Journal of Function Spaces

Sequence Spaces, Function Spaces and Approximation Theory

Lead Guest Editor: Mohammad Mursaleen
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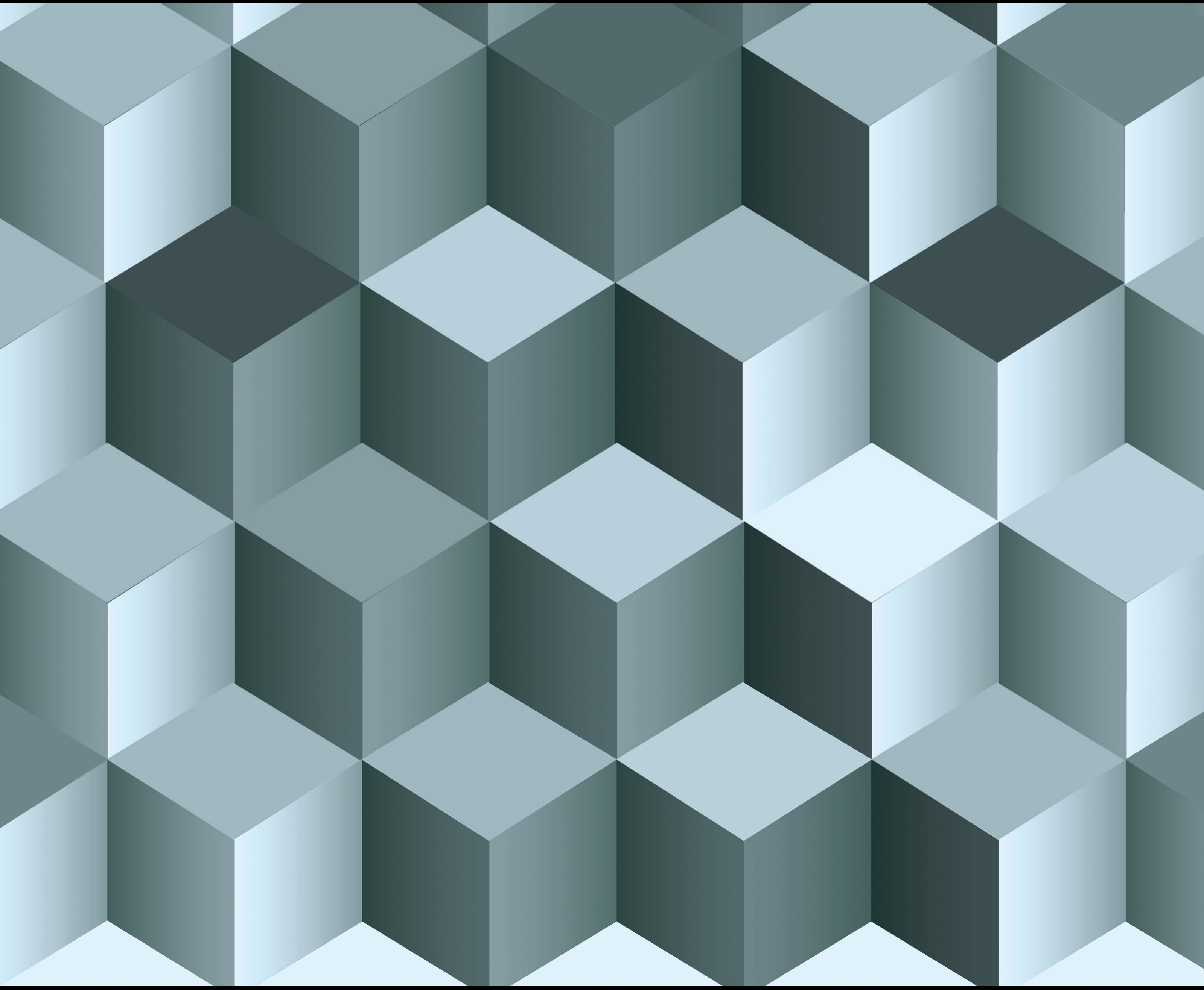

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Corrigendum

Corrigendum to "Rate of Approximation for Modified Lupaş-Jain-Beta Operators"

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In the article titled "Rate of Approximation for Modified Lupaş-Jain-Beta Operators" [1], a reference was omitted in error. The reference is shown below:

"P. Patel and V. N. Mishra, "On new class of linear and positive operators," Bollettino dell'Unione Matematica Italiana, vol. 8, no. 2, pp. 81–96, 2015" [2].

This reference should have been included as reference 4. The following sentence should have been included at the beginning of the fourth paragraph in the introduction:

"Motivated by the work of Jain in 2015, Patel and Mishra [4] introduce a new generalization of Lupa\c{s} operators and also gave Kantorovich and Durmeyer type modifications of these operators".

References

[1] M. Qasim, A. Khan, Z. Abbas, P. Raina, and Q.-B. Cai, "Rate of approximation for modified Lupaş-Jain-Beta operators," *Journal of Function Spaces*, vol. 2020, Article ID 5090282, 2020.

[2] P. Patel and V. N. Mishra, "On new class of linear and positive operators," *Bollettino dell'Unione Matematica Italiana*, vol. 8, no. 2, pp. 81–96, 2015.

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Research Article

On Generalized (p, q)-Euler Matrix and Associated Sequence Spaces

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In this study, we introduce new BK-spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ derived by the domain of (p, q)-analogue $B^{r,t}(p, q)$ of the binomial matrix in the spaces ℓ_s and ℓ_∞, respectively. We study certain topological properties and inclusion relations of these spaces. We obtain a basis for the space $b_s^{r,t}(p, q)$ and obtain Köthe-Toeplitz duals of the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$. We characterize certain classes of matrix mappings from the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ to space $\mu \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$. Finally, we investigate certain geometric properties of the space $b_s^{r,t}(p, q)$.

1. Introduction and Preliminaries

The (p, q)-calculus has been a wide and interesting area of research in recent times. Several researchers have worked in the field of (p, q)-calculus due to its vast applications in mathematics, physics, and engineering sciences. In the field of mathematics, it is widely used by researchers in operator theory, approximation theory, hypergeometric functions, special functions, quantum algebras, combinatorics, etc. By (p, q)-analogue of a known mathematical expression, we mean the generalization of that expression using two independent variables p and q rather than a single variable q as in q-calculus. If we put $p = 1$ in the (p, q)-analogue of a known mathematical expression, we get q-analogue of that expression. Furthermore, when $q \longrightarrow 1$, we receive the original expression. Chakrabarti and Jagannathan [1] introduced (p, q)-number to generalize several forms of q-oscillator algebras. Since then, several researchers used (p, q)-theory in different fields of mathematics to extend the theory of single parameter q-calculus. We strictly refer to [1–8] for studies in (p, q)-calculus and [9] in q-calculus.

1.1. Notations and Definitions on (p, q)-Calculus

Definition 1 (see [5]). Let $0 < q < p \le 1$. Then, twin basic number or (p, q)-number is defined by

$$[i]_{pq} = \begin{cases} \dfrac{p^i - q^i}{p - q} & (i > 0), \\ 0 & (i = 0). \end{cases} \tag{1}$$

Clearly, when $p = 1, [i]_{pq}$ reduces to its q version $[i]_q$.

Definition 2 (see [4]). The (p, q)-analogue of binomial coefficient or (p, q)-binomial coefficient is defined by

$$\begin{bmatrix} i \\ j \end{bmatrix}_{pq} = \begin{cases} \dfrac{[i]_{pq}!}{[i-j]_{pq}![j]!} & (i \ge j), \\ 0 & (j > i), \end{cases} \tag{2}$$

where (p, q)-factorial $[i]_{pq}$ of i is given by

$$[i]_{pq}! = [i]_{pq}[i-1]_{pq} \cdots [2]_{pq}[1]_{pq}. \quad (3)$$

Lemma 3. *The* (p, q) *-binomial formula is defined by*

$$(x \oplus y)^i = \begin{cases} (x+y)(qx+py)\left(q^2x+p^2y\right)\cdots\left(q^{(i-1)}x+p^{i-1}y\right) & (i \geq 1), \\ 1 & (i=0) \end{cases}$$

$$= \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} x^j y^{i-j}. \quad (4)$$

1.2. Sequence Spaces. Let w denote the set of all real-valued sequences. Any linear subspace of w is called sequence space. The following are some sequence spaces which we shall be frequently used throughout this paper:

$$\begin{aligned} \ell_\infty &= \left\{x=(x_i)\in w : \sup_{i\in\mathbb{N}_0}|x_i|<\infty\right\}, \ell_s = \left\{x=(x_i)\in w : \sum_{i=0}^{\infty}|x_i|^s<\infty\right\}, (1\leq s<\infty), \\ c &= \left\{x=(x_i)\in w : \lim_{i\longrightarrow\infty} x_i \text{ exists}\right\}, c_0 = \left\{x=(x_i)\in w : \lim_{i\longrightarrow\infty} x_i = 0\right\}, \\ cs &= \left\{x=(x_i)\in w : \lim_{i\longrightarrow\infty}\sum_{j=0}^{i} x_j \text{ exists}\right\}, cs_0 = \left\{x=(x_i)\in w : \lim_{i\longrightarrow\infty}\sum_{j=0}^{i} x_j = 0\right\}, \end{aligned} \quad (5)$$

and bs denotes the space of all bounded series.

Here, $\mathbb{N}_0$ denotes the set of all natural numbers including zero. The sequence spaces ℓ_s and ℓ_∞ are Banach spaces equipped with the norms

$$\|x\|_{\ell_s} = \left(\sum_{i=0}^{\infty}|x_i|^s\right)^{1/s} \text{ and } \|x\|_{\ell_\infty} = \sup_{i\in\mathbb{N}_0}|x_i|, \quad (6)$$

respectively.

Let λ and μ be two sequence spaces and $\Phi = (\phi_{ij})$ be an infinite matrix of real entries. By Φ_i, we denote the i^{th} row of the matrix Φ. We say that Φ defines a matrix mapping from λ to μ if $\Phi x \in \mu$ for every $x = (x_j) \in \lambda$, where $\Phi x = \{(\Phi x)_i\} = \{\sum_{j=0}^{\infty}\phi_{ij}x_j\}$ is Φ-transform of the sequence x. The notation $(\lambda : \mu)$ will denote the family of all matrices that map from λ to μ.

The matrix domain λ_Φ of the matrix Φ in the space λ is defined by

$$\lambda_\Phi = \{x \in w : \Phi x \in \lambda\}, \quad (7)$$

which itself is a sequence space. Using this notation, several authors in the past have constructed sequence spaces using some special matrices. For relevant literature, we refer to the papers [10–15] and textbooks [16–18]. For some recent publications dealing with the domain of triangles in classical spaces, we refer [19–28].

1.3. Literature Review. We give a short survey of literature concerning Euler sequence spaces. Altay and Başar [10] introduced Euler sequence space $e_0^r = (c_0)_{E^r}$ and $e_\infty^r = (\ell_\infty)_{E^r}$, obtained their α-, β-, γ-, and continuous duals, and characterized certain class of matrix mappings on the space $(c)_{E^r}$, where $E^r = (e_{ij}^r)$ denotes the Euler matrix of order r and is defined by

$$e_{ij}^r = \begin{cases} \binom{i}{j}(1-r)^{i-j}r^j & (0\leq j\leq i), \\ 0 & (j>i), \end{cases} \quad (8)$$

for all $i, j \in \mathbb{N}_0$ and $0 < r < 1$. The Euler matrix E^r is regular for $0 < r < 1$ and is invertible with $(E^r)^{-1} = E^{1/r}$.

Altay et al. [11] introduced the Euler space $e_s^r = (\ell_s)_{E^r}$, $1 \leq s \leq \infty$, and obtained certain inclusion relations, Schauder basis and Köthe-Toeplitz duals of the space e_s^r. As a natural continuation of [11], Mursaleen et al. [14] characterized various classes of matrix mappings from the space e_s^r to other spaces and examined certain geometric properties of the space e_s^r. Further, Altay and Polat [29] introduced Euler difference spaces $e_0^r(\Delta^B) = (c_0)_{E^r\Delta^B}$ and $e_c^r(\Delta^B) = (c)_{E^r\Delta^B}$, where Δ^B is backward difference operator defined by $(\Delta^B v)_j = v_j - v_{j-1}$ for all $j \in \mathbb{N}_0$. Extending these spaces, Polat and Başar [30] studied Euler difference spaces $e_0^r(\Delta^{Bm})$,$e_c^r(\Delta^{Bm})$, and $e_\infty^r(\Delta^{Bm})$ of $m^{\text{th}}(m\in\mathbb{N})$ order defined as the set of all sequences whose m^{th} order backward differences are in the spaces e_0^r,e_c^r, and e_∞^r, respectively. Kadak and Baliarsingh [31] further generalized these spaces by introducing Euler difference spaces $e_s^r(\Delta^{Bq})$,$e_0^r(\Delta^{Bq})$,$e_c^r(\Delta^{Bq})$, and $e_\infty^r(\Delta^{Bq})$ of fractional order q, where Δ^{Bq} is the backward fractional difference operator defined by $(\Delta^{Bq}x)_j = \sum_{i=0}^{\infty}(-1)^i\Gamma(q+1)/i!\Gamma(q-i+1)x_{j-i}$. Kara et al. [32] introduced paranormed Euler space $e^r(p) = (\ell(p))_{E^r}$ and studied its topological and geometric properties. Aftermore, Karakaya and Polat [33] studied paranormed Euler difference sequence spaces $e_0^r(\Delta^B, p)$,

$e_c^r(\Delta^B, p)$, and $e_\infty^r(\Delta^B, p)$. Extending these spaces, Karakaya et al. [34] studied paranormed Euler backward difference spaces $e_0^r(\Delta^{Bm}, p)$,$e_c^r(\Delta^{Bm}, p)$, and $e_\infty^r(\Delta^{Bm}, p)$ of m^{th} order. Besides, Demiriz and Çakan [35] introduced paranormed Euler difference spaces $e_0^r(u,p)$ and $e_c^r(u,p)$. Furthermore, Kirisci [36] introduced Euler almost null $f_0(E^r)$ and Euler almost convergent $f(E^r)$ sequence spaces. Later on, Kara and Başar [37] introduced generalized difference Euler spaces $e_0^r(B^{(m)}) = (c_0)_{B^{(m)}}$,$e_c^r(B^{(m)}) = (c)_{B^{(m)}}$ and $e_\infty^r(B^{(m)}) = (\ell_\infty)_{B^{(m)}}$, where $B^{(m)} = (b_{ij}^{(m)})$ is a generalized difference matrix defined by

$$b_{ij}^{(m)} = \begin{cases} \begin{pmatrix} m \\ i-j \end{pmatrix} r^{m-i+j} s^{i-j} & (\max\{0, i-m\} \le j \le i), \\ 0 & (0 \le j \le \max\{0, i-m\}) \text{ or } (j > i), \end{cases} \tag{9}$$

and characterized certain classes of compact operators on the spaces $e_0^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$. Meng and Mei [38] gave a further generalization of [37] by introducing Euler difference spaces $e_0^r(B_v^{(m)})$ and $e_c^r(B_v^{(m)})$, where the difference operator $B_v^{(m)} = (b_{ij}^{(m),v})$ is defined by

$$b_{ij}^{(m),v} = \begin{cases} \begin{pmatrix} m \\ i-j \end{pmatrix} r^{m-i+j} s^{i-j} v_j & (\max\{0, i-m\} \le j \le i), \\ 0 & (0 \le j \le \max\{0, i-m\}) \text{ or } (j > i), \end{cases} \tag{10}$$

where $v = (v_j)$ is a fixed sequence of nonzero real numbers. Recently, Bisgin [39, 40] introduced more generalized Euler space by defining binomial spaces $b_s^{r,t} = (\ell_s)_{B^{r,t}}$, $b_0^{r,t} = (c_0)_{B^{r,t}}$,$b_c^{r,t} = (c)_{B^{r,t}}$, and $b_\infty^{r,t} = (\ell_\infty)_{B^{r,t}}$, and $B^{r,t} = (b_{ij}^{r,t})$ is the binomial matrix defined by

$$b_{ij}^{r,t} = \begin{cases} \dfrac{1}{(r+t)^i} \begin{pmatrix} i \\ j \end{pmatrix} r^i s^{i-j} & (0 \le j \le i), \\ 0 & (j > i). \end{cases} \tag{11}$$

Meng and Song [41] further generalized these spaces by introducing binomial $B^{(m)}$-difference sequence spaces $b_0^{r,t}(B^{(m)}) = (c_0)_{B^{r,t}B^{(m)}}$,$b_c^{r,t}(B^{(m)}) = (c)_{B^{r,t}B^{(m)}}$, and $e_\infty^r(B^{(m)}) = (\ell_\infty)_{B^{r,t}B^{(m)}}$. Meng and Mei [42] studied binomial backward difference sequence spaces $b_0^{r,t}(\Delta^{Bq}) = (c_0)_{B^{r,t}\Delta^{Bq}}$,$b_c^{r,t}(\Delta^{Bq}) = (c)_{B^{r,t}\Delta^{Bq}}$, and $b_\infty^{r,t}(\Delta^{Bq}) = (\ell_\infty)_{B^{r,t}\Delta^{Bq}}$ of fractional order q. Besides, Yaying and Hazarika [27] also studied binomial backward difference spaces $b_s^{r,t}(\Delta^{Bq}) = (\ell_s)_{B^{r,t}\Delta^{Bq}}$ of fractional order q.

For $0 < q < 1$, the q-Cesàro matrix $C(q) = (c_{ij}^q)$ [43, 44] is defined by

$$c_{ij}^q = \begin{cases} \dfrac{q^j}{[i+1]_q} & (0 \le j \le i), \\ 0 & (j > i). \end{cases} \tag{12}$$

Demiriz and Sahin [45] studied the domain of q-Cesàro mean in the spaces c and c_0. Very recently, Yaying et al. [28] studied Banach sequence spaces $\mathcal{X}_s^q$ and $\mathcal{X}_\infty^q$ defined as the domain of q-Cesàro mean in the spaces ℓ_s and ℓ_∞, respectively, and studied associated operator ideals.

Motivated by the above studies, we generalize Euler mean E^r and Binomial mean $B^{r,t}$ in the sense of (p,q)-theory to $B^{r,t}(p,q)$ and study its domain $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ in the spaces ℓ_s and ℓ_∞, respectively. We investigate some topological properties and inclusion relations of the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ and obtain a basis for the space $b_s^{r,t}(p,q)$. In Section 3, we obtain the Köthe-Toeplitz duals (α-, β-, and γ-duals) of the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$. In Section 4, we characterize some matrix mappings from $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ spaces to space $\mu \in \{\ell_\infty, c, c_0, \ell_1, cs, cs_0, bs\}$. Section 5 is devoted to investigation of certain geometric properties like Banach-Saks of type s and modulus of convexity of the space $b_s^{r,t}(p,q)$.

In the rest of the paper, $1 \le s < \infty$, unless stated otherwise.

2. Generalized Euler Sequence Spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$

In this section, we introduce sequence spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$, study their topological properties and some inclusion relations, and obtain a basis for the space $b_s^{r,t}(p,q)$.

Let r, t be nonnegative real numbers and $0 < q < p \le 1$ holds, then the generalized (p,q)-Euler matrix $B^{r,t}(p,q) = (b_{ij}^{r,t})$ of order (r,t) is defined by

$$b_{ij}^{r,t} = \begin{cases} \dfrac{1}{(r \oplus t)^i} \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\begin{pmatrix} i-j \\ 2 \end{pmatrix}} q^{\begin{pmatrix} j \\ 2 \end{pmatrix}} r^j t^{i-j} & (0 \le j \le i), \\ 0 & (j > i). \end{cases} \tag{13}$$

One can clearly observe that the matrix $B^{r,t}(p,q)$ reduces to the binomial matrix $B^{r,t}$ when $p = q = 1$. Thus, $B^{r,t}(p,q)$ generalizes binomial matrix $B^{r,t}$. We may call the matrix $B^{r,t}(p,q)$ as the (p,q)-analogue of the binomial matrix $B^{r,t}$. We also realise that when $p = 1$, the matrix $B^{r,t}(p,q)$ reduces to its q-version $B^{r,t}(q)$ with entries $\begin{bmatrix} i \\ j \end{bmatrix}_q q^{\begin{pmatrix} j \\ 2 \end{pmatrix}} r^j t^{i-j} / (r \oplus t)_q^i$ if $0 \le j \le i$ and 0, otherwise. We call $B^{r,t}(q)$ as the q-analogue of the binomial matrix $B^{r,t}$. Moreover, when $t=1-r$, then the matrix $B^{r,t}(p,q)$ reduces to $E^r(p,q)$ with entries $1/(r\oplus(1-r))^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j(1-r)^{i-j}$ if $0\le j\le i$ and 0, otherwise. The generalized (p,q)-Euler sequence spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ are defined by

$$\begin{aligned} b_s^{r,t}(p,q) &= \left\{ x=(x_j)\in w : \sum_{i=0}^{\infty}\left| \frac{1}{(r\oplus t)^i}\sum_{j=0}^{i}\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}\right|^s <\infty \right\}, \\ b_\infty^{r,t}(p,q) &= \left\{ x=(x_j)\in w : \sup_{i\in\mathbb{N}_0}\left| \frac{1}{(r\oplus t)^i}\sum_{j=0}^{i}\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}\right| <\infty \right\}. \end{aligned} \tag{14}$$

The above sequence spaces can be redefined in the notation of (7) by

$$b_s^{r,t}(p,q)=(\ell_s)_{B^{r,t}(p,q)} \text{ and } b_\infty^{r,t}=(\ell_\infty)_{B^{r,t}(p,q)}. \tag{15}$$

The spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ reduce to the following classes of spaces in the special cases of (p,q) and (r,t):

(1) When $p=1$, the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ reduce to q-binomial sequence spaces $b_s^{r,t}(q)=(\ell_s)_{B^{r,t}(q)}$ and $b_\infty^{r,t}(q)=(\ell_\infty)_{B^{r,t}(q)}$, respectively, which further reduce to binomial sequence spaces $b_s^{r,t}$ and $b_\infty^{r,t}$, respectively, when $q\longrightarrow 1$, as studied by Bisgin [40]

(2) When $p=1$ and $r+t=1$, the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ reduce to q-Euler space $e_s^r(q)=(\ell_s)_{E^r(q)}$ and $e_\infty^r(q)=(\ell_\infty)_{E^r(q)}$, respectively, which further reduce to well known Euler sequence spaces e_s^r and e_∞^r, respectively, when $q\longrightarrow 1$, as studied by Altay et al. [11]

(3) When $r+t=1$, the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ reduce to (p,q)-Euler sequence spaces $e_s^r(p,q)=(\ell_s)_{E^r(p,q)}$ and $e_\infty^r(p,q)=(\ell_\infty)_{E^r(p,q)}$

Let us define a sequence $y=(y_i)$ in terms of sequence $x=(x_j)$ by

$$y_i=\left(B^{r,t}(p,q)x\right)_i=\frac{1}{(r\oplus t)^i}\sum_{j=0}^{i}\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}x_j, \tag{16}$$

for each $i\in\mathbb{N}_0$. The sequence y is called $B^{r,t}(p,q)$-transform of the sequence x. Further, on using (16), we write

$$x_i=\sum_{j=0}^{i}(-1)^{i-j}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r\oplus t)^j}{r^i q^{\binom{i}{2}}}y_j, \tag{17}$$

for each $i\in\mathbb{N}_0$.

It is known that if λ is a BK-space and Φ is a triangle then the domain λ_Φ of the matrix Φ in the space λ is also a BK-space equipped with the norm $\|x\|_{\lambda_\Phi}=\|\Phi x\|_\lambda$. In the light of this, we have the following result.

Theorem 4. *The sequence spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ are BK-spaces equipped with the norms defined by*

$$\begin{aligned} \|x\|_{b_s^{r,t}(p,q)} &= \left\|B^{r,t}(p,q)x\right\|_{\ell_s} \\ &= \left(\sum_{i=0}^{\infty}\left|\frac{1}{(r\oplus t)_{pq}^i}\sum_{j=0}^{i}\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}x_j\right|^s\right)^{1/s}, \\ \|x\|_{b_\infty^{r,t}(p,q)} &= \left\|B^{r,t}(p,q)x\right\|_{\ell_\infty} \\ &= \sup_{i\in\mathbb{N}_0}\left|\frac{1}{(r\oplus t)_{pq}^i}\sum_{j=0}^{i}\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}x_j\right|, \end{aligned} \tag{18}$$

respectively.

Proof. The proof is a routine exercise and hence omitted. □

Theorem 5. *The sequence spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ are linearly isomorphic to ℓ_s and ℓ_∞, respectively.*

Proof. We provide the proof for the space $b_s^{r,t}(p,q)$. Define the mapping $T : b_s^{r,t}(p,q) \longrightarrow \ell_s$ by $Tx = B^{r,t}(p,q)x$ for all $x \in b_s^{r,t}(p,q)$. It is easy to observe that T is linear and one to one. Let $y=(y_i)\in \ell_s$ and $x=(x_i)$ is as defined in (17). Then, we have

$$\begin{aligned}\|x\|_{b_s^{r,t}(p,q)} &= \left(\sum_{i=0}^{\infty}\left|\frac{1}{(r\oplus t)_{pq}^i}\sum_{j=0}^{i}\begin{bmatrix} i\\ j\end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j} x_j\right|^s\right)^{1/s}\\ &= \left(\sum_{i=0}^{\infty}\left|\frac{1}{(r\oplus t)_{pq}^i}\sum_{j=0}^{i}\begin{bmatrix} i\\ j\end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j} x_j\right.\right.\\ &\quad\left.\left.\cdot\left(\sum_{k=0}^{j}(-1)^{j-k}\frac{\begin{bmatrix} i\\ j\end{bmatrix}_{pq} q^{\binom{j-k}{2}} t^{j-k}(r\oplus t)^k}{r^j q^{\binom{j}{2}}} y_k\right)\right|^s\right)^{1/s}\\ &= \left(\sum_{i=0}^{\infty}|y_i|^s\right)^{1/s} = \|y\|_{\ell_s} < \infty.\end{aligned} \tag{19}$$

Thus, $x \in b_s^{r,t}(p,q)$ and the mapping $T : b_s^{r,t}(p,q) \longrightarrow \ell_s$ is onto and norm preserving. Hence, the space $b_s^{r,t}(p,q)$ is linearly isomorphic to ℓ_s. This completes the proof. □

Theorem 6. *The space $b_s^{r,t}(p,q)$, $1\le s\le\infty$, is not a Hilbert space, except for the case $s=2$.*

Proof. Define the sequences $x=(x_i)$ and $y=(y_i)$ by

$$x_i = \begin{cases} 1 & (i=0),\\ (-1)^i\left(\left(\dfrac{t}{r}\right)^i - \dfrac{(r\oplus t)q^{\binom{i-1}{2}} t^{i-1}\begin{bmatrix} i\\ 1\end{bmatrix}_{pq}}{r^i q^{\binom{i}{2}}}\right) & (i>0),\end{cases}$$

$$y_i = \begin{cases} 1 & (i=0),\\ (-1)^i\left(\left(\dfrac{t}{r}\right)^i + \dfrac{(r\oplus t)q^{\binom{i-1}{2}} t^{i-1}\begin{bmatrix} i\\ 1\end{bmatrix}_{pq}}{r^i q^{\binom{i}{2}}}\right) & (i>0).\end{cases} \tag{20}$$

We realise that $(B^{r,t}(p,q)x)_i = (1,1,0,0,\cdots)$ and $(B^{r,t}(p,q)y)_i = (1,-1,0,0,\cdots)$. Then

$$\begin{aligned}\|x+y\|_{b_s^{r,t}(p,q)}^2 + \|x-y\|_{b_s^{r,t}(p,q)}^2 &= 8 \ne 2^{2+2/s}\\ &= 2\left(\|x\|_{b_s^{r,t}(p,q)}^2 + \|y\|_{b_s^{r,t}(p,q)}^2\right).\end{aligned} \tag{21}$$

Thus, $b_s^{r,t}(p,q)$ norm violates the parallelogram identity. Hence, $b_s^{r,t}(p,q)$ is not a Hilbert space, except for the case $s=2$. □

Now we give certain inclusion relations related to the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$.

Theorem 7. *The inclusion $\ell_s \subset b_s^{r,t}(p,q)$, $1\le s\le\infty$, strictly holds.*

Proof. We provide proof of the inclusion $\ell_s \subset b_s^{r,t}(p,q)$, $1\le s<\infty$. Let $x=(x_i)\in\ell_s$ for $1<s<\infty$. Applying Hölder's inequality, we have

$$\begin{aligned}\sum_{i=0}^{\infty}\left|(B^{r,t}(p,q)x)_i\right|^s &\le \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{\begin{bmatrix} i\\ j\end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}}{(r\oplus t)^i}|x_j|\right)^s\\ &\le \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{\begin{bmatrix} i\\ j\end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}}{(r\oplus t)^i}|x_j|^s\right.\end{aligned}$$

$$\cdot\left(\sum_{j=0}^{i}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}}{(r\oplus t)^i}\right)^{s-1}$$

$$=\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}}{(r\oplus t)^i}|x_j|^s\right)$$

$$=\sum_{j=0}^{\infty}|x_j|^s\left(\sum_{i=j}^{\infty}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}}{(r\oplus t)^i}\right). \tag{22}$$

Thus, $\|x\|^s_{b_s^{r,t}(p,q)} \le K\|x\|^s_{\ell_s} < \infty$, where $K=\sup_{j\in\mathbb{N}_0}\left(\sum_{i=j}^{\infty}\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j}/(r\oplus t)^i\right)$, provided K exists. This yields the fact that $x\in b_s^{r,t}(p,q)$. Thus, $\ell_s\subset b_s^{r,t}(p,q)$. Similarly, we can show that $\ell_1\subset b_1^{r,t}(p,q)$.

Now consider the sequence $x_i=(-1)^i$, then it is easy to see that $x_i\in b_s^{r,t}(p,q)\setminus\ell_s$. Hence, the inclusion $\ell_s\subset b_s^{r,t}(p,q)$ is strict. This completes the proof. □

Theorem 8. *The inclusion $b_s^{r,t}(p,q)\subset b_k^{r,t}(p,q), 1\le s<k<\infty$, strictly holds.*

Proof. It is known that inclusion $\ell_s\subset\ell_k$ holds for $1\le s<k<\infty$ and the mapping $B^{r,t}(p,q)$: $b_s^{r,t}(p,q)\longrightarrow\ell_s$ is isomorphic, therefore, the inclusion $b_s^{r,t}(p,q)\subset b_k^{r,t}(p,q)$ holds. To prove the strictness part, we recall that the inclusion $\ell_s\subset\ell_k$ strictly holds for $1\le s<k<\infty$. We choose $y\in\ell_k\setminus\ell_s$ and x as defined in (17). Then, $B^{r,t}(p,q)x=y\in\ell_k\setminus\ell_s$. This implies that $x\in b_k^{r,t}(p,q)\setminus b_s^{r,t}(p,q)$. Hence, the inclusion $b_s^{r,t}(p,q)\subset b_k^{r,t}(p,q)$ is strict. □

Theorem 9. *The inclusion $b_s^{r,t}(p,q)\subset b_\infty^{r,t}(p,q)$ strictly holds.*

Proof. The proof is similar to the proof of Theorem 8. To show the strictness part, we consider the sequence $x_k=(1,1,1,\cdots)$. Then, it is clear that $x\in b_\infty^{r,t}(p,q)\setminus b_s^{r,t}(p,q)$. Hence, the inclusion $b_s^{r,t}(p,q)\subset b_\infty^{r,t}(p,q)$ strictly holds. □

We recall that domain λ_Φ of a triangle Φ in space λ has a basis if and only if λ has a basis. This statement together with Theorem 5 gives us the following result.

Theorem 10. *Let $\xi_j=(B^{r,t}(p,q)x)_j$ for each $j\in\mathbb{N}_0$. Define the sequence $b^{(j)}(p,q)=(b_i^{(j)}(p,q))$ of elements of the space $b_s^{r,t}(p,q)$ for every fixed $j\in\mathbb{N}_0$ by*

$$b_i^{(j)}(p,q)=\begin{cases}(-1)^{i-j}\dfrac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r\oplus t)^j}{r^i q^{\binom{i}{2}}} & (j\le i),\\ 0 & (j>i).\end{cases} \tag{23}$$

Then, the sequence $\{b^{(j)}(p,q)\}$ forms a basis for the space $b_s^{r,t}(p,q)$ and every $x\in b_s^{r,t}(p,q)$ can be uniquely expressed in the form $x=\sum_{j=0}^{\infty}\xi_j b^{(j)}(p,q)$ for each $j\in\mathbb{N}_0$.

3. Köthe-Toeplitz Duals

In this section, we obtain Köthe-Toeplitz duals (α-, β-, and γ-duals) of the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$. We omit the proofs for cases $s=1$ and $s=\infty$ as these can be obtained by analogy and provide proofs for only the case $1<s<\infty$ in the current section. First, we recall the definitions of Köthe-Toeplitz duals.

Definition 11. The Köthe-Toeplitz duals or $\alpha-,\beta-$, and γ-duals of subset $\lambda\subset w$ are defined by

$$\begin{aligned}\lambda^\alpha&=\{a=(a_i)\in w: ax=(a_ix_i)\in\ell_1 \text{ for all } x\in\lambda\},\\ \lambda^\beta&=\{a=(a_i)\in w: ax=(a_ix_i)\in cs \text{ for all } x\in\lambda\} \text{ and},\\ \lambda^\gamma&=\{a=(a_i)\in w: ax=(a_ix_i)\in bs \text{ for all } x\in\lambda\},\end{aligned} \tag{24}$$

respectively.

Quite recently, Talebi [25] obtained Köthe-Toeplitz duals of the domain of an arbitrary invertible summability matrix in ℓ_s space. We follow his approach to find the Köthe-Toeplitz duals of the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$. In the rest of the paper, $\mathcal{N}$ will denote the family of all finite subsets of $\mathbb{N}_0$ and $k=s/1-s$ is the complement of s.

Theorem 12. *Define the sets* $d^{(k)}(p,q)$ *and* $d_{\infty}(p,q)$ *by*

$$d^{(k)}(p,q)=\left\{a=(a_i)\in w:\sup_{N\in\mathcal{N}}\sum_{j=0}^{\infty}\left|\sum_{i\in N}(-1)^{i-j}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r\oplus t)^{j}}{r^{i}q^{\binom{i}{2}}}a_i\right|^{k}<\infty\right\}\text{ and,}$$
$$d_{\infty}(p,q)=\left\{a=(a_i)\in w:\sup_{j\in\mathbb{N}_0}\sum_{i=0}^{\infty}\left|(-1)^{i-j}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r\oplus t)^{j}}{r^{i}q^{\binom{i}{2}}}a_i\right|<\infty\right\}. \tag{25}$$

Then, $[b_1^{r,t}(p,q)]^{\alpha}=d_{\infty}(p,q),[b_s^{r,t}(p,q)]^{\alpha}=d^{(k)}(p,q)$ *and* $[b_{\infty}^{r,t}(p,q)]^{\alpha}=d^{(1)}(p,q)$.

Proof. Let $1<s<\infty$. Let $(a_i)\in w$ and $y=(y_i)$ be the $B^{r,t}(p,q)$ -transform of sequence $x=(x_i)$. Then, from the equality (17), we have

$$a_i x_i=\sum_{j=0}^{i}(-1)^{i-j}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r\oplus t)^{j}}{r^{i}q^{\binom{i}{2}}}a_i y_j=\left(G^{r,t}(p,q)y\right)_i, \tag{26}$$

for all $i\in\mathbb{N}_0$, where the matrix $G^{r,t}(p,q)=(g_{ij}^{r,t}(p,q))$ is defined by

$$g_{ij}^{r,t}(p,q)=\begin{cases}(-1)^{i-j}\dfrac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r\oplus t)^{j}}{r^{i}q^{\binom{i}{2}}}a_i & (0\le j\le i),\\ 0 & (j>i).\end{cases} \tag{27}$$

Applying Theorem 2.1 of [25], we immediately obtained that

$$\left[b_s^{r,t}(p,q)\right]^{\alpha}=\left\{a=(a_i)\in w:\sup_{N\in\mathcal{N}}\sum_{j=0}^{\infty}\left|\sum_{i\in N}(-1)^{i-j}\frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r\oplus t)^{j}}{r^{i}q^{\binom{i}{2}}}a_i\right|^{k}<\infty\right\}. \tag{28}$$

This completes the proof. □

Theorem 13. *Define the sets* $d_1(p,q)$,$d_2(p,q)$,$d_3(p,q)$, *and* $d^{[k]}(p,q)$ *by*

$$d_1(p,q)=\left\{a=(a_l)\in w:\sum_{l=j}^{\infty}(-1)^{l-j}\frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^j}{r^l q^{\binom{l}{2}}}a_l \text{ exists for each } j\in\mathbb{N}_0\right\},$$

$$d_2(p,q)=\left\{a=(a_l)\in w:\sup_{i,j\in\mathbb{N}_0}\left|\sum_{l=j}^{i}(-1)^{l-j}\frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^j}{r^l q^{\binom{l}{2}}}a_l\right|<\infty\right\},$$

$$d_3(p,q)=\left\{a=(a_l)\in w:\lim_{i\longrightarrow\infty}\sum_{j=0}^{i}\left|\sum_{l=j}^{i}(-1)^{l-j}\frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^j}{r^l q^{\binom{l}{2}}}a_l\right|=\sum_{j=0}^{i}\left|\lim_{i\longrightarrow\infty}\sum_{l=j}^{i}(-1)^{l-j}\frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^j}{r^l q^{\binom{l}{2}}}a_l\right|\right\},$$

$$d^{[k]}(p,q)=\left\{a=(a_l)\in w:\sup_{i\in\mathbb{N}_0}\sum_{j=0}^{i}\left|\sum_{l=j}^{i}(-1)^{l-j}\frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^j}{r^l q^{\binom{l}{2}}}a_l\right|^k<\infty\right\}. \tag{29}$$

Then, $[b_l^{r,t}(p,q)]^{\beta}=d_1(p,q)\cap d_2(p,q)$,$[b_s^{r,t}]^{\beta}=d_1(p,q)\cap d^{[k]}(p,q)$ *and* $[b_{\infty}^{r,t}(p,q)]^{\beta}=d_1(p,q)\cap d_3(p,q)$.

Proof. Let $(a_i)\in w$ and $y=(y_i)$ be the $B^{r,t}(p,q)$-transform of sequence $x=(x_i)$. Then, from the equality (17), we get

$$\begin{aligned}\sum_{j=0}^{i}a_jx_j&=\sum_{j=0}^{i}\left\{\sum_{l=0}^{j}(-1)^{j-l}\frac{\begin{bmatrix} j \\ l \end{bmatrix}_{pq} q^{\binom{j-l}{2}} t^{j-l}(r\oplus t)^l}{r^j q^{\binom{j}{2}}}y_l\right\}a_j\\&=\sum_{j=0}^{i}\left\{\sum_{l=j}^{i}(-1)^{l-j}\frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^j}{r^l q^{\binom{l}{2}}}a_l\right\}y_j,\\&=\left(H^{r,t}(p,q)y\right)_i,\end{aligned} \tag{30}$$

for each $i\in\mathbb{N}_0$, where the matrix $H^{r,t}(p,q)=(h_{ij}^{r,t}(p,q))$ is defined by

$$h_{ij}^{r,t}(p,q)=\begin{cases}\sum_{l=j}^{i}(-1)^{l-j}\dfrac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^{j}}{r^{l}q^{\binom{l}{2}}}a_l & (0\le j\le i),\\ 0 & (j>i),\end{cases} \tag{31}$$

for all $i,j\in\mathbb{N}_0$.

Thus, by applying Theorem 2.2 of [25], we straightly get

$$\left[b_s^{r,t}(p,q)\right]^{\beta}=d_1(p,q)\cap d^{[k]}(p,q). \tag{32}$$

This completes the proof. □

Theorem 14. *Define the set $d_4(p,q)$ by*

$$d_4(p,q)=\left\{a=(a_i)\in w:\sup_{i\in\mathbb{N}_0}\sum_{j=0}^{i}\left|\sum_{l=j}^{i}(-1)^{l-j}\dfrac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^{j}}{r^{l}q^{\binom{l}{2}}}a_l\right|<\infty\right\}. \tag{33}$$

Then, $[b_1^{r,t}(p,q)]^{\gamma}=d_2(p,q)$, $[b_s^{r,t}(p,q)]^{\gamma}=d^{[k]}(p,q)$ *and* $[b_{\infty}^{r,t}(p,q)]^{\gamma}=d_4(p,q)$.

Proof. The proof is similar to the previous theorem except that Theorem 2.3 of [25] is employed instead of Theorem 2.2 of [25]. □

4. Matrix Mappings

In this section, we characterize a certain class of matrix mappings from the spaces $b_s^{r,t}(p,q)$ and $b_{\infty}^{r,t}(p,q)$ to space $\mu\in\{\ell_\infty,c,c_0,\ell_1,bs,cs,cs_0\}$. The following theorem is fundamental in our investigation.

Theorem 15. *Let $1\le s\le\infty$ and λ be an arbitrary subset of w. Then, $\Phi=(\phi_{ij})\in(b_s^{r,t}(p,q):\lambda)$ if and only if $\Psi^{(i)}=(\psi_{mj}^{(i)})\in(\ell_s:c)$ for each $i\in\mathbb{N}_0$, and $\Psi=(\psi_{ij})\in(\ell_s:\lambda)$, where*

$$\psi_{mj}^{(i)}=\begin{cases}0 & (j>m),\\ \sum_{l=j}^{m}(-1)^{l-j}\dfrac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^{j}}{r^{l}q^{\binom{l}{2}}}\phi_{il} & (0\le j\le m),\end{cases}$$

$$\psi_{ij}=\sum_{l=j}^{\infty}(-1)^{l-j}\dfrac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r\oplus t)^{j}}{r^{l}q^{\binom{l}{2}}}\phi_{il}. \tag{34}$$

for all $i,j\in\mathbb{N}_0$.

Proof. The proof is similar to the proof of Theorem 4.1 of [13]. Hence, we omit details. □

Now, using the results presented in Stielglitz and Tietz [46] together with Theorem 15, we obtain the following results:

Corollary 16. *The following statements hold:*

(1) $\Phi\in(b_1^{r,t}(p,q):\ell_\infty)$ if and only if

$$\lim_{m\to\infty}\psi_{mj}^{(i)}\ \text{exists for all}\ i,j\in\mathbb{N}_0, \tag{35}$$

$$\sup_{i,j\in\mathbb{N}_0}\left|\psi_{mj}^{(i)}\right|<\infty, \tag{36}$$

$$\sup_{i,j\in\mathbb{N}_0}\left|\psi_{ij}\right|<\infty, \tag{37}$$

(2) $\Phi \in (b_1^{r,t}(p,q): c)$ if and only if (35) and (36) hold, and (37) and

$$\lim_{i\longrightarrow\infty} \psi_{ij} \text{ exists for all } j \in \mathbb{N}_0, \quad (38)$$

also hold

(3) $\Phi \in (b_1^{r,t}(p,q): c_0)$ if and only if (35) and (36) hold, and (37) and

$$\lim_{i\longrightarrow\infty} \psi_{ij} = 0 \text{ for all } j \in \mathbb{N}_0, \quad (39)$$

also hold

(4) $\Phi \in (b_1^{r,t}(p,q): \ell_1)$ if and only if (35) and (36) hold, and

$$\sup_{j\in\mathbb{N}_0} \sum_{i=0}^{\infty} \left|\psi_{ij}\right| < \infty, \quad (40)$$

also holds

(5) $\Phi \in (b_1^{r,t}(p,q): bs)$ if and only if (35) and (36) hold, and (37) also holds with $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i}\psi_{lj}$

(6) $\Phi \in (b_1^{r,t}(p,q): cs)$ if and only if (35) and (36) hold, and (37) and (38) also hold with $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i}\psi_{lj}$

(7) $\Phi \in (b_1^{r,t}(p,q): cs_0)$ if and only if (35) and (36) hold, and (37) and (39) also hold with $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i}\psi_{lj}$

Corollary 17. *The following statements hold:*

(1) $\Phi \in (b_s^{r,t}(p,q): \ell_\infty)$ if and only if (35) holds, and

$$\sup_{m\in\mathbb{N}_0} \sum_{j=0}^{m} \left|\psi_{mj}^{(i)}\right|^k < \infty, \quad (41)$$

$$\sup_{i\in\mathbb{N}_0} \sum_{j=0}^{i} \left|\psi_{ij}\right|^k < \infty, \quad (42)$$

also hold

(2) $\Phi \in (b_s^{r,t}(p,q): c)$ if and only if (35) and (41) hold, and (38) and (42) also hold

(3) $\Phi \in (b_s^{r,t}(p,q): c_0)$ if and only if (35) and (41) hold, (39) and (42) also hold

(4) $\Phi \in (b_s^{r,t}(p,q): \ell_1)$ if and only if (35) and (41) hold, and

$$\sup_{N\in\mathcal{N}} \sum_{j=0}^{\infty} \left|\sum_{i\in N}\psi_{ij}\right|^k < \infty, \quad (43)$$

also holds

(5) $\Phi \in (b_s^{r,t}(p,q): bs)$ if and only if (35) and (41) hold, and (42) also holds with $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i}\psi_{lj}$

(6) $\Phi \in (b_s^{r,t}(p,q): cs)$ if and only if (35) and (41) hold, and (38) and (42) also hold

(7) $\Phi \in (b_s^{r,t}(p,q): cs_0)$ if and only if (35) and (41) hold, and (39) and (42) also hold with $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i}\psi_{lj}$

Corollary 18. *The following statements hold:*

(1) $\Phi \in (b_\infty^{r,t}(p,q): \ell_\infty)$ if and only if (35) and

$$\lim_{m\longrightarrow\infty} \sum_{j=0}^{m} \left|\psi_{mj}^{(i)}\right| = \sum_{j=0}^{m} \left|\lim_{m\longrightarrow\infty} \psi_{mj}^{(i)}\right| \text{ for each } i \in \mathbb{N}_0, \quad (44)$$

hold, and (42) also holds with $k = 1$

(2) $\Phi \in (b_\infty^{r,t}(p,q): c)$ if and only if (35) and (44) hold, and (38) and

$$\lim_{i\longrightarrow\infty} \sum_{j=0}^{i} \left|\psi_{ij}\right| = \sum_{j=0}^{i} \left|\lim_{i\longrightarrow\infty} \psi_{ij}\right|, \quad (45)$$

also hold

(3) $\Phi \in (b_\infty^{r,t}(p,q): c_0)$ if and only if (35) and (44) hold, and

$$\lim_{i\longrightarrow\infty} \sum_{j=0}^{i} \psi_{ij} = 0, \quad (46)$$

also holds

(4) $\Phi \in (b_\infty^{r,t}(p,q): \ell_1)$ if and only if (35) and (44) hold, and (43) also holds with $k = 1$

(5) $\Phi \in (b_\infty^{r,t}(p,q): bs)$ if and only if (35) and (44) hold, and (42) also hold with $k = 1$, and $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i} \psi_{lj}$

(6) $\Phi \in (b_\infty^{r,t}(p,q): cs)$ if and only if (35) and (44) hold, and (45) also holds with $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i} \psi_{lj}$

(7) $\Phi \in (b_\infty^{r,t}(p,q): cs_0)$ if and only if (35) and (44) hold, and (46) also holds with $\Psi(i,j)$ instead of ψ_{ij}, where $\Psi(i,j) = \sum_{l=0}^{i} \psi_{lj}$

We recall a basic lemma due to Basar and Altay [47] that will help in characterizing certain classes of matrix mappings from the spaces $b_s^{r,t}(p,q)$ and $b_\infty^{r,t}(p,q)$ to any arbitrary space μ.

Lemma 19 (see [47]). *Let λ and μ be any two sequence spaces, Φ be an infinite matrix and Ω be a triangular matrix. Then, $\Phi \in (\lambda : \mu_\Omega)$ if and only if $\Omega\Phi \in (\lambda : \mu)$.*

Now, by combining Lemma 19 with Corollaries 16, 17, and 18, we derive the following classes of matrix mappings:

Corollary 20. *Let $\Phi = (\phi_{ij})$ be an infinite matrix and define the matrix $C^\alpha = (c^\alpha_{ij})$ by*

$$c^\alpha_{ij} = \sum_{l=0}^{i} \frac{\binom{\alpha+i-l-1}{i-l}}{\binom{\alpha+i}{i}} \phi_{lj}, (\alpha > 1), \tag{47}$$

for all $i, j \in \mathbb{N}_0$. Then, the necessary and sufficient conditions that Φ belongs to any one of the classes $(b^{r,t}_1(p,q): C^\alpha_s)$, $(b^{r,t}_1(p,q): C^\alpha_\infty)$,$(b^{r,t}_s(p,q): C^\alpha_\infty)$,$(b^{r,t}_\infty(p,q): C^\alpha_s)$, and $(b^{r,t}_\infty(p,q): C^\alpha_\infty)$ can be obtained from the respective ones in Corollaries 16, 17, and 18, by replacing the entries of the matrix Φ by those of matrix C^α, where C^α_s and C^α_∞ are generalized Cesàro sequence spaces of order α defined by Roopaei et al. [48].

Corollary 21. *Let $\Phi = (\phi_{ij})$ be an infinite matrix and define the matrix $\tilde{C} = (C_{ij})$ by*

$$\tilde{C}_{ij} = \sum_{l=0}^{i} \frac{C_l C_{i-l}}{C_{i+1}} \phi_{lj}, (i, j \in \mathbb{N}_0), \tag{48}$$

where (C_i) is the sequence of Catalan numbers. Then, the necessary and sufficient conditions that Φ belongs to any one of the classes $(b^{r,t}_1(p,q): c_0(\tilde{C}))$,$(b^{r,t}_1(p,q): c(\tilde{C}))$,$(b^{r,t}_s(p,q): c_0(\tilde{C}))$, $(b^{r,t}_s(p,q): c(\tilde{C}))$,$(b^{r,t}_\infty(p,q): c_0(\tilde{C}))$, and $(b^{r,t}_\infty(p,q): c(\tilde{C}))$ can be obtained from the respective ones in Corollaries 16, 17, and 18, by replacing the entries of the matrix Φ by those of matrix $\tilde{C}$, where $c(\tilde{C})$ and $c_0(\tilde{C})$ are Catalan sequence spaces defined by İlkhan [49].

Corollary 22. *Let $\Phi = (\phi_{ij})$ be an infinite matrix and define the matrix $C^{(q)} = (c^{(q)}_{ij})$ by*

$$c^{(q)}_{ij} = \sum_{l=0}^{i} \frac{q^l}{[i+1]_q} \phi_{lj}, (i, j \in \mathbb{N}_0), \tag{49}$$

where $[i]_q$ is the q-analogue of $i \in \mathbb{N}_0$. Then, the necessary and sufficient conditions that Φ belongs to any one of the classes $(b^{r,t}_1(p,q): \mathscr{X}^q_s)$,$(b^{r,t}_1(p,q): \mathscr{X}^q_\infty)$,$(b^{r,t}_s(p,q): \mathscr{X}^q_s)$, $(b^{r,t}_\infty(p,q): \mathscr{X}^q_\infty)$,$(b^{r,t}_\infty(p,q): \mathscr{X}^q_s)$, and $(b^{r,t}_\infty(p,q): \mathscr{X}^q_\infty)$ can be obtained from the respective ones in Corollaries 16, 17, and 18, by replacing the entries of the matrix Φ by those of matrix $C^{(q)}$, where $\mathscr{X}^q_s$ and $\mathscr{X}^q_\infty$ are q-Cesàro sequence spaces defined by Yaying et al. [28].

5. Geometric Properties

In this section, we examine some geometric properties of the space $b^{r,t}_s(p,q)$. Before proceeding, we recall some notions in Banach spaces which are necessary for this investigation. We use the notation $B(\lambda)$ for unit ball in λ.

Definition 23 (see [50]). A Banach space λ has the weak Banach-Saks property if every weakly null sequence (x_i) in λ has a subsequence (x_{i_j}) whose Cesàro means sequence is norm convergent to zero, that is,

$$\lim_{i\longrightarrow\infty} \left\| \frac{1}{i+1} \sum_{j=0}^{i} x_{i_j} \right\| = 0. \tag{50}$$

Further, λ has the Banach-Saks property if every bounded sequence in λ has a subsequence whose Cesàro means sequence is norm convergent.

Definition 24 (see [51]). A Banach space λ has the Banach-Saks type s, if every weakly null sequence (x_i) has a subsequence (x_{i_j}) such that, for some $K > 0$,

$$\left\| \sum_{j=0}^{i} x_{i_j} \right\| \le K(i+1)^{1/s}, \tag{51}$$

for all $i \in \mathbb{N}_0$.

Theorem 25. *The sequence space $b^{r,t}_s(p,q)$ is of Banach-Saks type s.*

Proof. Let (ς_i) be a sequence of positive numbers satisfying $\sum_{i=0}^{\infty} \varsigma_i \le 1/2$. Let (x_i) be a weakly null sequence in $B(b^{r,t}_s(p,q))$. We set $z_0 = x_0 = 0$ and $z_1 = x_{i_1} = x_1$. Then, there exists $u_1 \in \mathbb{N}_0$ such that

$$\left\| \sum_{j=u_1+1}^{\infty} z_1(j) e^{(j)} \right\|_{b^{r,t}_s(p,q)} < \varsigma_1. \tag{52}$$

Since (x_i) is a weakly null sequence, we realise that $x_i \longrightarrow 0$ coordinatewise. Thus, there exists an $i_2 \in \mathbb{N}_0$ such that

$$\left\| \sum_{j=0}^{u_1} x_i(j) e^{(j)} \right\|_{b^{r,t}_s(p,q)} < \varsigma_1, \tag{53}$$

when $i \ge i_2$. We again set $z_2 = x_{i_2}$. Then, there exists $u_2 > u_1$ such that

$$\left\| \sum_{j=u_2+1}^{\infty} z_2(j) e^{(j)} \right\|_{b^{r,t}_s(p,q)} < \varsigma_2. \tag{54}$$

We again use the fact that $x_i \longrightarrow 0$ coordinatewise, which implies that there exists $i_3 > i_2$ such that

$$\left\| \sum_{j=0}^{u_2} x_i(j) e^{(j)} \right\|_{b_s^{r,t}(p,q)} < \varsigma_2, \tag{55}$$

when $i \geq i_3$.

Continuing this process will lead us to two increasing sequences (i_j) and (u_j) such that

$$\left\| \sum_{j=0}^{u_l} x_i(j) e^{(j)} \right\|_{b_s^{r,t}(p,q)} < \varsigma_l, \tag{56}$$

for all $i \geq i_{j+1}$ and

$$\left\| \sum_{j=u_l+1}^{\infty} z_l(j) e^{(j)} \right\|_{b_s^{r,t}(p,q)} < \varsigma_l. \tag{57}$$

where $z_l = x_{i_l}$. Thus

$$\begin{aligned} &\left\| \sum_{l=0}^{i} z_l \right\|_{b_s^{r,t}(p,q)} \\ &= \left\| \sum_{l=0}^{i} \left(\sum_{j=0}^{u_{l-1}} z_l(j) e^{(j)} + \sum_{j=u_{l-1}+1}^{u_l} z_l(j) e^{(j)} + \sum_{j=u_l}^{\infty} z_l(j) e^{(j)} \right) \right\|_{b_s^{r,t}(p,q)} \\ &\leq \left\| \sum_{l=0}^{i} \left(\sum_{j=u_{l-1}+1}^{u_l} z_l(j) e^{(j)} \right) \right\|_{b_s^{r,t}(p,q)} + 2 \sum_{l=0}^{i} \varsigma_l. \end{aligned} \tag{58}$$

Now, since $x_i \in B(b_s^{r,t}(p,q))$ and $\|x\|_{b_s^{r,t}(p,q)} = \sum_{i=0}^{\infty} |\sum_{j=0}^{i} b_{ij}^{r,t}(p,q) x_j|$, we realise that $\|x\|_{b_s^{r,t}(p,q)} \leq 1$. Therefore, we have

$$\begin{aligned} &\left\| \sum_{l=0}^{i} \left(\sum_{j=u_{l-1}+1}^{u_l} z_l(j) e^{(j)} \right) \right\|_{b_s^{r,t}(p,q)}^{p} \\ &= \sum_{l=0}^{i} \sum_{j=u_{l-1}+1}^{u_l} \left| \sum_{m=0}^{j} b_{jm}^{r,t}(p,q) z_l(m) \right|^{p} \\ &\leq \sum_{l=0}^{i} \sum_{j=0}^{\infty} \left| \sum_{m=0}^{j} b_{jm}^{r,t}(p,q) z_l(m) \right|^{p} \leq i+1. \end{aligned} \tag{59}$$

Now using the fact that $1 \leq (i+1)^{1/s}$ for all $i \in \mathbb{N}_0$ and $1 \leq s < \infty$, we obtain

$$\left\| \sum_{l=0}^{i} z_l \right\|_{b_s^{r,t}(p,q)} \leq (i+1)^{1/s} + 1 \leq 2(i+1)^{1/s}. \tag{60}$$

Thus, we conclude that $b_s^{r,t}(p,q)$ is of the Banach-Saks type s. □

Definition 26. The Gurarii's modulus of convexity of a normed linear space λ is defined by

$$\begin{aligned} \beta_\lambda(\varsigma) = \inf \Big\{ 1 - \inf_{0 \leq \iota \leq 1} \|\iota x + (1-\iota) y\| &: x, y \\ \in B(\lambda), \|x - y\| = \varsigma \Big\}, &\text{ where } 0 \leq \varsigma \leq 2. \end{aligned} \tag{61}$$

Theorem 27. *The Gurarii's modulus of convexity of the normed space $b_s^{r,t}(p,q)$ is*

$$\beta_{b_s^{r,t}(p,q)} \leq 1 - \left(1 - \left(\frac{\varsigma}{2} \right)^{s} \right)^{1/s}, \text{ where } 0 \leq \varsigma \leq 2. \tag{62}$$

Proof. Let $x \in b_s^{r,t}(p,q)$. Then

$$\|x\|_{b_s^{r,t}(p,q)} = \|B^{r,t}(p,q) x\|_{\ell_s} = \left(\sum_{i=0}^{\infty} \left| \left(B^{r,t}(p,q) x \right)_i \right|^{s} \right)^{1/s}. \tag{63}$$

Let $0 \leq \varsigma \leq 2$ and consider the following two sequences:

$$\begin{aligned} x &= \left(\left(C^{r,t}(p,q) \left(1 - \left(\frac{\varsigma}{2} \right)^{s} \right) \right)^{1/s}, C^{r,t}(p,q) \left(\frac{\varsigma}{2} \right), 0, 0, \cdots \right), \\ y &= \left(\left(C^{r,t}(p,q) \left(1 - \left(\frac{\varsigma}{2} \right)^{s} \right) \right)^{1/s}, C^{r,t}(p,q) \left(\frac{-\varsigma}{2} \right), 0, 0, \cdots \right), \end{aligned} \tag{64}$$

where the matrix $C^{r,t}(p,q) = (c_{ij}^{r,t}(p,q))$ is the inverse of the matrix $B^{r,t}(p,q)$. Then, we observe that

$$\begin{aligned} \|x\|_{b_s^{r,t}(p,q)} &= \|B^{r,t}(p,q) x\|_{\ell_s} = \left| \left(1 - \left(\frac{\varsigma}{2} \right)^{s} \right)^{1/s} \right|^{s} + \left| \frac{\varsigma}{2} \right|^{s} \\ &= 1 - \left(\frac{\varsigma}{2} \right)^{s} + \left(\frac{\varsigma}{2} \right)^{s} = 1, \\ \|y\|_{b_s^{r,t}(p,q)} &= \|B^{r,t}(p,q) y\|_{\ell_s} = \left| \left(1 - \left(\frac{\varsigma}{2} \right)^{s} \right)^{1/s} \right|^{s} + \left| \frac{-\varsigma}{2} \right|^{s} \\ &= 1 - \left(\frac{\varsigma}{2} \right)^{s} + \left(\frac{\varsigma}{2} \right)^{s} = 1. \\ \|x - y\|_{b_s^{r,t}(p,q)} &= \|B^{r,t}(p,q) x - B^{r,t}(p,q) y\|_{\ell_s} \\ &= \left(\left| \left(1 - \left(\frac{\varsigma}{2} \right)^{s} \right)^{1/s} - \left(1 - \left(\frac{\varsigma}{2} \right)^{s} \right)^{1/s} \right|^{s} \right. \\ &\quad \left. + \left| \frac{\varsigma}{2} - \left(\frac{-\varsigma}{2} \right) \right|^{s} \right)^{1/s} = \varsigma. \end{aligned} \tag{65}$$

Finally, for $0 \le \iota \le 1$, we have

$$\begin{aligned}
&\inf_{0\le\iota\le1} \|\iota x + (1-\iota)y\|_{b_s^{r,t}(p,q)} \\
&\quad = \inf_{0\le\iota\le1} \left\|\iota B^{r,t}(p,q)x + (1-\iota)B^{r,t}(p,q)y\right\|_{\ell_s} \\
&\quad = \inf_{0\le\iota\le1} \left\{ \left|\iota\left(1-\left(\frac{\varsigma}{2}\right)^s\right)^{1/s} + (1-\iota)\left(1-\left(\frac{\varsigma}{2}\right)^s\right)^{1/s}\right|^s \right. \\
&\qquad \left. + \left|\iota\left(\frac{\varsigma}{2}\right) + (1-\iota)\left(\frac{-\varsigma}{2}\right)\right|\right\}^{1/s} \\
&\quad = \inf_{0\le\iota\le1} \left\{1-\left(\frac{\varsigma}{2}\right)^s + |2\iota-1|^s\left(\frac{\varsigma}{2}\right)^s\right\}^{1/s} \\
&\quad = \left(1-\left(\frac{\varsigma}{2}\right)^s\right)^{1/s}.
\end{aligned} \tag{66}$$

Consequently, $\beta_{b_s^{r,t}(p,q)}(\varsigma) \le 1 - (1-(\varsigma/2)^s)^{1/s}$. This completes the proof. □

Corollary 28. *The following results hold:*

(1) If $\varsigma = 2$, then $\beta_{b_s^{r,t}(p,q)}(\varsigma) \le 1$. Hence, $b_s^{r,t}(p,q)$ is strictly convex

(2) If $0 < \varsigma < 2$, then $0 < \beta_{b_s^{r,t}(p,q)}(\varsigma) < 1$. Hence, $b_s^{r,t}(p,q)$ is uniformly convex

6. Conclusion

The (p,q)-Euler matrix $B^{r,t}$ of order (r,t) generalizes some of the well-known matrices presented in the literature, for instance, Binomial matrix $B^{r,t}$ of order (r,t) [39, 40], Euler matrix of order r [10, 11], etc. Thus, the results presented in this paper strengthen the results of [11, 14, 40, 52–55]. As for future scope, we shall study the domain of the matrix $B^{r,t}(p,q)$ in the spaces c and c_0 of convergent and null sequences, respectively.

Data Availability

All the data are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Research Article

Approximation Properties of (p, q)-Szász-Mirakjan-Durrmeyer Operators

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In this article, we introduce a new Durrmeyer-type generalization of (p, q)-Szász-Mirakjan operators using the (p, q)-gamma function of the second kind. The moments and central moments are obtained. Then, the Voronovskaja-type asymptotic formula is investigated and point-wise estimates of these operators are studied. Also, some local approximation properties of these operators are investigated by means of modulus of continuity and Peetre $\mathscr{K}$-functional. Finally, the rate of convergence and weighted approximation of these operators are presented.

1. Introduction

In recent years, (p, q)-analogues of well-known positive operators have been widely constructed and researched since Mursaleen et al. first introduced (p, q)-Berstein operators [1] and (p, q)-Bernstein-Stancu operators [2]. In [3], Acar first introduced (p, q)-Szász-Mirakjan operators and gave a recurrence relation for the moments of these operators. In [4], Mursaleen et al. proposed a Kantorovich variant of the (p, q)-analogue of Szász-Mirakjan operators under the nondecreasing condition. In [5], Sharma and Gupta introduced (p, q)-Szász-Mirakjan-Kantorovich operators and studied their approximation properties. In [6], Acar et al. constructed King's type (p, q)-Szász-Mirakjan operators preserving x^2 and discussed the order of approximation and weighted approximation properties of these operators. In [7], Aral and Gupta constructed (p, q)-Szász-Mirakjan-Durrmeyer operators using (p, q)-gamma function of the first kind and estimated moments and established some direct results of these operators. In [8], Mursaleen et al. introduced a new modification of Szász-Mirakjan operators based on the (p, q)-calculus and investigated their approximation properties including weighted approximation and Voronovskaya-type theorem. In [9], Mursaleen et al. proposed two different Kantorovich-type (p, q)-Szász-Mirakjan operators and discussed their error estimated. In [10], Kara and Mahmudov constructed a new (p, q)-Szász-Mirakjan operators as

Definition 1. Let $0 < q < p \leq 1$ and $m \in \mathbb{N}$. For $g : [0,\infty) \longrightarrow \mathbb{R}$, (p, q) -Szász-Mirakjan operators can be defined as

$$S_m^{p,q}(g\,; t) = \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t)\, g\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right), \tag{1}$$

where $s_{m,k}^{p,q}(t) = \left(p^{k(k-m)}/q^{(k(k-1))k(k-1)/2}\right)\left([m]_{p,q}^k t^k/[k]_{p,q}!\right)e_{p,q}(-[m]_{p,q}p^{k-m+1}q^{-k}t)$.

Meantime, quantitative estimates for the convergence in the polynomial weighted spaces and Voronovskaya theorem for new (p, q)-Szász-Mirakjan operators (1) were given. All these achievement motivates us to construct the Durrmeyer analogue of the (p, q)-Szász-Mirakjan operators defined by (1).

Definition 2. Let $0<q<p\leq 1$ and $m\in\mathbb{N}$. For $g:[0,\infty)\longrightarrow\mathbb{R}$, we construct the (p,q)-Szász-Mirakjan-Durrmeyer operators as

$$D_m^{p,q}(g;t)=[m]_{p,q}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{-k}p^{k-m}s_{m,k}^{p,q}(u)g(u)\mathrm{d}_{p,q}u. \tag{2}$$

Let us recall the basic notations of (p,q)-calculus which can be found in [11]. For any fixed real number $p>0$ and $q>0$, the (p,q)-integers $[m]_{p,q}$ are defined as

$$[m]_{p,q}=p^{m-1}+p^{m-2}q+p^{m-3}q^2+\cdots+pq^{m-2}+q^{m-1}=\begin{cases}\dfrac{p^m-q^m}{p-q}, & p\neq q\neq 1,\\ mp^{m-1}, & p=q\neq 1,\\ [m]_q, & p=1,\\ m, & p=q=1,\end{cases} \tag{3}$$

where $[m]_q$ denotes the q-integers and $m=0,1,2,\cdots$. Also (p,q)-factorial $[m]_{p,q}!$ is defined as follows:

$$[m]_{p,q}!=\begin{cases}[1]_{p,q}[2]_{p,q}\cdots[m]_{p,q}, & m\geq 1,\\ 1, & m=0.\end{cases} \tag{4}$$

Now, we introduce two types of (p,q)-analogues of exponential function $e_{p,q}(t)$ and $E_{p,q}(t)$ (see [7]):

$$\begin{aligned}e_{p,q}(t)&=\sum_{m=0}^{\infty}\frac{p^{(m(m-1))/2}t^m}{[m]_{p,q}!},\ t\in\mathbb{R},\ |p|<1 \text{ and } |q|<1,\\ E_{p,q}(t)&=\sum_{m=0}^{\infty}\frac{q^{(m(m-1))/2}t^m}{[m]_{p,q}!},\ t\in\mathbb{R},\ |p|<1 \text{ and } |q|<1.\end{aligned} \tag{5}$$

Let g be an arbitrary function. The improper (p,q)-integral of $g(t)$ on $[0,\infty)$ was defined as (see [12])

$$\int_0^{\infty}g(t)\mathrm{d}_{p,q}t=(p-q)\sum_{i=-\infty}^{\infty}\frac{q^i}{p^{i+1}}g\left(\frac{q^i}{p^{i+1}}\right),\ 0<\frac{q}{p}<1. \tag{6}$$

The (p,q)-gamma function of the second kind was defined in [12] as follows:

$$\gamma_{p,q}(z)=\int_0^{\infty}q^{(z(z-1))/2}t^{z-1}e_{p,q}(-pt)\mathrm{d}_{p,q}t,\ \Re(z)>0. \tag{7}$$

Meantime, the (p,q)-gamma function fulfills the following relation:

$$\gamma_{p,q}(z+1)=[z]_{p,q}\gamma_{p,q}(z), \tag{8}$$

moreover, for any nonnegative integer $m>0$, the following relation holds:

$$\gamma_{p,q}(m+1)=[m]_{p,q}!. \tag{9}$$

2. Auxiliary Results

In order to discuss the approximation properties of the operators $D_m^{p,q}(g;t)$, we need the following lemmas.

