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Convex Geometry in Orlicz Space

Lead Guest Editor: Chang Jian Zhao
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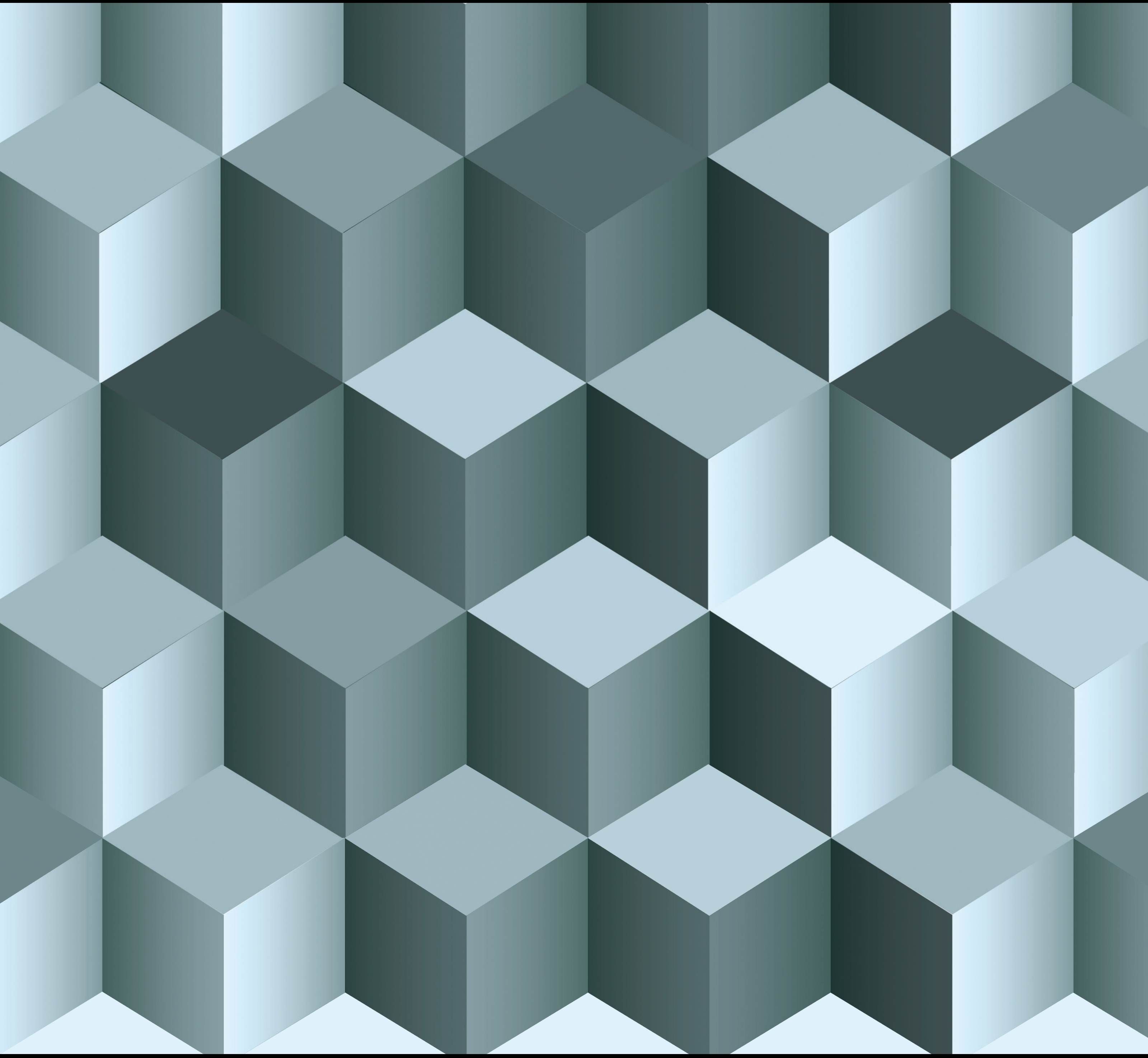

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Research Article

Some Integral Inequalities for n-Polynomial ζ-Preinvex Functions

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In this paper, we study the properties of n-polynomial ζ-preinvex functions and establish some integral inequalities of Hermite-Hadamard type via this class of convex functions. Moreover, we discuss some special cases which provide a significant complement to the integral estimations of preinvex functions. Applications of the obtained results to the inequalities for special means are also considered.

1. Introduction and Preliminaries

The geometric inequalities involving volume, surface area, mean width, etc. in the Orlicz space have attracted considerable attention of researchers, and the convexity properties of functions have been a powerful tool for dealing with various problems of convex geometry (see [1, 2]). This suggests that it is a significant work to develop new inequalities for generalized convex functions. For this purpose, let us start with recalling some concepts and notations on the convexity of functions.

A set $\mathscr{C} \subset \mathbb{R}$ is said to be convex if

$$(1-t)x + ty \in \mathscr{C}, \tag{1}$$

for any $x, y \in \mathscr{C}$ and $t \in [0, 1]$.

A function $\mathscr{F} : \mathscr{C} \mapsto \mathbb{R}$ is said to be convex if the inequality

$$\mathscr{F}((1-t)x + ty) \leq (1-t)\mathscr{F}(x) + t\mathscr{F}(y) \tag{2}$$

holds for any $x, y \in \mathscr{C}$ and $t \in [0, 1]$.

In recent years, the classical concept of convexity has been extended and generalized in different directions. Mititelu [3] introduced the notion of invex set, as follows.

Definition 1 [3]. Let $\mathscr{X} \subset \mathbb{R}$ be a nonempty set and $\eta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a real-valued function. A set $\mathscr{X}$ is said to be invex with respect to η if

$$x + t\eta(y, x) \in \mathscr{X}, \tag{3}$$

for all $x, y \in \mathscr{X}$ and $t \in [0, 1]$.

The invexity would reduce to the classical convexity if $\eta(y, x) = y - x$. Weir and Mond [4] defined the class of preinvex functions as follows.

Definition 2 [4]. Let $\mathscr{X} \subset \mathbb{R}$ be a nonempty invex set with respect to $\eta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$. A function $\mathscr{F} : \mathscr{X} \mapsto \mathbb{R}$ is said to be preinvex with respect to η if the inequality

$$\mathscr{F}(x + t\eta(y, x)) \leq (1-t)\mathscr{F}(x) + t\mathscr{F}(y) \tag{4}$$

holds for all $x, y \in \mathscr{X}$ and $t \in [0, 1]$.

As a generalization of convex functions, Gordji et al. [5] introduced the notion of ζ-convex function.

Definition 3 [5]. A function $\mathscr{F} : \mathscr{I} \subset \mathbb{R} \mapsto \mathbb{R}$ is said to be ζ-convex function with respect to $\zeta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ if the inequality

$$\mathscr{F}(tx+(1-t)y) \le \mathscr{F}(y) + t\zeta(\mathscr{F}(x), \mathscr{F}(y)) \tag{5}$$

holds for all $x, y \in \mathscr{I}$ and $t \in [0, 1]$.

The properties of convexity have numerous applications in different fields of pure and applied mathematics; especially, the concept of convexity has close relation with the theory of inequalities. Many inequalities are direct consequences of the applications of classical convexity. As is known to us, the Hermite-Hadamard inequality is one of the most significant result associated with convex functions, it reads as follows.

Let $\mathscr{F} : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\mathscr{F}\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \mathscr{F}(x)dx \le \frac{\mathscr{F}(a)+\mathscr{F}(b)}{2}. \tag{6}$$

Noor [6] obtained a generalization of classical Hermite-Hadamard's inequality using the class of preinvex functions, as follows.

Let $\mathscr{F} : [a, a+\eta(b, a)] \to \mathbb{R}$ be a preinvex function, then

$$\mathscr{F}\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mathscr{F}(x)dx \le \frac{\mathscr{F}(a)+\mathscr{F}(b)}{2}. \tag{7}$$

The result of Noor has inspired a lot of investigators to deal with new generalizations and refinements of Hermite-Hadamard's inequality via preinvexity. For example, Barani et al. [7] obtained the generalizations of Hermite-Hadamard's inequality for functions whose derivative absolute values are preinvex. Du et al. [8] and Noor et al. [9] obtained several generalizations of Hermite-Hadamard's inequality via (s, m)-preinvex functions and h-preinvex functions, respectively. Park [10, 11] derived several variations of Hermite-Hadamard's inequality from differentiable preinvex functions. Sarikaya et al. [12] and Wu et al. [13] established the Hermite-Hadamard-like type inequalities via log-preinvex functions and harmonically (p, h, m)-preinvex functions, respectively. Wang and Liu [14] and Li [15] obtained different refinements of Hermite-Hadamard's inequality using s-preinvex functions. Deng et al. [16, 17] and Wu et al. [18] deduced some quantum Hermite-Hadamard-type inequalities by using generalized (s, m)-preinvex functions and strongly preinvex functions, respectively.

Recently, Toplu et al. [19] proposed the concept of n-polynomial convex functions and investigated their properties.

In this paper, we shall introduce a new class of n-polynomial convex functions based on a different form of inequality in the definition compared with [19], which is convenient to the generalizations and applications of n-polynomial convexity. More specifically, we will define a class of convex functions called as n-polynomial ζ-preinvex functions. We then show that this class of convex functions contains a number of other classes of convex functions. Furthermore, we establish some new integral inequalities of Hermite-Hadamard type for n-polynomial ζ-preinvex functions. Finally, we apply the obtained inequalities to establish two inequalities for special means.

Firstly, we introduce the notion of n-polynomial ζ-preinvex functions.

Definition 4. Let $n \in \mathbb{N}$. A nonnegative function $\mathscr{F} : \mathscr{X} \mapsto \mathbb{R}$ is said to be n-polynomial ζ-preinvex with respect to bifunctions $\eta, \zeta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ if the inequality

$$\mathscr{F}(a+t\eta(b,a)) \le \mathscr{F}(a) + \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\zeta(\mathscr{F}(b), \mathscr{F}(a)) \tag{8}$$

holds for all $a, b \in \mathscr{X}$ and $t \in [0, 1]$.

Note that if we take $n = 1$, then we have 1-polynomial ζ-preinvexity, which is just the ζ-preinvex functions defined by the inequality

$$\mathscr{F}(a+t\eta(b,a)) \le \mathscr{F}(a) + t\zeta(\mathscr{F}(b), \mathscr{F}(a)), \quad \forall a, b \in \mathscr{X}, t \in [0, 1]. \tag{9}$$

If we take $\zeta(\mathscr{F}(b), \mathscr{F}(a)) = \mathscr{F}(b) - \mathscr{F}(a)$, then we obtain the class of n-polynomial preinvex functions, which is defined by the inequality

$$\mathscr{F}(a+t\eta(b,a)) \le \frac{1}{n}\sum_{s=1}^{n}(1-t)^s\mathscr{F}(a) + \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\mathscr{F}(b), \quad \forall a, b \in \mathscr{X},\ t \in [0, 1]. \tag{10}$$

If we take $\eta(b, a) = b - a$, then we get the class of n-polynomial ζ-convex functions, which is defined by the inequality

$$\mathscr{F}(a+t(b-a)) \le \mathscr{F}(a) + \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\zeta(\mathscr{F}(b), \mathscr{F}(a)), \quad \forall a, b \in \mathscr{X},\ t \in [0, 1]. \tag{11}$$

If we take $n = 1$ in inequality (11), then we have the class of ζ-convex functions. Furthermore, we obtain the classical convex functions by setting $\zeta(\mathscr{F}(b), \mathscr{F}(a)) = \mathscr{F}(b) - \mathscr{F}(a)$.

If we take $n = 2$ in Definition 4, then we have the class of 2-polynomial ζ-preinvex functions, which is defined by the following inequality:

$$\mathscr{F}(a+t\eta(b,a)) \le \mathscr{F}(a) + \frac{3t-t^2}{2}\zeta(\mathscr{F}(b), \mathscr{F}(a)), \quad \forall a, b \in \mathscr{X},\ t \in [0, 1]. \tag{12}$$

Note that $0 \le t \le 3t - t^2/2$, this shows that, for every nonnegative bifunction ζ, the ζ-preinvex function is also the 2-polynomial ζ-preinvex functions. More generally, we have the following result.

Proposition 5. *For every nonnegative bifunction ζ and $n \geq 2$, if $\mathcal{F} : \mathcal{X} \mapsto \mathbb{R}$ is a $(n-1)$-polynomial ζ-preinvex function, then $\mathcal{F}$ is a n-polynomial ζ-preinvex function.*

To verify the validity of Proposition 5, it is enough to show that

$$\frac{1}{n-1}\sum_{s=1}^{n-1}\left[1-(1-t)^s\right] \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right], \tag{13}$$

for any $n \geq 2$ and $t \in [0,1]$.

Direct computation gives

$$\begin{aligned}
&\frac{1}{n-1}\sum_{s=1}^{n-1}\left[1-(1-t)^s\right] - \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right] \\
&\quad = \left(\frac{1-t}{t}\right)\left[\frac{1-(1-t)^n}{n} - \frac{1-(1-t)^{n-1}}{n-1}\right] \\
&\quad = \left(\frac{1-t}{t}\right)\left[\frac{(1-t)^n + nt(1-t)^{n-1} - 1}{n(n-1)}\right] \\
&\quad = \left(\frac{1-t}{t}\right)\left[\frac{(1-t)^n + nt(1-t)^{n-1} - ((1-t)+t)^n}{n(n-1)}\right] \\
&\quad = -\left(\frac{1-t}{t}\right)\left[\frac{C_n^2 t^2(1-t)^{n-2} + C_n^3 t^3 (1-t)^{n-3} + \cdots + C_n^n t^n}{n(n-1)}\right] \\
&\quad \leq 0,
\end{aligned} \tag{14}$$

which implies the required inequality (13).

As a consequence, we obtain the following.

Proposition 6. *For every nonnegative bifunction ζ, if $\mathcal{F} : \mathcal{X} \mapsto \mathbb{R}$ is a ζ-preinvex function, then $\mathcal{F}$ is a n-polynomial ζ-preinvex function.*

Choosing $\zeta(\mathcal{F}(b), \mathcal{F}(a)) = \mathcal{F}(b) - \mathcal{F}(a)$ in Proposition 6 gives the following.

Proposition 7. *If $\mathcal{F} : [a, a+\eta(b,a)] \to \mathbb{R}$ is a preinvex function with $\mathcal{F}(b) - \mathcal{F}(a) \geq 0$, then $\mathcal{F}$ is a n-polynomial preinvex function.*

2. Main Results

In this section, we establish some new Hermite-Hadamard-type inequalities using the class of n-polynomial ζ-preinvex functions. We first need to introduce the notation called Condition C, which was presented by Mohan and Neogy in [20].

Condition C. Let $\mathcal{X} \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(.,.)$, we say that the bifunction $\eta(.,.)$ satisfies the Condition C, if for any $x, y \in \mathcal{X}$ and $t \in [0,1]$, we have

$$\begin{aligned}
\eta(x, x+t\eta(y,x)) &= -t\eta(y,x), \\
\eta(y, x+t\eta(y,x)) &= (1-t)\eta(y,x).
\end{aligned} \tag{15}$$

Note that for any $x, y \in \mathcal{X}$, $t_1, t_2 \in [0,1]$ and from Condition C, we can deduce

$$\eta(x+t_2\eta(y,x), x+t_1\eta(y,x)) = (t_2-t_1)\eta(y,x). \tag{16}$$

Throughout the paper we assume that Condition C is satisfied for the domain with respect to bifunction $\eta(.,.)$ as a precondition.

Theorem 8. *Let $\mathcal{F} : [a, a+\eta(b,a)] \to \mathbb{R}$ be a n-polynomial ζ-preinvex function. If $\eta(b,a) > 0$ and $\mathcal{F} \in L[a, a+\eta(b,a)]$, then we have*

$$\begin{aligned}
&\mathcal{F}\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{n+2^{-n}-1}{n} M_\zeta \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mathcal{F}(x)dx \\
&\leq \mathcal{F}(a) + \frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\zeta(\mathcal{F}(b), \mathcal{F}(a)),
\end{aligned} \tag{17}$$

where M_ζ is the upper bound of bifunction ζ.

Proof. Using the definition of n-polynomial ζ-preinvex function and Condition C, we have

$$\begin{aligned}
\mathcal{F}\left(\frac{2a+\eta(b,a)}{2}\right) &= \mathcal{F}\left(a + (1-t)\eta(b,a) + \frac{1}{2}(t-(1-t))\eta(b,a)\right) \\
&= \mathcal{F}\left(a + (1-t)\eta(b,a) + \frac{1}{2}\eta(a+t\eta(b,a), a\right. \\
&\quad \left. + (1-t)\eta(b,a))\right) \leq \mathcal{F}(a+(1-t)\eta(b,a)) \\
&\quad + \frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{1}{2}\right)^s\right)\zeta(\mathcal{F}(a+t\eta(b,a)), \\
&\quad \mathcal{F}(a+(1-t)\eta(b,a))) \leq \mathcal{F}(a+(1-t)\eta(b,a)) \\
&\quad + \frac{n+2^{-n}-1}{n} M_\zeta.
\end{aligned} \tag{18}$$

Hence, we obtain

$$\mathcal{F}(a+(1-t)\eta(b,a)) \geq \mathcal{F}\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{n+2^{-n}-1}{n} M_\zeta. \tag{19}$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, it follows that

$$\int_0^1 \mathcal{F}(a+(1-t)\eta(b,a))dt \geq \mathcal{F}\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{n+2^{-n}-1}{n} M_\zeta, \tag{20}$$

that is,

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx \geq \mathcal{F}\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{n+2^{-n}-1}{n}M_{\zeta}. \tag{21}$$

The left-hand side inequality of (17) is proved.

On the other hand, from the definition of n-polynomial ζ-preinvex function, one has

$$\mathcal{F}(a+t\eta(b,a)) \leq \mathcal{F}(a) + \frac{1}{n}\sum_{s=1}^{n}(1-(1-t)^s)\zeta(\mathcal{F}(b),\mathcal{F}(a)). \tag{22}$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we obtain

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx &= \int_0^1 \mathcal{F}(a+t\eta(b,a))dt \\ &\leq \int_0^1\left(\mathcal{F}(a) + \frac{1}{n}\sum_{s=1}^{n}(1-(1-t)^s)\zeta(\mathcal{F}(b),\mathcal{F}(a))\right)dt \\ &= \mathcal{F}(a) + \frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\zeta(\mathcal{F}(b),\mathcal{F}(a)).\end{aligned} \tag{23}$$

This proves the right-hand side inequality of (17). The proof of Theorem 8 is complete.

Before we put forward another kind of integral inequality of Hermite-Hadamard type, we need to prove an auxiliary result, which will play a key role in deducing subsequent results. For the sake of simplicity, we let $\mathcal{I} = [a, a+\eta(b,a)]$ and let $\mathcal{I}^\circ$ be the interior of $\mathcal{I}$.

Lemma 9. *Let $\mathcal{F}:\mathcal{I}\mapsto\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ with $\eta(b,a)>0$, $\min\{\lambda,\mu\}\geq t>0$. If $\mathcal{F}'\in L[\mathcal{I}]$, then*

$$\begin{aligned}&\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx \\ &\quad= \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}(-t)\,\mathcal{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)dt \right. \\ &\qquad \left. + \int_0^{\lambda} t\,\mathcal{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)dt\right].\end{aligned} \tag{24}$$

Proof. Let

$$\begin{aligned}I &= \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}(-t)\mathcal{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)dt \right. \\ &\quad \left. + \int_0^{\lambda} t\mathcal{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)dt\right] = I_1 + I_2.\end{aligned} \tag{25}$$

Integrating by parts yields

$$\begin{aligned}I_1 &= \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}(-t)\mathcal{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)dt\right] \\ &= \frac{1}{\lambda+\mu}\left(\mu\mathcal{F}(a) - \int_0^{\mu}\mathcal{F}\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)dt\right) \\ &= \frac{\mu}{\lambda+\mu}\mathcal{F}(a) - \frac{1}{\eta(b,a)}\int_a^{a+\frac{\mu}{\lambda+\mu}\eta(b,a)}\mathcal{F}(x)dx.\end{aligned} \tag{26}$$

Similarly,

$$I_2 = \frac{\lambda}{\lambda+\mu}\mathcal{F}(a+\eta(b,a)) - \frac{1}{\eta(b,a)}\int_{a+(\mu/(\lambda+\mu))\eta(b,a)}^{a+\eta(b,a)}\mathcal{F}(x)dx. \tag{27}$$

Substituting the formulations of I_1 and I_2 in (25) leads to the desired identity (24). The proof of Lemma 9 is complete.

We shall now give some estimations of bounds for Hermite-Hadamard-type inequalities.

Theorem 10. *Let $\mathcal{F}:\mathcal{I}\longrightarrow\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ with $\eta(b,a)>0$, $\lambda>0$, $\mu>0$, and let $\mathcal{F}'\in L[\mathcal{I}]$. If $|\mathcal{F}'|$ is n-polynomial ζ-preinvex function, then*

$$\begin{aligned}&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\ &\quad\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\frac{\lambda^2+\mu^2}{2}\,|\mathcal{F}'(a)| + \frac{1}{n}\sum_{s=1}^{n}K_1\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right) \right. \\ &\qquad \left. + \frac{1}{n}\sum_{s=1}^{n}K_2\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)\right],\end{aligned} \tag{28}$$

where

$$\begin{aligned}K_1 &= \frac{\mu^2}{2} - \frac{\mu(s+2)(\lambda+\mu)^{s+1} - (\lambda+\mu)^{s+2} + \lambda^{s+2}}{(s+1)(s+2)(\lambda+\mu)^s}, \\ K_2 &= \frac{\lambda^2}{2} - \frac{\lambda^{s+2}}{(s+1)(s+2)(\lambda+\mu)^s}.\end{aligned} \tag{29}$$

Proof. Using Lemma 9 and the assumption that $|\mathcal{F}'|$ is n-polynomial ζ-preinvex function, we have

$$\begin{aligned}
&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\
&\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}|-t|\left|\mathcal{F}'\left(a+\frac{\mu-t}{\mu+\lambda}\eta(b,a)\right)\right|dt+\int_0^{\lambda}|t|\left|\mathcal{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)\right|dt\right] \\
&\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}t\left\{|\mathcal{F}'(a)|+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(1-\frac{\mu-t}{\mu+\lambda}\right)^s\right)\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)\right\}dt\right. \\
&\quad\left.+\int_0^{\lambda}t\left\{|\mathcal{F}'(a)|+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(1-\frac{\mu+t}{\mu+\lambda}\right)^s\right)\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)\right\}dt\right] \\
&= \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}t|\mathcal{F}'(a)|dt+\frac{1}{n}\sum_{s=1}^{n}\int_0^{\mu}t\left(1-\left(\frac{\lambda+t}{\lambda+\mu}\right)^s\right)\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)dt\right. \\
&\quad\left.+\int_0^{\lambda}t|\mathcal{F}'(a)|dt+\frac{1}{n}\sum_{s=1}^{n}\int_0^{\lambda}t\left(1-\left(\frac{\lambda-t}{\lambda+\mu}\right)^s\right)\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)dt\right] \\
&= \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\frac{\lambda^2+\mu^2}{2}|\mathcal{F}'(a)|+\frac{1}{n}\sum_{s=1}^{n}\int_0^{\mu}t\left(1-\left(\frac{\lambda+t}{\lambda+\mu}\right)^s\right)\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)dt\right. \\
&\quad\left.+\frac{1}{n}\sum_{s=1}^{n}\int_0^{\lambda}t\left(1-\left(\frac{\lambda-t}{\lambda+\mu}\right)^s\right)\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)dt\right],
\end{aligned} \tag{30}$$

which implies the desired inequality (28) since

$$\begin{aligned}
\int_0^{\mu}t\left(1-\left(\frac{\lambda+t}{\lambda+\mu}\right)^s\right)dt &= \frac{\mu^2}{2}-\frac{\mu(s+2)(\lambda+\mu)^{s+1}-(\lambda+\mu)^{s+2}+\lambda^{s+2}}{(s+1)(s+2)(\lambda+\mu)^s} \\
&= K_1,
\end{aligned}$$

$$\int_0^{\lambda}t\left(1-\left(\frac{\lambda-t}{\lambda+\mu}\right)^s\right)dt=\frac{\lambda^2}{2}-\frac{\lambda^{s+2}}{(s+1)(s+2)(\lambda+\mu)^s}=K_2. \tag{31}$$

This completes the proof of Theorem 10.

Next, we discuss some special cases of Theorem 10.

(I) If we consider $\lambda=\mu=1$ in Theorem 10, then we have

$$\begin{aligned}
&\left|\frac{\mathcal{F}(a)+\mathcal{F}(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\
&\leq \frac{\eta(b,a)}{4}\left[|\mathcal{F}'(a)|+\frac{1}{n}\sum_{s=1}^{n}\frac{2^s(s+1)(s+2)-2\left(1+2^{s+1}s\right)}{2^{s+1}(s+1)(s+2)}\zeta\right. \\
&\quad\left.\cdot\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)+\frac{1}{n}\sum_{s=1}^{n}\frac{2^s(s+1)(s+2)-2}{(s+1)(s+2)2^{s+1}}\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)\right].
\end{aligned} \tag{32}$$

(II) If we take $\eta(b,a)=b-a$ in Theorem 10, then we get

$$\begin{aligned}
&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^{b}\mathcal{F}(x)dx\right| \\
&\leq \frac{b-a}{(\lambda+\mu)^2}\left[\frac{\lambda^2+\mu^2}{2}|\mathcal{F}'(a)|+\frac{1}{n}\sum_{s=1}^{n}K_1\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)\right. \\
&\quad\left.+\frac{1}{n}\sum_{s=1}^{n}K_2\zeta\left(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|\right)\right].
\end{aligned} \tag{33}$$

(III) If we choose $\zeta(|\mathcal{F}'(b)|,|\mathcal{F}'(a)|)=|\mathcal{F}'(b)|-|\mathcal{F}'(a)|$ in Theorem 10, then we obtain

$$\begin{aligned}
&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\
&\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\frac{\lambda^2+\mu^2}{2}|\mathcal{F}'(a)|+\frac{1}{n}\sum_{s=1}^{n}(K_1+K_2)\left(|\mathcal{F}'(b)|-|\mathcal{F}'(a)|\right)\right].
\end{aligned} \tag{34}$$

Theorem 11. *Let $\mathcal{F}:\mathcal{I}\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ with $\eta(b,a)>0$, $\lambda>0$, $\mu>0$, and let $\mathcal{F}'\in L[\mathcal{I}]$, $(1/p)+(1/q)=1$, $p>1, q>1$. If $|\mathcal{F}'|^q$ is n-polynomial ζ-preinvex function, then*

$$\begin{aligned}
&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\
&\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^{p+1}}{p+1}\right)^{1/p}\left(\mu|\mathcal{F}'(a)|^q+\frac{1}{n}\sum_{s=1}^{n}K_3\zeta\left(|\mathcal{F}'(b)|^q,|\mathcal{F}'(a)|^q\right)\right)^{1/q}\right. \\
&\quad\left.+\left(\frac{\lambda^{p+1}}{p+1}\right)^{1/p}\left(\lambda|\mathcal{F}'(a)|^q+\frac{1}{n}\sum_{s=1}^{n}K_4\zeta\left(|\mathcal{F}'(b)|^q,|\mathcal{F}'(a)|^q\right)\right)^{1/q}\right],
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
K_3&=\mu-\frac{(\lambda+\mu)^{s+1}-\lambda^{s+1}}{(s+1)(\lambda+\mu)^s}, \\
K_4&=\lambda-\frac{\lambda^{s+1}}{(s+1)(\lambda+\mu)^s}.
\end{aligned} \tag{36}$$

Proof. Using Lemma 9, Hölder's inequality, and the fact that $|\mathcal{F}'|^q$ is n-polynomial ζ-preinvex function, it follows that

$$\begin{aligned}
&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\
&\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}|-t|\left|\mathcal{F}'\left(a+\frac{\mu-t}{\mu+\lambda}\eta(b,a)\right)\right|dt\right. \\
&\quad\left.+\int_0^{\lambda}|t|\left|\mathcal{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)\right|dt\right] \\
&\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\int_0^{\mu}|-t|^p dt\right)^{1/p}\left(\int_0^{\mu}\left|\mathcal{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)\right|^q dt\right)^{1/q}\right. \\
&\quad\left.+\left(\int_0^{\lambda}|t|^p dt\right)^{1/p}\left(\int_0^{\lambda}\left|\mathcal{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)\right|^q dt\right)^{1/q}\right] \\
&\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^{p+1}}{p+1}\right)^{1/p}\left(\int_0^{\mu}\left(|\mathcal{F}'(a)|^q+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{\lambda+t}{\lambda+\mu}\right)^s\right)\zeta\right.\right.\right. \\
&\quad\left.\left.\cdot\left(|\mathcal{F}'(b)|^q,|\mathcal{F}'(a)|^q\right)\right)dt\right)^{1/q}+\left(\frac{\lambda^{p+1}}{p+1}\right)^{1/p}\left(\int_0^{\lambda}\left(|\mathcal{F}'(a)|^q\right.\right. \\
&\quad\left.\left.\left.+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{\lambda-t}{\lambda+\mu}\right)^s\right)\zeta\left(|\mathcal{F}'(b)|^q,|\mathcal{F}'(a)|^q\right)\right)dt\right)^{1/q}\right] \\
&= \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^{p+1}}{p+1}\right)^{1/p}\left(\mu|\mathcal{F}'(a)|^q+\frac{1}{n}\sum_{s=1}^{n}K_3\zeta\left(|\mathcal{F}'(b)|^q,|\mathcal{F}'(a)|^q\right)\right)^{1/q}\right. \\
&\quad\left.+\left(\frac{\lambda^{p+1}}{p+1}\right)^{1/p}\left(\lambda|\mathcal{F}'(a)|^q+\frac{1}{n}\sum_{s=1}^{n}K_4\zeta\left(|\mathcal{F}'(b)|^q,|\mathcal{F}'(a)|^q\right)\right)^{1/q}\right],
\end{aligned} \tag{37}$$

where

$$K_3 = \int_0^{\mu}\left(1-\left(\frac{\lambda+t}{\lambda+\mu}\right)^s\right)dt = \mu - \frac{(\lambda+\mu)^{s+1}-\lambda^{s+1}}{(s+1)(\lambda+\mu)^s},$$
$$K_4 = \int_0^{\lambda}\left(1-\left(\frac{\lambda-t}{\lambda+\mu}\right)^s\right)dt = \lambda - \frac{\lambda^{s+1}}{(s+1)(\lambda+\mu)^s}. \tag{38}$$

The proof of Theorem 11 is complete.

We now discuss some special cases of Theorem 11.

(I) If we choose $\lambda = \mu = 1$ in Theorem 11, then

$$\begin{aligned}&\left|\frac{\mathcal{F}(a)+\mathcal{F}(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\ &\le \frac{\eta(b,a)}{4(p+1)^{1/p}}\left[\left\{\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}\frac{2^s(s+1)-\left(2^{s+1}-1\right)}{2^s(s+1)}\right.\right. \\ &\left.\cdot\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right\}^{1/q} + \left\{\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}\frac{2^s(s+1)-1}{2^s(s+1)}\right. \\ &\left.\left.\cdot\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right\}^{1/q}\right].\end{aligned} \tag{39}$$

(II) If we take $\eta(b,a) = b-a$ in Theorem 11, then

$$\begin{aligned}&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^b\mathcal{F}(x)dx\right| \\ &\le \frac{b-a}{(\lambda+\mu)^2}\left[\left(\frac{\mu^{p+1}}{p+1}\right)^{1/p}\left(\mu\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}K_3\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right)^{1/q}\right. \\ &\left. + \left(\frac{\lambda^{p+1}}{p+1}\right)^{1/p}\left(\lambda\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}K_4\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right)^{1/q}\right].\end{aligned} \tag{40}$$

(III) If we put $\zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) = |\mathcal{F}'(b)|^q - |\mathcal{F}'(a)|^q$ in Theorem 11, then

$$\begin{aligned}&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\ &\le \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^{p+1}}{p+1}\right)^{1/p}\left(\left(\mu - \frac{1}{n}\sum_{s=1}^{n}K_3\right)\left|\mathcal{F}'(a)\right|^q\right.\right. \\ &\left. + \frac{1}{n}\sum_{s=1}^{n}K_3\left|\mathcal{F}'(b)\right|^q\right)^{1/q} + \left(\frac{\lambda^{p+1}}{p+1}\right)^{1/p}\left(\left(\lambda - \frac{1}{n}\sum_{s=1}^{n}K_4\right)\left|\mathcal{F}'(a)\right|^q\right. \\ &\left.\left. + \frac{1}{n}\sum_{s=1}^{n}K_4\left|\mathcal{F}'(b)\right|^q\right)^{1/q}\right].\end{aligned} \tag{41}$$

Theorem 12. *Let $\mathcal{F}:\mathcal{I}\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^{\circ}$ with $\eta(b,a)>0$, $\lambda>0$, $\mu>0$, and let $\mathcal{F}'\in L[\mathcal{I}]$, $q\ge 1$. If $|\mathcal{F}'|^q$ is n-polynomial ζ-preinvex function, then*

$$\begin{aligned}&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\ &\le \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^2}{2}\right)^{1-(1/q)}\left(\frac{\mu^2}{2}\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}K_1\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right)^{1/q}\right. \\ &\left. + \left(\frac{\lambda^2}{2}\right)^{1-(1/q)}\left(\frac{\lambda^2}{2}\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}K_2\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right)^{1/q}\right],\end{aligned} \tag{42}$$

where K_1 and K_2 are the expressions as described in Theorem 10.

Proof. Note that $|\mathcal{F}'|^q$ is n-polynomial ζ-preinvex function, by using the power mean inequality, we have

$$\begin{aligned}&\left|\frac{\mu\mathcal{F}(a)+\lambda\mathcal{F}(a+\eta(b,a))}{\lambda+\mu} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathcal{F}(x)dx\right| \\ &\le \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\int_0^{\mu}|-t|\left|\mathcal{F}'\left(a+\frac{\mu-t}{\mu+\lambda}\eta(b,a)\right)\right|dt\right. \\ &\left. + \int_0^{\lambda}|t|\left|\mathcal{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)\right|dt\right] \\ &\le \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\int_0^{\mu}|-t|\,dt\right)^{1-(1/q)}\left(\int_0^{\mu}t\left|\mathcal{F}'\left(a+\frac{\mu-t}{\mu+\lambda}\eta(b,a)\right)\right|^q dt\right)^{1/q}\right. \\ &\left. + \left(\int_0^{\lambda}|t|\,dt\right)^{1-(1/q)}\left(\int_0^{\lambda}t\left|\mathcal{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)\right|^q dt\right)^{1/q}\right] \\ &\le \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^2}{2}\right)^{1-(1/q)}\left(\int_0^{\mu}t\left|\mathcal{F}'(a)\right|^q dt + \frac{1}{n}\sum_{s=1}^{n}\int_0^{\mu}t\left(1-\left(\frac{\lambda+t}{\lambda+\mu}\right)^s\right)\zeta\right.\right. \\ &\left.\cdot\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)dt\right)^{1/q} + \left(\frac{\lambda^2}{2}\right)^{1-(1/q)} \\ &\cdot\left(\int_0^{\lambda}t\left|\mathcal{F}'(a)\right|^q dt + \frac{1}{n}\sum_{s=1}^{n}\int_0^{\lambda}t\left(1-\left(\frac{\lambda-t}{\lambda+\mu}\right)^s\right)\zeta\right. \\ &\left.\left.\cdot\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)dt\right)^{1/q}\right] \\ &= \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^2}{2}\right)^{1-(1/q)}\left(\frac{\mu^2}{2}\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}K_1\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right)^{1/q}\right. \\ &\left. + \left(\frac{\lambda^2}{2}\right)^{1-(1/q)}\left(\frac{\lambda^2}{2}\left|\mathcal{F}'(a)\right|^q + \frac{1}{n}\sum_{s=1}^{n}K_2\zeta\left(\left|\mathcal{F}'(b)\right|^q,\left|\mathcal{F}'(a)\right|^q\right)\right)^{1/q}\right].\end{aligned} \tag{43}$$

Here, K_1 and K_2 are formulated as that of Theorem 10. This completes the proof of Theorem 12.

We now discuss some special cases of Theorem 12.

