Availability

All the data used to support the findings of this study are included in Section 3.2 of this article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# Divisor Problems Related to Hecke Eigenvalues in Three Dimensions

Jing Huang and Huafeng Liu 

School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong 250014, China

Correspondence should be addressed to Huafeng Liu; hflui\_sdu@hotmail.com

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In this paper, we consider divisor problems related to Hecke eigenvalues in three dimensions. We establish upper bounds and asymptotic formulas for these problems on average.

## 1. Introduction

Let  $\Gamma = SL_2(\mathbb{Z})$  be the full modular group. Let  $H_k^*$  denote the set of primitive holomorphic forms with even integral weight  $k \geq 2$  for  $\Gamma$ . Then  $H_k^*$  consists of common eigenfunctions  $f$  of all Hecke operators  $T_n$ . The Hecke eigenfunction  $f$  has the following Fourier series expansion:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}, \quad (\Im z > 0), \quad (1)$$

where  $\lambda_f(n)$  denotes the  $n$ -th normalized eigenvalue. It is known that  $\lambda_f(n)$ , as a function of  $n$ , is real-valued and multiplicative. Moreover, for all integers  $n \geq 1$ , Deligne [1] showed that

$$|\lambda_f(n)| \leq d(n), \quad (2)$$

where  $d(n)$  is the divisor function. Also, for every prime  $p$ ,

$$\begin{aligned} \alpha_f(p) &= \alpha_f(p) + \beta_f(p), \\ \alpha_f(p)\beta_f(p) &= |\alpha_f(p)| = |\beta_f(p)| = 1. \end{aligned} \quad (3)$$

Now we introduce some specific automorphic  $L$ -functions. Define the Hecke  $L$ -function attached to  $f$  as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left( 1 - \frac{\alpha_f(p)}{p^s} \right)^{-1} \left( 1 - \frac{\alpha_f(p)^{-1}}{p^s} \right)^{-1}, \quad (4)$$

for  $\Re s > 1$ . Moreover, the Rankin-Selberg  $L$ -function attached to  $f$  can be defined as

$$\begin{aligned} L(s, f \times f) &= \prod_p \left( 1 - \frac{\alpha_f(p)^2}{p^s} \right)^{-1} \left( 1 - \frac{1}{p^s} \right)^{-2} \\ &\quad \cdot \left( 1 - \frac{\alpha_f(p)^{-2}}{p^s} \right)^{-1}, \end{aligned} \quad (5)$$

$\Re s > 1$ .

Then,  $L(s, f \times f)$  can be rewritten as

$$L(s, f \times f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)^2}{n^s} = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}(n)}{n^s}. \quad (6)$$

As usual,  $\zeta(s)$  denotes the Riemann zeta-function. The symmetric square  $L$ -function attached to  $f$  can be defined as, for  $\Re s > 1$ ,

$$L(s, \text{sym}^2 f) = \prod_p \left(1 - \frac{\alpha_f(p)^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)^{-2}}{p^s}\right)^{-1}, \quad (7)$$

which can also be expressed in the Dirichlet series

$$\begin{aligned} L(s, \text{sym}^2 f) &= \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f}(n)}{n^s} \\ &= \prod_p \left(1 + \frac{\lambda_{\text{sym}^2 f}(p)}{p^s} + \frac{\lambda_{\text{sym}^2 f}(p^2)}{p^{2s}} + \frac{\lambda_{\text{sym}^2 f}(p^3)}{p^{3s}} + \dots\right), \quad \Re s > 1. \end{aligned} \quad (8)$$

The symmetric square  $L$ -function  $L(s, \text{sym}^2 f)$  has a functional equation and could be analytic continued to an entire function over the whole complex plane. We refer to works of Hecke in [2], Gelbert and Jacquet [3], Kim [4], and Kim and Shahidi [5,6] for these properties. Therefore, the symmetric square  $L$ -function  $L(s, \text{sym}^2 f)$  could be seen as a general  $L$ -function in the sense of Perelli [7].

In number theory, considering the properties and average behaviors of the Fourier coefficients is meaningful and interesting. Some classical problems concern mean values of these Fourier coefficients and related problems with the corresponding automorphic  $L$ -functions (see [1, 8–29], etc.). Here, we just give a brief history for general divisor problems related to these Fourier coefficients.

Let  $\omega$  be an integer greater than one, and

$$\begin{aligned} \lambda_{\omega, f}(n) &= \sum_{n=n_1 n_2, \dots, n_\omega} \lambda_f(n_1) \lambda_f(n_2), \dots, \lambda_f(n_\omega), \\ \lambda_{\omega, f \times f}(n) &= \sum_{n=n_1 n_2, \dots, n_\omega} \lambda_{f \times f}(n_1) \lambda_{f \times f}(n_2), \dots, \lambda_{f \times f}(n_\omega). \end{aligned} \quad (9)$$

In particular, we have  $\lambda_{1, f}(n) = \lambda_f(n)$  and  $\lambda_{1, f \times f}(n) = \lambda_{f \times f}(n)$ . In 1927, Hecke [30] showed that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{(1/2)}. \quad (10)$$

Subsequently, this upper bound was improved by many scholars (for example, see [12, 21, 24]). In this direction, the best result so far was obtained by Wu [24] who showed that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{(1/3)} (\log x)^{\rho_{(1/2)}^+}, \quad (11)$$

where

$$\begin{aligned} \rho_{(1/2)}^+ &= \frac{102 + 7\sqrt{21}}{210} \left(\frac{6 - \sqrt{21}}{5}\right)^{(1/2)} \\ &\quad + \frac{102 - 7\sqrt{21}}{210} \left(\frac{6 + \sqrt{21}}{5}\right)^{(1/2)} - \frac{33}{55}, \\ &= -0.118 \dots \end{aligned} \quad (12)$$

Rankin [20] and Selberg [22] established

$$\sum_{n \leq x} \lambda_{f \times f}(n) = C_f x + O_f(x^{(3/5)}), \quad (13)$$

where  $C_f$  is a positive constant depending on  $f$ . Later, Kanemitsu, Sankaranarayanan, and Tanigawa [31] studied general divisor problems and proved

$$\begin{aligned} \sum_{n \leq x} \lambda_{\omega, f}(n) &\ll x^{1 - (3/2k+2)+\varepsilon}, \\ \sum_{n \leq x} \lambda_{\omega, f \times f}(n) &= M_\omega(x) + O(x^{1 - (1/2k)+\varepsilon}), \end{aligned} \quad (14)$$

where  $\omega$  is an integer and  $\omega \geq 2$ ;  $M_\omega(x)$  comes from a residue and takes the form  $xP_{\omega-1}(\log x)$ ;  $P_{\omega-1}(t)$  denotes a polynomial of  $t$  with degree  $\omega - 1$ . Fomenko [32] also proved the same result for the sum  $\sum_{n \leq x} \lambda_{\omega, f}(n)$ . After that, Kanemitsu, Sankaranarayanan, and Tanigawa's result was improved by Lü [33], and some general results were obtained (see [34–37], etc.).

In this paper, we consider divisor problems related to Hecke eigenvalues in three dimensions motivated by the above results and Ivić's work on three-dimensional divisor problems (see, e.g., [38]). We first introduce some notation. For any fixed integer  $1 < a < b < c$ , we define

$$\begin{aligned} \lambda_f^{a,b,c}(n) &:= \sum_{n=n_1^a n_2^b n_3^c} \lambda_f(n_1) \lambda_f(n_2) \lambda_f(n_3), \\ \lambda_{f \times f}^{a,b,c}(n) &:= \sum_{n=n_1^a n_2^b n_3^c} \lambda_{f \times f}(n_1) \lambda_{f \times f}(n_2) \lambda_{f \times f}(n_3). \end{aligned} \quad (15)$$

We are interested in studying the average behaviors of sums

$$\begin{aligned} S_f(a, b, c; x) &:= \sum_{n \leq x} \lambda_f^{a,b,c}(n), \\ S_{f \times f}(a, b, c; x) &:= \sum_{n \leq x} \lambda_{f \times f}^{a,b,c}(n), \end{aligned} \quad (16)$$

which can be seen as divisor problems related to Hecke eigenvalues in three dimensions. We establish the following results.

**Theorem 1.** Let  $a, b$ , and  $c$  be fixed integers satisfying  $1 < a < b < c$ . Then, for any  $\epsilon > 0$ , one has

$$S_f(a, b, c; x) \ll \begin{cases} x^{(1/a) - (3/2)(7a-b-c) + \epsilon}, & \text{if } c \leq 2a, \\ x^{(1/a) - (3/2)(5a-b) + \epsilon}, & \text{if } b < 2a < c, \\ x^{(1/2a) + \epsilon}, & \text{if } 2a \leq b. \end{cases} \quad (17)$$

**Theorem 2.** Let  $a, b$ , and  $c$  be fixed integers satisfying  $1 < a < b < c$ . Then, for any  $\epsilon > 0$ , one has

$$S_{f \times f}(a, b, c; x) = \begin{cases} M_1 x^{(1/a)} + M_2 x^{(1/b)} + M_3 x^{(1/c)} + O(x^{(1/a) - (84/748a - 131(b+c)) + \epsilon}), & \text{if } c \leq 2a, \\ M_1 x^{(1/a)} + M_2 x^{(1/b)} + O(x^{(1/a) - (84/486a - 131b) + \epsilon}), & \text{if } b < 2a < c, \\ M_1 x^{(1/a)} + O(x^{(5/8a) + \epsilon}), & \text{if } 2a \leq b, \end{cases} \quad (18)$$

where

$$\begin{aligned} M_1 &= L\left(\frac{b}{a}, f \times f\right) L\left(\frac{c}{a}, f \times f\right) L(1, \text{sym}^2 f), \\ M_2 &= L\left(\frac{a}{b}, f \times f\right) L\left(\frac{c}{b}, f \times f\right) L(1, \text{sym}^2 f), \\ M_3 &= L\left(\frac{a}{c}, f \times f\right) L\left(\frac{b}{c}, f \times f\right) L(1, \text{sym}^2 f). \end{aligned} \quad (19)$$

## 2. Some Lemmas

To prove our theorems, we need to quote some lemmas, which include the individual and averaged subconvexity bounds for the Riemann zeta-function and the symmetric square  $L$ -function. And from the following Lemma 1, we know that the Rankin–Selberg  $L$ -function  $L(s, f \times f)$  could be decomposed into the product of the Riemann zeta-function and the corresponding symmetric square  $L$ -function.

**Lemma 1.** For  $\Re s > 1$ , one has

$$L(s, f \times f) = \zeta(s) L(s, \text{sym}^2 f). \quad (20)$$

*Proof.* By comparing the Euler products of two sides of (20) and applying (3), we can get this lemma. This lemma can also be found in [33].  $\square$

**Lemma 2.** For any  $\epsilon > 0$ , one has

$$\int_1^T \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \sim CT \log T, \quad (21)$$

uniformly for  $T \geq 1$ , and the subconvexity bound in the critical strip

$$L(\sigma + it, f) \ll \begin{cases} (1 + |t|)^{(2(1-\sigma)/3) + \epsilon}, & \text{if } \frac{1}{2} < \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \quad (22)$$

where  $|t| \geq 1$ .

*Proof.* These results are proved by Good [11].  $\square$

**Lemma 3.** For any  $\epsilon > 0$ , one has

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{12} dt \ll T^{2+\epsilon}, \quad (23)$$

uniformly for  $T \geq 1$ , and the subconvexity bound in the critical strip

$$\zeta(\sigma + it) \ll \begin{cases} (1 + |t|)^{(13/42)(1-\sigma) + \epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \quad (24)$$

where  $|t| \geq 1$ .

*Proof.* The twelfth mean value estimate (23) is due to Heath-Brown [39]. The subconvexity bound (24) is due to Bourgain [40].  $\square$

**Lemma 4.** For any  $\epsilon > 0$ , one has

$$\int_1^T \left| L(\sigma + it, \text{sym}^2 f) \right|^2 dt \ll T^{3(1-\sigma) + \epsilon}, \quad (25)$$

uniformly for  $T \geq 1$ , and the subconvexity bound in the critical strip

$$L(\sigma + it, \text{sym}^2 f) \ll \begin{cases} (1 + |t|)^{(5/4)(1-\sigma) + \epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \quad (26)$$

where  $|t| \geq 1$ .

*Proof.* The first result follows from Perelli's mean value theorem with  $L(s, \text{sym}^2 f)$  (see [7]), and the second one is due to Nunes [19].  $\square$

### 3. Proof of Theorem 1

In this section, we shall complete the proof of Theorem 1. Note that

$$L(as, f)L(bs, f)L(cs, f) = \sum_{n=1}^{\infty} \frac{\lambda_f^{a,b,c}(n)}{n^s}. \quad (27)$$

Then, by (27) and Perron's formula (see Proposition 5.54 in [41]), we can obtain

$$S_f(a, b, c; x) = \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} L(as, f)L(bs, f)L(cs, f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right), \quad (28)$$

where  $s = \sigma + it$ ,  $\eta = (1/a) + \varepsilon$  and  $1 \leq T \leq x$  is a parameter to be chosen later.

We shift the line of the integral of (28) to the line  $\Re s = (1/2a)$ . Then, Cauchy's residue theorem shows that

$$\begin{aligned} S_f(a, b, c; x) &= \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta+iT}^{(1/2a)-iT} \right\} \\ &\quad L(as, f)L(bs, f)L(cs, f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\ &:= G_1 + G_2 + G_3 + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right). \end{aligned} \quad (29)$$

The following work is to estimate  $G_1, G_2$ , and  $G_3$ . The estimates for the integrals over the horizontal segments are similar, so we handle  $G_2$  and  $G_3$  first. To get this goal, we consider three cases  $c \leq 2a$ ,  $b < 2a < c$ , and  $2a \leq b$ .

We first consider the case  $c \leq 2a$ . To estimate  $G_2$  and  $G_3$ , we divide the integral interval into the following eight intervals  $I_1, \dots, I_4$ , some of which may be empty, and use Lemma 2.

Interval  $I_1$ :

$$\begin{aligned} I_1 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{2a} \leq \sigma \leq \frac{1}{c} \right\}. \end{aligned} \quad (30)$$

In this interval, we have

$$\begin{aligned} T^{-1} \times \int_{I_1} x^\sigma |L(as + iat, f)L(bs + ibt, f)L(cs + ict, f)| d\sigma \\ \ll \max_{(1/2a) \leq \sigma \leq (1/c)} x^\sigma T^{(2/3)(1-a\sigma)+(2/3)(1-b\sigma)+(2/3)(1-c\sigma)+\varepsilon} T^{-1} \\ \ll \max_{(1/2a) \leq \sigma \leq (1/c)} T^{1+\varepsilon} \left( \frac{x}{T^{(2/3)(a+b+c)}} \right)^\sigma \\ \ll x^{(1/c)} T^{(1/3)-(2(a+b)/3c)+\varepsilon} + x^{(1/2a)} T^{(2/3)-(b+c/3a)+\varepsilon}. \end{aligned} \quad (31)$$

Interval  $I_2$ :

$$\begin{aligned} I_2 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, 1 < c\sigma \leq c\eta \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{c} < \sigma \leq \frac{1}{b} \right\}. \end{aligned} \quad (32)$$

In this interval, we have

$$\begin{aligned} T^{-1} \times \int_{I_2} x^\sigma |L(as + iat, f)L(bs + ibt, f)L(cs + ict, f)| d\sigma \\ \ll \max_{(1/c) < \sigma \leq (1/b)} x^\sigma T^{(2/3)(1-a\sigma)+(2/3)(1-b\sigma)+\varepsilon} T^{-1} \\ \ll \max_{(1/c) < \sigma \leq (1/b)} T^{(1/3)+\varepsilon} \left( \frac{x}{T^{(2/3)(a+b)}} \right)^\sigma \\ \ll x^{1/b} T^{-(1/3)-(2a/3b)+\varepsilon} + x^{(1/c)} T^{(1/3)-(2(a+b)/3c)+\varepsilon}. \end{aligned} \quad (33)$$

Interval  $I_3$ :

$$\begin{aligned} I_3 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{c} \right\}. \end{aligned} \quad (34)$$

This interval is an empty set noting that  $(1/b) > (1/c)$ .

Interval  $I_4$ :

$$\begin{aligned} I_4 &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, 1 < c\sigma \leq c\eta \right\} \\ &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{a} \right\}. \end{aligned} \quad (35)$$

In this interval, we have

$$T^{-1} \times \int_{I_4} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)L(c\sigma + ict, f)| d\sigma$$

Thus, by (31)–(36), we have

$$\begin{aligned} &\ll \max_{(1/b) < \sigma \leq (1/a)} x^\sigma T^{(2/3)(1-a\sigma)+\varepsilon} T^{-1} \\ &\ll x^{(1/a)} T^{-1+\varepsilon} + x^{(1/b)} T^{-(1/3)-\frac{2a}{3b}+\varepsilon}. \end{aligned} \quad (36)$$

$$\begin{aligned} |G_2 + G_3| &\ll T^{-1} \int_{(1/2a)}^\eta x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)L(c\sigma + ict, f)| d\sigma \\ &= T^{-1} \int_{I_1 \cup \dots \cup I_4} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)L(c\sigma + ict, f)| d\sigma \\ &\ll x^{(1/2a)} T^{(2/3)-(b+c/3a)+\varepsilon} + x^{(1/b)} T^{-(1/3)-(2a/3b)+\varepsilon} + x^{(1/c)} T^{(1/3)-(2(a+b)/3c)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \end{aligned} \quad (37)$$

Next, we handle  $G_1$ . We have

$$\begin{aligned} |G_1| &\ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| t^{-1} dt + x^{1/2a} \\ &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| dt + x^{1/2a}. \end{aligned} \quad (38)$$

Then, by Lemma 2 and applying Cauchy's inequality, we can obtain

$$\begin{aligned} |G_1| &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} T_1^{(2/3)(1-(b/2a))} T_1^{(2/3)(1-(c/2a))} \\ &\quad \cdot \left( \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \right)^{1/2} \times \left( \int_{T_1/2}^{T_1} 1 dt \right)^{1/2} + x^{1/2a} \\ &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{(4/3)-(b+c/3a)+\varepsilon} \\ &\ll x^{1/2a} T^{(4/3)-(b+c/3a)+\varepsilon}. \end{aligned} \quad (39)$$

According to (29), (37), and (39), we have

$$\begin{aligned} S_f(a, b, c; x) &\ll x^{1/2a} T^{(4/3)-(b+c/3a)+\varepsilon} + x^{1/b} T^{-(1/3)-(2a/3b)} \\ &\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \end{aligned} \quad (40)$$

By taking  $T = x^{(3/2)(7a-b-c)}$  in (40), we can obtain

$$S_f(a, b, c; x) \ll x^{(1/a)-(3/2)(7a-b-c)+\varepsilon}, \quad (41)$$

which proves the first result of Theorem 1.

For the case  $b < 2a < c$ , to estimate  $G_2$  and  $G_3$ , we still divide the integral interval into four corresponding short intervals  $I_1'', \dots, I_4''$ , which are different from ones for the case  $c \leq 2a$ . In fact, the corresponding short intervals  $I_1'', I_3''$  become empty sets in the current situation. However, we still can estimate  $G_2 + G_3$  similar to the corresponding parts of the case  $c \leq 2a$  and get

$$\begin{aligned} |G_2 + G_3| &\ll x^{1/2a} T^{-(b/3a)+\varepsilon} + x^{1/b} T^{-(1/3)-(2a/3b)+\varepsilon} \\ &\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \end{aligned} \quad (42)$$

The estimate of  $G_1$  becomes the following at the current case by noting  $(c/2a) > 1$ .

$$\begin{aligned}
|G_1| &\ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| t^{-1} dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \lim_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \lim_{T_1 \leq T} T_1^{-1+(2/3)(1-(b/2a))} \left( \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \right)^{1/2} \left( \int_{T_1/2}^{T_1} 1 dt \right)^{1/2} + x^{1/2a} \\
&\ll x^{1/2a} \log T \lim_{T_1 \leq T} T_1^{-1+(2/3)(1-(b/2a))+(1/2)+(1/2)+\varepsilon} + x^{1/2a} \\
&\ll x^{1/2a} T^{(2/3)-(b/3a)+\varepsilon}.
\end{aligned} \tag{43}$$

Recalling (29), we have

$$\begin{aligned}
S_f(a, b, c; x) &\ll x^{1/2a} T^{(2/3)-(b/3a)+\varepsilon} + x^{1/b} T^{-(1/3)-(2a/3b)+\varepsilon} \\
&\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}.
\end{aligned} \tag{44}$$

By taking  $T = x^{(3/2)(5a-b)}$  in (44), we can get

$$S_f(a, b, c; x) \ll x^{(1/a)-(3/2)(5a-b)+\varepsilon}, \tag{45}$$

which proves the second result of Theorem 1.

For the case  $2a \leq b$ , to estimate  $G_2$  and  $G_3$ , we also divide the integral interval into four corresponding short intervals  $I'_1, \dots, I'_4$ , which are different from ones for the case  $c \leq 2a$ . In fact, the corresponding short intervals  $I'_1, I'_2, I'_3$  become empty sets in the current situation. However, we still can estimate  $G_2 + G_3$  similarly to the corresponding parts of the case  $c \leq 2a$  and get

$$|G_2 + G_3| \ll x^{1/2a} T^{-(2/3)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \tag{46}$$

The estimate of  $G_1$  becomes the following by noting  $(c/2a) > (b/2a) > 1$ .

$$\begin{aligned}
|G_1| &\ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| t^{-1} dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) L\left(\frac{c}{2a} + ict, f\right) \right| dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \left( \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \right)^{1/2} \left( \int_{T_1/2}^{T_1} 1 dt \right)^{1/2} + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+(1/2)+(1/2)+\varepsilon} + x^{1/2a} \\
&\ll x^{1/2a} T^\varepsilon.
\end{aligned} \tag{47}$$

Recalling (29), we have

$$S_f(a, b, c; x) \ll x^{1/2a} T^\varepsilon + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \tag{48}$$

By taking  $T = x^{1/2a}$  in (48), we can get

$$S_f(a, b, c; x) \ll x^{(1/2a)+\varepsilon}, \tag{49}$$

which proves the third result of Theorem 1.

#### 4. Proof of Theorem 2

In this section, we shall prove Theorem 2, the process of which is more complicated than Theorem 1. Note that

$$L(as, f \times f) L(bs, f \times f) L(cs, f \times f) = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}^{a,b,c}(n)}{n^s}. \tag{50}$$

Then, by (50) and Perron's formula (see Proposition 5.54 in [41]), we have

\openup3

$$\begin{aligned}
S_{f \times f}(a, b, c; x) &= \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} L(as, f \times f) L(bs, f \times f) L \\
&\quad \cdot (cs, f \times f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right),
\end{aligned} \tag{51}$$

where  $s = \sigma + it$ ,  $\eta = (1/a) + \varepsilon$  and  $1 \leq T \leq x$  is a parameter which will be decided later.

In view of (20), the points  $s = (1/a)$ ,  $s = (1/b)$  and  $s = (1/c)$  are the only three possible simple poles of the integrand in the rectangle  $I_T = \{s = \sigma + it: (1/2a) \leq \sigma \leq \eta, |t| \leq T\}$ . The corresponding possible residues at  $s = (1/a)$ ,  $s = (1/b)$ , and  $s = (1/c)$  are equal to

$$\begin{aligned} M_1 &:= L\left(\frac{b}{a}, f \times f\right) L\left(\frac{c}{a}, f \times f\right) L(1, \text{sym}^2 f), \\ M_2 &:= L\left(\frac{a}{b}, f \times f\right) L\left(\frac{c}{b}, f \times f\right) L(1, \text{sym}^2 f), \\ M_3 &:= L\left(\frac{a}{c}, f \times f\right) L\left(\frac{b}{c}, f \times f\right) L(1, \text{sym}^2 f), \end{aligned} \quad (52)$$

respectively.

We move the integral line of the integral in (28) to the parallel segment with  $\Re s = (1/2a)$ . We first consider the case  $c \leq 2a$ . In this situation, the points  $s = (1/a)$ ,  $s = (1/b)$ , and  $s = (1/c)$  are all poles of the integrand in the rectangle  $I_T$ . Therefore, by Cauchy's residue theorem, we can obtain

$$\begin{aligned} S_{f \times f}(a, b, c; x) &= \left\{ \text{Res}_{s=(1/a)} + \text{Res}_{s=(1/b)} + \text{Res}_{s=(1/c)} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} \\ &\quad + \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta+iT}^{(1/2a)-iT} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\ &:= M_1 x^{(1/a)} + M_2 x^{(1/b)} + M_3 x^{(1/c)} + J_1 + J_2 + J_3 + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right). \end{aligned} \quad (53)$$

Now, the remaining work is to handle these three terms  $J_1$ ,  $J_2$ , and  $J_3$ . In addition, the estimates of these integrals on the horizontal parts are analogous, and so we deal with  $J_2$  and  $J_3$  firstly. To estimate  $J_2$  and  $J_3$ , similarly to the method estimating  $G_2$  and  $G_3$ , we still divide the integral interval into the following four short intervals  $I_1^*, \dots, I_4^*$  and apply Lemmas 3 and 4.

Interval  $I_1^*$ :

$$\begin{aligned} I_1^* &:= \left\{ s = \sigma + iT: \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\ &= \left\{ s = \sigma + iT: \frac{1}{2a} \leq \sigma \leq \frac{1}{c} \right\}. \end{aligned} \quad (54)$$

In this interval, we have

$$\begin{aligned} T^{-1} \times \int_{I_1^*} x^\sigma &\left| \zeta(a\sigma + iat) L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\ &\ll \max_{(1/2a) \leq \sigma \leq (1/c)} x^\sigma T^{((13/42)+(5/4))(1-a\sigma)} T^{((13/42)+(5/4))(1-b\sigma)} T^{((13/42)+(5/4))(1-c\sigma)} T^{-1+\varepsilon} \\ &\ll \max_{(1/2a) \leq \sigma \leq (1/c)} T^{(309/84)+\varepsilon} \left( \frac{x}{T^{(131/84)(a+b+c)}} \right)^\sigma \\ &\ll x^{1/c} T^{(89/42)-(131(a+b)/84c)+\varepsilon} + x^{1/2a} T^{(487/168)-(131(b+c)/168a)+\varepsilon}. \end{aligned} \quad (55)$$

Interval  $I_2^*$ :

$$\begin{aligned} I_2^* &:= \left\{ s = \sigma + iT: \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1, 1 < c\sigma \leq c\eta \right\} \\ &= \left\{ s = \sigma + iT: \frac{1}{c} < \sigma \leq \frac{1}{b} \right\}. \end{aligned} \quad (56)$$

In this interval, we have

$$\begin{aligned}
 & T^{-1} \times \int_{I_2^*} x^\sigma \left| \zeta(a\sigma + iat) \left( L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & \ll \max_{(1/c) < \sigma \leq (1/b)} x^\sigma T^{((13/42)+(5/4))(1-a\sigma)} T^{((13/42)+(5/4))(1-b\sigma)} T^{-1+\varepsilon} \\
 & \ll \max_{(1/c) < \sigma \leq (1/b)} T^{(89/42)+\varepsilon} \left( \frac{x}{T^{(131/84)(a+b)}} \right)^\sigma \\
 & \ll x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} + x^{1/c} T^{(89/42)-(131(a+b)/84c)+\varepsilon}.
 \end{aligned} \tag{57}$$

Interval  $I_3^*$ :

$$\begin{aligned}
 I_3^* &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, \frac{c}{2a} \leq c\sigma \leq 1 \right\} \\
 &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{c} \right\}.
 \end{aligned} \tag{58}$$

This interval is an empty set noting that  $(1/b) > (1/c)$ .

Interval  $I_4^*$ :

$$\begin{aligned}
 I_4^* &:= \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta, 1 < c\sigma \leq c\eta \right\} \\
 &= \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{a} \right\}.
 \end{aligned} \tag{59}$$

In this interval, we have

$$\begin{aligned}
 & T^{-1} \times \int_{I_4^*} x^\sigma \left| \zeta(a\sigma + iat) \left( L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & \ll \max_{(1/b) < \sigma \leq (1/a)} x^\sigma T^{((13/42)+(5/4))(1-a\sigma)} T^{-1+\varepsilon} \\
 & \ll x^{1/a} T^{-1+\varepsilon} + x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon}.
 \end{aligned} \tag{60}$$

From (55)–(60), we can obtain

$$\begin{aligned}
 & |J_2 + J_3| \\
 & \ll T^{-1} \int_{1/2a}^\eta x^\sigma \left| \zeta(a\sigma + iat) \left( L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & = T^{-1} \int_{I_1^* \cup \dots \cup I_4^*} x^\sigma \left| \zeta(a\sigma + iat) \left( L(a\sigma + iat, \text{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \text{sym}^2 f) \right) \right| \cdot \left| \zeta(c\sigma + ict) L(c\sigma + ict, \text{sym}^2 f) \right| d\sigma \\
 & \ll x^{1/2a} T^{(487/168)-(131(b+c)/168a)+\varepsilon} + x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} + x^{1/c} T^{(89/42)-(131(a+b)/84c)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}.
 \end{aligned} \tag{61}$$

Now, we turn to estimate  $J_1$ , and we have

$$\begin{aligned}
 |J_1| & \ll x^{1/2a} \int_1^T \left| L\left(\frac{1}{2} + iat, f \times f\right) L\left(\frac{b}{2a} + ibt, f \times f\right) L\left(\frac{c}{2a} + ict, f \times f\right) \right| t^{-1} dt + x^{1/2a} \\
 & \ll x^{1/2a} + x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| \\
 & \quad \cdot \left| \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) \zeta\left(\frac{c}{2a} + ict\right) L\left(\frac{c}{2a} + ict, \text{sym}^2 f\right) \right| dt.
 \end{aligned} \tag{62}$$



Then, using Lemmas 3 and 4 and Hölder's inequality, we obtain

$$\begin{aligned}
 |J_1| &\ll x^{1/2a} + x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1} T_1^{(13/42)(1-(b/2a))} T_1^{(5/4)(1-(b/2a))} T_1^{(13/42)(1-(c/2a))} T_1^{(5/4)(1-(c/2a))} \\
 &\quad \times \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| dt \\
 &\ll x^{1/2a} + x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} T_1^{(13/42)(1-(b/2a))} T_1^{(5/4)(1-(b/2a))} T_1^{(13/42)(1-(c/2a))} T_1^{(5/4)(1-(c/2a))} \\
 &\quad \times \left( \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{1/12} \left( \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right|^2 dt \right)^{1/2} \left( \int_{T_1/2}^{T_1} 1 dt \right)^{5/12} \\
 &\ll x^{(1/2a)+\varepsilon} T^{(145/42)-(131(b+c)/168a)+\varepsilon}.
 \end{aligned} \tag{63}$$

By putting (53), (61), and (63) together, we have

$$\begin{aligned}
 S_{f \times f}(a, b, c; x) &= M_1 x^{1/a} + M_2 x^{1/b} + M_3 x^{1/c} + O\left(x^\varepsilon \left(x^{1/2a} T^{(145/42)-(131(b+c)/168a)} + x^{1/b} T^{(47/84)-(131a/84b)} + x^{1/c} T^{(89/42)-(131(a+b)/84c)} + x^{1/a} T^{-1+\varepsilon}\right)\right).
 \end{aligned} \tag{64}$$

By taking  $T = x^{(84/748a - 131(b+c))}$  in (64), we can obtain

$$\begin{aligned}
 S_{f \times f}(a, b, c; x) &= M_1 x^{1/a} + M_2 x^{1/b} + M_3 x^{1/c} \\
 &\quad + O\left(x^{(1/a)-(84/748a - 131(b+c))+\varepsilon}\right),
 \end{aligned} \tag{65}$$

which proves the first result in Theorem 2.

For the case  $b < 2a < c$ , we use a similar argument to the first corresponding case. In this case, the points  $s = (1/a)$  and  $s = (1/b)$  are the two simple poles of the integrand in the rectangle  $I_T$ . Then, by Cauchy's residue theorem we have

$$\begin{aligned}
 S_{f \times f}(a, b, c; x) &= \left\{ \text{Res}_{s=(1/a)} + \text{Res}_{s=(1/b)} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} \\
 &\quad + \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta-iT}^{(1/2a)-iT} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} ds + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\
 &= M_1 x^{(1/a)} + M_2 x^{(1/b)} + J_1'' + J_2'' + J_3'' + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right).
 \end{aligned} \tag{66}$$

To estimate  $J_2'' + J_3''$ , we still divide the integral interval into four short intervals  $I_1^{**}, \dots, I_4^{**}$ , which are different from ones for the case  $c \leq 2a$ . In fact, the corresponding short intervals  $I_1^{**}, I_3^{**}$  become empty sets in this situation. However, we still can estimate  $J_2'' + J_3''$  by following a similar argument to the corresponding parts of the case  $c \leq 2a$  and get

$$\begin{aligned}
 |J_2'' + J_3''| &\ll x^{1/2a} T^{(75/56)-(131b/168a)+\varepsilon} + x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} \\
 &\quad + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}.
 \end{aligned} \tag{67}$$

The estimate of  $J_1''$  becomes the following at the current case by noting  $(c/2a) > 1 > (b/2a)$ .

$$\begin{aligned}
|J'_1| &\ll x^{1/2a} \int_1^T \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) \right| \cdot \left| \zeta\left(\frac{c}{2a} + ict\right) L\left(\frac{c}{2a} + ict, \text{sym}^2 f\right) \right| t^{-1} dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} T_1^{(131/84)(1-(b/2a))} \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| dt + x^{1/2a} \\
&\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} T_1^{(131/84)(1-(b/2a))} \left( \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{1/12} \times \left( \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right|^2 dt \right)^{1/2} \left( \int_{T_1/2}^{T_1} 1 dt \right)^{5/12} + x^{1/2a} \\
&\ll x^{1/2a} T^{(159/84)-(131b/168a)+\varepsilon}.
\end{aligned} \tag{68}$$

Thus, recalling (67), we have

$$\begin{aligned}
S_{f \times f}(a, b, c; x) &= M_1 x^{1/a} + M_2 x^{1/b} \\
&\quad + O\left(x^{1/2a} T^{(159/84)-(131b/168a)+\varepsilon}\right. \\
&\quad \left.+ x^{1/b} T^{(47/84)-(131a/84b)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}\right).
\end{aligned} \tag{69}$$

By taking  $T = x^{(84/486a-131b)}$  in (69), we have

$$S_{f \times f}(a, b, c; x) = M_1 x^{1/a} + M_2 x^{1/b} + O\left(x^{1/a-(84/486a-131b)+\varepsilon}\right), \tag{70}$$

which proves the second result of Theorem 2.

For the case  $2a \leq b$ , we use a similar argument to the first corresponding case. In this situation, the point  $s = (1/a)$  is the only simple pole of the integrand in the rectangle  $I_T$ . Then, Cauchy's residue theorem shows

$$\begin{aligned}
S_{f \times f}(a, b, c; x) &= \text{Res}_{s=(1/a)} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right) \\
&\quad + \frac{1}{2\pi i} \left\{ \int_{(1/2a)-iT}^{(1/2a)+iT} + \int_{(1/2a)+iT}^{\eta+iT} + \int_{\eta-iT}^{(1/2a)-iT} \right\} L(as, f \times f) L(bs, f \times f) L(cs, f \times f) \frac{x^s}{s} ds \\
&= M_1 x^{(1/a)} + J'_1 + J'_2 + J'_3 + O\left(\frac{x^{(1/a)+\varepsilon}}{T}\right).
\end{aligned} \tag{71}$$

To estimate  $J'_2 + J'_3$ , we still divide the integral interval into four short intervals  $I_1^{***}, \dots, I_4^{***}$ , which are different from ones for the case  $c \leq 2a$ . In fact, the corresponding short intervals  $I_1^{***}, I_2^{***}, I_3^{***}$  become empty sets in this situation. However, we still can estimate  $J'_2 + J'_3$  by following

a similar argument to the corresponding parts of the case  $c \leq 2a$  and get

$$|J'_2 + J'_3| \ll x^{1/2a} T^{-(37/168)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1+\varepsilon}. \tag{72}$$

The estimate of  $J_1'$  becomes the following by noting  $(c/2a) > (b/2a) > 1$ .

$$\begin{aligned}
 |J_1'| &\ll x^{1/2a} \int_1^T \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) \right| \cdot \left| \zeta\left(\frac{c}{2a} + ict\right) L\left(\frac{c}{2a} + ict, \text{sym}^2 f\right) \right| t^{-1} dt + x^{1/2a} \\
 &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| dt + x^{1/2a} \\
 &\ll x^{1/2a} \log T \max_{T_1 \leq T} T_1^{-1+\varepsilon} \left( \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{1/12} \\
 &\quad \times \left( \int_{T_1/2}^{T_1} \left| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right|^2 dt \right)^{1/2} \left( \int_{T_1/2}^{T_1} 1 dt \right)^{5/12} + x^{1/2a} \\
 &\ll x^{1/2a} T^{(1/3)+\varepsilon}.
 \end{aligned} \tag{73}$$

Thus, recalling (71), we have

$$S_{f \times f}(a, b, c; x) = M_1 x^{1/a} + O\left(x^{1/2a} T^{(1/3)+\varepsilon} + x^{(1/a)+\varepsilon} T^{-1}\right). \tag{74}$$

By taking  $T = x^{(3/8a)}$  in (74), we have

$$S_{f \times f}(a, b, c; x) = M_1 x^{1/a} + O\left(x^{(5/8a)+\varepsilon}\right), \tag{75}$$

which proves the third result of Theorem 2.

## Data Availability

The data supporting the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# On a Certain Quadratic Character Sums of Ternary Symmetry Polynomials mod $p$

Yuanyuan Meng 

School of Mathematics, Northwest University, Xi'an, Shaanxi, China

Correspondence should be addressed to Yuanyuan Meng; yymeng@stumail.nwu.edu.cn

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In this article, we are using the elementary methods and the properties of the classical Gauss sums to study the calculating problem of a certain quadratic character sums of a ternary symmetry polynomials modulo  $p$  and obtain some interesting identities for them.

## 1. Introduction

Let  $p$  be an odd prime,  $(*/p)$  denotes the Legendre symbol mod  $p$ , i.e., for any integer  $n$ , one has

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if } n \text{ is a quadratic residue mod } p, \\ -1, & \text{if } n \text{ is a quadratic nonresidue mod } p, \\ 0, & \text{if } p \mid n. \end{cases} \quad (1)$$

Some of the most commonly used properties of the Legendre symbol are as follows (see [1, 2]):

$$\begin{aligned} \left(\frac{-1}{p}\right) &= (-1)^{(p-1)/2}, \\ \left(\frac{2}{p}\right) &= (-1)^{(p^2-1)/8}, \\ \left(\frac{q}{p}\right) \cdot \left(\frac{p}{q}\right) &= (-1)^{((p-1)(q-1))/4}, \end{aligned} \quad (2)$$

where  $p$  and  $q$  are two different odd primes.

The introduction of the Legendre symbol has not only enriched the content of number theory but also greatly promoted the development of elementary and analytic number theory, especially the research on the properties of

primes. For example, if  $p$  is a prime with  $p \equiv 1 \pmod{4}$ , then for any integers  $r$  and  $s$  with  $(rs/p) = -1$ , one has the identity (see Theorems 4–11 in [2])

$$\begin{aligned} p &= \alpha^2(p) + \beta^2(p) \\ &= \left( \sum_{a=1}^{(p-1)/2} \left( \frac{a+r\bar{a}}{p} \right) \right)^2 + \left( \sum_{b=1}^{(p-1)/2} \left( \frac{b+s\bar{b}}{p} \right) \right)^2. \end{aligned} \quad (3)$$

From (3), we naturally wonder, for other forms of primes  $p$ , can they also be expressed in terms of Legendre's symbol?

In particular, if  $p$  is an odd prime with  $p \equiv 1 \pmod{3}$ , then there are two integers  $d$  and  $b$  such that the identity (see [3])

$$4p = d^2 + 27b^2, \quad (4)$$

where  $d$  is uniquely determined by  $d \equiv 1 \pmod{3}$ .

In addition, if  $p$  is an odd with  $p \equiv 3 \pmod{4}$ , then there are two integers  $x$  and  $y$  such that

$$p = x^2 - 2 \cdot \left( \frac{2}{p} \right) \cdot y^2. \quad (5)$$

Although we have not found the representations of  $d$  and  $b$  or  $x$  and  $y$  in terms of the Legendre symbol modulo  $p$ , we found that a certain quadratic character sum of the ternary symmetry polynomials are closely related to the numbers  $d$  and  $b$ .

In this paper, we shall use elementary methods and the properties of the classical Gauss sums to study the calculating problem of a certain quadratic character sum of binary symmetry polynomials modulo  $p$  and obtain several interesting identities for them. That is, we shall prove the following results.

**Theorem 1.** *Let  $p$  be an odd prime with  $(3, p-1) = 1$ . Then, we have*

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4 bc + b^4 ac + c^4 ab + abc}{p} \right) = - \left( \frac{-1}{p} \right) \cdot p. \quad (6)$$

**Theorem 2.** *Let  $p$  be an odd prime with  $p \equiv 1 \pmod{6}$ . If 2 is a cubic residue modulo  $p$ , then we have the identity*

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4 bc + b^4 ac + c^4 ab + abc}{p} \right) \\ &= \left( \frac{-1}{p} \right) \cdot (9p d - 5p - d^2), \end{aligned} \quad (7)$$

where  $d$  is the same as defined in (4).

**Theorem 3.** *Let  $p$  be an odd prime with  $p \equiv 1 \pmod{6}$ . If 2 is not a cubic residue modulo  $p$ , then we have the identity*

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4 bc + b^4 ac + c^4 ab + abc}{p} \right) \\ &= -\frac{1}{2} \cdot \left( \frac{-1}{p} \right) \cdot (9pd + 9pb + 10p + 2d^2), \\ & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4 bc + b^4 ac + c^4 ab + abc}{p} \right) \\ &= -\frac{1}{2} \cdot \left( \frac{-1}{p} \right) \cdot (9pd - 9pb + 10p + 2d^2). \end{aligned} \quad (8)$$

From these theorems, we may immediately deduce the following three corollaries.

**Corollary 1.** *Let  $p$  be an odd prime with  $(3, p-1) = 1$ . Then, we have*

$$\left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4 bc + b^4 ac + c^4 ab + abc}{p} \right) \right| = p. \quad (9)$$

**Corollary 2.** *Let  $p$  be a prime with  $p \equiv 1 \pmod{6}$ . If 2 is a cubic residue modulo  $p$ , then we have*

$$\left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4 bc + b^4 ac + c^4 ab + abc}{p} \right) \right| = |9pd - 5p - d^2|. \quad (10)$$

**Corollary 3.** *If  $p$  is an odd prime with  $p \equiv 1 \pmod{6}$ , if 2 is not a cubic residue modulo  $p$ , then we have*

$$\begin{aligned} & \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4 bc + b^4 ac + c^4 ab + abc}{p} \right) \right| \\ &= \frac{1}{2} \cdot |9pd + 9pb + 10p + 2d^2| \text{ or } \\ & \frac{1}{2} \cdot |9pd - 9pb + 10p + 2d^2|. \end{aligned} \quad (11)$$

Notes: it is clear that our method is applicable to multi-variate symmetry polynomials  $f(x_1, x_2, \dots, x_k)$ . But, when  $k$  is larger, the calculation is more complicated, so we do not give it.

Theorem 3 is flawed. In other words, it gives us two possibilities. We do not know for sure which one is the exact value. How to determine its exact value is an open problem. Interested readers are encouraged to join us in the research.

## 2. Several Lemmas

In this section, we first give several simple lemmas. Of course, the proofs of these lemmas need some knowledge of elementary and analytic number theory. They can be found in many number theory books, such as [1, 2]. Other papers related to Gauss sums and character sums can also be found in [4–11]; here, we do not need to list.

**Lemma 1.** *Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . Then, for any third-order character  $\lambda \pmod{p}$ , one has the identity*

$$\tau^3(\lambda) + \tau^3(\bar{\lambda}) = dp, \quad (12)$$

where  $d$  is the same as defined in (4).

*Proof.* See the work of Zhang and Hu [12].  $\square$

**Lemma 2.** *Let  $p$  be a prime with  $p \equiv 1 \pmod{6}$ . Then, for any sixth-order character  $\psi \pmod{p}$ , one has the identity*

$$\tau^3(\psi) + \tau^3(\bar{\psi}) = \begin{cases} p^{(1/2)} \cdot (d^2 - 2p), & \text{if } p \equiv 1 \pmod{12}; \\ -i \cdot p^{(1/2)} \cdot (d^2 - 2p), & \text{if } p \equiv 7 \pmod{12}, \end{cases} \quad (13)$$

where  $i^2 = -1$  and  $d$  is the same as defined in (4).

*Proof.* This result is Lemma 3 in the work of Chen [4], so we omit the proof process.  $\square$

**Lemma 3.** *Let  $p$  be a prime with  $p \equiv 1 \pmod{6}$ . Then for any third-order character  $\lambda \pmod{p}$ , we have the identity*

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= \left( \frac{-1}{p} \right) \cdot (-p + \bar{\lambda}(2) \cdot \tau^3(\lambda) + \lambda(2) \tau^3(\bar{\lambda})), \end{aligned} \quad (14)$$

where  $(*/p) = \chi_2$  denotes the Legendre symbol modulo  $p$ .

*Proof.* Note that  $\chi_2^2 = \chi_0$  and  $\tau^2(\chi_2) = \chi_2(-1) \cdot p$ ; from the properties of Gauss sums and the Legendre symbol modulo  $p$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} \chi_2(a+b+c+1) &= - \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \chi_2(a+b+1) \\ &= \frac{-1}{\tau(\chi_2)} \cdot \sum_{d=1}^{p-1} \chi_2(d) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) e\left(\frac{d(a+b+1)}{p}\right), \\ &= -\tau(\chi_2) \cdot \sum_{d=1}^{p-1} \chi_2(d) e\left(\frac{d}{p}\right) = -\tau^2(\chi_2). \end{aligned} \quad (15)$$

It is clear that  $\lambda(-1) = 1$  and

and from the properties of the Gauss sums, we can get

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1 + \left(\frac{a}{p}\right)\right) e\left(\frac{ba}{p}\right) = \chi_2(b) \tau(\chi_2), \quad (16)$$

$$\begin{aligned} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} \lambda(c) \chi_2(a+b+c+1) \\ &= \frac{1}{\tau(\chi_2)} \cdot \sum_{d=1}^{p-1} \chi_2(d) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{d(a+b+c+1)}{p}\right), \\ &= \frac{\tau^2(\chi_2) \cdot \tau(\lambda)}{\tau(\chi_2)} \sum_{d=1}^{p-1} \chi_2(d) \bar{\lambda}(d) e\left(\frac{d}{p}\right) = \tau(\chi_2) \cdot \tau(\lambda) \cdot \tau(\chi_2 \bar{\lambda}), \end{aligned} \quad (17)$$

$$\begin{aligned} \sum_{a=0}^{p-1} \chi_2 \lambda(a^2 - 1) &= \frac{1}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2 \bar{\lambda}(b) \sum_{a=0}^{p-1} e\left(\frac{b(a^2 - 1)}{p}\right), \\ &= \frac{\tau(\chi_2)}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2 \bar{\lambda}(b) \chi_2(b) e\left(\frac{-b}{p}\right) = \frac{\tau(\chi_2) \cdot \tau(\bar{\lambda})}{\tau(\chi_2 \bar{\lambda})}. \end{aligned} \quad (18)$$

On the other hand, note that  $\bar{\lambda}^2 = \lambda$ ; we also have

$$\begin{aligned} \sum_{a=0}^{p-1} \chi_2 \lambda(a^2 - 1) &= \sum_{a=0}^{p-1} \chi_2 \lambda((a+1)^2 - 1) = \sum_{a=1}^{p-1} \chi_2 \lambda(a^2 + 2a), \\ &= \frac{1}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2 \bar{\lambda}(b) \sum_{a=1}^{p-1} \chi_2 \lambda(a) e\left(\frac{b(a+2)}{p}\right), \\ &= \frac{\tau(\chi_2 \lambda)}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2 \bar{\lambda}(b) \chi_2(b) \bar{\lambda}(b) e\left(\frac{2b}{p}\right) = \frac{\bar{\lambda}(2) \cdot \tau(\lambda) \cdot \tau(\chi_2 \lambda)}{\tau(\chi_2 \bar{\lambda})}. \end{aligned} \quad (19)$$

Note that  $\tau(\lambda) \cdot \tau(\bar{\lambda}) = p$ ; from (17)–(19), we have the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} \lambda(c) \chi_2(a+b+c+1) = \left(\frac{-1}{p}\right) \cdot \bar{\lambda}(2) \cdot \tau^3(\lambda). \quad (20)$$

Similarly, we also get

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} \bar{\lambda}(c) \chi_2(a+b+c+1) = \left(\frac{-1}{p}\right) \cdot \lambda(2) \cdot \tau^3(\bar{\lambda}). \quad (21)$$

Now, combining (15), (20), and (21), we have the identity

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= -\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot (\bar{\lambda}(2) \cdot \tau^3(\lambda) + \lambda(2) \tau^3(\bar{\lambda})). \end{aligned} \quad (22)$$

This proves Lemma 3.  $\square$

**Lemma 4.** Let  $p$  be a prime with  $p \equiv 1 \pmod{6}$ . Then, for any third-order character  $\lambda \pmod{p}$ , we have the identity

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= -\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot (\lambda(2) + \bar{\lambda}(2)) \cdot \tau^3(\bar{\lambda}), \\ & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \bar{\lambda}(a) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= -\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot (\lambda(2) + \bar{\lambda}(2)) \cdot \tau^3(\bar{\lambda}). \end{aligned} \quad (23)$$

*Proof.* Note that  $\tau(\chi_2 \lambda) \cdot \tau(\chi_2 \bar{\lambda}) = \chi_2(-1) \cdot p$ ; from the methods of proving Lemma 3, we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \sum_{c=1}^{p-1} \chi_2(a+b+c+1) \\ &= -\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \chi_2(a+b+1), \\ &= -\frac{1}{\tau(\chi_2)} \sum_{c=1}^{p-1} \chi_2(c) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) e\left(\frac{c(a+b+1)}{p}\right) \\ &= -\tau(\chi_2 \lambda) \cdot \tau(\chi_2 \bar{\lambda}) = -\left(\frac{-1}{p}\right) \cdot p. \end{aligned} \quad (24)$$

Applying (18) and (19), we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \sum_{c=1}^{p-1} \lambda(c) \chi_2(a+b+c+1) \\ &= \frac{1}{\tau(\chi_2)} \sum_{d=1}^{p-1} \chi_2(d) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \\ & \quad \cdot \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{d(a+b+c+1)}{p}\right), \\ &= \tau(\lambda) \cdot \tau(\chi_2 \lambda) \cdot \sum_{d=1}^{p-1} \chi_2(d) \lambda(d) e\left(\frac{d}{p}\right) \\ &= \tau(\lambda) \cdot \tau^2(\chi_2 \lambda), \\ &= \left(\frac{-1}{p}\right) \cdot \bar{\lambda}(2) \cdot \tau^3(\bar{\lambda}). \end{aligned} \quad (25)$$

Similarly, we also have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \sum_{c=1}^{p-1} \bar{\lambda}(c) \chi_2(a+b+c+1) \\ &= \left(\frac{-1}{p}\right) \cdot \lambda(2) \cdot \tau^3(\bar{\lambda}). \end{aligned} \quad (26)$$

Combining (24)–(26), we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= -\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot (\lambda(2) + \bar{\lambda}(2)) \cdot \tau^3(\bar{\lambda}). \end{aligned} \quad (27)$$

Taking the conjugate in (27), we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \bar{\lambda}(a) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= -\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot (\lambda(2) + \bar{\lambda}(2)) \cdot \tau^3(\lambda). \end{aligned} \quad (28)$$

Now, Lemma 4 follows from (27) and (28).  $\square$

**Lemma 5.** Let  $p$  be a prime with  $p \equiv 1 \pmod{6}$ . Then, for any third-order character  $\lambda \pmod{p}$ , we have the identities



$$\begin{aligned}
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\
&= -\frac{\tau^3(\chi_2 \lambda)}{\tau(\chi_2)} + \left(\frac{-1}{p}\right) \cdot (\bar{\lambda}(2) \cdot \tau^3(\bar{\lambda}) + \lambda(2) \cdot \tau^3(\lambda)), \\
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \bar{\lambda}(ab) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\
&= -\frac{\tau^3(\chi_2 \lambda)}{\tau(\chi_2)} + \left(\frac{-1}{p}\right) \cdot (\bar{\lambda}(2) \cdot \tau^3(\bar{\lambda}) + \lambda(2) \cdot \tau^3(\lambda)).
\end{aligned} \tag{29}$$

*Proof.* Note that  $\bar{\lambda}^2 = \lambda$ ; from (18), (19), and the methods of proving Lemma 3, we have

$$\begin{aligned}
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \sum_{c=1}^{p-1} \chi_2(a+b+c+1) \\
&= -\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \chi_2(a+b+1), \\
&= -\frac{1}{\tau(\chi_2)} \sum_{c=1}^{p-1} \chi_2(c) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) e\left(\frac{c(a+b+1)}{p}\right) = -\frac{\tau^3(\chi_2 \lambda)}{\tau(\chi_2)},
\end{aligned} \tag{30}$$

$$\begin{aligned}
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \sum_{c=1}^{p-1} \lambda(c) \chi_2(a+b+c+1) \\
&= \frac{1}{\tau(\chi_2)} \sum_{d=1}^{p-1} \chi_2(d) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{d(a+b+c+1)}{p}\right), \\
&= \tau(\lambda) \cdot \tau^2(\chi_2 \lambda) = \left(\frac{-1}{p}\right) \cdot \bar{\lambda}(2) \cdot \tau^3(\bar{\lambda}),
\end{aligned} \tag{31}$$

$$\begin{aligned}
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \sum_{c=1}^{p-1} \bar{\lambda}(c) \chi_2(a+b+c+1) \\
&= \frac{1}{\tau(\chi_2)} \sum_{d=1}^{p-1} \chi_2(d) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \sum_{c=1}^{p-1} \bar{\lambda}(c) e\left(\frac{d(a+b+c+1)}{p}\right), \\
&= \frac{1}{\tau(\chi_2)} \cdot \tau(\bar{\lambda}) \cdot \tau^2(\chi_2 \lambda) \cdot \tau(\chi_2 \bar{\lambda}) = \left(\frac{-1}{p}\right) \cdot \lambda(2) \cdot \tau^3(\lambda).
\end{aligned} \tag{32}$$

Combining (30)–(32), we have

$$\begin{aligned}
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(ab) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\
&= -\frac{\tau^3(\chi_2 \lambda)}{\tau(\chi_2)} + \left(\frac{-1}{p}\right) \cdot (\bar{\lambda}(2) \cdot \tau^3(\bar{\lambda}) + \lambda(2) \cdot \tau^3(\lambda)).
\end{aligned} \tag{33}$$

Taking the conjugate given above, we can deduce the other identity. This proves Lemma 5.  $\square$

**Lemma 6.** *Let  $p$  be a prime with  $p \equiv 1 \pmod{6}$ . Then, for any third-order character  $\lambda \pmod{p}$ , we have the identity*

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= -\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot (\lambda(2) \cdot \tau^3(\bar{\lambda}) + \bar{\lambda}(2) \cdot \tau^3(\lambda)). \end{aligned} \quad (34)$$

*Proof.* From (18), (19), and the methods of proving Lemma 3, we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \sum_{c=1}^{p-1} \chi_2(a+b+c+1) \\ &= -\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \chi_2(a+b+1), \\ &= -\frac{1}{\tau(\chi_2)} \sum_{c=1}^{p-1} \chi_2(c) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) e\left(\frac{c(a+b+1)}{p}\right), \\ &= -\tau(\chi_2 \lambda) \tau(\chi_2 \bar{\lambda}) = -\left(\frac{-1}{p}\right) \cdot p, \end{aligned} \quad (35)$$

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \sum_{c=1}^{p-1} \lambda(c) \chi_2(a+b+c+1) \\ &= \frac{1}{\tau(\chi_2)} \sum_{d=1}^{p-1} \chi_2(d) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{d(a+b+c+1)}{p}\right), \\ &= \frac{\tau(\lambda) \cdot \tau^2(\chi_2 \bar{\lambda}) \cdot \tau(\chi_2 \lambda)}{\tau(\chi_2)} = \left(\frac{-1}{p}\right) \cdot \bar{\lambda}(2) \cdot \tau^3(\lambda), \end{aligned} \quad (36)$$

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \sum_{c=1}^{p-1} \bar{\lambda}(c) \chi_2(a+b+c+1) \\ &= \frac{1}{\tau(\chi_2)} \sum_{d=1}^{p-1} \chi_2(d) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \sum_{c=1}^{p-1} \bar{\lambda}(c) e\left(\frac{d(a+b+c+1)}{p}\right), \\ &= \frac{\tau(\lambda) \cdot \tau^2(\chi_2 \bar{\lambda}) \cdot \tau(\chi_2 \lambda)}{\tau(\chi_2)} = \left(\frac{-1}{p}\right) \cdot \lambda(2) \cdot \tau^3(\lambda). \end{aligned} \quad (37)$$

Combining (35)–(37), we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \lambda(a) \bar{\lambda}(b) \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a+b+c+1) \\ &= -\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot (\lambda(2) \cdot \tau^3(\bar{\lambda}) + \bar{\lambda}(2) \cdot \tau^3(\lambda)). \end{aligned} \quad (38)$$

This proves Lemma 6.  $\square$

### 3. Proofs of the Theorems

Now, we prove our theorems. From the properties of the reduced residue system modulo  $p$ , we have

$$\begin{aligned}
 & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi_2(a^4 bc + b^4 ac + c^4 ab + abc) \\
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi_2(a^4 bc^6 + b^4 ac^6 + c^6 ab + abc^3), \\
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} \chi_2(a^3 + b^3 + 1 + \bar{c}^3), \\
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(a^3 b^3) \sum_{c=1}^{p-1} \chi_2(a^3 + b^3 + c^3 + 1).
 \end{aligned} \tag{39}$$

If  $(3, p-1) = 1$ , when  $a$  passes through a reduced residue system modulo  $p$ , then  $a^3$  also passes through a reduced residue system modulo  $p$ . So, from (39) and (15), we have the identity

$$\begin{aligned}
 & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi_2(a^4 bc + b^4 ac + c^4 ab + abc) \\
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{c=1}^{p-1} \chi_2(a + b + c + 1) = -\tau^2(\chi_2) \\
 &= -\left(\frac{-1}{p}\right) \cdot p.
 \end{aligned} \tag{40}$$

This proves Theorem 1.

If  $p \equiv 1 \pmod{6}$ , let  $\lambda$  be a third-order character modulo  $p$ ; then, for any integer  $a$  with  $(a, p) = 1$ , we have the identity

$$1 + \lambda(a) + \bar{\lambda}(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue modulo } p; \\ 0, & \text{if } a \text{ is not a cubic residue modulo } p. \end{cases} \tag{41}$$

So, from the symmetry of  $a$  and  $b$ , (39), and Lemmas 3–6, we have

$$\begin{aligned}
 & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi_2(a^4 bc + b^4 ac + c^4 ab + abc) \\
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(a) \chi_2(b) (1 + \lambda(a) + \bar{\lambda}(a)) \cdot (1 + \lambda(b) + \bar{\lambda}(b)) \\
 &\quad \times \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a + b + c + 1), \\
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(a) \chi_2(b) (1 + 2\lambda(a) + 2\bar{\lambda}(a) + \lambda(ab) + \bar{\lambda}(ab) + 2\lambda(a)\bar{\lambda}(b)) \\
 &\quad \times \sum_{c=1}^{p-1} (1 + \lambda(c) + \bar{\lambda}(c)) \chi_2(a + b + c + 1), \\
 &= \left(\frac{-1}{p}\right) \cdot (-p + \bar{\lambda}(2) \cdot \tau^3(\lambda) + \lambda(2) \tau^3(\bar{\lambda})) \\
 &\quad - 2\left(\frac{-1}{p}\right) \cdot p + 2\left(\frac{-1}{p}\right) (\bar{\lambda}(2) + \lambda(2)) \cdot \tau^3(\bar{\lambda}) \\
 &\quad - 2\left(\frac{-1}{p}\right) \cdot p + 2\left(\frac{-1}{p}\right) (\bar{\lambda}(2) + \lambda(2)) \cdot \tau^3(\lambda) \\
 &\quad - \frac{\tau^3(\chi_2 \lambda)}{\tau(\chi_2)} + \left(\frac{-1}{p}\right) \cdot (\bar{\lambda}(2) \tau^3(\bar{\lambda}) + \lambda(2) \cdot \tau^3(\lambda)) \\
 &\quad - \frac{\tau^3(\chi_2 \bar{\lambda})}{\tau(\chi_2)} + \left(\frac{-1}{p}\right) \cdot (\bar{\lambda}(2) \cdot \tau^3(\bar{\lambda}) + \lambda(2) \cdot \tau^3(\lambda)) \\
 &\quad - 2\left(\frac{-1}{p}\right) \cdot p + 2\left(\frac{-1}{p}\right) \cdot (\lambda(2) \cdot \tau^3(\bar{\lambda}) + \bar{\lambda}(2) \cdot \tau^3(\lambda)), \\
 &= -7\left(\frac{-1}{p}\right) \cdot p + \left(\frac{-1}{p}\right) \cdot ((4\lambda(2) + 5\bar{\lambda}(2)) \cdot \tau^3(\lambda) + (5\lambda(2) + 4\bar{\lambda}(2)) \tau^3(\bar{\lambda})) \\
 &\quad - \frac{\tau^3(\chi_2 \lambda) + \tau^3(\chi_2 \bar{\lambda})}{\tau(\chi_2)}.
 \end{aligned} \tag{42}$$

If 2 is a cubic residue modulo  $p$ , then  $\lambda(2) = \bar{\lambda}(2) = 1$ . Note that  $\chi_2\lambda = \psi$  is a sixth-order character modulo  $p$ ,  $\chi_2(-1) = 1$ , and  $\tau(\chi_2) = \sqrt{p}$ , if  $p \equiv 1 \pmod{12}$ ;  $\chi_2(-1) = -1$  and  $\tau(\chi_2) = i \cdot \sqrt{p}$ , if  $p \equiv 7 \pmod{12}$ . From (42) and Lemmas 1 and 2, we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4bc + b^4ac + c^4ab + abc}{p} \right) \\ &= \left( \frac{-1}{p} \right) \cdot (9pd - 5p - d^2). \end{aligned} \quad (43)$$

This proves Theorem 2.

Now, we prove Theorem 3. If 2 is not a cubic residue modulo  $p$ , then we have  $1 + \lambda(2) + \bar{\lambda}(2) = 0$  or  $\lambda(2) + \bar{\lambda}(2) = -1$ . From (42) and Lemmas 1 and 2, we have

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4bc + b^4ac + c^4ab + abc}{p} \right) \\ &= -\left( \frac{-1}{p} \right) \cdot (4pd + 5p + d^2) \\ &+ \left( \frac{-1}{p} \right) \cdot (\bar{\lambda}(2) \cdot \tau^3(\lambda) + \lambda(2)\tau^3(\bar{\lambda})). \end{aligned} \quad (44)$$

Since  $1 + \lambda(2) + \bar{\lambda}(2) = 0$  or  $\lambda^2(2) + \lambda(2) + 1 = 0$ , we have

$$\lambda(2) = \frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot i, \quad (45)$$

$$\bar{\lambda}(2) = \frac{1}{2} \mp \frac{\sqrt{3}}{2} \cdot i, \quad \text{where } i^2 = -1,$$

$$\begin{aligned} (\tau^3(\lambda) - \tau^3(\bar{\lambda}))^2 &= (\tau^3(\lambda) + \tau^3(\bar{\lambda}))^2 - 4p^3, \\ &= (dp)^2 - 4p^3 = p^2(d^2 - 4p) = -27p^2b^2, \end{aligned} \quad (46)$$

or

$$\tau^3(\lambda) - \tau^3(\bar{\lambda}) = \pm 3 \cdot \sqrt{3} \cdot p \cdot b \cdot i. \quad (47)$$

So, from (45)–(47) and Lemma 1, we have

$$\bar{\lambda}(2) \cdot \tau^3(\lambda) + \lambda(2)\tau^3(\bar{\lambda}) = -\frac{dp}{2} \pm \frac{9pb}{2}. \quad (48)$$

Combining (44) and (48), we have the identity

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left( \frac{a^4bc + b^4ac + c^4ab + abc}{p} \right) \\ &= -\left( \frac{-1}{p} \right) \cdot (4pd + 5p + d^2) - \left( \frac{-1}{p} \right) \cdot \frac{p}{2} \cdot (d \pm 9b). \end{aligned} \quad (49)$$

This completes the proof of Theorem 3.

## 4. Conclusions

The main results of this paper are three identities involving a certain quadratic character sums of ternary symmetry polynomials modulo  $p$ . Theorem 1 proved an exact identity for the sum, if  $(3, p-1) = 1$ . Theorem 2 discussed the case  $p \equiv 1 \pmod{3}$  and obtained an exact identity, provided 2 is a cubic residue modulo  $p$ . Theorem 3 also discussed the case  $p \equiv 1 \pmod{3}$ , where 2 is not a cubic nonresidue modulo  $p$ . In this case, two possibilities are given. These results not only give the exact values of a certain quadratic character sums of ternary polynomials modulo  $p$  but also some new contribution to research in related fields.

## Data Availability

No data were used in this paper.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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## Research Article

# The Recursive Properties of the Error Term of the Fourth Power Mean of the Generalized Cubic Gauss Sums

Zhang Jin<sup>1</sup> and Zhang Jiafan<sup>2</sup> 

<sup>1</sup>School of Information Engineering, Xi'an University, Xi'an 710065, China

<sup>2</sup>School of Mathematics, Northwest University, Xi'an 710127, China

Correspondence should be addressed to Zhang Jiafan; zhangjiafan@stumail.nwu.edu.cn

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In this paper, we use the analytic methods and the properties of the classical Gauss sums to study the properties of the error term of the fourth power mean of the generalized cubic Gauss sums and give two recurrence formulae for it.

## 1. Introduction

For any integer  $q \geq 2$  and any Dirichlet character  $\chi \bmod q$ , the definition of the classical Gauss sums  $G(m, \chi; q)$  is

$$G(m, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma}{q}\right), \quad (1)$$

where  $m$  is an integer and  $e(y) = e^{2\pi i y}$ .

This sum and its properties are of great significance to the analytic number theory, and many number theory problems are closely related to them. Therefore, it is necessary to study the various properties of  $G(m, \chi; q)$  and related sums. In this paper, we consider the generalized  $k$ -th Gauss sums  $G(m, k, \chi; q)$ , which is defined as follows:

$$G(m, k, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma^k}{q}\right), \quad (2)$$

where  $k$  is any positive integer and  $m$  is an integer with  $(m, q) = 1$ .