Lemma 3 (see ([10], Lemma 4)). *For $0<q<p\leq 1$, $m\in\mathbb{N}$, and $t\in[0,\infty)$, we have*

$$\begin{aligned}&S_m^{p,q}(1;t)=1,\ S_m^{p,q}(u;t)=t,\ S_m^{p,q}(u^2;t)=t^2+\frac{p^{m-1}}{[m]_{p,q}}t,\\ &S_m^{p,q}(u^3;t)=t^3+\frac{(2p+q)p^{m-2}}{[m]_{p,q}}t^2+\frac{p^{2m-2}}{[m]_{p,q}^2}t,\\ &S_m^{p,q}(u^4;t)=t^4+\frac{(3p^2+2qp+q^2)p^{m-3}}{[m]_{p,q}}t^3+\frac{(3p^2+3qp+q^2)p^{2m-4}}{[m]_{p,q}^2}t^2+\frac{p^{3m-3}}{[m]_{p,q}^3}t.\end{aligned} \tag{10}$$

Lemma 4. *For $0<q<p\leq 1$, $r=p/q$, $m\in\mathbb{N}$, and $t\in[0,\infty)$, we have*

$$\begin{aligned}&D_m^{p,q}(1;t)=1,\ D_m^{p,q}(u;t)=t+\frac{p^mq^{-1}}{[m]_{p,q}},\\ &D_m^{p,q}(u^2;t)=t^2+\frac{p^m\left([2]_{p,q}^2p^{-1}q^{-2}\right)}{[m]_{p,q}}t+\frac{p^{2m}q^{-3}[2]_{p,q}}{[m]_{p,q}^2},\\ &D_m^{p,q}(u^3;t)=t^3+\frac{(q^{-1}g_1(r)+2p^{-1}+p^{-2}q)p^m}{[m]_{p,q}}t^2+\frac{(p^{-2}+g_2(r)q^{-2}+g_1(r)p^{-1}q^{-1})p^{2m}}{[m]_{p,q}^2}t+\frac{p^{3m}q^{-3}g_3(r)}{[m]_{p,q}^3},\\ &D_m^{p,q}(u^4;t)=t^4+\frac{(3p^{-1}+2qp^{-2}+q^2p^{-3}+q^{-1}h_1(r))p^m}{[m]_{p,q}}t^3\\ &\quad+\frac{((3+(2r+1)h_1(r))p^{-2}+3p^{-3}q+p^{-4}q^2+q^{-2}h_2(r))p^{2m}}{[m]_{p,q}^2}t^2\\ &\quad+\frac{(p^{-3}+p^{-2}q^{-1}h_1(r)+p^{-1}q^{-2}h_2(r)+q^{-3}h_3(r))p^{3m}}{[m]_{p,q}^3}t\\ &\quad+\frac{q^{-4}h_4(r)p^{4m}}{[m]_{p,q}^4},\end{aligned} \tag{11}$$

where $g_1(r)=\sum_{i=1}^{3}[i]_r$, $g_2(r)=\sum_{1\le i<j\le 3}[i]_r[j]_r$, $g_3(r)=[2]_r[3]_r$, $h_1(r)=\sum_{i=1}^{4}[i]_r$, $h_2(r)=\sum_{1\le i<j\le 4}[i]_r[j]_r$, $h_3(r)=\sum_{1\le i<j<k\le 4}[i]_r[j]_r[k]_r$, *and* $h_4(r)=[2]_r[3]_r[4]_r$.

Proof. Using the (p,q)-gamma function of the second kind, we can obtain

$$\begin{aligned}D_m^{p,q}(1;t)&=[m]_{p,q}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{-k}p^{k-m}s_{m,k}^{p,q}(u)\mathrm{d}_{p,q}u\\&=[m]_{p,q}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{-(k(k+1))/2}p^{(k-m)(k+1)}\\&\quad\cdot\frac{\left([m]_{p,q}u\right)^k}{[k]_{p,q}!}e_{p,q}\left(-[m]_{p,q}p^{k-m+1}q^{-k}u\right)\mathrm{d}_{p,q}u\\&=[m]_{p,q}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{-(k(k+1))/2}p^{(k-m)(k+1)}\\&\quad\cdot\frac{\left([m]_{p,q}u\right)^k}{[k]_{p,q}!}e_{p,q}\left(-p\left([m]_{p,q}p^{k-m}q^{-k}u\right)\right)\mathrm{d}_{p,q}u\\&=\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{(k(k+1))/2}\frac{\left([m]_{p,q}p^{k-m}q^{-k}u\right)^k}{[k]_{p,q}!}e_{p,q}\\&\quad\cdot\left(-p\left([m]_{p,q}p^{k-m}q^{-k}u\right)\right)\mathrm{d}_{p,q}\left([m]_{p,q}p^{k-m}q^{-k}u\right)\\&=\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\frac{\gamma_{p,q}(k+1)}{[k]_{p,q}!}=\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)=1.\end{aligned}\tag{12}$$

and next using $[k+1]_{p,q}=p^k+q[k]_{p,q}$, we have

$$\begin{aligned}D_m^{p,q}(u;t)&=[m]_{p,q}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{-k(k+1)/2}p^{(k-m)(k+1)}\\&\quad\cdot\frac{\left([m]_{p,q}u\right)^k}{[k]_{p,q}!}e_{p,q}\left(-[m]_{p,q}p^{k-m+1}q^{-k}u\right)u\mathrm{d}_{p,q}u\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{m-k}}{[m]_{p,q}[k]_{p,q}!}\int_0^{\infty}q^{-k(k+1)/2+k(k+2)}\\&\quad\cdot\left([m]_{p,q}p^{k-m}q^{-k}u\right)^{k+1}e_{p,q}\left(-p\left([m]_{p,q}p^{k-m}q^{-k}u\right)\right)\\&\quad\times\mathrm{d}_{p,q}\left([m]_{p,q}p^{k-m}q^{-k}u\right)\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{m-k}q^{-1}}{[m]_{p,q}[k]_{p,q}!}\int_0^{\infty}q^{(k+1)(k+2)/2}s^{k+1}e_{p,q}(-ps)\mathrm{d}_{p,q}s\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{m-k}q^{-1}}{[m]_{p,q}[k]_{p,q}!}\gamma_{p,q}(k+2)=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{m-k}q^{-1}[k+1]_{p,q}}{[m]_{p,q}}\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{m-k}q^{-1}\left(p^k+q[k]_{p,q}\right)}{[m]_{p,q}}=\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\\&\quad+\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\frac{p^mq^{-1}}{[m]_{p,q}}=S_m^{p,q}(u;t)+S_m^{p,q}(1;t)=t+\frac{p^mq^{-1}}{[m]_{p,q}}.\end{aligned}\tag{13}$$

Using the equality $[k+1]_{p,q}[k+2]_{p,q}=q^3[k]_{p,q}^2+q(2q+p)p^k[k]_{p,q}+[2]_{p,q}p^{2k}$ and Lemma 3, we have

$$\begin{aligned}D_m^{p,q}(u^2;t)&=[m]_{p,q}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{-k}p^{k-m}s_{m,k}^{p,q}(u)u^2\mathrm{d}_{p,q}u\\&=[m]_{p,q}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\int_0^{\infty}q^{-(k(k+1))/2}p^{(k-m)(k+1)}\\&\quad\cdot\frac{\left([m]_{p,q}u\right)^k}{[k]_{p,q}!}e_{p,q}\left(-[m]_{p,q}p^{k-m+1}q^{-k}u\right)u^2\mathrm{d}_{p,q}u\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{2m-2k}}{[m]_{p,q}^2[k]_{p,q}!}\int_0^{\infty}q^{-(k(k+1))/2+k(k+3)}\\&\quad\cdot\left([m]_{p,q}p^{k-m}q^{-k}u\right)^{k+2}e_{p,q}\left(-p\left([m]_{p,q}p^{k-m}q^{-k}u\right)\right)\\&\quad\times\mathrm{d}_{p,q}\left([m]_{p,q}p^{k-m}q^{-k}u\right)\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{2m-2k}q^{-3}}{[m]_{p,q}^2[k]_{p,q}!}\int_0^{\infty}q^{((k+2)(k+3))/2}s^{k+2}e_{p,q}(-ps)\mathrm{d}_{p,q}s\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{2m-2k}q^{-3}}{[m]_{p,q}^2[k]_{p,q}!}\gamma_{p,q}(k+3)\\&=\sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{2m-2k}q^{-3}[k+1]_{p,q}[k+2]_{p,q}}{[m]_{p,q}^2}\\&=\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right)^2+\frac{p^mq^{-2}(p+2q)}{[m]_{p,q}}\\&\quad\cdot\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}+\frac{p^{2m}q^{-3}[2]_{p,q}}{[m]_{p,q}^2}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\\&=S_m^{p,q}(u^2;t)+\frac{p^mq^{-2}(p+2q)}{[m]_{p,q}}S_m^{p,q}(u;t)\\&\quad+\frac{p^{2m}q^{-3}[2]_{p,q}}{[m]_{p,q}^2}S_m^{p,q}(1;t)=t^2+\left(p^{-1}+2q^{-1}+pq^{-2}\right)\\&\quad\cdot\frac{p^m}{[m]_{p,q}}t+\frac{p^{2m}q^{-3}[2]_{p,q}}{[m]_{p,q}^2}=t^2+\frac{p^m\left([2]_{p,q}^2p^{-1}q^{-2}\right)}{[m]_{p,q}}t\\&\quad+\frac{p^{2m}q^{-3}[2]_{p,q}}{[m]_{p,q}^2}.\end{aligned}\tag{14}$$

Using $[m]_{p,q}=q^{m-1}[m]_r$, we have

$$\begin{aligned}[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}&=\left(q[k]_{p,q}+p^k\right)\left(q^2[k]_{p,q}+[2]_{p,q}p^k\right)\\&\quad\cdot\left(q^3[k]_{p,q}+[3]_{p,q}p^k\right)=q^6[k]_{p,q}^3\\&\quad+\left(q^3[3]_{p,q}+q^4[2]_{p,q}+q^5\right)[k]_{p,q}^2p^k\\&\quad+\left(q[2]_{p,q}[3]_{p,q}+q^2[3]_{p,q}+q^3[2]_{p,q}\right)[k]_{p,q}p^{2k}+[2]_{p,q}[3]_{p,q}p^{3k}\\&=q^6[k]_{p,q}^3+q^5\left([3]_r+[2]_r+1\right)[k]_{p,q}^2p^k\\&\quad+q^4\left([3]_r[2]_r+[3]_r+[2]_r\right)[k]_{p,q}p^{2k}+q^3[3]_r[2]_rp^{3k}\\&\coloneqq q^6[k]_{p,q}^3+q^5g_1(r)[k]_{p,q}^2p^k+q^4g_2(r)[k]_{p,q}p^{2k}+q^3g_3(r)p^{3k}.\end{aligned}\tag{15}$$

Thus,

$$
\begin{aligned}
D_m^{p,q}(u^3;t) &= [m]_{p,q}\sum_{k=0}^{\infty} s_{m,k}^{p,q}(t)\int_0^{\infty} q^{-k}p^{k-m}s_{m,k}^{p,q}(u)u^3\,\mathrm{d}_{p,q}u \\
&= [m]_{p,q}\sum_{k=0}^{\infty} s_{m,k}^{p,q}(t)\int_0^{\infty} q^{-k(k+1)/2}p^{(k-m)(k+1)} \\
&\quad\cdot \frac{\left([m]_{p,q}u\right)^k}{[k]_{p,q}!}e_{p,q}\left(-[m]_{p,q}p^{k-m+1}q^{-k}u\right)u^3\,\mathrm{d}_{p,q}u \\
&= \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{3m-3k}}{[m]_{p,q}^3[k]_{p,q}!}\int_0^{\infty} q^{-k(k+1)/2+k(k+4)} \\
&\quad\cdot\left([m]_{p,q}p^{k-m}q^{-k}u\right)^{k+3}e_{p,q}\left(-p\left([m]_{p,q}p^{k-m}q^{-k}u\right)\right) \\
&\quad\times \mathrm{d}_{p,q}\left([m]_{p,q}p^{k-m}q^{-k}u\right) = \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{3m-3k}q^{-6}}{[m]_{p,q}^3[k]_{p,q}!} \\
&\quad\cdot\int_0^{\infty} q^{((k+3)(k+4))/2}s^{k+3}e_{p,q}(-ps)\,\mathrm{d}_{p,q}s \\
&= \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{3m-3k}q^{-6}}{[m]_{p,q}^3[k]_{p,q}!}\gamma_{p,q}(k+4) \\
&= \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{3m-3k}q^{-6}[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}}{[m]_{p,q}^3} \\
&= \sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right)^3 + \frac{p^mq^{-1}g_1(r)}{[m]_{p,q}}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t) \\
&\quad\cdot\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right)^2 + \frac{p^{2m}q^{-2}g_2(r)}{[m]_{p,q}^2}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}} \\
&\quad+\frac{p^{3m}q^{-3}g_3(r)}{[m]_{p,q}^3}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t) = S_m^{p,q}(u^3;t) \\
&\quad+\frac{p^mq^{-1}g_1(r)}{[m]_{p,q}}S_m^{p,q}(u^2;t)+\frac{p^{2m}q^{-2}g_2(r)}{[m]_{p,q}^2}S_m^{p,q}(u;t) \\
&\quad+\frac{p^{3m}q^{-3}g_3(r)}{[m]_{p,q}^3}S_m^{p,q}(1;t) \\
&= t^3+\frac{\left(q^{-1}g_1(r)+2p^{-1}+p^{-2}q\right)p^m}{[m]_{p,q}}t^2 \\
&\quad+\frac{\left(p^{-2}+g_2(r)q^{-2}+g_1(r)p^{-1}q^{-1}\right)p^{2m}}{[m]_{p,q}^2}t+\frac{p^{3m}q^{-3}g_3(r)}{[m]_{p,q}^3}.
\end{aligned}
\tag{16}
$$

Similarly, using $\prod_{i=1}^{4}[k+i]_{p,q} = q^{10}[k]_{p,q}^4 + h_1(r)q^9[k]_{p,q}^3p^k + h_2(r)q^8[k]_{p,q}^2p^{2k} + h_3(r)q^7[k]_{p,q}p^{3k} + h_4(r)q^6p^{4k}$, we can obtain

$$
\begin{aligned}
D_m^{p,q}(u^4;t) &= [m]_{p,q}\sum_{k=0}^{\infty} s_{m,k}^{p,q}(t)\int_0^{\infty} q^{-k}p^{k-m}s_{m,k}^{p,q}(u)u^4\,\mathrm{d}_{p,q}u \\
&= [m]_{p,q}\sum_{k=0}^{\infty} s_{m,k}^{p,q}(t)\int_0^{\infty} q^{-(k(k+1))/2}p^{(k-m)(k+1)} \\
&\quad\cdot \frac{\left([m]_{p,q}u\right)^k}{[k]_{p,q}!}e_{p,q}\left(-[m]_{p,q}p^{k-m+1}q^{-k}u\right)u^4\,\mathrm{d}_{p,q}u \\
&= \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{4m-4k}}{[m]_{p,q}^3[k]_{p,q}!}\int_0^{\infty} q^{-((k(k+1))/2)+(k(k+5))} \\
&\quad\cdot\left([m]_{p,q}p^{k-m}q^{-k}u\right)^{k+4}e_{p,q}\left(-p\left([m]_{p,q}p^{k-m}q^{-k}u\right)\right) \\
&\quad\times \mathrm{d}_{p,q}\left([m]_{p,q}p^{k-m}q^{-k}u\right) = \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{4m-4k}q^{-10}}{[m]_{p,q}^4[k]_{p,q}!} \\
&\quad\cdot\int_0^{\infty} q^{((k+4)(k+5))/2}s^{k+4}e_{p,q}(-ps)\,\mathrm{d}_{p,q}s \\
&= \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{4m-4k}q^{-10}}{[m]_{p,q}^4[k]_{p,q}!}\gamma_{p,q}(k+5) \\
&= \sum_{k=0}^{\infty}\frac{s_{m,k}^{p,q}(t)p^{4m-4k}q^{-10}[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}{[m]_{p,q}^4} \\
&= \sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right)^4 + \frac{p^mq^{-1}h_1(r)}{[m]_{p,q}} \\
&\quad\cdot\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right)^3 + \frac{p^{2m}q^{-2}h_2(r)}{[m]_{p,q}^2} \\
&\quad\cdot\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right)^2 + \frac{p^{3m}q^{-3}h_3(r)}{[m]_{p,q}^3} \\
&\quad\cdot\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t)\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}} + \frac{p^{3m}q^{-4}h_4(r)}{[m]_{p,q}^4}\sum_{k=0}^{\infty}s_{m,k}^{p,q}(t) \\
&= S_m^{p,q}(u^4;t)+\frac{p^mq^{-1}h_1(r)}{[m]_{p,q}}S_m^{p,q}(u^3;t) \\
&\quad+\frac{p^{2m}q^{-2}h_2(r)}{[m]_{p,q}^2}S_m^{p,q}(u^2;t)+\frac{p^{3m}q^{-3}h_3(r)}{[m]_{p,q}^3}S_m^{p,q}(u;t) \\
&\quad+\frac{p^{3m}q^{-4}h_4(r)}{[m]_{p,q}^4}S_m^{p,q}(1;t) \\
&= t^4+\frac{\left(3p^{-1}+2qp^{-2}+q^2p^{-3}+q^{-1}h_1(r)\right)p^m}{[m]_{p,q}}t^3 \\
&\quad+\frac{\left((3+(2r+1)h_1(r))p^{-2}+3p^{-3}q+p^{-4}q^2+q^{-2}h_2(r)\right)p^{2m}}{[m]_{p,q}^2}t^2 \\
&\quad+\frac{\left(p^{-3}+p^{-2}q^{-1}h_1(r)+p^{-1}q^{-2}h_2(r)+q^{-3}h_3(r)\right)p^{3m}}{[m]_{p,q}^3}t \\
&\quad+\frac{q^{-4}h_4(r)p^{4m}}{[m]_{p,q}^4}.
\end{aligned}
\tag{17}
$$

Lemma 5. *Using Lemma 4, we immediately have the following explicit formulas for the central moments:*

$$
\begin{aligned}
A_m^{p,q}(t) &:= D_m^{p,q}(u-t;t) = \frac{q^{-1}p^m}{[m]_{p,q}}, \\
B_m^{p,q}(t) &:= D_m^{p,q}\left((u-t)^2;t\right) = \frac{\left(p^{-1}+pq^{-2}\right)p^m}{[m]_{p,q}}t + \frac{q^{-3}[2]_{p,q}p^{2m}}{[m]_{p,q}^2}.
\end{aligned}
\tag{18}
$$

Lemma 6. *The sequences (p_m), (q_m) satisfy $0 < q_m < p_m \le 1$ such that $p_m \longrightarrow 1$, $q_m \longrightarrow 1$ and $p_m^m \longrightarrow \beta \in [0,1]$, $[m]_{p_m,q_m} \longrightarrow \infty$ as $m \longrightarrow \infty$, $r_m = p_m/q_m$, then*

$$\lim_{m\longrightarrow\infty}[m]_{p_m,q_m}A_m^{p_m,q_m}(t)=\beta, \quad (19)$$

$$\lim_{m\longrightarrow\infty}[m]_{p_m,q_m}B_m^{p_m,q_m}(t)=2\beta t, \quad (20)$$

$$\lim_{m\longrightarrow\infty}[m]_{p_m,q_m}D_m^{p_m,q_m}\left((u-t)^4;t\right)=0. \quad (21)$$

Proof. Applying Lemma 5, we can easily obtain (19) and (20). While $m\longrightarrow\infty$, we can rewrite

$$D_m^{p_m,q_m}(1;t)=1\,D_m^{p_m,q_m}(u;t)t+\frac{\beta}{[m]_{p_m,q_m}};\quad D_m^{p_m,q_m}\left(u^2;t\right)\sim t^2+\frac{4\beta}{[m]_{p_m,q_m}}t,$$

$$D_m^{p_m,q_m}\left(u^3;t\right)\sim t^3+\frac{9\beta}{[m]_{p_m,q_m}}t^2;\quad D_m^{p_m,q_m}\left(u^4;t\right)\sim t^4+\frac{16\beta}{[m]_{p_m,q_m}}t^3. \quad (22)$$

Using $D_m^{p_m,q_m}((u-t)^4;t)=D_m^{p_m,q_m}(u^4;t)-4D_m^{p_m,q_m}(u^3;t)t+6D_m^{p_m,q_m}(u^2;t)t^2-4D_m^{p_m,q_m}(u;t)t^3+D_m^{p_m,q_m}(1;t)t^4$ and $16-9\times4+4\times6-4\times1=0$, we can get (21).

3. Voronovskaja-Type Theorem

Theorem 7. *Let (p_m), (q_m) be the sequences defined in Lemma 6 and $g\in C_B[0,\infty)$. Supposing that $g''(t)$ exists at a point $t\in[0,\infty)$, then, we can obtain*

$$\lim_{m\longrightarrow\infty}[m]_{p_m,q_m}\left(D_m^{p_m,q_m}(g;t)-g(t)\right)=\beta\Big(g(t)+tg'(t)\Big), \quad (23)$$

where $C_B[0,\infty)$ denotes the set of all real-valued bounded and continuous functions defined on $[0,\infty)$, endowed with the norm $\|g\|=\sup_{t\in[0,\infty)}|g(t)|$.

Proof. By the Taylor's expansion theorem of function $g\in C_B[0,\infty)$, we can obtain

$$g(u)-g(t)=g'(t)(u-t)+\frac{1}{2}g''(t)(u-x)^2+\Theta(u,t)(u-t)^2, \quad (24)$$

where $u,t\in[0,\infty)$, $\Theta(u,t)$ is bounded and $\lim_{u\longrightarrow t}\Theta(u,t)=0$. Applying the operator $D_m^{p_m,q_m}$ to the equality above, we can obtain

$$D_m^{p_m,q_m}(g;t)-g(t)=g'(t)A_m^{p_m,q_m}(t)+\frac{1}{2}g''(t)B_m^{p_m,q_m}(t)+D_m^{p_m,q_m}\left(\Theta(u,t)(u-t)^2;t\right) \quad (25)$$

Since $\lim_{u\longrightarrow t}\Theta(u,t)=0$, then for all $\varepsilon>0$, there exists $\delta>0$ such that $|u-t|<\delta$ and it will imply $|\Theta(u,t)|<\varepsilon$ for all fixed $t\in[0,\infty)$ as m sufficiently large. While if $|u-t|\geq\delta$, then $|\Theta(u,t)|\leq C/\delta^2(u-t)^2$, where $C>0$ is a constant. Using Lemma 6

$$\begin{aligned}&[m]_{p_m,q_m}\left|D_m^{p_m,q_m}\left(\Theta(u,t)(u-t)^2;t\right)\right|\\&\leq\varepsilon[m]_{p_m,q_m}\left|D_m^{p_m,q_m}\left((u-t)^2;t\right)\right|\\&\quad+\frac{C}{\delta^2}[m]_{p_m,q_m}D_m^{p_m,q_m}\left((u-t)^4;t\right)\longrightarrow0\,(m\longrightarrow\infty).\end{aligned} \quad (26)$$

The proof is completed.

4. Point-Wise Estimate

In this section, we establish two point-wise estimate of the operators $D_m^{p,q}(g;t)$. First, a function $g\in C[0,\infty)$ is said to satisfy the Lipschitz condition Lip_γ on D (named $g\in\mathrm{Lip}_{C_{g,\gamma}}(\gamma,E)$), $\gamma\in(0,1]$, $D\subset[0,\infty)$ if

$$|g(u)-g(t)|\leq C_{g,\gamma}|u-t|^\gamma,\ u\in[0,\infty)\text{ and }t\in D, \quad (27)$$

where $C_{g,\gamma}$ is an absolute positive constant depending only on g and γ.

Theorem 8. *Let $0<q<p\leq1$, $\gamma\in(0,1]$ and E be any bounded subset on $[0,\infty)$. If $g\in C_B[0,\infty)\cap Lip_{C_{g,\gamma}}(\gamma,E)$, then, for all $t\in[0,\infty)$, we have*

$$|D_m^{p,q}(g;t)-g(t)|\leq C_{g,\gamma}\left(\left(B_m^{p,q}(t)\right)^{\gamma/2}+2d^\gamma(t;E)\right), \quad (28)$$

where $d(t;E)$ denotes the distance between t and E defined by

$$d(t;E)=\inf\{|u-t|:u\in E\}. \quad (29)$$

Proof. Let $\bar{E}$ be the closure of E. Using the properties of infimum, there is at least a point $u_0\in\bar{E}$ such that $\mathrm{d}(t;E)=|t-u_0|$. By the triangle inequality

$$|g(u)-g(t)|\leq|g(u)-g(u_0)|+|g(t)-g(u_0)|, \quad (30)$$

we can obtain

$$\begin{aligned}|D_m^{p,q}(g;t)-g(t)|&\leq D_m^{p,q}(|g(u)-g(u_0)|;t)+D_m^{p,q}(|g(t)-g(u_0)|;t)\\&\leq C_{g,\gamma}\{D_m^{p,q}(|u-u_0|^\gamma;x)+|t-u_0|^\gamma\}\\&\leq C_{g,\gamma}\{D_m^{p,q}(|u-t|^\gamma+|t-u_0|^\gamma;t)+|t-u_0|^\gamma\}\\&=C_{g,\gamma}\{D_m^{p,q}(|u-t|^\gamma;t)+2\,|\,t-u_0|^\gamma\}.\end{aligned} \quad (31)$$

Choosing $p_1=2/\gamma$ and $p_2=2/(2-\gamma)$ and using the well-known Hölder inequality, we have

$$|D_m^{p,q}(f;x)-f(x)|\le C_{g,\gamma}\left\{\left(D_m^{p,q}\left(|t-x|^{p_1\gamma};x\right)\right)^{1/p_1}\left(D_m^{p,q}(1^{p_2};x)\right)^{1/p_2}+2d^\gamma(x;E)\right\}$$
$$\le C_{g,\gamma}\left\{\left(D_m^{p,q}\left((t-x)^2;x\right)\right)^{\gamma/2}+2d^\gamma(x;E)\right\}$$
$$\le C_{g,\gamma}\left(\left(B_m^{p,q}(t)\right)^{\gamma/2}+2d^\gamma(x;E)\right). \quad (32)$$

Next, we obtain the local direct estimate of the operators $D_m^{p,q}$, using the Lipcshitz-type maximal function of the order γ introduced by Lenze [13] as

$$\tilde{\omega}_\gamma(g;t)=\sup_{t,u\in(0,\infty),t\ne u}\frac{|g(u)-g(t)|}{|u-t|^\gamma},\gamma\in(0,1]. \quad (33)$$

Theorem 9. *Let $g\in C_B[0,\infty)$ and $\gamma\in(0,1]$. Then, for all $t\in[0,\infty)$, we have*

$$|D_m^{p,q}(g;t)-g(t)|\le\tilde{\omega}_\gamma(g;t)\left(B_m^{p,q}(t)\right)^{\gamma/2}. \quad (34)$$

Proof. From equation (33), we have

$$|D_m^{p,q}(g;t)-g(t)|\le\tilde{\omega}_\gamma(g;t)D_m^{p,q}(|u-t|^\gamma;t). \quad (35)$$

Applying the well-known Hölder inequality, we can get

$$|D_m^{p,q}(g;t)-g(t)|\le\tilde{\omega}_\gamma(g;t)\left(D_m^{p,q}\left((u-t)^2;t\right)\right)^{\gamma/2}\le\tilde{\omega}_\gamma(g;t)\left(B_m^{p,q}(t)\right)^{\gamma/2}. \quad (36)$$

5. Local Approximation

In this section, we establish local approximation theorem for (p,q)-Szász-Mirakjan-Durrmeyer operators. Let us consider the following $\mathscr{K}$-functional:

$$\mathscr{K}(g;\delta)=\inf_{h\in S^2}\left\{\|g-h\|+\delta\|h''\|\right\}, \quad (37)$$

where $\delta\in[0,\infty)$ and $S^2=\{h\in C_B[0,\infty): h',h''\in C_B[0,\infty)\}$. The usual modulus of continuity and the second-order modulus of smoothness of g can be defined by

$$\omega(g;\delta)=\sup_{0<|u|\le\delta}\sup_{t\in[0,\infty)}|g(t+u)-g(t)|,$$
$$\omega_2(g;\delta)=\sup_{0<|u|\le\delta}\sup_{t\in[0,\infty)}|g(t+2u)-2g(t+u)+g(t)|. \quad (38)$$

By ([14], p.177, Theorem 2.4), there exists an absolute constant $C>0$ such

$$\mathscr{K}\left(g;\delta^2\right)\le C\omega_2(g;\delta),\delta>0. \quad (39)$$

Theorem 10. *Let (p_m), (q_m) be the sequences defined in Lemma 6 and $g\in C_B[0,\infty)$. Then, for all $m\in\mathbb{N}$, there exists an absolute positive $C_1=4C$ such that*

$$|D_m^{p_m,q_m}(g;t)-g(t)|\le C_1\omega_2\left(g;\sqrt{\left(A_m^{p_m,q_m}(t)\right)^2+B_m^{p_m,q_m}(t)}\right)\cdot+\omega\left(g;|A_m^{p_m,q_m}(t)|\right). \quad (40)$$

Proof. For a given function $g\in C_B[0,\infty)$, let us define the following new operators:

$$\mathbf{D}_m^{p_m,q_m}(g;t)=D_m^{p_m,q_m}(g;t)-g\left(A_m^{p_m,q_m}(t)+t\right)+g(t),g\in[0,\infty). \quad (41)$$

By Lemma 4 and Lemma 5, we obtain

$$\mathbf{D}_m^{p_m,q_m}(1;t)=1 \text{ and } \mathbf{D}_m^{p_m,q_m}(u;t)=t. \quad (42)$$

Let $t,u\in[0,\infty)$ and $h\in S^2$. By Taylor's expansion formula, we get

$$h(u)=h(t)+h'(t)(u-t)+\int_t^u h''(v)(u-v)dv. \quad (43)$$

Applying $\mathbf{D}_m^{p_m,q_m}$ to the above equality, we can write

$$\mathbf{D}_m^{p_m,q_m}(h;t)-h(t)=\mathbf{D}_m^{p_m,q_m}\left(\int_t^u h''(v)(u-v)dv;t\right)$$
$$\le D_m^{p_m,q_m}\left(\left|\int_t^u|u-v|h''(v)dv\right|;t\right)+\left|\int_t^{A_m^{p_m,q_m}(t)+t}h''(u)\left(A_m^{p_m,q_m}(t)+t-v\right)dv\right|$$
$$\le D_m^{p_m,q_m}\left((u-t)^2;t\right)\|h''\|+\left(A_m^{p_m,q_m}(t)\right)^2\|h''\|$$
$$=\left(A_m^{p_m,q_m}(t)+B_m^{p_m,q_m}(t)\right)^2\|h''\|. \quad (44)$$

On the other hand, since $|\mathbf{D}_m^{p_m,q_m}(g;t)|\le3\|g\|$. Hence,

$$|D_m^{p_m,q_m}(g;t)-g(t)|=|\mathbf{D}_m^{p_m,q_m}(g;t)+g\left(A_m^{p_m,q_m}(t)+t\right)-2g(t)|$$
$$\le|\mathbf{D}_m^{p_m,q_m}(g-h;t)-(g-h)(t)|+|\mathbf{D}_m^{p_m,q_m}(h;x)-h(x)|+|g\left(A_m^{p_m,q_m}(x)+x\right)-g(x)|$$
$$\le4\|g-h\|+\left(\left(A_m^{p_m,q_m}(t)+B_m^{p_m,q_m}(t)\right)^2+\right)\|h''\|+\omega\left(g;|A_m^{p_m,q_m}(t)|\right). \quad (45)$$

Taking infinum on the right hand side over all $h\in S^2$ and using (39), we obtain the desired assertion.

Corollary 11. *Let (p_m), (q_m) be the sequences defined in Lemma 6 and $g\in C_B[0,\infty)$. Then, for any finite interval $I\subset[0,\infty)$, the sequence $\{D_m^{p_m,q_m}(g;t)\}$ converges to g uniformly on I.*

6. Rate of Convergence

Let

$$\begin{aligned} &B_2[0,\infty) = \left\{g : |g(t)| \le M_g\left(1+t^2\right)\right\},\\ &C_2[0,\infty) = \left\{g : g \in B_2[0,\infty) \cap C[0,\infty)\right\},\\ &C_2^0[0,\infty) = \left\{g : g \in C_2[0,\infty) \text{and} \lim_{t\to\infty} \frac{|g(t)|}{1+t^2} < \infty\right\}, \end{aligned} \tag{46}$$

where M_g is an absolute constant depending only on g. $C_2^0[0,\infty)$ is equipped with the norm $\|g\|_2 = \sup_{t\in[0,\infty)} |g(t)|/(1+t^2)$. As is known, if $f \in C[0,\infty)$ is not uniform, we cannot obtain $\lim_{\delta\longrightarrow 0^+} \omega(g;\delta) = 0$. In [15], Ispir defined the following weighted modulus of continuity

$$\Omega(g;\delta) = \sup_{0<u\le\delta, t\in(0,\infty)} \frac{|g(u+t)-g(t)|}{(1+u^2)(1+t^2)}, \tag{47}$$

and proved the properties of monotone increasing about $\Omega(g;\delta)$ as $\delta > 0$, $\lim_{\delta\longrightarrow 0^+} \Omega(g;\delta) = 0$ and the inequality

$$\Omega(g;\lambda\delta) \le 2(1+\lambda)\left(1+\delta^2\right)\Omega(g;\delta), \tag{48}$$

while $\lambda > 0$ and $g \in C_2^0[0,\infty)$. Meantime, we recall the modulus of continuity of g on the interval $[0,\kappa] \subset [0,\infty)$ by

$$\omega_\kappa(g;\delta) = \sup_{u,t\in(0,\kappa], |u-t|\le\delta} |g(u)-g(t)|, \delta > 0. \tag{49}$$

Theorem 12. *Let $g \in C_2[0,\infty)$, $0 < q < p \le 1$, and $\kappa > 0$, we have*

$$\|D_m^{p,q}(g;t) - g\|_{C[0,\kappa]} \le 4M_g\left(3+2\kappa^2\right)B_m^{p,q}(\kappa) + 2\omega_{\kappa+1}\left(f;\sqrt{B_m^{p,q}(\kappa)}\right). \tag{50}$$

Proof. For any $t \in [0,\kappa]$ and $u > \kappa+1$, we easily have $1 \le (u-\kappa)^2 \le (u-t)^2$, thus

$$\begin{aligned} |g(u)-g(t)| &\le |g(u)| + |g(t)| \le M_g\left(2+u^2+t^2\right)\\ &= M_g\left(2+t^2+(u-t+t)^2\right)\\ &\le M_g\left(2+2t^2+(u-t)^2\right) \le M_g\left(3+2t^2\right)(u-t)^2\\ &\le M_g\left(3+2\kappa^2\right)(u-t)^2, \end{aligned} \tag{51}$$

and for any $t \in [0,\kappa]$, $u \in [0,\kappa+1]$ and $\delta > 0$, we have

$$|g(u)-g(t)| \le \omega_{\kappa+1}(|u-t|;t) \le \left(1+\frac{|u-t|}{\delta}\right)\omega_{\kappa+1}(g;\delta). \tag{52}$$

For (51) and (52), we can get

$$|g(u)-g(t)| \le M_g\left(3+2\kappa^2\right)(u-t)^2 + \left(1+\frac{|u-t|}{\delta}\right)\omega_{\kappa+1}(g;\delta). \tag{53}$$

By Schwarz's inequality, for any $t \in [0,\kappa]$, we can get

$$\begin{aligned} |D_m^{p,q}(g;t)-g(t)| &\le D_m^{p,q}(|g(u)-g(t)|;t)\\ &\le M_g\left(3+2\kappa^2\right)D_m^{p,q}\left((u-t)^2;t\right)\\ &\quad + D_m^{p,q}\left(\left(1+\frac{|u-t|}{\delta}\right);t\right)\omega_{\kappa+1}(g;\delta)\\ &\le M_g\left(3+2\kappa^2\right)D_m^{p,q}\left((u-t)^2;t\right)\\ &\quad + \omega_{\kappa+1}(g;\delta)\left(1+\frac{1}{\delta}\sqrt{D_m^{p,q}\left((u-t)^2;t\right)}\right)\\ &\le M_g\left(3+2\kappa^2\right)B_m^{p,q}(t) + \omega_{\kappa+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{B_m^{p,q}(t)}\right)\\ &\le M_g\left(3+2\kappa^2\right)B_m^{p,q}(\kappa) + \omega_{\kappa+1}(g;\delta)\left(1+\frac{1}{\delta}\sqrt{B_m^{p,q}(\kappa)}\right). \end{aligned} \tag{54}$$

By taking $\delta = \sqrt{B_m^{p,q}(\kappa)}$ and supremum over all $t \in [0,\kappa]$, we accomplish the proof of Theorem 12.

7. Weighted Approximation

In this section, we will discuss the following three theorems about weighted approximation for the operators $D_m^{p_m,q_m}(g;t)$.

Theorem 13. *Let $f \in C_2^0[0,\infty)$ and the sequences (p_m), (q_m) be the sequences defined in Lemma 6; then, for any $t \in [0,\infty)$, there exists $N \in \mathbb{N}_+$ such that for all $m > N$, the inequality*

$$\frac{\left|D_m^{p_m,q_m}(g;t)-g(t)\right|}{\left(1+t^2\right)^2} \le 10\Omega\left(f;\frac{1}{\sqrt{[m]_{p_m,q_m}}}\right) \tag{55}$$

holds.

Proof. Using (47) and (48), we can write

$$\begin{aligned} |g(u)-g(t)| &\le \left(1+(u-t)^2\right)\left(1+t^2\right)\Omega(f;|u-t|)\\ &\le 2\left(1+\frac{|u-t|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1+(u-t)^2\right)\left(1+t^2\right)\\ &\le \begin{cases} 4\left(1+\delta^2\right)^2\left(1+t^2\right)\Omega(f;\delta), & |u-t|\le\delta,\\ 4\left(1+\delta^2\right)\left(1+t^2\right)\Omega(f;\delta)\dfrac{|u-t|+|u-t|^3}{\delta}, & |u-t|>\delta. \end{cases} \end{aligned} \tag{56}$$

For any $\delta \in (0,1/2)$ and $t, u \in [0,\infty)$, (56) can be rewritten

$$|g(t)-g(x)| \le 5\left(1+t^2\right)\Omega(f;\delta)\left(\frac{5}{4}+\frac{|u-t|+|u-t|^3}{\delta}\right). \tag{57}$$

Using (20) and (21), there exists $N \in \mathbb{N}_+$ such that for any $m > N$,

$$D_m^{p_m,q_m}\left((u-t)^2;t\right) \le \frac{2}{[m]_{p_m,q_m}} t, D_m^{p_m,q_m}\left((u-t)^4;t\right) \le \frac{t^3}{2}. \quad (58)$$

By Schwarz's inequality, we can obtain

$$D_m^{p_m,q_m}(|u-t|;t) \le \sqrt{D_m^{p_m,q_m}\left((u-t)^2;t\right)} \le \frac{\sqrt{2t}}{\sqrt{[m]_{p_m,q_m}}}, \quad (59)$$

$$\begin{aligned} D_m^{p_m,q_m}\left(|u-t|^3;t\right) &\le \sqrt{D_m^{p_m,q_m}\left((u-t)^2;t\right)}\sqrt{D_m^{p_m,q_m}\left((u-t)^4;t\right)} \\ &\le \frac{t^2}{\sqrt{[m]_{p_m,q_m}}}. \end{aligned} \quad (60)$$

Since $D_m^{p_m,q_m}$ is linear and positive, using (58)–(60), we can obtain

$$\begin{aligned} |D_m^{p_m,q_m}(g;t) - g(t)| &\le 5\left(1+t^2\right)\Omega(g;\delta)\left(\frac{5}{4} + \frac{D_m^{p_m,q_m}\left(|u-t|+|u-t|^3;t\right)}{\delta}\right) \\ &\le 5\left(1+t^2\right)\left(\frac{5}{4} + \frac{\sqrt{2t}+t^2}{\delta\sqrt{[m]_{p_m,q_m}}}\right)\Omega(f;\delta). \end{aligned} \quad (61)$$

Choosing $\delta = 1/\sqrt{[m]_{p_m,q_m}}$, we have

$$\begin{aligned} |D_m^{p_m,q_m}(g;t) - g(t)| &\le 5\left(1+t^2\right)\left(\frac{5}{4} + \sqrt{2t} + t^2\right)\Omega\left(f;\frac{1}{\sqrt{[m]_{p_m,q_m}}}\right) \\ &\le 10\left(1+t^2\right)^2\Omega\left(f;\frac{1}{\sqrt{[m]_{p_m,q_m}}}\right). \end{aligned} \quad (62)$$

the conclusion holds.

Theorem 14. *Let (p_m), (q_m) be the sequences defined in Lemma 6. Then, for any $g \in C_2^0[0,\infty)$, we have*

$$\lim_{m\longrightarrow\infty} \|D_m^{p_m,q_m}(g;t) - g\|_2 = 0. \quad (63)$$

Proof. By weighted Korovkin theorem in [16], we see that it is sufficient to verify the following three conditions:

$$\lim_{m\longrightarrow\infty} \left\|D_m^{p_m,q_m}\left(u^k;t\right) - t^k\right\|_2 = 0, k = 0,1,2. \quad (64)$$

Since $D_m^{p_m,q_m}(1;x) = 1$, then (64) holds true for $k = 0$. By Lemma 4, we can obtain

$$\begin{aligned} \|D_m^{p_m,q_m}(u;t) - t\|_2 &= \sup_{t\in[0,\infty)} \frac{\left|D_m^{p_m,q_m}(u;t) - t\right|}{1+t^2} \\ &\le \frac{p_m^m q_m^{-1}}{[m]_{p_m,q_m}} \sup_{t\in[0,\infty)} \frac{t}{1+x^2} \\ &= \frac{p_m^m q_m^{-1}}{[m]_{p_m,q_m}} \longrightarrow 0, m \longrightarrow \infty, \end{aligned}$$

$$\begin{aligned} \left\|D_m^{p_m,q_m}\left(u^2;t\right) - t^2\right\|_2 &= \sup_{t\in[0,\infty)} \frac{\left|D_m^{p_m,q_m}\left(u^2;t\right) - t^2\right|}{1+t^2} \\ &\le \sup_{t\in[0,\infty)} \frac{t}{1+t^2} \frac{p_m^m\left([2]_{p_m,q_m}^2 p_m^{-1} q_m^{-2}\right)}{[m]_{p_m,q_m}} \\ &\quad + \sup_{t\in[0,\infty)} \frac{t}{1+t^2} \frac{p_m^{2m} q_m^{-3} [2]_{p_m,q_m}}{[m]_{p_m,q_m}^2} \\ &= \frac{p_m^m\left([2]_{p_m,q_m}^2 p_m^{-1} q_m^{-2}\right)}{[m]_{p_m,q_m}} \\ &\quad + \frac{p_m^{2m} q_m^{-3} [2]_{p_m,q_m}}{[m]_{p_m,q_m}^2} \longrightarrow 0, n \longrightarrow \infty. \end{aligned} \quad (65)$$

Thus, the proof of Theorem 14 is completed.

Theorem 15. *Let (p_m), (q_m) be the sequences defined in Lemma 6. Then, for any $g \in C_2^0[0,\infty)$ and $\lambda > 0$, we have*

$$\lim_{m\longrightarrow\infty} \sup_{t\in[0,\infty)} \frac{\left|D_m^{p_m,q_m}(g;t) - g(t)\right|}{\left(1+t^2\right)^{1+\lambda}} = 0. \quad (66)$$

Proof. Let $t_0 \in (0,\infty)$ be arbitrary but fixed. Then,

$$\begin{aligned} \sup_{t\in[0,\infty)} \frac{\left|D_m^{p_m,q_m}(g;t) - g(t)\right|}{\left(1+t^2\right)^{1+\lambda}} &\le \sup_{t\in[0,t_0]} \frac{\left|D_m^{p_m,q_m}(g;t) - g(t)\right|}{\left(1+t^2\right)^{1+\lambda}} \\ &\quad + \sup_{t\in(t_0,\infty)} \frac{\left|D_m^{p_m,q_m}(g;t) - g(t)\right|}{\left(1+t^2\right)^{1+\lambda}} \\ &\le \|D_m^{p_m,q_m}(g;t) - g\|_{C[0,t_0]} \\ &\quad + \|g\|_2 \sup_{t\in(t_0,\infty)} \frac{\left|D_m^{p_m,q_m}\left(1+u^2;t\right)\right|}{\left(1+t^2\right)^{1+\lambda}} \\ &\quad + \sup_{t\in(t_0,\infty)} \frac{|g(t)|}{\left(1+t^2\right)^{1+\lambda}}. \end{aligned} \quad (67)$$

Since $|g(t)| \le M_g(1+t^2)$, we have $\sup_{t\in(t_0,\infty)} |g(t)|/$

$(1+t^2)^{1+\lambda} \leq M_g\|g\|_2/(1+t_0^2)^{\lambda}$. Let $\varepsilon > 0$ be arbitrary, we can choose t_0 to be so large that

$$\frac{M_g\|g\|_2}{\left(1+t_0^2\right)^{\lambda}} < \frac{\varepsilon}{3}. \tag{68}$$

In view of Lemma 4, while $t \in (t_0,\infty)$, we can obtain

$$\|f\|_2 \lim_{t\longrightarrow\infty}\lim_{m\longrightarrow\infty} \frac{\left|D_m^{p_m,q_m}\left(1+u^2;x\right)\right|}{\left(1+t^2\right)^{1+\lambda}} = \|f\|_2 \lim_{t\longrightarrow\infty}\frac{1}{\left(1+t^2\right)^{\lambda}} = 0. \tag{69}$$

Hence, we can choose N and t_0 to be so large such that for any $m > N$ the inequality

$$\sup_{t\in[t_0,\infty)} \|g\|_2 \frac{\left|D_m^{p_m,q_m}\left(1+u^2;t\right)\right|}{\left(1+t^2\right)^{1+\lambda}} < \frac{\varepsilon}{3}. \tag{70}$$

holds. Also, the first term of the above inequality tends to zero by Theorem 12, that is

$$\|D_m^{p_m,q_m}(g;t) - g\|_{C[0,t_0]} < \frac{\varepsilon}{3}. \tag{71}$$

Thus, combining (68)–(71), we obtain the desired result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors read and approved the final manuscript.

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Research Article

BMO Functions Generated by $A_X(\mathbb{R}^n)$ Weights on Ball Banach Function Spaces

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Let X be a ball Banach function space on $\mathbb{R}^n$. We introduce the class of weights $A_X(\mathbb{R}^n)$. Assuming that the Hardy-Littlewood maximal function M is bounded on X and X', we obtain that $\mathrm{BMO}(\mathbb{R}^n) = \{\alpha \ln \omega : \alpha \geq 0, \omega \in A_X(\mathbb{R}^n)\}$. As a consequence, we have $\mathrm{BMO}(\mathbb{R}^n) = \{\alpha \ln \omega : \alpha \geq 0, \omega \in A_{L^{p(\cdot)}(\mathbb{R}^n)}(\mathbb{R}^n)\}$, where $L^{p(\cdot)}(\mathbb{R}^n)$ is the variable exponent Lebesgue space. As an application, if a linear operator T is bounded on the weighted ball Banach function space $X(\omega)$ for any $\omega \in A_X(\mathbb{R}^n)$, then the commutator $[b, T]$ is bounded on X with $b \in \mathrm{BMO}(\mathbb{R}^n)$.

1. Introduction

It is well known that there is a relation between $A_\infty(\mathbb{R}^n)$ weights and $\mathrm{BMO}(\mathbb{R}^n)$, i.e., for any $p \in (1, \infty]$,

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{\alpha \ln W : \alpha \geq 0, W \in A_p(\mathbb{R}^n)\right\}. \tag{1}$$

See, for instance, [1] (p. 409). The purpose of this note is to reveal the relation between $\mathrm{BMO}(\mathbb{R}^n)$ and $A_X(\mathbb{R}^n)$ weights over the ball Banach function space X.

To state our results, we begin with the definition of the ball Banach function space. Denote by the symbol $\mathscr{M}(\mathbb{R}^n)$ the set of all measurable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and

$$\mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\}. \tag{2}$$

Definition 1. A Banach space $X \subset \mathscr{M}(\mathbb{R}^n)$ is called a ball Banach function space if it satisfies that

(i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere

(ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_X \leq \|f\|_X$

(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$

(iv) $B \in \mathbb{B}$ implies that $\mathbf{1}_B \in X$, where $\mathbb{B}$ is as in (2);

(v) for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on B, such that, for any $f \in X$,

$$\int_B |f(x)|\, dx \leq C_{(B)} \|f\|_X \tag{3}$$

For any ball Banach function space X, the *associate space* (*Köthe dual*) X' is defined by setting

$$X' := \left\{f \in \mathscr{M}(\mathbb{R}^n) : \|f\|_X{}' := \sup\left\{\|fg\|_{L^1(\mathbb{R}^n)} : g \in X, \|g\|_X = 1\right\} < \infty\right\}, \tag{4}$$

where $\|\cdot\|_X{}'$ is called the *associate norm* of $\|\cdot\|_X$ (see, for instance, [2] (Chapter 1, Definitions 2.1 and 2.3)).

Remark 2. By [3] (Proposition 2.3), we know that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

Now, we introduce the class of weights $A_X(\mathbb{R}^n)$ and recall the function space BMO. A weight ω is a locally integrable function such that $0 < \omega(x) < \infty$ almost everywhere $x \in \mathbb{R}^n$.

Definition 3. Let X be a ball Banach function space. We say that a weight ω belongs to $A_X(\mathbb{R}^n)$ if

$$\sup_{B\in\mathbb{B}} \frac{\|\omega \mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \frac{\|\omega^{-1}\mathbf{1}_B\|_{X'}}{\|\mathbf{1}_B\|_{X'}} < \infty, \quad (5)$$

here and hereafter $\mathbf{1}_B$ is the characteristic function for B.

Remark 4.

(1) There is an immediate consequence. Let X be a ball Banach function space. If $\omega \in A_X(\mathbb{R}^n)$, then $\omega^{-1} \in A_{X'}(\mathbb{R}^n)$

(2) We recall that the definition of $A_p(\mathbb{R}^n)$. Let $p \in [1, \infty]$. A weight W belongs to $A_p(\mathbb{R}^n)$ if

$$\sup_{B\in\mathbb{B}} \left\{ \frac{1}{|B|} \int_B W(x)\, dx \right\} \left\{ \frac{1}{|B|} \int_B W(x)^{1-p'}\, dx \right\}^{p-1} < \infty \quad (6)$$

By the definition of $A_X(\mathbb{R}^n)$ and $A_p(\mathbb{R}^n)$, $W \in A_p(\mathbb{R}^n)$ if and only if $\omega := W^{1/p} \in A_{L^p(\mathbb{R}^n)}(\mathbb{R}^n)$ for any $p \in [1, \infty]$.

The classical function space $\mathrm{BMO}(\mathbb{R}^n)$ is the collection of all locally integrable functions f such that

$$\mathrm{BMO}(\mathbb{R}^n) := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx, \quad (7)$$

where the supremum is taking all balls B in $\mathbb{R}^n$ and f_B is the mean value of the function f on B, namely,

$$f_B := \frac{1}{|B|} \int_B f(y)\, dy. \quad (8)$$

By the well-known John-Nirenberg inequality, John and Nirenberg [4] proved that there exists a positive constant C such that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \le \|f\|_{\mathrm{BMO}_{L^p(\mathbb{R}^n)}} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}, \quad (9)$$

where $p \in [1, \infty)$ and

$$\mathrm{BMO}_{L^p(\mathbb{R}^n)} := \sup_B \left\{ \frac{1}{|B|} \int_B |f(x) - f_B|^p\, dx \right\}^{1/p}. \quad (10)$$

We also recall that the Hardy-Littlewood maximal function M is defined by setting, for any locally integrable function f and $x \in \mathbb{R}^n$,

$$Mf(x) := \sup_{B\in\mathbb{B}} \frac{\mathbf{1}_B(x)}{|B|} \int_B |f(y)|\, dy. \quad (11)$$

Now, we state our result as the following theorem.

Theorem 5. *Let X be ball Banach function spaces. If the Hardy-Littlewood maximal function M is bounded on X and X', then*

$$BMO(\mathbb{R}^n) := \{\alpha \ln \omega : \alpha \ge 0, \omega \in A_X(\mathbb{R}^n)\}. \quad (12)$$

Remark 6. Let $p \in (1, \infty)$, Theorem5goes back to the classical result for $X := L^p(\mathbb{R}^n)$.

As an example, let $\mathscr{P} = \mathscr{P}(\mathbb{R}^n)$ be the collection of all measurable functions $p(\cdot)$: $\mathbb{R}^n \longrightarrow [1, \infty]$. Then, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty): \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda} \right]^{p(x)} dx \le 1 \right\} < \infty. \quad (13)$$

Denote $p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$. A measurable function $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ is said to be *globally log-Hölder continuous* if there exists a $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e + (1/|x-y|))}, \qquad |p(x) - p_\infty| \lesssim \frac{1}{\log(e + |x|)}, \quad (14)$$

where the implicit positive constants are independent of x and y.

Definition 7 ([5], Definition 1.4.). Given an exponent function $p(\cdot)$: $\mathbb{R}^n \longrightarrow [1, \infty)$ *and a weight ω , we say that $\omega \in A_{p(\cdot)}$ if there exists a constant K such that for every ball B,*

$$\|\omega \mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\omega^{-1}\mathbf{1}_B\|_{L^{p(\cdot)'}(\mathbb{R}^n)} \le C|B|, \quad (15)$$

where $1/p(x) + 1/p(x)' = 1$ for almost everywhere $x \in \mathbb{R}^n$.

Remark 8. Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \le p_+ < \infty$. By [3] (Lemma 2.5 and Proposition 3.8.), for any ball $B \subset \mathbb{R}^n$, $|B| = \|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\mathbf{1}_B\|_{L^{p(\cdot)'}(\mathbb{R}^n)}$. This shows that for $X = L^{p(\cdot)}(\mathbb{R}^n)$

$$A_{L^{p(\cdot)}(\mathbb{R}^n)}(\mathbb{R}^n) = A_{p(\cdot)}(\mathbb{R}^n). \quad (16)$$

Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \le p_+ < \infty$. We know that M is bounded on

$L^{p(\cdot)}(\mathbb{R}^n)$ and its duality $L^{p(\cdot)'}(\mathbb{R}^n)$; see, for instance, [6, 7] and their references.

Corollary 9. *Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \le p_+ < \infty$. Then, $BMO(\mathbb{R}^n) = \{\alpha \ln \omega : \alpha \ge 0, \omega \in A_{p(\cdot)}(\mathbb{R}^n)\}$.*

2. Proof of Theorem 5

The following lemmas give two elementary properties of ball Banach function spaces, whose proof is similar to the one corresponding to Banach function spaces; see [2].

Lemma 10 (Holder's inequality). *Let X be a ball Banach function space with the associate space X'. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le \|f\|_X \|g\|_X{}'. \tag{17}$$

Lemma 11 (G. G. Lorentz, W. A. J. Luxembourg). *Every ball Banach function space X coincides with its second associate space X''. In other words, a function f belongs to X if and only if it belongs to X'' and, in that case,*

$$\|f\|_X = \|f\|_X{}''. \tag{18}$$

Under weak boundedness of the Hardy-Littlewood maximal function M on X, the norm $\|\cdot\|_X$ enjoys the following property; see [8] (Lemma 2.2).

Lemma 12. *Let X be a ball Banach function space and suppose that the Hardy-Littlewood maximal operator M is weakly bounded on X or X' , that is, there exists a positive constant C such that*

$$\left\| 1_{\{x\in\mathbb{R}^n : Mf(x)>\lambda\}} \right\|_X \le C\lambda^{-1} \|f\|_X \tag{19}$$

or

$$\left\| 1_{\{x\in\mathbb{R}^n : Mf(x)>\lambda\}} \right\|_{X'} \le C\lambda^{-1} \|f\|_X{}' \tag{20}$$

holds for all $\lambda > 0$ and all $f \in X$. Then, there exists a positive constant C such that for all balls $B \in \mathbb{B}$, $\|1_B\|_X \|1_B\|_X{}' \le C|B|$.

Remark 13. By Lemma10, we have $|B| \le \|\mathbf{1}_B\|_X \|\mathbf{1}_B\|_X{}'$ for any ball $B \in \mathbb{B}$.

Lemma 14. *Let $\varphi \in L^1_{loc}(\mathbb{R}^n)$ and X be a ball Banach function space. Suppose that the Hardy-Littlewood maximal operator M is weakly bounded on X. Then, $e^\varphi \in A_X(\mathbb{R}^n)$ if and only if there exists a positive constant C such that for any ball $B \in \mathbb{B}$*

$$\begin{aligned} \frac{\|e^{\varphi-\varphi_B} 1_B\|_X}{\|1_B\|_X} &\le C, \\ \frac{\left\|e^{-(\varphi-\varphi_B)} 1_B\right\|_X{}'}{\|1_B\|_X{}'} &\le C. \end{aligned} \tag{21}$$

Proof. We first prove the sufficiency. In fact, by the definition of $A_X(\mathbb{R}^n)$, we have

$$\frac{\|e^\varphi \mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \frac{\|e^{-\varphi}\mathbf{1}_B\|_X{}'}{\|\mathbf{1}_B\|_X{}'} = \frac{\|e^{\varphi-\varphi_B}\mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \frac{\left\|e^{-(\varphi-\varphi_Q)}\mathbf{1}_B\right\|_X{}'}{\|\mathbf{1}_B\|_X{}'} \le C. \tag{22}$$

Conversely, suppose that $e^\varphi \in A_X(\mathbb{R}^n)$. Then by Lemmas 10 and 12

$$\begin{aligned} \frac{\|e^{\varphi-\varphi_B}\mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} &= e^{-\varphi_Q} \frac{\|e^\varphi \mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \le \frac{1}{|B|}\int_B e^{-\varphi(x)}\, dx \frac{\|e^\varphi \mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \\ &\le \frac{\|e^{-\varphi}\|_X{}' \|\mathbf{1}_B\|_X}{|B|} \frac{\|e^\varphi \mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \le C. \end{aligned} \tag{23}$$

Also,

$$\begin{aligned} \frac{\left\|e^{-(\varphi-\varphi_B)}\mathbf{1}_B\right\|_X{}'}{\|\mathbf{1}_B\|_X{}'} &= e^{\varphi_Q} \frac{\|e^{-\varphi}\mathbf{1}_B\|_X{}'}{\|\mathbf{1}_B\|_X{}'} \le \frac{1}{|B|}\int_B e^{\varphi(x)}\, dx \frac{\|e^\varphi \mathbf{1}_B\|_X{}'}{\|\mathbf{1}_B\|_X{}'} \\ &\le \frac{\|e^\varphi\|_X \|\mathbf{1}_B\|_X{}'}{|B|} \frac{\|e^{-\varphi}\mathbf{1}_B\|_X{}'}{\|\mathbf{1}_B\|_X{}'} \le C. \end{aligned} \tag{24}$$

The John-Nirenberg inequality for ball Banach function spaces X was established by Izuki et al. ([9], Theorem 3.1).

Lemma 15. *Let X be a ball Banach function space such that M is bounded on X' and write $C_0 := \|M\|_X{}'^{\to X'}$. Then, there exists a positive constant C_1 such that for all balls B, $f \in BMO(\mathbb{R}^n)$ and $\lambda \ge 0$,*

$$\left\| 1_{\{x\in B: |f-f_B|>\lambda\}} \right\|_X \le C_1 2^{-\left(\lambda/\left(2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{BMO}\right) 2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{BMO}\right)} \|1_B\|_X. \tag{25}$$

As a consequence of Lemma 15, we have the following inequality.

Lemma 16. *Let X be a ball Banach function space. Suppose that M is bounded on X'. Suppose that $\varphi \in BMO$. Then for any $\alpha \in [0, (\ln 2/(2^{n+2}(1 + 2^{n+4}C_0)\|f\|_{BMO})))$ and ball $B \in \mathbb{B}$, we have*

$$\left\| e^{\alpha|\varphi-\varphi_B|} 1_B \right\|_X \le C_1 \left(1 - 2^{(\alpha/\ln 2) - \left(1/\left(2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{BMO}\right)\right)}\right) 2^{1/\left(2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{BMO}\right)} \|1_B\|_X, \tag{26}$$

where C_1 is as in Lemma 15.

Proof. By Lemma 15, we have

$$\begin{aligned}\left\|e^{\alpha|\varphi-\varphi_B|}\right\|_X &\le \sum_{k=0}^{\infty}\left\|\mathbf{1}_{\{x\in B:k+1\ge|\varphi-\varphi_B|>k\}}e^{\alpha|\varphi-\varphi_B|}\right\|_X \\ &\le \sum_{k=0}^{\infty}e^{\alpha(k+1)}\left\|\mathbf{1}_{\{x\in B:|\varphi-\varphi_B|>k\}}\right\|_X \\ &\le \sum_{k=0}^{\infty}C_1e^{\alpha(k+1)}2^{-k/\left(2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{\mathrm{BMO}}\right)}\|\mathbf{1}_B\|_X \\ &= \sum_{k=0}^{\infty}C_1e^{\alpha}2^{\left((\alpha k/\ln 2)-\left(k/\left(2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{\mathrm{BMO}}\right)\right)\right)}\|\mathbf{1}_B\|_X \\ &\le C_1\left(1-2^{(\alpha/\ln 2)-\left(1/\left(2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{\mathrm{BMO}}\right)\right)}\right) \\ &\quad\cdot 2^{1/\left(2^{n+2}\left(1+2^{n+4}C_0\right)\|f\|_{\mathrm{BMO}}\right)}\|\mathbf{1}_B\|_X. \end{aligned} \tag{27}$$

Lemma 17. *Let X be a ball Banach function space. If $\omega\in A_X(\mathbb{R}^n)$, then $\ln\omega\in BMO(\mathbb{R}^n)$.*

Proof. Let $\varphi:=\ln\omega$. Then, $\omega=e^{\varphi}$. By Lemmas 10, 12, and 14, we obtain that

$$\begin{aligned}\|\varphi\|_{\mathrm{BMO}(\mathbb{R}^n)} &\le C\sup_{B\in\mathbb{B}}\frac{1}{|B|}\int_B e^{|\varphi(x)-\varphi_B|}dx \\ &\le C\sup_{B\in\mathbb{B}}\frac{1}{|B|}\left[\int_{\{x\in B:\varphi(x)-\varphi_B\ge 0\}}e^{\varphi(x)-\varphi_B}dx\right. \\ &\quad\left.+\int_{\{x\in B:\varphi(x)-\varphi_B<0\}}e^{-(\varphi(x)-\varphi_B)}dx\right] \\ &\le C\sup_{B\in\mathbb{B}}\frac{1}{|B|}\left[\|e^{\varphi-\varphi_B}\mathbf{1}_B\|_X\|\mathbf{1}_B\|_{X}{}'\right. \\ &\quad\left.+\|e^{-(\varphi-\varphi_B)}\mathbf{1}_B\|_{X}{}'\|\mathbf{1}_B\|_X\right]<\infty. \end{aligned} \tag{28}$$

Proof of Theorem 18. By Lemma 17, for any $\omega\in A_X(\mathbb{R}^n)$ and $\alpha\ge 0$, $\varphi:=\alpha\ln\omega\in\mathrm{BMO}(\mathbb{R}^n)$. Conversely, suppose that $\varphi\in\mathrm{BMO}(\mathbb{R}^n)$. Since M is bounded on X', by Lemma 16, we know that there exist $\beta_1\in[0,\ln 2/(2^{n+2}(1+2^{n+4}\|M\|_{X}{}'_{\to X'})\|\varphi\|_{\mathrm{BMO}}))$ and $C_3\in(0,\infty)$ such that, for any ball $B\in\mathbb{B}$,

$$\frac{\|e^{\beta_1|\varphi-\varphi_B|}\mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X}\le C_3. \tag{29}$$

Similarly, since M is bounded on X, by Lemmas 11 and 16, we know that there exist $\beta_2\in[0,\ln 2/(2^{n+2}(1+2^{n+4}\|M\|_{X\to X})\|\varphi\|_{\mathrm{BMO}}))$ and $C_4\in(0,\infty)$ such that, for any ball $B\in\mathbb{B}$,

$$\frac{\|e^{\beta_2|\varphi-\varphi_B|}\mathbf{1}_B\|_{X}{}'}{\|\mathbf{1}_B\|_{X}{}'}\le C_4. \tag{30}$$

Taking $\alpha=\min\{\beta_1,\beta_2\}$ and $C=\max\{C_3,C_4\}$ and applying Lemma 14, we get the desired result.

3. Applications

In this section, we will show that the boundedness of the commutator of a linear operator T on X with the BMO function can be derived from the weighted boundedness of T on X. We first establish the following Minkowski-type inequality.

Lemma 19. *Let X be a Banach function space and F a measurable function on $\mathbb{R}^n\times\mathbb{R}^m$. If, for almost every $x\in\mathbb{R}^n$, $F(x,\cdot)\in L^1(\mathbb{R}^m)$ and, for almost every $y\in\mathbb{R}^m$, $F(\cdot,y)\in X$, then*

$$\left\|\int_{\mathbb{R}^m}|F(\cdot,y)|\,dy\right\|_X\le\int_{\mathbb{R}^m}\|F(\cdot,y)\|_X\,dy. \tag{31}$$

Proof. By Lemma 11, we have

$$\begin{aligned}\left\|\int_{\mathbb{R}^m}|F(\cdot,y)|\,dy\right\|_X &= \left\|\int_{\mathbb{R}^m}|F(\cdot,y)|\,dy\right\|_X'' \\ &= \sup\left\{\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|F(x,y)|\,dy g(x)\,dx\right|: g\in X' \text{ such that} \|g\|_{X}{}'=1\right\}. \end{aligned} \tag{32}$$

From the Fubini theorem and Lemma 10, it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|F(x,y)|dyg(x)dx\right| &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|F(x,y)||g(x)|dydx \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n}|F(x,y)||g(x)|dxdy \\ &\lesssim \int_{\mathbb{R}^m}\|F(\cdot,y)\|_X\|g\|_{X'}\,dy \\ &= \int_{\mathbb{R}^m}\|F(\cdot,y)\|_X dy, \end{aligned} \tag{33}$$

which implies the desired conclusion. This finishes the proof of Lemma 19.