(I) Choosing $\lambda = \mu = 1$ in Theorem 12, we get

$$\left|\frac{\mathscr{F}(a)+\mathscr{F}(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathscr{F}(x)dx\right| \\ \leq \frac{\eta(b,a)}{2^{3-1/q}}\left[\left(\frac{\left|\mathscr{F}'(a)\right|^q}{2}+\frac{1}{n}\sum_{s=1}^{n}\left(\frac{(s+1)(s+2)2^s-2^{s+2}s-2}{2^{s+1}(s+1)(s+2)}\right)\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right. \\ \left.+\left(\frac{\left|\mathscr{F}'(a)\right|^q}{2}+\frac{1}{n}\sum_{s=1}^{n}\frac{(s+1)(s+2)2^s-2}{(s+1)(s+2)2^{s+1}}\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right]. \tag{44}$$

(II) Choosing $\eta(b,a)=b-a$ in Theorem 12, we have

$$\left|\frac{\mu\mathscr{F}(a)+\lambda\mathscr{F}(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b\mathscr{F}(x)dx\right| \\ \leq \frac{b-a}{(\lambda+\mu)^2}\left[\left(\frac{\mu^2}{2}\right)^{1-(1/q)}\left(\frac{\mu^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_1\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right. \\ \left.+\left(\frac{\lambda^2}{2}\right)^{1-(1/q)}\left(\frac{\lambda^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_2\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right]. \tag{45}$$

(III) Taking $\zeta(|\mathscr{F}'(b)|^q,|\mathscr{F}'(a)|^q)=|\mathscr{F}'(b)|^q-|\mathscr{F}'(a)|^q$ in Theorem 12, we obtain

$$\left|\frac{\mu\mathscr{F}(a)+\lambda\mathscr{F}(a+\eta(b,a))}{\lambda+\mu}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathscr{F}(x)dx\right| \\ \leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\left(\frac{\mu^2}{2}\right)^{1-(1/q)}\left(\left(\frac{\mu^2}{2}-\frac{1}{n}\sum_{s=1}^{n}K_1\right)\left|\mathscr{F}'(a)\right|^q\right.\right. \\ \left.+\frac{1}{n}\sum_{s=1}^{n}K_1\left|\mathscr{F}'(b)\right|^q\right)^{1/q}+\left(\frac{\lambda^2}{2}\right)^{1-(1/q)}\left(\left(\frac{\lambda^2}{2}-\frac{1}{n}\sum_{s=1}^{n}K_2\right)\left|\mathscr{F}'(a)\right|^q\right. \\ \left.\left.+\frac{1}{n}\sum_{s=1}^{n}K_2\left|\mathscr{F}'(b)\right|^q\right)^{1/q}\right]. \tag{46}$$

Theorem 13. *Let $\mathscr{F}:\mathscr{I}\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^\circ$ with $\eta(b.a)\lambda>0$, $\lambda>0$, $\mu>0$, and let $\mathscr{F}'\in L[\mathscr{I}]$, $1/p+1/q=1$, $p>1$, $q>1$. If $|\mathscr{F}'|^q$ is n-polynomial ζ-preinvex function, then*

$$\left|\frac{\mu\mathscr{F}(a)+\lambda\mathscr{F}(a+\eta(b,a))}{\lambda+\mu}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathscr{F}(x)dx\right| \\ \leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\frac{1}{\mu}\left\{\left(\frac{\mu^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\frac{\mu^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_5\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right.\right. \\ \left.+\left(\frac{\mu^{p+2}}{p+2}\right)^{1/p}\left(\frac{\mu^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_6\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right\} \\ +\frac{1}{\lambda}\left\{\left(\frac{\lambda^{p+2}}{(p+1)(p+2)}\right)^{1/p}\times\left(\frac{\lambda^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_7\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right. \\ \left.\left.+\left(\frac{\lambda^{p+2}}{p+2}\right)^{1/p}\left(\frac{\lambda^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_8\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right\}\right], \tag{47}$$

where

$$K_5=\frac{\mu^2}{2}-\frac{\mu^{s+2}}{(s+2)(\lambda+\mu)^s}, \\ K_6=\frac{\mu^2}{2}-\frac{\mu^{s+2}}{(s+1)(s+2)(\lambda+\mu)^s}, \\ K_7=\frac{\lambda^2}{2}-\frac{(\mu+\lambda)^{s+2}-\mu^{s+2}-\lambda(s+2)\mu^{s+1}}{(s+1)(s+2)(\lambda+\mu)^s}, \\ K_8=\frac{\lambda^2}{2}-\frac{(\mu+\lambda)^{s+2}-\mu^{s+2}-\lambda(s+2)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^s}. \tag{48}$$

Proof. Note that $|\mathscr{F}'|^q$ is n-polynomial ζ-preinvex function, by using the refined Hölder inequality (see [19]), we obtain

$$\left|\frac{\mu\mathscr{F}(a)+\lambda\mathscr{F}(a+\eta(b,a))}{\lambda+\mu}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mathscr{F}(x)dx\right| \leq \frac{\eta(b,a)}{(\lambda+\mu)^2} \\ \cdot\left[\int_0^\mu|-t|\left|\mathscr{F}'\left(a+\frac{\mu-t}{\mu+\lambda}\eta(b,a)\right)\right|dt+\int_0^\lambda|t|\left|\mathscr{F}'\left(a+\frac{\mu+t}{\lambda+\mu}\eta(b,a)\right)\right|dt\right] \\ \leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\frac{1}{\mu}\left\{\left(\int_0^\mu(\mu-t)|-t|^pdt\right)^{1/p}\left(\int_0^\mu(\mu-t)\left|\mathscr{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)\right|^qdt\right)^{1/q}\right.\right. \\ \left.+\left(\int_0^\mu t|-t|^pdt\right)^{1/p}\left(\int_0^\mu t\left|\mathscr{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)\right|^qdt\right)^{1/q}\right\} \\ +\frac{1}{\lambda}\left\{\left(\int_0^\lambda(\lambda-t)|t|^pdt\right)^{1/p}\left(\int_0^\lambda(\lambda-t)\left|\mathscr{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)\right|^qdt\right)^{1/q}\right. \\ \left.\left.+\left(\int_0^\lambda t\left|\mathscr{F}'\left(a+\frac{\mu-t}{\lambda+\mu}\eta(b,a)\right)\right|^qdt\right)^{1/q}\right\}\right] \leq \frac{\eta(b,a)}{(\lambda+\mu)^2} \\ \cdot\left[\frac{1}{\mu}\left\{\left(\frac{\mu^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\int_0^\mu\mu-t\tilde{n}\left(\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}\right.\right.\right.\right. \\ \left.\left.\cdot\left(1-\left(\frac{\mu-t}{\lambda+\mu}\right)^s\right)\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)dt\right)^{1/q}+\left(\frac{\mu^{p+2}}{p+2}\right)^{1/q} \\ \left.\cdot\left(\int_0^\mu t\left(\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{\mu-t}{\lambda+\mu}\right)^s\right)\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)dt\right)^{1/q}\right\} \\ +\frac{1}{\lambda}\left\{\left(\frac{\lambda^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\int_0^\lambda\lambda-t\tilde{n}\left(\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{\mu-t}{\lambda+\mu}\right)^s\right)\right.\right.\right. \\ \left.\left.\cdot\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)dt\right)^{1/q}+\left(\frac{\lambda^{p+2}}{p+2}\right)^{1/q} \\ \left.\left.\cdot\left(\int_0^\lambda t\left(\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{\mu-t}{\lambda+\mu}\right)^s\right)\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)dt\right)^{1/q}\right\}\right] \\ =\frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\frac{1}{\mu}\left\{\left(\frac{\mu^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\frac{\mu^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_5\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right.\right. \\ \left.+\left(\frac{\mu^{p+2}}{p+2}\right)^{1/p}\left(\frac{\mu^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_6\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right\} \\ +\frac{1}{\lambda}\left\{\left(\frac{\lambda^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\frac{\lambda^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_7\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right. \\ \left.\left.+\left(\frac{\mu^{p+2}}{p+2}\right)^{1/p}\left(\frac{\lambda^2}{2}\left|\mathscr{F}'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}K_8\zeta\left(\left|\mathscr{F}'(b)\right|^q,\left|\mathscr{F}'(a)\right|^q\right)\right)^{1/q}\right\}\right] \tag{49}$$

A direct computation gives

$$K_5=\int_0^\mu(\mu-t)\left(1-\left(\frac{\mu-t}{\lambda+\mu}\right)^s\right)dt=\frac{\mu^2}{2}-\frac{\mu^{s+2}}{(s+2)(\lambda+\mu)^s},$$

$$
\begin{aligned}
K_6 &= \int_0^{\mu} t\left(1-\left(\frac{\mu-t}{\lambda+\mu}\right)^s\right)dt = \frac{\mu^2}{2} - \frac{\mu^{s+2}}{(s+1)(s+2)(\lambda+\mu)^s}, \\
K_7 &= \int_0^{\lambda} (\lambda-t)\left(1-\left(\frac{\mu+t}{\lambda+\mu}\right)^s\right)dt \\
&= \frac{\lambda^2}{2} - \frac{(\mu+\lambda)^{s+2} - \mu^{s+2} - \lambda(s+2)\mu^{s+1}}{(s+1)(s+2)(\lambda+\mu)^s}, \\
K_8 &= \int_0^{\lambda} t\left(1-\left(\frac{\mu+t}{\lambda+\mu}\right)^s\right)dt \\
&= \frac{\lambda^2}{2} - \frac{(\mu+\lambda)^{s+2} - \mu^{s+2} - \lambda(s+2)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^s}.
\end{aligned} \tag{50}
$$

This completes the proof of Theorem 13.

Let us now discuss some special cases of Theorem 13.

(I) If we take $\lambda = \mu = 1$ in Theorem 13, we get

$$
\begin{aligned}
&\left|\frac{\mathscr{F}(a)+\mathscr{F}(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mathscr{F}(x)dx\right| \\
&\le \frac{\eta(b,a)}{4}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}\right. \\
&\quad \times \left\{\left(\frac{|\mathscr{F}'(a)|^q}{2} + \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2} - \frac{1}{(s+2)2^s}\right)\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right. \\
&\quad \left. + \left(\frac{|\mathscr{F}'(a)|^q}{2} + \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2} - \frac{1}{(s+1)(s+2)2^s}\right)\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right\} \\
&\quad + \left(\frac{1}{p+2}\right)^{1/p}\left\{\left(\frac{|\mathscr{F}'(a)|^q}{2} + \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2} - \frac{2^{s+2}-s-3}{(s+1)(s+2)2^s}\right)\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right. \\
&\quad \left.\left. + \left(\frac{|\mathscr{F}'(a)|^q}{2} + \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2} + \frac{1+s2^{s+1}}{(s+1)(s+2)2^s}\right)\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right\}\right].
\end{aligned} \tag{51}
$$

(II) If we put $\eta(b,a) = b-a$ in Theorem 13, then

$$
\begin{aligned}
&\left|\frac{\mu\mathscr{F}(a)+\lambda\mathscr{F}(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^{b} \mathscr{F}(x)dx\right| \\
&\le \frac{b-a}{(\lambda+\mu)^2}\left[\frac{1}{\mu}\left\{\left(\frac{\mu^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\frac{\mu^2}{2}|\mathscr{F}'(a)|^q + \frac{1}{n}\sum_{s=1}^{n} K_5\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right.\right. \\
&\quad \left. + \left(\frac{\mu^{p+2}}{p+2}\right)^{1/p}\left(\frac{\mu^2}{2}|\mathscr{F}'(a)|^q + \frac{1}{n}\sum_{s=1}^{n} K_6\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right\} \\
&\quad + \frac{1}{\lambda}\left\{\left(\frac{\lambda^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\frac{\lambda^2}{2}|\mathscr{F}'(a)|^q + \frac{1}{n}\sum_{s=1}^{n} K_7\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right. \\
&\quad \left.\left. + \left(\frac{\lambda^{p+2}}{p+2}\right)^{1/p}\left(\frac{\lambda^2}{2}|\mathscr{F}'(a)|^q + \frac{1}{n}\sum_{s=1}^{n} K_8\zeta\left(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q\right)\right)^{1/q}\right\}\right].
\end{aligned} \tag{52}
$$

(III) If we choose $\zeta(|\mathscr{F}'(b)|^q, |\mathscr{F}'(a)|^q) = |\mathscr{F}'(b)|^q - |\mathscr{F}'(a)|^q$ in Theorem 13, then

$$
\begin{aligned}
&\left|\frac{\mu\mathscr{F}(a)+\lambda\mathscr{F}(a+\eta(b,a))}{\lambda+\mu} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mathscr{F}(x)dx\right| \\
&\le \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[\frac{1}{\mu}\left\{\left(\frac{\mu^{p+2}}{(p+1)(p+2)}\right)^{1/p}\left(\left(\frac{\mu^2}{2} - \frac{1}{n}\sum_{s=1}^{s} K_5\right)|\mathscr{F}'(a)|^q\right.\right.\right. \\
&\quad \left. + \frac{1}{n}\sum_{s=1}^{n} K_5|\mathscr{F}'(b)|^q\right)^{1/q} + \left(\frac{\mu^{p+2}}{p+2}\right)^{1/p}\left(\left(\frac{\mu^2}{2} - \frac{1}{n}\sum_{s=1}^{n} K_6\right)|\mathscr{F}'(a)|^q\right. \\
&\quad \left.\left. + \frac{1}{n}\sum_{s=1}^{n} K_6|\mathscr{F}'(b)|^q\right)^{1/q}\right\} + \frac{1}{\lambda}\left\{\left(\frac{\lambda^{p+2}}{(p+1)(p+2)}\right)^{1/p}\right. \\
&\quad \cdot\left(\left(\frac{\lambda^2}{2} - \frac{1}{n}\sum_{s=1}^{n} K_7\right)|\mathscr{F}'(a)|^q + \frac{1}{n}\sum_{s=1}^{n} K_7|\mathscr{F}'(b)|^q\right)^{1/q} \\
&\quad \left.\left. + \left(\frac{\lambda^{p+2}}{p+2}\right)^{1/p}\left(\left(\frac{\lambda^2}{2} - \frac{1}{n}\sum_{s=1}^{n} K_8\right)|\mathscr{F}'(a)|^q + \frac{1}{n}\sum_{s=1}^{n} K_8|\mathscr{F}'(b)|^q\right)^{1/q}\right\}\right].
\end{aligned} \tag{53}
$$

3. Application to Special Means

Let us recall the definitions of the arithmetic mean, weighted arithmetic mean, and the mean for functions, as follows:

(1) The arithmetic mean

$$
\mathscr{A}(a_1, a_2, \cdots, a_n) = \frac{a_1 + a_2 + \cdots + a_n}{n}. \tag{54}
$$

(2) The weighted arithmetic mean

$$
\mathscr{A}(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n) = \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}. \tag{55}
$$

(3) The mean of the function Φ on $[a, b]$

$$
\mathscr{A}_{\Phi}(a, b) = \frac{1}{b-a}\int_a^b \Phi(x)dx. \tag{56}
$$

We establish the following inequalities for special means.

Proposition 14. *Let $\Phi : [a, a+\eta(b,a)] \to \mathbb{R}$ be a preinvex function with $\Phi(b) - \Phi(a) \ge 0$. If $\eta(b,a) > 0$ and $\Phi \in L[a, a+\eta(b,a)]$, then we have the following inequality*

$$
\mathscr{A}_{\Phi}(a, a+\eta(b,a)) \le \Phi(a) + (\Phi(b) - \Phi(a))\mathscr{A}\left(\frac{1}{2}, \frac{2}{3}, \cdots, \frac{n}{n+1}\right). \tag{57}
$$

Proof. Taking $\mathscr{F}(x) = \Phi(x), x \in [a, a+\eta(b,a)]$ and $\zeta(\mathscr{F}(b), \mathscr{F}(a)) = \mathscr{F}(b) - \mathscr{F}(a)$. Since $\Phi(x)$ is a preinvex function with $\Phi(b) - \Phi(a) \ge 0$, we deduce from Proposition 7 that $\Phi(x)$ is a n-polynomial preinvex function. Using Theorem 8, we obtain the desired inequality (57).

Proposition 15. *Let Φ be a differentiable function, and let $|\Phi'|: [a, a+\eta(b,a)] \to \mathbb{R}$ be a preinvex function with $|\Phi'(b)| - |\Phi'(a)| \geq 0$. If $\eta(b,a) > 0$, $\lambda > 0$, $\mu > 0$, and $|\Phi'| \in L[a, a+\eta(b,a)]$, then we have the following inequality*

$$\begin{aligned}&|\mathscr{A}(\Phi(a), \Phi(a+\eta(b,a)) ; \mu, \lambda) - \mathscr{A}_{\Phi}(a, a+\eta(b,a))| \\ &\leq \frac{\eta(b,a)}{(\lambda+\mu)^2}\left[|\Phi'(a)|\mathscr{A}(\lambda^2, \mu^2) + \frac{2}{n}\left(|\Phi'(b)| - |\Phi'(a)|\right)\sum_{s=1}^{n}\mathscr{A}(K_1, K_2)\right],\end{aligned} \tag{58}$$

where K_1 and K_2 are the expressions as that described in Theorem 10.

Proof. Choosing $\mathscr{F}(x) = \Phi(x), x \in [a, a+\eta(b,a)]$ and $\zeta(\mathscr{F}(b), \mathscr{F}(a)) = \mathscr{F}(b) - \mathscr{F}(a)$. Since $|\Phi'(x)|$ is a preinvex function with $|\Phi'(b)| - |\Phi'(a)| \geq 0$, it follows from Proposition 7 that $|\Phi'(x)|$ is a n-polynomial preinvex function. Using Theorem 10 leads to the desired inequality (58).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

S.W. and M.U.A. finished the proofs of the main results and the writing work. M.U.U., S.T., and A.K. gave lots of advice on the proofs of the main results and the writing work. All authors read and approved the final manuscript.

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References

[1] R. J. Gardner, D. Hug, and W. Weil, "The Orlicz Brunn-Minkowski theory: a general framework, additions, and inequalities," *Journal of Differential Geometry*, vol. 97, no. 3, pp. 427–476, 2014.

[2] P. M. Gruber, *Convex and Discrete Geometry*, Springer, Berlin, 2007.

[3] S. Mititelu, "Invex sets," *Studii şi Cercetări Matematice*, vol. 46, no. 5, pp. 529–532, 1994.

[4] T. Weir and B. Mond, "Pre-invex functions in multiple objective optimization," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 29–38, 1988.

[5] M. E. Gordji, M. R. Delavar, and M. D. L. Sen, "On φ-convex functions," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 173–183, 2016.

[6] M. A. Noor, "Hermite-Hadamard integral inequalities for log -preinvex functions," *Journal of Mathematical Analysis and Approximation Theory*, vol. 2, pp. 126–131, 2007.

[7] A. Barani, A. G. Ghazanfari, and S. S. Dragomir, "Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex," *Journal of Inequalities and Applications*, vol. 2012, no. 1, Article ID 247, 2012.

[8] T.-S. Du, J.-G. Liao, and Y.-J. Li, "Properties and integral inequalities of Hadamard- Simpson type for the generalized (s, m)-preinvex functions," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 5, pp. 3112–3126, 2016.

[9] M. A. Noor, K. I. Noor, M. U. Awan, and J. Li, "On Hermite-Hadamard inequalities for h-preinvex functions," *Univerzitet u Nišu*, vol. 28, no. 7, pp. 1463–1474, 2014.

[10] J. Park, "Simpson-like and Hermite-Hadamard-like type integral inequalities for twice differentiable preinvex functions," *International Journal of Pure and Applied Mathematics*, vol. 79, pp. 623–640, 2012.

[11] J. Park, "Hermite-Hadamard-like type integral inequalities for functions whose derivatives of n-th order are preinvex," *Applied Mathematical Sciences*, vol. 7, pp. 6637–6650, 2013.

[12] M. Z. Sarikaya, N. Alp, and H. Bozkurt, "On Hermite-Hadamard type integral inequalities for preinvex and log -preinvex functions," *Contemporary Analysis and Applied Mathematics*, vol. 1, pp. 237–252, 2013.

[13] S.-H. Wu, I. A. Baloch, and İ. İşcan, "On harmonically -preinvex functions," *Journal of Function Spaces*, vol. 2017, Article ID 2148529, 9 pages, 2017.

[14] S. H. Wang and X. M. Liu, "Hermite-Hadamard type inequalities for operator s-preinvex functions," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1070–1081, 2015.

[15] J. Y. Li, "On Hadamard-type inequalities for s-preinvex functions," *Journal of Chongqing Normal University (Natural Science)*, vol. 27, pp. 5–8, 2010.

[16] Y. Deng, H. Kalsoom, and S. Wu, "Some new quantum Hermite-Hadamard-type estimates within a class of generalized (s,m)-preinvex functions," *Symmetry*, vol. 11, no. 10, article 1283, 2019.

[17] Y. Deng, M. U. Awan, and S. Wu, "Quantum integral inequalities of Simpson-type for strongly preinvex functions," *Mathematics*, vol. 7, no. 8, p. 751, 2019.

[18] S. Wu, M. U. Awan, M. V. Mihai, M. A. Noor, and S. Talib, "Estimates of upper bound for a kth order differentiable functions involving Riemann–Liouville integrals via higher order strongly h-preinvex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, Article ID 227, 2019.

[19] T. Toplu, M. Kadakal, and İ. İşcan, "On n-polynomial convexity and some related inequalities," *AIMS Mathematics*, vol. 5, no. 2, pp. 1304–1318, 2020.

[20] S. R. Mohan and S. K. Neogy, "On invex sets and preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 3, pp. 901–908, 1995.

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Research Article

Multiplication Operators on Orlicz Generalized Difference (sss)

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In this article, we inspect the sufficient conditions on the Orlicz generalized difference sequence space to be premodular Banach (sss). We look at some topological and geometrical structures of the multiplication operators described on Orlicz generalized difference prequasi normed (sss).

1. Introduction

The multiplication operators have a large subject of mathematics in functional analysis, namely, in eigenvalue distribution theorem, geometric structure of Banach spaces, and theory of fixed point. For more technicalities (see [1–6]), by $\mathbb{C}^{\mathbb{N}}$, c, ℓ_∞, ℓ_r, and c_0, we mean the spaces of each, convergent, bounded, r-absolutely summable and convergent to zero sequences of complex numbers. $\mathbb{N}$ displays the set of nonnegative integers. Tripathy et al. [7] popularized and measured the forward and backward generalized difference sequence spaces:

$$G\left(\Delta_n^{(m)}\right) = \left\{(w_k) \in \mathbb{C}^{\mathbb{N}} : \left(\Delta_n^{(m)} w_k\right) \in G\right\},$$
$$G(\Delta_n^m) = \left\{(w_k) \in \mathbb{C}^{\mathbb{N}} : (\Delta_n^m w_k) \in G\right\}, \tag{1}$$

where $m, n \in \mathbb{N}$, $G = \ell_\infty$, c, or c_0, with

$$\Delta_n^{(m)} w_k = \sum_{\nu=0}^{m} (-1)^\nu C_\nu^m w_{k+\nu n}, \text{ and } \Delta_n^m w_k = \sum_{\nu=0}^{m} (-1)^\nu C_\nu^m \, w_{k-\nu n}, \tag{2}$$

successively. When $n = 1$, the generalized difference sequence spaces concentrated to $G(\Delta^{(m)})$ defined and examined by Et and Çolak [8]. If $m = 1$, the generalized difference sequence spaces diminished to $G(\Delta_n)$ constructed and studied by Tripathy and Esi [9]. While if $n = 1$ and $m = 1$, the generalized difference sequence spaces reduced to $G(\Delta)$ defined and investigated by Kizmaz [10].

An Orlicz function [11] is a function $\psi : [0,\infty) \to [0,\infty)$, which is convex, continuous, and nondecreasing with $\psi(0) = 0$, $\psi(u) > 0$, for $u > 0$ and $\psi(u) \to \infty$, as $u \to \infty$. In [12], an Orlicz function ψ is called to satisfy the δ_2-condition for each values of $x \geq 0$, if there is $k > 0$, such that $\psi(2x) \leq k\psi(x)$. The δ_2-condition is equivalent to $\psi(lx) \leq kl\,\psi(x)$, for every values of x and $l > 1$. Lindentrauss and Tzafriri [13] used the idea of an Olicz function to construct the Orlicz sequence space:

$$\ell_\psi = \left\{u \in \mathbb{C}^{\mathbb{N}} : \rho(\beta u) < \infty, \text{for some } \beta > 0\right\}, \text{where } \rho(u) = \sum_{k=0}^{\infty} \psi(|u_k|), \tag{3}$$

$(\ell_\psi, \|.\|)$ is a Banach space with the Luxemburg norm:

$$\|u\| = \inf\left\{\beta > 0 : \rho\left(\frac{u}{\beta}\right) \leq 1\right\}. \tag{4}$$

Every Orlicz sequence space includes a subspace that is isomorphic to c_0 or ℓ^q, for some $1 \le q < \infty$.

Recently, different classes of sequences have been examined the usage of Orlicz functions via Et et al. [14], Mursaleen et al. [15–17], and Alotaibi et al. [18].

Let $r = (r_j) \in \mathbb{R}^{+\mathbb{N}}$, where $\mathbb{R}^{+\mathbb{N}}$ denotes the space of sequences with positive reals, and we define the Orlicz backward generalized difference sequence space as follows:

$$\left(\ell_\psi(\Delta_{n+1}^m)\right)_\tau = \left\{w = (w_j) \in \mathbb{C}^{\mathbb{N}} : \exists \sigma > 0 \text{ with } \tau(\sigma w) < \infty\right\}, \tag{5}$$

where $\tau(w) = \sum_{j=0}^{\infty} \psi(|\Delta_{n+1}^m |w_j||)$, $w_j = 0$, for $j < 0$, $\Delta_{n+1}^m |w_j| = \Delta_{n+1}^{m-1} |w_j| - \Delta_{n+1}^{m-1} |w_{j-1}|$, and $\Delta^0 w_j = w_j$, for all $j, n, m \in \mathbb{N}$. It is a Banach space, with

$$\|w\| = \inf \left\{\sigma > 0 : \tau\left(\frac{w}{\sigma}\right) \le 1\right\}. \tag{6}$$

When $\psi(w) = w^r$, then $\ell_\psi(\Delta_{n+1}^m) = \ell_r(\Delta_{n+1}^m)$ investigated via many authors (see [19–21]). By $\mathfrak{B}(W, Z)$, we will denote the set of every operators which are linear and bounded between Banach spaces W and Z, and if $W = Z$, we write $\mathfrak{B}(W)$. On sequence spaces, Basarir and Kara examined the compact operators on some Euler $B(m)$-difference sequence spaces [22], some difference sequence spaces of weighted means [23], the Riesz $B(m)$-difference sequence space [24], the B-difference sequence space derived by weighted mean [25], and the m^{th} order difference sequence space of generalized weighted mean [26]. Mursaleen and Noman [27, 28] investigated the compact operators on some difference sequence spaces. The multiplication operators on $(ces(r), \|.\|)$ with the Luxemburg norm $\|.\|$ elaborated by Komal et al. [29]. İlkhan et al. [30] studied the multiplication operators on Cesáro second order function spaces. Bakery et al. [31] examined the multiplication operators on weighted Nakano (sss). The aim of this article is to explain some results of $(\ell_\psi(\Delta_{n+1}^m))_\tau$ equipped with the prequasi norm τ. Firstly, we accord the sufficient conditions on the Orlicz generalized difference sequence space to become premodular Banach (sss). Secondly, we provide with the necessity and sufficient conditions on the Orlicz generalized difference sequence space provided with the prequasi norm so that the multiplication operator defined on it is bounded, approximable, invertible, Fredholm, and closed range operator.

2. Preliminaries and Definitions

Definition 1 [32]. An operator $V \in \mathfrak{B}(W)$ is known as approximable if there are $D_r \in F(W)$, for every $r \in \mathbb{N}$ and $\lim_{r\to\infty} \|V - D_r\| = 0$.

By $\Upsilon(W, Z)$, we will denote the space of all approximable operators from W to Z.

Theorem 2 [32]. *Let W be a Banach space with* $\dim(W) = \infty$*; then,*

$$F(W) \underset{\neq}{\subset} \Upsilon(W) \underset{\neq}{\subset} \mathfrak{B}_c(W) \underset{\neq}{\subset} \mathfrak{B}(W). \tag{7}$$

Definition 3 [33]. An operator $V \in \mathfrak{B}(W)$ is named Fredholm if $\dim (R(V))^c < \infty$, $\dim (ker V) < \infty$, and $R(V)$ are closed, where $(R(V))^c$ indicates the complement of range V.

The sequence $e_j = (0, 0, \ldots, 1, 0, 0, \cdots)$ with 1 in the j^{th} coordinate, for all $j \in \mathbb{N}$, will be used in the sequel.

Definition 4 [34]. The space of linear sequence spaces $\mathbb{Y}$ is called (sss) if

(1) $e_r \in \mathbb{Y}$ with $r \in \mathbb{N}$

(2) Let $u = (u_r) \in \mathbb{C}^{\mathbb{N}}$, $v = (v_r) \in \mathbb{Y}$, and $|u_r| \le |v_r|$, for every $r \in \mathbb{N}$, then $u \in \mathbb{Y}$. This means $\mathbb{Y}$ be solid

(3) If $(u_r)_{r=0}^{\infty} \in \mathbb{Y}$, then $(u_{[r/2]})_{r=0}^{\infty} \in \mathbb{Y}$, wherever $[r/2]$ indicates the integral part of $r/2$

Definition 5 [35]. A subspace of the (sss) $\mathbb{Y}_\tau$ is named a premodular (sss) if there is a function $\tau : \mathbb{Y} \to [0, \infty)$ confirming the conditions:

(i) $\tau(y) \ge 0$ for each $y \in \mathbb{Y}$ and $\tau(y) = 0 \Leftrightarrow y = \theta$, where θ is the zero element of $\mathbb{Y}$

(ii) There exists $a \ge 1$ such that $\tau(\eta y) \le a|\eta|\tau(y)$, for all $y \in \mathbb{Y}$, and $\eta \in \mathbb{C}$

(iii) For some $b \ge 1$, $\tau(y + z) \le b(\tau(y) + \tau(z))$, for every $y, z \in \mathbb{Y}$

(iv) $|y_r| \le |z_r|$ with $r \in \mathbb{N}$ implies $\tau((y_r)) \le \tau((z_r))$

(v) For some $b_0 \ge 1$, $\tau((y_r)) \le \tau((y_{[r/2]})) \le b_0 \tau((y_i))$

(vi) If $y = (y_r)_{r=o}^{\infty} \in \mathbb{Y}$ and $d > 0$, then there is $r_0 \in \mathbb{N}$ with $\tau((y_r)_{r=r_0}^{\infty}) < d$

(vii) There is $t > 0$ with $\tau(v, 0, 0, 0, \cdots) \ge t|v|\tau(1, 0, 0, 0, \cdots)$, for any $v \in \mathbb{C}$

The (sss) $\mathbb{Y}_\tau$ is known as prequasi normed (sss) if τ administers the parts (i)-(iii) of Definition 5 and when the space $\mathbb{Y}$ is complete under τ, then $\mathbb{Y}_\tau$ is named a prequasi Banach (sss).

Theorem 6 [35]. *A prequasi norm (sss) $\mathbb{Y}_\tau$ if it is premodular (sss).*

The inequality [36], $|a_i + b_i|^{q_i} \le H(|a_i|^{q_i} + |b_i|^{q_i})$, where $q_i \ge 0$ for all $i \in \mathbb{N}$, $H = \max\{1, 2^{h-1}\}$ and $h = \sup_i q_i$, will be used in the sequel.

3. Main Results

3.1. Prequasi Norm on $\ell_\psi(\Delta^m_{n+1})$. In this section, we explain the conditions on the Orlicz backward generalized difference sequence space to form premodular Banach (sss).

Definition 7. The backward generalized difference Δ^m_{n+1} is named an absolute nondecreasing, if $|x_i| \le |y_i|$, for all $i \in \mathbb{N}$, then $|\Delta^m_{n+1}|x_i|| \le |\Delta^m_{n+1}|y_i||$.

Theorem 8. *Let ψ be an Orlicz function fulfilling the δ_2-condition and Δ^m_{n+1} be an absolute nondecreasing, then the space $(\ell_\psi(\Delta^m_{n+1}))_\tau$ can be a premodular Banach (sss), where*

$$\tau(w) = \sum_{j=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}|w_j|\right|\right), \text{for all } w \in \ell_\psi(\Delta^m_{n+1}). \tag{8}$$

Proof. (1-i) Assume $v, w \in \ell_\psi(\Delta^m_{n+1})$. Since ψ can be nondecreasing, convex, agreeable δ_2-condition, and Δ^m_{n+1} can be an absolute nondecreasing, then there is $b > 0$ such that

$$\begin{aligned} \tau(v+w) &= \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|v_i + w_i||) \\ &\le \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|v_i|| + |\Delta^m_{n+1}|w_i||) \\ &\le \frac{1}{2}\left(\sum_{i=0}^{\infty} \psi(2|\Delta^m_{n+1}|v_i||) + \sum_{i=0}^{\infty} \psi(2|\Delta^m_{n+1}|w_i||)\right. \\ &\le \frac{b}{2}(\tau(v) + \tau(w)) \le B(\tau(v) + \tau(w)) < \infty, \end{aligned} \tag{9}$$

for some $B = \max\{1, (b/2)\}$. Then, $v + w \in \ell_\psi(\Delta^m_{n+1})$.

(1) (1-ii) Suppose $\lambda \in \mathbb{C}$ and $v \in \ell_\psi(\Delta^m_{n+1})$. Since ψ is fulfilling the δ_2-condition, we obtain

$$\begin{aligned} \tau(\lambda v) &= \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|\lambda v_r||) \le d|\lambda| \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|v_r||) \\ &\le D|\lambda|\tau(v) < \infty, \end{aligned} \tag{10}$$

where $D = \max\{1, d\}$. Then, $\lambda v \in \ell_\psi(\Delta^m_{n+1})$. So, from parts (1-i) and (1-ii), the space $\ell_\psi(\Delta^m_{n+1})$ is linear. Since $e_r \in \ell_q \subseteq \ell_\psi \subseteq \ell_\psi(\Delta^m_{n+1})$, for every $r \in \mathbb{N}$ and $q \ge 1$, hence, $e_r \in \ell_\psi(\Delta^m_{n+1})$, for each $r \in \mathbb{N}$.

(2) Let $|x_i| \le |y_i|$, for every $i \in \mathbb{N}$ and $y \in \ell_\psi(\Delta^m_{n+1})$. Since ψ is nondecreasing and Δ^m_{n+1} is an absolute nondecreasing, therefore, we get

$$\tau(x) = \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|x_i||) \le \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|y_i||) = \tau(y) < \infty, \tag{11}$$

hence $x \in \ell_\psi(\Delta^m_{n+1})$

(3) Suppose $(v_r) \in \ell_\psi(\Delta^m_{n+1})$, one has

$$\tau\left(\left(v_{\left[\frac{r}{2}\right]}\right)\right) = \sum_{r=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left|v_{\left[\frac{r}{2}\right]}\right|\right|\right) \le 2\sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|v_r||) = 2\tau(v), \tag{12}$$

then $(v_{[r/2]}) \in \ell_\psi(\Delta^m_{n+1})$

(i) Evidently, $\tau(w) \ge 0$ and $\tau(w) = 0 \Leftrightarrow w = \theta$

(ii) There is $D \ge 1$ where $\tau(\eta w) \le D|\eta|\tau(w)$, for every $w \in \ell_\psi(\Delta^m_{n+1})$ and $\eta \in \mathbb{C}$

(iii) For some $B \ge 1$, we obtain $\tau(v + w) \le B(\tau(v) + \tau(w))$, for all $v, w \in \ell_\psi(\Delta^m_{n+1})$

(iv) Plainly from (2).

(v) From (3), we have that $b_0 = 2 \ge 1$

(vi) It is apparent that $\bar{F} = \ell_\psi(\Delta^m_{n+1})$

(vii) Since ψ is verifying the δ_2-condition, there is ζ with $0 < \zeta \le \psi(|\eta|)/|\eta|$ such that $\tau(\eta, 0,0,0, \cdots) \ge \zeta|\eta|\tau(1,0,0,0,\cdots)$, for each $\eta \ne 0$ and $\zeta > 0$, if $\eta = 0$

Therefore, the space $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is premodular (sss). To show that $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is a premodular Banach (sss), Suppose $x^i = (x^i_k)^\infty_{k=0}$ is a Cauchy sequence in $(\ell_\psi(\Delta^m_{n+1}))_\tau$, then for all $\varepsilon \in (0, 1)$, there is $i_0 \in \mathbb{N}$ such that for all $i, j \ge i_0$, we get

$$\tau\left(x^i - x^j\right) = \sum_{k=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left|x^i_k - x^j_k\right|\right|\right) < \psi(\varepsilon). \tag{13}$$

Since ψ is nondecreasing; hence, for $i, j \ge i_0$ and $k \in \mathbb{N}$, we obtain

$$\left|\Delta^m_{n+1}\left|x^i_k\right| - \Delta^m_{n+1}\left|x^j_k\right|\right| < \varepsilon. \tag{14}$$

Hence, $(\Delta^m_{n+1}|x^j_k|)$ is a Cauchy sequence in $\mathbb{C}$ for fixed $k \in \mathbb{N}$, so $\lim_{j\to\infty} \Delta^m_{n+1} x^j_k = \Delta^m_{n+1} x^0_k$ for fixed $k \in \mathbb{N}$. Therefore, $\tau(x^i - x^0) < \psi(\varepsilon)$, for each $i \ge i_0$. Finally, to explain that $x^0 \in \ell_\psi(\Delta^m_{n+1})$, we have

$$\tau\left(x^0\right) = \tau\left(x^0 - x^n + x^n\right) \le B\left(\tau\left(x^n - x^0\right) + \tau(x^n)\right) < \infty. \tag{15}$$

So, $x^0 \in \ell_\psi(\Delta^m_{n+1})$. This implies that $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is a premodular Banach (sss).