It is clear that this sum is a generalization of the classical Gauss sums  $G(m, \chi; q)$ . In fact,  $G(m, 1, \chi; q) = G(m, \chi; q)$ . Of course, the value of  $G(m, k, \chi; q)$  is irregular as  $\chi$  varies. However, some scholars have found that  $G(m, k, \chi; q)$  has good value distribution properties in some problems of weighted mean value, even if we can get their exact calculation formulae for some  $2k$ th power mean. In addition, there are some good upper bound estimates for  $|G(m, k, \chi; q)|$ . For example, for any integer  $n$  with  $(n, q) = 1$ , from the general result of Cochrane and Zheng [1], we can deduce

$$|G(m, k, \chi; q)| \leq 2^{\omega(q)} q^{(1/2)}, \quad (3)$$

where  $\omega(q)$  denotes the number of distinct prime divisors of  $q$ . The case that  $q$  is a prime is due to Weil [2].

For  $k = 2$ , by the results of Zhang [3], let  $n$  be any integer with  $(n, p) = 1$ , and there are the following two identities:

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(n, 2, \chi; p)|^4 = \begin{cases} 3p^2 - 6p - 1 + 4\left(\frac{n}{p}\right)\sqrt{p}, & \text{if } p \equiv 1 \pmod{4}, \\ 3p^2 - 6p - 1, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (4)$$

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(n, 2, \chi; p)|^6 = 10p^3 - 25p^2 - 4p - 1, \quad \text{if } p \equiv 3 \pmod{4},$$

where  $(*/p)$  denotes the Legendre symbol modulo  $p$ .

Zhang and Liu [4] have studied the sum

$$\sum_{\chi \bmod p} |G(n, 3, \chi; q)|^4 \quad (5)$$

and obtained the following calculation formula:

$$\sum_{\chi \bmod p} |G(1, 3, \chi; p)|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U, \quad (6)$$

where  $p$  is a prime with  $3|p-1$  and  $U = \sum_{a=1}^p e(a^3/p)$  is a real constant.

However, the value of  $U$  was not given in [4], and the form of  $U$  was not concise. Now, for any integer  $1 \leq m \leq p-1$ , we let

$$E(m, p) = \frac{1}{p-1} \sum_{\chi \bmod p} |G(m, 3, \chi; p)|^4 - 5p^2 + 13p + 1. \quad (7)$$

In this paper, we use the analytic methods and the properties of the classical Gauss sums to study the

calculating problem of the  $n$ th power mean of  $E(m, p)$  and give two recurrence formulae for it. That is, we shall prove the following main results.

**Theorem 1.** *Let  $p$  be an odd prime with  $3|p-1$ . Then, for any positive integer  $n$  and integer  $m$  with  $(m, p) = 1$ , we have the third-order linear recurrence formula:*

$$E^n(m, p) = U(p, d) \cdot E^{n-3}(m, p) + V(p, d) \cdot E^{n-2}(m, p), \quad (8)$$

where  $U(p, d)$  and  $V(p, d)$  are defined as

$$\begin{aligned} U(p, d) &= 64dp + (d^5 - 12d^4 + 60d^3 - 160d^2 + 240d - 384)p^2 \\ &\quad - (5d^3 - 48d^2 + 180d - 320)p^3 + (5d - 24)p^4, \\ V(p, d) &= 3p \cdot (p^2 + 8p - 4dp + 4d^2 - 16d + 16), \end{aligned} \quad (9)$$

and  $4p = d^2 + 27b^2$ , where  $d$  is uniquely determined by  $d \equiv 1 \pmod{3}$  (see [5]).

**Theorem 2.** *Let  $p$  be an odd prime with  $3|p-1$ . Then, for any positive integer  $n \geq 3$ , we have the recurrence formula*

$$\begin{aligned} \sum_{m=1}^{p-1} E^n(m, p) &= U(p, d) \sum_{m=1}^{p-1} E^{n-3}(m, p) \\ &\quad + V(p, d) \sum_{m=1}^{p-1} E^{n-2}(m, p), \end{aligned} \quad (10)$$

where the first three terms of  $\sum_{m=1}^{p-1} E^n(m, p)$  are

$$\begin{aligned} \sum_{m=1}^{p-1} E(m, p) &= 0, \\ \sum_{m=1}^{p-1} E^2(m, p) &= 2p(p-1)(p-2d+4)^2, \\ \sum_{m=1}^{p-1} E^3(m, p) &= (p-1)U(p, d). \end{aligned} \quad (11)$$

**Theorem 3.** *Let  $p$  be an odd prime with  $3|p-1$ . Then, for any positive integer  $n \geq 1$ , we have the third-order linear recurrence formula*

$$\sum_{m=1}^{p-1} E^{-n}(m, p) = \frac{1}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-3)}(m, p) - \frac{V(p, d)}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-1)}(m, p), \quad (12)$$

where the initial values of  $\sum_{m=1}^{p-1} E^{-n}(m, p)$  are

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-1}(m, p) &= \frac{2p(p-1)(p-2d+4)^2 - (p-1)V(p, d)}{U(p, d)}, \\ \sum_{m=1}^{p-1} E^{-2}(m, p) &= \frac{(p-1)V^2(p, d) - 2p(p-1)(p-2d+4)^2 V(p, d)}{U^2(p, d)}, \\ \sum_{m=1}^{p-1} E^{-3}(m, p) &= \frac{p-1}{U(p, d)} + \frac{2p(p-1)(p-2d+4)^2 V^2(p, d) - (p-1)V^3(p, d)}{U^3(p, d)}.\end{aligned}\tag{13}$$

Taking  $n = 4$  in Theorem 3, we may immediately deduce the following corollary.

**Corollary 1.** Let  $p$  be an odd prime with  $3|p-1$ ; then, we have the identity

$$\begin{aligned}\frac{1}{p-1} \sum_{m=1}^{p-1} \frac{1}{|E(m, p)|^4} \\ = \frac{2p(p-2d+4)^2 - 2V(p, d)}{U^2(p, d)} + \frac{V^4(p, d) - 2p(p-2d+4)^2 V^3(p, d)}{U^4(p, d)}.\end{aligned}\tag{14}$$

## 2. Several Lemmas

In this section, we give three lemmas which are necessary in the proofs of our theorems. In the process of proving our lemmas, we need some knowledge of the analytic number theory; all of which can be found in [6–8], so it is not necessary to repeat them here.

**Lemma 1.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{3}$ . Then, for any third-order character  $\lambda \pmod{p}$ , we have

$$\tau^3(\lambda) + \tau^3(\bar{\lambda}) = dp, \tag{15}$$

where  $d$  is uniquely determined by  $4p = d^2 + 27b^2$  and  $d \equiv 1 \pmod{3}$ .

*Proof.* This result can be found in [9] or [10].  $\square$

**Lemma 2.** Let  $p$  be an odd prime with  $3|p-1$ . Then, for any cubic character  $\lambda \pmod{p}$ , we have the identities

$$\begin{aligned}E(m, p) &= \bar{\lambda}(m) \left( \sum_{a=1}^{p-1} \bar{\lambda}(a^3 - 1) \right)^2 \cdot \tau(\lambda) + \lambda(m) \left( \sum_{b=1}^{p-1} \lambda(b^3 - 1) \right)^2 \cdot \tau(\bar{\lambda}) \\ &= \bar{\lambda}(m) \tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2.\end{aligned}\tag{16}$$

*Proof.* For any integer  $1 \leq a \leq p-1$ , it is easy to show that

From the properties of the cubic character modulo  $p$ , we have

$$1 + \lambda(a) + \lambda^2(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue mod } p, \\ 0, & \text{otherwise.} \end{cases}\tag{17}$$

$$\begin{aligned}\lambda^2 &= \bar{\lambda}, \\ \lambda(-1) &= 1, \\ \overline{\tau(\bar{\lambda})} &= \tau(\bar{\lambda}).\end{aligned}\tag{18}$$

So, we have the identity

$$\begin{aligned}
 \sum_{a=1}^{p-1} \lambda(a^3 - 1) &= \frac{1}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \sum_{a=1}^{p-1} \bar{\lambda}(b) e\left(\frac{b(a^3 - 1)}{p}\right) \\
 &= \frac{1}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \bar{\lambda}(b) e\left(\frac{-b}{p}\right) \sum_{a=1}^{p-1} (1 + \lambda(a) + \bar{\lambda}(a)) e\left(\frac{ba}{p}\right) \\
 &= \frac{1}{\tau(\bar{\lambda})} \sum_{b=1}^{p-1} \bar{\lambda}(b) e\left(\frac{-b}{p}\right) (-1 + \bar{\lambda}(b)\tau(\lambda) + \lambda(b)\tau(\bar{\lambda})) \\
 &= \frac{1}{\tau(\bar{\lambda})} \left( -\lambda(-1)\tau(\bar{\lambda}) + \tau(\lambda) \sum_{b=1}^{p-1} \lambda(b) e\left(\frac{-b}{p}\right) + \tau(\bar{\lambda}) \sum_{b=1}^{p-1} e\left(\frac{-b}{p}\right) \right) \\
 &= \frac{1}{\tau(\bar{\lambda})} (-\tau(\bar{\lambda}) + \tau^2(\lambda) - \tau(\bar{\lambda})) = \frac{\tau^3(\lambda)}{p} - 2.
 \end{aligned} \tag{19}$$

Therefore,

$$\begin{aligned}
 E(m, p) &= \bar{\lambda}(m)\tau(\lambda) \left( \sum_{a=1}^{p-1} \bar{\lambda}(a^3 - 1) \right)^2 + \lambda(m)\tau(\bar{\lambda}) \left( \sum_{a=1}^{p-1} \lambda(a^3 - 1) \right)^2 \\
 &= \bar{\lambda}(m)\tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m)\tau(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2.
 \end{aligned} \tag{20}$$

This proves Lemma 2.  $\square$

$$E^3(m, p) = U(p, d) + V(p, d) \cdot E(m, p), \tag{21}$$

where

**Lemma 3.** Let  $p$  be an odd prime with  $3|p-1$ . Then, for any cubic character  $\lambda$  modulo  $p$ , we have the identity

$$\begin{aligned}
 U(p, d) &= 64dp + (d^5 - 12d^4 + 60d^3 - 160d^2 + 240d - 384)p^2 \\
 &\quad - (5d^3 - 48d^2 + 180d - 320)p^3 + (5d - 24)p^4, \\
 V(p, d) &= 3p \cdot (p^2 + 8p - 4dp + 4d^2 - 16d + 16),
 \end{aligned} \tag{22}$$

and  $E(m, p)$  is defined as the same as in Lemma 2.



*Proof.* From Lemma 2, we have

$$\begin{aligned}
 E^3(m, p) &= \left( \bar{\lambda}(m) \tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^3 \\
 &= \left( \tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \right)^3 + \left( \tau(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^3 \\
 &\quad + 3p \cdot \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \cdot E(m, p).
 \end{aligned} \tag{23}$$

Note that  $\tau(\lambda)\tau(\bar{\lambda}) = p$ , and from (17), we have

Therefore,

$$\begin{aligned}
 \tau^6(\lambda) + \tau^6(\bar{\lambda}) &= \left( \tau^3(\lambda) + \tau^3(\bar{\lambda}) \right)^2 - 2\tau^3(\lambda)\tau^3(\bar{\lambda}) \\
 &= d^2 p^2 - 2p^3.
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 &\left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \\
 &= \left( 4 - \frac{4\tau^3(\bar{\lambda})}{p} + \frac{\tau^6(\bar{\lambda})}{p^2} \right) \cdot \left( 4 - \frac{4\tau^3(\lambda)}{p} + \frac{\tau^6(\lambda)}{p^2} \right) \\
 &= 16 - \frac{16}{p} \left( \tau^3(\lambda) + \tau^3(\bar{\lambda}) \right) + \frac{4}{p^2} \left( \tau^6(\lambda) + \tau^6(\bar{\lambda}) \right) + \frac{16}{p^2} \tau^3(\lambda) \tau^3(\bar{\lambda}) \\
 &\quad - \frac{4}{p^3} \left( \tau^3(\lambda) \tau^6(\bar{\lambda}) + \tau^3(\bar{\lambda}) \tau^6(\lambda) \right) + \frac{1}{p^4} \tau^6(\lambda) \tau^6(\bar{\lambda}) \\
 &= 16 - \frac{16}{p} \cdot dp + \frac{4}{p^2} \cdot (d^2 p^2 - 2p^3) + \frac{16}{p^2} \cdot p^3 - \frac{4}{p^3} \cdot p^3 \cdot dp + \frac{1}{p^4} \cdot p^6 \\
 &= p^2 + 8p - 4dp + 4d^2 - 16d + 16.
 \end{aligned} \tag{25}$$

In addition,

$$\begin{aligned}
 \left( \tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \right)^3 &= \tau^3(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^6 \\
 &= \tau^3(\lambda) \left( \frac{\tau^{18}(\bar{\lambda})}{p^6} + 36 \frac{\tau^{12}(\bar{\lambda})}{p^4} + 144 \frac{\tau^6(\bar{\lambda})}{p^2} + 64 - 12 \frac{\tau^{15}(\bar{\lambda})}{p^5} \right. \\
 &\quad \left. + 24 \frac{\tau^{12}(\bar{\lambda})}{p^4} - 16 \frac{\tau^9(\bar{\lambda})}{p^3} - 144 \frac{\tau^9(\bar{\lambda})}{p^3} + 96 \frac{\tau^6(\bar{\lambda})}{p^2} - 192 \frac{\tau^3(\bar{\lambda})}{p} \right) \\
 &= 64 \tau^3(\lambda) + 240 p \tau^3(\bar{\lambda}) - 160 \tau^6(\bar{\lambda}) + \frac{60}{p} \tau^9(\bar{\lambda}) \\
 &\quad - \frac{12}{p^2} \tau^{12}(\bar{\lambda}) + \frac{1}{p^3} \tau^{15}(\bar{\lambda}) - 192 p^2.
 \end{aligned} \tag{26}$$

Using the method similar to (24), we obtain

$$\begin{aligned}
 \tau^9(\lambda) + \tau^9(\bar{\lambda}) &= (\tau^6(\lambda) + \tau^6(\bar{\lambda}))(\tau^3(\lambda) + \tau^3(\bar{\lambda})) - p^3(\tau^3(\lambda) + \tau^3(\bar{\lambda})) \\
 &= d^3 p^3 - 3 d p^4, \\
 \tau^{12}(\lambda) + \tau^{12}(\bar{\lambda}) &= d^4 p^4 - 4 d^2 p^5 + 2 p^6, \\
 \tau^{15}(\lambda) + \tau^{15}(\bar{\lambda}) &= d^5 p^5 - 5 d^3 p^6 + 5 d p^7.
 \end{aligned} \tag{27}$$

Combining formulae (23)–(27), we have

$$\begin{aligned}
 E^3(m, p) &= 64(\tau^3(\lambda) + \tau^3(\bar{\lambda})) + 240 p(\tau^3(\bar{\lambda}) + \tau^3(\lambda)) - 160(\tau^6(\bar{\lambda}) + \tau^6(\lambda)) \\
 &\quad + \frac{60}{p}(\tau^9(\bar{\lambda}) + \tau^9(\lambda)) - \frac{12}{p^2}(\tau^{12}(\bar{\lambda}) + \tau^{12}(\lambda)) + \frac{1}{p^3}(\tau^{15}(\bar{\lambda}) + \tau^{15}(\lambda)) \\
 &\quad - 384 p^2 + 3 p \cdot (p^2 + 8 p - 4 d p + 4 d^2 - 16 d + 16) \cdot E(m, p) \\
 &= (5 d - 24) p^4 - (5 d^3 - 48 d^2 + 180 d - 320) p^3 \\
 &\quad + (d^5 - 12 d^4 + 60 d^3 - 160 d^2 + 240 d - 384) p^2 + 64 d p \\
 &\quad + 3 p \cdot (p^2 + 8 p - 4 d p + 4 d^2 - 16 d + 16) \cdot E(m, p) \\
 &= U(p, d) + V(p, d) \cdot E(m, p).
 \end{aligned} \tag{28}$$

This proves Lemma 3.

□

### 3. Proofs of the Theorems

Now, we shall complete the proofs of our main results. Firstly, we prove Theorem 1. Let  $p$  be an odd prime with

$3|p-1$ ,  $\chi$  be any Dirichlet character modulo  $p$ , and  $\lambda$  be a cubic character modulo  $p$ . Then, from the properties of the classical Gauss sums and (17), we have

$$\begin{aligned}
 \sum_{\chi \bmod p} |G(m, 3, \chi; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p}\right) \right|^4 \\
 &= \sum_{\chi \bmod p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi(abc\bar{d}) e\left(\frac{m(a^3 + b^3 - c^3 - d^3)}{p}\right) \\
 &= (p-1) \sum_{\substack{a=1 \\ ab \equiv c \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{m(a^3 + b^3 - c^3 - d^3)}{p}\right) \\
 &= (p-1) \sum_{\substack{a=1 \\ ab \equiv c \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 + b^3 - c^3 - 1)}{p}\right) \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) \tag{29} \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} (1 + \lambda(d) + \bar{\lambda}(d)) e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) + (p-1)\bar{\lambda}(m)\tau(\lambda) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \bar{\lambda}((a^3 - 1)(b^3 - 1)) \\
 &\quad + (p-1)\lambda(m)\tau(\bar{\lambda}) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \lambda((a^3 - 1)(b^3 - 1)) \\
 &= (p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md^3(a^3 - 1)(b^3 - 1)}{p}\right) + (p-1)E(m, p),
 \end{aligned}$$

where  $E(m, p)$  is the same as in Lemma 2.

For any integer  $n$ , we have the trigonometric identity

$$\sum_{a=1}^{p-1} e\left(\frac{na}{p}\right) = \begin{cases} p-1, & \text{if } (n, p) = p, \\ -1, & \text{if } (n, p) = 1. \end{cases} \tag{30}$$

From (30), we have the identity

$$\begin{aligned}
 & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{m d(a^3 - 1)(b^3 - 1)}{p}\right) \\
 &= 2(p-1) \sum_{\substack{a=1 \\ a^3 \equiv 1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ b^3 \equiv 1 \pmod{p}}}^{p-1} 1 - (p-1) \sum_{\substack{a=1 \\ a^3 \equiv 1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ b^3 \equiv 1 \pmod{p}}}^{p-1} 1 - \sum_{\substack{a=1 \\ a^3 \equiv 1 \pmod{p}}}^{p-1} \sum_{\substack{b=1 \\ b^3 \equiv 1 \pmod{p}}}^{p-1} 1 \\
 &= 6(p-1)^2 - 9(p-1) - (p-4)^2 = 5p^2 - 13p - 1.
 \end{aligned} \tag{31}$$

Combining (29)–(31), we have

$$\begin{aligned}
 \sum_{\chi \bmod p} |G(m, 3, \chi; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p}\right) \right|^4 \\
 &= (p-1)(5p^2 - 13p - 1) + (p-1)E(m, p).
 \end{aligned} \tag{32}$$

For any positive integer  $n$ , from Lemma 3, we have

where

$$\begin{aligned}
 E^n(m, p) &= E^3(m, p) \cdot E^{n-3}(m, p) \\
 &= [U(p, d) + V(p, d) \cdot E(m, p)] \cdot E^{n-3}(m, p) \\
 &= U(p, d) \cdot E^{n-3}(m, p) + V(p, d) \cdot E^{n-2}(m, p),
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 U(p, d) &= 64 dp + (d^5 - 12d^4 + 60d^3 - 160d^2 + 240d - 384)p^2 \\
 &\quad - (5d^3 - 48d^2 + 180d - 320)p^3 + (5d - 24)p^4, \\
 V(p, d) &= 3p \cdot (p^2 + 8p - 4dp + 4d^2 - 16d + 16).
 \end{aligned} \tag{34}$$

This proves Theorem 1.

Now, we prove Theorem 2. From Theorem 1, we have

$$\sum_{m=1}^{p-1} E^n(m, p) = U(p, d) \sum_{m=1}^{p-1} E^{n-3}(m, p) + V(p, d) \sum_{m=1}^{p-1} E^{n-2}(m, p). \tag{35}$$

From Lemma 2, we have

$$\sum_{m=1}^{p-1} E(m, p) = \sum_{m=1}^{p-1} \left( \bar{\lambda}(m) \tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right) = 0, \quad (36)$$

$$\begin{aligned} \sum_{m=1}^{p-1} E^2(m, p) &= \sum_{m=1}^{p-1} \left( \bar{\lambda}(m) \tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^2 \\ &= \sum_{m=1}^{p-1} \lambda(m) \tau^2(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^4 + \sum_{m=1}^{p-1} \bar{\lambda}(m) \tau^2(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^4 \\ &\quad + \sum_{m=1}^{p-1} 2p \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \\ &= 2p(p-1)(p-2d+4)^2. \end{aligned} \quad (37)$$

From (36) and Lemma 3, we have

$$\begin{aligned} \sum_{m=1}^{p-1} E^3(m, p) &= \sum_{m=1}^{p-1} \left( \bar{\lambda}(m) \tau(\lambda) \left( \frac{\tau^3(\bar{\lambda})}{p} - 2 \right)^2 + \lambda(m) \tau(\bar{\lambda}) \left( \frac{\tau^3(\lambda)}{p} - 2 \right)^2 \right)^3 \\ &= \sum_{m=1}^{p-1} (U(p, d) + V(p, d)E(m, p)) = (p-1)U(p, d). \end{aligned} \quad (38)$$

Now, Theorem 2 follows from (35)–(38).

Finally, we prove Theorem 3. For any integer  $n \geq 1$ , from Lemma 3, we have

$$\sum_{m=1}^{p-1} E^{-n}(m, p) = \frac{1}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-3)}(m, p) - \frac{V(p, d)}{U(p, d)} \cdot \sum_{m=1}^{p-1} E^{-(n-1)}(m, p), \quad (39)$$

$$\begin{aligned} E^2(m, p) &= U(p, d) \cdot E^{-1}(m, p) + V(p, d), \\ E(m, p) &= U(p, d) \cdot E^{-2}(m, p) + V(p, d) \cdot E^{-1}(m, p), \\ E^0(m, p) &= U(p, d) \cdot E^{-3}(m, p) + V(p, d) \cdot E^{-2}(m, p), \\ E^{-1}(m, p) &= U(p, d) \cdot E^{-4}(m, p) + V(p, d) \cdot E^{-3}(m, p). \end{aligned} \quad (40)$$

Therefore, we have

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-1}(m, p) &= \sum_{m=1}^{p-1} \frac{E^2(m, p) - V(p, d)}{U(p, d)} \\ &= \frac{2p(p-1)(p-2d+4)^2 - (p-1)V(p, d)}{U(p, d)},\end{aligned}\quad (41)$$

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-2}(m, p) &= \sum_{m=1}^{p-1} \frac{E(m, p) - V(p, d) \cdot E^{-1}(m, p)}{U(p, d)} \\ &= \frac{(p-1)V^2(p, d) - 2p(p-1)(p-2d+4)^2V(p, d)}{U^2(p, d)},\end{aligned}\quad (42)$$

$$\begin{aligned}\sum_{m=1}^{p-1} E^{-3}(m, p) &= \sum_{m=1}^{p-1} \frac{1 - V(p, d) \cdot E^{-2}(m, p)}{U(p, d)} \\ &= \frac{p-1}{U(p, d)} + \frac{2p(p-1)(p-2d+4)^2V^2(p, d) - (p-1)V^3(p, d)}{U^3(p, d)}.\end{aligned}\quad (43)$$

Combining (39)–(43), we may immediately complete the proof of Theorem 3.

#### 4. Conclusion

The main results of this paper give two third-order linear recurrence formulae for the error term of the fourth power mean of the generalized cubic Gauss sums, and these results are the improvement and generalization of [4]. They are some new contributions in the relevant fields.

#### Data Availability

No data were used to support this study.

#### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

#### Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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## Research Article

# On a Sum Involving the Sum-of-Divisors Function

Feng Zhao<sup>1</sup> and Jie Wu<sup>2</sup>

<sup>1</sup>Department of Mathematics and Statistics, North China University of Water Resources and Electric Power, Jinshui E Road, Zhengzhou 450046, Henan, China

<sup>2</sup>CNRS LAMA 8050, Laboratoire D'Analyse et de Mathématiques Appliquées, Université Paris-Est Créteil, Créteil Cedex 94010, France

Correspondence should be addressed to Feng Zhao; zhaofeng@ncwu.edu.cn

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Let  $\sigma(n)$  be the sum of all divisors of  $n$  and let  $[t]$  be the integral part of  $t$ . In this paper, we shall prove that  $\sum_{n \leq x} \sigma([x/n]) = (\pi^2/6)x \log x + O(x(\log x)^{(2/3)}(\log_2 x)^{(4/3)})$  for  $x \rightarrow \infty$ , and that the error term of this asymptotic formula is  $\Omega(x)$ .

## 1. Introduction

As usual, denote by  $\varphi(n)$  the Euler function and by  $[t]$  the integral part of real  $t$ , respectively. Recently, Bordellès et al. [1] studied the asymptotic behaviour of the quantity

$$S_\varphi(x) := \sum_{n \leq x} \varphi\left(\left[\frac{x}{n}\right]\right), \quad (1)$$

for  $x \rightarrow \infty$ . By exponential sum technique, they proved that

$$\begin{aligned} &\left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1)\right)x \log x \leq S_\varphi(x) \\ &\leq \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + \frac{1380}{4009} + o(1)\right)x \log x, \end{aligned} \quad (2)$$

and conjectured that

$$S_\varphi(x) \sim \frac{6}{\pi^2}x \log x, \quad \text{as } x \rightarrow \infty. \quad (3)$$

Very recently, Wu [2] improved (2) and Zhai [3] resolved conjecture (3) by showing

$$S_\varphi(x) = \frac{6}{\pi^2}x \log x + O(x(\log x)^{(2/3)}(\log_2 x)^{(1/3)}), \quad (4)$$

and also proved that the error term in (4) is  $\Omega(x)$ , where  $\log_2$  denotes the iterated logarithm. Some related works can be found in [4, 5]. Since the sum-of-divisors function  $\sigma(n) := \sum_{d|n} d$  has similar properties as the Euler function  $\varphi(n)$  in many cases, it seems natural and interesting to consider its analogy of (3).

Our result is as follows.

### Theorem 1

(i) For  $x \rightarrow \infty$ , we have

$$\begin{aligned} S_\sigma(x) &:= \sum_{n \leq x} \sigma\left(\left[\frac{x}{n}\right]\right) = \frac{\pi^2}{6}x \log x \\ &+ O(x(\log x)^{(2/3)}(\log_2 x)^{(4/3)}). \end{aligned} \quad (5)$$

(ii) Let  $E(x)$  be the error term in (5). Then, for  $x \rightarrow \infty$ , we have

$$E(x) = \Omega(x), \quad \text{i.e. } \limsup_{x \rightarrow \infty} \frac{|E(x)|}{x} > 0. \quad (6)$$

Let  $\mu(n)$  be the Möbius function and define  $\text{id}(n) = n$  and  $1(n) = 1$  for all integers  $n \geq 1$ . Then,  $\varphi = \text{id} * \mu$  and  $\sigma = \text{id} * 1$ . In Zhai's approach proving (4), the inequality

$$\sum_{n \leq x} \mu(n) \ll x \exp\left\{-c\sqrt{\log x}\right\}, \quad (x \geq 1), \quad (7)$$

plays a key role, where  $c > 0$  is a positive constant. Clearly, such a bound is not true for 1. By refining Zhai's approach, we shall prove our result.

## 2. Preliminary Lemmas

As in [3], we need some bounds on exponential sums of the type  $\sum_{N \leq n < N'} e(T/n)$  where  $N < N' \leq 2N$ . For large values of  $N$ , Zhai used the theory of exponent pair, and for smaller ones the Vinogradov method. Both estimates are contained in the following general theorem of Karatsuba [6, Theorem 1], which will be a key tool for proving Theorem 1.

**Lemma 1.** *Let  $k \geq 2$  and  $M$  and  $P$  be integers,  $P$  being positive. Let  $f \in \mathcal{C}^{k+1}([M, M+P]; \mathbb{R})$ . Suppose that there exist positive absolute constants  $c_0, c_1, c_2, c_3$ , and  $c_4$  such that  $c_0 < 1$ ,  $c_1 < 1$ , and  $c_2 + c_4 < c_1$ ; an integer  $r$  such that  $c_0 k \leq r \leq k$ ; and distinct numbers  $s_j \geq 2$  ( $j = 1, \dots, r$ ) not exceeding  $k$ , such that for  $M \leq t \leq M+P$  the following inequalities are satisfied:*

- (i)  $|f^{(k+1)}(t)/(k+1)!| \leq P^{-c_1(k+1)}$ .
- (ii)  $P^{-c_2 s_j} \leq |f^{(s_j)}(t)/s_j!| \leq P^{-c_3 s_j}$ , ( $j = 1, \dots, r$ ).

Then, for each positive integer  $P_1$  not exceeding  $P$ , we have

$$\left| \sum_{M \leq m \leq M+P_1-1} e(f(m)) \right| \leq AP^{1-(c/k^2)}, \quad (8)$$

where  $e(t) := e^{2\pi i t}$  and  $A > 0$ ,  $c > 0$  are absolute constants.

The next two lemmas are essentially a special case of [7, Lemmas 2.5 and 2.6] with  $a = 1$ . The only difference is that the ranges of  $T$  and  $N$  here are slightly larger than those of [7, Lemmas 2.5 and 2.6] ( $T \geq N^2$  in place of  $T \geq N^{(3/2)}$  and  $N \leq x^{(2/3)}$  in place of  $N \leq x^{(1/2)}$ , respectively). Although the proof is completely similar, for the convenience of readers, we still reproduce a proof here.

**Lemma 2.** *Let  $e^{100} \leq N < N' \leq 2N$  and  $T \geq N^{(3/2)}$ . Then, there exists an absolute positive constant  $c_5$  such that*

$$\sum_{N \leq n < N'} e\left(\frac{T}{n}\right) \ll N \exp\left\{-\left(\frac{c_5 \log^3 N}{\log^2 T}\right)\right\}, \quad (9)$$

where the implied constant is absolute.

*Proof.* We apply Lemma 1 to  $f(t) := (T/t)$  with  $M = N$ ,  $P = N$ ,  $P_1 = N' - N$ . For this, we choose

$$\begin{aligned} c_0 &= \frac{1}{100}, \\ c_1 &= \frac{99}{100}, \\ c_2 &= \frac{87}{100}, \\ c_3 &= \frac{3}{4}, \\ c_4 &= \frac{1}{100}, \\ k &= \left\lceil 100 \frac{\log(T/n)}{\log N} \right\rceil, \end{aligned} \quad (10)$$

and take the  $s_j$  to be all integers  $s$  such that

$$4 \frac{\log(T/n)}{\log N} \leq s \leq 5 \frac{\log(T/n)}{\log N}. \quad (11)$$

Obviously the number  $r$  of  $s_j$  is between  $c_0 k$  and  $k$ . Next we shall verify that  $f(t)$  satisfies the conditions (i) and (ii) of Lemma 1 with the parameters chosen above.

For  $N \leq t \leq 2N$ , we have

$$\left| \frac{f^{(k+1)}(t)}{(k+1)!} \right| = T t^{-k-2} \leq T N^{-k-2} = N^{-\eta_1}, \quad (12)$$

where

$$\eta_1 := k + 1 - \frac{\log(T/n)}{\log N} \geq k + 1 - \frac{1}{100} k \geq \frac{99}{100} (k + 1) = c_1 (k + 1). \quad (13)$$

Similarly for  $N \leq t \leq 2N$ , we find the inequality  $|f^{(s_j)}(t)/s_j!| \leq N^{-\eta_3}$ , where

$$\eta_3 := s_j - \frac{\log(T/n)}{\log N} \geq \frac{3}{4} s_j = c_3 s_j. \quad (14)$$

For the lower bound of (ii), we have

$$\left| \frac{f^{(s_j)}(t)}{s_j!} \right| = T t^{-1-s_j} \geq T (2N)^{-1-s_j} = N^{-\eta_2}, \quad (15)$$

where

$$\begin{aligned} \eta_2 &:= s_j - \frac{\log(T/n)}{\log N} + \frac{\log 2}{\log N} (s_j + 1) \\ &\leq \frac{4}{5} s_j + \frac{\log 2}{100} (s_j + 1) \leq \frac{87}{100} s_j = c_2 s_j. \end{aligned} \quad (16)$$

From Lemma 1, there exist two positive constants  $c$  and  $A$  such that



$$\left| \sum_{N \leq n < N'} e\left(\frac{T}{n}\right) \right| \leq AN^{1-(c/k^2)} \leq AN \exp\left\{-\frac{c_5 \log^3 N}{\log^2(T/n)}\right\}, \quad (17)$$

with  $c_5 := 10^{-4}c$ . This completes the proof of Lemma 2.  $\square$

**Lemma 3.** Define  $\psi(t) := t - [t] - (1/2)$ . Let  $c_5$  be the constant defined by Lemma 2 and  $c_6 := (8/9)^2 c_5$ ,  $c^* := ((3/5)c_6)^{-(1/3)}$ . Then, we have

$$\sum_{N \leq n < N'} \frac{1}{n} \psi\left(\frac{x}{n}\right) \ll e^{-c_6 (\log N)^3 / (\log x)^2} \frac{(\log N)^3}{(\log x)^2}, \quad (18)$$

uniformly for  $x \geq 10$ ,  $\exp\{c^* (\log x)^{(2/3)}\} \leq N \leq x^{(2/3)}$  and  $N < N' \leq 2N$ .

*Proof.* By invoking a classical result on  $\psi(t)$  (see 8, page 39]) we can write, for any  $H \geq 1$ ,

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll NH^{-1} + \sum_{1 \leq h \leq H} h^{-1} \left| \sum_{N \leq n < N'} e\left(\frac{hx}{n}\right) \right|. \quad (19)$$

An application of Lemma 2 with  $T = hx \geq x \geq N^{(3/2)}$  yields

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N \left( H^{-1} + e^{-c_5 (\log N)^3 / \log^2(Hx)} \log H \right). \quad (20)$$

Taking  $H = \exp\{(\log N)^3 / (\log x)^2\} \leq x^{(8/27)}$ , we easily deduce that

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N \left( e^{-(\log N)^3 / (\log x)^2} + e^{-c_6 (\log N)^3 / (\log x)^2} \frac{(\log N)^3}{(\log x)^2} \right). \quad (21)$$

The first term can be absorbed by the second, since  $c_5$  can be chosen small enough to ensure that  $c_6 < 1$  and since  $\exp\{c^* (\log x)^{(2/3)}\} \leq N$  implies  $(\log N)^3 / (\log x)^2 \geq c^*$ . Hence,

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N e^{-c_6 (\log N)^3 / (\log x)^2} \frac{(\log N)^3}{(\log x)^2}, \quad (22)$$

and an Abel summation produces the required result.  $\square$

**Lemma 4.** Let  $2 \leq z_1 < z_2 \leq x$  and  $F_x(t) := (1/t)\psi(x/t)$ . Denote by  $V_{F_x}[z_1, z_2]$  the total variation of  $F_x$  on  $[z_1, z_2]$ . Then,

$$V_{F_x}[z_1, z_2] \ll \frac{x}{z_1^2} + \frac{1}{z_1}, \quad (23)$$

where the implied constant is absolute.

*Proof.* If  $z_1 = t_0 < t_1 < \dots < t_n = z_2$  is a partition of the interval  $[z_1, z_2]$ , then

$$\begin{aligned} \sum_{k=1}^n |F_x(t_k) - F_x(t_{k-1})| &= \sum_{k=1}^n \left| \frac{1}{t_k} \psi\left(\frac{x}{t_k}\right) - \frac{1}{t_{k-1}} \psi\left(\frac{x}{t_{k-1}}\right) \right| \\ &\leq \sum_{k=1}^n \left| \left( \frac{1}{t_k} - \frac{1}{t_{k-1}} \right) \psi\left(\frac{x}{t_k}\right) \right| + \sum_{k=1}^n \frac{1}{t_{k-1}} \left| \psi\left(\frac{x}{t_k}\right) - \psi\left(\frac{x}{t_{k-1}}\right) \right|. \end{aligned} \quad (24)$$

Since  $|\psi(t)| \leq 1$  for all  $t$ , we have

$$\sum_{k=1}^n \left| \left( \frac{1}{t_k} - \frac{1}{t_{k-1}} \right) \psi\left(\frac{x}{t_k}\right) \right| \leq \frac{1}{z_1} - \frac{1}{z_2} < \frac{1}{z_1}. \quad (25)$$

On the other hand, since  $\psi(u)$  is of period 1, we have

$$\begin{aligned} \sum_{k=1}^n \frac{1}{t_{k-1}} \left| \psi\left(\frac{x}{t_k}\right) - \psi\left(\frac{x}{t_{k-1}}\right) \right| &\leq \frac{1}{z_1} \left( \frac{x}{z_1} + 1 \right) V_\psi[0, 1] \leq 2 \left( \frac{x}{z_1^2} + \frac{1}{z_1} \right). \end{aligned} \quad (26)$$

Inserting these two bounds into (24), we obtain the required result.  $\square$

### 3. Proof of Theorem 1

#### 3.1. A Formula on the Mean Value of $\sigma(n)$

##### Lemma 5

(i) For  $x \geq 2$  and  $1 \leq z \leq x^{(1/3)}$ , we have

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 - x \frac{(z - [z])^2 + [z]}{2z} + O\left(\frac{x}{z}\right) - \Delta(x, z), \quad (27)$$

where

$$\Delta(x, z) := \sum_{d \leq (x/z)} \frac{x}{d} \psi\left(\frac{x}{d}\right). \quad (28)$$

(ii) For  $x \rightarrow \infty$ , we have

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x). \quad (29)$$

*Proof.* Using  $\sigma(n) = \sum_{dm=n} m$ , the hyperbole principle of Dirichlet allows us to write

$$\sum_{n \leq x} \sigma(n) = \sum_{dm \leq x} m = S_1 + S_2 - S_3, \quad (30)$$

where

$$\begin{aligned}
S_1 &:= \sum_{d \leq (x/z)} \sum_{m \leq (x/d)} m, \\
S_2 &:= \sum_{m \leq z} \sum_{d \leq (x/m)} m, \\
S_3 &:= \sum_{d \leq (x/z)} \sum_{m \leq z} m.
\end{aligned} \tag{31}$$

Firstly we have

$$S_2 = \sum_{m \leq z} m \left\lfloor \frac{x}{m} \right\rfloor = x[z] + O(z^2), \tag{32}$$

$$S_3 = \left\lfloor \frac{x}{z} \right\rfloor \frac{[z]([z] + 1)}{2} = x \frac{[z]([z] + 1)}{2z} + O(z^2). \tag{33}$$

Secondly we can write

$$\begin{aligned}
S_1 &= \frac{1}{2} \sum_{d \leq (x/z)} \left( \frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{1}{2} \right) \left( \frac{x}{d} - \psi\left(\frac{x}{d}\right) + \frac{1}{2} \right) \\
&= \frac{1}{2} \sum_{d \leq (x/z)} \left( \frac{x^2}{d^2} - 2 \frac{x}{d} \psi\left(\frac{x}{d}\right) + \psi\left(\frac{x}{d}\right)^2 - \frac{1}{4} \right) \\
&= \frac{\pi^2}{12} x^2 - \frac{1}{2} xz - \Delta(x, z) + O(x/z),
\end{aligned} \tag{34}$$

where  $\Delta(x, z)$  is as in (28). Inserting (32), (33), and (34) into (30) and using  $z^2 \leq (x/z)$ , we get (27).

Taking  $z = 1$  in (27) and noticing that

$$\begin{aligned}
\sum_{d \leq x} \frac{1}{d^2} &= \frac{\pi^2}{6} + O\left(\frac{1}{x}\right), \\
\left| \sum_{d \leq x} \frac{x}{d} \psi\left(\frac{x}{d}\right) \right| &\ll x \log x,
\end{aligned} \tag{35}$$

we obtain the required bound. This completes the proof.  $\square$

### 3.2. Estimates of Error Terms

**Lemma 6.** Let  $N_0 := \exp\{(6/c_6)(\log x)^{(2/3)}(\log_2 x)^{(1/3)}\}$ , where  $c_6$  is given as in Lemma 3. Let  $\Delta(x, z)$  be defined by (28). Then, for  $x \geq 10$  and  $2 \leq z \leq \sqrt{N_0}$ , we have

$$\begin{aligned}
&\left| \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n}, z\right) \right| + \left| \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n} - 1, z\right) \right| \\
&\ll x \left( \frac{1}{(\log x)^3} + \frac{\log x}{z} \right).
\end{aligned} \tag{36}$$

*Proof.* Denote by  $\Delta_1(x, z)$  and  $\Delta_2(x, z)$  two sums on the left-hand side of (36), respectively. By (28) of Lemma 5, we can write

$$\begin{aligned}
\Delta_1(x, z) &= x \sum_{N_0 < n \leq \sqrt{x}} \sum_{d \leq (x/(nz))} \frac{1}{dn} \psi\left(\frac{x}{dn}\right) \\
&= x \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{N_0 < n \leq \min(\sqrt{x}, x/(dz))} \frac{1}{n} \psi\left(\frac{x}{dn}\right) \\
&= x \Delta_1^\dagger(x, z) + x \Delta_1^\sharp(x, z),
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\Delta_1^\dagger(x, z) &:= \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{N_0 < n \leq (x/d)^{(2/3)}} \frac{1}{n} \psi\left(\frac{x}{dn}\right), \\
\Delta_1^\sharp(x, z) &:= \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{(x/d)^{(2/3)} < n \leq \min\{\sqrt{x}, x/(dz)\}} \frac{1}{n} \psi\left(\frac{x}{dn}\right).
\end{aligned} \tag{38}$$

For  $0 \leq k \leq (\log((x/d)^{(2/3)}/N_0))/\log 2$ , let  $N_k := 2^k N_0$  and define

$$\mathfrak{S}_k(d) := \sum_{N_k < n \leq 2N_k} \frac{1}{n} \psi\left(\frac{x}{dn}\right). \tag{39}$$

Noticing that  $N_0 \leq N_k \leq (x/d)^{(2/3)}$ , we can apply Lemma 3 to derive that

$$\mathfrak{S}_k(d) \ll e^{-\vartheta((\log N_k)^3/(\log(x/d))^2)}, \tag{40}$$

with  $\vartheta(t) := c_6 t - \log t$ . It is clear that  $\vartheta(t)$  is increasing on  $[c_6, \infty)$ . On the other hand, for  $k \geq 0$  and  $d \geq 1$ , we have

$$(\log N_k)^3/(\log(x/d))^2 \geq (\log N_0)^3/(\log x)^2 = (6/c_6) \log_2 x. \tag{41}$$

Thus,

$$\begin{aligned}
\vartheta\left(\frac{(\log N_k)^3}{(\log(x/d))^2}\right) &\geq \vartheta\left(\left(\frac{6}{c_6}\right) \log_2 x\right) \\
&= 6 \log_2 x - \log\left(\left(\frac{6}{c_6}\right) \log_2 x\right) \geq 5 \log_2 x,
\end{aligned} \tag{42}$$

which implies that  $\mathfrak{S}_k(d) \ll (\log x)^{-5}$ . Inserting this into the expression of  $\Delta_1^\dagger(x, z)$ , we get

$$\Delta_1^\dagger(x, z) \ll \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{2^k N_0 \leq (x/d)^{(2/3)}} |\mathfrak{S}_k(d)| \ll (\log x)^{-3}. \tag{43}$$

Next we bound  $\Delta_1^\sharp(x, z)$ . Let  $F(t)$  be a function of bounded variation on  $[n, n+1]$  for each integer  $n$  and let  $V_F[n, n+1]$  be the total variation of  $F$  on  $[n, n+1]$ . Integrating by parts, we have

$$\int_n^{n+1} \left(t - n - \frac{1}{2}\right) dF(t) = \frac{1}{2}(F(n+1) + F(n)) - \int_n^{n+1} F(t) dt. \quad (44)$$

From this, we can derive that

$$\frac{1}{2}(F(n+1) + F(n)) = \int_n^{n+1} F(t) dt + O(V_F[n, n+1]), \quad (45)$$

for  $n \geq 1$ . Summing over  $n$ , we find that

$$\begin{aligned} \sum_{N_1 < n \leq N_2} F(n) &= \int_{N_1}^{N_2} F(t) dt \\ &+ \frac{1}{2}(F(N_1) + F(N_2)) + O(V_F[N_1, N_2]). \end{aligned} \quad (46)$$

We apply this formula to

$$F_{(x/d)}(t) = \frac{1}{t} \psi\left(\frac{(x/d)}{t}\right), \quad (47)$$

$$N_1 = \left\lfloor (x/d)^{(2/3)} \right\rfloor,$$

$$N_2 = \left\lceil \min \left\{ \sqrt{x}, \frac{x}{(dz)} \right\} \right\rceil.$$

According to Lemma 4, we have  $V_{F_{(x/d)}}[N_1, N_2] \ll (x/d)^{-(1/3)}$ , and thus by putting  $u = (x/d)/t$ , we obtain, with the notation  $x_{d,1} = \max(\sqrt{x}/d, tz)$  and  $x_{d,2} = (x/d)^{(1/3)}$ ,

$$\begin{aligned} \sum_{(x/d)^{(2/3)} < n \leq \min\{\sqrt{x}, (x/(dz))\}} \frac{1}{n} \psi\left(\frac{x}{dn}\right) &= \int_{x_{d,1}}^{x_{d,2}} \frac{\psi(u)}{u} du + O\left(\left(\frac{x}{d}\right)^{-(1/3)}\right) \\ &\ll z^{-1} + \left(\frac{x}{d}\right)^{-(1/3)} \ll z^{-1}, \end{aligned} \quad (48)$$

where we have used the fact that  $z \leq \sqrt{N_0}$  and  $d \leq (x/(N_0 z)) \Rightarrow z \leq (x/d)^{(1/3)}$  and the bound

$$\begin{aligned} \int_{x_{d,1}}^{x_{d,2}} \frac{\psi(u)}{u} du &= \int_{x_{d,1}}^{x_{d,2}} \left( \int_{x_{d,1}}^u \psi(t) dt \right) \frac{du}{u^2} - \frac{1}{x_{d,2}^{(2/3)}} \int_{x_{d,1}}^{x_{d,2}} \psi(t) dt \\ &\ll x_{d,1}^{-1} + (x_{d,2})^{-(2/3)} \ll z^{-1} + (x/d)^{-(2/3)} \ll z^{-1}. \end{aligned} \quad (49)$$

Using (48), a simple partial integration allows us to derive that

$$\Delta_1^\#(x, z) \ll z^{-1} \sum_{d \leq x/(N_0 z)} d^{-1} \ll z^{-1} \log x. \quad (50)$$

Combining (43) and (50), it follows that

$$|\Delta_1(x, z)| \ll x(\log x)^{-3} + xz^{-1} \log x. \quad (51)$$

Similarly, we can prove the same bound for  $|\Delta_2(x, z)|$ . This completes the proof.  $\square$

**3.3. End of the Proof of Theorem 1.** Let  $c_6$  be the constant given as in Lemma 3 and  $N_0 := \exp\{(6/c_6)(\log x)^{(2/3)}(\log_2 x)^{(1/3)}\}$ . Let  $z \in [2, \sqrt{N_0}]$  be a parameter to be chosen later.

Putting  $d = \lfloor x/n \rfloor$ , we have  $(x/n) - 1 < d \leq (x/n)$  and  $x/(d+1) < n \leq (x/d)$ . We have, with the convention  $\sigma(0) = 0$ ,

$$\begin{aligned} S_\sigma(x) &= \sum_{d \leq x} \sigma(d) \sum_{(x/(d+1)) < n \leq (x/d)} 1 \\ &= \sum_{dn \leq x} \sigma(d) - \sum_{dn \leq x, d \geq 2} \sigma(d-1) \\ &= \sum_{dn \leq x} (\sigma(d) - \sigma(d-1)). \end{aligned} \quad (52)$$

By the hyperbole principle of Dirichlet, we can write

$$S_\sigma(x) = S_1(x, \sigma) + S_2(x, \sigma) - S_3(x, \sigma), \quad (53)$$

where

$$\begin{aligned} S_1(x, \sigma) &:= \sum_{d \leq \sqrt{x}, dn \leq x} (\sigma(d) - \sigma(d-1)), \\ S_2(x, \sigma) &:= \sum_{n \leq \sqrt{x}, dn \leq x} (\sigma(d) - \sigma(d-1)), \\ S_3(x, \sigma) &:= \sum_{d \leq \sqrt{x}, n \leq \sqrt{x}} (\sigma(d) - \sigma(d-1)). \end{aligned} \quad (54)$$

With the help of the bound  $\sigma(n) \ll n \log_2 n$ , we can derive that

$$S_3(x, \sigma) = \lfloor \sqrt{x} \rfloor \sigma(\lfloor \sqrt{x} \rfloor) \ll x \log_2 x. \quad (55)$$

For evaluating  $S_1(x, \sigma)$ , we write

$$\begin{aligned} S_1(x, \sigma) &= \sum_{d \leq \sqrt{x}} (\sigma(d) - \sigma(d-1)) \left\lfloor \frac{x}{d} \right\rfloor \\ &= x \sum_{d \leq \sqrt{x}} \frac{\sigma(d) - \sigma(d-1)}{d} + O\left( \sum_{d \leq \sqrt{x}} |\sigma(d) - \sigma(d-1)| \right). \end{aligned} \quad (56)$$

With the help of Lemma 5 (ii), a simple partial integration gives us

$$\begin{aligned}
\sum_{d \leq \sqrt{x}} \frac{\sigma(d) - \sigma(d-1)}{d} &= \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d(d+1)} \\
&= \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d^2} - \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d^2(d+1)} \\
&= \int_{1-}^{\sqrt{x}} t^{-2} d\left(\frac{\pi^2}{12} t^2 + O(t \log t)\right) + O(1) \\
&= \frac{\pi^2}{12} \log x + O(1), \\
\sum_{d \leq \sqrt{x}} |\sigma(d) - \sigma(d-1)| \\
&\ll \sum_{d \leq \sqrt{x}} \sigma(d) \ll x.
\end{aligned} \tag{57}$$

Inserting these estimates into (56), we find that

$$S_1(x, \sigma) = \frac{\pi^2}{12} x \log x + O(x). \tag{58}$$

Finally, we evaluate  $S_2(x, \sigma)$ . For this, we write

$$S_2(x, \sigma) = S_2^\dagger(x, \sigma) + S_2^\sharp(x, \sigma), \tag{59}$$

where

$$\begin{aligned}
S_2^\dagger(x, \sigma) &:= \sum_{n \leq N_0, dn \leq x} (\sigma(d) - \sigma(d-1)), \\
S_2^\sharp(x, \sigma) &:= \sum_{N_0 < n \leq \sqrt{x}, dn \leq x} (\sigma(d) - \sigma(d-1)).
\end{aligned} \tag{60}$$

By the bound that  $\sigma(n) \ll n \log_2 n$ , we have

$$\begin{aligned}
S_2^\dagger(x, \sigma) &= \sum_{n \leq N_0} \sigma\left(\left[\frac{x}{n}\right]\right) \ll \sum_{n \leq N_0} \left(\frac{x}{n}\right) \log_2 x \\
&\ll x (\log x)^{(2/3)} (\log_2 x)^{(4/3)}.
\end{aligned} \tag{61}$$

On the other hand, (27) of Lemma 5 allows us to derive that

$$\begin{aligned}
\sum_{d \leq x} \sigma(d) - \sum_{d \leq x-1} \sigma(d) &= \frac{\pi^2}{12} (x^2 - (x-1)^2) - \Delta(x, z) \\
&\quad + \Delta(x-1, z) + O\left(\frac{x}{z}\right) \\
&= \frac{\pi^2}{6} x - \Delta(x, z) + \Delta(x-1, z) + O\left(\frac{x}{z}\right),
\end{aligned} \tag{62}$$

where  $\Delta(x, z)$  is given by (28). Thus,

$$\begin{aligned}
S_2^\sharp(x, \sigma) &= \sum_{N_0 < n \leq \sqrt{x}} \left( \frac{\pi^2}{6} \cdot \frac{x}{n} - \Delta\left(\frac{x}{n}, z\right) + \Delta\left(\frac{x}{n} - 1, z\right) + O\left(\frac{x}{nz}\right) \right) \\
&= \frac{\pi^2}{12} x \log x + O\left(x (\log x)^{(2/3)} (\log_2 x)^{(1/3)} + xz^{-1} \log x\right) - \Delta_1(x, z) + \Delta_2(x, z),
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\Delta_1(x, z) &:= \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n}, z\right) \ll x (\log x)^{-3} + xz^{-1} \log x, \\
\Delta_2(x, z) &:= \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n} - 1, z\right) \ll x (\log x)^{-3} + xz^{-1} \log x,
\end{aligned} \tag{64}$$

thanks to Lemma 6. Inserting these estimates into (59), we find that

$$S_2(x, \sigma) = \frac{\pi^2}{12} x \log x + O\left(x (\log x)^{(2/3)} (\log_2 x)^{(4/3)} + xz^{-1} \log x\right), \tag{65}$$

Now (5) follows from (53), (55), (58), and (66) with the choice of  $z = (\log x)^{(1/3)}$ .

**3.4. Proof of Theorem 1.** (ii) For any odd prime  $p$ , (52) allows us to write

$$\begin{aligned}
\sum_{d|p} (\sigma(d) - \sigma(d-1)) &= S_\sigma(p) - S_\sigma(p-1) \\
&= \frac{\pi^2}{6} (\log p - \log(p-1)) + E(p) - E(p-1) \\
&\geq E(p) - E(p-1) \geq -2E^*(p),
\end{aligned} \tag{66}$$

where  $E^*(p) := \max\{|E(p)|, |E(p-1)|\}$ . On the other hand, we have

$$\begin{aligned}
\sum_{d|p} (\sigma(d) - \sigma(d-1)) &= \sigma(p) - \sigma(p-1) + 1 \\
&\leq p + 1 - \left(p - 1 + \frac{1}{2}(p-1) + 2 + 1\right) + 1 \leq -\frac{1}{4}p.
\end{aligned} \tag{67}$$

Thus,  $E^*(p) \geq (1/8)p$  for all odd primes.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# A Four-Order Linear Recurrence Formula Involving the Quartic Gauss Sums and One Kind Two-Term Exponential Sums

Lan Qi<sup>1</sup> and Xingxing Lv<sup>2</sup>

<sup>1</sup>School of Mathematics and Statistics, Yulin University, Yulin, Shaanxi, China

<sup>2</sup>School of Mathematics, Northwest University, Xi'an, Shaanxi, China

Correspondence should be addressed to Xingxing Lv; lvxingxing@stumail.nwu.edu.cn

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The main purpose of this paper is using the analytic method and the properties of the classical Gauss sums to study the computational problem of one kind hybrid power mean involving the quartic Gauss sums and two-term exponential sums and give an interesting four-order linear recurrence formula for it. As an application, we can obtain all values of this kind hybrid power mean with mathematica software.

## 1. Introduction

In the number theory textbooks, especially the elementary number theory and the analytic number theory textbooks, there are many contents related to primes, such as famous prime number theorem and Dirichlet's theorem. However, maybe the most important and most profound property related to primes is that any prime  $p$  with  $p = 4k + 1$  can be expressed as the square sums of two positive integers. That is,  $p = \alpha^2 + \beta^2$ . And, more precisely (see Theorems 4–11 of [1]),

$$p = \alpha^2 + \beta^2 \equiv \left( \sum_{a=1}^{(p-1)/2} \left( \frac{a+r\bar{a}}{p} \right) \right)^2 + \left( \sum_{a=1}^{(p-1)/2} \left( \frac{a+n\bar{a}}{p} \right) \right)^2, \quad (1)$$

where  $(*/p)$  denotes Legendre's symbol modulo  $p$ ,  $r$  and  $n$  are any two integers such that  $(rn/p) = -1$ , and  $\bar{a}$  satisfies the congruence equation  $a \cdot \bar{a} \equiv 1 \pmod{p}$ .

This conclusion has been extended by Professor W. P. Zhang in an unpublished paper. That is, he proved the following two conclusions. Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, for any nonprincipal even character  $\chi \pmod{p}$ , one has the identity

$$p = \left| \sum_{a=1}^{(p-1)/2} \chi(a+r\bar{a}) \right|^2 + \left| \sum_{a=1}^{(p-1)/2} \chi(a+n\bar{a}) \right|^2, \quad (2)$$

where  $r$  and  $n$  are any two integers such that  $(rn/p) = -1$ .

Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . Then, for any nonreal character  $\chi \pmod{p}$  with  $\chi^3 \neq \chi_0$ , the principal character  $\pmod{p}$ , one has

$$p = \left| \sum_{a=1}^{(p-1)/3} \chi^3(a^2 + \bar{a}) \right|^2 + \left| \sum_{a=1}^{(p-1)/3} \chi^3(a^2 + t\bar{a}) \right|^2 + \left| \sum_{a=1}^{(p-1)/3} \chi^3(a^2 + t^2\bar{a}) \right|^2, \quad (3)$$

where  $t$  is any integer such that  $p \nmid t^3 - 1$  and  $p \nmid t - 1$ .

Unfortunately, formulas (1) and (2) are only suitable for the primes  $p$  with  $p = 4k + 1$ . If  $p = 4k + 3$ , we still have not found any similar representation. Some related works can also be found in [2–12].

On the other hand, in [10, 11], X. X. Li and S. M. Shen studied the mean value properties of the quartic Gauss sums and two-term exponential sums and proved an interesting recurrence formula for it. That is, they give a recurrence formula for

$$M_k(p) = \sum_{m=1}^{p-1} \left( \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right)^k \cdot \left( \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right)^h, \quad (4)$$

with  $h = 2$  and  $3$ , respectively.

Inspired by literatures [10, 11], we considered the calculating problem of the hybrid power mean

$$H_k(p) = \sum_{m=1}^{p-1} \left( \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right)^k \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2. \quad (5)$$

Of course, the contents in [10] and  $H_k(p)$  look a little similar, but they are very different. The main difference between them is the absolute value of the two-term exponential sums. In fact, for all positive integers  $k \geq 1$ , we have

$$\left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^k \neq \left( \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right)^k. \quad (6)$$

Our goal is to obtain a sharp asymptotic formula for  $H_k(p)$ . About this problem, it seems that none had studied it yet, at least we have not seen related papers before. The problem is interesting because it may reveal the regularity of the distribution of values related to the quartic Gauss sums and two-term exponential sums.

In this paper, we will use the analytic methods and the properties of the classical Gauss sums to study this problem and prove an interesting four-order linear recurrence formula for it. That is, we will prove the following.

**Theorem 1.** *Let  $p$  be an odd prime with  $p \equiv 1 \pmod{8}$ . Then, for any integer  $k \geq 1$ , we have the four-order linear recurrence formula:*

$$\sum_{m=1}^{p-1} \left| \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right|^2 \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 = 3p^3 - 20p^2 - 4p\alpha^2 + 12p\alpha - 3p. \quad (11)$$

**Corollary 2.** *Let  $p$  be an odd prime with  $p \equiv 1 \pmod{8}$ . Then, we have*

$$\sum_{m=1}^{p-1} \left| \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right|^4 \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 = 17p^4 + 4p^3\alpha^2 - 88p^2\alpha^2 - 116p^3 - 17p^2 + 120p^2\alpha - 4p\alpha^2. \quad (12)$$

Notes: in Theorems 1 and 2, we only considered the two-term exponential sums:

$$S(m, k, p) = \sum_{a=1}^{p-1} e\left(\frac{ma^k+a}{p}\right), \quad (13)$$

with  $k = 4$ . Maybe we can use a similar method to study the general case  $k \geq 3$ , but it is hard to get an exact value for the

$$H_{k+4}(p) = 6pH_{k+2}(p) + 8p\alpha H_{k+1}(p) - (p^2 - 4p\alpha^2)H_k(p), \quad (7)$$

where the first four terms of  $H_k(p)$  are  $H_1(p) = 6p \cdot (1 - \alpha)$ ,

$$\begin{aligned} H_2(p) &= 3p^3 - 20p^2 - 4p\alpha^2 + 12p\alpha - 3p, \\ H_3(p) &= 6p^3\alpha - 66p^2\alpha - 6p\alpha + 8p\alpha^2 + 34p^2, \\ H_4(p) &= 17p^4 + 4p^3\alpha^2 - 88p^2\alpha^2 - 116p^3 \\ &\quad - 17p^2 + 120p^2\alpha - 4p\alpha^2. \end{aligned} \quad (8)$$

**Theorem 2.** *Let  $p$  be an odd prime with  $p \equiv 5 \pmod{8}$ . Then, for any integer  $k \geq 1$ , we have the recurrence formula*

$$H_{k+4}(p) = -2pH_{k+2}(p) + 8p\alpha H_{k+1}(p) - (9p^2 - 4p\alpha^2)H_k(p), \quad (9)$$

where the first four terms of  $H_k(p)$  are  $H_1(p) = 2p \cdot (\alpha + 1)$ ,

$$\begin{aligned} H_2(p) &= -p^3 + 4p^2 + p + 4p\alpha^2 + 4p\alpha, \\ H_3(p) &= -10p^2 + 2\alpha \cdot (3p^3 - 17p^2 - 3p), \\ H_4(p) &= 4p^3\alpha^2 - 7p^4 - 8p^2\alpha^2 + 28p^3 - 4p\alpha^2 + 7p^2 + 8p^2\alpha. \end{aligned} \quad (10)$$

If  $p \equiv 1 \pmod{8}$ , then there exists an integer  $c$  such that  $c^4 \equiv -1 \pmod{p}$ . So, from Theorem 1, we may immediately deduce the following two corollaries.

**Corollary 1.** *Let  $p$  be an odd prime with  $p \equiv 1 \pmod{8}$ . Then, we have*

hybrid power mean as in  $H_k(p)$ , except  $k = 4$ . Therefore, we will not discuss the general integer  $k$  in this paper.

Papers [10, 11] and our results are independent of each other. That is, they cannot be deduced from each other.

Even if our corollaries are used, they cannot be deduced with the results in [10].

In addition, our methods are not suitable for studying the mean value,

$$\sum_{m=1}^{p-1} \left( \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right)^k \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^3, \quad (14)$$

because we cannot deal with the cubic power mean  $|\sum_{a=1}^{p-1} e(ma^4 + a/p)|^3$ .

Maybe we can use our methods to study the fourth power mean:

$$\sum_{m=1}^{p-1} \left( \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right)^k \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^4. \quad (15)$$

This is definitely a challenge, which is the goal of our further study.

## 2. Several Lemmas

In this section, we give some lemmas which are necessary in the proof of our theorems. Hereinafter, we shall use some properties of the classical Gauss sums and quadratic residue mod  $p$ , and all of them can be found in [1, 13], so they will not be repeated here. First, we have the following.

**Lemma 1.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, for any four-order character  $\psi$  modulo  $p$ , we have the identity

$$\tau^2(\psi) + \tau^2(\bar{\psi}) = 2\sqrt{p} \cdot \alpha, \quad (16)$$

where  $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e(a/p)$  denotes the classical Gauss sums,  $\alpha = \alpha(p) = (1/2)\sum_{a=1}^{p-1} (a + \bar{a}/p)$ , and  $(*/p) = \chi_2$  denotes Legendre's symbol modulo  $p$ .

*Proof.* See Lemma 2.2 in [9].

**Lemma 2.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ , and  $(*/p) = \chi_2$  denotes the Legendre symbol mod  $p$ . Then, for any integer  $b$  with  $(b, p) = 1$ , we have the identities

$$A(b) = \sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right) = \left(\frac{b}{p}\right)\sqrt{p} + \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)e\left(\frac{ba^2}{p}\right). \quad (17)$$

If  $p = 8k + 5$ , then we have

$$\begin{aligned} A^2(b) &= -p + 2\left(\frac{b}{p}\right)\sqrt{p} \left( \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)e\left(\frac{ba^2}{p}\right) \right) + \left(\frac{b}{p}\right)\sqrt{p} \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right), \\ A^3(b) &= -5\left(\frac{b}{p}\right)p^{(3/2)} + p \left( \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)e\left(\frac{ba^2}{p}\right) \right) + 3p \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right) \\ &\quad + \left(\frac{b}{p}\right)\sqrt{p} \left( \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)e\left(\frac{ba^2}{p}\right) \right) \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right), \\ A^4(b) &= -2pA^2(b) + 4pA(b) \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right) - 9p^2 + p \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right)^2. \end{aligned} \quad (18)$$

If  $p = 8k + 1$ , then we have

$$\begin{aligned} A^2(b) &= 3p + 2\left(\frac{b}{p}\right)\sqrt{p} \left( \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)e\left(\frac{ba^2}{p}\right) \right) + \left(\frac{b}{p}\right)\sqrt{p} \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right), \\ A^3(b) &= 7\left(\frac{b}{p}\right)p^{(3/2)} + 5p \left( \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)e\left(\frac{ba^2}{p}\right) \right) + 3p \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right) \\ &\quad + \left(\frac{b}{p}\right)\sqrt{p} \left( \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)e\left(\frac{ba^2}{p}\right) \right) \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right), \\ A^4(b) &= 6pA^2(b) + 4pA(b) \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right) - p^2 + p \left( \sum_{a=1}^{p-1} \left(\frac{a+\bar{a}}{p}\right) \right)^2. \end{aligned} \quad (19)$$



*Proof.* See Lemmas 1 and 3 in [12].

**Lemma 3.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = p^2 - 4p - 1. \quad (20)$$

*Proof.* Since  $p \equiv 1 \pmod{4}$ , the congruence equation  $x^4 \equiv 1 \pmod{p}$  has four solutions. From the properties of reduced residue system mod  $p$  and the trigonometric identity,

$$\sum_{m=1}^{p-1} e\left(\frac{nm}{p}\right) = \begin{cases} p-1, & \text{if } p|n, \\ -1, & \text{if } p \nmid n, \end{cases} \quad (21)$$

we have

$$\begin{aligned} \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} e\left(\frac{mb^4(a^4 - 1) + b(a-1)}{p}\right) \\ &= \sum_{a=1}^{p-1} \left( \sum_{m=1}^{p-1} e\left(\frac{m(a^4 - 1)}{p}\right) \right) \left( \sum_{b=1}^{p-1} e\left(\frac{b(a-1)}{p}\right) \right) = (p-1)^2 - 3(p-1) + (p-5) = p^2 - 4p - 1. \end{aligned} \quad (22)$$

This proves Lemma 3.