Let T be a linear operator defined by

$$Tf(x):=\int_{\mathbb{R}^n}K(x,y)f(y)dy. \tag{34}$$

Given a symbol b, we define the commutator $[b,T]f=b\,Tf-T(bf)$.

Let ω be a weight. Define $X(\omega):=\{f\in\mathscr{M}(\mathbb{R}^n): f\omega\in X\}$ and $\|f\|_{X(\omega)}:=\|f\omega\|_X$. We say that T is bounded on $X(\omega)$ if there exists a positive constant C such that for all $f\in X(\omega)$,

$$\|(Tf)\omega\|_X\le C\|f\omega\|_X. \tag{35}$$

Theorem 20. *Let X be a ball Banach function space. Suppose that M is bounded on X and X' . If, for any $\omega\in A_X(\mathbb{R}^n)$, T is* *bounded on* $X(\omega)$ *then, for all* $b \in BMO(\mathbb{R}^n)$, $[b, T]$ *is bounded on* X *, i.e.,*

$$\| [b, T]f \|_X \le C \|f\|_X, \tag{36}$$

where C is independent of f.

Proof. We adapt the idea from [10, 11]. Without loss of generality, we assume that $b \neq 0$ in $\mathrm{BMO}(\mathbb{R}^n)$. By Theorem 5, there exists a $\alpha \in (0,\infty)$ such that $e^{\alpha b} \in A_X(\mathbb{R}^n)$. As well known, for every $\theta \in [0, 2\pi]$, $b \cos \theta \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|b \cos \theta\|_{\mathrm{BMO}(\mathbb{R}^n)} = \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Thus,

$$e^{\alpha b \cos \theta} \in A_X(\mathbb{R}^n). \tag{37}$$

For any $z \in \mathbb{C}$, define $g(z) \coloneqq e^{z\alpha[b(x)-b(y)]}$. Then, $g(z)$ is analytic on $\mathbb{C}$ and the Cauchy integral formula implies that

$$\begin{aligned} b(x) - b(y) &= \frac{g'(0)}{\alpha} = \frac{1}{2\alpha\pi i} \int_{|z|=1} \frac{g(z)}{|z|^2} \, dz \\ &= \frac{1}{2\alpha\pi} \int_0^{2\pi} e^{i\theta\alpha[b(x)-b(y)]} e^{-i\theta} \, d\theta. \end{aligned} \tag{38}$$

For any $\theta \in [0, 2\pi]$, set $h_\theta(x) \coloneqq f(x) e^{-\alpha b(x) e^{i\theta}}$. Since $f \in X$, we have

$$\|h_\theta\|_{X\left(e^{\alpha b \cos \theta}\right)} = \left\| f(x) e^{-\alpha b e^{i\theta}} e^{\alpha b \cos \theta} \right\|_X = \|f\|_X. \tag{39}$$

By this and (38), we conclude that

$$\begin{aligned} [b, T]f(x) &= \int_{\mathbb{R}^n} K(x, y) \left[\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta\alpha[b(x)-b(y)]} e^{-i\theta} \, d\theta \right] f(y) \, dy \\ &= \frac{1}{2\pi} \int_0^{2\pi} T(h_\theta)(x) e^{\alpha b(x) e^{i\theta}} e^{-i\theta} \, d\theta. \end{aligned} \tag{40}$$

Applying Lemma 19 and the weighted boundedness of T, we have

$$\| [b, T]f \|_X \le \frac{1}{2\alpha\pi} \int_0^{2\pi} \| T(h_\theta) \|_{X\left(e^{\alpha b \cos \theta}\right)} \, d\theta \le C \|f\|_X. \tag{41}$$

We complete the proof of Theorem 20.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Convergence of Some Iterative Algorithms for System of Generalized Set-Valued Variational Inequalities

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In this article, we consider and study a system of generalized set-valued variational inequalities involving relaxed cocoercive mappings in Hilbert spaces. Using the projection method and Banach contraction principle, we prove the existence of a solution for the considered problem. Further, we propose an iterative algorithm and discuss its convergence. Moreover, we establish equivalence between the system of variational inequalities and altering points problem. Some parallel iterative algorithms are proposed, and the strong convergence of the sequences generated by these iterative algorithms is discussed. Finally, a numerical example is constructed to illustrate the convergence analysis of the proposed parallel iterative algorithms.

1. Introduction

The theory of variational inequality was planted in the early 1960's by Stampacchia [1] in the framework of obstacle constraint minimization problems. The first evolutionary variational inequality was solved in the seminal paper of Lions and Stampacchia [2]. Since its inception, it has enjoyed a vigorous development for the last few decades. This subject has developed in multiple directions using innovative techniques to solve fundamental problems to be insurmountable previously. This field is influential and experiencing an explosive growth in theory as well as applications. Consequently, some of these developments in this area enriched other mutual areas of mathematical and engineering sciences such as economics, transportation, nonlinear programming, and operations research. It has been shown that this theory provides the most natural, direct, simple, unified, and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems; see, for example, [3–9] and the references cited therein.

Recently, fixed-point methods have been extensively investigated for solving monotone variational inequalities. Among the fixed point algorithms, Mann-like iterative algorithms are useful for solving several nonlinear problems. The proximal point algorithm is a widely used tool for solving a variety of (single objective) convex optimization problems such as finding zeros of maximal monotone operators and fixed points of nonexpansive mappings, as well as minimizing convex functions. In 2007, Agarwal et al. [10] introduced the following iteration process. Let C be a nonempty convex subset of a normed linear space X and let $T : C \to C$ be an operator. Then, for arbitrary $x_0 \in C$, estimate $x_n \in C$ by the following S-iterative scheme:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0,1) satisfying some suitable conditions. Sahu [11] proved that the S-iteration process is more applicable than the Picard [12], Mann [13], and Ishikawa [14] iteration algorithms because it converges faster than these iteration processes for contraction mappings and also works for nonexpansive mappings. Recently, Cholamjiak et al. [15] applied the S-iteration process for finding a minimizer of a convex function and fixed points of nonexpansive mappings in CAT(0) spaces. In [16], Amir et al. studied the convex constraint multiobjective optimization problem as the constrained set of fixed points of the nonexpansive mapping.

On the other hand, a number of numerical methods such as auxiliary principle, projection method, Wiener-Hopf equations, dynamical systems, and decomposition have been developed for solving the variational inequalities and related optimization problems. Among these methods, the projection method and its variant forms have been proved innovative and important tools for finding the approximate solutions of variational inequalities. In this technique, the concept of projection is used for the fixed-point formulation of variational inequality. This alternative formulation has played a significant role in developing various projection-type methods for solving variational inequalities. It is well known that the convergence of the projection methods requires that the operator must be strongly monotone and Lipschitz continuous. It is also known that the relaxed cocoercive mappings are more general than strongly monotone mappings, and under some mild conditions, relaxed cocoercive mappings can be reduced to strongly monotone mappings. For some related works, see, [17–24].

Following the facts and discussion mentioned above, in this paper, we consider a system of generalized set-valued variational inequalities defined over closed and convex subsets of a real Hilbert space. We establish an equivalence between the system of generalized set-valued variational inequalities and nonlinear projection equations using the projection technique. Further, by virtue of the projection method and Banach contraction principle, we prove an existence result. Furthermore, we propose an iterative algorithm and show that the approximate solution generated by the proposed algorithm converges strongly to the unique solution of the system of generalized set-valued variational inequalities involving relaxed cocoercive and relaxed monotone mappings in Hilbert space. Moreover, we establish equivalence between the system of variational inequalities and altering points problem. Parallel Mann and parallel S-iterative algorithms [11] have been proposed for solving the considered system of variational inequalities. Finally, the convergence analysis of the proposed parallel iterative algorithms is discussed. A numerical example is constructed to illustrate the convergence analysis of the proposed parallel iterative algorithms. The results presented in this paper can be viewed as generalizations and refinements of several results existing in the literature and include general variational inequality and some other classes of variational inequalities as special cases.

Now, we enumerate some basic notions, definitions, and results that are worthwhile tools in succeeding analysis and will be utilized in the rest of this paper.

Throughout the paper, unless otherwise specified, let $\mathscr{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$.

Definition 1 [25]. A mapping $T : \mathscr{H} \to \mathscr{H}$ is said to be

(i) γ-strongly monotone, if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \gamma \|x - y\|^2, \forall x, y \in \mathscr{H}. \quad (2)$$

(ii) μ_1-cocoercive, if there exists a constant $\mu_1 > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \mu_1 \|Tx - Ty\|^2, \forall x, y \in \mathscr{H}. \quad (3)$$

(iii) Relaxed μ_2-cocoercive, if there exists a constant $\mu_2 > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\mu_2) \|Tx - Ty\|^2, \forall x, y \in \mathscr{H}. \quad (4)$$

(iv) Relaxed (μ, γ)-cocoercive, if there exist constants μ, $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\mu) \|Tx - Ty\|^2 + \gamma \|x - y\|^2, \forall x, y \in \mathscr{H}. \quad (5)$$

(v) L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\|, \forall x, y \in \mathscr{H}. \quad (6)$$

(vi) α-contraction, if there exists a constant $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in \mathscr{H}. \quad (7)$$

T is called nonexpansive, if $\alpha = 1$.

Remark 2. We remark that every γ-strongly monotone mapping is relaxed (μ, γ)-cocoercive mapping and every γ-cocoercive mapping is $1/\gamma$-Lipschitz continuous.

Define the norm $\|\cdot\|_*$ on $\mathscr{H} \times \mathscr{H}$ by

$$\|(p, q)\|_* = \|p\| + \|q\|, \forall p, q \in \mathscr{H}. \quad (8)$$

Note that $(\mathscr{H} \times \mathscr{H}, \|\cdot\|_*)$ is a Banach space.

Definition 3 [26]. Let $T : \mathscr{H} \to \mathscr{H}$ be a single-valued mapping. A set-valued mapping $G : H \to 2^H$ is said to be a relaxed monotone with respect to T if and only if there exists a constant $c > 0$ such that

$$\langle Tu - Tv, x - y \rangle \geq (-c) \|x - y\|^2, \forall x, y \in \mathscr{H} \text{ and for some } u \in G(x), v \in G(y). \quad (9)$$

Let Ω be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. Then, for any $x \in \mathscr{H}$, there exists a unique nearest point $P_\Omega(x)$ of Ω such that

$$\|x - P_\Omega(x)\| \le \|x - y\|, \forall y \in \Omega. \quad (10)$$

The mapping P_Ω is called the metric projection [27] from $\mathscr{H}$ onto Ω. Note that the metric projection mapping P_Ω is nonexpansive from $\mathscr{H}$ onto Ω (see, Agarwal et al. [28]), i.e.,

$$\|P_\Omega(x) - P_\Omega(y)\| \le \|x - y\|, \forall x, y \in \mathscr{H}. \quad (11)$$

Lemma 4 [27]. *Let Ω be a closed and convex subset in $\mathscr{H}$. Then, for any $s \in \mathscr{H}$, the projection $P_\Omega : \mathscr{H} \to \Omega$ of $\mathscr{H}$ onto Ω satisfies*

$$\langle t - s, r - t\rangle \ge 0, \forall r \in \Omega, \quad (12)$$

if and only if

$$t = P_\Omega(s), \forall t \in \Omega. \quad (13)$$

Lemma 5. *[29]. Let $\{\omega_n\}$ be a nonnegative real sequence satisfying the following condition*

$$\omega_{n+1} \le (1 - \alpha_n)\omega_n + \varepsilon_n, \forall n \ge n_0, \quad (14)$$

where $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, and $\varepsilon_n = o(\alpha_n)$. Then, $\lim_{n\to\infty} \omega_n = 0$.

2. System of Generalized Set-Valued Variational Inequalities

Let $\mathscr{H}$ be a real Hilbert space and Ω_1 and Ω_2 be the nonempty closed and convex subsets of $\mathscr{H}$. Let $I = \{1, 2\}$ be an index set; for each $i \in I$, let $\varphi_i, \psi_i : \mathscr{H} \to \mathscr{H}$; $M_i : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ be nonlinear single-valued mappings and $P_i : \mathscr{H} \to CB(\mathscr{H})$ be a set-valued mapping. We consider the problem of finding $(x, y) \in \Omega_1 \times \Omega_2$ with $u \in P_1(x)$ and $v \in P_2(y)$ such that

$$\begin{cases} \langle \varrho_1 M_1(u, v) + \varphi_1(y) - \psi_1(x), \psi_1(y^*) - \varphi_1(y)\rangle \ge 0, \forall y^* \in \Omega_2, \\ \langle \varrho_2 M_2(v, u) + \varphi_2(x) - \psi_2(y), \psi_2(x^*) - \varphi_2(x)\rangle \ge 0, \forall x^* \in \Omega_1, \end{cases} \quad (15)$$

where ϱ_1 and ϱ_2 are positive constants. We call problem (15) a system of generalized set-valued variational inequalities. Some special cases of problem (15) are listed below.

(i) If $P_i : \mathscr{H} \to \mathscr{H}$ are the single-valued mappings and $\varphi_i = I$ are the identity mappings, then problem (15) reduces to the equivalent problem of finding $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\begin{cases} \langle \varrho_1 M_1(y, x) + y - \psi_1(x), \psi_1(y^*) - y\rangle \ge 0, \forall y^* \in \Omega_2, \\ \langle \varrho_2 M_2(x, y) + x - \psi_2(y), \psi_2(x^*) - x\rangle \ge 0, \forall x^* \in \Omega_1. \end{cases} \quad (16)$$

Problem (16) is called the system of variational inequalities.

(ii) If $M_i(\cdot, \cdot) = M_i(\cdot)$, then problem (16) coincides with the following problem of finding $(x, y) \in H \times H$ such that

$$\begin{cases} \langle \varrho_1 M_1(x) + y - \psi_1(x), \psi_1(y^*) - y\rangle \ge 0, \forall y^* \in \Omega_2, \\ \langle \varrho_2 M_2(y) + x - \psi_2(y), \psi_2(x^*) - x\rangle \ge 0, \forall x^* \in \Omega_1. \end{cases} \quad (17)$$

Problem (17) was studied by Sahu et al. [11].

(iii) If $M_1 = M_2 = T, \psi_1 = \psi_2 = I$ and $\Omega_1 = \Omega_2 = C$, then problem (17) reduces to the following system of nonlinear variational inequalities of finding $x, y \in C$ such that

$$\begin{cases} \langle \varrho T(y) + x - y, x^* - x\rangle \ge 0, \forall x^* \in C, \\ \langle \eta T(x) + y - x, x^* - y\rangle \ge 0, \forall x^* \in C. \end{cases} \quad (18)$$

Problem (18) was studied by Verma [30]. He extended the concept of variational inequalities to the system of nonlinear variational inequalities.

(iv) If $M_i(\cdot, \cdot) = M_i(\cdot), \psi_1(x) = \varphi_1(y), \varphi_2(x) = \psi_2(y), \varrho_1 = \varrho_2 = 1$ and $P_i : \mathscr{H} \to \mathscr{H}$ are the single-valued mappings, then problem (15) coincides to the following system of extended general variational inequalities of finding $x, y \in \mathscr{H}$ such that

$$\begin{cases} \langle M_1(x), \psi_1(y^*) - \varphi_1(y)\rangle \ge 0, \forall \psi_1(y^*) \in \Omega_1, \\ \langle M_2(y), \psi_2(x^*) - \varphi_2(x)\rangle \ge 0, \forall \psi_2(x^*) \in \Omega_2. \end{cases} \quad (19)$$

An equivalent form of problem (19) was studied by Noor et al. [31].

(v) If $\psi_1(y^*) = \psi_2(x^*) = g(v), \varphi_1 = \varphi_2 = \varphi$ and $\Omega_1 = \Omega_2 = \Omega$, then problem (19) reduces to the following system of variational inequalities of finding $x, y \in \mathscr{H}$, $\varphi(x), \varphi(y) \in \Omega$ such that

$$\begin{cases} \langle M_1(x), g(v) - \varphi(y)\rangle \ge 0, \forall g(v) \in \Omega, \\ \langle M_2(y), g(v) - \varphi(x)\rangle \ge 0, \forall g(v) \in \Omega. \end{cases} \quad (20)$$

The problem of type (20) is called a system of extended general variational inequalities with four nonlinear operators.

Next, we establish the following lemma which plays a crucial role to prove the existence result for the unique solution of the system of generalized set-valued variational inequalities (15) and to propose an iterative algorithm for studying convergence analysis.

Lemma 6. *Let Ω_1 and Ω_2 be the nonempty closed convex subsets of a real Hilbert space $\mathscr{H}$. Let $I = \{1, 2\}$ be an index set; for each $i \in I$, let $\varphi_i, \psi_i : \mathscr{H} \to \mathscr{H}$ and $M_i : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ be the nonlinear single-valued mappings. Let $P_i : \mathscr{H} \to CB(\mathscr{H})$ be a set-valued mapping. Then, the system of generalized set-valued variational inequalities (15) has a solution (x, y, u, v), if and only if $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$, and $v \in P_2(y)$ satisfies*

$$\varphi_1(y) = P_{\Omega_1}[\psi_1(x) - \varrho_1 M_1(u, v)], \tag{21}$$

$$\varphi_2(x) = P_{\Omega_2}[\psi_2(y) - \varrho_2 M_2(v, u)], \tag{22}$$

where ϱ_1 and ϱ_2 are positive constants.

Proof. Let $(x, y) \in \Omega_1 \times \Omega_2, u \in P_1(x)$ and $v \in P_2(y)$ is a solution of the system of generalized set-valued variational inequalities (15). Then

$$\begin{cases} \langle \varrho_1 M_1(u, v) + \varphi_1(y) - \psi_1(x), \psi_1(y^*) - \varphi_1(y)\rangle \geq 0, \forall y^* \in \Omega_2, \\ \langle \varrho_2 M_2(v, u) + \varphi_2(x) - \psi_2(y), \psi_2(x^*) - \varphi_2(x)\rangle \geq 0, \forall x^* \in \Omega_1. \end{cases} \tag{23}$$

Then, from Lemma 4, we have

$$\begin{aligned} \varphi_1(y) &= P_{\Omega_1}[\psi_1(x) - \varrho_1 M_1(u, v)], \\ \varphi_2(x) &= P_{\Omega_2}[\psi_2(y) - \varrho_2 M_2(v, u)]. \end{aligned} \tag{24}$$

Conversely, suppose that $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$ and $v \in P_2(y)$ satisfies

$$\begin{aligned} \varphi_1(y) &= P_{\Omega_1}[\psi_1(x) - \varrho_1 M_1(u, v)], \\ \varphi_2(x) &= P_{\Omega_2}[\psi_2(y) - \varrho_2 M_2(v, u)]. \end{aligned} \tag{25}$$

Again, it follows from Lemma 4 that

$$\begin{aligned} &\langle \varrho_1 M_1(u, v) + \varphi_1(y) - \psi_1(x), \psi_1(y^*) - \varphi_1(y)\rangle \geq 0, \forall y^* \in \Omega_2, \\ &\langle \varrho_2 M_2(v, u) + \varphi_2(x) - \psi_2(y), \psi_2(x^*) - \varphi_2(x)\rangle \geq 0, \forall x^* \in \Omega_1. \end{aligned} \tag{26}$$

Thus, $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$, and $v \in P_2(y)$ is solution of the system of generalized set-valued variational inequalities (15).

Now, by virtue of the Banach contraction principle, we shall show the existence of the unique solution for the system of generalized set-valued variational inequalities (15).

Theorem 7. *Let $\mathscr{H}$ be a real Hilbert space and $I = \{1, 2\}$ be an index set, for each $i \in I$; let Ω_i be a closed and convex subset of $\mathscr{H}$. Let $\varphi_i, \psi_i : \mathscr{H} \to \mathscr{H}$ and $M_i : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ be nonlinear single-valued mappings such that φ_i is relaxed $(\epsilon_{\varphi_i}, \varepsilon_{\varphi_i})$ -cocoercive and L_{φ_i} -Lipschitz type continuous, and ψ_i is relaxed $(\epsilon_{\psi_i}, \varepsilon_{\psi_i})$ -cocoercive and L_{ψ_i} -Lipschitz type continuous. Let $P_i : \mathscr{H} \to CB(\mathscr{H})$ be a set-valued mapping such that P_i is a relaxed monotone with respect to M_i with constant c_i, $\mathscr{D}$ -Lipschitz type continuous with constant ς_i, and M_i is L_{M_i} -Lipschitz type continuous in the first argument and L'_{M_i} -Lipschitz type continuous in the second argument. In addition, $\mu_i > 0$ satisfies*

$$0 < \mu_1, \mu_2 < 1, \tag{27}$$

where $\mu_1 = \kappa_1 + \iota_1 + \omega_2, \mu_2 = \kappa_2 + \iota_2 + \omega_1$, and

$$\begin{aligned} \kappa_1 &= \sqrt{1 - 2\varepsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right) L^2_{\psi_1}}, \iota_1 = \sqrt{1 + 2\varrho_1 c_1 + \varrho_1^2 L^2_{M_1} \varsigma_1^2}, \\ \kappa_2 &= \sqrt{1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right) L^2_{\psi_2}}, \iota_2 = \sqrt{1 + 2\varrho_2 c_2 + \varrho_2^2 L^2_{M_2} \varsigma_2^2}, \\ \omega_1 &= \sqrt{1 - 2\varepsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right) L^2_{\varphi_1}} + L'_{M_1}\varsigma_2, \omega_2 \\ &= \sqrt{1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right) L^2_{\varphi_2}} + L'_{M_2}\varsigma_1. \end{aligned} \tag{28}$$

Then, the system of generalized set-valued variational inequalities (15) admits unique solution.

Proof. For any given $x \in \Omega_1, y \in \Omega_2$, we define a mapping $\vartheta : \mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$ by

$$\vartheta(x, y) = (\Phi(x, y), Y(x, y)), \tag{29}$$

where $\Phi, Y : \Omega_1 \times \Omega_2 \to \mathscr{H}$ are the single-valued mappings defined by

$$\Phi(x, y) = x - \varphi_2(x) + P_{\Omega_2}[\psi_2(y) - \varrho_2 M_2(v, u)], \tag{30}$$

$$Y(x, y) = y - \varphi_1(y) + P_{\Omega_1}[\psi_1(x) - \varrho_1 M_1(u, v)], \tag{31}$$

where $u \in P_1(x), v \in P_2(y)$ and ϱ_1, ϱ_2 are positive constants. Utilizing the fact that the projection mapping P_{Ω_1} is nonexpansive, it follows from (31) that

$$\begin{aligned} &\|Y(x_1, y_1) - Y(x_2, y_2)\| \\ &\quad = \|y_1 - \varphi_1(y_1) + P_{\Omega_1}[\psi_1(x_1) - \varrho_1 M_1(u_1, v_1)] \\ &\qquad - \left(y_2 - \varphi_1(y_2) + P_{\Omega_1}[\psi_1(x_2) - \varrho_1 M_1(u_2, v_2)]\right)\| \\ &\quad \leq \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| + \|P_{\Omega_1}[\psi_1(x_1) \\ &\qquad - \varrho_1 M_1(u_1, v_1)] - P_{\Omega_1}[\psi_1(x_2) - \varrho_1 M_1(u_2, v_2)]\| \\ &\quad \leq \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| + \|\psi_1(x_1) - \varrho_1 M_1(u_1, v_1) \\ &\qquad - [\psi_1(x_2) - \varrho_1 M_1(u_2, v_2)]\| \\ &\quad \leq \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| + \|x_1 - x_2 - (\psi_1(x_1) \\ &\qquad - \psi_1(x_2))\| + \|x_1 - x_2 - \varrho_1(M_1(u_1, v_1) - M_1(u_2, v_1))\| \\ &\qquad + \varrho_1\|M_1(u_2, v_1) - M_1(u_2, v_2)\|. \end{aligned} \tag{32}$$

Since the mapping φ_1 is relaxed $(\epsilon_{\varphi_1}, \varepsilon_{\varphi_1})$-cocoercive and

L_{φ_1}-Lipschitz type continuous, then we have

$$\begin{aligned}
&\|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\|^2 \\
&\quad = \|y_1 - y_2\|^2 - 2\langle \varphi_1(y_1) - \varphi_1(y_2), y_1 - y_2\rangle \\
&\qquad + \|\varphi_1(y_1) - \varphi_1(y_2)\|^2 \\
&\quad \le \|y_1 - y_2\|^2 + 2\epsilon_{\varphi_1}\|\varphi_1(y_1) - \varphi_1(y_2)\|^2 \\
&\qquad - 2\varepsilon_{\varphi_1}\|y_1 - y_2\|^2 + \|\varphi_1(y_1) - \varphi_1(y_2)\|^2 \\
&\quad \le \left[1 - 2\varepsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2\right]\|y_1 - y_2\|^2,
\end{aligned} \tag{33}$$

which implies that

$$\begin{aligned}
&\|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| \\
&\quad \le \sqrt{1 - 2\varepsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2}\,\|y_1 - y_2\|.
\end{aligned} \tag{34}$$

Again, using the fact that the mapping ψ_1 is relaxed $(\epsilon_{\psi_1}, \varepsilon_{\psi_1})$-cocoercive and L_{ψ_1}-Lipschitz type continuous, then we have

$$\begin{aligned}
&\|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\|^2 \\
&\quad = \|x_1 - x_2\|^2 - 2\langle \psi_1(x_1) - \psi_1(x_2), x_1 - x_2\rangle \\
&\qquad + \|\psi_1(x_1) - \psi_1(x_2)\|^2 \\
&\quad \le \|x_1 - x_2\|^2 + 2\epsilon_{\psi_1}\|\psi_1(x_1) - \psi_1(x_2)\|^2 \\
&\qquad - 2\varepsilon_{\psi_1}\|x_1 - x_2\|^2 + \|\psi_1(x_1) - \psi_1(x_2)\|^2 \\
&\quad \le \left[1 - 2\varepsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2\right]\|x_1 - x_2\|^2,
\end{aligned} \tag{35}$$

which implies that

$$\begin{aligned}
&\|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\| \\
&\quad \le \sqrt{1 - 2\varepsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2}\,\|x_1 - x_2\|.
\end{aligned} \tag{36}$$

Since P_1 is relaxed monotone with respect to M_1 with constant c_1 in the first argument, $\mathscr{D}$-Lipschitz type continuous with constant ς_1 and M_1 is L_{M_1}-Lipschitz type continuous in the first argument, then we have

$$\begin{aligned}
&\|x_1 - x_2 - \varrho_1(M_1(u_1, v_1) - M_1(u_2, v_1))\|^2 \\
&\quad = \|x_1 - x_2\|^2 - 2\varrho_1\langle M_1(u_1, v_1) - M_1(u_2, v_1), x_1 - x_2\rangle \\
&\qquad + \varrho_1^2\|M_1(u_1, v_1) - M_1(u_2, v_1)\|^2 \\
&\quad \le \|x_1 - x_2\|^2 + 2\varrho_1 c_1\|x_1 - x_2\|^2 + \varrho_1^2 L_{M_1}^2\|u_1 - u_2\|^2 \\
&\quad \le \|x_1 - x_2\|^2 + 2\varrho_1 c_1\|x_1 - x_2\|^2 + \varrho_1^2 L_{M_1}^2\varsigma_1^2\|x_1 - x_2\|^2 \\
&\quad = \left(1 + 2\varrho_1 c_1 + \varrho_1^2 L_{M_1}^2\varsigma_1^2\right)\|x_1 - x_2\|^2,
\end{aligned} \tag{37}$$

which implies that

$$\begin{aligned}
&\|x_1 - x_2 - \varrho_1(M_1(u_1, v_1) - M_1(u_2, v_1))\| \\
&\quad \le \sqrt{1 + 2\varrho_1 c_1 + \varrho_1^2 L_{M_1}^2\varsigma_1^2}\,\|x_1 - x_2\|.
\end{aligned} \tag{38}$$

Since M_1 is L'_{M_1}-Lipschitz type continuous in the second argument and P_2 is $\mathscr{D}$-Lipschitz type continuous with constant ς_2, then we have

$$\|M_1(u_2, v_1) - M_1(u_2, v_2)\| \le L'_{M_1}\|v_1 - v_2\| \le L'_{M_1}\varsigma_2\|y_1 - y_2\|. \tag{39}$$

Thus, from (34), (36), (38), and (39), (32) becomes

$$\begin{aligned}
&\|Y(x_1, y_1) - Y(x_2, y_2)\| \\
&\quad \le \left[\sqrt{1 - 2\varepsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2} + L'_{M_1}\varsigma_2\right]\|y_1 - y_2\| \\
&\qquad + \left[\sqrt{1 - 2\varepsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2}\right. \\
&\qquad \left. + \sqrt{1 + 2\varrho_1 c_1 + \varrho_1^2 L_{M_1}^2\varsigma_1^2}\right]\|x_1 - x_2\|.
\end{aligned} \tag{40}$$

Again, utilizing the fact that the projection mapping P_{Ω_2} is nonexpansive, it follows from (30) that

$$\begin{aligned}
&\|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| \\
&\quad = \|x_1 - \varphi_2(x_1) + P_{\Omega_2}[\psi_2(y_1) - \varrho_2 M_2(v_1, u_1)] \\
&\qquad - \left(x_2 - \varphi_2(x_2) + P_{\Omega_2}[\psi_2(y_2) - \varrho_2 M_2(v_2, u_2)]\right)\| \\
&\quad \le \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| + \\
&\qquad \|P_{\Omega_2}[\psi_2(y_1) - \varrho_2 M_2(v_1, u_1)] \\
&\qquad - P_{\Omega_2}[\psi_2(y_2) - \varrho_2 M_2(v_2, u_2)]\| \\
&\quad \le \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| \\
&\qquad + \|\psi_2(y_1) - \varrho_2 M_2(v_1, u_1) - [\psi_2(y_2) - \varrho_2 M_2(v_2, u_2)]\| \\
&\quad \le \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| + \\
&\qquad \|y_1 - y_2 - (\psi_2(y_1) - \psi_2(y_2))\| + \\
&\qquad \|y_1 - y_2 - \varrho_2(M_2(v_1, u_1) - M_2(v_2, u_1))\| \\
&\qquad + \varrho_2\|M_2(v_2, u_1) - M_2(v_2, u_2)\|.
\end{aligned} \tag{41}$$

Since the mapping φ_2 is relaxed $(\epsilon_{\varphi_2}, \varepsilon_{\varphi_2})$-cocoercive and L_{φ_2}-Lipschitz type continuous, then we have

$$\begin{aligned}
&\|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\|^2 \\
&\quad = \|x_1 - x_2\|^2 - 2\langle \varphi_2(x_1) - \varphi_2(x_2), x_1 - x_2\rangle \\
&\qquad + \|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\quad \le \|x_1 - x_2\|^2 + 2\epsilon_{\varphi_2}\|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\qquad - 2\varepsilon_{\varphi_2}\|x_1 - x_2\|^2 + \|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\quad \le \left[1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2\right]\|x_1 - x_2\|^2,
\end{aligned} \tag{42}$$

which implies that

$$\begin{aligned}
&\|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| \\
&\quad \le \sqrt{1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2}\,\|x_1 - x_2\|.
\end{aligned} \tag{43}$$

Again, using the fact that the mapping ψ_2 is relaxed $(\epsilon_{\psi_2}, \varepsilon_{\psi_2})$-cocoercive and L_{ψ_2}-Lipschitz type continuous, then we have

$$\begin{aligned}
&\|y_1 - y_2 - (\psi_2(y_1) - \psi_2(y_2))\|^2 \\
&\quad = \|y_1 - y_2\|^2 - 2\langle \psi_2(y_1) - \psi_2(y_2), y_1 - y_2\rangle \\
&\qquad + \|\psi_2(y_1) - \psi_2(y_2)\|^2 \\
&\quad \le \|y_1 - y_2\|^2 + 2\epsilon_{\psi_2}\|\psi_2(y_1) - \psi_2(y_2)\|^2 \\
&\qquad - 2\varepsilon_{\psi_2}\|y_1 - y_2\|^2 + \|\psi_2(y_1) - \psi_2(y_2)\|^2 \\
&\quad \le \left[1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2\right]\|y_1 - y_2\|^2,
\end{aligned} \tag{44}$$

which implies that

$$\begin{aligned}
&\|y_1 - y_2 - (\psi_2(y_1) - \psi_2(y_2))\| \\
&\quad \le \sqrt{1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2}\,\|y_1 - y_2\|.
\end{aligned} \tag{45}$$

Since P_2 is relaxed monotone with respect to M_2 with constant c_2 in the first argument, $\mathscr{D}$-Lipschitz type continuous with constant ς_2 and M_2 is L_{M_2}-Lipschitz type continuous in the first argument, then we have

$$\begin{aligned}
&\|y_1 - y_2 - \varrho_2(M_2(v_1, u_1) - M_2(v_2, u_1))\|^2 \\
&\quad = \|y_1 - y_2\|^2 - 2\varrho_2\langle M_2(v_1, u_1) - M_2(v_2, u_1), y_1 - y_2\rangle \\
&\qquad + \varrho_2^2\|M_2(v_1, u_1) - M_2(v_2, u_1)\|^2 \\
&\quad \le \|y_1 - y_2\|^2 + 2\varrho_2 c_2\|y_1 - y_2\|^2 + \varrho_2^2 L_{M_2}^2\|v_1 - v_2\|^2 \\
&\quad \le \|y_1 - y_2\|^2 + 2\varrho_2 c_2\|y_1 - y_2\|^2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2\|y_1 - y_2\|^2 \\
&\quad = \left[1 + 2\varrho_2 c_2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2\right]\|y_1 - y_2\|^2,
\end{aligned} \tag{46}$$

which implies that

$$\begin{aligned}
&\|y_1 - y_2 - \varrho_2(M_2(v_1, u_1) - M_2(v_2, u_1))\| \\
&\quad \le \sqrt{1 + 2\varrho_2 c_2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2}\,\|y_1 - y_2\|.
\end{aligned} \tag{47}$$

Since M_2 is L'_{M_2}-Lipschitz type continuous in the second argument and P_1 is $\mathscr{D}$-Lipschitz type continuous with constant ς_1, then we have

$$\|M_2(v_2, u_1) - M_2(v_2, u_2)\| \le L'_{M_2}\|u_1 - u_2\| \le L'_{M_2}\varsigma_1\|x_1 - x_2\|. \tag{48}$$

Thus, from (43), (45), (47), and (48), (41) becomes

$$\begin{aligned}
&\|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| \\
&\quad \le \left[\sqrt{1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2} + L'_{M_2}\varsigma_1\right] \\
&\qquad \cdot \|x_1 - x_2\| + \left[\sqrt{1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2}\right. \\
&\qquad \left. + \sqrt{1 + 2\varrho_2 c_2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2}\right]\|y_1 - y_2\|.
\end{aligned} \tag{49}$$

Now, it follows from (40) and (49) that

$$\begin{aligned}
&\|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| + \|Y(x_1, y_1) - Y(x_2, y_2)\| \\
&\quad \le \mu_1\|x_1 - x_2\| + \mu_2\|y_1 - y_2\| \\
&\quad \le \max\{\mu_1, \mu_2\}(\|x_1 - x_2\| + \|y_1 - y_2\|),
\end{aligned} \tag{50}$$

where $\mu_1 = \kappa_1 + \iota_1 + \omega_2, \mu_2 = \kappa_2 + \iota_2 + \omega_1$, and

$$\begin{aligned}
\kappa_1 &= \sqrt{1 - 2\varepsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2}, \iota_1 = \sqrt{1 + 2\varrho_1 c_1 + \varrho_1^2 L_{M_1}^2\varsigma_1^2}, \\
\kappa_2 &= \sqrt{1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2}, \iota_2 = \sqrt{1 + 2\varrho_2 c_2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2}, \\
\omega_1 &= \sqrt{1 - 2\varepsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2} + L'_{M_1}\varsigma_2, \omega_2 \\
&= \sqrt{1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2} + L'_{M_2}\varsigma_1.
\end{aligned} \tag{51}$$

It follows from (8), (29), and (50) that

$$\|\vartheta(x_1, y_1) - \vartheta(x_2, y_2)\|_* \le \max\{\mu_1, \mu_2\}\|(x_1, y_1) - (x_2, y_2)\|. \tag{52}$$

It follows from conditions (27) and (52) that ϑ is a contraction mapping. Therefore, there exists unique $(x, y) \in \Omega_1 \times \Omega_2$ such that $\vartheta(x, y) = (x, y)$. Thus, we have

$$\begin{aligned}
\varphi_1(y) &= P_{\Omega_1}[\psi_1(x) - \varrho_1 M_1(u, v)], \\
\varphi_2(x) &= P_{\Omega_2}[\psi_2(y) - \varrho_2 M_2(v, u)].
\end{aligned} \tag{53}$$

Therefore, by Lemma 6, one can conclude that $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$, and $v \in P_2(y)$ is the unique solution of the system of generalized set-valued variational inequalities (15).

3. Iterative Algorithm and Convergence Result

In this section, we suggest an iterative algorithm to analyse the convergence of the system of generalized set-valued variational inequalities (15).

By utilizing (21) and (22) of Lemma 6, we can offer the following iterative forms:

$$y = (1-\gamma_n)y + \gamma_n\left(y - \varphi_1(y) + P_{\Omega_1}[\psi_1(x) - \varrho_1 M_1(u,v)]\right), \tag{54}$$

$$x = (1-\delta_n)x + \delta_n\left(x - \varphi_2(x) + P_{\Omega_2}[\psi_2(y) - \varrho_2 M_2(v,u)]\right), \tag{55}$$

where ϱ_1 and ϱ_2 are positive constants and the real sequences $\gamma_n, \delta_n \in [0,1]$. Now, we propose the following iterative algorithm.

Algorithm 8. For any given $(x_0, y_0) \in \Omega_1 \times \Omega_2$, we choose $u_0 \in P_1(x_0)$ and $v_0 \in P_2(y_0)$. For $\varrho_1, \varrho_2 > 0$ and $\gamma_n, \delta_n \in [0,1]$ and from (54) and (55), let

$$\begin{aligned} y_1 &= (1-\gamma_n)y_0 + \gamma_n\left(y_0 - \varphi_1(y_0) + P_{\Omega_1}[\psi_1(x_0) - \varrho_1 M_1(u_0,v_0)]\right),\\ x_1 &= (1-\delta_n)x_0 + \delta_n\left(x_0 - \varphi_2(x_0) + P_{\Omega_2}[\psi_2(y_0) - \varrho_2 M_2(v_0,u_0)]\right). \end{aligned} \tag{56}$$

Since $u_0 \in P_1(x_0)$ and $v_0 \in P_2(y_0)$, by Nadler's theorem [32], there exist $u_1 \in P_1(x_1)$ and $v_1 \in P_2(y_1)$ such that

$$\begin{aligned} \|u_0 - u_1\| &\le (1+1)D(P_1(x_0), P_1(x_1)),\\ \|v_0 - v_1\| &\le (1+1)D(P_2(y_0), P_2(y_1)), \end{aligned} \tag{57}$$

where D is the Hausdorff metric. Let

$$\begin{aligned} y_2 &= (1-\gamma_n)y_1 + \gamma_n\left(y_1 - \varphi_1(y_1) + P_{\Omega_1}[\psi_1(x_1) - \varrho_1 M_1(u_1,v_1)]\right),\\ x_2 &= (1-\delta_n)x_1 + \delta_n\left(x_1 - \varphi_2(x_1) + P_{\Omega_2}[\psi_2(y_1) - \varrho_2 M_2(v_1,u_1)]\right). \end{aligned} \tag{58}$$

Again, it follows from Nadler's theorem [32] that there exist $u_2 \in P_1(x_2)$ and $v_2 \in P_2(y_2)$ such that

$$\begin{aligned} \|u_1 - u_2\| &\le \left(1+2^{-1}\right)D(P_1(x_1), P_1(x_2)),\\ \|v_1 - v_2\| &\le \left(1+2^{-1}\right)D(P_2(y_1), P_2(y_2)). \end{aligned} \tag{59}$$

Continuing in the same manner, we can figure out the sequences $\{x_n\}, \{y_n\}, \{u_n\}$, and $\{v_n\}$ by the following iterative process:

$$y_{n+1} = (1-\gamma_n)y_n + \gamma_n\left(y_n - \varphi_1(y_n) + P_{\Omega_1}[\psi_1(x_n) - \varrho_1 M_1(u_n,v_n)]\right), \tag{60}$$

$$x_{n+1} = (1-\delta_n)x_n + \delta_n\left(x_n - \varphi_2(x_n) + P_{\Omega_2}[\psi_2(y_n) - \varrho_2 M_2(v_n,u_n)]\right), \tag{61}$$

and for $n = 0, 1, ,2, \cdots$, choose $u_{n+1} \in P_1(x_{n+1})$ and $v_{n+1} \in P_2(y_{n+1})$ such that

$$\|u_{n+1} - u_n\| \le \left(1 + (n+1)^{-1}\right)D(P_1(x_{n+1}), P_1(x_n)), \tag{62}$$

$$\|v_{n+1} - v_n\| \le \left(1 + (n+1)^{-1}\right)D(P_2(y_{n+1}), P_2(y_n)). \tag{63}$$

Now, we are accessible to study the convergence of the proposed iterative algorithm for the system of generalized set-valued variational inequalities (15).

Theorem 9. *Let $\mathscr{H}$ be a real Hilbert space and $I = \{1,2\}$ be an index set, for each $i \in I$; let Ω_i be a closed convex subset of $\mathscr{H}$. Let $\varphi_i, \psi_i : \mathscr{H} \to \mathscr{H}$ and $M_i : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ be nonlinear single-valued mappings such that φ_i is relaxed $(\epsilon_{\varphi_i}, \varepsilon_{\varphi_i})$-cocoercive, L_{φ_i}-Lipschitz type continuous; ψ_i is relaxed $(\epsilon_{\psi_i}, \varepsilon_{\psi_i})$-cocoercive, L_{ψ_i}-Lipschitz type continuous; and M_i is L_{M_i}-Lipschitz type continuous in the first argument and L'_{M_i}-Lipschitz type continuous in the second argument. Let $P_i : \mathscr{H} \to CB(\mathscr{H})$ be a set-valued mapping such that P_i is a relaxed monotone with respect to M_i with constant c_i and D-Lipschitz type continuous with constant ς_i. In addition, for each $i = 1, 2$, $\rho_i, c_i > 0$ satisfy the following conditions:*

(i) $0 < n_i, l_i < 1$,

(ii) $2\varepsilon_{\psi_i} - (1 + 2\epsilon_{\psi_i})L^2_{\psi_i} < 1$,

(iii) $\delta_n(1-l_1) - \gamma_n(m_2+n_2) \ge 0, \gamma_n(1-l_2) - \delta_n(m_1+n_1) \ge 0$, such that

$$\sum_{n=0}^{\infty} \delta_n(1-l_1) - \gamma_n(m_2+n_2) = \infty, \sum_{n=0}^{\infty} \gamma_n(1-l_2) - \delta_n(m_1+n_1) = \infty, \tag{64}$$

where

$$\begin{aligned} l_1 &= \sqrt{1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L^2_{\varphi_2}} + \varrho_2 L'_{M_2}\varsigma_1, m_1 = \sqrt{1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L^2_{\psi_2}},\\ l_2 &= \sqrt{1 - 2\varepsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L^2_{\varphi_1}} + \varrho_1 L'_{M_1}\varsigma_2, m_2 = \sqrt{1 - 2\varepsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L^2_{\psi_1}},\\ n_1 &= \sqrt{1 + 2\varrho_2 c_2 + \varrho_2^2 L^2_{M_2}\varsigma_2^2}, n_2 = \sqrt{1 + 2\varrho_1 c_1 + \varrho_1^2 L^2_{M_1}\varsigma_1^2}. \end{aligned} \tag{65}$$

Then, the approximate solution $\{(x_n, y_n, u_n, v_n)\}$

generated by Algorithm 8 converges strongly to the unique solution (x, y, u, v) of the system of generalized set-valued variational inequalities (15).

Proof. Let $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$ and $v \in P_2(y)$ be a solution of (15); then, from (55) and (61) of Algorithm 8 and utilizing the fact that the projection mapping P_{Ω_2} is nonexpansive, we have

$$\begin{aligned}\|x_{n+1} - x\| = {} & \|(1-\delta_n)x_n + \delta_n(x_n - \varphi_2(x_n) \\ & + P_{\Omega_2}[\psi_2(y_n) - \varrho_2 M_2(v_n, u_n)]) \\ & - [(1-\delta_n)x + \delta_n(x - \varphi_2(x) + P_{\Omega_2}[\psi_2(y) \\ & - \varrho_2 M_2(v, u)])]\| \le (1-\delta_n)\|x_n - x\| + \delta_n\|x_n \\ & - x - (\varphi_2(x_n) - \varphi_2(x))\| \\ & + \delta_n\|P_{\Omega_2}[\psi_2(y_n) - \varrho_2 M_2(v_n, u_n)] \\ & - P_{\Omega_2}[\psi_2(y) - \varrho_2 M_2(v, u)]\| \\ \le {} & (1-\delta_n)\|x_n - x\| + \delta_n\|x_n - x - (\varphi_2(x_n) - \varphi_2(x))\| \\ & + \delta_n\|\psi_2(y_n) - \varrho_2 M_2(v_n, u_n) - (\psi_2(y) \\ & - \varrho_2 M_2(v, u))\| \le (1-\delta_n)\|x_n - x\| + \delta_n\|x_n \\ & - x - (\varphi_2(x_n) - \varphi_2(x))\| + \delta_n\|y_n - y - (\psi_2(y_n) \\ & - \psi_2(y))\| + \delta_n\|y_n - y - \varrho_2(M_2(v_n, u_n) \\ & - M_2(v, u_n))\| + \delta_n\varrho_2\|M_2(v, u_n) - M_2(v, u)\|.\end{aligned} \tag{66}$$

Since the mapping φ_2 is relaxed $(\epsilon_{\varphi_2}, \varepsilon_{\varphi_2})$-cocoercive and L_{φ_2}-Lipschitz type continuous, then we have

$$\begin{aligned}&\|x_n - x - (\varphi_2(x_n) - \varphi_2(x))\|^2 \\ &\quad = \|x_n - x\|^2 - 2\langle\varphi_2(x_n) - \varphi_2(x), x_n - x\rangle \\ &\qquad + \|\varphi_2(x_n) - \varphi_2(x)\|^2 \\ &\quad \le \|x_n - x\|^2 + 2\varepsilon_{\varphi_2}\|\varphi_2(x_n) - \varphi_2(x)\|^2 \\ &\qquad - 2\varepsilon_{\varphi_2}\|x_n - x\|^2 + \|\varphi_2(x_n) - \varphi_2(x)\|^2 \\ &\quad \le \left[1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\varepsilon_{\varphi_2}\right)L_{\varphi_2}^2\right]\|x_n - x\|^2,\end{aligned} \tag{67}$$

which implies that

$$\begin{aligned}&\|x_n - x - (\varphi_2(x_n) - \varphi_2(x))\| \\ &\quad \le \sqrt{1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\varepsilon_{\varphi_2}\right)L_{\varphi_2}^2}\|x_n - x\|.\end{aligned} \tag{68}$$

Again, using the fact that mapping ψ_2 is relaxed $(\varepsilon_{\psi_2}, \varepsilon_{\psi_2})$-cocoercive and L_{ψ_2}-Lipschitz type continuous, then we have

$$\begin{aligned}&\|y_n - y - (\psi_2(y_n) - \psi_2(y))\|^2 \\ &\quad = \|y_n - y\|^2 - 2\langle\psi_2(y_n) - \psi_2(y), y_n - y\rangle \\ &\qquad + \|\psi_2(y_n) - \psi_2(y)\|^2 \\ &\quad \le \|y_n - y\|^2 + 2\epsilon_{\psi_2}\|\psi_2(y_n) - \psi_2(y)\|^2 \\ &\qquad - 2\varepsilon_{\psi_2}\|y_n - y\|^2 + \|\psi_2(y_n) - \psi_2(y)\|^2 \\ &\quad \le \left[1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2\right]\|y_n - y\|^2,\end{aligned} \tag{69}$$

which implies that

$$\begin{aligned}&\|y_n - y - (\psi_2(y_n) - \psi_2(y))\| \\ &\quad \le \sqrt{1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2}\|y_n - y\|.\end{aligned} \tag{70}$$

Since P_2 is relaxed monotone with respect to M_2 with constant c_2 in the first argument, D-Lipschitz type continuous with constant ς_2 and M_2 is L_{M_2}-Lipschitz type continuous in the first argument, then we have

$$\begin{aligned}&\|y_n - y - \varrho_2(M_2(v_n, u_n) - M_2(v, u_n))\|^2 \\ &\quad = \|y_n - y\|^2 - 2\varrho_2\langle M_2(v_n, u_n) - M_2(v, u_n), y_n - y\rangle \\ &\qquad + \varrho_2^2\|M_2(v_n, u_n) - M_2(v, u_n)\|^2 \\ &\quad \le \|y_n - y\|^2 + 2\varrho_2 c_2\|y_n - y\|^2 + \varrho_2^2 L_{M_2}^2\|v_n - v\|^2 \\ &\quad \le \|y_n - y\|^2 + 2\varrho_2 c_2\|y_n - y\|^2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2\|y_n - y\|^2 \\ &\quad = \left(1 + 2\varrho_2 c_2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2\right)\|y_n - y\|^2,\end{aligned} \tag{71}$$

which implies that

$$\begin{aligned}&\|y_n - y - \varrho_2(M_2(v_n, u_n) - M_2(v, u_n))\| \\ &\quad \le \sqrt{1 + 2\varrho_2 c_2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2}\|y_n - y\|.\end{aligned} \tag{72}$$

Since M_2 is L'_{M_2}-Lipschitz type continuous in the second argument and P_1 is D-Lipschitz type continuous with constant ς_1, then, we have

$$\|M_2(v, u_n) - M_2(v, u)\| \le L'_{M_2}\|u_n - u\| \le L'_{M_2}\varsigma_1\|x_n - x\|. \tag{73}$$

Thus, from (68), (70), (72), and (73), (66) becomes

$$\begin{aligned}\|x_{n+1} - x\| \le {} & \left[1 - \delta_n\left(1 - \sqrt{1 - 2\varepsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2} - \varrho_2 L'_{M_2}\varsigma_1\right)\right] \\ & \cdot \|x_n - x\| + \delta_n\left[\sqrt{1 - 2\varepsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2}\right. \\ & \left. + \sqrt{1 + 2\varrho_2 c_2 + \varrho_2^2 L_{M_2}^2\varsigma_2^2}\right]\|y_n - y\|.\end{aligned} \tag{74}$$

Again, from (54) and (60) of Algorithm 8 and utilizing the fact that the projection mapping P_{Ω_1} is nonexpansive, we have

$$\begin{aligned}\|y_{n+1}-y\| = {} & \|(1-\gamma_n)y_n+\gamma_n(y_n-\varphi_1(y_n)+P_{\Omega_1}[\psi_1(x_n) \\ & -\varrho_1 M_1(u_n,v_n)])-[(1-\gamma_n)y+\gamma_n(y-\varphi_1(y) \\ & +P_{\Omega_1}[\psi_1(x)-\varrho_1 M_1(u,v)])]\| \\ \le {} & (1-\gamma_n)\|y_n-y\|+\gamma_n\|y_n-y-(\varphi_1(y_n)-\varphi_1(y))\| \\ & +\gamma_n\|P_{\Omega_1}[\psi_1(x_n)-\varrho_1 M_1(u_n,v_n)] \\ & -P_{\Omega_1}[\psi_1(x)-\varrho_1 M_1(u,v)]\| \\ \le {} & (1-\gamma_n)\|y_n-y\|+\gamma_n\|y_n-y-(\varphi_1(y_n)-\varphi_1(y))\| \\ & +\gamma_n\|\psi_1(x_n)-\varrho_1 M_1(u_n,v_n)-[\psi_1(x)-\varrho_1 M_1(u,v)]\| \\ \le {} & (1-\gamma_n)\|y_n-y\|+\gamma_n\|y_n-y-(\varphi_1(y_n)-\varphi_1(y))\| \\ & +\gamma_n\|x_n-x-(\psi_1(x_n)-\psi_1(x))\|+\gamma_n\|x_n-x \\ & -\varrho_1(M_1(u_n,v_n)-M_1(u,v_n))\| \\ & +\gamma_n\varrho_1\|M_1(u,v_n)-M_1(u,v)\|. \end{aligned} \tag{75}$$

Since the mapping φ_1 is relaxed $(\epsilon_{\varphi_1},\varepsilon_{\varphi_1})$-cocoercive and L_{φ_1}-Lipschitz type continuous, then we have

$$\begin{aligned}&\|y_n-y-(\varphi_1(y_n)-\varphi_1(y))\|^2 \\ &\quad=\|y_n-y\|^2-2\langle\varphi_1(y_n)-\varphi_1(y),y_n-y\rangle \\ &\qquad+\|\varphi_1(y_n)-\varphi_1(y)\|^2 \\ &\quad\le\|y_n-y\|^2+2\epsilon_{\varphi_1}\|\varphi_1(y_n)-\varphi_1(y)\|^2 \\ &\qquad-2\varepsilon_{\varphi_1}\|y_n-y\|^2+\|\varphi_1(y_n)-\varphi_1(y)\|^2 \\ &\quad\le\left[1-2\varepsilon_{\varphi_1}+\left(1+2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2\right]\|y_n-y\|^2,\end{aligned} \tag{76}$$

which implies that

$$\begin{aligned}&\|y_n-y-(\varphi_1(y_n)-\varphi_1(y))\| \\ &\quad\le\sqrt{1-2\varepsilon_{\varphi_1}+\left(1+2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2}\|y_n-y\|.\end{aligned} \tag{77}$$

Again, using the fact that mapping ψ_1 is relaxed $(\epsilon_{\psi_1},\varepsilon_{\psi_1})$-cocoercive and L_{ψ_1}-Lipschitz type continuous, then we have

$$\begin{aligned}&\|x_n-x-(\psi_1(x_n)-\psi_1(x))\|^2 \\ &\quad=\|x_n-x\|^2-2\langle\psi_1(x_n)-\psi_1(x),x_n-x\rangle \\ &\qquad+\|\psi_1(x_n)-\psi_1(x)\|^2 \\ &\quad\le\|x_n-x\|^2+2\epsilon_{\psi_1}\|\psi_1(x_n)-\psi_1(x)\|^2 \\ &\qquad-2\varepsilon_{\psi_1}\|x_n-x\|^2+\|\psi_1(x_n)-\psi_1(x)\|^2 \\ &\quad\le\left[1-2\varepsilon_{\psi_1}+\left(1+2\epsilon_{\psi_1}\right)L_{\psi_1}^2\right]\|x_n-x\|^2.\end{aligned} \tag{78}$$

which implies that

$$\begin{aligned}&\|x_n-x-(\psi_1(x_n)-\psi_1(x))\| \\ &\quad\le\sqrt{1-2\varepsilon_{\psi_1}+\left(1+2\epsilon_{\psi_1}\right)L_{\psi_1}^2}\|x_n-x\|.\end{aligned} \tag{79}$$

Since P_1 is relaxed monotone with respect to M_1 with constant c_1 in the first argument, D-Lipschitz type continuous with constant ς_1 and M_1 is L_{M_1}-Lipschitz type continuous in the first argument, then we have

$$\begin{aligned}&\|x_n-x-\varrho_1(M_1(u_n,v_n)-M_1(u,v_n))\|^2 \\ &\quad=\|x_n-x\|^2-2\varrho_1\langle M_1(u_n,v_n)-M_1(u,v_n),x_n-x\rangle \\ &\qquad+\varrho_1^2\|M_1(u_n,v_n)-M_1(u,v_n)\|^2 \\ &\quad\le\|x_n-x\|^2+2\varrho_1c_1\|x_n-x\|^2+\varrho_1^2L_{M_1}^2\|u_n-u\|^2 \\ &\quad\le\|x_n-x\|^2+2\varrho_1c_1\|x_n-x\|^2+\varrho_1^2L_{M_1}^2\varsigma_1^2\|x_n-x\|^2 \\ &\quad=\left[1+2\varrho_1c_1+\varrho_1^2L_{M_1}^2\varsigma_1^2\right]\|x_n-x\|^2,\end{aligned} \tag{80}$$

which implies that

$$\begin{aligned}&\|x_n-x-\varrho_1(M_1(u_n,v_n)-M_1(u,v_n))\| \\ &\quad\le\sqrt{1+2\varrho_1c_1+\varrho_1^2L_{M_1}^2\varsigma_1^2}\|x_n-x\|.\end{aligned} \tag{81}$$

Since M_1 is L'_{M_1}-Lipschitz type continuous in the second argument and P_2 is D-Lipschitz type continuous with constant ς_2, then, we have

$$\|M_1(u,v_n)-M_1(u,v)\|\le L'_{M_1}\|v_n-v\|\le L'_{M_1}\varsigma_2\|y_n-y\|. \tag{82}$$

Thus, from (77), (79), (81), and (82), (75) becomes

$$\begin{aligned}\|y_{n+1}-y\|\le{} & \left[1-\gamma_n\left(1-\sqrt{1-2\varepsilon_{\varphi_1}+\left(1+2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2}-\varrho_1L'_{M_1}\varsigma_2\right)\right]\|y_n-y\|+\gamma_n \\ & \cdot\left[\sqrt{1-2\varepsilon_{\psi_1}+\left(1+2\epsilon_{\psi_1}\right)L_{\psi_1}^2}+\sqrt{1+2\varrho_1c_1+\varrho_1^2L_{M_1}^2\varsigma_1^2}\right]\|x_n-x\|.\end{aligned} \tag{83}$$

For the sake of simplicity, we put

$$\begin{aligned}l_1&=\sqrt{1-2\varepsilon_{\varphi_2}+\left(1+2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2}+\varrho_2L'_{M_2}\varsigma_1,\ m_1 \\ &=\sqrt{1-2\varepsilon_{\psi_2}+\left(1+2\epsilon_{\psi_2}\right)L_{\psi_2}^2}, \\ l_2&=\sqrt{1-2\varepsilon_{\varphi_1}+\left(1+2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2}+\varrho_1L'_{M_1}\varsigma_2,\ m_2 \\ &=\sqrt{1-2\varepsilon_{\psi_1}+\left(1+2\epsilon_{\psi_1}\right)L_{\psi_1}^2},\end{aligned}$$

$$n_1=\sqrt{1+2\varrho_2c_2+\varrho_2^2L_{M_2}^2\varsigma_2^2},\ n_2=\sqrt{1+2\varrho_1c_1+\varrho_1^2L_{M_1}^2\varsigma_1^2}. \tag{84}$$

Then, it follows from (74) and (83) that

$$\begin{aligned}\|x_{n+1}-x\|+\|y_{n+1}-y\| &\le [1-\delta_n(1-l_1)]\|x_n-x\| \\ &\quad+\delta_n(m_1+n_1)\|y_n-y\| \\ &\quad+[1-\gamma_n(1-l_2)]\|y_n-y\| \\ &\quad+\gamma_n(m_2+n_2)\|x_n-x\| \\ &=[1-\delta_n(1-l_1)+\gamma_n(m_2+n_2)] \\ &\quad\cdot\|x_n-x\|+[1-\gamma_n(1-l_2) \\ &\quad+\delta_n(m_1+n_1)]\|y_n-y\|,\end{aligned} \tag{85}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x\|+\|y_{n+1}-y\| &\le \tau_1\|x_n-x\|+\tau_2\|y_n-y\| \\ &\le \omega(\|x_n-x\|+\|y_n-y\|),\end{aligned} \tag{86}$$

where
$\omega=\max\{\tau_1,\tau_2\}, \tau_1=1-\delta_n(1-l_1)+\gamma_n(m_2+n_2)$ and $\tau_2=1-\gamma_n(1-l_2)+\delta_n(m_1+n_1)$.

Thus, from (86), Lemma 5, and conditions, we have

$$\lim_{n\to\infty}(\|x_n-x\|+\|y_n-y\|)=0, \tag{87}$$

which implies that

$$\lim_{n\to\infty}\|x_n-x\|=\lim_{n\to\infty}\|y_n-y\|=0. \tag{88}$$

Hence, $x_n\to x$ and $y_n\to y$. It follows from (62) and (63) that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences; we can assume that $u_n\to u$ and $v_n\to v$, strongly. Next, we show that $u\in P_1(x)$ and $v\in P_2(y)$. Since $u_n\in P_1(x_n)$, then we have

$$\begin{aligned}d(u,P_1(x)) &\le \|u-u_n\|+d(u_n,P_1(x)) \\ &\le \|u-u_n\|+\left(1+n^{-1}\right)D(P_1(x_n),P_1(x)) \\ &\le \|u-u_n\|+\left(1+n^{-1}\right)\varsigma_1\|x_n-x\|\to 0 \text{ as } n\to\infty.\end{aligned} \tag{89}$$

Hence, $d(u,P_1(x))\to 0$, so $u\in P_1(x)$ as $P_1(x)\in CB(\mathscr{H})$. Similarly, it is easy to show that $v\in P_2(y)$. Thus, by Lemma 6, one can deduce that $(x,y,u,v), x\in\Omega_1, y\in\Omega_2, u\in P_1(x)$, and $v\in P_2(y)$ is a solution to the system of generalized set-valued variational inequalities (15).

4. Altering Points Problem

In this section, the concept of altering points problem is used to find the solution of the considered system of variational inequalities (16). We propose parallel Mann and parallel S-iterative algorithms, and the strong convergence of the sequences generated by these parallel iterative algorithms is discussed.

Definition 10. [11]. Let Ω_1 and Ω_2 be nonempty subsets of a metric space X. Then, the points $x\in\Omega_1$ and $y\in\Omega_2$ are the altering points of mappings $A_1:\Omega_1\to\Omega_2$ and $A_2:\Omega_2\to\Omega_1$, if

$$\begin{aligned}A_1(x)&=y,\\ A_2(y)&=x.\end{aligned} \tag{90}$$

Alt-$(A_1,A_2)=\{(x,y)\in\Omega_1\times\Omega_2 : A_1(x)=y \text{ and } A_2(y)=x\}$ is called the set of altering points of the mappings A_1 and A_2.

Lemma 11 [33]. *Let Ω_1 and Ω_2 be nonempty closed subsets of a complete metric space X. Let $A_1:\Omega_1\to\Omega_2$ and $A_2:\Omega_2\to\Omega_1$ be Lipschitz continuous mappings with constants κ_1 and κ_2, respectively, such that $\kappa_1\kappa_2<1$. Then, the following conditions hold:*

(i) There exists unique point $(x,y)\in\Omega_1\times\Omega_2$ such that x and y are altering points of the mappings A_1 and A_2.

(ii) For arbitrary $x_1\in\Omega_1$, the sequence $\{(x_n,y_n)\}\in\Omega_1\times\Omega_2$ generated by

$$\begin{cases} y_n=A_1x_n,\\ x_{n+1}=A_2y_n, n\in\mathbb{N},\end{cases} \tag{91}$$

converges to (x,y).

Lemma 12. *Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space $\mathscr{H}$. Let $I=\{1,2\}$ be an index set, for each $i\in I$; let $\varphi_i,\psi_i:\mathscr{H}\to\mathscr{H}$ and $M_i:\mathscr{H}\times\mathscr{H}\to\mathscr{H}$ be nonlinear single-valued mappings. Then, $(x,y)\in\Omega_1\times\Omega_2$ is the solution to the system of variational inequalities (16), if and only if (x,y) satisfies*

$$\begin{aligned}y&=P_{\Omega_2}[\psi_1(x)-\varrho_1M_1(y,x)],\\ x&=P_{\Omega_1}[\psi_2(y)-\varrho_2M_2(x,y)],\end{aligned} \tag{92}$$

where ϱ_1 and ϱ_2 are positive constants.

Define the mappings $A_1:\Omega_1\to\Omega_2$ and $A_2:\Omega_2\to\Omega_1$ as follows:

$$A_1:=P_{\Omega_2}[\psi_1-\varrho_1M_1(y,\cdot)], \tag{93}$$

$$A_2:=P_{\Omega_1}[\psi_2-\varrho_2M_2(x,\cdot)], \tag{94}$$

where ϱ_1,ϱ_2 are constants in $(0,1]$. By virtue of Lemma 4, we can formulate the system of variational inequalities (16) into the following equivalent altering points problem.

Find $(x,y)\in\Omega_1\times\Omega_2$ such that

$$\begin{cases} x=P_{\Omega_1}[\psi_2-\varrho_2M_2(x,\cdot)](y),\\ y=P_{\Omega_2}[\psi_1-\varrho_1M_1(y,\cdot)](x).\end{cases} \tag{95}$$

Now, we propose the following parallel Mann iterative algorithm to solve the system of variational inequalities (16) as follows.

Algorithm 13. Let Ω_1, Ω_2 be nonempty closed convex subsets of a real Hilbert space H. For any $(x_0, y_0) \in \Omega_1 \times \Omega_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $\Omega_1 \times \Omega_2$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n A_2(y_n), \tag{96}$$

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n A_1(x_n), \tag{97}$$

where $\alpha_n \in [0, 1]$, $n \in \mathbb{N}$ and A_1, A_2 are the mappings defined by (93) and (94), respectively.

Also, we propose the following parallel S-iterative algorithm to solve the system of variational inequalities (16). Note that the parallel S-iterative algorithm is more general than the parallel Mann iterative algorithm.

Algorithm 14. Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space H. For any $(x_0, y_0) \in \Omega_1 \times \Omega_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $\Omega_1 \times \Omega_2$ defined by

$$x_{n+1} = A_2[(1-\alpha_n)y_n + \alpha_n A_1(x_n)], \tag{98}$$

$$y_{n+1} = A_1[(1-\alpha_n)x_n + \alpha_n A_2(y_n)], \tag{99}$$

where $\alpha_n \in (0, 1)$, $n \in \mathbb{N}$ and A_1, A_2 are the mappings defined by (93) and (94), respectively.

Theorem 15 [33]. *Let Ω_1 and Ω_2 be nonempty closed convex subsets of a Banach space X. Let $A_1 : \Omega_1 \to \Omega_2$ and $A_2 : \Omega_2 \to \Omega_1$ be Lipschitz continuous mappings with constants $\kappa_1 < 1$ and $\kappa_2 < 1$, respectively. Then, the sequence $\{(x_n, y_n)\} \in \Omega_1 \times \Omega_2$ generated by the parallel S -iterative algorithm (98)–(99) converges strongly to the unique point $(x, y) \in \Omega_1 \times \Omega_2$ such that x and y are altering points of the mappings A_1 and A_2.*

Next, we prove the following proposition, which plays a crucial role to prove the convergence of parallel iterative algorithms.