Taking into consideration (Theorem 6), we be over the following theorem.

Theorem 9. *If ψ is an Orlicz function satisfying the δ_2 -condition and Δ^m_{n+1} is an absolute nondecreasing, then the space $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is prequasi Banach (sss), where*

$$\tau(x) = \sum_{j=0}^{\infty} \psi\left(|\Delta^m_{n+1}|x_j||\right), \text{for all } x \in \ell_\psi(\Delta^m_{n+1}). \tag{16}$$

Corollary 10. *If $0 < p < \infty$ and Δ^m_{n+1} is an absolute nondecreasing, then $(\ell_p(\Delta^m_{n+1}))_\tau$ is a premodular Banach (sss), where $\tau(x) = \sum_{i=0}^{\infty} | \Delta^m_{n+1} | x_i||^p$, for all $x \in \ell_p(\Delta^m_{n+1})$.*

4. Bounded Multiplication Operator on $\ell_\psi(\Delta^m_{n+1})$

Here and after, we explain some geometric and topological structures of the multiplication operator reserve on $\ell_\psi(\Delta^m_{n+1})$.

Definition 11. Let $\kappa \in \mathbb{C}^{\mathbb{N}} \cap \ell_\infty$ and W_τ be a prequasi normed (sss). An operator $V_\kappa : W_\tau \to W_\tau$ is named multiplication operator if $V_\kappa w = \kappa w = (\kappa_r w_r)_{r=0}^{\infty} \in W$, for every $w \in W$. If $V_\kappa \in \mathfrak{B}(W)$, we call it a multiplication operator generated by κ.

Theorem 12. *If $\kappa \in \mathbb{C}^{\mathbb{N}}$, ψ is an Orlicz function verifying the δ_2-condition, and Δ^m_{n+1} is an absolute nondecreasing, then $\kappa \in \ell_\infty$, if and only if, $V_\kappa \in \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$, where $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for each $x \in \ell_\psi(\Delta^m_{n+1})$.*

Proof. Assume the conditions can be satisfied. Let $\kappa \in \ell_\infty$. So, there is $\varepsilon > 0$ with $|\kappa_r | \leq \varepsilon$, for each $r \in \mathbb{N}$, for $x \in (\ell_\psi(\Delta^m_{n+1})_\tau$. Since Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing verifying the δ_2-condition, then

$$\begin{aligned}\tau(V_\kappa x) &= \tau(\kappa x) = \tau\left((\kappa_r x_r)_{r=0}^{\infty}\right) = \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}(|\kappa_r||x_r|)|) \\ &\leq \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}(\varepsilon|x_r|)|) \leq d\varepsilon \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|x_r||) \leq D\tau(x),\end{aligned} \tag{17}$$

where $D = \max\{1, d\varepsilon\}$. This implies $V_\kappa \in \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$. Inversely, suppose that $V_\kappa \in \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$. Let us suppose $\kappa \notin \ell_\infty$, hence, for all $j \in \mathbb{N}$, there is $i_j \in \mathbb{N}$ so as to $\kappa_{i_j} > j$. Since Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing, one has

$$\begin{aligned}\tau\left(V_\kappa e_{i_j}\right) &= \tau\left(\kappa e_{i_j}\right) = \tau\left(\left(\kappa_r \left(e_{i_j}\right)_r\right)_{r=0}^{\infty}\right) \\ &= \sum_{r=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left(|\kappa_r|\left|\left(e_{i_j}\right)_r\right|\right)\right|\right) = \psi\left(\left|\Delta^m_{n+1}\left|\kappa_{i_j}\right|\right|\right) \\ &> \psi(|\Delta^m_{n+1}|j||) = \psi(|\Delta^m_{n+1}|j||)\tau\left(e_{i_j}\right).\end{aligned} \tag{18}$$

This proves that $V_\kappa \notin \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$. Therefore, $\kappa \in \ell_\infty$.

Theorem 13. *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi normed (sss), with $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$. Then, $|\kappa_r| = 1$, for every $r \in \mathbb{N}$, if and only if, V_κ is an isometry.*

Proof. Presume $|\kappa_r| = 1$, for each $r \in \mathbb{N}$, we have

$$\begin{aligned}\tau(V_\kappa x) &= \tau(\kappa x) = \tau\left((\kappa_r x_r)_{r=0}^{\infty}\right) \\ &= \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}(|\kappa_r||x_r|)|) \\ &= \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|x_r||) = \tau(x),\end{aligned} \tag{19}$$

for each $x \in (\ell_\psi(\Delta^m_{n+1}))_\tau$. Therefore, V_κ is an isometry. Inversely, suppose that $|\kappa_i| < 1$, for some $i = i_0$, given that Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing, we get

$$\begin{aligned}\tau\left(V_\kappa e_{i_0}\right) &= \tau\left(\kappa e_{i_0}\right) = \tau\left(\left(\kappa_r\left(e_{i_0}\right)_r\right)_{r=0}^{\infty}\right) \\ &= \sum_{r=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left(|\kappa_r|\left|\left(e_{i_0}\right)_r\right|\right)\right|\right) \\ &< \sum_{r=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left|\left(e_{i_0}\right)_r\right|\right|\right) = \tau\left(e_{i_0}\right).\end{aligned} \tag{20}$$

While $|\kappa_{i_0}| > 1$, we can show that $\tau(V_\kappa e_{i_0}) > \tau(e_{i_0})$. As a result, in both cases, we obtain a contradiction. Therefore, $|\kappa_r| = 1$, for all $r \in \mathbb{N}$.

5. Approximable Multiplication Operator on $\ell_\psi(\Delta^m_{n+1})$

In this section, we investigate the sufficient conditions on the Orlicz backward generalized difference sequence space equipped with prequasi norm τ so that the multiplication operator acting on $\ell_\psi(\Delta^m_{n+1})$ is an approximable and compact.

By $\mathrm{card}(A)$, we denote the cardinality of the set A.

Theorem 14. *If $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is a prequasi normed (sss), where $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$, then $V_\kappa \in \Upsilon((\ell_\psi(\Delta^m_{n+1}))_\tau)$ if and only if $(\kappa_r)_{r=0}^{\infty} \in c_0$.*

Proof. Let $V_\kappa \in \Upsilon((\ell_\psi(\Delta^m_{n+1}))_\tau)$. So, $V_\kappa \in B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$, to show that $(\kappa_r)_{r=0}^{\infty} \in c_0$. Assume $(\kappa_r)_{r=0}^{\infty} \notin c_0$, therefore there is $\delta > 0$ so that $A_\delta = \{r \in \mathbb{N} : |\kappa_r| \geq \delta\}$ has $\mathrm{card}(A_\delta) = \infty$. Suppose $a_i \in A_\delta$, for each $i \in \mathbb{N}$, then $\{e_{a_i} : a_i \in A_\delta\}$ is an infinite bounded set in $(\ell_\psi(\Delta^m_{n+1}))_\tau$. Suppose

$$\begin{aligned}\tau\left(V_\kappa e_{a_i} - V_\kappa e_{a_j}\right) &= \tau\left(\kappa e_{a_i} - \kappa e_{a_j}\right) = \tau\left(\left(\kappa_r\left(\left(e_{a_i}\right)_r - \left(e_{a_j}\right)_r\right)\right)_{r=0}^{\infty}\right)\\ &= \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\kappa_r\left(\left(e_{a_i}\right)_r - \left(e_{a_j}\right)_r\right)\right|\right|\right)\\ &\geq \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\delta\left(\left(e_{a_i}\right)_r - \left(e_{a_j}\right)_r\right)\right|\right|\right)\\ &= \tau\left(\delta e_{a_i} - \delta e_{a_j}\right),\end{aligned} \tag{21}$$

for every $a_i, a_j \in A_\delta$. This proves $\{e_{a_i} : a_i \in B_\delta\} \in \ell_\infty$ which cannot have a convergent subsequence under V_κ. This gives that $V_\kappa \notin B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$. Then, $V_\kappa \notin \Upsilon((\ell_\psi(\Delta^m_{n+1}))_\tau)$, and this gives a contradiction. So, $\lim_{i\to\infty}\kappa_i = 0$. Contrarily, assume $\lim_{i\to\infty}\kappa_i = 0$, then for all $\delta > 0$, the set $A_\delta = \{i \in \mathbb{N} : |\kappa_i| \geq \delta\}$ has card $(A_\delta) < \infty$. So, for all $\delta > 0$, the space

$$\left(\left(\ell_\psi(\Delta^m_{n+1})\right)_\tau\right)_{A_\delta} = \left\{x = (x_i) \in \mathbb{C}^{A_\delta}\right\} \tag{22}$$

is finite dimensional. Then, $V_\kappa \mid ((\ell_\psi(\Delta^m_{n+1}))_\tau)_{A_\delta}$ is a finite rank operator. For all $i \in \mathbb{N}$, illustrate $\kappa_i \in \mathbb{C}^{\mathbb{N}}$ by

$$(\kappa_i)_j = \begin{cases}\kappa_j, & j \in A_{\frac{1}{i}}\\ 0, & \text{otherwise.}\end{cases} \tag{23}$$

Evidently, V_{κ_i} has rank $(V_{\kappa_i}) < \infty$ as dim $((\ell_\psi(\Delta^m_{n+1}))_\tau)_{A_{1/i}} < \infty$, for all $i \in \mathbb{N}$. Hence, since Δ^m_{n+1} is an absolute nondecreasing and ψ is convex and nondecreasing, we obtain

$$\begin{aligned}\tau\left(\left(V_\kappa - V_{\kappa_i}\right)x\right) &= \tau\left(\left(\left(\kappa_j - (\kappa_i)_j\right)x_j\right)_{j=0}^{\infty}\right)\\ &= \sum_{j=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left(\left|\left(\kappa_j - (\kappa_i)_j\right)x_j\right|\right)\right|\right)\\ &= \sum_{j=0, j\in A_{\frac{1}{i}}}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left(\left|\left(\kappa_j - (\kappa_i)_j\right)x_j\right|\right)\right|\right)\\ &\quad + \sum_{j=0, j\notin A_{\frac{1}{i}}}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left(\left|\left(\kappa_j - (\kappa_i)_j\right)x_j\right|\right)\right|\right)\\ &= \sum_{j=0, j\notin A_{\frac{1}{i}}}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\kappa_j x_j\right|\right|\right) \leq \frac{1}{i}\sum_{j=0, j\notin A_{\frac{1}{i}}}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|x_j\right|\right|\right)\\ &< \frac{1}{i}\sum_{j=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|x_j\right|\right|\right) = \frac{1}{i}\tau(x).\end{aligned} \tag{24}$$

This gives that $\|V_\kappa - V_{\kappa_i}\| \leq 1/i$, and that V_κ is a limit of finite rank operators. So, V_κ is an approximable operator.

Theorem 15. *Pick up $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi normed (sss), where $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for every $x \in \ell_\psi(\Delta^m_{n+1})$. Therefore, $V_\kappa \in B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$, if and only if, $(\kappa_i)_{i=0}^{\infty} \in c_0$.*

Proof. Clearly, since every approximable operator is compact.

Corollary 16. *If $\kappa \in \mathbb{C}^{\mathbb{N}}$, ψ is an Orlicz function satisfying the δ_2-condition, and Δ^m_{n+1} is an absolute nondecreasing, then $\mathfrak{B}_c((\ell_\psi(\Delta^m_{n+1}))_\tau) \subsetneq \mathfrak{B}((\ell_\psi(\Delta^m_{n+1}))_\tau)$, where $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$.*

Proof. In view of I that is a multiplication operator on $(\ell_\psi(\Delta^m_{n+1}))_\tau$ generated by $\kappa = (1, 1,)$. So, $I \notin B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$ and $I \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$.

6. Fredholm Multiplication Operator on $\ell_\psi(\Delta^m_{n+1})$

In this section, we introduce the sufficient conditions on the sequence space $\ell_\psi(\Delta^m_{n+1})$ equipped with prequasi norm τ so that the multiplication operator acting on it has closed range, invertible, and Fredholm.

Theorem 17. *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$, $(\ell_\psi(\Delta^m_{n+1}))_\tau$, be prequasi Banach (sss), where $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$, and $V_\kappa \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$. Then, κ be bounded away from zero on $(\ker(\kappa))^c$, if and only if, $R(V_\kappa)$ is closed.*

Proof. Suppose the sufficient condition be satisfied, so, there is $\varepsilon > 0$ with $|\kappa_i| \geq \varepsilon$, for every $i \in (\ker(\kappa))^c$, to prove that $R(V_\kappa)$ is closed. Let d be a limit point of $R(V_\kappa)$. Hence, there is $V_\kappa x_i$ in $(\ell_\psi(\Delta^m_{n+1}))_\tau$, for each $i \in \mathbb{N}$ so that $\lim_{i\to\infty}V_\kappa x_i = d$. Clearly, $(V_\kappa x_i)$ is a Cauchy sequence. As Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing, we have

$$\begin{aligned}\tau\left(V_\kappa x_i - V_\kappa x_j\right) &= \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\kappa_r(x_i)_r - \kappa_r\left(x_j\right)_r\right|\right|\right)\\ &= \sum_{r=0, r\in(\ker(\kappa))^c}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\kappa_r(x_i)_r - \kappa_r\left(x_j\right)_r\right|\right|\right)\\ &\quad + \sum_{r=0, r\notin(\ker(\kappa))^c}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\kappa_r(x_i)_r - \kappa_r\left(x_j\right)_r\right|\right|\right)\\ &\geq \sum_{r=0, r\in(\ker(\kappa))^c}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left(|\kappa_r|\left|(x_i)_r - \left(x_j\right)_r\right|\right)\right|\right)\\ &= \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left(|\kappa_r|\left|(y_i)_r - \left(y_j\right)_r\right|\right)\right|\right)\\ &> \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\varepsilon(y_i)_r - \left(y_j\right)_r\right|\right|\right) = \tau(\varepsilon(y_n - y_m)),\end{aligned} \tag{25}$$

where

$$(y_i)_r = \begin{cases} (x_i)_r, & r \in (\ker(\kappa))^c \\ 0, & r \notin (\ker(\kappa))^c \end{cases}. \tag{26}$$

This gives that (y_i) is a Cauchy sequence in $(\ell_\psi(\Delta^m_{n+1}))_\tau$. As $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is complete, there is $x \in (\ell_\psi(\Delta^m_{n+1}))_\tau$ so that $\lim_{i\to\infty} y_i = x$. As V_κ is continuous, then $\lim_{i\to\infty} V_\kappa y_i = V_\kappa x$. Although $\lim_{i\to\infty} V_\kappa x_i = \lim_{i\to\infty} V_\kappa y_i = d$, therefore, $V_\kappa x = d$. So, $d \in R(V_\kappa)$. This implies that $R(V_\kappa)$ is closed. Inversely, assume $R(V_\kappa)$ be closed, hence, V_κ be bounded away from zero on $((\ell_\psi(\Delta^m_{n+1}))_\tau)_{(\ker(\kappa))^c}$. So, there is $\varepsilon > 0$ so that $\tau(V_\kappa x) \geq \varepsilon\tau(x)$, for every $x \in ((\ell_\psi(\Delta^m_{n+1}))_\tau)_{(\ker(\kappa))^c}$.

Let $B = \{r \in (\ker(\kappa))^c : |\kappa_r| < \varepsilon\}$ as Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing verifying the δ_2-condition, if $B \neq \varphi$; then for $i_0 \in B$, one has

$$\begin{aligned} \tau\left(V_\kappa e_{i_0}\right) &= \tau\left(\left(\kappa_r\left(e_{i_0}\right)_r\right)_{r=0}^{\infty}\right) = \sum_{r=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\right|\kappa_r\left(e_{n_0}\right)_r\right|\right|\right) \\ &< \sum_{r=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\right|\varepsilon\left(e_{n_0}\right)_r\right|\right|\right) \leq d\varepsilon\tau\left(e_{n_0}\right), \end{aligned} \tag{27}$$

for some $d \geq 1$. This implies a contradiction. Therefore, $B = \varphi$ so that $|\kappa_r| \geq \varepsilon$, for each $r \in (\ker(\kappa))^c$. This completes the proof of the theorem.

Theorem 18. *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi Banach (sss), with $\tau(w) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|w_r||)$, for every $w \in \ell_\psi(\Delta^m_{n+1})$. There are $b > 0$ and $B > 0$ so that $b < \kappa_r < B$, for every $r \in \mathbb{N}$, if and only if, $V_\kappa \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$ be invertible.*

Proof. Assume the conditions be established, define $\gamma \in \mathbb{C}^{\mathbb{N}}$ by $\gamma_r = 1/\kappa_r$, from Theorem 12, we obtain $V_\kappa, V_\gamma \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$ and $V_\kappa . V_\gamma = V_\gamma . V_\kappa = I$. Therefore, V_γ is the inverse of V_κ. Conversely, let V_κ be invertible. Hence, $R(V_\kappa) = ((\ell_\psi(\Delta^m_{n+1}))_\tau)_{\mathbb{N}}$. This gives $R(V_\kappa)$ which is closed. From Theorem 17, there is $b > 0$ so that $|\kappa_r| \geq b$, for each $r \in (\ker(\kappa))^c$. Now, $\ker(\kappa) = \varphi$, else $\kappa_{r_0} = 0$, for several $r_0 \in \mathbb{N}$, we have $e_{r_0} \in \ker(V_\kappa)$. This implies a contradiction, as $\ker(V_\kappa)$ is trivial. Therefore, $|\kappa_r| \geq a$, for every $r \in \mathbb{N}$. Because V_κ is bounded, so from Theorem 12, there is $B > 0$ so that $|\kappa_r| \leq B$, for each $r \in \mathbb{N}$. Hence, we have proved that $b \leq |\kappa_r| \leq B$, for every $r \in \mathbb{N}$.

Theorem 19. *Pick up $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi Banach (sss), where $\tau(w) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|w_r||)$, for every $w \in \ell_\psi(\Delta^m_{n+1})$. Then, $V_\kappa \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$ be the Fredholm operator, if and only if, (i) card $(\ker(\kappa)) < \infty$ and (ii) $|\kappa_r| \geq \varepsilon$, for each $r \in (\ker(\kappa))^c$.*

Proof. Assume V_κ be Fredholm, Let card $(\ker(\kappa)) = \infty$. Therefore, $e_n \in \ker(V_\kappa)$, for every $n \in \ker(\kappa)$. As e_n's is linearly independent, this implies card $(\ker(V_\kappa) = \infty$. This gives a contradiction. Hence, card $(\ker(\kappa)) < \infty$. From Theorem 17, condition (ii) is verified. Next, if the necessary conditions are satisfied, to prove that V_κ is Fredholm, from Theorem 17, condition (ii) implies that $R(V_\kappa)$ is closed. Condition (i) gives that $\dim(\ker(V_\kappa)) < \infty$ and $\dim((R(V_\kappa))^c) < \infty$. So, V_κ is Fredholm.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] R. K. Singh and A. Kumar, "Multiplication operators and composition operators with closed ranges," *Bulletin of the Australian Mathematical Society*, vol. 16, no. 2, pp. 247–252, 1977.

[2] M. B. Abrahmse, "Multiplication operators," in *Lecture Notes in Mathematics*, vol. 693, pp. 17–36, Springer Verlag, 1978.

[3] K. Raj, S. K. Sharma, and A. Kumar, "Multiplication operator on Musielak-Orlicz spaces of Bochner type," *Nonlinear Analysis*, vol. 73, pp. 2541–2557, 2010.

[4] A. K. Sharma, K. Raj, and S. K. Sharma, "Products of multiplication composition and differentiation operators from H^∞ to weighted Bloch spaces," *Indian Journal of Mathematics*, vol. 54, pp. 159–179, 2012.

[5] R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, Elsevier, Amsterdam, Netherlands, 1993.

[6] H. Takagi and K. Yokouchi, "Multiplication and composition operators between two L^p-spaces," *Contemporary Mathematics*, vol. 232, pp. 321–338, 1999.

[7] B. C. Tripathy, A. Esi, and B. K. Tripathy, "On a new type of generalized difference Cesáro sequence spaces," *Soochow Journal of Mathematics*, vol. 31, no. 3, pp. 333–340, 2005.

[8] M. Et and R. Çolak, "On some generalized difference spaces," *Soochow Journal of Mathematics*, vol. 21, pp. 377–386, 1995.

[9] B. C. Tripathy and A. Esi, "A new type of difference sequence spaces," *International Journal of Science & Technology*, vol. 1, no. 1, pp. 11–14, 2006.

[10] H. Kizmaz, "On certain sequence spaces," *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

[11] M. A. Krasnoselskii and Y. B. Rutickii, *Convex Functions and Orlicz Spaces*, Groningen, Netherlands, 1961, translated from Russian.

[12] W. Orlicz and Ü. Raume, "L^m," in *Bull. Int. Acad Polon. Sci. A*, pp. 93–107, PWN, Warszawa, 1936, Collected Papers, 1988, 345–359.

[13] J. Lindenstrauss and L. Tzafriri, "On Orlicz sequence spaces," *Israel Journal of Mathematics*, vol. 10, no. 3, pp. 379–390, 1971.

[14] E. Mikail, M. Mursaleen, and M. Isık, "On a class of fuzzy sets defined by Orlicz functions," *Filomat*, vol. 27, no. 5, pp. 789–796, 2013.

[15] M. Mursaleen, K. Raj, and S. K. Sharma, "Some spaces of difference sequences and lacunary statistical convergence in n -normed space defined by sequence of Orlicz functions," *Miskolc Mathematical Notes*, vol. 16, no. 1, pp. 283–304, 2015.

[16] M. Mursaleen, S. K. Sharma, and A. Kiliçman, "Sequence spaces defined by Musielak-Orlicz function over -normed spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 364743, 10 pages, 2013.

[17] M. Mursaleen, S. K. Sharma, S. A. Mohiuddine, and A. Kiliçman, "New difference sequence spaces defined by Musielak-Orlicz function," *Abstract and Applied Analysis*, vol. 2014, Article ID 691632, 9 pages, 2014.

[18] A. Alotaibi, M. Mursaleen, S. Sharma K., and S. A. Mohiuddine, "Sequence spaces of fuzzy numbers defined by a Musielak-Orlicz function," *Filomat*, vol. 29, no. 7, pp. 1461–1468, 2015.

[19] A. Esi, B. C. Tripathy, and B. Sarm, "On some new type generalized sequence spaces," *Mathematica Slovaca*, vol. 57, pp. 475–482, 2007.

[20] M. Yesilkayagil and F. Basar, "Domain of the Nörlund matrix in some of Maddox's spaces," *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, vol. 87, no. 3, pp. 363–371, 2017.

[21] A. Kiliçman and K. Raj, "Matrix transformations of Norlund–Orlicz difference sequence spaces of nonabsolute type and their Toeplitz duals," *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 110, 2020.

[22] E. E. Kara and M. Basarir, "On compact operators and some Euler $B(m)$-difference sequence spaces," *Journal of Mathematical Analysis and Applications*, vol. 379, no. 2, pp. 499–511, 2011.

[23] M. Başarır and E. E. Kara, "On some difference sequence spaces of weighted means and compact operators," *Annals of Functional Analysis*, vol. 2, no. 2, pp. 114–129, 2011.

[24] M. Basarir and E. E. Kara, "On compact operators on the Riesz $B(m)$-difference sequence space," *Iran. J. Sci. Technol. Trans. A Sci*, vol. 35, no. A4, pp. 279–285, 2011.

[25] M. Basarir and E. E. Kara, "On the B-difference sequence space derived by generalized weighted mean and compact operators," *Journal of Mathematical Analysis and Applications*, vol. 391, no. 1, pp. 67–81, 2012.

[26] M. Basarir and E. E. Kara, "On the mth order difference sequence space of generalized weighted mean and compact operators," *Acta Mathematica Scientia*, vol. 33, no. 3, pp. 797–813, 2013.

[27] M. Mursaleen and A. K. Noman, "Compactness by the Hausdorff measure of noncompactness," *Nonlinear Analysis*, vol. 73, no. 8, pp. 2541–2557, 2010.

[28] M. Mursaleen and A. K. Noman, "Compactness of matrix operators on some new difference sequence spaces," *Linear Algebra and its Applications*, vol. 436, no. 1, pp. 41–52, 2012.

[29] B. S. Komal, S. Pandoh, and K. Raj, "Multiplication operators on Cesàro sequence spaces," *Demonstratio Mathematica*, vol. 49, no. 4, pp. 430–436, 2016.

[30] M. Ilkhan, S. Demiriz, and E. E. Kara, "Multiplication operators on Cesàro second order function spaces," *Positivity*, vol. 24, no. 3, pp. 605–614, 2020.

[31] A. A. Bakery, A. R. Abou Elmatty, and O. M. K. S. K. Mohamed, "Multiplication operators on weighted Nakano (sss)," *Journal of Mathematics*, vol. 2020, Article ID 1608631, 7 pages, 2020.

[32] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980, MR582655 (81j:47001).

[33] T. Mrowka, "A Brief Introduction to Linear Analysis: Fredholm Operators," in *Geometry of Manifolds. Fall 2004*, (Massachusetts Institute of Technology: MIT OpenCouseWare), 2004.

[34] N. F. Mohamed and A. A. Bakery, "Mappings of type Orlicz and generalized Cesáro sequence space," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[35] A. A. Bakery and M. M. Mohammed, "Small pre-quasi Banach operator ideals of type Orlicz-Cesáro mean sequence spaces," *Journal of Function Spaces*, vol. 2019, Article ID 7265010, 9 pages, 2019.

[36] B. Altay and F. Basar, "Generalization of the sequence space $\ell(p)$ derived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 147–185, 2007.

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Research Article

On Mixed Quermassintegral for Log-Concave Functions

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In this paper, the functional Quermassintegral of log-concave functions in $\mathbb{R}^n$ is discussed. We obtain the integral expression of the ith functional mixed Quermassintegral, which is similar to the integral expression of the ith mixed Quermassintegral of convex bodies.

1. Introduction

Let $\mathscr{K}^n$ be the set of convex bodies (compact convex subsets with nonempty interiors) in $\mathbb{R}^n$, the fundamental Brunn-Minkowski inequality for convex bodies states that for $K, L \in \mathscr{K}^n$, the volume of the bodies and of their Minkowski sum $K+L=\{x+y : x \in K, y \in L\}$ is given by

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1}$$

with equality if and only if K and L are homothetic; namely, they agree up to a translation and a dilation. Another geometric quantity related to the convex bodies K and L is the mixed volume. The most important result concerning the mixed volume is Minkwoski's first inequality:

$$V_1(K, L) := \frac{1}{n} \lim_{t \to 0^+} \frac{V(K+tL) - V(K)}{t} \geq V(K)^{(n-1)/n} V(L)^{1/n}, \tag{2}$$

for $K, L \in \mathscr{K}^n$. In particular, when choosing L to be a unit ball, up to a factor, $V_1(K, L)$ is exactly the perimeter of K, and inequality (2) turns out to be the isoperimetric inequality in the class of convex bodies. The mixed volume $V_1(K, L)$ admits a simple integral representation (see [1, 2]):

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L dS_K, \tag{3}$$

where h_L is the support function of L and S_K is the area measure of K.

The Quermassintegrals $W_i(K)(i=0,1,\cdots,n)$ of K, which are defined by letting $W_0(K)=V_n(K)$, the volume of K; $W_n(K)=\omega_n$, the volume of the unit ball B_2^n in $\mathbb{R}^n$ and for general $i=1,2,\cdots,n-1$,

$$W_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{\mathscr{G}_{i,n}} \mathrm{vol}_i\left(K|_{\xi_i}\right) d\mu(\xi_i), \tag{4}$$

where $\mathscr{G}_{i,n}$ is the Grassmannian manifold of i-dimensional linear subspaces of $\mathbb{R}^n$, $d\mu(\xi_i)$ is the normalized Haar measure on $\mathscr{G}_{i,n}$, $K|_{\xi_i}$ denotes the orthogonal projection of K onto the i-dimensional subspaces ξ_i, and vol_i is the i-dimensional volume on space ξ_i.

In the 1930s, Aleksandrov and Fenchel and Jessen (see [3, 4]) proved that for a convex body K in $\mathbb{R}^n$, there exists a regular Borel measure $S_{n-1-i}(K)$ $(i=0,1,\cdots,n-1)$ on S^{n-1}, the unit sphere in $\mathbb{R}^n$, for $K, L \in \mathscr{K}^n$, the following representation holds

$$\begin{aligned} W_i(K, L) &= \frac{1}{n-i} \lim_{t \to 0^+} \frac{W_i(K+tL) - W_i(K)}{\varepsilon} \\ &= \frac{1}{n} \int_{S^{n-1}} h_L(u) dS_{n-1-i}(K, u). \end{aligned} \tag{5}$$

The quantity $W_i(K, L)$ is called the ith mixed Quermassintegral of K and L.

In the 1960s, the Minkowski addition was extended to the $L^p (p \geq 1)$ Minkowski sum $h^p_{K+_p t\cdot L} = h^p_K + t h^p_L$. The extension of the mixed Quermassintegral to the L^p mixed Quermassintegral due to Lutwak [1], the L^p mixed Quermassintegral inequalities, and the L^p Minkowski problem are established. (See [2, 5–13] for more about the L^p Minkowski theory.) The L^p mixed Quermassintegrals are defined by

$$W_{p,i}(K, L) := \frac{p}{n-i} \lim_{t \to 0^+} \frac{W_i(K +_p t \cdot L) - W_i(L)}{t}, \qquad (6)$$

for $i = 0, 1, \cdots, n-1$. In particular, for $p = 1$ in (6), it is $W_i(K, L)$, and $W_{p,0}(K, L)$ is denoted by $V_p(K, L)$, which is called the L_p mixed volume of K and L. Similarly, the L^p mixed Quermassintegral has the following integral representation (see [1]):

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h^p_L(u) dS_{p,i}(K, u). \qquad (7)$$

The measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$ and has Radon-Nikodym derivative $dS_{p,i}(K, \cdot)/dS_i(K, \cdot) = h_K(\cdot)^{1-p}$. In particular, $p = 1$ in (7) yields the representation (5).

Most recently, the interest in the log-concave functions has been considerably increasing, motivated by the analogy properties between the log-concave functions and the volume convex bodies in $\mathcal{K}^n$. The classical Prékopa-Leindler inequality (see [14–18]) firstly shows the connections of the volume of convex bodies and log-concave functions. The Blaschke-Santaló inequality for even log-concave functions is established in [19, 20] by Ball (for the general case, see [21–24]). The mean width for log-concave function is introduced by Klartag and Milman and Rotem [25–27]. The affine isoperimetric inequality for log-concave functions is proved by Avidan et al. [28]. The John ellipsoid for log-concave functions has been establish by Alonso-Gutiérrez et al. [29]; the LYZ ellipsoid for log-concave functions is established by Fang and Zhou [30]. (See [31–37] for more about the pertinent results.)

Let $f = e^{-u}$, $g = e^{-v}$ be log-concave functions, $\alpha, \beta > 0$, the "sum" and "scalar multiplication" of log-concave functions are defined as

$$\alpha \cdot f \oplus \beta \cdot g := e^{-w}, \quad w^* = \alpha u^* + \beta v^*, \qquad (8)$$

where w^* denotes as usual the Fenchel conjugate of the convex function ω. The total mass integral $J(f)$ of f is defined by $J(f) = \int_{\mathbb{R}^n} f(x)dx$. In paper [38] of Colesanti and Fragalà, the quantity $\delta J(f, g)$, which is called as the first variation of J at f along g, $\delta J(f, g) = \lim_{t \to 0^+} (J(f \oplus t \cdot g) - J(f))/t$, is discussed. It has been shown that $\delta J(f, g)$ is finite and has the following integral expression:

$$\delta J(f, g) = \int_{\mathbb{R}^n} v^* d\mu(f), \qquad (9)$$

where $\mu(f)$ is the measure of f on $\mathbb{R}^n$.

Inspired by the paper [38] of Colesanti and Fragalà, in this paper, we define the ith functional Quermassintegrals $W_i(f)$ as the i-dimensional average total mass of f:

$$W_i(f) := \frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n-i,n}} J_{n-i}(f) d\mu(\xi_{n-i}), \quad i = 0, 1, \cdots, n-1, \qquad (10)$$

where $J_i(f)$ denotes the i-dimensional total mass of f defined in Section 4, $\mathcal{G}_{i,n}$ is the Grassmannian manifold of $\mathbb{R}^n$, and $d\mu(\xi_{n-i})$ is the normalized measure on $\mathcal{G}_{i,n}$. Moreover, we define the first variation of W_i at f along g, which is

$$W_i(f, g) = \lim_{t \to 0^+} \frac{W_i(f \oplus t \cdot g) - W_i(f)}{t}. \qquad (11)$$

It is a natural extension of the Quermassintegral of convex bodies in $\mathbb{R}^n$; we call it the ith functional mixed Quermassintegral. In fact, if one takes $f = \chi_K$, and dom $(f) = K \in \mathbb{R}^n$, then $W_i(f)$ turns out to be $W_i(K)$, and $W_i(\chi_K, \chi_L)$ equals to $W_i(K, L)$. The main result in this paper is to show that the ith functional mixed Quermassintegral has the following integral expressions.

Theorem 1. *Let $f, g \in \mathcal{A}'$, be integrable functions, $\mu_i(f)$ be the i-dimensional measure of f, and $W_i(f, g)$ be the ith functional mixed Quermassintegral of f and g. Then,*

$$W_i(f, g) = \frac{1}{n-i} \int_{\mathbb{R}^n} h_{g|_{\xi_{n-i}}} d\mu_{n-i}(f), \quad i = 0, 1, \cdots, n-1, \qquad (12)$$

where $h_{g|_{\epsilon_{n-i}}}$ is the support function of $g|_{\epsilon_{n-i}}$.

The paper is organized as follows: In Section 2, we introduce some notations about the log-concave functions. In Section 3, the projection of a log-concave function onto subspace is discussed. In Section 4, we focus on how we can represent the ith functional mixed Quermassintegral $W_i(f, g)$ similar as $W_i(K, L)$. Owing to the Blaschke-Petkantschin formula and the similar definition of the support function of f, we obtain the integral representation of the ith functional mixed Quermassintegral $W_i(f, g)$.

2. Preliminaries

Let $u : \Omega \to (-\infty, +\infty]$ be a convex function; that is, $u((1-t)x + ty) \leq (1-t)u(x) + tu(y)$ for $t \in (0, 1)$, where $\Omega = \{x \in \mathbb{R}^n : u(x) \in \mathbb{R}\}$ is the domain of u. By the convexity of u, Ω is a convex set in $\mathbb{R}^n$. We say that u is proper if $\Omega \neq \emptyset$, and u is of class $\mathcal{C}^2_+$ if it is twice differentiable on int (Ω), with a positive definite Hessian matrix. In the following, we define the subclass of u:

$$\mathscr{L} = \left\{ u : \Omega \to (-\infty,+\infty]:\ u \text{ is convex, low semicontinuous, } \lim_{\|x\|\to+\infty} u(x) = +\infty \right\}. \tag{13}$$

Recall that the Fenchel conjugate of u is the convex function defined by

$$u^*(y) = \sup_{x\in\mathbb{R}^n} \{\langle x, y\rangle - u(x)\}. \tag{14}$$

It is obvious that $u(x) + u^*(y) \ge \langle x, y\rangle$ for all $x, y \in \Omega$, and there is an equality if and only if $x \in \Omega$ and y is in the subdifferential of u at x, which means

$$u^*(\nabla u(x)) + u(x) = \langle x,\nabla u(x)\rangle. \tag{15}$$

Moreover, if u is a lower semicontinuous convex function, then also u^* is a lower semicontinuous convex function, and $u^{**} = u$.

The infimal convolution of u and v from Ω to $(-\infty, +\infty]$ is defined by

$$u\square v(x) = \inf_{y\in\Omega} \{u(x-y) + v(y)\}. \tag{16}$$

The right scalar multiplication by a nonnegative real number α is

$$(u\alpha)(x) := \begin{cases} \alpha u\left(\dfrac{x}{\alpha}\right), & \text{if } \alpha > 0, \\ I_{\{0\}}, & \text{if } \alpha = 0. \end{cases} \tag{17}$$

The following proposition below gathers some elementary properties of the Fenchel conjugate and the infimal convolution of u and v, which can be found in [38, 39].

Proposition 2. *Let $u, v : \Omega \to (-\infty,+\infty]$ be convex functions. Then,*

$$(u\square v)^* = u^* + v^* \tag{18}$$

(1) $(u\alpha)^* = \alpha u^*, \alpha > 0$

(2) $\text{dom}\ (u\square v) = \text{dom}\ (u) + \text{dom}\ (v)$

(3) it holds $u^*(0) = -\inf\ (u)$; in particular, if u is proper, then $u^*(y) > -\infty$; $\inf\ (u) > -\infty$ implies u^* is proper

The following proposition about the Fenchel and Legendre conjugates is obtained in [39].