**Lemma 4.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, for any four-order character  $\psi \pmod{p}$  if  $p = 8k + 1$ , we have the identity

$$\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = 2\tau(\psi) - 2\sqrt{p} \cdot \tau(\bar{\psi}). \quad (23)$$

If  $p = 8k + 5$ , then we have

$$\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = 0. \quad (24)$$

*Proof.* For any integer  $m$  with  $(m, p) = 1$ , note that  $1 + \psi(m) + \psi^2(m) + \bar{\psi}(m) = 4$  if  $m^4 \equiv 1 \pmod{p}$ ;  $1 + \psi(m) + \psi^2(m) + \bar{\psi}(m) = 0$ , otherwise. From (21) and the properties of Gauss sums, we have

$$\begin{aligned} \sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \psi(m) e\left(\frac{mb^4(a^4 - 1) + b(a-1)}{p}\right) \\ &= \tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a^4 - 1) \sum_{b=1}^{p-1} e\left(\frac{b(a-1)}{p}\right) = -\tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a^4 - 1) \\ &= -\tau(\psi) \sum_{a=1}^{p-1} (1 + \psi(a) + \psi^2(a) + \bar{\psi}(a)) \bar{\psi}(a-1) \\ &= -\tau(\psi) \left( \sum_{a=1}^{p-1} \bar{\psi}(a-1) + \sum_{a=1}^{p-1} \bar{\psi}(1 - \bar{a}) \right) - \tau(\psi) \sum_{a=1}^{p-1} \chi_2(a) \bar{\psi}(a-1) - \tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a(a-1)), \end{aligned} \quad (25)$$

where we have used the identity  $\chi_2 = \psi^2$ .

If  $p = 8k + 1$ , then  $\psi(-1) = 1$  and

$$\sum_{a=1}^{p-1} \bar{\psi}(a-1) + \sum_{a=1}^{p-1} \bar{\psi}(1 - \bar{a}) = 2 \sum_{a=1}^{p-1} \bar{\psi}(a-1) = -2. \quad (26)$$

From the properties of Gauss sums,  $\tau(\chi_2) = \sqrt{p}$  and  $\psi^3 = \bar{\psi}$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi_2(a) \bar{\psi}(a-1) &= \frac{1}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{b(a-1)}{p}\right) \\ &= \frac{\sqrt{p}}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \chi_2(b) e\left(\frac{-b}{p}\right) = \sqrt{p} \cdot \frac{\tau(\bar{\psi})}{\tau(\psi)}, \end{aligned} \quad (27)$$

$$\begin{aligned} \sum_{a=1}^{p-1} \bar{\psi}(a(a-1)) &= \frac{1}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \sum_{a=1}^{p-1} \bar{\psi}(a) e\left(\frac{b(a-1)}{p}\right) \\ &= \frac{\tau(\bar{\psi})}{\tau(\psi)} \sum_{b=1}^{p-1} \psi^2(b) e\left(\frac{-b}{p}\right) = \sqrt{p} \cdot \frac{\tau(\bar{\psi})}{\tau(\psi)}. \end{aligned} \quad (28)$$

If  $p = 8k + 1$ , then combining (25)–(28), we have

$$\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = 2\tau(\psi) - 2\sqrt{p} \cdot \tau(\bar{\psi}). \quad (29)$$

If  $p = 8k + 5$ , then  $\psi(-1) = -1$ . Thus,

$$\sum_{a=1}^{p-1} \bar{\psi}(a-1) + \sum_{a=1}^{p-1} \bar{\psi}(1-a) = \sum_{a=1}^{p-1} \bar{\psi}(a-1) + \sum_{a=1}^{p-1} \bar{\psi}(1-a) = 0, \quad (30)$$

$$\sum_{a=1}^{p-1} \chi_2(a) \bar{\psi}(a-1) = \frac{\sqrt{p}}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \chi_2(b) e\left(\frac{-b}{p}\right) = -\sqrt{p} \cdot \frac{\tau(\bar{\psi})}{\tau(\psi)}, \quad (31)$$

$$\sum_{a=1}^{p-1} \bar{\psi}(a(a-1)) = \frac{\tau(\bar{\psi})}{\tau(\psi)} \sum_{b=1}^{p-1} \psi^2(b) e\left(\frac{-b}{p}\right) = \sqrt{p} \cdot \frac{\tau(\bar{\psi})}{\tau(\psi)}. \quad (32)$$

Combining (25) and (30)–(32), we know that if  $p = 8k + 5$ , then

$$\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = 0. \quad (33)$$

Now, Lemma 4 follows from (26) and (33).

**Lemma 5.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ ; then, we have

$$\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = 2\sqrt{p} - \psi(-1)(\tau^2(\psi) + \tau^2(\bar{\psi})). \quad (34)$$

*Proof.* Note that  $\chi_2$  is a real character mod  $p$  and  $\tau(\chi_2) = \sqrt{p}$ , and from the method of proving Lemma 3, we have

$$\begin{aligned} \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \chi_2(m) e\left(\frac{mb^4(a^4 - 1) + b(a-1)}{p}\right) \\ &= \sqrt{p} \cdot \sum_{a=1}^{p-1} \chi_2(a^4 - 1) \sum_{b=1}^{p-1} e\left(\frac{b(a-1)}{p}\right) = -\sqrt{p} \cdot \sum_{a=1}^{p-1} \chi_2(a^4 - 1) \\ &= -\sqrt{p} \cdot \sum_{a=1}^{p-1} (1 + \psi(a) + \chi_2(a) + \bar{\psi}(a)) \chi_2(a-1) \\ &= -\sqrt{p} \cdot \left( -2 + \sum_{a=1}^{p-1} \psi(a) \chi_2(a-1) + \sum_{a=1}^{p-1} \bar{\psi}(a) \chi_2(a-1) \right) = 2\sqrt{p} - \psi(-1)(\tau^2(\psi) + \tau^2(\bar{\psi})). \end{aligned} \quad (35)$$

This proves Lemma 5.

### 3. Proofs of the Theorems

In this section, we shall complete the proofs of our theorems. First, we prove Theorem 1. For any integer  $m$  with  $(m, p) = 1$ , note that  $\psi^2 = \chi_2$  and  $\tau(\chi_2) = \sqrt{p}$ , and we have

$$\begin{aligned} A(m) &= \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1} (1 + \psi(a) + \chi_2(a) + \bar{\psi}(a)) e\left(\frac{ma}{p}\right) \\ &= \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) + \bar{\psi}(m) \tau(\psi) + \chi_2(m) \sqrt{p} + \psi(m) \tau(\bar{\psi}) \\ &= \bar{\psi}(m) \tau(\psi) + \chi_2(m) \sqrt{p} + \psi(m) \tau(\bar{\psi}). \end{aligned} \quad (36)$$

If  $p$  is an odd prime with  $p \equiv 1 \pmod{8}$ , then, applying (36) and Lemmas 1, 4, and 5 and noting that  $\tau(\psi)\tau(\bar{\psi}) = p$ , we have

$$\begin{aligned}
 H_1(p) &= \sum_{m=1}^{p-1} \left( \bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}) \right) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
 &= \tau(\psi)(2\tau(\bar{\psi}) - 2\sqrt{p}\tau(\psi)) + \tau(\bar{\psi})(2\tau(\psi) - 2\sqrt{p}\tau(\bar{\psi})) + \sqrt{p}(2\sqrt{p} - \tau^2(\psi) - \tau^2(\bar{\psi})) \\
 &= 6p - 3\sqrt{p} \cdot (\tau^2(\psi) + \tau^2(\bar{\psi})) = 6p \cdot (1 - \alpha).
 \end{aligned} \tag{37}$$

Similarly, note that the identity

$$\tau^4(\psi) + \tau^4(\bar{\psi}) = (\tau^2(\psi) + \tau^2(\bar{\psi}))^2 - 2p^2 = 2p \cdot (2\alpha^2 - p). \tag{38}$$

From the method of proving (37) and Lemmas 1 and 2, we also have

$$\begin{aligned}
 H_2(p) &= \sum_{m=1}^{p-1} \left( \bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}) \right)^2 \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
 &= \sum_{m=1}^{p-1} \left( 3p + \chi_2(m)\tau^2(\psi) + \chi_2(m)\tau^2(\bar{\psi}) + 2\sqrt{p}\psi(m)\tau(\psi) + 2\sqrt{p}\bar{\psi}(m)\tau(\bar{\psi}) \right) \times \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
 &= 3p(p^2 - 4p - 1) + (\tau^2(\psi) + \tau^2(\bar{\psi}))(2\sqrt{p} - \tau^2(\psi) - \tau^2(\bar{\psi})) + 2\sqrt{p}\tau(\psi) \\
 &\quad (2\tau(\psi) - 2\sqrt{p}\tau(\bar{\psi})) + 2\sqrt{p}\tau(\bar{\psi})(2\tau(\bar{\psi}) - 2\sqrt{p}\tau(\psi)) \\
 &= 3p^3 - 22p^2 - 3p + 6\sqrt{p}(\tau^2(\psi) + \tau^2(\bar{\psi})) - (\tau^4(\psi) + \tau^4(\bar{\psi})) = 3p^3 - 20p^2 - 4p\alpha^2 + 12p\alpha - 3p, \\
 H_3(p) &= \sum_{m=1}^{p-1} \left( \bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}) \right)^3 \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
 &= \sum_{m=1}^{p-1} \left( \psi(m)\tau^3(\psi) + \bar{\psi}(m)\tau^3(\bar{\psi}) + \chi_2(m)p\sqrt{p} + 3p\bar{\psi}(m)\tau(\psi) + 3p\psi(m)\tau(\bar{\psi}) \right. \\
 &\quad \left. + 3\sqrt{p}\tau^2(\psi) + 3\sqrt{p}\tau^2(\bar{\psi}) + 3p\bar{\psi}(m)\tau(\psi) + 3p\psi(m)\tau(\bar{\psi}) + 6p^{(3/2)}\chi_2(m) \right) \times \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
 &= \tau^3(\psi)(2\tau(\psi) - 2\sqrt{p}\tau(\bar{\psi})) + \tau^3(\bar{\psi})(2\tau(\bar{\psi}) - 2\sqrt{p}\tau(\psi)) + p^{(3/2)}(2\sqrt{p} - \tau^2(\psi) - \tau^2(\bar{\psi})) \\
 &\quad + 3\sqrt{p}(p^2 - 4p - 1)(\tau^2(\psi) + \tau^2(\bar{\psi})) + 6p\tau(\psi)(2\tau(\bar{\psi}) - 2\sqrt{p}\tau(\psi)) + 6p\tau(\bar{\psi})(2\tau(\psi) - 2\sqrt{p}\tau(\bar{\psi})) \\
 &\quad + 6p^{(3/2)}(2\sqrt{p} - \tau^2(\psi) - \tau^2(\bar{\psi})) = 3p^{(1/2)}(p^2 - 11p - 1)(\tau^2(\psi) + \tau^2(\bar{\psi})) + 2(\tau^4(\psi) + \tau^4(\bar{\psi})) + 38p^2 \\
 &= 6p^3\alpha - 66p^2\alpha - 6p\alpha + 8p\alpha^2 + 34p^2,
 \end{aligned} \tag{39}$$

(40)

$$\begin{aligned}
H_4(p) &= \sum_{m=1}^{p-1} \left( \bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}) \right)^4 \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} \left( \begin{aligned} &\tau^4(\psi) + \tau^4(\bar{\psi}) + p^2 + 6p^2 + 6p\chi_2(m)\tau^2(\psi) + 6p\chi_2(m)\tau^2(\bar{\psi}) \\ &+ 12p^2 + 12p^{(3/2)}\psi(m)\tau(\psi) + 12p^{(3/2)}\bar{\psi}(m)\tau(\bar{\psi}) + 4\sqrt{p}\bar{\psi}(m)\tau^3(\psi) + \\ &4\sqrt{p}\psi(m)\tau^3(\bar{\psi}) + 4p\chi_2(m)\tau^2(\psi) + 4p\chi_2(m)\tau^2(\bar{\psi}) + 4p^{(3/2)}\psi(m)\tau(\psi) \\ &+ 4p^{(3/2)}\bar{\psi}(m)\tau(\bar{\psi}) \end{aligned} \right) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 \\
&= (\tau^4(\psi) + \tau^4(\bar{\psi}) + 19p^2)(p^2 - 4p - 1) + 10p(\tau^2(\psi) + \tau^2(\bar{\psi}))(2\sqrt{p} - \tau^2(\psi) - \tau^2(\bar{\psi})) \\
&\quad + 16p^{(3/2)}\tau(\psi)(2\tau(\psi) - 2\sqrt{p} \cdot \tau(\bar{\psi})) + 16p^{(3/2)}\tau(\bar{\psi})(2\tau(\bar{\psi}) - 2\sqrt{p} \cdot \tau(\psi)) + 4\sqrt{p}\tau^3(\bar{\psi}) \\
&\quad (2\tau(\psi) - 2\sqrt{p} \cdot \tau(\bar{\psi})) + 4\sqrt{p}\tau^3(\psi)(2\tau(\bar{\psi}) - 2\sqrt{p} \cdot \tau(\psi)) \\
&= (\tau^4(\psi) + \tau^4(\bar{\psi}))(p^2 - 22p - 1) + 60p^{(3/2)}(\tau^2(\psi) + \tau^2(\bar{\psi})) + p^2(19p^2 - 160p - 19) \\
&= 17p^4 + 4p^3\alpha^2 - 88p^2\alpha^2 - 116p^3 - 17p^2 + 120p^2\alpha - 4p\alpha^2.
\end{aligned} \tag{41}$$

If  $k \geq 1$ , then, from Lemmas 1 and 2, we have

$$\begin{aligned}
H_{k+4}(p) &= \sum_{m=1}^{p-1} A^k(m) \cdot A^4(m) \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} A^k(m) \cdot (6pA^2(m) + 8p\alpha A(m) - (p^2 - 4p\alpha^2)) \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 \\
&= 6pH_{k+2}(p) + 8p\alpha H_{k+1}(p) - (p^2 - 4p\alpha^2)H_k(p).
\end{aligned} \tag{42}$$

Now, Theorem 1 follows from (37)–(42).

*Proof of Theorem 2.* If the prime  $p$  satisfies  $p = 8k + 5$ , then note that  $\psi(-1) = -1$ , and from (13) and Lemmas 1, 3, 4, and 5, we have

$$\begin{aligned}
H_1(p) &= \sum_{m=1}^{p-1} (\bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi})) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 \\
&= \sqrt{p}(2\sqrt{p} + \tau^2(\psi) + \tau^2(\bar{\psi})) = 2p \cdot (\alpha + 1).
\end{aligned} \tag{43}$$

Similarly, note that  $\tau(\psi)\tau(\bar{\psi}) = -p$ ; from Lemmas 1–5, we also have

$$\begin{aligned}
H_2(p) &= \sum_{m=1}^{p-1} (\bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}))^2 \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} (-p + \chi_2(m)\tau^2(\psi) + \chi_2(m)\tau^2(\bar{\psi}) + 2\sqrt{p}\psi(m)\tau(\psi) + 2\sqrt{p}\bar{\psi}(m)\tau(\bar{\psi})) \times \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= -p(p^2 - 4p - 1) + (\tau^2(\psi) + \tau^2(\bar{\psi}))(2\sqrt{p} + \tau^2(\psi) + \tau^2(\bar{\psi})) = -p^3 + 4p^2 + p + 4p\alpha^2 + 4p\alpha,
\end{aligned} \tag{44}$$

$$\begin{aligned}
H_3(p) &= \sum_{m=1}^{p-1} (\bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}))^3 \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} \left( \begin{aligned} &\psi(m)\tau^3(\psi) + \bar{\psi}(m)\tau^3(\bar{\psi}) + \chi_2(m)p\sqrt{p} - 3p\bar{\psi}(m)\tau(\psi) - 3p\psi(m)\tau(\bar{\psi}) \\ &+ 3\sqrt{p}\tau^2(\psi) + 3\sqrt{p}\tau^2(\bar{\psi}) + 3p\bar{\psi}(m)\tau(\psi) + 3p\psi(m)\tau(\bar{\psi}) - 6p^{(3/2)}\chi_2(m) \end{aligned} \right) \times \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= -5p^{(3/2)}(2\sqrt{p} + G(2, \psi)) + 3\sqrt{p}G(2, \psi)(p^2 - 4p - 1) = -10p^2 + 2\alpha \cdot (3p^3 - 17p^2 - 3p),
\end{aligned} \tag{45}$$

$$\begin{aligned}
H_4(p) &= \sum_{m=1}^{p-1} (\bar{\psi}(m)\tau(\psi) + \chi_2(m)\sqrt{p} + \psi(m)\tau(\bar{\psi}))^4 \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} \left( \begin{aligned} &\tau^4(\psi) + \tau^4(\bar{\psi}) + p^2 + 6p^2 + 6p\chi_2(m)\tau^2(\psi) + 6p\chi_2(m)\tau^2(\bar{\psi}) \\ &- 12p^2 - 12p^{(3/2)}\psi(m)\tau(\psi) - 12p^{(3/2)}\bar{\psi}(m)\tau(\bar{\psi}) + 4\sqrt{p}\bar{\psi}(m)\tau^3(\psi) \\ &+ 4\sqrt{p}\psi(m)\tau^3(\bar{\psi}) - 4p\chi_2(m)\tau^2(\psi) - 4p\chi_2(m)\tau^2(\bar{\psi}) \\ &+ 4p^{(3/2)}\psi(m)\tau(\psi) + 4p^{(3/2)}\bar{\psi}(m)\tau(\bar{\psi}) \end{aligned} \right) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= (\tau^4(\psi) + \tau^4(\bar{\psi}) - 5p^2)(p^2 - 4p - 1) + 2p(\tau^2(\psi) + \tau^2(\bar{\psi}))(2\sqrt{p} + \tau^2(\psi) + \tau^2(\bar{\psi})) \\
&= G(4, \psi)(p^2 - 2p - 1) + 4p^{(3/2)}G(2, \psi) - p^2(5p^2 - 24p - 5) \\
&= 4p^3\alpha^2 - 7p^4 - 8p^2\alpha^2 + 28p^3 - 4p\alpha^2 + 7p^2 + 8p^2\alpha.
\end{aligned} \tag{46}$$

If  $k \geq 1$ , then, from Lemmas 1 and 2, we have

$$\begin{aligned}
H_{k+4}(p) &= \sum_{m=1}^{p-1} A^k(m) \cdot A^4(m) \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} A^k(m) \cdot (-2pA^2(m) + 8p\alpha A(m) - (9p^2 - 4p\alpha^2)) \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= -2pH_{k+2}(p) + 8p\alpha H_{k+1}(p) - (9p^2 - 4p\alpha^2)H_k(p).
\end{aligned} \tag{47}$$

Now, Theorem 2 follows from (43)–(47).

This completes the proofs of our all theorems.

#### 4. Conclusion

The main results of this paper are two theorems. Theorem 1 establishes a four-order linear recurrence formula for the hybrid power mean involving the quartic Gauss sums and the two-term exponential sums for the case  $p \equiv 1 \pmod{8}$ . Theorem 2 establishes a similar conclusion for the case  $p \equiv 5 \pmod{8}$ . These achievements represent new

contributions to research in the relevant fields, and it also has a good reference function to the research of related problems.

#### Data Availability

No data were used to support the findings of the study.

#### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors have equally contributed to this work and have read and approved the final manuscript.

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## Research Article

# A Hybrid Mean Value Involving Dedekind Sums and the Generalized Kloosterman Sums

Xiaowei Pan<sup>1</sup> and Xiaoyan Guo<sup>2</sup>

<sup>1</sup>School of Health Management, Xi'an Medical University, Xi'an, Shaanxi, China

<sup>2</sup>School of Mathematics, Northwest University, Xi'an, Shaanxi, China

Correspondence should be addressed to Xiaowei Pan; panxiaowei@xiji.edu.cn

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In this paper, we use the mean value theorem of Dirichlet  $L$ -functions and the properties of Gauss sums and Dedekind sums to study the hybrid mean value problem involving Dedekind sums and the general Kloosterman sums and give an interesting identity for it.

## 1. Introduction

Let  $q$  be a natural number and  $h$  be an integer coprime to  $q$ . The classical Dedekind sums

$$S(h, q) = \sum_{a=1}^q \left( \left( \frac{a}{q} \right) \right) \left( \left( \frac{ah}{q} \right) \right), \quad (1)$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer,} \end{cases} \quad (2)$$

describes the behaviour of the logarithm of the  $\eta$ -function (see [1, 2]) under modular transformations. There are many

papers written on their various properties (see the examples in [3–10] and [11]).

In particular, Zhang and Liu [12] studied the hybrid mean value problems related to Dedekind sums and Kloosterman sums:

$$K(m, n; q) = \sum_{a=1}^q e\left(\frac{ma + n\bar{a}}{q}\right), \quad (3)$$

where  $q \geq 3$  is an integer,  $\sum_{a=1}^{q'}$  denotes the summation over all  $1 \leq a \leq q$  with  $(a, q) = 1$ ,  $e(y) = e^{2\pi i y}$ , and  $\bar{a}$  denotes the multiplicative inverse of  $a \bmod q$ . They proved the following results:

**Theorem 1.** Let  $p$  be an odd prime, then one has the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1; p)|^2 \cdot |K(b, 1; p)|^2 \cdot S(a\bar{b}, p) = \begin{cases} \frac{1}{12} \cdot p^2 \cdot ((p-1)(p-2) - 12 \cdot h_p^2), & \text{if } p \equiv 3 \pmod{4}, \\ \frac{1}{12} \cdot p^2 \cdot (p-1)(p-2), & \text{if } p \equiv 1 \pmod{4}, \end{cases} \quad (4)$$

where  $h_p$  denotes the class number of the quadratic field  $\mathbf{Q}(\sqrt{-p})$ .

**Theorem 2.** Let  $p$  be an odd prime, then one has the asymptotic formula:

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1; p)|^2 \cdot |K(b, 1; p)|^2 \cdot |S(a\bar{b}, q)|^2 = \frac{1}{24}p^5 + O\left(p^4 \cdot \exp\left(\frac{3 \ln \ln p}{\ln p}\right)\right), \quad (5)$$

where  $\exp(y) = e^y$ .

It is natural that people will ask, for the general Kloosterman sums

$$K(m, n, \chi; p) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma + n\bar{a}}{p}\right), \quad (6)$$

what will happen? Whether there exists an identity similar to Theorem 1? Here,  $\chi$  denotes any Dirichlet character mod  $p$ .

The main purpose of this paper is to answer these questions. That is, we shall use the mean value theorem of Dirichlet  $L$ -functions and the properties of Gauss sums and Dedekind sums to prove the following.

**Theorem 3.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, for any Dirichlet character  $\chi \pmod{p}$ , we have the identity:

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) = \begin{cases} \frac{1}{12} \cdot p^2 (p-1)(p-2), & \text{if } \chi(-1) = 1, \\ \frac{1}{12} \cdot p^2 (p-1)(p-2) - \frac{p^3}{\pi^2} \cdot |L(1, \chi)|^2, & \text{if } \chi(-1) = -1. \end{cases} \quad (7)$$

**Theorem 4.** Let  $p$  be an odd prime with  $p \equiv 3 \pmod{4}$ . Then, for any Dirichlet character  $\chi \pmod{p}$ , we have the identity:

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) = \begin{cases} \frac{1}{12} \cdot p^2 (p-1)(p-2) - p^2 \cdot h_p^2, & \text{if } \chi(-1) = 1, \\ \frac{p^2 (p-1)(p-2)}{12} - p^2 \cdot h_p^2 - \frac{2p^3}{\pi^2} \cdot |L(1, \chi)|^2, & \text{if } \chi(-1) = -1 \text{ and } \chi \neq \chi_2, \\ \frac{p^2 (p-1)(p-2)}{12} + p^2 \cdot (p-4) \cdot h_p^2, & \text{if } \chi = \chi_2, \end{cases} \quad (8)$$

where  $\chi_2 = (*/p)$  denotes the Legendre symbol and  $h_p$  denotes the class number of the quadratic field  $\mathbf{Q}(\sqrt{-p})$ .

It is clear that if  $\chi = \chi_0$ , then  $K(a, 1, \chi; p) = K(a, 1; p)$ . Note that  $\chi_0(-1) = 1$ , from Theorems 3 and 4, we may immediately deduce Theorem 1 in [12], so our results are the generalization of [12].

## 2. Several Lemmas

In this section, we shall give several simple lemmas, which are necessary to the proofs of our theorems. Hereafter, we shall use many properties of character sums and Gauss sums, and all of these can be found in reference [13]. First, we have the following.

**Lemma 1.** Let  $p > 3$  be a prime,  $\chi$  be any fixed Dirichlet character mod  $p$ . Then, for any nonprincipal character  $\chi_1 \pmod{p}$  with  $\chi\chi_1 \neq \chi_0$ , we have the identity:

$$\sum_{m=1}^{p-1} \chi_1(m) \cdot |K(m, 1, \chi; p)|^2 = \frac{\bar{\chi}_1(-1) \cdot \tau^2(\chi_1) \cdot \tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\chi_1)}, \quad (9)$$

where  $\chi_0$  denotes the principal character mod  $p$ ,  $\tau(\chi)$  denotes the Gauss sums defined as  $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e(a/p)$ , and  $\bar{\chi}$  denotes the complex conjugate of  $\chi$ .

*Proof.* From the definition of Kloosterman sums and the properties of Gauss sums, we have



$$\begin{aligned}
\sum_{m=1}^{p-1} \chi_1(m) \cdot |K(m, 1, \chi; p)|^2 &= \sum_{m=1}^{p-1} \chi_1(m) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \bar{\chi}(b) e\left(\frac{am - mb + \bar{a} - \bar{b}}{p}\right) \\
&= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \sum_{m=1}^{p-1} \chi_1(m) e\left(\frac{mb(a-1) + \bar{b} \cdot (\bar{a} - 1)}{p}\right) = \tau(\chi_1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \bar{\chi}_1(b) \bar{\chi}_1(a-1) e\left(\frac{\bar{b} \cdot (\bar{a} - 1)}{p}\right) \\
&= \tau(\chi_1) \sum_{a=1}^{p-1} \chi(a) \bar{\chi}_1(a-1) \sum_{b=1}^{p-1} \bar{\chi}_1(b) e\left(\frac{\bar{b} \cdot (\bar{a} - 1)}{p}\right) = \tau^2(\chi_1) \sum_{a=1}^{p-1} \chi(a) \bar{\chi}_1(a-1) \bar{\chi}_1(\bar{a} - 1) \\
&= \bar{\chi}_1(-1) \tau^2(\chi_1) \sum_{a=1}^{p-1} \chi(a) \chi_1(a) \bar{\chi}_1^2(a-1) = \bar{\chi}_1(-1) \tau^2(\chi_1) \sum_{a=1}^{p-1} \chi(a+1) \chi_1(a+1) \bar{\chi}_1^2(a).
\end{aligned} \tag{10}$$

On the other hand, from the properties of Gauss sums, we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \chi(a+1) \chi_1(a+1) \bar{\chi}_1^2(a) &= \frac{1}{\tau(\bar{\chi}\chi_1)} \sum_{a=1}^{p-1} \bar{\chi}_1^2(a) \sum_{b=1}^{p-1} \bar{\chi}\chi_1(b) e\left(\frac{b(a+1)}{p}\right) \\
&= \frac{1}{\tau(\bar{\chi}\chi_1)} \sum_{b=1}^{p-1} \bar{\chi}\chi_1(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \bar{\chi}_1^2(a) e\left(\frac{ba}{p}\right) = \frac{\tau(\bar{\chi}_1^2)}{\tau(\bar{\chi}\chi_1)} \sum_{b=1}^{p-1} \bar{\chi}(b) \chi_1(b) e\left(\frac{b}{p}\right) = \frac{\tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\chi_1)}.
\end{aligned} \tag{11}$$

Combining (10) and (11), we may immediately deduce the identity:

$$\sum_{m=1}^{p-1} \chi_1(m) \cdot |K(m, 1, \chi; p)|^2 = \frac{\bar{\chi}_1(-1) \cdot \tau^2(\chi_1) \cdot \tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\chi_1)}. \tag{12}$$

This proves Lemma 1.

**Lemma 2.** Let  $p$  be a prime with  $p \equiv 1 \pmod{4}$  and  $\chi$  be any odd character mod  $p$ . Then, we have the identity:

$$\sum_{m=1}^{p-1} \bar{\chi}(m) \cdot |K(m, 1, \chi; p)|^2 = \tau^2(\bar{\chi}). \tag{13}$$

*Proof.* Since  $p \equiv 1 \pmod{4}$  and  $\chi$  is an odd character mod  $p$ , we know  $\chi$  is not the Legendre symbol and  $\chi^2 \neq \chi_0$ . Note that  $\chi(-1) = -1$ , from (10), we have

$$\begin{aligned}
\sum_{m=1}^{p-1} \bar{\chi}(m) \cdot |K(m, 1, \chi; p)|^2 &= \tau^2(\bar{\chi}) \sum_{a=1}^{p-1} \chi(a) \chi(a-1) \chi(\bar{a} - 1) \\
&= \chi(-1) \tau^2(\bar{\chi}) \sum_{a=1}^{p-1} \chi^2(a-1) = -\tau^2(\bar{\chi}) \left( \sum_{a=0}^{p-1} \chi^2(a) - 1 \right) = \tau^2(\bar{\chi}).
\end{aligned} \tag{14}$$

This proves Lemma 2.

**Lemma 3.** Let  $p$  be a prime with  $p \equiv 3 \pmod{4}$  and  $\chi_2$  be the Legendre symbol. Then, we have the identity:

$$\sum_{m=1}^{p-1} \chi_2(m) |K(m, 1, \chi_2; p)|^2 = -(p-2) \cdot \tau^2(\chi_2). \quad (15)$$

*Proof.* Note that  $\chi_2 = \bar{\chi}_2$ ,  $\chi_2^2 = \chi_0$ , and  $\chi_2(-1) = -1$ , and from the definition of  $K(m, 1, \chi_2; p)$  and the properties of Gauss sums, we have

$$\begin{aligned} \sum_{m=1}^{p-1} \chi_2(m) |K(m, 1, \chi_2; p)|^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) \sum_{m=1}^{p-1} \chi_2(m) e\left(\frac{m(a-b) + \bar{a} - \bar{b}}{p}\right) \\ &= \tau(\chi_2) \sum_{a=1}^{p-1} \chi_2(a) \sum_{b=1}^{p-1} \chi_2(b(a-1)) e\left(\frac{\bar{b}(\bar{a}-1)}{p}\right) = \tau^2(\chi_2) \sum_{a=1}^{p-1} \chi_2(a) \chi_2(a-1) \chi_2(\bar{a}-1) = -(p-2) \cdot \tau^2(\chi_2). \end{aligned} \quad (16)$$

This proves Lemma 3.

**Lemma 4.** Let  $q > 2$  be an integer, then for any integer  $a$  with  $(a, q) = 1$ , we have the identity:

$$S(a, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \pmod{d} \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2, \quad (17)$$

where  $L(1, \chi)$  denotes the Dirichlet  $L$ -function corresponding to the character  $\chi \pmod{d}$ .

*Proof.* See Lemma 2 of [7].

### 3. Proof of the Theorems

In this section, we will complete the proof of our theorems. First we prove Theorem 3. From Lemma 4 and the definition of  $S(a, p)$ , we have

$$S(a, p) = \frac{1}{\pi^2} \cdot \frac{p}{p-1} \cdot \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2, \quad (18)$$

and (with  $a = 1$ )

$$\sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^2 = \frac{\pi^2 \cdot (p-1)}{p} \cdot S(1, p) = \frac{\pi^2 \cdot (p-1)}{p} \cdot \sum_{a=1}^{p-1} \left(\frac{a}{p} - \frac{1}{2}\right)^2 = \frac{\pi^2}{12} \cdot \frac{(p-1)^2(p-2)}{p^2}. \quad (19)$$

Since  $p \equiv 1 \pmod{4}$ , we know the Legendre symbol  $(\cdot/p) = \chi_2$  is an even character mod  $p$ . Note that, for any nonprincipal character  $\chi \pmod{p}$ ,  $|\tau(\chi)| = \sqrt{p}$ . So, if  $\chi$  is an

even character mod  $p$ , then from Lemma 1, (18), and (19), we have

$$\begin{aligned} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} \left| \sum_{a=1}^{p-1} \chi_1(a) |K(a, 1, \chi; p)|^2 \right|^2 \cdot |L(1, \chi_1)|^2 \\ &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} \left| \frac{\tau^2(\chi_1) \cdot \tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\bar{\chi}_1)} \right|^2 \cdot |L(1, \chi_1)|^2 = \frac{p^4 \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 \\ &= \frac{1}{12} \cdot p^2(p-1)(p-2). \end{aligned} \quad (20)$$

If  $\chi$  is an odd character mod  $p$ , then note that the identity:

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} f(\chi_1) = \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \chi \neq \chi_0}} f(\chi_1) + f(\bar{\chi}), \quad (21)$$

from (18), (19), Lemmas 1 and 2, and the method of proving (20), we have

$$\begin{aligned} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \sum_{a=1}^{p-1} \chi_1(a) |K(a, 1, \chi; p)|^2 \right|^2 \cdot |L(1, \chi_1)|^2 \\ &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \chi \neq \chi_0}} \left| \sum_{a=1}^{p-1} \chi_1(a) |K(a, 1, \chi; p)|^2 \right|^2 \cdot |L(1, \chi_1)|^2 + \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \chi = \chi_0}} \left| \sum_{a=1}^{p-1} \chi_1(a) |K(a, 1, \chi; p)|^2 \right|^2 \cdot |L(1, \chi_1)|^2 \\ &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \chi \neq \chi_0}} \left| \frac{\tau^2(\chi_1) \cdot \tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\bar{\chi}_1)} \right|^2 \cdot |L(1, \chi_1)|^2 + \frac{p \cdot \pi^{-2}}{p-1} |\tau^2(\bar{\chi})|^2 \cdot |L(1, \bar{\chi})|^2 \\ &= \frac{p^4 \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 - \frac{p^4 \cdot \pi^{-2}}{p-1} |L(1, \chi)|^2 + \frac{p^3 \cdot \pi^{-2}}{p-1} |L(1, \chi)|^2 = \frac{1}{12} \cdot p^2(p-1)(p-2) - \frac{p^3}{\pi^2} \cdot |L(1, \chi)|^2, \end{aligned} \quad (22)$$

where  $\sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \chi \neq \chi_0}}$  denotes the summation over all odd characters  $\chi_1$  with  $\chi_1 \neq \bar{\chi}$ .

Combining (20) and (22), we may immediately deduce the identity:

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) = \begin{cases} \frac{1}{12} \cdot p^2(p-1)(p-2), & \text{if } \chi(-1) = 1, \\ \frac{1}{12} \cdot p^2(p-1)(p-2) - \frac{p^3}{\pi^2} \cdot |L(1, \chi)|^2, & \text{if } \chi(-1) = -1. \end{cases} \quad (23)$$

This proves Theorem 3.

Now, we prove Theorem 4. Since  $p \equiv 3 \pmod{4}$ , we know the Legendre symbol  $\chi_2$  is an odd character mod  $p$  and

$|\tau(\bar{\chi}_2^2)| = 1$ . If  $\chi$  is an even character mod  $p$ , then note that  $L(1, \chi_2) = (\pi \cdot h_p / \sqrt{p})$  (see reference [14]), from (18), (19), Lemma 1 and the properties of Gauss sums, we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \frac{\tau^2(\chi_1) \cdot \tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\bar{\chi}_1)} \right|^2 \cdot |L(1, \chi_1)|^2 \\
&= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \neq \chi_2}} \left| \frac{\tau^2(\chi_1) \cdot \tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\bar{\chi}_1)} \right|^2 \cdot |L(1, \chi_1)|^2 + \frac{p^3 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi_2)|^2 \\
&= \frac{p^4 \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 + \frac{p^3 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi_2)|^2 - \frac{p^4 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi_2)|^2 \\
&= \frac{1}{12} \cdot p^2 (p-1)(p-2) - p^2 \cdot h_p^2,
\end{aligned} \tag{24}$$

where  $h_p$  denotes the class number of the quadratic field  $\mathbf{Q}(\sqrt{-p})$ .

If  $\chi$  is an odd nonreal character mod  $p$ , then from (18), (19), Lemmas 1 and 2, and the method of proving (22) and (24), we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \sum_{a=1}^{p-1} \chi_1(a) |K(a, 1, \chi; p)|^2 \right|^2 \cdot |L(1, \chi_1)|^2 \\
&= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \chi \neq \chi_0}} \left| \frac{\tau^2(\chi_1) \cdot \tau(\bar{\chi}_1^2) \cdot \tau(\bar{\chi}\chi_1)}{\tau(\bar{\chi}\bar{\chi}_1)} \right|^2 \cdot |L(1, \chi_1)|^2 + \frac{p \cdot \pi^{-2}}{p-1} \left| \sum_{a=1}^{p-1} \bar{\chi}(a) |K(a, 1, \chi; p)|^2 \right|^2 \cdot |L(1, \bar{\chi})|^2 \\
&= \frac{p^4 \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 + \frac{p^3 \cdot \pi^{-2}}{p-1} \cdot |L(1, \bar{\chi})|^2 + \frac{p^3 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi)|^2 + \frac{p^3 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi_2)|^2 \\
&\quad - \frac{2p^4 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi)|^2 - \frac{p^4 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi_2)|^2 = \frac{1}{12} \cdot p^2 (p-1)(p-2) \\
&\quad - p^2 \cdot h_p^2 - \frac{2p^3}{\pi^2} \cdot |L(1, \chi)|^2.
\end{aligned} \tag{25}$$

If  $\chi = \chi_2$  is the Legendre symbol, then from (18), (19), Lemmas 1 and 3, and the method of proving (25), we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) &= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left| \sum_{a=1}^{p-1} \chi_1(a) |K(a, 1, \chi; p)|^2 \right|^2 \cdot |L(1, \chi_1)|^2 \\
&= \frac{p \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \chi_1 \neq \chi_2}} \left| \frac{\tau^2(\chi_1) \cdot \tau(\bar{\chi}_1) \cdot \tau(\bar{\chi}_2 \chi_1)}{\tau(\bar{\chi}_2 \bar{\chi}_1)} \right|^2 \cdot |L(1, \chi_1)|^2 + \frac{p \cdot \pi^{-2}}{p-1} \left| \sum_{a=1}^{p-1} \bar{\chi}_2(a) |K(a, 1, \chi_2; p)|^2 \right|^2 \cdot |L(1, \bar{\chi}_2)|^2 \\
&= \frac{p^4 \cdot \pi^{-2}}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1, \chi_1)|^2 - \frac{p^4 \cdot \pi^{-2}}{p-1} \cdot |L(1, \chi_2)|^2 + \frac{p^3 \cdot \pi^{-2}}{p-1} \cdot (p-2)^2 \cdot |L(1, \chi_2)|^2 \\
&= \frac{p^2(p-1)(p-2)}{12} + p^2 \cdot (p-4) \cdot h_p^2.
\end{aligned} \tag{26}$$

Combining (24), (25), and (26), we can deduce the identity:

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(a\bar{b}, p) = \begin{cases} \frac{1}{12} \cdot p^2(p-1)(p-2) - p^2 \cdot h_p^2, & \text{if } \chi(-1) = 1, \\ \frac{p^2(p-1)(p-2)}{12} - p^2 \cdot h_p^2 - \frac{2p^3}{\pi^2} \cdot |L(1, \chi)|^2, & \text{if } \chi(-1) = -1 \text{ and } \chi \neq \chi_2, \\ \frac{p^2(p-1)(p-2)}{12} + p^2 \cdot (p-4) \cdot h_p^2, & \text{if } \chi = \chi_2. \end{cases} \tag{27}$$

This completes the proof of Theorem 4.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# On the Sixth Residues and Some New Properties of Their Distribution

Juanli Su<sup>1</sup> and Jin Zhang<sup>2</sup> 

<sup>1</sup>Department of Arts and Sciences, Yangling Vocational and Technical College, Yangling, China

<sup>2</sup>School of Information Engineering, Xi'an University, Xi'an, Shaanxi, China

Correspondence should be addressed to Jin Zhang; zhangjin@xawl.edu.cn

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In this paper, we use the analytic methods, the properties of the sixth-order characters, and the classical Gauss sums to study the computational problems of a certain special sixth residues' modulo  $p$  and give two exact calculating formulas for them.

## 1. Introduction

Let  $p$  be an odd prime and  $k$  be a fixed positive integer. For any integer  $a$  with  $(a, p) = 1$ , if the congruence equation  $x^k \equiv a \pmod{p}$  has a solution  $x$ , then we call  $a$  is a  $k$ th residue modulo  $p$ . Otherwise,  $a$  is called a  $k$ th nonresidue modulo  $p$ . In particular, if  $k = 2, 3$ , and  $4$ , we call  $a$  is a quadratic residue, cubic residue, and quartic residue modulo  $p$ , respectively. Undoubtedly, the research of quadratic residue is the most concerned topic. Legendre first introduced the characteristic function of the quadratic residues  $(a/p)$  modulo  $p$ , which later was called Legendre's symbol. It is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo } p, \\ 0, & \text{if } p|a. \end{cases} \quad (1)$$

Sometimes, we write Legendre's symbol  $(*/p)$  as  $\chi_2$  for the sake of writing. This is because the introduction of this function greatly facilitated the study of quadratic residue properties and promoted the development of elementary number theory and analytic number theory. This is especially true in the study of primes and related problems. For

example, if  $p$  is a prime with  $p \equiv 1 \pmod{4}$ , then one has (see Theorem 4–11 in [1])

$$p = \left(\frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a+r\bar{a}}{p}\right)\right)^2 + \left(\frac{1}{2} \sum_{b=1}^{p-1} \left(\frac{b+s\bar{b}}{p}\right)\right)^2, \quad (2)$$

where  $\bar{a}$  denotes the inverse of  $a$ . That is,  $a \cdot \bar{a} \equiv 1 \pmod{p}$ , and  $(rs/p) = -1$ .

Of course, there are many papers involving quadratic residues and primes, so we cannot cover all of them. Those who are interested can refer to [2–9].

In this paper, we are concerned with the problem of whether the special integers  $a + \bar{a}$  and  $a - \bar{a}$  both are  $k$ th residues' modulo  $p$ . Let  $N_k(p)$  denote the number of all integers  $1 < a < p - 1$  such that  $a + \bar{a}$  and  $a - \bar{a}$  both are  $k$ th residues' modulo  $p$ . Then, how are the values of  $N_k(p)$  distributed?

Very recently, some authors had studied the calculating problem of  $N_k(p)$  and obtained a series of interesting results. For example, Wang and Lv [10] obtained the identity

$$N_2(p) = \begin{cases} \frac{1}{8}(p-3), & \text{if } p \equiv 3 \pmod{8}, \\ \frac{1}{8}(p-7), & \text{if } p \equiv 7 \pmod{8}. \end{cases} \quad (3)$$

Hu and Chen [11] proved the following result: let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . If 2 is a cubic residue mod  $p$ , then one has the identity

$$N_3(p) = \frac{1}{9} \cdot (p + 4d - 11). \quad (4)$$

If 2 is not a cubic residue mod  $p$ , then one has the asymptotic formula

$$N_3(p) = \frac{1}{9} \cdot (p - 5) + E(p), \quad (5)$$

where  $d$  is defined in (7) and  $E(p)$  satisfies the estimates  $|E(p)| \leq (2/3) \cdot \sqrt{p}$ .

Su and Zhang [12] considered the case  $p \equiv 5 \pmod{8}$  and proved the identity

$$N_4(p) = \frac{1}{16} \cdot \left( p - 7 - 2 \sum_{a=1}^{p-1/2} \left( \frac{a + \bar{a}}{p} \right) \right). \quad (6)$$

As an extension of [10–12], a natural problem is what about sixth residues modulo  $p$ ? It is clear that if  $(p-1, 6) = 2$ , then the problem is trivial. That is, any quadratic residue  $a$  modulo  $p$  is a sixth residue modulo  $p$ . So, we just consider the nontrivial case  $p \equiv 1 \pmod{6}$ . In this case, we know that there are two integers  $d$  and  $b$  such that the identity

$$4p = d^2 + 27 \cdot b^2, \quad (7)$$

where  $d$  is uniquely determined by  $d \equiv 1 \pmod{3}$  (see [13]).

And, it is clear from (7) that the value of  $N_6(p)$  must be related to  $d$  and  $b$ .

In this paper, we will use the analytic methods, the properties of the classical Gauss sums, and the estimate for some special character sums to study the computational problems of  $N_6(p)$  and give an exact calculating formula for it. That is, we will prove the following two results.

**Theorem 1.** *Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . If 2 is a cubic residue modulo  $p$ , then we have the identity*

$$N_6(p) = \frac{1}{36} \cdot (p + 4d - 11), \quad (8)$$

where  $d$  is the same as defined in (7).

**Theorem 2.** *Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . If 2 is not a cubic residue modulo  $p$ , then we have*

$$N_6(p) = \frac{1}{72} \cdot (2p + 5d - 10 + 9b) \quad (9)$$

$$\text{or } \frac{1}{72} \cdot (2p + 5d - 10 - 9b).$$

From our theorems, we may immediately deduce the following two corollaries.

**Corollary 1.** *Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . If 2 is a cubic residue modulo  $p$ , then we have the congruence*

$$p + 4d \equiv 11 \pmod{36}. \quad (10)$$

**Corollary 2.** *Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . If 2 is not a cubic residue modulo  $p$ , then we have the congruence*

$$2p + 5d \equiv 10 + 9b \pmod{72} \quad (11)$$

$$\text{or } 2p + 5d \equiv 10 - 9b \pmod{72}.$$

First, in Theorems 1 and 2, we must distinguish whether 3 is a cubic residue modulo  $p$  because of the need to calculate the character sums. In different cases, the values of character sums are different.

Second, if  $p$  is a prime with  $p \equiv 1 \pmod{12}$ , then, for some character sums, we can only use Weil's classical work [14, 15] to get some upper bound estimates and we cannot get their exact values. So, in this case, we can only deduce a sharp asymptotic formula for  $N_6(p)$ . That is,

$$N_6(p) = \frac{1}{36} \cdot p + O(p^{1/2}). \quad (12)$$

Third, if  $p$  is an odd prime with  $p \equiv 7 \pmod{12}$  and 2 is not a cubic residue modulo  $p$ , then our Theorem 2 also obtained an exact calculating formula for  $N_6(p)$ , which is obviously better than the corresponding result in [11].

Of course, our Theorem 2 is flawed, and it presents two possibilities. How to determine its correct value is an interesting open problem.

Finally, if  $p$  is a prime with  $p \equiv 7 \pmod{12}$ , then we know that 2 is a cubic residue modulo  $p$  if and only if  $2|d$ . That is,  $d$  is an even number. Otherwise,  $d$  is an odd number. Especially for primes  $p = 7, 19, 67, 79, 103, 139, 151$ , after some simple calculations, we have  $4 \times 7 = 1^2 + 27 \cdot 1^2$ ,  $4 \times 19 = 7^2 + 27 \cdot 1^2$ ,  $4 \times 67 = (-5)^2 + 27 \cdot 3^2$ ,  $4 \times 79 = (-17)^2 + 27 \cdot 1^2$ ,  $4 \times 103 = 13^2 + 27 \cdot 3^2$ ,  $4 \times 139 = (-23)^2 + 27 \cdot 1^2$ , and  $4 \times 151 = 19^2 + 27 \cdot 3^2$ . Since  $N_6(p)$  is an integer, so applying Corollary 2, we can get the congruences:  $2 \cdot 7 + 5 \cdot 1 \equiv 10 + 9 \cdot 1 \pmod{72}$ ,  $2 \cdot 19 + 5 \cdot 7 \equiv 10 - 9 \cdot 1 \pmod{72}$ ,  $2 \cdot 67 + 5 \cdot (-5) \equiv 10 + 9 \cdot 3 \pmod{72}$ ,  $2 \cdot 79 + 5 \cdot (-17) \equiv 10 - 9 \cdot 1 \pmod{72}$ ,  $2 \cdot 103 + 5 \cdot 13 \equiv 10 - 9 \cdot 3 \pmod{72}$ ,  $2 \cdot 139 + 5 \cdot (-23) \equiv 10 + 9 \cdot 1 \pmod{72}$ , and  $2 \cdot 151 + 5 \cdot 19 \equiv 10 + 9 \cdot 3 \pmod{72}$ .

Now, we consider Legendre's symbol  $(d + b/p)$ . Note that  $(1 + 1/7) = 1$ ,  $(7 + 1/19) = -1$ ,  $(-5 + 3/67) = 1$ ,  $(-17 + 1/79) = -1$ ,  $(13 + 3/103) = -1$ ,  $(-23 + 1/139) = 1$ , and  $(19 + 3/151) = 1$ . From the above congruences and these values, we have a reason to believe the following.

**Conjecture.** *Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . If 2 is not a cubic residue modulo  $p$ , then we have the identity*

$$N_6(p) = \frac{1}{72} \cdot \left( 2p + 5d - 10 + \left( \frac{d+b}{p} \right) \cdot 9b \right). \quad (13)$$

## 2. Several Lemmas

In this section, we decompose the proofs of our theorems into the following several lemmas. For the sake of simplicity, the basic knowledge required in this section is not listed, and



only three necessary references [1, 16, 17] are provided here. First, we have the following.

**Lemma 1.** *Let  $p$  be an odd prime with  $p \equiv 1 \pmod{3}$ . Then, for any third-order character  $\lambda$  modulo  $p$  (i.e.,  $\lambda \neq \chi_0$  and  $\lambda^3 = \chi_0$ , the principal character modulo  $p$ ), we have the identity*

$$\tau^3(\lambda) + \tau^3(\bar{\lambda}) = dp, \quad (14)$$

where  $4p = d^2 + 27 \cdot b^2$ ,  $d$  is uniquely determined by  $d \equiv 1 \pmod{3}$ ,  $\tau(\lambda) = \sum_{a=1}^{p-1} \lambda(a)e(a/p)$  denotes the classical Gauss sums, and  $e(y) = e^{2\pi i y}$ .

*Proof.* For the proof of this lemma, see Zhang and Hu [18] or Berndt and Evans [19].  $\square$

**Lemma 2.** *Let  $p$  be a prime with  $p \equiv 1 \pmod{6}$ . Then, for any third-order character  $\lambda \pmod{p}$  and sixth-order character  $\psi = \chi_2 \lambda$  (i.e.,  $\psi^i \neq \chi_0$ ,  $1 \leq i \leq 5$ , and  $\psi^6 = \chi_0$ ), we have the identity*

$$\tau(\psi) = \frac{\lambda(2) \cdot \tau(\chi_2) \cdot \tau^2(\bar{\lambda})}{p}. \quad (15)$$

*Proof.* From the properties of the Gauss sums and the reduced residue system modulo  $p$ , note that the identity

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = 1 + \sum_{a=1}^{p-1} (1 + \chi_2(a))e\left(\frac{ba}{p}\right) = \chi_2(b) \cdot \tau(\chi_2), \quad (16)$$

and we have

$$\begin{aligned} \sum_{a=0}^{p-1} \chi_2 \lambda(a^2 - 1) &= \frac{1}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2 \bar{\lambda}(b) \sum_{a=0}^{p-1} e\left(\frac{b(a^2 - 1)}{p}\right) \\ &= \frac{\tau(\chi_2)}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2(b) \bar{\lambda}(b) \chi_2(b) e\left(\frac{-b}{p}\right) = \frac{\tau(\chi_2) \cdot \tau(\bar{\lambda})}{\tau(\chi_2 \bar{\lambda})}. \end{aligned} \quad (17)$$

On the contrary, we also have

$$\begin{aligned} \sum_{a=0}^{p-1} \chi_2 \lambda(a^2 - 1) &= \sum_{a=1}^{p-1} \chi_2 \lambda(a^2 + 2a) \\ &= \frac{1}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2 \bar{\lambda}(b) \sum_{a=1}^{p-1} \chi_2(a) \lambda(a) e\left(\frac{b(a+2)}{p}\right) \\ &= \frac{\tau(\chi_2 \lambda)}{\tau(\chi_2 \bar{\lambda})} \sum_{b=1}^{p-1} \chi_2(b) \bar{\lambda}(b) \chi_2(b) \bar{\lambda}(b) e\left(\frac{2b}{p}\right) \\ &= \frac{\bar{\lambda}(2) \cdot \tau(\chi_2 \lambda) \cdot \tau(\lambda)}{\tau(\chi_2 \bar{\lambda})}. \end{aligned} \quad (18)$$

Note that identity  $\tau(\lambda) \cdot \tau(\bar{\lambda}) = p$ , and from (17) and (18), we deduce the identity

$$\tau(\chi_2 \lambda) = \tau(\psi) = \frac{\lambda(2) \cdot \tau(\chi_2) \cdot \tau^2(\bar{\lambda})}{p}. \quad (19)$$

This proves Lemma 2.  $\square$

**Lemma 3.** *Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . Then, for any third-order character  $\lambda \pmod{p}$ , we have the identity*

$$\sum_{a=1}^{p-1} (\lambda(a^2 - \bar{a}^2) + \bar{\lambda}(a^2 - \bar{a}^2)) = \frac{1 + \lambda(2)}{p} \cdot (\tau^3(\lambda) + \bar{\lambda}(2) \cdot \tau^3(\bar{\lambda})). \quad (20)$$

*Proof.* Note that  $\lambda^2 = \bar{\lambda}$ ,  $\lambda(-1) = 1$ , and  $\chi_2(-1) = -1$ , and from Lemma 2, properties of the Gauss sums, and Legendre's symbol mod  $p$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \lambda(a^2 - \bar{a}^2) &= \sum_{a=1}^{p-1} \bar{\lambda}^2(a) \lambda(a^4 - 1) = \sum_{a=1}^{p-1} (1 + \chi_2(a)) \cdot \bar{\lambda}(a) \lambda(a^2 - 1) \\ &= \sum_{a=1}^{p-1} \bar{\lambda}(a) \lambda(a^2 - 1) + \sum_{a=1}^{p-1} \chi_2(-a) \cdot \bar{\lambda}(-a) \lambda((-a)^2 - 1) \\ &= \sum_{a=1}^{p-1} \lambda^2(a) \lambda(a^2 - 1) = \sum_{a=1}^{p-1} (1 + \chi_2(a)) \lambda(a) \lambda(a - 1) \\ &= \frac{1}{\tau(\bar{\lambda})} \sum_{a=1}^{p-1} \lambda(a) \sum_{b=1}^{p-1} \bar{\lambda}(b) e\left(\frac{b(a-1)}{p}\right) + \frac{1}{\tau(\bar{\lambda})} \sum_{a=1}^{p-1} \psi(a) \sum_{b=1}^{p-1} \bar{\lambda}(b) e\left(\frac{b(a-1)}{p}\right) \\ &= \frac{\tau^2(\lambda)}{\tau(\bar{\lambda})} - \frac{\tau^2(\psi)}{\tau(\bar{\lambda})} = \frac{\tau^3(\lambda)}{p} - \frac{\tau^2(\psi) \cdot \tau(\lambda)}{p} = \frac{\tau^3(\lambda) + \bar{\lambda}(2) \cdot \tau^3(\bar{\lambda})}{p}, \end{aligned} \quad (21)$$

where we have used the identity  $\tau^2(\chi_2) = \chi_2(-1) \cdot p = -p$ . Similarly, we can also deduce that

$$\sum_{a=1}^{p-1} \bar{\lambda}(a^2 - \bar{a}^2) = \frac{\tau^3(\bar{\lambda}) + \lambda(2) \cdot \tau^3(\lambda)}{p}. \quad (22)$$

It is clear that Lemma 3 follows from (21) and (22).  $\square$

**Lemma 4.** Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . Then, we have the identity

$$\sum_{a=1}^{p-1} (\lambda(a + \bar{a}) + \bar{\lambda}(a + \bar{a})) = \frac{(1 - \lambda(2)) \cdot \tau^3(\lambda) + (1 - \bar{\lambda}(2)) \cdot \tau^3(\bar{\lambda})}{p}. \quad (23)$$

*Proof.* From the methods of proving Lemma 2 and the properties of the Gauss sums, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \lambda(a + \bar{a}) &= \sum_{a=1}^{p-1} \bar{\lambda}(a) \lambda(a^2 + 1) = \sum_{a=1}^{p-1} \lambda(a^2) \lambda(a^2 + 1) \\ &= \sum_{a=1}^{p-1} (1 + \chi_2(a)) \cdot \lambda(a) \lambda(a + 1) = \frac{\tau^3(\lambda)}{p} + \frac{\tau^2(\chi_2 \lambda) \cdot \tau(\lambda)}{p} \\ &= \frac{\tau^3(\lambda)}{p} - \frac{\bar{\lambda}(2) \cdot \tau^3(\bar{\lambda})}{p}. \end{aligned} \quad (24)$$

So, from (24), we have

$$\begin{aligned} \sum_{a=1}^{p-1} (\lambda(a + \bar{a}) + \bar{\lambda}(a + \bar{a})) &= \frac{\tau^3(\lambda)}{p} - \frac{\bar{\lambda}(2) \cdot \tau^3(\bar{\lambda})}{p} + \frac{\tau^3(\bar{\lambda})}{p} - \frac{\lambda(2) \cdot \tau^3(\lambda)}{p} \\ &= \frac{(1 - \lambda(2)) \cdot \tau^3(\lambda) + (1 - \bar{\lambda}(2)) \cdot \tau^3(\bar{\lambda})}{p}. \end{aligned} \quad (25)$$

This proves Lemma 4.  $\square$

**Lemma 5.** Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . Then, we have

$$\sum_{a=1}^{p-1} (\lambda(a - \bar{a}) + \bar{\lambda}(a - \bar{a})) = \frac{(1 + \lambda(2)) \cdot \tau^3(\lambda) + (1 + \bar{\lambda}(2)) \cdot \tau^3(\bar{\lambda})}{p}. \quad (26)$$

*Proof.* It is the same as the proof of Lemma 4, so it is omitted.  $\square$

**Lemma 6.** Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . Then, we have the identities

$$\sum_{a=1}^{p-1} \psi(a + \bar{a}) \bar{\psi}(a - \bar{a}) = \sum_{a=1}^{p-1} \psi(a - \bar{a}) \bar{\psi}(a + \bar{a}) = 0. \quad (27)$$

*Proof.* Note that  $\psi(-1) = \bar{\psi}(-1) = -1$ , and from the properties of the reduced residue system modulo  $p$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \psi(a + \bar{a}) \bar{\psi}(a - \bar{a}) &= \sum_{a=1}^{p-1} \psi(a^2 + 1) \bar{\psi}(a^2 - 1) \\ &= \sum_{a=1}^{p-1} \psi(\bar{a}^2 + 1) \bar{\psi}(\bar{a}^2 - 1) \\ &= \sum_{a=1}^{p-1} \psi(a^2 + 1) \bar{\psi}(1 - a^2) \\ &= - \sum_{a=1}^{p-1} \psi(a^2 + 1) \bar{\psi}(a^2 - 1) \\ &= - \sum_{a=1}^{p-1} \psi(a + \bar{a}) \bar{\psi}(a - \bar{a}), \end{aligned} \quad (28)$$

which implies that

$$\sum_{a=1}^{p-1} \psi(a + \bar{a}) \bar{\psi}(a - \bar{a}) = 0. \quad (29)$$

This proves Lemma 6.  $\square$

**Lemma 7.** Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . For any sixth-order character  $\psi \pmod{p}$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \chi_2(a - \bar{a}) &= \sum_{a=1}^{p-1} \bar{\psi}(a + \bar{a}) \cdot \chi_2(a - \bar{a}) = 0, \\ \sum_{a=1}^{p-1} \psi(a - \bar{a}) \cdot \chi_2(a + \bar{a}) &= \sum_{a=1}^{p-1} \bar{\psi}(a - \bar{a}) \cdot \chi_2(a + \bar{a}) = 0. \end{aligned} \quad (30)$$

*Proof.* Since  $\psi = \chi_2 \lambda$ ,  $\chi_2(-1) = -1$ , so, from the reduced residue system modulo  $p$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \chi_2(a - \bar{a}) &= \sum_{a=1}^{p-1} \bar{\lambda}(a) \psi(a^2 + 1) \cdot \chi_2(a^2 - 1) \\ &= \sum_{a=1}^{p-1} \lambda(a) \psi(\bar{a}^2 + 1) \cdot \chi_2(\bar{a}^2 - 1) \\ &= \sum_{a=1}^{p-1} \lambda^2(a) \psi(a^2 + 1) \cdot \chi_2(1 - a^2) \\ &= - \sum_{a=1}^{p-1} \bar{\lambda}(a) \psi(a^2 + 1) \cdot \chi_2(a^2 - 1) \\ &= - \sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \chi_2(a - \bar{a}), \end{aligned} \quad (31)$$

which implies that

$$\sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \chi_2(a - \bar{a}) = 0. \quad (32)$$

Similarly, we can also deduce the identity

$$\sum_{a=1}^{p-1} \psi(a - \bar{a}) \cdot \chi_2(a + \bar{a}) = 0. \quad (33)$$

This proves Lemma 7.  $\square$

**Lemma 8.** Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . Then, for any third-order character  $\lambda \pmod{p}$  and  $\psi = \chi_2 \lambda$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \lambda(a - \bar{a}) &= \sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \bar{\lambda}(a - \bar{a}) = 0, \\ \sum_{a=1}^{p-1} \psi(a - \bar{a}) \cdot \lambda(a + \bar{a}) &= \sum_{a=1}^{p-1} \psi(a - \bar{a}) \cdot \bar{\lambda}(a + \bar{a}) = 0. \end{aligned} \quad (34)$$

*Proof.* Note that  $\lambda(-1) = 1$  and  $\psi(-1) = -1$ , we have

$$\begin{aligned} \sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \lambda(a - \bar{a}) &= \sum_{a=1}^{p-1} \psi(-a - \bar{a}) \cdot \lambda(-a + \bar{a}) \\ &= - \sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \lambda(a - \bar{a}). \end{aligned} \quad (35)$$

So, we have the identity

$$\sum_{a=1}^{p-1} \psi(a + \bar{a}) \cdot \lambda(a - \bar{a}) = 0. \quad (36)$$

Similarly, we can also deduce the identity

$$\sum_{a=1}^{p-1} \psi(a - \bar{a}) \cdot \lambda(a + \bar{a}) = 0. \quad (37)$$

This proves Lemma 8.  $\square$

**Lemma 9.** Let  $p$  be an odd prime with  $p \equiv 7 \pmod{12}$ . Then, for any third-order character  $\lambda \pmod{p}$ , we have

$$\sum_{a=1}^{p-1} (\lambda(a + \bar{a}) \cdot \bar{\lambda}(a - \bar{a}) + \bar{\lambda}(a + \bar{a}) \cdot \lambda(a - \bar{a})) = -4. \quad (38)$$

*Proof.* From the properties of Legendre's symbol  $\pmod{p}$ , we have

$$1 + \psi(a) + \psi^2(a) + \psi^3(a) + \bar{\psi}^2(a) + \bar{\psi}(a) = \begin{cases} 6, & \text{if } a \text{ is a sixth residue modulo } p, \\ 0, & \text{if } a \text{ is not a sixth residue modulo } p, \end{cases} \quad (42)$$

and note that  $\psi^2 = \bar{\lambda}$ ,  $\psi^3 = \chi_2$ ,  $\chi_2(-1) = -1$ ,  $\lambda(-1) = 1$ ,  $\psi(-1) = -1$ , and  $(a^2 + 1, p) = 1$ , and we have the identity

$$\begin{aligned} \sum_{a=1}^{p-1} \lambda(a + \bar{a}) \cdot \bar{\lambda}(a - \bar{a}) &= \sum_{a=1}^{p-1} \lambda(a^2 + 1) \cdot \bar{\lambda}(a^2 - 1) \\ &= \sum_{a=1}^{p-1} (1 + \chi_2(a)) \lambda(a + 1) \cdot \bar{\lambda}(a - 1) \\ &= -1 + \sum_{a=0}^{p-1} \lambda(a + 1) \cdot \bar{\lambda}(a - 1) + \sum_{a=1}^{p-1} \chi_2(a) \lambda(a + 1) \cdot \bar{\lambda}(a - 1) \\ &= -1 + \sum_{a=1}^{p-1} \lambda(a + 2) \cdot \bar{\lambda}(a) - \sum_{a=1}^{p-1} \chi_2(a) \lambda(a - 1) \cdot \bar{\lambda}(a + 1) \\ &= -1 + \sum_{a=1}^{p-1} \lambda(1 + 2 \cdot \bar{a}) - \sum_{a=1}^{p-1} \chi_2(a) \lambda(a - 1) \cdot \bar{\lambda}(a + 1) \\ &= -2 + \sum_{a=0}^{p-1} \lambda(1 + 2 \cdot a) - \sum_{a=1}^{p-1} \chi_2(a) \lambda(a - 1) \cdot \bar{\lambda}(a + 1) \\ &= -2 - \sum_{a=1}^{p-1} \chi_2(a) \lambda(a - 1) \cdot \bar{\lambda}(a + 1). \end{aligned} \quad (39)$$

Similarly, we also have

$$\begin{aligned} \sum_{a=1}^{p-1} \bar{\lambda}(a + \bar{a}) \cdot \lambda(a - \bar{a}) &= \sum_{a=1}^{p-1} \bar{\lambda}(a^2 + 1) \cdot \lambda(a^2 - 1) \\ &= \sum_{a=1}^{p-1} (1 + \chi_2(a)) \bar{\lambda}(a + 1) \cdot \lambda(a - 1) \\ &= -2 + \sum_{a=1}^{p-1} \chi_2(a) \bar{\lambda}(a + 1) \cdot \lambda(a - 1). \end{aligned} \quad (40)$$

Combining (39) and (40), we have the identity

$$\sum_{a=1}^{p-1} (\lambda(a + \bar{a}) \cdot \bar{\lambda}(a - \bar{a}) + \bar{\lambda}(a + \bar{a}) \cdot \lambda(a - \bar{a})) = -4. \quad (41)$$

This proves Lemma 9.  $\square$

### 3. Proofs of the Theorems

In this section, we shall complete the proofs of our main results. First, we prove Theorem 1. For any prime  $p$  with  $p \equiv 7 \pmod{12}$ , let  $\lambda$  denote a third-order character modulo  $p$ ; then,  $\psi = \chi_2 \lambda$  is a sixth-order character modulo  $p$ . So, for any integer  $a$  with  $(a, p) = 1$ , from the characteristic function

$$\begin{aligned}
N_6(p) &= \frac{1}{36} \sum_{a=2}^{p-2} \left( 1 + \psi(a + \bar{a}) + \bar{\lambda}(a + \bar{a}) + \chi_2(a + \bar{a}) + \lambda(a + \bar{a}) + \bar{\psi}(a + \bar{a}) \right) \\
&\quad \times \left( 1 + \psi(a - \bar{a}) + \bar{\lambda}(a - \bar{a}) + \chi_2(a - \bar{a}) + \lambda(a - \bar{a}) + \bar{\psi}(a - \bar{a}) \right) \\
&= \frac{1}{36} \sum_{a=1}^{p-1} \left( 1 + \psi(a + \bar{a}) + \bar{\lambda}(a + \bar{a}) + \chi_2(a + \bar{a}) + \lambda(a + \bar{a}) + \bar{\psi}(a + \bar{a}) \right) \\
&\quad \times \left( 1 + \psi(a - \bar{a}) + \bar{\lambda}(a - \bar{a}) + \chi_2(a - \bar{a}) + \lambda(a - \bar{a}) + \bar{\psi}(a - \bar{a}) \right) - \frac{1}{18} \cdot (1 + \lambda(2) + \bar{\lambda}(2)) \\
&= \frac{p-1}{36} + \frac{1}{36} \sum_{a=1}^{p-1} \left( \psi(a + \bar{a}) + \bar{\lambda}(a + \bar{a}) + \chi_2(a + \bar{a}) + \lambda(a + \bar{a}) + \bar{\psi}(a + \bar{a}) \right) \\
&\quad + \frac{1}{36} \sum_{a=1}^{p-1} \left( \psi(a - \bar{a}) + \bar{\lambda}(a - \bar{a}) + \chi_2(a - \bar{a}) + \lambda(a - \bar{a}) + \bar{\psi}(a - \bar{a}) \right) \\
&\quad + \frac{1}{36} \sum_{a=1}^{p-1} \left( \psi(a^2 - \bar{a}^2) + \bar{\lambda}(a^2 - \bar{a}^2) + \chi_2(a^2 - \bar{a}^2) + \lambda(a^2 - \bar{a}^2) + \bar{\psi}(a^2 - \bar{a}^2) \right) \\
&\quad + \frac{1}{36} \sum_{a=1}^{p-1} \psi(a + \bar{a}) \left( \bar{\lambda}(a - \bar{a}) + \chi_2(a - \bar{a}) + \lambda(a - \bar{a}) + \bar{\psi}(a - \bar{a}) \right) \\
&\quad + \frac{1}{36} \sum_{a=1}^{p-1} \bar{\lambda}(a + \bar{a}) \left( \psi(a - \bar{a}) + \lambda(a - \bar{a}) + \chi_2(a - \bar{a}) + \bar{\psi}(a - \bar{a}) \right) \\
&\quad + \frac{1}{36} \sum_{a=1}^{p-1} \chi_2(a + \bar{a}) \left( \psi(a - \bar{a}) + \bar{\lambda}(a - \bar{a}) + \lambda(a - \bar{a}) + \bar{\psi}(a - \bar{a}) \right) \\
&\quad + \frac{1}{36} \sum_{a=1}^{p-1} \lambda(a + \bar{a}) \left( \psi(a - \bar{a}) + \bar{\lambda}(a - \bar{a}) + \chi_2(a - \bar{a}) + \bar{\psi}(a - \bar{a}) \right) \\
&\quad + \frac{1}{36} \sum_{a=1}^{p-1} \bar{\psi}(a + \bar{a}) \left( \psi(a - \bar{a}) + \bar{\lambda}(a - \bar{a}) + \chi_2(a - \bar{a}) + \lambda(a - \bar{a}) \right) \\
&\quad - \frac{1}{18} \cdot (1 + \lambda(2) + \bar{\lambda}(2)).
\end{aligned} \tag{43}$$

Note that  $p \equiv 7 \pmod{12}$  and  $\chi_2(-1) = -1$ , so we have

$$\sum_{a=1}^{p-1} \psi(a \pm \bar{a}) = \sum_{a=1}^{p-1} \bar{\psi}(a \pm \bar{a}) = \sum_{a=1}^{p-1} \chi_2(a \pm \bar{a}) = 0, \tag{44}$$

$$\sum_{a=1}^{p-1} \psi(a^2 - \bar{a}^2) = \sum_{a=1}^{p-1} \bar{\psi}(a^2 - \bar{a}^2) = \sum_{a=1}^{p-1} \chi_2(a^2 - \bar{a}^2) = 0. \tag{45}$$

From Lemma 1, 4, and 5, we have

$$\begin{aligned}
&\sum_{a=1}^{p-1} (\lambda(a + \bar{a}) + \bar{\lambda}(a + \bar{a}) + \lambda(a - \bar{a}) + \bar{\lambda}(a - \bar{a})) \\
&= \frac{2}{p} \cdot (\tau^3(\lambda) + \tau^3(\bar{\lambda})) = 2d.
\end{aligned} \tag{46}$$

Now, if 2 is a cubic residue modulo  $p$ , then  $1 + \lambda(2) + \bar{\lambda}(2) = 3$ . Combining (43)–(46), Lemma 3, and Lemma 6–9, we have

$$N_6(p) = \frac{p-1}{36} + \frac{1}{36} \cdot (2d+2d-4) - \frac{3}{18} = \frac{1}{36} \cdot (p+4d-11). \quad (47)$$

This proves Theorem 1.