Proposition 16. *Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space $\mathscr{H}$. For each $i = 1, 2$, let $\psi_i : \mathscr{H} \to \mathscr{H}$ be the single valued mappings such that ψ_i is δ_{ψ_i} -strongly monotone and L_{ψ_i} -Lipschitz type continuous. Let $M_1 : \Omega_2 \times \Omega_1 \to \mathscr{H}$ and $M_2 : \Omega_1 \times \Omega_2 \to \mathscr{H}$ be the single-valued mappings such that M_i is δ_{M_i} -strongly monotone and L_{M_i} -Lipschitz type continuous with respect to the second argument. Then, the mappings A_1 and A_2 defined by (93) and (94) are Lipschitz type continuous with constant $(\mho_1 + \Theta_1)$ and $(\mho_2 + \Theta_2)$, respectively, where $\mho_i = \sqrt{1 - 2\delta_{\psi_i} + L_{\psi_i}^2}$ and $\Theta_i = \sqrt{1 - 2\varrho_i \delta_{M_i} + \rho_i^2 L_{M_i}^2}$.*

Proof. For any given $x_1, x_2 \in \Omega_1$, (93), and using the nonexpansiveness of the projection mapping P_{Ω_2}, we have

$$\begin{aligned}\|A_1(x_1) - A_1(x_2)\| &= \|P_{\Omega_2}[\psi_1 - \varrho_1 M_1(y, \cdot)](x_1) \\ &\quad - P_{\Omega_2}[\psi_1 - \varrho_1 M_1(y, \cdot)](x_2)\| \\ &\le \|\psi_1(x_1) - \varrho_1 M_1(y, (x_1))] \\ &\quad - [\psi_1(x_2) - \varrho_1 M_1(y, (x_2))]\| \\ &\le \|\psi_1(x_1) - \psi_1(x_2) - (x_1 - x_2)\| \\ &\quad + \|(x_1 - x_2) - \varrho_1(M_1(y, (x_1)) \\ &\quad - M_1(y, (x_2)))\|.\end{aligned} \tag{100}$$

Since ψ_1 is δ_{ψ_1}-strongly monotone and L_{ψ_1}-Lipschitz type continuous, then, we get

$$\begin{aligned}&\|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\|^2 \\ &\quad = \|x_1 - x_2\|^2 - 2\langle \psi_1(x_1) - \psi_1(x_2), x_1 - x_2\rangle \\ &\qquad + \|\psi_1(x_1) - \psi_1(x_2)\|^2 \\ &\quad \le \|x_1 - x_2\|^2 - 2\delta_{\psi_1}\|x_1 - x_2\|^2 + L_{\psi_1}^2\|x_1 - x_2\|^2 \\ &\quad = \left(1 - 2\delta_{\psi_1} + L_{\psi_1}^2\right)\|x_1 - x_2\|^2 \\ &\quad = \mho_1^2\|x_1 - x_2\|^2,\end{aligned} \tag{101}$$

which implies that

$$\|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\| \le \mho_1\|x_1 - x_2\|. \tag{102}$$

Again, using the fact that M_1 is δ_{M_1}-strongly monotone and L_{M_1}-Lipschitz type continuous with respect to the second argument, we have

$$\begin{aligned}&\|x_1 - x_2 - \varrho_1(M_1(y, (x_1)) - M_1(y, (x_2)))\|^2 \\ &\quad = \|x_1 - x_2\|^2 - 2\varrho_1\langle M_1(y, (x_1)) - M_1(y, (x_2)), x_1 - x_2\rangle \\ &\qquad + \rho_1^2\|M_1(y, (x_1)) - M_1(y, (x_2))\|^2 \\ &\quad \le \|x_1 - x_2\|^2 - 2\varrho_1\delta_{M_1}\|x_1 - x_2\|^2 + \rho_1^2 L_{M_1}^2\|x_1 - x_2\|^2 \\ &\quad = \left(1 - 2\varrho_1\delta_{M_1} + \varrho_1^2 L_{M_1}^2\right)\|x_1 - x_2\|^2 \\ &\quad = \Theta_1^2\|x_1 - x_2\|^2,\end{aligned} \tag{103}$$

which implies that

$$\|x_1 - x_2 - \varrho_1(M_1(y, (x_1)) - M_1(y, (x_2)))\| \le \Theta_1\|x_1 - x_2\|. \tag{104}$$

Utilizing (102) and (104), (100) becomes

$$\|A_1(x_1) - A_1(x_2)\| \le (\mho_1 + \Theta_1)\|x_1 - x_2\|, \tag{105}$$

i.e., A_1 is $(\mho_1 + \Theta_1)$-Lipschitz continuous, where $\mho_1 =$

$\sqrt{1-2\delta_{\psi_1}+L_{\psi_1}^2}$ and $\Theta_1=\sqrt{1-2\varrho_1\delta_{\psi_1}+\varrho_1^2 L_{M_1}^2}$. Similarly, one can prove that A_2 is $(\mho_2+\Theta_2)$-Lipschitz continuous.

Now, we prove the convergence of the parallel Mann iterative scheme (96)–(97).

Theorem 17. *Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space $\mathscr{H}$. For each $i=1,2$, let $\psi_i:\mathscr{H}\rightarrow\mathscr{H}$ be the single-valued mapping such that ψ_i is δ_{ψ_i}-strongly monotone and L_{ψ_i}-Lipschitz type continuous. Let $M_1:\Omega_2\times\Omega_1\rightarrow\mathscr{H}$ and $M_2:\Omega_1\times\Omega_2\rightarrow\mathscr{H}$ be the single-valued mappings such that M_i is δ_{M_i}-strongly monotone and L_{M_i}-Lipschiz type continuous with respect to the second argument. For any $(x_0,y_0)\in\Omega_1\times\Omega_2$, let $\{(x_n,y_n)\}$ be an iterative sequence in $\Omega_1\times\Omega_2$ generated by Algorithm 13, where $\alpha_n\in[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$ and A_1,A_2 are the mappings defined by (93) and (94), respectively. In addition, the following condition holds:*

$$0<\mho_i+\Theta_i<1, \tag{106}$$

where $\mho_i=\sqrt{1-2\delta_{\psi_i}+L_{\psi_i}^2}$ and $\Theta_i=\sqrt{1-2\varrho_i\delta_{M_i}+\varrho_i^2L_{M_i}^2}$, for $i=1,2$. Then

(i) There exists unique point $(x,y)\in\Omega_1\times\Omega_2$, which is the solution of the system of variational inequalities (16).

(ii) The sequence $\{(x_n,y_n)\}$ generated by parallel Mann iterative Algorithm 13 converges strongly to the point (x,y).

Proof.

(i) Conclusion follows from Lemma 11 and (95).

(ii) It follows from (94), (95), (96), and Lipschitz continuity of mapping A_2 that

$$\begin{aligned}\|x_{n+1}-x\|&=\|(1-\alpha_n)x_n+\alpha_n\mathrm{A}_2(y_n)-x\|\\&=\|(1-\alpha_n)x_n+\alpha_nA_2(y_n)-(1-\alpha_n)x-\alpha_nx\|\\&\le(1-\alpha_n)\|x_n-x\|+\alpha_n\|A_2(y_n)-x\|\\&=(1-\alpha_n)\|x_n-x\|+\alpha_n\|A_2(y_n)-A_2(y)\|\\&\le(1-\alpha_n)\|x_n-x\|+\alpha_n(\mho_2+\Theta_2)\|y_n-y\|.\end{aligned} \tag{107}$$

Again, utilizing (93), (95), (97), and Lipschitz continuity of mapping A_1, we get

$$\begin{aligned}\|y_{n+1}-y\|&=\|(1-\alpha_n)y_n+\alpha_nA_1(x_n)-y\|\\&=\|(1-\alpha_n)y_n+\alpha_nA_1(x_n)-(1-\alpha_n)y-\alpha_ny\|\\&\le(1-\alpha_n)\|y_n-y\|+\alpha_n\|A_1(x_n)-y\|\\&=(1-\alpha_n)\|y_n-y\|+\alpha_n\|A_1(x_n)-A_1(x)\|\\&\le(1-\alpha_n)\|y_n-y\|+\alpha_n(\mho_1+\Theta_1)\|x_n-x\|.\end{aligned} \tag{108}$$

Letting $\Theta=\max\{\mho_1+\Theta_1,\mho_2+\Theta_2\}$; then, from (107) and (108), we have

$$\begin{aligned}&\|x_{n+1}-x\|+\|y_{n+1}-y\|\\&\quad\le(1-\alpha_n)\|x_n-x\|+\alpha_n(\mho_2+\Theta_2)\|y_n-y\|\\&\qquad+(1-\alpha_n)\|y_n-y\|+\alpha_n(\mho_1+\Theta_1)\|x_n-x\|\\&\quad\le(1-\alpha_n)\|x_n-x\|+\alpha_n\Theta\|y_n-y\|\\&\qquad+(1-\alpha_n)\|y_n-y\|+\alpha_n\Theta\|x_n-x\|\\&\quad=(1-\alpha_n(1-\Theta))[\|x_n-x\|+\|y_n-y\|].\end{aligned} \tag{109}$$

Thus, from (8) and (109), we have

$$\begin{aligned}\|(x_{n+1},y_{n+1})-(x,y)\|_*&=\|(x_{n+1}-x,y_{n+1}-y)\|_*\\&=\|x_{n+1}-x\|+\|y_{n+1}-y\|\\&\le(1-\alpha_n(1-\Theta))[\|x_n-x\|+\|y_n-y\|]\\&\le(1-\alpha_n(1-\Theta))\|(x_n,y_n)-(x,y)\|_*.\end{aligned} \tag{110}$$

Since $\sum_{n=0}^{\infty}\alpha_n=\infty$ and from condition (106), we have $\Theta\in(0,1)$. Thus, from Lemma 5, we can conclude that $\lim_{n\to\infty}\|(x_n,y_n)-(x,y)\|_*=0$. Therefore, $\lim_{n\to\infty}\|x_n-x\|=0=\lim_{n\to\infty}\|y_n-y\|$. Hence, the sequences $\{x_n\}$ and $\{y_n\}$ generated by parallel Mann iterative algorithm converge strongly to x and y, respectively.

Now, we prove the convergence of the parallel S-iterative algorithm (98)–(99).

Theorem 18. *Let Ω_1 and Ω_2 be nonempty closed and convex subsets of a real Hilbert space $\mathscr{H}$. For each $i=1,2$, let $\psi_i:\mathscr{H}\rightarrow\mathscr{H}$ be the single-valued mapping such that ψ_i is δ_{ψ_i}-strongly monotone and L_{ψ_i}-Lipschitz type continuous. Let $M_1:\Omega_2\times\Omega_1\rightarrow\mathscr{H}$ and $M_2:\Omega_1\times\Omega_2\rightarrow\mathscr{H}$ be the single-valued mappings such that M_i is δ_{M_i}-strongly monotone and L_{M_i}-Lipschitz type continuous with respect to the second argument. For any $(x_0,y_0)\in\Omega_1\times\Omega_2$, let $\{(x_n,y_n)\}$ be an iterative sequence in $\Omega_1\times\Omega_2$ generated by Algorithm 14, where $\alpha_n\in(0,1)$, A_1, and A_2 are the mappings defined by (93) and (94), respectively. In addition, the following condition holds:*

$$0<\mho_i+\Theta_i<1, \tag{111}$$

where $\mho_i = \sqrt{1 - 2\delta_{\psi_i} + L_{\psi_i}^2}$ *and* $\Theta_i = \sqrt{1 - 2\rho_i \delta_{M_i} + \rho_i^2 L_{M_i}^2}$, *for* $i = 1, 2$. *Then*

(i) *There exists unique point* $(x, y) \in \Omega_1 \times \Omega_2$, *which is the solution of the system of variational inequalities (16).*

(ii) *The sequence* $\{(x_n, y_n)\}$ *generated by parallel S-iterative Algorithm 14 converges strongly to the point* (x, y).

Proof.

(i) The conclusion follows from Lemma 11 and (95).

(ii) It follows from (94), (95), (98), and Lipschitz continuity of A_2, we get

$$\begin{aligned} \|x_{n+1} - x\| &= \|A_2[(1-\alpha_n)y_n + \alpha_n A_1(x_n)] - x\| \\ &= \|A_2[(1-\alpha_n)y_n + \alpha_n A_1(x_n)] - A_2(y)\| \\ &\le (\mho_2 + \Theta_2)\|(1-\alpha_n)y_n + \alpha_n A_1(x_n) - y\| \\ &\le (\mho_2 + \Theta_2)[(1-\alpha_n)\|y_n - y\| + \alpha_n\|A_1(x_n) - y\|] \\ &= (\mho_2 + \Theta_2)[(1-\alpha_n)\|y_n - y\| + \alpha_n\|A_1(x_n) - A_1(x)\|] \\ &\le (\mho_2 + \Theta_2)[(1-\alpha_n)\|y_n - y\| + \alpha_n(\mho_1 + \Theta_1)\|x_n - x\|]. \end{aligned} \tag{112}$$

Using the same facts, we have

$$\begin{aligned} \|y_{n+1} - y\| \le (\mho_1 + \Theta_1)[(1-\alpha_n)\|x_n - x\| \\ + \alpha_n(\mho_2 + \Theta_2)\|y_n - y\|]. \end{aligned} \tag{113}$$

Letting $\Theta = \max\{\mho_1 + \Theta_1, \mho_2 + \Theta_2\}$, then from (112) and (113), we have

$$\begin{aligned} \|x_{n+1} - x\| + \|y_{n+1} - y\| &\le (\mho_2 + \Theta_2)[(1-\alpha_n)\|y_n - y\| + \alpha_n \\ &\quad \cdot (\mho_1 + \Theta_1)\|x_n - x\|] + (\mho_1 + \Theta_1) \\ &\quad \cdot [(1-\alpha_n)\|x_n - x\| + \alpha_n(\mho_2 + \Theta_2)\|y_n - y\|] \\ &\le \Theta[(1-\alpha_n)\|y_n - y\| + \alpha_n\Theta\|x_n - x\|] + \Theta \\ &\quad \cdot [(1-\alpha_n)\|x_n - x\| + \alpha_n\Theta\|y_n - y\|] \\ &= \Theta[(1-\alpha_n(1-\Theta))][\|x_n - x\| + \|y_n - y\|]. \end{aligned} \tag{114}$$

Thus, from (8) and (114), we have

$$\begin{aligned} \|(x_{n+1}, y_{n+1}) - (x, y)\|_* &= \|(x_{n+1} - x, y_{n+1} - y)\|_* \\ &= \|x_{n+1} - x\| + \|y_{n+1} - y\| \\ &\le \Theta[(1-\alpha_n(1-\Theta))][\|x_n - x\| + \|y_n - y\|] \\ &= \Theta[(1-\alpha_n(1-\Theta))]\|(x_n, y_n) - (x, y)\|_*. \end{aligned} \tag{115}$$

Notice that $\Theta[(1-\alpha_n(1-\Theta))] < \Theta$ and from condition (111), we have $\Theta \in (0, 1)$. Thus, we can conclude that $\lim_{n\to\infty}\|(x_n, y_n) - (x, y)\|_* = 0$. Therefore, $\lim_{n\to\infty}\|x_n - x\| = 0 = \lim_{n\to\infty}\|y_n - y\|$. Hence, the sequences $\{x_n\}$ and $\{y_n\}$ generated by parallel S-iterative algorithm converge strongly to x and y, respectively.

Finally, we discuss an example which illustrates the convergence analysis of parallel Mann and parallel S-iterative algorithms.

Example 1. Let $\mathscr{H} = \mathbb{R}$, $\Omega_1 = (-\infty, 0]$ and $\Omega_2 = [0, \infty)$. Let $\psi_1, \psi_2 : \mathscr{H} \to \mathscr{H}$ be the single-valued mappings defined by

$$\psi_1(x) = \frac{2x - 1}{3}, \psi_2(x) = \frac{5x - 2}{6}, \forall x \in \mathscr{H}. \tag{116}$$

Suppose that the mappings $M_1, M_2 : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ are defined by

$$M_1(x, y) = \frac{4x + 3y - 2}{3}, \; M_2(x, y) = \frac{17x + 17y + 1}{12}, \forall x, y \in \mathscr{H}. \tag{117}$$

Then, it is easy to check that for each $i = 1, 2$, ψ_i is δ_{ψ_i}-strongly monotone and L_{ψ_i}-Lipschitz continuous with $\delta_{\psi_1} = 2/3 = L_{\psi_1}$ and $\delta_{\psi_2} = 5/6 = L_{\psi_2}$ and M_i is δ_{M_i}-strongly monotone and L_{M_i}-Lipschitz continuous in the second argument with $\delta_{M_1} = 4/3 = L_{M_1}$ and $\delta_{M_2} = 17/12 = L_{M_2}$. Let the mappings $A_1 : \Omega_1 \to \Omega_2$ and $A_2 : \Omega_2 \to \Omega_1$ be defined as follows:

$$\begin{aligned} A_1 &:= P_{\Omega_2}[\psi_1 - \varrho_1 M_1(y, \cdot)], \\ A_2 &:= P_{\Omega_1}[\psi_2 - \varrho_2 M_2(x, \cdot)]. \end{aligned} \tag{118}$$

Then, for $\varrho_1, \varrho_2 = 1$,

$$\begin{aligned} A_1(x) &= P_{\Omega_2}[\psi_1(x) - M_1(y, x)] = P_{\Omega_2}\left[\frac{2x - 1}{3} - \frac{4x + 3y - 2}{3}\right] \\ &= \frac{-2x - 3y + 1}{3}, \forall x, y \in \mathscr{H}, \end{aligned}$$

$$\begin{aligned} A_2(y) &= P_{\Omega_1}[\psi_1(y) - M_2(x, y)] = P_{\Omega_1}\left[\frac{5y - 2}{6} - \frac{17x + 17y + 1}{12}\right] \\ &= \frac{-17x - 7y - 5}{12}, \forall x, y \in \mathscr{H}. \end{aligned} \tag{119}$$

It is easy to verify that A_1 is 2/3-Lipschitz continuous and A_2 is 7/12-Lipschitz continuous. Also,

$$\begin{aligned}\mho_1+\Theta_1&=\sqrt{1-2\delta_{\psi_1}+L^2_{\psi_1}}+\sqrt{1-2\rho_1\delta_{M_1}+\rho_1^2L^2_{M_1}}\\&=\sqrt{1-\frac{4}{3}+\frac{4}{9}}+\sqrt{1-\frac{8}{3}+\frac{16}{9}}=\frac{2}{3}<1,\\ \mho_2+\Theta_2&=\sqrt{1-2\delta_{\psi_2}+L^2_{\psi_2}}+\sqrt{1-2\rho_2\delta_{M_2}+\rho_2^2L^2_{M_2}}\\&=\sqrt{1-\frac{10}{6}+\frac{25}{36}}+\sqrt{1-\frac{34}{12}+\frac{289}{144}}=\frac{7}{12}<1.\end{aligned}\tag{120}$$

Thus, all the conditions of Theorem 17 and Theorem 18 are satisfied. Hence, by using Proposition 16, Algorithm 13, and Algorithm 14, the conclusions of Theorem 17 and Theorem 18 follow.

5. Concluding Remarks

In this paper, a system of generalized set-valued variational inequalities involving relaxed cocoercive mappings in Hilbert spaces is considered. Using the projection method and Banach contraction principle, an existence result is proved. Also, we proposed an iterative algorithm, and its convergence is discussed. Moreover, we established an equivalence between the system of variational inequalities and altering points problem. Some parallel iterative algorithms are proposed, and the strong convergence of the sequences generated by these iterative algorithms is discussed. Finally, a numerical example is constructed to illustrate the convergence analysis of the proposed parallel iterative algorithms.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to the writing of this manuscript. All authors read and approve the final version.

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Research Article

Some Nonlinear Integral Inequalities Connected with Retarded Terms on Time Scales

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The objective of this research is to formulate a specific class of integral inequalities of Gronwall kind concerning retarded term and nonlinear integrals with time scales theory. Our results generate several new inequalities that reflect continuous and discrete form, as well as giving the unknown function an upper bound estimate. The effectiveness of such inequalities arises from the belief that it is widely relevant in unique circumstances where there is no valid utilization of various available inequalities. Applications are additionally represented to display the legitimacy of built-up hypotheses.

1. Introduction

Over the last decades, a variety of basic and critical inequalities have been inspected with improvement in the methods of differential and integral equations, which are anticipating a great deal of study in the analysis of boundedness, global existence, and stability of solutions of differential and integral equations as well as difference equations [1, 2]. Among others, the inequalities of the Gronwall-Bellman type are of unique significance for providing explicit estimates for unknown functions.

Bellman [3] has proven the integral inequality

$$\mathfrak{L}(\hbar) \le \mathfrak{m} + \int_{\hbar_1}^{\hbar} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})d\mathscr{P}, \hbar \in [\hbar_1, \hbar], \tag{1}$$

for some $\mathfrak{m} \ge 0$, which is significantly dedicated to evaluate the equilibrium and asymptotic behavior with a view to find solutions for integral equations. Such inequalities have been remarkably strengthened over a very long period of time by verifying their importance and intrinsic potential in the diverse fields of applied sciences.

Afterward, Pachpatte [4] replaced the constant $\mathfrak{m}$ from the prior integral inequality by a nondecreasing function $\mathfrak{m}(\hbar)$ and contemplated

$$\mathfrak{L}(\hbar) \le \mathfrak{m}(\hbar) + \mathscr{H}(\hbar)\int_{0}^{\hbar} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})d\mathscr{P}, \hbar \in [0,\infty). \tag{2}$$

The inequality (2) encourages fresh speculations and can be classified as incredible techniques in the understanding of specific differential and integral equations.

El-Owaidy et al. [5] introduced yet another basic type of inequality

$$\mathfrak{L}(\hbar) \le \mathfrak{m}_0 + \int_{0}^{\hbar} \mathscr{G}(\mathscr{P})\left[\mathfrak{L}^{\Psi}(\mathscr{P}) + \int_{0}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})d\mathfrak{x}\right]d\mathscr{P}, \hbar \in [0,\infty), \tag{3}$$

in which integrand accommodates the power and allows it more challenging to determine the unknown function where $0 \le \Psi < 1$ is included.

Several scholars researched linear and nonlinear modifications for booming such kinds of inequalities (see [6–12]). Most articles tend to do with retarded nonlinear integral inequalities. In this database, the retarded integral inequality was identified by Lipovan [13].

$$\mathfrak{L}(\hbar) \le \mathfrak{m} + \int_{\hbar_1}^{\hbar} \mathcal{G}(\mathcal{P})\rho(\mathfrak{L}(\mathcal{P}))d\mathcal{P} + \int_{\omega(\hbar_1)}^{\omega(\hbar)} \mathcal{J}(\mathcal{P})\rho(\mathfrak{L}(\mathcal{P}))d\mathcal{P}, \hbar_1 \le \hbar < \hbar_2, \tag{4}$$

$\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\rho(\mathfrak{L}) > 0$ for $\mathfrak{L} > 0$, and the integral consists of nonlinear equations in its more standard way. Over the span of late years, multiple retarded integral inequalities have been discussed by numerous researchers [14, 15] and the references therein.

Remarkably, the dynamic inequalities predict a necessary position within the production of the basic principle of time-scale dynamic equations. Hilger [16] was named to be the primary researcher who started the growth of time scale analytics. The ultimate point is to study an equation or an inequality which can be dynamic such that a time scale $\mathbb{T}$ be a domain of an unknown function. The motivation that guides the time scale theory is to unify continuous and discrete inspection. Several authors have approached and finished proper assessment of the characterizations and utilization of different types of inequalities on a timely basis [17–21] and the references in that. Around the beginning, Bohner and Peterson [22] tested the integral inequality

$$\mathfrak{L}(\hbar) \le \mathfrak{m}(\hbar) + \int_{\hbar_1}^{\hbar} \mathcal{J}(\mathcal{P})\mathfrak{L}(\mathcal{P})\Delta\mathcal{P}, \hbar \in \mathbb{T}. \tag{5}$$

Later in 2010, Li [23] has assessed the consequent nonlinear integral inequality of one variable

$$\mathfrak{L}^{\gamma}(\hbar) \le \mathfrak{m}(\hbar) + \mathcal{H}(\hbar)\int_{\hbar_1}^{\hbar} [\mathcal{G}(\mathcal{P})\mathfrak{L}(\rho(\mathcal{P})) + \mathcal{J}(\mathcal{P})]\Delta\mathcal{P}, \tag{6}$$

for $\hbar \in \hbar_1$ with initial conditions $\mathfrak{L}(\hbar) = \Omega(\hbar), \hbar \in [\beta, \hbar_1] \cap \mathbb{T}$, $Y(\rho(\hbar)) \le (\mathfrak{m}(\hbar))^{1/\gamma}$ for $\hbar \in \hbar_1$, $\rho(\hbar) \le \hbar_1$, where $\gamma \ge 1$ is a constant, $\rho(\hbar) \le \hbar$, $-\infty < \beta = \inf \{\rho(\hbar), \hbar \in \mathbb{T}_0\} \le \hbar_1$, and $\Omega(\hbar) \in C_{rd}([\beta, \hbar_1] \cap \mathbb{T}, \mathbb{R}_+)$. Besides that, Pachpatte [24] has attempted to see the augmentation of the integral inequality

$$\mathfrak{L}(\hbar) \le \mathfrak{m}(\hbar)\int_{\hbar_1}^{\hbar} \mathcal{G}(\mathcal{P})\left[\mathfrak{L}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{J}(\mathcal{P}, \mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right]\Delta\mathcal{P}, \tag{7}$$

such that $\mathcal{J}(\mathcal{P}, \mathfrak{x}) \ge 0$, $\mathcal{J}^{\Delta}(\mathcal{P}, \mathfrak{x}) \ge 0$ for $\hbar, \mathfrak{x} \in \mathbb{T}$ and $\sigma \le \hbar$. Further, in 2017, Haidong [25] suggested that nonlinear integral inequality be generalized as follows

$$\begin{aligned}\mathfrak{L}(\hbar) \le{}& \mathfrak{m}(\hbar) + \mathcal{G}(\hbar)\int_{\omega(\hbar_1)}^{\omega(\hbar)} \left[\mathcal{G}_1(\mathcal{P})\mathfrak{L}(\mathcal{P}) + \mathcal{G}_2(\mathcal{P})\int_{\omega(\hbar_1)}^{\hbar} \mathcal{J}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right]\Delta\mathcal{P} \\ &+ \lambda\mathfrak{m}(T)\int_{\omega(\hbar_1)}^{\omega(T)} \left[\mathcal{G}_1(\mathcal{P})\mathfrak{L}(\mathcal{P}) + \mathcal{G}_2(\mathcal{P})\int_{\omega(\hbar_1)}^{\hbar} \mathcal{J}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right]\Delta\mathcal{P},\end{aligned} \tag{8}$$

where $\lambda \ge 0$. Although a lot of studies have been conducted on integral inequalities related to time scales, there is not that much research on retarded integral inequalities on time scales has been performed. For certain cases, however, specific types of integral and differential equations via power are required to investigate time scale delay inequalities in order to thrive and meet the desired targets.

Our primary concern of this work is not only to analyze nonlinear integral inequalities with retarded term but also to explore the well-known existing results which determine the explicit bounds of the solutions of the unknown functions of the particular dynamic equations on time scales. The new speculations are used as supportive tools to exhibit the description of integral inequalities and equations. The offered dynamic integral strategy for acquiring new results is clear and effective. There are other benefits of this technique: it is fast and short. Moreover, the proposed procedure can be modified to solve various systems with nonlinear fractional partial differential equations.

The rest of the manuscript is arranged as follows. In Section 2, we have some preliminary data which is an essential element for our key studies. Section 3 is devoted to theoretical discussions on the immense solutions of the problem beneath consideration. The examples supporting the theoretical consequences are given in Section 4. Finally, a few concluding feedback and suggestions for future research are provided in Section 5 and thus completes this work.

2. Basic Material on Time Scales

Below, we interpret some basic definitions and valuable theorem regarding time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition 1 (see [22]). The forward jump operator σ on $\mathbb{T}$ be defined by $\sigma(\hbar) = \inf \{\mathcal{P} \in \mathbb{T} : \mathcal{P} > \hbar\}$ for all $\hbar \in \mathbb{T}$. If $\sigma(\hbar) = \hbar$, then $\hbar$ is said to be right-dense and right-scattered if $\sigma(\hbar) > \hbar$. The backward jump operator and left-scattered and left-dense points are defined in a similar way.

Definition 2 (see [22]). $\xi : \mathbb{T} \to \mathbb{R}_+$ is called the graininess function if $\xi(\hbar) = \sigma(\hbar) - \hbar$. Also, $\xi(\hbar) = 0$ for $\mathbb{T} = \mathbb{R}$ and $\xi(\hbar) = \mathcal{P}$, with a positive constant $\mathcal{P}$ for $\mathbb{T} = \mathcal{P}\mathbb{Z}$.

Definition 3 (see [22]). The set $\mathbb{T}^{\mathfrak{k}}$ is derived from T as follows: if $\mathbb{T}$ has a left-scattered maximum $\mathfrak{m}$, then $\mathbb{T}^{\mathfrak{k}} = \mathbb{T} - \mathfrak{m}$; otherwise, $\mathbb{T}^{\mathfrak{k}} = \mathbb{T}$.

Definition 4 (see [22]). For some $\hbar \in \mathbb{T}^{\mathfrak{k}}$ and a function $\mathfrak{L} \in (\mathbb{T}, \mathbb{R})$, the delta derivative of $\mathfrak{L}$ is denoted by $\mathfrak{L}^{\Delta}(\hbar)$ and satisfies

$$\left|\mathfrak{L}(\sigma(\hbar)) - \mathfrak{L}(\mathscr{P}) - \mathfrak{L}^{\Delta}(\hbar)(\sigma(\hbar) - \mathscr{P})\right| \le \varepsilon|\sigma(\hbar) - \mathscr{P}|, \forall \varepsilon > 0, \mathscr{P} \in \aleph, \tag{9}$$

and a neighborhood of $\hbar$ is $\aleph$. Also, $\mathfrak{L}$ is the delta differential function at $\hbar$.

Definition 5 (see [22]). An antiderivative of $\mathfrak{L}$ is $\mathfrak{L}_1$ if $\mathfrak{L}_1^{\Delta}(\hbar) = \mathfrak{L}(\hbar)$, $\hbar \in \mathbb{T}^{\mathfrak{k}}$, whereas

$$\int_{\varsigma_1}^{\varsigma} \mathfrak{L}(\hbar)\Delta\hbar = \mathfrak{L}_1(\varsigma) - \mathfrak{L}_1(\varsigma_1), \varsigma_1, \varsigma \in \mathbb{T}, \tag{10}$$

is the Cauchy integral of $\mathfrak{L}$.

Definition 6 (see [22]). The set of all regressive and rd-continuous functions $\mathscr{G} : \mathbb{T} \longrightarrow \mathbb{R}$ will be represented by $\mathfrak{R}$ provided $1 + \xi(\hbar)\mathscr{G}(\hbar) \neq 0$ for all $\hbar \in \mathbb{T}^{\mathfrak{k}}$ holds.

Definition 7. If $\mathbb{T} = \mathbb{R}$ and $\mathscr{G} : \mathbb{R} \longrightarrow \mathbb{R}$ are a continuous function, then the exponential function is given by

$$e_{\mathscr{G}}(\hbar, \mathscr{P}) = e^{\int_{\mathscr{P}}^{\hbar} \mathscr{G}(\tau^*)d\tau^*}, e_{\alpha}(\hbar, \mathscr{P}) = e^{\alpha(\hbar - \mathscr{P})}, e_{\alpha}(\hbar, 0) = e^{\alpha(\hbar)}, \mathscr{P}, \hbar, \alpha \in \mathbb{R}. \tag{11}$$

Theorem 8 (see [22]). *If $\mathscr{G} \in \mathfrak{R}$, then*

(i) $e_{\mathscr{G}}(\hbar, \hbar) \equiv 1$ *and* $e_0(\hbar, r) \equiv 1$

$$\begin{aligned} e_{\mathscr{G}}(\sigma(\hbar), \mathscr{P}) &\equiv (1 + \xi(\hbar))\mathscr{G}(\hbar)e_{\mathscr{G}}(\hbar, \mathscr{P}), \\ \frac{1}{e_{\mathscr{G}}(\hbar, \mathscr{P})} &= e_{\ominus\mathscr{G}}(\hbar, \mathscr{P}) = e_{\mathscr{G}}(\mathscr{P}, \hbar). \end{aligned} \tag{12}$$

(ii) *If $\mathscr{J} \in \mathfrak{R}^+$, then $\ominus \mathscr{J} \in \mathfrak{R}^+$*

(iii) *If $\mathscr{J} \in \mathfrak{R}^+$, then $e_{\mathscr{J}}(\hbar, \hbar_0) > 0$, for all $\hbar_0, \hbar \in \mathbb{T}$, where $\ominus \mathscr{J} = -(\mathscr{J}/(1 + \xi \mathscr{J}))$.*

To more descriptions of the study of the time scale, we direct the reader to Bohner and Peterson [22] excellent monograph, which describes and organizes most of the time scale logic.

3. Nonlinear Powered Integral Inequalities via Retarded Term

Given the documentations all through the content for the simplicity of perusing: $\mathbb{R}$ stands for the set of real numbers, $\mathbb{C}_{rd}$ is the set of rd continuous functions, $\mathbb{R}_+ = [0,\infty)$, $\hbar_1 \in \mathbb{T}$, $\hbar_1 \ge 0$, $\mathbb{T}_0 = [\hbar_1,\infty) \cap \mathbb{T}$, and

(H1). The continuous function $\mathfrak{L}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is a nonnegative.

(H2). $\xi(\hbar) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is a function which is increasing and differentiable on $[\hbar_1, \infty)$ so that $\xi(\hbar) \le \hbar$, $\xi(\hbar_1) = \hbar_1$.

(H3). $\mathscr{G}(\hbar), \mathscr{J}(\hbar), \mathscr{B}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$.

(H4). $\mathfrak{m}(\hbar) \in \mathbb{C}_{rd}([\mathbb{T}_0, (0,\infty))$ is nondecreasing.

(H5). $\mu, \kappa \in \mathbb{C}_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ is a nondecreasing function and $\mu(\mathscr{J}) > 0$, $\kappa(\mathscr{J}) > 0$ for $\mathscr{J} > 0$.

Specifically, the fundamental lemma to be used afterwards is presented below:

Lemma 9 (see [26]). *Let $\psi \ge 0$, $j \ge \vartheta \ge 0$, and $j \neq 0$, then*

$$\psi^{\vartheta/j} \le \frac{\vartheta}{j}\psi + \frac{j - \vartheta}{j}. \tag{13}$$

Proof. For $\vartheta = 0$, inequality (13) is accurate, unless $\delta = 1$ if $\vartheta > 0$, $\delta = \vartheta/j$, and

$$\psi^{\vartheta/j} \le \frac{\vartheta}{j} Y^{\vartheta j/j}\psi + \frac{j - \vartheta}{j} Y^{\vartheta/j}, Y > 0. \tag{14}$$

We obtain (13) for $Y = 1$.

Theorem 10. *Suppose that (H1), (H2), (H3), and (H4) be satisfied. Moreover*

$$\begin{aligned} \mathfrak{L}(\hbar) \le \mathfrak{m}(\hbar) &+ \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})\left(\mathfrak{L}(\mathscr{P}) + \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right)^{\aleph}\Delta\mathscr{P} \\ &+ \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})\left(\int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right)\Delta\mathscr{P}. \end{aligned} \tag{15}$$

$\forall \hbar \in \mathbb{T}_0$. *Then*

$$\mathfrak{L}(\hbar) \le \mathfrak{m}(\hbar) + \frac{\mathscr{H}_1(\hbar)e_{\ominus(-Y)}(\sigma(\hbar), \hbar_1)}{1 - \mathscr{H}_1(\hbar)\int_{\hbar_1}^{\xi(\hbar)} \chi(\mathscr{P})e_{\ominus(-Y)}(\mathscr{P}, \mathscr{P}_1)\Delta\mathscr{P}}, \tag{16}$$

where $\aleph \in (0, 1]$ is a positive constant,

$$\begin{aligned} \mathscr{H}_1(\hbar) = \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathscr{H}(\mathscr{P})\Bigg[\Bigg(\aleph\Bigg(\mathscr{H}(\mathscr{P}) &+ \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\mathscr{H}(\mathfrak{x})\Delta\mathfrak{x}\Bigg) \\ &+ (1 - \aleph) + \int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})\mathscr{H}(\mathfrak{x})\Delta\mathfrak{x}\Bigg]\Delta\mathscr{P}, \end{aligned} \tag{17}$$

$$Y(\hbar) = \mathscr{G}(\hbar)\left[2\aleph(\mathscr{H}(\hbar))\left(1 + \int_{\hbar_1}^{\hbar} \mathscr{J}(\mathscr{P})\Delta\mathscr{P}\right) + (1-\aleph) + 2\mathscr{H}(\hbar)\int_{\hbar_1}^{\hbar} \mathscr{B}(\mathscr{P})\Delta\mathscr{P}\right], \quad (18)$$

$$\chi(\hbar) = \mathscr{G}(\hbar)\left[\aleph\left(1 + \int_{\hbar_1}^{\hbar} \mathscr{J}(\mathscr{P})\Delta\mathscr{P}\right) + \int_{\hbar_1}^{\hbar} \mathscr{B}(\mathscr{P})\Delta\mathscr{P}\right]. \quad (19)$$

Proof. Let us define

$$\mathcal{Q}(\hbar) = \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})\left(\mathfrak{L}(\mathscr{P}) + \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right)^{\aleph}\Delta\mathscr{P} + \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})\left(\int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right)\Delta\mathscr{P}, \quad (20)$$

therefore, (15) reaches to

$$\mathfrak{L}(\hbar) \le \mathfrak{m}(\hbar) + \mathcal{Q}(\hbar), \mathcal{Q}(\hbar_1) = 0, \quad (21)$$

by executing Lemma 9 and (21) into (20), we deduce

$$\begin{aligned}
\mathcal{Q}(\hbar) &\le \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})\left[\aleph\left(\mathfrak{L}(\mathscr{P}) + \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right) + (1-\aleph)\right]\Delta\mathscr{P} \\
&\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}(\mathscr{P})\left(\int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})\mathfrak{L}(\mathfrak{x})\Delta\mathfrak{x}\right)\Delta\mathscr{P} \\
&\le \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})[m(\mathscr{P}) + \mathcal{Q}(\mathscr{P})] \\
&\quad \cdot \left[\aleph\left((\mathfrak{m}(\mathscr{P}) + \mathcal{Q}(\mathscr{P}) + \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})(m(\mathfrak{x}) + \mathcal{Q}(\mathfrak{x}))\Delta\mathfrak{x} + (1-\aleph)\right]\Delta\mathscr{P} \\
&\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})[m(\mathscr{P}) + \mathcal{Q}(\mathscr{P})]\left[\int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})(\mathfrak{m}(\mathfrak{x}) + \mathcal{Q}(\mathfrak{x}))\Delta\mathfrak{x}\right]\Delta\mathscr{P} \\
&\le \int_{\hbar_1}^{\xi(\hbar)} \left[\mathscr{G}(\mathscr{P})\mathfrak{m}(\mathscr{P})(\aleph(\mathfrak{m}(\mathscr{P})) + \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\mathfrak{m}(\mathfrak{x})\Delta\mathfrak{x}\right)\Bigg) \\
&\quad + (1-\aleph)\mathscr{G}(\mathscr{P})\mathfrak{m}(\mathscr{P}) + \mathscr{G}(\mathscr{P})\mathfrak{m}(\mathscr{P})\left(\int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})\mathfrak{m}(\mathfrak{x})\Delta\mathfrak{x}\right)\Bigg]\Delta\mathscr{P} \\
&\quad + \int_{\hbar_1}^{\xi(\hbar)} \left[\mathscr{G}(\mathscr{P})\left(2(\aleph(\mathfrak{m}(\mathscr{P})))\left(1 + \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\Delta(\mathfrak{x})\right)\right)\right. \\
&\quad \left. + (1-\aleph)\mathscr{G}(\mathscr{P}) + 2\mathscr{G}(\mathscr{P})\mathfrak{m}(\mathscr{P})\left(\int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})\Delta(\mathfrak{x})\right)\right]\mathcal{Q}(\mathscr{P})\Delta\mathscr{P} \\
&\quad + \int_{\hbar_1}^{\xi(\hbar)} \left[\mathscr{G}(\mathscr{P})\left(\aleph\left(1 + \int_{\hbar_1}^{\mathscr{P}} \mathscr{J}(\mathfrak{x})\Delta\mathfrak{x}\right)\right)\right. \\
&\quad \left. + \mathscr{G}(\mathscr{P})\left(\int_{\hbar_1}^{\mathscr{P}} \mathscr{B}(\mathfrak{x})\Delta\mathfrak{x}\right)\right]\mathcal{Q}^2(\mathscr{P})\Delta\mathscr{P} \le \mathscr{H}_1(\hbar) \\
&\quad + \int_{\hbar_1}^{\xi(\hbar)} Y(\mathscr{P})\mathcal{Q}(\mathscr{P})\Delta\mathscr{P} + \int_{\hbar_1}^{\xi(\hbar)} \chi(\mathscr{P})\mathcal{Q}^2(\mathscr{P})\Delta\mathscr{P},
\end{aligned} \quad (22)$$

where $\mathscr{H}_1(\hbar)$, $Y(\hbar)$, and $\chi(\hbar)$ are quoted in (17), (18), and (19), respectively. Fixing $\hbar_2 \in \mathbb{T}_0$ for an arbitrary $\hbar \in [\hbar_1, \hbar_2] \cap \mathbb{T}$ and taking $\mathcal{Q}_1(\hbar)$ by

$$\mathcal{Q}_1(\hbar) \le \mathscr{H}_1(\hbar_2) + \int_{\hbar_1}^{\xi(\hbar)} Y(\mathscr{P})\mathcal{Q}(\mathscr{P})\Delta\mathscr{P} + \int_{\hbar_1}^{\xi(\hbar)} \chi(\mathscr{P})\mathcal{Q}^2(\mathscr{P})\Delta\mathscr{P}, \quad (23)$$

(22) can be carried out as

$$\mathcal{Q}(\hbar) \le \mathcal{Q}_1(\hbar), \mathcal{Q}(\xi(\hbar)) \le \mathcal{Q}_1(\xi(\hbar)) \le \mathcal{Q}_1(\hbar), \mathcal{Q}_1(\hbar_1) = \mathscr{H}_1(\hbar_2), \quad (24)$$

since $\mathcal{Q}_1(\hbar)$ is nondecreasing, hence we can compose (23) and (24) as

$$\mathcal{Q}_1^{\Delta}(\hbar) \le Y(\xi(\hbar))\xi^{\Delta}(\hbar)\mathcal{Q}_1(\hbar) + \chi(\xi(\hbar))\xi^{\Delta}(\hbar)\mathcal{Q}_1^2(\hbar), \quad (25)$$

take $\mathfrak{Z}(\hbar) = \mathcal{Q}_1^{-1}(\hbar)$, then $\mathfrak{Z}^{\Delta}(\hbar) = -\mathcal{Q}_1^{-2}(\hbar)\mathcal{Q}_1^{\Delta}(\hbar)$; therefore, the last inequality with Theorem 8 yields

$$\mathfrak{Z}^{\Delta}(\hbar) + Y(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}(\hbar) \ge -\chi(\xi(\hbar))\omega^{\Delta}(\hbar), \quad (26)$$

or

$$\begin{aligned}
\left(\frac{\mathfrak{Z}(\hbar)}{e_{\ominus(Y)}(\sigma(\hbar), \hbar_1)}\right)^{\Delta} &= \frac{\mathfrak{Z}^{\Delta}(\hbar) + Y(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}(\hbar)}{e_{\ominus(Y)}(\sigma(\hbar), \hbar_1)} \\
&\ge -\chi(\xi(\hbar))\xi^{\Delta}(\hbar)e_{!(-Y)}(\sigma(\hbar), \hbar_1).
\end{aligned} \quad (27)$$

Integrating (27) and applying $\mathfrak{Z}(\hbar_1) = \mathscr{H}_1^{-1}(\hbar_2)$, we attain

$$\mathfrak{Z}(\hbar) \ge \frac{1 - \mathscr{H}_1(\hbar_2)\int_{\hbar_1}^{\hbar} \chi(\xi(\mathscr{P}))\xi^{\Delta}(\mathscr{P})e_{\ominus(-Y)}(\sigma(\hbar), \hbar_1)\Delta\mathscr{P}}{\mathscr{H}_1(\hbar_2)e_{\ominus(-Y)}(\sigma(\hbar), \hbar_1)}. \quad (28)$$

Substituting $\mathfrak{Z}(\hbar) = \mathcal{Q}_1^{-1}(\hbar)$ in (28) and from (24), we get

$$\mathcal{Q}(\hbar) \le \frac{\mathscr{H}_1(\hbar_2)e_{\ominus(-Y)}(\sigma(\hbar), \hbar_1)}{1 - \mathscr{H}_1(\hbar_2)\int_{\hbar_1}^{\hbar} \chi(\xi(\mathscr{P}))\xi^{\Delta}(\mathscr{P})e_{\ominus(-Y)}(\sigma(\hbar), \hbar_1)\Delta\mathscr{P}}. \quad (29)$$

Substituting the previous value in (21), we have

$$\mathfrak{L}(\hbar_2) \le m(\hbar_2) + \frac{\mathscr{H}_1(\hbar_2)e_{\ominus(-Y)}(\sigma(\hbar), \hbar_1)}{1 - \mathscr{H}_1(\hbar_2)\int_{\hbar_1}^{\xi(\hbar)} \chi(\mathscr{P})e_{\ominus(-Y)}(\mathscr{P}, \mathscr{P}_1)\Delta\mathscr{P}}, \quad (30)$$

the arbitrariness of $\hbar_2$ in the previous inequality claims the optimal bound in (16).

Remark 11. Choose $\mathfrak{m}(\hbar) = u_0, \gamma = 1, \mathfrak{L}(\hbar) \le \phi(u(t)), \mathscr{J}(\hbar) = g(t)$, and $\mathscr{G}(\hbar) = f(t)$, then Theorem 10 will become a small deviation of Theorem 2.2 studied by Abdeldaim and EI-Deeb [14], if $\mathbb{T} = \mathbb{R}$ and $q(t) = 0$.

Theorem 12. *The presumption (H1), (H2), (H4), and (H5), $\mathcal{G}(\hbar)$, $\mathcal{J}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, and*

$$\mathfrak{L}(\hbar) \leq \mathfrak{m}(\hbar) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mu(\mathfrak{L}(\mathcal{P})) \cdot \left(\mathfrak{L}^{\lambda}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{J}(\mathfrak{x})\kappa(\mathfrak{L}(\mathfrak{x}))\Delta\mathfrak{x}\right)^{\aleph} \Delta\mathcal{P}, \forall \hbar \in \mathbb{T}_0, \tag{31}$$

satisfy, then

$$\mathfrak{L}(\hbar) \leq \Sigma^{-1}\left(\bar{M}(\hbar)\right), \tag{32}$$

$\forall \hbar \in [\hbar_1, \hbar_2]$*, where* $\aleph$ *and* λ *are constants,* $\aleph > 0$*,* $\lambda \geq 1$*,* $\aleph + \lambda > 1$*, and*

$$\bar{M}(\hbar) = \Sigma(m(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mathcal{F}(\mathcal{P})\Delta\mathcal{P}, \mathcal{F}(\hbar) = \left(G^{-1}\left[\mathcal{P}^{-1}\left(\mathcal{P}\left(G\left(\mathfrak{m}^{\lambda}(\hbar)\right) + \int_{\hbar_1}^{\hbar} \mathcal{J}(\mathcal{P})\Delta\mathcal{P}\right) + \lambda\int_{\hbar_1}^{\omega(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P}\right)\right]\right)^{\aleph}, \tag{33}$$

$$G(\ell) = \int_1^{\ell} \frac{\Delta\mathcal{P}}{\kappa(\mathcal{P})}, \ell > 0, \tag{34}$$

$$\mathcal{P}(\ell) = \int_1^{\ell} \frac{\phi\left(G^{-1}(\mathcal{P})\right)\Delta\mathcal{P}}{(\mu(G(\mathcal{P})))\left(G^{-1}(\mathcal{P})\right)^{\aleph+\lambda-1}}, \ell > 0, \tag{35}$$

$$\Sigma(\ell) = \int_1^{\ell} \frac{\Delta\mathcal{P}}{\mu(\mathcal{P})}, \ell > 0, \tag{36}$$

the inverses of G, $\mathcal{P}$, *and* Σ *are* G^{-1}, $\mathcal{P}^{-1}$, *and* Σ^{-1}, *and select* $\hbar_3$ *in such a way that*

$$\begin{aligned}
&\mathcal{P}\left(G\left(\mathfrak{m}^{\lambda}(\hbar)\right) + \int_{\hbar_1}^{\hbar} \mathcal{J}(\mathcal{P})\Delta\mathcal{P}\right) + \lambda\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P} \\
&\quad \leq \int_1^{\ell} \frac{\phi\left(G^{-1}(\mathcal{P})\right)\Delta\mathcal{P}}{\left(\mu\left(G^{-1}(\mathcal{P})\right)\right)\left(G^{-1}(\mathcal{P})\right)^{\aleph+\lambda-1}}, \xi(\mathfrak{m}(\hbar)) \\
&\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mathcal{F}(\mathcal{P})\Delta\mathcal{P} \leq \frac{\Delta\mathcal{P}}{\mu(\mathcal{P})}, \mathcal{P}^{-1} \\
&\quad \times \left(\mathcal{P}\left(G\left(\mathfrak{m}^{\lambda}(\hbar)\right) + \int_{\hbar_1}^{\hbar} \mathcal{J}(\mathcal{P})\Delta\mathcal{P}\right)\right. \\
&\quad \left. + \lambda\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P}\right) \leq \frac{\Delta\mathcal{P}}{\phi(\mathcal{P})}.
\end{aligned} \tag{37}$$

Proof. Let $\hbar_3 \in \mathbb{T}_0$ be fixed for $\hbar \in [\hbar_1, \hbar_3] \cap \mathbb{T}$ and letting

$$\mathcal{Q}_2(\hbar) = \mathfrak{m}(\hbar_3) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mu(\mathfrak{L}(\mathcal{P})) \cdot \left(\mathfrak{L}^{\lambda}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{J}(\mathfrak{x})\kappa(\mathfrak{L}(\mathfrak{x}))\Delta\mathfrak{x}\right)^{\aleph} \Delta\mathcal{P}, \tag{38}$$

(31) and (38) imply that

$$\mathfrak{L}(\hbar) \leq \mathcal{Q}_2(\hbar), \quad \mathcal{Q}_2(\hbar_1) = \mathfrak{m}(\hbar_3), \tag{39}$$

since $\mathcal{Q}_2(\hbar)$ is nondecreasing, then (38) equals to

$$\begin{aligned}
\mathcal{Q}_2^{\Delta}(\hbar) &= \mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mu(\mathfrak{L}(\xi(\hbar))) \cdot \left(\mathfrak{L}^{\lambda}(\xi(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{J}(\mathcal{P})\kappa(\mathfrak{L}(\mathcal{P}))\Delta\mathcal{P}\right)^{\aleph} \\
&\leq \mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mu(\mathfrak{L}(\xi(\hbar))) \cdot \left(\mathcal{Q}_2^{\lambda}(\xi(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{J}(\mathcal{P})\kappa(\mathcal{Q}_2(\mathcal{P}))\Delta\mathcal{P}\right)^{\gamma} \\
&\leq \mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mu(\mathfrak{L}(\xi(\hbar)))\mathfrak{Z}_1^{\aleph}(\xi(\hbar)),
\end{aligned} \tag{40}$$

where

$$\mathfrak{Z}_1(\xi(\hbar)) = \mathcal{Q}_2^{\lambda}(\xi(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{J}(\mathcal{P})\kappa(\mathcal{Q}_2(\mathcal{P}))\Delta\mathcal{P}, \tag{41}$$

$$\mathfrak{Z}_1(\xi(\hbar_1)) = \mathfrak{m}^{\lambda}(\hbar_3), \mathcal{Q}_2(\xi(\hbar)) \leq \mathfrak{Z}_1(\xi(\hbar)). \tag{42}$$

By the definition of $\mathfrak{Z}_1(\xi(\hbar))$, utilizing (34), (42), and $\mathfrak{Z}_1(\xi(\hbar)) > 0$, we have

$$\mathfrak{Z}_1^{\Delta}(\xi(\hbar))\xi^{\Delta}(\hbar) = \lambda\mathcal{Q}_2^{\lambda-1}\mathcal{Q}_2^{\Delta}(\xi(\hbar))\xi^{\Delta}(\hbar) + \mathcal{J}(\xi(\hbar))\xi^{\Delta}(\hbar)\kappa(\mathcal{Q}_2(\xi\hbar))), \tag{43}$$

so that

$$\begin{aligned}
\mathfrak{Z}_1^{\Delta}(\xi(\hbar)) &\leq \lambda\mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mu(\mathfrak{Z}_1(\xi(\hbar)))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\hbar)) + \mathcal{J}(\xi(\hbar))\kappa(\mathfrak{Z}_1(\xi(\hbar)))\frac{\mathfrak{Z}_1^{\Delta}(\xi(\hbar))}{\kappa(\mathfrak{Z}_1(\xi(\hbar)))} \\
&\leq \frac{\lambda\mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mu(W(\xi(\hbar)))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\hbar))}{\kappa(\mathfrak{Z}_1(\xi(\hbar)))} + \mathcal{J}(\xi(\hbar)).
\end{aligned} \tag{44}$$

Integrating the above inequality, using (34) and (42), we get

$$\begin{aligned} G(\mathfrak{Z}_1(\xi(\hbar))) &\le G\left(\mathbf{m}^{\lambda}(\hbar_3)\right) + \int_{\hbar_1}^{\hbar} \mathcal{J}(\xi(\mathcal{P}))\Delta\mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \frac{\lambda\mathcal{G}(\xi(\mathcal{P}))\xi^{\Delta}(\mathcal{P})\mu(\mathfrak{Z}_1(\xi(\mathcal{P})))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\mathcal{P}))}{\kappa(\mathfrak{Z}_1(\xi(\mathcal{P})))}\Delta\mathcal{P} \\ &\le G\left(\mathbf{m}^{\lambda}(\hbar_3)\right) + \int_{\hbar_1}^{\hbar_2} \mathcal{J}(\xi(\mathcal{P}))\Delta\mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \frac{\lambda\mathcal{G}(\xi(\mathcal{P}))\xi^{\Delta}(\mathcal{P})\mu(\mathfrak{Z}_1(\omega(\mathcal{P})))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\mathcal{P}))}{\kappa(\mathfrak{Z}_1(\xi(\mathcal{P})))}\Delta\mathcal{P}, \end{aligned} \tag{45}$$

for $\hbar < \hbar_3$. Denoting

$$\mathcal{Q}_3(\hbar) = \int_{\hbar_1}^{\hbar} \frac{\lambda\mathcal{G}(\xi(\mathcal{P}))\xi^{\Delta}(\mathcal{P})\mu(\mathfrak{Z}_1(\xi(\mathcal{P})))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\mathcal{P}))}{\kappa(\mathfrak{Z}_1(\xi(\mathcal{P})))}\Delta\mathcal{P}, \tag{46}$$

(45) and (46) give

$$\begin{aligned} \mathfrak{Z}_1(\xi(\hbar)) \le G^{-1}(\mathcal{Q}_3(\hbar)), \mathcal{Q}_3(\hbar_1) &= G\left(\mathbf{m}^{\lambda}(\hbar_3)\right) \\ &\quad + \int_{\hbar_1}^{\hbar_3} \mathcal{J}(\xi(\mathcal{P}))\Delta\mathcal{P}. \end{aligned} \tag{47}$$

Differentiating (46) and applying (47), we observe that

$$\begin{aligned} \mathcal{Q}_3^{\Delta}(\hbar) &= \frac{\lambda\mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mu(\mathfrak{Z}_1(\xi(\hbar)))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\hbar))}{\kappa(\mathfrak{Z}_1(\xi(\hbar)))} \\ &\le \frac{\lambda\mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mu(G^{-1}(\mathcal{Q}_3(\hbar)))G^{-1}\left(\mathcal{Q}_3^{\aleph+\lambda-1}(\hbar)\right)}{\kappa\left(G^{-1}(\mathcal{Q}_3(\hbar))\right)}, \end{aligned} \tag{48}$$

implies

$$\frac{\kappa\left(G^{-1}(\mathcal{Q}_3(\hbar))\right)\mathcal{Q}_3^{\Delta}(\hbar)}{\mu\left(G^{-1}(\mathcal{Q}_3(\hbar))\right)G^{-1}\left(\mathcal{Q}_3^{\aleph+\lambda-1}(\hbar)\right)} \le \lambda\mathcal{G}(\xi(\hbar))\xi^{\Delta}(\hbar). \tag{49}$$

Inequality (49) with integration and (35) and (47) generate the approximation

$$\begin{aligned} \mathcal{P}(\mathcal{Q}_3(\hbar)) &\le \mathcal{P}\left(G\left(\mathbf{m}^{\lambda}(\hbar_3)\right) + \int_{\hbar_1}^{\hbar_3} \mathcal{J}(\xi(\mathcal{P}))\Delta\mathcal{P}\right) \\ &\quad + \lambda\int_{\hbar_1}^{\hbar} \mathcal{G}(\xi(\mathcal{P}))\xi^{\Delta}(\mathcal{P})\Delta\mathcal{P}, \end{aligned} \tag{50}$$

$$\begin{aligned} \mathcal{Q}_3(\hbar) \le \mathcal{P}^{-1}\Bigg(\mathcal{P}\left(G\left(\mathbf{m}^{\lambda}(\hbar_3)\right) + \int_{\hbar_1}^{\hbar_2} \mathcal{J}(\xi(\mathcal{P}))\Delta\mathcal{P}\right) \\ + \lambda\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P}\Bigg). \end{aligned} \tag{51}$$

In view of (47) and (51), we derive

$$\begin{aligned} \mathfrak{Z}_1^{\aleph}(\xi(\hbar)) \le \Bigg(G^{-1}\Bigg[\mathcal{P}^{-1}\Bigg(\mathcal{P}\left(G\left(\mathbf{m}^{\lambda}(\hbar_3)\right) + \int_{\hbar_1}^{\hbar_2} \mathcal{J}(\xi(\mathcal{P}))\Delta\mathcal{P}\right) \\ + \lambda\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P}\Bigg)\Bigg]\Bigg)^{\aleph} = \mathcal{F}(\hbar). \end{aligned} \tag{52}$$

Substitute (52) in (40), integrate from $\hbar_1$ to $\hbar$ in the resultant inequality, employ (36), and put $\hbar = \hbar_3$, we attain

$$\mathcal{Q}_2(\hbar) \le \Sigma^{-1}\left(\Sigma(m(\hbar_3)) + \int_{\hbar_1}^{\xi(\hbar_2)} \mathcal{G}(\mathcal{P})\mathcal{F}(\mathcal{P})\Delta\mathcal{P}\right). \tag{53}$$

The acquired inequality in (32) can be produced by the arbitrary $\hbar_3$ to the last inequality and from (39).

Remark 13. As a special case on $\mathbb{T} = \mathbb{R}$, if we take $\mathcal{L}(\hbar) = u(x)$, $\mathbf{m}(\hbar) = g(x)$, $\mathcal{G}(\hbar) = k(x, t)$, $\mathcal{J} = 0$, $\aleph = 1$, $\mu(\mathcal{L}(\hbar)) = \phi(u(t))$, $\hbar_1 = x_0$, and $\xi(\hbar) = x$, then Theorem 12 changes to Theorem 2.1 of [27] due to Oguntuase.

Theorem 14. *If $\mathcal{G}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, the conditions (H1), (H2), (H4) and*

$$\begin{aligned} \mathfrak{L}^{\aleph+1}(\hbar) \le \mathbf{m}(\hbar) + \left(\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mathfrak{L}^{\aleph}(\mathcal{P})\Delta\mathcal{P}\right)^2 + 2\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mathfrak{L}^{\aleph}(\mathcal{P}) \\ \cdot\left(\mathfrak{L}^{\aleph}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{G}(\mathfrak{x})\mathfrak{L}^{\aleph}(\mathfrak{x})\Delta\mathfrak{x}\right)\Delta\mathcal{P}, \quad \forall\hbar \in \mathbb{T}_0, \end{aligned} \tag{54}$$

hold. Then

$$\mathfrak{L}(\hbar) \le \mathbf{m}^{1/(\gamma+1)}(\hbar) + \frac{2}{\aleph+1}\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Pi(\mathcal{P})e_{\ominus(-\mathcal{G})}(\sigma(\mathcal{P}), \mathcal{P}_1)\Delta\mathcal{P}, \tag{55}$$

provided with $\aleph \in (0, 1)$ is a constant and

$$\Pi(\hbar) = \left[m^{(1-\aleph)/(\aleph+1)}(\hbar) + \frac{2(1-\aleph)}{\aleph+1}\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})e\ominus(-\mathcal{G})\frac{1-\aleph}{\aleph}(\mathcal{P}, \mathcal{P}_1)\Delta\mathcal{P}\right]^{\aleph/(1-\aleph)}, \tag{56}$$

such that

$$\mathfrak{m}^{(1-\aleph)/(\aleph+1)}(\hbar) + \frac{2(1-\aleph)}{\aleph+1}\int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P}) e_{\ominus(-\mathscr{G})}^{(1-\aleph)/\aleph}(\mathscr{P}, \mathscr{P}_1)\Delta\mathscr{P} > 0, \forall \hbar \in \mathbb{T}. \tag{57}$$

Proof. Fix $\hbar_4 \in \mathbb{T}_0$ an arbitrary $\hbar \in [\hbar_1, \hbar_4] \cap \mathbb{T}$ and denote the positive and nondecreasing function $\mathscr{Q}_4(\hbar)$ by

$$\mathscr{Q}_4^{\aleph+1}(\hbar) = \mathfrak{m}(\hbar_4) + \left(\int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}^{\aleph}(\mathscr{P})\Delta\mathscr{P}\right)^2 + 2\int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}^{\aleph}(\mathscr{P}) \cdot \left(\mathfrak{L}^{\aleph}(\mathscr{P}) + \int_{\hbar_1}^{\mathscr{P}} \mathscr{G}(\mathfrak{x})\mathfrak{L}^{\aleph}(\mathfrak{x})\Delta\mathfrak{x}\right)\Delta\mathscr{P}. \tag{58}$$

(54) restates as

$$\mathfrak{L}(\hbar) \le \mathscr{Q}_4(\hbar), \mathfrak{L}(\xi(\hbar)) \le \mathscr{Q}_4(\xi(\hbar)) \le \mathscr{Q}_4(\hbar), \mathscr{Q}_4(\hbar_1) = \mathfrak{m}^{1/(1+\aleph)}(\hbar_4). \tag{59}$$

Differentiating (58) and employing (59), we get

$$(\aleph+1)\mathscr{Q}_4^{\aleph}(\hbar)\mathscr{Q}_4^{\Delta}(\hbar) = 2\left(\int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}^{\aleph}(\mathscr{P})\Delta\mathscr{P}\right)\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{L}^{\aleph}(\xi(\hbar)) + 2\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{L}^{\aleph}(\xi(\hbar)) \cdot \left(\mathfrak{L}^{\aleph}(\xi(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathfrak{L}^{\aleph}(\mathscr{P})\Delta\mathscr{P}\right), \tag{60}$$

which becomes

$$\begin{aligned}\mathscr{Q}_4^{\Delta}(\hbar) &\le \frac{2}{\aleph+1}\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\left(\mathscr{Q}_4^{\aleph}(\hbar) + 2\int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathscr{Q}_4^{\aleph}(\mathscr{P})\Delta\mathscr{P}\right) \\ &\le \frac{2}{\aleph+1}\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}_2(\hbar), \quad \mathfrak{Z}_2(\hbar_1) = \mathfrak{m}^{\aleph/(1+\aleph)}(\hbar_4),\end{aligned} \tag{61}$$

so that

$$\mathfrak{Z}_2(\hbar) = \mathscr{Q}_4^{\aleph}(\hbar) + 2\int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})\mathscr{Q}_4^{\aleph}(\mathscr{P})\Delta\mathscr{P}, \tag{62}$$

$$\mathscr{Q}_4(\hbar) \le \mathfrak{Z}_2^{1/\aleph}(\hbar), \quad \mathscr{Q}_4(\xi(\hbar)) \le \mathfrak{Z}_2^{1/\aleph}(\xi(\hbar)) \le \mathfrak{Z}_2^{1/\aleph}(\hbar). \tag{63}$$

Taking derivative (62) and from (63), we deduce

$$\begin{aligned}\mathfrak{Z}_2^{\Delta}(\hbar) &= \aleph\mathscr{Q}_4^{\aleph-1}(\hbar)\mathscr{Q}_4^{\Delta}(\hbar) + 2\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathscr{Q}_4^{\aleph}(\xi(\hbar)) \\ &\le \frac{2\aleph}{\aleph+1}\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}_2^{(\aleph-1)/\aleph}(\hbar)\mathfrak{Z}_2(\hbar) + 2\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}_2(\hbar) \\ &\le \frac{2\aleph}{\aleph+1}\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}_2^{(2\aleph-1)/\aleph}(\hbar) + 2\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}_2(\hbar),\end{aligned} \tag{64}$$

or equivalently,

$$\mathfrak{Z}_2^{(1-2\aleph)/\aleph}(\hbar)\mathfrak{Z}_2^{\Delta}(\hbar) - 2\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)\mathfrak{Z}_2^{(1-\aleph)/\aleph}(\hbar) \le \frac{2\aleph}{\aleph+1}\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar), \tag{65}$$

which by using Theorem 8, (61), and (63) leads to bound

$$\mathfrak{Z}_2(\hbar) \le e_{\ominus(-\mathscr{G})}(\sigma(\hbar), \hbar_1)\left[\mathfrak{m}^{(1-\aleph)/(\aleph+1)}(\hbar_4) + \frac{2(1-\aleph)}{\aleph+1}\int_{\hbar_1}^{\hbar} \mathscr{G}(\xi(\mathscr{P}))\xi^{\Delta}(\mathscr{P})e_{\ominus(-\mathscr{G})}^{(1-\aleph)/\aleph}(\sigma(\mathscr{P}), \mathscr{P}_1)\Delta\mathscr{P}\right]^{\aleph/(1-\aleph)}, \tag{66}$$

where $\mathscr{G} = -2\mathscr{G}(\hbar)$. From (61) and (66), it is noticed that

$$\mathscr{Q}_4^{\Delta}(\hbar) \le \frac{2}{\aleph+1}\mathscr{G}(\xi(\hbar))\xi^{\Delta}(\hbar)e_{\ominus(-\mathscr{G})}(\sigma(\hbar), \hbar_1)\left[\mathfrak{m}^{(1-\aleph)/(\aleph+1)}(\hbar_4) + \frac{2(1-\aleph)}{\aleph+1}\int_{\hbar_1}^{\xi(\hbar)} \mathscr{G}(\mathscr{P})e_{\ominus(-\mathscr{G})}^{(1-\aleph)/\aleph}(\mathscr{P}, \mathscr{P}_1)\Delta\mathscr{P}\right]^{\aleph/(1-\aleph)}, \tag{67}$$

$\hbar_4 \in \mathbb{T}_0$ is chosen; therefore, the required estimation in (55) can be obtained by integrating the above inequality from $\hbar_1$ to $\hbar$ and then combining the obtained inequality with (59).