Proposition 3 (see [39]). *Let $u : \Omega \to (-\infty,+\infty]$ be a closed convex function, and set $\mathscr{C} := \text{int}\ (\Omega)$, $\mathscr{C}^* := \text{int}\ (\text{dom}\ (u^*))$. Then, $(\mathscr{C}, u)$ is a convex function of Legendre type if and only if $\mathscr{C}^*, u^*$ is. In this case, $(\mathscr{C}^*, u^*)$ is the Legendre conjugate of $(\mathscr{C}, u)$ (and conversely). Moreover, $\nabla u := \mathscr{C} \to \mathscr{C}^*$ is a continuous bijection, and the inverse map of ∇u is precisely ∇u^*.*

A function $f : \mathbb{R}^n \to (-\infty,+\infty]$ is called log-concave if for all $x, y \in \mathbb{R}^n$ and $0 < t < 1$, we have $f((1-t)x + ty) \ge f^{1-t}(x) f^t(y)$. If f is a strictly positive log-concave function on $\mathbb{R}^n$, then there exists a convex function $u : \Omega \to (-\infty,+\infty]$ such that $f = e^{-u}$. The log-concave function is closely related to the convex geometry of $\mathbb{R}^n$. An example of a log-concave function is the characteristic function χ_K of a convex body K in $\mathbb{R}^n$, which is defined by

$$\chi_K(x) = e^{-I_K(x)} = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \notin K, \end{cases} \tag{19}$$

where I_K is a lower semicontinuous convex function, and the indicator function of K is

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ \infty, & \text{if } x \notin K. \end{cases} \tag{20}$$

In the later sections, we also use f to denote f being extended to $\mathbb{R}^n$:

$$\bar{f} = \begin{cases} f, & x \in \Omega, \\ 0, & x \in \dfrac{R^n}{\Omega}. \end{cases} \tag{21}$$

Let $\mathscr{A} = \{f : \mathbb{R}^n \to (0,+\infty]: f = e^{-u}, u \in \mathscr{L}\}$ be the subclass of f in $\mathbb{R}^n$. The addition and multiplication by nonnegative scalars in $\mathscr{A}$ are defined by the following (see [38]).

Definition 4. Let $f = e^{-u}$, $g = e^{-v} \in \mathscr{A}$, and $\alpha, \beta \ge 0$. The sum and multiplication of f and g *are* defined as

$$\alpha \cdot f \oplus \beta \cdot g = e^{-[(u\alpha)\square(v\beta)]}. \tag{22}$$

That means,

$$(\alpha \cdot f \oplus \beta \cdot g)(x) = \sup_{y\in\mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^{\alpha} g\left(\frac{y}{\beta}\right)^{\beta}. \tag{23}$$

In particular, when $\alpha = 0$ and $\beta > 0$, we have $(\alpha \cdot f \oplus \beta \cdot g)(x) = g(x/\beta)^{\beta}$; when $\alpha > 0$ and $\beta = 0$, then $(\alpha \cdot f \oplus \beta \cdot g)(x) = f(x/\alpha)^{\alpha}$; finally, when $\alpha = \beta = 0$, we have $(\alpha \cdot f \oplus \beta \cdot g) = I_{\{0\}}$.

The following lemma is obtained in [38].

Lemma 5 (see [38]). *Let $u \in \mathscr{L}$, then there exist constants a and b, with $a > 0$, such that, for $x \in \Omega$,*

$$u(x) \ge a\|x\| + b. \tag{24}$$

Moreover, u^* is proper and satisfies $u^*(y) > -\infty$, $\forall y \in \Omega$.

Lemma 5 grants that $\mathscr{L}$ is closed under the operations of infimal convolution and right scalar multiplication defined in (16) and (17) which are closed.

Proposition 6 (see [38]). *Let u and v belong both to the same class $\mathscr{L}$, and $\alpha, \beta \geq 0$. Then, $u\alpha\Box v\beta$ belongs to the same class as u and v.*

Let $f \in \mathscr{A}$, according to papers of [26, 40], the support function of $f = e^{-u}$ is defined as

$$h_f(x) = (-\log f(x))^* = u^*(x), \tag{25}$$

where u^* is the Legendre transform of u. The definition of h_f is a proper generalization of the support function h_K. In fact, one can easily check $h_{\chi_K} = h_K$. Obviously, the support function h_f share the most of the important properties of support functions h_K. Specifically, it is easy to check that the function $h : \mathscr{A} \to \mathscr{L}$ has the following properties [27]:

(1) h is a bijective map from $\mathscr{A} \to \mathscr{L}$

(2) h is order preserving: $f \leq g$ if and only if $h_f \leq h_g$

(3) h is additive: for every $f, g \in \mathscr{A}$, we have $h_{f\oplus g} = h_f + h_g$

The following proposition shows that h_f is $GL(n)$ covariant.

Proposition 7 (see [30]). *Let $f \in \mathscr{A}$, $A \in GL(n)$ and $x \in \mathbb{R}^n$. Then,*

$$h_{f\circ A}(x) = h_f\left(A^{-t}x\right). \tag{26}$$

Let $u, v \in \mathscr{L}$, denote by $u_t = u\Box vt(t > 0)$, and $f_t = e^{-u_t}$. The following lemmas describe the monotonicity and convergence of u_t and f_t, respectively.

Lemma 8 (see [38]). *Let $f = e^{-u}$, $g = g^{-v} \in \mathscr{A}$. For $t > 0$, set $u_t = u\Box(vt)$ and $f_t = e^{-u_t}$. Assume that $v(0) = 0$, then for every fixed $x \in \mathbb{R}^n$, $u_t(x)$ and $f_t(x)$ are, respectively, pointwise decreasing and increasing with respect to t; in particular, it holds*

$$u_1(x) \leq u_t(x) \leq u(x), f(x) \leq f_t(x) \leq f_1(x) \quad \forall x \in \mathbb{R}^n, \forall t \in [0, 1]. \tag{27}$$

Lemma 9 (see [38]). *Let u and v belong both to the same class $\mathscr{L}$ and, for any $t > 0$, set $u_t := u\Box(vt)$. Assume that $v(0) = 0$, then*

(1) $\forall x \in \Omega$, $\lim_{t\to 0^+} u_t(x) = u(x)$

(2) $\forall E \subset\subset \Omega$, $\lim_{t\to 0^+} \nabla u_t(x) = \nabla u$ uniformly on E

Lemma 10 (see [38]). *Let u and v belong both to the same class $\mathscr{L}$ and for any $t > 0$, let $u_t := u\Box(vt)$. Then, $\forall x \in \operatorname{int}(\Omega_t)$, and $\forall t > 0$,*

$$\frac{d}{dt}(u_t(x)) = -\psi(\nabla u_t(x)), \tag{28}$$

where $\psi := v^*$.

3. Projection of Functions onto Linear Subspace

Let $\mathscr{G}_{i,n}(0 \leq i \leq n)$ be the Grassmannian manifold of i-dimensional linear subspace of $\mathbb{R}^n$. The elements of $\mathscr{G}_{i,n}$ will usually be denoted by ξ_i, and $\xi_i^\perp$ stands for the orthogonal complement of ξ_i which is a $(n-i)$-dimensional subspace of $\mathbb{R}^n$. Let $\xi_i \in \mathscr{G}_{i,n}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The projection of f onto ξ_i is defined by (see [25, 41])

$$f|_{\xi_i}(x) := \max\left\{f(y): y \in x + {\xi_i}^\perp\right\}, \quad \forall x \in \Omega|_{\xi_i}, \tag{29}$$

where $\xi_i^\perp$ is the orthogonal complement of ξ_i in $\mathbb{R}^n$ and $\Omega|_{\xi_i}$ is the projection of Ω onto ξ_i. By the definition of the log-concave function $f = e^{-u}$, for every $x \in \Omega|_{\xi_i}$, one can rewrite (29) as

$$f|_{\xi_i}(x) = \exp\left\{\max\left\{-u(y): y \in x + \xi_i^\perp\right\}\right\} = e^{-u|_{\xi_i}}(x). \tag{30}$$

Regarding the "sum" and "multiplication" of f, we say that the projection keeps the structure on $\mathbb{R}^n$. In other words, we have the following proposition.

Proposition 11. *Let $f, g \in \mathscr{A}$, $\xi_i \in \mathscr{G}_{i,n}$, and $\alpha, \beta > 0$. Then,*

$$(\alpha\cdot f \oplus \beta\cdot g)|_{\xi_i} = \alpha\cdot f|_{\xi_i} \oplus \beta\cdot g|_{\xi_i}. \tag{31}$$

Proof. Let $f, g \in \mathscr{A}$, let $x_1, x_2, x \in \xi_i$ such that $x = \alpha x_1 + \beta x_2$, then we have

$$\begin{aligned}(\alpha\cdot f \oplus \beta\cdot g)|_{\xi_i}(x) &\geq (\alpha\cdot f \oplus \beta\cdot g)\left(\alpha x_1 + \beta x_2 + \xi_i^\perp\right)\\ &\geq f\left(x_1 + \xi_i^\perp\right)^\alpha g\left(x_2 + \xi_i^\perp\right)^\beta.\end{aligned} \tag{32}$$

Taking the supremum of the second right-hand inequality over all $\xi_i^\perp$, we obtain $(\alpha\cdot f \oplus \beta\cdot g)|_{\xi_i} \geq \alpha\cdot f|_{\xi_i} \oplus \beta\cdot g|_{\xi_i}$. On the other hand, for $x \in \xi_i$, $x_1, x_2 \in \xi_i$ such that $x_1 + x_2 = x$, then

$$\begin{aligned}\left(\alpha\cdot f\,|_{\xi_i} \oplus \beta\cdot g_{\xi_i}\right)(x) &= \sup_{x_1+x_2=x}\left\{\max\left\{f^\alpha\left(\frac{x_1}{\alpha} + \xi_i^\perp\right)\right\}\max\right.\\ &\quad\left.\cdot\left\{g^\beta\left(\frac{x_2}{\beta} + \xi_i^\perp\right)\right\}\right\}\\ &\geq \sup_{x_1+x_2=x}\left\{\max\left(f^\alpha\left(\frac{x_1}{\alpha} + \xi_i^\perp\right)g^\beta\left(\frac{x_2}{\beta} + \xi_i^\perp\right)\right)\right\}\\ &= \max\left\{\sup_{x_1+x_2=x}\left(f^\alpha\left(\frac{x_1}{\alpha} + \xi_i^\perp\right)g^\beta\left(\frac{x_2}{\beta} + \xi_i^\perp\right)\right)\right\}\\ &= (\alpha\cdot f \oplus \beta\cdot g)|_{\xi_i}(x).\end{aligned} \tag{33}$$

Since $f, g \geq 0$, the inequality $\max\{f \cdot g\} \leq \max\{f\} \cdot \max\{g\}$ holds. So, we complete the proof.

Proposition 12. *Let $\xi_i \in \mathscr{G}_{i,n}$, f and g are functions on $\mathbb{R}^n$, such that $f(x) \leq g(x)$ holds. Then,*

$$f|_{\xi_i} \leq g|_{\xi_i} \tag{34}$$

holds for any $x \in \xi_i$.

Proof. For $y \in x + \xi_i^{\perp}$, since $f(y) \leq g(y)$, then $f(y) \leq \max\{g(y): y \in x + \xi_i^{\perp}\}$. So, $\max\{f(y): y \in x + L_i^{\perp}\} \leq \max\{g(y) : y \in x + \xi_i^{\perp}\}$. By the definition of the projection, we complete the proof.

For the convergence of f, we have the following.

Proposition 13. *Let $\{f_i\}$ be functions such that $\lim_{n\to\infty} f_n = f_0$, $\xi_i \in \mathscr{G}_{i,n}$, then $\lim_{n\to\infty}(f_{n\xi_i}) = f_0|_{\xi_i}$.*

Proof. Since $\lim_{n\to\infty} f_n = f_0$, it means that $\forall \varepsilon > 0$, there exist N_0, $\forall n > N_0$, such that $f_0 - \varepsilon \leq f_n \leq f_0 + \varepsilon$. By the monotonicity of the projection, we have $f_0|_{\xi_i} - \varepsilon \leq f_n|_{\xi_i} \leq f_0|_{\xi_i} + \varepsilon$. Hence, each $\{f_n|_{\xi_i}\}$ has a convergent subsequence; we denote it also by $\{f_n|_{\xi_i}\}$, converging to some $f'_0|_{\xi_i}$. Then, for $x \in \xi_i$, we have

$$f_0|_{\xi_i}(x) - \varepsilon \leq f'_0|_{\xi_i}(x) = \lim_{n\to\infty}\left(f_n|_{\xi_i}\right)(x) \leq f_0|_{\xi_i}(x) + \varepsilon. \tag{35}$$

By the arbitrary of ε, we have $f'_0|_{\xi_i} = f_0|_{\xi_i}$, so we complete the proof.

Combining with Proposition 13 and Lemma 9, it is easy to obtain the following proposition.

Proposition 14. *Let u and v belong both to the same class $\mathscr{L}$ and $\Omega \in \mathbb{R}^n$ be the domain of u, for any $t > 0$, set $u_t = u\Box(vt)$. Assume that $v(0) = 0$ and $\xi_i \in \mathscr{G}_{i,n}$, then*

(1) $\forall x \in \Omega|_{\xi_i}$, $\lim_{t\to 0^+} u_t|_{\xi_i}(x) = u|_{\xi_i}(x)$

$$\forall x \in \operatorname{int}\left(\Omega|_{\xi_i}\right), \lim_{t\to 0^+} \nabla u_t|_{\xi_i} = \nabla u|_{\xi_i} \tag{36}$$

Now, let us introduce some facts about the functions $u_t = u\Box(vt)$ with respect to the parameter t.

Lemma 15. *Let $\xi_i \in \mathscr{G}_{i,n}$, u and v belong both to the same class $\mathscr{L}$, $u_t := u\Box(vt)$ and Ω_t be the domain of u_t ($t > 0$). Then, for $x \in \Omega_t|_{\xi_i}$,*

$$\frac{d}{dt}\left(u_t|_{\xi_i}\right)(x) = -\psi\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right), \tag{37}$$

where $\psi := v^*|_{\xi_i}$.

Proof. Set $D_t := \Omega_t|_{\xi_i} \subset \xi_i$, for fixed $x \in \operatorname{int}(D_t)$, the map $t \to \nabla(u_t|_{\xi_i})(x)$ is differentiable on $(0, +\infty)$. Indeed, by the definition of Fenchel conjugate and the definition of projection u, it is easy to see that $(u_{\xi_i})^* = u^*|_{\xi_i}$ and $(u\Box ut)|_{\xi_i} = u|_{\xi_i}\Box ut|_{\xi_i}$ hold. Proposition 6 and the property of the projection grant the differentiability. Set $\varphi := u^*|_{\xi_i}$ and $\psi := v^*|_{\xi_i}$, and $\varphi_t = \varphi + t\psi$, then φ_t belongs to the class $\mathscr{C}_+^2$ on ξ_i. Then, $\nabla^2\varphi_t = \nabla^2\varphi + t\nabla^2\psi$ is nonsingular on ξ_i. So, the equation

$$\nabla\varphi(y) + t\nabla\psi(y) - x = 0 \tag{38}$$

locally defines a map $y = y(x, t)$ which is of class $\mathscr{C}^1$. By Proposition 3, we have $\nabla(u_t|_{\xi_i})$ is the inverse map of $\nabla\varphi_t$, that is, $\nabla\varphi_t(\nabla(u_t|_{\xi_i}(x)) = x$, which means that for every $x \in \operatorname{int}(D_t)$ and every $t > 0$, $t \to \nabla(u_t|_{\xi_i})$ is differentiable. Using equation (15) again, we have

$$u_t|_{\xi_i}(x) = \left\langle x, \nabla\left(u_t|_{\xi_i}\right)(x)\right\rangle - \varphi_t\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right), \quad \forall x \in \operatorname{int}(D_t). \tag{39}$$

Moreover, note that $\varphi_t = \varphi + t\psi$, we have

$$\begin{aligned} u_t|_{\xi_i}(x) &= \left\langle x, \nabla\left(u_{t\xi_i}\right)(x)\right\rangle - \varphi\left(\nabla\left(u_{t\xi_i}\right)(x)\right) - t\psi\left(\nabla\left(u_{t\xi_i}\right)(x)\right) \\ &= u_t\Big|_{\xi_i}\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right) - t\psi\left(\nabla\left(u_t|_{\xi_i}\right)(x)\right). \end{aligned} \tag{40}$$

Differential the above formal we obtain, $d/dt(u_t|_{\xi_i})(x) = -\psi(\nabla(u_t|_{\xi_i})(x))$. Then, we complete the proof of the result.

4. Functional Quermassintegrals of Log-Concave Function

A function $f \in \mathscr{A}$ is nondegenerate and integrable if and only if $\lim_{\|x\|\to+\infty} u(x)/\|x\| = +\infty$. Let $\mathscr{L}' = \{u \in \mathscr{L} : u \in \mathscr{C}_+^2(\mathbb{R}^n), \lim_{\|x\|\to+\infty} u(x)/\|x\| = +\infty\}$, and $\mathscr{A}' = \{f : \mathbb{R}^n \to (0,+\infty]: f = e^{-u}, u \in \mathscr{L}'\}$. Now, we define the ith total mass of f.

Definition 16. Let $f \in \mathscr{A}'$, $\xi_i \in \mathscr{G}_{i,n}(i = 1, 2, \cdots, n-1)$, and $x \in \Omega|_{\xi_i}$. The ith total mass of f is defined as

$$J_i(f) := \int_{\xi_i} f\Big|_{\xi_i}(x)dx, \tag{41}$$

where $f|_{\xi_i}$ is the projection of f onto ξ_i defined by (29) and dx is the i-dimensional volume element in ξ_i.

Remark 17.

(1) The definition of $J_i(f)$ follows the i-dimensional volume of the projection a convex body. If $i = 0$, we defined $J_0(f) := \omega_n$, the volume of the unit ball in $\mathbb{R}^n$, for the completeness

(2) When taking $f = \chi_K$, the characteristic function of a convex body K, one has $J_i(f) = V_i(K)$, the i-dimensional volume in ξ_i

Definition 18. Let $f \in \mathscr{A}'$. Set $\xi_i \in \mathscr{G}_{i,n}$ be a linear subspace and for $x \in \Omega|_{\xi_i}$, the ith functional Quermassintegrals of f (or the i-dimensional mean projection mass of f) are defined as

$$W_{n-i}(f) := \frac{\omega_n}{\omega_i} \int_{\mathscr{G}_{i,n}} J_i(f) d\mu(\xi_i), \quad i = 1, 2, \cdots, n, \tag{42}$$

where $J_i(f)$ is the ith total mass of f defined by (41) and $d\mu$ (ξ_i) is the normalized Haar measure on $\mathscr{G}_{i,n}$.

Remark 19.

(1) The definition of $W_i(f)$ follows the definition of the i th Quermassintegrals $W_i(K)$, that is, the ith mean total mass of f on $\mathscr{G}_{i,n}$. Also, in a recent paper [42], the authors give the same definition by defining the Quermassintegral of the support set for the quasiconcave functions

(2) When i equals to n in (42), we have $W_0(f) = \int_{\mathbb{R}^n} f(x)dx = J(f)$, the total mass function of f defined by Colesanti and Fragalá [38]. Then, we can say that our definition of $W_i(f)$ is a natural extension of the total mass function of $J(f)$

(3) From the definition of the Quermassintegrals $W_i(f)$, the following properties are obtained (see also [42]):

$$\text{Positivity::} 0 \leq W_i(f) \leq +\infty \tag{43}$$

(i) Monotonicity: $W_i(f) \leq W_i(g)$, if $f \leq g$

(ii) Generally speaking, $W_i(f)$ has no homogeneity under dilations. That is, $W_i(\lambda \cdot f) = \lambda^{n-i} W_i(f^\lambda)$, where $\lambda \cdot f(x) = \lambda f(x/\lambda)$, $\lambda > 0$

Definition 20. Let $f, g \in \mathscr{A}'$, $\oplus$, and $\cdot$ denote the operations of "sum" and "multiplication" in $\mathscr{A}'$. $W_i(f)$ and $W_i(g)$ are, respectively, the ith Quermassintegrals of f and g. Whenever the following limit exists,

$$W_i(f, g) = \frac{1}{(n-i)} \lim_{t \to 0^+} \frac{W_i(f \oplus t \cdot g) - W_i(f)}{t}, \tag{44}$$

we denote it by $W_i(f, g)$ and call it as the first variation of W_i at f along g, or the ith functional mixed Quermassintegrals of f and g.

Remark 21. Let $f = \chi_K$ and $g = \chi_L$, with $K, L \in \mathscr{K}^n$. In this case $W_i(f \oplus t \cdot g) = W_i(K + tL)$, then $W_i(f, g) = W_i(K, L)$. In general, $W_i(f, g)$ has no analog properties of $W_i(K, L)$; for example, $W_i(f, g)$ is not always nonnegative and finite.

The following is devoted to proving that $W_i(f, g)$ exists under the fairly weak hypothesis. First, we prove that the first i-dimensional total mass of f is translation invariant.

Lemma 22. *Let $\xi_i \in \mathscr{G}_{i,n}$, $f = e^{-u}, g = e^{-v} \in \mathscr{A}'$. Let $c = \inf u|_{\xi_i} =: u(0)$, $d = \inf v|_{\xi_i} := v(0)$, and set $\tilde{u}_i(x) = u|_{\xi_i}(x) - c$, $\tilde{v}_i(x) = v|_{\xi_i}(x) - d$, $\tilde{\varphi}_i(y) = (\tilde{u}_i)^*(y)$, $\tilde{\psi}_i(y) = (\tilde{v}_i)^*(y)$, $\tilde{f}_i = e^{-\tilde{u}_i}$, $\tilde{g}_i = e^{-\tilde{v}_i}$, and $\tilde{f}_t|_i = \tilde{f} \oplus t \cdot \tilde{g}$. Then, if $\lim_{t \to 0^+} ((J_i(\tilde{f}_t) - J_i(\tilde{f}))/t) = \int_{\xi_i} \tilde{\psi}_i d\mu_i(\tilde{f})$ holds, then we have $\lim_{t \to 0^+} ((J_i(f_t) - J_i(f))/t) = \int_{\xi_i} \psi_i d\mu_i(f)$.*

Proof. By the construction, we have $\tilde{u}_i(0) = 0$, $\tilde{v}_i(0) = 0$, and $\tilde{v}_i \geq 0$, $\tilde{\varphi}_i \geq 0$, $\tilde{\psi}_i \geq 0$. Further, $\tilde{\psi}_i(y) = \psi_i(y) + d$, and $\tilde{f}_i = e^c f_i$. So,

$$\lim_{t \to 0^+} \frac{J_i\left(\tilde{f}_t\right) - J_i\left(\tilde{f}\right)}{t} = \int_{\xi_i} \tilde{\psi}_i d\mu_i\left(\tilde{f}\right) = e^c \int_{\xi_i} \psi_i d\mu_i(f) + de^c \int_{\xi_i} d\mu_i(f). \tag{45}$$

On the other hand, since $f_i \oplus t \cdot g_i = e^{-(c+dt)}(\tilde{f}_i \oplus t \cdot \tilde{g}_i)$, we have, $J_i(f \oplus t \cdot g) = e^{-(c+dt)} J_i(\tilde{f}_i \oplus t \cdot \tilde{g}_i)$. By derivation of both sides of the above formula, we obtain

$$\begin{aligned}\lim_{t \to 0^+} \frac{J_i(f \oplus t \cdot g) - J_i(f)}{t} &= -de^{-c} \lim_{t \to 0^+} J_i\left(\tilde{f}_i \oplus t\tilde{g}_i\right) dx + e^{-c} \lim_{t \to 0^+} \\ &\quad \cdot \left[\frac{J_i\left(\tilde{f}_t\right) - J_i\left(\tilde{f}\right)}{t}\right] = -de^{-c} J_i\left(\tilde{f}_i\right) \\ &\quad + \int_{\xi_i} \psi_i d\mu_i(f) + d \int_{\xi_i} d\mu_i(f) \\ &= \int_{\xi_i} \psi_i d\mu_i(f).\end{aligned} \tag{46}$$

So, we complete the proof.

Theorem 23. *Let $f, g \in \mathscr{A}'$, with $-\infty \leq \inf (\log g) \leq +\infty$ and $W_i(f) > 0$. Then, $W_j(f, g)$ is differentiable at f along g, and it holds*

$$W_j(f, g) \in [-k, +\infty], \tag{47}$$

where $k = \max\{d, 0\} W_i(f)$.

Proof. Let $\xi_i \in \mathscr{G}_{i,n}$, since $u|_{\xi_i} := -\log(f_{\xi_i}) = -(\log f)|_{\xi_i}$ and $v|_{\xi_i} := -\log(g_{\xi_i}) = -(\log f)|_{\xi_i}$. By the definition of f_t and Proposition 11, we obtain $f_t|_{\xi_i} = (f \oplus t\cdot g)|_{\xi_i} = f|_{\xi_i} \oplus t\cdot g|_{\xi_i}$. Notice that $v|_{\xi_i}(0) = v(0)$, set $d := v(0)$, $\tilde{v}|_{\xi_i}(x) := v|_{\xi_i}(x) - d$, $\tilde{g}|_{\xi_i}(x) := e^{-\tilde{v}|_{\xi_i}(x)}$, and $\tilde{f}_t|_{\xi_i} := f|_{\xi_i} \oplus t\cdot \tilde{g}|_{\xi_i}$. Up to a translation of coordinates, we may assume $\inf(v) = v(0)$. Lemma 8 says that for every $x \in \xi_i$,

$$f|_{\xi_i} \le \tilde{f}_t\Big|_{\xi_i} \le \tilde{f}_1\Big|_{\xi_i}, \quad \forall x \in \mathbb{R}^n, \forall t \in [0,1]. \tag{48}$$

Then, there exists $\tilde{f}|_{\xi_i}(x) := \lim_{t\to 0^+} \tilde{f}_t\big|_{\xi_i}(x)$. Moreover, it holds $\tilde{f}|_{\xi_i}(x) \ge f|_{\xi_i}(x)$ and $\tilde{f}_t|_{\xi_i}$ is pointwise decreasing as $t \to 0^+$. Lemma 5 and Proposition 6 show that $f|_{\xi_i} \oplus t\cdot \tilde{g}|_{\xi_i} \in \mathscr{A}'$, $\forall t \in [0,1]$. Then, $J_i(f) \le J_i(\tilde{f}_t) \le J_i(\tilde{f}_1)$, $-\infty \le J_i(f)$, $J_i(\tilde{f}_1) < \infty$. Hence, by monotonicity and convergence, we have $\lim_{t\to 0^+} W_i(\tilde{f}_t) = W_i(\tilde{f})$. In fact, by definition, we have $\tilde{f}_t|_{\xi_i}(x) = e^{-\inf\{u|_{\xi_i}(x-y) + tv|_{\xi_i}(y/t)\}}$,

$$-\inf\left\{u|_{\xi_i}(x-y) + tv|_{\xi_i}\left(\frac{y}{t}\right)\right\} \le -\inf u|_{\xi_i}(x-y) - t\inf v|_{\xi_i}\left(\frac{y}{t}\right). \tag{49}$$

Note that $-\infty \le \inf(v|_{\xi_i}) \le +\infty$, then $-\inf u|_{\xi_i}(x-y) - t\inf v|_{\xi_i}(y/t)$ is a continuous function of variable t, then

$$\tilde{f}\Big|_{\xi_i}(x) := \lim_{t\to 0^+} \tilde{f}_t|_{\xi_i}(x) = f\big|_{\xi_i}(x). \tag{50}$$

Moreover, $W_i(\tilde{f}_t)$ is a continuous function of $(t \in [0,1])$; then, $\lim_{t\to 0^+} W_i(\tilde{f}_t) = W_i(f)$. Since $f_t|_{\xi_i} = e^{-dt}\tilde{f}|_{\xi_i}(x)$, we have

$$\frac{W_i(f_t) - W_i(f)}{t} = W_i(f)\frac{e^{-dt}-1}{t} + e^{-dt}\frac{W_i\left(\tilde{f}_t\right) - W_i(f)}{t}. \tag{51}$$

Notice that, $\tilde{f}_t|_{\xi_i} \ge f|_{\xi_i}$, we have the following two cases, that is, $\exists t_0 > 0 : W_i(\tilde{f}_{t_0}) = W_i(f)$ or $W_i(\tilde{f}_t) = W_i(f)$, $\forall t > 0$.

For the first case, since $W_i(\tilde{f}_t)$ is a monotone increasing function of t, it must hold $W_i(\tilde{f}_t) = W_i(f)$ for every $t \in [0, t_0]$. Hence, we have $\lim_{t\to 0^+}(W_i(f_t) - W_i(f))/t = -dW_i(f)$; the statement of the theorem holds true.

In the latter case, since $\tilde{f}_t|_{\xi_i}$ is an increasing nonnegative function, it means that $\log(W_i(\tilde{f}_t))$ is an increasing concave function of t. Then, $\exists(\log(W_i(\tilde{f}_t)) - \log(W_i(f)))/t \in [0,+\infty]$. On the other hand, since

$$\log'\left(W_i\left(\tilde{f}_t\right)\right)\Big|_{t=0} = \lim_{t\to 0^+}\frac{\log\left(W_i\left(\tilde{f}_t\right)\right) - \log(W_i(f))}{W_i\left(\tilde{f}_t\right) - W_i(f)} = \frac{1}{W_i(f)}. \tag{52}$$

Then,

$$\lim_{t\to 0^+}\frac{W_i\left(\tilde{f}_t\right) - W_i(f)}{\log\left(W_i\left(\tilde{f}_t\right)\right) - \log(W_i(f))} = W_i(f) > 0. \tag{53}$$

From above, we infer that $\exists \lim_{t\to 0^+}(W_i(\tilde{f}_t) - W_i(f))/t \in [0,+\infty]$. Combining the above formulas, we obtain

$$\lim_{t\to 0^+}\frac{W_i(f_t) - W_i(f)}{t} \in [-\max\{d,0\}W_i(f), +\infty]. \tag{54}$$

So, we complete the proof.

In view of the example of the mixed Quermassintegral, it is natural to ask whether, in general, $W_i(f,g)$ has some kind of integral representation.

Definition 24. Let $\xi_i \in \mathscr{G}_{i,n}$ and $f = e^{-u} \in \mathscr{A}'$. Consider the gradient map $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$, the Borel measure $\mu_i(f)$ *on* ξ_i is defined by

$$\mu_i(f) := \frac{\left(\nabla u|_{\xi_i}\right)_{\#}}{\|x\|^{n-i}}\left(f|_{\xi_i}\right). \tag{55}$$

Recall that the following Blaschke-Petkantschin formula is useful.

Proposition 25 (see [43]). *Let $\xi_i \in \mathscr{G}_{i,n}(i = 1, 2, \cdots, n)$ be linear subspace of $\mathbb{R}^n$ and f be a nonnegative bounded Borel function on $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} f(x)dx = \frac{\omega_n}{\omega_i}\int_{\mathscr{G}_{i,n}}\int_{\xi_i} f(x)\|x\|^{n-i}dx\,d\mu(\xi_i). \tag{56}$$

Now, we give a proof of Theorem 1.

Proof of Theorem 1. By the definition of the ith Quermassintegral of f, we have

$$\frac{W_i(f_t) - W_i(f)}{t} = \frac{\omega_n}{\omega_{n-i}}\int_{\mathscr{G}_{n-i,n}}\frac{J_{n-i}(f_t) - J_{n-i}(f)}{t}d\mu(\xi_{n-i}). \tag{57}$$

Let $t > 0$ be fixed, take $C \subset\subset \Omega|_{\xi_{n-i}}$, and by reduction $0 \in \operatorname{int}(\Omega)|_{\xi_{n-i}}$, we have $C \subset\subset \Omega|_{\xi_{n-i}}$, by Lemma 15, we obtain

$$\lim_{h\to 0}\frac{J_{n-i}(f_{t+h})(x)-J_{n-i}(f_t(x))}{h}=\int_{\xi_{n-i}}\psi\left(\nabla u_t\mid_{\xi_{n-i}}(x)\right)f_t\Big|_{\xi_{n-i}}(x)dx, \tag{58}$$

where $\psi = h_{g|_{\xi_{n-i}}} = \nu|^*_{\xi_{n-i}}$. Then, we have

$$\begin{aligned}\lim_{h\to 0}\frac{W_i(f_{t+h})-W_i(f_t)}{h}&=\frac{\omega_n}{\omega_{n-i}}\int_{\mathscr{G}_{n-i,n}}\int_{\xi_{n-i}}\frac{\psi\left(\nabla u_t\mid_{\xi_{n-i}}(x)\right)f_t|_{\xi_{n-i}}(x)}{\|x\|^{n-i}}\\&\quad\cdot\|x\|^{n-i}dxd\mu(\xi_{n-i}),\\&=\int_{\mathbb{R}^n}\frac{\psi\left(\nabla u_t\mid_{\xi_{n-i}}(x)\right)f_t|_{\xi_{n-i}}(x)}{\|x\|^{n-i}}dx\\&=\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_t).\end{aligned} \tag{59}$$

So, we have $W_i(f_{t+h})-W_i(f_t)=\int_0^t\{\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_s)\}ds$. The continuity of ψ implies $\lim_{s\to 0^+}\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_s)ds=\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f)ds$. Therefore,

$$\begin{aligned}\lim_{t\to 0^+}\frac{W_i(f_t)-W_i(f)}{t}&=\frac{d}{dt}W_i(f_t)\Big|_{t=0^+}=\lim_{s\to 0^+}\frac{d}{dt}W_i(f_t)\Big|_{t=s}\\&=\lim_{s\to 0^+}\frac{d}{dt}\int_0^t\left\{\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f_s)\right\}ds\\&=\int_{\mathbb{R}^n}\psi d\mu_{n-i}(f).\end{aligned} \tag{60}$$

Since $\psi = h_{g|_\xi}$, we have

$$W_i(f,g)=\frac{1}{n-i}\lim_{t\to 0^+}\frac{W_i(f_t)-W_i(f)}{t}=\frac{1}{n-i}\int_{\mathbb{R}^n}h_{g|_{\xi_{n-i}}}d\mu_{n-i}(f). \tag{61}$$

So, we complete the proof.

Remark 26. From the integral representation (12), the ith functional mixed Quermassintegral is linear in its second argument, with the sum in $\mathscr{A}'$, *for* $f, g, h\in\mathscr{A}'$, then we have $W_i(f, g\oplus h)=W_i(f,g)+W_i(f,h)$.

Data Availability

No data were used to support this study.

Disclosure

This paper is presented as Arxiv in the following link: https://arxiv.org/abs/2003.11367.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of this manuscript.

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References

[1] E. Lutwak, "The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem," *Journal of Differential Geometry*, vol. 38, no. 1, pp. 131–150, 1993.

[2] E. Lutwak, "The Brunn-Minkowski-Firey theory II: affine and geominimal surface area," *Advances in Mathematics*, vol. 118, pp. 224–294, 1996.

[3] A. Aleksandrov, "On the theory of mixed volumes. I: extensions of certain concepts in the theory of convex bodies," *Matematicheskii Sbornik*, vol. 2, pp. 947–972, 1937.

[4] W. Fenchel and B. Jessen, "Mengenfunktionen und konvexe Körper," *Danske Videnskabernes Selskab Mathematisk-fysiske Meddelelser*, vol. 16, pp. 1–31, 1938.

[5] C. Haberl and F. E. Schuster, "Asymmetric affine Lp Sobolev inequalities," *Journal of Functional Analysis*, vol. 257, no. 3, pp. 641–658, 2009.

[6] E. Lutwak, D. Yang, and G. Zhang, "Lp affine isoperimetric inequalities," *Journal of Differential Geometry*, vol. 56, no. 1, pp. 111–132, 2000.

[7] E. Lutwak, D. Yang, and G. Zhang, "Sharp affine L_p sobolev inequalities," *Journal of Differential Geometry*, vol. 62, no. 1, pp. 17–38, 2002.

[8] E. Lutwak, D. Yang, and G. Zhang, "On the L_p-Minkowski problem," *Transactions of the American Mathematical Society*, vol. 356, no. 11, pp. 4359–4370, 2004.

[9] E. Lutwak, D. Yang, and G. Zhang, "Volume inequalities for subspaces of L_p," *Journal of Differential Geometry*, vol. 68, no. 1, pp. 159–184, 2004.

[10] E. Lutwak, D. Yang, and G. Zhang, "Optimal Sobolev norms and the Lp Minkowski problem," *International Mathematics Research Notices*, vol. 2006, article 62987, pp. 1–21, 2006.

[11] E. Werner, "On L_p affine surface areas," *Indiana University Mathematics Journal*, vol. 56, no. 5, pp. 2305–2324, 2007.

[12] E. Werner and D. Ye, "New Lp affine isoperimetric inequalities," *Advances in Mathematics*, vol. 218, no. 3, pp. 762–780, 2008.

[13] E. Werner and D. Ye, "Inequalities for mixed p-affine surface area," *Mathematische Annalen*, vol. 347, no. 3, pp. 703–737, 2010.

[14] H. J. Brascamp and E. H. Lieb, "On extensions of the Brunn-Minkowski and Prekopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation," *Journal of Functional Analysis*, vol. 22, no. 4, pp. 366–389, 1976.