Now, we prove Theorem 2. If 2 is not a cubic residue modulo  $p$ , then  $1 + \lambda(2) + \bar{\lambda}(2) = 0$ . That is to say,

$$\begin{aligned} \lambda(2) &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot i, \\ \bar{\lambda}(2) &= -\frac{1}{2} \mp \frac{\sqrt{3}}{2} \cdot i, \end{aligned} \quad (48)$$

where  $i^2 = -1$ .

In this case, from Lemma 1, we have

$$\begin{aligned} &(\tau^3(\lambda) - \tau^3(\bar{\lambda}))^2 \\ &= d^2 \cdot p^2 - 4p^3 \\ &= -p^2 \cdot (4p - d^2) = -27 \cdot p^2 \cdot b^2, \end{aligned} \quad (49)$$

or

$$\tau^3(\lambda) - \tau^3(\bar{\lambda}) = \pm 3 \cdot \sqrt{3} \cdot p \cdot b \cdot i. \quad (50)$$

From (48), (50), and Lemma 3, we have

$$\begin{aligned} &\sum_{a=1}^{p-1} (\lambda(a^2 - \bar{a}^2) + \bar{\lambda}(a^2 - \bar{a}^2)) \\ &= \frac{1 + \lambda(2)}{p} \cdot (\tau^3(\lambda) + \bar{\lambda}(2) \cdot \tau^3(\bar{\lambda})) \\ &= -\frac{1}{p} \cdot (\bar{\lambda}(2) \cdot \tau^3(\lambda) + \lambda(2) \cdot \tau^3(\bar{\lambda})) \\ &= \frac{1}{2p} \cdot (pd \pm 9 \cdot p \cdot b) = \frac{1}{2} \cdot (d \pm 9b). \end{aligned} \quad (51)$$

Combining (43)–(46), (51), and Lemma 6–9, we have

$$\begin{aligned} N_6(p) &= \frac{p-1}{36} + \frac{1}{36} \cdot (2d-4) + \frac{1}{72} \cdot (d+9b) \\ &= \frac{1}{72} \cdot (2p+5d-10+9b), \end{aligned} \quad (52)$$

or

$$\begin{aligned} N_6(p) &= \frac{p-1}{36} + \frac{1}{36} \cdot (2d-4) + \frac{1}{72} \cdot (d-9b) \\ &= \frac{1}{72} \cdot (2p+5d-10-9b). \end{aligned} \quad (53)$$

This completes the proofs of our all results.

## 4. Conclusion

The main results of this paper are two theorems and two corollaries. Theorem 1 gives an exact computing formula for

$N_6(p)$  with  $p \equiv 7 \pmod{12}$  and 2 is a cubic residue modulo  $p$ . If  $p \equiv 7 \pmod{12}$  and 2 is not a cubic residue modulo  $p$ , then Theorem 2 established an identity for  $N_6(p)$  and there are two possibilities. As some applications of these theorems, we also deduced two interesting congruences. For example, one of them is

$$p + 4d \equiv 11 \pmod{36}, \quad (54)$$

where  $p \equiv 7 \pmod{12}$  and 2 is a cubic residue modulo  $p$ .

In addition, if  $p \equiv 7 \pmod{12}$  and 2 is not a cubic residue modulo  $p$ , then we also have an interesting conjecture. That is,

$$N_6(p) = \frac{1}{72} \cdot \left( 2p + 5d - 10 + \left( \frac{d+b}{p} \right) \cdot 9b \right). \quad (55)$$

## Data Availability

No data were used to support the findings of the study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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## Research Article

# On the Hybrid Fourth Power Mean Involving Legendre's Symbol and One Kind Two-Term Exponential Sums

Jingzhe Wang 

School of Statistics and Mathematics, Inner Mongolia University of Finance and Economics, Hohhot 010070, Inner Mongolia, China

Correspondence should be addressed to Jingzhe Wang; wangjingzhe729@126.com

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The main purpose of this article is using the elementary methods and the properties of the quadratic residue modulo an odd prime  $p$  to study the calculating problem of the fourth power mean of one kind two-term exponential sums and give an interesting calculating formula for it.

## 1. Introduction

Let  $q \geq 3$  be a fixed integer. For any integer  $k \geq 2$  and integer  $m$  with  $(m, q) = 1$ , we define the two-term exponential sums  $G(m, k; q)$  as follows:

$$G(m, k; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + a}{q}\right), \quad (1)$$

where, as usual,  $e(y) = e^{2\pi i y}$  and  $i$  denotes the imaginary unit, that is  $i^2 = -1$ .

Since this kind of sums play a very important role in the study of analytic number theory, so many number theorists and scholars had studied the various properties of  $G(m, k; q)$  and obtained a series of meaningful research results, we do not want to enumerate here, and interested readers can refer to [1–16]. For example, Zhang and Zhang [1] proved that for any odd prime  $p$ , one has

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases} \quad (2)$$

where  $n$  represents any integer with  $(n, p) = 1$ .

Shen and Zhang [2] obtained an interesting recurrence formula for

$$A(m, 4; p) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right), \quad (3)$$

where  $p$  is an odd prime with  $p \equiv 1 \pmod{4}$ .

Chen and Zhang [3] proved that for any prime  $p$  with  $p \equiv 5 \pmod{8}$ , one has the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^2 \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma + \bar{a}}{p}\right) \right|^2 = 3p^3 - 3p^2 + 2p^{(3/2)}\alpha - 3p, \quad (4)$$

where  $\alpha = \alpha(p) = \sum_{a=1}^{(p-1/2)} (a + \bar{a}/p)$ ,  $(*/p)$  denotes Legendre's symbol modulo  $p$ , and  $a \cdot \bar{a} \equiv 1 \pmod{p}$ .

Zhang and Han [4] used the elementary method to obtain the identity

$$\sum_{a=1}^{p-1} \left| \sum_{n=0}^{p-1} e\left(\frac{n^3 + an}{p}\right) \right|^6 = 5p^4 - 8p^3 - p^2, \quad (5)$$

where  $p$  denotes an odd prime with  $3 \nmid (p-1)$ .

Chen and Wang [5] studied the calculating problem of the fourth power mean of  $G(m, 4; p)$  and proved the following conclusion.

Let  $p > 3$  be an odd prime, then one has the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^4 = \begin{cases} 2p^3, & \text{if } p = 12k + 11; \\ 2p^2(p-2), & \text{if } p = 12k + 7; \\ 2p(p^2 - 4p - 2\alpha^2), & \text{if } p = 24k + 5; \\ 2p(p^2 - 6p - 2\alpha^2), & \text{if } p = 24k + 13; \\ 2p(p^2 - 10p - 2\alpha^2), & \text{if } p = 24k + 1; \\ 2p(p^2 - 8p - 2\alpha^2), & \text{if } p = 24k + 17, \end{cases} \quad (6)$$

where  $\alpha = \sum_{a=1}^{(p-1/2)} (a + \bar{a}/p)$  and  $(*/p)$  denotes Legendre's symbol modulo  $p$ .

Zhang and Zhang [6] proved that for any prime  $p$ , one has the identity

$$\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 = \begin{cases} p^2(\delta - 3), & \text{if } p \equiv 1 \pmod{6}; \\ p^2(\delta + 3), & \text{if } p \equiv -1 \pmod{6}, \end{cases} \quad (7)$$

where  $\delta = \sum_{d=1}^{p-1} (d - 1 + \bar{d}/p)$  is an integer which satisfies the estimate  $|\delta| \leq 2\sqrt{p}$ .

Liu and Zhang [7] proved that for any prime  $p$  with  $3 \nmid (p-1)$ , one has the identity

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3 + a}{p}\right) \right|^6 = p(p-1)(6p^3 - 28p^2 + 39p + 5), \quad (8)$$

where  $\sum_{\chi \bmod p}$  denotes the summation over all Dirichlet characters, modulo  $p$ .

Inspired by the works in [5, 6], in this paper, we consider the following calculating problem of the  $2h$ -th power mean of the two-term exponential sums:

$$\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + na}{p}\right) \right|^{2h}, \quad (9)$$

where  $p$  is an odd prime and  $h \geq 2$  is an integer.

About this problem, it seems that none had studied it before; at least we have not seen such a result at present. In this paper, we will use the properties of the solutions of the congruence equations and the quadratic residue to study this problem and give an interesting calculating formula for (9) with  $h = 2$ . That is, we will prove the following result.

**Theorem 1.** *Let  $p > 3$  be an odd prime, then we have the identity*

$$\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^4 = \begin{cases} 2p^{(3/2)} \left(6\alpha - \left(\frac{3}{p}\right) \cdot p\right), & \text{if } p = 8k + 5; \\ 2p^{(3/2)} \left(\left(\frac{3}{p}\right) \cdot p - 10\alpha\right), & \text{if } p = 8k + 1; \\ 0, & \text{if } p = 4k + 3, \end{cases} \quad (10)$$

where  $\alpha = \sum_{a=1}^{(p-1/2)} (a + \bar{a}/p)$  and  $(*/p)$  denotes Legendre's symbol modulo  $p$ .

Note that the estimate  $|\alpha| \leq \sqrt{p}$  (see [17] for general results), and from this theorem and [5], we may immediately deduce the following several corollaries.

**Corollary 1.** *Let  $p > 3$  be an odd prime with  $p \equiv 1 \pmod{4}$ , then we have the asymptotic formula*

$$\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^4 = 2 \cdot \left(\frac{6}{p}\right) \cdot p^{(5/2)} + O(p^2). \quad (11)$$



**Corollary 2.** Let  $p > 3$  be an odd prime with  $p \equiv 1 \pmod{4}$ , then we have the asymptotic formula

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{m^2 a^4 + a}{p}\right) \right|^4 = 2p^3 + 2 \cdot \left(\frac{6}{p}\right) \cdot p^{(5/2)} + O(p^2). \quad (12)$$

Some notes: the constant  $\alpha = \alpha(p)$  in our theorem has a special meaning. In fact, for any prime  $p$  with  $p \equiv 1 \pmod{4}$ , one has the identity (see [18])

$$p = \left( \frac{1}{2} \sum_{a=1}^{p-1} \left( \frac{a + \bar{a}}{p} \right) \right)^2 + \left( \frac{1}{2} \sum_{a=1}^{p-1} \left( \frac{a + r\bar{a}}{p} \right) \right)^2 = \alpha^2 + \beta^2, \quad (13)$$

where  $r$  is any quadratic nonresidue modulo  $p$ . That is,  $(r/p) = -1$ .

For any prime  $p = 4k + 1$  and integer  $h \geq 3$ , we naturally ask whether there is an exact calculating formula for (9)?

This is an open problem. We believe this to be true. We even have the following.

**Conjecture 1.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, for any integer  $h \geq 2$ , there are two integers  $C = C(h, p)$  and  $D = D(h, p)$  depending only on  $h$  and  $p$ , such that the identity

$$\sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^{2h} = C \cdot p^{(2h+1/2)} + D \cdot p^{(2h-1/2)} \cdot \alpha. \quad (14)$$

## 2. Several Lemmas

In this section, we will give several necessary lemmas. Of course, the proofs of some lemmas need the knowledge of elementary and analytic number theory. In particular, the properties of the quadratic residues and the Legendre's symbol modulo  $p$  are going to be used. All these can be

found in [15, 18–20], and we do not repeat them. First, we have the following lemma.

**Lemma 1.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, for any fourth-order character  $\lambda \pmod{p}$ , we have the identity

$$\tau^2(\lambda) + \tau^2(\bar{\lambda}) = 2\sqrt{p} \cdot \alpha, \quad (15)$$

where  $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e(a/p)$  denotes the classical Gauss sums,  $\alpha = \alpha(p) = \sum_{a=1}^{(p-1/2)} (a + \bar{a}/p)$ , and  $(*/p) = \chi_2$  denotes Legendre's symbol modulo  $p$ .

*Proof.* See Lemma 2 in Chen and Zhang [3].  $\square$

**Lemma 2.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ , then we have the identity

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) = \begin{cases} -p \cdot (12\alpha + 3), & \text{if } p = 8k + 5; \\ p \cdot (20\alpha - 7), & \text{if } p = 8k + 1, \end{cases} \quad (16)$$

where  $\alpha = \alpha(p)$  is defined as in Lemma 1.

*Proof.* For any odd prime  $p$  with  $p \equiv 1 \pmod{4}$ , let  $\lambda$  be any fourth-order character modulo  $p$ , and

$$A(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right). \quad (17)$$

Then, from the properties of the classical Gauss sums, we have

$$\begin{aligned} A(m) &= 1 + \sum_{a=1}^{p-1} (1 + \lambda(a) + \chi_2(a) + \bar{\lambda}(a)) \cdot e\left(\frac{ma}{p}\right) \\ &= \bar{\lambda}(m) \cdot \tau(\lambda) + \chi_2(m) \cdot \tau(\chi_2) + \lambda(m) \cdot \tau(\bar{\lambda}), \end{aligned} \quad (18)$$

where  $(m, p) = 1$ . Note that  $\tau(\chi_2) = \sqrt{p}$ ,  $\chi_2 \cdot \lambda = \bar{\lambda}$ , from (18) and Lemma 1, we have

$$\begin{aligned} A^2(m) &= (\bar{\lambda}(m) \cdot \tau(\lambda) + \chi_2(m) \cdot \tau(\chi_2) + \lambda(m) \cdot \tau(\bar{\lambda}))^2 \\ &= p + 2\sqrt{p} \cdot (\lambda(m) \cdot \tau(\lambda) + \bar{\lambda}(m) \cdot \tau(\bar{\lambda})) + 2\tau(\lambda) \cdot \tau(\bar{\lambda}) + \chi_2(m) \cdot 2 \cdot \sqrt{p} \cdot \alpha, \end{aligned} \quad (19)$$

where  $p \equiv 1 \pmod{4}$ . If  $p = 8k + 5$ , then  $\lambda(-1) = -1$  and  $\tau(\lambda) \cdot \tau(\bar{\lambda}) = -p$ , so we have

$$\begin{aligned} A^2(m) \cdot A(-m) &= (-p + 2\sqrt{p} \cdot (\lambda(m) \cdot \tau(\lambda) + \bar{\lambda}(m) \cdot \tau(\bar{\lambda})) + \chi_2(m) \cdot 2 \cdot \sqrt{p} \cdot \alpha) \times (-\bar{\lambda}(m) \cdot \tau(\lambda) + \chi_2(m) \cdot \sqrt{p} - \lambda(m) \cdot \tau(\bar{\lambda})) \\ &= 3\chi_2(m)p^{(3/2)} + 3p(\bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda})) - 2\sqrt{p}\alpha(\lambda(m)\tau(\lambda) + \bar{\lambda}(m)\tau(\bar{\lambda})) - 2p\alpha. \end{aligned} \quad (20)$$

If  $p = 8k + 1$ , then  $\lambda(-1) = 1$  and  $\tau(\lambda) \cdot \tau(\bar{\lambda}) = p$ , so we have

$$\begin{aligned}
A^2(m) \cdot A(-m) &= (3p + 2\sqrt{p} \cdot (\lambda(m) \cdot \tau(\lambda) + \bar{\lambda}(m) \cdot \tau(\bar{\lambda})) + \chi_2(m) \cdot 2 \cdot \sqrt{p} \cdot \alpha) \times (\bar{\lambda}(m) \cdot \tau(\lambda) + \chi_2(m) \cdot \sqrt{p} + \lambda(m) \cdot \tau(\bar{\lambda})) \\
&= 7\chi_2(m)p^{(3/2)} + 5p(\bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda})) + 6p\alpha + 2\sqrt{p}\alpha(\lambda(m)\tau(\lambda) + \bar{\lambda}(m)\tau(\bar{\lambda})).
\end{aligned} \tag{21}$$

Now, if  $p = 8k + 5$ , then from (20) and Lemma 1, we have

$$\begin{aligned}
&\tau(\chi_2) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) \\
&= \sum_{m=1}^{p-1} \chi_2(m) \cdot A^2(m) \cdot A(-m) \cdot e\left(\frac{-m}{p}\right) \\
&= -3p^{(3/2)} + 3p \cdot \sum_{m=1}^{p-1} (\lambda(m)\tau(\lambda) + \bar{\lambda}(m)\tau(\bar{\lambda})) \cdot e\left(\frac{-m}{p}\right) \\
&\quad - 2\sqrt{p}\alpha \cdot \sum_{m=1}^{p-1} (\bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda})) \cdot e\left(\frac{-m}{p}\right) - 2p^{(3/2)} \cdot \alpha \\
&= -3p^{(3/2)} - 3p \cdot (\tau^2(\lambda) + \tau^2(\bar{\lambda})) - 4p^{(3/2)} \cdot \alpha - 2p^{(3/2)} \cdot \alpha \\
&= -p^{(3/2)} \cdot (12\alpha + 3).
\end{aligned} \tag{22}$$

If  $p = 8k + 1$ , then from (21) and Lemma 1, we have

$$\begin{aligned}
&\tau(\chi_2) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) \\
&= \sum_{m=1}^{p-1} \chi_2(m) \cdot A^2(m) \cdot A(-m) \cdot e\left(\frac{-m}{p}\right) \\
&= -7p^{(3/2)} + 5p \cdot \sum_{m=1}^{p-1} (\lambda(m)\tau(\lambda) + \bar{\lambda}(m)\tau(\bar{\lambda})) \cdot e\left(\frac{-m}{p}\right) \\
&\quad + 2\sqrt{p}\alpha \cdot \sum_{m=1}^{p-1} (\bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda})) \cdot e\left(\frac{-m}{p}\right) + 6p^{(3/2)} \cdot \alpha \\
&= -7p^{(3/2)} + 5p \cdot (\tau^2(\lambda) + \tau^2(\bar{\lambda})) + 4p^{(3/2)} \cdot \alpha + 6p^{(3/2)} \cdot \alpha \\
&= p^{(3/2)} \cdot (20 \cdot \alpha - 7).
\end{aligned} \tag{23}$$

Combining (22) and (23) and  $\tau(\chi_2) = \sqrt{p}$ , we may immediately deduce Lemma 2.  $\square$

**Lemma 3.** Let  $p$  be an odd prime with  $p \equiv 1 \pmod{4}$ . Then, we have the identity

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2(a^4 + 1 - b^4) = \begin{cases} 3p + 2\alpha, & \text{if } p = 8k + 5; \\ 7p - 2\alpha, & \text{if } p = 8k + 1. \end{cases} \tag{24}$$

*Proof.* From the methods of proving Lemma 2, we have

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2(a^4 + 1 - b^4) = \frac{1}{\sqrt{p}} \cdot \sum_{m=1}^{p-1} \chi_2(m) |A(m)|^2 \cdot e\left(\frac{m}{p}\right). \tag{25}$$

If  $p = 8k + 5$ , then note that  $\overline{\tau(\bar{\lambda})} = -\tau(\bar{\lambda})$ , and from (18), we have

$$|A(m)|^2 = 3p - \chi_2(m) \cdot (\tau^2(\lambda) + \tau^2(\bar{\lambda})) = 3p - 2\chi_2(m) \cdot \sqrt{p} \cdot \alpha, \tag{26}$$

Then, applying (25), we have

$$\begin{aligned}
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2(a^4 + 1 - b^4) &= \frac{1}{\sqrt{p}} \cdot \sum_{m=1}^{p-1} \chi_2(m) (3p - 2\chi_2(m) \\
&\quad \cdot \sqrt{p} \cdot \alpha) \cdot e\left(\frac{m}{p}\right) \\
&= 3p + 2\alpha.
\end{aligned} \tag{27}$$

If  $p = 8k + 1$ , then  $|A(m)|^2 = A^2(m)$ , and from (19) and (25), we have

$$\begin{aligned}
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2(a^4 + 1 - b^4) &= \frac{1}{\sqrt{p}} \cdot \sum_{m=1}^{p-1} \chi_2(m) |A(m)|^2 \cdot e\left(\frac{m}{p}\right) \\
&= \sum_{m=1}^{p-1} \chi_2(m) (3\sqrt{p} + 2 \cdot (\lambda(m) \cdot \tau(\lambda) + \bar{\lambda}(m) \cdot \tau(\bar{\lambda})) + \chi_2(m) \cdot 2 \cdot \alpha) \cdot e\left(\frac{m}{p}\right) \\
&= 7p - 2\alpha.
\end{aligned} \tag{28}$$

Now, Lemma 3 follows from (27) and (28).  $\square$

**Lemma 4.** Let  $p > 3$  be a prime with  $p \equiv 1 \pmod{4}$ . Then, we have the identity

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b \equiv c+1 \pmod{p}}}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) = 2 \cdot \chi_2(6) \cdot p - \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2). \quad (29)$$

*Proof.* From the properties of the complete residue system modulo  $p$ , we have

$$\begin{aligned} & \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b \equiv c+1 \pmod{p}}}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) \\ &= \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b \equiv 0 \pmod{p}}}^{p-1} \chi_2((a+c)^4 + (b+1)^4 - c^4 - 1) \\ &= \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + 4a^3c + 6a^2c^2 + 4ac^3 + a^4 - 4a^3 + 6a^2 - 4a) \\ &= \sum_{a=1}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + 4a^4c + 6a^4c^2 + 4a^4c^3 + a^4 - 4a^3 + 6a^2 - 4a) \\ &= \sum_{a=1}^{p-1} \sum_{c=0}^{p-1} \chi_2(2 + 4c + 6c^2 + 4c^3 - 4a + 6a^2 - 4a^3) \\ &= \chi_2(2) \cdot \sum_{a=1}^{p-1} \sum_{c=0}^{p-1} \chi_2(4 + 4c + 3c^2 + c^3 - 4a + 3a^2 - a^3) \\ &= \chi_2(2) \cdot \sum_{a=1}^{p-1} \sum_{c=0}^{p-1} \chi_2((c+1)^3 + c + 1 - (a-1)^3 - (a-1)) \\ &= \chi_2(2) \cdot \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(c^3 + c - a^3 - a) - \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) \\ &= \chi_2(2) \cdot \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(c^3 + 3c^2a + 3ca^2 + c) - \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) \\ &= \chi_2(2) \cdot \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(c^3 + 3c(2a+c)^2 + 4c) - \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) \\ &= \chi_2(2) \cdot \sum_{a=0}^{p-1} \sum_{c=1}^{p-1} \chi_2(c^3 + 3ca^2 + 4c) - \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2). \end{aligned} \quad (30)$$

Note that the identity

$$\sum_{a=0}^{p-1} \left( \frac{a^2 + n}{p} \right) = \begin{cases} p-1, & \text{if } (n, p) = p, \\ -1, & \text{if } (n, p) = 1, \end{cases} \quad (31)$$

and  $\chi_2(2) = -1$ , if  $p = 8k + 5$ ;  $\chi_2(2) = 1$ , if  $p = 8k + 1$ . We have

$$\begin{aligned} \sum_{a=0}^{p-1} \sum_{c=1}^{p-1} \chi_2(c^3 + 3ca^2 + 4c) &= p \cdot \sum_{\substack{c=1 \\ c^3+4c \equiv 0 \pmod{p}}}^{p-1} \chi_2(3c) - \sum_{c=1}^{p-1} \chi_2(3c) \\ &= p \cdot \sum_{\substack{c=1 \\ c^2+1 \equiv 0 \pmod{p}}}^{p-1} \chi_2(6c) = 2 \cdot \chi_2(3) \cdot p. \end{aligned} \quad (32)$$

From (30) and (32), we have

$$\begin{aligned} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b \equiv c+1 \pmod{p}}}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) &= 2 \cdot \chi_2(6) \cdot p - \chi_2(2) \\ &\cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2). \end{aligned} \quad (33)$$

This proves Lemma 4.  $\square$

**Lemma 5.** Let  $p > 3$  be a prime with  $p \equiv 1 \pmod{4}$ . Then, we have the identity

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4) e\left(\frac{a+b-c}{p}\right) = \begin{cases} \chi_2(2) \cdot p \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) - 3p, & \text{if } p = 8k + 5; \\ \chi_2(2) \cdot p \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) - 7p, & \text{if } p = 8k + 1. \end{cases} \quad (34)$$

*Proof.* It is clear that

$$\begin{aligned}
& \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4) e\left(\frac{a+b-c}{p}\right) \\
&= \sum_{a=1}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4) e\left(\frac{a+b-c}{p}\right) + \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(b^4 - c^4) e\left(\frac{b-c}{p}\right) \\
&= p \cdot \sum_{\substack{b=0 \\ b+1 \equiv c \pmod p}}^{p-1} \sum_{c=0}^{p-1} \chi_2(1 + b^4 - c^4) - \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(1 + b^4 - c^4) - 1 \\
&\quad + \sum_{b=0}^{p-1} \chi_2(b^4 - 1) \sum_{c=0}^{p-1} e\left(\frac{c(b-1)}{p}\right) - \sum_{b=0}^{p-1} \chi_2(b^4 - 1) \\
&= p \cdot \sum_{b=0}^{p-1} \chi_2(1 + b^4 - (b+1)^4) - \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(1 + b^4 - c^4) - 1 - \sum_{b=0}^{p-1} \chi_2(b^4 - 1) \\
&= p \cdot \sum_{b=0}^{p-1} \chi_2(4b^3 + 6b^2 + 4b) - \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(1 + b^4 - c^4) - 1 - \sum_{b=0}^{p-1} \chi_2(b^4 - 1).
\end{aligned} \tag{35}$$

From (31), we have

$$\begin{aligned}
\sum_{b=0}^{p-1} \chi_2(b^4 - 1) &= \sum_{b=0}^{p-1} (1 + \chi_2(b)) \cdot \chi_2(b^2 - 1) \\
&= \sum_{b=0}^{p-1} \chi_2(b^2 - 1) + \sum_{b=1}^{p-1} \chi_2(b) \cdot \chi_2(b^2 - 1) \\
&= -1 + \sum_{b=1}^{p-1} \chi_2(b - \bar{b}) = -1 + 2 \cdot (-1)^{(p-1)/4} \cdot \alpha.
\end{aligned} \tag{36}$$

Note that  $\chi_2(4) = 1$  and  $(2, p) = 1$ , and from the properties of complete residue system modulo  $p$  and  $\chi_2(-1) = 1$ , we have

$$\begin{aligned}
\sum_{b=0}^{p-1} \chi_2(4b^3 + 6b^2 + 4b) &= \chi_2(4) \sum_{b=0}^{p-1} \chi_2(4b^3 + 6b^2 + 4b) \\
&= \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2((2b)^3 + 3(2b)^2 + 4(2b)) \\
&= \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + 3b^2 + 4b) \\
&= \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2((b+1)^3 + b + 1 - 2) \\
&= \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b - 2) \\
&= \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2).
\end{aligned} \tag{37}$$

Combining (35)–(37) and Lemma 3, we have

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4) e\left(\frac{a+b-c}{p}\right) = \begin{cases} \chi_2(2) \cdot p \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) - 3p, & \text{if } p = 8k + 5; \\ \chi_2(2) \cdot p \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) - 7p, & \text{if } p = 8k + 1. \end{cases} \tag{38}$$

This proves Lemma 5.  $\square$

### 3. Proof of the Theorem

Applying several basic lemmas in Section 2, we can easily complete the proof of our theorem. In fact, for any odd

prime  $p > 3$ , if  $p = 4k + 3$ , then from the properties of the reduced residue system modulo  $p$ , we have

$$\begin{aligned} \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right|^4 &= \sum_{m=1}^{p-1} \left( \frac{-m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e\left( \frac{-ma^4 + a}{p} \right) \right|^4 \\ &= - \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e\left( \frac{-ma^4 - a}{p} \right) \right|^4 = - \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right|^4. \end{aligned} \quad (39)$$

So, in this case, we have the identity

If  $p = 4k + 1$ , then we have

$$\sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right|^4 = 0. \quad (40)$$

$$\begin{aligned} &\sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right|^4 \\ &= \tau(\chi_2) \cdot \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} \chi_2(a^4 + b^4 - c^4 - d^4) e\left( \frac{a + b - c - d}{p} \right) \\ &= \tau(\chi_2) \cdot \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=1}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) e\left( \frac{d(a + b - c - 1)}{p} \right) \\ &\quad + \tau(\chi_2) \cdot \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4) e\left( \frac{a + b - c}{p} \right) \\ &= \tau(\chi_2) \cdot p \cdot \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) \\ &\quad \quad \quad a+b \equiv c+1 \pmod{p} \\ &\quad - \tau(\chi_2) \cdot \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4 - 1) \\ &\quad + \tau(\chi_2) \cdot \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \chi_2(a^4 + b^4 - c^4) e\left( \frac{a + b - c}{p} \right). \end{aligned} \quad (41)$$

If  $p = 8k + 5$ , then note that  $\chi_2(2) = -1$ , and from (41), Lemma 2, Lemma 4, and Lemma 5, we may immediately deduce

$$\begin{aligned}
& \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right|^4 \\
&= p^{(3/2)} \cdot \left( 2 \cdot \chi_2(6) \cdot p - \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) \right) + p^{(3/2)} \cdot (12\alpha + 3) + p^{(1/2)} \cdot \left( \chi_2(2) \cdot p \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) - 3p \right) \\
&= 2 \cdot p^{(3/2)} \cdot \left( 6\alpha - \left( \frac{3}{p} \right) \cdot p \right).
\end{aligned} \tag{42}$$

Similarly, if  $p = 8k + 1$ , then note that  $\chi_2(2) = 1$ , and from (41), Lemma 2, Lemma 4, and Lemma 5, we may immediately deduce

$$\begin{aligned}
& \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \cdot \left| \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right|^4 \\
&= p^{(3/2)} \cdot \left( 2 \cdot \chi_2(6) \cdot p - \chi_2(2) \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) \right) - p^{(3/2)} \cdot (20\alpha - 7) + p^{(1/2)} \cdot \left( \chi_2(2) \cdot p \cdot \sum_{b=0}^{p-1} \chi_2(b^3 + b + 2) - 7p \right) \\
&= 2 \cdot p^{(3/2)} \cdot \left( \left( \frac{3}{p} \right) \cdot p - 10\alpha \right).
\end{aligned} \tag{43}$$

Now, the theorem follows from (40), (42), and (43). This completes the proof of our main result.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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## Research Article

# A Note on Cube-Full Numbers in Arithmetic Progression

Mingxuan Zhong <sup>1</sup> and Yuankui Ma <sup>2</sup>

<sup>1</sup>School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710119, Shaanxi, China

<sup>2</sup>School of Science, Xi'an Technological University, Xi'an 710021, Shaanxi, China

Correspondence should be addressed to Yuankui Ma; mayuankui@xatu.edu.cn

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We obtain an asymptotic formula for the cube-full numbers in an arithmetic progression  $n \equiv l \pmod{q}$ , where  $(q, l) = 1$ . By extending the construction derived from Dirichlet's hyperbola method and relying on Kloosterman-type exponential sum method, we improve the very recent error term with  $x^{(118/4029)} < q$ .

## 1. Introduction and Main Results

Let  $k > 1$  be a fixed integer and  $n$  be a positive integer. We call  $n$  a powerful number (or  $k$ -full number) if  $n = 1$  or for a prime  $p$  dividing  $n$ ,  $p^k$  also divides  $n$ . Let  $\mathcal{P}_k$  denote the set of powerful numbers. Suppose  $k = 2, 3$ , and this defines square-full numbers and cube-full numbers, respectively. Erdős and Szekeres [1] first introduced powerful numbers and gave

$$\sum_{n \leq x, n \in \mathcal{P}_k} 1 = \sum_{m=k}^{2k-1} c_{k,m} x^{(1/m)} + \Delta_k(x), \quad (1)$$

where  $c_{k,m}$  are effective constants and  $\Delta_k(x) \ll x^{(1/(k+1))}$ . From then on, many authors have studied the powerful numbers and got a lot of relevant conclusions (see [2–18] and references therein).

In 2013, Liu and Zhang [19] investigated the distribution of square-full numbers in arithmetic progressions and got an asymptotic formula

$$\sum_{\substack{n \leq x, n \in \mathcal{P}_2 \\ n \equiv l \pmod{q}}} 1 = \alpha(l, q) x^{(1/2)} + O\left(q^{(49/141)+\varepsilon} x^{(19/47)+\varepsilon}\right), \quad (2)$$

under the condition of  $(q, l) = 1$ . By utilizing the method of exponent pairs, Srichan [20] then obtained

$$\sum_{\substack{n \leq x, n \in \mathcal{P}_2 \\ n \equiv l \pmod{q}}} 1 = \sum_{k=2}^3 \beta_k(\ell, q) x^{(1/k)} + O\left(q^{(3/2)+\varepsilon} x^{(1/6)}\right), \quad (3)$$

$$\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 = \sum_{k=3}^5 \gamma_k(\ell, q) x^{(1/k)} + O\left(q^{(127/92)+\varepsilon} x^{(7/46)}\right),$$

where the error terms had been corrected by Watt [MR3265055].

Recently, Chan [21] got a new asymptotic formula

$$\sum_{\substack{n \leq x, n \in \mathcal{P}_2 \\ n \equiv l \pmod{q}}} 1 = \sum_{k=2}^3 \delta_k(\ell, q) x^{(1/k)} + O_\varepsilon\left(\left(x^{(1/6)} q^{(1/12)} + \frac{x^{(1/5)}}{q^{(1/5)}}\right) q^\varepsilon\right), \quad (4)$$

which improved his own result with Tsang [22]. As a critical step, he [21] mainly dealt with a sum in the form of by following closely Montgomery and Vaughan's construction [23]. It is somewhat similar to Dirichlet's hyperbola method shown in Figure 1.

$$\sum_{\substack{a^2 b^3 \leq x \\ a^2 b^3 \equiv l \pmod{q}}} 1. \quad (5)$$



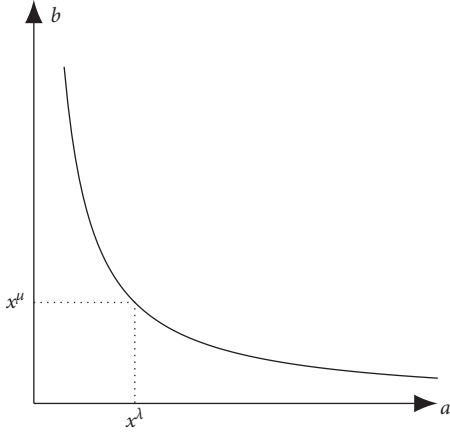


FIGURE 1: Dirichlet's hyperbola method.

Actually, they divided the above sum into four parts as shown in Figure 2 and then discussed them separately.

Motivated by this idea, we turn to discuss the following sum with three parameters:

$$\sum_{\substack{a^3 b^4 c^5 \leq x \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1. \quad (6)$$

By extending the construction from Montgomery and Vaughan [23], the summation (5) is divided into eight parts as shown in Figure 3.

Then, relying again on Kloosterman-type exponential sum method, an asymptotic formula of (5) is obtained. Finally, an asymptotic formula of cube-full numbers in an arithmetic progression is derived.

Before we formulate our result we need to give some definitions that will be used below. For modulus  $q$ , we define

$$\begin{aligned} N_3(n; q) &:= \#\{x \pmod{q}: x^3 \equiv n \pmod{q}, (x, q) = 1\}, \\ N_4(n; q) &:= \#\{x \pmod{q}: x^4 \equiv n \pmod{q}, (x, q) = 1\}, \\ N_5(n; q) &:= \#\{x \pmod{q}: x^5 \equiv n \pmod{q}, (x, q) = 1\}. \end{aligned} \quad (7)$$

It is clear that  $N_3(na^3; q) = N_3(n; q)$ ,  $N_4(na^4; q) = N_4(n; q)$ , and  $N_5(na^5; q) = N_5(n; q)$  for any  $(a, q) = 1$ , and  $N_3(n; q) = N_4(n; q) = N_5(n; q) = 0$  if  $(n, q) > 1$ .

**Theorem 1.** For  $(l, q) = 1$ ,

$$\begin{aligned} \sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 &= A_q(l) \frac{x^{(1/3)}}{q} + B_q(l) \frac{x^{(1/4)}}{q} + C_q(l) \frac{x^{(1/5)}}{q} \\ &+ O\left(\left(x^{(49/360)} q^{(1/2)} + \frac{x^{(71/357)}}{q^{(3/14)}}\right) q^\varepsilon\right), \end{aligned} \quad (8)$$

where

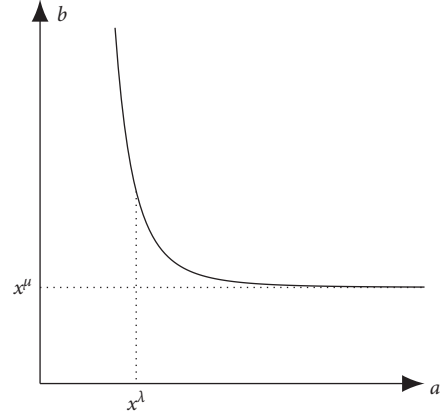


FIGURE 2: Generalized Dirichlet's hyperbola method (proof of Theorem 3 in [7]).

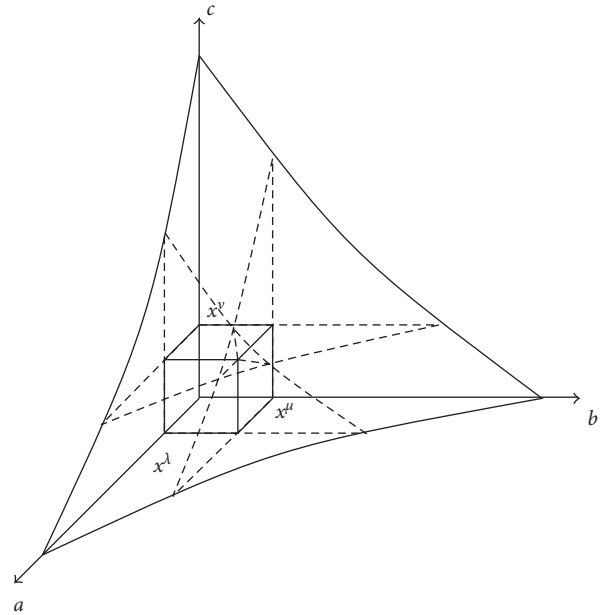


FIGURE 3: Generalized Dirichlet's hyperbola method in three dimensional space (proof of Theorem 2 below).

$$\begin{aligned} A_q(l) &= A'_q(l) \left( \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d, e, f, q) = 1}} \frac{\mu(d)\mu(f)\mu(e)}{d^{(8/3)} e^3 f^{(10/3)}} \sum_{\substack{g|ed \\ (g, q) = 1}} \frac{\mu(g)}{g^{(5/3)}} \right), \\ B_q(l) &= B'_q(l) \left( \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d, e, f, q) = 1}} \frac{\mu(d)\mu(f)\mu(e)}{d^2 e^{(9/4)} f^{(5/2)}} \sum_{\substack{g|ed \\ (g, q) = 1}} \frac{\mu(g)}{g^{(5/4)}} \right), \\ B_q(l) &= C'_q(l) \left( \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d, e, f, q) = 1}} \frac{\mu(d)\mu(f)\mu(e)}{d^{(8/5)} e^{(9/5)} f^2} \sum_{\substack{g|ed \\ (g, q) = 1}} \frac{\mu(g)}{g} \right), \end{aligned} \quad (9)$$

in which  $A'_q(l)$ ,  $B'_q(l)$ , and  $C'_q(l)$  are defined below.

Note that compared with the result in [20], we improve the error term when  $x^{(118/4029)} < q$ .

The key in our proof of Theorem 1 is the following.

**Theorem 2.** Let  $(l, q) = 1$ , and  $\eta, \lambda$  be two parameters such that

$$\begin{cases} \frac{1}{10} < \eta < \frac{46}{51}, \\ \frac{5}{51} < \lambda < \frac{13}{132}, \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{10} < \eta < \frac{67}{660}, \\ \frac{13}{132} < \lambda < \frac{1}{10}. \end{cases} \quad (10)$$

Then, we have

$$\sum_{\substack{a^3 b^4 c^5 \leq x \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 = A'_q(l) \frac{x^{(1/3)}}{q} + B'_q(l) \frac{x^{(1/4)}}{q} + C'_q(l) \frac{x^{(1/5)}}{q} + O\left(\frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon}} + \max(x^\eta, q^{(1/2)}) q^\varepsilon x^\lambda\right), \quad (11)$$

where

$$\begin{aligned} A'_q(l) &= \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{b^{(4/3)} c^{(5/3)}}, \\ B'_q(l) &= \frac{\phi(q)}{q} \left( 5 - \frac{5}{4} \int_1^{\infty} \frac{t - [t]}{t^{(9/4)}} dt \right) \left( -3 - \frac{3}{4} \int_1^{\infty} \frac{t - [t]}{t^{(7/4)}} dt \right) \\ &\quad - \sum_{c=1}^{\infty} \frac{N_4(lc^3; q) - ((\phi(q))/q)}{c^{(5/4)}} \\ &\quad + \frac{3}{4} \int_1^{\infty} \left( \sum_{c=1}^{\infty} \frac{\sum_{a \leq \mu} N_4(lac^3; q) - ((\phi(q))/q)\mu}{c^{(5/4)}} \right) \mu^{-(7/4)} d\mu, \\ C'_q(l) &= \left( -4 - \frac{4}{5} \int_1^{\infty} \frac{t - [t]}{t^{(9/5)}} dt \right) \left( \frac{3}{2} - \frac{3}{5} \int_1^{\infty} \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q} \\ &\quad - \frac{4}{5} \int_1^{\infty} \frac{\sum_{b \leq \mu} N_5(lb; q) - ((\phi(q))/q)\mu}{\mu^{(8/5)}} d\mu \\ &\quad + N_5(l; q) - \frac{\phi(q)}{q} - \frac{3}{5} \int_1^{\infty} \frac{\sum_{a \leq \mu} N_5(la^2; q) - ((\phi(q))/q)\mu}{\mu^{(8/5)}} d\mu \\ &\quad + \frac{12}{25} \int_1^{\infty} \int_1^{\infty} \frac{\sum_{b \leq \lambda} \sum_{a \leq \mu} N_5(la^2b; q) - ((\phi(q))/q)\lambda\mu}{\mu^{(8/5)} \lambda^{(9/5)}} d\lambda d\mu. \end{aligned} \quad (12)$$

## 2. Some Lemmas

Before we start the proof, let us give a few lemmas which are needed later.

**Lemma 1.** For  $(l, q) = 1$ ,

$$\begin{aligned} \sum_{a \leq u} N_i(la; q) &= \frac{\phi(q)}{q} u + O_\varepsilon(q^{(1/2)+\varepsilon}), \quad i = 3, 4, 5, \\ \sum_{a \leq u} N_i(la^2; q) &= \frac{\phi(q)}{q} u + O_\varepsilon(q^{(1/2)+\varepsilon}), \quad i = 3, 5, \\ \sum_{a \leq u} N_4(la^3; q) &= \frac{\phi(q)}{q} u + O_\varepsilon(q^{(1/2)+\varepsilon}). \end{aligned} \quad (13)$$

*Proof.* The first result can be found in Lemma 3 of [21]. Note that

$$\sum_{a \leq q} N_i(la; q) = \phi(q), \quad i = 3, 4, 5. \quad (14)$$

The second and third one can be proved in the same way. The proofs of the last three are slightly different. For example, by orthogonal property of additive characters, we have

$$\begin{aligned} \sum_{a \leq u} N_3(la^2; q) &= \frac{1}{q} \sum_{a \leq u} \sum_{n=1}^q \sum_{x=1}^q e\left(\frac{nx^3 - \ln a^2}{q}\right) \\ &= \frac{1}{q} \sum_{a \leq u} \sum_{n=1}^q e\left(\frac{-\ln a^2}{q}\right) \sum_{y=1}^q e\left(\frac{ny}{q}\right) \sum_{\chi \in G_3} \chi(y) \\ &= \frac{1}{q} \sum_{\chi \in G_3} \sum_{a \leq u} \sum_{y=1}^q \chi(y) \sum_{n=1}^q e\left(\frac{ny - \ln a^2}{q}\right) \\ &= \sum_{\chi \in G_3} \sum_{a \leq u} \sum_{\substack{y \leq q \\ y \equiv la^2 \pmod{q}}} \chi(y) \\ &= \sum_{a \leq u} \sum_{\substack{y \leq q \\ y \equiv la^2 \pmod{q} \\ (y, q)=1}} 1 + \sum_{\substack{\chi \in G_3 \\ \chi \neq \chi_0}} \sum_{a \leq u} \sum_{\substack{y \leq q \\ y \equiv la^2 \pmod{q}}} \chi(y) \\ &= \sum_{\substack{a \leq u \\ (a, q)=1}} 1 + O\left(\sum_{\substack{\chi \in G_3 \\ \chi \neq \chi_0}} \left| \sum_{a \leq u} \chi^2(a) \right| \right) \\ &= \sum_{d|q} \mu(d) \sum_{\substack{a \leq u \\ d|a}} 1 + O(|G_3| q^{(1/2)+\varepsilon}) \\ &= \frac{\phi(q)}{q} u + O(q^{(1/2)+\varepsilon}), \end{aligned} \quad (15)$$

where  $G_3$  is the set of all characters  $\chi(\bmod q)$  such that  $\chi^3 = \chi_0$ , the principal character.  $\square$

*Proof.* Let  $\|x\|$  be the distance from  $x$  to the nearest integer; then, we have

**Lemma 2.** For  $q \geq 1$  and  $(l, q) = 1$ ,

$$\sum_{n=1}^{q-1} (n, q) \left| \sum_{b=B+1}^{B+L} \sum_{c=C+1}^{C+M} e\left(\frac{-nlb^2c}{q}\right) \right| \ll q \cdot d(q) \log q. \quad (16)$$

$$\begin{aligned} \sum_{n=1}^{q-1} (n, q) \left| \sum_{b=B+1}^{B+L} \sum_{c=C+1}^{C+M} e\left(\frac{-nlb^2c}{q}\right) \right| &= \sum_{d|q} d \sum_{n_f=1}^{(q/d-1)} \left| \sum_{b=B+1}^{B+L} \sum_{c=C+1}^{C+M} e\left(\frac{-n_f lb^2c}{(q/d)}\right) \right|, \\ &\ll \sum_{d|q} d \sum_{n_f=1}^{(q/d-1)} \sum_{b=B+1}^{B+L} \min\left(L, \frac{1}{\|(n_f lb^2)/(q/d)\|}\right) \\ &\ll \sum_{d|q} d \sum_{b=B+1}^{B+L} \left( \sum_{n_f \leq (q/L db^2)} L + \sum_{(q/L db^2) < n_f \leq (q/d)} \frac{(q/d)}{n_f b^2} \right) \\ &\ll \sum_{d|q} d \left( \frac{q}{d} + \log q \cdot \frac{q}{d} \right), \\ &\ll q \cdot d(q) \log q. \end{aligned} \quad (17)$$

**Lemma 3.** For  $(l, q) = 1$ ,

$$\begin{aligned} \sum_{\substack{b \leq x^\mu \\ (b, q)=1}} \sum_{\substack{c \leq x^\nu \\ (c, q)=1}} \sum_{\substack{a \leq x^\lambda \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 &= \frac{\phi(q)}{q^2} x^{\lambda+\mu+\nu} + O_\varepsilon\left(\frac{x^\lambda}{q^{(1/2)-\varepsilon}}\right) \\ &\quad + O_\varepsilon(x^{\mu+\nu} q^\varepsilon). \end{aligned} \quad (18)$$

*Proof.* Using the trivial estimation of the innermost sum, we have  $\square$

$$\begin{aligned} \sum_{\substack{b \leq x^\mu \\ (b, q)=1}} \sum_{\substack{c \leq x^\nu \\ (c, q)=1}} \sum_{\substack{a \leq x^\lambda \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 &= \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} N_3(lb^2c; q) \left( \frac{x^\lambda}{q} + O(1) \right), \\ &= \frac{x^\lambda}{q} \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} N_3(lb^2c; q) + O_\varepsilon\left( \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} N_3(lb^2c; q) \right). \end{aligned} \quad (19)$$

Then, by orthogonal property of additive characters, we obtain

$$\begin{aligned} \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} N_3(lb^2c; q) &= \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} \frac{1}{q} \sum_{n=1}^q \sum_{x=1}^q \imath e\left(\frac{n(x^3 - lb^2c)}{q}\right) \\ &= \frac{\phi(q)}{q} (x^{\mu+\nu} + O(1)) + \frac{1}{q} \sum_{n=1}^{q-1} \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} e\left(\frac{-nlb^2c}{q}\right) \sum_{x=1}^q \imath e\left(\frac{nx^3}{q}\right) \\ &= \frac{\phi(q)}{q} (x^{\mu+\nu} + O(1)) + \frac{1}{q} \sum_{n=1}^{q-1} \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} e\left(\frac{-nlb^2c}{q}\right) \sum_{y \pmod{q}} e\left(\frac{ny}{q}\right) \sum_{\chi \in G_3} \chi(y). \end{aligned} \quad (20)$$

Interchanging the order of summations and combining Lemma 2 and Eq. (12.48) on page 324 of [24], we have

$$\begin{aligned} \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} N_3(lb^2c; q) &= \frac{\phi(q)}{q} (x^{\mu+\nu} + O(1)) + O\left(\frac{1}{q} \sum_{\chi \in G_3} \sum_{n=1}^{q-1} \left| \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} e\left(\frac{-nlb^2c}{q}\right) \right| (n, q) d(n, q) \sqrt{q}\right) \\ &= \frac{\phi(q)}{q} (x^{\mu+\nu} + O(1)) + O\left(\frac{d(q)}{\sqrt{q}} |G_3| \sum_{n=1}^{q-1} \left| \sum_{b \leq x^\mu} \sum_{c \leq x^\nu} e\left(\frac{-nlb^2c}{q}\right) \right| (n, q)\right) \\ &= \frac{\phi(q)}{q} x^{\mu+\nu} + O_\epsilon(q^{(1/2)+\epsilon}). \end{aligned} \quad (21)$$

Finally, we get

$$\sum_{\substack{b \leq x^\mu \\ (b, q)=1}} \sum_{\substack{c \leq x^\nu \\ (c, q)=1}} \sum_{\substack{a \leq x^\lambda \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 = \frac{\phi(q)}{q^2} x^{\lambda+\mu+\nu} + O_\epsilon\left(\frac{x^\lambda}{q^{(1/2)-\epsilon}}\right) + O_\epsilon(x^{\mu+\nu} q^\epsilon). \quad (22)$$

By definition of  $N_3(n; q)$ ,  $N_3(n; q) \ll_\epsilon q^\epsilon$  (see Lemma 2.2 in [25]).  $\square$

**Lemma 4.** *If we define*

$$S(a, b; q) = \sum_{n=1}^q \imath e\left(\frac{an^3 + b\bar{n}^4}{q}\right), \quad (23)$$

*then we have  $|S(a, b; q)| \ll (a, b, q)^{(1/2)} q^{(1/2)+\epsilon}$ .*

*Proof.* Following much the same way as Lemma 4 of [21], we first suppose  $q = rs$  with  $(r, s) = 1$ . By the “reciprocity” formula  $s\bar{s} + r\bar{r} \equiv 1 \pmod{q}$ , where  $s\bar{s} \equiv 1 \pmod{r}$  and  $r\bar{r} \equiv 1 \pmod{s}$ , and the additive multiplicity of exponential function

$$e\left(\frac{an^3 + b\bar{n}^4}{q}\right) = e\left(\frac{a\bar{s}n^3 + b\bar{s}\bar{n}^4}{r}\right) e\left(\frac{a\bar{r}n^3 + b\bar{r}\bar{n}^4}{s}\right), \quad (24)$$

with  $n = sx + ry$ ,  $1 \leq x \leq r$ ,  $1 \leq y \leq s$ ,  $(x, r) = 1$ , and  $(y, s) = 1$ , it follows that

$$\begin{aligned} S(a, b; q) &= \sum_{x=1}^r \imath \sum_{y=1}^s \imath e\left(\frac{a\bar{s}x^3 + b\bar{s}\bar{x}^4}{r}\right) e\left(\frac{a\bar{r}y^3 + b\bar{r}\bar{y}^4}{s}\right) \\ &= S(a\bar{s}, b\bar{s}; r) S(a\bar{r}, b\bar{r}; s). \end{aligned} \quad (25)$$

Now we just need to discuss the argument in the following cases:

(I) Prime moduli  $q = p$  case. Now Theorem 2 obtained by Moreno and Moreno [26], which is a special form of the Bombieri-Weil bound [27], implies

$$|S(a, b; p)| \leq 10(a, b, p)^{(1/2)} p^{(1/2)}, \quad (26)$$

provided that  $((ax^7 + b)/x^4)$  is not the form of  $h^p(x) - h(x)$  with  $h(x) \in \bar{F}_p[x]$ , where  $\bar{F}_p$  is the algebraic closure of  $F_p$ . Let

$$\frac{ax^7 + b}{x^4} = \frac{f^p(x)}{g^p(x)} - \frac{f(x)}{g(x)}, \quad (27)$$

with  $f(x), g(x) \in \bar{F}_p[x]$  and  $(f(x), g(x)) = 1$ . Then, we have  $g^p(x) | x^4$ , derived from

$$g^p(x)(ax^7 + b) = x^4(f^p(x) - g^{p-1}(x)f(x)). \quad (28)$$

This is impossible if  $p > 4$ , by comparing the degrees of both sides of the above. If  $p \leq 4$ , the validity of (26) can be easily checked.

(II) Prime power moduli  $q = p^\beta$  case with  $\beta > 1$ . Obviously, we only need to consider it with the assumption  $(a, b, p) = 1$ . Following the proofs of Lemma 12.2 and 12.3 in [24] with the equation of  $S(a, b; q)$ , we obtain

$$S(a, b; p^{2\alpha}) = p^\alpha \sum_{\substack{y=1 \\ g_I(y) \equiv 0 \pmod{p^\alpha}}}^{p^\alpha} \imath e\left(\frac{g(y)}{p^{2\alpha}}\right), \quad (29)$$

$$S(a, b; p^{2\alpha+1}) = p^\alpha \sum_{\substack{y=1 \\ g_I(y) \equiv 0 \pmod{p^\alpha}}}^{p^\alpha} \imath e\left(\frac{g(y)}{p^{2\alpha+1}}\right) G_p(y), \quad (30)$$

where

$$\begin{aligned} g(y) &= \frac{ay^7 + b}{y^4}, \\ G_p(y) &= \sum_{z=1}^p e\left(\frac{h(y)z^2 + g_I(y)p^{-\alpha}z}{p}\right), \end{aligned} \quad (31)$$

with  $h(y) = (g''(y)/2)$ .

Note that  $g'(y) = ((3ay^{10} - 4by^3)/y^8)$  and  $h(y) = ((3ay^{11} + 10by^4)/y^{10})$ . Now we concentrate on the number of solutions of congruence equation  $g'(y) \equiv 0 \pmod{p^\alpha}$  with  $(y, p) = 1$ .

$\beta = 2\alpha$  with  $\alpha \geq 1$ . Then, the congruence equation is

$$3ay^7 - 4b \equiv 0 \pmod{p^\alpha}. \quad (32)$$

If  $(b, p) = p$ , then  $(a, p) = 1$ . Relying on the properties of indices, we deduce that (32) has no solution when  $p > 3$  and one solution when  $p = 3$ . Next, we assume  $(b, p) = 1$ . If  $p = 2$  and  $4 \nmid a$ , then (32) has at most seven solutions. And if  $p > 3$  and  $(a, p) = 1$ , (32) also has at most seven solutions. Then, we have

$$|S(a, b; p^{2\alpha})| \leq 7p^\alpha, \quad \text{if } (a, b, p) = 1, \quad (33)$$

where  $\beta = 2\alpha + 1$  with  $\alpha \geq 1$ . Firstly, in the same way, if  $(b, p) = p$ , by the analysis in the case  $\beta = 2\alpha$ , the sum in (30) is empty unless  $p = 3$  in which case one has  $|S(a, b; 3^{2\alpha+1})| \leq 3^{\alpha+1}$ . Now suppose  $(b, p) = 1$ ; according to Chapter 3 in [24], we know if  $p \nmid 2h(y)$ , then we have  $|G_p(y)| \leq p^{(1/2)}$ . Therefore, if  $p \neq 2, 7$  (for otherwise  $p \mid 3ay^7 + 10b$  and  $p \mid 3ay^7 - 4b$  imply  $p \mid 14b$ , a contradiction), we have  $|G_p(y)| \leq p^{(1/2)}$ ; hence,  $|S(a, b; p^{2\alpha+1})| \leq 7p^{\alpha+(1/2)}$  as there are at most seven solutions to (32). If  $p = 2$  or  $7$ ,  $|G_p(y)| \leq 7$ , and hence  $|S(a, b; p^{2\alpha+1})| \leq 49p^\alpha$ .

In any case, we have

$$|S(a, b; p^{2\alpha+1})| \leq 49p^{(\alpha+1)/2}, \quad \text{if } (a, b, p) = 1. \quad (34)$$

Combining (25), (26), (33), and (34), we finally obtain

$$|S(a, b; q)| \leq 49^{\omega(q)} (a, b, q)^{(1/2)} q^{(1/2)} \ll (a, b, q)^{(1/2)} q^{(1/2)+\varepsilon}, \quad (35)$$

which completes the proof of Lemma 4.

By applying Lemma 4, we have the following.  $\square$

**Lemma 5.** For  $(l, q) = 1$ ,

$$N_{A,B}(K, L) := \#\{(a, b) : a^3 b^4 \equiv l \pmod{q}, A < a \leq A + K, B < b \leq B + L\}, \quad (36)$$

$$= \frac{\phi(q)KL}{q^2} + O\left(\frac{K+L}{q^{(1/2)-\varepsilon}} + q^{(1/2)+\varepsilon}\right).$$

*Proof.* Note that if  $a^3 b^4 \equiv l \pmod{q}$ , then  $a^4 b^4 \equiv la \pmod{q}$  and  $b \equiv l(\overline{ab})^3 \pmod{q}$ . So,

$$N_{A,B}(K, L) = \#\{(a, b, w) : a \equiv lw^4 \pmod{q}, b \equiv l\overline{w}^3 \pmod{q}, A < a \leq A + K, B < b \leq B + L, 1 \leq w \leq q, (w, q) = 1\}. \quad (37)$$

The remaining part of the proof is similar to Theorem 4 in [21].  $\square$

### 3. Proof of Theorem 2

Consider three positive parameters  $\lambda, \mu$ , and  $\nu$ . By extending the construction from Montgomery and Vaughan [23] as shown in Figure 3, we have

$$\begin{aligned} \sum_{\substack{a^3 b^4 c^5 \leq x \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 &= \sum_{\substack{a \leq x^\lambda \\ (a,q)=1}} \sum_{\substack{b^4 c^5 \leq (x/a^3) \\ b^4 c^5 \equiv l\overline{a^3} \pmod{q}}} 1 + \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{a^3 c^5 \leq (x/b^4) \\ a^3 c^5 \equiv l\overline{b^4} \pmod{q}}} 1 + \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{a^3 b^4 \leq (x/c^5) \\ a^3 b^4 \equiv l\overline{c^5} \pmod{q}}} 1 - \sum_{\substack{a \leq x^\lambda \\ (a,q)=1}} \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{c^5 \leq (x/(a^3 b^4)) \\ c^5 \equiv l\overline{a^3 b^4} \pmod{q}}} 1 \\ &\quad - \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{a^3 \leq (x/(b^4 c^5)) \\ a^3 \equiv l\overline{b^4 c^5} \pmod{q}}} 1 - \sum_{\substack{a \leq x^\lambda \\ (a,q)=1}} \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{b^4 \leq (x/(a^3 c^5)) \\ b^4 \equiv l\overline{a^3 c^5} \pmod{q}}} 1 + \sum_{\substack{a \leq x^\lambda \\ (a,q)=1}} \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{a^3 b^4 c^5 \equiv l \pmod{q}}} 1 \\ &\quad + \sum_{\substack{a > x^\lambda \\ (a,q)=1}} \sum_{\substack{b > x^\mu \\ (b,q)=1}} \sum_{\substack{c > x^\nu \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 \\ &:= T_1 + T_2 + T_3 - T_4 - T_5 - T_6 + T_7 + T_8. \end{aligned} \quad (38)$$

First, we estimate  $T_3$ .

$$\begin{aligned}
 T_3 &= \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{a^3 b^4 \leq (x/c^5) \\ a^3 b^4 \equiv l c^5 \pmod{q}}} 1, \\
 &= \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{b \leq x^{\mu_1} \\ (b,q)=1}} \sum_{\substack{a^3 \leq (x/(b^4 c^5)) \\ a^3 \equiv l b^4 c^5 \pmod{q}}} 1 + \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{a \leq x^{\lambda_1} \\ (a,q)=1}} \sum_{\substack{b^4 \leq (x/(a^3 c^5)) \\ b^4 \equiv l a^3 c^5 \pmod{q}}} 1 \\
 &\quad - \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{a \leq x^{\lambda_1} \\ (a,q)=1}} \sum_{\substack{b \leq x^{\mu_1} \\ b^4 \equiv l a^3 c^5 \pmod{q}}} 1 + \sum_{\substack{c \leq x^\nu \\ (c,q)=1}} \sum_{\substack{a > x^{\lambda_1} \\ a^3 b^4 \leq (x/c^5)}} \sum_{\substack{b > x^{\mu_1} \\ a^3 b^4 \equiv l c^5 \pmod{q}}} 1 \\
 &:= T_{31} + T_{32} - T_{33} + T_{34}.
 \end{aligned} \tag{39}$$

For  $T_{33}$ , we have

$$T_{33} = \frac{\phi(q)}{q^2} x^{\lambda_1 + \mu_1 + \nu} + O\left(\frac{x^{\lambda_1}}{q^{(1/2)-\varepsilon}}\right) + O(x^{\mu_1 + \nu} q^\varepsilon), \tag{40}$$

by Lemma 3.