4. Enforcement on Theoretical Results

This segment is about to discuss the immediate utilization of Theorem 14 by assessing the boundedness and uniqueness of the retarded nonlinear integrodifferential equation on time scales. For this, let us consider

$$(\gamma+1)\mathfrak{L}^{\gamma}(\hbar)\mathfrak{L}^{\Delta}(\hbar) = \mathscr{H}(\xi(\hbar), \mathfrak{L}(\xi(\hbar))) + \mathscr{A} \cdot \left(\xi(\hbar), \mathfrak{L}(\xi(\hbar)), \int_{\hbar_1}^{\hbar} \mathscr{I}(\mathscr{P}, \mathfrak{L}(\mathscr{P}))\Delta\mathscr{P}\right), \mathfrak{L}(\hbar_1) = \hbar_1, \tag{68}$$

$$\forall \hbar \in \mathbb{T}_0, \text{where } \mathscr{H}, \mathscr{I} : \mathbb{T}_0 \times \mathbb{R} \longrightarrow \mathbb{R}, \mathscr{A} : \mathbb{T}_0 \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \text{ and } \omega(\hbar) \le \hbar, 0 < \omega^{\Delta}(\hbar) \le \hbar, \omega(\hbar_1) = \hbar_1. \tag{69}$$

We suggest the boundedness on the solution of (68) in the first illustration.

For example, if $\mathfrak{L}(\hbar)$ is the solution of (68) with conditions

$$|\mathscr{H}(\hbar, \mathfrak{L}(\hbar))| \le \mathscr{G}^2(\hbar)\left|\mathfrak{L}^{2\aleph}(\hbar)\right|, \tag{70}$$

$$\left|\mathscr{A}\left(\hbar, \mathfrak{L}(\hbar), \int_{\hbar_1}^{\hbar} \mathscr{I}(\mathscr{P}, \mathfrak{L}(\mathscr{P}))\Delta\mathscr{P}\right)\right| \le |\mathscr{I}(\mathscr{P}, \mathfrak{L}(\mathscr{P}))| \cdot \left[\left|\mathfrak{L}^{\aleph}(\hbar)\right| + \int_{\hbar_1}^{\hbar} |\mathscr{I}(\mathscr{P}, \mathfrak{L}(\mathscr{P}))|\Delta\mathscr{P}\right], \tag{71}$$

then

$$|\mathfrak{L}(\hbar)| \le \hbar_1 + \frac{2}{\aleph+1}\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Pi_1(\mathcal{P})e_{\ominus(-\mathcal{G})}(\sigma(\mathcal{P}),\mathcal{P}_1)\Delta\mathcal{P}, \tag{72}$$

where $\mathcal{G}$, $\aleph$, and ξ are the same as in Theorem 14 and

$$\Pi_1(\hbar) = \left[\hbar_1^{1-\aleph} + \frac{2(1-\aleph)}{\aleph+1}\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})e_{!(-\mathcal{G})}^{(1-\aleph)/\aleph}(\mathcal{P},\mathcal{P}_1)\Delta\mathcal{P}\right]^{\aleph/(1-\aleph)}. \tag{73}$$

Proof. Keeping $\hbar = \mathcal{P}$ in (68) and integrating from $\hbar_1$ to $\hbar$, we have

$$\begin{aligned}\mathfrak{L}^{\aleph+1}(\hbar) = \mathfrak{L}^{\aleph+1}(\hbar) + \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}(\xi(\mathcal{P})))\Delta\mathcal{P} \\ + \int_{\hbar_1}^{\hbar} \mathcal{A}\left(\xi(\mathcal{P}),\mathfrak{L}(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}(\mathfrak{x}))\Delta\mathfrak{x}\right)\Delta\mathcal{P},\end{aligned} \tag{74}$$

which with the help of (70) and (71) takes the form

$$\begin{aligned}\left|\mathfrak{L}^{\aleph+1}(\hbar)\right| &= \left|\hbar_1^{\aleph+1} + \int_{\hbar_1}^{\hbar} \mathcal{H}((\xi(\mathcal{P}),\mathfrak{L}(\xi(\mathcal{P})))\Delta\mathcal{P} \right.\\ &\quad \left. + \int_{\hbar_1}^{\hbar} \mathcal{A}(\omega(\mathcal{P}),\mathfrak{L}(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}(\mathfrak{x}))\Delta\mathfrak{x})\Delta\mathcal{P}\right| \\ &\le \left|\hbar_1^{\aleph+1}\right| + \int_{\hbar_1}^{\hbar} |\mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}(\xi(\mathcal{P})))|\Delta\mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \left|\mathcal{A}(\xi(\mathcal{P}),\mathfrak{L}(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}(\mathfrak{x}))\Delta\mathfrak{x}\right)\Big|\Delta\mathcal{P} \\ &\le \hbar_1^{\aleph+1} + \left(\int_{\hbar_1}^{\hbar} \mathcal{G}(\xi(\mathcal{P}))\mathfrak{L}^{\aleph}(\xi(\mathcal{P}))\Delta\mathcal{P}\right)^2 \\ &\quad + 2\int_{\hbar_1}^{\hbar} G(\xi(\mathcal{P}))\mathfrak{L}^{\aleph}(\xi(\mathcal{P})) \\ &\quad \cdot \left[\mathfrak{L}^{\aleph}(\xi(\mathcal{P})) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{G}(\mathfrak{x}) \mid \mathfrak{L}^{\aleph}(\mathfrak{x})\Delta\mathfrak{x}\right].\end{aligned} \tag{75}$$

Let $\bar{\mathcal{G}}(\hbar) = (\mathcal{G}(\hbar))/(\xi^{\Delta}(\xi^{-1}(\hbar)))$, then $\mathcal{G}(\hbar) \le \bar{\mathcal{G}}(\hbar)$ for $\hbar \in \mathbb{T}_0$; therefore, (75) yields

$$\begin{aligned}\left|\mathfrak{L}^{\aleph+1}(\hbar)\right| &\le \hbar_1^{\aleph+1} + \left(\int_{\hbar_1}^{\xi(\hbar)} \bar{\mathcal{G}}(\mathcal{P})\mathfrak{L}^{\aleph}(\mathcal{P})\Delta\mathcal{P}\right)^2 \\ &\quad + 2\int_{\hbar_1}^{\xi(\hbar)} \bar{\mathcal{G}}(\mathcal{P})\mathfrak{L}^{\aleph}(\mathcal{P})\left[\mathfrak{L}^{\aleph}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \bar{\mathcal{G}}(\mathfrak{x})\mathfrak{L}^{\aleph}(\mathfrak{x})\Delta\mathfrak{x}\right].\end{aligned} \tag{76}$$

The requisite bound (72) can be accomplished by the reasonable implementation of Theorem 14 with some alterations in the previous inequality; hence, here, we remove the information.

The second example is based on the uniqueness on the solution of (66).

For example, we list the following hypotheses as below

$$|\mathcal{H}(\hbar,\mathfrak{L}_1(\hbar)) - \mathcal{H}(\hbar,\mathfrak{L}_2(\hbar))| \le \mathcal{G}^2(\hbar)\left|\mathfrak{L}_1^{2\aleph}(\hbar) - \mathfrak{L}_2^{2\aleph}(\hbar)\right|, \tag{77}$$

$$\begin{aligned}&\left|\mathcal{A}\left(\hbar,\mathfrak{L}_1(\hbar),\int_{\hbar_1}^{\hbar} \mathcal{I}(\mathcal{P},\mathfrak{L}_1(\mathcal{P}))\Delta\mathcal{P}\right)\right. \\ &\quad \left. - \mathcal{A}\left(\hbar,\mathfrak{L}_2(\hbar),\int_{\hbar_1}^{\hbar} \mathcal{I}(\mathcal{P},\mathfrak{L}_2(\mathcal{P}))\Delta\mathcal{P}\right)\right| \\ &\le |\mathcal{I}(\mathcal{P},\mathfrak{L}_1(\mathcal{P})) - \mathcal{I}(\mathcal{P},\mathfrak{L}_2(\mathcal{P}))| \\ &\quad \times \left[\left|\mathfrak{L}_1^{\aleph}(\hbar) - \mathfrak{L}_2^{\aleph}(\hbar)\right| + \int_{\hbar_1}^{\hbar} |\mathcal{I}(\mathcal{P},\mathfrak{L}_1(\mathcal{P})) - \mathcal{I}(\mathcal{P},\mathfrak{L}_2(\mathcal{P}))|\Delta\mathcal{P}\right],\end{aligned} \tag{78}$$

then (68) has at most one solution.

Proof. The solutions $\mathfrak{L}_1(\hbar)$ and $\mathfrak{L}_2(\hbar)$ of (68) transform into

$$\begin{aligned}\mathfrak{L}_1^{\aleph+1}(\hbar) - \mathfrak{L}_2^{\aleph+1}(\hbar) &= \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}_1(\xi(\mathcal{P})))\Delta\mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \mathcal{A}\left(\xi(\mathcal{P}),\mathfrak{L}_1(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}_1(\mathfrak{x}))\Delta\mathfrak{x}\right)\Delta\mathcal{P} \\ &\quad - \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}_2(\xi(\mathcal{P})))\Delta\mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \mathcal{A}\left(\xi(\mathcal{P}),\mathfrak{L}_2(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}_2(\mathfrak{x}))\Delta\mathfrak{x}\right)\Delta\mathcal{P}.\end{aligned} \tag{79}$$

Equivalently

$$\begin{aligned}\left|\mathfrak{L}_1^{\aleph+1}(\hbar) - \mathfrak{L}_2^{\aleph+1}(\hbar)\right| &\le \left|\int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}_1(\xi(\mathcal{P})))\Delta\mathcal{P}\right. \\ &\quad + \int_{\hbar_1}^{\hbar} \mathcal{A}(\xi(\mathcal{P}),\mathfrak{L}_1(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}_1(\mathfrak{x}))\Delta\mathfrak{x})\Delta\mathcal{P} \\ &\quad - \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}_2(\xi(\mathcal{P})))\Delta\mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \mathcal{A}(\xi(\mathcal{P}),\mathfrak{L}_2(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}_2(\mathfrak{x}))\Delta\mathfrak{x})\Delta\mathcal{P} \\ &\le \int_{\hbar_1}^{\hbar} |\mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}_1(\xi(\mathcal{P}))) - \mathcal{H}(\xi(\mathcal{P}),\mathfrak{L}_2(\xi(\mathcal{P})))|\Delta\mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} |\mathcal{A}(\xi(\mathcal{P}),\mathfrak{L}_1(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}_1(\mathfrak{x})))\Delta\mathfrak{x}) \\ &\quad - \mathcal{A}(\xi(\mathcal{P}),\mathfrak{L}_2(\xi(\mathcal{P})),\int_{\hbar_1}^{\mathcal{P}} \mathcal{I}(\mathfrak{x},\mathfrak{L}_2(\mathfrak{x}))\Delta\mathfrak{x})|\Delta\mathcal{P}.\end{aligned} \tag{80}$$

Applying (77) and (78) in the above inequality, we get

$$\left|\mathfrak{L}_1^{\aleph+1}(\hbar) - \mathfrak{L}_2^{\aleph+1}(\hbar)\right| \leq \left(\int_{\hbar_1}^{\xi(\hbar)} \bar{\mathscr{G}}(\mathscr{P})\left(\mathfrak{L}_1^{\aleph}(\mathscr{P}) - \mathfrak{L}_2^{\aleph}(\mathscr{P})\right)\Delta\mathscr{P}\right)^2 + 2\int_{\hbar_1}^{\xi(\hbar)} \bar{\mathscr{G}}(\mathscr{P})\left(\mathfrak{L}_1^{\aleph}(\mathscr{P}) - \mathfrak{L}_2^{\aleph}(\mathscr{P})\right) \times \left[\left(\mathfrak{L}_1^{\aleph}(\mathscr{P}) - \mathfrak{L}_2^{\aleph}(\mathscr{P})\right) + \int_{\hbar_1}^{\mathscr{P}} \bar{\mathscr{G}}(\mathfrak{x})\left(\mathfrak{L}_1^{\aleph}(\mathfrak{x}) - \mathfrak{L}_2^{\aleph}(\mathfrak{x})\Delta\mathfrak{x}\right]. \tag{81}$$

The last inequality by making few modifications to the Theorem 14 for the function

$$\left|\mathfrak{L}_1^{\aleph+1}(\hbar) - \mathfrak{L}_2^{\aleph+1}(\hbar)\right|, \tag{82}$$

induces

$$\left[\mathfrak{L}_1^{\aleph+1}(\hbar) - \mathfrak{L}_2^{\aleph+1}(\hbar)\right] \leq 0. \tag{83}$$

Thus, $\mathfrak{L}_1(\hbar) = \mathfrak{L}_2(\hbar)$, and there exists at least one solution of (68).

5. Conclusion

Unlike some proven and defined inequalities in the literature, Theorem 10, Theorem 12, and Theorem 14 have examined some dynamic integral inequalities of the Gronwall-Bellman form in a single independent variable with a retarded and nonlinear term that can be used to overcome the qualitative properties of integral equations. Our observations may be used to solve the difficulty of measuring the explicit bounds of undefined functions and to expand and unify continuous inequalities by the use of basic technologies. We believe that the findings obtained here are of a general kind and offer many contributions to statistical data and are useful to identify the existence and uniqueness of the integrodifferential equations. As should be obvious, the provided results present a helpful resource in the study of solutions of certain delay dynamic equations on time scales. The inequalities that have been created unify some known continuous and discrete inequalities. One can say that it will be attractive for the researchers to generalize our results for further exploration.

Data Availability

The data are available on request.

Conflicts of Interest

The authors declare that there are no competing interests.

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Research Article

Strong Converse Results for Linking Operators and Convex Functions

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We consider a family $B_{n,\rho}^{[c]}$ of operators which is a link between classical Baskakov operators (for $\rho = \infty$) and their genuine Durrmeyer type modification (for $\rho = 1$). First, we prove that for fixed n, c and a fixed convex function f, $B_{n,\rho}^{[c]} f$ is decreasing with respect to ρ. We give two proofs, using various probabilistic considerations. Then, we combine this property with some existing direct and strong converse results for classical operators, in order to get such results for the operators $B_{n,\rho}^{[c]}$ applied to convex functions.

1. Introduction

The Baskakov-type operators depending on a real parameter c were introduced by Baskakov in [1]. This class of operators includes the classical Bernstein, Szász-Mirakjan, and Baskakov operators as special cases for $c = -1$, $c = 0$, and $c = 1$, respectively.

Let $c \in \mathbb{R}$, $n \in \mathbb{R}$, $n > c$ for $c \geq 0$, and $-n/c \in \mathbb{N}$ for $c < 0$. Furthermore, let $I_c = [0,\infty)$ for $c \geq 0$ and $I_c = [0,-1/c]$ for $c < 0$. Consider $f : I_c \to \mathbb{R}$ given in such a way that the corresponding integrals and series are convergent.

The Baskakov-type operators are defined as follows:

$$\left(B_{n,\infty}^{[c]} f\right)(x) = \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) f\left(\frac{j}{n}\right), \quad (1)$$

where

$$p_{n,j}(x) = \begin{cases} \dfrac{n^j}{j!} x^j e^{-nx} & , c = 0, \\ \dfrac{n^{c,\bar{j}}}{j!} x^j (1+cx)^{-(n/c+j)} & , c \neq 0, \end{cases} = \begin{cases} \dfrac{1}{n-c} \dfrac{(-c)^{j+1}}{B(j+1, -n/c-j+1)} x^j (1+cx)^{-(n/c+j)} & , c < 0, \\ \dfrac{n^j}{\Gamma(j+1)} x^j e^{-nx} & , c = 0, \\ \dfrac{1}{n-c} \dfrac{c^{j+1}}{B(j+1, n/c-1)} x^j (1+cx)^{-(n/c+j)} & , c > 0, \end{cases} \quad (2)$$

and $a^{c,\bar{j}} := \prod_{l=0}^{j-1}(a+cl)$, $a^{c,\bar{0}} := 1$.

We remark that (2) is well defined also for $j \in \mathbb{R}$, $j \geq 0$, which will be considered below.

The genuine Baskakov-Durrmeyer-type operators are given by

$$\left(B_{n,1}^{[c]}f\right)(x) = f(0)p_{n,0}^{[c]}(x) + f\left(-\frac{1}{c}\right)p_{n,-n/c}^{[c]}(x) + \sum_{j=1}^{-n/c-1} p_{n,j}^{[c]}(x)(n+c)\int_0^{-1/c} p_{n+2c,j-1}^{[c]}(t)f(t)dt, for c < 0,$$

$$\left(B_{n,1}^{[c]}f\right)(x) = f(0)p_{n,0}^{[c]}(x) + \sum_{j=1}^{\infty} p_{n,j}^{[c]}(x) \cdot (n+c)\int_0^{\infty} p_{n+2c,j-1}^{[c]}(t)f(t)dt, for c \geq 0. \quad (3)$$

In the last years, a nontrivial link between classical Baskakov-type operators and their genuine Durrmeyer-type modification came into the focus of research. Depending on a parameter $\rho \in \mathbb{R}^+$, the linking operators are given by

$$\left(B_{n,1}^{[c]}f\right)(x) = \sum_{j=0}^{\infty} F_{n,j,\rho}^{[c]}(f)p_{n,j}^{[c]}(x), \quad (4)$$

where

$$F_{n,j,\rho}^{[c]}(f) = \begin{cases} f(0), & j=0, c \in \mathbb{R}, \\ f\left(-\frac{1}{c}\right). & j = -\frac{n}{c}, c < 0, \\ \int_{I_c} \mu_{n,j,\rho}^{[c]}(t)f(t)dt, & \text{otherwise}, \end{cases} \quad (5)$$

with $\mu_{n,j,\rho}^{[c]}(t) = (n\rho + c)p_{n\rho+2c,j\rho-1}^{[c]}(t)$.

For $c < 0$, $B_{n,\rho}^{[c]}f$ is well defined if $f \in L_1[0,1]$ with finite limits at the endpoints of the interval $[0,-1/c]$, i.e., $f(0) = \lim_{x\to 0^+} f(x)$ and $f(-1/c) = \lim_{x\to -1/c^-} f(x)$.

For $c \geq 0$, the operators $B_{n,\rho}^{[c]}$ are well defined for functions $f \in W_n^\rho$ having a finite limit $f(0) = \lim_{x\to 0^+} f(x)$ where W_n^ρ denotes the space of functions $f \in L_{1,loc}[0,\infty)$ satisfying certain growth conditions, i. e., there exist constants $M > 0$, $0 \leq q < n\rho + c$, such that a. e. on $[0,\infty)$.

$$|f(t)| \leq Me^{qt} f or c = 0,$$
$$|f(t)| \leq Mt^{q/c} f or c > 0. \quad (6)$$

First, we prove that for fixed n, c and a fixed convex function f, $B_{n,\rho}^{[c]}f$ is decreasing with respect to ρ. We give two proofs, using various probabilistic considerations. Then, we combine this property with some existing direct and strong converse results for classical operators, in order to get such results for the operators $B_{n,\rho}^{[c]}$ applied to convex functions.

2. The Case $c = -1$

For the linking Bernstein operator, i.e., $c = -1$, Rasa and Sta nila [2], (10) proved that for a convex function $f \in C[0,1]$,

$$B_{n,\rho}^{[-1]}(f;x) \geq B_{n,\sigma}^{[-1]}(f;x) \geq f(x), 1 \leq \rho < \sigma \leq \infty. \quad (7)$$

For the proof, they used that $B_{n,\rho}$ can be written as a combination of the classical Bernstein operator and Beta operator and some corresponding results for the Beta operator from Adell et al. [3], Theorem 1. For the case $\rho = 1$ and the case $\rho = \infty$, strong converse results are known [4], Theorem 1.1, [5], p.117 [6], and [7], Theorem 3.2, Theorem 5:

$$\left\|B_{n,\rho}^{[-1]}f - f\right\|_\infty \sim \omega_\varphi^2\left(f, n^{-1/2}\right), \rho \in \{1,\infty\}, \quad (8)$$

where (see [5])

$$\omega_\varphi^2(f,t) := \sup_{0<h\leq t}\left\|\Delta_{h\varphi}^2 f\right\|, \quad (9)$$

with φ^2 a weight function and $\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h)$.

Thus, for f convex, $1 \leq \rho \leq \infty$

$$0 \leq B_{n,\infty}^{[-1]}(f,x) - f(x) \leq B_{n,\rho}^{[-1]}(f;x) - f(x) \leq B_{n,1}^{[-1]}(\mathrm{f};x) - f(x), \quad (10)$$

leading to

$$C_1^{-1}\omega_\varphi^2\left(f,n^{-1/2}\right) \leq \left\|B_{n,\infty}^{[-1]}f - f\right\| \leq \left\|B_{n,\rho}^{[-1]}f - f\right\| \leq \left\|B_{n,1}^{[-1]}f - f\right\| \leq C_1\omega_\varphi^2\left(f,n^{-1/2}\right), \quad (11)$$

i.e.,

$$\left\|B_{n,\rho}^{[-1]}f - f\right\|_\infty \sim \omega_\varphi^2\left(f,n^{-1/2}\right), 1 \leq \rho \leq \infty. \quad (12)$$

3. The Case $c = 0$

Consider the classical Szász-Mirakjan operators

$$B_{n,\infty}^{[0]}(f;x) := \sum_{j=0}^{\infty} p_{n,j}^{[0]}(x)f\left(\frac{j}{n}\right) \quad (13)$$

and also the operators

$$\mathcal{S}_r(f;x) = \begin{cases} \dfrac{r^{rx}}{\Gamma(rx)}\displaystyle\int_0^\infty t^{rx-1}e^{-rt}f(t)dt, & x > 0, \\ f(0), & x = 0, \end{cases} \quad (14)$$

where $r > 0$.

Moreover, for $r>0$, let (see [8]).

$$G_r(f;x)=\begin{cases}\frac{(r/x)^r}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-rt/x}f(t)dt, & x>0,\\ f(0), & x=0.\end{cases} \tag{15}$$

Theorem 1 (see [8], Theorem 5 and Remark 6). *Let f and x be fixed and f convex, such that $G_r(|f|;x)<\infty$ for all $r>0$. Then, $G_r(f;x)$ is nonincreasing with respect to r.*

Then,

$$\mathcal{S}_r(f;x)=G_{rx}(f;x). \tag{16}$$

For $c=0$,

$$B_{n,\rho}^{[0]}=B_{n,\infty}^{[0]}\circ\mathcal{S}_{n\rho}. \tag{17}$$

Let f be convex and n and x be fixed, such that $G_r(|f|;x)<\infty$, for all $r>0$.

Let $1\le\rho\le\sigma$. Then, by (16) and Theorem 1,

$$\mathcal{S}_{n\sigma}(f;x)=G_{n\sigma x}(f;x)\le G_{n\rho x}(f;x)=\mathcal{S}_{n\rho}(f;x). \tag{18}$$

Now by (17),

$$B_{n,\sigma}^{[0]}f=B_{n,\infty}^{[0]}(\mathcal{S}_{n\sigma})f\le B_{n,\infty}^{[0]}(\mathcal{S}_{n\rho})f=B_{n,\rho}^{[0]}f. \tag{19}$$

Thus,

$$B_{n,\sigma}^{[0]}f\le B_{n,\rho}^{[0]}f. \tag{20}$$

Strong converse results are known also in this case (see [6], Theorem 1.2 and [9], Theorem 5.1, Theorem 5.2):

$$\left\|B_{n,\rho}^{[0]}f-f\right\|_\infty\sim\omega_\varphi^2\left(f,n^{-1/2}\right),\ \rho\in\{1,\infty\}. \tag{21}$$

Thus, for f convex,

$$\left\|B_{n,\rho}^{[0]}f-f\right\|_\infty\sim\omega_\varphi^2\left(f,n^{-1/2}\right),\ 1\le\rho\le\infty. \tag{22}$$

4. The Case $c=1$

To treat this case, we need some preliminaries.

If X and Y are independent random variables with densities $f(\theta)$, $g(\theta)$, and $\theta>0$, then X/Y has density

$$w(t)=\int_0^\infty f(t\theta)g(\theta)\theta d\theta. \tag{23}$$

Let $(U_\tau)_{\tau\ge0}$ and $(V_\tau)_{\tau\ge0}$ be two independent gamma processes (see [8], p.129), i. e., U_τ has density $1/\Gamma(\tau)\theta^{\tau-1}e^{-\theta}$, $\theta>0$. Let $x>0$ and $r>0$. Then, $Z_r^x:=U_{rx}/V_{r+1}$ has density

$$\begin{aligned}w_{r,x}(t)&=\int_0^\infty\frac{1}{\Gamma(rx)}(t\theta)^{rx-1}e^{-t\theta}\frac{1}{\Gamma(r+1)}\theta^{r+1}e^{-\theta}d\theta\\&=\frac{t^{rx-1}}{\Gamma(rx)\Gamma(r+1)}\int_0^\infty\theta^{rx+r}e^{-\theta(t+1)}d\theta,\ \text{for } t>0.\end{aligned} \tag{24}$$

Substitute $s=\theta(t+1)$. Then,

$$\begin{aligned}w_{r,x}(t)&=\frac{t^{rx-1}}{\Gamma(rx)\Gamma(r+1)}\int_0^\infty\frac{s^{rx+r}}{(t+1)^{rx+r}}e^{-s}\frac{ds}{t+1}\\&=\frac{t^{rx-1}}{(t+1)^{rx+r+1}}\frac{\Gamma(rx+r+1)}{\Gamma(rx)\Gamma(r+1)}\\&=\frac{1}{B(rx,r+1)}\frac{t^{rx-1}}{(t+1)^{rx+r+1}}.\end{aligned} \tag{25}$$

Let $\mathcal{B}_r(f;x)=\int_0^\infty w_{r,x}(t)f(t)dt$.

Consequently,

$$\mathcal{B}_r(f;x)=Ef\left(\frac{U_{rx}}{V_{r+1}}\right)=Ef(Z_r^x), \tag{26}$$

compare with [8], (9).

As in [10], Proof of Lemma 2, let $1\le r\le s$, $x>0$. Since the random vectors (U_{rx},U_{sx}) and (V_{r+1},V_{s+1}) are independent, we have

$$\begin{aligned}E(Z_r^x|U_{sx},V_{s+1})&=E(U_{rx}\mid U_{sx})E\left(V_{r+1}^{-1}\mid V_{s+1}\right)\\&=\frac{r}{s}U_{sx}E\left(V_{r+1}^{-1}\mid V_{s+1}\right),\end{aligned} \tag{27}$$

with [10], Lemma 1. Moreover, as in [10], (19), we get

$$E\left(V_{r+1}^{-1}\mid V_{s+1}\right)=V_{s+1}^{-1}\cdot\frac{s}{r}. \tag{28}$$

Thus,

$$E(Z_r^x|U_{sx},V_{s+1})=\frac{U_{sx}}{V_{s+1}}=Z_s^x. \tag{29}$$

As at the end of [10], Proof of Lemma 2, taking here the conditional expectation w.r.t. Z_s^x, we get

$$E(Z_r^x\mid Z_s^x)=Z_s^x,\ a.s., 1\le r\le s,\ x>0. \tag{30}$$

Now (30) is exactly the assumption of [8], Theorem 5 (a). Accordingly, [8], Theorem 5 (a) and Remark 6 show that

$$\mathcal{B}_rf\ge\mathcal{B}_sf,\ 1\le r\le s,\ f\text{ convex}. \tag{31}$$

This implies

$$\begin{aligned}&\mathcal{B}_{n\rho}f\ge\mathcal{B}_{n\sigma}f,\ 1\le\rho\le\sigma,\ f\text{ convex},\\&B_{n,\infty}^{[1]}\mathcal{B}_{n\rho}f\ge B_{n,\infty}^{[1]}\mathcal{B}_{n\sigma}f,\ 1\le\rho\le\sigma,\ f\text{ convex},\end{aligned} \tag{32}$$

where $B_{n,\infty}^{[1]}$ are the classical Baskakov operators.

Since, $B^{[1]}_{n,\rho} = B^{[1]}_{n,\infty} \circ \mathscr{B}_{n\rho}$, we infer that

$$B^{[1]}_{n,\sigma} f \le B^{[1]}_{n,\rho} f, \text{fconvex}. \tag{33}$$

The direct and strong converse results are known also in this case (see [11], Theorem 1.2, Theorem 1.3 and [12], Theorem 1.1):

$$\left\| B^{[1]}_{n,\rho} f - f \right\|_{\infty} \sim \omega^2_{\varphi}\left(f, n^{-1/2}\right), \rho \in \{1, \infty\}. \tag{34}$$

Thus, for f convex,

$$\left\| B^{[1]}_{n,\rho} f - \mathrm{f} \right\|_{\infty} \sim \omega^2_{\varphi}\left(f, n^{-1/2}\right), 1 \le \rho \le \infty. \tag{35}$$

5. An Application of Ohlin's Lemma

For more details about the techniques used in this section, the reader is referred to [13] and the references therein.

Lemma 2 (Ohlin's Lemma) (see [14]). *Let X and Y be two random variables on the same probability space such that $E X = EY$. If the distribution functions F_X and F_Y cross exactly one time, i.e., for some x_0 holds*

$$F_X(x) \le F_Y(x) if x < x_0 and F_X(x) \ge F_Y(x) if x > x_0, \tag{36}$$

then $Ef(X) \le Ef(Y)$, for all convex functions $f : \mathbb{R} \to \mathbb{R}$.

We have $\int_{I_c} \mu^{[c]}_{n,j,\rho}(t)dt = 1$ and $\int_{I_c} t\mu_{n,j,\rho}(t)dt = j/n$.

Therefore, $\mu^{[c]}_{n,j,\rho}$ is the probability density function of a random variable $X^{[c]}_{n,j,\rho}$ with expectation $EX^{[c]}_{n,j,\rho} = j/n$ and probability distribution function $G^{[c]}_{n,j,\rho}(x) = \int_0^x \mu^{[c]}_{n,j,\rho}(t)dt$.

Let $\rho < \sigma$. We will apply Ohlin's Lemma to the random variables $X^{[c]}_{n,j,\rho}$ and $X^{[c]}_{n,j,\sigma}$. Since their expectation is equal, we have to prove that $G^{[c]}_{n,j,\rho}$ and $G^{[c]}_{n,j,\sigma}$ cross exactly ones.

Let

$$g(x) = G^{[c]}_{n,j,\rho}(x) - G^{[c]}_{n,j,\sigma}(x), x \in I_c. \tag{37}$$

Then, $g'(x) = \mu^{[c]}_{n,j,\rho}(x) - \mu^{[c]}_{n,j,\sigma}(x)$.

In what follows, we suppose $c \ne 0$; the case $c = 0$ can be treated similarly or we can consider $c \to 0$ in the computations below. For $c \ne 0$, we have

$$\begin{aligned} g'(x) &= K_1 x^{j\rho-1}(1+cx)^{-(np/c+j\rho+1)} - K_2 x^{j\sigma-1}(1+cx)^{-(n\sigma/c+j\sigma+1)} \\ &= K_2 x^{j\rho-1}(1+cx)^{-\left(np/c\frac{n\rho}{c}+j\rho+1\right)} \left\{ \frac{K_1}{K_2} - \left[x^j (1+cx)^{-(n/c+j)} \right]^{\sigma-\rho} \right\} \end{aligned} \tag{38}$$

with positive constants K_1 and K_2.

On $\operatorname{int}(I_c)$, the equation $g'(x) = 0$ is equivalent to $(K_1/K_2)^{1/\sigma-\rho}(1+cx)^{n/c+j} = x^j$.

First, suppose that $j > 0$ and, if $c < 0$, $j < -n/c$. Then, on $\operatorname{int}(I_c)$, the equation $g'(x) = 0$ is equivalent to $h(x) = x$, where

$$h(x) := \left(\frac{K_1}{K_2}\right)^{1/j(\sigma-\rho)} (1+cx)^{1+n/cj}, x \in \operatorname{int}(I_c) \tag{39}$$

is a strictly convex function. The equation $h(x) = x$ has at most two roots in $\operatorname{int}(I_c)$, and so the derivative g' has at most two zeroes in $\operatorname{int}(I_c)$. If $c < 0$, $g(0) = g(-1/c) = 0$; if $c > 0$, $g(0) = \lim_{x\to\infty} g(x) = 0$. Therefore, g' has at least one zero in $\operatorname{int}(I_c)$.

Suppose that g' has exactly one zero in $\operatorname{int}(I_c)$, let it be x_0. Then, g' has opposite signs in the two intervals determined by x_0. But (38) shows that g' is positive near the endpoints of I_c. This contradiction leads us to the conclusion that g' has exactly two zeroes $x_1 < x_2$ in $\operatorname{int}(I_c)$; they are also roots of the equation $x = h(x)$, $h(x)$ being a strictly convex function. Moreover, g' is positive outside of (x_1, x_2) and negative inside it. Therefore, $g(x)$ is strictly increasing for $x < x_1$ and for $x > x_2$, and strictly decreasing for $x_1 < x < x_2$, with $g(x_1) > 0$ and $g(x_2) < 0$.

We conclude that there exists $x_0 \in (x_1, x_2)$ with $g(x_0) = 0$, $g(x) \ge 0$ for $x < x_0$, and $g(x) \le 0$ for $x > x_0$. Remembering that $g(x) = G^{[c]}_{n,j,\rho}(x) - G^{[c]}_{n,j,\sigma}(x)$, we see that

$$\begin{aligned} G^{[c]}_{n,j,\sigma}(x) &\le G^{[c]}_{n,j,\rho}(x), \ x < x_0, \\ G^{[c]}_{n,j,\sigma}(x) &\ge G^{[c]}_{n,j,\rho}(x), \ x > x_0. \end{aligned} \tag{40}$$

Now Ohlin's Lemma shows that

$$Ef\left(X^{[c]}_{n,j,\sigma}\right) \le Ef\left(X^{[c]}_{n,j,\rho}\right), \tag{41}$$

i.e.,

$$\int_{I_c} \mu^{[c]}_{n,j,\sigma}(t) f(t) dt \le \int_{I_c} \mu^{[c]}_{n,j,\rho}(t) f(t) dt, \tag{42}$$

for f convex, $j > 0$, and if $c < 0, j < -n/c$.

With notation from (5), this means that

$$F^{[c]}_{n,j,\sigma}(f) \le F^{[c]}_{n,j,\rho}(f), f \text{convex}. \tag{43}$$

Moreover, (5) shows that the above relation is an equality if $j = 0$ and, for $c < 0$, if $j = -n/c$. Now according to (4),

$$B^{[c]}_{n,\sigma} f \le B^{[c]}_{n,\rho} f, f \text{convex}, 0 < \rho < \sigma. \tag{44}$$

Letting $\sigma \to \infty$, we see that the above inequality is valid for $0 < \rho < \sigma \le \infty$. This was proved with other methods, for $c \in \{-1, 0, 1\}$, in the preceding sections.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no competing financial interests.

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Research Article

Global Existence of Solutions to a System of Integral Equations Related to an Epidemic Model

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A system of integral equations related to an epidemic model is investigated. Namely, we derive sufficient conditions for the existence and uniqueness of global solutions to the considered system. The proof is based on Perov's fixed point theorem and some integral inequalities.

1. Introduction

Many phenomena related to infectious diseases can be modeled as an integral equation (see e.g., [1–4] and the references therein). In [3], Gripenberg investigated the large time behavior of solutions to the integral equation

$$x(t) = k\left(p(t) - \int_0^t A(t-s)x(s)\,ds\right)\left(f(t) + \int_0^t a(t-s)x(s)\,ds\right), t \geq 0, \tag{1}$$

which arises in the study of the spread of an infectious disease that does not induce permanent immunity. Namely, sufficient conditions were provided so that (1) admits nonnegative, continuous, and bounded solution. Using the comparison method and some integral estimates, Pachpatte [5] established the convergence of solutions to (1) to 0 as $t \longrightarrow \infty$. In [6], Brestovanská studied the integral equation

$$\begin{aligned} x(t) = &\left(g_1(t) + \int_0^t A_1(t-s)F_1(s, x(s))\,ds\right)\cdots \\ &\cdot\left(g_p(t) + \int_0^t A_p(t-s)F_p(s, x(s))ds\right), \end{aligned} \tag{2}$$

for all $t \geq 0$. Namely, sufficient criteria for the global existence and uniqueness of global solutions to (2) were derived. Moreover, under certain conditions, the convergence of solutions to (2) to 0 as $t \longrightarrow \infty$ was proved. In [7], using weakly Picard technique operators in a gauge space, Olaru investigated the qualitative behavior of solutions to the integral equation

$$x(t) = \left(g_1(t) + \int_0^t K_1(t, s, x(s))ds\right)\left(g_2(t) + \int_0^t K_2(t, s, x(s))ds\right), t \geq 0. \tag{3}$$

In this paper, we consider the system of integral equations

$$\begin{cases} x(t) &= \prod_{i=1}^{2}\left(f_i(t) + \int_0^t A_i(t-s)F_i(s, x(s), y(s))\,ds\right), t \geq 0, \\ y(t) &= \prod_{i=1}^{2}\left(g_i(t) + \int_0^t B_i(t-s)G_i(s, x(s), y(s))\,ds\right), t \geq 0, \end{cases} \tag{4}$$

where $f_i, A_i, g_i, B_i \in C([0,\infty))$ and $F_i, G_i \in C([0,\infty) \times \mathbb{R} \times \mathbb{R})$. Namely, we are concerned with the global existence of solutions to the considered system. Using Perov's fixed point theorem, sufficient conditions are derived for which the system (4) admits one and only one continuous global solution.

The rest of the paper is organized as follows. In Section 2, we recall some notions on fixed point theory including Perov's fixed point theorem. In Section 3, we state and prove our main result.

2. Preliminaries

Let n be a positive natural number and define the partial order $\preceq_n$ in $\mathbb{R}^n$ by

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \preceq_n z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \Leftrightarrow y_i \leq z_i, i = 1, 2, \cdots, n, \tag{5}$$

for all $y, z \in \mathbb{R}^n$. We denote by $0_{\mathbb{R}^n}$ the zero vector in $\mathbb{R}^n$, i.e.,

$$0_{\mathbb{R}^n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6}$$

Let S be a nonempty set and $d : S \times S \longrightarrow S$ be a given mapping. We say that d is a vector-valued metric on S (see, e.g., [8]), if for all $x, y, z \in S$,

(i) $0_{\mathbb{R}^n} \preceq_n d(x, y)$

(ii) $d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \preceq_n d(x, y) + d(y, z)$

In this case, we say that (S, d) is a generalized metric space. In such spaces, the notions of convergent sequence, Cauchy sequence, and completeness are similar to those for usual metric spaces.

Let $M_n(\mathbb{R}_+)$ be set of square matrices of size n with nonnegative coefficients. Given $\mathscr{K} \in M_n(\mathbb{R}_+)$, we denote by $\rho(\mathscr{K})$ its spectral radius.

Lemma 1 (Perov's fixed point theorem, see [9]). *Let (S, d) be a complete generalized metric space and $\mathscr{H} : S \longrightarrow S$ be a given mapping. Suppose that there exists $\mathscr{K} \in M_n(\mathbb{R}_+)$ with $\rho(\mathscr{K}) < 1$ such that*

$$d(\mathscr{H}(x), \mathscr{H}(y)) \preceq_n \mathscr{K} d(x, y), \tag{7}$$

for all $x, y \in S$. Then, the mapping $\mathscr{H}$ admits a unique fixed point in S.

3. Global Existence

The system (4) is investigated under the following assumptions:

(i) $f_i, A_i, g_i, B_i \in C([0,\infty))$ and $F_i, G_i \in C([0,\infty) \times \mathbb{R} \times \mathbb{R})$, $i = 1, 2$

(ii) f_i, g_i, $i = 1, 2$, are bounded functions

(iii) $A_i, B_i \in L^1((0,\infty)) \cap L^\varsigma((0,\infty))$, $i = 1, 2$, for some $\varsigma > 1$

(iv) For all $i = 1, 2$, there exist positive constants $L_1^{(i)}$ and $L_2^{(i)}$ such that

$$|F_i(t, u, v) - F_i(t, \bar{u}, \bar{v})| \leq L_1^{(i)} |u - \bar{u}| + L_2^{(i)} |v - \bar{v}|, \tag{8}$$

for all $t \geq 0$ and $(u, v), (\bar{u}, \bar{v}) \in \mathbb{R}^2$

(v) For all $i = 1, 2$, there exist positive constants $Q_1^{(i)}$, $Q_2^{(i)}$, $\ell_1^{(i)}$, and $\ell_2^{(i)}$ such that

$$|F_i(t, u, v)| \leq Q_1^{(i)} \left| u \right|^{\ell_1^{(i)}} + Q_2^{(i)} \left| v \right|^{\ell_2^{(i)}}, \tag{9}$$

for all $t \geq 0$ and $(u, v) \in \mathbb{R}^2$

(vi) For all $i = 1, 2$, there exist positive constants $M_1^{(i)}$ and $M_2^{(i)}$ such that

$$|G_i(t, u, v) - G_i(t, \bar{u}, \bar{v})| \leq M_1^{(i)} |u - \bar{u}| + M_2^{(i)} |v - \bar{v}|, \tag{10}$$

for all $t \geq 0$ and $(u, v), (\bar{u}, \bar{v}) \in \mathbb{R}^2$

(vii) For all $i = 1, 2$, there exist positive constants $P_1^{(i)}$, $P_2^{(i)}$, $\sigma_1^{(i)}$, and $\sigma_2^{(i)}$ such that

$$|G_i(t, u, v)| \leq P_1^{(i)} \left| u \right|^{\sigma_1^{(i)}} + P_2^{(i)} \left| v \right|^{\sigma_2^{(i)}}, \tag{11}$$

for all $t \geq 0$ and $(u, v) \in \mathbb{R}^2$

(viii) There exist positive constants ρ_1 and ρ_2 satisfying

$$\left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}}\right)\delta\right]\left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}}\right)\delta\right] \leq \rho_1, \tag{12}$$

$$\left[\mu + \left(P_1^{(1)} \rho_1^{\sigma_1^{(1)}} + P_2^{(1)} \rho_2^{\sigma_2^{(1)}}\right)\delta\right]\left[\mu + \left(P_1^{(2)} \rho_1^{\sigma_1^{(2)}} + P_2^{(2)} \rho_2^{\sigma_2^{(2)}}\right)\delta\right] \leq \rho_2, \tag{13}$$

where

$$\mu = \max\left\{\mu_f, \mu_g\right\}, \tag{14}$$

$$\mu_f = \max\left\{|f_1(t)|, |f_2(t)| : t \geq 0\right\}, \tag{15}$$

$$\mu_g = \max\left\{|g_1(t)|, |g_2(t)| : t \geq 0\right\}, \tag{16}$$

$$\delta = \max_{i=1,2}\left\{\int_0^\infty |A_i(s)|\,ds, \int_0^\infty |B_i(s)|\,ds\right\}. \tag{17}$$

Remark 2. Notice that from (i) and (ii), one has $\mu < \infty$. Moreover, by (iii), one has $\delta < \infty$.

Our main result is given by the following theorem.

Theorem 3. *Under assumptions (i)–(viii), system (4) admits one and only one solution $(x^*, y^*) \in C([0,\infty)) \times C([0,\infty))$ satisfying $|x^*(t)| \le \rho_1$ and $|y^*(t)| \le \rho_2$, for all $t \ge 0$.*

Proof. Let T be an arbitrary positive number and $I_T = [0, T]$. For $i = 1, 2$, let

$$C_{T,\rho_i} = \{x \in C(I_T) : |x(t)| \le \rho_i, t \in I_T\}. \tag{18}$$

We introduce the mapping $\mathscr{H} : C_{T,\rho_1} \times C_{T,\rho_2} \longrightarrow C(I_T) \times C(I_T)$ defined by

$$\begin{aligned}\mathscr{H}(x,y)(t) &= (\mathscr{H}_1(x,y)(t), \mathscr{H}_2(x,y)(t)) \\ &= \left(\prod_{i=1}^{2} \mathscr{H}_1^{(i)}(x,y)(t), \prod_{i=1}^{2} \mathscr{H}_2^{(i)}(x,y)(t)\right), t \in I_T,\end{aligned} \tag{19}$$

where

$$\mathscr{H}_1^{(i)}(x,y)(t) = f_i(t) + \int_0^t A_i(t-s)F_i(s,x(s),y(s))ds, \tag{20}$$

$$\mathscr{H}_2^{(i)}(x,y)(t) = g_i(t) + \int_0^t B_i(t-s)G_i(s,x(s),y(s))ds, \tag{21}$$

for all $i = 1, 2$.

Let $(x,y) \in C_{T,\rho_1} \times C_{T,\rho_2}$. For all $i = 1, 2$ and $t \in I_T$, using (i), (ii), (iii), and (v), and taking in consideration Remark 2, one obtains

$$\begin{aligned}\left|\mathscr{H}_1^{(i)}(x,y)(t)\right| &\le |f_i(t)| + \int_0^t |A_i(t-s)||F_i(s,x(s),y(s))|ds \le \mu \\ &+ \int_0^t |A_i(t-s)|\left(Q_1^{(i)}|x(s)|^{\ell_1^{(i)}} + Q_2^{(i)}|y(s)|^{\ell_2^{(i)}}\right)ds \le \mu \\ &+ \left(Q_1^{(i)}\rho_1^{\ell_1^{(i)}} + Q_2^{(i)}\rho_2^{\ell_2^{(i)}}\right)\int_0^t |A_i(t-s)|ds = \mu \\ &+ \left(Q_1^{(i)}\rho_1^{\ell_1^{(i)}} + Q_2^{(i)}\rho_2^{\ell_2^{(i)}}\right)\int_0^\infty |A_i(\tau)|d\tau \le \mu \\ &+ \left(Q_1^{(i)}\rho_1^{\ell_1^{(i)}} + Q_2^{(i)}\rho_2^{\ell_2^{(i)}}\right)\delta.\end{aligned} \tag{22}$$

Therefore, using (12), it holds that

$$\begin{aligned}|\mathscr{H}_1(x,y)(t)| &= \prod_{i=1}^{2}\left|\mathscr{H}_1^{(i)}(x,y)(t)\right| \\ &\le \left[\mu + \left(Q_1^{(1)}\rho_1^{\ell_1^{(1)}} + Q_2^{(1)}\rho_2^{\ell_2^{(1)}}\right)\delta\right] \\ &\cdot \left[\mu + \left(Q_1^{(2)}\rho_1^{\ell_1^{(2)}} + Q_2^{(2)}\rho_2^{\ell_2^{(2)}}\right)\delta\right] \le \rho_1,\end{aligned} \tag{23}$$

which yields

$$\mathscr{H}_1\left(C_{T,\rho_1} \times C_{T,\rho_2}\right) \subset C_{T,\rho_1}. \tag{24}$$

Similarly, for all $i = 1, 2$ and $t \in I_T$, using (i), (ii), (iii), and (vii), and taking in consideration Remark 2, one obtains

$$\begin{aligned}\left|\mathscr{H}_2^{(i)}(x,y)(t)\right| &\le |g_i(t)| + \int_0^t |B_i(t-s)||G_i(s,x(s),y(s))|ds \\ &\le \mu + \int_0^t |B_i(t-s)|\left(P_1^{(i)}|x(s)|^{\sigma_1^{(i)}} + P_2^{(i)}|y(s)|^{\sigma_2^{(i)}}\right)ds \\ &\le \mu + \left(P_1^{(i)}\rho_1^{\sigma_1^{(i)}} + P_2^{(i)}\rho_2^{\sigma_2^{(i)}}\right)\int_0^t |B_i(t-s)|ds \\ &\le \mu + \left(P_1^{(i)}\rho_1^{\sigma_1^{(i)}} + P_2^{(i)}\rho_2^{\sigma_2^{(i)}}\right)\delta.\end{aligned} \tag{25}$$

Hence, using (13), it holds that

$$\begin{aligned}|\mathscr{H}_2(x,y)(t)| &= \prod_{i=1}^{2}\left|\mathscr{H}_2^{(i)}(x,y)(t)\right| \\ &\le \left[\mu + \left(P_1^{(1)}\rho_1^{\sigma_1^{(1)}} + P_2^{(1)}\rho_2^{\sigma_2^{(1)}}\right)\delta\right] \\ &\cdot \left[\mu + \left(P_1^{(2)}\rho_1^{\sigma_1^{(2)}} + P_2^{(2)}\rho_2^{\sigma_2^{(2)}}\right)\delta\right] \le \rho_2,\end{aligned} \tag{26}$$

which yields

$$\mathscr{H}_2\left(C_{T,\rho_1} \times C_{T,\rho_2}\right) \subset C_{T,\rho_2}. \tag{27}$$

Therefore, it follows from (24) and (27) that the mapping $\mathscr{H}$ maps the set $C_{T,\rho_1} \times C_{T,\rho_2}$ into itself, i.e.,

$$\mathscr{H} : C_{T,\rho_1} \times C_{T,\rho_2} \longrightarrow C_{T,\rho_1} \times C_{T,\rho_2}. \tag{28}$$

Next, let us introduce the metric

$$d_r : C(I_T) \times C(I_T) \longrightarrow \mathbb{R}, \tag{29}$$

defined by

$$d_r(x,y) = \max_{t\in I_T} e^{-rt}|x(t) - y(t)|, (x,y) \in C(I_T) \times C(I_T), \tag{30}$$

where $r > 0$ will be specified later. Moreover, we introduce the vector-valued metric

$$D_r : \left(C_{T,\rho_1} \times C_{T,\rho_2}\right) \times \left(C_{T,\rho_1} \times C_{T,\rho_2}\right) \longrightarrow \mathbb{R}^2, \tag{31}$$

defined by

$$D_r((x,y),(\bar{x},\bar{y})) = \begin{pmatrix} d_r(x,\bar{x}) \\ d_r(y,\bar{y}) \end{pmatrix}, (x,y),(\bar{x},\bar{y}) \in C_{T,\rho_1} \times C_{T,\rho_2}. \tag{32}$$

It can be easily seen that $(C_{T,\rho_1} \times C_{T,\rho_2}, D_r)$ is a complete generalized metric space. On the other hand, for all (x,y), $(\bar{x},\bar{y}) \in C_{T,\rho_1} \times C_{T,\rho_2}$, and $t \in I_T$, using (22), one has

$$\begin{aligned}
|\mathscr{H}_1(x,y)(t) - \mathscr{H}_1(\bar{x},\bar{y})(t)| &= \left|\prod_{i=1}^{2} \mathscr{H}_1^{(i)}(x,y)(t) - \prod_{i=1}^{2} \mathscr{H}_1^{(i)}(\bar{x},\bar{y})(t)\right| \\
&= \left|\mathscr{H}_1^{(1)}(x,y)(t)\mathscr{H}_1^{(2)}(x,y)(t) - \mathscr{H}_1^{(1)}(\bar{x},\bar{y})(t)\mathscr{H}_1^{(2)}(\bar{x},\bar{y})(t)\right| \\
&\le \left|\mathscr{H}_1^{(1)}(x,y)(t)\right|\left|\mathscr{H}_1^{(2)}(x,y)(t) - \mathscr{H}_1^{(2)}(\bar{x},\bar{y})(t)\right| \\
&\quad + \left|\mathscr{H}_1^{(2)}(\bar{x},\bar{y})(t)\right|\left|\mathscr{H}_1^{(1)}(x,y)(t) - \mathscr{H}_1^{(1)}(\bar{x},\bar{y})(t)\right| \\
&\le \left[\mu + \left(Q_1^{(1)}\rho_1^{\ell_1^{(1)}} + Q_2^{(1)}\rho_2^{\ell_2^{(1)}}\right)\delta\right]\left|\mathscr{H}_1^{(2)}(x,y)(t) - \mathscr{H}_1^{(2)}(\bar{x},\bar{y})(t)\right| \\
&\quad + \left[\mu + \left(Q_1^{(2)}\rho_1^{\ell_1^{(2)}} + Q_2^{(2)}\rho_2^{\ell_2^{(2)}}\right)\delta\right]\left|\mathscr{H}_1^{(1)}(x,y)(t) - \mathscr{H}_1^{(1)}(\bar{x},\bar{y})(t)\right|.
\end{aligned} \tag{33}$$

Moreover, using (iii), (iv), (20) and Hölder's inequality, for all $i = 1, 2$, one obtains

$$\begin{aligned}
&\left|\mathscr{H}_1^{(i)}(x,y)(t) - \mathscr{H}_1^{(i)}(\bar{x},\bar{y})(t)\right| \\
&= \left|f_i(t) + \int_0^t A_i(t-s)F_i(s,x(s),y(s))\,ds - f_i(t)\right. \\
&\quad \left. - \int_0^t A_i(t-s)F_i(s,\bar{x}(s),\bar{y}(s))\,ds\right| \le \int_0^t |A_i(t-s)||F_i(s,x(s),y(s)) \\
&\quad - F_i(s,\bar{x}(s),\bar{y}(s))|\,ds \le \int_0^t |A_i(t-s)|\left(L_1^{(i)}\,|\,x(s) - \bar{x}(s)|\right. \\
&\quad \left. + L_2^{(i)}\,|\,y(s) - \bar{y}(s)\,|\,\right)ds = L_1^{(i)}\int_0^t |A_i(t-s)|e^{rs}(e^{-rs}\,|\,x(s) \\
&\quad - \bar{x}(s)\,|\,)\,ds + L_2^{(i)}\int_0^t |A_i(t-s)|e^{rs}(e^{-rs}\,|\,y(s) - \bar{y}(s)\,|\,)\,ds \\
&\le \left(L_1^{(i)}\int_0^t |\,A_i(t-s)\,|\,e^{rs}\,ds\right)d_r(x,\bar{x}) + \left(L_2^{(i)}\int_0^t |\,A_i(t-s)\,|\,e^{rs}\,ds\right)d_r(y,\bar{y}) \\
&= \left(\int_0^t |\,A_i(t-s)\,|\,e^{rs}\,ds\right)\left(L_1^{(i)}d_r(x,\bar{x}) + L_2^{(i)}d_r(y,\bar{y})\right) \\
&\le \delta_\varsigma\left(\int_0^t e^{rs\varsigma'}\,ds\right)^{\frac{1}{\varsigma'}}\left(L_1^{(i)}d_r(x,\bar{x}) + L_2^{(i)}d_r(y,\bar{y})\right),
\end{aligned} \tag{34}$$

where

$$\varsigma' = \frac{\varsigma}{\varsigma - 1}, \tag{35}$$

$$\delta_\varsigma = \max_{i=1,2}\left\{\left(\int_0^\infty |A_i(s)|^\varsigma\,ds\right)^{1/\varsigma}, \left(\int_0^\infty |B_i(s)|^\varsigma\,ds\right)^{1/\varsigma}\right\}. \tag{36}$$

Notice that by (iii), one has $\delta_\varsigma < \infty$. Hence, it holds that

$$\begin{aligned}
\left|\mathscr{H}_1^{(i)}(x,y)(t) - \mathscr{H}_1^{(i)}(\bar{x},\bar{y})(t)\right| &\le \delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'} \\
&\quad \cdot \left(e^{rt\varsigma'} - 1\right)^{1/\varsigma'}\left(L_1^{(i)}d_r(x,\bar{x}) + L_2^{(i)}d_r(y,\bar{y})\right).
\end{aligned} \tag{37}$$

Therefore, by (33), one obtains

$$\begin{aligned}
&|\mathscr{H}_1(x,y)(t) - \mathscr{H}_1(\bar{x},\bar{y})(t)| \\
&\le \left[\mu + \left(Q_1^{(1)}\rho_1^{\ell_1^{(1)}} + Q_2^{(1)}\rho_2^{\ell_2^{(1)}}\right)\delta\right]\delta_\varsigma \varsigma'^{-1/\varsigma'} r^{\frac{-1}{\varsigma'}}\left(e^{rt\varsigma'} - 1\right)^{1/\varsigma'} \\
&\quad \times \left(L_1^{(2)}d_r(x,\bar{x}) + L_2^{(2)}d_r(y,\bar{y})\right) \\
&\quad + \left[\mu + \left(Q_1^{(2)}\rho_1^{\ell_1^{(2)}} + Q_2^{(2)}\rho_2^{\ell_2^{(2)}}\right)\delta\right]\delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'}\left(e^{rt\varsigma'} - 1\right)^{1/\varsigma'} \\
&\quad \times \left(L_1^{(1)}d_r(x,\bar{x}) + L_2^{(1)}d_r(y,\bar{y})\right) = \delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'}\left(e^{rt\varsigma'} - 1\right)^{1/\varsigma'} \\
&\quad \times \left(L_1^{(2)}\left[\mu + \left(Q_1^{(1)}\rho_1^{\ell_1^{(1)}} + Q_2^{(1)}\rho_2^{\ell_2^{(1)}}\right)\delta\right]\right. \\
&\quad \left. + L_1^{(1)}\left[\mu + \left(Q_1^{(2)}\rho_1^{\ell_1^{(2)}} + Q_2^{(2)}\rho_2^{\ell_2^{(2)}}\right)\delta\right]\right)d_r(x,\bar{x}) \\
&\quad + \delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'}\left(e^{rt\varsigma'} - 1\right)^{1/\varsigma'}\left(L_2^{(2)}\left[\mu + \left(Q_1^{(1)}\rho_1^{\ell_1^{(1)}} + Q_2^{(1)}\rho_2^{\ell_2^{(1)}}\right)\delta\right]\right. \\
&\quad \left. + L_2^{(1)}\left[\mu + \left(Q_1^{(2)}\rho_1^{\ell_1^{(2)}} + Q_2^{(2)}\rho_2^{\ell_2^{(2)}}\right)\delta\right]\right)d_r(y,\bar{y}),
\end{aligned} \tag{38}$$

which yields

$$\begin{aligned}
&d_r(\mathscr{H}_1(x,y), \mathscr{H}_1(\bar{x},\bar{y})) \\
&\quad \le \left(1 - e^{-rT\varsigma'}\right)^{1/\varsigma'}\delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'}(\alpha_{11}d_r(x,\bar{x}) + \alpha_{12}d_r(y,\bar{y})),
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
\alpha_{11} = L_1^{(2)}&\left[\mu + \left(Q_1^{(1)}\rho_1^{\ell_1^{(1)}} + Q_2^{(1)}\rho_2^{\ell_2^{(1)}}\right)\delta\right] \\
&+ L_1^{(1)}\left[\mu + \left(Q_1^{(2)}\rho_1^{\ell_1^{(2)}} + Q_2^{(2)}\rho_2^{\ell_2^{(2)}}\right)\delta\right],
\end{aligned} \tag{40}$$

$$\begin{aligned}
\alpha_{12} = L_2^{(2)}&\left[\mu + \left(Q_1^{(1)}\rho_1^{\ell_1^{(1)}} + Q_2^{(1)}\rho_2^{\ell_2^{(1)}}\right)\delta\right] \\
&+ L_2^{(1)}\left[\mu + \left(Q_1^{(2)}\rho_1^{\ell_1^{(2)}} + Q_2^{(2)}\rho_2^{\ell_2^{(2)}}\right)\delta\right].
\end{aligned} \tag{41}$$

Similarly, using (iii), (vi), (21), and (25), one obtains

$$d_r(\mathscr{H}_2(x,y),\mathscr{H}_2(\bar{x},\bar{y})) \le \left(1-e^{-rT\varsigma'}\right)^{1/\varsigma'} \delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'} (\alpha_{21} d_r(x,\bar{x}) + \alpha_{22} d_r(y,\bar{y})), \quad (42)$$

where

$$\alpha_{21} = M_1^{(2)}\left[\mu + \left(P_1^{(1)}\rho_1^{\sigma_1^{(1)}} + P_2^{(1)}\rho_2^{\sigma_2^{(1)}}\right)\delta\right] + M_1^{(1)}\left[\mu + \left(P_1^{(2)}\rho_1^{\sigma_1^{(2)}} + P_2^{(2)}\rho_2^{\sigma_2^{(2)}}\right)\delta\right], \quad (43)$$

$$\alpha_{22} = M_2^{(2)}\left[\mu + \left(P_1^{(1)}\rho_1^{\sigma_1^{(1)}} + P_2^{(1)}\rho_2^{\sigma_2^{(1)}}\right)\delta\right] + M_2^{(1)}\left[\mu + \left(P_1^{(2)}\rho_1^{\sigma_1^{(2)}} + P_2^{(2)}\rho_2^{\sigma_2^{(2)}}\right)\delta\right]. \quad (44)$$

Therefore, it follows from (19), (39), and (42) that

$$D_r(\mathscr{H}(x,y),\mathscr{H}(\bar{x},\bar{y})) \circ_{\mathbb{R}^2} \mathscr{K} D_r((x,y),(\bar{x},\bar{y})), (x,y),(\bar{x},\bar{y}) \in C_{T,\rho_1} \times C_{T,\rho_2}, \quad (45)$$

where $\mathscr{K}$ is the square matrix of size 2 defined by

$$\mathscr{K} = \left(1-e^{-rT\varsigma'}\right)^{1/\varsigma'} \delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}. \quad (46)$$

On the other hand, one has

$$\rho(\mathscr{K}) = \left(1-e^{-rT\varsigma'}\right)^{1/\varsigma'} \delta_\varsigma \varsigma'^{-1/\varsigma'} r^{-1/\varsigma'} \rho\left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\right]. \quad (47)$$

Therefore, taking

$$r^{1/\varsigma'} \ge \delta_\varsigma \varsigma'^{-1/\varsigma'} \rho\left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\right], \quad (48)$$

one obtains

$$\rho(\mathscr{K}) \le \left(1-e^{-rT\varsigma'}\right)^{1/\varsigma'} < 1. \quad (49)$$

Then, by Lemma 1, one deduces that the mapping $\mathscr{H}$ defined by (19) admits a unique fixed point in $C_{T,\rho_1} \times C_{T,\rho_2}$, which is the unique solution to (4) in $C_{T,\rho_1} \times C_{T,\rho_2}$. On the other hand, since the real number $T > 0$ is arbitrary, it holds that (4) admits a unique continuous global solution (x^*, y^*) satisfying $|x^*(t)| \le \rho_1$ and $|y^*(t)| \le \rho_2$, for all $t \ge 0$. The proof is completed.

We end the paper with the following example.

Example 4. Consider the system of integral equations

$$\begin{aligned} x(t) &= \left(\frac{2}{t+1} + \int_0^t \frac{e^{-2(t-s)}}{8} \sin\left(\frac{x(s)}{4} + \frac{3y(s)}{\sqrt{s+16}}\right) ds\right) \\ &\quad \cdot \int_0^t \frac{3e^{-(t-s)^2}}{16\sqrt{\pi}} \left(\frac{x(s)}{s^2+2} + \frac{y(s)}{2}\right) ds, t \ge 0, \\ y(t) &= \left(\frac{t}{t+1} + \int_0^t \frac{e^{-(t-s)}}{16} \arctan\left(\frac{2x(s)+y(s)}{5}\right) ds\right) \\ &\quad \cdot \left(1 + \int_0^t \frac{e^{-(t-s)^2}}{16\sqrt{\pi}} \left(\frac{x(s)}{6} + \frac{2y(s)}{s^2+3}\right) ds\right), t \ge 0. \end{aligned} \quad (50)$$

System (50) is a special case of System (4), where

$$f_1(t) = \frac{2}{t+1}, f_2(t) = 0, A_1(t) = \frac{e^{-2t}}{8}, A_2(t) = \frac{3e^{-t^2}}{16\sqrt{\pi}}, \quad (51)$$

$$F_1(t,u,v) = \sin\left(\frac{u}{4} + \frac{3v}{\sqrt{t+16}}\right), F_2(t,u,v) = \frac{u}{t^2+2} + \frac{v}{2}, \quad (52)$$

$$g_1(t) = \frac{t}{t+1}, g_2(t) = 1, B_1(t) = \frac{e^{-t}}{16}, B_2(t) = \frac{e^{-t^2}}{16\sqrt{\pi}}, \quad (53)$$

$$G_1(t,u,v) = \arctan\left(\frac{2u+v}{5}\right), G_2(t,u,v) = \frac{u}{6} + \frac{2v}{t^2+3}. \quad (54)$$

Let us check the validity of assumptions (i)–(viii). It can be easily seen that

$$f_i, A_i, g_i, B_i \in C([0,\infty)), F_i, G_i \in C([0,\infty) \times \mathbb{R} \times \mathbb{R}), i = 1, 2, \quad (55)$$

$$\mu_f = \max\{|f_1(t)|, |f_2(t)| : t \ge 0\} = 2, \quad (56)$$

$$\mu_g = \max\{|g_1(t)|, |g_2(t)| : t \ge 0\} = 1, \quad (57)$$

$$\mu = \max\{\mu_f, \mu_g\} = 2. \quad (58)$$

Moreover, one has

$$\int_0^\infty |A_1(t)| dt = \frac{1}{16}, \int_0^\infty |A_1(t)|^\varsigma dt = \frac{1}{2\varsigma 8^\varsigma} (\text{for all } \varsigma > 1), \quad (59)$$

$$\int_0^\infty |A_2(t)| dt = \frac{3}{32}, \int_0^\infty |A_2(t)|^\varsigma dt = \left(\frac{3}{16\sqrt{\pi}}\right)^\varsigma \frac{\sqrt{\pi}}{2\sqrt{\varsigma}} (\text{for all } \varsigma > 1), \quad (60)$$

$$\int_0^\infty |B_1(t)|dt = \frac{1}{16}, \int_0^\infty |B_1(t)|^\varsigma dt = \frac{1}{\varsigma 16^\varsigma} \,(\text{for all } \varsigma > 1), \quad (61)$$

$$\int_0^\infty |B_2(t)|dt = \frac{1}{32}, \int_0^\infty |B_2(t)|^\varsigma dt = \frac{\sqrt{\pi}}{2\sqrt{\varsigma}(16\sqrt{\pi})^\varsigma} \,(\text{for all } \varsigma > 1), \quad (62)$$

$$\delta = \max_{i=1,2}\left\{\int_0^\infty |A_i(s)|\,ds, \int_0^\infty |B_i(s)|\,ds\right\} = \frac{3}{32}. \quad (63)$$

Therefore, assumptions (i)–(iii) of Theorem 3 are satisfied. For all $t \geq 0$ and $(u, v), (\bar{u}, \bar{v}) \in \mathbb{R}^2$, one has

$$\begin{aligned}|F_1(t,u,v) - F_1(t,\bar{u},\bar{v})| &= \left|\sin\left(\frac{u}{4} + \frac{3v}{\sqrt{t+16}}\right) - \sin\left(\frac{\bar{u}}{4} + \frac{3\bar{v}}{\sqrt{t+16}}\right)\right| \leq \frac{1}{4}|u-\bar{u}| + \frac{3}{4}|v-\bar{v}|\end{aligned} \quad (64)$$

$$\begin{aligned}|F_2(t,u,v) - F_2(t,\bar{u},\bar{v})| &= \left|\frac{u}{t^2+2} + \frac{v}{2} - \frac{\bar{u}}{t^2+2} - \frac{\bar{v}}{2}\right| \\ &\leq \frac{1}{2}|u-\bar{u}| + \frac{1}{2}|v-\bar{v}|,\end{aligned} \quad (65)$$

which shows that assumption (iv) of Theorem 3 is satisfied with

$$L_1^{(1)} = \frac{1}{4}, \quad L_2^{(1)} = \frac{3}{4}, \quad L_1^{(2)} = L_2^{(2)} = \frac{1}{2}. \quad (66)$$

Similarly, one has

$$\begin{aligned}|G_1(t,u,v) - G_1(t,\bar{u},\bar{v})| &= \left|\arctan\left(\frac{2u+v}{5}\right) - \arctan\left(\frac{2\bar{u}+\bar{v}}{5}\right)\right| \\ &\leq \frac{2}{5}|u-\bar{u}| + \frac{1}{5}|v-\bar{v}|\end{aligned} \quad (67)$$

$$\begin{aligned}|G_2(t,u,v) - G_2(t,\bar{u},\bar{v})| &= \left|\frac{u}{6} + \frac{2v}{t^2+3} - \frac{\bar{u}}{6} - \frac{2\bar{v}}{t^2+3}\right| \\ &\leq \frac{1}{6}|u-\bar{u}| + \frac{2}{3}|v-\bar{v}|,\end{aligned} \quad (68)$$

which shows that assumption (vi) of Theorem 3 is satisfied with

$$M_1^{(1)} = \frac{2}{5}, M_2^{(1)} = \frac{1}{5}, M_1^{(2)} = \frac{1}{6}, M_2^{(2)} = \frac{2}{3}. \quad (69)$$

For all $t \geq 0$ and $(u, v) \in \mathbb{R}^2$, one has

$$|F_1(t,u,v)| = \left|\sin\left(\frac{u}{4} + \frac{3v}{\sqrt{t+16}}\right)\right| \leq \frac{1}{4}|u| + \frac{3}{4}|v|, \quad (70)$$

$$|F_2(t,u,v)| = \left|\frac{u}{t^2+2} + \frac{v}{2}\right| \leq \frac{1}{2}|u| + \frac{1}{2}|v|, \quad (71)$$

which shows that assumption (v) of Theorem 3 is satisfied with

$$Q_1^{(1)} = \frac{1}{4}, Q_2^{(1)} = \frac{3}{4}, Q_1^{(2)} = Q_2^{(2)} = \frac{1}{2}, \ell_1^{(1)} = \ell_2^{(1)} = \ell_1^{(2)} = \ell_2^{(2)} = 1. \quad (72)$$

Similarly, one has

$$|G_1(t,u,v)| = \left|\arctan\left(\frac{2u+v}{5}\right)\right| \leq \frac{2}{5}|u| + \frac{1}{5}|v|, \quad (73)$$

$$|G_2(t,u,v)| = \left|\frac{u}{6} + \frac{2v}{t^2+3}\right| \leq \frac{1}{6}|u| + \frac{2}{3}|v|, \quad (74)$$

which shows that assumption (VII) is satisfied with

$$P_1^{(1)} = \frac{2}{5}, P_2^{(1)} = \frac{1}{5}, P_1^{(2)} = \frac{1}{6}, P_2^{(2)} = \frac{2}{3}, \sigma_1^{(1)} = \sigma_2^{(1)} = \sigma_1^{(2)} = \sigma_2^{(2)} = 1. \quad (75)$$

From the above estimates, one deduces that the system of inequalities (12) and (13) is equivalent to

$$\begin{cases}\left[2 + \frac{3}{32}\left(\frac{1}{4}\rho_1 + \frac{3}{4}\rho_2\right)\right]\left[2 + \frac{3}{64}(\rho_1 + \rho_2)\right] \leq \rho_1, \\ \left[2 + \frac{3}{160}(2\rho_1 + \rho_2)\right]\left[2 + \frac{1}{64}(\rho_1 + 4\rho_2)\right] \leq \rho_2.\end{cases} \quad (76)$$

Taking $\rho_1 = \rho_2 = \rho > 0$, (76) reduces to

$$\begin{cases}\left(2 + \frac{3}{32}\rho\right)^2 \leq \rho, \\ \left(2 + \frac{9}{160}\rho\right)\left(2 + \frac{5}{64}\rho\right) \leq \rho.\end{cases} \quad (77)$$

On the other hand, one observes easily that

$$\left(2 + \frac{9}{160}\rho\right)\left(2 + \frac{5}{64}\rho\right) \leq \left(2 + \frac{3}{32}\rho\right)^2, \rho > 0. \quad (78)$$

Therefore, any $\rho > 0$ satisfying

$$\left(2 + \frac{3}{32}\rho\right)^2 \leq \rho, \quad (79)$$

is a solution to (77). In particular, for $\rho = 36$, one has

$$\left(2 + \frac{3}{32}\rho\right)^2 = 28.890625 < 36 = \rho. \quad (80)$$

Therefore, $\rho = 36$ is a solution to (77), which shows that assumption (viii) of Theorem 3 is satisfied with $(\rho_1, \rho_2) = (36, 36)$.