[15] L. Leindler, "On a certain converse of Hölder inequality II," *Acta Scientiarum Mathematicarum*, vol. 33, pp. 217–223, 1972.

[16] A. Prékopa, "Logarithmic concave measures with applications to stochastic programming," *Acta Scientiarum Mathematicarum*, vol. 32, pp. 335–343, 1971.

[17] A. Prékopa, "On logarithmic concave measures with applications to stochastic programming," *Acta Scientiarum Mathematicarum*, vol. 34, pp. 335–343, 1973.

[18] A. Prékopa, "New proof for the basic theorem of logconcave measures," *Alkalmazott Matematikai Lapja*, vol. 1, pp. 385–389, 1975.

[19] K. Ball, *Isometric Problems in* l_p *and Section of Convex Sets*, Ph.D dissertation, Cambridge, 1986.

[20] K. Ball, "Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$," *Studia Mathematica*, vol. 88, no. 1, pp. 69–84, 1988.

[21] F. Barthe, K. J. Böröczky, and M. Fradelizi, "Stability of the functional forms of the Blaschke-Santaló inequality," *Monatshefte für Mathematik*, vol. 173, no. 2, pp. 135–159, 2014.

[22] M. Fradelizi, Y. Gordon, M. Meyer, and S. Reisner, "The case of equality for an inverse Santaló functional inequality," *Advances in Geometry*, vol. 10, no. 4, pp. 621–630, 2010.

[23] J. Haddad, C. H. Jiménez, and M. Montenegro, "Asymmetric Blaschke-Santalo functional inequalities," *Journal of Functional Analysis*, vol. 278, no. 2, article 108319, 2020.

[24] L. Rotem, "A sharp Blaschke-Santaló inequality for $$\alpha $$ α -concave functions," *Geometriae Dedicata*, vol. 172, no. 1, pp. 217–228, 2014.

[25] B. Klartag and V. D. Milman, "Geometry of log-concave functions and measures," *Geometriae Dedicata*, vol. 112, no. 1, pp. 169–182, 2005.

[26] L. Rotem, "On the mean width of log-concave functions, in: Geometric Aspects of Functional Analysis," in *Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics, vol 2050*Springer, Berlin, Heidelberg.

[27] L. Rotem, "Support functions and mean width forα-concave functions," *Advances in Mathematics*, vol. 243, pp. 168–186, 2013.

[28] S. Artstein-Avidan, B. Klartag, C. Schütt, and E. Werner, "Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality," *Journal of Functional Analysis*, vol. 262, no. 9, pp. 4181–4204, 2012.

[29] D. Alonso-Gutiérrez, B. G. Merino, C. H. Jiménez, and R. Villa, "John's ellipsoid and the integral ratio of a log-concave function," *Journal of Geometric Analysis*, vol. 28, no. 2, pp. 1182–1201, 2018.

[30] N. Fang and J. Zhou, "LYZ ellipsoid and petty projection body for log-concave functions," *Advances in Mathematics*, vol. 340, pp. 914–959, 2018.

[31] D. Alonso-Gutiérrez, B. González Merino, C. H. Jiménez, and R. Villa, "Rogers-Shephard inequality for log-concave functions," *Journal of Functional Analysis*, vol. 271, no. 11, pp. 3269–3299, 2016.

[32] S. Artstein-Avidan and B. A. Slomka, "A note on Santaló inequality for the polarity transform and its reverse," *Proceedings of the American Mathematical Society*, vol. 143, no. 4, pp. 1693–1704, 2015.

[33] M. Barchiesi, G. M. Capriani, N. Fusco, and G. Pisante, "Stability of Polya-Szegő inequality for log-concave functions," *Journal of Functional Analysis*, vol. 267, no. 7, pp. 2264–2297, 2014.

[34] U. Caglar and E. M. Werner, "Divergence for *s*-concave and log concave functions," *Advances in Mathematics*, vol. 257, pp. 219–247, 2014.

[35] U. Caglar and E. M. Werner, "Mixedf-divergence and inequalities for log-concave functions," *Proceedings of the London Mathematical Society*, vol. 110, no. 2, pp. 271–290, 2015.

[36] U. Caglar and D. Ye, "Affine isoperimetric inequalities in the functional Orlicz-Brunn-Minkowski theory," *Advances in Applied Mathematics*, vol. 81, pp. 78–114, 2016.

[37] Y. Lin, "Affine Orlicz Pólya-Szegö principle for log-concave functions," *Journal of Functional Analysis*, vol. 273, no. 10, pp. 3295–3326, 2017.

[38] A. Colesanti and I. Fragalà, "The first variation of the total mass of log-concave functions and related inequalities," *Advances in Mathematics*, vol. 244, pp. 708–749, 2013.

[39] T. Rockafellar, *Convex Analysis*, Princeton Press, Princeton, NJ, USA, 1970.

[40] S. Artstein-Avidan and V. Milman, "A characterization of the support map," *Advances in Mathematics*, vol. 223, no. 1, pp. 379–391, 2010.

[41] D. Alonso-Gutiérrez, S. Artstein-Avidan, B. González Merino, C. H. Jiménez, and R. Villa, "Rogers-Shephard and local Loomis-Whitney type inequalities," *Mathematische Annalen*, vol. 374, no. 3-4, pp. 1719–1771, 2019.

[42] S. G. Bobkov, A. Colessanti, and I. Fragalà, "Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities," *Manuscripta Mathematica*, vol. 143, no. 1-2, pp. 131–169, 2014.

[43] E. Jensen, *Local Stereology*, World Scientific, New York, MY, USA, 1998.

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Research Article

The Functional Orlicz Brunn-Minkowski Inequality for q-Capacity

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In this paper, we establish functional forms of the Orlicz Brunn-Minkowski inequality and the Orlicz-Minkowski inequality for the electrostatic q-capacity, which generalize previous results by Zou and Xiong. We also show that these two inequalities are equivalent.

1. Introduction

The classical Brunn-Minkowski inequality was inspired by questions around the isoperimetric problem. It is viewed as one of cornerstones of the Brunn-Minkowski theory, which is a beautiful and powerful tool to conquer all sorts of geometrical problems involving metric quantities such as volume, surface area, and mean width.

An excellent reference on this inequality is provided by Gardner [1].

In 2015, Colesanti, Nyström, Salani, Xiao, Yang, and Zhang (CNSXYZ) [2] introduced the electrostatic q-capacity. Let K be a compact set in the n-dimensional Euclidean space R^n. For $1 < q < n$, the electrostatic q-capacity, $C_q(K)$, of K is defined by

$$C_q(K) = \inf \left\{ \int_{R^n} |\nabla u|^q dx : u \in C_c^\infty(R^n) \text{ and } u \geq \chi_K \right\}, \quad (1)$$

where $C_c^\infty(R^n)$ denotes the set of functions from $C^\infty(R^n)$ with compact supports and χ_K is the characteristic function of K. If $q = 2$, then $C_2(K)$ is the classical electrostatic (or Newtonian) capacity of K. The Minkowski-type problems for the electrostatic q-capacity have attracted increasing attention [2–10]. The electrostatic q-capacity also has applications in analysis, mathematical physics, and partial differential equations (see [11–13]).

The electrostatic q-capacity can be extended on function spaces. Let $C(S^{n-1})$ denote the set of continuous functions defined on S^{n-1}, which is equipped with the metric induced by the maximal norm. Write $C_+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$. For $1 < q < n$ and $f \in C_+(S^{n-1})$, define the electrostatic q-capacity $C_q(f)$ by

$$C_q(f) = C_q([f]), \quad (2)$$

where $[f]$ denotes the Aleksandrov body (also known as the Wulff shape) associated with f. For nonnegative $f \in C(S^{n-1})$, the Aleksandrov body $[f]$ is defined by

$$[f] = \bigcap_{\xi \in S^{n-1}} \{x \in R^n : x \cdot \xi \leq f(\xi)\}. \quad (3)$$

Obviously, $[f]$ is a compact convex set containing the origin and $h_{[f]} \leq f$, where $h_{[f]}$ denotes the support function of $[f]$. Moreover,

$$K = [h_K], \quad (4)$$

for every compact convex set K containing the origin. If $f \in C_+(S^{n-1})$, then$[f]$ is a convex body in R^n containing the origin in its interior. The Aleksandrov convergence lemma reads: if the sequence $\{f_j\}_j \subset C_+(S^{n-1})$ converges uniformly to $f \in C_+(S^{n-1})$, then, $\lim_{j\to\infty}[f_j] = [f]$.

Suppose $a, b \geq 0$ (not both zero). If $f, g \in C_+(S^{n-1})$, then, the L_p Minkowski sum $a \cdot g +_p b \cdot f$ is defined by

$$a \cdot f +_p b \cdot g = \left(af^p + bg^p\right)^{1/p}, \tag{5}$$

where the L_p scalar multiplication $a \cdot f$ is defined by $a^{a/p}f$. By the definition of the Aleksandrov body (3), we have $[a \cdot h_K +_p b \cdot h_L] = a \cdot K +_p b \cdot L$ for convex bodies K and L containing the origin in their interiors. Here, $a \cdot K +_p b \cdot L$ denotes the L_p Minkowski sum of K and L, i.e.,

$$h_{a\cdot K +_p b\cdot L}{}^p = ah_K{}^p + bh_L{}^p, \tag{6}$$

for every $u \in S^{n-1}$, which was defined by Firey [14]. In the mid 1990s, it was shown in [15, 16] that when L_p Minkowski sum is combined with volume the result is an embryonic L_p-Brunn-Minkowski theory. Zou and Xiong ([7], Theorem 3.11) established the functional form of the L_p Brunn-Minkowski inequality for the electrostatic q-capacity. Suppose $1 < p < \infty$ and $1 < q < n$.

If $f, g \in C_+(S^{n-1})$, then

$$C_q(f +_p g)^{p/(n-q)} \geq C_q(f)^{p/(n-q)} + C_q(g)^{p/(n-q)}, \tag{7}$$

with equality if and only if $[f]$ and $[g]$ are dilates.

The Orlicz Brunn-Minkowski theory which was launched by Lutwak et al. in a series of papers [17–19] is an extension of the L_p Brunn-Minkowski theory. This theory has been considerably developed in the recent years. The Orlicz sum was introduced by Gardner et al. [20]. Let Φ be the class of convex, strictly increasing functions, $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$. Suppose $\phi \in \Phi$ and $a, b \geq 0$ (not both zero). If K and L are convex bodies that contain the origin in their interiors in R^n, then, the Orlicz sum $a \cdot K +_\phi b \cdot L$ is the convex body defined by

$$h_{a\cdot K +_\phi b\cdot L}(u) = \inf\left\{\tau > 0 : a\phi\left(\frac{h_K(u)}{\tau}\right) + b\phi\left(\frac{h_L(u)}{\tau}\right) \leq \phi(1)\right\}, \tag{8}$$

for every $u \in S^{n-1}$. Gardner et al. ([20], Corollary 7.5) established the Orlicz Brunn-Minkowski inequality (see also ([21], Theorem 1). Same as the Orlicz sum of convex bodies, we extend the L_p Minkowski sum of functions to the Orlicz sum. For $\phi \in \Phi$, $f, g \in C_+(S^{n-1})$, and $a, b \geq 0$ (not both zero), the Orlicz sum $a \cdot f +_\phi b \cdot g$ is defined by

$$a \cdot f +_\phi b \cdot g = \inf\left\{\tau > 0 : a\phi\left(\frac{f}{\tau}\right) + b\phi\left(\frac{g}{\tau}\right) \leq \phi(1)\right\}. \tag{9}$$

If we take $\phi(t) = t^p (p \geq 1)$ in (9), then it, induces the L_p Minkowski sum (5). By the definition of the Aleksandrov body (3), (8), (9), and (4), we have $[a \cdot h_K +_\phi b \cdot h_L] = a \cdot K +_\phi b \cdot L$ for convex bodies K and L containing the origin in their interiors.

The main aim of this paper is to establish the functional form of the Orlicz Brunn-Minkowski inequality for the electrostatic q-capacity.

Theorem 1. *Suppose $\phi \in \Phi$ and $1 < q < n$. If $f, g \in C_+(S^{n-1})$, then,*

$$\phi\left(\left(\frac{C_q(f)}{C_q(f +_\phi g)}\right)^{1/(n-q)}\right) + \phi\left(\left(\frac{C_q(g)}{C_q(f +_\phi g)}\right)^{1/(n-q)}\right) \leq \phi(1). \tag{10}$$

If ϕ is strictly convex, equality holds if and only if $[f]$ and $[g]$ are dilates.

2. Notation and Preliminary Results

For excellent references on convex bodies, we recommend the books by Gardner [22], Gruber [23], and Schneider [24].

We will work in R^n equipped with the standard Euclidean norm. Let $x \cdot y$ denote the standard inner product of $x, y \in R^n$. For $x \in R^n$, $|x| = \sqrt{x \cdot x}$ denotes the Euclidean norm of x. We write $B = \{x \in R^n : |x| \leq 1\}$ and S^{n-1} for the standard unit ball of R^n and its surface, respectively. Each compact convex set K is uniquely determined by its support function $h_K : R^n \longrightarrow R$, which is defined by $h_K(x) = \max\{x \cdot y : y \in R^n\}$, for $x \in R^n$. Obviously, the support function is positively homogeneous of order 1.

The class of compact convex sets in R^n is often equipped with the Hausdorff metric δ_H, which is defined for compact convex sets K and L by

$$\delta_H(K, L) = \max\left\{|h_K(u) - h_L(u)|: u \in S^{n-1}\right\}. \tag{11}$$

Denote by K^n the set of convex bodies in R^n and by K_o^n the set of convex bodies which contain the origin in their interiors. For $s > 0$, the set $sK = \{sx : x \in K\}$ is called a dilate of convex body K. Convex bodies K and L are said to be homothetic, provided $K = sL + x$ for some $s > 0$ and $x \in R^n$. Let $K, L \in K^n$, the Minkowski sum of K and L is the convex body

$$K + L = \{x, y : x \in K, y \in L\}. \tag{12}$$

Some properties of the electrostatic q-capacitary measure are required [2, 3, 7, 8, 11]. The electrostatic q-capacitary measure, $\mu_q(E, \cdot)$, of a bounded open convex set E in $R^n$is the measure on the unit sphere S^{n-1} defined for $\omega \subset S^{n-1}$ and $1 < q < n$ by

$$\mu_q(E, \omega) = \int_{g^{-1}(\omega)} |\nabla U|^q dH^{n-1}, \tag{13}$$

where $g^{-1} : S^{n-1} \longrightarrow \partial E$ (the set of boundary points of E) denotes the inverse Gauss map, H^{n-1} the $(n-1)$-dimensional

Hausdorff measure, and U the q-equilibrium potential of E. If $K \in K^n$, then the electrostatic q-capacitary measure $\mu_q(K,\cdot)$ has the following properties. First, it is positively homogeneous of degree $(n-q-1)$, i.e., $\mu_q(sK,\cdot) = s^{n-q-1}\mu_q(K,\cdot)$ for $s > 0$. Second, it is translation invariant, i.e., $\mu_q(K+x,\cdot) = \mu_q(K,\cdot)$ for $x \in R^n$. Third, its centroid is at the origin, i.e., $\int_{S^{n-1}} u d\mu_q(K,u) = 0$. Moreover, it is absolutely continuous with respect to the surface area measure $S(K,\cdot)$. The weak convergence of the electrostatic q-capacitary measure is proved by CNSXYZ ([2], Lemma 4.1): if $\{K_j\} \subset K^n$ converges to $K \in K^n$, then$\mu_q(K_j,\cdot)$ converges weakly to $\mu_q(K,\cdot)$.

CNSXYZ [2] showed the Hadamard variational formula for the electrostatic q-capacity: for $K, L \in K^n$ and $1 < q < n$,

$$\left.\frac{dC_q(K+tL)}{dt}\right|_{t=0^+} = (q-1)\int_{S^{n-1}} h_L(u) d\mu_q(K,u). \tag{14}$$

And variational formula (14) leads to the following Poincare q-capacity formula:

$$C_q(K) = \frac{q-1}{n-q}\int_{S^{n-1}} h_K(u) d\mu_q(K,u). \tag{15}$$

The electrostatic q-capacity $C_q(K)$ has the following properties. First, it is increasing with respect to set inclusion; that is, if $K_1 \subseteq K_2$, then $C_q(K_1) \le C_q(K_2)$. Second, it is positively homogeneous of degree $(n-q)$, i.e., $C_q(sK) = s^{n-q}C_q(K)$ for $s > 0$. Third, it is a rigid motion invariant, i.e.,

$C_q(\varphi K + x) = C_q(K)$ for $x \in R^n$ and $\varphi \in O(n)$. If $q = 2$, then (15) induces the Poincare capacity formula

$$C_2(K) = \frac{1}{n-2}\int_{S^{n-1}} h_K(u) d\mu_2(K,u). \tag{16}$$

Let $C(S^{n-1})$ denote the set of continuous functions defined on S^{n-1}, which is equipped with the metric induced by the maximal norm. Write $C_+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$. Let $K \in K_o^n$ and $g \in C(S^{n-1})$. There is a $t_0 > 0$ such that $h_K + tg \in C_+(S^{n-1})$ for $|t| < t_0$. The Aleksandrov body $[h_K + tg]$ is continuous in $t \in (-t_0, t_0)$. The Hadamard variational formula for the electrostatic q-capacity [2] states the following:

$$\left.\frac{dC_q(h_K+tg)}{dt}\right|_{t=0^+} = (q-1)\int_{S^{n-1}} g(u) d\mu_q(K,u). \tag{17}$$

For $f \in C_+(S^{n-1})$, define

$$C_q(f) = C_q([f]). \tag{18}$$

Obviously, $C_q(h_K) = C_q(K)$ for every $K \in K_o^n$. By the Aleksandrov convergence lemma and the continuity of C_q on K^n, the functional $C_q : C_+(S^{n-1}) \longrightarrow (0,\infty)$ is continuous. For $K \in K_o^n$ and $g \in C(S^{n-1})$, the mixed electrostatic q-capacity $C_q(K,g)$ is defined by

$$C_q(K,g) = \frac{1}{n-q}\left.\frac{dC_q(h_K+tg)}{dt}\right|_{t=0^+}. \tag{19}$$

Applying the Hadamard variational formula for the electrostatic q-capacity, the mixed electrostatic q-capacity $C_q(K,g)$ has the following integral representation:

$$C_q(K,g) = \frac{q-1}{n-q}\int_{S^{n-1}} g(u) d\mu_q(K,u). \tag{20}$$

Let $L \in K^n$. If $g = h_L$, then, $C_q(K,g)$ is the mixed electrostatic q-capacity $C_q(K,L)$, which has the following integral representation:

$$C_q(K,L) = \frac{q-1}{n-q}\int_{S^{n-1}} h_L(u) d\mu_q(K,u). \tag{21}$$

The Minkowski inequality for the electrostatic q-capacity ([2], Theorem 3.6) states the following: let$1 < q < n$.

If $K, L \in K^n$, then,

$$C_q(K,L) \ge C_q(K)^{(n-q-1)/(n-q)} C_q(L)^{1/(n-q)}, \tag{22}$$

with equality if and only if K and L are homothetic.

Let $1 \le p < \infty$ and $1 < q < n$. For $K \in K_o^n$ and $g \in C(S^{n-1})$, the L_p Hadamard variational formula for the electrostatic q-capacity [7] states the following:

$$\left.\frac{dC_q(h_K +_p t\cdot g)}{dt}\right|_{t=0^+} = \frac{q-1}{p}\int_{S^{n-1}} g(u)^p h_K(u)^{1-p} d\mu_q(K,u). \tag{23}$$

The L_p mixed electrostatic q-capacity $C_{p,q}(K,g)$ is defined by

$$C_{p,q}(K,g) = \frac{1}{n-q}\left.\frac{dC_q(h_K +_p t\cdot g)}{dt}\right|_{t=0^+} = \frac{q-1}{n-q}\int_{S^{n-1}} g(u)^p h_K(u)^{1-p} d\mu_q(K,u). \tag{24}$$

Take $g = h_K$ in (24), and combine $C_q(K,g) = C_q(K)$ to obtain the Poincare q-capacity formula (15). Zou and Xiong ([7], Theorem 3.9) established the L_p Minkowski inequality for the L_p electrostatic q-capacity: let $1 < p < \infty$ and $1 < q < n$. If $K \in K_o^n$ and $g \in C(S^{n-1})$, then,

$$C_q(K,g) \ge C_q(K)^{(n-q-p)/(n-q)} C_q(L)^{p/(n-q)}, \tag{25}$$

with equality if and only if K and $[g]$ are dilates.

Based on the Orlicz sum (9), we define the Orlicz mixed electrostatic q-capacity as follows. For $K \in K_o^n$ and $g \in C(S^{n-1})$, the Orlicz mixed electrostatic q-capacity $C_{\phi,q}(K,g)$ is defined by

$$C_{\phi,q}(K,g)=\frac{1}{n-q}\frac{dC_q\left(h_K+_\phi t\cdot g\right)}{dt}\bigg|_{t=0^+}. \quad (26)$$

Indeed, the Orlicz mixed electrostatic q-capacity can be extended on function spaces. Let $\phi\in\Phi$ and $1<q<n$. For $f\in C_+(S^{n-1})$ and $g\in C(S^{n-1})$, the Orlicz mixed electrostatic q-capacity $C_{\phi,q}([f],g)$ is defined by

$$C_{\phi,q}([f],g)=\frac{1}{n-q}\frac{dC_q\left(f+_\phi t\cdot g\right)}{dt}\bigg|_{t=0^+}. \quad (27)$$

If $f=h_K$ with $K\in K_o^n$, then, $C_{\phi,q}([h_K],g)=C_{\phi,q}(K,g)$. If $g=h_L$ with $L\in K_o^n$, then, $C_{\phi,q}([f],h_L)$ is the Orlicz mixed electrostatic q-capacity $C_{\phi,q}([f],L)$. In particular, $C_{\phi,q}([h_K],h_K)=\phi(1)C_q(K)$ for every $K\in K_o^n$.

3. Main Results

The following variational formula of electrostatic q-capacity plays a crucial role in our proof.

Lemma 2 ([2], Lemma 5.1). *Let $I\subset R$ be an interval containing 0 in its interior, and let $h_t(u)=h(t,u)$: $I\times S^{n-1}\longrightarrow[0,\infty)$ be continuous such that the convergence in*

$$h'(0,u)=\lim_{t\to 0}\frac{h(t,u)-h(0,u)}{t} \quad (28)$$

is uniformly on S^{n-1}. Then,

$$\frac{dC_q([h_t])}{dt}\bigg|_{t=0^+}=(q-1)\int_{S^{n-1}}h'(0,u)d\mu_q([h_0],u). \quad (29)$$

Suppose $\phi\in\Phi$, $f,g\in C_q(S^{n-1})$, and $a,b\geq 0$ (not both zero). For every given $u\in S^{n-1}$, the function $t\mapsto a\phi(f(u)/t)+b\phi(g(u)/t)$ is strictly decreasing. By the definition of the Orlicz sum (9), we have $(a\cdot f+_\phi b\cdot g)(u)=t$ and only if $a\phi(f(u)/t)+b\phi(g(u)/t)=\phi(1)$.

for every $u\in S^{n-1}$.

The continuity properties of the Orlicz sum were established by Xi et al. [21].

Lemma 3 ([21], Lemma 3.1). *Suppose,$\phi\in\Phi f\in C_+(S^{n-1})$, $g\in C(S^{n-1})$, and $a,b\geq 0$ (not both zero).*

(i) Let $\{f_i\},\{g_i\}\subset C_+(S^{n-1})$ and $\{g_i\}\subset C(S^{n-1})$ such that $f_i\longrightarrow f$ and $g_i\longrightarrow g$, respectively. Then, $a\cdot f_i+_\phi b\cdot g_i\longrightarrow a\cdot f+_\phi b\cdot g$

(ii) Let $\{\phi_i\}\subset\Phi$ such that $\phi_i\to\phi$. Then, $a\cdot f+_{\phi_i}b\cdot g\longrightarrow a\cdot f+_\phi b\cdot g$

(iii) Let $a_i,b_i\geq 0$ (not both zero) such that $a_i\longrightarrow a$ and $b_i\longrightarrow b$. Then

$$a_i\cdot f+_\phi b_i\cdot g\longrightarrow a\cdot f+_\phi b\cdot g \quad (30)$$

Due to Lemma 2, the integral representation of the Orlicz mixed electrostatic q-capacity is given.

Lemma 4. *Suppose $\phi\in\Phi$ and $1<q<n$. If $f\in C_+(S^{n-1})$ and $g\in C(S^{n-1})$, then*

$$C_{\varphi,q}([f],g)=\frac{q-1}{n-q}\int_{S^{n-1}}\varphi\left(\frac{g(u)}{f(u)}\right)f(u)d\mu_q([f],u). \quad (31)$$

Proof. Take an interval $I=[0,t_0]$ for $0<t_0<\infty$. Denote $h_t(u)$: $I\times S^{n-1}\longrightarrow(0,\infty)$ by

$$h_t(u)=h(t,u)=\left(f+_\phi t\cdot g\right)(u). \quad (32)$$

Then, the definition of the Orlicz sum (9) and Lemma 3 imply that the function $h_t(u)$: $I\times S^{n-1}\longrightarrow(0,\infty)$ is continuous. By (9), we have

$$\phi\left(\frac{f(u)}{h_t(u)}\right)+t\phi\left(\frac{g(u)}{h_t(u)}\right)=\phi(1), \quad (33)$$

for every $u\in S^{n-1}$. Since $\varphi\in\phi$, we obtain

$$\frac{dh_t}{dt}=\frac{h_t^2\phi(g/h_t)}{f\phi'(f/h_t)+tf\phi'(f/h_t)}. \quad (34)$$

Note that $(f/h)\longrightarrow 1^-$ as $t\longrightarrow 0^+$ and the fact that $h_0=f$. Thus,

$$\lim_{t\to 0^+}\frac{h_t-h_0}{t}=\frac{f\phi(g/f)}{\phi_l'(1)}, \quad (35)$$

uniformly on S^{n-1}, where $\phi_l'(1)$ denotes the left derivative of $\phi(t)$ at $t=1$. Apply Lemma 2 and (35) to get

$$\frac{dC_q\left(f+_\phi t\cdot g\right)}{dt}\bigg|_{t=0^+}=\frac{q-1}{\varphi_l'(1)}\int_{S^{n-1}}\phi\left(\frac{g(u)}{f(u)}\right)d\mu_q([f],u). \quad (36)$$

Thus, (27) and (36) yield the desired lemma.

Indeed, (36) can be considered as the Orlicz Hadamard variational formula for the electrostatic q-capacity. If we take $\phi(t)=t^p(1\leq p<\infty)$ and $f=h_K$ with $K\in K_o^n$ in (36), then, we obtain the L_p Hadamard variational formula (23).

Note that $[h_K]=K$ for every $K\in K_o^n$. Take $f=h_K$ in Lemma 4 to get

Lemma 5. *Suppose $\phi\in\Phi$ and $1<q<n$. If $K\in K_o^n$ and $g\in C(S^{n-1})$, then,*

$$C_{\phi,q}(K,g)=\frac{q-1}{n-q}\int_{S^{n-1}}\phi\left(\frac{g(u)}{h_K(u)}\right)h_K(u)d\mu_q(K,u). \quad (37)$$

A direct consequence of Lemma 4 and the homogeneity of the electrostatic q-capacitary measure can be obtained.

Corollary 6. *Suppose $\phi \in \Phi$ and $1 < q < n$. If $f \in C_+(S^{n-1})$, then*

$$C_{\varphi,q}(c[f], f) = \varphi\left(\frac{1}{c}\right)c^{n-q}C_q([f]) = \varphi\left(\frac{1}{c}\right)c^{n-q}C_q(f), \quad (38)$$

for every $c > 0$.

Let $\phi \in \Phi$, $1 < q < n$, and $K, L \in K_o^n$. Note that $K +_\phi t \cdot L = [h_K +_\phi t \cdot h_L]$, and apply (18) and (36) to obtain

$$\left.\frac{dC_q(K +_\phi t \cdot L)}{dt}\right|_{t=0^+} = \frac{q-1}{\phi_l'(1)}\int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)d\mu_q(K, u). \quad (39)$$

Based on (39), one can define the Orlicz mixed electrostatic q-capacity $C_{\phi,q}(K, L)$ of convex bodies K and L as follows:

$$C_{\phi,q}(K, L) = \frac{q-1}{n-q}\int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)d\mu_q(K, u), \quad (40)$$

which was first defined by Hong et al. ([10], Definition 3.1).

Lemma 7. *Suppose $\phi \in \Phi$, $f \in C_+(S^{n-1})$, $g \in C(S^{n-1})$, and $1 < q < n$.*

(i) Let $\phi_1, \phi_2 \in \Phi$. If $\phi_1 \le \phi_2$, then $C_{\phi_1,q}([f], g) \longrightarrow C_{\phi_2,q}([f], g)$

(ii) Let $\{f_i\} \subset C_+(S^{n-1})$ and $\{g_i\} \subset C(S^{n-1})$ such that $f_i \longrightarrow f$ and $g_i \longrightarrow g$, respectively. Then, $C_{\phi,q}([f_i], g_i) \longrightarrow C_{\phi,q}([f], g)$

(iii) Let $\{\phi_i\} \subset \Phi$ such that $\phi_i \longrightarrow \phi$. Then, $C_{\phi_i,q}([f], g) \longrightarrow C_{\phi,q}([f], g)$

Proof. It follows from (31) that (i) holds if $\phi_1 \le \phi_2$.

Since $f > 0, f_i > 0, g \ge 0, g_i \ge 0$ and $f_i \longrightarrow f$, $g_i \longrightarrow g$ uniformly on S^{n-1}; it follows that $g_i/f_i \longrightarrow g/f$ uniformly on S^{n-1}. Note that $\phi \in \Phi$, we have $\phi(g_i/f_i)f_i \longrightarrow \phi(g/f)f$ uniformly on S^{n-1}. The Aleksandrov convergence lemma implies that $[f_i] \longrightarrow [f]$ uniformly on S^{n-1}. Meanwhile, the convergence$[f_i] \longrightarrow [f]$ implies that $\mu_q([f_i], \cdot) \longrightarrow \mu_q([f], \cdot)$ weakly. Applying Lemma 4, one concludes that (ii) holds.

Clearly, there exists a compact interval $I \subset (0, \infty)$ such that $g/f \in I$ for all $u \in S^{n-1}$.

(iii) directly follows from Lemma 4 and the fact that the sequence $\{\phi_i(t)\}$ converges uniformly to $\phi(t)$ on I.

Next, we show that there is a natural Orlicz extension of the Minkowski inequality for the electrostatic q-capacity.

Theorem 8. *Suppose $\phi \in \Phi$ and $1 < q < n$. If $f, g \in C_+(S^{n-1})$, then,*

$$\frac{C_{\phi,q}([f], (g))}{C_q(f)} \ge \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right). \quad (41)$$

If φ is strictly convex, then equality holds if and only if $[f]$ and $[g]$ are dilates.

Proof. By the definition of the mixed electrostatic q-capacity (20) and the fact that $h_{[g]} \le g$, we have

$$C_q([f], [g]) = C_q\left([f], h_{[g]}\right) \le C_q([f], g). \quad (42)$$

for every $f, g \in C_+(S^{n-1})$. From (31), Jensen's inequality, (20), (42), (22), and (18), it follows that

$$\begin{aligned}\frac{C_{\phi,q}([f], g)}{C_q(f)} &= \frac{((q-1)/(n-q))\int_{S^{n-1}}\phi(g(u)/f(u))f(u)d\mu_q([f], u)}{C_q(f)} \\ &\ge \phi\left(\frac{C_q([f], g)}{C_q(f)}\right) \ge \phi\left(\frac{C_q([f], [g])}{C_q(f)}\right) \\ &\ge \phi\left(\frac{C_q(g)^{1/(n-q)}}{C_q(f)^{1/(n-q)}}\right) = \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right).\end{aligned} \quad (43)$$

It remains to prove the equality condition. Now, suppose ϕ is strictly convex. If equality in (41) holds, then, by the equality condition of Jensen's inequality, there exists an $s > 0$ such that $g = sf$ for almost every $u \in S^{n-1}$with respect to the measure $(f(\cdot)d\mu_q([f], \cdot))/(C_q(f))$. Then, we have

$$\begin{aligned}s &= \frac{(q-1)/(n-q)\int_{S^{n-1}}(g(u))/(f(u))f(u)d\mu_q([f], u)}{C_q(f)} \\ &= \frac{C_q([f], g)}{C_q(f)} = \frac{C_q([f], [g])}{C_q(f)},\end{aligned} \quad (44)$$

where the last step is from the equality condition of (42). The definition of Aleksandrov body implies that $h_{[g]} = sh_{[f]}$ for almost every $u \in S^{n-1}$ with respect to the measure $(f(\cdot)d\mu_q([f], \cdot))/(C_q(f))$. Thus,

$$C_q([f], [g])h_{[f]}(u) = C_q(f)h_{[g]}(u), \quad (45)$$

for almost every $u \in S^{n-1}$ with respect to the measure $(f(\cdot)d\mu_q([f], \cdot))/(C_q(f))$. By the equality condition of the Minkowski inequality for the electrostatic q-capacity, there exists $x \in R^n$ such that $[g] = s[f] + x$.

Hence, for almost every $u \in S^{n-1}$ with respect to the measure $f(\cdot)d\mu_q([f],\cdot)$,

$$\left(sC_q(f) + \frac{q-1}{n-q}x \cdot \int_{S^{n-1}} u d\mu_q([f],u)\right)h_{[f]}(u) = C_q(f)\left(sh_{[f]}(u) + x \cdot u\right). \tag{46}$$

Since the centroid of $\mu_q([f],\cdot)$ is at the origin, we have that $x \cdot u = 0$ for almost every $u \in S^{n-1}$ with respect to the measure $(f(\cdot)d\mu_q([f],\cdot))/(C_q(f))$. Note that the electrostatic q-capacitary measure $\mu_q([f],\cdot)$ is not concentrated on any great subsphere of S^{n-1}. Hence, $x = 0$, which in turn implies that $[f]$ and $[g]$ are dilates.

Conversely, assume that $[f]$ and $[g]$ are dilates, say, $[f] = c[g]$ for some $c > 0$. From our assumption, Corollary 6, (18), and the fact that $C_q(c[g]) = c^{n-q}C_q([g])$, it follows that

$$\frac{C_{\phi,q}([f],g)}{C_q(f)} = \frac{C_{\phi,q}(c[g],g)}{C_q(c[g])} = \frac{\phi(1/c)c^{n-q}C_q([g])}{c^{n-q}C_q([g])} = \phi\left(\frac{1}{c}\right) = \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right). \tag{47}$$

This completes the proof.

By using the Orlicz-Minkowski inequality for the electrostatic q-capacity, we establish the following Orlicz Brunn-Minkowski inequality for the electrostatic q-capacity which is the general version of Theorem 1.

Theorem 9. *Suppose $\phi \in \Phi$ and $1 < q < n$. If $f, g \in C_+(S^{n-1})$ and $a, b \geq 0$ (not both zero); then,*

$$a\phi\left(\left(\frac{C_q(f)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right) + b\phi\left(\left(\frac{C_q(f)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right) \leq \phi(1). \tag{48}$$

If ϕ is strictly convex, then equality holds if and only if $[f]$ and $[g]$ are dilates.

Proof. By (31), (9), and the Orlicz-Minkowski inequality for the electrostatic q-capacity (41), we have

$$\begin{aligned}\phi(1) &= \frac{C_{\phi,q}([a \cdot f +_\phi b \cdot g], a \cdot f +_\phi b \cdot g)}{C_q(a \cdot f +_\phi b \cdot g)} \\ &= a\frac{C_{\phi,q}([a \cdot f +_\phi b \cdot g], f)}{C_q(a \cdot f +_\phi b \cdot g)} + b\frac{C_{\phi,q}([a \cdot f +_\phi b \cdot g], g)}{C_q(a \cdot f +_\phi b \cdot g)} \\ &\geq a\phi\left(\left(\frac{C_q(f)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right) \\ &\quad + b\phi\left(\left(\frac{C_q(g)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right).\end{aligned} \tag{49}$$

By the equality condition of the Orlicz-Minkowski inequality for the electrostatic q-capacity, we have that if ϕ is strictly convex, then equality in (48) holds if and only if $[f]$ and $[g]$ are dilates of $[a \cdot f +_\varphi b \cdot g]$.

Remark 1. The case $\phi(t) = t^p (1 \leq p < \infty)$ of Theorem 9 was established by Zou and Xiong [7].

For $K, L \in K_o^n$, take $f = h_K$ and $g = h_L$ in Theorem 9 to obtain the following Orlicz-Brunn-Minkowski inequality for the electrostatic q-capacity, which was established by Hong et al. [10].

Corollary 10 ([10], Theorem 4.2). *Suppose $\phi \in \Phi$ and $1 < q < n$. IfK, $L \in K_o^n$, then*

$$a\phi\left(\left(\frac{C_q(K)}{C_q(a \cdot K +_\phi b \cdot L)}\right)^{1/(n-q)}\right) + b\phi\left(\left(\frac{C_q(L)}{C_q(a \cdot K +_\phi b \cdot L)}\right)^{1/(n-q)}\right) \leq \phi(1). \tag{50}$$

If φ is strictly convex, then equality holds if and only if K and L are dilates.