Then, we estimate  $T_{31}$  and  $T_{32}$ . For  $T_{31}$ , we know

$$\begin{aligned}
 T_{31} &= \sum_{c \leq x^\nu} \sum_{b \leq x^{\mu_1}} N_3(lb^2 c; q) \left( \frac{x^{(1/3)}}{b^{(4/3)} c^{(5/3)} q} + O(1) \right), \\
 &= \sum_{c \leq x^\nu} \frac{1}{c^{(5/3)}} \left( \sum_{b=1}^{\infty} \frac{N_3(lb^2 c; q)}{b^{(4/3)}} \right) \frac{x^{(1/3)}}{q} - \sum_{c \leq x^\nu} \frac{1}{c^{(5/3)}} \frac{3\phi(q)}{q^2} x^{(1/3)-(1/3)\mu_1} \\
 &\quad + O\left( \frac{x^{(1/3)-(1/3)\mu_1}}{q^{(1/2)-\varepsilon}} \sum_{c \leq x^\nu} \frac{1}{c^{(5/3)}} \right) + O(x^{\mu_1 + \nu} q^\varepsilon).
 \end{aligned} \tag{41}$$

The first term in the above formula is

$$\begin{aligned}
 &\frac{x^{(1/3)}}{q} \sum_{b=1}^{\infty} \frac{1}{b^{(4/3)}} \left( \sum_{c=1}^{\infty} \frac{N_3(lb^2 c; q)}{c^{(5/3)}} \right) - \frac{3}{2} \frac{\phi(q)}{q^2} x^{(1/3)-(2/3)\nu} \sum_{b=1}^{\infty} \frac{1}{b^{(4/3)}} + O\left( \frac{x^{(1/3)-(5/3)\nu}}{q^{1/2-\varepsilon}} \right) \\
 &= \frac{x^{(1/3)}}{q} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{N_3(lb^2 c; q)}{b^{(4/3)} c^{(5/3)}} - \frac{3}{2} \frac{\phi(q)}{q^2} x^{(1/3)-(2/3)\nu} \sum_{b=1}^{\infty} \frac{1}{b^{(4/3)}} + O\left( \frac{x^{(1/3)-(5/3)\nu}}{q^{(1/2)-\varepsilon}} \right).
 \end{aligned} \tag{42}$$

In order to simplify our final result, by using Euler's summation formula which can be found in

Theorem 3.2 in [28], the constant  $\sum_{b=1}^{\infty} (1/b^{(4/3)})$  can be rewritten as

$$\begin{aligned}
\sum_{b=1}^{\infty} \frac{1}{b^{(4/3)}} &= \lim_{x \rightarrow \infty} \sum_{b \leq x} \frac{1}{b^{(4/3)}} \\
&= \lim_{x \rightarrow \infty} \left( -3x^{-(1/3)} + 4 - \frac{4}{3} \int_1^{\infty} \frac{t - [t]}{t^{(7/3)}} dt + O(x^{-(4/3)}) \right) \\
&= 4 - \frac{4}{3} \int_1^{\infty} \frac{t - [t]}{t^{(7/3)}} dt \\
&= 6 - 2 \int_1^{\infty} \frac{t - [t]}{t^{(7/3)}} dt.
\end{aligned} \tag{43}$$

Thus, we obtain the asymptotic formula of  $T_{31}$  as

$$\begin{aligned}
T_{31} &= \frac{x^{(1/3)}}{q} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{b^{(4/3)} c^{(5/3)}} - \left( 6 - 2 \int_1^{\infty} \frac{t - [t]}{t^{(7/3)}} dt \right) \frac{\phi(q)}{q} x^{(1/3) - (2/3)\nu} \\
&\quad - \sum_{c \leq x^\nu} \frac{1}{c^{(5/3)}} \frac{3\phi(q)}{q^2} x^{(1/3) - (1/3)\mu_1} + O\left( \frac{x^{(1/3) - (5/3)\nu}}{q^{(1/2) - \varepsilon}} \right) + O\left( \frac{x^{(1/3) - (4/3)\mu_1}}{q^{(1/2) - \varepsilon}} + x^{\mu_1 + \nu} q^\varepsilon \right).
\end{aligned} \tag{44}$$

Next, we deal with  $T_{32}$ .

$$\begin{aligned}
T_{32} &= \sum_{\substack{c \leq x^\nu \\ (c, q) = 1}} \sum_{\substack{a \leq x^{\lambda_1} \\ (a, q) = 1}} \sum_{\substack{b^4 \leq (x/(a^3 c^5)) \\ b^4 \equiv la^3 c^5 \pmod{q}}} 1 \\
&= \frac{x^{(1/4)}}{q} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} \sum_{a \leq x^{\lambda_1}} \frac{N_4(lac^3; q)}{a^{(3/4)}} + O(x^{\nu + \lambda_1} q^\varepsilon) \\
&= \frac{x^{(1/4)}}{q} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} \left( \sum_{a \leq x^{\lambda_1}} \frac{N_4(lac^3; q) - ((\phi(q))/q)}{a^{(3/4)}} + \frac{\phi(q)}{q} \sum_{a \leq x^{\lambda_1}} \frac{1}{a^{(3/4)}} \right) + O(x^{\nu + \lambda_1} q^\varepsilon) \\
&= \frac{x^{(1/4)}}{q} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} \left( \int_1^{x^{\lambda_1}} \frac{1}{\mu^{(3/4)}} d \left( \sum_{a \leq \mu} N_4(lac^3; q) - \frac{\phi(q)}{q} \mu \right) + \frac{\phi(q)}{q} \left( 4x^{(1/4)\lambda_1} - 3 - \frac{3}{4} \int_1^{\infty} \frac{t - [t]}{t^{(7/4)}} dt + O\left( x^{-\frac{3}{4}\lambda_1} \right) \right) \right) \\
&\quad + O(x^{\nu + \lambda_1} q^\varepsilon) \\
&= \frac{x^{(1/4)}}{q} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} \left( O\left( \frac{q^{(1/2) - \varepsilon}}{x^{(3/4)\lambda_1}} \right) - N_4(lc^3; q) + \frac{\phi(q)}{q} + \frac{3}{4} \int_1^{\infty} \frac{\sum_{a \leq \mu} N_4(lac^3; q) - ((\phi(q))/q)\mu}{\mu^{(7/4)}} d\mu \right) \\
&\quad + \frac{\phi(q)}{q^2} x^{(1/4)} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} \left( 4x^{(1/4)\lambda_1} - 3 - \frac{3}{4} \int_1^{\infty} \frac{t - [t]}{t^{(7/4)}} dt + O(x^{-(3/4)\lambda_1}) \right) + O(x^{\nu + \lambda_1} q^\varepsilon).
\end{aligned} \tag{45}$$

If we let  $F_c(\mu) = \sum_{a \leq \mu} N_4(lac^3; q) - ((\phi(q))/q)\mu$ , then the first term in the above formula is

$$\begin{aligned}
 & -\frac{x^{(1/4)}}{q} \sum_{c \leq x^\nu} \frac{N_4(lc^3; q) - ((\phi(q))/q)}{c^{5/4}} + \frac{3}{4} \frac{x^{(1/4)}}{q} \int_1^\infty \frac{\sum_{c \leq x^\nu} (F_c(\mu)/c^{(5/4)})}{\mu^{(7/4)}} d\mu + O\left(\frac{x^{(1/4)-(3/4)\lambda_1}}{q^{(1/2)-\varepsilon}}\right) \\
 & = -\frac{x^{(1/4)}}{q} \sum_{c=1}^\infty \frac{N_4(lc^3; q) - ((\phi(q))/q)}{c^{(5/4)}} + O\left(\frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}}\right) + \frac{3}{4} \frac{x^{(1/4)}}{q} \int_1^\infty \left(\sum_{c=1}^\infty \frac{F_c(\mu)}{c^{(5/4)}}\right) \mu^{-(7/4)} d\mu \\
 & \quad + O\left(\frac{x^{(1/4)-(3/4)\lambda_1}}{q^{(1/2)-\varepsilon}}\right) + O\left(\frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}}\right) \\
 & = \frac{x^{(1/4)}}{q} \left( -\sum_{c=1}^\infty \frac{N_4(lc^3; q) - ((\phi(q))/q)}{c^{(5/4)}} + \frac{3}{4} \frac{x^{(1/4)}}{q} \int_1^\infty \left(\sum_{c=1}^\infty \frac{F_c(\mu)}{c^{(5/4)}}\right) \mu^{-(7/4)} d\mu \right) + O\left(\frac{x^{(1/4)-(3/4)\lambda_1}}{q^{(1/2)-\varepsilon}}\right) + O\left(\frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}}\right).
 \end{aligned} \tag{46}$$

And the second term is

$$\begin{aligned}
 & \frac{\phi(q)}{q^2} x^{(1/4)} \left( -4x^{-(1/4)\nu} + 5 - \frac{5}{4} \int_1^\infty \frac{t - [t]}{t^{(9/4)}} dt + O(x^{-(1/4)\nu}) \right) \left( 4x^{(1/4)\lambda_1} - 3 - \frac{3}{4} \int_1^\infty \frac{t - [t]}{t^{(7/4)}} dt + O(x^{-(3/4)\lambda}) \right) \\
 & = 4 \frac{\phi(q)}{q^2} x^{(1/4)+(1/4)\lambda_1} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} + \frac{\phi(q)}{q^2} x^{(1/4)-(1/4)\nu} \left( 12 + 3 \int_1^\infty \frac{t - [t]}{t^{(7/4)}} dt \right) \\
 & \quad + \frac{\phi(q)}{q^2} x^{(1/4)} \left( 5 - \frac{5}{4} \int_1^\infty \frac{t - [t]}{t^{(9/4)}} dt \right) \left( -3 - \frac{3}{4} \int_1^\infty \frac{t - [t]}{t^{(7/4)}} dt \right) + O\left(\frac{\phi(q)}{q^2} x^{(1/4)-(5/4)\nu}\right) + O\left(\frac{\phi(q)}{q^2} x^{(1/4)-(3/4)\lambda_1}\right).
 \end{aligned} \tag{47}$$

So, we can get

$$\begin{aligned}
 T_{32} & = \frac{x^{(1/4)}}{q} \left[ \frac{\phi(q)}{q} \left( 5 - \frac{5}{4} \int_1^\infty \frac{t - [t]}{t^{(9/4)}} dt \right) \left( -3 - \frac{3}{4} \int_1^\infty \frac{t - [t]}{t^{(7/4)}} dt \right) - \sum_{c=1}^\infty \frac{N_4(lc^3; q) - ((\phi(q))/q)}{c^{(5/4)}} \right. \\
 & \quad \left. + \frac{3}{4} \frac{x^{(1/4)}}{q} \int_1^\infty \left( \sum_{c=1}^\infty \frac{F_c(\mu)}{c^{(5/4)}} \right) \mu^{-(7/4)} d\mu \right] + 4 \frac{\phi(q)}{q^2} x^{(1/4)+(1/4)\lambda_1} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} + \frac{\phi(q)}{q^2} x^{(1/4)-(1/4)\nu} \left( 12 + 3 \int_1^\infty \frac{t - [t]}{t^{(7/4)}} dt \right) \\
 & \quad + O\left(\frac{\phi(q)}{q^2} x^{(1/4)-(5/4)\nu}\right) + O\left(\frac{\phi(q)}{q^2} x^{(1/4)-(3/4)\lambda_1}\right) + O\left(\frac{x^{(1/4)-(3/4)\lambda_1}}{q^{(1/2)-\varepsilon}}\right) + O\left(\frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}}\right) + O(x^{\nu+\lambda_1} q^\varepsilon).
 \end{aligned} \tag{48}$$

And for  $T_{34}$ , following closely Chan [21] as shown in Figure 2 instead of Dirichlet's hyperbola method shown in Figure 1, we just need to divide the interval  $\{(a, b): a > x^n, b > x^m, a^3 b^4 \leq x\}$  in the same way. Note that the sum

$$\sum_{\substack{a > x^{\lambda_1} \\ a^3 b^4 \leq (x/c^5)}} \sum_{\substack{b > x^{\mu_1} \\ a^3 b^4 \equiv lc^5 \pmod{q}}} 1 \tag{49}$$



can be estimated with the help of asymptotic formula given at the end of page 101 in [21]; then, using Lemma 5, we can get

$$T_{34} = \sum_{c \leq x^\nu} \frac{1}{c^{(5/3)}} \cdot \frac{3\phi(q)}{q^2} x^{(1/3)-(1/3)\mu_1} - \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} \cdot \frac{3\phi(q)}{q^2} x^{(1/4)-(1/4)\lambda_1} + \frac{\phi(q)}{q^2} x^{\lambda_1+\mu_1+\nu} + O\left(\frac{x^{(1/3)-(1/3)\mu_1}}{2^{J_1} \cdot q} \sum_{c \leq x^\nu} \frac{1}{c^{(5/4)}} + 2^{J_1} q^{(1/2)+\varepsilon} x^\nu \log x\right), \quad (50)$$

where  $2^{J_1}$  satisfies  $(X^{\mu_1}/(2^{J_1} q^{(1/2)})) \ll 1$ .

Picking  $\lambda_1$  to have the same size as  $\mu_1$  and combining the previous results, we have

$$T_3 = \frac{x^{(1/3)}}{q} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{b^{(4/3)} c^{(5/3)}} - \left(6 - 2 \int_1^{\infty} \frac{t - [t]}{t^{(7/3)}} dt\right) \frac{\phi(q)}{q} x^{(1/3)-(2/3)\nu} + \frac{\phi(q)}{q^2} x^{(1/4)-(1/4)\nu} \left(12 + 3 \int_1^{\infty} \frac{t - [t]}{t^{(7/4)}} dt\right) + \frac{x^{(1/4)}}{q} \left[\frac{\phi(q)}{q} \left(5 - \frac{5}{4} \int_1^{\infty} \frac{t - [t]}{t^{(9/4)}} dt\right) \left(-3 - \frac{3}{4} \int_1^{\infty} \frac{t - [t]}{t^{(7/4)}} dt\right) - \sum_{c=1}^{\infty} \frac{N_4(lc^3; q) - ((\phi(q))/q)}{c^{(5/4)}}\right] + \frac{3}{4} \frac{x^{(1/4)}}{q} \int_1^{\infty} \left(\sum_{c=1}^{\infty} \frac{F_c(\mu)}{c^{(5/4)}}\right) \mu^{-(7/4)} d\mu + O\left(x^{\mu_1+\nu} q^\varepsilon + 2^{J_1} q^{(1/2)+\varepsilon} x^\nu + \frac{x^{(1/3)-(5/3)\nu}}{q^{(1/2)-\varepsilon}} + \frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}} + \frac{x^{(1/4)-(3/4)\lambda_1}}{q^{(1/2)-\varepsilon}} + \frac{x^{(1/3)-(4/3)\mu_1}}{q^{(1/2)-\varepsilon}}\right). \quad (51)$$

For  $T_2$ , in the same way as  $T_3$ , we have

$$T_2 = \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{a^3 c^5 \leq (x/b^4) \\ a^3 c^5 \equiv lb^4 \pmod{q}}} 1, \\ = \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{c \leq x^{\mu/2} \\ (c,q)=1}} \sum_{\substack{a^3 \leq (x/b^4 c^5) \\ a^3 \equiv lb^4 c^5 \pmod{q}}} 1 + \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{a \leq x^{\lambda_2} \\ (a,q)=1}} \sum_{\substack{c^5 \leq (x/a^3 b^4) \\ c^5 \equiv la^3 b^4 \pmod{q}}} 1 \\ - \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{a \leq x^{\lambda_2} \\ (a,q)=1}} \sum_{\substack{c \leq x^{\mu/2} \\ b^4 \equiv la^3 c^5 \pmod{q}}} 1 + \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{a > x^{\lambda_2} \\ a^3 c^5 \leq (x/b^4)}} \sum_{\substack{c > x^{\mu/2} \\ a^3 c^5 \equiv lb^4 \pmod{q}}} 1 \\ := T_{21} + T_{22} - T_{23} + T_{24}.$$

And we can obtain

$$\begin{aligned}
 T_{21} &= \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \left( \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{c^{(5/3)}} \right) \frac{x^{(1/3)}}{q} - \frac{3}{2} \frac{\phi(q)}{q^2} \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} x^{(1/3)-(2/3)\mu_2} + O\left(x^{\mu+\mu_2} q^\varepsilon + \frac{x^{(1/3)-(5/3)\mu_2}}{q^{(1/2)-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}}\right), \\
 T_{23} &= \frac{\phi(q)}{q^2} x^{\lambda_2+\mu_2+\mu} + O\left(\frac{x^{\lambda_2}}{q^{(1/2)-\varepsilon}}\right) + O(x^{\mu_2+\mu} q^\varepsilon), \\
 T_{24} &= \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \cdot \frac{3}{2} \frac{\phi(q)}{q^2} x^{(1/3)-(2/3)\mu_2} - \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \cdot \frac{5}{2} \frac{\phi(q)}{q^2} x^{(1/5)-(2/5)\lambda_2} + \frac{\phi(q)}{q^2} x^{\lambda_2+\mu_2+\mu} \\
 &\quad + O\left(\frac{x^{(1/3)-(2/3)\mu_2}}{2^{I_2} \cdot q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} + 2^{I_2} q^{(1/2)+\varepsilon} x^\mu \log x\right).
 \end{aligned} \tag{53}$$

For  $T_{22}$ , again in the same way as the proof of  $T_3$ , we have

$$\begin{aligned}
 T_{22} &= \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{a \leq x^{\lambda_2} \\ (a,q)=1}} \sum_{\substack{c^5 \leq \frac{x}{a^3 b^4} \\ c^5 \equiv la^3 b^4 \pmod{q}}} 1, \\
 &= \frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \sum_{a \leq x^{\lambda_2}} \frac{N_5(la^2b; q)}{a^{(3/5)}} + O(x^{\mu+\lambda_2} q^\varepsilon) \\
 &= \frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \left( \sum_{a \leq x^{\lambda_2}} \frac{N_5(la^2b; q) - ((\phi(q))/q)}{a^{(3/5)}} + \sum_{a \leq x^{\lambda_2}} \frac{1}{a^{(3/5)}} \frac{\phi(q)}{q} \right) + O(x^{\mu+\lambda_2} q^\varepsilon) \\
 &= \frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \left( \int_1^{x^{\lambda_2}} \frac{1}{\mu^{(3/5)}} d\left( \sum_{a \leq \mu} N_5(la^2b; q) - \frac{\phi(q)}{q} \mu \right) \right) \\
 &\quad + \frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \frac{\phi(q)}{q} \left( \frac{5}{2} x^{(2/5)\lambda_2} - \frac{3}{2} - \frac{3}{5} \int_1^\infty \frac{t - [t]}{t^{(8/5)}} dt + O(x^{-(3/5)\lambda_2}) \right) + O(x^{\mu+\lambda_2} q^\varepsilon).
 \end{aligned} \tag{54}$$

If we let  $F_b(\mu) = \sum_{a \leq \mu} N_5(la^2b; q) - ((\phi(q))/q)\mu$ , then the first term in the above formula is

$$\begin{aligned}
& \frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \left( O\left( \frac{q^{(1/2)+\varepsilon}}{x^{(3/5)\lambda_2}} \right) - N_5(lb; q) + \frac{\phi(q)}{q} + \frac{3}{5} \int_1^\infty \frac{F_b(\mu)}{\mu^{(8/5)}} d\mu \right) \\
&= -\frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{N_5(lb; q) - ((\phi(q))/q)}{b^{(4/5)}} + \frac{3}{5} \frac{x^{(1/5)}}{q} \int_1^\infty \frac{\sum_{b \leq x^\mu} (F_b(\mu)/b^{(4/5)})}{\mu^{(8/5)}} + O\left( \frac{x^{1/5-3/5\lambda_2}}{q^{1/2-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{4/5}} \right) \\
&= \left( N_5(l; q) - \frac{\phi(q)}{q} - \frac{4}{5} \int_1^\infty \frac{\sum_{b \leq \mu} N_5(lb; q) - ((\phi(q))/q)}{\mu^{(9/5)}} d\mu \right) \frac{x^{(1/5)}}{q} \\
&\quad + \frac{12}{25} \int_1^\infty \int_1^\infty \frac{\sum_{b \leq \lambda} F_b(\mu)}{\mu^{(8/5)} \lambda^{(9/5)}} d\lambda d\mu \frac{x^{(1/5)}}{q} \\
&\quad - \frac{3}{5} \frac{x^{(1/5)}}{q} \int_1^\infty \frac{\sum_{a \leq \mu} N_5(la^2; q) - ((\phi(q))/q)}{\mu^{(8/5)}} d\mu \\
&\quad + O\left( \frac{x^{(1/5)-(3/5)\lambda_2}}{q^{(1/2)-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \right) + O\left( \frac{x^{(1/5)-(4/5)\mu}}{q^{(1/2)-\varepsilon}} \right),
\end{aligned} \tag{55}$$

and the second term is

$$\begin{aligned}
& \frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \frac{\phi(q)}{q} \frac{5}{2} x^{(2/5)\lambda_2} + \frac{x^{(1/5)}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \frac{\phi(q)}{q} \left( -\frac{3}{2} - \frac{3}{5} \int_1^\infty \frac{t - [t]}{t^{(8/5)}} dt \right) + O\left( \frac{\phi(q)}{q} x^{(1/5)-(3/5)\lambda_2} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \right) \\
&= \frac{5}{2} \frac{x^{(1/5)+(2/5)\lambda}}{q} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \frac{\phi(q)}{q} - \frac{x^{(1/5)}}{q} \frac{\phi(q)}{q} \left( 5x^{(1/5)\mu} - 4 - \frac{4}{5} \int_1^\infty \frac{t - [t]}{t^{(9/5)}} dt + O(x^{-(4/5)\mu}) \right) \left( \frac{3}{2} + \frac{3}{5} \int_1^\infty \frac{t - [t]}{t^{(8/5)}} dt \right) \\
&\quad + O\left( \frac{\phi(q)}{q} x^{(1/5)-(3/5)\lambda_2} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \right) \\
&= \frac{5}{2} \frac{\phi(q)}{q^2} x^{(1/5)+(2/5)\lambda_2} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} + \left( -4 - \frac{4}{5} \int_1^\infty \frac{t - [t]}{t^{(9/5)}} dt \right) \left( -\frac{3}{2} - \frac{3}{5} \int_1^\infty \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q^2} x^{(1/5)} \\
&\quad + \left( -\frac{15}{2} - 3 \int_1^\infty \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q^2} x^{(1/5)+(1/5)\mu} + O\left( \frac{\phi(q)}{q} x^{(1/5)-(3/5)\lambda_2} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} \right) + O\left( \frac{\phi(q)}{q^2} x^{(1/5)-(4/5)\mu} \right).
\end{aligned} \tag{56}$$

Then, we can obtain

$$\begin{aligned}
T_{22} &= \frac{x^{(1/5)}}{q} \left[ \left( -4 - \frac{4}{5} \int_1^\infty \frac{t - [t]}{t^{(9/5)}} dt \right) \left( -\frac{3}{2} - \frac{3}{5} \int_1^\infty \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q} - \frac{4}{5} \int_1^\infty \frac{\sum_{b \leq \mu} N_5(lb; q) - ((\phi(q))/q)\mu}{\mu^{(8/5)}} d\mu \right. \\
&\quad \left. - \frac{3}{5} \frac{x^{(1/5)}}{q} \int_1^\infty \frac{\sum_{a \leq \mu} N_5(la^2; q) - ((\phi(q))/q)}{\mu^{(8/5)}} d\mu + N_5(l; q) - \frac{\phi(q)}{q} + \frac{12}{25} \int_1^\infty \int_1^\infty \frac{\sum_{b \leq \lambda} F_b(\mu)}{\mu^{(8/5)} \lambda^{(9/5)}} d\lambda d\mu \right] \\
&\quad + \frac{5}{2} \frac{\phi(q)}{q^2} x^{(1/5)+(2/5)\lambda_2} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} + \left( -\frac{15}{2} - 3 \int_1^\infty \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q^2} x^{(1/5)+(1/5)\mu} \\
&\quad + O\left( \frac{x^{(1/5)-(3/5)\lambda_2}}{q^{(1/2)-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} + \frac{x^{(1/5)-(4/5)\mu}}{q^{(1/2)-\varepsilon}} + \frac{\phi(q)}{q^2} x^{(1/5)-(4/5)\mu} + x^{\mu+\lambda_2} q^\varepsilon \right).
\end{aligned} \tag{57}$$

If we pick  $\lambda_1$  to have the same size as  $\mu_1$ , we can get

$$\begin{aligned}
 T_2 = & \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \left( \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{c^{(5/3)}} \right) \frac{x^{(1/3)}}{q} + \frac{x^{(1/5)}}{q} \left[ \left( -4 - \frac{4}{5} \int_1^{\infty} \frac{t - [t]}{t^{(9/5)}} dt \right) \left( -\frac{3}{2} - \frac{3}{5} \int_1^{\infty} \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q} \right. \\
 & - \frac{4}{5} \int_1^{\infty} \frac{\sum_{b \leq \mu} N_5(lb; q) - ((\phi(q))/q)\mu}{\mu^{(8/5)}} d\mu + N_5(l; q) - \frac{\phi(q)}{q} - \frac{3}{5} \frac{x^{(1/5)}}{q} \int_1^{\infty} \frac{\sum_{a \leq \mu} N_5(la^2; q) - ((\phi(q))/q)\mu}{\mu^{(8/5)}} d\mu \\
 & \left. + \frac{12}{25} \int_1^{\infty} \int_1^{\infty} \frac{\sum_{b \leq \lambda} F_b(\mu)}{\mu^{(8/5)} \lambda^{(9/5)}} d\lambda d\mu \right] + \left( -\frac{15}{2} - 3 \int_1^{\infty} \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q^2} x^{(1/5)+(1/5)\mu} \\
 & + O \left( \frac{x^{(1/3)-(5/3)\mu_2}}{q^{(1/2)-\varepsilon}} + \frac{x^{(1/5)-(3/5)\lambda_2}}{q^{(1/2)-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{(4/5)}} + \frac{x^{(1/5)-(4/5)\mu}}{q^{(1/2)-\varepsilon}} + x^{\mu+\mu_2} q^\varepsilon + 2^{J_2} q^{(1/2)+\varepsilon} x^\mu \right),
 \end{aligned} \tag{58}$$

where  $2^{J_2}$  satisfies  $(X^{\mu_1}/2^{J_2} q^{(1/2)}) \ll 1$ .

Similar to the proofs of  $T_2$  and  $T_3$ , we can get the following:

$$\begin{aligned}
 T_1 = & \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} \left( \sum_{c=1}^{\infty} \frac{N_4(lac^3; q)}{c^{(5/4)}} \right) \frac{x^{(1/4)}}{q} \\
 & + \sum_{a \leq x^\lambda} \frac{1}{a^{(3/5)}} \left( -4 \frac{\phi(q)}{q} - N_5(la^2; q) + \frac{4}{5} \int_1^{\infty} \frac{\sum_{l_1 \leq \mu} N_5(l_1 a^2; q) - ((\phi(q))/q)\mu}{\mu^{(9/5)}} d\mu \right) \frac{x^{(1/5)}}{q} \\
 & + O \left( \frac{x^{(1/4)-(5/4)\mu_3}}{q^{(1/2)-\varepsilon}} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} + x^{\lambda+\mu_3} q^\varepsilon + 2^{J_3} q^{(1/2)+\varepsilon} x^\lambda + \frac{x^{(1/5)-(4/5)\lambda_3}}{q^{(1/2)-\varepsilon}} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/5)}} \right), \\
 T_4 = & \left( \frac{5}{2} x^{(2/5)\lambda} - \frac{3}{2} - \frac{3}{5} \int_1^{\infty} \frac{t - [t]}{t^{(8/5)}} dt + O(x^{-(3/5)\lambda}) \right) \frac{5\phi(q)}{q^2} x^{(1/5)+(1/5)\mu} + O \left( \frac{x^{(1/5)-(4/5)\mu}}{q^{(1/2)-\varepsilon}} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/5)}} \right) \\
 & + \sum_{a \leq x^\lambda} \frac{1}{a^{(3/5)}} \left( -4 \frac{\phi(q)}{q} - N_5(la^2; q) + \frac{4}{5} \int_1^{\infty} \frac{\sum_{l_1 \leq \mu} N_5(l_1 a^2; q) - ((\phi(q))/q)\mu}{\mu^{(9/5)}} d\mu \right) \frac{x^{(1/5)}}{q} \\
 = & \frac{25}{2} \frac{\phi(q)}{q^2} x^{(1/5)+(\mu/5)+(2/5)\lambda} + \left( -\frac{15}{2} - 3 \int_1^{\infty} \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q^2} x^{(1/5)+(1/5)\mu} \\
 & + \sum_{a \leq x^\lambda} \frac{1}{a^{(3/5)}} \left( -4 \frac{\phi(q)}{q} - N_5(la^2; q) + \frac{4}{5} \int_1^{\infty} \frac{\sum_{l_1 \leq \mu} N_5(l_1 a^2; q) - ((\phi(q))/q)\mu}{\mu^{(9/5)}} d\mu \right) \frac{x^{(1/5)}}{q} \\
 & + O \left( \frac{\phi(q)}{q^2} x^{(1/5)+(\mu/5)-(3/5)\lambda} \right) + O \left( \frac{x^{(1/5)-(4/5)\mu}}{q^{(1/2)-\varepsilon}} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/5)}} \right),
 \end{aligned}$$

$$\begin{aligned}
T_5 &= \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \left( \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{c^{(5/3)}} \right) \frac{x^{(1/3)}}{q} - \frac{3}{2} \frac{\phi(q)}{q^2} x^{(1/3)-(2/3)\nu} \sum_{b \leq x^\mu} \frac{1}{b^{4/3}} + O\left( \frac{x^{(1/3)-(2/3)\mu}}{q^{(1/2)-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \right) \\
&= \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \left( \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{c^{(5/3)}} \right) \frac{x^{(1/3)}}{q} + \frac{3}{2} \frac{\phi(q)}{q^2} x^{(1/3)-(2/3)\nu} \left( 3x^{-(1/3)\mu} - 4 + \frac{4}{3} \int_1^\infty \frac{t - [t]}{t^{(7/3)}} dt + O(x^{-(4/3)\mu}) \right) \\
&\quad + O\left( \frac{x^{(1/3)-(5/3)\mu}}{q^{(1/2)-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \right) \\
&= \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \left( \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{c^{(5/3)}} \right) \frac{x^{(1/3)}}{q} + \frac{9}{2} \frac{\phi(q)}{q^2} x^{(1/3)-(2/3)\nu-(1/3)\mu} + \left( -6 + 2 \int_1^\infty \frac{t - [t]}{t^{(7/3)}} dt \right) \frac{\phi(q)}{q} x^{(1/3)-(2/3)\nu} \\
&\quad + O\left( \frac{\phi(q)}{q^2} x^{(1/3)-(2/3)\nu-(4/3)\mu} \right) + O\left( \frac{x^{(1/3)-(5/3)\mu}}{q^{(1/2)-\varepsilon}} \sum_{b \leq x^\mu} \frac{1}{b^{(4/3)}} \right), \\
T_6 &= \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} \left( \sum_{c=1}^{\infty} \frac{N_4(lac^3; q)}{c^{(5/4)}} \right) \frac{x^{(1/4)}}{q} - \frac{4\phi(q)}{q^2} x^{(1/4)-(1/4)\nu} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} + O\left( \frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} \right) \\
&= \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} \left( \sum_{c=1}^{\infty} \frac{N_4(lac^3; q)}{c^{(5/4)}} \right) \frac{x^{(1/4)}}{q} - \frac{4\phi(q)}{q^2} x^{(1/4)-(21/4)\nu} \left( 4x^{(1/4)\lambda} - 3 - \frac{3}{4} \int_1^\infty \frac{t - [t]}{t^{(7/4)}} dt + O(x^{-(3/4)\lambda}) \right) \\
&\quad + O\left( \frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} \right) \\
&= \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} \left( \sum_{c=1}^{\infty} \frac{N_4(lac^3; q)}{c^{(5/4)}} \right) \frac{x^{(1/4)}}{q} - 16 \frac{\phi(q)}{q^2} x^{(1/4)-(1/4)\nu+(1/4)\lambda} + \left( 12 + 3 \int_1^\infty \frac{t - [t]}{t^{(7/4)}} dt \right) \frac{\phi(q)}{q} x^{(1/4)-(1/4)\nu} \\
&\quad + O\left( \frac{\phi(q)}{q^2} x^{(1/4)-(1/4)\nu-(1/4)\lambda} \right) + O\left( \frac{x^{(1/4)-(5/4)\nu}}{q^{(1/2)-\varepsilon}} \sum_{a \leq x^\lambda} \frac{1}{a^{(3/4)}} \right), \\
T_7 &= \sum_{\substack{a \leq x^\lambda \\ (a,q)=1}} \sum_{\substack{b \leq x^\mu \\ (b,q)=1}} \sum_{\substack{c \leq x^\nu \\ (c,q)=1 \\ a^3b^4c^5 \equiv l \pmod{q}}} 1 = \frac{\phi(q)}{q} x^{\lambda+\mu+\nu} + O\left( \frac{x^\lambda}{q^{(1/2)-\varepsilon}} x^{\mu+\nu} \right).
\end{aligned}$$

(59)

Finally, we discuss  $T_8$  as follows: dividing interval  $[x^\nu, x^{((1-3\lambda-4\mu)/5)}]$  into  $m$  parts with the length of each one being  $((x^{((1-3\lambda-4\mu)/5)} - x^\nu)/m)$ . The  $n$ th part is

$$\left[ x^\nu + (n-1) \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m}, x^\nu + n \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m} \right]. \quad (60)$$

Let  $a_0$  be the intersection of the plane  $b = x^\mu$ ,  $c = x^\nu + n((x^{((1-3\lambda-4\mu)/5)} - x^\nu)/m)$ , and curved surface  $a^3b^4c^5 = x$ , which is

$$a_0^3 x^{4\mu} \left( x^\nu + n \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m} \right)^5 = x, \quad (61)$$

and then we obtain

$$a_0 = \frac{x^{((1-4\mu)/3)}}{\left( x^\nu + n \left( (x^{((1-3\lambda-4\mu)/5)} - x^\nu)/m \right) \right)^{(5/3)}}. \quad (62)$$

Similarly, we have

$$b_0 = \frac{x^{((1-3\lambda)/4)}}{\left( x^\nu + n \left( (x^{((1-3\lambda-4\mu)/5)} - x^\nu)/m \right) \right)^{(5/4)}}. \quad (63)$$

We define the intersection of  $a^3b^4c^5 = x$  and  $x^\nu + n((x^{((1-3\lambda-4\mu)/5)} - x^\nu)/m)$  as

$$a^3 b^4 = \frac{x}{\left(x^y + n \left(x^{((1-3\lambda-4\mu)/5)} - x^y\right)/m\right)^5} := x_n. \quad (64)$$

Now we divide the interval in the  $n$ th part and follow the construction in [23]; firstly, we have rectangles

$$R_i = \left[2^{i-1}x^\lambda, 2^i x^\lambda\right] \times \left[x^\mu, \left(\frac{x_n}{(2^i x^\lambda)^3}\right)^{(1/4)}\right], 1 \leq i \leq I = \log_2\left(\frac{a_0}{x^\lambda}\right). \quad (65)$$

In the remaining regions

$$S_i := \left\{ (a, b), 2^{i-1}x^\lambda < a \leq 2^i x^\lambda, b > \left(\frac{x_n}{(2^i x^\lambda)^3}\right)^{(1/4)}, a^3 b^4 \leq x_n \right\}, \quad (66)$$

we place additional rectangles  $R_{ijk}$ . If we let  $1 \leq k \leq 2^{j-1}$ , in the same way we place a further rectangles and so on, then we can get  $R_{ijk}$  which is

$$R_{ijk} = \left[2^{i-1} \left(1 + \frac{2k-2}{2^j}\right) x^\lambda, 2^{i-1} \left(1 + \frac{2k-1}{2^j}\right) x^\lambda\right] \times \left[\left(\frac{x_n}{(2^{i-1} (1 + (2k/2^j)) x^\lambda)^3}\right)^{(1/4)}, \left(\frac{x_n}{(2^{i-1} (1 + (2k-1/2^j)) x^\lambda)^3}\right)^{(1/4)}\right]. \quad (67)$$

The remaining regions are

$$\begin{aligned} S_{ijk} &= \left\{ (a, b): 2^{i-1} \left(1 + \frac{2k-2}{2^j}\right) x^\lambda < a \leq 2^{i-1} \left(1 + \frac{2k-1}{2^j}\right) x^\lambda, \right. \\ &\quad \left. b > \left(\frac{x_n}{(2^{i-1} (1 + ((2k-1)/2^j)) x^\lambda)^3}\right)^{(1/4)}, a^3 b^4 \leq x_n \right\}, \\ S'_{ijk} &= \left\{ (a, b): 2^{i-1} \left(1 + \frac{2k-1}{2^j}\right) x^\lambda < a \leq 2^{i-1} \left(1 + \frac{2k}{2^j}\right) x^\lambda, \right. \\ &\quad \left. b > \left(\frac{x_n}{(2^{i-1} (1 + (2k/2^j)) x^\lambda)^3}\right)^{(1/4)}, a^3 b^4 \leq x_n \right\}. \end{aligned} \quad (68)$$

In this part, expanding  $R_i$  and  $R_{ijk}$  to a cube with height  $((x^{((1-3\lambda-4\mu)/5)} - x^y)/m)$ , we can get  $R_{ni}$ ,  $R_{nijk}$ ,  $S_{ni}$ ,  $S_{nijk}$ ,  $S'_{nijk}$ ,

and the remaining region  $S_n$  correspondingly. Then, we have the asymptotic formula

$$\begin{aligned} T_8 &= \sum_n \sum_{i=1}^{I-1} \sum_{\substack{(a,b,c) \in R_{ni} \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 + \sum_n \sum_{j=1}^J \sum_{i=1}^I \sum_{k=1}^{2^{j-1}} \sum_{\substack{(a,b,c) \in R_{nijk} \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 + \sum_n \sum_{i=1}^I \sum_{k=1}^{2^{j-1}} \sum_{\substack{(a,b,c) \in S_{nijk} \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 \\ &\quad + \sum_n \sum_{i=1}^I \sum_{k=1}^{2^{j-1}} \sum_{\substack{(a,b,c) \in S'_{nijk} \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 + \sum_n \sum_{\substack{(a,b,c) \in S_n \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1. \end{aligned} \quad (69)$$

If we let  $N_{A,B,C}(K, L, M) := \#\{(a, b, c): a^3 b^4 c^5 \equiv l \pmod{q}, A < a \leq A + K, B < b \leq B + L, C < c \leq C + M\}$ , by Lemma 5, we know

$$N_{A,B,C}(K, L, M) = \frac{\phi(q)}{q^2} KLM + O\left(\frac{M}{q^{(1/2)-\varepsilon}} + KLq^\varepsilon\right). \quad (70)$$

Then, we further obtain

$$\begin{aligned}
T_8 &= \frac{\phi(q)}{q^2} \text{Area of } \{(a, b, c): a > x^\lambda, b > x^\mu, c > x^\nu, a^3 b^4 c^5 \leq x\} \\
&+ O\left(\sum_n \sum_{i=1}^{I_n} \sum_{k=1}^{2^{j-1}} \frac{\text{Area of } S_{nijk}}{q}\right) + O\left(\sum_n \sum_{i=1}^{I_n} \sum_{k=1}^{2^{j-1}} \frac{\text{Area of } S'_{nijk}}{q}\right) \\
&+ O\left(\sum_n \sum_{i=1}^I \sum_{k=1}^{2^{j-1}} \sum_{\substack{(a,b,c) \in S_{nijk} \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1\right) + O\left(\sum_n \sum_{i=1}^I \sum_{k=1}^{2^{j-1}} \sum_{\substack{(a,b,c) \in S'_{nijk} \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1\right) \\
&+ O\left(\sum_n \sum_{i=1}^{I_n} \frac{1}{q^{(1/2)-\varepsilon}} \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m} + \sum_n \sum_{i=1}^{I_n} 2^{i-1} x^\lambda \left(\frac{x_n^{(1/4)}}{(2^i x^\lambda)^{(3/4)}} - x^\mu\right)\right) \\
&+ \sum_n \sum_{j=1}^J \sum_{i=1}^{I_n} \sum_{k=1}^{2^{j-1}} x^{\frac{((1-3\lambda-4\mu)/5)}{m}} - x^\nu \frac{1}{q^{(1/2)}} + \sum_n \sum_{j=1}^J \sum_{i=1}^{I_n} \sum_{k=1}^{2^{j-1}} 2^{i-1} x^\lambda \frac{1}{2^{2j}} \frac{x_n^{(1/4)}}{(2^{i-1} x^\lambda)^{(1/4)}} + O\left(\sum_n \sum_{\substack{(a,b,c) \in S_n \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1\right),
\end{aligned} \tag{71}$$

in which the main term is

$$\begin{aligned}
&\frac{\phi(q)}{q^2} \int_{x^\nu}^{x^{((1-3\lambda-4\mu)/5)}} \int_{x^\lambda}^{(x^{((1-4\mu)/3)}/c^{(5/3)})} \left(\left(\frac{x}{a^3 c^5}\right)^{(1/4)} - x^\mu\right) da dc \\
&= \frac{\phi(q)}{q^2} \left(\frac{25}{2} x^{((1+2\lambda+\mu)/5)} + \frac{9}{2} x^{((1-\mu-2\nu)/3)} - 16 x^{((1+\lambda-\nu)/4)} - x^{\lambda+\mu+\nu}\right).
\end{aligned} \tag{72}$$

Now we deal with the error term of  $T_8$ . Note that

$$\begin{aligned}
S_{nijk} &\subseteq \left[2^{i-1} \left(1 + \frac{2k-2}{2^j}\right) x^\lambda, 2^{i-1} \left(1 + \frac{2k-1}{2^j}\right) x^\lambda\right] \\
&\times \left[\left(\frac{x_n}{(2^{i-1} (1 + ((2k-1)/2^j)) x^\lambda)^3}\right)^{(1/4)}, \left(\frac{x_n}{(2^{i-1} (1 + ((2k-2)/2^j)) x^\lambda)^3}\right)^{(1/4)}\right] \\
&\times \left[x^\nu + (n-1) \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m}, x^\nu + n \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m}\right].
\end{aligned} \tag{73}$$

Thus, the area of  $S_{nijk}$  is at most

$$\frac{2^{i-1} x^\lambda}{2^j} \cdot \frac{x_n^{(1/4)}}{(2^{i-1} x^\lambda)^{(3/4)}} \cdot \frac{1}{2^j} \cdot \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m}, \tag{74}$$

which implies that the estimation of the first error term is

$$\sum_n \sum_{i=1}^{I_n} \sum_{k=1}^{2^{j-1}} \frac{\text{Area of } S_{nijk}}{q} \ll \frac{1}{2^j} \cdot \frac{1}{q} \cdot \frac{m^{(2/3)} x^{((1-\mu)/3)}}{(x^{((1-3\lambda-4\mu)/5)})^{(2/3)}}. \tag{75}$$

Similarly, we have the same error bound for the second, third, and the fourth error term. The error terms in the fifth are

$$\begin{aligned}
 & \sum_n \sum_{i=1}^{I_n} \frac{1}{q^{(1/2)-\varepsilon}} \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m} \ll \frac{1}{q^{(1/2)}} (x^{((1-3\lambda-4\mu)/5)} - x^\nu), \\
 & \sum_n \sum_{i=1}^{I_n} 2^{i-1} x^\lambda \left( \frac{x_n^{(1/4)}}{(2^i x^\lambda)^{(3/4)}} - x^\mu \right) \ll \frac{m^{(5/3)} x^{((1-\mu)/3)}}{(x^{((1-3\lambda-4\mu)/5)})^{(5/3)}}, \\
 & \sum_n \sum_{j=1}^J \sum_{i=1}^{I_n} \sum_{k=1}^{2^{j-1}} \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m} \frac{1}{q^{(1/2)}} \ll 2^J \frac{1}{q^{(1/2)}} (x^{((1-3\lambda-4\mu)/5)} - x^\nu), \\
 & \sum_n \sum_{j=1}^J \sum_{i=1}^{I_n} \sum_{k=1}^{2^{j-1}} 2^{i-1} x^\lambda \frac{1}{2^{2j}} \frac{x_n^{(1/4)}}{(2^{i-1} x^\lambda)^{(1/4)}} \ll \frac{m^{(5/3)} x^{((1-\mu)/3)}}{(x^{((1-3\lambda-4\mu)/5)})^{(5/3)}}.
 \end{aligned} \tag{76}$$

For the last error term, we extend  $S_n$  into a ring sector which volume is

$$\frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m} \left( \int_{x^\lambda}^{a_1} \left[ \left( \frac{x_{n-1}}{a^3} \right)^{(1/4)} - x^\mu \right] da - \int_{x^\lambda}^{a_0} \left[ \left( \frac{x_n}{a^3} \right)^{(1/4)} - x^\mu \right] da \right), \tag{77}$$

where

$$a_1^3 x^{4\mu} \left( x^\nu + (n-1) \frac{x^{((1-3\lambda-4\mu)/5)} - x^\nu}{m} \right)^5 = x. \tag{78}$$

$$\sum_n \sum_{\substack{(a,b,c) \in S_n \\ a^3 b^4 c^5 \equiv I \pmod{q}}} 1 \ll \frac{1}{m} x^{((1-3\lambda-4\mu)/5)} x^{((1-\mu-5\nu)/3)} = \frac{1}{m} x^{((8-9\lambda-17\mu-25\nu)/15)}. \tag{79}$$

Then, we can add  $n$  and get

Therefore, we have

$$\begin{aligned}
 T_8 &= \frac{\phi(q)}{q^2} \left( \frac{25}{2} x^{((1+2\lambda+\mu)/5)} + \frac{9}{2} x^{((1-\mu-2\nu)/3)} - 16 x^{((1+\lambda-\nu)/4)} - x^{\lambda+\mu+\nu} \right) \\
 &+ O \left( \frac{1}{2^J} \cdot \frac{1}{q} \cdot \frac{m^{(2/3)} x^{((1-\mu)/3)}}{(x^{((1-3\lambda-4\mu)/5)})^{(2/3)}} + \frac{1}{q^{(1/2)}} (x^{((1-3\lambda-4\mu)/5)} - x^\nu) + \frac{m^{(5/3)} x^{((1-\mu)/3)}}{(x^{((1-3\lambda-4\mu)/5)})^{(5/3)}} \right. \\
 &\left. + 2^J \frac{1}{q^{(1/2)}} (x^{((1-3\lambda-4\mu)/5)} - x^\nu) + \frac{1}{m} x^{((8-9\lambda-17\mu-25\nu)/15)} \right).
 \end{aligned} \tag{80}$$

Now we come to simplify the above error terms. Firstly, we have

$$\frac{1}{q^{(1/2)}} (x^{((1-3\lambda-4\mu)/5)} - x^\nu) \ll \frac{x^{(1/5)-(2/5)\lambda}}{q^{(1/2)-\varepsilon}}. \tag{81}$$

When  $m = O(1)$  and  $\lambda = \mu$ , we know that

$$\frac{m^{(5/3)} x^{((1-\mu)/3)}}{(x^{((1-3\lambda-4\mu)/5)})^{(5/3)}} \ll x^{2\lambda}. \tag{82}$$



If  $2^J \ll x^\lambda$  and  $\lambda = \mu$ , then

$$2^J \frac{1}{q^{(1/2)}} \left( x^{((1-3\lambda-4\mu)/5)} - x^\nu \right) \ll \frac{x^{(1/5)-(2/5)\lambda}}{q^{(1/2)-\varepsilon}}. \quad (83)$$

If we pick  $2^J = x^\lambda$  and  $\lambda = \mu$ , we get

$$\frac{1}{2^J} \cdot \frac{1}{q} \cdot \frac{m^{(2/3)} x^{((1-\mu)/3)}}{(x^{((1-3\lambda-4\mu)/5)})^{(2/3)}} \ll \frac{x^{(1/5)-(2/5)\lambda}}{q^{(1/2)-\varepsilon}}, \quad (84)$$

with the condition  $m = O(1)$ .

Therefore, if we choose  $\lambda = \mu = \nu$ ,  $\mu_i = \lambda_i$  ( $i = 1, 2, 3$ ) and  $m = O(1)$ , then we get all error terms from  $T_1$  to  $T_8$ :

---


$$\begin{aligned} O \left( \frac{x^{(1/3)-(4/3)\mu_1}}{q^{(1/2)-\varepsilon}} + x^{\mu_1+\lambda} q^\varepsilon + 2^{J_1} q^{(1/2)+\varepsilon} x^\lambda + \frac{x^{(1/3)-(5/3)\mu_2+(1/5)\lambda}}{q^{(1/2)-\varepsilon}} + x^{\mu_2+\lambda} q^\varepsilon + 2^{J_2} q^{(1/2)+\varepsilon} x^\lambda \right. \\ \left. + \frac{x^{(1/4)-(5/4)\mu_3+(2/5)\lambda}}{q^{(1/2)-\varepsilon}} + x^{\mu_3+\lambda} q^\varepsilon + 2^{J_3} q^{(1/2)+\varepsilon} x^\lambda + \frac{x^{(1/3)-(5/3)\lambda}}{q^{(1/2)-\varepsilon}} + \frac{x^{(1/5)-(2/5)\lambda}}{q^{(1/2)-\varepsilon}} + x^{(8/15)-(51/15)\lambda} \right), \end{aligned} \quad (85)$$


---

where  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$  satisfy the conditions

$$\begin{aligned} \frac{1}{10} < \mu_1 < \frac{1}{5} - \lambda, \quad \frac{x^{\mu_1}}{2^{J_1} q^{(1/2)}} \ll 1, \\ \frac{2}{25} + \frac{3}{25} \lambda < \mu_2 < \frac{1}{5} - \lambda, \quad \frac{x^{\mu_2}}{2^{J_2} q^{(1/2)}} \ll 1, \\ \frac{1}{25} + \frac{8}{25} \lambda < \mu_3 < \frac{1}{5} - \lambda, \quad \frac{x^{\mu_3}}{2^{J_3} q^{(1/2)}} \ll 1. \end{aligned} \quad (86)$$

Further if we choose  $\eta = \mu_1 = \mu_2 = \mu_3$ , then we obtain

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$$O \left( \frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon}} + x^{\eta+\lambda} q^\varepsilon + 2^{J_1} q^{(1/2)+\varepsilon} x^\lambda + \frac{x^{(1/5)-(2/5)\lambda}}{q^{(1/2)-\varepsilon}} + x^{(8/15)-(51/15)\lambda} + \frac{x^{(1/3)-(5/3)\lambda}}{q^{(1/2)-\varepsilon}} \right), \quad (87)$$


---

where  $(x^\eta / (2^{J_1} q^{(1/2)})) \ll 1$ ,  $(5/51) < \lambda < (1/10)$  and  $(1/10) < \eta < (1/5) - \lambda$ . Continue to simplify the error term and get

$$O \left( \frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon}} + x^{\eta+\lambda} q^\varepsilon + 2^{J_1} q^{(1/2)+\varepsilon} x^\lambda \right), \quad (88)$$

where  $(x^\eta / (2^{J_1} q^{(1/2)})) \ll 1$  and  $\begin{cases} (8/15) - (66/15)\lambda < \eta < (1/5) - \lambda, \\ (5/51) < \lambda < (13/132), \end{cases}$  or

$\begin{cases} (1/10) < \eta < (1/5) - \lambda \\ (13/132) < \lambda < (1/10) \end{cases}$ . If  $q < x^{2\eta}$ , we can choose  $2^{J_1} = (x^\eta / q^{1/2})$ ; otherwise, we can choose  $2^{J_1} = 2$ . Finally, we have

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$$\sum_{\substack{a^3 b^4 c^5 \leq x \\ a^3 b^4 c^5 \equiv l \pmod{q}}} 1 = A'_q(l) \frac{x^{(1/3)}}{q} + B'_q(l) \frac{x^{(1/4)}}{q} + C'_q(l) \frac{x^{(1/5)}}{q} + O \left( \frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon}} + \max(x^\eta, q^{(1/2)}) q^\varepsilon x^\lambda \right), \quad (89)$$

where

$$\begin{aligned}
 A'_q(l) &= \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{N_3(lb^2c; q)}{b^{(4/3)} c^{(5/3)}}, \\
 B'_q(l) &= \frac{\phi(q)}{q} \left( 5 - \frac{5}{4} \int_1^{\infty} \frac{t - [t]}{t^{(9/4)}} dt \right) \left( -3 - \frac{3}{4} \int_1^{\infty} \frac{t - [t]}{t^{(7/4)}} dt \right) - \sum_{c=1}^{\infty} \frac{N_4(lc^3; q) - ((\phi(q))/q)}{c^{(5/4)}} \\
 &\quad + \frac{3}{4} \frac{x^{(1/4)}}{q} \int_1^{\infty} \left( \sum_{a \leq \mu} \frac{N_4(lac^3; q) - ((\phi(q))/q)\mu}{c^{(5/4)}} \right) \mu^{-(7/4)} d\mu, \\
 C'_q(l) &= \left( -4 - \frac{4}{5} \int_1^{\infty} \frac{t - [t]}{t^{(9/5)}} dt \right) \left( -\frac{3}{2} - \frac{3}{5} \int_1^{\infty} \frac{t - [t]}{t^{(8/5)}} dt \right) \frac{\phi(q)}{q} - \frac{4}{5} \int_1^{\infty} \frac{\sum_{b \leq \mu} N_5(lb; q) - ((\phi(q))/q)\mu}{\mu^{(8/5)}} d\mu \\
 &\quad + N_5(l; q) - \frac{\phi(q)}{q} - \frac{3}{5} \frac{x^{(1/5)}}{q} \int_1^{\infty} \frac{\sum_{a \leq \mu} N_5(la^2; q) - ((\phi(q))/q)\mu}{\mu^{(8/5)}} d\mu \\
 &\quad + \frac{12}{25} \int_1^{\infty} \int_1^{\infty} \frac{\sum_{b \leq \lambda} \sum_{a \leq \mu} N_5(la^2b; q) - ((\phi(q))/q)\lambda\mu}{\mu^{(8/5)} \lambda^{(9/5)}} d\lambda d\mu.
 \end{aligned} \tag{90}$$

#### 4. Proof of Theorem 1

First recall that any cube-full number can be written uniquely as  $n = a^3b^4c^5$  where  $b, c$  are square-free numbers

and  $(b, c) = 1$ . Following the proof of Theorem 2.3 in [20], we have

$$\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 = \sum_{\substack{a^3b^4c^5 \leq x \\ (b,c)=1, b,c \text{ squarefree} \\ a^3b^4c^5 \equiv l \pmod{q}}} 1 = \sum_{\substack{a^3b^4c^5 \leq x \\ a^3b^4c^5 \equiv l \pmod{q} \\ (b,c)=1}} \mu^2(b)\mu^2(c). \tag{91}$$

In view of the identity  $\mu^2(b) = \sum_{d^2|b} \mu(d)$ , we get

$$\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 = \sum_{\substack{a^3b^4c^5d^8 \leq x \\ a^3b^4c^5d^8 \equiv l \pmod{q} \\ (b,c)=1}} \mu(d)\mu^2(c). \tag{92}$$

$$\sum_{\substack{l|b \\ l|c}} \mu(l) = \begin{cases} 1 & \text{if } (b, c) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{93}$$

and  $\mu^2(cl) = \mu^2(c)\mu^2(l)$  for  $(c, l) = 1$ , otherwise  $\mu^2(cl) = 0$  and  $\mu^2(l)\mu(l) = \mu(l)$ , we get

From the identity

$$\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 = \sum_{\substack{a^3b^4c^5d^8 \leq x \\ a^3b^4c^5d^8 \equiv l \pmod{q} \\ (c,d)=1}} \mu(d)\mu^2(c) \sum_{\substack{e|b \\ e|c}} \mu(e) = \sum_{\substack{a^3b^4c^5d^8e^9 \leq x \\ a^3b^4c^5d^8e^9 \equiv l \pmod{q} \\ (c,e,d)=1}} \mu(d)\mu^2(c)\mu(e). \tag{94}$$

Using again  $\mu^2(c) = \sum_{f^2|c} \mu(f)$  and

we have

$$\sum_{\substack{g|c \\ g|ed}} \mu(g) = \begin{cases} 1, & \text{if } (ed, c) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{95}$$

$$\begin{aligned}
\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 &= \sum_{\substack{a^3 b^4 c^5 d^8 e^9 f^{10} \leq x \\ a^3 b^4 c^5 d^8 e^9 \equiv l \pmod{q} \\ (c, e, d) = 1}} \mu(d) \mu(e) \sum_{f^2 | c} \mu(f), \\
&= \sum_{\substack{a^3 b^4 c^5 d^8 e^9 f^{10} \leq x \\ a^3 b^4 c^5 d^8 e^9 f^{10} \equiv l \pmod{q} \\ (c, f, e, d) = 1}} \mu(d) \mu(f) \mu(e) \\
&= \sum_{\substack{a^3 b^4 c^5 d^8 e^9 f^{10} \leq x \\ a^3 b^4 c^5 d^8 e^9 f^{10} \equiv l \pmod{q} \\ (f, e, d) = 1}} \mu(d) \mu(f) \mu(e) \sum_{\substack{g | c \\ g | ed}} \mu(g) \\
&= \sum_{\substack{a^3 b^4 c^5 g^5 d^8 e^9 f^{10} \leq x \\ a^3 b^4 c^5 g^5 d^8 e^9 f^{10} \equiv l \pmod{q} \\ (f, e, d) = 1}} \mu(d) \mu(f) \mu(e) \sum_{g | ed} \mu(g) \\
&= \sum_{\substack{g^5 d^8 e^9 f^{10} \leq x \\ (d, e, f) = 1}} \mu(d) \mu(f) \mu(e) \sum_{g | ed} \mu(g) \sum_{\substack{a^3 b^4 c^5 \leq (x/g^5 d^8 e^9 f^{10}) \\ a^3 b^4 c^5 \equiv l g^5 d^8 e^9 f^{10} \pmod{q}}} 1.
\end{aligned} \tag{96}$$

From  $(x^\eta / (2^{I_1} q^{(1/2)})) \ll 1$ , we know that if  $q \leq x^{2\eta}$ , then we can pick  $2^{I_1}$  of size  $(x^\eta / q^{(1/2)})$ , and if  $q > x^{2\eta}$ , we simply pick  $2^{I_1} = 2$ . Then, we obtain

$$\begin{aligned}
\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 &= \sum_{\substack{g^5 d^8 e^9 f^{10} \leq x q^{-(1/2\eta)} \\ (d, e, f, q) = 1}} \mu(d) \mu(f) \mu(e) \sum_{\substack{g | ed \\ (g, q) = 1}} \mu(g) \sum_{\substack{a^3 b^4 c^5 \leq (x/g^5 d^8 e^9 f^{10}) \\ a^3 b^4 c^5 \equiv l g^5 d^8 e^9 f^{10} \pmod{q}}} 1 \\
&+ \sum_{\substack{x q^{-(1/2\eta)} < g^5 d^8 e^9 f^{10} \leq x \\ (d, e, f, q) = 1}} \mu(d) \mu(f) \mu(e) \sum_{\substack{g | ed \\ (g, q) = 1}} \mu(g) \sum_{\substack{a^3 b^4 c^5 \leq (x/g^5 d^8 e^9 f^{10}) \\ a^3 b^4 c^5 \equiv l g^5 d^8 e^9 f^{10} \pmod{q}}} 1.
\end{aligned} \tag{97}$$

From the result of Theorem 2, we have

$$\begin{aligned}
\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 &= A'_q(l) \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d, e, f, q) = 1}} \frac{\mu(d) \mu(f) \mu(e)}{d^{(8/3)} e^3 f^{(10/3)}} \sum_{\substack{g | ed \\ (g, q) = 1}} \frac{\mu(g)}{g^{(5/3)}} \cdot \frac{x^{(1/3)}}{q} \\
&+ B'_q(l) \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d, e, f, q) = 1}} \frac{\mu(d) \mu(f) \mu(e)}{d^2 e^{(9/4)} f^{(5/2)}} \sum_{\substack{g | ed \\ (g, q) = 1}} \frac{\mu(g)}{g^{(5/4)}} \cdot \frac{x^{(1/4)}}{q} \\
&+ C'_q(l) \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d, e, f, q) = 1}} \frac{\mu(d) \mu(f) \mu(e)}{d^{(8/5)} e^{(9/5)} f^2} \sum_{\substack{g | ed \\ (g, q) = 1}} \frac{\mu(g)}{g} \cdot \frac{x^{(1/5)}}{q} \\
&+ O\left( \sum_{d^8 e^9 f^{10} \leq x q^{-(1/2\eta)}} \frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon} (d^8 e^9 f^{10})^{(1/3)-(4/3)\eta}} + \sum_{d^8 e^9 f^{10} \leq x q^{-(1/2\eta)}} \frac{x^{\eta+\lambda}}{(d^8 e^9 f^{10})^{\eta+\lambda}} q^\varepsilon \right. \\
&\left. + \sum_{x q^{-(1/2\eta)} \leq d^8 e^9 f^{10} \leq x} \frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon} (d^8 e^9 f^{10})^{(1/3)-(4/3)\eta}} + \sum_{x q^{-(1/2\eta)} \leq d^8 e^9 f^{10} \leq x} \frac{x^\lambda}{(d^8 e^9 f^{10})^\lambda} q^{(1/2)+\varepsilon} \right).
\end{aligned} \tag{98}$$

For the final term of the O-term, we have

$$\begin{aligned}
& \sum_{xq^{-(1/2)\eta} \leq d^8 e^9 f^{10} \leq x} \frac{x^\lambda}{(d^8 e^9 f^{10})^\lambda} q^{(1/2)+\varepsilon} \\
& \ll x^\lambda q^{(1/2)+\varepsilon} \sum_{d \leq x^{(1/8)}} \sum_{e \leq x^{(1/9)}} \sum_{xq^{-(1/2)\eta} d^{-8} e^{-9} \leq f^{10} \leq \frac{x}{d^8 e^9}} \frac{1}{d^{8\lambda} e^{9\lambda} f^{10\lambda}} \\
& \ll x^\lambda q^{(1/2)+\varepsilon} \sum_{d \leq x^{(1/8)}} \frac{1}{d^{8\lambda}} \sum_{e \leq x^{(1/9)}} \frac{1}{e^{9\lambda}} \sum_{xq^{-(1/2)\eta} d^{-8} e^{-9} \leq f^{10} \leq \frac{x}{d^8 e^9}} \frac{1}{f^{10\lambda}} \\
& \ll x^\lambda q^{(1/2)+\varepsilon} \sum_{d \leq x^{(1/8)}} \frac{1}{d^{8\lambda}} \sum_{e \leq x^{(1/9)}} \frac{1}{e^{9\lambda}} \frac{x^{(1/10)-\lambda}}{d^{(8/10)-8\lambda} e^{(9/10)-9\lambda}} \\
& \ll x^{(1/10)} q^{(1/2)+\varepsilon} \sum_{d \leq x^{(1/8)}} \frac{1}{d^{(8/10)}} \sum_{e \leq x^{(1/9)}} \frac{1}{e^{(9/10)}} \\
& = x^{(49/360)} q^{(1/2)+\varepsilon}.
\end{aligned} \tag{99}$$

In the same way, we have

$$\sum_{xq^{-(1/2)\eta} \leq d^8 e^9 f^{10} \leq x} \frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon} (d^8 e^9 f^{10})^{(1/3)-(4/3)\eta}} \ll x^{(49/360)} q^{(7/60\mu_1)-(7/6)} \ll x^{(49/360)} q^{(1/2)+\varepsilon}. \tag{100}$$

The first two terms of the error term are

$$\frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon}} + x^{\eta+\lambda} q^\varepsilon. \tag{101}$$

If we take

$$\frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon}} = x^{\eta+\lambda} q^\varepsilon, \tag{102}$$

and assume  $q = x^\theta$ , then we have  $\eta = (1/7) - (3/7)\lambda - (3/14)\theta$  and

$$x^{\eta+\lambda} q^\varepsilon = \frac{x^{(1/7)+(4/7)\lambda}}{q^{(3/14)}} q^\varepsilon. \tag{103}$$

Now we pick  $\lambda = (5/51) + \varepsilon$  and get

$$\frac{x^{(1/3)-(4/3)\eta}}{q^{(1/2)-\varepsilon}} + x^{\eta+\lambda} q^\varepsilon \ll \frac{x^{(71/357)}}{q^{(3/14)}} q^\varepsilon. \tag{104}$$

Consequently, we obtain

$$\begin{aligned}
\sum_{\substack{n \leq x, n \in \mathcal{P}_3 \\ n \equiv l \pmod{q}}} 1 &= A'_q(l) \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d,e,f,q)=1}} \frac{\mu(d)\mu(f)\mu(e)}{d^{(8/3)} e^3 f^{(10/3)}} \sum_{\substack{g|ed \\ (g,q)=1}} \frac{\mu(g)}{g^{(5/3)}} \cdot \frac{x^{(1/3)}}{q} \\
&+ B'_q(l) \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d,e,f,q)=1}} \frac{\mu(d)\mu(f)\mu(e)}{d^2 e^{(9/4)} f^{(5/2)}} \sum_{\substack{g|ed \\ (g,q)=1}} \frac{\mu(g)}{g^{(5/4)}} \cdot \frac{x^{(1/4)}}{q} \\
&+ C'_q(l) \sum_{\substack{d^8 e^9 f^{10} \geq 1 \\ (d,e,f,q)=1}} \frac{\mu(d)\mu(f)\mu(e)}{d^{(8/5)} e^{(9/5)} f^2} \sum_{\substack{g|ed \\ (g,q)=1}} \frac{\mu(g)}{g} \cdot \frac{x^{(1/5)}}{q} + O\left(\left(x^{(49/360)} q^{(1/2)} + \frac{x^{(71/357)}}{q^{(3/14)}}\right) q^\varepsilon\right),
\end{aligned} \tag{105}$$

which is Theorem 2.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# Distribution of $\alpha p^2$ Modulo One with Prime Variable $p$ of a Special Form

Fei Xue<sup>1</sup>, Jinjiang Li<sup>1</sup> and Min Zhang<sup>2</sup>

<sup>1</sup>Department of Mathematics, China University of Mining and Technology, Beijing 100083, China

<sup>2</sup>School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China

Correspondence should be addressed to Jinjiang Li; [jinjiang.li.math@gmail.com](mailto:jinjiang.li.math@gmail.com)

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Let  $\mathcal{P}_r$  denote an almost-prime with at most  $r$  prime factors, counted according to multiplicity. In this paper, it is proved that, for  $\alpha \in (\mathbb{R}/\mathbb{Q})$ ,  $\beta \in \mathbb{R}$ , and  $0 < \theta < (10/1561)$ , there exist infinitely many primes  $p$ , such that  $\|\alpha p^2 + \beta\| < p^{-\theta}$  and  $p + 2 = \mathcal{P}_4$ , which constitutes an improvement upon the previous result.