Finally, by Theorem 3, one deduces that system (50) admits one and only one solution $(x^*, y^*) \in C([0,\infty)) \times C([0,\infty))$ satisfying

$$\sup_{t\geq 0}\{|x^*(t)|,|y^*(t)|\} \leq 36. \tag{81}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

On Durrmeyer Type λ-Bernstein Operators via (p, q)-Calculus

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In the present paper, Durrmeyer type λ-Bernstein operators via (p, q)-calculus are constructed, the first and second moments and central moments of these operators are estimated, a Korovkin type approximation theorem is established, and the estimates on the rate of convergence by using the modulus of continuity of second order and Steklov mean are studied, a convergence theorem for the Lipschitz continuous functions is also obtained. Finally, some numerical examples are given to show that these operators we defined converge faster in some λ cases than Durrmeyer type (p, q)-Bernstein operators.

1. Introduction

In 2016, Mursaleen et al. [1] proposed the following $(p, q$)-analogue of Bernstein operators:

$$B_{n,p,q}(f;x)=\sum_{k=0}^{n} b_{n,k}(x;p,q)f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), x\in[0,1], \quad (1)$$

where $b_{n,k}(x;p,q)(k=0,1,\cdots,n)$ are (p, q)-Bernstein basis functions and defined as

$$b_{n,k}(x;p,q)=\frac{1}{p^{n(n-1)/2}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2}x^k(1\ominus x)_{p,q}^{n-k}, x\in[0,1]. \quad (2)$$

They also introduced and studied some important approximation properties of the Stancu type of operators (1) in [2]. After their construction, there are more and more papers on the study of (p, q)-analogue of Bernstein type operators, we mention some of them as [3–11], we also mention some other positive linear operators as [12–19].

Very recently, Cai et al. [20] proposed the following λ -Bernstein operators based on (p, q)-integers as

$$B_{n,p,q}^{\lambda}(f;x)=\sum_{k=0}^{n} b_{n,k}^{\lambda}(x;p,q)f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), x\in[0,1], \quad (3)$$

where

$$\begin{cases} b^{\lambda}_{n,0}(x;p,q) = b_{n,0}(x;p,q) - \dfrac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,1}(x;p,q), \\ b^{\lambda}_{n,k}(x;p,q) = b_{n,k}(x;p,q) + \lambda\left(\dfrac{p^{1-n}[n]_{p,q} - 2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]^2_{p,q}-1} b_{n+1,k}(x;p,q) - \dfrac{p^{1-n}[n]_{p,q} - 2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]^2_{p,q}-1} b_{n+1,k+1}(x;p,q)\right), (k=1,2,\cdots,n-1) \\ b^{\lambda}_{n,n}(x;p,q) = b_{n,n}(x;p,q) - \dfrac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,n}(x;p,q), \end{cases} \tag{4}$$

$b_{n,k}(x;p,q)(k=0,1,\cdots,n)$ are defined in (2), $\lambda \in [-1,1]$, $n \geq 2$, $x \in [0,1]$, and $0 < q < p \leq 1$. They also constructed the (p,q)-analogue of Kantorovich type λ-Bernstein operators and investigated their A-statistical convergence properties in [21].

Inspired by above research, based on (3), we introduce Durrmeyer type λ-Bernstein operators via (p,q)-calculus as

$$D^{\lambda}_{n,p,q}(f;x) = [n+1]_{p,q} p^{-n} \sum_{k=0}^{n} \frac{p^k}{q^k} b^{\lambda}_{n,k}(x;p,q) \cdot \int_0^1 b_{n,k}(qt;p,q) f(t) d_{p,q}t, \tag{5}$$

where $x \in [0,1]$, $0 < q < p \leq 1$, $b^{\lambda}_{n,k}(x;p,q)(k=0,1,\cdots,n)$ are defined in (4) and $b_{n,k}(\cdot;p,q)(k=0,1,\cdots,n)$ are defined in (2). Apparently, when $\lambda = 0$, operators $D^{0}_{n,p,q}(f;x)$ become to $D^{(p,q)}_{n}(f;x)$ which are defined as follows in [11],

$$D^{(p,q)}_{n}(f;x) = [n+1]_{p,q} p^{-n^2} \sum_{k=0}^{n} b^{(p,q)}_{n,k}(x) \left(\frac{q}{p}\right)^{-k} \cdot \int_0^1 b^{(p,q)}_{n,k}(qt) f(t) d_{p,q}t, \tag{6}$$

where $b^{(p,q)}_{n,k}(x) = p^{n(n-1)/2} b_{n,k}(x;p,q)$. We will show that operators $D^{\lambda}_{n,p,q}(f;x)$ converge to f faster than operators $D^{(p,q)}_{n}(f;x)$ in some cases of λ by some numerical examples in Section 4, say, we have more modeling flexibility when adding the parameter λ.

We first mention some definitions based on (p,q)-integers; details can be found in [22–26]. For any fixed real number $p > 0$ and $q > 0$, the (p,q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{p-q}, & p \neq q \neq 1; \\ \dfrac{1-q^n}{1-q}, & p = 1; \\ n, & p = q = 1. \end{cases} \tag{7}$$

(p,q)-factorial and (p,q)-binomial coefficients are defined as follows:

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}, & n = 1,2,\cdots; \\ 1, & n = 0, \end{cases} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}. \tag{8}$$

The (p,q)-power basis $(x \oplus t)^n_{p,q}$ and $(x \ominus t)^n_{p,q}$ are defined by

$$(x \oplus t)^n_{p,q} = (x+t)(px+qt)(p^2x+q^2t)\cdots(p^{n-1}x+q^{n-1}t),$$
$$(x \ominus t)^n_{p,q} = (x-t)(px-qt)(p^2x-q^2t)\cdots(p^{n-1}x-q^{n-1}t). \tag{9}$$

Let $f : [0,a] \to \mathbb{R}$, then (p,q)-integration of a funciton f is defined by

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} a\right), \left|\frac{p}{q}\right| > 1. \tag{10}$$

This paper is mainly organized as follows: in Section 2, we estimate some moments and central moments of $D^{\lambda}_{n,p,q}(f;x)$ in order to obtain our main results; in Section 3, we study a Korovkin type approximation theorem and estimate the rate of convergence of $D^{\lambda}_{n,p,q}(f)$ to f by using the second order modulus of smoothness, Peetre's K-functional, Steklov mean function, and Lipschitz class function; in Section 4, we give some numerical experiments to verify our theoretical results; in the final section, a conclusion is given.

2. Some Lemmas

Before giving our main results, we need the following lemmas.

Lemma 1. *Let $0 < q < p \leq 1$, we have*

$$\int_0^1 b_{n,k}(qt\,;p,q)d_{p,q}t = \left(\frac{q}{p}\right)^k \frac{p^n}{[n+1]_{p,q}}\,; \tag{11}$$

$$\int_0^1 t b_{n,k}(qt\,;p,q)d_{p,q}t = \left(\frac{q}{p}\right)^k p^{-k} \frac{p^{2n}[k+1]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}}\,; \tag{12}$$

$$\int_0^1 t^2 b_{n,k}(qt\,;p,q)d_{p,q}t = \left(\frac{q}{p}\right)^k p^{-2k} \frac{p^{3n}[k+1]_{p,q}[k+2]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}. \tag{13}$$

Proof. According to the following equations of Lemma 1 in [11]

$$\int_0^1 b_{n,k}^{(p,q)}(qt)t^s d_{p,q}t = \left(\frac{q}{p}\right)^k p^{-ks} p^{n(n+2s+1)/2} \frac{[n]_{p,q}![k+s]_{p,q}!}{[k]_{p,q}![n+s+1]_{p,q}!},\ s = 0, 1, 2, 3\cdots, \tag{14}$$

and the fact that $b_{n,k}^{(p,q)}(x) = p^{n(n-1)/2} b_{n,k}(x\,;p,q)$, we get the proof of Lemma 1 easily.

Lemma 2. *Let $e_k(t) = t^k (k = 0, 1, 2)$, $\lambda \in [-1, 1]$, $x \in [0, 1]$, and $0 < q < p \leq 1$, then for the operators $D_{n,p,q}^{\lambda}(f\,;x)$, we have*

$$D_{n,p,q}^{\lambda}(e_0\,;x) = 1\,; \tag{15}$$

$$\begin{aligned} D_{n,p,q}^{\lambda}(e_1\,;x) = {} & \frac{x}{q} + \frac{p^n\left(q - [2]_{p,q}x\right)}{q[n+2]_{p,q}} \\ & + \frac{2\lambda q[n]_{p,q}[n+1]_{p,q}x^2\left(1 - x^{n-1}\right)(1 - q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2 - 1\right)} \\ & + \frac{\lambda p^n\left(1 - x^{n+1}\right)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q} - 1\right)} \\ & - \frac{\lambda[n+1]_{p,q}x(1 - x^n)(1 - q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q} - 1\right)} \\ & - \frac{2\lambda q[n+1]_{p,q}x(1 - x^n)}{p[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2 - 1\right)} \\ & - \frac{\lambda(1 \ominus x)_{p,q}^{n+1}}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q} - 1\right)}\,; \end{aligned} \tag{16}$$

$$\begin{aligned} D_{n,p,q}^{\lambda}(e_2\,;x) = {} & \frac{q^3[n]_{p,q}^2x^2}{[n+2]_{p,q}[n+3]_{p,q}} \\ & + \frac{p^{n-1}q[n]_{p,q}x\left([2]_{p,q}^2 - q^2x\right) + [2]_{p,q}p^{2n}}{[n+2]_{p,q}[n+3]_{p,q}} \\ & + \frac{\lambda q^2\left(1 - q^2/p^2\right)[n]_{p,q}[n+1]_{p,q}x^2}{\left(p^{1-n}[n]_{p,q} + 1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ & \cdot \left[\frac{2q^2[n-1]_{p,q}x\left(1 - x^{n-2}\right)}{p[n]_{p,q} - p^n} - \left(1 - x^{n-1}\right)\right] \\ & + \frac{2\lambda q[n]_{p,q}[n+1]_{p,q}x^2\left(1 - x^{n-1}\right)\left(p - [3]_{p,q}q^2/p^3\right)}{\left(p^{2-2n}[n]_{p,q}^2 - 1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ & + \frac{[2]_{p,q}\lambda p^{2n}\left(1 - x^{n+1}\right)}{\left(p^{1-n}[n]_{p,q} - 1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ & + \frac{\lambda p^{n-2}q[n+1]_{p,q}x(1 - x^n)}{[n+2]_{p,q}[n+3]_{p,q}} \\ & \cdot \left[\frac{p^2 + q^2}{p^{1-n}[n]_{p,q} + 1} + \frac{p(1 - p/q)(p + 2q)}{p^{1-n}[n]_{p,q} - 1}\right] \\ & - \frac{[2]_{p,q}\lambda p^n(1 \ominus x)_{p,q}^{n+1}}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q} - 1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ & - \frac{8\lambda p^{n-1}q^2[n+1]_{p,q}x(1 - x^n)}{\left(p^{2-2n}[n]_{p,q}^2 - 1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ & + \frac{2\lambda p^{n-1}q(2q - p)[n+1]_{p,q}x(1!x)_{p,q}^n}{p^{n(n-1)/2}\left(p^{2-2n}[n]_{p,q}^2 - 1\right)[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \tag{17}$$

Proof. By (5), (11), and Lemma 2 of [20], we have

$$\begin{aligned} D_{n,p,q}^{\lambda}(1\,;x) &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\int_0^1 b_{n,k}(qt\,;p,q)d_{p,q} \\ &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\frac{q^k}{p^k}\frac{p^n}{[n+1]_{p,q}} \\ &= \sum_{k=0}^{n} b_{n,k}^{\lambda}(x\,;p,q) = 1. \end{aligned} \tag{18}$$

Next, by the fact that $[k+1]_{p,q} = p^k + q[k]_{p,q}$, (5) and (12), we get

$$\begin{aligned} D_{n,p,q}^{\lambda}(t\,;x) &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\int_0^1 b_{n,k}(qt\,;p,q)t d_{p,q}t \\ &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\frac{q^k}{p^k}p^{-k}\frac{p^{2n}[k+1]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} \\ &= \sum_{k=0}^{n} b_{n,k}^{\lambda}(x\,;p,q)\frac{p^n[k+1]_{p,q}}{p^k[n+2]_{p,q}} \\ &= \frac{q[n]_{p,q}}{[n+2]_{p,q}}B_{n,p,q}^{\lambda}(t\,;x) + \frac{p^n}{[n+2]_{p,q}}B_{n,p,q}^{\lambda}(1\,;x). \end{aligned} \tag{19}$$

Then, the desired of (16) can be obtained by Lemma 2 and Lemma 3 of [20] and easy computations. Finally, by (5) and (13), we have

$$\begin{aligned} D^{\lambda}_{n,p,q}\left(t^2;x\right) &= [n+1]_{p,q} p^{-n} \sum_{k=0}^{n} \frac{p^k}{q^k} b^{\lambda}_{n,k}(x;p,q) \int_0^1 b_{n,k}(qt;p,q) t^2 d_{p,q} t \\ &= [n+1]_{p,q} p^{-n} \sum_{k=0}^{n} \frac{p^k}{q^k} b^{\lambda}_{n,k}(x;p,q) \frac{q^k}{p^k} p^{-2k} \\ &\quad \cdot \frac{p^{3n}[k+1]_{p,q}[k+2]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}} \\ &= \sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q) \frac{p^{2n}[k+1]_{p,q}[k+2]_{p,q}}{p^{2k}[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \quad (20)$$

Using $[k+1]_{p,q}[k+2]_{p,q} = q^3[k]^2_{p,q} + p^k q(2q+p)[k]_{p,q} + [2]_{p,q}p^{2k}$, we obtain

$$\begin{aligned} D^{\lambda}_{n,p,q}\left(t^2;x\right) = {} & \frac{q^3[n]^2_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} B^{\lambda}_{n,p,q}\left(t^2;x\right) \\ & + \frac{p^n q(p+2q)[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} B^{\lambda}_{n,p,q}(t;x) \\ & + \frac{[2]_{p,q}p^{2n}}{[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \quad (21)$$

We can get (17) by Lemma 2–4 of [20] and some computations. Lemma 2 is proved.

Lemma 3. *Let $\Phi^k_x(t) = (t-x)^k (k=1,2)$, $\lambda \in [-1,1]$, $x \in [0,1]$, and $0 < q < p \leq 1$, then we have*

$$\begin{aligned} D^{\lambda}_{n,p,q}\left(\Phi^1_x;x\right) = {} & \frac{1-q}{q}x + \frac{p^n\left(q-[2]_{p,q}x\right)}{q[n+2]_{p,q}} \\ & + \frac{2\lambda q[n]_{p,q}[n+1]_{p,q}x^2\left(1-x^{n-1}\right)(1-q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]^2_{p,q}-1\right)} \\ & + \frac{\lambda p^n\left(1-x^{n+1}\right)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ & - \frac{\lambda[n+1]_{p,q}x(1-x^n)(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ & - \frac{2\lambda q[n+1]_{p,q}x(1-x^n)}{p[n+2]_{p,q}\left(p^{2-2n}[n]^2_{p,q}-1\right)} \\ & - \frac{\lambda(1\ominus x)^{n+1}_{p,q}}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} := \Omega^{\lambda}_{n,p,q}(x); \end{aligned} \quad (22)$$

$$\begin{aligned} & \leq \frac{1-q}{q} + \frac{1}{[n+2]_{p,q}} + \frac{2}{p^{1-n}[n]_{p,q}-1} \\ & \quad + \frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ & := \Theta(n;p,q); \end{aligned} \quad (23)$$

$$\begin{aligned} D^{\lambda}_{n,p,q}\left(\Phi^2_x;x\right) \leq {} & \frac{(1-q)^2}{q^2} + \frac{8}{q[n+2]_{p,q}} + \frac{4}{p^{1-n}[n]_{p,q}-1} \\ & + \frac{1}{[n+3]_{p,q}} + \frac{8}{q^2[n+2]_{p,q}[n+3]_{p,q}} \\ & + \frac{11}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ & + \frac{12}{p^{n(n-1)/2}[n+2]_{p,q}[n+3]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ & := \Psi(n;p,q). \end{aligned} \quad (24)$$

Proof. We can obtain (22) easily by (15) and (16). For $\lambda \in [0,1]$, we have

$$\begin{aligned} D^{\lambda}_{n,p,q}\left(\Phi^1_x;x\right) \leq {} & \frac{1-q}{q} + \frac{p^n}{[n+2]_{p,q}} \\ & + \frac{2q[n]_{p,q}[n+1]_{p,q}(1-q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]^2_{p,q}-1\right)} \\ & + \frac{p^n}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \end{aligned} \quad (25)$$

For $\lambda \in [-1,0]$, we have

$$\begin{aligned} D^{\lambda}_{n,p,q}\left(\Phi^1_x;x\right) \leq {} & \frac{1-q}{q} + \frac{p^n}{[n+2]_{p,q}} \\ & + \frac{[n+1]_{p,q}(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ & + \frac{2[n+1]_{p,q}}{[n+2]_{p,q}\left(p^{2-2n}[n]^2_{p,q}-1\right)} \\ & + \frac{1}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \end{aligned} \quad (26)$$

On one hand, since

$$\frac{[n]_{p,q}}{p^n\left(p^{1-n}[n]_{p,q}+1\right)} = \frac{[n]_{p,q}}{p[n]_{p,q}+p^n} \leq \frac{[n]_{p,q}}{p[n]_{p,q}+q^n} \leq \frac{[n]_{p,q}}{[n+1]_{p,q}} \leq 1, \quad (27)$$

we get

$$\frac{2q[n]_{p,q}[n+1]_{p,q}(1-q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \le \frac{2q[n+1]_{p,q}(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \tag{28}$$

On the other hand, we have

$$\frac{[n+1]_{p,q}}{[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \le \frac{p^n}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \le \frac{1}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \tag{29}$$

with the fact that $[n+1]_{p,q}/p^{1-n}[n]_{p,q}+1 = p^n[n+1]_{p,q}/p[n]_{p,q} + p^n \le p^n$. Combing (25)–(29), we have

$$\begin{aligned} D_{n,p,q}^{\lambda}(\Phi_x^1;x) &\le \frac{1-q}{q} + \frac{p^n}{[n+2]_{p,q}} + \frac{2[n+1]_{p,q}(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad + \frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\le \frac{1-q}{q} + \frac{1}{[n+2]_{p,q}} + \frac{2}{p^{1-n}[n]_{p,q}-1} \\ &\quad + \frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \end{aligned} \tag{30}$$

Thus, the desired result of (23) is proved. Finally, by Lemma 2 and the linear property of $D_{n,p,q}^{\lambda}(f)$, we have

$$\begin{aligned} D_{n,p,q}^{\lambda}(\Phi_x^2;x) &= D_{n,p,q}^{\lambda}(e_2;x) - 2xD_{n,p,q}^{\lambda}(e_1;x) + x^2 \\ &\le \left(\frac{q^3[n]_{p,q}^2}{[n+2]_{p,q}[n+3]_{p,q}} - \frac{2}{q} + 1\right)x^2 \\ &\quad + \frac{p^{n-1}q[2]_{p,q}^2[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{[2]_{p,q}p^{2n}}{[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad + \frac{2[2]_{p,q}p^n x^2}{q[n+2]_{p,q}} + \Lambda_{n,p,q}^{\lambda}(x) \\ &\le \frac{(1-q)^2}{q^2} + \frac{q[2]_{p,q}^2 + 2[2]_{p,q}}{q[n+2]_{p,q}} \\ &\quad + \frac{[2]_{p,q}\left(q^2+[3]_{p,q}\right)}{q^2[n+2]_{p,q}[n+3]_{p,q}} + \Lambda_{n,p,q}^{\lambda}(x), \end{aligned} \tag{31}$$

where $\Lambda_{n,p,q}^{\lambda}(x)$ is some function related to λ, $[n]_{p,q}$ and x, and we will estimate it in two cases. For $\lambda \in [0,1]$, we have

$$\begin{aligned} \Lambda_{n,p,q}^{\lambda}(x) &\le \frac{4}{p^{1-n}[n]_{p,q}-1} + \frac{[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad + \frac{4}{p[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad + \frac{2}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad + \frac{[2]_{p,q}}{\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad + \frac{2}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \tag{32}$$

For $\lambda \in [-1,0]$, we have

$$\begin{aligned} \Lambda_{n,p,q}^{\lambda}(x) &\le \frac{4}{p^{1-n}[n]_{p,q}-1} + \frac{1}{[n+3]_{p,q}} \\ &\quad + \frac{2[3]_{p,q}+2}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad + \frac{3}{q\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad + \frac{8}{\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad + \frac{2+[2]_{p,q}}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \tag{33}$$

From the above two equations (32) and (33), we obtain

$$\begin{aligned} \Lambda_{n,p,q}^{\lambda}(x) &\le \frac{4}{p^{1-n}[n]_{p,q}-1} + \frac{1}{[n+3]_{p,q}} \\ &\quad + \frac{2[3]_{p,q}+2}{p[\mathrm{n}+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad + \frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad + \frac{8}{\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad + \frac{2+[2]_{p,q}}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \tag{34}$$

Combing (31), (33), and (34), we get

$$D^{\lambda}_{n,p,q}(\Phi_x^2;x) \le \frac{(1-q)^2}{q^2} + \frac{8}{q[n+2]_{p,q}} + \frac{4}{p^{1-n}[n]_{p,q}-1} + \frac{1}{[n+3]_{p,q}} + \frac{8}{q^2[n+2]_{p,q}[n+3]_{p,q}} + \frac{11}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} + \frac{12}{p^{n(n-1)/2}[n+2]_{p,q}[n+3]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \quad (35)$$

Thus, we arrive at (24). Lemma 3 is proved.

Lemma 4. *(See [6]).*
Let sequences $q := \{q_n\} = \{1-\alpha_n\}$, $p := \{p_n\} = \{1-\beta_n\}$ *such that* $0 \le \beta_n < \alpha_n < 1$, $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$. *The following statements are true*

(A) *If* $\lim_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ *and* $e^{n\beta_n}/n \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

(B) *If* $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$ *and* $e^{n\beta_n}(\alpha_n-\beta_n) \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

(C) *If* $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ *and* max $\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n-\beta_n)\} \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

3. Rate of Convergence

In the sequel, let sequences $q := \{q_n\}$ and $p := \{p_n\}$ satisfy the conditions of Lemma 4. We first give a Korovkin type approximation theorem for $D^{\lambda}_{n,p,q}(f)$.

Theorem 5. *Let* f *be a continuous function on* $[0,1]$, $\lambda \in [-1,1]$ *and* $n > 1$, *then* $D^{\lambda}_{n,p,q}(f;x)$ *converge uniformly to* f *on* $[0,1]$.

Proof. Since the hypothesis of sequences p and q, we know that $[n+i]_{p,q} \to \infty (i=1,2,3)$ as $n \to \infty$. It is easy to get $D^{\lambda}_{n,p,q}(e_k;x) \to x^k (k=0,1,2)$ combining the relation $[n+i]_{p,q} = [i]_{p,q}p^n + q^i[n]_{p,q} (i=0,1,2)$. Therefore, we obtain the desired result due to the well-known Korovkin theorem (see [27], pp. 8-9).

Let f be a continuous function on $[0,1]$ and endowed with the norm $\|f\| = \sup_{x\in[0,1]}|f(x)|$. Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C^2}\left\{\|f-g\| + \delta\|g''\|\right\}, \quad (36)$$

where $\delta > 0$ and $C^2 = \{g \in C[0,1]: g', g'' \in C[0,1]\}$. The second order modulus of smoothness is defined as

$$\omega_2(f;\delta) = \sup_{0<h\le\delta}\ \sup_{x,x+h,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|. \quad (37)$$

We know that there is a relationship between $K_2(f;\delta)$ and $\omega_2(f;\sqrt{\delta})$, that is

$$K_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right), \quad (38)$$

where C is a positive constant. The modulus of continuity is denoted by

$$\omega(f;\delta) = \sup_{0<h\le\delta}\ \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|. \quad (39)$$

Then, the rate of convergence of $D^{\lambda}_{n,p,q}(f)$ to f is given as follows.

Theorem 6. *Let* f *be a continuous function on* $[0,1]$, $\lambda \in [-1,1]$, *and* $n > 1$, *we have*

$$\left|D^{\lambda}_{n,p,q}(f;x) - f(x)\right| \le C\omega_2\left(f;\frac{1}{2}\sqrt{(\Theta(n;p,q))^2 + \Phi(n;p,q)}\right) + \omega(f;\Theta(n;p,q)), \quad (40)$$

where C *is a positive constant,* $\Theta(n;p,q)$ *and* $\Psi(n;p,q)$ *are defined in (23) and (24).*

Proof. Let us define auxiliary operators $\widehat{D}^{\lambda}_{n,p,q}(f;x)$ which preserve linear functions as

$$\widehat{D}^{\lambda}_{n,p,q}(f;x) = D^{\lambda}_{n,p,q}(f;x) - f\left(x + \Omega^{\lambda}_{n,p,q}(x)\right) + f(x), \quad (41)$$

where $\Omega^{\lambda}_{n,p,q}(x)$ is defined in (22). Obviously,

$$\widehat{D}^{\lambda}_{n,p,q}(t-x;x) = 0. \quad (42)$$

Set $g \in C^2$, by Taylor's expansion, we have

$$g(t) = g(x) + g'(t-x) + \int_x^t (t-u)g''(u)du, x, t \in [0,1]. \quad (43)$$

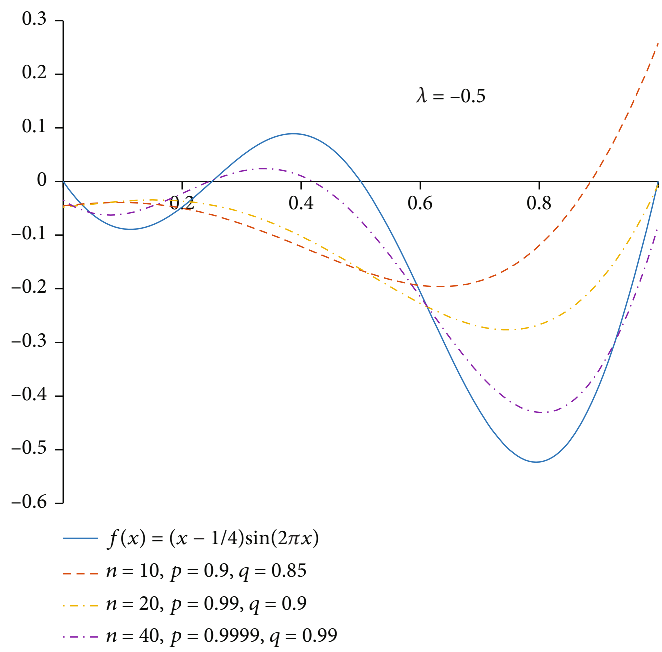

Figure 1: The convergence of $D^{-0.5}_{n,p,q}(f\,;x)$ to $f(x)$.

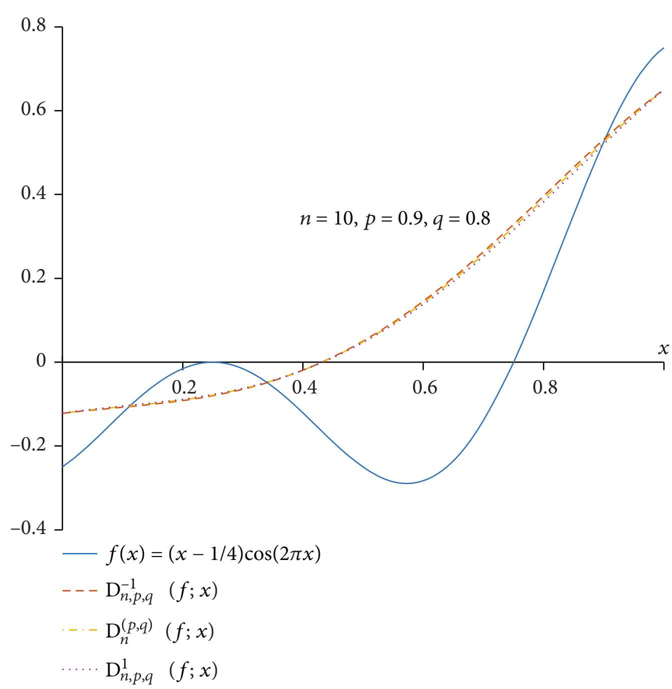

Figure 2: The approximation graphs of $D^{(p,q)}_{n}(f\,;x)$, $D^{-1}_{n,p,q}(f\,;x)$, and $D^{1}_{n,p,q}(f\,;x)$.

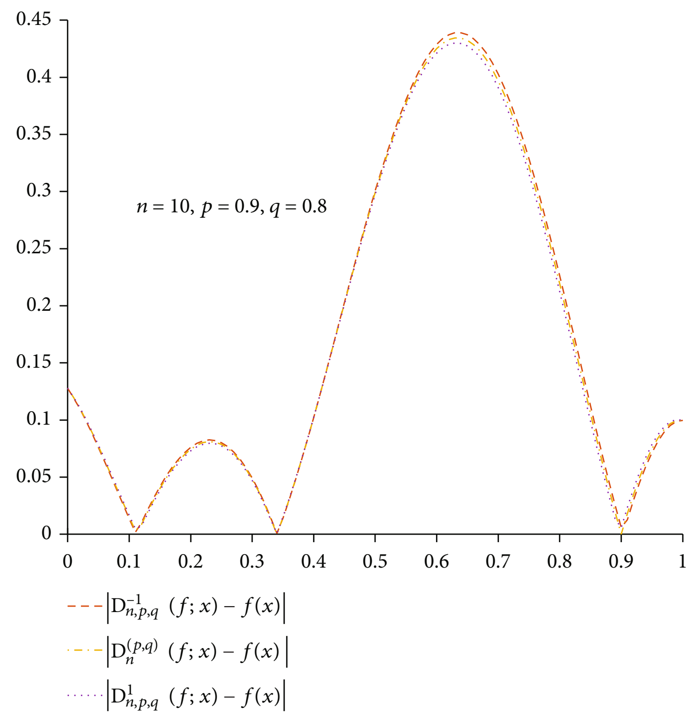

Figure 3: The error graphs of $|D_n^{(p,q)}(f\,;x)-f(x)|$, $|D^{-1}_{n,p,q}(f\,;x)-f(x)|$, and $|D^{1}_{n,p,q}(f\,;x)-f(x)|$.

Table 1: The maximum error of $|D_n^{(p,q)}(f\,;x)-f(x)|$, $|D^{-1}_{n,p,q}(f\,;x)-f(x)|$ and $|D^{1}_{n,p,q}(f\,;x)-f(x)|$ with $p=1-1/n$, $q=1-2/n$.

n	10	20	30	40	50	70	100
$\max\left(\left\|D^{-1}_{n,p,q}(f\,;x)-f(x)\right\|\right)$	0.4390	0.2868	0.2110	0.1672	0.1385	0.1033	0.0749
$\max\left(\left\|D_n^{(p,q)}(f\,;x)-f(x)\right\|\right)$	0.4345	0.2848	0.2101	0.1667	0.1382	0.1031	0.0748
$\max\left(\left\|D^{1}_{n,p,q}(f\,;x)-f(x)\right\|\right)$	0.4301	0.2827	0.2091	0.1661	0.1379	0.1030	0.0747

Applying $\widehat{D}^{\lambda}_{n,p,q}(g\,;x)$ to (43) and by (42), we obtain

$$\widehat{D}^{\lambda}_{n,p,q}(g\,;x)-g(x)=\widehat{D}^{\lambda}_{n,p,q}\left(\int_x^t (t-u)g''(u)du\,;x\right). \quad (44)$$

Thus, by (41), we have

$$\begin{aligned}\left|\widehat{D}^{\lambda}_{n,p,q}(g\,;x)-g(x)\right| &\le \left|D^{\lambda}_{n,p,q}\left(\int_x^t (t-u)g''(u)du\,;x\right)\right| \\ &\quad+\left|\int_x^{x+\Omega^{\lambda}_{n,p,q}(x)}\left(x+\Omega^{\lambda}_{n,p,q}(x)-u\right)g''(u)du\right| \\ &\le \left[D^{\lambda}_{n,p,q}(\Phi_x^2\,;x)+\left(\Omega^{\lambda}_{n,p,q}(x)\right)^2\right]\|g''\| \\ &\le \left[(\Theta(n\,;p,q))^2+\Phi(n\,;p,q)\right]\|g''\|,\end{aligned} \quad (45)$$

where $\Theta(n\,;p,q)$ and $\Psi(n\,;p,q)$ are defined in (23) and (24). According to (41), (5), and (15), we have

$$\left|\widehat{D}^{\lambda}_{n,p,q}(f\,;x)\right| \le \left|D^{\lambda}_{n,p,q}(f\,;x)\right|+2\|f\|\le 3\|f\|. \quad (46)$$

Using (41), (45), and (46), we get

$$\begin{aligned}\left|D^{\lambda}_{n,p,q}(f\,;x)-f(x)\right| &\le \left|\widehat{D}^{\lambda}_{n,p,q}(f-g\,;x)-(f-g)(x)\right| \\ &\quad+\left|\widehat{D}^{\lambda}_{n,p,q}(g\,;x)-g(x)\right| \\ &\quad+\left|f\left(x+\Omega^{\lambda}_{n,p,q}(x)\right)-f(x)\right| \\ &\le 4\|f-g\|+\left[(\Theta(n\,;p,q))^2\right. \\ &\quad\left.+\Phi(n\,;p,q)\right]\|g''\|+\omega(f\,;\Theta(n\,;p,q)).\end{aligned} \quad (47)$$

Taking the infimum on the right hand side over all $g \in C^2$, we have

$$\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| \leq 4K_2\left(f;\frac{1}{4}\left[(\Theta(n;p,q))^2+\Phi(n;p,q)\right]\right) + \omega(f;\Theta(n;p,q)). \quad (48)$$

Therefore, we obtain

$$\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| \leq C\omega_2\left(f;\frac{1}{2}\sqrt{(\Theta(n;p,q))^2+\Phi(n;p,q)}\right) + \omega(f;\Theta(n;p,q)). \quad (49)$$

where C is a positive constant, $\Theta(n;p,q)$ and $\Psi(n;p,q)$ are defined in (23) and (24). Theorem 6 is proved.

Let f be a continuous function on [0, 1], the Steklov mean function is defined as

One can write

$$f_h(x)-f(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left[2f(x+u+v) - f(x+2(u+v))-f(x)\right]dudv \quad (50)$$

by the fact that f_h is continuous on [0, 1]. It is obvious that

$$||f_h-f|| \leq \tilde{\omega}_2(f,h), \quad (51)$$

where $\tilde{\omega}_2(f,\delta)=\sup_{x,u,v\geq 0}\sup_{|u-v|\leq\delta}|f(x+2u)-2f(x+u+v)+f(x+2v)|$. If f is continuous on [0, 1], so are f'_h, f''_h and

$$||f'_h|| \leq \frac{5}{h}\tilde{\omega}(f,h), \; ||f''_h|| \leq \frac{9}{h^2}\tilde{\omega}_2(f,h), \quad (52)$$

where $\tilde{\omega}(f,h)=\sup_{x,u,v\geq 0,|u-v|<h}|f(x+u)-f(x+v)|$. Details can be found in [28].

Now, we apply Steklov mean to prove the following theorem.

Theorem 7. *Let f be a continuous function on [0, 1], $\lambda \in [-1, 1]$, and $n > 1$, we have*

$$\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| \leq 5\tilde{\omega}(f,\Theta(n;p,q)) + \frac{13}{2}\tilde{\omega}_2\left(f,\sqrt{\Phi(n;p,q)}\right), \quad (53)$$

where $\Theta(n;p,q)$ and $\Psi(n;p,q)$ are defined in (23) and (24).

Proof. Since

$$\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| \leq D^{\lambda}_{n,p,q}(|f-f_h|;x) + \left|D^{\lambda}_{n,p,q}(f_h-f_h(x);x)\right| + |f_h(x)-f(x)|. \quad (54)$$

By (5), (15), and (51), we have

$$D^{\lambda}_{n,p,q}(|f-f_h|;x) \leq \left|\left|D^{\lambda}_{n,p,q}(f-f_h)\right|\right| \leq ||f-f_h|| \leq \tilde{\omega}_2(f,h). \quad (55)$$

By Taylor's expansion, Lemma 3 and (52), we obtain

$$\begin{aligned}\left|D^{\lambda}_{n,p,q}(f_h-f_h(x);x)\right| &\leq |f'_h(x)|D^{\lambda}_{n,p,q}(\Phi^1_x;x) + \frac{1}{2}||f''_h||D^{\lambda}_{n,p,q}(\Phi^2_x;x) \\ &\leq \frac{5\Theta(n;p,q)}{h_1}\tilde{\omega}(f,h_1) + \frac{9\Psi(n;p,q)}{2h_2^2}\tilde{\omega}_2(f,h_2).\end{aligned} \quad (56)$$

Therefore, by (54)–(56), we have

$$\begin{aligned}\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| &\leq \frac{5\Theta(n;p,q)}{h_1}\tilde{\omega}(f,h_1) + \left(\frac{9\Psi(n;p,q)}{2h_2^2}+2\right)\tilde{\omega}_2(f,h_2) \\ &= 5\tilde{\omega}(f,\Theta(n;p,q)) + \frac{13}{2}\tilde{\omega}_2\left(f,\sqrt{\Phi(n;p,q)}\right),\end{aligned} \quad (57)$$

by choosing $h_1=\Theta(n;p,q)$, $h_2=\sqrt{\Psi(n;p,q)}$. Theorem 7 is proved.

Finally, we study the rate of convergence of $D^{\lambda}_{n,p,q}(f)$ with the help of functions of Lipschitz class $\mathrm{Lip}_M(\xi)$, where M is a positive constant, $0<\xi\leq 1$. A function f belongs to $\mathrm{Lip}_M(\xi)$ if

$$|f(t)-f(x)| \leq M|t-x|^{\xi}(t,x\in[0,1]). \quad (58)$$

We have the following theorem.

Theorem 8. *Let $f \in Lip_M(\xi)$, $\lambda \in [-1, 1]$, and $n > 1$, we have*

$$\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| \leq M(\Psi(n;p,q))^{\xi/2}, \quad (59)$$

where $\Psi(n;p,q)$ is defined in (24).

Proof. Since $f \in \mathrm{Lip}_M(\xi)$ and $D^{\lambda}_{n,p,q}(f)$ are linear positive operators, using Hölder's inequality, we have

$$\begin{aligned}\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| &\le D^{\lambda}_{n,p,q}(|f(t)-f(x)|;x)\\ &\le MD^{\lambda}_{n,p,q}\left(|t-x|^{\xi};x\right)\\ &= M[n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\left[\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right]^{2-\xi/2}\\ &\quad\cdot\left[\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right]^{\xi/2}\int_0^1 b_{n,k}(qt;p,q)|t-x|^{\xi}d_{p,q}t\\ &\le M\left[[n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right.\\ &\quad\left.\int_0^1 b_{n,k}(qt;p,q)d_{p,q}t\right]^{2-\xi/2}\\ &\quad\left[[n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right.\\ &\quad\left.\int_0^1 b_{n,k}(qt;p,q)(t-x)^2d_{p,q}t\right]^{\xi/2}\\ &= MD^{\lambda}_{n,p,q}\left((t-x)^2;x\right)^{\xi/2}.\end{aligned} \tag{60}$$

Thus, Theorem 8 can be obtained by (24).

4. Numerical Examples

In this section, we give several numerical examples to show the convergence of $D^{\lambda}_{n,p,q}(f;x)$ and $D^{(p,q)}_{n}(f;x)$ to $f(x)$ with different values of parameters.

Example 9. Let $f(x)=(x-1/4)\sin(2\pi x)$ and $\lambda=-0.5$. The graphs of $f(x)$ and $D^{\lambda}_{n,p,q}(f;x)$ with different values of parameters $(n=10, p=0.9, q=0.85;n=20, p=0.99, q=0.9;n=40, p=0.9999, q=0.99)$ are shown in Figure 1.

Example 10. Let $f(x)=(x-1/4)\cos(2\pi x)$ and $n=10, p=0.9, q=0.8$. The graphs of $f(x)$, $D^{(p,q)}_{n}(f;x)$ and $D^{\lambda}_{n,p,q}(f;x)$ for $\lambda=-1$ and $\lambda=1$ are given in Figure 2. The error graphs of $|D^{(p,q)}_{n}(f;x)-f(x)|$ and $|D^{\lambda}_{n,p,q}(f;x)-f(x)|$ for $\lambda=-1$ and $\lambda=1$ are given in Figure 3. Moreover, in Table 1, there are given the maximum errors of $|D^{(p,q)}_{n}(f;x)-f(x)|$, $|D^{-1}_{n,p,q}(f;x)-f(x)|$, and $|D^{1}_{n,p,q}(f;x)-f(x)|$ with different values of parameters, where $p=1-1/n, q=1-2/n$.

5. Conclusion

In the present paper, we proposed a class of Durrmeyer type λ-Bernstein operators based on (p, q)-calculus. Due to the parameter λ, we have more flexibility in modeling. We studied the Korovkin type theorem, the estimated rate of convergence by using Peetres K-functional, the modulus of continuity of second order and Steklov mean; we also obtained a convergence theorem for the Lipschitz continuous functions. To make things more intuitive, we also give some numerical examples.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

We declare that there is no conflict of interest.

Acknowledgments

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Research Article

Approximation by Parametric Extension of Szász-Mirakjan-Kantorovich Operators Involving the Appell Polynomials

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The purpose of this article is to introduce a Kantorovich variant of Szász-Mirakjan operators by including the Dunkl analogue involving the Appell polynomials, namely, the Szász-Mirakjan-Jakimovski-Leviatan-type positive linear operators. We study the global approximation in terms of uniform modulus of smoothness and calculate the local direct theorems of the rate of convergence with the help of Lipschitz-type maximal functions in weighted space. Furthermore, the Voronovskaja-type approximation theorems of this new operator are also presented.

1. Introduction

In the year 1950, a famous mathematician Szász [1] invented the positive linear operators for the continuous function f on $[0,\infty)$ and that were extensively searched rather than Bernstein operators [2]. For $z\in[0,\infty)$ and $f\in C[0,\infty)$, Szász introduced the operators as follows:

$$S_r(f;z)=e^{-rz}\sum_{k=0}^{\infty}\frac{(rz)^k}{k!}f\left(\frac{k}{r}\right), \tag{1}$$

where $C[0,\infty)$ is the space of continuous functions on $[0,\infty)$. In recent years, Szász-Mirakjan operators were introduced by Sucu [3] by proposing an exponential function on Dunkl generalization by including a nonnegative number $\eta\geq 0$, such that

$$\mathcal{S}_r^*(f;z)=\frac{1}{e_\eta(rz)}\sum_{\kappa=0}^{\infty}\frac{(rz)^\kappa}{\gamma_\eta(\kappa)}f\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right), \tag{2}$$

where $e_\eta(z)=\sum_{\kappa=0}^{\infty}z^\kappa/\gamma_\eta(\kappa)$ and a recursion formula for $s=0,1,2,\cdots$.

$$\frac{\gamma_\eta(\kappa+1)}{(\kappa+1+2\eta\theta_{\kappa+1})}=\gamma_\eta(\kappa),$$
$$\theta_\kappa=\begin{cases}0 & \text{if } \kappa=2r,\ r\in\mathbb{N}\cup\{0\},\\ 1 & \text{if } \kappa=2r+1,\ r\in\mathbb{N}\cup\{0\}.\end{cases} \tag{3}$$

In 1969, Jakimovski and Leviatan introduced the sequence of Szász-Mirakjan-type positive linear operators by the use of Appell polynomials [4], $L(u)e^{uz}=\sum_{\kappa=0}^{\infty}H_\kappa(z)u^\kappa$ such that

$$J_r(h;z)=\frac{e^{-rz}}{L(1)}\sum_{\kappa=0}^{\infty}H_\kappa(rz)f\left(\frac{\kappa}{r}\right), \tag{4}$$

where $L(1)\neq 0$, $L(u)=\sum_{\kappa=0}^{\infty}b_\kappa u^\kappa$, $H_\kappa(z)=\sum_{j=0}^{\kappa}b_j(z^{\kappa-j}/(\kappa-j)!)(\kappa\in\mathbb{N})$. Note that, if $L(1)=1$ in (4), the Szász-Mirakjan operator (1) is obtained. Most recently, in [5], Nasiruzzaman and Aljohani have introduced the Szász-Mirakjan-Jakimovski-Leviatan-type operators involving the Dunkl generalization for the function $f \in C[0,\infty)$ by

$$\mathscr{P}^{*}_{r,\eta}(f\,;z)=\frac{1}{L(1)e_{\eta}(rz)}\sum_{\kappa=0}^{\infty}H_{\kappa}(rz)f\left(\frac{\kappa+2\eta\theta_{\kappa}}{r}\right). \quad (5)$$

Lemma 1. [5]. *For the test function $\gamma_j=t^j$, if $j=0,1,2,3,4$, the operators $\mathscr{P}^{*}_{r,\eta}$ have $\mathscr{P}^{*}_{r,\eta}(\gamma_0\,;z)=1$, $\mathscr{P}^{*}_{r,\eta}(\gamma_1\,;z)=z+(1/r)((L'(1)/L(1))+2\eta)$, and the following identities:*

$$\mathscr{P}^{*}_{r,\eta}(\gamma_2\,;z)=z^2+\frac{1}{r}\left(\frac{2L'(1)}{L(1)}+4\eta+1\right)z+\frac{1}{r^2}\cdot\left(\frac{L''(1)}{L(1)}+(1+4\eta)\frac{L'(1)}{L(1)}+4\eta^2\right),$$

$$\mathscr{P}^{*}_{r,\eta}(\gamma_3\,;z)=z^3+\frac{3}{r}\left(\frac{L'(1)}{L(1)}+2\eta+1\right)z^2+\frac{1}{r^2}\cdot\left(\frac{3L''(1)}{L(1)}+6(1+2\eta)\frac{L'(1)}{L(1)}+2+6\eta+12\eta^2\right)z+\frac{1}{r^3}\cdot\left(\frac{3L'''(1)}{L(1)}+3(1+2\eta)\frac{L''(1)}{L(1)}+2\left(1+3\eta+6\eta^2\right)\frac{L'(1)}{L(1)}+8\eta^3\right),$$

$$\mathscr{P}^{*}_{r,\eta}(\gamma_4\,;z)=z^4+\frac{1}{r}\left(\frac{4L'(1)}{L(1)}+8\eta+6\right)z^3+\frac{1}{r^2}\cdot\left(\frac{6L''(1)}{L(1)}+8(1+3\eta)\frac{L'(1)}{L(1)}+11+24\eta+24\eta^2\right)z^2+\frac{1}{r^3}\left(\frac{4L'''(1)}{L(1)}+8(1+3\eta)\frac{L''(1)}{L(1)}+2\left(11+24\eta+24\eta^2\right)\cdot\frac{L'(1)}{L(1)}+6+16\eta+24\eta^{+}32\eta^3\right)z+\frac{1}{r^4}\cdot\left(\frac{L'v(1)}{L(1)}+2(3+4\eta)\frac{L'''(1)}{L(1)}+\left(11+24\eta+24\eta^2\right)\cdot\frac{L''(1)}{L(1)}+\left(6+16\eta+24\eta^2+32\eta^3\right)\frac{L'(1)}{L(1)}+16\eta^4\right). \quad (6)$$

There are several research articles mentioned regarding the Szász-Mirakjan-type operators, for instance, [6–13]. For some further related concepts and approximation, we refer to see [9, 10, 14–20].

2. Kantorovich Operators Involving Appell Polynomials and Their Moments

In this section, we construct the generalized operators of recent investigation [5] including the Kantorovich polynomial. For this purpose, we let $f \in C_{\Phi}[0,\infty)=\{f\in C[0,\infty): f(t)=O(t^{\Phi})\}$ as $t\to\infty$; then, for all $z\in[0,\infty)$, $\Phi>r$, $r\in\mathbb{N}$, $L(1)\neq 0$, and $\eta\geq 0$, we define the operators as follows:

$$\mathscr{R}^{*}_{r,\eta}(f\,;z)=\frac{r}{L(1)e_{\eta}(rz)}\sum_{\kappa=0}^{\infty}H_{\kappa}(rz)\int_{(\kappa+2\eta\theta_{\kappa})/r}^{(\kappa+1+2\eta\theta_{\kappa})/r}f(t)\mathrm{d}t. \quad (7)$$

Lemma 2. *For $j=0,1,2,3,4$, let the test function be $\gamma_j=t^j$. Then, operators $\mathscr{R}^{*}_{r,\eta}(\cdot;\cdot)$ have the following identities:*

$$\mathscr{R}^{*}_{r,\eta}(\gamma_0\,;z)=1,$$

$$\mathscr{R}^{*}_{r,\eta}(\gamma_1\,;z)=z+\frac{1}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right),$$

$$\mathscr{R}^{*}_{r,\eta}(\gamma_2\,;z)=z^2+\frac{1}{r}\left(2\frac{L'(1)}{L(1)}+4\eta+2\right)z+\frac{1}{3r^2}\cdot\left(3\frac{L''(1)}{L(1)}+6(1+2\eta)\frac{L'(1)}{L(1)}+12\eta^2+6\eta+1\right),$$

$$\mathscr{R}^{*}_{r,\eta}(\gamma_3\,;z)=z^3+\frac{3}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+3\right)z^2+\frac{3}{2r^2}\cdot\left(2\frac{L''(1)}{L(1)}+2(3+4\eta)\frac{L'(1)}{L(1)}+8\eta^2+8\eta+3\right)z+\frac{1}{4r^3}\left(12\frac{L'''(1)}{L(1)}+6(3+4\eta)\frac{L''(1)}{L(1)}+6\cdot\left(8\eta^2+8\eta+3\right)\frac{L'(1)}{L(1)}+32\eta^3+8\eta^2+8\eta+1\right),$$

$$\mathscr{R}^{*}_{r,\eta}(\gamma_4\,;z)=z^4+\frac{1}{r}\left(4\frac{L'(1)}{L(1)}+8\eta+8\right)z^3+\frac{1}{r^2}\cdot\left(6\frac{L''(1)}{L(1)}+(14+24\eta)\frac{L'(1)}{L(1)}+24\eta^2+36\eta+9\right)z^2+\frac{1}{r^3}\left(4\frac{L'''(1)}{L(1)}+(14+24\eta)\frac{L''(1)}{L(1)}+\left(38+72\eta+48\eta^2\right)\frac{L'(1)}{L(1)}+32\eta^3+48\eta^2+36\eta+13\right)z+\frac{1}{r^4}\left(\frac{L'v(1)}{L(1)}+4(3+2\eta)\frac{L'''(1)}{L(1)}+\left(19+36\eta+24\eta^2\right)\cdot\frac{L''(1)}{L(1)}+\left(13+36\eta+48\eta^2+32\eta^3\right)\frac{L'(1)}{L(1)}+16\eta^4+16\eta^3+8\eta^2+2\eta+1\right). \quad (8)$$

Proof. To prove this Lemma, we take into account [5] Lemma 1. Thus, for all $j=0,1,2,3,4$ and $\gamma_j=t^j$, we can conclude that

$$\int_{(\kappa+2\eta\theta_\kappa)/r}^{(\kappa+1+2\eta\theta_\kappa)/r} t^j \mathrm{d}t = \begin{cases} \frac{1}{r} & \text{for } j=0, \\ \frac{1}{2r^2}+\frac{1}{r}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right) & \text{for } j=1, \\ \frac{1}{3r^3}+\frac{1}{r^2}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)+\frac{1}{r}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)^2 & \text{for } j=2, \\ \frac{1}{4r^4}+\frac{1}{r^3}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)+\frac{3}{2r^2}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)^2+\frac{1}{r}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)^3 & \text{for } j=3, \\ \frac{1}{5r^5}+\frac{1}{r^4}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)+\frac{2}{r^3}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)^2+\frac{2}{r^2}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)^3+\frac{1}{r}\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right)^4 & \text{for } j=4. \end{cases} \tag{9}$$

Thus, from (7) and (9), clearly we can write

$$\begin{aligned}
\mathscr{R}^*_{r,\eta}(\gamma_0; z) &= \mathscr{P}^*_{r,\eta}(\gamma_0; z) = 1, \\
\mathscr{R}^*_{r,\eta}(\gamma_1; z) &= \mathscr{P}^*_{r,\eta}(\gamma_1; z) + \frac{1}{2r}\mathscr{P}^*_{r,\eta}(\gamma_0; z), \\
\mathscr{R}^*_{r,\eta}(\gamma_2; z) &= \mathscr{P}^*_{r,\eta}(\gamma_2; z) + \frac{1}{r}\mathscr{P}^*_{r,\eta}(\gamma_1; z) + \frac{1}{3r^2}\mathscr{P}^*_{r,\eta}(\gamma_0; z), \\
\mathscr{R}^*_{r,\eta}(\gamma_3; z) &= \mathscr{P}^*_{r,\eta}(\gamma_3; z) + \frac{3}{2r}\mathscr{P}^*_{r,\eta}(\gamma_2; z) + \frac{1}{r^2}\mathscr{P}^*_{r,\eta}(\gamma_1; z) \\
&\quad + \frac{1}{4r^3}\mathscr{P}^*_{r,\eta}(\gamma_0; z), \\
\mathscr{R}^*_{r,\eta}(\gamma_4; z) &= \mathscr{P}^*_{r,\eta}(\gamma_4; z) + \frac{2}{r}\mathscr{P}^*_{r,\eta}(\gamma_3; z) + \frac{2}{r^2}\mathscr{P}^*_{r,\eta}(\gamma_2; z) \\
&\quad + \frac{1}{r^3}\mathscr{P}^*_{r,\eta}(\gamma_1; z) + \frac{1}{r^4}\mathscr{P}^*_{r,\eta}(\gamma_0; z).
\end{aligned} \tag{10}$$

Therefore, by applying Lemma 1, we get the required results.

Lemma 3. *For the central moments $(\gamma_1 - z)^i, i = 1, 2, 4,$ we have the following identities:*

$$\mathscr{R}^*_{r,\eta}((\gamma_1 - z); z) = \frac{1}{2r}\left(2\frac{L'(1)}{L(1)} + 4\eta + 1\right),$$

$$\mathscr{R}^*_{r,\eta}\left((\gamma_1 - z)^2; z\right) = \frac{z}{r} + \frac{1}{3r^2}\left(3\frac{L''(1)}{L(1)} + 6(1+2\eta)\frac{L'(1)}{L(1)} + 12\eta^2 + 6\eta + 1\right),$$

$$\begin{aligned}
\mathscr{R}^*_{r,\eta}\left((\gamma_1 - z)^4; z\right) &= -\frac{7}{r^2}z^2 + \frac{1}{r^3}\left(-8\frac{L'''(1)}{L(1)} - 4\frac{L''(1)}{L(1)} + (24\eta+20)\frac{L'(1)}{L(1)} + 40\eta^2 + 28\eta + 12\right)z \\
&\quad + \frac{1}{r^4}\left(\frac{L'v(1)}{L(1)} + 4(3+2\eta)\frac{L'''(1)}{L(1)} + \left(19 + 36\eta + 24\eta^2\right)\frac{L''(1)}{L(1)} + (13 + 36\eta \right. \\
&\quad \left. + 48\eta^2 + 32\eta^3)\frac{L'(1)}{L(1)} + 16\eta^4 + 16\eta^3 + 8\eta^2 + 2\eta + 1\right).
\end{aligned} \tag{11}$$

3. Approximations in Weighted Space

In the present section, we follow the well-known results by Gadziev [21] and recall the results in weighted spaces with some additional conditions precisely, under the analogous of P.P. Korovkin's theorem holds. In order to define the uniformly approximations, we take $z \to \varphi(z)$ be the kind of functions which is continuous and strictly increasing with the assumptions $\Phi(z) = 1 + \varphi^2(z)$ and $\lim_{z\to\infty}\Phi(z) = \infty$. For this reason, we let $B_\Phi[0,\infty)$ be a set of all such functions which are defined on $[0,\infty)$ and verifying the results

$$B_\Phi[0,\infty) = \{f : |f(z)| \le K_f\Phi(z)\}, \tag{12}$$

where K_f is a constant and depending only on function f and $B_\Phi[0,\infty)$ equipped the norm with

$$\|f\|_\Phi = \sup_{z\in[0,\infty)} \frac{|f(z)|}{\Phi(z)}. \tag{13}$$

Furthermore, we denote the set all continuous functions on $[0,\infty)$ by $C[0,\infty)$ and its subsets be $C_\Phi[0,\infty)$ defined as $C_\Phi[0,\infty) = B_\Phi[0,\infty) \cap C[0,\infty)$.

It is well known for the sequence of linear positive operators $\{K_r\}_{r\geq 1}$ (see [21]) maps $C_\Phi[0,\infty)$ into $B_\Phi[0,\infty)$ if and only if

$$|K_r(\Phi\,;z)| \leq M\Phi(z), \tag{14}$$

where M is a positive constant. For $m \in \mathbb{N}$, let us denote

$$C_\Phi^m[0,\infty) = \left\{ f \in C_\Phi[0,\infty):\ \lim_{z\to\infty} \frac{f(z)}{\Phi(z)} = c, \text{exists and is finite} \right\}. \tag{15}$$

Theorem 4. *Let $H_f = \{f$: such that $f(z)/\Phi(z)$ is convergent when $z \to \infty\}$. Then, for every $f \in H_f \cap C_\Phi[0,\infty)$, operators (7) are uniformly convergent on each compact subset of $[0,\infty)$ such that*

$$\mathcal{R}_{r,\eta}^*(f\,;z) \rightrightarrows f, \tag{16}$$

where $\rightrightarrows$ denotes the uniform convergence.

Proof. In view of Lemma 2, we use Korovkin's theorem by [22]; then, it is enough to see that for each $j = 0, 1, 2$

$$\mathcal{R}_{r,\eta}^*\left(\gamma_j\,;z\right) \to z^j \tag{17}$$

uniformly. Thus obviously, we get $\lim_{r\to\infty}\mathcal{R}_{r,\eta}^*(\gamma_0\,;z) = 1$, $\lim_{r\to\infty}\mathcal{R}_{r,\eta}^*(\gamma_1\,;z) = z$, and $\lim_{r\to\infty}\mathcal{R}_{r,\eta}^*(\gamma_2\,;z) = z^2$, which completes the proof of Theorem 4.

Theorem 5 [21, 23]. *Let the positive linear operators $\{J_r\}_{r\geq 1}$ acting from $C_\Phi[0,\infty)$ to $B_\Phi[0,\infty)$ and for $j = 0, 1, 2$ if it verifies that $\lim_{r\to\infty}\|J_r(\gamma_j) - z^j\|_\Phi = 0$, then for every $f \in C_\Phi^m[0,\infty)$ it satisfies*

$$\lim_{r\to\infty}\|J_r(f) - f\|_\Phi = 0. \tag{18}$$

Theorem 6. *For every $\varphi \in C_\Phi^m[0,\infty)$, operators $\mathcal{R}_{r,\eta}^*$ satisfy*

$$\lim_{r\to\infty}\left\|\mathcal{R}_{r,\eta}^*(\varphi) - \varphi\right\|_\Phi = 0. \tag{19}$$

Proof. It is enough to prove Theorem 6; we use the well-known Korovkin theorem and show

$$\lim_{r\to\infty}\|\mathcal{R}_{r,\eta}^*\left(\gamma_j\right) - z^j\|_\Phi = 0, \quad j = 0, 1, 2. \tag{20}$$

Taking into account Lemma 2, then it is easy to see that

$$\|\mathcal{R}_{r,\eta}^*(\gamma_0) - 1\|_\Phi = \sup_{z\in[0,\infty)} \frac{|\mathcal{R}_{r,\eta}^*(1\,;z) - 1\,|}{\Phi(z)} = 0. \tag{21}$$

For $j = 1$, we can write here

$$\begin{aligned}\|\mathcal{R}_{r,\eta}^*(\gamma_1) - z\|_\Phi &= \sup_{z\in[0,\infty)} \frac{\left|\mathcal{R}_{r,\eta}^*(\gamma_1\,;z) - z\right|}{\Phi(z)} \\ &= \sup_{z\in[0,\infty)} \frac{1}{\Phi(z)}\left|\frac{1}{2r}\left(\frac{2L'(1)}{L(1)} + 4\eta + 1\right)\right|.\end{aligned} \tag{22}$$

If $r \to \infty$, then easily we get $\|\mathcal{R}_{r,\eta}^*(\gamma_1) - z\|_\Phi \to 0$. Similarly, for $j = 2$, we conclude that

$$\begin{aligned}\|\mathcal{R}_{r,\eta}^*(\gamma_2) - z^2\|_\Phi &= \sup_{z\in[0,\infty)} \frac{|\mathcal{R}_{r,\eta}^*(\gamma_2\,;z) - z^2\,|}{\Phi(z)} \\ &= \sup_{z\in[0,\infty)} \frac{z}{1+z^2}\left|\frac{1}{r}\left(2\frac{L'(1)}{L(1)} + 4\eta + 2\right)\right| \\ &\quad + \sup_{z\in[0,\infty)} \frac{1}{\Phi(z)}\left|\frac{1}{3r^2}\left(3\frac{L''(1)}{L(1)} + 6(1+2\eta)\right.\right. \\ &\quad \left.\left.\cdot\frac{L'(1)}{L(1)} + 12\eta^2 + 6\eta + 1\right)\right|.\end{aligned} \tag{23}$$

Thus, we easily get $\|\mathcal{R}_{r,\eta}^*(\gamma_2) - z^2\|_\Phi \to 0$, as $r \to \infty$.

Theorem 7. *If $\varphi \in C_\Phi^m[0,\infty)$. Then, operators $\mathcal{R}_{r,\eta}^*$ follow that*

$$\lim_{r\to\infty}\sup_{z\in[0,\infty)} \frac{|\mathcal{R}_{r,\eta}^*(\varphi\,;z) - \varphi(z)\,|}{(\Phi(z))^{1+\xi}} = 0, \tag{24}$$

where the number $\xi \in [0,\infty)$.