Remark 2. The case $\varphi(t) = t$ of Corollary 10 was obtained by Colesanti and Salani [25]. Borell [26] first established the Brunn-Minkowski inequality for the classical electrostatic capacity, while its equality condition was shown by Caffarelli et al. [4].

Theorem 11. *Suppose $\phi \in \Phi$, $1 < q < n$, and $f, g \in C_+(S^{n-1})$. Then, the Orlicz-Brunn-Minkowski inequality for the electrostatic q-capacity implies the Orlicz-Minkowski inequality for the electrostatic q-capacity.*

Proof. For $t \geq 0$ and $f, g \in C_+(S^{n-1})$, define the function $G(t)$ by

$$G(t) = \phi(1) - \phi\left(\left(\frac{C_q(f)}{C_q(f +_\phi t \cdot g)}\right)^{1/(n-q)}\right) - t\phi\left(\left(\frac{C_q(g)}{C_q(f +_\phi t \cdot g)}\right)^{1/(n-q)}\right). \tag{51}$$

The Orlicz-Brunn-Minkowski inequality for the electrostatic q-capacity implies that $G(t)$ is nonnegative. Obviously, $G(0) = 0$. Thus,

$$\lim_{t \to 0^+} \frac{G(t) - G(0)}{t} \geq 0. \tag{52}$$

On the other hand, by (51) and the continuity of C_q, we have

$$\begin{aligned}\lim_{t\to 0^+}\frac{G(t)-G(0)}{t}&=\lim_{t\to 0^+}\frac{\phi(1)-\phi\left(\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}\right)-t\phi\left(\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}\right)}{t}\\&=\lim_{t\to 0^+}\frac{\phi(1)-\phi\left(\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}\right)}{t}-\phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right)\\&=\lim_{t\to 0^+}\frac{\phi(1)-\phi\left(\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}\right)}{1-\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}}\cdot\frac{1-\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}}{t}\\&\quad-\phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right).\end{aligned}\tag{53}$$

Let $s=((C_q(f))/(C_q(f+_\phi t\cdot g)))^{1/(n-q)}$. Note that $s\longrightarrow 1^-$ as $t\longrightarrow 0^+$. Consequently,

$$\lim_{t\to 0^+}\frac{\phi(1)-\phi\left(\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}\right)}{1-\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}}=\phi_l'(1).\tag{54}$$

The continuity of C_q and (27) imply

$$\begin{aligned}&\lim_{t\to 0^+}\frac{1-\left((C_q(f))/(C_q(f+_\phi t\cdot g))\right)^{1/(n-q)}}{t}\\&=\lim_{t\to 0^+}\frac{C_q(f+_\varphi t\cdot g)-(C_q(f))^{1/(n-q)}}{t}\\&\quad\cdot\lim_{t\to 0^+}\left(C_q(f+_\phi t\cdot g)\right)^{-1/(n-q)}\\&=\frac{1}{n-q}\left(C_q(f)\right)^{(1/(n-q))-1}\cdot\lim_{t\to 0^+}\frac{\left(C_q(f+_\phi t\cdot g)\right)^{1/(n-q)}-C_q(f)}{t}\\&\quad\cdot\left(C_q(f)\right)^{-1/(n-q)}=\frac{C_{\phi,q}([f],g)}{\phi_l'(1)C_q(f)}.\end{aligned}\tag{55}$$

From (53), (54), (55), and (52), it follows that

$$\lim_{t\to 0^+}\frac{G(t)-G(0)}{t}=\left(\frac{C_{\phi,q}([f],g)}{C_q(f)}\right)-\phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right)\geq 0,\tag{56}$$

which implies the Orlicz-Minkowski inequality for the electrostatic q-capacity (41).

Finally, we show an immediate application of the Orlicz-Minkowski inequality for the electrostatic q-capacity.

Lemma 12. *Suppose $\phi\in\Phi$ and $1<q<n$. If $f,g\in C_+(S^{n-1})$ and C is a subset of $C_+(S^{n-1})$ such that $f,g\in C$, then the following assertions hold:*

(i) $C_{\phi,q}([h],f)=C_{\phi,q}([h],g)$ for all $h\in C$; then $[f]=[g]$

(ii) $(C_{\phi,q}([f],h))/(C_q(f))=(C_{\phi,q}([g],h))/(C_q(f))$ for all $h\in C$; then $[f]=[g]$

Proof. We first show that (i) holds. Since $C_{\phi,q}([f],f)=\phi(1)C_q(f)$, it follows that $\phi(1)=(C_{\phi,q}([f],g))/(C_q(g))$ by the assumption. By the Orlicz-Minkowski inequality for the electrostatic q-capacity, we have $\phi(1)\geq\phi(((C_q(f))/(C_q(g)))^{1/n-q})$. The monotonicity of ϕ and $1<q<n$ imply that

$$\frac{C_q(f)}{C_q(g)}<1,\tag{57}$$

with equality if and only if $[f]$ and $[g]$ are dilates. This inequality is reversed if interchanging f and g. So,$C_q(f)=C_q(g)$ and $[f]$ and $[g]$ are dilates. Assume that $s[f]=s[g]$ for some $s>0$. The homogeneity of C_q implies $s=1$. Thus, $[f]=[g]$.

Then, we can prove (ii) with the similar arguments in (i).

If the Orlicz mixed electrostatic q-capacity $C_{\phi,q}$ is restricted on convex bodies, then we obtain the following characterizations for identity of convex bodies, which were proved by Hong et al. [13].

Corollary 13 ([10], Theorem 3.3). *Suppose $\phi\in\Phi$ and $1<q<n$. If $K,L\in K_o^n$ and C is a subset of K_o^n such that $K,L\in C$, then the following assertions hold:*

(i) $C_{\phi,q}(Q,K)=C_{\phi,q}(Q,L)$ for all $Q\in C$; then $K=L$

(ii) $(C_{\phi,q}(K,Q))/(C_q(K)) = (C_{\phi,q}(L,Q))/(C_q(L))$ *for all* $Q \in C$*; then* $K = L$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and approved the final manuscript.

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References

[1] R. J. Gardner, "The Brunn-Minkowski inequality," *Bulletin of the American Mathematical Society*, vol. 39, no. 3, pp. 355–406, 2002.

[2] A. Colesanti, K. Nyström, P. Salani, J. Xiao, D. Yang, and G. Zhang, "The Hadamard variational formula and the Minkowski problem for p-capacity," *Advances in Mathematics*, vol. 285, pp. 1511–1588, 2015.

[3] D. Jerison, "A Minkowski problem for electrostatic capacity," *Acta Mathematica*, vol. 176, no. 1, pp. 1–47, 1996.

[4] L. A. Caffarelli, D. Jerison, and E. H. Lieb, "On the case of equality in the Brunn-Minkowski inequality for capacity," *Advances in Mathematics*, vol. 117, no. 2, pp. 193–207, 1996.

[5] L. A. Caffarelli, "Some regularity properties of solutions of Monge Ampère equation," *Communications on Pure and Applied Mathematics*, vol. 44, no. 8-9, pp. 965–969, 1991.

[6] M. Akman, J. Gong, J. Hineman, J. Lewis, and A. Vogel, "The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity," arXiv: 1709.00447v1.

[7] D. Zou and G. Xiong, "The Minkowski problem for the electrostatic p-capacity," *Journal of Differential Geometry*, 2020, arXiv: 1702.08120v1.

[8] G. Xiong, J. Xiong, and L. Xu, "The L capacitary Minkowski problem for polytopes," *Journal of Functional Analysis*, vol. 277, no. 9, pp. 3131–3155, 2019.

[9] Y. Feng, Y. Zhou, and B. He, "The L_p electrostatic q-capacitary Minkowski problem for general measures," *Journal of Mathematical Analysis and Applications*, vol. 487, no. 1, article 123959, 2020.

[10] H. Hong, D. Ye, and N. Zhang, "The p-capacitary Orlicz-Hadamard variational formula and Orlicz-Minkowski problems," *Calculus of Variations and Partial Differential Equations*, vol. 57, no. 1, 2018.

[11] L. Evans and R. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press LLC, Boca Raton, 1992.

[12] V. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[13] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equation*, Springer, New York, 2nd edition, 2011.

[14] W. J. Firey, "p-means of convex bodies," *Mathematica Scandinavica*, vol. 10, pp. 17–24, 1962.

[15] E. Lutwak, "The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem," *Journal of Differential Geometry*, vol. 38, no. 1, pp. 131–150, 1993.

[16] E. Lutwak, "The Brunn-Minkowski-Firey theory II," *Advances in Mathematics*, vol. 118, no. 2, pp. 244–294, 1996.

[17] C. Haberl, E. Lutwak, D. Yang, and G. Zhang, "The even Orlicz Minkowski problem," *Advances in Mathematics*, vol. 224, no. 6, pp. 2485–2510, 2010.

[18] E. Lutwak, D. Yang, and G. Zhang, "Orlicz projection bodies," *Advances in Mathematics*, vol. 223, no. 1, pp. 220–242, 2010.

[19] E. Lutwak, D. Yang, and G. Zhang, "Orlicz centroid bodies," *Journal of Differential Geometry*, vol. 84, no. 2, pp. 365–387, 2010.

[20] R. J. Gardner, D. Hug, and W. Weil, "The Orlicz Brunn-Minkowski theory: a general framework, additions, and inequalities," *Journal of Differential Geometry*, vol. 97, no. 3, pp. 427–476, 2014.

[21] D. Xi, H. Jin, and G. Leng, "The Orlicz Brunn-Minkowski inequality," *Advances in Mathematics*, vol. 260, pp. 350–374, 2014.

[22] R. J. Gardner, *Geometric Tomography*, Cambridge University Press, New York, second edition, 2006.

[23] P. M. Gruber, *Convex and Discrete Geometry*, Springer, Berlin, 2007.

[24] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory, Second Expanded Edition*, Cambridge University Press, Cambridge, 2014.

[25] A. Colesanti and P. Salani, "The Brunn-Minkowski inequality for p-capacity of convex bodies," *Mathematische Annalen*, vol. 327, no. 3, pp. 459–479, 2003.

[26] C. Borell, "Capacitary inequalities of the Brunn-Minkowski type," *Mathematische Annalen*, vol. 263, no. 2, pp. 179–184, 1983.

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Research Article

On the Discrete Orlicz Electrostatic q-Capacitary Minkowski Problem

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We establish the existence of solutions to the Orlicz electrostatic q-capacitary Minkowski problem for polytopes. This contains a new result of the discrete L_p electrostatic q-capacitary Minkowski problem for $p < 0$and $1 < q < n$.

1. Introduction

The Orlicz Brunn-Minkowski theory was originated from the works of Ludwig [1], Ludwig and Reitzner [2], and Lutwak et al. [3, 4]. Hereafter, the new theory has quickly become an important branch of convex geometry (see, e.g., [5–10]). A special case of the theory is the L_p Brunn-Minkowski theory which is credited to Lutwak [11, 12] and attracted increasing interest in recent years (see, e.g., [13–20].

It is well known that the L_1 Brunn-Minkowski theory is the classical Brunn-Minkowski theory. One of the cornerstones of the classical Brunn-Minkowski theory is the Minkowski problem. More than a century ago, Minkowski himself solved the Minkowski problem for discrete measures [21]. The complete solution for arbitrary measures was given by Aleksandrov [22] and Fenchel and Jessen [23]. The regularity was studied by, e.g., Lewy [24], Nirenberg [25], Pogorelov [26], Cheng and Yau [27], and Caffarelli et al. [28].

A generalization of the Minkowski problem is the L_p Minkowski problem in the L_p Brunn-Minkowski theory, which has been extensively studied (see, e.g., [29–49]. Naturally, the corresponding Minkowski problem in the Orlicz Brunn-Minkowski theory is called the Orlicz Minkowski problem which was first investigated by Haberl et al. [50] for even measures. Today, great progress has been made on it (see, e.g., [51–60]). The present paper is aimed at dealing with the Orlicz capacitary Minkowski problem.

The electrostatic q-capacitary measure $\mu_q(\Omega, \cdot)$ (see [61]) of a bounded open convex set Ω in $\mathbb{R}^n$ is the measure on the unit sphere S^{n-1} defined for $\omega \subset S^{n-1}$ and $1 < q < n$ by

$$\mu_q(\Omega, \omega) = \int_{g^{-1}(\omega)} |\nabla U|^q d\mathscr{H}^{n-1}, \tag{1}$$

where $g^{-1} : S^{n-1} \to \partial\Omega$ (the boundary of Ω) denotes the inverse Gauss map, $\mathscr{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure, and U the q-equilibrium potential of Ω.

A convex body K is a compact convex set with nonempty interior in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$, and let $\mathscr{K}^n_o$ denote the set of convex bodies with the origin in their interiors. The support function (see [62, 63]) of $K \in \mathscr{K}^n$ is defined for $u \in S^{n-1}$ by

$$h_K(u) = h(K, u) = \max\{x \cdot u : x \in K\}, \tag{2}$$

where $x \cdot u$ denotes the standard inner product of x and u. Note that $h(cK, u) = ch(K, u)$ for $c > 0$.

Let $\varphi : (0,\infty) \to (0,\infty)$ be a given continuous function. For $1 < q < n$ and $K \in \mathscr{K}^n_o$, the Orlicz electrostatic q

-capacitary measure, $\mu_{\varphi,q}(K,\cdot)$, of K is defined by

$$d\mu_{\varphi,q}(K,\cdot)=\varphi(h_K)d\mu_q(K,\cdot). \tag{3}$$

When $\varphi(s)=s^{1-p}$ with $p\in\mathbb{R}$, the Orlicz electrostatic q-capacitary measure becomes the following L_p electrostatic q-capacitary measure introduced by Zou and Xiong [64]:

$$d\mu_{p,q}(K,\cdot)=h_K^{1-p}d\mu_q(K,\cdot). \tag{4}$$

The Minkowski problem characterizing the Orlicz electrostatic q-capacitary measure, proposed in [65], is the following.

1.1. The Orlicz Electrostatic q-Capacitary Minkowski Problem. Let $1<q<n$. Given a continuous function $\varphi:(0,\infty)\to(0,\infty)$ and a finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions so that $\mu=c\mu_{\varphi,q}(K,\cdot)$ for some convex body K and constant $c>0$?

Let φ be a constant function. When $q=2$, the Orlicz Minkowski-type problem is the classical electrostatic capacitary Minkowski problem. In the paper [66], Jerison established the existence of a solution to the electrostatic capacitary Minkowski problem. In a subsequent paper [67], he gave a new proof of the existence using a variational approach. The uniqueness was proved by Caffarelli et al. [68], and the regularity was given in [66]. When $1<q<n$, the Orlicz Minkowski-type problem is the electrostatic q-capacitary Minkowski problem posed in [61]. The existence and regularity for $1<q<2$ and the uniqueness for $1<q<n$ of its solutions were proved in [61], and the existence for $2<q<n$ was very recently solved in [69].

Let $\varphi(s)=s^{1-p}$ with $p\in\mathbb{R}$. Then, the Orlicz Minkowski-type problem is the L_p electrostatic q-capacitary Minkowski problem introduced by Zou and Xiong [64]. In [64], they completely solved the L_p electrostatic q-capacitary Minkowski problem for the case $p>1$ and $1<q<n$. It is generally known that when $p<1$, the L_p Minkowski problem becomes much harder. Actually, the L_p electrostatic q-capacitary Minkowski problem for the case $p<1$ and $1<q<n$ is also very difficult. Therefore, it is worth mentioning that an important breakthrough of the problem for the case $0<p<1$ and $1<q<2$ was made by Xiong et al. [70] for discrete measures.

The existence of the Orlicz electrostatic q-capacitary Minkowski problem was first investigated by Hong et al. [65]. As a consequence, in [65], they obtained a complete solution (including both existence and uniqueness) to the L_p electrostatic q-capacitary Minkowski problem for the case $p>1$ and $1<q<n$, which was independently solved by Zou and Xiong [64].

We observe the statement above. At present, there is no result about the L_p electrostatic q-capacitary Minkowski problem for the case $p<0$ and $1<q<n$. In this paper, we study the Orlicz electrostatic q-capacitary Minkowski problem including it.

A finite set E of S^{n-1} is said to be in general position if E is not contained in a closed hemisphere of S^{n-1} and any n elements of E are linearly independent.

A polytope in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided that it has positive n-dimensional volume. The convex hull of a subset of these points is called a facet of the polytope if it lies entirely on the boundary of the polytope and has positive $(n-1)$-dimensional volume.

Our main theorem is stated as follows.

Theorem 1. *Suppose $\varphi:(0,\infty)\to(0,\infty)$ is continuously differentiable and strictly increasing with $\varphi(s)\to\infty$ as $s\to\infty$ such that $\phi(t)=\int_t^\infty(1/\varphi(s))ds$ exists for $t>0$ and $\lim_{t\to 0^+}\phi(t)=\infty$. Let $\mu=\sum_{i=1}^N\alpha_i\delta_{u_i}$, where $\alpha_1,\cdots,\alpha_N>0$, the unit vectors $u_1,\cdots,u_N\in S^{n-1}$ are in general position, and δ_{u_i} is the Dirac delta. Then, for $1<q<n$, there exist a polytope P and constant $c>0$ such that*

$$\mu=c\mu_{\varphi,q}(P,\cdot). \tag{5}$$

When $\varphi(s)=s^{1-p}$ with $p<0$, and $\phi(t)=-(1/p)t^p$, which satisfy the assumptions of Theorem 1, we obtain the following.

Corollary 2. *Let $p<0$ and $1<q<n$. Suppose μ is a discrete measure on S^{n-1}, and its supports are in general position. If $p+q\neq n$, then there exists a polytope P_0 such that $\mu=\mu_{p,q}(P_0,\cdot)$; if $p+q=n$, then there exist a polytope P and constant $c>0$ such that $\mu=c\mu_{p,q}(P,\cdot)$.*

Obviously, this corollary makes up for the existing results for the L_p electrostatic q-capacitary Minkowski problem, to some extent.

The rest of this paper is organized as follows. In Section 2, some of the necessary facts about convex bodies and capacity are presented. In Section 3, a maximizing problem related to the Orlicz electrostatic q-capacitary Minkowski problem is considered and its corresponding solution is given. In Section 4, we give the proofs of Theorem 1 and Corollary 2.

2. Preliminaries

2.1. Basics regarding Convex Bodies. For quick later reference, we list some basic facts about convex bodies. Good general references are the books of Gardner [62] and Schneider [63].

The boundary and interior of $K\in\mathscr{K}^n$ will be denoted by ∂K and int K, respectively. $B=\{x\in\mathbb{R}^n:\sqrt{x\cdot x}\le 1\}$ denotes the unit ball. The volume, the n-dimensional Lebesgue measure, of a convex body $K\in\mathrm{K}^n$ is denoted by $V(K)$, and the volume of B is denoted by ω_n. We will write $C(S^{n-1})$ for the set of continuous functions on S^{n-1} and $C^+(S^{n-1})$ for the set of positive functions in $C(S^{n-1})$.

For $x\in\partial K$ with $K\in\mathrm{K}^n$, $g_K(x)$ is the Gauss map of K which is the family of all unit exterior normal vectors at x. In particular, $g_K(x)$ consists of a unique vector for H^{n-1}

-almost all $x \in \partial K$. The surface area measure of K is a Borel measure on S^{n-1} defined for a Borel set $\omega \subset S^{n-1}$ by

$$S(K,\omega) = \int_{x \in g_K^{-1}(\omega)} d\mathrm{H}^{n-1}(x). \tag{6}$$

For $f \in C^+(S^{n-1})$, the Aleksandrov body associated with f, denoted by $[f]$, is the convex body defined by

$$[f] = \bigcap_{u \in S^{n-1}} \{\xi \in \mathbb{R}^n : \xi \cdot u \le f(u)\}. \tag{7}$$

It is easy to see that $h_{[f]} \le f$ and $[h_K] = K$ for $K \in \mathrm{K}_o^n$.

The Hausdorff distance of two convex bodies $K, L \in \mathrm{K}^n$ is defined by

$$\delta(K,L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|. \tag{8}$$

For a sequence of convex bodies $K_i \in \mathrm{K}^n$ and a convex body $K \in \mathrm{K}^n$, we have $\lim_{i\to\infty} K_i = K$ provided that

$$\delta(K_i, K) \to 0, \tag{9}$$

as $i \to \infty$.

For $K \in \mathrm{K}^n$ and $u \in S^{n-1}$, the support hyperplane $H(K,u)$ of K at u is defined by

$$H(K,u) = \{x \in \mathbb{R}^n : x \cdot u = h(K,u)\}, \tag{10}$$

the half-space $H^-(K,u)$ at u is defined by

$$H^-(K,u) = \{x \in \mathbb{R}^n : x \cdot u \le h(K,u)\}, \tag{11}$$

and the support set $F(K,u)$ at u is defined by

$$F(K,u) = K \cap H(K,u). \tag{12}$$

Suppose that P is the set of polytopes in $\mathbb{R}^n$ and the unit vectors $u_1, \cdots, u_N$ are in general position. Let $\mathrm{P}(u_1, \cdots, u_N)$ be the subset of P. If $P \in \mathrm{P}$ with

$$P = \bigcap_{k=1}^{N} H^-(P, u_k), \tag{13}$$

then $P \in \mathrm{P}(u_1, \cdots, u_N)$. Obviously, if $P_i \in \mathrm{P}(u_1, \cdots, u_N)$ and P_i converges to a polytope P, then $P \in \mathrm{P}(u_1, \cdots, u_N)$. Let $\mathrm{P}_N(u_1, \cdots, u_N)$ be the subset of $\mathrm{P}(u_1, \cdots, u_N)$ that any polytope in $\mathrm{P}_N(u_1, \cdots, u_N)$ has exactly N facets.

2.2. Electrostatic q-Capacity and q-Capacitary Measure. Here, we collect some notion and basic facts on electrostatic q-capacity and q-capacitary measure (see [61, 64, 70]).

Let E be a compact set in n-dimensional Euclidean space $\mathbb{R}^n$. For $1 < q < n$, the electrostatic q-capacity, $C_q(E)$, of E is defined (see [61]) by

$$C_q(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^q dx : u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \ge 1 \text{ on } E \right\}, \tag{14}$$

where $C_c^\infty(\mathbb{R}^n)$ is the set of smooth functions with compact supports. When $q = 2$, the electrostatic q-capacity becomes the classical electrostatic capacity $C_2(E)$.

For $K \in \mathrm{K}^n$ and $1 < q < n$, we need the following isocapacitary inequality which is due to Maźya [71]:

$$V(K)^{(n-q)/n} \le \left(\frac{q-1}{n-q}\right)^{q-1} \left(n\omega_n^{q/n}\right)^{-1} C_q(K). \tag{15}$$

The following lemma (see [64, 70]) gives some basic properties of the electrostatic q-capacity.

Lemma 3. *Let E and F be two compact sets in $\mathbb{R}^n$ and $1 < q < n$.*

(i) If $E \subset F$, then

$$C_q(E) \le C_q(F) \tag{16}$$

(ii) For $\lambda > 0$,

$$C_q(\lambda E) = \lambda^{n-q} C_q(E) \tag{17}$$

(iii) For $x_0 \in \mathbb{R}^n$,

$$C_q(E + x_0) = C_q(E) \tag{18}$$

(iv) The functional $C_q(\cdot)$ is continuous on K^n with respect to the Hausdorff metric

The following lemma is some basic properties of the electrostatic q-capacitary measure (compare [64, 70]).

Lemma 4. *Let $K \in K^n$ and $1 < q < n$.*

(i) For $\lambda > 0$,

$$\mu_q(\lambda K, \cdot) = \lambda^{n-q-1} \mu_q(K, \cdot) \tag{19}$$

(ii) For $x_0 \in \mathbb{R}^n$,

$$\mu_q(K+x_0,\cdot\,)=\mu_q(K,\cdot\,) \tag{20}$$

(iii) For $K_j, K \in \mathscr{K}^n$, if $K_j \to K$, then

$$\mu_q\left(K_j,\cdot\,\right)\to\mu_q(K,\cdot\,) \tag{21}$$

weakly as $j \to +\infty$

(iv) The measure $\mu_q(K,\cdot)$ is absolutely continuous with respect to the surface area measure $S(K,\cdot)$

The following variational formula given in [61] of electrostatic q-capacity is critical.

Lemma 5. *Let $I \subset \mathbb{R}$ be an interval containing 0 in its interior, and let $h_t(u)=h(t,u)$: $I\times S^{n-1}\to(0,\infty)$ be continuous such that the convergence in*

$$h'(0,u)=\lim_{t\to 0}\frac{h(t,u)-h(0,u)}{t} \tag{22}$$

is uniform on S^{n-1}. Then,

$$\left.\frac{dC_q([h_t])}{dt}\right|_{t=0}=(q-1)\int_{S^{n-1}}h'(0,u)d\mu_q([h_0],u). \tag{23}$$

3. An Associated Maximization Problem

In this section, we solve a maximization problem, and its solution is exactly the solution in Theorem 1.

Suppose ϕ satisfies the assumptions of Theorem 1 and the unit vectors $u_1,\cdots,u_N$ are in general position. For $\alpha_1,\cdots,\alpha_N>0$ and $P\in \mathrm{P}(u_1,\cdots,u_N)$, define the function, $\Phi_P:\operatorname{int}P\to\mathbb{R}$, by

$$\Phi_P(\xi)=\sum_{k=1}^{N}\alpha_k\phi(h(P,u_k)-\xi\cdot u_k). \tag{24}$$

Let $1<q<n$. We consider the following maximization problem:

$$\sup\left\{\min_{\xi\in\operatorname{int}Q}\Phi_Q(\xi)\colon C_q(Q)=1,Q\in \mathrm{P}(u_1,\cdots,u_N)\right\}. \tag{25}$$

The solution to problem (25) is given in Theorem 9. Its proof requires the following three lemmas which are similar to those in [58].

Lemma 6. *Suppose $\varphi:(0,\infty)\to(0,\infty)$ is continuously differentiable and strictly increasing with $\varphi(s)\to\infty$ as $s\to\infty$ such that $\phi(t)=\int_t^\infty(1/\varphi(s))ds$ exists for $t>0$ and $\lim_{t\to0}\phi(t)=\infty$. For $\alpha_1,\cdots,\alpha_N>0$, if the unit vectors $u_1,\cdots,u_N\in S^{n-1}$ are in general position, then there exists a unique $\xi_\phi(P)\in\operatorname{int}P$ such that*

$$\Phi_P\left(\xi_\phi(P)\right)=\min_{\xi\in\operatorname{int}P}\Phi_P(\xi). \tag{26}$$

Proof. Since $\varphi:(0,\infty)\to(0,\infty)$ is continuously differentiable and strictly increasing, we have for $t>0$,

$$\phi''(t)=\frac{\varphi'(t)}{\varphi^2(t)}>0. \tag{27}$$

Therefore, ϕ is strictly convex on $(0,\infty)$.
Let $0<\lambda<1$ and $\xi_1,\xi_2\in\operatorname{int}P$. Then,

$$\begin{aligned}\lambda\Phi_P(\xi_1)+(1-\lambda)\Phi_P(\xi_2)&=\lambda\sum_{k=1}^{N}\alpha_k\phi(h(P,u_k)-\xi_1\cdot u_k)\\&\quad+(1-\lambda)\sum_{k=1}^{N}\alpha_k\phi(h(P,u_k)-\xi_2\cdot u_k)\\&=\sum_{k=1}^{N}\alpha_k[\lambda\phi(h(P,u_k)-\xi_1\cdot u_k)\\&\quad+(1-\lambda)\phi(h(P,u_k)-\xi_2\cdot u_k)]\geq\sum_{k=1}^{N}\alpha_k\phi(h(P,u_k)\\&\quad-(\lambda\xi_1+(1-\lambda)\xi_2)\cdot u_k)=\Phi_P(\lambda\xi_1+(1-\lambda)\xi_2).\end{aligned} \tag{28}$$

Equality holds if and only if $\xi_1\cdot u_k=\xi_2\cdot u_k$ for all $k=1,\cdots,N$. Since $u_1,\cdots,u_N$ are in general position, $\mathbb{R}^n=\operatorname{lin}\{u_1,\cdots,u_N\}$ which is the smallest linear subspace of $\mathbb{R}^n$ containing $\{u_1,\cdots,u_N\}$. Thus, $\xi_1=\xi_2$. Namely, Φ_P is strictly convex on int P.

Since $P\in\mathrm{P}(u_1,\cdots,u_N)$, it follows that for any $x\in\partial P$, there exists a $u_{i_0}\in\{u_1,\cdots,u_N\}$ such that

$$h\left(P,u_{i_0}\right)=x\cdot u_{i_0}. \tag{29}$$

Note that ϕ is strictly decreasing on $(0,\infty)$ and $\lim_{t\to0}\phi(t)=\infty$. Then, $\Phi_P(\xi)\to\infty$ whenever $\xi\in\operatorname{int}P$ and $\xi\to x$. This together with the strict convexity of Φ_p means that there exists a unique interior point $\xi_\phi(P)$ of P such that

$$\Phi_P\left(\xi_\phi(P)\right)=\min_{\xi\in\operatorname{int}P}\Phi_P(\xi). \tag{30}$$

Lemma 7. *Suppose $\alpha_1,\cdots,\alpha_N>0$, the unit vectors $u_1,\cdots,u_N\in S^{n-1}$ are in general position, and $\varphi:(0,\infty)\to(0,\infty)$ is continuously differentiable and strictly increasing with $\varphi(s)\to\infty$ as $s\to\infty$ such that $\phi(t)=\int_t^\infty(1/\varphi(s))ds$ exists for $t>0$ and $\lim_{t\to0}\phi(t)=\infty$. If $P_i\in P(u_1,\cdots,u_N)$ converges to a polytope P, then $\lim_{i\to\infty}\xi_\phi(P_i)=\xi_\phi(P)$ and*

$$\lim_{i\to\infty}\Phi_{P_i}\left(\xi_\phi(P_i)\right)=\Phi_P\left(\xi_\phi(P)\right). \tag{31}$$

Proof. Since $P_i\to P$ and $\xi_\phi(P_i)\in\operatorname{int}P_i$, it follows that $\xi_\phi(P_i)$ is bounded. Let $\xi_\phi(P_{i_j})$ be a subsequence of $\xi_\phi(P_i)$ with $\lim_{j\to\infty}\xi_\phi(P_{i_j})=\xi_0$. We first show that $\xi_0\in\operatorname{int}P$ by contradiction.

Assume $\xi_0 \in \partial P$. Then, $\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) = \infty$, which contradicts the fact that

$$\lim_{j\to\infty} \Phi_{P_{i_j}}\left(\xi_\phi\left(P_{i_j}\right)\right) \le \lim_{j\to\infty} \Phi_{P_{i_j}}\left(\xi_\phi(P)\right) = \Phi_P\left(\xi_\phi(P)\right) < \infty. \tag{32}$$

We next show that $\xi_0 = \xi_\phi(P)$. Let $\xi_0 \neq \xi_\phi(P)$. Then,

$$\lim_{j\to\infty} \Phi_{P_{i_j}}\left(\xi_\phi\left(P_{i_j}\right)\right) = \Phi_P(\xi_0) > \Phi_P\left(\xi_\phi(P)\right) = \lim_{j\to\infty} \Phi_{P_{i_j}}\left(\xi_\phi(P)\right). \tag{33}$$

This contradicts the fact that

$$\lim_{j\to\infty} \Phi_{P_{i_j}}\left(\xi_\phi\left(P_{i_j}\right)\right) \le \lim_{j\to\infty} \Phi_{P_{i_j}}\left(\xi_\phi(P)\right). \tag{34}$$

This means that $\lim_{i\to\infty} \xi_\phi(P_i) = \xi_\phi(P)$ and

$$\lim_{i\to\infty} \Phi_{P_i}\left(\xi_\phi(P_i)\right) = \Phi_P\left(\xi_\phi(P)\right). \tag{35}$$

Lemma 8. *Suppose $\alpha_1, \cdots, \alpha_N > 0$, the unit vectors $u_1, \cdots, u_N \in S^{n-1}$ are in general position, and $\varphi : (0,\infty) \to (0,\infty)$ is continuously differentiable and strictly increasing with $\varphi(s) \to \infty$ as $s \to \infty$ such that $\phi(t) = \int_t^\infty (1/\varphi(s))ds$ exists for $t > 0$ and $\lim_{t\to 0}\phi(t) = \infty$. Let $P \in P(u_1, \cdots, u_N)$ and $\delta \ge 0$ be small enough such that for $k_0 \in \{1, \cdots, N\}$,*

$$P_\delta = P \cap \left\{x : x \cdot u_{k_0} \le h\left(P, u_{k_0}\right) - \delta\right\} \in P(u_1, \cdots, u_N). \tag{36}$$

If the continuous function $\lambda : [0,\infty) \to (0,\infty)$ is continuously differentiable on $(0,\infty)$ and $\lim_{\delta\to 0}\lambda'(\delta)$ exists, then $\xi(\delta) := \xi_\phi(\lambda(\delta)P_\delta)$ has a right derivative, denoted by $\xi'_+(0)$, at 0.

Proof. The proof is based on the ideas of Wu et al. [58]. Let $\delta \ge 0$ be small enough and

$$\begin{aligned}\Phi(\delta) &= \min_{\xi \in \text{int }(\lambda(\delta)P_\delta)} \sum_{k=1}^N \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k) \\ &= \sum_{k=1}^N \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k).\end{aligned} \tag{37}$$

From this and the fact that $\xi(\delta)$ is an interior point of $\lambda(\delta)P_\delta$, it follows that for $i = 1, \cdots, n$,

$$\sum_{k=1}^N \alpha_k \phi'(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)u_{k,i} = 0, \tag{38}$$

where $u_k = (u_{k,1}, \cdots, u_{k,n})^T$.

Let

$$F_i(\delta, \xi_1, \cdots, \xi_n) = \sum_{k=1}^N \alpha_k \phi'(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k)u_{k,i} \tag{39}$$

for $i = 1, \cdots, n$, where $\xi = (\xi_1, \cdots, \xi_n)$. Then,

$$\frac{\partial F_i}{\partial \xi_j} = -\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k)u_{k,i}u_{k,j}, \tag{40}$$

$$\begin{aligned}\frac{\partial F_i}{\partial \delta} = &\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k)u_{k,i}\lambda'(\delta)h(P_\delta, u_k) \\ &- \alpha_{k_0}\phi''\left(\lambda(\delta)h\left(P, u_{k_0}\right) - \delta\lambda(\delta) - \xi \cdot u_{k_0}\right)u_{k_0,i}\lambda(\delta).\end{aligned} \tag{41}$$

Let $F = (F_1, \cdots, F_n)$. Then,

$$\left(\left.\frac{\partial F}{\partial \xi}\right|_{(\delta, \xi_1(\delta), \cdots, \xi_n(\delta))}\right)_{n\times n} = -\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)u_k u_k^T, \tag{42}$$

where $u_k u_k^T$ is an $n \times n$ matrix.

Since $u_1, \cdots, u_N$ are in general position, $\mathbb{R}^n = \text{lin}\{u_1, \cdots, u_N\}$. Thus, for any $x \in \mathbb{R}^n$ with $x \neq 0$, there exists a $u_{i_0} \in \{u_1, \cdots, u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Note that $\phi'' > 0$. Then, we have

$$\begin{aligned}&x^T\left(-\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)u_k u_k^T\right)x \\ &= -\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)(x \cdot u_k)^2 \\ &\le -\alpha_{i_0}\phi''\left(\lambda(\delta)h\left(P_\delta, u_{i_0}\right) - \xi(\delta) \cdot u_{i_0}\right)\left(x \cdot u_{i_0}\right)^2 < 0.\end{aligned} \tag{43}$$

Therefore, $(\partial F/\partial \xi|_{\delta, \xi_1(\delta), \cdots, \xi_n(\delta)})$ is negative definite. Thus,

$$\det\left(\left.\frac{\partial F}{\partial \xi}\right|_{\delta, \xi_1(\delta), \cdots, \xi_n(\delta)}\right) \neq 0. \tag{44}$$

From this, the fact that for $i = 1, \cdots, n$, $F_i(\delta, \xi_1(\delta), \cdots, \xi_n(\delta)) = 0$ follows by (38), the fact that $\partial F_i/\partial \xi_j$ is continuous on ξ and δ for all $1 \le i, j \le n$, and the implicit function theorem, it follows that $\xi(\delta) = \xi_\phi(\lambda(\delta)P_\delta)$ is continuously differentiable on a neighbourhood of δ small enough. Thus, $\xi(\delta)$ is continuously differentiable for small enough $\delta > 0$, and

$$\begin{pmatrix} \frac{d\xi_1}{d\delta} \\ \frac{d\xi_2}{d\delta} \\ \vdots \\ \frac{d\xi_n}{d\delta} \end{pmatrix} = -\begin{pmatrix} \frac{\partial F_1}{\partial \xi_1} & \frac{\partial F_1}{\partial \xi_2} & \cdots & \frac{\partial F_1}{\partial \xi_n} \\ \frac{\partial F_2}{\partial \xi_1} & \frac{\partial F_2}{\partial \xi_2} & \cdots & \frac{\partial F_2}{\partial \xi_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F_n}{\partial \xi_1} & \frac{\partial F_n}{\partial \xi_2} & \cdots & \frac{\partial F_n}{\partial \xi_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial \delta} \\ \frac{\partial F_2}{\partial \delta} \\ \vdots \\ \frac{\partial F_n}{\partial \delta} \end{pmatrix}. \tag{45}$$

This together with (40) and (41) implies $\lim_{\delta\to 0^+} \xi'(\delta)$ exists.