## 1. Introduction and Main Result

Let  $\mathcal{P}_r$  denote an almost-prime with at most  $r$  prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes  $p$  such that  $p + 2$  is a prime too. Up to now, this conjecture is still open, but many approximations about this conjecture were established. One of the most interesting results is due to Chen [1], who showed, in 1973, that there exist infinitely many primes  $p$  such that  $p + 2 = \mathcal{P}_2$ .

In 1981, Heath-Brown [2] showed that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is  $\mathcal{P}_2$ . In 2006, Green and Tao [3] established that there exist infinitely many arithmetic progressions consisting of three different primes  $p_1 < p_2 < p_3$  such that  $p_j + 2 = \mathcal{P}_2$  for each  $j = 1, 2, 3$ . Later, in 2008, Green and Tao [4] showed that, for any  $k \geq 3$ , there exist infinitely many arithmetic progressions consisting of  $k$  different primes  $p_1 < p_2 < \dots < p_k$  such that  $p_j + 2 = \mathcal{P}_2$  for each  $j = 1, 2, \dots, k$ .

Suppose that there is a problem including primes and let  $r \geq 2$  be an integer. Having in mind Chen's result, one may consider the problem with primes  $p$ , such that  $p + 2 = \mathcal{P}_r$ . Many authors investigated several kinds of problems of this type, such as Peneva and Tolev [5], Peneva [6], and Tolev [7–9].

Let  $\alpha$  be an irrational real number and  $\|x\|$  denote the distance from  $x$  to the nearest integer. Earlier work about the distribution of the fractional parts of the sequence  $\{\alpha p\}$  was first considered by Vinogradov [10], who showed that, for any real number  $\beta$ , there are infinitely many primes  $p$  such that for  $\theta = (1/5) - \varepsilon$ ; then,

$$\|\alpha p + \beta\| < p^{-\theta}, \quad (1)$$

where  $\varepsilon$  denotes arbitrarily small positive number. After that, the first improvement on (1) was due to Vaughan [11], who obtained  $\theta = (1/4)$  in (1) and who also required an additional factor  $(\log p)^8$  on the right-hand side of (1). Since then, many authors improved the upper bound of the exponent  $\theta$ , such as Harman [12, 13], Jia [14, 15], and Heath-Brown and Jia [16]. So far, the best result is given by Matomäki [17] with  $\theta = (1/3) - \varepsilon$ . Moreover, it seems very natural to consider the sequence  $\{\alpha p^k\}$  for  $k \geq 2$ , where  $p$  denotes a prime variable. Also, many authors studied the fractional parts of the sequence  $\{\alpha p^k\}$  for  $k \geq 2$ , such as Baker and Harman [18], Harman [19], and Wong [20].

In 2010, Todorova and Tolev [21] considered the distribution of  $\alpha p$  modulo one with primes of the form specified above and showed that, for  $\theta = (1/100)$ , there are infinitely many solutions in primes  $p$  to (1) such that

$p + 2 = \mathcal{P}_4$ . Later, Matomäki [22] showed that this result actually holds with  $p + 2 = \mathcal{P}_2$  and  $\theta = (1/1000)$ . After that, Shi [23] continued to improve the result of Matomäki [22] and showed that there are infinitely many solutions in primes  $p$  to (1) such that  $p + 2 = \mathcal{P}_2$  and  $\theta = (3/200)$ .

Moreover, for the case  $k = 2$ , Shi and Wu [24] established the result that there exist infinitely many primes  $p$ , which satisfy  $\|\alpha p^2 + \beta\| < p^{-\theta}$ , such that  $p + 2 = \mathcal{P}_4$  and  $\theta = (2/375) - \varepsilon$ .

In this paper, we shall continue to improve the result of Shi and Wu [24] and establish the following theorem.

**Theorem 1.** Suppose that  $\alpha \in (\mathbb{R}/\mathbb{Q})$ ,  $\beta \in \mathbb{R}$ , and  $0 < \theta < (10/1561)$ . Then, there exist infinitely many primes  $p$ , which satisfy  $p + 2 = \mathcal{P}_4$ , such that

$$\|\alpha p^2 + \beta\| < p^{-\theta}. \quad (2)$$

*Remark 1.* According to the work of Shi and Wu [24], our improvement comes from using the methods developed by Tolev [9] with more delicate iterative techniques and various bounds for exponential sums, combining with a version of Lemma 2.2 of [25], while the previous method, in dealing exponential sum, e.g., [24], is based on the traditional pattern of exponential sum estimates.

## 2. Notation

Let  $X$  be a sufficiently large real number. Set

$$\begin{aligned} \delta &= 0.307708, \\ \rho &= 0.23077, \\ \eta &= 0.076928, \\ \kappa &= 1.4999676, \\ 0 < \theta &< \frac{10}{1561}. \end{aligned} \quad (3)$$

Also, we put

$$\begin{aligned} z &= X^\eta, \\ y &= X^\rho, \\ D &= X^\delta, \\ \Delta &= \Delta(X) = X^{-\theta}, \\ H &= \Delta^{-1} \log^2 X. \end{aligned} \quad (4)$$

Throughout this paper, we always denote primes by  $p$  and  $q$ .  $\varepsilon$  always denotes an arbitrary small positive constant, which may not be the same at different occurrences. As usual, we use  $\Omega(n)$ ,  $\varphi(n)$ ,  $\mu(n)$ , and  $\Lambda(n)$  to denote the number of prime factors of  $n$  counted according to multiplicity, Euler's function, Möbius' function, and Mangold's function, respectively. We denote by  $\tau_k(n)$  the number of solutions of the equation  $m_1 m_2 \dots m_k = n$  in natural variables  $m_1, \dots, m_k$ . Especially, we write  $\tau_2(n) = \tau(n)$ . Let

$(m_1, m_2, \dots, m_k)$  and  $[m_1, m_2, \dots, m_k]$  be the greatest common divisor and the least common multiple of  $m_1, m_2, \dots, m_k$ , respectively. Also, we use  $[x]$  and  $\|x\|$ , respectively, to denote the integer part of  $x$  and the distance from  $x$  to the nearest integer.  $f(x) \ll g(x)$  means that  $f(x) = O(g(x))$ ;  $f(x) \asymp g(x)$  means that  $f(x) \ll g(x) \ll f(x)$ ;  $e(x) = e^{2\pi i x}$ ;  $\mathcal{L} = \log X$ .  $\mathcal{P}_r$  always denotes an almost-prime with at most  $r$  prime factors, counted according to multiplicity.

## 3. Preliminary Lemmas

**Lemma 1.** Let  $M \leq N < N_1 \leq M_1$  and  $a_n$  be any complex numbers. Then, we have

$$\left| \sum_{N < n \leq N_1} a_n \right| \leq \int_{-\infty}^{+\infty} \mathcal{K}(\theta) \left| \sum_{M < m \leq M_1} a_m e(\theta m) \right| d\theta, \quad (5)$$

where

$$\mathcal{K}(\theta) = \min \left( M_1 - M + 1, \frac{1}{\pi|\theta|}, \frac{1}{\pi^2 \theta^2} \right), \quad (6)$$

which satisfies

$$\int_{-\infty}^{+\infty} \mathcal{K}(\theta) d\theta \leq 3 \log(2 + M_1 - M). \quad (7)$$

*Proof.* See Lemma 2.2 of [25].  $\square$

**Lemma 2.** Let  $3 \leq u < v < w < X$ , and suppose that  $w - (1/2) \in \mathbb{N}$  and that  $w \geq 4u^2$ ,  $X \geq 64w^2u$ , and  $v^3 \geq 32X$ . Assume further that  $f(n)$  is a complex-valued function. Then, the sum

$$\sum_{(X/2) < n \leq X} \Lambda(n) f(n), \quad (8)$$

can be decomposed into  $O(\log^{10} X)$  sums, each of which is either of Type I:

$$\sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1} f(m\ell), \quad (9)$$

with  $M < M_1 \leq 2M$ ,  $L < L_1 \leq 2L$ ,  $L \geq w$ ,  $a_m \ll m^\varepsilon$ ,  $ML \asymp X$ , or of Type II:

$$\sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1} b_\ell f(m\ell), \quad (10)$$

with  $M < M_1 \leq 2M$ ,  $L < L_1 \leq 2L$ ,  $u \leq L \leq v$ ,  $a_m \ll m^\varepsilon$ ,  $b_\ell \ll \ell^\varepsilon$ , and  $ML \asymp X$ .

*Proof.* See Lemma 3 of [26].  $\square$

**Lemma 3.** For  $P \geq 1$ , we have

$$\sum_{1 \leq n \leq P} e(\alpha n) \leq \min \left( P, \frac{1}{2\|\alpha\|} \right). \quad (11)$$

*Proof.* See Lemma 4 of Chapter VI of [27].  $\square$



**Lemma 4.** Suppose that  $Y_1, Y_2$ , and  $\alpha$  are real numbers with  $Y_1 \geq 1$  and  $Y_2 \geq 1$  and that  $|\alpha - (a/q)| \leq q^{-2}$  with  $(a, q) = 1$ . Then, we have

$$\sum_{n \leq Y_1} \min\left(\frac{Y_1 Y_2}{n}, \frac{1}{\|\alpha n\|}\right) \ll Y_1 Y_2 \left(\frac{1}{q} + \frac{1}{Y_2} + \frac{q}{Y_1 Y_2}\right) \log(2Y_1 q). \quad (12)$$

*Proof.* See Lemma 2.2 of [28].  $\square$

#### 4. Proof of Theorem 1

As shown in [21], we take a periodic function  $\chi(t)$  with period 1 such that

$$\begin{cases} 0 < \chi(t) < 1, & \text{if } -\Delta < t < \Delta, \\ \chi(t) = 0, & \text{if } \Delta \leq t \leq 1 - \Delta, \end{cases} \quad (13)$$

which has a Fourier series,

$$\chi(t) = \Delta + \sum_{|k| > 0} g(k) e(kt), \quad (14)$$

with coefficients satisfying

$$\begin{aligned} g(0) &= \Delta, \\ g(k) &\ll \Delta, \quad \text{for all } k, \\ \sum_{|k| > H} |g(k)| &\ll X^{-1}. \end{aligned} \quad (15)$$

The existence of such a function is a consequence of a well-known lemma of Vinogradov. For instance, one can see Chapter I, §2 in [27]. Consider the sum

$$\Gamma := \Gamma(X) = \sum_{(X/2) < p \leq X, (p+2, P(z))=1} \chi(\alpha p^2 + \beta) \mathcal{W}_p \log p, \quad (16)$$

where

$$P(z) = \prod_{2 < p \leq z} p, \quad (17)$$

$$\mathcal{W}_p = 1 - \kappa \sum_{z < q \leq yq | p+2} \left(1 - \frac{\log q}{\log y}\right). \quad (18)$$

Let  $\Gamma_1$  denote the sum of the terms of  $\Gamma(X)$  in which  $\mathcal{W}_p > 0$ . Then, we have

$$\Gamma(X) \leq \Gamma_1. \quad (19)$$

If we denote by  $\Gamma_2$  the sum of the terms of  $\Gamma_1$  in which  $\mu(p+2) = 0$ , it is easy to see that

$$\begin{aligned} 0 \leq \Gamma_2 &\ll \sum_{q \geq z} \sum_{n \leq Xn+2 \equiv 0 \pmod{q^2}} \log n \ll (\log X) \sum_{z \leq q \leq \sqrt{X+2}} \left(\frac{X}{q^2} + 1\right) \\ &\ll X^{1+\varepsilon} z^{-1} + X^{(1/2)+\varepsilon} \ll X^{1-\eta+\varepsilon}. \end{aligned} \quad (20)$$

By noting the fact that the contribution of the terms (if such terms exist) in  $\Gamma_1$ , for which  $X - 2 < p \leq X$ , is  $O(\log X)$ , we deduce that

$$\Gamma \leq \Gamma_3 + O(X^{1-\eta+\varepsilon}), \quad (21)$$

where

$$\Gamma_3 = \sum_{\substack{(X/2) < p \leq X-2 \\ (p+2, P(z))=1}} \chi(\alpha p^2 + \beta) \mathcal{W}_p \log p. \quad (22)$$

On the one hand, if we assume that

$$\Gamma(X) \gg \frac{\Delta X}{\log X}, \quad (23)$$

then from (21), we obtain

$$\Gamma_3 \gg \frac{\Delta X}{\log X}, \quad (24)$$

and thus  $\Gamma_3 > 0$ . Hence, there exists a prime  $p$ , which satisfies

$$\frac{X}{2} < p \leq X - 2, \quad \mathcal{W}_p > 0, \quad (25)$$

$$\mu^2(p+2) = 1, \quad (p+2, P(z)) = 1,$$

and such that

$$\chi(\alpha p^2 + \beta) > 0. \quad (26)$$

Combining (13), (25), and (26), we can see that this prime  $p$  satisfies

$$\|\alpha p^2 + \beta\| \ll p^{-\theta}. \quad (27)$$

On the other hand, by the properties of the weights  $\mathcal{W}_p$  (for example, one can see Chapter 9 of [29]), it is easy to see that if  $p$  satisfies (25), then

$$\begin{aligned} \Omega(p+2) &= \sum_{q > z, q | p+2} 1 < \frac{1}{\kappa} + \sum_{q > z, q | p+2} \frac{\log q}{\log y} = \frac{1}{\kappa} + \frac{\log(p+2)}{\log y} \\ &\leq \frac{1}{\kappa} + \frac{1}{\rho} < 5, \end{aligned} \quad (28)$$

which implies  $p+2 = \mathcal{P}_4$ . Therefore, in order to prove Theorem 1, it is sufficient to show that there exists a sequence  $\{X_j\}_{j=1}^{\infty}$ , which satisfies

$$\lim_{j \rightarrow \infty} X_j = +\infty, \quad (29)$$

$$\Gamma(X_j) \gg \frac{\Delta(X_j) X_j}{\log X_j}, \quad j = 1, 2, 3, \dots$$

By (16) and (18), we can write  $\Gamma$  as follows:

$$\Gamma = \Psi - \kappa \Phi, \quad (30)$$



where

$$\Psi = \sum_{(X/2) < p \leq X, P(p)=1} \chi(\alpha p^2 + \beta) \log p, \quad (31)$$

$$\Phi = \sum_{(X/2) < p \leq X, P(p)=1} \chi(\alpha p^2 + \beta) (\log p) \sum_{z < q \leq yq, p+2} \left(1 - \frac{\log q}{\log y}\right). \quad (32)$$

Next, we shall give lower bound estimate of  $\Psi$  and upper bound estimate of  $\Phi$  by using lower bound linear sieve and upper bound linear sieve, respectively. First, we consider  $\Psi$ . Let  $\lambda^-(d)$  be the lower bounds for Rosser's weights of level  $D$ . Hence, for any positive integer  $d$ , there holds

$$\begin{aligned} |\lambda^-(d)| &\leq 1, \\ \lambda^-(d) &= 0, \quad \text{if } d > D \text{ or } \mu(d) = 0, \end{aligned} \quad (33)$$

$$\sum_{d|n} \lambda^-(d) \leq \sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \in \mathbb{N}, n > 1. \end{cases} \quad (34)$$

Also, we shall use the fact if  $2 < s < 4$ , then there holds

$$\sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} \geq \Pi(z) \left( \frac{2e^{\gamma} \log(s-1)}{s} + O((\log X)^{(-1/3)}) \right), \quad (35)$$

where

$$\Pi(z) = \prod_{2 < p \leq z} \left(1 - \frac{1}{p-1}\right). \quad (36)$$

Now, we take

$$s = \frac{\log D}{\log z} = \frac{\delta}{\eta} = \frac{76927}{19232} \in (2, 4), \quad (37)$$

in (35). By (31) and (34), we obtain

$$\begin{aligned} \Psi &= \sum_{(X/2) < p \leq X} \chi(\alpha p^2 + \beta) (\log p) \sum_{d|(p+2, P(z))} \mu(d) \geq \sum_{(X/2) < p \leq X} \chi(\alpha p^2 + \beta) (\log p) \sum_{d|(p+2, P(z))} \lambda^-(d) \\ &= \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} \chi(\alpha p^2 + \beta) \log p = \Psi_1, \text{ say.} \end{aligned} \quad (38)$$

From (32), we have

$$\begin{aligned} \Psi_1 &= \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} \left( \Delta + \sum_{|k| > 0} g(k) e(\alpha p^2 k + \beta k) \right) \log p \\ &= \Delta \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} \log p + \sum_{d|P(z)} \lambda^-(d) \sum_{|k| > 0} g(k) e(\beta k) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} e(\alpha p^2 k) \log p \\ &= \Delta \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} \log p + \sum_{d|P(z)} \lambda^-(d) \sum_{0 < |k| \leq H} (\Delta^{-1} g(k) e(\beta k)) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} e(\alpha p^2 k) \log p \\ &\quad + \sum_{d|P(z)} \lambda^-(d) \sum_{|k| > H} g(k) e(\beta k) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} e(\alpha p^2 k) \log p. \end{aligned} \quad (39)$$

By (33) and the fact that  $\lambda^-(d) = 0$  for  $d > D$ , we obtain

$$\begin{aligned} &\sum_{d|P(z)} \lambda^-(d) \sum_{|k| > H} g(k) e(\beta k) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} e(\alpha p^2 k) \log p \\ &\ll \sum_{d|P(z)} |\lambda^-(d)| \sum_{|k| > H} |g(k)| \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} \log p \ll \sum_{d \leq D} \frac{1}{\varphi(d)} \ll \log D \ll \log X. \end{aligned} \quad (40)$$

Therefore, we obtain

$$\Psi_1 = \Delta(\Psi_2 + \Psi_3) + O(\log X), \quad (41)$$

where

$$\begin{aligned} \Psi_2 &= \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{d}} \log p, \\ \Psi_3 &= \sum_{d|P(z)} \lambda^-(d) \sum_{0 < |k| \leq H} c(k) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{d}} e(\alpha p^2 k) \log p, \\ c(k) &= \Delta^{-1} g(k) e(\beta k) \ll 1. \end{aligned} \quad (42)$$

For  $\Psi_2$ , by Bombieri–Vinogradov’s mean value theorem (see Chapter 28 of [30]) and (33), we derive that

$$\Psi_2 = \frac{X}{2} \sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} + O\left(\frac{X}{\log^2 X}\right). \quad (43)$$

It follows from Mertens’ prime number theorem (see [31]) that

$$\Pi(z) \asymp \frac{1}{\log z}. \quad (44)$$

Then, from (35), (43), and (44), we obtain

$$\Psi_2 \geq e^\gamma X \Pi(z) \frac{\log(s-1)}{s} + O\left(\frac{X}{\log^{(4/3)} X}\right), \quad (45)$$

where  $s$  is defined by (37). For  $\Psi_3$ , we shall investigate it in Section 5.

Now, we study the sum  $\Phi$ , which is defined by (32). We rewrite  $\Phi$  in the following form:

$$\Phi = \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{\substack{(X/2) < p \leq Xp+2 \equiv 0 \pmod{d} \\ (p+2, P(z))=1}} \chi(\alpha p^2 + \beta) \log p. \quad (46)$$

In order to give upper bound estimate of  $\Phi$ , we shall apply an upper bound linear sieve. Let  $\lambda_q^+(d)$  be the upper bounds for Rosser’s weights of level  $(D/q)$ . Hence, for any positive integer  $d$ , we have

$$\begin{aligned} |\lambda_q^+(d)| &\leq 1, \\ \lambda_q^+(d) &= 0, \quad \text{if } d > \frac{D}{q} \text{ or } \mu(d) = 0, \end{aligned} \quad (47)$$

$$\sum_{d|n} \lambda_q^+(d) \geq \sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \in \mathbb{N}, n > 1. \end{cases} \quad (48)$$

Also, we shall use the fact, for  $1 < s_1 < 3$ , and there holds

$$\sum_{d|P(z)} \frac{\lambda_q^+(d)}{\varphi(d)} \leq \Pi(z) \left( \frac{2e^\gamma}{s_1} + O((\log X)^{-1/3}) \right). \quad (49)$$

For prime  $q$  in the sum  $\Phi$ , we take

$$s_1 = \frac{\log(D/q)}{\log z}. \quad (50)$$

Then, it is easy to check that  $1 < s_1 < 3$ , and thus, (49) holds. By (46)–(48), we obtain

$$\begin{aligned} \Phi &= \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{d}} \chi(\alpha p^2 + \beta) (\log p) \sum_{d|(p+2, P(z))} \mu(d) \\ &\leq \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{d}} \chi(\alpha p^2 + \beta) (\log p) \sum_{d|(p+2, P(z))} \lambda_q^+(d) \\ &= \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{d|P(z)} \lambda_q^+(d) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{d}} \chi(\alpha p^2 + \beta) (\log p) \\ &= \sum_{m \leq D} \nu(m) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} \chi(\alpha p^2 + \beta) (\log p) =: \Phi_1, \end{aligned} \quad (51)$$

where

$$\nu(m) = \sum_{\substack{z < q < y \\ d|P(z) \\ m=dq}} \left(1 - \frac{\log q}{\log y}\right) \lambda_q^+(d). \quad (52)$$

If  $m \leq z$ , then  $\nu(m) = 0$ . If  $z < m \leq D$ , by (17) and (52), we know that the representation  $m = dq$  with  $z < q < y$  and  $d|P(z)$  is unique. Thus, it is easy to see that

$$|\nu(m)| \leq 1. \quad (53)$$

From (14), we obtain

$$\begin{aligned} \Phi_1 &= \sum_{m \leq D} \nu(m) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} \left( \Delta + \sum_{|k| > 0} g(k) e(\alpha p^2 k + \beta k) \right) \log p \\ &= \Delta \sum_{m \leq D} \nu(m) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} e(\alpha p^2 k) \log p + \sum_{m \leq D} \nu(m) \sum_{|k| > 0} g(k) e(\beta k) \\ &\quad \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} e(\alpha p^2 k) \log p = \Delta \sum_{m \leq D} \nu(m) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} \log p + \Delta \sum_{m \leq D} \nu(m) \\ &\quad \sum_{0 < |k| \leq H} (\Delta^{-1} g(k) e(\beta k)) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} e(\alpha p^2 k) \log p \\ &\quad + \sum_{m \leq D} \nu(m) \sum_{|k| > H} g(k) e(\beta k) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} e(\alpha p^2 k) \log p. \end{aligned} \quad (54)$$

By (15) and (53), we obtain

$$\begin{aligned} &\sum_{m \leq D} \nu(m) \sum_{|k| > H} g(k) e(\beta k) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} e(\alpha p^2 k) \log p \\ &\ll \sum_{|k| > H} |g(k)| \sum_{m \leq D} \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} \log p \ll \sum_{m \leq D} \frac{1}{\varphi(m)} \ll \log X. \end{aligned} \quad (55)$$

Thus, we derive that

$$\Phi_1 = \Delta(\Phi_2 + \Phi_3) + O(\log X), \quad (56)$$

where

$$\begin{aligned} \Phi_2 &= \sum_{m \leq D} \nu(m) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} \log p, \\ \Phi_3 &= \sum_{m \leq D} \nu(m) \sum_{0 < |k| \leq H} c(k) \sum_{(X/2) < p \leq Xp+2 \equiv 0 \pmod{m}} e(\alpha p^2 k) \log p, \\ c(k) &= \Delta^{-1} g(k) e(\beta k) \ll 1. \end{aligned} \quad (57)$$

By Bombieri–Vinogradov’s mean value theorem and (53), we have

$$\Phi_2 = \frac{X}{2} \sum_{m \leq D} \frac{\nu(m)}{\varphi(m)} + O\left(\frac{X}{\log^2 X}\right). \quad (58)$$

Using (49) and (52), we obtain

$$\begin{aligned} \sum_{m \leq D} \frac{\nu(m)}{\varphi(m)} &= \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{d|P(z)} \frac{\lambda_q^+(d)}{\varphi(qd)} = \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \sum_{d|P(z)} \frac{\lambda_q^+(d)}{\varphi(d)} \\ &\leq \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \Pi(z) \left(2e^\gamma \left(\frac{\log(D/q)}{\log z}\right)^{-1} + O((\log X)^{(-1/3)})\right). \end{aligned} \quad (59)$$

Therefore, by (44), (58), and (59), we have

$$\begin{aligned} \Phi_2 &\leq e^\gamma X \Pi(z) \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \left(\frac{\log(D/q)}{\log z}\right)^{-1} \\ &\quad + O\left(\frac{X}{(\log X)^{(4/3)-\varepsilon}}\right). \end{aligned} \quad (60)$$

Now, we find a lower bound for the sum  $\Gamma$ . From (30), (38), (41), (45), (51), (56), and (60), we derive that

$$\Gamma \geq e^\gamma \Delta X \Pi(z) \mathfrak{S} + O\left(\frac{\Delta X}{(\log X)^{(4/3)-\varepsilon}}\right) + O(\Delta |\Psi_3 - \kappa \Phi_3|), \quad (61)$$

where

$$\begin{aligned} \mathfrak{S} &= \frac{\log(s-1)}{s} - \kappa \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \left(\frac{\log(D/q)}{\log z}\right)^{-1}, \\ s &= \frac{\log D}{\log z}. \end{aligned} \quad (62)$$

Moreover, by partial summation and the prime number theorem, it is easy to show that

$$\mathfrak{S} = \mathfrak{S}_0 + O\left(\frac{1}{\log X}\right), \quad (63)$$

where

$$\mathfrak{S}_0 = \frac{\log(s-1)}{s} - \kappa \eta \int_\eta^\rho \left(\frac{1}{u} - \frac{1}{\rho}\right) \frac{1}{\delta - u} du. \quad (64)$$

According to simple numerical calculation, we know that

$$\mathfrak{S}_0 \geq 0.000032113949. \quad (65)$$

From (44), (61), and (63), we obtain

$$\Gamma \geq e^\gamma \Delta X \Pi(z) \mathfrak{S}_0 + O\left(\frac{\Delta X}{(\log X)^{(4/3)-\varepsilon}}\right) + O(\Delta |\Psi_3 - \kappa \Phi_3|). \quad (66)$$

We shall illustrate that if  $X$  runs over a suitable sequence, which tends to infinity, then the second error term in (66) can be absorbed. Hence, we need the following lemma.

**Lemma 5.** Suppose that  $\alpha \in (\mathbb{R}/\mathbb{Q})$  and  $\delta, \theta, D$ , and  $H$  are defined in (3) and (4). Let  $\xi(d)$  and  $c(k)$  be complex numbers defined for  $d \leq D$  and  $0 < |k| \leq H$ , respectively, which satisfy

$$\begin{aligned} \xi(d) &\ll 1, \\ c(k) &\ll 1. \end{aligned} \quad (67)$$

Then, there exists a sequence  $\{X_j\}_{j=1}^\infty$  satisfying  $\lim_{j \rightarrow \infty} X_j = +\infty$ , such that the sum  $S(X)$  is defined by

$$S(X) = \sum_{d \leq D} \xi(d) \sum_{1 \leq |k| \leq H} c(k) \sum_{(X/2) < p \leq X, p+2 \equiv 0 \pmod{d}} (\log p) e(\alpha p^2 k), \quad (68)$$

which satisfies

$$S(X_j) \ll \frac{X_j}{\log^2 X_j}, \quad j = 1, 2, 3, \dots \quad (69)$$

The proof of Lemma 5 will be given in Section 5. From (42) and (57), we know that  $\Psi_3 - \kappa \Phi_3$  can be represented as a sum of type (68) with

$$\xi(d) = \lambda^*(d) - \kappa \nu(d), \quad (70)$$

where

$$\lambda^*(d) = \begin{cases} \lambda^-(d), & \text{if } d|P(z), \\ 0, & \text{otherwise.} \end{cases} \quad (71)$$

According to Lemma 5 and (66), there exists a sequence  $\{X_j\}_{j=1}^\infty$ , which tends to infinity, such that

$$\Gamma(X_j) \geq e^\gamma \Delta X_j \Pi(z) \mathfrak{S}_0 + O\left(\frac{\Delta X_j}{(\log X_j)^{(4/3)-\varepsilon}}\right). \quad (72)$$

From (44) and (72), we know that there exists a positive constant  $c > 0$  such that

$$\Gamma(X_j) \geq \frac{c\Delta(X_j)X_j}{\log X_j} > 0, \quad j = 1, 2, 3, \dots \quad (73)$$

This completes the proof of Theorem 1.

## 5. Proof of Lemma 5

In this section, we shall prove Lemma 5. Since  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , by Dirichlet's approximation theorem, there exist infinitely many integers  $A$  and natural numbers  $Q$  with  $(A, Q) = 1$  such that

$$\left| \alpha - \frac{A}{Q} \right| < \frac{1}{Q^2}. \quad (74)$$

For each such  $Q$ , we choose  $X$  in a suitable way, i.e., as in (144). In this way, we construct our sequence  $\{X_j\}_{j=1}^{\infty}$ .

First, we have

$$S(X) = W + O\left(HX^{(1/2)+\varepsilon}\right), \quad (75)$$

where

$$W = \sum_{(X/2) < n \leq X} \Lambda(n) \sum_{1 \leq |k| \leq H} c(k)e(\alpha n^2 k) \sum_{d \leq Dd|m\ell+22|d} \xi(d). \quad (76)$$

According to Lemma 2, by taking  $u = 2^{-7}X^{(\delta/2)}$ ,  $v = 2^7X^{(1/3)}$ , and  $w = X^{(1/2)-(\delta/4)}$ , it is easy to see that the sum  $W$  can be decompose into  $O(\log^{10} X)$  sums, each of which is either of Type I,

$$S_I = \sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1} \sum_{(X/2) < m\ell \leq X} \sum_{1 \leq |k| \leq H} c(k)e(\alpha m^2 \ell^2 k) \sum_{d \leq Dd|m\ell+22|d} \xi(d), \quad (77)$$

with  $M_1 \leq 2M$ ,  $L_1 \leq 2L$ ,  $L \geq w$ ,  $a_m \ll m^\varepsilon$ , and  $ML \asymp X$ , or of Type II

$$S_{II} = \sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1} \sum_{(X/2) < m\ell \leq X} b_\ell \sum_{1 \leq |k| \leq H} c(k)e(\alpha m^2 \ell^2 k) \sum_{d \leq Dd|m\ell+22|d} \xi(d), \quad (78)$$

with  $M_1 \leq 2M$ ,  $L_1 \leq 2L$ ,  $u \leq L \leq v$ ,  $a_m \ll m^\varepsilon$ ,  $b_\ell \ll \ell^\varepsilon$ , and  $ML \asymp X$ .

Next, we shall deal with the sums of Type I and Type II in the following sections, respectively.

**5.1. The Estimate of Type II Sums.** In this section, we shall deal with the estimate of the sums of Type II. First, we have

$$\begin{aligned} S_{II} &= \sum_{1 \leq |k| \leq H} c(k) \sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1} \sum_{(X/2) < m\ell \leq X} b_\ell e(\alpha m^2 \ell^2 k) \sum_{d \leq Dd|m\ell+22|d} \xi(d) \\ &\ll X^\varepsilon \sum_{1 \leq |k| \leq H} \sum_{M < m \leq M_1} \left| \sum_{L < \ell \leq L_1} \sum_{(X/2) < m\ell \leq X} b_\ell e(\alpha m^2 \ell^2 k) \sum_{d \leq Dd|m\ell+22|d} \xi(d) \right|. \end{aligned} \quad (79)$$

By Cauchy's inequality, we obtain

$$\begin{aligned}
 |S_{II}|^2 &\ll X^\varepsilon HM \sum_{1 \leq |k| \leq H} \sum_{M < m \leq M_1} \left| \sum_{L < \ell \leq L_1} \sum_{(X/2) < m\ell \leq X} b_\ell e(\alpha m^2 \ell^2 k) \sum_{d \leq D} \xi(d) \right|^2 \\
 &= X^\varepsilon HM \sum_{1 \leq |k| \leq H} \sum_{M < m \leq M_1} \sum_{L < \ell \leq L_1} \sum_{(X/2) < m\ell \leq X} \sum_{\substack{d_1, d_2 \leq Dm \\ \ell_1 + 2 \equiv 0 \pmod{d_1} \\ m\ell_2 + 2 \equiv 0 \pmod{d_2} \\ (d_1, d_2) = 1}} b_{\ell_1} \overline{b_{\ell_2}} \xi(d_1) \overline{\xi(d_2)} e(\alpha m^2 k(\ell_1^2 - \ell_2^2)) \\
 &\ll X^\varepsilon HM \sum_{1 \leq |k| \leq H} \sum_{L < \ell_1, \ell_2 \leq L_1} \sum_{\substack{d_1, d_2 \leq D \\ =1(\ell_1, d_1) = (\ell_2, d_2) = 1}} \left| \sum_{\substack{M' < m \leq M'_1 \\ m\ell_1 + 2 \equiv 0 \pmod{d_1} \\ m\ell_2 + 2 \equiv 0 \pmod{d_2}}} e(\alpha m^2 k(\ell_1^2 - \ell_2^2)) \right| \\
 &\ll X^\varepsilon HM \sum_{1 \leq |k| \leq H} \sum_{L < \ell_1, \ell_2 \leq L_1} \sum_{\substack{d_1, d_2 \leq D \\ =1(\ell_1, d_1) = (\ell_2, d_2) = 1}} |\mathcal{V}|,
 \end{aligned} \tag{80}$$

where

$$\begin{aligned}
 \mathcal{V} &= \sum_{\substack{M' < m \leq M'_1 \\ m\ell_1 + 2 \equiv 0 \pmod{d_1} \\ m\ell_2 + 2 \equiv 0 \pmod{d_2}}} e(\alpha m^2 k(\ell_1^2 - \ell_2^2)), \\
 M' &= \max\left(M, \frac{X}{2\ell_1}, \frac{X}{2\ell_2}\right), \\
 M'_1 &= \min\left(M_1, \frac{X}{\ell_1}, \frac{X}{\ell_2}\right).
 \end{aligned} \tag{81}$$

If the system of the congruence,

$$\begin{cases} m\ell_1 + 2 \equiv 0 \pmod{d_1}, \\ m\ell_2 + 2 \equiv 0 \pmod{d_2}, \end{cases} \tag{82}$$

has no solution, then  $\mathcal{V} = 0$ . Assume that (82) has a solution. Then, there exists an  $f_0 = f_0(\ell_1, \ell_2, d_1, d_2)$  such that (82) is equivalent to  $m \equiv f_0 \pmod{[d_1, d_2]}$ . In this case, we have

$$\begin{aligned}
 |\mathcal{V}| &= \left| \sum_{\substack{M' < m \leq M'_1 \\ m \equiv f_0 \pmod{[d_1, d_2]}}} e(\alpha m^2 k(\ell_1^2 - \ell_2^2)) \right| = \left| \sum_{\substack{\frac{M' - f_0}{[d_1, d_2]} < r \leq \frac{M'_1 - f_0}{[d_1, d_2]}}} e(\alpha (f_0 + r[d_1, d_2])^2 k(\ell_1^2 - \ell_2^2)) \right| \\
 &= \left| \sum_{\substack{\frac{M' - f_0}{[d_1, d_2]} < r \leq \frac{M'_1 - f_0}{[d_1, d_2]}}} e(\alpha (r^2 [d_1, d_2]^2 + 2f_0 r [d_1, d_2]) k(\ell_1^2 - \ell_2^2)) \right| = \left| \sum_{R < r \leq R_1} e(\alpha (r^2 [d_1, d_2]^2 + 2f_0 r [d_1, d_2]) k(\ell_1^2 - \ell_2^2)) \right|,
 \end{aligned} \tag{83}$$

where

$$\begin{aligned}
 R &= \frac{M' - f_0}{[d_1, d_2]}, \\
 R_1 &= \frac{M'_1 - f_0}{[d_1, d_2]}.
 \end{aligned} \tag{84}$$

$$\begin{aligned}
 &\ll X^\varepsilon H M^2 \sum_{1 \leq k \leq H} \sum_{L < \ell_1, \ell_2 \leq L_1} \sum_{\ell_1 = \ell_2} \sum_{d_1, d_2 \leq D} \frac{1}{[d_1, d_2]} \\
 &\ll X^\varepsilon H^2 M^2 L \sum_{d_1, d_2 \leq D} \frac{1}{[d_1, d_2]} \ll X^\varepsilon H^2 M^2 L \sum_{h \leq D^2} \frac{\tau^2(h)}{h} \\
 &\ll X^\varepsilon H^2 M^2 L \ll X^{1+\varepsilon} H^2 M.
 \end{aligned} \tag{85}$$

The contribution of  $\mathcal{V}$  with  $\ell_1 = \ell_2$  to  $|S_{II}|^2$  is

Therefore, we have

$$|S_{II}|^2 \ll X^{1+\varepsilon} H^2 M + X^\varepsilon H M \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ (\ell_1, d_1) = (\ell_2, d_2) = 1 \\ \ell_1 \neq \ell_2}} |\mathcal{V}|. \quad (86)$$

Moreover, by Cauchy's inequality again, we obtain

$$\begin{aligned} |S_{II}|^4 &\ll X^{2+\varepsilon} H^4 M^2 + X^\varepsilon H^3 M^2 \sum_{1 \leq k \leq H} \left( \sum_{d_1, d_2 \leq D} \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ (\ell_1, d_1) = (\ell_2, d_2) = 1 \\ \ell_1 \neq \ell_2}} |\mathcal{V}| \right)^2 \\ &\ll X^{2+\varepsilon} H^4 M^2 + X^\varepsilon H^3 M^2 \sum_{1 \leq k \leq H} \left( \sum_{d_1, d_2 \leq D} \frac{1}{[d_1, d_2]} \right) \times \sum_{d_1, d_2 \leq D} [d_1, d_2] \left( \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ (\ell_1, d_1) = (\ell_2, d_2) = 1 \\ \ell_1 \neq \ell_2}} |\mathcal{V}| \right)^2 \\ &\ll X^{2+\varepsilon} H^4 M^2 + X^\varepsilon H^3 M^2 L^2 \left( \sum_{d_1, d_2 \leq D} \frac{1}{[d_1, d_2]} \right) \times \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ (\ell_1, d_1) = (\ell_2, d_2) = 1 \\ \ell_1 \neq \ell_2}} |\mathcal{V}|^2 \\ &\ll X^{2+\varepsilon} H^4 M^2 + X^\varepsilon H^3 M^2 L^2 \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ (\ell_1, d_1) = (\ell_2, d_2) = 1 \\ \ell_1 \neq \ell_2}} |\mathcal{V}|^2 \\ &\ll X^{2+\varepsilon} H^4 M^2 + X^\varepsilon H^3 M^2 L^2 \cdot \Sigma_0, \end{aligned} \quad (87)$$

where

$$\Sigma_0 = \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ (\ell_1, d_1) = (\ell_2, d_2) = 1 \\ \ell_1 \neq \ell_2}} \times \sum_{R < r_1, r_2 \leq R_1} e\left(\alpha\left((r_1^2 - r_2^2)[d_1, d_2]^2 + 2f_0(r_1 - r_2)[d_1, d_2]\right)k(\ell_1^2 - \ell_2^2)\right). \quad (88)$$

For  $\Sigma_0$ , we have

$$\begin{aligned}
 \Sigma_0 &= \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} [d_1, d_2] \\
 &\quad \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ \ell_1 \neq \ell_2}} \sum_{\substack{(\ell_1, d_1) = (\ell_2, d_2) = 1}} \times \sum_{s_1, s_2} \left( \sum_{R < r_1, r_2 \leq R_1} 1 \right) e \left( \alpha (s_1 s_2 [d_1, d_2]^2 + 2 f_0 s_1 [d_1, d_2]) k (\ell_1^2 - \ell_2^2) \right) \\
 &= \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} [d_1, d_2] \\
 &\quad \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ \ell_1 \neq \ell_2}} \sum_{\substack{(\ell_1, d_1) = (\ell_2, d_2) = 1}} \times \sum_{\substack{s_1, s_2: s_1 \equiv s_2 \pmod{2} \\ 2R_1 2R < s_2 - s_1 \leq 2R_1}} e \left( \alpha (s_1 s_2 [d_1, d_2]^2 + 2 f_0 s_1 [d_1, d_2]) k (\ell_1^2 - \ell_2^2) \right) \quad (89) \\
 &= \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ \ell_1 \neq \ell_2}} \sum_{\substack{(\ell_1, d_1) = (\ell_2, d_2) = 1}} \sum_{|s_1| \leq 2R_1 - 2R} e \left( 2\alpha f_0 s_1 [d_1, d_2] k (\ell_1^2 - \ell_2^2) \right) \\
 &\quad \times \sum_{\substack{s_2: s_2 \equiv s_1 \pmod{2} \\ 2R_1 2R < s_2 - s_1 \leq 2R_1}} e \left( \alpha s_1 s_2 [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) \right).
 \end{aligned}$$

Set

$$D_0 = X^{(50/3)\theta}. \quad (90)$$

Then, we divide  $\Sigma_0$  into two parts

$$\Sigma_0 = \Sigma_1 + \Sigma_2, \quad (91)$$

where  $\Sigma_1$  denotes the part of  $\Sigma_0$  which satisfies  $[d_1, d_2] \leq D_0$ , while  $\Sigma_2$  denotes the remaining part of  $\Sigma_0$  which satisfies  $[d_1, d_2] > D_0$ . We set  $s_2 = s_1 + 2t$  in  $\Sigma_1$  and  $\Sigma_2$  and derive that

$$\Sigma_1 \leq \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D_0} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ \ell_1 \neq \ell_2}} \sum_{\substack{(\ell_1, d_1) = (\ell_2, d_2) = 1}} \times \sum_{|s_1| \leq 2R_1 - 2R} \left| \sum_{R' < t \leq R'_1} e \left( 2\alpha s_1 t [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) \right) \right|, \quad (92)$$

$$\Sigma_2 \leq \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ \ell_1 \neq \ell_2}} \sum_{\substack{(\ell_1, d_1) = (\ell_2, d_2) = 1}} \times \sum_{|s_1| \leq 2R_1 - 2R} \left| \sum_{R' < t \leq R'_1} e \left( 2\alpha s_1 t [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) \right) \right|, \quad (93)$$

where

$$\begin{aligned}
 R' &= \max(R - s_1, R), \\
 R'_1 &= \min(R_1 - s_1, R_1).
 \end{aligned} \quad (94)$$

First, we consider the upper bound for  $\Sigma_1$ . Let  $\Sigma_1^{(1)}$  and  $\Sigma_1^{(2)}$  denote the contribution of the right-hand side of (92) for  $s_1 \neq 0$  and  $s_1 = 0$ , respectively. Trivially, there holds



$$\Sigma_1^{(2)} \ll \text{HML}^2 \sum_{d_1, d_2 \leq D_0, (d_1, d_2, 2)=1} 1 \ll \text{HML}^2 D_0^2 \ll D_0^2 \text{HXL}. \quad (95)$$

For  $\Sigma_1^{(1)}$ , by Lemma 3, we have

$$\begin{aligned} \Sigma_1^{(1)} &\ll \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D_0} [d_1, d_2] \sum_{L < \ell_1, \ell_2 \leq L_1, \ell_1 \neq \ell_2} \times \sum_{0 < |s| \leq (2M/[d_1, d_2])} \min\left(\frac{M}{[d_1, d_2]}, \frac{1}{\|2\alpha s [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2)\|}\right) \\ &\ll \sum_{1 \leq k \leq H} \sum_{h \leq D_0^2} h \left( \sum_{d_1, d_2 \leq D_0, [d_1, d_2]=h} 1 \right) \sum_{L < \ell_1, \ell_2 \leq L_1, \ell_1 \neq \ell_2} \sum_{0 < |s| \leq (2M/h)} \min\left(\frac{M}{h}, \frac{1}{\|2\alpha s h^2 k (\ell_1^2 - \ell_2^2)\|}\right) \\ &\ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \leq D_0^2} \sum_{L < \ell_1, \ell_2 \leq L_1, \ell_1 \neq \ell_2} \sum_{0 < |s| \leq 2M} \min\left(M, \frac{1}{\|2\alpha s h^2 k (\ell_1^2 - \ell_2^2)\|}\right) \\ &\ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \leq D_0^2} \sum_{t_1, t_2} \left( \sum_{L < \ell_1, \ell_2 \leq L_1, \ell_1 - \ell_2 = t_1 \ell_1 + \ell_2 = t_2, \ell_1 \neq \ell_2} 1 \right) \sum_{0 < |s| \leq 2M} \min\left(M, \frac{1}{\|2\alpha s h^2 k t_1 t_2\|}\right) \\ &\ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \leq D_0^2} \sum_{1 \leq |t_1| \leq L_1, 1 \leq |t_2| \leq 4L} \sum_{0 < |s| \leq 2M} \min\left(M, \frac{1}{\|2\alpha s h^2 k t_1 t_2\|}\right) \\ &\ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \leq D_0^2} \sum_{1 \leq t_1, t_2 \leq 4L} \sum_{1 \leq |s| \leq 2M} \min\left(M, \frac{1}{\|2\alpha s h^2 k t_1 t_2\|}\right) \\ &\ll D_0^2 \sum_{1 \leq m \leq 64 D_0^4 \text{HML}^2} \tau_7(m) \min\left(M, \frac{1}{\|\alpha m\|}\right). \end{aligned} \quad (96)$$

By Lemma 4, we have

$$\begin{aligned} \sum_{1 \leq m \leq 64 D_0^4 \text{HML}^2} \tau_7(m) \min\left(M, \frac{1}{\|\alpha m\|}\right) &\ll X^\epsilon \sum_{1 \leq m \leq 64 D_0^4 \text{HML}^2} \min\left(\frac{64 D_0^4 H M^2 L^2}{m}, \frac{1}{\|\alpha m\|}\right) \\ &\ll X^\epsilon D_0^4 H M^2 L^2 \left( \frac{1}{Q} + \frac{1}{M} + \frac{Q}{D_0^4 H M^2 L^2} \right) \ll X^\epsilon \left( \frac{H X^2 D_0^4}{Q} + H X L D_0^4 + Q \right). \end{aligned} \quad (97)$$

Combining (92), (93), (96), and (97) and by noting the fact that  $ML \asymp X$ , we obtain

$$\Sigma_1 \ll X^\epsilon (H X^2 D_0^6 Q^{-1} + H X L D_0^6 + Q D_0^2). \quad (98)$$

Now, we consider the estimate of  $\Sigma_2$ . According to (93), by a splitting argument, we have

$$\Sigma_2 \ll \mathcal{L} \max_{D_0 \ll T \ll D^2} (T \Sigma_2^{(1)}), \quad (99)$$

where

$$\Sigma_2^{(1)} = \Sigma_2^{(1)}(T) = \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D, (d_1, d_2, 2)=1, T < [d_1, d_2]} \ll 2T \sum_{L < \ell_1, \ell_2 \leq L_1, (\ell_1, d_1) = (\ell_2, d_2) = 1, \ell_1 \neq \ell_2} \times \sum_{|s_1| \leq 2R_1 - 2R} \left| \sum_{R' < t \leq R'_1} e(2\alpha s_1 t [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2)) \right|. \quad (100)$$

By Lemma 1, we have

$$\begin{aligned}
 \Sigma_2^{(1)} &= \Sigma_2^{(1)}(T) \leq \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq DT < [d_1, d_2]} \\
 &\leq 2T \sum_{L < \ell_1, \ell_2 \leq L_1 \ell_1 \neq \ell_2} \sum_{|s| \leq (2M/T)} \times \int_{-\infty}^{+\infty} \mathcal{K}(\theta) \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha st [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) + \theta t) \right| d\theta \\
 &=: \int_{-\infty}^{+\infty} \mathcal{K}(\theta) \cdot \Sigma_2^{(2)}(\theta, T) d\theta,
 \end{aligned} \tag{101}$$

where

$$\begin{aligned}
 \mathcal{K}(\theta) &= \min\left(\frac{15M}{4T} + 1, \frac{1}{\pi|\theta|}, \frac{1}{\pi^2\theta^2}\right), \\
 \Sigma_2^{(2)}(\theta, T) &= \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq DT < [d_1, d_2]} \sum_{L < \ell_1, \ell_2 \leq L_1 \ell_1 \neq \ell_2} \sum_{|s| \leq (2M/T)} \times \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha st [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) + \theta t) \right|.
 \end{aligned} \tag{102}$$

According to (7) and (101), it is easy to see that

$$\Sigma_2^{(1)} \ll \mathcal{L} \max_{0 \leq \theta \leq 1} \Sigma_2^{(2)}(\theta, T). \tag{103}$$

For  $\Sigma_2^{(2)}(\theta, T)$ , we have

$$\begin{aligned}
 \Sigma_2^{(2)}(\theta, T) &= \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \left( \sum_{d_1, d_2 \leq D[d_1, d_2]=h} 1 \right) \sum_{L < \ell_1, \ell_2 \leq L_1 \ell_1 \neq \ell_2} \sum_{|s| \leq (2M/T)} \times \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha st h^2 k (\ell_1^2 - \ell_2^2) + \theta t) \right| \\
 &\ll \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \tau^2(h) \sum_{L < \ell_1, \ell_2 \leq L_1 \ell_1 \neq \ell_2} \sum_{|s| \leq (2M/T)} \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha st h^2 k (\ell_1^2 - \ell_2^2) + \theta t) \right| \\
 &= \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \tau^2(h) \sum_{t_1, t_2} \left( \sum_{L < \ell_1, \ell_2 \leq L_1 \ell_1 - \ell_2 = t_1 \ell_1 + \ell_2 = t_2 \ell_1 \neq \ell_2} 1 \right) \times \sum_{|s| \leq (2M/T)} \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha st h^2 k t_1 t_2 + \theta t) \right| \\
 &\ll \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \tau^2(h) \sum_{1 \leq |t_1|, |t_2| \leq 4L} \sum_{|s| \leq (2M/T)} \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha st h^2 k t_1 t_2 + \theta t) \right| \\
 &\ll \mathcal{L}^3 H M L^2 + \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \tau^2(h) \sum_{1 \leq |t_1|, |t_2|} \leq 4L \sum_{1 \leq |s|} \leq \frac{2M}{T} \times \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha st h^2 k t_1 t_2 + \theta t) \right| \\
 &\ll \mathcal{L}^3 H M L^2 + \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \tau^2(h) \sum_{1 \leq |m| \leq (32M L^2/T)} \tau_3(|m|) \times \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha t h^2 k m + \theta t) \right| \\
 &= \mathcal{L}^3 H M L^2 + \Sigma_2^{(3)}, \text{ say.}
 \end{aligned} \tag{104}$$

It follows from Cauchy's inequality that

$$\begin{aligned}
 (\Sigma_2^{(3)})^2 &\ll H \sum_{1 \leq k \leq H} \left( \sum_{T < h \leq 2T} \tau^2(h) \sum_{1 \leq |m| \leq (32ML^2/T)} \tau_3(|m|) \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha h^2 km + \theta t) \right| \right)^2 \\
 &\ll H \sum_{1 \leq k \leq H} \left( \sum_{T < h \leq 2T} \tau^4(h) \right) \sum_{T < h \leq 2T} \left( \sum_{1 \leq |m| \leq (32ML^2/T)} \tau_3(|m|) \times \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha h^2 km + \theta t) \right| \right)^2 \\
 &\ll H \left( \sum_{T < h \leq 2T} \tau^4(h) \right) \left( \sum_{1 \leq |m| \leq (32ML^2/T)} \tau_3^2(|m|) \right) \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \times \sum_{1 \leq |m| \leq (32ML^2/T)} \left| \sum_{(M/4T) < t \leq (4M/T)} e(2\alpha h^2 km + \theta t) \right|^2 \quad (105) \\
 &\ll \mathcal{L}^{23} H M L^2 \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \sum_{1 \leq |m| \leq (32ML^2/T)} \left| \sum_{(M/4T) < t \leq (4M/T)} (2\alpha h^2 km + \theta t) e \right|^2 \\
 &= \mathcal{L}^{23} H M L^2 \cdot \Sigma_2^{(4)}, \text{ say.}
 \end{aligned}$$

For  $\Sigma_2^{(4)}$ , we have

$$\begin{aligned}
 \Sigma_2^{(4)} &= \sum_{1 \leq k \leq H} \sum_{T < h \leq 2T} \sum_{1 \leq |m| \leq (32ML^2/T)} \sum_{(M/4T) < t_1, t_2 \leq (4M/T)} e((2\alpha h^2 km + \theta)(t_1 - t_2)) \\
 &\ll \sum_{1 \leq k \leq H} \sum_{1 \leq |m| \leq (32ML^2/T)} \sum_{(M/4T) < t_1, t_2 \leq (4M/T)} \left| \sum_{T < h \leq 2T} e((2\alpha h^2 km)(t_1 - t_2)) \right| \\
 &\ll \frac{H M^2 L^2}{T} + \frac{M}{T} \sum_{1 \leq k \leq H} \sum_{1 \leq |m| \leq (32ML^2/T)} \sum_{1 \leq |n| \leq (4M/T)} \left| \sum_{T < h \leq 2T} e(2\alpha h^2 kmn) \right| \quad (106) \\
 &\ll \frac{H M^2 L^2}{T} + \frac{M}{T} \sum_{1 \leq k \leq H} \sum_{1 \leq |s| \leq (256M^2 L^2/T^2)} \tau_3(|s|) \left| \sum_{T < h \leq 2T} e(\alpha h^2 ks) \right| \\
 &= \frac{H M^2 L^2}{T} + \frac{M}{T} \cdot \Sigma_2^{(5)}, \text{ say.}
 \end{aligned}$$

By Cauchy's inequality, we deduce that

$$\begin{aligned}
 (\Sigma_2^{(5)})^2 &\ll H \sum_{1 \leq k \leq H} \left( \sum_{1 \leq s \leq (256M^2 L^2/T^2)} \tau_3(s) \left| \sum_{T < h \leq 2T} e(\alpha h^2 ks) \right| \right)^2 \\
 &\ll H \left( \sum_{1 \leq s \leq (256M^2 L^2/T^2)} \tau_3^2(s) \right) \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq (256M^2 L^2/T^2)} \left| \sum_{T < h \leq 2T} e(\alpha h^2 ks) \right|^2 \\
 &\ll \frac{\mathcal{L}^8 H M^2 L^2}{T^2} \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq (256M^2 L^2/T^2)} \sum_{T < h_1, h_2 \leq 2T} e(\alpha ks(h_1^2 - h_2^2)) \quad (107) \\
 &\ll \frac{\mathcal{L}^8 H^2 M^4 L^4}{T^3} + \frac{\mathcal{L}^8 H M^2 L^2}{T^2} \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq (256M^2 L^2/T^2)} \sum_{T < h_1, h_2 \leq 2T, h_1 \neq h_2} e(\alpha ks(h_1^2 - h_2^2)) \\
 &= \frac{\mathcal{L}^8 H^2 M^4 L^4}{T^3} + \frac{\mathcal{L}^8 H M^2 L^2}{T^2} \cdot \Sigma_2^{(6)}, \text{ say.}
 \end{aligned}$$

For  $\Sigma_2^{(6)}$ , from Lemma 3, we have

$$\begin{aligned}\Sigma_2^{(6)} &= \sum_{1 \leq k \leq H} \sum_{t_1, t_2} \left( \sum_{T < h_1, h_2 \leq 2Th_1 - h_2 = t_1 h_1 + h_2 = t_2 h_1 \neq h_2} 1 \right) \sum_{1 \leq s \leq (256M^2 L^2 / T^2)} e(\alpha k s t_1 t_2) \\ &\ll \sum_{1 \leq k \leq H} \sum_{1 \leq t_1, t_2 \leq 4T} \left| \sum_{1 \leq s \leq (256M^2 L^2 / T^2)} e(\alpha k s t_1 t_2) \right| \ll \sum_{1 \leq k \leq H} \sum_{1 \leq t_1, t_2 \leq 4T} \min \left( \frac{M^2 L^2}{T^2}, \frac{1}{\|\alpha k t_1 t_2\|} \right) \\ &\ll \sum_{1 \leq n \leq 16HT^2} \tau_3(n) \min \left( \frac{M^2 L^2}{T^2}, \frac{1}{\|\alpha n\|} \right).\end{aligned}\tag{108}$$

It follows from Lemma 4 that

$$\begin{aligned}\sum_{1 \leq n \leq 16HT^2} \tau_3(n) \min \left( \frac{M^2 L^2}{T^2}, \frac{1}{\|\alpha n\|} \right) &\ll X^\varepsilon \sum_{1 \leq n \leq 16HT^2} \min \left( \frac{16HM^2 L^2}{n}, \frac{1}{\|\alpha n\|} \right) \\ &\ll X^\varepsilon HM^2 L^2 \left( \frac{1}{Q} + \frac{T^2}{M^2 L^2} + \frac{Q}{HM^2 L^2} \right) \ll X^\varepsilon (HX^2 Q^{-1} + HT^2 + Q).\end{aligned}\tag{109}$$

From (107), (108), and (109), we obtain

Putting (110) into (106), we obtain

$$\Sigma_2^{(5)} \ll X^\varepsilon \left( \frac{HX^2}{T^{(3/2)}} + \frac{HX^2}{TQ^{(1/2)}} + HX + \frac{H^{(1/2)}XQ^{(1/2)}}{T} \right).\tag{110}$$

$$\Sigma_2^{(4)} \ll X^\varepsilon \left( \frac{HX^2}{T} + \frac{HX^2 M}{T^{(5/2)}} + \frac{HX^2 M}{T^2 Q^{1/2}} + \frac{HXM}{T} + \frac{H^{(1/2)}XQ^{(1/2)}M}{T^2} \right).\tag{111}$$

Combining (105) and (111), one has

$$\Sigma_2^{(3)} \ll X^\varepsilon \left( \frac{HX^{(3/2)}L^{(1/2)}}{T^{(1/2)}} + \frac{HX^2}{T^{(5/4)}} + \frac{H^{(1/2)}X^2}{TQ^{(1/4)}} + \frac{H^{(3/4)}X^{(3/2)}Q^{(1/4)}}{T} \right).\tag{112}$$

Inserting (112) into (104), we derive that

$$\Sigma_2^{(2)}(\theta, T) \ll X^\varepsilon \left( HXL + \frac{HX^{(3/2)}L^{(1/2)}}{T^{(1/2)}} + \frac{HX^2}{T^{(5/4)}} + \frac{H^{(1/2)}X^2}{TQ^{1/4}} + \frac{H^{(3/4)}X^{(3/2)}Q^{(1/4)}}{T} \right),\tag{113}$$

which combines (99) and (103) to obtain

$$\begin{aligned} \Sigma_2 &\ll X^\varepsilon \max_{D_0 \ll T \ll D^2} \left( HXL T + HX^{(3/2)} L^{(1/2)} T^{(1/2)} + HX^2 T^{-(1/4)} + H^{(1/2)} X^2 Q^{-(1/4)} + H^{(3/4)} X^{(3/2)} Q^{(1/4)} \right) \\ &\ll X^\varepsilon \left( HXLD^2 + HX^{(3/2)} L^{(1/2)} D + HX^2 D_0^{-(1/4)} + H^{(1/2)} X^2 Q^{-(1/4)} + H^{(3/4)} X^{(3/2)} Q^{(1/4)} \right). \end{aligned} \quad (114)$$

From (91), (98), and (114), we obtain

$$\Sigma_0 \ll X^\varepsilon \left( D_0^6 HX^2 Q^{-1} + D_0^6 HXL + D_0^2 Q + HXLD^2 + HX^{(3/2)} L^{(1/2)} D + HX^2 D_0^{-(1/4)} + H^{(1/2)} X^2 Q^{-(1/4)} + H^{(3/4)} X^{(3/2)} Q^{(1/4)} \right), \quad (115)$$

which combines (87) yields

$$\begin{aligned} S_{II} &\ll X^\varepsilon \left( \begin{aligned} &HXL^{-(1/2)} + D_0^{(3/2)} HXQ^{-(1/4)} + D_0^{(3/2)} HX^{(3/4)} L^{(1/4)} + D_0^{(1/2)} Q^{(1/4)} H^{(3/4)} X^{(1/2)} + \\ &HX^{(3/4)} L^{(1/4)} D^{(1/2)} + HX^{(7/8)} L^{(1/8)} D^{(1/4)} + HXD_0^{-(1/16)} + H^{(7/8)} XQ^{-(1/16)} + H^{(15/16)} X^{(7/8)} Q^{(1/16)} \end{aligned} \right) \\ &\ll X^\varepsilon \left( \begin{aligned} &HXu^{-(1/2)} + D_0^{(3/2)} HXQ^{-(1/4)} + D_0^{(3/2)} HX^{(3/4)} v^{(1/4)} + D_0^{(1/2)} Q^{(1/4)} H^{(3/4)} X^{(1/2)} + HX^{(3/4)} v^{(1/4)} D^{(1/2)} \\ &+ HX^{(7/8)} v^{(1/8)} D^{(1/4)} + HXD_0^{-(1/16)} + H^{(7/8)} XQ^{-(1/16)} + H^{(15/16)} X^{(7/8)} Q^{(1/16)} \end{aligned} \right). \end{aligned} \quad (116)$$

**5.2. The Estimate of Type I Sums.** In this section, we shall deal with the estimate of the sums of Type I. First, we have

$$S_I = \sum_{1 \leq |k| \leq H} c(k) \sum_{d \leq D(d,2)=1} \xi(d) \sum_{M < m \leq M_1} a_m \sum_{L' < \ell \leq L'_1 m \ell + 2 \equiv 0 \pmod{d}} e(\alpha m^2 \ell^2 k), \quad (117)$$

where

$$\begin{aligned} L' &= \max\left(L, \frac{X}{2m}\right), \\ L'_1 &= \min\left(L_1, \frac{X}{m}\right). \end{aligned} \quad (118) \quad \text{where} \quad S_I \ll X^\varepsilon \cdot \max_{1 \leq T \leq D} \Sigma_3, \quad (119)$$

$$\Sigma_3 = \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \left| \sum_{L' < \ell \leq L'_1 m \ell + 2 \equiv 0 \pmod{d}} e(\alpha m^2 \ell^2 k) \right|. \quad (120)$$

For  $(m, d) = 1$ , there exists  $\bar{m}$ , which satisfies  $0 \leq \bar{m} \leq d-1$ , such that  $m\bar{m} \equiv 1 \pmod{d}$ . Therefore, the equation  $m\ell + 2 \equiv 0 \pmod{d}$  is equivalent to  $\ell \equiv -2\bar{m} \pmod{d}$ , i.e.,  $\ell = -2\bar{m} + dr$  for some  $r \in \mathbb{Z}$ . Then, it follows from Cauchy's inequality that

$$\begin{aligned}
(\Sigma_3)^2 &\ll HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \left| \sum_{L' < \ell \leq L'_1 m \ell + 2 \equiv 0 \pmod{d}} e(\alpha m^2 \ell^2 k) \right|^2 \\
&\ll HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)} \left| \sum_{(L' + 2\overline{m}/d) < r \leq (L'_1 + 2\overline{m}/d)} e(\alpha m^2 (-2\overline{m} + dr)^2 k) \right|^2 \\
&= HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \left| \sum_{(L' + 2\overline{m}/d) < r \leq (L'_1 + 2\overline{m}/d)} e(\alpha m^2 (d^2 r^2 - 4\overline{m}dr)k) \right|^2 \\
&= HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} = 1 \sum_{M < m \leq M_1(m,d)=1} = 1 \times \sum_{(L' + 2\overline{m}/d) < r \leq (L'_1 + 2\overline{m}/d)} e(\alpha m^2 (d^2 (r_1^2 - r_2^2) - 4\overline{m}d(r_1 - r_2))k).
\end{aligned} \tag{121}$$

Set

Then, we have

$$\begin{aligned}
R &= \frac{L' + 2\overline{m}}{d}, \\
R_1 &= \frac{L'_1 + 2\overline{m}}{d}.
\end{aligned} \tag{122}$$

$$\begin{aligned}
(\Sigma_3)^2 &\ll X^\varepsilon H^2 M^2 LT + HMT \left| \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \times \sum_{R < r_1, r_2 \leq R_1, r_1 \neq r_2} e(\alpha m^2 (d^2 (r_1^2 - r_2^2) - 4\overline{m}d(r_1 - r_2))k) \right| \\
&\ll X^\varepsilon H^2 M^2 LT + HMT \cdot |\Sigma_3^{(1)}|,
\end{aligned} \tag{123}$$

where

$$\Sigma_3^{(1)} = \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \sum_{R < r_1, r_2 \leq R_1, r_1 \neq r_2} e(\alpha m^2 (d^2 (r_1^2 - r_2^2) - 4\overline{m}d(r_1 - r_2))k). \tag{124}$$

For  $\Sigma_3^{(1)}$ , we have

$$\begin{aligned}
\Sigma_3^{(1)} &= \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \sum_{s_1, s_2} \left( \sum_{R < r_1, r_2 \leq R_1, r_1 - r_2 = s_1 r_1 + r_2 = s_2 r_1 \neq r_2} 1 \right) e(\alpha m^2 (d^2 s_1 s_2 - 4\overline{m}d s_1)k) \\
&= \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \sum_{s_1, s_2} \sum_{2R < s_1 + s_2 \leq 2R_1, 2R < s_2 - s_1 \leq 2R_1, s_1 \equiv s_2 \pmod{2}, s_1 \neq 0} e(\alpha m^2 (d^2 s_1 s_2 - 4\overline{m}d s_1)k) \\
&\ll \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \sum_{1 \leq |s_1| \leq (4L/T)} \left| \sum_{s_2: s_2 \equiv s_1 \pmod{2}, 2R - s_1 < s_2 \leq 2R_1 - s_1, 2R + s_1 < s_2 \leq 2R_1 + s_1} e(\alpha m^2 d^2 s_1 s_2 k) \right| \\
&\ll \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{(d,2)=1} \sum_{M < m \leq M_1(m,d)=1} \sum_{1 \leq |s_1| \leq (4L/T)} \left| \sum_{R - s_1 < t \leq R_1 - s_1, R < t \leq R_1} e(2\alpha m^2 d^2 s_1 t k) \right|.
\end{aligned} \tag{125}$$

Next, we will discuss the estimate of the right-hand side of (125) in two cases.