Proof. By the virtue of $|\varphi(z)\,| \leq \|\varphi\|_\Phi(1+z^2)$ and for any positive real z_0, we easily obtain

$$\begin{aligned}&\lim_{r\to\infty}\sup_{z\in[0,\infty)} \frac{|\mathcal{R}_{r,\eta}^*(\varphi\,;z) - \varphi(z)\,|}{(\Phi(z))^{1+\xi}} \\ &\quad\leq \sup_{z\leq z_0} \frac{|\mathcal{R}_{r,\eta}^*(\varphi\,;z) - \varphi(z)\,|}{(\Phi(z))^{1+\xi}} + \sup_{z\geq z_0} \frac{|\mathcal{R}_{r,\eta}^*(\varphi\,;z) - \varphi(z)\,|}{(\Phi(z))^{1+\xi}} \\ &\quad\leq \left\|\mathcal{R}_{r,\eta}^*(\varphi\,;z) - \varphi(z)\right\|_{C[0,z_0]} + \|\varphi\|_\Phi \sup_{z\geq z_0} \\ &\qquad\times \frac{|\mathcal{R}_{r,\eta}^*\left(1+t^2\,;z\right) - \varphi(z)\,|}{(\Phi(z))^{1+\xi}} + \sup_{z\geq z_0} \frac{|\varphi(z)\,|}{(\Phi(z))^{1+\xi}} \\ &\quad= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3, \text{(suppose)}.\end{aligned} \tag{25}$$

Thus,

$$\mathcal{J}_3 = \sup_{z\geq z_0} \frac{|\varphi(z)\,|}{(\Phi(z))^{1+\xi}} \leq \sup_{z\geq z_0} \frac{\|\varphi\|_\Phi\left(1+z^2\right)}{(\Phi(z))^{1+\xi}} \leq \frac{\|\varphi\|_\Phi}{\left(1+z_0^2\right)^\xi}. \tag{26}$$

From Lemma 2, it follows that

$$\lim_{r\to\infty}\sup_{z\geq z_0}\frac{\mathscr{R}^*_{r,\eta}\left(1+t^2;z\right)}{\Phi(z)}=1. \tag{27}$$

Now, for each $\varepsilon>0$, there exists $r_1\in\mathbb{N}$ for all $r\geq r_1$ such that

$$\sup_{z\geq z_0}\frac{\mathscr{R}^*_{r,\eta}\left(1+t^2;z\right)}{\Phi(z)}\leq\frac{\left(1+z_0^2\right)^{\xi}}{\|\varphi\|_{\Phi}}\frac{\varepsilon}{3}+1. \tag{28}$$

Therefore, for all $r\geq r_1$

$$\mathscr{I}_2=\|\varphi\|_{\Phi}\sup_{z\geq z_0}\frac{\mathscr{R}^*_{r,\eta}\left(1+t^2;z\right)}{\left(\Phi(z)\right)^{1+\xi}}\leq\frac{\|\varphi\|_{\Phi}}{\left(1+z_0^2\right)^{\xi}}+\frac{\varepsilon}{3}. \tag{29}$$

In view of (26) and (29), we get

$$\mathscr{I}_2+\mathscr{I}_3\leq 2\frac{\|\varphi\|_{\Phi}}{\left(1+z_0^2\right)^{\xi}}+\frac{\varepsilon}{3}. \tag{30}$$

If we choose any z_0 so large, such that $\|\varphi\|_{\Phi}/(1+z_0^2)^{\xi}\leq\varepsilon/6$, then we get

$$\mathscr{I}_2+\mathscr{I}_3\leq\frac{2\varepsilon}{3},\quad\text{for all } r\geq r_1. \tag{31}$$

On the other hand, there exists $r_2\geq r$ such that

$$\mathscr{I}_1=\left\|\mathscr{R}^*_{r,\eta}(\varphi;z)-\varphi(z)\right\|_{C[0,z_0]}\leq\frac{\varepsilon}{3}. \tag{32}$$

Finally, take $r_3=\max(r_1,r_2)$ and on combining (31) and (32) with the above expression, we get

$$\sup_{z\in[0,\infty)}\frac{\left|\mathscr{R}^*_{r,\eta}(\varphi;z)-\varphi(z)\right|}{\left(\Phi(z)\right)^{1+\xi}}<\varepsilon. \tag{33}$$

This completes the proof of Theorem 7.

Definition 8. For every $\bar{\delta}>0$ and all $f\in C[0,\infty)$, the modulus of continuity of the uniformly continuous function f on $[0,\infty)$ defined as

$$\bar{\omega}\left(f;\bar{\delta}\right)=\sup_{|t_1-t_2|\leq\bar{\delta}}\left|f(t_1)-f(t_2)\right|,\ t_1,t_2\in[0,\infty),$$
$$\left|f(t_1)-f(t_2)\right|\leq\left(1+\frac{|t_1-t_2|}{\bar{\delta}^2}\right)\bar{\omega}\left(f;\bar{\delta}\right). \tag{34}$$

Theorem 9 [24]. *Let the sequence of positive linear operators $\{K\}_{r\geq 1}:[x,y]\to C[u,v]$ and $[u,v]\subseteq[x,y]$, then*

(1) for any $f\in C[x,y]$ and $z\in[u,v]$, it follows that

$$\begin{aligned}&\left|K_r(f;z)-f(z)\right|\\&\quad\leq\left|f(z)\right|\left|K_r(1;z)-1\right|\\&\qquad+\left\{K_r(1;z)+\frac{1}{\bar{\delta}}\sqrt{K_r\left((t-z)^2;z\right)}\sqrt{K_r(1;z)}\right\}\bar{\omega}\left(f;\bar{\delta}\right),\end{aligned} \tag{35}$$

(2) if any $\varphi'\in C[x,y]$, then for all $z\in[u,v]$ one has

$$\begin{aligned}&\left|K_r(\phi;z)-\phi(z)\right|\\&\quad\leq\left|\phi(z)\right|\left|K_r(1;z)-1\right|+\left|\phi'(z)\right|\\&\qquad\times\left|K_r(t-z;z)\right|+K_r\left((t-z)^2;z\right)\\&\qquad\times\left\{\sqrt{K_r(1;z)}+\frac{1}{\bar{\delta}}\sqrt{K_r\left((t-z)^2;z\right)}\right\}\bar{\omega}\left(\phi';\bar{\delta}\right).\end{aligned} \tag{36}$$

Theorem 10. *Let $f\in C_{\Phi}[0,\infty)$, then for all $z\in[0,\infty)$ it follows the inequality*

$$\left|\mathscr{R}^*_{r,\eta}(f;z)-f(z)\right|\leq 2\bar{\omega}\left(f;\sqrt{\bar{\delta}^*_{r,\eta}(z)}\right), \tag{37}$$

where $\bar{\delta}=\sqrt{\bar{\delta}^*_{r,\eta}(z)}=\sqrt{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)}$.

Proof. If we consider Lemma 2 and Theorem 9, then we can obtain

$$\begin{aligned}&\left|\mathscr{R}^*_{r,\eta}(f;z)-f(z)\right|\\&\quad\leq\left|f(z)\right|\left|\mathscr{R}^*_{r,\eta}(\gamma_0;z)-1\right|\\&\qquad+\left\{\mathscr{R}^*_{r,\eta}(\gamma_0;z)+\frac{1}{\bar{\delta}}\sqrt{\mathscr{R}^*_{r,\eta}\left((\gamma_0-z)^2;z\right)}\sqrt{\mathscr{R}^*_{r,\eta}(\gamma_0;z)}\right\}\bar{\omega}\\&\qquad\times\left(f;\bar{\delta}\right),\end{aligned} \tag{38}$$

where if we take $\bar{\delta}=\sqrt{\bar{\delta}^*_{r,\eta}(z)}=\sqrt{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)}$ then we are easily denumerable to get results.

Theorem 11. *For any $z\in[0,\infty)$, if $\phi\in C'_{\Phi}[0,\infty)$, then we have the inequality*

$$\begin{aligned}\left|\mathscr{R}^*_{r,\eta}(\phi;z)-\phi(z)\right|&\leq\left|\frac{1}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)\right|\left|\phi'(z)\right|\\&\quad+2\bar{\delta}^*_{r,\eta}(z)\bar{\omega}\left(\phi';\sqrt{\bar{\delta}^*_{r,\eta}(z)}\right),\end{aligned} \tag{39}$$

where $\bar{\delta}=\sqrt{\bar{\delta}^*_{r,\eta}(z)}=\sqrt{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)}$.

Proof. If we consider Lemmas 2 and 3 and (2) of Theorem 9, then it is obvious to get that

$$\begin{aligned}&\left|\mathscr{R}^*_{r,\eta}(\phi;z)-\phi(z)\right|\\&\le\left|\mathscr{R}^*_{r,\eta}(\gamma_0;z)-1\right||\phi(z)|+\left|\phi'(z)\right|\\&\quad\times\left|\mathscr{R}^*_{r,\eta}(\gamma_1-z;z)\right|+\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)\\&\quad\times\left\{\sqrt{\mathscr{R}^*_{r,\eta}(\gamma_0;z)}+\frac{1}{\bar\delta}\sqrt{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)}\right\}\bar\omega\left(\phi';\bar\delta\right).\end{aligned}\tag{40}$$

Put $\bar\delta=\sqrt{\bar\delta^*_{r,\eta}(z)}=\sqrt{\mathscr{R}^*_{r,\eta}((\gamma_1-z)^2;z)}$, then we easily get our desired results of Theorem 11.

From [25] for an arbitrary $f\in C^m_\Phi[0,\infty)$, $m\in\mathbb{N}\cup\{0\}$, the weighted modulus of continuity introduced such that

$$\bar\Omega\left(f;\bar\delta\right)=\sup_{z\in[0,\infty),|h|\le\bar\delta}\frac{|f(z+h)-f(z)|}{\left(1+h^2\right)\left(1+z^2\right)}.\tag{41}$$

Two main properties of this modulus of continuity are $\lim_{\bar\delta\to0}\bar\Omega(f;\bar\delta)=0$ and

$$\begin{aligned}|f(t)-f(z)|&\le2\left(1+\frac{|t-z|}{\bar\delta}\right)\left(1+\bar\delta^2\right)\left(1+z^2\right)\\&\quad\cdot\left(1+(t-z)^2\right)\bar\Omega\left(f;\bar\delta\right),\end{aligned}\tag{42}$$

where $t,z\in[0,\infty)$ and $\bar\Omega$ weighted modulus of continuity of the function for $f\in C^m_\Phi[0,\infty)$.

Theorem 12. *Let $f\in C^m_\Phi[0,\infty)$, then for all $z\in[0,\infty)$ we have the inequality*

$$\sup_{z\in[0,\alpha_r(\eta))}\frac{\left|\mathscr{R}^*_{r,\eta}(f;z)-f(z)\right|}{1+z^2}\le M(1+\alpha_r(\eta))\Omega\left(f;\sqrt{\alpha_r(\eta)}\right),\tag{43}$$

where $M=2(2+M_1+\sqrt{M_2})>0$, for $M_1,M_2>0$ and

$$\begin{aligned}\alpha_r(\eta)=\max&\left\{\frac{1}{r},\frac{1}{3r^2}\left(3\frac{L''(1)}{L(1)}+6(1+2\eta)\frac{L'(1)}{L(1)}\right.\right.\\&\left.\left.+12\eta^2+6\eta+1\right)\right\}.\end{aligned}\tag{44}$$

Proof. We use expressions (41) and (42) and applying the Cauchy-Schwarz inequality to operators $\mathscr{R}^*_{r,\eta}$, we get

$$\begin{aligned}\left|\mathscr{R}^*_{r,\eta}(f;z)-f(z)\right|&\le2\left(1+\bar\delta^2\right)\left(1+z^2\right)\bar\Omega\left(f;\bar\delta\right)\\&\quad\cdot\left\{1+\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)+\mathscr{R}^*_{r,\eta}\right.\\&\quad\left.\cdot\left(\left(1+(\gamma_1-z)^2\right)\frac{|\gamma_1-z|}{\bar\delta};z\right)\right\}.\end{aligned}\tag{45}$$

We know the expression

$$\begin{aligned}&\mathscr{R}^*_{r,\eta}\left(\left(1+(\gamma_1-z)^2\right)\frac{|\gamma_1-z|}{\bar\delta};z\right)\\&=\frac{1}{\bar\delta}\mathscr{R}^*_{r,\eta}(|\gamma_1-z|;z)+\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2\frac{|\gamma_1-z|}{\bar\delta};z\right)\\&\le\frac{1}{\bar\delta}\left(\mathscr{R}^*_{r,\eta}(\gamma_1-z)^2;z\right)^{1/2}+\left(\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^4;z\right)\right)^{1/2}\\&\quad\times\left\{\mathscr{R}^*_{r,\eta}\left(\frac{(\gamma_1-z)^2}{\bar\delta^2};z\right)\right\}^{1/2}\\&=\frac{1}{\bar\delta}\left(\mathscr{R}^*_{r,\eta}(\gamma_1-z)^2;z\right)^{1/2}\left\{1+\sqrt{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^4;z\right)}\right\}.\end{aligned}\tag{46}$$

In view of Lemma 3, we can obtain

$$\begin{aligned}&\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)\le\alpha_r(\eta)(z+1)\le M_1(z+1)\text{ as }r\to\infty,\\&\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^4;z\right)\le\beta_r(\eta)\left(z^2+z+1\right)\\&\qquad\le M_2\left(z^2+z+1\right)\text{ as }r\to\infty,\end{aligned}\tag{47}$$

where M_1 and M_2 are positive constant and

$$\begin{aligned}\alpha_r(\eta)=\max&\left\{\frac{1}{r},\frac{1}{3r^2}\left(3\frac{L''(1)}{L(1)}+6(1+2\eta)\frac{L'(1)}{L(1)}\right.\right.\\&\left.\left.+12\eta^2+6\eta+1\right)\right\},\\ \beta_r(\eta)=\max&\left\{-\frac{7}{r^2},\frac{1}{r^3}\left(-8\frac{L'''(1)}{L(1)}-4\frac{L''(1)}{L(1)}+(24\eta+20)\right.\right.\\&\left.\cdot\frac{L'(1)}{L(1)}+40\eta^2+28\eta+12\right),\frac{1}{r^4}\left(\frac{L'v(1)}{L(1)}+4(3+2\eta)\right.\\&\cdot\frac{L'''(1)}{L(1)}+\left(19+36\eta+24\eta^2\right)\frac{L''(1)}{L(1)}\\&+\left(13+36\eta+48\eta^2+32\eta^3\right)\frac{L'(1)}{L(1)}+16\eta^4+16\eta^3\\&\left.\left.+8\eta^2+2\eta+1\right)\right\}.\end{aligned}\tag{48}$$

Thus, from inequality (45), we get

$$\begin{aligned}&\left|\mathscr{R}^{*}_{r,\eta}(f\,;z)-f(z)\right|\\&\quad\le 2\left(1+\bar{\delta}^{2}\right)\left(1+z^{2}\right)\bar{\Omega}\left(f\,;\bar{\delta}\right)\Big[1+\mathscr{R}^{*}_{r,\eta}\left((\gamma_1-z)^2\,;z\right)\\&\qquad+\frac{1}{\bar{\delta}}\left(\mathscr{R}^{*}_{r,\eta}(\gamma_1-z)^2\,;z\right)^{1/2}\left\{1+\sqrt{\mathscr{R}^{*}_{r,\eta}\left((\gamma_1-z)^4\,;z\right)}\right\}\Big]\\&\quad\le 2\left(1+\bar{\delta}^{2}\right)\left(1+z^{2}\right)\bar{\Omega}\left(f\,;\bar{\delta}\right)\Big[1+M_1(z+1)\\&\qquad+\frac{1}{\bar{\delta}}\sqrt{\alpha_r(\eta)(z+1)}\left\{1+\sqrt{M_2(z^2+z+1)}\right\}\Big].\end{aligned}\tag{49}$$

If we choose $\bar{\delta}=\sqrt{\alpha_r(\eta)}$ and taking supremum $z\in[0,\alpha_r(\eta))$, then we easily get the result.

4. Direct Approximation Results of $\mathscr{R}^{*}_{r,\eta}$

The present section gives some direct approximation results in space of K-functional and in Lipschitz spaces. We take $C_b[0,\infty)$ be the set of all continuous and bounded functions defined on $[0,\infty)$.

Definition 13. For every $\bar{\delta}>0$ and $f\in C[0,\infty)$ the K -functional is defined such that

$$\begin{aligned}&\mathscr{K}_{\psi}\left(f\,;\bar{\delta}\right)\\&\quad=\inf\left\{\left(\|f-\psi\|_{C_b[0,\infty)}+\bar{\delta}\|\psi''\|_{C_b[0,\infty)}\right):\psi,\psi'\in C_b^2[0,\infty)\right\},\end{aligned}\tag{50}$$

$$\begin{aligned}&C_b^k[0,\infty)\\&\quad=\left\{f:f\in C_b[0,\infty),k\in\mathbb{N}\,;\text{such that}\lim_{z\to\infty}\frac{f(z)}{1+z^2}=m_f<\infty\right\}.\end{aligned}\tag{51}$$

For an absolute constant $M>0$, one has

$$\mathscr{K}_{\psi}\left(f\,;\bar{\delta}\right)\le M\left\{\bar{\omega}_2\left(f\,;\sqrt{\bar{\delta}}\right)+\min\left(1,\bar{\delta}\right)\|f\|_{C_b[0,\infty)}\right\}.\tag{52}$$

Let $\bar{\omega}_2(f\,;\bar{\delta})$ denote the modulus of continuity of order two such that

$$\bar{\omega}_2\left(f\,;\bar{\delta}\right)=\sup_{0<h<\bar{\delta}}\sup_{z[\in 0,\infty)}|f(z+2h)-2f(z+h)+f(z)|,\tag{53}$$

while the classical modulus of continuity is given by

$$\bar{\omega}\left(f\,;\bar{\delta}\right)=\sup_{0<h<\bar{\delta}}\sup_{z\in[0,\infty)}|f(z+h)-f(z)|.\tag{54}$$

Theorem 14. *For an arbitrary $\varphi\in C_b^2[0,\infty)$, let an auxiliary operator $\mathscr{S}^{*}_{r,\eta}$ be such that*

$$\begin{aligned}&\mathscr{K}^{*}_{r,\eta}(\varphi\,;z)\\&\quad=\mathscr{R}^{*}_{r,\eta}(\varphi\,;z)+\varphi(z)-\varphi\left\{z+\frac{1}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)\right\}.\end{aligned}\tag{55}$$

Then, for any $\phi\in C_b^2[0,\infty)$ operators (55), verify the inequality

$$\begin{aligned}&\left|\mathscr{K}^{*}_{r,\eta}(\phi\,;z)-\phi(y)\right|\\&\quad\le\left\{\bar{\delta}^{*}_{r,\eta}(z)++\frac{1}{4r^2}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)^2\right\}\|\psi''\|,\end{aligned}\tag{56}$$

where $\bar{\delta}^{*}_{r,\eta}(z)$ is defined by Theorem 10.

Proof. For any $\phi\in C_b^2[0,\infty)$, it is easy to verify that $\mathscr{K}^{*}_{r,\eta}(\gamma_0\,;z)=1$ and

$$\begin{aligned}&\mathscr{K}^{*}_{r,\eta}(\gamma_1\,;z)\\&\quad=\mathscr{R}^{*}_{r,\eta}(\gamma_1\,;z)+z-\left\{z+\frac{1}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)\right\}=z.\end{aligned}\tag{57}$$

We have

$$\begin{aligned}&\left\|\mathscr{R}^{*}_{r,\eta}(\varphi\,;z)\right\|\le\|\varphi\|,\\&|\mathscr{K}^{*}_{r,\eta}(\varphi\,;z)|\le|\mathscr{R}^{*}_{r,\eta}(\varphi\,;z)|+|\varphi(z)|+|\varphi\\&\qquad\cdot\left\{z+\frac{1}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)\right\}|\le 3\|\varphi\|.\end{aligned}\tag{58}$$

For any $\phi\in C_b^2[0,\infty)$, the Taylor series expression gives us

$$\phi(t)=\phi(z)+(t-z)\phi'(z)+\int_z^t(t-\chi)\phi''(\chi)\mathrm{d}\chi.\tag{59}$$

Therefore, after applying the operators $\mathscr{K}^{*}_{r,\eta}$, on both sides we get

$$\begin{aligned}\mathscr{K}^*_{r,\eta}(\phi;z)-\phi(z)&=\phi'(z)\mathscr{K}^*_{r,\eta}(\gamma_1-z;z)+\mathscr{K}^*_{r,\eta}\\&\cdot\left(\int_z^{\gamma_1}(\gamma_1-\chi)\phi''(\chi)\mathrm{d}\chi;z\right)\\&=\mathscr{K}^*_{r,\eta}\left(\int_z^{t}(\gamma_1-\chi)\phi''(\chi)\mathrm{d}\chi;z\right)\\&=\mathscr{R}^*_{r,\eta}\left(\int_z^{\gamma_1}(\gamma_1-\chi)\phi''(\chi)\mathrm{d}\chi;z\right)\\&+\int_z^{z}(z-\chi)\phi''(\chi)\mathrm{d}\chi;z\\&-\int_z^{z+(1/2r)\left(2\left(L'(1)/\mathrm{L}(1)\right)+4\eta+1\right)}\\&\cdot\left(z+\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)-\chi\right)\phi''\\&\cdot(\chi)\mathrm{d}\chi;\left|\mathscr{K}^*_{r,\eta}(\phi;z)-\phi(z)\right|\\&\le\left|\mathscr{R}^*_{r,\eta}\left(\int_z^{\gamma_1}(\gamma_1-\chi)\phi''(\chi)\mathrm{d}\chi;z\right)\right|\\&+\left|\int_z^{z+(1/2r)\left(2\left(L'(1)/\mathrm{L}(1)\right)+4\eta+1\right)}\right.\\&\left.\cdot\left(z+\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)-\chi\right)\phi''(\chi)\mathrm{d}\chi\right|.\end{aligned}\tag{60}$$

We know the inequality

$$\left|\int_z^t(t-\chi)\phi''(\chi)\mathrm{d}\chi\right|\le(t-z)^2\|\phi''\|,$$

$$\begin{aligned}&\left|\int_z^{z+(1/2r)\left(2\left(L'(1)/\mathrm{L}(1)\right)+4\eta+1\right)}\right.\\&\left.\times\left(z+\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)-\chi\right)\phi''(\chi)\mathrm{d}\chi\right|\\&\le\left(\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)\right)^2\|\phi''\|.\end{aligned}\tag{61}$$

Thus, we get

$$\begin{aligned}&\left|\mathscr{K}^*_{r,\eta}(\phi;z)-\phi(y)\right|\\&\le\left\{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2;z\right)+\frac{1}{4r^2}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)^2\right\}\|\phi''\|.\end{aligned}\tag{62}$$

This gives the complete proof.

Theorem 15. *If $\phi\in C_b^2[0,\infty)$, then for any $f\in C_b[0,\infty)$ operators $\mathscr{R}^*_{r,\eta}$ by (7) satisfying*

$$\begin{aligned}&\left|\mathscr{R}^*_{r,\eta}(f;z)-f(z)\right|\\&\le\mathscr{M}\left[\bar{\omega}_2\left\{f;\frac{1}{2}\sqrt{\bar{\delta}^*_{r,\eta}(z)+\frac{1}{4r^2}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)^2}\right\}\right.\\&+\min\left\{1;\frac{1}{4}\left(\bar{\delta}^*_{r,\eta}(z)+\frac{1}{4r^2}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)^2\right)\right\}\\&\left.\times\|f\|_{C_b[0,\infty)}\right]+\bar{\omega}\left(f;\frac{1}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)\right),\end{aligned}\tag{63}$$

where $\bar{\delta}^*_{r,\eta}(z)$ is defined by Theorem 10.

Proof. We prove Theorem 15 in view of Theorem 14. Therefore, for all $f\in C_b[0,\infty)$ and $\phi\in C_b^2[0,\infty)$, we get

$$\begin{aligned}&\left|\mathscr{R}^*_{r,\eta}(f;z)-f(z)\right|\\&=\left|\mathscr{K}^*_{r,\eta}(f;z)-f(z)+f\left(z+\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)\right)-f(z)\right|\\&\le|\mathscr{K}^*_{r,\eta}(f-\phi;z)|+|\mathscr{K}^*_{r,\eta}(\phi;z)-\phi(z)|+|\phi(\mathrm{z})-f(z)|\\&+\left|f\left(z+\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)\right)-f(z)\right|\\&\le4\|f-\phi\|+\omega\left(f;\left|\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)\right|\right)\\&+\left\{\bar{\delta}^*_{r,\eta}(z)+\frac{1}{4r^2}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)^2\right\}\|\psi''\|.\end{aligned}\tag{64}$$

If we take infimum for all $\phi\in C_b^2[0,\infty)$, then in view of (50) it is easy to conclude that

$$\begin{aligned}&\left|\mathscr{R}^*_{r,\eta}(f;z)-f(z)\right|\\&\le4K_2\left\{f;\frac{1}{4}\left(\bar{\delta}^*_{r,\eta}(z)+\frac{1}{4r^2}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)^2\right)\right\}\\&+\bar{\omega}\left(f;\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)\right)\\&\le\mathscr{M}\left[\bar{\omega}_2\left\{f;\frac{1}{2}\sqrt{\bar{\delta}^*_{r,\eta}(z)+\frac{1}{4r^2}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)^2}\right\}\right.\\&+\min\left\{1;\frac{1}{4}\left(\bar{\delta}^*_{r,\eta}(z)+\frac{1}{4r^2}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)^2\right)\right\}\\&\left.\times\|f\|_{C_b[0,\infty)}\right]+\bar{\omega}\left(f;\frac{1}{2r}\left(2\frac{L'(1)}{\mathrm{L}(1)}+4\eta+1\right)\right).\end{aligned}\tag{65}$$

The proof is completed here.

Now, we give the local direct estimate for the operators $\mathscr{R}^*_{r,\eta}$ defined by (7) via the well-known Lipschitz-type maximal function involving the parameters $\mu, \nu > 0$ and number $\lambda \in (0,1]$. Thus, from [26], we recall that

$$\mathrm{Lip}^{\lambda}_{\mathscr{L}} = \left\{ f \in C_b[0,\infty) : |f(t) - f(z)| \leq \mathscr{L} \frac{|t-z|^{\lambda}}{(\mu z^2 + \nu z + t)^{\lambda/2}} ; z, t \in [0,\infty) \right\}, \tag{66}$$

where $\mathscr{L}$ is a positive constant.

Theorem 16. *For any $f \in Lip^{\lambda}_{\mathscr{L}}$ satisfied by (70), operators $\mathscr{R}^*_{r,\eta}$ hold the inequality*

$$|\mathscr{R}^*_{r,\eta}(f;z) - f(z)| \leq \mathscr{L} \left(\frac{\bar{\delta}^*_{r,\eta}(z)}{(\mu z^2 + \nu z)} \right)^{\lambda/2}, \tag{67}$$

where $\bar{\delta}^*_{r,\eta}(z)$ is obtained by Theorem 10.

Proof. Let $f \in \mathrm{Lip}^{\lambda}_{\mathscr{L}}$ for $0 < \lambda \leq 1$; then, first we verify the results are true when $\lambda = 1$. For any $\mu, \nu \geq 0$, it is easy to use the result $(\mu z^2 + \nu z + t)^{-1/2} \leq (\mu z^2 + \nu z)^{-1/2}$ and then we apply the Cauchy-Schwarz inequality. Thus, we can write

$$\begin{aligned} \left|\mathscr{R}^*_{r,\eta}(f;z) - f(z)\right| &\leq \left|\mathscr{R}^*_{r,\eta}(|f(t) - f(z)|;z)| + f(z)|(1;z) - 1\right| \\ &\leq \mathscr{R}^*_{r,\eta}\left(\frac{|t-z|}{(\mu z^2 + \nu z + t)^{1/2}} ; z \right) \\ &\leq \mathscr{L}(\mu z^2 + \nu z)^{-1/2} \mathscr{R}^*_{r,\eta}(|t-z|;z) \\ &\leq \mathscr{L}(\mu z^2 + \nu z)^{-1/2} \mathscr{R}^*_{r,\eta}(\gamma_1 - z)^2 ; z\big|^{1/2}. \end{aligned} \tag{68}$$

From these conclusions, we get that the statement holds for $\lambda = 1$. Now, we check if the statement is valid if $0 < \lambda < 1$. For this reason, we use monotonicity property to $\mathscr{R}^*_{r,\eta}$ and apply the well-known Hölder inequality

$$\begin{aligned} \left|\mathscr{R}^*_{r,\eta}(f;z) - f(z)\right| &\leq \mathscr{R}^*_{r,\eta}(|f(t) - f(z)|;z) \\ &\leq \left(\mathscr{R}^*_{r,\eta}\left(|f(t) - f(z)|^{2/\lambda} ; z \right) \right)^{\lambda/2} \\ &\quad \cdot \left(\mathscr{R}^*_{r,\eta}(\gamma_0 ; z) \right)^{(2-\lambda)/2} \\ &\leq \mathscr{L} \left\{ \frac{\mathscr{R}^*_{r,\eta}\left((\gamma_1 - z)^2 ; z\right)}{t + \mu z^2 + \nu z} \right\}^{\lambda/2} \\ &\leq \mathscr{L}(\mu z^2 + \nu z)^{-\lambda/2} \left\{ \mathscr{R}^*_{r,\eta}\left((\gamma_1 - z)^2 ; z\right) \right\}^{\lambda/2} \\ &\leq \mathscr{L}(\mu z^2 + \nu z)^{-\lambda/2} \left(\mathscr{R}^*_{r,\eta}(\gamma_1 - z)^2 ; z \right)^{\lambda/2} \\ &= \mathscr{L} \left(\frac{\bar{\delta}^*_{r,\eta}(z)}{(\mu z^2 + \nu z)} \right)^{\lambda/2}, \end{aligned} \tag{69}$$

which completes the proof.

Here, we obtain the other local approximation results of $\mathscr{R}^*_{r,\eta}$ in Lipschitz spaces. For all Lipschitz maximal function $f \in C_b[0,\infty)$, $0 < \lambda \leq 1$ and $t, z \in [0,\infty)$, from [27] we recall that

$$\bar{\omega}_{\lambda}(f;z) = \sup_{t \neq z,\, t \in [0,\infty)} \frac{|f(t) - f(z)|}{|t-z|^{\lambda}} \tag{70}$$

Theorem 17. *Let $f \in C_b[0,\infty)$, then for all $z \in [0,\infty)$,*

$$\left|\mathscr{R}^*_{r,\eta}(f;z) - f(z)\right| \leq \left(\bar{\delta}^*_{r,\eta}(z)\right)^{\lambda/2} \bar{\omega}_{\lambda}(f;z), \tag{71}$$

where $\bar{\omega}_{\lambda}(f;z)$ is obtained in and $\bar{\delta}^*_{r,\eta}(z)$ is defined by Theorem 10.

Proof. From the well-known Hölder inequality, we get

$$\begin{aligned} \left|\mathscr{R}^*_{r,\eta}(f;z) - \mathrm{f}(z)\right| &\leq \mathscr{R}^*_{r,\eta}(|f(t) - f(z)|;z) \\ &\leq \bar{\omega}_{\lambda}(f;z)|\mathscr{R}^*_{r,\eta}\left(|t-z|^{\lambda};z\right) \\ &\leq \bar{\omega}_{\lambda}(f;z)\left(\mathscr{R}^*_{r,\eta}(\gamma_0;z)\right)^{(2-\lambda)/2} \\ &\quad \cdot \left(\mathscr{R}^*_{r,\eta}\left(|t-z|^2;z\right)\right)^{\lambda/2} \\ &= \bar{\omega}_{\lambda}(f;z)\left(\mathscr{R}^*_{r,\eta}\left((\gamma_1 - z)^2;z\right)\right)^{\lambda/2}. \end{aligned} \tag{72}$$

Thus, we get the proof.

5. Voronovskaja-Type Approximation Theorems

In this section, we establish a quantitative Voronovskaja-type theorem for the operators $\mathscr{R}^*_{r,\eta}(f;z)$.

Theorem 18. *Let $f \in C_b[0,\infty)$, then for each $z \in [0,\infty)$*

$$\lim_{r\to\infty} r\left\{\mathscr{R}^*_{r,\eta}(\psi\,;z)-\psi(z)\right\} = \left(2\frac{L'(1)}{L(1)}+4\eta+1\right)\frac{\psi'(z)}{2} + \frac{z\psi''(z)}{2}, \tag{73}$$

where $\psi'(z), \psi''(z) \in C_b[0,\infty)$.

Proof. From the expression of Taylor's expansion of function $\psi(z)$ in $C_b[0,\infty)$, we write

$$\psi(t)=\psi(z)+(t-z)\psi'(z)+\frac{1}{2}(t-z)^2\psi''(z)+(t-z)^2Q_z(t), \tag{74}$$

where $Q_z(t)$ is the remainder term and $Q_z \in [0,\infty)$ with $Q_z(t)\to 0$ as $t\to z$. On applying the operators $\mathscr{R}^*_{r,\eta}(\cdot;z)$ to (74), then use the Cauchy-Schwarz inequality. Thus, we get

$$\begin{aligned}\mathscr{R}^*_{r,\eta}(\psi\,;z)-\psi(z) &= \psi'(z)\mathscr{R}^*_{r,\eta}(\gamma_1-z\,;z)+\frac{\psi''(z)}{2}\mathscr{R}^*_{r,\eta}\\ &\quad\cdot\left((\gamma_1-z)^2\,;z\right)+\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2Q_z(\gamma_1)\,;z\right)\\ &\le \psi'(z)\mathscr{R}^*_{r,\eta}(\gamma_1-z\,;z)+\frac{\psi''(z)}{2}\mathscr{R}^*_{r,\eta}\\ &\quad\cdot\left((\gamma_1-z)^2\,;z\right)\\ &\quad+\sqrt{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^4\,;z\right)}\sqrt{\mathscr{R}^*_{r,\eta}\left(Q_z^2(\gamma_1)\,;z\right)}.\end{aligned} \tag{75}$$

Since we have $\lim_{r\to\infty}\mathscr{R}^*_{r,\eta}(Q_z^2(\gamma_1)\,;z)=0$, therefore

$$\lim_{r\to\infty} r\left\{\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2Q_z(\gamma_1)\,;z\right)\right\}=0. \tag{76}$$

Thus, we have

$$\begin{aligned}&\lim_{r\to\infty} r\left\{\mathscr{R}^*_{r,\eta}(\psi\,;z)-\psi(z)\right\}\\ &\quad=\lim_{r\to\infty} r\left\{\mathscr{R}^*_{r,\eta}(\gamma_1-z\,;z)\psi'(z)+\frac{\psi''(z)}{2}\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2\,;z\right)\right.\\ &\qquad\left.+\mathscr{R}^*_{r,\eta}\left((\gamma_1-z)^2Q_z(\gamma_1)\,;z\right)\right\},\end{aligned} \tag{77}$$

which completes the proof.

As a consequence of Theorem 18, we immediately get the corollary.

Corollary 19. *For any $\psi \in C[0,\infty)$, we have*

$$\lim_{r\to\infty} r\left[\mathscr{R}^*_{r,\eta}(\psi\,;z)-\psi(z)-\frac{1}{2r}\left(2\frac{L'(1)}{L(1)}+4\eta+1\right)\psi'(z) - \frac{\bar{\delta}^*_{r,\eta}(z)}{2}\psi''(z)\right]=0. \tag{78}$$

6. Conclusion

Motivated by article [5], we have introduced a Kantorovich generalization of the Szász-Mirakjan operators by Dunkl analogue involving the Appell polynomials. These types of generalizations enable to give the generalized results rather than the earlier study demonstrations by [3, 5, 7]. Lastly, we have also discussed the Voronovskaja-type approximation theorems of these new operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors are very grateful and declare that they have no competing interest.

Authors' Contributions

All authors read and agreed to the contents of this research article.

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Research Article

On Uniqueness of New Orthogonality via 2-HH Norm in Normed Linear Space

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This paper generalizes the special case of the Carlsson orthogonality in terms of the 2-HH norm in real normed linear space. Dragomir and Kikianty (2010) proved in their paper that the Pythagorean orthogonality is unique in any normed linear space, and isosceles orthogonality is unique if and only if the space is strictly convex. This paper deals with the complete proof of the uniqueness of the new orthogonality through the medium of the 2-HH norm. We also proved that the Birkhoff and Robert orthogonality via the 2-HH norm are equivalent, whenever the underlying space is a real inner-product space.

1. Introduction

Different notions of orthogonality in normed linear spaces have been developed by various mathematicians. As a generalization of orthogonality from inner product space to normed linear space "x is orthogonal to y if and only if $\|x+\lambda y\| = \|x-\lambda y\|$ identically in λ" was suggested by Robert ([1, 2]). However, it has the weakness that for some normed linear space, at least one of every pair of orthogonal elements would have to be zero, i.e., $\|x+\lambda y\| = \|x-\lambda y\|$ for all λ only if $x=0$ or $y=0$. This difficulty is not experienced in the isosceles, Pythagorean, and Birkhoff orthogonalities.

To study the difference of orthogonality in the complex case in comparison with the real case, Paul et al. in 2018 came with a new concept of Birkhoff-James orthogonality by introducing new definitions on complex reflexive Banach spaces and introduced more than one equivalent characterization of Birkhoff-James orthogonality of compact linear operators in the complex case [3]. In 1945, James came with the concept of the Pythagorean and isosceles orthogonalities, which characterize inner product space via their homogeneity and additivity [4]. James also discussed the existence property of isosceles orthogonality type. The property of the uniqueness of isosceles orthogonality was not discussed until Kapoor and Prasad's paper was published. They proved that the Pythagorean orthogonality is unique in any normed linear space; however, the isosceles orthogonality is unique if and only if the space is strictly convex [5].

Carlsson introduced a more general type of orthogonality treating the isosceles and Pythagorean orthogonalities are special cases [6]. Martini and Wu showed many interesting connections between the Birkhoff and isosceles orthogonality. They proved that if a linear map preserves the Birkhoff orthogonality, then it also preserves the isosceles orthogonality [7]. In 2007, Alsina and Tomas gave a different characterization of the inner product space with the help of weaker linearity axioms of the scalar product and Pythagoras/isosceles orthogonality [8].

Using the concept of the p-HH norm as described in the paper [9], Kikianty and Dragomir came up with a new notion of orthogonality with the help of the 2-HH norm, which is closely related to the Pythagorean and isosceles orthogonalities [10]. They proved that the Pythagorean orthogonality via 2-HH norm satisfies the nondegeneracy, continuity, and symmetry properties; however, it is neither additive nor homogeneous in normed linear space, but it satisfies the property of existence and uniqueness in any normed linear spaces. Isosceles orthogonality via the 2-HH norm also satisfies the nondegeneracy, continuity, and symmetry properties but neither additive nor homogeneous in general. If the normed linear space X is convex, then the isosceles orthogonality via 2-HH norm satisfies the property of uniqueness, but the existent property holds in any normed linear space [10].

According to Carlsson's result described in [6], the isosceles and Pythagorean orthogonalities are special cases of the generalized Carlsson orthogonality. We introduced a new special case of the Carlsson orthogonality which satisfies all requirements as stated in Carlsson's orthogonality as well as the nondegeneracy, simplification, and continuity property of the inner product space. Furthermore, we proved that such orthogonality is homogeneous if and only if the underlying space is an inner product space [11]. Motivated by the results of Kikianty and Dragomir, and our previous result, we have attempted to introduce a new notion of orthogonality through the medium of the 2-HH norm, which we denote by the 2-HH-N orthogonality. We have proved that the 2-HH-N orthogonality is unique in any normed linear space. If the norm on X is induced by an inner product, then the Robert and Birkhoff orthogonality via the 2-HH norm is equivalent.

2. Definition Notation and Preliminary Results

Let us first establish the notations and terminologies used in this paper. Let X be the normed linear space which we consider to be real. For any $x, y \in X$, 2-HH-N denotes$x$ as the orthogonal to$y$ via the 2-HH norm, which we defined with the help of the new special case of the Carlsson orthogonality discussed in [11]. The Pythagorean orthogonality plays an important role in describing new orthogonality through the medium of the 2-HH norm. Given any two elements $x, y \in X$, we say that x is Pythagorean orthogonal to y, written as $x \perp_P y$, if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ [4]. Kikianty and Dragomir introduced the Pythagorean orthogonality via the 2-HH norm and using a similar idea to that of Kapoor and Prasad, they proved that " the Pythagorean orthogonality via 2-HH norm is unique in any normed space" [9]. Besides that, they also define the Carlsson orthogonality via the 2-HH norm in the paper [12].

For any $(x, y) \in X^2$, Kikianty and Dragomir defined the p-norm on X^2 as follows [10]:

$$\|(x, y)\|_p = \begin{cases} \|x\|^p + \|y\|^p, & 1 \le p < \infty \\ \max\{\|x\|, \|y\|\}, & p = \infty. \end{cases} \quad (1)$$

From (1), it is obvious that $\|(x, y)\|_p = \|(y, x)\|_p$, and therefore, the p-norm is symmetric. Using the concepts of Hermite-Hadamard's inequality, we have

$$\int_0^1 \|(1-t)x + ty\|^p dt \le \frac{\|x\|^p + \|y\|^p}{2} = \frac{\|(x, y)\|_p^p}{2} < \infty. \quad (2)$$

With the help of (2), they defined the p-HH norm on X^2 in the following ways [10]:

$$\|(x, y)\|_{p-HH} = \begin{cases} \left(\int_0^1 \|(1-t)x + ty\|^p\right) \frac{1}{p}, & \text{if } 1 \le p < \infty \\ \sup_{t \in [0,1]} \|(1-t)x + ty\|, & \text{if } p = \infty. \end{cases} \quad (3)$$

For all $x, y \in X$, it is obvious that $\|(x, y)\|_{p-HH} = \|(y, x)\|_{p-HH}$. Therefore, the p-HH norm is symmetric. They proved that $(X^2, \|(.,.)\|)$ is a normed linear space because the nondegeneracy and homogeneity of the norm can be derived from (3) and the triangle inequality follows from Minkowski's inequality. If the norm on X is induced by an inner product $(.,.)$, then as a special case of the p-HH norm, it is denoted by the 2-HH norm. It is defined in the paper [9] as follows:

$$\|(x, y)\|_{2-HH} = \int_0^1 \|(1-t)x + ty\|^2 dt. \quad (4)$$

For any $p \ge 1$, the p-norm and p-HH norm are equivalent in X^2.

Definition 1. A real-valued function f defined on a nonempty subset $X \subset \mathbb{R}^n$ is called convex if

(i) The domain X of the function is convex

(ii) For any $x, y \in X$ and $\lambda \in [0, 1]$

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y). \quad (5)$$

If the inequality ((5)) is strict whenever $x \ne y$ and $0 < \lambda < 1$, it is called strictly convex. To study the properties of orthogonality in normed linear space, it is interesting to investigate the following properties of orthogonality in ordinary Euclidean space as applied to normed linear space. For any Euclidean space X, let $x, y, z \in X$. Then, the following are considered as the main properties of orthogonality [9].

(i) *Nondegeneracy*: if $x \perp x$, then $x = 0$.

(ii) *Simplification*: if $x \perp y$, then for any $\alpha \in \mathbb{R}$, $\alpha x \perp \alpha y$.

(iii) *Continuity*: if $\{x_n\}, \{y_n\} \subset X$ such that $x_n \perp y_n$ for every $n \in \mathbb{N}$, $x_n \to x$ and $y_n \to y$, then $x \perp y$.

(iv) *Homogeneity*: if $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$.

(v) *Symmetry*: if $x \perp y$, then $y \perp x$.

(vi) *Additivity*: if $x\perp y$ and $x\perp z$, then $x\perp y+z$.

(vii) *Existence*: if $x\neq 0$, then there exist $\lambda\in\mathbb{R}$ such that $x\perp\lambda x+y$.

(viii) *Uniqueness*: for any $x\neq 0$, if there exists $\lambda\in\mathbb{R}$ such that $x\perp\lambda x+y$, then such λ is unique.

In this paper, we mainly focus on the last property. Kapoor and Prasad proved that the Pythagorean orthogonality is unique in any normed linear space, but the isosceles orthogonality is unique if and only if the normed linear space is strictly convex [5]. Regarding the Robert orthogonality, the property of existence is satisfied only in an inner-product space [1].

Definition 2 [4]. A vector x is said to be isosceles orthogonal to y if and only if

$$\|x-y\|=\|x+y\|. \tag{6}$$

Definition 3 [13]. A vector x is said to be the Birkhoff-James orthogonal to y if and only if

$$\|x+\lambda y\|\geq\|x\| \text{ for all } \lambda\in\mathbb{R}. \tag{7}$$

3. Main Result

Definition 4 [11]. A vector x is orthogonal to y if

$$\|x+\frac{1}{2}y\|^2+\|x-\frac{1}{2}y\|^2=\frac{1}{2}\|\sqrt{2}x+y\|^2+\|x\|^2. \tag{8}$$

If the underlying space X is a real inner product space and the relation (8) holds a. e on $[0,1]$. Then, using the concept of the 2-HH norm, we have

$$\begin{aligned}\int_0^1\|(1-t)x+\frac{1}{2}ty\|^2dt+\int_0^1\|(1-t)x-\frac{1}{2}ty\|^2dt\\ =\frac{1}{2}\int_0^1\|\sqrt{2}(1-t)x+ty\|^2dt\\ +\int_0^1\|(1-t)x\|^2dt.\end{aligned} \tag{9}$$

Now, the left-hand side of relation (9)

$$\begin{aligned}\int_0^1\|(1-t)x+\frac{1}{2}ty\|^2dt+\int_0^1\|(1-t)x-\frac{1}{2}ty\|^2dt\\ =\int_0^1\left\langle(1-t)x+\frac{1}{2}ty,(1-t)x+\frac{1}{2}ty\right\rangle dt\\ +\int_0^1\left\langle(1-t)x-\frac{1}{2}ty,(1-t)x-\frac{1}{2}ty\right\rangle dt\\ =\frac{1}{3}\|x\|^2+\frac{1}{12}\|y\|^2+\frac{1}{3}\|x\|^2+\frac{1}{12}\|y\|^2\\ =\frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2.\end{aligned} \tag{10}$$

Again, the right-hand side of relation (9)

$$\begin{aligned}\frac{1}{2}\int_0^1\|\sqrt{2}(1-t)x+ty\|^2dt+\int_0^1\|(1-t)x\|^2dt\\ =\frac{1}{2}\int_0^1\left\langle\sqrt{2}(1-t)x+ty,\sqrt{2}(1-t)x+ty\right\rangle dt\\ +\frac{1}{3}\|x\|^2=\frac{1}{2}\left(\frac{2}{3}\|x\|^2+\frac{1}{3}\|y\|^2\right)+\frac{1}{3}\|x\|^2\\ =\frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2.\end{aligned} \tag{11}$$

Now, we consider a notion of orthogonality as follows: let $(X.\|.\|)$ be a normed space. A vector $x\in X$ is said to be 2-HH-N orthogonal to $y\in X$ (denoted by $x\perp_{2-HH-N}y$) if and only if

$$\int_0^1\|(1-t)x+\frac{1}{2}ty\|^2dt+\int_0^1\|(1-t)x-\frac{1}{2}ty\|^2dt=\frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2. \tag{12}$$

Kikianty and Dragomir in [9] proved that " the Pythagorean orthogonality via 2-HH norm is unique in any normed space X". To prove this, they use the following lemma by omitting the proof. We give a detailed proof of the lemma as they stated in the paper [9].

Lemma 5. *Let $x,y\in X$, where X is the normed linear space. Let h be a function on $\mathbb{R}$ defined by*

$$h(\mu):=\int_0^1\|(1-t)y+\mu(tx)\|^2dt. \tag{13}$$

Then, h is a convex function on $\mathbb{R}$, and for any $r\in(0,1)$ and $\mu_1,\mu_2\in\mathbb{R}$ where $h(\mu_1)\neq h(\mu_2)$, we have

$$h[r\mu_1+(1-r)\mu_2]<rh(\mu_1)+(1-r)h(\mu_2). \tag{14}$$

Proof. Let $r\in(0,1)$ and $\mu_1,\mu_2\in\mathbb{R}$ such that $h(\mu_1)\neq h(\mu_2)$. Then,

$$\begin{aligned}h[r\mu_1+(1-r)\mu_2]=\int_0^1\|(1-t)y+[r\mu_1+(1-r)\mu_2](tx)\|^2dt\\ =\int_0^1\|(1-t)y+r\mu_1(tx)+\mu_2(tx)-r\mu_2(tx)\|^2dt\\ =\int_0^1\|(1-t)y+r\mu_1(tx)+\mu_2(tx)-r\mu_2(tx)\\ -r(1-t)y+r(1-t)y\|^2dt\\ =\int_0^1\|r[(1-t)y+\mu_1(tx)]+(1-r)[(1-t)y\\ +\mu_2(tx)]\|^2dt\leq\int_0^1r^2\|(1-t)y+\mu_1(tx)\|^2dt\\ +(1-r)^2\int_0^1\|(1-t)y+\mu_2(tx)\|^2dt\\ +2r(1-r)\int_0^1\|(1-t)y+\mu_1(tx)\|\,\|(1-t)y+\mu_2(tx)\|dt\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 r\|(1-t)y + \mu_1(tx)\|^2 dt \\
&\quad + (1-r)\int_0^1 \|(1-t)y + \mu_2(tx)\|^2 dt \\
&\quad + \left(r^2 - r\right)\int_0^1 \left[\|(1-t)y + \mu_1(tx)\|^2 + \|(1-t)y\right. \\
&\quad \left. + \mu_2(tx)\|^2 - 2\|(1-t)y + \mu_1(tx)\| \, \|(1-t)y\right. \\
&\quad \left. + \mu_2(tx)\|\right] dt = rh(\mu_1) + (1-r)h(\mu_2) \\
&\quad - r(1-r)\int_0^1 \left[\|(1-t)y + \mu_1(tx)\| - \|(1-t)y\right. \\
&\quad \left. + \mu_2(tx)\|\right]^2 dt \le rh(\mu_1) + (1-r)h(\mu_2),
\end{aligned} \tag{15}$$

which shows that h is a convex function. Since $h(\mu_1) \neq h(\mu_2)$, then the inequality will be strict and therefore

$$h[r\mu_1 + (1-r)\mu_2] < rh(\mu_1) + (1-r)h(\mu_2). \tag{16}$$

As a similar concept of Lemma 5, we also prove the following lemma which is useful to prove the uniqueness property of new orthogonality via the 2-HH norm.

Lemma 6. *Let $(X.\|.\|)$ be a normed space and $x, y \in X$. Let h be a function defined on $\mathbb{R}$ by*

$$h(\mu) \coloneqq \int_0^1 \|(1-t)y + \frac{\mu}{2}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{\mu}{2}(tx)\|^2 dt. \tag{17}$$

Then, h is a convex function on $\mathbb{R}$ and for any $r \in (0,1)$, and $\mu_1, \mu_2 \in \mathbb{R}$ where $h(\mu_1) \neq h(\mu_2)$, we have

$$h[r\mu_1 + (1-r)\mu_2] < rh(\mu_1) + (1-r)h(\mu_2). \tag{18}$$

Proof. Suppose $h(\mu) = f(\mu) + g(\mu)$, where

$$\begin{aligned}
f(\mu) &= \int_0^1 \|(1-t)y + \frac{\mu}{2}(tx)\|^2 dt, \\
g(\mu) &= \int_0^1 \|(1-t)y - \frac{\mu}{2}(tx)\|^2 dt.
\end{aligned} \tag{19}$$

First, we show that $f(k)$ is a convex function. Let $r \in (0,1)$ and $\mu_1, \mu_2 \in \mathbb{R}$ such that $h(\mu_1) \neq h(\mu_2)$.

$$\begin{aligned}
f[r\mu_1 + (1-r)\mu_2] &= \int_0^1 \|(1-t)y + \frac{1}{2}[r\mu_1 + (1-r)\mu_2](tx)\|^2 dt \\
&= \int_0^1 \|(1-t)y + \frac{1}{2}r\mu_1(tx) + \frac{1}{2}\mu_2(tx) - \frac{1}{2}r\mu_2(tx)\|^2 dt \\
&= \int_0^1 \|(1-t)y + \frac{r\mu_1}{2}(tx) + \frac{\mu_2}{2}(tx) - r\frac{\mu_2}{2}(tx) \\
&\quad + r(1-t)y - r(1-t)y\|^2 dt \\
&= \int_0^1 \|r\left[(1-t)y + \frac{\mu_1}{2}(tx)\right] + (1-r) \\
&\quad \cdot \left[(1-t)y + \frac{\mu_2}{2}(tx)\right]\|^2 dt \\
&\le r^2\int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt \\
&\quad + (1-r)^2\int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad + 2r(1-r)\int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\| \|(1-t)y \\
&\quad + \frac{\mu_2}{2}(tx)\| dt = r\int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt \\
&\quad + (1-r)\int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad + r^2\int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt \\
&\quad + (1-r)^2\int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad - r\int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt \\
&\quad - (1-r)\int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad + 2r(1-r)\int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\| \|(1-t)y \\
&\quad + \frac{\mu_2}{2}(tx)\| dt = rf(\mu_1) + (1-r)f(\mu_2) \\
&\quad + \left(r^2 - r\right)\int_0^1 \left[\|(1-t)y + \frac{\mu_1}{2}(tx)\| - \|(1-t)y\right. \\
&\quad \left. + \frac{\mu_1}{2}(tx)\|\right]^2 dt = rf(\mu_1) + (1-r)f(\mu_2) \\
&\quad - \left(r - r^2\right)\int_0^1 \left[\|(1-t)y + \frac{\mu_1}{2}(tx)\| - \|(1-t)y\right. \\
&\quad \left. + \frac{\mu_1}{2}(tx)\|\right]^2 dt \le rf(\mu_1) + (1-r)f(\mu_2),
\end{aligned} \tag{20}$$

which shows that f is a convex function. Similarly, for the function

$$g(\mu) = \int_0^1 \|(1-t)t - \frac{\mu}{2}(tx)\|^2 dt, \tag{21}$$

we can show that

$$g[r\mu_1 + (1-r)\mu_2] \le rg(\mu_1) + (1-r)g(\mu_2), \tag{22}$$

and we conclude that g is also a convex function. Also, we know that the sum of two convex functions is also convex. Then, $h(\mu) = f(\mu) + g(\mu)$ is convex. Since $h(\mu_1) \neq h(\mu_2)$, then the inequality will be strict, and therefore

$$h[r\mu_1 + (1-r)\mu_2] < rh(\mu_1) + (1-r)h(\mu_2). \quad (23)$$

Theorem 7. *2-HH-N orthogonality is unique in any normed space X.*

Proof. The proof has a similar idea to that of Kapoor and Prasad (pp. 406) and Kikianty and Dragomir (pp. 41). Suppose 2-HH-N orthogonality is not unique. Then, we must have elements $x \neq 0$ and $y \in X$, and a $\lambda > 0$ such that $x \perp_{2-HH-N} y$ and $x \perp_{2-HH-N} \lambda x + y$. Define a convex function

$$h(\mu) = \int_0^1 \|(1-t)y + \frac{\mu}{2}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{\mu}{2}(tx)\|^2 dt. \quad (24)$$

Now,

$$h(1) = \int_0^1 \|(1-t)y + \frac{1}{2}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{1}{2}(tx)\|^2 dt = \frac{2}{3}\|y\|^2 + \frac{1}{6}\|x\|^2 = h(0) + \frac{1}{6}\|x\|^2. \quad (25)$$

Setting $\beta = (2(1-t)\lambda)/t$ and note that

$$\begin{aligned} h(\beta) &= \int_0^1 \left\|(1-t)y + \frac{1}{2}\cdot\frac{2(1-t)\lambda}{t}(tx)\right\|^2 dt + \int_0^1 \left\|(1-t)y - \frac{1}{2}\cdot\frac{2(1-t)\lambda}{t}(tx)\right\|^2 d \\ &= \int_0^1 \|(1-t)y + (1-t)\lambda x\|^2 dt + \int_0^1 \|(1-t)y - (1-t)\lambda x\|^2 dt \\ &= \int_0^1 \|(1-t)(y+\lambda x)\|^2 dt + \int_0^1 \|(1-t)(y-\lambda x)\|^2 dt \\ &= \frac{\|y+\lambda x\|^2}{3} + \frac{\|y-\lambda x\|^2}{3}, \end{aligned} \quad (26)$$

and

$$\begin{aligned} h(\beta+1) &= \int_0^1 \|(1-t)y + \frac{1}{2}\cdot\left[\frac{2(1-t)\lambda}{t} + 1\right](tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{1}{2}\cdot\left[\frac{2(1-t)\lambda}{t} + 1\right](tx)\|^2 dt \\ &= \int_0^1 \|(1-t)(y+\lambda x) + \frac{tx}{2}\|^2 dt + \int_0^1 \|(1-t)(y-\lambda x) - \frac{tx}{2}\|^2 dt = \frac{\|y+\lambda x\|^2}{3} + \frac{\|y-\lambda x\|^2}{3} + \frac{\|x\|^2}{6} = h(\beta) + \frac{\|x\|^2}{6}. \end{aligned} \quad (27)$$

Now, suppose that $0 < \beta < 1$ and note that $h(1) \neq h(0)$ (since $x \neq 0$), Lemma 6 gives

$$h(\beta) < \beta h(1) + (1-\beta)h(0). \quad (28)$$

Also, $h(\beta+1) \neq h(\beta)$ (since $x \neq 0$), and with the help of Lemma 6

$$\begin{aligned} h(1) &< \beta h(\beta) + (1-\beta)h(\beta+1) = \beta h(\beta) + (1-\beta) \cdot \left[h(\beta) + \frac{\|x\|^2}{6}\right] = \beta h(\beta) + (1-\beta) \\ &\cdot [h(\beta) + h(1) - h(0)] \Rightarrow h(\beta) > \beta h(1) + (1-\beta)h(0), \end{aligned} \quad (29)$$

which contradicts (28). Now, consider the case $\beta > 1$, we have

$$h(1) \le \frac{\beta-1}{\beta}h(0) + \frac{1}{\beta}h(\beta) = h(0) + \frac{1}{\beta}[h(\beta) - h(0)] \Rightarrow h(1) - h(0) \le \frac{1}{\beta}[h(\beta) - h(0)] \Rightarrow \frac{\|x\|^2}{6} \le \frac{1}{\beta}[h(\beta) - h(0)]. \quad (30)$$

Since $x \neq 0$, then, $h(\beta) \neq h(0)$, and using the Lemma 6, we have

$$h(1) < \frac{\beta-1}{\beta}h(0) + \frac{1}{\beta}h(\beta). \quad (31)$$

Also, $h(1) \neq h(\beta+1)$ and Lemma 6 gives us

$$\begin{aligned} h(\beta) &< \frac{1}{\beta}h(1) + \frac{\beta-1}{\beta}h(\beta+1) = \frac{1}{\beta}h(1) \cdot + \frac{\beta-1}{\beta}[h(\beta) + h(1) - h(0)] \Rightarrow h(1) \\ &> \frac{1}{\beta}h(\beta) + \frac{\beta-1}{\beta}h(0), \end{aligned} \quad (32)$$

which contradicts the relation (31). For the case $\beta = 1$, we have

$$h(2) = h(1) + \frac{\|x\|^2}{6} = h(0) + \frac{\|x\|^2}{3}. \quad (33)$$

This shows that $h(2) \neq h(0)$ (since $x \neq 0$). Then, we have

$$h(1) < \frac{1}{2}h(0) + \frac{1}{2}h(2) = \frac{1}{2}\left[h(0) + h(0) + \frac{\|x\|^2}{3}\right] \Rightarrow h(1) < h(0) + \frac{\|x\|^2}{6}, \quad (34)$$

which contradicts (25). Thus, in all cases, we get a contradiction. Hence, 2-HH-N orthogonality is unique in any normed space.

In the following theorem, 2-HH-R and 2-HH-B denote the Robert orthogonality and Birkhoff-James orthogonality via the 2-HH norm, respectively.

Theorem 8. *Let $x, y \in X$, where X is a real normed linear space equipped with an inner-product space over the field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ and $\mu = \lambda t$. Then, 2-HH-R orthogonality implies 2-HH-B orthogonality and conversely.*

Proof. Assume $x \perp_{2-HH-R} y$. Then, for any $\mu \in \mathbb{R}$,

$$\begin{aligned}\int_0^1 \|(1-t)x + \mu y\|^2 dt &= \int_0^1 \|(1-t)x - \mu y\|^2 dt \\ &= \int_0^1 \langle (1-t)x - \mu y, (1-t)x - \mu y \rangle dt \\ &= \int_0^1 \|(1-t)x\|^2 - (1-t)\langle x, y\rangle \\ &\quad - \mu(1-t)\langle y, x\rangle + \|\mu y\|^2 dt \\ &= \int_0^1 \|(1-t)x\|^2 dt + \int_0^1 \|\mu y\|^2 dt \\ &\geq \int_0^1 \|(1-t)x\|^2 dt,\end{aligned} \tag{35}$$

which shows that x is 2-HH-B orthogonal to y. To prove the converse part, it is enough to show that $x \perp_{2-HH-B} y \Rightarrow \langle x, y\rangle = 0 \Rightarrow x \perp_{2-HH-R} y$. Let $x \perp_{2-HH-B} y$. Then, for any $\lambda \in \mathbb{K}$,

$$\begin{aligned}\int_0^1 \|(1-t)x + \lambda t y\|^2 dt \geq \int_0^1 \|(1-t)x\|^2 dt \Rightarrow \int_0^1 \|(1-t)x\|^2 dt \\ + \left[\lambda\langle y, x\rangle + \overline{\lambda y, x}\right] \int_0^1 t(1-t) dt \\ + |\lambda|^2 \|y\|^2 \int_0^1 t^2 dt \geq \int_0^1 \|(1-t)x\|^2 dt \Rightarrow \mathrm{Re} \\ \cdot [\lambda\langle y, x\rangle] + |\lambda|^2 \|y\|^2 \geq 0.\end{aligned} \tag{36}$$

Now, for $\lambda = -\langle x, y\rangle / 2\|y\|^2$, inequality (36) becomes $-|\langle y, x\rangle|^2 / 4\|y\|^2 \geq 0$. Therefore, we have $\langle x, y\rangle = 0$. On the other hand, it is easy to show that $\langle x, y\rangle = 0 \Rightarrow x \perp_{2-HH-R} y$.

4. Conclusion

We conclude that the new orthogonality via the 2-HH norm is unique in any normed linear space. Moreover, if the underlying space is a real inner product space, the Robert and Birkhoff orthogonality via the 2-HH norm are equivalent.

Data Availability

There is no use of any data to support this study.

Conflicts of Interest

The authors declare that they do not have any conflict of interest for the publication of the article.

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Research Article

Applications of Modified Sigmoid Functions to a Class of Starlike Functions

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The main focus of this investigation is the applications of modified sigmoid functions. Due to its various uses in physics, engineering, and computer science, we discuss several geometric properties like necessary and sufficient conditions in the form of convolutions for functions to be in the special class $\mathcal{S}^*_{SG}$ earlier introduced by Goel and Kumar and obtaining third-order Hankel determinant for this class using modified sigmoid functions. Also, the third-order Hankel determinant for 2- and 3-fold symmetric functions of this class is evaluated.

1. Introduction

In this section, we present the related material for better understanding of the concepts discussed later in this article. We start with the notation of $\mathcal{A}$, the class of functions f which are analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and its series representation is

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbb{U}. \quad (1)$$

Further, a subclass of class $\mathcal{A}$ which is denoted by $\mathcal{S}$ contains all univalent functions in $\mathbb{U}$. Bieberbach conjectured in 1916 that $|a_n| \leq n$, $n = 2, 3, \cdots$. De Branges proved this in 1985; see [1]. During this period, a lot of coefficient results were established for some subfamilies of $\mathcal{S}$. Some of these classes are the class $\mathcal{S}^*$, known as the class of starlike functions, the class $\mathcal{K}$, known as class of convex functions, and $\mathcal{R}$ of bounded turning functions. These are defined as

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \psi = \frac{1+z}{1-z}, z \in \mathbb{U} \right\}, \quad (2)$$

$$\mathcal{K}(\psi) = \left\{ f \in \mathcal{S} : \frac{\left(zf'(z)\right)'}{f'(z)} \prec \psi = \frac{1+z}{1-z}, z \in \mathbb{U} \right\}, \quad (3)$$

$$\mathcal{R}(\psi) = \left\{ f \in \mathcal{S} : f'(z) \prec \psi = \frac{1+z}{1-z}, z \in \mathbb{U} \right\}. \quad (4)$$

Now, recall the subordination definition; we say that an analytic function $f_1(z)$ is subordinate to $f_2(z)$ in $\mathbb{U}$ and is symbolically written as $f_1(z) \prec f_2(z)$ if there occurs a Schwarz function $u(z)$ with properties that $|u(z)| \leq 1$ and $u(0) = 1$ such that $f_1(z) = f_2(u(z))$. Moreover, if $f_2(z)$ is in the class $\mathcal{S}$, then we have the following equivalency, due to [2, 3],

$$f_1(0)=f_2(0),$$
$$f_1(\mathbb{U})\subseteq f_2(\mathbb{U}). \tag{5}$$

For two functions $f_1(z)=z+\sum_{n=2}^{\infty}a_{n,1}z^n$ and $f_2(z)=z+\sum_{n=2}^{\infty}a_{n,2}z^n$ in $\mathbb{U}$, then the convolution or Hadamard product is defined by

$$(f_1 * f_2)(z)=z+\sum_{n=2}^{\infty}a_{n,1}a_{n,2}z^n. \tag{6}$$

By varying the right-hand side of subordinated inequality in (2), several familiar classes can be obtained such as the following:

(1) For $\psi=(1+Az)/(1+Bz)$, we get the class $\mathcal{S}^*(A,B)$; see [4] for details

(2) While for different values of A and B the class $\mathcal{S}^*(\alpha)=\mathcal{S}^*(1-2\alpha,-1)$ is obtained and investigated in [5]

(3) For $\psi=1+(2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$, the class was defined and studied in [6]

(4) For $\psi=\sqrt{1+z}$, the class is denoted by $\mathcal{S}_L^*$; details can be seen in [7, 8], and for further study, see [9]

(5) For $\psi=\cosh(z)$, the class is denoted by $\mathcal{S}_{\cosh}^*$; see [10]

(6) For $\psi=1+\sin(z)$, the class is denoted by $\mathcal{S}_{\sin}^*$; see [11] for details, and for further investigation, see [12]

(7) While for $\psi=z+\sqrt{1+z^2}$, the class is denoted by $\mathcal{S}_l^*$; see [13]

(8) For $\psi=e^z$, the class denoted by $\mathcal{S}_e^*$ was defined and studied in [14, 15]

(9) Similarly, if $\psi=1+(4/3)z+(2/3)z^2$, then such a class is denoted by $\mathcal{S}_C^*$ and was introduced in [16], and for further study, the reader is referred to [17]

Also, several other subclasses of starlike functions were introduced recently in [18–22] by choosing some particular function for ψ such functions are associated with Bell numbers, shell-like curve connected with Fibonacci numbers, and functions connected with the conic domains.

In this paper, we investigate starlike functions associated with a kind of special functions known as modified sigmoid function $\psi(z)=2/(1+e^{-z})$. In mathematics, the theory of special functions is the most important for scientists and engineers who are concerned with actual mathematical calculations. To be specific, it has applications in problems of physics, engineering, and computer science. The activation function is an example of special function. These functions act as a squashing function which is the output of a neuron in a neural network between certain values (usually 0 and 1 and -1 and 1). There are three types of functions, namely, piecewise linear function, threshold function, and sigmoid function. In the hardware implementation of neural network, the most important and popular activation function is the sigmoid function. The sigmoid function is often used with gradient descendent type learning algorithm. Due to differentiability of the sigmoid function, it is useful in weight-learning algorithm. The sigmoid function increases the size of the hypothesis space that the network can represent. Some of its advantages are the following:

(1) It gives real numbers between 0 and 1

(2) It maps a very large output domain to a small range of outputs

(3) It never loses information because it is a one-to-one function

(4) It increases monotonically

For more details, see [23].

The class $\mathcal{S}_{SG}^*$ defined by Goel and Kumar in [24] is defined as

$$\frac{zf'(z)}{f(z)}\prec\frac{2}{1+e^{-z}},\ (z\in\mathbb{U}). \tag{7}$$

For a parameter q, with $n\in\mathbb{N}=\{1,2,3,\cdots\}$, Pommerenke [25, 26] defined Hankel determinant $H_{q,n}(f)$ for functions $f\in\mathcal{S}$ of the form (1) as follows:

$$H_{q,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}. \tag{8}$$

The growth of $H_{q,n}(f)$ has been evaluated for different subcollections of univalent functions. Exceptionally, for each of the sets $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$, the sharp bound of the determinant $H_{2,2}(f)=|a_2a_4-a_3^2|$ was found by Jangteng et al. [7, 27], while for the family of close-to-convex functions the sharp estimate is still unknown (see [28]). On the other hand, for the set of Bazilevic functions, the best estimate of $|H_{2,2}(f)|$ was proved by Krishna et al. [29]. For more work on $H_{2,2}(f)$, see [30–34].