From the Lagrange mean value theorem, we obtain that for $\delta > 0$ and $i = 1, \cdots, n$, there exists a $\delta_i(\delta)$ with $0 < \delta_i(\delta) < \delta$ such that

$$\frac{\xi_i(\delta) - \xi_i(0)}{\delta} = \xi_i'(\delta_i(\delta)). \tag{46}$$

Thus,

$$\left.\frac{d\xi_i(\delta)}{d\delta}\right|_{\delta=0^+} = \lim_{\delta\to 0^+} \frac{\xi_i(\delta) - \xi_i(0)}{\delta} = \lim_{\delta\to 0^+} \xi_i'(\delta_i(\delta)) \tag{47}$$

exists. Namely, $\xi_+'(0)$ exists.

We are ready to show the existence of a maximizer to problem (25).

Theorem 9. *Suppose $\alpha_1, \cdots, \alpha_N > 0$ and the unit vectors $u_1, \cdots, u_N \in S^{n-1}$ are in general position. Let $\varphi : (0,\infty) \to (0,\infty)$ be continuously differentiable and strictly increasing with $\varphi(s) \to \infty$ as $s \to \infty$ such that $\phi(t) = \int_t^\infty (1/\varphi(s))ds$ exists for $t > 0$ and $\lim_{t\to 0} \phi(t) = \infty$. Then, there exists a polytope $P \in P_N(u_1, \cdots, u_N)$ such that $\xi_\phi(P) = o$, $C_q(P) = 1$, and*

$$\Phi_P(o) = \sup \left\{ \min_{\xi\in\mathrm{int}\, Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in P(u_1, \cdots, u_N) \right\}. \tag{48}$$

Proof. For $x \in \mathbb{R}^n$ and $P \in \mathrm{P}(u_1, \cdots, u_N)$, we first show

$$\Phi_{P+x}\left(\xi_\phi(P+x)\right) = \Phi_P\left(\xi_\phi(P)\right). \tag{49}$$

From Lemma 6 and definition (24), we have

$$\begin{aligned} \Phi_{P+x}\left(\xi_\phi(P+x)\right) &= \min_{\xi\in\mathrm{int}\,(P+x)} \Phi_{P+x}(\xi) \\ &= \min_{\xi\in\mathrm{int}\,(P+x)} \sum_{k=1}^N \alpha_k \phi(h(P+x, u_k) - \xi \cdot u_k) \\ &= \min_{\xi-x\in\mathrm{int}\, P} \sum_{k=1}^N \alpha_k \phi(h(P, u_k) - (\xi - x) \cdot u_k) \\ &= \min_{\xi'\in\mathrm{int}\, P} \sum_{k=1}^N \alpha_k \phi\left(h(P, u_k) - \xi' \cdot u_k\right) \\ &= \Phi_P\left(\xi_\phi(P)\right). \end{aligned} \tag{50}$$

Therefore, by (49) and (iii) of Lemma 2.1, we can choose a sequence $P_i \in \mathrm{P}(u_1, \cdots, u_N)$ with $\xi_\phi(P_i) = o$ and $C_q(P_i) = 1$ such that

$$\lim_{i\to\infty} \Phi_{P_i}(o) = \sup \left\{ \min_{\xi\in\mathrm{int}\, Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in \mathrm{P}(u_1, \cdots, u_N) \right\}. \tag{51}$$

We next prove that P_i is bounded. Assume that P_i is not bounded. Since the unit vectors $u_1, \cdots, u_N$ are in general position, from the proof of ([45], Theorem 4.3), we see $V(P_i)$ is not bounded. However, from (15), and noting that $C_q(P_i) = 1$, we have

$$V(P_i) \le \left(\frac{q-1}{n-q}\right)^{n(q-1)/(n-q)} \left(n\omega_n^{q/n}\right)^{-n/(n-q)}, \tag{52}$$

which is a contradiction. Therefore, P_i is bounded.

By the Blaschke selection theorem, we can assume that a subsequence of P_i converges to a polytope $P \in \mathrm{P}(u_1, \cdots, u_N)$. Thus, from (iv) of Lemma 3 and Lemma 7, it follows that $C_q(P) = 1$, $\xi_\phi(P) = o$, and

$$\Phi_P(o) = \sup \left\{ \min_{\xi\in\mathrm{int}\, Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in \mathrm{P}(u_1, \cdots, u_N) \right\}. \tag{53}$$

We now prove that $P \in \mathrm{P}_N(u_1, \cdots, u_N)$, i.e., $F(P, u_i)$ are facets for all $i = 1, \cdots, N$. If not, there exists an $i_0 \in \{1, \cdots, N\}$ such that $F(P, u_{i_0})$ is not a facet of P. Choose $\delta \ge 0$ small enough so that the polytope

$$P_\delta = P \cap \left\{ x : x \cdot u_{i_0} \le h\left(P, u_{i_0}\right) - \delta \right\} \in \mathrm{P}(u_1, \cdots, u_N). \tag{54}$$

Let $\lambda(\delta) = C_q(P_\delta)^{-(1/(n-q))}$. Then, $\lambda(\delta)P_\delta \in \mathrm{P}(u_1, \cdots, u_N)$, $C_q(\lambda(\delta)P_\delta) = 1$ follows by (ii) of Lemma 3, and $\lambda(\delta)$ is continuous in $[0, \infty)$. Since for any $\delta_i \to 0$, there is that $\lambda(\delta_i) P_{\delta_i} \to P$, it follows from Lemma 7 that $\xi_\phi(\lambda(\delta_i)P_{\delta_i}) \to \xi_\phi(P) = o$. This implies

$$\lim_{\delta\to 0} \xi_\phi(\lambda(\delta)P_\delta) = o. \tag{55}$$

Let

$$r_{i_0}(u) = \begin{cases} 1, & u = u_{i_0}, \\ 0, & u \ne u_{i_0} \end{cases} \tag{56}$$

for $u \in S^{n-1}$. Then, from Lemma 5, we have for small enough $\delta \ge 0$,

$$\begin{aligned} \frac{dC_q(P_\delta)}{d\delta} &= \frac{dC_q\left(\left[h_{P_\delta}\right]\right)}{d\delta} = \lim_{t\to 0} \frac{C_q\left(\left[h_{P_\delta} + t r_{i_0}\right]\right) - C_q\left(\left[h_{P_\delta}\right]\right)}{t} \\ &= (q-1)\int_{S^{n-1}} r_{i_0}(u) d\mu_q(P_\delta, u) = (q-1)\mu_q\left(P_\delta, u_{i_0}\right). \end{aligned} \tag{57}$$

Thus, from (iii) and (iv) of Lemma 4, it follows that $C_q(P_\delta)$ is continuously differentiable for every $\delta > 0$, and

$$\lim_{\delta\to 0^+} \frac{dC_q(P_\delta)}{d\delta} = 0. \tag{58}$$

These imply that

$$\lambda'(\delta) = -\frac{1}{n-q} C_q(P_\delta)^{-(1/(n-q))-1} \frac{dC_q(P_\delta)}{d\delta} \tag{59}$$

is continuous for every $\delta > 0$, and

$$\lim_{\delta\to 0^+} \lambda'(\delta) = 0. \tag{60}$$

Therefore, $\lambda(\delta) = C_q(P_\delta)^{-(1/n-q)}$ satisfies the conditions of Lemma 8. Noting that $\xi(\delta) = \xi_\phi(\lambda(\delta)P_\delta)$, we see $\xi'_+(0)$ exists.

Recall

$$\Phi(\delta) = \sum_{k=1}^{N} \alpha_k \phi(\lambda(\delta) h(P_\delta, u_k) - \xi(\delta)\cdot u_k). \tag{61}$$

From this and (38), we have

$$\sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) u_k = 0. \tag{62}$$

Thus, it follows from (60), (61), and (62) that

$$\begin{aligned} \left.\frac{d\Phi(\delta)}{d\delta}\right|_{\delta=0^+} &= -\alpha_{i_0}\phi'\left(h\left(P, u_{i_0}\right)\right) \\ &\quad - \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k))\left(\xi'_+(0)\cdot u_k\right) \\ &= -\alpha_{i_0}\phi'\left(h\left(P, u_{i_0}\right)\right) - \xi'_+(0)\cdot \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) u_k \\ &= -\alpha_{i_0}\phi'\left(h\left(P, u_{i_0}\right)\right) > 0. \end{aligned} \tag{63}$$

This means

$$\lim_{\delta\to 0^+} \frac{\Phi(\delta) - \Phi(0)}{\delta} > 0. \tag{64}$$

Therefore, there exists a $\delta_0 > 0$ small enough such that

$$\Phi(\delta_0) > \Phi(0). \tag{65}$$

This together with (61) has

$$\Phi_{\lambda(\delta_0)P_{\delta_0}}\left(\xi_\phi\left(\lambda(\delta_0)P_{\delta_0}\right)\right) > \Phi_P\left(\xi_\phi(P)\right) = \Phi_P(o). \tag{66}$$

Note that $\lambda(\delta_0) = C_q(P_{\delta_0})^{-(1/(n-q))}$. Let $P_0 = \lambda(\delta_0)P_{\delta_0} - \xi_\phi(\lambda(\delta_0)P_{\delta_0})$. Then, $P_0 \in \mathrm{P}(u_1, \cdots, u_N)$, $C_q(P_0) = 1$, $\xi_\phi(P_0) = o$, and

$$\Phi_{P_0}(o) > \Phi_P(o). \tag{67}$$

This contradicts (53). Thus, $P \in \mathrm{P}_N(u_1, \cdots, u_N)$.

4. Solving the Orlicz Electrostatic q-Capacitary Minkowski Problem

Proof of Theorem 1. By Theorem 9, there exists a polytope $P \in \mathrm{P}_N(u_1, \cdots, u_N)$ with $\xi_\phi(P) = o$ and $C_q(P) = 1$ such that

$$\Phi_P(o) = \sup\left\{\min_{\xi\in \mathrm{int}\, Q} \Phi_Q(\xi) \colon C_q(Q) = 1, Q \in \mathrm{P}(u_1, \cdots, u_N)\right\}. \tag{68}$$

For $\gamma_1, \cdots, \gamma_N \in \mathbb{R}$, choose $|t|$ small enough so that the polytope P_t defined by

$$P_t = \bigcap_{i=1}^{N} \{x : x\cdot u_i \le h(P, u_i) + t\gamma_i\} \tag{69}$$

has exactly N facets. Then, $h(P_t, u_i) = h(P, u_i) + t\gamma_i$ for $i = 1, \cdots, N$.Let

$$\beta(t) = C_q(P_t)^{-(1/n-q)}. \tag{70}$$

Then, $\beta(t)P_t \in \mathrm{P}_N(u_1, \cdots, u_N)$ and $C_q(\beta(t)P_t) = 1$. By Lemma 5 and (iv) of Lemma 4, we obtain

$$\beta'(0) = -\frac{1}{n-q}\left.\frac{dC_q(P_t)}{dt}\right|_{t=0} = -\frac{q-1}{n-q}\sum_{i=1}^{N} \gamma_i \mu_q(P, u_i). \tag{71}$$

Define $\xi(t) = \xi_\phi(\beta(t)P_t)$ and

$$\begin{aligned} \Phi(t) &= \min_{\xi\in \mathrm{int}\,(\beta(t)P_t)} \sum_{k=1}^{N} \alpha_k \phi(\beta(t) h(P_t, u_k) - \xi\cdot u_k) \\ &= \sum_{k=1}^{N} \alpha_k \phi(\beta(t) h(P_t, u_k) - \xi(t)\cdot u_k). \end{aligned} \tag{72}$$

Since $\xi(t)$ is an interior point of $\beta(t)P_t$, this has

$$\sum_{k=1}^{N} \alpha_k \phi'(\beta(t) h(P_t, u_k) - \xi(t)\cdot u_k) u_{k,i} = 0, \tag{73}$$

for $i = 1, \cdots, n$, where $u_k = (u_{k,1}, \cdots, u_{k,n})^T$. Note that $\xi(0)$ is the origin. Then, letting $t = 0$ in (73), we have

$$\sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) u_{k,i} = 0, \tag{74}$$

for $i=1,\cdots,n$. Hence,

$$\sum_{k=1}^{N}\alpha_k\phi'(h(P,u_k))u_k=0. \tag{75}$$

Let

$$F_i(t,\xi_1,\cdots,\xi_n)=\sum_{k=1}^{N}\alpha_k\phi'(\beta(t)h(P_t,u_k)-(\xi_1u_{k,1}+\cdots+\xi_nu_{k,n}))u_{k,i} \tag{76}$$

for $i=1,\cdots,n$. Then,

$$\begin{aligned}\frac{\partial F_i}{\partial t}\bigg|_{(t,\xi_1,\cdots,\xi_n)}&=\sum_{k=1}^{N}\alpha_k\phi''(\beta(t)h(P_t,u_k)\\&\quad-(\xi_1u_{k,1}+\cdots+\xi_nu_{k,n}))\left[\beta'(t)h(P_t,u_k)+\beta(t)\gamma_k\right]u_{k,i},\ \frac{\partial F_i}{\partial\xi_j}\Big|_{(t,\xi_1,\cdots,\xi_n)}\\&=-\sum_{k=1}^{N}\alpha_k\phi''(\beta(t)h(P_t,u_k)-(\xi_1u_{k,1}+\cdots+\xi_nu_{k,n}))u_{k,i}u_{k,j}.\end{aligned} \tag{77}$$

Thus,

$$\left(\frac{\partial F}{\partial\xi}\bigg|_{(0,\cdots,0)}\right)_{n\times n}=-\sum_{k=1}^{N}\alpha_k\phi''(h(P,u_k))u_ku_k^T, \tag{78}$$

where $u_ku_k^T$ is an $n\times n$ matrix.

Since $u_1,\cdots,u_N$ are in general position, $\mathbb{R}^n=\mathrm{lin}\{u_1,\cdots,u_N\}$. Thus, for any $x\in\mathbb{R}^n$ with $x\neq0$, there exists a $u_{i_0}\in\{u_1,\cdots,u_N\}$ such that $u_{i_0}\cdot x\neq0$. Note that $\phi''>0$. Then, we have

$$\begin{aligned}x^T\left(-\sum_{k=1}^{N}\alpha_k\phi''(h(P,u_k))u_ku_k^T\right)x&\\=-\sum_{k=1}^{N}\alpha_k\phi''(h(P,u_k))(x\cdot u_k)^2&\\\le-\alpha_{i_0}\phi''\left(h\left(P,u_{i_0}\right)\right)\left(x\cdot u_{i_0}\right)^2<0.&\end{aligned} \tag{79}$$

Hence, $(\partial F/\partial\xi|_{(0,\cdots,0)})$ is negative definite. This implies $\det(\partial F/\partial\xi|_{(0,\cdots,0)})\neq0$. By this, the facts that for all $i=1,\cdots,n$, $F_i(0,\cdots,0)=0$ follows by (74) and $\partial F_i/\partial\xi_j$ is continuous on t and ξ for all $1\le i,j\le n$, and for the implicit function theorem, it follows that

$$\xi'(0)=\left(\xi_1'(0),\cdots,\xi_n'(0)\right) \tag{80}$$

exists.

Since $\Phi(0)$ is a maximizer of $\Phi(t)$, from (71), (72), and (75), we get

$$\begin{aligned}0=\Phi'(0)&=\sum_{k=1}^{N}\alpha_k\phi'(h(P,u_k))\left(\beta'(0)h(P,u_k)+\gamma_k-\xi'(0)\cdot u_k\right)\\&=\sum_{k=1}^{N}\alpha_k\phi'(h(P,u_k))\left[-\frac{q-1}{n-q}\left(\sum_{i=1}^{N}\gamma_i\mu_q(P,u_i)\right)h(P,u_k)+\gamma_k\right]\\&\quad-\xi'(0)\cdot\sum_{k=1}^{N}\alpha_k\phi'(h(P,u_k))u_k=\sum_{k=1}^{N}\alpha_k\phi'(h(P,u_k))\gamma_k\\&\quad-\frac{q-1}{n-q}\left(\sum_{i=1}^{N}\gamma_i\mu_q(P,u_i)\right)\left(\sum_{k=1}^{N}\alpha_k\phi'(h(P,u_k))h(P,u_k)\right)\\&=\sum_{k=1}^{N}\left[\alpha_k\phi'(h(P,u_k))-\frac{q-1}{n-q}\left(\sum_{j=1}^{N}\alpha_j\phi'\left(h\left(P,u_j\right)\right)h\left(P,u_j\right)\right)\mu_q(P,u_k)\right]\gamma_k.\end{aligned} \tag{81}$$

Since $\gamma_1,\cdots,\gamma_N$ are arbitrary,

$$\alpha_k=-\frac{q-1}{n-q}\left(\sum_{j=1}^{N}\alpha_j\phi'\left(h\left(P,u_j\right)\right)h\left(P,u_j\right)\right)\frac{1}{\phi'(h(P,u_k))}\mu_q(P,u_k), \tag{82}$$

for $k=1,\cdots,N$. Let

$$c=\frac{q-1}{n-q}\left(\sum_{j=1}^{N}\alpha_j\phi'\left(h\left(P,u_j\right)\right)h\left(P,u_j\right)\right). \tag{83}$$

Then, for $k=1,\cdots,N$,

$$\alpha_k=c\varphi(h(P,u_k))\mu_q(P,u_k), \tag{84}$$

i.e.,

$$\mu=c\mu_{\varphi,q}(P,\cdot). \tag{85}$$

This completes the proof.

Proof of Corollary 2. Let $\varphi(s)=s^{1-p}$ with $p<0$ in Theorem 1. Then, $\phi(t)=-(1/p)t^p$ for $t>0$ and $\lim_{t\to0^+}\phi(t)=\infty$. Therefore, we see φ and ϕ satisfy the conditions of Theorem 1. Thus, from the theorem, (3), and (4), we obtain

$$\mu=ch_P^{1-p}(\cdot)\mu_q(P,\cdot)=c\mu_{p,q}(P,\cdot). \tag{86}$$

If $p+q\neq n$, then from (i) of Lemma 4, we have

$$\mu=\mu_{p,q}\left(c^{1/(n-p-q)}P,\cdot\right). \tag{87}$$

Let $P_0=c^{1/(n-p-q)}P$. Then, our desired result is given. If $p+q=n$, then (86) is just the desired result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and approved the final manuscript.

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References

[1] M. Ludwig, "General affine surface areas," *Advances in Mathematics*, vol. 224, no. 6, pp. 2346–2360, 2010.

[2] M. Ludwig and M. Reitzner, "A classification of SL(n) invariant valuations," *Annals of Mathematics*, vol. 172, no. 2, pp. 1219–1267, 2010.

[3] E. Lutwak, D. Yang, and G. Zhang, "Orlicz centroid bodies," *Journal of Differential Geometry*, vol. 84, no. 2, pp. 365–387, 2010.

[4] E. Lutwak, D. Yang, and G. Zhang, "Orlicz projection bodies," *Advances in Mathematics*, vol. 223, no. 1, pp. 220–242, 2010.

[5] R. J. Gardner, D. Hug, and W. Weil, "The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities," *Journal of Differential Geometry*, vol. 97, no. 3, pp. 427–476, 2014.

[6] R. J. Gardner, D. Hug, W. Weil, and D. Ye, "The dual Orlicz-Brunn-Minkowski theory," *Journal of Mathematical Analysis and Applications*, vol. 430, no. 2, pp. 810–829, 2015.

[7] D. Xi, H. Jin, and G. Leng, "The Orlicz Brunn-Minkowski inequality," *Advances in Mathematics*, vol. 260, pp. 350–374, 2014.

[8] B. Zhu, J. Zhou, and W. Xu, "Dual Orlicz-Brunn-Minkowski theory," *Advances in Mathematics*, vol. 264, pp. 700–725, 2014.

[9] D. Zou and G. Xiong, "Orlicz-John ellipsoids," *Advances in Mathematics*, vol. 265, pp. 132–168, 2014.

[10] D. Zou and G. Xiong, "Orlicz-Legendre ellipsoids," *Journal of Geometric Analysis*, vol. 26, no. 3, pp. 2474–2502, 2016.

[11] E. Lutwak, "The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem," *Journal of Differential Geometry*, vol. 38, no. 1, pp. 131–150, 1993.

[12] E. Lutwak, "The Brunn-Minkowski-Firey Theory II," *Advances in Mathematics*, vol. 118, no. 2, pp. 244–294, 1996.

[13] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, "The log-Brunn-Minkowski inequality," *Advances in Mathematics*, vol. 231, no. 3-4, pp. 1974–1997, 2012.

[14] S. Campi and P. Gronchi, "The $_{cc}$ Busemann-Petty centroid inequality," *Advances in Mathematics*, vol. 167, pp. 128–142, 2002.

[15] S. Campi and P. Gronchi, "On the reverse L_p Busemann-Petty centroid inequality," *Mathematika*, vol. 49, pp. 1–11, 2002.

[16] C. Haberl and F. Schuster, "General L_p affine isoperimetric inequalities," *Journal of Differential Geometry*, vol. 83, pp. 1–26, 2009.

[17] C. Haberl and F. E. Schuster, "Asymmetric affine L_p Sobolev inequalities," *Journal of Functional Analysis*, vol. 257, no. 3, pp. 641–658, 2009.

[18] E. Lutwak, D. Yang, and G. Zhang, "L_p Affine Isoperimetric Inequalities," *Journal of Differential Geometry*, vol. 56, no. 1, pp. 111–132, 2000.

[19] E. Lutwak, D. Yang, and G. Zhang, "L_p John Ellipsoids," *Proceedings of the London Mathematical Society*, vol. 90, no. 2, pp. 497–520, 2005.

[20] E. Werner and D. Ye, "New L_p affine isoperimetric inequalities," *Advances in Mathematics*, vol. 218, no. 3, pp. 762–780, 2008.

[21] H. Minkowski, "Allgemeine Lehrsätze über die konvexen Polyeder," *Nachrichten von der Königlichen Gesellschaft der Wissenschaften*, vol. 1897, pp. 198–219, 1897.

[22] A. D. Aleksandrov, "On the theory of mixed volumes. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies," *Matematicheskii Sbornik*, vol. 3, pp. 27–46, 1938.

[23] W. Fenchel and B. Jessen, "Mengenfunktionen und konvexe Körper," *Danske Videnskabernes Selskab Mathematisk-fysiske Meddelelser*, vol. 16, pp. 1–31, 1938.

[24] H. Lewy, "On differential geometry in the large, I (Minkowski's problem)," *Transactions of the American Mathematical Society*, vol. 43, no. 2, pp. 258–270, 1938.

[25] L. Nirenberg, "The Weyl and Minkowski problems in differential geometry in the large," *Communications on Pure and Applied Mathematics*, vol. 6, no. 3, pp. 337–394, 1953.

[26] A. V. Pogorelov, *The Minkowski Multidimensional Problem*, V. H. Winston & Sons, Washington, D.C., 1978.

[27] S. Y. Cheng and S. T. Yau, "On the regularity of the solution of then-dimensional Minkowski problem," *Communications on Pure and Applied Mathematics*, vol. 29, no. 5, pp. 495–516, 1976.

[28] L. Caffarelli, L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations I. Monge-ampégre equation," *Communications on Pure and Applied Mathematics*, vol. 37, no. 3, pp. 369–402, 1984.

[29] G. Bianchi, K. J. Böröczky, A. Colesanti, and D. Yang, "The L_p -Minkowski problem for $-n < p < 1$," *Advances in Mathematics*, vol. 341, pp. 493–535, 2019.

[30] K. J. Böröczky and F. Fodor, "The L_p dual Minkowski problem for $p > 1$ and $q > 0$," *Journal of Differential Equations*, vol. 266, no. 12, pp. 7980–8033, 2019.

[31] K. J. Böröczky and H. T. Trinh, "The planar L_p-Minkowski problem for $0 < p < 1$," *Advances in Applied Mathematics*, vol. 87, pp. 58–81, 2017.

[32] K. J. Böröczky, P. Hegedüs, and G. Zhu, "On the discrete logarithmic Minkowski problem," *International Mathematics Research Notices*, vol. 2016, no. 6, pp. 1807–1838, 2016.

[33] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, "The logarithmic Minkowski problem," *Journal of the American Mathematical Society*, vol. 26, no. 3, pp. 831–852, 2013.

[34] W. Chen, "L_p Minkowski problem with not necessarily positive data," *Advances in Mathematics*, vol. 201, no. 1, pp. 77–89, 2006.

[35] S. Chen, Q.-R. Li, and G. Zhu, "On the L_p Monge-Ampère equation," *Journal of Differential Equations*, vol. 263, pp. 4997–5011, 2017.

[36] K. S. Chou and X. J. Wang, "The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry," *Advances in Mathematics*, vol. 205, no. 1, pp. 33–83, 2006.

[37] P. Guan and C. Lin, "On equation det $(u_{ij} + \delta_{ij}u) = u^p f$ on S^n," preprint.

[38] Y. Huang and Q. Lu, "On the regularity of the L_p Minkowski problem," *Advances in Applied Mathematics*, vol. 50, pp. 268–280, 2013.

[39] Y. Huang and Y. Zhao, "On the L_p dual Minkowski problem," *Advances in Mathematics*, vol. 332, pp. 57–84, 2018.

[40] D. Hug, E. Lutwak, D. Yang, and G. Zhang, "On the L_p Minkowski Problem for Polytopes," *Discrete & Computational Geometry*, vol. 33, no. 4, pp. 699–715, 2005.

[41] H. Jian, J. Lu, and X. J. Wang, "Nonuniqueness of solutions to the L_p-Minkowski problem," *Advances in Mathematics*, vol. 281, pp. 845–856, 2015.

[42] J. Lu and X.-J. Wang, "Rotationally symmetric solution to the L_p Minkowski problem," *Journal of Differential Equations*, vol. 254, pp. 983–1005, 2013.

[43] E. Lutwak, D. Yang, and G. Zhang, "On the L_p-Minkowski problem," *Transactions of the American Mathematical Society*, vol. 356, no. 11, pp. 4359–4370, 2004.

[44] E. Lutwak, D. Yang, and G. Zhang, "L_p dual curvature measures," *Advances in Mathematics*, vol. 329, pp. 85–132, 2018.

[45] G. Zhu, "The logarithmic Minkowski problem for polytopes," *Advances in Mathematics*, vol. 262, pp. 909–931, 2014.

[46] G. Zhu, "The centro-affine Minkowski problem for polytopes," *Journal of Differential Geometry*, vol. 101, no. 1, pp. 159–174, 2015.

[47] G. Zhu, "The L_p Minkowski problem for polytopes for $0 < p < 1$," *Journal of Functional Analysis*, vol. 269, no. 4, pp. 1070–1094, 2015.

[48] G. Zhu, "The L_p Minkowski problem for polytopes for $p < 0$," *Indiana University Mathematics Journal*, vol. 66, no. 4, pp. 1333–1350, 2017.

[49] G. Zhu, "Continuity of the solution to the L_p Minkowski problem," *Proceedings of the American Mathematical Society*, vol. 145, no. 1, pp. 379–386, 2017.

[50] C. Haberl, E. Lutwak, D. Yang, and G. Zhang, "The even Orlicz Minkowski problem," *Adv. Math.*, vol. 224, no. 6, pp. 2485–2510, 2010.

[51] G. Bianchi, K. J. Böröczky, and A. Colesanti, "The Orlicz version of the L_p Minkowski problem on S^{n-1} for $-n < p < 1$," *Advances in Applied Mathematics*, vol. 111, article 101937, 2019.

[52] R. J. Gardner, D. Hug, S. Xing, and D. Ye, "General volumes in the Orlicz–Brunn–Minkowski theory and a related Minkowski problem II," *Calculus of Variations and Partial Differential Equations*, vol. 59, no. 1, p. 15, 2020.

[53] R. J. Gardner, D. Hug, W. Weil, S. Xing, and D. Ye, "General volumes in the Orlicz–Brunn–Minkowski theory and a related Minkowski problem I," *Calculus of Variations and Partial Differential Equations*, vol. 58, no. 1, p. 12, 2019.

[54] Q. Huang and B. He, "On the Orlicz Minkowski problem for polytopes," *Discrete & Computational Geometry*, vol. 48, no. 2, pp. 281–297, 2012.

[55] H. Jian and J. Lu, "Existence of solutions to the Orlicz-Minkowski problem," *Advances in Mathematics*, vol. 344, pp. 262–288, 2019.

[56] S. Yijing, "Existence and uniqueness of solutions to Orlicz Minkowski problems involving $0 < p < 1$," *Advances in Applied Mathematics*, vol. 101, pp. 184–214, 2018.

[57] Y. Wu, D. Xi, and G. Leng, "On the discrete Orlicz Minkowski problem," *Transactions of the American Mathematical Society*, vol. 371, no. 3, pp. 1795–1814, 2019.

[58] Y. Wu, D. Xi, and G. Leng, "On the discrete Orlicz Minkowski problem II," *Geometriae Dedicata*, vol. 205, no. 1, pp. 177–190, 2020.

[59] S. Xing and D. Ye, "On the general dual Orlicz Minkowski problem," *Indiana University Mathematics Journal*in press.

[60] B. Zhu, S. Xing, and D. Ye, "The dual Orlicz Minkowski problem," *Journal of Geometric Analysis*, vol. 28, no. 4, pp. 3829–3855, 2018.

[61] A. Colesanti, K. Nyström, P. Salani, J. Xiao, D. Yang, and G. Zhang, "The Hadamard variational formula and the Minkowski problem for p-capacity," *Advances in Mathematics*, vol. 285, pp. 1511–1588, 2015.

[62] R. J. Gardner, *Geometric Tomography*, Cambridge Univ. Press, New York, Second edition, 2006.

[63] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. . Press, New York, Second edition, 2014.

[64] D. Zou and G. Xiong, "The L_p Minkowski problem for the electrostatic p-capacity," *Journal of Differential Geometry*in press.

[65] H. Hong, D. Ye, and N. Zhang, "The p-capacitary Orlicz Hadamard variational formula and Orlicz Minkowski problems," *Calculus of Variations and Partial Differential Equations*, vol. 57, pp. 1–31, 2018.

[66] D. Jerison, "A Minkowski problem for electrostatic capacity," *Acta Mathematica*, vol. 176, no. 1, pp. 1–47, 1996.

[67] D. Jerison, "The direct method in the calculus of variations for convex bodies," *Advances in Mathematics*, vol. 122, no. 2, pp. 262–279, 1996.

[68] L. A. Caffarelli, D. Jerison, and E. H. Lieb, "On the case of equality in the Brunn-Minkowski inequality for capacity," *Advances in Mathematics*, vol. 117, no. 2, pp. 193–207, 1996.

[69] M. Akman, J. Gong, J. Hineman, J. Lewis, and A. Vogel, "The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity," preprint.

[70] G. Xiong, J. Xiong, and L. Xu, "The L_p capacitary Minkowski problem for polytopes," *Journal of Functional Analysis*, vol. 277, no. 9, pp. 3131–3155, 2019.

[71] V. Maz'ya, "Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings," *Journal of Functional Analysis*, vol. 224, no. 2, pp. 408–430, 2005.

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Research Article

Functional Geominimal Surface Area and Its Related Affine Isoperimetric Inequality

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The first variation of the total mass of log-concave functions was studied by Colesanti and Fragalà, which includes the L_p mixed volume of convex bodies. Using Colesanti and Fragalà's first variation formula, we define the geominimal surface area for log-concave functions, and its related affine isoperimetric inequality is also established.

1. Introduction

As we have known, Minkowski addition (the vector addition of convex bodies) is the cornerstone in the classical Brunn-Minkowski theory. Combining with volume, it leads to the Brunn-Minkowski inequality that is one of the most important results in convex geometry. The first variation of volume with respect to Minkowski addition is named the first mixed volume, and its related inequality is the Minkowski inequality. For more history and developments of the Brunn-Minkowski inequality, one may refer to the excellent survey [1]. For instance, the Prékopa-Leindler inequality [2–8] is known as the functional version of the Brunn-Minkowski inequality. In recent years, finding the functional counterparts of existing geometric results, especially for log-concave functions, has been receiving intensive attentions (see, e.g., [9–34]).

In 2013, Colesanti and Fragalà [35] introduced the "Minkowski addition" and "scalar multiplication," $\alpha \cdot f \oplus \beta \cdot g$ (where $\alpha, \beta > 0$), of log-concave functions f and g as

$$\alpha \cdot f \oplus \beta \cdot g(x) = \sup_{y \in \mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^{\alpha} g\left(\frac{y}{\beta}\right)^{\beta}. \tag{1}$$

We remark that a function $f : \mathbb{R}^n \longrightarrow [0,\infty)$ is log-concave if it has the form $f(x) = e^{-u(x)}$, where $u : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex. The total mass of f is defined as

$$J(f) = \int_{\mathbb{R}^n} f(x)dx. \tag{2}$$

Similar to the case of convex bodies, Colesanti and Fragalà [35] considered the following variational

$$\delta J(f, g) = \lim_{t \to 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t}, \tag{3}$$

and it is called the first variation of J at f along g. The first variation,$\delta J(f, g)$, includes the$L_p$mixed volume when it restricted$f$and$g$to the subclass of log-concave functions (see [35], Proposition 3.12).

Colesanti and Fragalà's work inspired us a natural way to extend the L_p geominimal surface area for convex bodies to the class of log-concave functions. For convenience, we recall the definition of L_p geominimal surface area. For a convex body K containing the origin in its interior, its L_p geominimal surface area, $G_p(K)$, is defined as (the case $p = 1$, see Petty [36], and $p > 1$, see Lutwak [37])

$$\omega_n^{p/n} G_p(K) = \inf \left\{ nV_p(K, Q) V(Q^\circ)^{p/n} : Q \in \mathcal{K}_o^n \right\}, \tag{4}$$

where ω_n is the volume of the unit ball in n-dimensional Euclidean space $\mathbb{R}^n$, Q° is the polar body of Q defined by $Q^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1, \forall y \in Q\}$, $\mathcal{K}_o^n$ denotes the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and $V_p(K, Q)$ is the L_p mixed volume (for detailed definition, see Section 2). The fundamental inequality for L_p geominimal surface area is the following affine isoperimetric inequality (see, e.g., [37], Theorem 3.12):

$$G_p(K)^n \le n^n \omega_n^p V(K)^{n-p}, \tag{5}$$

with equality if and only if K is an ellipsoid.

The L_p geominimal surface area, $G_p(K)$, is an important notation in the L_p Brunn-Minkowski theory, which serves as a bridge connecting affine differential geometry, relative differential geometry, and Minkowski geometry. In the past three decades, the L_p geominimal surface area has developed rapidly (see [25, 38–42] for some of the pertinent results).

Since $\delta J(f, g)$ includes the L_p mixed volume, we extend the L_p geominimal surface area to the functional version as follows.

Definition 1. Let $f : \mathbb{R}^n \longrightarrow [0,\infty)$ be an integral log-concave function and $p > 0$. The L_p geominimal surface area of f is defined as

$$c_n^{p/n} G_p(f) = \inf \left\{\delta J(f, g) J(g^\circ)^{p/n} : g \text{ is a log-concave function}\right\}, \tag{6}$$

where $c_n = (2\pi)^{n/2}$, and $g^\circ(x) = \inf_{y\in\mathbb{R}^n} e^{-\langle x,y\rangle}/g(y)$ is the polar function of g.

In Lemma 5, we prove that the above definition includes the L_p geominimal surface area (4) when $p \ge 1$ and restricted f, g to the subclass of log-concave functions.

In order to study the functional geominimal surface area, we need the integral formula of $\delta J(\cdot, \cdot)$. Hence, we need some notations. We write $\langle x, y\rangle$ for the usual inner product of $x, y \in \mathbb{R}^n$, and $\|x\|$ denotes the Euclidean normal of $x \in \mathbb{R}^n$. We say that $g = e^{-v}$ is an admissible perturbation for $f = e^{-u}$ if there exists a constant $c > 0$ such that $u^* - cv^*$ is convex, where $u^*(y) = \sup_{x\in\mathbb{R}^n}\{\langle x, y\rangle - u(x)\}$ is the Legendre conjugate of u. Let $\mathcal{A}'$ denote the set of log-concave functions given by function f such that $u = -\log f$ belongs to

$$\mathcal{L}' = \left\{u \in \mathcal{L} : \text{dom }(u) = \mathbb{R}^n,\ u \in \mathcal{C}_+^2(\mathbb{R}^n),\ \lim_{\|x\|\to+\infty} \frac{u(x)}{\|x\|} = +\infty\right\}. \tag{7}$$

Here, $\text{dom }(u) = \{x \in \mathbb{R}^n : u(x) < +\infty\}$ and

$$\mathcal{L} = \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\} \mid \quad u \text{ is convex, dom } (u) \ne \varnothing, \quad \lim_{\|x\|\to+\infty} u(x) = +\infty \right\}. \tag{8}$$

Colesanti and Fragalà ([35], Theorem 4.5) provided an integral formula for the first variation $\delta J(f, g)$ when $f, g \in \mathcal{A}'$ and g is an admissible perturbation for f. For our aims, we consider the following optimization problem:

$$\inf \left\{\delta J(f, g) J(g^\circ)^{p/n} : f, g \in \mathcal{A}', p > 0 \text{ and } g \text{ is an admissible perturbation for } f\right\}. \tag{9}$$

If the extremum in (9) exists, then it is denoted by $c_n^{p/n} G_p^{(1)}(f)$.