*Case 1.* Suppose that  $MT \leq D_0$ , and under this condition, there holds  $1 \ll M, T \ll D_0$ . By Lemma 4, we have

$$\begin{aligned}
 \Sigma_3^{(1)} &\ll \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1} \sum_{1 \leq s \leq 8L} \min \left( L, \frac{1}{\|2\alpha m^2 d^2 sk\|} \right) \\
 &\ll \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1} \sum_{1 \leq s \leq 8L} \min \left( \frac{256HM^2T^2L^2}{2m^2d^2sk}, \frac{1}{\|2\alpha m^2 d^2 sk\|} \right) \\
 &\ll \sum_{1 \leq n \leq 256HM^2T^2L} \tau_7(n) \min \left( \frac{HM^2T^2L^2}{n}, \frac{1}{\|an\|} \right) \ll X^\varepsilon HX^2T^2 \left( \frac{1}{Q} + \frac{1}{L} + \frac{Q}{HX^2T^2} \right) \\
 &\ll X^\varepsilon \left( \frac{HX^2D_0^2}{Q} + HX(MT)T + Q \right) \ll X^\varepsilon \left( \frac{HX^2D_0^2}{Q} + HXD_0^2 + Q \right).
 \end{aligned} \tag{126}$$

From (119), (123), and (126), we derive that, under the condition  $MT \leq D_0$ , there holds

$$\begin{aligned}
 S_I &\ll X^\varepsilon \left( HX^{(1/2)}D_0^{(1/2)} + HXD_0^{(3/2)}Q^{-(1/2)} + HX^{(1/2)}D_0^{(3/2)} \right. \\
 &\quad \left. + H^{(1/2)}D_0^{(1/2)}Q^{(1/2)} \right).
 \end{aligned} \tag{127}$$

*Case 2.* Now, we suppose that  $MT > D_0$ . Set

$$\begin{aligned}
 R' &= \max(R, R - s_1), \\
 R'_1 &= \min(R_1, R_1 - s_1).
 \end{aligned} \tag{128}$$

Applying Lemma 1 to (125), we have

$$\begin{aligned}
 \Sigma_3^{(1)} &\ll \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1} \sum_{1 \leq |s_1| \leq (4L/T)} \left| \sum_{R' < t \leq R'_1} e(2\alpha m^2 d^2 s_1 tk) \right| \\
 &\ll \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1} \sum_{1 \leq |s_1| \leq (4L/T)} \int_{-\infty}^{+\infty} \mathcal{K}(\theta)_1 \left| \sum_{(L/4T) < t \leq (4L/T)} e(2\alpha m^2 d^2 s_1 tk + \theta t) \right| d\theta \\
 &= \int_{-\infty}^{+\infty} \mathcal{K}_1(\theta) \cdot \Sigma_3^{(2)}(\theta, T) d\theta,
 \end{aligned} \tag{129}$$

where

$$\begin{aligned}
 \mathcal{K}_1(\theta) &= \min \left( \frac{15L}{4T} + 1, \frac{1}{\pi|\theta|}, \frac{1}{\pi^2\theta^2} \right), \\
 \Sigma_3^{(2)}(\theta, T) &= \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1} \sum_{1 \leq |s_1| \leq (4L/T)} \left| \sum_{(L/4T) < t \leq (4L/T)} e(2\alpha m^2 d^2 s_1 tk + \theta t) \right|.
 \end{aligned} \tag{130}$$

According to (7) and (129), it is easy to see that

$$\Sigma_3^{(1)} \ll \mathcal{L} \cdot \max_{0 \leq \theta \leq 1} \Sigma_3^{(2)}(\theta, T). \quad (131)$$

For  $\Sigma_3^{(2)}(\theta, T)$ , we have

$$\Sigma_3^{(2)}(\theta, T) \ll \sum_{1 \leq k \leq H} \sum_{MT < h \leq 4MT} \tau(h) \sum_{1 \leq |s_1| \leq (4L/T)} \left| \sum_{(L/4T) < t \leq (4L/T)} e(2\alpha h^2 stk + \theta t) \right|. \quad (132)$$

It follows from Cauchy's inequality that

$$\begin{aligned} (\Sigma_3^{(2)}(\theta, T))^2 &\ll H \left( \sum_{MT < h \leq 4MT} \tau^2(h) \right) \left( \sum_{1 \leq |s| \leq (4L/T)} 1 \right) \sum_{1 \leq k \leq H} \sum_{MT < h \leq 4MT} \times \sum_{1 \leq |s| \leq (4L/T)} \left| \sum_{(L/4T) < t_1, t_2 \leq (4L/T)} e(2\alpha h^2 stk + \theta t) \right|^2 \\ &\ll X^\varepsilon HML \sum_{1 \leq k \leq H} \sum_{MT < h \leq 4MT} \sum_{1 \leq |s| \leq (4L/T)} \sum_{(L/4T) < t_1, t_2 \leq (4L/T)} e((2\alpha h^2 sk + \theta)(t_1 - t_2)) \\ &\ll X^\varepsilon HML \sum_{1 \leq k \leq H} \sum_{1 \leq |s| \leq (4L/T)} \sum_{(L/4T) < t_1, t_2 \leq (4L/T)} \left| \sum_{MT < h \leq 4MT} e(2\alpha h^2 sk(t_1 - t_2)) \right| \\ &\ll \frac{X^\varepsilon H^2 M^2 L^3}{T} + X^\varepsilon HML \cdot \Sigma_3^{(3)}, \end{aligned} \quad (133)$$

where

$$\Sigma_3^{(3)} = \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq (4L/T)} \sum_{(L/4T) < t_1, t_2 \leq (4L/T)} \left| \sum_{MT < h \leq 4MT} e(2\alpha h^2 sk(t_1 - t_2)) \right|. \quad (134)$$

For  $\Sigma_3^{(3)}$ , we have

$$\begin{aligned} \Sigma_3^{(3)} &= \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq (4L/T)} \sum_{1 \leq |r_1| \leq (4L/T)} \left( \sum_{(4L/T) < t_1, t_2 \leq (4L/T)} 1 \right) \sum_{t_1 - t_2 = r_1 t_1 + t_2 = r_2} \left| \sum_{MT < h \leq 4MT} e(2\alpha h^2 skr_1) \right| \\ &\ll \frac{L}{T} \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq (4L/T)} \sum_{1 \leq s \leq (4L/T)} \left| \sum_{MT < h \leq 4MT} e(2\alpha h^2 skr_1) \right| \\ &\ll \frac{L}{T} \sum_{1 \leq n \leq (32HL^2/T^2)} \tau_4(n) \left| \sum_{MT < h \leq 4MT} e(\alpha h^2 n) \right|. \end{aligned} \quad (135)$$



Therefore, by Cauchy's inequality, one has

$$\begin{aligned} (\Sigma_3^{(3)})^2 &\ll \frac{L^2}{T^2} \left( \sum_{1 \leq n \leq (32HL^2/T^2)} \tau_4^2(n) \right) \left( \sum_{1 \leq n \leq (32HL^2/T^2)} \left| \sum_{MT < h \leq 4MT} e(\alpha h^2 n) \right|^2 \right) \\ &\ll \frac{X^\varepsilon HL^4}{T^4} \sum_{1 \leq n \leq (32HL^2/T^2)} \sum_{MT < h_1, h_2 \leq 4MT} e(\alpha(h_1^2 - h_2^2)n) \ll \frac{X^\varepsilon H^2 ML^6}{T^5} + \frac{X^\varepsilon HL^4}{T^4} \cdot \Sigma_3^{(4)}, \end{aligned} \quad (136)$$

where

For  $\Sigma_3^{(4)}$ , by Lemma 3, we have

$$\Sigma_3^{(4)} = \sum_{MT < h_1, h_2 \leq 4MT, h_1 \neq h_2} \left| \sum_{1 \leq n \leq (32HL^2/T^2)} e(\alpha(h_1^2 - h_2^2)n) \right|. \quad (137)$$

$$\begin{aligned} \Sigma_3^{(4)} &= \sum_{1 \leq t_1, |t_2| \leq 8MT} \left( \sum_{MT < h_1, h_2 \leq 4MT, h_1 - h_2 = t_1, h_1 + h_2 = t_2} 1 \right) \left| \sum_{1 \leq n \leq (32HL^2/T^2)} e(\alpha t_1 t_2 n) \right| \\ &\ll \sum_{1 \leq t_1, t_2 \leq 8MT} \left| \sum_{1 \leq n \leq (32HL^2/T^2)} e(\alpha t_1 t_2 n) \right| \ll \sum_{1 \leq t_1, t_2 \leq 8MT} \min\left(\frac{HL^2}{T^2}, \frac{1}{\|\alpha t_1 t_2\|}\right) \\ &\ll \sum_{1 \leq t \leq 64M^2 T^2} \tau(t) \min\left(\frac{HL^2}{T^2}, \frac{1}{\|\alpha t\|}\right). \end{aligned} \quad (138)$$

It follows from Lemma 4 that

$$\begin{aligned} \sum_{1 \leq t \leq 64M^2 T^2} \tau(t) \min\left(\frac{HL^2}{T^2}, \frac{1}{\|\alpha t\|}\right) &\ll X^\varepsilon \sum_{1 \leq t \leq 64M^2 T^2} \min\left(\frac{64HM^2 L^2}{t}, \frac{1}{\|\alpha t\|}\right) \\ &\ll X^\varepsilon HM^2 L^2 \left( \frac{1}{Q} + \frac{T^2}{HL^2} + \frac{Q}{HM^2 L^2} \right) \ll X^\varepsilon \left( \frac{HX^2}{Q} + M^2 T^2 + Q \right). \end{aligned} \quad (139)$$

From (136), (138), and (139), we derive that

$$\Sigma_3^{(3)} \ll X^\varepsilon \left( \frac{HX^{(1/2)} L^{(5/2)}}{T^{(5/2)}} + \frac{HXL^2}{T^2 Q^{(1/2)}} + \frac{H^{(1/2)} XL}{T} + \frac{H^{(1/2)} L^2 Q^{(1/2)}}{T^2} \right), \quad (140)$$

which combines (133) yields

$$\Sigma_3^{(2)}(\theta, T) \ll X^\varepsilon \left( \frac{HXL^{(1/2)}}{T^{(1/2)}} + \frac{HX^{(3/4)} L^{(5/4)}}{T^{(5/4)}} + \frac{HXL}{TQ^{(1/4)}} + \frac{H^{(3/4)} X^{(1/2)} LQ^{(1/4)}}{T} \right). \quad (141)$$

From (123), (131), and (141), we obtain

$$\Sigma_3 \ll X^\varepsilon \left( HXL^{-(1/2)}T^{(1/2)} + HXL^{-(1/4)}T^{(1/4)} + HX^{(7/8)}L^{(1/8)}T^{-(1/8)} + HXQ^{-(1/8)} + H^{(7/8)}Q^{(1/8)}X^{(3/4)} \right), \quad (142)$$

from which and (139), we derive that, under the condition  $MT > D_0$ , and there holds

$$\begin{aligned} S_I &\ll X^\varepsilon \max_{1 \leq T \leq D} \left( \frac{HXT^{(1/2)}}{L^{(1/2)}} + \frac{HXT^{(1/4)}}{L^{(1/4)}} + \frac{HX^{(7/8)}L^{(1/8)}}{T^{(1/8)}} + \frac{HX}{Q^{(1/8)}} + H^{(7/8)}Q^{(1/8)}X^{(3/4)} \right) \\ &\ll X^\varepsilon \left( \frac{HXD^{(1/2)}}{w^{(1/2)}} + \frac{HXD^{(1/4)}}{w^{(1/4)}} + \frac{HX^{(7/8)}L^{(1/8)}M^{(1/8)}}{(MT)^{(1/8)}} + \frac{HX}{Q^{(1/8)}} + H^{(7/8)}Q^{(1/8)}X^{(3/4)} \right) \\ &\ll X^\varepsilon \left( HXw^{-(1/2)}D^{(1/2)} + HXw^{-(1/4)}D^{(1/4)} + HXD_0^{-(1/8)} + HXQ^{-(1/8)} + H^{(7/8)}Q^{(1/8)}X^{(3/4)} \right). \end{aligned} \quad (143)$$

5.3. *Proof of Lemma 5.* From (116), (127), and (143), by taking

$$Q = X^{(4138/15)\theta}, \quad (144)$$

then we deduce that, under conditions (3) and (4), there holds

$$S_I \ll X^{1-\omega} \text{ and } S_{II} \ll X^{1-\omega}, \quad (145)$$

for some  $\omega > 0$ . This completes the proof of Lemma 5.

## Data Availability

The data used to support the findings of the study available within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# A Diophantine Problem with Unlike Powers of Primes

Quanwu Mu  and Liyan Xi

School of Science, Xi'an Polytechnic University, Xi'an, 710048, Shaanxi, China

Correspondence should be addressed to Quanwu Mu; [muquanwu@163.com](mailto:muquanwu@163.com)

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Let  $k$  be an integer with  $4 \leq k \leq 6$  and  $\eta$  be any real number. Suppose that  $\lambda_1, \lambda_2, \dots, \lambda_5$  are nonzero real numbers, not all of them have the same sign, and  $\lambda_1/\lambda_2$  is irrational. It is proved that the inequality  $|\lambda_1 p_1 + \lambda_2 p_2^k + \lambda_3 p_3^k + \lambda_4 p_4^k + \lambda_5 p_5^k + \eta| < (\max_{1 \leq j \leq 5} p_j)^{-\sigma(k)}$  has infinitely many solutions in prime variables  $p_1, p_2, p_3, p_4$ , and  $p_5$ , where  $0 < \sigma(4) < 1/36$ ,  $0 < \sigma(5) < 4/189$ , and  $0 < \sigma(6) < 1/54$ . This gives an improvement of the recent results.

## 1. Introduction

The determination of the minimal  $s$  such that the Diophantine equation

$$N = \sum_{i=1}^s x_i^{i+1}, \quad (1)$$

is solvable in positive integers  $x_1, \dots, x_s$ , for all sufficiently large integers  $N$  is an interesting problem in additive number theory. In 1951, Roth [1] proved that  $s = 50$  is acceptable. This result was subsequently improved by Thanigasalam et al. [2–4], Vaughan and Vaughan [5, 6], Brüdern and Brüdern [7, 8] and Ford and Ford [9, 10]. The best currently known result is due to Ford [10], with  $s = 14$ . Schwarz [11] suggested to analyze the related Diophantine inequality. The first result was obtained by Brüdern [12], who showed that the values of

$$\sum_{i=1}^{22} \lambda_{i+1} x_i^{i+1}, \quad (2)$$

at integer points  $(x_1, \dots, x_{22})$  are dense on the real line provided that  $\lambda_2, \dots, \lambda_{23}$  are nonzero real numbers and  $\lambda_2/\lambda_3$  is irrational. Thanks to a pruning technique, Brüdern [13] proved that the values taken by

$$\sum_{i=1}^{16} \lambda_{i+1} x_i^{i+1}, \quad (3)$$

at integer points  $(x_1, \dots, x_{16})$  are dense on the real line if  $\lambda_2, \dots, \lambda_{17}$  are nonzero real numbers and at least one of the ratios  $\lambda_i/\lambda_j$  is irrational.

Suppose that  $x_1, \dots, x_s$  are prime variables,  $N$  is a sufficiently large integer, and  $N + s$  is even. In 1969, Vaughan proved in his doctoral thesis that (1) is solvable if  $s = 31$ . Later, Vaughan [5] improved upon his own result by taking  $s = 30$  in place of  $s = 31$ . By calculating, the exponential density more accurately, Shan [14] showed that  $s = 23$  is acceptable. In addition, Prachar [15] established that each sufficiently large odd integer  $N$  can be represented as

$$N = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5, \quad (4)$$

where  $p_1, p_2, \dots, p_5$  are prime numbers. As a corollary of [16] in Theorem 1, Ren and Tsang obtained the same result as Prachar. It is of some interest to consider the analogous form for Diophantine inequalities. Let  $\lambda_1, \lambda_2, \dots, \lambda_5$  be nonzero real numbers, not all of them have the same sign and  $\lambda_1/\lambda_2$  as irrational. In 2016, Ge and Li [17] proved that, for any given real numbers  $\eta$  and  $\sigma$ ,  $0 < \sigma < 1/720$ , there exist infinitely many solutions in prime numbers  $p_j$  to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \left( \max_{1 \leq j \leq 5} p_j \right)^{-\sigma}. \quad (5)$$

Let  $k \geq 4$  be an integer. The first author [18] investigated the solvability of more general Diophantine inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^k + \eta| < \left( \max_{1 \leq j \leq 5} p_j \right)^{-\sigma(k)}, \quad (6)$$

and proved that (6) has infinitely many solutions in prime variables  $p_j$  for  $0 < \sigma(4) < 5/288$  and  $0 < \sigma(k) < 5/(6k^2(k+1))$  with  $k \geq 5$ . Subsequently, Liu [19] obtained  $0 < \sigma(5) < 5/288$ . In [20], the first author and Qu showed that  $0 < \sigma(5) < 5/252$  is acceptable. Very recently, this result was improved by Zhu [21], who obtained  $0 < \sigma(5) < 1/48$ . In [22], Gao and Liu gave an improvement ([18] in Theorem 1.2) in case  $k \geq 6$ , and they proved  $0 < \sigma(6) < 1/56$  particularly.

The main purpose of this paper is to sharpen the above results in case  $4 \leq k \leq 6$ . We obtain the following theorem.

**Theorem 1.** *Let  $k$  be an integer with  $4 \leq k \leq 6$  and  $\eta$  be any given real number. Suppose that  $\lambda_1, \lambda_2, \dots, \lambda_5$  are nonzero real numbers, not all of them are same sign, and  $\lambda_1/\lambda_2$  is irrational. Then, inequality (6) has infinitely many solutions in prime variables  $p_1, p_2, \dots, p_5$ , where  $0 < \sigma(4) < 1/36$ ,  $0 < \sigma(5) < 4/189$ , and  $0 < \sigma(6) < 1/54$ .*

The improvement derives not only from the use of the function  $\rho(m)$  constructed by Harman and Kumchev (see Section 8 in [23] and Section 5 in [24], for details) but also from some ingredients in [21]. It is worth remarking that Ge et al. [25] obtained  $0 < \sigma(5) < 1/32$ , if the condition “ $\lambda_1/\lambda_2$  is irrational” in Theorem 1 is replaced by “ $\lambda_1/\lambda_2$  is irrational and  $\lambda_2/\lambda_4$  and  $\lambda_3/\lambda_5$  are rational.”

*Notation.* Throughout the paper,  $\varepsilon$  and  $\delta$  are arbitrarily small, fixed positive real numbers. Any statement in which  $\varepsilon$  occurs holds for each positive  $\varepsilon$ . The implicit constants in  $O$ -term,  $\ll$ - and  $\gg$ -symbols depend at most on  $\lambda_1, \lambda_2, \dots, \lambda_5$  and  $\varepsilon$ . The letter  $p$ , with or without subscript, is reserved for a prime number. By  $A \asymp B$ , we mean that  $A \ll B$  and  $A \gg B$ . For simplicity, we write  $\mathcal{L} = \log X$  and  $e(\alpha) = \exp(2\pi i \alpha)$ .

## 2. Preliminaries

We apply the Davenport–Heilbronn circle method (see [26] and Chapter 11 in [27]) to prove Theorem 1. Since  $\lambda_1/\lambda_2$  is irrational, there are infinitely many convergents to its continued fraction. Let  $q$  be any denominator of a convergent to  $\lambda_1/\lambda_2$ . As in [20], let  $X$  run through the sequences:

$$X = q^{(7/3)}. \quad (7)$$

We set

$$\mathcal{J} = \left[ \frac{1}{3} X^{(1/2)}, \frac{2}{3} X^{(1/2)} \right), \quad (8)$$

$$S_2^*(\alpha) = \sum_{m \in \mathcal{J}} \rho(m) e(\alpha m^2),$$

where the function  $\rho(m)$  is defined by 5.2 in [24]. According to [24],  $\rho(m)$  is a nontrivial lower bound for the characteristic function of the set of primes in  $\mathcal{J}$ , and it satisfies

$$\rho(m) \leq \begin{cases} 1, & \text{if } m \text{ is a prime,} \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

For further properties of  $\rho(m)$ , see Lemma 1 and (4.2)–(4.4) in [24]. Let

$$I_j = \left[ (\delta X)^{(1/j)}, X^{(1/j)} \right], \quad S_j(\alpha) = \sum_{p \in I_j} (\log p) e(\alpha p^j). \quad (10)$$

By the prime number theorem, it is easy to show that  $S_j(\alpha) \ll X^{(1/j)}$ . For any fixed  $\tau > 0$ , set  $K_\tau(\alpha) = (\pi\alpha)^{-2} \sin^2(\pi\tau\alpha)$  for  $\alpha \neq 0$  and  $K_\tau(0) = \tau^2$ . Clearly, we have

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}). \quad (11)$$

A straightforward application of the Cauchy integral formula gives

$$\int_{-\infty}^{+\infty} e(x\alpha) K_\tau(\alpha) d\alpha = \max(0, \tau - |x|). \quad (12)$$

Identity (12) is also a corollary of Lemma 4 in [26]. For  $4 \leq k \leq 6$ , put

$$G(\alpha) = S_1(\lambda_1 \alpha) S_2^*(\lambda_2 \alpha) \left( \prod_{j=3}^4 S_j(\lambda_j \alpha) \right) S_k(\lambda_5 \alpha) e(\alpha \eta) K_\tau(\alpha). \quad (13)$$

We write

$$I(\tau, \eta, \mathfrak{X}) = \int_{\mathfrak{X}} G(\alpha) d\alpha, \quad (14)$$

for any measurable subset  $\mathfrak{X}$  of  $\mathbb{R}$ . It follows from (9) and (12) that

$$I(\tau, \eta, \mathbb{R}) = \sum_{\substack{p_j \in I_j, j=1,3,4, \\ p_5 \in I_k, m_2 \in \mathcal{J}}} \rho(m_2) \prod_{\substack{1 \leq j \leq 5 \\ j \neq 2}} \log p_j \\ \times \int_{-\infty}^{+\infty} e \left( \left( \lambda_1 p_1 + \lambda_2 m_2^2 + \sum_{j=3}^4 \lambda_j p_j^j + \lambda_5 p_5^k + \eta \right) \alpha \right)$$

$$\begin{aligned} & K_\tau(\alpha) d\alpha \\ &= \sum_{\substack{p_j \in I_j, j=1,3,4, \\ p_5 \in I_k, m_2 \in \mathcal{J}}} \rho(m_2) \prod_{\substack{1 \leq j \leq 5 \\ j \neq 2}} \log p_j \\ & \times \max \left( 0, \tau - \left| \lambda_1 p_1 + \lambda_2 m_2^2 + \sum_{j=3}^4 \lambda_j p_j^j + \lambda_5 p_5^k + \eta \right| \right) \\ & \leq \tau \mathcal{L}^4 \mathcal{N}(X), \end{aligned} \quad (15)$$

where  $\mathcal{N}(X)$  denotes the number of solutions of the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^k + \eta| < \tau, \quad (16)$$

with  $p_2 \in \mathcal{J}$ ,  $p_5 \in I_k$ , and  $p_j \in I_j$  for  $j \in \{1, 3, 4\}$ . In what follows, we take

$$\tau = \begin{cases} X^{-(1/36)+20\epsilon} & \text{if } k = 4, \\ X^{-(4/189)+20\epsilon} & \text{if } k = 5, \\ X^{-(1/54)+20\epsilon} & \text{if } k = 6, \end{cases} \quad (17)$$

actually. We now divide the real line into three disjoint parts:

$$\begin{aligned} \mathfrak{M} &= \{\alpha: |\alpha| \leq X^{-(1/8)}\}, \\ \mathfrak{m} &= \{\alpha: X^{-(1/8)} < |\alpha| \leq \xi\}, \\ \mathfrak{t} &= \{\alpha: |\alpha| > \xi\}, \end{aligned} \quad (18)$$

where  $\xi = \tau^{-2} X^{(1/16) - (1/4k) + 10\epsilon}$ . These sets are called the major arc, the minor arcs, and the trivial regions, respectively.

In the following sections, we shall prove that the dominant contribution to  $I(\tau, \eta, \mathbb{R})$  is from the major arc, and the contribution from the minor arcs and the trivial region can be neglected.

### 3. The Major Arc

Our first goal is to show that

$$|I(\tau, \eta, \mathfrak{M})| \gg \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-1}. \quad (19)$$

The proof of (19) is quite similar to that given in Section 3 in [20]. For completeness of exposition, we briefly present the proof procedure below.

Let

$$\begin{aligned} \mathfrak{M}_1 &= \{\alpha: |\alpha| \leq X^{-1+(5/12k)-\epsilon}\}, \\ \mathfrak{M}_2 &= \{\alpha: X^{-1+(5/12k)-\epsilon} < |\alpha| \leq X^{-(7/8)}\}, \\ \mathfrak{M}_3 &= \{\alpha: X^{-(7/8)} < |\alpha| \leq X^{-(1/8)}\}. \end{aligned} \quad (20)$$

Then, we have  $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3$  and

$$I(\tau, \eta, \mathfrak{M}) = I(\tau, \eta, \mathfrak{M}_1) + I(\tau, \eta, \mathfrak{M}_2) + I(\tau, \eta, \mathfrak{M}_3). \quad (21)$$

By a similar argument as that in pp. 1656–1657 in [20], we can obtain

$$|I(\tau, \eta, \mathfrak{M}_1)| \gg \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-1}. \quad (22)$$

To estimate the integrals  $I(\tau, \eta, \mathfrak{M}_2)$  and  $I(\tau, \eta, \mathfrak{M}_3)$ , we need the following two lemmas.

**Lemma 1.** *Let  $j \geq 2$  be an integer. Then, for nonzero real number  $\lambda$  and any  $\epsilon > 0$ , we have*

$$S_j(\lambda\alpha) \ll \begin{cases} X^{(1/j)(1-j4^{1-j})+\epsilon} |\alpha|^{-4^{1-j}} & \text{for } X^{-1} \leq |\alpha| \leq X^{-1+(1/2j)}, \\ \frac{1}{X^j} \left(1 - (1/2) \cdot 4^{1-j}\right) + \epsilon & \text{for } X^{-1+(1/2j)} < |\alpha| \leq X^{-(1/2j)}. \end{cases} \quad (23)$$

*Proof.* It follows from Theorem 1 in [28].  $\square$

**Lemma 2.** *For  $4 \leq k \leq 6$ , suppose that*

$$\begin{aligned} F(\alpha) \in \{ & S_1^2(\lambda_1\alpha), S_3^8(\lambda_3\alpha), (S_2^*(\lambda_2\alpha))^2 S_3^4(\lambda_3\alpha), \\ & (S_2^*(\lambda_2\alpha))^2 S_4^4(\lambda_4\alpha), \\ & (S_2^*(\lambda_2\alpha))^2 S_5^6(\lambda_5\alpha), (S_2^*(\lambda_2\alpha))^2 S_6^8(\lambda_5\alpha), \\ & (S_2^*(\lambda_2\alpha) S_3(\lambda_3\alpha) S_k(\lambda_5\alpha))^2, (S_2^*(\lambda_2\alpha))^2 S_4^2(\lambda_4\alpha) S_5^4(\lambda_5\alpha) \}. \end{aligned} \quad (24)$$

*Then, we have*

$$\begin{aligned} \int_{-1}^1 |F(\alpha)| d\alpha &\ll X^{-1} F(0)^{1+\epsilon}, \\ \int_{\mathbb{R}} |F(\alpha)| K_\tau(\alpha) d\alpha &\ll \tau X^{-1} F(0)^{1+\epsilon}. \end{aligned} \quad (25)$$

*Proof.* See Lemma 3.7 in [20].  $\square$

When  $\alpha \in \mathfrak{M}_2$ , it follows from (23) that

$$S_4(\lambda_4\alpha) \ll X^{(15/64)+\epsilon} |\alpha|^{-(1/64)} \ll X^{(1/4)-(5/768k)+2\epsilon}. \quad (26)$$

Combining this with the Cauchy–Schwarz inequality and Lemma 2 gives

$$\begin{aligned}
|I(\tau, \eta, \mathfrak{M}_2)| &\ll \tau^2 \sup_{\alpha \in \mathfrak{M}_2} |S_4(\lambda_4 \alpha)| \int_{\mathfrak{M}_2} |S_1(\lambda_1 \alpha) S_2^*(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_k(\lambda_5 \alpha)| d\alpha \\
&\ll \tau^2 \sup_{\alpha \in \mathfrak{M}_2} |S_4(\lambda_4 \alpha)| \left( \int_{-1}^1 |S_1(\lambda_1 \alpha)|^2 d\alpha \right)^{(1/2)} \\
&\quad \times \left( \int_{-1}^1 |S_2^*(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_k(\lambda_5 \alpha)|^2 d\alpha \right)^{(1/2)} \\
&\ll \tau^2 X^{(1/4) - (5/768k) + 2\epsilon} \cdot (X^{1+2\epsilon})^{(1/2)} \cdot (X^{(2/3) + (2/k) + \epsilon})^{(1/2)} \\
&\ll \tau^2 X^{(13/12) + (1/k) - \epsilon},
\end{aligned} \tag{27}$$

where (11) is used.

When  $\alpha \in \mathfrak{M}_3$ , (23) implies

$$S_4(\lambda_4 \alpha) \ll X^{(1/4) - (1/512) + \epsilon}. \tag{28}$$

Proceeding as in the proof of (27), we have

$$|I(\tau, \eta, \mathfrak{M}_3)| \ll \tau^2 X^{(13/12) + (1/k) - \epsilon}. \tag{29}$$

This with (27), (22), and (21) yields (19).

#### 4. The Minor Arcs

The next thing to do in the proof is to establish that

$$|I(\tau, \eta, \mathfrak{m})| \ll \tau^2 X^{(13/12) + (1/k)} \mathcal{L}^{-2}. \tag{30}$$

This work forms the bulk of the present paper. We subdivide  $\mathfrak{m}$  into four disjoint parts:  $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3 \cup \mathfrak{m}_4$ , where

$$\begin{aligned}
\mathfrak{m}_1 &= \{\alpha \in \mathfrak{m} : |S_1(\lambda_1 \alpha)| \leq X^{(6/7) + 2\epsilon}\}, \\
\mathfrak{m}_2 &= \{\alpha \in \mathfrak{m} : |S_1(\lambda_1 \alpha)| > X^{(6/7) + 2\epsilon}, |S_2^*(\lambda_2 \alpha)| \leq X^{(3/7) + 2\epsilon}, |S_3(\lambda_3 \alpha)| > X^{(11/36) + 2\epsilon}\}, \\
\mathfrak{m}_3 &= \{\alpha \in \mathfrak{m} : |S_1(\lambda_1 \alpha)| > X^{(6/7) + 2\epsilon}, |S_2^*(\lambda_2 \alpha)| \leq X^{(3/7) + 2\epsilon}, |S_3(\lambda_3 \alpha)| \leq X^{(11/36) + 2\epsilon}\}, \\
\mathfrak{m}_4 &= \{\alpha \in \mathfrak{m} : |S_1(\lambda_1 \alpha)| > X^{(6/7) + 2\epsilon}, |S_2^*(\lambda_2 \alpha)| > X^{(3/7) + 2\epsilon}\}.
\end{aligned} \tag{31}$$

Therefore,

$$I(\tau, \eta, \mathfrak{m}) = \sum_{j=1}^4 I(\tau, \eta, \mathfrak{m}_j). \tag{32}$$

To prove (30), it suffices to show that  $|I(\tau, \eta, \mathfrak{m}_j)| \ll \tau^2 X^{(13/12) + (1/k)} \mathcal{L}^{-2}$  holds for  $1 \leq j \leq 4$ .

We apply Hölder's inequality and Lemma 2 to estimate  $|I(\tau, \eta, \mathfrak{m}_1)|$ . When  $k = 4$ , we have

$$\begin{aligned}
|I(\tau, \eta, \mathfrak{m}_1)| &\ll \left( \sup_{\alpha \in \mathfrak{m}_1} |S_1(\lambda_1 \alpha)| \right)^{(1/4)} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(3/8)} \\
&\quad \times \left( \int_{\mathbb{R}} |S_3(\lambda_3 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(1/8)} \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_4(\lambda_4 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/4)} \\
&\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_k(\lambda_5 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/4)} \\
&\ll (X^{(6/7) + 2\epsilon})^{(1/4)} (\tau X^{1+\epsilon})^{(3/8)} (\tau X^{(5/3) + \epsilon})^{(1/8)} (\tau X^{1+\epsilon})^{(1/4)} (\tau X^{(4/k) + \epsilon})^{(1/4)} \\
&\ll \tau X^{(13/12) + (1/k) - (1/28) + 2\epsilon}.
\end{aligned} \tag{33}$$

If  $k = 5$ , then

$$\begin{aligned}
& |I(\tau, \eta, \mathbf{m}_1)| \\
& \ll \left( \sup_{\alpha \in \mathbf{m}_1} |S_1(\lambda_1 \alpha)| \right)^{(3/16)} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(13/32)} \\
& \times \left( \int_{\mathbb{R}} |S_3(\lambda_3 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(3/32)} \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_4(\lambda_4 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/4)} \\
& \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_k(\lambda_5 \alpha)|^6 K_\tau(\alpha) d\alpha \right)^{(1/8)} \\
& \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_k(\lambda_5 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(1/8)} \\
& \ll \left( X^{(6/7)+2\epsilon} \right)^{(3/16)} \left( \tau X^{1+\epsilon} \right)^{(13/32)} \left( \tau X^{(5/3)+\epsilon} \right)^{(3/32)} \left( \tau X^{1+\epsilon} \right)^{(1/4)} \left( \tau X^{(6/k)+\epsilon} \right)^{(1/8)} \left( \tau X^{(2/3)+(2/k)+\epsilon} \right)^{(1/8)} \\
& \ll \tau X^{(13/12)+(1/k)-(3/112)+2\epsilon}.
\end{aligned} \tag{34}$$

In case  $k = 6$ , we obtain

$$\begin{aligned}
& |I(\tau, \eta, \mathbf{m}_1)| \\
& \ll \left( \sup_{\alpha \in \mathbf{m}_1} |S_1(\lambda_1 \alpha)| \right)^{(1/6)} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(5/12)} \\
& \times \left( \int_{\mathbb{R}} |S_3(\lambda_3 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(1/12)} \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_4(\lambda_4 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/4)} \\
& \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_k(\lambda_5 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(1/12)} \\
& \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha) S_3(\lambda_3 \alpha) S_k(\lambda_5 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(1/6)} \\
& \ll \left( X^{(6/7)+2\epsilon} \right)^{(1/6)} \left( \tau X^{1+\epsilon} \right)^{(5/12)} \left( \tau X^{(5/3)+\epsilon} \right)^{(1/12)} \left( \tau X^{1+\epsilon} \right)^{(1/4)} \left( \tau X^{(8/k)+\epsilon} \right)^{(1/12)} \left( \tau X^{(2/3)+(2/k)+\epsilon} \right)^{(1/6)} \\
& \ll \tau X^{(13/12)+(1/k)-(1/42)+2\epsilon}.
\end{aligned} \tag{35}$$

It follows from (33)–(35) and (17) that

$$|I(\tau, \eta, \mathbf{m}_1)| \ll \tau^2 X^{(13/12)+(1/k)-\epsilon} \ll \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-2}. \tag{36}$$

In order to establish an upper bound for  $|I(\tau, \eta, \mathbf{m}_2)|$  as small as possible, we need the following lemma.

**Lemma 3** (Lemma 3.4 in [21]). *Let*

$$\mathfrak{N} = \left\{ \alpha: X^{(11/36)+2\epsilon} < |S_3(\lambda_3 \alpha)| \leq X^{(1/3)} \right\}. \tag{37}$$

Then, we have

$$\int_{\mathfrak{N}} |S_3(\lambda_3 \alpha)|^2 |S_4(\lambda_4 \alpha)|^2 K_\tau(\alpha) d\alpha \ll \tau X^{(1/6)+4\epsilon}. \tag{38}$$

For  $4 \leq k \leq 6$ , by the Cauchy–Schwarz inequality, Lemmas 2 and 3, we obtain

$$\begin{aligned}
|I(\tau, \eta, \mathbf{m}_2)| & \ll X^{(1/k)} \left( \sup_{\alpha \in \mathbf{m}_2} |S_2^*(\lambda_2 \alpha)| \right) \left( \int_{\mathfrak{N}} |S_3(\lambda_3 \alpha)|^2 |S_4(\lambda_4 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(1/2)} \\
& \times \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(1/2)} \\
& \ll \tau X^{(1/k)+(3/7)+(1/12)+(1/2)+4\epsilon} \\
& \ll \tau X^{(13/12)+(1/k)-(1/14)+5\epsilon},
\end{aligned} \tag{39}$$



where the trivial upper bound  $S_k(\lambda_5\alpha) \ll X^{(1/k)}$  is used. It is easily derived from (17) that

$$|I(\tau, \eta, \mathbf{m}_2)| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \ll \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-2}. \quad (40)$$

The upper bound estimation of  $|I(\tau, \eta, \mathbf{m}_3)|$  plays a crucial role in the proof. The parameter  $\tau$ , which is given by (17), is determined in this step. When  $k = 4$ , by Hölder's inequality and Lemma 2, we have

$$\begin{aligned} |I(\tau, \eta, \mathbf{m}_3)| &\ll \left( \sup_{\alpha \in \mathbf{m}_3} |S_3(\lambda_3\alpha)| \right) \left( \int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_{\tau}(\alpha) d\alpha \right)^{(1/2)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_4(\lambda_4\alpha)|^4 K_{\tau}(\alpha) d\alpha \right)^{(1/4)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_k(\lambda_5\alpha)|^4 K_{\tau}(\alpha) d\alpha \right)^{(1/4)} \\ &\ll \left( X^{(11/36)+2\varepsilon} \right) \left( \tau X^{1+\varepsilon} \right)^{(1/2)} \left( \tau X^{1+\varepsilon} \right)^{(1/4)} \left( \tau X^{(4/k)+\varepsilon} \right)^{(1/4)} \\ &\ll \tau X^{(13/12)+(1/k)-(1/36)+4\varepsilon}. \end{aligned} \quad (41)$$

In the case of  $k = 5$ , we obtain

$$\begin{aligned} |I(\tau, \eta, \mathbf{m}_3)| &\ll \left( \sup_{\alpha \in \mathbf{m}_3} |S_2^*(\lambda_2\alpha)| \right)^{(1/6)} \left( \sup_{\alpha \in \mathbf{m}_3} |S_3(\lambda_3\alpha)| \right)^{(1/3)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_{\tau}(\alpha) d\alpha \right)^{(1/2)} \left( \int_{\mathbb{R}} |S_3(\lambda_3\alpha)|^8 K_{\tau}(\alpha) d\alpha \right)^{(1/12)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_4(\lambda_4\alpha)|^4 K_{\tau}(\alpha) d\alpha \right)^{(1/4)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_k(\lambda_5\alpha)|^6 K_{\tau}(\alpha) d\alpha \right)^{(1/6)} \\ &\ll \left( X^{(3/7)+2\varepsilon} \right)^{(1/6)} \left( X^{(11/36)+2\varepsilon} \right)^{(1/3)} \left( \tau X^{1+\varepsilon} \right)^{(1/2)} \left( \tau X^{(5/3)+\varepsilon} \right)^{(1/12)} \\ &\quad \times \left( \tau X^{1+\varepsilon} \right)^{(1/4)} \left( \tau X^{(6/k)+\varepsilon} \right)^{(1/6)} \\ &\ll \tau X^{(13/12)+(1/k)-(4/189)+4\varepsilon}. \end{aligned} \quad (42)$$

If  $k = 6$ , we deduce that

$$\begin{aligned} |I(\tau, \eta, \mathbf{m}_3)| &\ll \left( \sup_{\alpha \in \mathbf{m}_3} |S_3(\lambda_3\alpha)| \right)^{(2/3)} \left( \int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_{\tau}(\alpha) d\alpha \right)^{(1/2)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_4(\lambda_4\alpha)|^4 K_{\tau}(\alpha) d\alpha \right)^{(1/4)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_k(\lambda_5\alpha)|^8 K_{\tau}(\alpha) d\alpha \right)^{(1/12)} \\ &\quad \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_3(\lambda_3\alpha)|^2 |S_k(\lambda_5\alpha)|^2 K_{\tau}(\alpha) d\alpha \right)^{(1/6)} \\ &\ll \left( X^{(11/36)+2\varepsilon} \right)^{(2/3)} \left( \tau X^{1+\varepsilon} \right)^{(1/2)} \left( \tau X^{1+\varepsilon} \right)^{(1/4)} \left( \tau X^{(8/k)+\varepsilon} \right)^{(1/12)} \left( \tau X^{(2/3)+(2/k)+\varepsilon} \right)^{(1/6)} \\ &\ll \tau X^{\frac{13}{12} + \frac{1}{k} - \frac{1}{54} + 4\varepsilon}. \end{aligned} \quad (43)$$

Inequalities (41)–(43) and (17) together give

$$|I(\tau, \eta, \mathbf{m}_3)| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \ll \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-2}. \quad (44)$$

In the remainder of this section, we shall be trying to estimate  $|I(\tau, \eta, \mathbf{m}_4)|$ . By a familiar dyadic dissection argument, we divide  $\mathbf{m}_4$  into at most  $\ll \mathcal{L}^3$  disjoint sets  $E(Z_1, Z_2, y)$ . For  $\alpha \in E(Z_1, Z_2, y)$ , we have

$$Z_1 < |S_1(\lambda_1\alpha)| \leq 2Z_1, Z_2 < |S_2^*(\lambda_2\alpha)| \leq 2Z_2, y < |\alpha| \leq 2y, \quad (45)$$

where  $Z_1 = 2^{k_1} X^{(6/7)+2\varepsilon}$ ,  $Z_2 = 2^{k_2} X^{(3/7)+2\varepsilon}$ , and  $y = 2^{k_3} X^{-(1/8)}$  for some nonnegative integers  $k_1, k_2$ , and  $k_3$ . For the sake of convenience, we take the notation  $\mathcal{A}$  as a shortcut for  $E(Z_1, Z_2, y)$ , and let  $m(\mathcal{A})$  stand for the Lebesgue measure of  $\mathcal{A}$ .

**Lemma 4** (Lemma 4.3 in [20]). *We have*

$$m(\mathcal{A}) \ll y X^{(18/7)+9\varepsilon} Z_1^{-2} Z_2^{-4}. \quad (46)$$

When  $k = 4$ , it follows from (11) and Hölder's inequality that

$$\begin{aligned}
& |I(\tau, \eta, \mathcal{A})| \\
& \ll \left( \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)|^2 |S_2^*(\lambda_2 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/12)} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(5/12)} \\
& \times \left( \int_{\mathbb{R}} |S_3(\lambda_3 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(1/8)} \left( \int_{\mathbb{R}} |S_4(\lambda_4 \alpha)|^{16} K_\tau(\alpha) d\alpha \right)^{(1/48)} \\
& \times \left( \int_{\mathbb{R}} |S_k(\lambda_5 \alpha)|^{16} K_\tau(\alpha) d\alpha \right)^{(1/48)} \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_4(\lambda_4 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/6)} \\
& \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_k(\lambda_5 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/6)} \\
& \ll (Z_1^2 Z_2^4 \cdot m(\mathcal{A}) \cdot \min(\tau^2, y^{-2}))^{(1/12)} (\tau X^{1+\varepsilon})^{(5/12)} (\tau X^{(5/3)+\varepsilon})^{(1/8)} \\
& \times (\tau X^{3+\varepsilon})^{(1/48)} (\tau X^{(12/k)+\varepsilon})^{(1/48)} (\tau X^{1+\varepsilon})^{(1/6)} (\tau X^{(4/k)+\varepsilon})^{(1/6)} \\
& \ll (y X^{(18/7)+9\varepsilon} \cdot \min(\tau^2, y^{-2}))^{(1/12)} \tau^{(11/12)} X^{(41/48)+(11/12k)+\varepsilon} \\
& \ll \tau X^{(13/12)+(1/k)-(1/28)+2\varepsilon},
\end{aligned} \tag{47}$$

where Lemmas 2 and 3 are used.

When  $k = 5$ , by the similar argument as in the proof of (47), we obtain

$$\begin{aligned}
& |I(\tau, \eta, \mathcal{A})| \\
& \ll \left( \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)|^2 |S_2^*(\lambda_2 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/16)} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(7/16)} \\
& \times \left( \int_{\mathbb{R}} |S_3(\lambda_3 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(1/8)} \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_4(\lambda_4 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/8)} \\
& \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_4(\lambda_4 \alpha)|^2 |S_k(\lambda_5 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/4)} \\
& \ll (Z_1^2 Z_2^4 \cdot m(\mathcal{A}) \cdot \min(\tau^2, y^{-2}))^{(1/16)} (\tau X^{1+\varepsilon})^{(7/16)} \\
& \times (\tau X^{(5/3)+\varepsilon})^{(1/8)} (\tau X^{1+\varepsilon})^{(1/8)} (\tau X^{(1/2)+(4/k)+\varepsilon})^{(1/4)} \\
& \ll (y X^{(18/7)+9\varepsilon} \cdot \min(\tau^2, y^{-2}))^{(1/16)} \tau^{(15/16)} X^{(43/48)+(1/k)+\varepsilon} \\
& \ll \tau X^{(13/12)+(1/k)-(3/112)+2\varepsilon}.
\end{aligned} \tag{48}$$

When  $k = 6$ , we have

$$\begin{aligned}
& |I(\tau, \eta, \mathcal{A})| \\
& \ll \left( \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)|^2 |S_2^*(\lambda_2 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/16)} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\tau(\alpha) d\alpha \right)^{(7/16)} \\
& \times \left( \int_{\mathbb{R}} |S_3(\lambda_3 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(1/8)} \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_4(\lambda_4 \alpha)|^4 K_\tau(\alpha) d\alpha \right)^{(1/4)} \\
& \times \left( \int_{\mathbb{R}} |S_2^*(\lambda_2 \alpha)|^2 |S_k(\lambda_5 \alpha)|^8 K_\tau(\alpha) d\alpha \right)^{(1/8)} \\
& \ll (Z_1^2 Z_2^4 \cdot m(\mathcal{A}) \cdot \min(\tau^2, y^{-2}))^{(1/16)} (\tau X^{1+\varepsilon})^{(7/16)} (\tau X^{(5/3)+\varepsilon})^{(1/8)} (\tau X^{1+\varepsilon})^{(1/4)} (\tau X^{(8/k)+\varepsilon})^{(1/8)} \\
& \ll (y X^{(18/7)+9\varepsilon} \cdot \min(\tau^2, y^{-2}))^{(1/16)} \tau^{(15/16)} X^{(43/48)+(1/k)+\varepsilon} \\
& \ll \tau X^{(13/12)+(1/k)-(3/112)+2\varepsilon}.
\end{aligned} \tag{49}$$

Thanks to (17) and (47)–(49), we are led to the conclusion that

$$\begin{aligned} |I(\tau, \eta, \mathbf{m}_4)| &\ll \mathcal{L}^3 \cdot \max_{\mathcal{A}} |I(\tau, \eta, \mathcal{A})| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \\ &\ll \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-2}. \end{aligned} \quad (50)$$

This together with (36), (40), (44), and (32) gives (30).

## 5. The Trivial Regions

Finally, it only remains to treat  $I(\tau, \eta, \mathbf{t})$ . Suppose that  $r$  and  $j$  are positive integers with  $r \leq j$ . For any  $\xi \in [X^\varepsilon, +\infty)$  and nonzero real  $\lambda$ , we have (see (5.1) and (5.2) in [20])

$$\begin{aligned} \int_{\xi}^{+\infty} |S_j(\lambda\alpha)|^{2^r} K_{\tau}(\alpha) d\alpha &\ll \xi^{-1} X^{((2^r-r)/j)+\varepsilon}, \\ \int_{\xi}^{+\infty} |S_2^*(\lambda_2\alpha)|^4 K_{\tau}(\alpha) d\alpha &\ll \xi^{-1} X^{1+\varepsilon}. \end{aligned} \quad (51)$$

It follows from Hölder's inequality that

$$\begin{aligned} |I(\tau, \eta, \mathbf{t})| &\ll \int_{\xi}^{+\infty} \left| S_k(\lambda_5\alpha) S_2^*(\lambda_2\alpha) \prod_{\substack{1 \leq j \leq 4 \\ j \neq 2}} S_j(\lambda_j\alpha) \right| K_{\tau}(\alpha) d\alpha \\ &\ll \left( \int_{\xi}^{+\infty} |S_1(\lambda_1\alpha)|^2 K_{\tau}(\alpha) d\alpha \right)^{(1/2)} \left( \int_{\xi}^{+\infty} |S_2^*(\lambda_2\alpha)|^4 K_{\tau}(\alpha) d\alpha \right)^{(1/4)} \\ &\quad \times \left( \int_{\xi}^{+\infty} |S_3(\lambda_3\alpha)|^8 K_{\tau}(\alpha) d\alpha \right)^{(1/8)} \left( \int_{\xi}^{+\infty} |S_4(\lambda_4\alpha)|^{16} K_{\tau}(\alpha) d\alpha \right)^{(1/16)} \\ &\quad \times \left( \int_{\xi}^{+\infty} |S_k(\lambda_5\alpha)|^{16} K_{\tau}(\alpha) d\alpha \right)^{(1/16)} \\ &\ll (\xi^{-1} X^{1+\varepsilon})^{(1/2)} (\xi^{-1} X^{1+\varepsilon})^{(1/4)} (\xi^{-1} X^{(5/3)+\varepsilon})^{(1/8)} (\xi^{-1} X^{3+\varepsilon})^{(1/16)} (\xi^{-1} X^{(12/k)+\varepsilon})^{(1/16)} \\ &\ll \xi^{-1} X^{(55/48)+(3/4k)+2\varepsilon}. \end{aligned} \quad (52)$$

Recalling that  $\xi = \tau^{-2} X^{(1/16)-(1/4k)+10\varepsilon}$  and inserting this expression into (52) yields

$$|I(\tau, \eta, \mathbf{t})| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \ll \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-2}. \quad (53)$$

## 6. Completion of the Proof

We are now in a position to get the desired conclusion. It should be noted that

$$I(\tau, \eta, \mathbb{R}) = I(\tau, \eta, \mathfrak{M}) + I(\tau, \eta, \mathbf{m}) + I(\tau, \eta, \mathbf{t}). \quad (54)$$

From this and (19), (30), and (53), we infer that  $I(\tau, \eta, \mathbb{R}) \gg \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-1}$ . Hence, by (15),

$$\mathcal{N}(X) \gg \tau X^{(13/12)+(1/k)} \mathcal{L}^{-5}. \quad (55)$$

This implies inequality (16) has  $\gg \tau X^{(13/12)+(1/k)} \mathcal{L}^{-5}$  solutions in quintuples of primes  $(p_1, p_2, \dots, p_5)$  with  $p_2 \in \mathcal{J}$ ,  $p_5 \in I_k$ , and  $p_j \in I_j$ , for  $j \in \{1, 3, 4\}$ . Notice that  $\lambda_1/\lambda_2$  is irrational,  $q$  is any denominator of a convergent to  $\lambda_1/\lambda_2$  and  $X = q^{7/3}$ . By substituting (17) into (55), we deduce that  $\mathcal{N}(X) \rightarrow +\infty$  as  $q \rightarrow +\infty$ . In view of

$$\max_{1 \leq j \leq 5} p_j \asymp X, \quad (56)$$

and (17), we obtain the required range of  $\sigma(k)$  in Theorem 1. This completes the proof of Theorem 1.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# A System of Two Diophantine Inequalities with Primes

Xue Han, Huafeng Liu , and Deyu Zhang

School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong 250358, China

Correspondence should be addressed to Huafeng Liu; hfliu\_sdu@hotmail.com

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Let  $1 < d < c < 128/119$ ,  $1 < \alpha < \beta < 6^{1-d/c}$ . In this paper, we prove that there exist positive real numbers  $N_1^{(0)}$  and  $N_2^{(0)}$  depending on  $c, d, \alpha, \beta$  such that for all real numbers  $N_1 > N_1^{(0)}$ ,  $N_2 > N_2^{(0)}$  and  $\alpha \leq N_2/N_1^{d/c} \leq \beta$ , the system of two Diophantine inequalities  $|p_1^c + \dots + p_6^c - N_1| < N_1^{-(1/c)(128/119-c)} \log^{109} N_1$ ,  $|p_1^d + \dots + p_6^d - N_2| < N_2^{-(1/d)(128/119-d)} \log^{109} N_2$  is solvable in prime variables  $p_1, \dots, p_6$ .

## 1. Introduction

Assume that  $c > 1$  is not an integer and  $\varepsilon$  is a sufficiently small positive number. Let  $H(c)$  denote the least integer  $s$  such that the Diophantine inequality

$$|p_1^c + p_2^c + \dots + p_s^c - N| < \varepsilon \quad (1)$$

is solvable in primes  $p_1, p_2, \dots, p_s$  for sufficiently large  $N$ . In 1952, Piatetski-Shapiro [1] proved that

$$\limsup_{c \rightarrow \infty} \frac{H(c)}{c \log c} \leq 4. \quad (2)$$

Piatetski-Shapiro also proved  $H(c) \leq 5$  for  $1 < c < 3/2$ . Tolev [2] first improved this result for  $c$  close to one. More precisely, Tolev proved that if  $1 < c < 15/14$ , the Diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon(N) \quad (3)$$

has prime solutions  $p_1, p_2, p_3$  for large  $N$ , where  $\varepsilon(N) = N^{-(1/c)(15/14-c)} \log^9 N$ . Later, this result was improved by many authors (see [3–9]), and many analogous problems of this type were studied (for example, see [10–17]).

In 1995, Tolev [18] first considered the system of two Diophantine inequalities with primes

$$\begin{aligned} |p_1^c + \dots + p_5^c - N_1| &< \varepsilon_1^*(N_1), \\ |p_1^d + \dots + p_5^d - N_2| &< \varepsilon_2^*(N_2). \end{aligned} \quad (4)$$

He established that for all real numbers  $N_1, N_2$  satisfying  $N_1 > N_1^{(0)}$ ,  $N_2 > N_2^{(0)}$  and  $\alpha \leq N_2/N_1^{d/c} \leq \beta$ , system (4) with

$$\begin{aligned} \varepsilon_1^*(N_1) &= N_1^{-(1/c)(35/34-c)} \log^{12} N_1, \varepsilon_2^*(N_2) \\ &= N_2^{-(1/d)(35/34-d)} \log^{12} N_2 \end{aligned} \quad (5)$$

has solutions in primes  $p_1, \dots, p_5$ , where  $c, d, \alpha, \beta$  are real numbers satisfying the conditions

$$1 < d < c < 35/34, 1 < \alpha < \beta < 5^{1-d/c}, \quad (6)$$

and  $N_1^{(0)}, N_2^{(0)}$  depending on  $c, d, \alpha, \beta$  are sufficiently large numbers. Subsequently, Tolev's result was improved by Zhai [19] and Zhai and Cao [20]. Now the best result is due to Zhai and Cao [20] who proved that system (4) with

$$\begin{aligned} \varepsilon_1^*(N_1) &= N_1^{-(1/c)(27/26-c)} \log^{100} N_1, \varepsilon_2^*(N_2) \\ &= N_2^{-(1/d)(27/26-d)} \log^{100} N_2 \end{aligned} \quad (7)$$

is solvable for  $1 < d < c < 27/26$ .

In this paper, we consider the following system of two Diophantine inequalities over primes  $p_1, \dots, p_6$ :

$$\begin{aligned} |p_1^c + \cdots + p_6^c - N_1| &< \varepsilon_1(N_1), \\ |p_1^d + \cdots + p_6^d - N_2| &< \varepsilon_2(N_2), \end{aligned} \quad (8)$$

where  $c > 1$ ,  $d > 1$  are different numbers but close to 1 and  $\varepsilon_1(N_1), \varepsilon_2(N_2)$  satisfy

$$\begin{aligned} \varepsilon_1(N_1) &\longrightarrow 0, \text{ as } N_1 \longrightarrow \infty, \\ \varepsilon_2(N_2) &\longrightarrow 0, \text{ as } N_2 \longrightarrow \infty. \end{aligned} \quad (9)$$

We have to impose a condition on the orders of  $N_1$  and  $N_2$  due to the inequality

$$(x_1^c + \cdots + x_6^c)^{d/c} \leq x_1^d + \cdots + x_6^d \leq 6^{1-d/c} (x_1^c + \cdots + x_6^c)^{d/c}, \quad (10)$$

which holds for every positive  $x_1, \dots, x_6$  provided  $1 < d < c$ . Our result is as follows.

**Theorem 1.** Suppose that  $c, d, \alpha, \beta$  are real numbers and satisfy conditions

$$1 < d < c < 128/119, \quad (11)$$

$$1 < \alpha < \beta < 6^{1-d/c}. \quad (12)$$

Then, there exist positive real numbers  $N_1^{(0)}, N_2^{(0)}$  depending on  $c, d, \alpha, \beta$  such that for all real numbers  $N_1 > N_1^{(0)}, N_2 > N_2^{(0)}$  and

$$\alpha \leq N_2/N_1^{d/c} \leq \beta, \quad (13)$$

system (8) with

$$\begin{aligned} \varepsilon_1(N_1) &= N_1^{-(1/c)(128/119-c)} \log^{109} N_1, \\ \varepsilon_2(N_2) &= N_2^{-(1/d)(128/119-d)} \log^{109} N_2 \end{aligned} \quad (14)$$

has solutions in primes  $p_1, \dots, p_6$ .

*Notation.* Throughout this paper, the letter  $p$ , with or without a subscript, always represents a prime,  $c$  and  $d$  are real numbers satisfying (11), and  $\alpha$  and  $\beta$  are real numbers satisfying (12).  $\eta$  denotes a sufficiently small positive number depending on  $c$  and  $d$ .  $\chi(t)$  denotes the characteristic function over the interval  $[-1, 1]$ .  $\rho = \beta' + iy'$  is the non-trivial zero of the Riemann zeta function  $\zeta(s)$ . As usual,  $\Lambda(n)$  and  $\tau(n)$  are the von Mangoldt function and the divisor function, respectively.  $N_1$  and  $N_2$  are sufficiently large numbers. We set

$$\begin{aligned} X &= N_1^{1/c}, \tau_1 = X^{3/4-c-\eta}, \tau_2 = X^{3/4-d-\eta}, \\ \varepsilon_1 &= X^{-(128/119-c)} \log^{107} X, \varepsilon_2 = X^{-(128/119-d)} \log^{107} X, \\ K_1 &= X^{(128/119-c)} \log^{-106} X, K_2 = X^{(128/119-d)} \log^{-106} X, \\ e(t) &= e^{2\pi it}, \varphi(t) = e^{-\pi t^2}, \varphi_\delta(t) = \delta \varphi(\delta t). \end{aligned} \quad (15)$$

## 2. Outline of the Proof

Let  $\lambda$  denote a sufficiently small positive number, whose value depends on  $c, d, \alpha, \beta$  and will be determined more precisely in Lemma 1. Let

$$B = \sum_{\lambda X < p_1, \dots, p_6 \leq X} (\log p_1) \cdots (\log p_6) \chi\left(\frac{p_1^c + \cdots + p_6^c - N_1}{\varepsilon_1 \log X}\right) \chi\left(\frac{p_1^d + \cdots + p_6^d - N_2}{\varepsilon_2 \log X}\right), \quad (16)$$

$$S(x, y) = \sum_{\lambda X < p \leq X} (\log p) e(xp^c + yp^d), \quad (17)$$

$$D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S^6(x, y) e(-N_1 x - N_2 y) \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy. \quad (18)$$

We divide the plane into three regions: the neighbourhood of origin  $\Omega_1$ , the intermediate region  $\Omega_2$ , and the trivial region  $\Omega_3$ , which are defined as

$$\begin{aligned} \Omega_1 &= \{(x, y): \max(|x|/\tau_1, |y|/\tau_2) < 1\}, \\ \Omega_2 &= \{(x, y): \max(|x|/\tau_1, |y|/\tau_2) \geq 1, \\ &\quad \max(|x|/K_1, |y|/K_2) \leq 1\}, \\ \Omega_3 &= \{(x, y): \max(|x|/K_1, |y|/K_2) > 1\}. \end{aligned} \quad (19)$$

Thus, the integral  $D$  can be represented as

$$D = D_1 + D_2 + D_3, \quad (20)$$

where

$$\begin{aligned} D_i &= \int \int_{\Omega_i} S^6(x, y) e(-N_1 x - N_2 y) \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy \\ &\quad (i = 1, 2, 3). \end{aligned} \quad (21)$$

If we show that  $B$  tends to infinity as  $X$  tends to infinity, then Theorem 1 holds. From Lemma 4, it is sufficient to prove that  $D$  tends to infinity as  $X$  tends to infinity. To do this, noting (20), we shall prove the following:

$$|D_1| \gg \varepsilon_1 \varepsilon_2 X^{6-c-d}, \quad (22)$$

$$|D_2| \ll \frac{\varepsilon_1 \varepsilon_2 X^{6-c-d}}{\log X}, \quad (23)$$

$$|D_3| \ll 1. \quad (24)$$

In Section 3, we first give some auxiliary lemmas. Inequality (22) is proved in Section 4. Inequality (23), from which we can get the range of  $c$  and  $d$ , is proved in Section 5. In Section 6, we complete the proof of Theorem 1.

### 3. Auxiliary Lemmas

**Lemma 1.** Let  $\delta \in [\alpha, \beta]$ . There exists  $\lambda > 0$  depending on  $c, d, \alpha, \beta$  such that for the volume  $V$  of the domain in six-dimensional space defined by

$$t_1, \dots, t_6 > \lambda, |t_1^c + \dots + t_6^c - 1| < \mu_1, |t_1^d + \dots + t_6^d - \delta| < \mu_2, \quad (25)$$

we have

$$V \gg \mu_1 \mu_2, \quad (26)$$

provided  $\mu_1, \mu_2$  are sufficiently small.

*Proof.* This lemma is similar to Lemma 1 in Tolev [18]. We can write the volume of  $V$  as

$$V = \int_{t_1, \dots, t_6 > \lambda} \dots \int_{\substack{|t_1^c + \dots + t_6^c - 1| < \mu_1 \\ |t_1^d + \dots + t_6^d - \delta| < \mu_2}} 1 dt_1 \dots dt_6. \quad (27)$$

Then, we can fix  $t_1, \dots, t_5$  to get the range of  $t_6$  from last two inequalities and get

$$(1 - t_1^c + \dots + t_5^c - \mu_1)^{1/c} < t_6 < (1 - t_1^c + \dots + t_5^c + \mu_1)^{1/c}, \quad (28)$$

$$(\delta - t_1^d + \dots + t_5^d - \mu_2)^{1/d} < t_6 < (\delta - t_1^d + \dots + t_5^d + \mu_2)^{1/d}. \quad (29)$$

Since  $\mu_1$  and  $\mu_2$  are sufficiently small, we can adjust the value of  $\lambda$  to ensure that there are intersections between (28) and (29). We may also get this lemma by adjusting the value of  $\lambda$  and using circle method to estimate

$$|t_1^c + \dots + t_6^c - 1| < \mu_1, |t_1^d + \dots + t_6^d - \delta| < \mu_2. \quad (30)$$

**Lemma 2** (see [18], Lemma 2). The function  $\varphi(t) = e^{-\pi t^2}$  has the following properties:

$$\varphi(x) = \int_{-\infty}^{\infty} \varphi(t) e(-xt) dt, \quad (31)$$

$$\chi(t/q) \geq \varphi(t) - e^{-\pi q^2} \quad \text{for } q > 0, \quad (32)$$

$$\varphi(t) \geq e^{-\pi} \quad \text{for } |t| \leq 1. \quad (33)$$

**Lemma 3.** Let  $B > 1$  denote a real number and  $f$  be a smooth real function on  $[B, 2B]$ . Suppose that there exists a positive constant  $A = A(f)$  such that

$$AB^{1-j} \ll_j |f^j(x)| \ll_j AB^{1-j}, \quad \text{for } x \sim B \text{ and } j \in \mathbb{N}, \quad (34)$$

where the implied absolute constant depends only on  $j$ . Then, there exists an exponent pair  $(\kappa, \iota)$  with

$$0 \leq \kappa \leq 1/2 \leq \iota < 1, \quad (35)$$

such that

$$\sum_{B < n \leq B+h} e(f(n)) \ll A^\kappa B^\iota, \quad \text{for } 1 < h \leq B. \quad (36)$$

*Proof.* We can find this lemma in Ivić ([21], pp. 72–79).

**Lemma 4.** The quantities  $B$  and  $D$  satisfy

$$B \geq D + O(1). \quad (37)$$

*Proof.* From (31) and (32) in Lemma 2, we have

$$\begin{aligned} \chi\left(\frac{p_1^c + \dots + p_6^c - N_1}{\varepsilon_1 \log X}\right) &\geq \varphi\left(\frac{p_1^c + \dots + p_6^c - N_1}{\varepsilon_1}\right) - e^{-\pi(\log X)^2} \\ \varphi\left(\frac{p_1^c + \dots + p_6^c - N_1}{\varepsilon_1}\right) &= \varphi\left(\frac{N_1 - (p_1^c + \dots + p_6^c)}{\varepsilon_1}\right) \\ &= \int_{-\infty}^{\infty} \varphi(x) e\left(-x \frac{N_1 - (p_1^c + \dots + p_6^c)}{\varepsilon_1}\right) dx \\ &= \int_{-\infty}^{\infty} e(x(p_1^c + \dots + p_6^c)) e(-xN_1) \varphi_{\varepsilon_1}(x) dx, \end{aligned} \quad (38)$$

where we substitute  $\varepsilon_1 x$  for  $x$ . Similarly, we have

$$\begin{aligned} \chi\left(\frac{p_1^d + \dots + p_6^d - N_2}{\varepsilon_2 \log X}\right) &\geq \varphi\left(\frac{p_1^d + \dots + p_6^d - N_2}{\varepsilon_2}\right) - e^{-\pi(\log X)^2} \\ \varphi\left(\frac{p_1^d + \dots + p_6^d - N_2}{\varepsilon_2}\right) &= \int_{-\infty}^{\infty} e(y(p_1^d + \dots + p_6^d)) e(-yN_2) \varphi_{\varepsilon_2}(y) dy. \end{aligned} \quad (39)$$

Then,



$$\begin{aligned}
B &\geq \sum_{\lambda X < p_1, \dots, p_6 \leq X} (\log p_1) \cdots (\log p_6) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e(x(p_1^c + \cdots + p_6^c) + y(p_1^d + \cdots + p_6^d)) e(-xN_1 - yN_2) \\
&\quad \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy + O(1) \\
&\geq D + O(1).
\end{aligned} \tag{40}$$

**Lemma 5.** *There are two real functions  $G(x), F(x)$  defined in  $[a, b]$ ,  $|G(x)| \leq H$  for  $a \leq x \leq b$  and  $G(x)/F(x)$  is a monotonous function. Write*

$$I = \int_a^b G(x) e(F(x)) dx. \tag{41}$$

*If  $F'(x) \geq h > 0$  or  $F'(x) \leq -h < 0$  for all  $x \in [a, b]$ , then*

$$|I| \ll H/h. \tag{42}$$

*If  $F''(x) \geq h > 0$  for all  $x \in [a, b]$ , then*

$$|I| \ll H/\sqrt{h}. \tag{43}$$

*Proof.* This lemma can be found in Ivić ([21], pp. 56-57).

**Lemma 6.** *Let  $\Psi(u) = \sum_{n \leq u} \Lambda(n)$  and  $2 \leq t \leq u$ ; then,*

$$\Psi(u) = u - \sum_{|\gamma'| \leq t} \frac{u^\rho}{\rho} + O\left(\frac{u \log^2 u}{t}\right), \tag{44}$$

where  $\rho = \beta' + i\gamma'$  is the nontrivial zero of  $\zeta(s)$ .

*Proof.* This is a well-known explicit formula, which can be found in Karatsuba ([22], p. 80).

#### 4. The Estimate of the Integral $D_1$

In this section, we give the estimate of the integral  $D_1$  and set  $T = X^{3/4+119c/512-\eta}$ .

**Lemma 7.** *If  $1 \leq T_1 \leq T$ , then*

$$\frac{1}{\sqrt{T_1}} \sum_{0 < \gamma' \leq T_1} X^{\beta'} \ll X e^{-(\log X)^{1/4}}, \tag{45}$$

and

$$\frac{1}{T_1} \sum_{0 < \gamma' \leq T_1} X^{\beta'} \ll X e^{-(\log X)^{1/4}}. \tag{46}$$

*Proof.* We can get this lemma from Lemma 11 and Lemma 12 of Tolev [2].

**Lemma 8.** *If  $\max(|x|/\tau_1, |y|/\tau_2) < 1$ , then*

$$|J(x, y)| \ll X e^{-(\log X)^{1/5}}, \tag{47}$$

where

$$J(x, y) = \sum_{|\gamma'| \leq T} I_\rho(x, y), I_\rho(x, y) = \int_{\lambda X}^X e(t^c x + t^d y) t^{\rho-1} dt. \tag{48}$$

*Proof.* Without loss of generality, we consider the case  $0 \leq x < \tau_1, 0 \leq y < \tau_2$ . From  $\rho = \beta' + i\gamma'$ , we have

$$I_\rho(x, y) = \int_{\lambda X}^X t^{\beta'-1} e\left(t^c x + t^d y + \frac{\gamma'}{2\pi} \log t\right) dt. \tag{49}$$

Let

$$F(t) = t^c x + t^d y + \frac{\gamma'}{2\pi} \log t. \tag{50}$$

We define three sets of nontrivial zeroes of  $\zeta(s)$  as

$$M_1 = \left\{ \rho: |\gamma'| \leq T, -\gamma'/2\pi > 3(cX^c x + dX^d y)/2 \right\},$$

$$\begin{aligned}
M_2 &= \left\{ \rho: |\gamma'| \leq T, (c(\lambda X)^c x + d(\lambda X)^d y)/2 \right. \\
&\quad \left. \leq -\gamma'/2\pi \leq 3(cX^c x + dX^d y)/2 \right\},
\end{aligned}$$

$$M_3 = \left\{ \rho: |\gamma'| \leq T, -\gamma'/2\pi < (c(\lambda X)^c x + d(\lambda X)^d y)/2 \right\}, \tag{51}$$

and the set  $M_2$  may be empty.

We first consider  $X^{-c} \leq x \leq \tau_1$  or  $X^{-d} \leq y \leq \tau_2$ . In this situation, we consider the following three cases.

*Case 1.* When  $\rho \in M_1$ . In this case, we have

$$\begin{aligned}
F'(t) &= ct^{c-1} x + dt^{d-1} y + \frac{\gamma'}{2\pi t} \\
&\leq \frac{1}{\lambda X} \left( c(\lambda X)^c x + d(\lambda X)^d y - \frac{\gamma'}{2\pi} \right) < 0.
\end{aligned} \tag{52}$$

Applying Lemma 5, we have

$$|I_\rho(x, y)| \ll \frac{X^{\beta'}}{-cX^c x - dX^d y - \gamma'/2\pi} \ll \frac{X^{\beta'}}{|\gamma'|}. \tag{53}$$

Therefore, by (46), we can obtain



$$\begin{aligned}
\sum_{\rho \in M_1} |I_\rho(x, y)| &\ll \sum_{0 < \gamma' \leq T} \frac{X^{\beta'}}{\gamma'} \\
&\ll (\log X) \max_{1 \leq T_1 \leq T} \left( \frac{1}{T_1} \sum_{0 < \gamma' \leq T_1} X^{\beta'} \right) \\
&\ll X e^{-(\log X)^{1/5}}.
\end{aligned} \tag{54}$$

Case 2. When  $\rho \in M_2$ . In this case, we have

$$\begin{aligned}
F''(t) &= c(c-1)t^{c-2}x + d(d-1)t^{d-2}y - \frac{\gamma'}{2\pi t^2} \\
&= (X^c x + X^d y)X^{-2},
\end{aligned} \tag{55}$$

$$|I_\rho(x, y)| \ll \frac{X^{\beta'}}{\sqrt{X^c x + X^d y}}.$$

Hence, we use (45) and get

$$\begin{aligned}
\sum_{\rho \in M_2} |I_\rho(x, y)| &\ll \frac{1}{\sqrt{X^c x + X^d y}} \sum_{0 < \gamma' \leq X^c x + X^d y} X^{\beta'} \\
&\ll X e^{-(\log X)^{1/4}}.
\end{aligned} \tag{56}$$

If the set  $M_2$  is empty, then the upper bound is trivial.

Case 3. When  $\rho \in M_3$ . In this case, we have

$$\begin{aligned}
F'(t) &= ct^{c-1}x + dt^{d-1}y + \frac{\gamma'}{2\pi t} \gg \frac{1}{X} \left( c(\lambda X)^c x + d(\lambda X)^d y - \frac{\gamma'}{2\pi} \right) > 0, \\
|I_\rho(x, y)| &\ll \frac{X^{\beta'}}{c(\lambda X)^c x + d(\lambda X)^d y + \gamma'/2\pi}.
\end{aligned} \tag{57}$$

From (46), we have

$$\begin{aligned}
\sum_{\rho \in M_3} |I_\rho(x, y)| &\ll \sum_{-\pi c(\lambda X)^c x - \pi d(\lambda X)^d y < \gamma' \leq T} \frac{X^{\beta'}}{c(\lambda X)^c x + d(\lambda X)^d y + \gamma'/2\pi} \\
&\ll \sum_{-\pi c(\lambda X)^c x - \pi d(\lambda X)^d y < \gamma' \leq X^c x + X^d y} \frac{X^{\beta'}}{X^c x + X^d y} + \sum_{X^c x + X^d y < \gamma' \leq T} \frac{X^{\beta'}}{\gamma'} \\
&\ll (\log X) \max_{1 \leq T_1 \leq T} \left( \frac{1}{T_1} \sum_{0 < \gamma' \leq T_1} X^{\beta'} \right) \\
&\ll X e^{-(\log X)^{1/5}}.
\end{aligned} \tag{58}$$

Therefore, from (54)–(58), we obtain

$$|J(x, y)| \ll X e^{-(\log X)^{1/5}}. \tag{59}$$

Now we consider the remaining case  $0 \leq x \leq X^{-c}$  and  $0 \leq y \leq X^{-d}$ . We use the trivial estimate of  $|I_\rho(x, y)|$  and get

$$\sum_{\rho \in M_2} |I_\rho(x, y)| \ll \sum_{\rho \in M_2} X^{\beta'} \ll X^{\beta_0}, \tag{60}$$

where  $\beta_0 = \max_{\rho \in M_2} \beta' < 1$ .  $\sum_{\rho \in M_1} |I_\rho(x, y)|$  and  $\sum_{\rho \in M_3} |I_\rho(x, y)|$  are estimated analogically as in the previous case. Then, the estimate for  $|J(x, y)|$  is established.

**Lemma 9.** If  $\max(|x|/\tau_1, |y|/\tau_2) < 1$ , then

$$S(x, y) = I(x, y) + O\left(Xe^{-(\log X)^{1/5}}\right), \quad (61)$$

where

*Proof.* Noting

$$\sum_{\substack{\lambda X < p^v \leq X \\ v > 1}} (\log p) e(p^{cv}x + p^{dv}y) \ll \sum_{p^2 \leq X} \log p \sum_{\substack{\log(\lambda X) \\ \log p} < v < \frac{\log X}{\log p}} 1 \ll X^{1/2}, \quad (63)$$

we have

$$\begin{aligned} S(x, y) &= \sum_{\lambda X < n \leq X} \Lambda(n) e(n^c x + n^d y) - \sum_{\substack{\lambda X < p^v \leq X \\ v > 1}} (\log p) e(p^{cv}x + p^{dv}y) \\ &= \sum_{\lambda X < n \leq X} \Lambda(n) e(n^c x + n^d y) + O(X^{1/2}) := U(x, y) + O(X^{1/2}), \end{aligned} \quad (64)$$

where  $v$  denotes an integer number. Applying Abel's transformation, we get

Then, Lemma 6 implies

$$\begin{aligned} U(x, y) &= - \int_{\lambda X}^X (\Psi(t) - \Psi(\lambda X)) \frac{d}{dt} (e(t^c x + t^d y)) dt \\ &\quad + (\Psi(X) - \Psi(\lambda X)) e(X^c x + X^d y). \end{aligned} \quad (65)$$

$$\begin{aligned} U(x, y) &= - \int_{\lambda X}^X \left( t - \lambda X - \sum_{|y'| \leq T} \frac{t^\rho - (\lambda X)^\rho}{\rho} + O\left(\frac{X \log^2 X}{T}\right) \right) \frac{d}{dt} (e(t^c x + t^d y)) dt \\ &\quad + \left( X - \lambda X - \sum_{|y'| \leq T} \frac{X^\rho - (\lambda X)^\rho}{\rho} + O\left(\frac{X \log^2 X}{T}\right) \right) e(X^c x + X^d y) \\ &= - \int_{\lambda X}^X \left( t - \lambda X - \sum_{|y'| \leq T} \frac{t^\rho - (\lambda X)^\rho}{\rho} \right) \frac{d}{dt} (e(t^c x + t^d y)) dt \\ &\quad + \left( X - \lambda X - \sum_{|y'| \leq T} \frac{X^\rho - (\lambda X)^\rho}{\rho} \right) e(X^c x + X^d y) + O\left(\frac{X \log^2 X (X^c \tau_1 + X^d \tau_2)}{T}\right). \end{aligned} \quad (66)$$

Using integration by parts, we have

$$\begin{aligned}
 U(x, y) &= \int_{\lambda X}^X e(t^c x + t^d y) \left( 1 - \sum_{|y'| \leq T} t^{\rho-1} \right) dt + O\left( \frac{X \log^2 X (X^c \tau_1 + X^d \tau_2)}{T} \right) \\
 &= I(x, y) - J(x, y) + O\left( \frac{X \log^2 X (X^c \tau_1 + X^d \tau_2)}{T} \right).
 \end{aligned} \tag{67}$$

From Lemma 8, this lemma follows.

**Lemma 10** (see [18], Lemma 8). For  $S(x, y)$  defined by (17), we have

$$\int_{\Omega_1} \int |S(x, y)|^4 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy \ll \varepsilon_1 \varepsilon_2 X^{4-c-d} \log^8 X. \tag{68}$$

**Lemma 11** (see [18], Lemma 9). For  $I(x, y)$  defined in Lemma 9, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |I(x, y)|^4 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy \ll \varepsilon_1 \varepsilon_2 X^{4-c-d} \log^4 X. \tag{69}$$

Write

$$H = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I^6(x, y) e(-N_1 x - N_2 y) \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy, \tag{70}$$

and

$$H_1 = \int_{\Omega_1} \int I^6(x, y) e(-N_1 x - N_2 y) \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy. \tag{71}$$

**Lemma 12.** For integrals  $H$  and  $H_1$  defined by (70) and (71), respectively, we have

$$|H - H_1| \ll \frac{\varepsilon_1 \varepsilon_2 X^{6-c-d}}{\log X}. \tag{72}$$

*Proof.* We have

$$\begin{aligned}
 |H - H_1| &\ll \int \int_{\mathbb{R}^2 / \Omega_1} |I(x, y)|^6 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy \\
 &\ll \max_{\mathbb{R}^2 \setminus \Omega_1} |I(x, y)|^2 \int \int_{\mathbb{R}^2} |I(x, y)|^4 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy.
 \end{aligned} \tag{73}$$

Applying Lemma 5 with  $G(t) = 1$  and  $F(t) = xt^c + yt^d$ , from  $F''(t) = c(c-1)xt^{c-2} + d(d-1)yt^{d-2}$ , we can get

$$\max_{\mathbb{R}^2 \setminus \Omega_1} |I(x, y)| \ll X^{5/8+\eta/2}. \tag{74}$$

Then, by Lemma 11, we can get this lemma.

**Lemma 13.** For the integral  $H$  defined by (70), we have

$$H \gg \varepsilon_1 \varepsilon_2 X^{6-c-d}. \tag{75}$$

*Proof.* We have

$$\begin{aligned}
 H &= \int \cdots \int_{\lambda X < t_1, \dots, t_6 \leq X} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e(x(t_1^c + \cdots + t_6^c - N_1)) e(y(t_1^d + \cdots + t_6^d - N_2)) \times \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy dt_1 \cdots dt_6 \\
 &= \int \cdots \int_{C_1} \Delta dt_1 \cdots dt_6 + \int \cdots \int_{C_2} \Delta dt_1 \cdots dt_6,
 \end{aligned} \tag{76}$$

where

$$\begin{aligned}
C_1 &= \left\{ \lambda X < t_1, \dots, t_6 \leq X : \left| \frac{t_1^c + \dots + t_6^c - N_1}{\varepsilon_1} \right| < 1 \text{ and } \left| \frac{t_1^d + \dots + t_6^d - N_2}{\varepsilon_2} \right| < 1 \right\}, \\
C_2 &= \left\{ \lambda X < t_1, \dots, t_6 \leq X : \left| \frac{t_1^c + \dots + t_6^c - N_1}{\varepsilon_1} \right| \geq 1 \text{ or } \left| \frac{t_1^d + \dots + t_6^d - N_2}{\varepsilon_2} \right| \geq 1 \right\}, \\
\Delta &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e(x(t_1^c + \dots + t_6^c - N_1)) e(y(t_1^d + \dots + t_6^d - N_2)) \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy.
\end{aligned} \tag{77}$$

From (31), we get

$$\begin{aligned}
\int_{-\infty}^{\infty} e(x(t_1^c + \dots + t_6^c - N_1)) \varphi_{\varepsilon_1}(x) dx &= \varphi\left(\frac{t_1^c + \dots + t_6^c - N_1}{\varepsilon_1}\right), \\
\int_{-\infty}^{\infty} e(x(t_1^d + \dots + t_6^d - N_2)) \varphi_{\varepsilon_2}(x) dx &= \varphi\left(\frac{t_1^d + \dots + t_6^d - N_2}{\varepsilon_2}\right).
\end{aligned} \tag{78}$$

In (76), the integral over  $C_2$  is convergent, and hence  $\int \dots \int \Delta dt_1 \dots dt_6 = O(1)$ . Applying Lemma 2, we have

$$H \gg \int_{C_1} \dots \int 1 dt_1 \dots dt_6 \gg X^6 \int_{C'_1} \dots \int 1 d(t_1/X) \dots d(t_6/X), \tag{79}$$

where

$$\begin{aligned}
C'_1 &= \left\{ \lambda < t_1/X, \dots, t_6/X : \left| (t_1/X)^c + \dots + (t_6/X)^c - 1 \right| \right. \\
&\quad < \varepsilon_1 X^{-c} \text{ and } \left| (t_1/X)^d + \dots + (t_6/X)^d - N_2/X^d \right| \\
&\quad < \varepsilon_2 X^{-d} \left. \right\}.
\end{aligned} \tag{80}$$

From (13) and Lemma 1, we have

$$H \gg \varepsilon_1 \varepsilon_2 X^{6-c-d}. \tag{81}$$

**Lemma 14.** For the integral  $D_1$  in (20), we have

$$|D_1| \gg \varepsilon_1 \varepsilon_2 X^{6-c-d}. \tag{82}$$

*Proof.* We have

$$\begin{aligned}
|D_1 - H_1| &\ll \int \int_{\Omega_1} |S^6(x, y) - I^6(x, y)| \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy \\
&\ll \max_{\Omega_1} |S(x, y) - I(x, y)|^2 \int \int_{\Omega_1} (|S(x, y)|^4 + |I(x, y)|^4) \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy.
\end{aligned} \tag{83}$$

From Lemmas 9–11, we can obtain

$$|D_1 - H_1| \ll \frac{\varepsilon_1 \varepsilon_2 X^{6-c-d}}{\log X}. \tag{84}$$

By Lemma 12, we have

$$D_1 = H + O\left(\frac{\varepsilon_1 \varepsilon_2 X^{6-c-d}}{\log X}\right). \tag{85}$$

Then, from Lemma 13, we can get this lemma.

## 5. The Estimate for the Integral $D_2$

**Lemma 15.** Let  $a_m, b_n$  be arbitrary complex numbers and

$$\max(|x|/\tau_1, |y|/\tau_2) \geq 1, \max(|x|/K_1, |y|/K_2) \leq 1, \tag{86}$$

$$X^{1/4} < R \leq X^{1/2}, X^{1/4} < L < L_1 \leq 2L, LR \leq X.$$

Write

$$W = \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq R} a_m b_n e((mn)^c x + (mn)^d y). \tag{87}$$

Then,

$$|W| \ll (\mathcal{B})^{1/2} X^{101/238} \log^{-13} X, \quad (88)$$

where

$$\mathcal{A} = \sum_{X^{1/4} < m \leq R} |a_m|^2, \mathcal{B} = \sum_{L < n \leq L_1} |b_n|^2. \quad (89)$$

*Proof.* Without loss of generality, we can consider the case  $\tau_1 \leq x \leq K_1, 0 < y \leq K_2$ . We define  $R_i (0 \leq i \leq Q)$ :

$$R_0 = X^{1/4}, R_{i+1} = \min(R_i + s, R), R_Q = R, \quad (90)$$

where  $s \in [1, R]$  will be determined later and  $Q \ll R/s$ . Thus,  $W$  can be rewritten as

$$W = \sum_{L < n \leq L_1} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i} a_m b_n e((mn)^c x + (mn)^d y). \quad (91)$$

By Cauchy's inequality, we have

$$|W|^2 \leq \mathcal{B}Q \sum_{L < n \leq L_1} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m_1, m_2 \leq R_i} a_{m_1} \bar{a}_{m_2} e((m_1^c - m_2^c)n^c x + (m_1^d - m_2^d)n^d y). \quad (92)$$

We rearrange the sums as follows:

$$\begin{aligned} |W|^2 &\ll \mathcal{B}Q \left( \sum_{L < n \leq L_1} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i} |a_m|^2 \right) \\ &\quad + \mathcal{B}Q \sum_{1 \leq i \leq Q} \sum_{\substack{R_{i-1} < m_1, m_2 \leq R_i \\ m_1 \neq m_2}} \left( |a_{m_1}| |a_{m_2}| \left| \sum_{L < n \leq L_1} e((m_1^c - m_2^c)n^c x + (m_1^d - m_2^d)n^d y) \right| \right) \\ &\ll \mathcal{B}Q \left( \mathcal{A}L + \sum_{1 \leq h \leq s} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i - h} |a_m| |a_{m+h}| \left| \sum_{L < n \leq L_1} e(f(n)) \right| \right), \end{aligned} \quad (93)$$

where

$$f(n) = ((m+h)^c - m^c)n^c x + ((m+h)^d - m^d)n^d y. \quad (94)$$

Next, we handle the exponential sum  $|\sum_{L < n \leq L_1} e(f(n))|$  of (93). From the derivatives of  $f$ , we get

$$f^j(n) = \left[ ((m+h)^c - m^c)L^{c-1}x + ((m+h)^d - m^d)L^{d-1}y \right] L^{1-j}. \quad (95)$$

Then, by Lemma 3 with the exponent pair (see also [21], p. 77)

$$BA^2BA^2B(0, 1) = (13/40, 22/40), \quad (96)$$

we have

$$\begin{aligned} \left| \sum_{L < n \leq L_1} e(f(n)) \right| &\ll \left[ ((m+h)^c - m^c)L^{c-1}x + ((m+h)^d - m^d)L^{d-1}y \right]^{13/40} L^{22/40} \\ &\ll \left[ ((m+h)^c - m^c)L^{c-1}K_1 + ((m+h)^d - m^d)L^{d-1}K_2 \right]^{13/40} L^{22/40} \\ &\ll s^{13/40} \left( R^{13(c-1)/40} L^{(13c+9)/40} K_1^{-13/40} + R^{13(d-1)/40} L^{(13d+9)/40} K_2^{-13/40} \right). \end{aligned} \quad (97)$$

Inserting this upper bound into (93), we have

$$|W|^2 \ll \mathcal{B}Q \left[ \mathcal{A}L + s^{13/40} \left( R^{13(c-1)/40} L^{(13c+9)/40} K_1^{13/40} + R^{13(d-1)/40} L^{(13d+9)/40} K_2^{13/40} \right) \sum_0 \right], \quad (98)$$

where

$$\sum_0 \sum_{1 \leq h \leq s} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i - h} |a_m| |a_{m+h}|. \quad (99)$$

By Cauchy's inequality, we obtain

$$\begin{aligned} \sum_0 &\leq \sum_{1 \leq h \leq s} \sum_{X^{1/4} < m \leq R-h} |a_m| |a_{m+h}| \\ &\leq \sum_{1 \leq h \leq s} \left( \sum_{X^{1/4} < m \leq R-h} |a_m|^2 \right)^{1/2} \left( \sum_{X^{1/4} < m \leq R-h} |a_{m+h}|^2 \right)^{1/2} \\ &\ll s \mathcal{A}. \end{aligned} \quad (100)$$

From (98) and  $Q \ll R/s$ , we have

$$|W|^2 \ll \mathcal{A} \mathcal{B} L R \left[ 1/s + s^{13/40} \left( R^{13(c-1)/40} L^{(13c-31)/40} K_1^{13/40} + R^{13(d-1)/40} L^{(13d-31)/40} K_2^{13/40} \right) \right]. \quad (101)$$

We take  $1/s = s^{13/40} (R^{13(c-1)/40} L^{(13c-31)/40} K_1^{13/40} + R^{13(d-1)/40} L^{(13d-31)/40} K_2^{13/40})$ , i.e.,

$$\begin{aligned} s &= R^{13(1-c)/53} L^{(31-13c)/53} K_1^{-13/53} \\ &\quad + R^{13(1-d)/53} L^{(31-13d)/53} K_2^{-13/53}, \end{aligned} \quad (102)$$

and then  $s \in [1, R]$ . Inserting this value of  $s$  into (101), we get

$$|W|^2 \ll \mathcal{A} \mathcal{B} X^{101/119} \log^{-26} X, \quad (103)$$

which yields this lemma.

**Lemma 16** (see [2], Lemma 9). Let  $a_m, b_n$  be arbitrary complex numbers and  $L < L_1 \leq 2L, L \leq X$ . Write

$$V = \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/L} a_m b_n e((mn)^c x + (mn)^d y). \quad (104)$$

Then, there exist  $a'_m, b'_n$  satisfying  $|a'_m| \leq |a_m|, |b'_n| \leq |b_n|$  such that

$$|V| \ll (\log X) \left| \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/L} a'_m b'_n e((mn)^c x + (mn)^d y) \right|. \quad (105)$$

**Lemma 17.** Assume that  $\max(|x|/\tau_1, |y|/\tau_2) \geq 1$ ,  $\max(|x|/K_1, |y|/K_2) \leq 1$ ; then,

$$|S(x, y)| \ll X^{110/119} \log^{-9} X. \quad (106)$$

*Proof.* Without loss of generality, we can consider the case  $\tau_1 \leq x \leq K_1, 0 < y \leq K_2$ . Clearly,

$$S(x, y) = V_0(x, y) - V_1(x, y) + O(X^{1/2}), \quad (107)$$

where

$$\begin{aligned} V_0(x, y) &= \sum_{X^{1/4} < n \leq X} \Lambda(n) e(n^c x + n^d y), \\ V_1(x, y) &= \sum_{X^{1/4} < n \leq \lambda X} \Lambda(n) e(n^c x + n^d y). \end{aligned} \quad (108)$$

Hence, it is sufficient to prove that

$$|V_0(x, y)|, |V_1(x, y)| \ll X^{110/119} \log^{-9} X. \quad (109)$$

The estimates of  $V_0(x, y)$  and  $V_1(x, y)$  are similar, and thus we focus on the estimate of  $V_0(x, y)$ . Using Vaughan's identity (see [23]), we have

$$V_0(x, y) = S_1 - S_2 - S_3, \quad (110)$$

where

$$\begin{aligned} S_1 &= \sum_{g \leq X^{1/4}} \mu(g) \sum_{l \leq X/g} (\log l) e((lg)^c x + (lg)^d y), \\ S_2 &= \sum_{k \leq X^{1/2}} \sum_{r \leq X/k} c_k e((kr)^c x + (kr)^d y), \\ S_3 &= \sum_{X^{1/4} < m \leq X^{3/4}} \sum_{X^{1/4} < n \leq X/m} a_m \Lambda(n) e((mn)^c x + (mn)^d y), \end{aligned} \quad (111)$$

where  $|c_k| \leq \log k$  and  $|a_m| \leq \tau(m)$ .

For  $S_2$ , noting

$$\sum_{X^{1/4} < k \leq X^{1/2}} \sum_{r \leq X^{1/4}} c_k e((kr)^c x + (kr)^d y) \ll X^{3/4} \log X, \quad (112)$$

we have

$$S_2 = S_2^{(1)} + S_2^{(2)} + O(X^{3/4} \log X), \quad (113)$$

where

$$\begin{aligned} S_2^{(1)} &= \sum_{k \leq X^{1/4}} \sum_{r \leq X/k} c_k e((kr)^c x + (kr)^d y), \\ S_2^{(2)} &= \sum_{X^{1/4} < k \leq X^{1/2}} \sum_{X^{1/4} < r \leq X/k} c_k e((kr)^c x + (kr)^d y). \end{aligned} \quad (114)$$

For  $S_1$ , we have

$$|S_1| \leq \sum_{g \leq X^{1/4}} \left| \sum_{l \leq X/g} (\log l) e((lg)^c x + (lg)^d y) \right|. \quad (115)$$

The second summation over  $l$  in (115) is

$$\ll (\log^2 X) \max_{L_2 \in [L, L_1]} \left| \sum_{L < l \leq L_2} e(h(l)) \right|, \quad (116)$$

where  $L < L_1 \leq 2L, L_1 \leq X/g$ , and  $h(l) = (lg)^c x + (lg)^d y$ . Noting  $h^j(l) = (L^{c-1} g^c x + L^{d-1} g^d y) L^{1-j}$ , Lemma 3 with exponent pair  $(13/40, 22/40)$  gives

$$\left| \sum_{L < l \leq L_1} e(h(l)) \right| \ll \left( L^{c-1} g^c K_1 + L^{d-1} g^d K_2 \right)^{13/40} L^{22/40} \\ \ll X^{13c/40} L^{9/40} K_1^{13/40} + X^{13d/40} L^{9/40} K_2^{13/40}. \quad (117)$$

Recalling (115) and (116), we have

$$|S_1| \ll X^{110/119} \log^{-9} X. \quad (118)$$

Arguing similarly, we can get the following same bound of  $S_2^{(1)}$ :

$$|S_2^{(1)}| \ll X^{110/119} \log^{-9} X. \quad (119)$$

For  $S_3$ , we divide it into three parts:

$$S_3 = W_1 + W_2 + W_3, \quad (120)$$

where

$$W_1 = \sum_{X^{1/2} < n \leq X^{3/4}} \sum_{X^{1/4} < m \leq X/n} a_m \Lambda(n) e((mn)^c x + (mn)^d y), \\ W_2 = \sum_{X^{1/2} < n \leq X^{3/4}} \sum_{X^{1/4} < m \leq X/n} a_n \Lambda(m) e((mn)^c x + (mn)^d y), \\ W_3 = \sum_{X^{1/4} < n \leq X^{1/2}} \sum_{X^{1/4} < m \leq X^{1/2}} a_m \Lambda(n) e((mn)^c x + (mn)^d y). \quad (121)$$

For  $W_1$ , using dyadic subdivision and Lemma 16, we have

$$|W_1| \ll (\log^2 X) |W_1'(L)|, \quad (122)$$

where

$$W_1'(L) = \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/L} a_m' b_n' e((mn)^c x + (mn)^d y), \quad (123)$$

where  $X^{1/2} \leq L < L_1 \leq 2L \leq X^{3/4}$ ,  $|a_m'| \leq |a_m| \leq \tau(m)$ , and  $|b_n'| \leq \Lambda(n)$ . Lemma 15 implies

$$|W_1'(L)| \ll \left( \frac{X}{L} \log^3 X \cdot L \log X \right)^{1/2} X^{101/238} \log^{-13} X \\ = X^{110/119} \log^{-11} X, \quad (124)$$

where we used the mean value estimates

$$\sum_{m \leq y} \tau^2(m) \ll y \log^3 y, \quad \sum_{n \leq y} \Lambda^2(n) \ll y \log y. \quad (125)$$

Therefore, we have

$$|W_1| \ll X^{110/119} \log^{-9} X. \quad (126)$$

We estimate  $W_2$  and  $W_3$  similar to  $W_1$  and get

$$|W_2|, |W_3| \ll X^{110/119} \log^{-9} X. \quad (127)$$

For  $S_2^{(2)}$ , we follow a similar argument to  $S_3$  and get

$$S_2^{(2)} = U_1 + U_2, \quad (128)$$

where

$$U_1 = \sum_{X^{1/4} < r \leq X^{1/2}} \sum_{X^{1/4} < k \leq X^{1/2}} c_k e((kr)^c x + (kr)^d y), \\ U_2 = \sum_{X^{1/2} < r \leq X^{3/4}} \sum_{X^{1/4} < k \leq X/r} c_k e((kr)^c x + (kr)^d y). \quad (129)$$

Using Lemmas 15 and 16 and the mean value estimate

$$\sum_{k \leq y} \log^2 k \ll y \log^2 y, \quad (130)$$

we get

$$|U_1|, |U_2| \ll X^{110/119} \log^{-9} X. \quad (131)$$

From (110) and (118)–(131), we can obtain (109).

**Lemma 18** (see [18], Lemma 14). *For  $S(x, y)$  defined by (17), we have*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S(x, y)|^4 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy \ll X^2 \log^6 X. \quad (132)$$

**Lemma 19.** *For the integral  $D_2$  in (20), we have*

$$|D_2| \ll \frac{\varepsilon_1 \varepsilon_2 X^{6-c-d}}{\log X}. \quad (133)$$

*Proof.* By Lemmas 17 and 18, we have

$$|D_2| \ll \max_{\Omega_2} |S(x, y)|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S(x, y)|^4 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y) dx dy \\ \ll (X^{110/119} \log^{-9} X)^2 X^2 \log^6 X \\ \ll X^{458/119} \log^{-12} X, \quad (134)$$

which yields this lemma.

## 6. Proof of Theorem 1

In this section, we first give the estimate of  $D_3$  in (20) and then complete the proof of Theorem 1.

For the integral  $D_3$  in (20), by Lemma 2, we can easily get

$$|D_3| \ll 1. \quad (135)$$

From Lemmas 14 and 19 and (135), we know that (22)–(24) follows, respectively. Therefore, recalling Lemma 4 and (20), we can get

$$B \gg \varepsilon_1 \varepsilon_2 X^{6-c-d}, \quad (136)$$

from which we complete the proof of Theorem 1.

## Data Availability

The data supporting the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# Resonance between the Representation Function and Exponential Functions over Arithmetic Progression

Li Ma  and Xiaofei Yan 

School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong 250100, China

Correspondence should be addressed to Xiaofei Yan; [xfyan.sdu@gmail.com](mailto:xfyan.sdu@gmail.com)

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Let  $r(n)$  denote the number of representations of a positive integer  $n$  as a sum of two squares, i.e.,  $n = x_1^2 + x_2^2$ , where  $x_1$  and  $x_2$  are integers. We study the behavior of the exponential sum twisted by  $r(n)$  over the arithmetic progressions  $\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e(\alpha n^\beta)$ , where  $0 \neq \alpha \in \mathbb{R}$ ,  $0 < \beta < 1$ ,  $e(x) = e^{2\pi i x}$ , and  $n \sim X$  means  $X < n \leq 2X$ . Here,  $X > 1$  is a large parameter,  $1 \leq l \leq q$  are integers, and  $(l, q) = 1$ . We obtain the upper bounds in different situations.

## 1. Introduction

In analytic number theory, the problems concerning nonlinear exponential twisting arithmetic functions arise naturally in investigating equidistribution theory, zero-distribution of  $L$ -functions, and so on. Let  $a_n$  be some arithmetic number-theoretic function. We usually consider the general nonlinear exponential sum of the form

$$S(X, \alpha) = \sum_{n \sim X} a_n e(\alpha n^\beta), \quad 0 \neq \alpha \in \mathbb{R}, \quad 0 \leq \beta \leq 1. \quad (1)$$

Here,  $n \sim X$  means  $X \leq n \leq 2X$ , and  $e(z) = e^{2\pi i z}$ . When  $\beta = 1/2$  and  $a_n = \Lambda(n)$  is the von Mangoldt function, the sum  $S(X, \alpha)$  was studied by Vinogradov [1]. For  $a_n = \Lambda(n)$  and  $a_n = \mu(n)$  ( $\mu$  is the Möbius function), the sums  $S(X, \alpha)$  were studied by Iwaniec, Luo, and Sarnak, and they showed these sums are intimately related to  $L$ -functions of  $GL_2$ . If  $f$  is a holomorphic cusp form of even weight on the upper half plane, they also proved that a good upper bound of  $S(X, \alpha)$  implies a quasi-Riemann hypothesis for  $L(s, f)$  [2]. In addition, studying the behavior of the Fourier coefficients of automorphic forms has great significance in modern number theory. Analytic number

theorists always estimate the mean value or the twisted sums such as  $S(X, \alpha)$  mentioned above to obtain some information about the Fourier coefficients (for examples, see [3–10]).

If  $\beta$  is a variable and  $a_n$  are the Fourier coefficients of automorphic forms, these sums were studied by Ren and Ye [7] and Sun and Wu [11]. They proved that the resonance phenomenon occurs only when  $\beta = 1/2$  and  $|\alpha|$  is close to  $2\sqrt{k}$ ,  $k \in \mathbb{Z}^+$ . Let  $r(n)$  denote the number of representations of a positive integer  $n$  as a sum of two squares, i.e.,  $n = x_1^2 + x_2^2$ , where  $x_1$  and  $x_2$  are integers. Sun and Wu [11] also studied the case that  $a_n = r(n)$  and obtained the resonance phenomenon. Yan [12] studied the nonlinear exponential sum twisting the Fourier coefficients of Maass forms over the arithmetic progress and obtained an asymptotic formula for the sum

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} \lambda_g(n) e(\alpha n^\beta), \quad 0 \neq \alpha \in \mathbb{R}, \quad 0 \leq \beta \leq 1, \quad (2)$$

where  $g$  is a Maass cusp form for  $SL(2, \mathbb{Z})$  and  $\lambda_g(n)$  is the  $n$ -th Fourier coefficient of  $g$ . These analogues to the arithmetic progression are the main motivation of this paper.

In this paper, we study the nonlinear exponential sum

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e(\alpha n^\beta), \quad (3)$$

where  $0 \neq \alpha \in \mathbb{R}$ ,  $0 < \beta < 1$ . Here,  $X > 1$  is a large parameter.  $1 \leq l \leq q$  are integers, and  $(l, q) = 1$ . We consider the case that  $q$  tends to infinity as  $X \rightarrow \infty$  and obtain analogues of the result of Sun and Wu [11].

The principal aim of this paper is to prove the following result.

**Theorem 1.** *Let  $X > 1$ ,  $0 < \beta < 1$ , and  $0 \neq \alpha \in \mathbb{R}$ . Let  $l, q \in \mathbb{N}$  and  $l \leq q \leq X^{1/2}$ . Let  $\delta = (q, 4)$ .*

(i) *If  $|\alpha| \beta q X^\beta < (1/4) \sqrt{\delta X}$ , then we have*

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e(\alpha n^\beta) \ll q^{(1/3)+\varepsilon} X (|\alpha| \beta X^\beta)^{-1}. \quad (4)$$

(ii) *If  $|\alpha| \beta q X^\beta \geq (1/4) \sqrt{\delta X}$  and  $\beta \neq (1/2)$ , then we have*

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e(\alpha n^\beta) \ll q^{(1/2)+\varepsilon} |2\beta - 1|^{-(1/2)} (|\alpha| \beta X^\beta)^{1+\varepsilon}. \quad (5)$$

(iii) *If  $|\alpha| \beta q X^\beta \geq (1/4) \sqrt{\delta X}$  and  $\beta = 1/2$ , then for  $|\alpha| < 1/q$  or  $|\alpha| > \sqrt{X}/q$ , we have*

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e(\alpha n^\beta) \ll q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}. \quad (6)$$

For  $(1/q) \leq |\alpha| \leq (\sqrt{X}/q)$ , we have

$$\begin{aligned} \sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e(\alpha n^\beta) &= \frac{1}{q} \sum_{c|q} \varepsilon(a, n_c) c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D_c} n_c; c)| n_c^{-1/4} G_Q(c, a) \\ &\quad + O(q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}), \end{aligned} \quad (7)$$

where

$$\varepsilon(a, n_c) = \frac{\delta_c \varepsilon_a a_1}{2} \int_1^2 u^{-1/4} e\left(\operatorname{sgn}(\alpha) \left(|\alpha| - \frac{2\sqrt{\delta n_c}}{2c}\right) \sqrt{Xu}\right) du, \quad (8)$$

and  $\delta_c = 1$  or  $0$  according to if there exists a positive integer  $n_c$  for  $c|q$  satisfying

$$|c|\alpha| - \sqrt{\delta n_c}| \leq X^{-1/2}, \quad (9)$$

or not.

(iv) *In particular, if  $|\alpha| = (\sqrt{\delta k}/q)$  with  $1 \leq k \leq (X/\delta)$ , then we have*

$$\begin{aligned} \sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e\left(\pm \frac{\sqrt{\delta k n}}{q}\right) &= \frac{2\sqrt{2}}{3} (2^{3/4} - 1) \delta^{-1/4} (1 \pm i) |S(-l, -\overline{D_c} k; q)| q^{-5/2} X^{3/4} r(k, Q^*) k^{-1/4} G_Q(q, a) \\ &\quad + O(k^{-1/4} X^{1/4} (qXk)^\varepsilon). \end{aligned} \quad (10)$$

To prove Theorem 1, we shall follow the steps in [7, 11, 12] first. Then, we will use a new Voronoi-type summation formula generalized by Hu et al. [13] to get the asymptotic formula, and this is the key to success. Thus, we can get the Kloosterman sum, use Weil's bound to get the saving in the  $q$ -aspect, and then obtain a similar main term as that in [12].

## 2. Some Lemmas

To prove Theorem 1, we need to quote some lemmas. First, we consider the Kloosterman sum, which is defined as

$$S(m, n; c) = \sum_{d \pmod{c}}^* e\left(\frac{md + n\overline{d}}{c}\right), \quad (11)$$

where  $\bar{d}$  denotes the inverse of  $d$  modulo  $c$ . The famous Weil's bound of the Kloosterman sum is

$$|S(m, n; c)| \leq (m, n, c)^{1/2} c^{1/2} \tau(c), \quad (12)$$

where  $d(c)$  denotes the divisor function.

Let  $J_\nu$  denote the standard J-Bessel function. Let  $r(n, Q)$  denote the number of representations of  $n$  by the quadratic form  $Q$ , namely,

$$r(n, Q) = \#\{m \in \mathbb{Z}^l : Q(m) = n\}. \quad (13)$$

If  $Q(x) = x_1^2 + x_2^2 + \cdots + x_l^2$ , we denote  $r(n, Q) = r_l(n)$ . Let  $A$  be a symmetric positive definite integral matrix

associated to  $Q$ , and let  $D = \det A$  denote the discriminant of  $A$ . Let  $G_Q(c, a) = \sum_{x \bmod c} e(aQ(x)/c)$ ,  $\delta = (c, D)$ ,  $D = \delta D_c$ , and  $Q^*$  be a positive definite integral quadratic form, which is defined in terms of the Smith normal form of  $Q^\dagger$  (see [14]), and  $Q^\dagger$  the adjoint form of  $Q$ .

Then, we have the following Voronoi summation formula [13].

**Lemma 1.** Let  $(c, a) = 1$  and  $F \in C_0^\infty(\mathbb{R}^+)$  be a smooth compact function. We have

$$\sum_{n=1}^{\infty} r(n, Q) e\left(\frac{an}{c}\right) F(n) = \frac{(2\pi)^{m/2} c^{-m}}{\Gamma(m/2) \sqrt{D}} G_Q(c, a) \Phi\left(\frac{m}{2}\right) + \sum_{n=1}^{\infty} r(n, Q^*) e\left(\frac{-a\bar{D}_c n}{c}\right) (\delta n)^{(1-(m/2))/2} G(n), \quad (14)$$

where

$$G(n) = \frac{4\pi D^{(m/4)-1}}{c^{(m/2)+1}} G_Q(c, a) \int_0^\infty F(x) x^{((m/2)-1)/2} J_{(m/2)-1}\left(\frac{4\pi\sqrt{\delta n x}}{c\sqrt{D}}\right) dx, \quad (15)$$

and  $\Phi(s)$  is the Mellin transform of  $F(x)$ , which is given by

$$\Phi(s) = \int_0^\infty F(x) x^{s-1} dx. \quad (16)$$

*Remark 1.* In our situation,  $D = 4$ , but we still want to compute the dependence of  $D$  in our proof. If one can obtain

the asymptotic formula for general  $r(n, Q)$ , then our result can be applied directly to get the analogues for  $r(n, Q)$ .

For asymptotic expansions of the Bessel functions, we quote the following lemma.

**Lemma 2.** For  $z > 0$  large, we have

$$J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z \mp \frac{\pi}{2} \nu - \frac{\pi}{4}\right) - \frac{\nu^2 - (1/4)}{2z} \sin\left(z \mp \frac{\pi}{2} \nu - \frac{\pi}{4}\right) + O(z^{-2}) \right\}. \quad (17)$$

For the mean value of  $r(n)$ , we have the classical result [16].

**Lemma 3.**

$$\sum_{n \leq x} r(n) = \pi x + O(x^{1/3}). \quad (18)$$

We also need the following result [17].

**Lemma 4.** Let  $G(x)$  and  $F(x)$  be real functions in  $[a, b]$  with  $G(x)/F'(x)$  being monotonic. Suppose that  $|G(x)| \leq M$ .

(a) If  $F'(x) \geq u > 0$  or  $F'(x) \leq -u < 0$ , then

$$\int_a^b G(x) e(F(x)) dx \ll \frac{M}{u}. \quad (19)$$

(b) If  $F''(x) \geq \nu > 0$  or  $F''(x) \leq -\nu < 0$ , then

$$\int_a^b G(x) e(F(x)) dx \ll \frac{M}{\sqrt{\nu}}. \quad (20)$$

### 3. Proof of Theorem 1

In this section, we will finish the proof of Theorem 1. By the formula of the Ramanujan sum

$$\sum_{d|q} \sum_{a=1, (a,d)=1}^d e\left(\frac{an}{d}\right) = \begin{cases} 0, & \text{if } q \nmid n, \\ q, & \text{if } q|n, \end{cases} \quad (21)$$

we get

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n) e(\alpha n^\beta) = \frac{1}{q} \sum_{c|q} \sum_{a \pmod{c}}^* e\left(-\frac{al}{c}\right) \sum_{n \sim X} r(n) e\left(\frac{an}{c}\right) e(\alpha n^\beta), \quad (22)$$

where  $\sum^*$  means the summation is restricted by  $(a, c) = 1$ .

Let  $\Delta > 1$ , and let  $0 \leq \phi(x) \leq 1$  be a  $C^\infty$  function supported on  $[1, 2]$ , which is identically 1 on  $[1 + \Delta^{-1}, 2 - \Delta^{-1}]$  and satisfies  $\phi^{(r)}(x) \ll \Delta^{-r}$  for  $r \geq 0$ . Using the bound in Lemma 3, we get

$$\sum_{n \sim X} r(n) e\left(\frac{an}{c}\right) e(\alpha n^\beta) = \sum_{n=1}^{\infty} r(n) e\left(\frac{an}{c}\right) W(n) + R_1(N, X, \Delta), \quad (23)$$

where

$$W(x) = \phi\left(\frac{x}{X}\right) e(\alpha x^\beta), \quad (24)$$

$$\begin{aligned} R_1(N, X, \Delta) &\ll \sum_{X < n \leq X + (X/\Delta)} |r(n)| \\ &\ll \left(X + \frac{X}{\Delta}\right) + \left(X + \frac{X}{\Delta}\right)^{(1/3)} - X - X^{(1/3)} \\ &\ll \left(\frac{X}{\Delta}\right)^{1+\varepsilon}. \end{aligned} \quad (25)$$

Applying Lemma 1 with  $m = 2$  and  $F(n) = W(n)$ , we have

$$\begin{aligned} \sum_{n=1}^{\infty} r(n) e\left(\frac{an}{c}\right) W(n) &= \frac{2\pi}{c^2 \sqrt{D}} G_Q(c, a) \int_0^\infty W(x) dx \\ &\quad + \sum_{n=1}^{\infty} r(n, Q^*) e\left(\frac{-aD_c n}{c}\right) G(n), \end{aligned} \quad (26)$$

where

$$G(n) = \frac{4\pi}{c^2 \sqrt{D}} G_Q(c, a) \int_0^\infty W(x) J_0\left(\frac{4\pi \sqrt{\delta n x}}{c \sqrt{D}}\right) dx. \quad (27)$$

For the first term, changing variables  $x = Xt$  and applying Lemma 4(a), we get

$$\int_0^\infty W(x) dx = X \int_1^2 \phi(t) e(\alpha (Xt)^\beta) dt \ll X(|\alpha| \beta X^\beta)^{-1}. \quad (28)$$

By [16], we have  $G_Q(c, a) \ll c$ . Thus, the contribution of the first term to (22) is

$$\begin{aligned} &\frac{1}{q} \sum_{c|q} \sum_{a \pmod{c}}^* e\left(-\frac{al}{c}\right) \frac{2\pi}{c^2 \sqrt{D}} G_Q(c, a) X(|\alpha| \beta X^\beta)^{-1} \\ &\ll \frac{1}{q} \sum_{c|q} \sum_{a \pmod{c}}^* e\left(-\frac{al}{c}\right) \frac{1}{c} X(|\alpha| \beta X^\beta)^{-1} \\ &\ll q^{-1+\varepsilon} X(|\alpha| \beta X^\beta)^{-1}. \end{aligned} \quad (29)$$

Next, we turn to estimate the contribution from the term involving  $G(n)$ . Using Lemma 2, we have

$$J_0(z) = \frac{1-i}{2\sqrt{\pi z}} (e^{iz} + ie^{-iz}) - \frac{1+i}{16\sqrt{\pi z}^3} (e^{iz} - ie^{-iz}) + O(z^{-5/2}). \quad (30)$$

Taking  $z = (4\pi \sqrt{\delta n x})/c \sqrt{D}$ , we obtain

$$\begin{aligned} J_0\left(\frac{4\pi \sqrt{\delta n x}}{c \sqrt{D}}\right) &= \frac{(1-i)c^{1/2} D^{1/4}}{4\pi \delta^{1/4}} (xy)^{-1/4} \left(e\left(\frac{2\sqrt{\delta x y}}{c \sqrt{D}}\right) + ie\left(-\frac{2\sqrt{\delta x y}}{c \sqrt{D}}\right)\right) - \frac{(1+i)c^{3/2} D^{3/4}}{128\pi^2 \delta^{3/4}} (xy)^{-3/4} \left(e\left(\frac{2\sqrt{\delta x y}}{c \sqrt{D}}\right) \right. \\ &\quad \left. - ie\left(-\frac{2\sqrt{\delta x y}}{c \sqrt{D}}\right)\right) + O(c^{5/2} (xy)^{-5/4}). \end{aligned} \quad (31)$$

Putting this in  $G(y)$ , we get

$$\begin{aligned}
G(y) &= \frac{(1-i)}{c^{3/2} D^{1/4} \delta^{1/4}} G_Q(c, a) \int_0^\infty (xy)^{-1/4} \phi\left(\frac{x}{X}\right) e(\alpha x^\beta) \left( e\left(\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right) + ie\left(-\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right) \right) dx \\
&\quad - \frac{(1+i)D^{1/4}}{32\pi c^{1/2} \delta^{3/4}} G_Q(c, a) \int_0^\infty (xy)^{-3/4} \phi\left(\frac{x}{X}\right) e(\alpha x^\beta) \left( e\left(\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right) - ie\left(-\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right) \right) dx \\
&\quad + O\left(c^{3/2} \int_0^\infty (xy)^{-5/4} \phi\left(\frac{x}{X}\right) dx\right).
\end{aligned} \tag{32}$$

Changing variable  $x = Xt^2$ , we obtain

$$\begin{aligned}
G(y) &= \frac{2(1-i)}{c^{3/2} D^{1/4} \delta^{1/4}} G_Q(c, a) X^{3/4} y^{-1/4} \int_0^\infty t^{-1/2} \phi(t^2) e(\alpha X^\beta t^{2\beta}) \left( e\left(\frac{2\sqrt{\delta X y}}{c\sqrt{D}} t\right) + ie\left(-\frac{2\sqrt{\delta X y}}{c\sqrt{D}} t\right) \right) dt \\
&\quad - \frac{(1+i)D^{1/4}}{16\pi c^{1/2} \delta^{3/4}} G_Q(c, a) X^{1/4} y^{-3/4} \int_0^\infty t^{-1/2} \phi(t^2) e(\alpha X^\beta t^{2\beta}) \left( e\left(\frac{2\sqrt{\delta X y}}{c\sqrt{D}} t\right) - ie\left(-\frac{2\sqrt{\delta X y}}{c\sqrt{D}} t\right) \right) dt \\
&\quad + O\left(c^{3/2} \int_0^\infty (xy)^{-5/4} \phi\left(\frac{x}{X}\right) dx\right) \\
&= G_1(y) + O\left(c^{3/2} X^{-1/4} y^{-5/4}\right),
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
G_1(y) &= a_1 c^{-3/2} X^{3/4} y^{-1/4} G_Q(c, a) \left( P_+ \left( \frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) + iP_+ \left( -\frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) \right) \\
&\quad + a_2 c^{-1/2} X^{1/4} y^{-3/4} G_Q(c, a) \left( P_- \left( \frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) - iP_- \left( -\frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) \right),
\end{aligned} \tag{34}$$

with

$$a_1 = \frac{2(1-i)}{D^{1/4} \delta^{1/4}}, \tag{35}$$

$$a_2 = -\frac{(1+i)D^{1/4}}{16\pi \delta^{3/4}},$$

$$P_\pm(w) = \int_0^\infty t^{\pm 1/2} \phi(t^2) e(\alpha X^\beta t^{2\beta} + wt) dt. \tag{36}$$

The  $O$ -term contributes

$$\frac{1}{q} \sum_{c|q} \sum_{n=1}^\infty r(n, Q^*) |S(-l, -\overline{D}_c n; c)| c^{3/2} X^{-1/4} n^{-5/4} \tag{37}$$

$$\ll q^{1+\varepsilon} X^{-(1/4)+\varepsilon}.$$

The integral  $P_\pm(w)$  defined in (36) was studied by Ren and Ye [7], Sun and Wu [11], and Yan [12]. Here, we follow their steps and choose the parameters with a few differences to get the  $q$ -aspect saving.

Apply the method given in [7], we obtain

$$P_\pm(w) \ll \frac{\Delta^{h-1}}{(\max\{|\alpha|\beta X^\beta, |w|\})^h}, \quad h = 1, 2. \tag{38}$$

We will take  $w = \pm (2\sqrt{\delta y X})/(c\sqrt{D})$  here. Then,  $|w| \in [(1/2)|\alpha|\beta X^\beta, 4|\alpha|\beta X^\beta]$  implies  $y \in I$ , where

$$I = \left[ \frac{1}{16} \delta^{-1} c^2 D (|\alpha|\beta)^2 X^{2\beta-1}, 4\delta^{-1} c^2 D (|\alpha|\beta)^2 X^{2\beta-1} \right]. \tag{39}$$

Thus, for  $y = n \notin I$ , one obtains by (34) and (38) that

$$G_1(y) \ll (c^{-3/2} X^{3/4} y^{-1/4} + c^{-1/2} X^{1/4} y^{-3/4}) G_Q(c, a) R_2(X, y), \tag{40}$$

where

$$R_2(X, y) \ll \frac{\Delta^{h-1}}{(\sqrt{Xy}/c)^h}, \quad h = 1, 2. \tag{41}$$

Thus, the contrition from  $G_1(n)$  with  $n \notin I$  is

$$\begin{aligned}
& \frac{1}{q} \sum_{c|q} \sum_{a \bmod c}^* e\left(-\frac{al}{c}\right) \sum_{n \notin I} r(n, Q^*) e\left(\frac{-a\overline{D_c}n}{c}\right) G_1(n) \\
& \ll \frac{1}{q} \sum_{c|q} \sum_{n=1}^{\infty} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| \left( c^{-1/2} X^{3/4} n^{-1/4} + c^{1/2} X^{1/4} n^{-3/4} \right) R_2(X, n).
\end{aligned} \tag{42}$$

Let

$$Y = \Delta^2 c^2 X^{-1}. \tag{43}$$

we get

$$h = \begin{cases} 1, & \text{if } n \leq Y, \\ 2, & \text{if } n > Y, \end{cases} \tag{44}$$

Applying (41) with

$$\begin{aligned}
& \ll \frac{1}{q} \sum_{c|q} \sum_{n \leq Y} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| \left( c^{-1/2} X^{3/4} n^{-1/4} + c^{1/2} X^{1/4} n^{-3/4} \right) (cX^{-1/2} n^{-1/2}) \\
& + \frac{1}{q} \sum_{c|q} \sum_{n > Y} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| \left( c^{-1/2} X^{3/4} n^{-1/4} + c^{1/2} X^{1/4} n^{-3/4} \right) (c^2 \Delta X^{-1} n^{-1}) \\
& =: \sum_1 + \sum_2.
\end{aligned} \tag{45}$$

By the bound  $\sum_{n \leq x} r(n) = \pi x + O(x^{1/3})$ , we obtain

$$\begin{aligned}
\sum_1 &= \frac{1}{q} \sum_{c|q} \sum_{n \leq Y} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| \left( c^{1/2} X^{1/4} n^{-3/4} + c^{3/2} X^{-1/4} n^{-5/4} \right) \\
&\ll \frac{1}{q} \sum_{c|q} d(c) c X^{1/4} \sum_{n \leq Y} r(n, Q^*) (l, \overline{D_c}n; c)^{1/2} n^{-3/4} \\
&+ \frac{1}{q} \sum_{c|q} d(c) c^2 X^{-1/4} \sum_{n \leq Y} r(n, Q^*) (l, \overline{D_c}n; c)^{1/2} n^{-5/4} \\
&\ll q^{(1/2)+\varepsilon} \Delta^{(1/2)+\varepsilon}.
\end{aligned} \tag{46}$$

Similarly, using the bound  $\sum_{n=1}^{\infty} (|r(n, Q^*)|^2 / n^{1+\varepsilon}) \ll 1$ , we get

$$\begin{aligned}
& \sum_2 = \frac{1}{q} \sum_{c|q} \sum_{n>Y} r(n, Q^*) |S(-l, -\overline{D_c n}; c)| \Delta(c^{3/2} X^{-1/4} n^{-5/4} + c^{5/2} X^{-3/4} n^{-7/4}) \\
& \ll \frac{1}{q} \Delta \sum_{c|q} d(c) c^{2+\varepsilon} X^{-1/4} \sum_{n>Y} r(n, Q^*) (l, \overline{D_c n}; c)^{1/2} n^{-5/4} \\
& \quad + \frac{1}{q} \Delta \sum_{c|q} d(c) c^{3+\varepsilon} X^{-3/4} \sum_{n>Y} r(n, Q^*) (l, \overline{D_c n}; c)^{1/2} n^{-7/4} \\
& \ll \frac{1}{q} \Delta \sum_{c|q} c^{2+\varepsilon} X^{-1/4} \left( \sum_{n>Y} \frac{|r(n, Q^*)|^2}{n^{1+\varepsilon}} \right)^{1/2} \left( \sum_{n>Y} n^{-3/2} \right)^{1/2} \\
& \quad + \frac{1}{q} \Delta \sum_{c|q} c^{3+\varepsilon} X^{-3/4} \left( \sum_{n>Y} \frac{|r(n, Q^*)|^2}{n^{1+\varepsilon}} \right)^{1/2} \left( \sum_{n>Y} n^{-5/2} \right)^{1/2} \\
& \ll \frac{1}{q} \Delta \sum_{c|q} c^{2+\varepsilon} X^{-1/4} Y^{-(1/4)+\varepsilon} + \frac{1}{q} \Delta \sum_{c|q} c^{3+\varepsilon} X^{-3/4} Y^{-(3/4)+\varepsilon} \\
& \ll q^{(1/2)+\varepsilon} \Delta^{(1/2)+\varepsilon}.
\end{aligned} \tag{47}$$

Finally, we obtain the contribution of (45) as

$$\ll q^{(1/2)+\varepsilon} \Delta^{(1/2)+\varepsilon}. \tag{48}$$

If  $|\alpha|\beta q X^\beta < (1/2)\sqrt{\delta X/D}$ , then  $I \cap Z^+ = \emptyset$ , by (25), (29), (37), and (48), and choosing

$$\Delta = X^{2/3} q^{-1/3}, \tag{49}$$

we get

$$\sum_{n \sim X, n \equiv l \pmod{q}} r(n) e(\alpha n^\beta) \ll q^{(1/3)+\varepsilon} X(|\alpha|\beta X^\beta)^{-1}. \tag{50}$$

Next, we will handle the situation that  $y = n \in I$ . By the trivial estimate  $\int_1^{\sqrt{2}} t^{\pm 1/2} \phi(t^2) e(f(t)) dt \ll 1$  and putting it into (22), we get the second term in the right-hand side of (34) which contributes

$$\begin{aligned}
& \ll \frac{1}{q} \sum_{c|q} \sum_{n \in I} r(n, Q^*) |S(-l, -\overline{D_c n}; c)| c^{-1/2} X^{1/4} n^{-3/4} G_Q(c, a) \\
& \ll q^{(1/2)+\varepsilon} (|\alpha|\beta X^\beta)^{(1/2)+\varepsilon}.
\end{aligned} \tag{51}$$

Applying (38) to bound the terms with  $\alpha w > 0$ , we have

$$P_+ \left( \frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) + iP_+ \left( -\frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) = \varepsilon_\alpha \int_0^\infty t^{1/2} \phi(t^2) e(f_1(t)) dt + O(R_2(X, y, c)), \tag{52}$$

where

$$\varepsilon_\alpha = \begin{cases} i, & \alpha > 0, \\ 1, & \alpha < 0, \end{cases} \tag{53}$$

$$f_1(t) = f_1(t, y) = \operatorname{sgn}(\alpha) \left( |\alpha| X^\beta t^{2\beta} - \frac{2\sqrt{\delta X y}}{c\sqrt{D}} t \right). \tag{54}$$

Since the contribution from the  $O$ -term is absorbed by (51), we only need to estimate

$$\varepsilon_\alpha \frac{a_1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{n \in I} r(n, Q^*) |S(-l, -\overline{D_c n}; c)| n^{-1/4} G_Q(c, a) \int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n)) dt. \tag{55}$$

We distinguish two cases according to  $\beta = 1/2$  or not.

If  $\beta \neq 1/2$ , we have

$$f_1''(t) = \operatorname{sgn}(\alpha)|\alpha|(2\beta)(2\beta-1)X^\beta t^{2\beta-2} \gg |\alpha|\beta|2\beta-1|X^\beta, \quad \text{for } t \in [1, \sqrt{2}]. \quad (56)$$

By integration by parts and [17], we have

$$\int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n)) dt \ll \max_{1 \leq t \leq \sqrt{2}} \left| \int_1^t e(f_1(u)) du \right| \ll (|\alpha|\beta|2\beta-1|X^\beta)^{-1/2}. \quad (57)$$

Hence, (55) is

$$\begin{aligned} & \ll \frac{1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{n \in I} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| n^{-1/4} G_Q(c, a) |\alpha|\beta|2\beta-1|X^\beta)^{-1/2} \\ & \ll q^{(1/2)+\varepsilon} |2\beta-1|^{-(1/2)} (|\alpha|\beta X^\beta)^{1+\varepsilon}. \end{aligned} \quad (58)$$

If  $\beta = 1/2$ , let

$$H_r = 2^{-r} c |\alpha| X^{1/2}, \quad 1 \leq r \leq r_0 = \lceil \log_2(c |\alpha| X^{1/2}) \rceil + 1, \quad (59)$$

and write

$$A(H_r) = \left\{ n: H_r X^{-1/2} < \left| c |\alpha| - 2 \sqrt{\frac{\delta n}{D}} \right| \leq 2 H_r X^{-1/2} \right\}. \quad (60)$$

Then,

$$|A(H_r)| \leq H_r c |\alpha| X^{-1/2}. \quad (61)$$

Moreover, for  $n \in A(H_r)$ , we have

$$|f_1'(t)| = \left| |\alpha| - \frac{2\sqrt{\delta n}}{c\sqrt{D}} \right| X^{1/2} > H_r c^{-1}. \quad (62)$$

Then, we obtain

$$\int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t)) dt \ll H_r^{-1} c. \quad (63)$$

Hence, in this situation, (55) contributes

$$\begin{aligned} & \ll \frac{1}{q} \sum_{c|q} c^{-1/2} X^{(3/4)+\varepsilon} \sum_{r=1}^{r_0} H_r^{-1} \sum_{n \in A(H_r)} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| n^{-1/4} G_Q(c, a) \\ & \ll \frac{1}{q} \sum_{c|q} c^{-1/2} X^{(3/4)+\varepsilon} \sum_{r=1}^{r_0} H_r^{-1} (|\alpha|^2 c^2)^{-(1/4)+\varepsilon} c^{(3/2)+\varepsilon} |A(H_r)| \\ & \ll q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}. \end{aligned} \quad (64)$$

It remains to estimate

$$\varepsilon_\alpha \frac{a_1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{n \in I \cap I_0} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| n^{-1/4} G_Q(c, a) \int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n)) dt, \quad (65)$$

where



$$I_0 = \left\{ n: \left| c|\alpha| - 2\sqrt{\frac{\delta n}{D}} \right| \leq X^{-1/2} \right\}. \quad (66)$$

Note that

$$\left| \left\{ n: \left| c|\alpha| - 2\sqrt{\frac{\delta n}{D}} \right| \leq X^{-1/2} \right\} \right| \leq c|\alpha|X^{-1/2}. \quad (67)$$

If  $|\alpha| \leq (1/q)$ , then  $I_0 = \emptyset$ , and we have (65) which vanishes. And if  $|\alpha| \geq (\sqrt{X}/q)$ , then by trivial estimate, (65) contributes

$$\begin{aligned} & \frac{1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{|c|\alpha - 2\sqrt{(\delta n/D)}| \leq X^{-1/2}} r(n, Q^*) |S(-l, -\overline{D}_c n; c)|^{n-1/4} G_Q(c, a) \\ & \ll \frac{1}{q} \sum_{c|q} c^{1/2} X^{3/4} |I_0| (c^2 |\alpha|^2)^{-(1/4)+\varepsilon} \\ & \ll q^\varepsilon X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}. \end{aligned} \quad (68)$$

If  $(1/q) \leq |\alpha| \leq (\sqrt{X}/q)$ , then there is at most one integer satisfying

$$\left| c|\alpha| - 2\sqrt{\frac{\delta n}{D}} \right| \leq X^{-1/2}, \quad (69)$$

for every  $c|q$ , say  $n = n_c$ . Hence, (55) becomes

$$\begin{aligned} & \varepsilon_\alpha \frac{a_1}{q} \sum_{c|q} \delta_c c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) \int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n_c)) dt \\ & = \frac{\varepsilon_\alpha a_1}{2q} \sum_{c|q} \delta_c c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) \int_1^2 u^{-1/4} e(f_1(u^{1/2}, n_c)) du + O(c^{-(1/2)+\varepsilon} \Delta^{-1} X^{3/4} r(n_c, Q^*) n_c^{-1/4}) \\ & = \frac{1}{q} \sum_{c|q} \varepsilon(a, n_c) c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) + O(1), \end{aligned} \quad (70)$$

where

$$\varepsilon(a, n_c) = \frac{\delta_c \varepsilon_\alpha a_1}{2} \int_1^2 u^{-1/4} e\left(\operatorname{sgn}(\alpha) \left( |\alpha| - \frac{2\sqrt{\delta n_c}}{c\sqrt{D}} \right) \sqrt{Xu} \right) du, \quad (71)$$

and  $\delta_c = 1$  or  $0$  according to if there exists such  $n_c$  or not. Recall

$$\begin{aligned} a_1 &= \frac{2(1-i)}{D^{1/4} \delta^{1/4}}, \\ \varepsilon_\alpha &= \begin{cases} i, & \alpha > 0, \\ 1, & \alpha < 0. \end{cases} \end{aligned} \quad (72)$$

When  $\beta = 1/2$  and  $(1/q) \leq |\alpha| \leq (\sqrt{X}/q)$  and applying (64) and (68), we derive that (55) equals to

$$\begin{aligned} & \frac{1}{q} \sum_{c|q} \varepsilon(a, n_c) c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) \\ & + O(q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}). \end{aligned} \quad (73)$$

This proves Theorem 1(iii).

In particular, if  $|\alpha| = (2/q)\sqrt{\delta k/D}$  with  $1 \leq k \leq (DX/4\delta)$ , then for  $c|q$ ,  $c \neq q$ , we have

$$\left| c|\alpha| - 2\sqrt{\frac{\delta k}{D}} \right| = 2\sqrt{\frac{\delta k}{D}} \left| \frac{c}{q} - 1 \right| \geq \frac{2}{q} \sqrt{\frac{\delta k}{D}} \geq 2\sqrt{\frac{\delta k}{DX}} \geq 1. \quad (74)$$

Thus,  $\delta_c = 0$  for  $c \neq q$  in this situation. Hence, (73) becomes

$$\sum_{n \sim Xn \equiv l \pmod{q}} r(n) e\left(\pm \frac{2}{q} \sqrt{\frac{\delta kn}{D}}\right) = \frac{4}{3} (2^{3/4} - 1) D^{-1/4} \delta^{-1/4} (1 \pm i) |S(-l, -\overline{D}_c k; q)| q^{-5/2} X^{3/4} r(k, Q^*) k^{-1/4} G_Q(q, a) \\ + O(k^{-1/4} X^{1/4} (qXk)^\varepsilon). \quad (75)$$

This proves Theorem 1(iv).

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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