In Section 3, we prove that for $p > 0$ and $f \in \mathcal{A}'$, if $J(f)$ is finite, then there exists a unique log-concave function $\bar{f} \in \mathcal{A}'$ such that

$$G_p^{(1)}(f) = \delta J(f, \bar{f}) \text{and} J(\bar{f}^\circ) = c_n. \tag{10}$$

Similar to the geometric case, the unique log-concave function $\bar{f}$ is called p -Petty functions of f and denoted by $T_p f$.

Using p-Petty functions, we obtain the following analytic inequality with equality conditions involving $G_p^{(1)}(f)$.

Theorem 2. *Suppose $f \in \mathcal{A}'$ and $p > 0$. If f has its barycenter at 0 (i.e., $\int_{\mathbb{R}^n} x f(x) dx = 0$), then*

$$J(f)^{p/n} G_p^{(1)}(f) \le c_n^{p/n} \left(nJ(f) + \int_{\mathbb{R}^n} f \log f dx\right), \tag{11}$$

with equality if $T_p f(x) = f(x)$ and $f(x) = ce^{-(\|Ax\|^2/2)}$ for $A \in SL(n)$ and $c > 0$.

2. Background

2.1. Functional Setting. Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ if for every $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ it satisfies

$$u((1-\lambda)x + \lambda y) \le (1-\lambda)u(x) + \lambda u(y), \tag{12}$$

we say u is a convex function; let

$$\text{dom }(u) = \{x \in \mathbb{R}^n : u(x) \in \mathbb{R}\}. \tag{13}$$

By the convexity of u, dom (u) is a convex set. We say that u is proper if dom $(u) \ne \varnothing$. The Legendre conjugate of u is the convex function defined by

$$u^*(y) = \sup_{x\in\mathbb{R}^n}\{\langle x, y\rangle - u(x)\} \quad \forall y \in \mathbb{R}^n. \tag{14}$$

Clearly, $u(x) + u^*(y) \ge \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$; there is an equality if and only if $x \in \text{dom }(u)$ and y is in the subdifferential of u at x. Hence, it can be checked that

$$u^*(\nabla u(x)) + u(x) = \langle x, \nabla u(x)\rangle. \tag{15}$$

On the class of convex functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, the infimal convolution is defined by

$$u\Box v(x)=\inf_{y\in\mathbb{R}^n}\{u(x-y)+v(y)\}\quad \forall x\in\mathbb{R}^n, \tag{16}$$

and the right scalar multiplication by a nonnegative real number $\alpha>0$,

$$(u\alpha)(x)=\alpha u\left(\frac{x}{\alpha}\right). \tag{17}$$

It was proved in [21] (Proposition 2.1) that if $u, v:\mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ are convex functions and $\alpha>0$, then

$$\begin{aligned}(u\Box v)^*&=u^*+v^*,\\(u\alpha)^*&=\alpha u^*.\end{aligned} \tag{18}$$

The following result will be used later.

Theorem 3 ([43], Theorem 10.9). *Let C be a relatively open convex set, and let $f_1, f_2, \cdots$ be a sequence of finite convex functions on C. Suppose that the real number $f_1(x), f_2(x), \cdots$ is bounded for each $x\in C$. It is then possible to select a subsequence of $f_1, f_2, \cdots$, which converges uniformly on closed bounded subsets of C to some finite convex function f.*

The functional Blaschke-Santaló inequality states that let f, g be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$f(x)g(y)\le e^{-\langle x,y\rangle},\quad \forall x,y\in\mathbb{R}^n. \tag{19}$$

If f has its barycenter at 0, which means that $\int_{\mathbb{R}^n} xf(x)\,dx=0$, then

$$\left(\int_{\mathbb{R}^n} f(x)dx\right)\left(\int_{\mathbb{R}^n} g(x)dx\right)\le c_n^2, \tag{20}$$

with equality if and only if there exists a positive definite matrix A and $C>0$ such that, a.e. in $\mathbb{R}^n$,

$$f(x)=Ce^{-\frac{\langle Ax,x\rangle}{2}},\quad g(y)=C^{-1}e^{-\frac{\langle A^{-1}x,x\rangle}{2}}. \tag{21}$$

2.2. The First Variation of the Total Mass of Log-Concave Functions. In this paper, we set

$$\begin{aligned}&\mathscr{L}=\left\{u:\mathbb{R}^n\longrightarrow\mathbb{R}\cup\{+\infty\}\mid\quad u\ \text{proper, convex,}\quad \lim_{\|x\|\to+\infty}u(x)=+\infty\right\},\\&\mathscr{A}=\{f:\mathbb{R}^n\longrightarrow\mathbb{R}\mid\quad f=e^{-u},u\in\mathscr{L}\}.\end{aligned} \tag{22}$$

The total mass functional of f is defined as

$$J(f)=\int_{\mathbb{R}^n} f(x)dx. \tag{23}$$

The Gaussian function

$$\gamma(x)=e^{-\frac{\|x\|^2}{2}} \tag{24}$$

plays within class $\mathscr{A}$ the role of the ball in the set of convex bodies, and $J(\gamma)=(2\pi)^{n/2}=c_n$. For every $A\in GL(n)$, we write

$$\gamma_A(x)=e^{-\frac{\|Ax\|^2}{2}}. \tag{25}$$

From the definition of polar function and Legendre conjugate of function, we note that if $f\in\mathscr{A}$, then

$$f^\circ=e^{-\varphi^*}. \tag{26}$$

The support function of log-concave function $f=e^{-\varphi}$ is (see [44])

$$h_f(x)=\varphi^*(x). \tag{27}$$

This is a proper generalization, in the sense that $h_{\chi_K}=h_K$. Let $f=e^{-u}$, $g=e^{-v}$, and let $\alpha,\beta>0$, then

$$\alpha\cdot f\oplus\beta\cdot g=e^{-[(u\alpha)\Box(v\beta)]}, \tag{28}$$

which in explicit form reads

$$(\alpha\cdot f\oplus\beta\cdot g)(x)=\sup_{y\in\mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^{\alpha} g\left(\frac{y}{\beta}\right)^{\beta}. \tag{29}$$

The support function of $\alpha\cdot f\oplus\beta\cdot g$ satisfies

$$h_{\alpha\cdot f\oplus\beta\cdot g}(x)=\alpha h_f(x)+\beta h_g(x). \tag{30}$$

In particular,

$$h_{\alpha\cdot f}(x)=\alpha h_f(x). \tag{31}$$

Let $f, g\in\mathscr{A}$. The first variation of J at f along g is defined as

$$\delta J(f,g)=\lim_{t\to0^+}\frac{J(f\oplus t\cdot g)-J(f)}{t}. \tag{32}$$

The existence of the above limit was proved by Colesanti and Fragalà [35], and $\delta J(f,g)\in[-k,+\infty]$ with $k=\max\{\inf(-\log g),0\}J(f)$. In particular, for every $f\in\mathscr{A}$ with $J(f)>0$, then

$$\delta J(f,f)=nJ(f)+\int_{\mathbb{R}^n} f\log f\,dx. \tag{33}$$

The functional version of Minkowski first inequality reads as follows (see, e.g., [35], Theorem 5.1): let $f, g\in\mathscr{A}$ and assume that $J(f)>0$. Then,

$$\delta J(f,g)\ge J(f)\left[\log\frac{J(g)}{J(f)}+n\right]+\int_{\mathbb{R}^n} f\log f\,dx, \tag{34}$$

with equality if and only if there exists $x_0\in\mathbb{R}^n$ such that $g(x)=f(x-x_0)$ for $\forall x\in\mathbb{R}^n$.

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space $\mathbb{R}^n$. We write $\mathscr{K}_o^n$ for the set of convex bodies that contain the origin in their interiors. Let $V(K)$ denote the n-dimensional volume of convex body K. The volume of the standard unit ball in $\mathbb{R}^n$ is denoted by $\omega_n = \pi^{n/2}/\Gamma((n/2)+1)$. A convex body $K \in \mathscr{K}^n$ is uniquely determined by its support function, which is defined as $h_K(x) = \max\{\langle x, y\rangle : y \in K\}$, where $\langle \cdot, \cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. The polar body of K is defined by $K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1, \forall y \in K\}$.

For real $p \ge 1$, $K, L \in \mathscr{K}^n$, and real $\varepsilon > 0$, the Minkowski-Firey L_p combination $K +_p \varepsilon \cdot L$ is a convex body whose support function is given by

$$h\left(K +_p \varepsilon \cdot L, \cdot\right)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p. \tag{35}$$

The L_p mixed volume $V_p(K, L)$ of convex bodies K and L is defined by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V\left(K +_p \varepsilon \cdot L\right) - V(K)}{\varepsilon}. \tag{36}$$

The existence of this limit is showed in [45].

The following result show that $\delta J(f, g)$ includes the L_p mixed volume for convex bodies.

Proposition 4 ([35], Proposition 3.12). *Let $q \in (1, +\infty)$, $p = q/(q-1)$ and $K, L \in \mathscr{K}_o^n$. Let $u = (h_{K^\circ}(x)^q)/q$, $v(x) = (h_{L^\circ}(x)^q)/q$, and $f = e^{-u}$, $g = e^{-v}$. Then, there exists a positive constant $c = c(n, q)$ such that*

$$J(f) = c(n, q) V(K), \tag{37}$$

with $c(n, q) = q^{n/q}\Gamma((n+q)/q)$, and

$$\frac{p}{n} \delta J(f, g) = c(n, q) V_p(K, L). \tag{38}$$

We set $\mathscr{A}'$ as the subclasses of $\mathscr{A}$ given by the function f such that $u = \log f$ belongs to

$$\mathscr{L}' = \left\{u \in \mathscr{L} : \operatorname{dom}(u) = \mathbb{R}^n, \quad u \in \mathscr{C}_+^2(\mathbb{R}^n), \quad \lim_{\|x\| \to +\infty} \frac{u(x)}{\|x\|} = +\infty\right\}. \tag{39}$$

For log-concave function $f = e^{-u}$, the Borel measure μ_f on $\mathbb{R}^n$ is defined by (see [35])

$$\mu_f = (\nabla u)_\# (f \mathscr{H}^n). \tag{40}$$

Here, $\mathscr{H}^n$ is the n-dimensional Hausdorff measure. We need the fact that the barycenter of μ_f is the origin; i.e.,

$$\int_{\mathbb{R}^n} x \mu_f(x) = 0. \tag{41}$$

We recall that the log-concave function $g = e^{-v}$ is an admissible perturbation for log-concave function $f = e^{-u}$ if

$$\exists a > 0 : u^* - a v^* \tag{42}$$

is convex.

Colesanti and Fragalà [35] provided an integral representation of the first variation $\delta J(\cdot, \cdot)$ (see, e.g., [35], Theorem 4.5): let $f = e^{-u}$ and $g = e^{-v} \in \mathscr{A}'$ and assume that g is an admissible perturbation for f. Then, $\delta J(f, g)$ is finite and is given by

$$\delta J(f, g) = \int_{\operatorname{dom}(u)} h_g(x) d\mu_f(x). \tag{43}$$

3. Functional L_p Geominimal Surface Areas

Analogy to convex bodies, for $f \in \mathscr{A}$ and $p \in \mathbb{R}$, we consider the following optimization problem:

$$c_n^{p/n} G_\lambda(f) = \inf\left\{\delta J(f, g) J(g^\circ)^{p/n} : g \in \mathscr{A}\right\}. \tag{44}$$

The following result shows that the above optimization problem includes Lutwak's L_p geominimal surface areas for convex bodies (4) when $p > 1$ (up to a constant which is dependent on n and p). This is one of the reasons why $G_p(f)$ is called the L_p geominimal surface area for log-concave function f.

Lemma 5. *Let $p > 1$, $q = p/(p-1)$, and $K \in \mathscr{K}_o^n$. If $f = e^{-h_{K^\circ(x)}^q/q}$, then*

$$G_p(f) = \alpha(n, p) G_p(K), \tag{45}$$

with $\alpha(n, p) = (1/p) c(n, q) c(n, p)^{p/n} \omega_n^{p/n} c_n^{-(p/n)}$ for $c(n, q) = q^{n/q}\Gamma((n+q)/q)$.

Proof. Let $K, L \in \mathscr{K}_o^n$, $u(x) = h_{K^\circ}(x)^q/q$, $v(x) = h_{L^\circ}(x)^q/q$, and $f = e^{-u}$, $g = e^{-v}$. It is not hard to see that

$$v^*(x) = \frac{h_L(x)^p}{p}. \tag{46}$$

Then, Proposition 4 tells us that

$$\begin{aligned} J(g^\circ) &= c(n, p) V(L^\circ), \\ \frac{p}{n} \delta J(f, g) &= c(n, q) V_p(K, L), \end{aligned} \tag{47}$$

with $c(n, q) = q^{n/q}\Gamma((n+q)/q)$.

From the definitions of L_p geominimal surface area of convex bodies (4) and log-concave functions (44), we have

$$\begin{aligned} c_n^{p/n} G_p(f) &= \inf\left\{\delta J(f,g) J(g^\circ)^{p/n} : g \in \mathscr{A}\right\} \\ &= \inf\left\{\frac{n}{p} c(n,q) c(n,p)^{p/n} V_p(K,L) V(L^\circ)^{p/n} : K, L \in \mathscr{K}_o^n\right\} \\ &= \frac{1}{p} c(n,q) c(n,p)^{p/n} \omega_n^{p/n} G_p(K). \end{aligned} \tag{48}$$

Since we need the integral representation of the first variation $\delta J(f,g)$ in dealing the problem (44), we focus on

$$c_n^{p/n} G_p^{(1)}(f) = \inf\left\{\delta J(f,g) J(g^\circ)^{p/n} : g \in \mathscr{A}' \text{ and } g \text{ is an admissible perturbation for f}\right\} \tag{49}$$

for $f \in \mathscr{A}'$ and $p \in \mathbb{R}$. Trivially, $G_p(f) \le G_p^{(1)}(f)$.

We need the next lemma.

Lemma 6. *Let $f, g \in \mathscr{A}'$ and assume that g is an admissible perturbation for f. If $A \in SL(n)$, then*

$$\delta J(f \circ A, g \circ A) = \delta J(f,g). \tag{50}$$

Proof. Let $f(x) = e^{-u(x)}$ and $g(x) = e^{-v(x)}$. We note that

$$\begin{aligned} (v \circ A)^*(x) &= \sup_{y \in \mathbb{R}^n}\left\{\langle x, y\rangle - v(Ay)\right\} \\ &= \sup_{y \in \mathbb{R}^n}\left\{\left\langle A^{-t}x, Ay\right\rangle - v(Ay)\right\} = v^*\left(A^{-t}x\right). \end{aligned} \tag{51}$$

Since $\nabla_x(u \circ A) = A^t \nabla_{Ax} u$, we have

$$\begin{aligned} \delta J(f \circ A, g \circ A) &= \int_{\mathbb{R}^n} (v \circ A)^*(\nabla(u \circ \mathrm{A})(x)) f \circ A(x) dx \\ &= \int_{\mathbb{R}^n} (v \circ A)^*\left(A^t \nabla_{Ax} u(Ax)\right) f(Ax) dx \\ &= \int_{\mathbb{R}^n} v^*(\nabla_{Ax} u(Ax)) f(Ax) d \\ &= \int_{\mathbb{R}^n} v^*(\nabla u(z)) f(z) dz = \delta J(f,g). \end{aligned} \tag{52}$$

The following result shows that the functional geominimal surface area is affine invariant.

Lemma 7. *Suppose $f \in \mathscr{A}'$ and $p > 0$. If $A \in SL(n)$, then*

$$G_p^{(1)}(f \circ A) = G_p^{(1)}(f). \tag{53}$$

Proof. By (51) and the definition of polar function (26), we have

$$J((g \circ A)^\circ) = J\left(g^\circ \circ A^{-t}\right) = J(g^\circ), \tag{54}$$

for $A \in \mathrm{SL}(n)$. Combing with Lemma 6, we have

$$\delta J(f \circ A, g) J(g^\circ)^{p/n} = \delta J\left(f, g \circ A^{-1}\right) J\left(\left(\left(g \circ A^{-1}\right)^\circ\right)\right)^{p/n}. \tag{55}$$

Therefore, we obtain

$$G_p^{(1)}(f \circ A) = G_p^{(1)}(f), \tag{56}$$

for $A \in SL(n)$.

The following lemma was proved by Cordero-Erausquin and Klartag ([46], Lemma 16).

Lemma 8. *Let μ be a finite Borel measure in $\mathbb{R}^n$, and let K be the interior of conv(Supp(μ)). If $x_0 \in K$ and the barycenter of μ lies at the origin, then there exists a constant $C_{\mu,x_0} > 0$ with the following property: for any nonnegative, μ -integrable, convex function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$,*

$$\varphi(x_0) \le C_{\mu,x_0} \int_{\mathbb{R}^n} \varphi d\mu. \tag{57}$$

The next proposition shows that the infimum in the definition of the p-geominimal surface area of log-concave function is a minimum.

Proposition 9. *Let $p > 0$ and $f \in \mathscr{A}'$. If $J(f)$ is finite, then there exists a unique log-concave function $\bar{f} \in \mathscr{A}$ such that*

$$G_p^{(1)}(f) = \delta J\left(f, \bar{f}\right) \text{ and } J\left(\bar{f}^\circ\right) = c_n. \tag{58}$$

Proof. From the definition of $G_p^{(1)}(f)$, there exists a sequence $g_i \in \mathscr{A}'$ such that $J(g_i^\circ) = c_n$, with $\delta J(f,\gamma) \ge \delta J(f, g_i)$ for all i, and

$$\delta J(\mathrm{f}, g_i) \longrightarrow G_p^{(1)}(f). \tag{59}$$

Let $g_i(x) = e^{-v_i(x)}$, then

$$\delta J(f,\gamma) \ge \delta J(f, g_i) = \int_{\mathbb{R}^n} v_i^*(x) d\mu_f(x). \tag{60}$$

First, we assume that v_i are nonnegative and $v_i(0) = 0$ for all i. In this case, from (14), we have

$$v_i^*(x) = \sup_{y \in \mathbb{R}^n}\left\{\langle y, x\rangle - v_i(y)\right\} \ge \langle 0, x\rangle - v_i(0) = 0, \tag{61}$$

and

$$v_i^*(0) = \sup_{y \in \mathbb{R}^n}\left\{\langle y, 0\rangle - v_i(y)\right\} = -\inf_{y \in \mathbb{R}^n} v_i(y) = 0. \tag{62}$$

Let$K$be the interior ofconv(Supp(μ_f)). By Lemma 8 and ((59)), we conclude that$v_i^*$are uniformly upper bound which is dependent only on$f$. According to Theorem 3, there exists a subsequence $\{v^*_{i_j}\}_{j=1,2,\cdots}$ that converges pointwise in K to a convex function $v^* : K \longrightarrow \mathbb{R}$. We extend the definition of v^* by setting $v^*(x) = +\infty$ for $x \in \bar{K}$ and for $x \in \partial K$,

$$v^*(x) = \lim_{\lambda\to 1^-} v^*(\lambda x). \tag{63}$$

This limit always exists in $[0, +\infty]$, since the function $\lambda \mapsto v^*(\lambda x)$ is nondecreasing for $\lambda \in (0, 1)$ following from the convexity of v^* and $v^*(0) = 0$. Moreover, we have that $v^*(\lambda x) Z v^*(x)$ as $\lambda \longrightarrow 1^-$ for any $x \in \bar{K}$. Because $v_i^* \longrightarrow v^*$ is equivalent to $v_i \longrightarrow v$ (here, $v = (v^*)^*$), hence, there exits a log-concave function $\bar{f} = e^{-v}$ which satisfies the claim.

In the general case, there exist $x_0^{(i)} \in \mathbb{R}^n$ and $\inf_{x\in\mathbb{R}^n} v_i(x) = d_i \in \mathbb{R}$ such that $\bar{v}_i(x) = v_i(x - x_0^{(i)}) - d_i$ are nonnegative and $\bar{v}_i(0) = 0$ for all $i = 1, 2, \cdots$. The convexity of v_i and $e^{-v_i} \in \mathscr{A}'$ ensures the finiteness of d_i; i.e.,$|d_i| <k$ for some $k > 0$. Similar to the first case, we have

$$\delta J(f, \gamma) \geq \delta J(f, \bar{g}_i) = \int_{\mathbb{R}^n} \bar{v}_i^*(x) d\mu_f(x), \tag{64}$$

where $\bar{g}_i = e^{-v_i}$. Lemma 8 deduces that

$$\delta J(f, \gamma) \geq \delta J(f, \bar{g}_i) = \int_{\mathbb{R}^n} \bar{v}_i^*(x) d\mu_f(x) \geq \frac{1}{C_{\mu_f, x_0}} \bar{v}_i^*(x_0) \tag{65}$$

holds for $x_0 \in K$. Moreover,

$$\begin{aligned} \bar{v}_i^*(x) &= \sup_{y\in\mathbb{R}^n} \{\langle y, x\rangle - \bar{v}_i(y)\} \\ &= \sup_{y\in\mathbb{R}^n} \left\{\langle y, x\rangle - v_i\left(y - x_0^{(i)}\right)\right\} + d_i \\ &= \sup_{y\in\mathbb{R}^n} \left\{\left\langle y + x_0^{(i)}, x\right\rangle - v_i(y)\right\} + d_i \\ &= v_i^*(x) + \left\langle x, x_0^{(i)}\right\rangle + d_i. \end{aligned} \tag{66}$$

Therefore,

$$C_{\mu_f, x_0} \delta J(f, \gamma) \geq \bar{v}_i^*(x_0) = v_i^*(x_0) + \left\langle x_0, x_0^{(i)}\right\rangle + d_i, \tag{67}$$

i.e.,

$$v_i^*(x_0) \leq C_{\mu_f, x_0} \delta J(f, \gamma) - \left\langle x_0, x_0^{(i)}\right\rangle - d_i, \tag{68}$$

for any $x_0 \in K$. Then, along the same line of the first case, we conclude that the claim of this proposition holds.

The uniqueness of the minimizing function is demonstrated as follows. Suppose $h_1, h_2 \in \mathscr{A}$, such that $J(h_1^\circ) = J(h_2^\circ) = c_n$, and

$$\begin{aligned} \delta J(f, h_1) J(h_1^\circ)^{p/n} &= \inf\left\{\delta J(f, g) J(g^\circ)^{p/n} : g \in \mathscr{A}'\right\} \\ &= \delta J(f, h_2) J(h_2^\circ)^{p/n}, \end{aligned} \tag{69}$$

i.e.,

$$\delta J(f, h_1) = \delta J(f, h_2). \tag{70}$$

Let $h_1 = e^{-v_1}$ and $h_2 = e^{-v_2}$. Define $h \in \mathscr{A}'$, by

$$h = \frac{1}{2} \cdot h_1 \oplus \frac{1}{2} \cdot h_2 = e^{-v_1(1/2)\square v_2(1/2)}. \tag{71}$$

Then, from (18) and (70), we have

$$\begin{aligned} \delta J(f, h) &= \int_{\mathbb{R}^n} \left(v_1 \frac{1}{2} \square v_2 \frac{1}{2}\right)^* (x) d\mu_f(x) \\ &= \frac{1}{2} \int_{\mathbb{R}^n} v_1^*(x) d\mu_f(x) + \frac{1}{2} \int_{\mathbb{R}^n} v_2^*(x) d\mu_f(\mathrm{x}) \\ &= \frac{1}{2} \delta J(f, h_1) + \frac{1}{2} \delta J(f, h_2) = \delta J(f, h_1) = \delta J(f, h_2), \end{aligned} \tag{72}$$

and by the basic inequality $\sqrt{ab} \leq (a + b)/2$ for $a, b > 0$ and (18), we have

$$\begin{aligned} J(h^\circ) &= \int_{\mathbb{R}^n} e^{-[(v_1 1/2)\square(v_2(1/2))]^*} dx = \int_{\mathbb{R}^n} e^{-[(1/2)v_1^*(x)+(1/2)v_2^*(x)]} \\ &\cdot dx \leq \frac{1}{2} J(h_1^\circ) + \frac{1}{2} J(h_2^\circ), \end{aligned} \tag{73}$$

with equality if and only if $h_1^\circ = h_2^\circ$. Therefore,

$$\delta J(f, h) J(h^\circ)^{p/n} \leq \delta J(f, h_1) J(h_1^\circ)^{p/n} \tag{74}$$

is the contradiction that would arise if it was the case that $h_1 \neq h_2$.

The unique function whose existence is guaranteed by Proposition 9 will be denoted by $T_p f$, and will be called the p-Petty body of log-concave function f (or the λ -Petty function). The polar function of $T_p f$ will be denoted by $T_p^\circ f$, rather than $(T_p f)^\circ$. For $f \in \mathscr{A}$ and $p > 0$, the log-concave function $T_p f$ is defined by

$$\begin{aligned} G_p^{(1)}(f) &= \delta J(f, T_p f), \\ J\left(T_p^\circ f\right) &= c_n. \end{aligned} \tag{75}$$

Lemma 10. *If $p > 0$ and $f \in \mathscr{A}$, then for $A \in SL(n)$,*

$$T_p(f \circ A) = T_p f \circ A. \tag{76}$$

Proof. From the definition of T_p and Lemma 7,

$$\delta J(f, T_p f) = G_p^{(1)}(f) = G_p^{(1)}(f \circ A) = \delta J(f \circ A, T_p(f \circ A)), \tag{77}$$

Lemma 6 deduces

$$\delta J(f, T_p f) = \delta J(f \circ A, T_p(f \circ A)) = \delta J(f, T_p(f \circ A) \circ A^{-1}). \tag{78}$$

The uniqueness of Proposition 9 ensures that $T_p f = T_p(f \circ A) \circ A^{-1}$.

By the Blaschke-Santaló inequality, we obtain the following affine isoperimetric inequality for the functional geominimal surface area.

Theorem 11. *Let $f \in \mathscr{A}'$ and $p > 0$. If f has its barycenter at 0, then*

$$J(f)^{p/n} G_p^{(1)}(f) \le c_n^{p/n} \left(nJ(f) + \int_{\mathbb{R}^n} f \log f dx \right), \tag{79}$$

with equality if $T_p f(x) = f(x)$ and $f(x) = ce^{-(\|Ax\|^2/2)}$ for $A \in SL(n)$ and $c > 0$.

Proof. Taking $g = f$ in (49), together with (33), we have,

$$c_n^{p/n} G_p^{(1)}(f) \le \delta J(f, f) J(f^\circ)^{p/n} = \left(nJ(f) + \int_{\mathbb{R}^n} f \log f dx \right) J(f^\circ)^{p/n}, \tag{80}$$

i.e.,

$$c_n \left(\frac{G_p^{(1)}(f)^n}{\left(n + (1/J(f)) \int_{\mathbb{R}^n} f \log f dx\right)^n J(f)^{n-p}} \right)^{1/p} \le J(f) J(f^\circ). \tag{81}$$

By Blaschke-Santaló inequality (20) and the above inequality, we have

$$\left(\frac{G_p^{(1)}(f)^n}{\left(n + (1/J(f)) \int_{\mathbb{R}^n} f \log f dx\right)^n J(f)^{n-p}} \right)^{1/p} \le c_n. \tag{82}$$

This is the desired inequality.

To obtain the equality condition, first assume that $T_p f = f$. Formula (77) tells us that

$$G_p^{(1)}(f) = \delta J(f, f) \text{ and } J(f^\circ) = c_n. \tag{83}$$

This shows that there is equality in (81). From the condition of Blaschke-Santaló inequality, we known that there exists a positive definite matrix A and $c > 0$ such that, a.e. in $\mathbb{R}^n$,

$$f(x) = ce^{-\frac{\|Ax\|^2}{2}}. \tag{84}$$

Therefore, we obtain the equality condition, namely, $T_p f = f$ and $f(x) = ce^{-(\|Ax\|^2/2)}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of this manuscript. The second author is the corresponding author.

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References

[1] R. Gardner, "The Brunn-Minkowski inequality," *Bulletin of the American Mathematical Society*, vol. 39, no. 3, pp. 355–405, 2002.

[2] H. Brascamp and E. Lieb, "On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation," *Journal of Functional Analysis*, vol. 22, no. 4, pp. 366–389, 1976.

[3] M. Fradelizi and M. Meyer, "Some functional inverse Santaló inequalities," *Advances in Mathematics*, vol. 218, no. 5, pp. 1430–1452, 2008.

[4] B. Klartag and V. D. Milman, "Geometry of log-concave functions and measures," *Geometriae Dedicata*, vol. 112, no. 1, pp. 169–182, 2005.

[5] L. Leindler, "On a certain converse of Hölder inequality II," *Acta Scientiarum Mathematicarum*, vol. 33, pp. 217–223, 1972.

[6] A. Prékopa, "Logarithmic concave measures with applications to stochastic programming," *Acta Scientiarum Mathematicarum*, vol. 32, pp. 335–343, 1971.

[7] A. Prékopa, "On logarithmic concave measures and functions," *Acta Scientiarum Mathematicarum*, vol. 34, pp. 335–343, 1973.

[8] A. Prékopa, "New proof for the basic theorem of logconcave measures Hungarian," *Alkalmazott Matematikai Lapja*, vol. 1, pp. 385–389, 1975.

[9] D. Alonso-Gutiérrez, "A reverse Rogers-Shephard inequality for log-concave functions," *The Journal of Geometric Analysis*, vol. 29, no. 1, pp. 299–315, 2019.

[10] D. Alonso-Gutiérrez, S. Artstein-Avidan, B. G. Merino, C. H. Jiménez, and R. Villa, "Rogers–Shephard and local Loomis–

Whitney type inequalities," *Mathematische Annalen*, vol. 374, no. 3-4, pp. 1719–1771, 2019.

[11] D. Alonso-Gutiérrez, J. Bernues, and B. G. Merino, "An extension of Berwald's inequality and its relation to Zhang's inequality," *Journal of Mathematical Analysis and Applications*, vol. 486, no. 1, p. 123875, 2020.

[12] D. Alonso-Gutiérrez, J. Bernués, and B. G. Merino, "Zhang's inequality for log-concave functions," https://arxiv.org/abs/1810.07507.

[13] D. Alonso-Gutiérrez, B. G. Merino, C. H. Jiménez, and R. Villa, "Rogers–Shephard inequality for log-concave functions," *Journal of Functional Analysis*, vol. 271, no. 11, pp. 3269–3299, 2016.

[14] S. Artstein-Avidan, B. Klartag, and V. Milman, "The Santaló point of a function, and a functional form of the Santaló inequality," *Mathematika*, vol. 51, no. 1-2, pp. 33–48, 2004.

[15] S. Artstein-Avidan, B. Klartag, C. Schütt, and E. M. Werner, "Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality," *Journal of Functional Analysis*, vol. 262, no. 9, pp. 4181–4204, 2012.

[16] S. Artstein-Avidan and V. Milman, "The concept of duality in convex analysis, and the characterization of the Legendre transform," *Annals of Mathematics*, vol. 169, no. 2, pp. 661–674, 2009.

[17] S. Artstein-Avidan and B. A. Slomka, "A note on Santaló inequality for the polarity transform and its reverse," *Proceedings of the American Mathematical Society*, vol. 143, no. 4, pp. 1693–1704, 2015.

[18] S. Artstein-Avidan and B. A. Slomka, "Functional covering numbers," *Journal of Geometric Analysis*, 2019.

[19] K. Ball, "Logarithmically concave functions and sections of convex sets in R^{n}," *Studia Mathematica*, vol. 88, no. 1, pp. 69–84, 1988.

[20] M. Barchiesia, G. M. Caprianib, N. Fusco, and G. Pisante, "Stability of Pólya–Szegő inequality for log-concave functions," *Journal of Functional Analysis*, vol. 267, no. 7, pp. 2264–2297, 2014.

[21] F. Barthe, K. J. Böröczky, and M. Fradelizi, "Stability of the functional forms of the Blaschke-Santaló inequality," *Monatshefte für Mathematik*, vol. 173, no. 2, pp. 135–159, 2014.

[22] U. Caglar, M. Fradelizi, O. Guedon, J. Lehec, C. Schütt, and E. M. Werner, "Functional versions of L_p-affine surface area and entropy inequalities," *International Mathematics Research Notices*, vol. 2016, no. 4, pp. 1223–1250, 2016.

[23] U. Caglar and E. M. Werner, "Divergence for s -concave and log concave functions," *Advances in Mathematics*, vol. 257, pp. 219–247, 2014.

[24] U. Caglar and E. M. Werner, "Mixed f-divergence and inequalities for log-concave functions," *Proceedings of the London Mathematical Society*, vol. 110, no. 2, pp. 271–290, 2015.

[25] U. Caglar and D. Ye, "Affine isoperimetric inequalities in the functional Orlicz-Brunn-Minkowski theory," *Advances in Applied Mathematics*, vol. 81, pp. 78–114, 2016.

[26] M. Fradelizi and M. Meyer, "Some functional forms of Blaschke-Santaló inequality," *Mathematische Zeitschrift*, vol. 256, no. 2, pp. 379–395, 2007.

[27] Q. Huang and A. Li, "The functional version of the Ball inequality," *Proceedings of the American Mathematical Society*, vol. 145, no. 8, pp. 3531–3541, 2017.

[28] J. Lehec, "A direct proof of the functional Santaló inequality," *Comptes Rendus Mathematique*, vol. 347, no. 1-2, pp. 55–58, 2009.

[29] J. Lehec, "Partitions and functional Santaló inequalities," *Archiv der Mathematik*, vol. 92, no. 1, pp. 89–94, 2009.

[30] B. Li, C. Schütt, and E. M. Werner, "Floating functions," *Israel Journal of Mathematics*, vol. 231, no. 1, pp. 181–210, 2019.

[31] Y. Lin, "Affine Orlicz Pólya-Szegö principle for log-concave functions," *Journal of Functional Analysis*, vol. 273, no. 10, pp. 3295–3326, 2017.

[32] V. Milman and L. Rotem, "Mixed integrals and related inequalities," *Journal of Functional Analysis*, vol. 264, no. 2, pp. 570–604, 2013.

[33] F. Mussnig, "Volume, polar volume and Euler characteristic for convex functions," *Advances in Mathematics*, vol. 344, pp. 340–373, 2019.

[34] L. Rotem, "Support functions and mean width for α-concave functions," *Advances in Mathematics*, vol. 243, pp. 168–186, 2013.

[35] A. Colesanti and I. Fragalà, "The first variation of the total mass of log-concave functions and related inequalities," *Advances in Mathematics*, vol. 244, pp. 708–749, 2013.

[36] C. M. Petty, "Geominimal surface area," *Geometriae Dedicata*, vol. 3, pp. 77–97, 1974.

[37] E. Lutwak, "The Brunn–Minkowski–Firey theory II," *Advances in Mathematics*, vol. 118, no. 2, pp. 244–294, 1996.

[38] D. Ye, "Lp geominimal surface areas and their inequalities," *International Mathematics Research Notices*, vol. 9, pp. 2465–2498, 2015.

[39] D. Ye, "Dual Orlicz–Brunn–Minkowski theory: dual Orlicz L_φ affine and geominimal surface areas," *Journal of Mathematical Analysis and Applications*, vol. 443, no. 1, pp. 352–371, 2016.

[40] D. Ye, B. Zhu, and J. Zhou, "The mixed Lp geominimal surface areas for multiple convex bodies," *Indiana University Mathematics Journal*, vol. 64, no. 5, pp. 1513–1552, 2015.

[41] B. Zhu, N. Li, and J. Zhou, "Isoperimetric inequalities for L_p geominimal surface area," *Glasgow Mathematical Journal*, vol. 53, no. 3, pp. 717–726, 2011.

[42] B. Zhu, J. Zhou, and W. Xu, "Lp mixed geominimal surface area," *Journal of Mathematical Analysis and Applications*, vol. 422, no. 2, pp. 1247–1263, 2015.

[43] T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[44] L. Rotem, "On the mean width of log-concave functions," in *Geometric Aspects of Functional Analysis*, B. Klartag, S. Mendelson, and V. Milman, Eds., vol. 2050 of Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2012.

[45] E. Lutwak, "The Brunn-Minkowski-Firey theory I Mixed volumes and the Minkowski problem," *Journal of Differential Geometry*, vol. 38, pp. 131–150, 1993.

[46] D. Cordero-Erausquin and B. Klartag, "Moment measures," *Journal of Functional Analysis*, vol. 268, no. 12, pp. 3834–3866, 2015.