The determinant

$$H_{3,1}(f)=\begin{vmatrix} 1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix} \tag{9}$$

is known as the third-order Hankel determinant, and the estimation of this determinant $|H_{3,1}(f)|$ is the focus of various researchers of this field. In 2010, the first article on $H_{3,1}(f)$ was published by Babalola [35], in which he obtained the upper bound of $|H_{3,1}(f)|$ for the classes of $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$. Later on, a few mathematicians extended this work for various subcollections of holomorphic and univalent functions; see [36–41]. In 2017, Zaprawa [42] improved their work by proving

$$\left|H_{3,1}(f)\right| \leq \begin{cases} 1, & \text{for } f \in \mathcal{S}^*, \\ \dfrac{49}{540}, & \text{for } f \in \mathcal{K}, \\ \dfrac{41}{60}, & \text{for } f \in \mathcal{R}. \end{cases} \quad (10)$$

And he asserted that these inequalities are not sharp as well. Additionally, for the sharpness, he investigated the subfamilies of $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{R}$ comprising functions with m-fold symmetry and acquired the sharp bounds. Recently, in 2018, Kowalczyk et al. [43] and Lecko et al. [44] got the sharp inequalities which are

$$\begin{aligned} \left|H_{3,1}(f)\right| &\leq 4/135, \\ \left|H_{3,1}(f)\right| &\leq 1/9, \end{aligned} \quad (11)$$

for the classes $\mathcal{K}$ and $\mathcal{S}^*(1/2)$, respectively, where the symbol $\mathcal{S}^*(1/2)$ indicates the family of starlike functions of order 1/2. Additionally, in 2018, the authors [45] got an improved bound $|H_{3,1}(f)| \leq 8/9$ for $f \in \mathcal{S}^*$, which is yet not the best possible. In this article, our main purpose is to study necessary and sufficient conditions for functions to be in the class $\mathcal{S}^*_{SG}$ in the form of convolutions results, coefficient inequality, and important third-order Hankel determinant for this class in (7) and also for its 2- and 3-fold symmetric functions.

2. A Set of Lemmas

Let $\mathcal{P}$ be the family of functions $p(z)$ that are holomorphic in $\mathbb{D}$ with $\mathfrak{Re}\,p(z) > 0$ and its series form is as follows:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n (z \in \mathbb{D}). \quad (12)$$

Lemma 1. *If $p(z) \in \mathcal{P}$ and it is of the form (12), then*

$$|c_n| \leq 2 \text{ for } n \geq 1, \quad (13)$$

$$|c_{n+k} - \delta c_n c_k| \leq 2 \text{ for } 0 \leq \delta \leq 1, \quad (14)$$

$$|c_n c_m - c_l c_k| \leq 4 \text{ for } n + m = l + k, \quad (15)$$

$$\left|c_{n+2k} - \delta c_n c_k^2\right| \leq 2(1 + 2\delta) \text{ for } \delta \in \mathbb{R}, \quad (16)$$

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}, \quad (17)$$

and for $\xi \in \mathbb{C}$.

$$\left|c_2 - \xi c_1^2\right| \leq 2 \max\{1; |2\xi - 1|\}. \quad (18)$$

For the results in (13), (14), (15), (16), and (17), see [46]. Also, see [47] for (18).

Lemma 2. *[48]. If $p(z) \in \mathcal{P}$ and is represented by (12), then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & (\nu \leq 0), \\ 2 & (0 \leq \nu \leq 1), \\ 4\nu - 2 & (\nu \geq 1). \end{cases} \quad (19)$$

Lemma 3. *Let $p \in \mathcal{P}$ have representation of the form (12), then*

$$\left|\alpha c_1^3 - \beta c_1 c_2 + \gamma c_3\right| \leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|. \quad (20)$$

Proof. Consider the left-hand side of (20) and then rearranging the terms, we have

$$\begin{aligned} \left|\alpha c_1^3 - \beta c_1 c_2 + \gamma c_3\right| &= \left|\alpha\left(c_1^3 - 2c_1 c_2 + c_3\right) - (\beta - 2\alpha)(c_1 c_2 - c_3) \right. \\ &\quad \left. + (\alpha - \beta + \gamma) c_3\right| \leq |\alpha|\left|c_1^3 - 2c_1 c_2 + c_3\right| \\ &\quad + |\beta - 2\alpha||c_1 c_2 - c_3| + |\alpha - \beta + \gamma||c_3| \\ &\leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|, \end{aligned} \quad (21)$$

where we have used (13) and (14).

3. Convolution Results for Class $\mathcal{S}^*_{SG}$

Theorem 4. *Let $f(z) \in \mathcal{A}$ be the form (1), then $f(z) \in \mathcal{S}^*_{SG}$, if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - \alpha z^2}{(1 - z)^2}\right] \neq 0, \quad (22)$$

for all $\alpha = \alpha_\xi = 2/(1 - e^{-\xi})$ and also for $\alpha = 1$.

Proof. Since $f(z) \in \mathcal{S}^*_{SG}$ is analytic in domain $\mathbb{U}$, so $f(z) \neq 0$, for all $z \in \mathbb{U}^* = \mathbb{U}\{0\}$, that is $(1/z)f(z) \neq 0$ for $z \in \mathbb{U}$, which is equivalent to (22) for $\alpha = 1$. In this case, the proof is completed. Now, from definition (7), there occurs a Schwarz function $u(z)$, such that $|u(z)| < 1$ and $u(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-u(z)}}. \quad (23)$$

Equivalently,

$$\frac{zf'(z)}{f(z)} \neq \frac{2}{1 + e^{-\xi}}, \quad |\xi| = 1, \quad (24)$$

which implies that

$$zf'(z) - f(z)\frac{2}{1 + e^{-\xi}} \neq 0. \quad (25)$$

We know that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}, \quad f(z) = f(z) * \frac{z}{1-z}. \tag{26}$$

By simple computation, equation (25) becomes

$$\frac{1}{z}\left[f(z) * \frac{z-\alpha z^2}{(1-z)^2}\right] \neq 0, \tag{27}$$

where α is given above.

Conversely, suppose equation (22) holds true for $\alpha = 1$, it implies that $(1/z)f(z) \neq 0$, for all $z \in \mathbb{U}$. Let $\Phi(z) = zf'(z)/f(z)$ be analytic in $\mathbb{U}$, with $\Phi(0) = 1$. Also, suppose that $\Psi(z) = 2/(1+e^{-z})$, $z \in \mathbb{U}$. It is clear from (24) that $\Psi(\partial\mathbb{U}) \cap \Phi(\mathbb{U}) = \phi$. Hence, the simply connected domain $\Phi(\mathbb{U})$ is contained in connected component of Ψ $(\partial\mathbb{U})$. The univalence of "Ψ", together with the fact $\Phi(0) = \Psi(0) = 1$, shows that $\Phi \prec \Psi$ and implies that $f(z) \in \mathcal{S}^*_{SG}$.

Theorem 5. *Let $f(z) \in \mathcal{A}$ be of the form (1), then the necessary and sufficient condition for function $f(z)$ that belongs to class $\mathcal{S}^*_{SG}$ is*

$$1 - \sum_{n=2}^{\infty}\left(\frac{n(1-e^{-\xi})-2}{1-e^{-\xi}}\right)a_n z^{n-1} \neq 0. \tag{28}$$

Proof. In the light of Theorem 4, we show that $\mathcal{S}^*_{SG}$ if and only if

$$\begin{aligned} 0 \neq \frac{1}{z}\left[f(z) * \frac{z-\alpha z^2}{(1-z)^2}\right] &= \frac{1}{z}\left[zf'(z) - \alpha\left(zf'(z) - f(z)\right)\right] \\ &= 1 - \sum_{n=2}^{\infty}((\alpha-1)n-\alpha)a_n z^{n-1} \\ &= 1 - \sum_{n=2}^{\infty}\left(\frac{n(1-e^{-\xi})-2}{1-e^{-\xi}}\right)a_n z^{n-1}. \end{aligned} \tag{29}$$

Hence, the proof is completed.

Theorem 6. *Let $f \in \mathcal{A}$ be of the form (1) and satisfies*

$$\sum_{n=2}^{\infty}\left|\frac{n(1-e^{-\xi})-2}{1-e^{-\xi}}\right||a_n| < 1, \tag{30}$$

then $f \in \mathcal{S}^*_{SG}$.

Proof. To show $f \in \mathcal{S}^*_{SG}$, we have to show that (28) is satisfied. Consider

$$\begin{aligned} \left|1 - \sum_{n=2}^{\infty}((\alpha-1)n-\alpha)a_n z^{n-1}\right| &> 1 - \sum_{n=2}^{\infty}\left|((\alpha-1)n-\alpha)a_n z^{n-1}\right| \\ &= 1 - \sum_{n=2}^{\infty}|((\alpha-1)n-\alpha)||a_n||z|^{k-1} \\ &> 1 - \sum_{n=2}^{\infty}|((\alpha-1)n-\alpha)||a_n| \\ &= 1 - \sum_{n=2}^{\infty}\left|\frac{n(1-e^{-\xi})-2}{1-e^{-\xi}}\right||a_n| > 0, \end{aligned} \tag{31}$$

so by Theorem 5, $f(z) \in \mathcal{S}^*_{SG}$.

4. Upper Bound $H_{3,1}(f)$ for Set $\mathcal{S}^*_{SG}$

Theorem 7. *Let $f \in \mathcal{S}^*_{SG}$ and is of the form (1), then*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{1}{4}\max\left\{1, \frac{|2\lambda-1|}{2}\right\}. \tag{32}$$

Proof. Since $f \in \mathcal{S}^*_{SG}$, then there exists an analytic function $w(z)$, $|w(z)| \leq 1$ and $w(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1+e^{-w(z)}}. \tag{33}$$

Denote

$$\Psi(w(z)) = \frac{2}{1+e^{-w(z)}}, \quad k(z) = 1 + c_1 z + c_2 z^2 + \cdots = \frac{1+w(z)}{1-w(z)}. \tag{34}$$

Obviously, the function $k(z) \in \mathcal{P}$ and $w(z) = (k(z)-1)/(k(z)+1)$. This gives

$$w(z) = \frac{k(z)-1}{k(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}, \tag{35}$$

$$\begin{aligned} \frac{2}{1+e^{-w(z)}} = 1 &+ \frac{1}{4}c_1 z + \left(\frac{1}{4}c_2 - \frac{1}{8}c_1^2\right)z^2 \\ &+ \left(\frac{11}{192}c_1^3 - \frac{1}{4}c_2 c_1 + \frac{1}{4}c_3\right)z^3 \\ &+ \left(\frac{1}{4}c_1^2 c_2 - \frac{1}{2}c_3 c_1 - \frac{1}{4}c_2^2 + \frac{1}{2}c_4\right)z^4 + \cdots, \end{aligned} \tag{36}$$

while

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2 a_3 + 3a_4\right)z^3 + \left(-a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2 + 4a_5\right)z^4 + \cdots. \quad (37)$$

On equating coefficients of (36) and (37), we get

$$a_2 = \frac{1}{4}c_1, \quad (38)$$

$$a_3 = \frac{1}{8}c_2 - \frac{1}{32}c_1^2, \quad (39)$$

$$a_4 = \frac{7}{1152}c_1^3 - \frac{5}{96}c_2 c_1 + \frac{1}{12}c_3, \quad (40)$$

$$a_5 = -\frac{1}{16}\left(\frac{17}{1152}c_1^4 - \frac{7}{24}c_1^2 c_2 + \frac{2}{3}c_3 c_1 + \frac{3}{8}c_2^2 - c_4\right). \quad (41)$$

Now from (38) and (39), we have

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{8}\left|c_2 - \frac{2\lambda + 1}{4}c_1^2\right|. \quad (42)$$

Now, using (18), we get the required result.

If we put $\lambda = 1$, the above result becomes as follows.

Corollary 8. *Let $f(z) \in \mathcal{S}_{SG}^*$ be of the form (1) then*

$$\left|a_3 - a_2^2\right| \le \frac{1}{4}. \quad (43)$$

The result is best possible for function

$$f(z) = z \exp\left(\int_0^z \frac{e^{t^2} - 1}{t\left(e^{t^2} + 1\right)}dt\right) = z + \frac{1}{4}z^3 + \cdots. \quad (44)$$

Theorem 9. *Let $f(z) \in \mathcal{S}_{SG}^*$ be of the form (1), then*

$$\left|a_2 a_3 - a_4\right| \le \frac{1}{6}. \quad (45)$$

The result is best possible for function defined as

$$f_n(z) = z \exp\left(\int_0^z \frac{e^{t^3} - 1}{t\left(e^{t^3} + 1\right)}dt\right) = z + \frac{1}{6}z^4 + \cdots. \quad (46)$$

Applying Lemma 3, we get the required result.

Proof. By using (38), (39), and (40), we get

$$\left|a_2 a_3 - a_4\right| = \left|\frac{1}{72}c_1^3 - \frac{1}{12}c_2 c_1 + \frac{1}{12}c_3\right|. \quad (47)$$

Applying Lemma 3, we get the required result.

Theorem 10. *Let $f(z) \in \mathcal{S}_{SG}^*$ be of the form, (1) then*

$$\left|a_2 a_4 - a_3^2\right| \le \frac{55}{576}. \quad (48)$$

Proof. With the help of (38), (39), and (40), we get

$$\left|a_2 a_4 - a_3^2\right| = \left|\frac{1}{48}c_3 c_1 - \frac{7}{9216}c_1^4 - \frac{1}{192}c_1^2 c_2 - \frac{1}{64}c_2^2\right|. \quad (49)$$

Now, rearranging the terms

$$\left|a_2 a_4 - a_3^2\right| = \left|\frac{c_1}{192}\left(c_3 - c_1 c_2\right) - \frac{c_1 c_3 - c_2^2}{64} - \frac{7}{9216}c_1^4\right|. \quad (50)$$

Using (13), (14), and (15), we get the required result.

For the third Hankel determinant, we need the following result.

Lemma 11. *[24]. Let $f(z) \in \mathcal{S}_{SG}^*$ be of the form (1). Then,*

$$\begin{aligned} |a_2| &\le \frac{1}{2}, \\ |a_3| &\le \frac{1}{4}, \\ |a_4| &\le \frac{1}{6}, \\ |a_5| &\le \frac{1}{8}. \end{aligned} \quad (51)$$

These results are sharp for function defined as

$$f_n(z) = z \exp\left(\int_0^z \frac{e^{t^{n-1}} - 1}{t\left(e^{t^{n-1}} + 1\right)}dt\right), \quad \text{for } a_n (n = 2, 3, 4, 5). \quad (52)$$

Theorem 12. *Let $f(z) \in \mathcal{S}_{SG}^*$ be of the form (1). Then,*

$$\left|H_{3,1}(f)\right| \le \frac{191}{2304} \simeq 0.0829. \quad (53)$$

Proof. Since

$$H_{3,1}(f) = a_3\left(a_2 a_4 - a_3^2\right) - a_4\left(a_4 - a_2 a_3\right) + a_5\left(a_3 - a_2^2\right), \quad (54)$$

by applying triangle inequality, we obtain

$$\left|H_{3,1}(f)\right| \le |a_3|\left|a_2 a_4 - a_3^2\right| + |a_4|\left|a_4 - a_2 a_3\right| + |a_5|\left|a_3 - a_2^2\right|. \quad (55)$$

Now, using Corollary 8, Theorems 9 and 10, and Lemma 11, we get the required result.

5. Bounds of $H_{3,1}(f)$ for 2-Fold and 3-Fold Symmetric Functions

Let $m \in \mathbb{N} = \{1, 2, 3, \cdots\}$, if a rotation of domain $\mathbb{D}$ about the origin through an angle $2\pi/m$ carries itself on the domain $\mathbb{D}$ is called m-fold symmetric. It is very much clear to see that an analytic function f is m-fold symmetric in $\mathbb{D}$, if

$$f\left(e^{\frac{2\pi}{m}}z\right) = e^{\frac{2\pi}{m}}f(z), \quad z \in \mathbb{D}. \tag{56}$$

By $\mathcal{S}^{(m)}$, we mean the set of m-fold symmetric univalent functions having the following series form

$$f(z) = z + \sum_{k=2}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in \mathbb{D}. \tag{57}$$

The subclass $\mathcal{S}^{*(m)}_{SG}$ is a set of m-fold symmetric starlike functions associated with modified sigmoid function. More precisely, an analytic function f of the form (57) belongs to class $\mathcal{S}^{*(m)}_{SG}$ if and only if

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-((p(z)-1)/(p(z)+1))}}, \quad p \in \mathcal{P}^{(m)}, \tag{58}$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \quad z \in \mathbb{D} \right\}. \tag{59}$$

Theorem 13. *If $f \in \mathcal{S}^{*(2)}_{SG}$ be of the form (57), then*

$$\left|H_{3,1}(f)\right| \leq \frac{1}{32}. \tag{60}$$

Proof. Since $f \in \mathcal{S}^{*(2)}_{SG}$; therefore, there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-((p(z)-1)/(p(z)+1))}}. \tag{61}$$

Using the series form (57) and (59), when $m = 2$ in the above relation, we have

$$a_3 = \frac{1}{8}c_2, \qquad a_5 = \frac{1}{16}c_4 - \frac{3}{128}c_2^2. \tag{62}$$

Now,

$$H_{3,1}(f) = a_3 a_5 - a_3^2. \tag{63}$$

Therefore,

$$H_{3,1}(f) = \frac{c_2}{128}\left(c_4 - \frac{3}{8}c_2^2\right). \tag{64}$$

Using (13) and (14) along with triangle inequality, we get

$$\left|H_{3,1}(f)\right| \leq \frac{1}{32}. \tag{65}$$

Theorem 14. *If $f \in \mathcal{S}^{*(3)}_{SG}$ be of the form (57), then*

$$\left|H_{3,1}(f)\right| \leq \frac{1}{36}. \tag{66}$$

Proof. Since $f \in \mathcal{S}^{*(3)}_{SG}$; therefore, there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-((p(z)-1)/(p(z)+1))}}. \tag{67}$$

Using the series form (57) and (59), when $m = 3$ in the above relation, we have

$$a_4 = \frac{1}{12}c_3. \tag{68}$$

Now,

$$H_{3,1}(f) = -a_4^2. \tag{69}$$

Therefore,

$$H_{3,1}(f) = \frac{1}{144}c_3^2. \tag{70}$$

Using (13), we get

$$\left|H_{3,1}(f)\right| \leq \frac{1}{36}. \tag{71}$$

The result is best possible for function defined as follows:

$$f_4(z) = z\ \exp\left(\int_0^z \frac{e^{t^3} - 1}{t\left(e^{t^3} + 1\right)}dt\right) = z + \frac{1}{6}z^4 + \cdots. \tag{72}$$

Data Availability

The data used in this article are artificial and hypothetical, and anyone can use these data before prior permission by just citing this article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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Research Article

Some Results on Wijsman Ideal Convergence in Intuitionistic Fuzzy Metric Spaces

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In the present work, we study and extend the notion of Wijsman $\mathcal{J}$–convergence and Wijsman $\mathcal{J}^*$–convergence for the sequence of closed sets in a more general setting, i.e., in the intuitionistic fuzzy metric spaces (briefly, IFMS). Furthermore, we also examine the concept of Wijsman $\mathcal{J}^*$–Cauchy and $\mathcal{J}$–Cauchy sequence in the intuitionistic fuzzy metric space and observe some properties.

1. Introduction

In 1951, Fast [1] initiated the theory of statistical convergence. It is an extremely effective tool to study the convergence of numerical problems in sequence spaces by the idea of density. Statistical convergence of the sequence of sets was examined by Nuray and Rhoades [2]. Ulusu and Nuray [3] studied the Wijsman lacunary statistical convergence sequence of sets and connected with the Wijsman statistical convergence. Esi et al. [4] introduced the Wijsman λ-statistical convergence of interval numbers. Kostyrko et al. [5] generalized the statistical convergence and introduced the notion of ideal $\mathcal{J}$–convergence. Salát et al. [6, 7] investigated it from the sequence space viewpoint and associated with the summability theory. Further, it was analyzed by Khan et al. [8] with the help of a bounded operator. In 2008, Das et al. [9] analyzed $\mathcal{J}$ and $\mathcal{J}^*$–convergence for double sequences. Kisi and Nuray [10] initiated new convergence definitions for the sequence of sets. Furthermore, Gümüş [11] studied the Wijsman ideal convergent sequence of sets using the Orlicz function.

In 1965, Zadeh [12] started the fuzzy sets theory. This theory has proved its usefulness and ability to solve many problems that classical logic was unable to handle. Karmosil et al. [13] introduced the fuzzy metric space, which has the most significant applications in quantum particle physics. Afterward, numerous researchers have studied the concept of fuzzy metric spaces in different ways. George et al. [14, 15] modified the notion of fuzzy metric space and determined a Hausdorff topology for fuzzy metric spaces. Atanassov [16] generalized the fuzzy sets and introduced the notion of intuitionistic fuzzy sets in 1986. Park [17] examined the notion of IFMS, and Saadati and Park [18] further analyzed the intuitionistic fuzzy topological spaces. Moreover, statistical convergence, ideal convergence, and different properties of sequences in intuitionistic fuzzy normed spaces were examined by Mursaleen et al. [19–21]. Also one can refer to Sengül and Et [22], Sengül et al. [23], Et and Yilmazer [24], Mohiuddine and Alamri [25], and Mohiuddine et al. [26, 27].

2. Preliminaries

We recall some concepts and results which are needed in sequel.

Definition 1 [5]. A family of subsets $\mathcal{J} \subseteq 2^{\mathbb{N}}$ is known as an ideal in a non-empty set $\mathbb{N}$, *if*

(1) $\emptyset \in \mathcal{J}$,

(2) for any $\mathcal{C}, \mathcal{D} \in \mathcal{J} \Rightarrow \mathcal{C} \cup \mathcal{D} \in \mathcal{J}$,

(3) for any $\mathcal{C} \in \mathcal{J}$ and $\mathcal{D} \subseteq \mathcal{C}, \Rightarrow \mathcal{D} \in \mathcal{J}$.

Remark 2 [5]. An ideal $\mathcal{J}$ is known as non-trivial if $\mathbb{N} \notin \mathcal{J}$. A nontrivial ideal $\mathcal{J}$ is known as admissible if $\{\{n\}: n \in \mathbb{N}\} \in \mathcal{J}$.

Definition 3 [5]. A nonempty subset $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is known as filter in $\mathbb{N}$ if

(1) for every $\emptyset \notin \mathcal{F}$,

(2) for every $\mathcal{C}, \mathcal{D} \in \mathcal{F} \Rightarrow \mathcal{C} \cap \mathcal{D} \in \mathcal{F}$,

(3) for every $\mathcal{C} \in \mathcal{F}$ with $\mathcal{C} \subseteq \mathcal{D}$, one obtain $\mathcal{D} \in \mathcal{F}$.

Proposition 4 [5]. *For every ideal $\mathcal{J}$, there is a filter $\mathcal{F}(\mathcal{J})$ associated with $\mathcal{J}$ defined as follows:*

$$\mathcal{F}(\mathcal{J}) = \{K \subseteq \mathbb{N} : K = \mathbb{N} \setminus A, \textit{for some } A \in \mathcal{J}\}. \quad (1)$$

Definition 5 [5]. Let $\{\mathcal{C}_1, \mathcal{C}_1, \cdots\}$ be a mutually disjoint sequence of sets of $\mathcal{J}$. Then, there is sequence of sets $\{\mathcal{D}_1, \mathcal{D}_1, \cdots\}$ so that $\cup_{j=1}^{\infty} \mathcal{D}_j \in \mathcal{J}$ and each symmetric difference $\mathcal{C}_j \triangle \mathcal{D}_j (j = 1, 2, \cdots)$ is finite. In this case, admissible ideal $\mathcal{J}$ is known as property (AP).

Lemma 6 [28]. *Suppose $\mathcal{J}$ be an admissible ideal alongside property (AP). Let a countable collection of subsets $\{\mathcal{C}_k\}_{k=1}^{\infty}$ of positive integer $\mathbb{N}$ in such a way that $\mathcal{C}_k \in \mathcal{F}(\mathcal{J})$. Then, there exists a set $\mathcal{C} \subset \mathbb{N}$ such that $\mathcal{C} \setminus \mathcal{C}_k$ is finite for all $k \in \mathbb{N}$ and $\mathcal{C} \in \mathcal{F}(\mathcal{J})$.*

Definition 7 [29]. Let $(\mathcal{M}, d)$ be a metric space and $\{\mathcal{C}_k\}$ be a sequence of nonempty closed subsets of $\mathcal{M}$ which is said to be Wijsman convergent to the closed $\mathcal{C}$ *of* $\mathcal{M}$, if

$$\lim_{k\to\infty} d(x, \mathcal{C}_k) = d(x, \mathcal{C}) \text{for every} \in \mathcal{M}. \quad (2)$$

In other words, $W - \lim_{k\to\infty} \mathcal{C}_k = \mathcal{C}$.

In 2012, Nuray and Rhoades [2] initiated the theory of Wijsman statistical convergence for a sequence of sets. Furthermore, Kisi and Nuray [10] extended it into $\mathcal{J}$ -convergence.

Definition 8 [10]. Suppose $(\mathcal{M}, d)$ is a metric space. A nonempty closed subset $\{\mathcal{C}_k\}$ of $\mathcal{M}$ is known as Wijsman $\mathcal{J}$ –convergent to a closed set $\mathcal{C}$, if for every $x \in \mathcal{M}$, one has

$$\{k \in \mathbb{N} : |d(x, \mathcal{C}_k) - d(x, \mathcal{C})| \geq \epsilon\} \in \mathcal{J}. \quad (3)$$

Hence, one writes $\mathcal{J}_W - \lim_{k\to\infty} \mathcal{C}_k = \mathcal{C}$.

Definition 9 [10]. Suppose $(\mathcal{M}, d)$ *is a metric space. A nonempty closed subset $\{\mathcal{C}_k\}$ of $\mathcal{M}$ is known as Wijsman $\mathcal{J}$ –Cauchy if for each $x \in \mathcal{M}$, there exists a positive integer $m = m(\epsilon)$ so that the set*

$$\left\{k \in \mathbb{N} : \left|d(x, \mathcal{C}_k) - d\left(x, \mathcal{C}_p\right)\right| \geq \epsilon\right\} \in \mathcal{J}, \text{for all } p \geq m. \quad (4)$$

Definition 10 [10]. Suppose $(\mathcal{M}, d)$ is a separable metric space and $\{\mathcal{C}_k\}, \mathcal{C}$ is nonempty closed subsets of $\mathcal{M}$. A sequence $\{\mathcal{C}_k\}$ is known as Wijsman $\mathcal{J}^*$ –convergent to $\mathcal{C}$ if and only if $\exists P \in \mathcal{F}(\mathcal{J})$ and $P = \{p = (p_j < p_{j+1}, j \in \mathbb{N})\} \subset \mathbb{N}$ in such a manner that

$$\lim_{k\to\infty} d\left(x, \mathcal{C}_{m_k}\right) = d(x, \mathcal{C}), \text{for every } x \in \mathcal{M}. \quad (5)$$

One writes $\mathcal{J}_W^* - \lim_{k\to\infty} \mathcal{C}_k = \mathcal{C}$.

Definition 11 [10]. Suppose $(\mathcal{M}, d)$ is a separable metric space and $\mathcal{J}$ is an admissible ideal. A sequence $\{\mathcal{C}_k\}$ of nonempty closed subsets of $\mathcal{M}$ is known as the Wijsman $\mathcal{J}^*$ –Cauchy sequence if there exists $P \in \mathcal{F}(\mathcal{J})$, where $P = \{p = (p_j < p_{j+1}, i \in \mathbb{N})\}$ in such a way that subsequence $\mathcal{C}_P = \{\mathcal{C}_{p_k}\}$ is Wijsman Cauchy in $\mathcal{M}$, i.e.,

$$\lim_{k,l\to\infty} |d\left(x, \mathcal{C}_{m_k}\right) - d\left(x, \mathcal{C}_{p_l}\right)| = 0. \quad (6)$$

Remark 12 [10]. In general, the Wijsman topology is not first-countable, if sequence of nonempty sets $\{\mathcal{C}_k\}$ is Wijsman convergent to set $\mathcal{C}$, then every subsequence of $\{\mathcal{C}_k\}$ may not be convergent to $\mathcal{C}$. Every subsequence of the convergent sequence $\{\mathcal{C}_k\}$ converges to the same limit provided that $\mathcal{M}$ is a separable metric space.

Definition 13 [17]. Let $\mathcal{M}$ be a nonempty set, η and φ be fuzzy sets on $\mathcal{M}^2 \times (0,\infty)$, $*$ be a continuous t-norm, and $\diamond$ be a continuous t -conorm. Then, the five-tuple $(\mathcal{M}, \eta, \varphi, *, \diamond)$ is known as an intuitionistic fuzzy metric space (for short, IFMS) if it fulfills the subsequent conditions for all $s, t > 0$ and for every $y, z, w \in \mathcal{M}$:

(a) $\eta(y, z, s) + \varphi(y, z, s) \leq 1$,

(b) $\eta(y, z, s) > 0$,

(c) $\eta(y, z, s) = 1$ if and only if $y = z$,

(d) $\eta(y, z, s) = \eta(z, y, s)$,

(e) $\eta(y, z, s) * \eta(z, w, t) \leq \eta(y, w, s + t)$,

(f) $\eta(y, z, .)$: $(0,\infty) \to 0, 1]$ is continuous,

(g) $\varphi(y, z, s) < 1$,

(h) $\varphi(y, z, s) = 0$ if and only if $y = z$,

(i) $\varphi(y, z, s) = \varphi(z, y, s)$,

(j) $\varphi(y, z, s) \diamond \varphi(z, w, t) \geq \varphi(y, w, s + t)$,

(k) $\varphi(y, .)$: $(0,\infty) \to 0, 1]$ is continuous.

In such situation, (η, φ) is called the intuitionistic fuzzy metric (briefly, IFM).

Example 14 [17]. Suppose $(\mathcal{M}, d)$ is a metric space. Define $c \diamond d = \min(c + d, 1)$ and $c * d = cd$ for all $c, d \in [0, 1]$, and suppose η and φ are fuzzy sets on $\mathcal{M}^2 \times (0,\infty)$ defined as

$$\eta(y,z,s)=\frac{s}{s+d(y,z)},\varphi(y,z,s)=\frac{d(y,z)}{s+d(y,z)}. \tag{7}$$

Then $(\mathscr{M},\eta,\varphi,*,\diamond)$ is an IFMS.

Definition 15 [18]. Let $(\mathscr{M},\eta,\varphi,*,\diamond)$ be an IFMS and $\mathscr{C}$ be a nonempty subset of $\mathscr{M}$. For all $s>0$ and $x\in\mathscr{M}$, we define

$$\eta(x,\mathscr{C},s)=\sup\{\eta(x,y,s): y\in\mathscr{C}\} \tag{8}$$

and

$$\varphi(x,\mathscr{C},s)=\inf\{\varphi(x,y,s): y\in\mathscr{C}\}, \tag{9}$$

where $\eta(x,\mathscr{C},s)$ and $\varphi(x,\mathscr{C},s)$ are the degree of nearness and nonnearness of x to $\mathscr{C}$ at s, respectively.

Saadati and Park [18] studied the notion of convergence sequence with respect to IFMS which are defined as follows:

Definition 16 [18]. Let $(\mathscr{M},\eta,\varphi,*,\diamond)$ be an IFMS. A sequence $x=(x_k)$ is convergent to ξ if for any $0<\epsilon<0$ *and* $s>0$ there exists $k_0\in\mathbb{N}$ in such a way that

$$\eta(x_k,\xi,s)>1-\epsilon \text{ and } \varphi(x_k,\xi,s)<\epsilon \text{ for all } k\geq k_0. \tag{10}$$

Definition 17 [20]. An IFMS $(\mathscr{M},\eta,\varphi,*,\diamond)$ is known as separable if it contains a countable dense subset, i.e., there is a countable set $\{x_k\}$ along with subsequent property: for any $s>0$ and for all $\xi\in\mathscr{M}$, there is at least one x_n in order that

$$\eta(x_n,\xi,s)\geq 1-\epsilon \text{ and } \varphi(x_n,\xi,s)\leq\epsilon, \text{ for each } \epsilon\in(0,1). \tag{11}$$

3. Wijsman $\mathscr{J}$ and $\mathscr{J}^*$–convergence in IFMS

Throughout this section, we denote $\mathscr{J}$ to be the admissible ideal in $\mathbb{N}$. We begin with the following definitions as follows.

Definition 18. Let $(\mathscr{M},\eta,\varphi,*,\diamond)$ be an IFMS. A sequence of sets $\{C_k\}$ is said be Wijsman convergent to $\mathscr{C}$ if for every $\epsilon>0$ and $s>0$ there exists $k_0\in\mathbb{N}$ such that

$$\lim_{k\to\infty}\eta(x,C_k,s)=\eta(x,\mathscr{C},s) \text{ and } \lim_{k\to\infty}\varphi(x,C_k,s)=\varphi(x,C,s) \text{ for all } k\geq k_0. \tag{12}$$

The set of all Wijsman limit point of the sequence $\{\mathscr{C}_k\}$ is denoted by $L_{\{\mathscr{C}_k\}}$.

Definition 19. Let $(\mathscr{M},\eta,\varphi,*,\diamond)$ be an IFMS and $\mathscr{J}$ be a proper ideal in $\mathbb{N}$. A sequence $\{\mathscr{C}_k\}$ of nonempty closed subsets of $\mathscr{M}$ is known as Wijsman $\mathscr{J}$ –convergent to $\mathscr{C}$ with respect to IFM (η,φ), if for every $0<\epsilon<1$, for each $x\in\mathscr{M}$ and for all $s>0$ such that

$$\{k\in\mathbb{N}: |\eta(x,\mathscr{C}_k,s)-\eta(x,\mathscr{C},s)| \leq 1-\epsilon \text{ or } |\varphi(x,\mathscr{C}_k,s)-\varphi(x,\mathscr{C},s)|\geq\epsilon\}\in\mathscr{J}. \tag{13}$$

We write $(\eta,\varphi)-\mathscr{J}_W-\lim_{k\to\infty}\mathscr{C}_k=\mathscr{C}$.

Example 20. Suppose $(\mathscr{M},\eta,\varphi,*,\diamond)$ is an IFMS and $\mathscr{C},\{\mathscr{C}_k\}$ is nonempty closed subsets of $\mathscr{M}$. Assume $\mathscr{M}=\mathbb{R}^2$ and $\{\mathscr{C}_k\}$ are sequence defined by

$$\mathscr{C}_k=\begin{cases}(x,y)\in\mathbb{R}^2: 0\leq x\leq k, 0\leq y\leq \frac{1}{k}.x, \text{ if } k\neq n^2\\ (x,y)\in\mathbb{R}^2: x\geq 0, y=1, \text{ if } k=n^2,\end{cases}$$
$$\mathscr{C}=\{(x,y)\in\mathbb{R}^2: x\geq 0, y=0\}. \tag{14}$$

Since

$$\lim_{k\to\infty}\frac{1}{k}|\{n\leq k: |\eta((x,y),\mathscr{C}_k,s)-\eta((x,y),\mathscr{C},s)| \leq 1-\epsilon \text{ or } |\varphi((x,y),\mathscr{C}_k,s)-\varphi((x,y),\mathscr{C},s)|\geq\epsilon\}|=0. \tag{15}$$

Therefore, the sequence of sets $\{\mathscr{C}_k\}$ is Wijsman statistical convergent to the set $\mathscr{C}$.

Now, define the set S as

$$S(\epsilon)=\{k\in\mathbb{N}: |\eta((x,y),\mathscr{C}_k,s)-\eta((x,y),\mathscr{C},s)| \leq 1-\epsilon \text{ or } |\varphi((x,y),\mathscr{C}_k,s)-\varphi((x,y),\mathscr{C},s)|\geq\epsilon\}. \tag{16}$$

If we assume $\mathscr{J}=\mathscr{J}_d$, then the Wijsman statistical convergence coincides with the Wijsman ideal convergence. Therefore,

$$\{k\in\mathbb{N}: |\eta((x,y),\mathscr{C}_k,s)-\eta((x,y),\mathscr{C},s)| \leq 1-\epsilon \text{ or } |\varphi((x,y),\mathscr{C}_k,s)-\varphi((x,y),\mathscr{C},s)|\geq\epsilon\} =\{k\in\mathbb{N}: k=n^2\}\subset\mathscr{J}_d. \tag{17}$$

Definition 21. Let $(\mathscr{M},\eta,\varphi,*,\diamond)$ be a separable IFMS and $\mathscr{J}$ be an admissible ideal in $\mathbb{N}$. A sequence $\{\mathscr{C}_k\}$ of nonempty closed subsets of $\mathscr{M}$ is known as Wijsman $\mathscr{J}$–Cauchy with respect to IFM (η,φ), if for each $0<\epsilon<1$, for each $x\in\mathscr{M}$ and for all $s>0$, $\exists$ $l=l(\epsilon)$ such that

$$\{k\in\mathbb{N}: |\eta(x,\mathscr{C}_k,s)-\eta(x,\mathscr{C}_l,s)| \leq 1-\epsilon \text{ or } |\varphi(x,\mathscr{C}_k,s)-\varphi(x,\mathscr{C}_l,t)|\geq\epsilon\}\in\mathscr{J}. \tag{18}$$

Definition 22. Let $(\mathscr{M},\eta,\varphi,*,\diamond)$ be a separable IFMS and $\{\mathscr{C}_k\}$ be any nonempty closed subset of $\mathscr{M}$. The sequence $\{\mathscr{C}_k\}$ is known as Wijsman $\mathscr{J}^*$–Cauchy with respect to IFM (η,φ), if there exists $P=\{p=(p_j): p_j<p_{j+1}, j\in\mathbb{N}\}\subset\mathbb{N}$ and $P\in\mathscr{F}(\mathscr{J})$ with the result that the subsequence $\mathscr{C}_P=\{\mathscr{C}_{p_k}\}$ is Wijsman Cauchy in $\mathscr{M}$, i.e.

$$\lim_{k,l\to\infty} |\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta\left(x, \mathscr{C}_{p_l}, s\right)| = 1 \quad (19)$$

and

$$\lim_{k,l\to\infty} |\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi\left(x, \mathscr{C}_{p_l}, s\right)| = 0. \quad (20)$$

Definition 23. Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS and $\mathscr{J}$ be an proper ideal in $\mathbb{N}$. Let $\{\mathscr{C}_k\}$ be nonempty closed subsets of $\mathscr{M}$. The sequence $\{\mathscr{C}_k\}$ is known as Wijsman $\mathscr{J}^*$–convergent to $\mathscr{C}$ with respect to (η, φ), if there exists $P \in \mathscr{F}(\mathscr{J})$, where $P = \{p = (p_j): p_j < p_{j+1}, j \in \mathbb{N}\} \subset \mathbb{N}$ such that for each $s > 0$, we have

$$\lim_{k\to\infty} \eta\left(x, \mathscr{C}_{p_k}, s\right) = \eta(x, \mathscr{C}, s), \quad (21)$$

and

$$\lim_{k\to\infty} \varphi\left(x, \mathscr{C}_{p_k}, s\right) = \varphi(x, \mathscr{C}, s). \quad (22)$$

In such case, we write $(\eta, \varphi) - \mathscr{J}_W^* - \lim \mathscr{C}_k = \mathscr{C}$.

In the following theorem, we prove that every Wijsman $\mathscr{J}$–convergent implies the Wijsman $\mathscr{J}$–Cauchy condition in IFMS:

Theorem 24. *Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS and let $\mathscr{J}$ be an arbitrary admissible ideal. Then, every Wijsman $\mathscr{J}$–convergent sequence of closet sets $\{\mathscr{C}_k\}$ is Wijsman $\mathscr{J}$ – Cauchy with respect to IFM (η, φ).*

Proof. Suppose $(\eta, \varphi) - \mathscr{J}_W - \lim_{k\to\infty} \mathscr{C}_k = \mathscr{C}$. Then, for every $0 < \epsilon < 1$, for all $s > 0$ and $x \in X$, the set

$$U(\epsilon, s) = \{k \in \mathbb{N} : |\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}, s)| \le 1 - \epsilon \text{ or } |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}, s)| \ge \epsilon\} \quad (23)$$

belongs to $\mathscr{J}$. Since $\mathscr{J}$ is an admissible ideal, then there exists $k_0 \in \mathbb{N}$ with the result that $k_0 \notin U(\epsilon, s)$. Now, suppose that

$$V(\epsilon, s) = \left\{k \in \mathbb{N} : |\eta(x, \mathscr{C}_k, s) - \eta\left(x, \mathscr{C}_{k_0}, s\right)| \le (1 - 2\epsilon) \text{ or } |\varphi(x, \mathscr{C}_k, s) - \varphi\left(x, \mathscr{C}_{k_0}, s\right)| \ge 2\epsilon\right\}. \quad (24)$$

Considering the inequality

$$\left|\eta(x, \mathscr{C}_k, s) - \eta\left(x, \mathscr{C}_{k_0}, s\right)\right| \le |\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}, s)| + \left|\eta\left(x, \mathscr{C}_{k_0}, s\right) - \eta(x, \mathscr{C}, s)\right|, \quad (25)$$

and

$$|\varphi(x, \mathscr{C}_k, s) - \varphi\left(x, \mathscr{C}_{k_0}, s\right)| \le |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}, s)| + \left|\varphi\left(x, \mathscr{C}_{k_0}, s\right) - \varphi(x, \mathscr{C}, s)\right|. \quad (26)$$

Observe that if $k \in V(\epsilon, s)$, therefore

$$|\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}, s)| + \left|\eta\left(x, \mathscr{C}_{k_0}, s\right) - \eta(x, \mathscr{C}, s)\right| \le (1 - 2\epsilon), \quad (27)$$

and

$$|\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}, s)| + |\varphi\left(x, \mathscr{C}_{k_0}, s\right) - \varphi(x, \mathscr{C}, s)| \ge 2\epsilon. \quad (28)$$

From another point of view, since $k_0 \notin U(\epsilon, s)$, we obtain

$$\left|\eta\left(x, \mathscr{C}_{k_0}, s\right) - \eta(x, \mathscr{C}, s)\right| > 1 - \epsilon \text{ and } |\varphi\left(x, \mathscr{C}_{k_0}, s\right) - \varphi(x, \mathscr{C}, s)| < \epsilon. \quad (29)$$

We achieve that

$$|\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}, s)| \le 1 - \epsilon \text{ or } |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}, s)| \ge \epsilon. \quad (30)$$

Hence, $k \in U(\epsilon, s)$. This implies that $U(\epsilon, s) \subset V(\epsilon, s) \in \mathscr{J}$ for every $0 < \epsilon < 1$ and for all $s > 0$ and $x \in \mathscr{M}$. Therefore, $V(\epsilon, s) \in \mathscr{J}$, so the sequence is $\{\mathscr{C}_k\}$ which is Wijsman $\mathscr{J}$–Cauchy.

Theorem 25. *Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS and let $\mathscr{J}$ be an admissible ideal. Then, every Wijsman $\mathscr{J}^*$–Cauchy sequence of closed sets is Wijsman $\mathscr{J}$–Cauchy.*

Proof. Suppose that sequence $\{\mathscr{C}_k\}$ is Wijsman $\mathscr{J}^*$–Cauchy with respect to IFM (η, φ). Then, for each $x \in \mathscr{M}$ and for each $0 < \epsilon < 1$, there exists $P \in \mathscr{F}(\mathscr{J})$, where $P = \{(p_j): p_j < p_{j+1}, j \in \mathbb{N}\}$ in such a way that

$$\left|\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta\left(x, \mathscr{C}_{p_l}, s\right)\right| \le 1 - \epsilon, \quad (31)$$

and

$$\left|\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi\left(x, \mathscr{C}_{p_l}, s\right)\right| \ge \epsilon, \quad \forall k, l > k_0 = k_0(\epsilon). \quad (32)$$

Suppose $N = N(\epsilon) = p_{k_0+1}$. Therefore, for each $\epsilon > 0$, one obtains

$$\left|\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta(x, \mathscr{C}_N, s)\right| \le 1 - \epsilon, \quad (33)$$

and

$$\left|\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi(x, \mathscr{C}_N, s)\right| \geq \epsilon \text{ for all } k > k_0. \tag{34}$$

Now, suppose that $K = \mathbb{N} \setminus P$. Clearly, $K \in \mathscr{J}$ and

$$Q(\epsilon, s) = \{k \in \mathbb{N} \mid \eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}_N, s)| \leq 1 - \epsilon \text{ or } |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}_N, s)| \geq \epsilon \subset K \cup \left\{p_1, p_2, \cdots, p_{k_0}\right\} \in \mathscr{J}. \tag{35}$$

Hence, for all $s > 0$ and for each $0 < \epsilon < 1$, one can determine $N = N(\epsilon)$ so that $Q(\epsilon, s) \in \mathscr{J}$, that is, sequence $\{\mathscr{C}_k\}$ is Wijsman $\mathscr{J}$–Cauchy.

Theorem 26. *Let $\mathscr{J}$ be an admissible ideal including property (AP) and $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS. Then, the notion of Wijsman $\mathscr{J}^*$–Cauchy sequence of sets coincides with Wijsman $\mathscr{J}$ –Cauchy with respect to (η, φ) and vice-versa.*

Proof. The direct part is already proven in Theorem 25.

Now, suppose that sequence $\{\mathscr{C}_k\}$ is Wijsman $\mathscr{J}$–Cauchy sequence with respect to IFM (η, φ). Then by definition, if for every $0 < \epsilon < 1$, for each $x \in X$ and for all $s > 0$, there exists a $m = m(\epsilon)$ such that

$$B(\epsilon, s) = \{k \in \mathbb{N} | \eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}_n, s)| \leq 1 - \epsilon \text{ or } |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}_n, s)| \geq \epsilon\} \in I. \tag{36}$$

Now, suppose that

$$P_j(\epsilon, s) = \left\{k \in \mathbb{N} \middle| \eta(x, \mathscr{C}_k, s) - \eta\left(x, \mathscr{C}_{m_j}, s\right)\right| > 1 - \frac{1}{j} \text{ or } |\,(x, \mathscr{C}_k, s) - \varphi\left(x, \mathscr{C}_{m_j}, s\right)| < \left.\frac{1}{j}\right\}, \tag{37}$$

where $m_j = m(1/j)$, $j = 1, 2, 3, \cdots$. Obviously, for $j = 1, 2, 3 \cdots$, $P_j(\epsilon, s) \in \mathscr{F}(\mathscr{J})$. Using Lemma 6, there exists $P \subset \mathbb{N}$ so that $P \in \mathscr{F}(\mathscr{J})$ and $P \setminus P_j$ are finite for all j.

Now, we prove that

$$\lim_{k,l \to \infty} |\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}_l, s)| = 1, \tag{38}$$

and

$$\lim_{k,l \to \infty} |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}_l, s)| = 0. \tag{39}$$

To show the above equations, let $\epsilon > 0$, and $r \in \mathbb{N}$ such that $r > 2/\epsilon$. If $k, l \in P$, then $P \setminus P_j$ is a finite set; therefore, there exists $w = w(r)$ in order that

$$\left|\eta(x, \mathscr{C}_k, s) - \eta\left(x, \mathscr{C}_{l_r}, s\right)\right| > 1 - \frac{1}{r}, \quad \left|\eta(x, \mathscr{C}_l, s) - \eta\left(x, \mathscr{C}_{l_r}, s\right)\right| > 1 - \frac{1}{r}, \tag{40}$$

and

$$\left|\varphi(x, \mathscr{C}_k, s) - \varphi\left(x, \mathscr{C}_{l_r}, s\right)\right| < \frac{1}{r}, \quad \left|\varphi(x, \mathscr{C}_l, s) - \varphi\left(x, \mathscr{C}_{l_r}, s\right)\right| < \frac{1}{r}, \tag{41}$$

for all $k, l > w(r)$. Then, the above inequalities follow that for $k, l > w(r)$

$$|\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}_l, s)| \leq \left|\eta(x, \mathscr{C}_k, s) - \eta\left(x, \mathscr{C}_{l_r}, s\right)\right| + \left|\eta(x, \mathscr{C}_l, s) - \eta\left(x, \mathscr{C}_{l_r}, s\right)\right| > \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \epsilon, \tag{42}$$

and

$$|\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}_l, s)| \leq \left|\varphi(x, \mathscr{C}_k, s) - \varphi\left(x, \mathscr{C}_{l_j}, s\right)\right| + \left|\varphi(x, \mathscr{C}_l, s) - \varphi\left(x, \mathscr{C}_{l_j}, s\right)\right| < \frac{1}{r} + \frac{1}{r} < \epsilon. \tag{43}$$

Therefore, for each $\epsilon > 0$, $\exists w = w(\epsilon)$ and $k, l \in P \in \mathscr{F}(I)$, we achieve

$$\{k \in \mathbb{N} : |\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}_l, s)| \leq 1 - \epsilon \text{ or } |\,\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}_l, s)| \geq \epsilon\} \in \mathscr{J}. \tag{44}$$

This proves that the sequence $\{\mathscr{C}_k\}$ is a Wijsman $\mathscr{J}^*$–Cauchy.

Theorem 27. *Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS and let $\mathscr{J}$ be an admissible ideal. Then*

$$(\eta, \varphi) - \mathscr{J}_W^* - \lim_{k \to \infty} \mathscr{C}_k = \mathscr{C} \tag{45}$$

implies that sequence $\{\mathscr{C}_k\}$ is a Wijsman $\mathscr{J}$–Cauchy sequence with respect to IFM (η, φ).

Proof. Suppose that $(\eta, \varphi) - \mathscr{J}_W^* - \lim_{k \to \infty} \mathscr{C}_k = \mathscr{C}$. Then, there exists $P = \{p = (p_j) : p_j < p_{j+1}, j \in \mathbb{N}\} \subset \mathbb{N}$ with $P \in \mathscr{F}(\mathscr{J})$ so that $\mathscr{C}_P = \{\mathscr{C}_{p_k}\}$

$$\lim_{k \to \infty} \eta\left(x, \mathscr{C}_{p_k}, s\right) = \eta(x, \mathscr{C}, s), \tag{46}$$

and

$$\lim_{k\to\infty} \varphi\left(x, \mathscr{C}_{p_k}, s\right) = \varphi(x, \mathscr{C}, s), \quad (47)$$

for any $\epsilon > 0$ and $k, l > k_0$.

Suppose $r \in \mathbb{N}$ and $\epsilon > 0$ in such a way that $r > 2/\epsilon$. If k, $l \in P$, then $P \setminus P_j$ is a finite set; therefore, there exists $k(r) = k$ so that

$$\left|\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta\left(x, \mathscr{C}_{p_l}, s\right)\right| \le \left|\eta\left(x, \mathscr{C}_{p_k}, s\right) - (x, \mathscr{C}, s)\right| + \left|\eta\left(x, \mathscr{C}_{p_l}, s\right) - \eta(x, \mathscr{C}, s)\right| > \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \epsilon, \quad (48)$$

and

$$\left|\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi\left(x, \mathscr{C}_{p_l}, s\right)\right| < \left|\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi(x, \mathscr{C}, s)\right| + \left|\varphi\left(x, \mathscr{C}_{p_l}, s\right) - \varphi(x, \mathscr{C}, s)\right| < \frac{1}{r} + \frac{1}{r} < \epsilon. \quad (49)$$

Therefore,

$$\lim_{k,l\to\infty} \left|\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta\left(x, \mathscr{C}_{p_l}, s\right)\right| = 1, \quad (50)$$

and

$$\lim_{k,l\to\infty} \left|\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi\left(x, \mathscr{C}_{p_l}, s\right)\right| = 0. \quad (51)$$

Hence, sequence $\{\mathscr{C}_k\}$ is Wijsman $\mathscr{J}$–Cauchy with respect to IFM (η, φ).

4. Wijsman $\mathscr{J}$–cluster points and Wijsman $\mathscr{J}$–limit points in IFMS

Throughout this section, we denote $\mathscr{J}$ to be the proper ideal in $\mathbb{N}$ and define Wijsman $\mathscr{J}$–cluster and $\mathscr{J}$–limit points of the sequence of sets in intuitionistic fuzzy metric space and obtain some results.

Definition 28. Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS. An element $\mathscr{C} \in \mathscr{M}$ is known as the Wijsman $\mathscr{J}$ -cluster point of $\{\mathscr{C}_k\}$ if and only if for any $x \in \mathscr{M}$ and for all $\epsilon, s > 0$, one has

$$\{k \in \mathbb{N} : |\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}, s)| < 1 - \epsilon \text{ or } |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}, s)| > \epsilon\} \notin \mathscr{J}. \quad (52)$$

We denote $\mathscr{J}_W^{(\eta,\varphi)}(\Gamma_{\{\mathscr{C}_k\}})$ as the collection of all Wijsman $\mathscr{J}$-cluster points.

Definition 29. Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS. An element $\mathscr{C} \in \mathscr{M}$ is known as Wijsman $\mathscr{J}$–limit point of sequence $\{\mathscr{C}_k\}$ of nonempty closed subsets of $\mathscr{M}$ provided that $P = \{p = (p_j): p_j < p_{j+1}, j \in \mathbb{N}\} \subset \mathbb{N}$ in such a way that $P \notin \mathscr{J}$, and for any $x \in \mathscr{M}$ and $s > 0$, we obtain

$$\lim_{k\to\infty} \eta(x, \mathscr{C}_k, s) = \eta(x, \mathscr{C}, s) \text{ and } \lim_{k\to\infty} \varphi(x, \mathscr{C}_k, s) = \varphi(x, \mathscr{C}, s). \quad (53)$$

We denote $\mathscr{J}_W^{(\eta,\varphi)}(\Lambda_{\{\mathscr{C}_k\}})$ as the collection of all Wijsman $\mathscr{J}$–limit points.

Theorem 30. *Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS. Then, for any sequence sets, $\{\mathscr{C}_k\} \subset \mathscr{M}$, $\mathscr{J}_W^{(\eta,\varphi)}(\Lambda_{\{\mathscr{C}_k\}}) \subset \mathscr{J}_W^{(\eta,\varphi)}(\Gamma_{\{\mathscr{C}_k\}})$.*

Proof. Suppose $\mathscr{C} \in \mathscr{J}_W^{(\eta,\varphi)}(\Lambda_{\{\mathscr{C}_k\}})$. Then, there exists $P = \{p_1 < p_2 < \cdots\} \subset \mathbb{N}$ such that $P = \{p = (p_j): p_j < p_{j+1}, j \in \mathbb{N}\} \notin \mathscr{J}$ and for all $s > 0$ and $x \in \mathscr{M}$, we have

$$\lim_{k\to\infty} \eta\left(x, \mathscr{C}_{p_k}, s\right) = \eta(x, \mathscr{C}, s), \quad (54)$$

and

$$\lim_{k\to\infty} \varphi\left(x, \mathscr{C}_{p_k}, s\right) = \varphi(x, \mathscr{C}, s). \quad (55)$$

According to Equations (54) and (55), there exists $k_0 \in \mathbb{N}$ so that for each $\epsilon > 0$ and for any $x \in X$ and $k > k_0$

$$\left|\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta(x, \mathscr{C}, s)\right| > 1 - \epsilon, \quad (56)$$

and

$$\left|\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi(x, \mathscr{C}, s)\right| < \epsilon. \quad (57)$$

Hence,

$$\left\{k \in \mathbb{N} : |\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta(x, \mathscr{C}, s)| > 1 - \epsilon, |\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi(x, \mathscr{C}, s)| < \epsilon\right\} \supseteq \left\{P \setminus p_1, p_1, \dot{;} p_{k_0}\right\}. \quad (58)$$

Then, the right-hand side of (58) does not belong to $\mathscr{J}$, and then

$$\left\{k \in \mathbb{N} : \left|\eta\left(x, \mathscr{C}_{p_k}, s\right) - \eta(x, \mathscr{C}, s)\right| > 1 - \epsilon, \left|\varphi\left(x, \mathscr{C}_{p_k}, s\right) - \varphi(x, \mathscr{C}, s)\right| < \epsilon\right\} \notin \mathscr{J}, \quad (59)$$

which means that $\mathscr{C} \in \mathscr{J}_W^{(\eta,\varphi)}(\Gamma_{\{\mathscr{C}_k\}})$.

Theorem 31. *Let $(\mathscr{M}, \eta, \varphi, *, \diamond)$ be a separable IFMS. Then, for any sequence $\{\mathscr{C}_k\} \subset \mathscr{M}$, $\mathscr{J}_W^{(\eta,\varphi)}(\Gamma_{\{\mathscr{C}_k\}}) \subset L_{\{\mathscr{C}_k\}}$.*

Proof. Let $\mathscr{C} \in \mathscr{J}_W^{(\eta,\varphi)}(\Gamma_{\{\mathscr{C}_k\}})$. Then, for each $\epsilon > 0$ and for all $s > 0$ and for each $x \in \mathscr{M}$, one has

$$\{k \in \mathbb{N} : |\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}, s)| < 1 - \epsilon \text{ or } |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}, s)| > \epsilon\} \notin \mathscr{J}. \quad (60)$$

Suppose

$$Q_k = \left\{k \in \mathbb{N} : |\eta(x, \mathscr{C}_k, s) - \eta(x, \mathscr{C}, s)| > 1 - \frac{1}{k}, |\varphi(x, \mathscr{C}_k, s) - \varphi(x, \mathscr{C}, s)| < \frac{1}{k}\right\}, \quad (61)$$

for $k \in \mathbb{N}$. $\{Q_k\}_{k=1}^{\infty}$ is a descending sequence of subsets of $\mathbb{N}$. Hence, $Q = \{k = (k_i)\colon k_i < k_{i+1}, i \in \mathbb{N}\} \notin \mathscr{J}$ so that

$$\lim_{k\to\infty} \eta\left(x, \mathscr{C}_{k_i}, s\right) = \eta(x, \mathscr{C}, s), \quad (62)$$

and

$$\lim_{k\to\infty} \varphi\left(x, \mathscr{C}_{k_i}, s\right) = \varphi(x, \mathscr{C}, s), \quad (63)$$

which means that $\mathscr{C} \in L_{\{\mathscr{C}_k\}}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Research Article

Criteria in Nuclear Fréchet Spaces and Silva Spaces with Refinement of the Cannon-Whittaker Theory

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Along with the theory of bases in function spaces, the existence of a basis is not always guaranteed. The class of power series spaces contains many classical function spaces, and it is of interest to look for a criterion for this class to ensure the existence of bases which can be expressed in an easier form than in the classical case given by Cannon or even by Newns. In this article, a functional analytical method is provided to determine a criterion for basis transforms in nuclear Fréchet spaces ((NF)-spaces), which is indeed a refinement and a generalization of those given in this concern through the theory of Whittaker on polynomial bases. The provided results are supported by illustrative examples. Then, we give the necessary and sufficient conditions for the existence of bases in Silva spaces. Moreover, a nuclearity criterion is given for Silva spaces with bases. Subsequently, we show that the presented results refine and generalize the fundamental theory of Cannon-Whittaker on the effectiveness property in the sense of infinite matrices.

1. Introduction

The existence of bases is one of the fundamental problems in the classical theory of analytic functions. A functional analytic approach to the theory of bases in function spaces emerges naturally when studying classes of functions which play a vital role in applied mathematics and mathematical physics. This paper is entirely devoted to the study of bases in the spaces of holomorphic functions in one or two complex variables. Let us consider two important problems which arise in the study of function spaces as follows:

(1) Does the space under consideration possess a basis?

(2) If this is the case, how can any other basis of this space be characterized?

Assume these problems are answered in a positive way. Then, when E denotes the space and $(x_n)_{n\in\mathbb{N}}$ stands for a basis in E, each element $x \in E$ admits a (unique) decomposition of the form $\sum_{n=1}^{\infty} a_n(x)x_n$ whereby for each $n \in \mathbb{N}$, a_n is a linear functional on E. In practice (e.g., in approximation theory), the choice of a suitable basis is very important.

The present work essentially deals with these two fundamental problems in the case where the considered function spaces admit a set of polynomials as a basis (or in the terminology of Cannon-Whittaker, a basic set of polynomials) [1–5]. Basic examples of such function spaces are given by the space of holomorphic functions in an open disc or the space of analytic functions on a closed disc. Of course, as the theory of holomorphic functions in the plane allows generalizations to higher dimensions, analogous problems may be considered in the corresponding function spaces, see [2, 6–14].

Cannon and Whittaker [5, 15–17] studied the existence of basic sets of polynomials of one complex variable (as bases) in the classical spaces $\mathcal{O}(B(r))$, and later, many authors considered such problem in the space $\mathcal{O}(\bar{B}(r) \times \bar{B}(r))$, see for example [2, 14, 18, 19]. The main tool in their investigations is a Cannon criterion, determining whether the considered set of polynomials forms a basis for the space (or in the Whittaker terminology, this set of polynomials is effective). So that by the effectiveness of the basic set of polynomials $\{P_n(z)\}$, $z \in \mathbb{C}$, in the closed ball $\bar{B}(r)$, it means that the set forms a base for the class E of holomorphic functions regular in $\bar{B}(r)$ with norm given by $M(f, r)$, $f \in E$.

In our approach here, we introduce the topic from a different point of views, as the change of basis problem presented in the abstract setting of nuclear Fréchet spaces ((NF)-spaces) and nuclear Silva spaces ((NS)-spaces) having in mind essentially the basic example $\mathcal{O}(B(r))$. Criteria are obtained which tell us under which conditions an infinite matrix P is a basis transform in both a (NF)-space and a NS-space with basis. These criteria are then applied to the case of power series spaces, thus yielding a refinement of Cannon's criterion in the case $\mathcal{O}(\bar{B}(r))$ or even those given in [4, 13, 18, 20]. It is worth mentioning that this problem has been treated by many authors from different angles for which we may mention [4, 5, 21, 22]. Furthermore, significant advances of the subject in higher dimensional spaces have been investigated as in the space of several complex variables $\mathbb{C}^n$ [2], monogenic function spaces [8–12, 23, 24], and the matrix function spaces [22]. Much close to the effectiveness problem is the study of spaces of entire functions having finite growth, wherein [25] the author showed that such spaces are also (NF)-spaces.

The main purpose of this paper is to find criteria under which a sequence of vectors $(x_n)_{n\in\mathbb{N}}$ in a (NF)-space E with basis is itself a basis for E. This leads to the notion of basis-preserving homeomorphisms between (NF)-spaces with bases. The analog study on (NS)-spaces is also provided. Our results are then applied to the case of power series spaces to get a link with the theory of Whittaker on polynomial bases [5] and the theory of Newns who treated the problem of effectiveness by using the topological method approach (see [4]). To begin our investigation, we first give a survey of all necessary definitions and basic results from the general theory of locally convex topological vector spaces and the theory of nuclear spaces. For a more general account, we refer to [26–29]. Then, we point out that the results obtained in this paper might form the starting point of further investigations concerning basis transforms in more general locally convex spaces and higher dimension context.

2. Preliminaries

In the sequel, $P = (P_{ij})$, $Q = (Q_{ij})$, and $T = (T_{ij})$ be infinite complex matrices and the infinite identity matrix will be denoted by I. All vector spaces will be Fréchet spaces over $\mathbb{C}$, although all results are without changes valid for vector spaces over $\mathbb{R}$. Since we are interested in basis transforms, it will be convenient to look upon a space together with a basis, and notation of the form (V, b) will be used, where V is the vector space and $b = \{b_n\}_{n=1}^{+\infty}$ is a topological basis for V. If V and W are isomorphic, then there exists an isomorphism φ, called a basis-preserving isomorphism, mapping the basis b of V to a basis c of W (i.e., $\varphi(b_n) = c_n$ for all n); (V, b) and (V, c) are called similar. This relation is written $(V, b) \simeq (V, c)$. Obviously, it is not necessarily true that if b and f are two bases for V then $(V, b) \simeq (V, f)$. If it is true however we speak of conjugate bases. If $x = (x_n)$ is a sequence of vectors in V and for each k, the series

$$y_k = \sum_{n\in\mathbb{N}} P_{kn} x_n, \quad k \in \mathbb{N} \tag{1}$$

converges, we simply write $y = Px$.

2.1. Köthe Sequence Spaces. Let ω denote the space of all infinite sequence of complex numbers and let $A = (a_n^k)$ be an infinite matrix of real nonnegative numbers such that $a_n^k \le a_n^{k+1}$ for all n, k and for each n, these exist k such that $a_n^k \ne 0$. Here, k and n stand for the row and the column indices, respectively. With such a matrix A, we associate the following subset of ω:

$$K(A) = \left\{ \xi = (\xi_n) \in \omega : \|\xi\|_k = \sum_{n=1}^{\infty} a_n^k |\xi_n| < +\infty, k \in \mathbb{N} \right\}, \tag{2}$$

with the topology given by the seminorms $\|\bullet\|_k$. Note that $\mathcal{P} = \{\|\bullet\|: k \in \mathbb{N}\}$ is a sequence of norms on $K(A)$, which is moreover a system of norms on $K(A)$. Putting for each $n \in \mathbb{N}$, $e_n = (\delta_{kn})_{k\in\mathbb{N}}$, we cite the following important result proved by Pietsch in [29].

Theorem 1 (see [29]). *$K(A)$ is an (F)-space and $e = (e_n)_{n\in\mathbb{N}}$ is a basis for $K(A)$.*

Remark 2. It is well known that if (V, x) is a (NF)-space and the topology is given by seminorms p_k, then $(V, x) \simeq (K(A), e)$, where A is given by the relation $a_n^k = p_k(x_n)$. This makes it possible to use alternatively the seminorms p_k, $\|\varphi(.)\|_k$, or $|\varphi(.)|_k$ where φ is the basis-preserving isomorphism.

2.2. Nuclear Fréchet Spaces with Basis. We shall write shortly (F)-space for Fréchet space and (NF)-space for nuclear Fréchet space, as is common in the literature (see [27, 30–32]). Now, let $(E, \mathcal{P})$ be an (F)-space where it is henceforth assumed that the countable system of seminorms $\mathcal{P} = \{p_k : k \in \mathbb{N}\}$ is in fact a sequence of norms. We therefore write $p_k = \|\bullet\|_k$.

The following result given in [28, 33] is useful in the sequel.

Theorem 3 (Banach's homeomorphism theorem). *Let $(E, \mathcal{P})$ and (F, Q) be (F)-spaces and let $T : E \to F$ be a bijective and bounded linear operator. Then, T is a homeomorphism and thus T^{-1} is also bounded.*

The definition of the (NF)-spaces was introduced in [26] as follows.

Definition 4 ((NF)-space with basis). Let $(E, \mathcal{P})$ be an (F)-space. Then, it is called a (NF)-space if for each $k \in \mathbb{N}$, there exist $\ell \in \mathbb{N}$ and a sequences $(y_n)_{n\in\mathbb{N}}$ in E and $(L_n)_{n\in\mathbb{N}}$ in E' such that for each $x \in E$,

(i) $x =_{\|.\|_k} \sum_{n\in\mathbb{N}} L_n(x) y_n$

(ii) $\sum_{n=1}^{\infty} \|L_n\|'_\ell \|y_n\|_k < +\infty$

where $\|L_n\|'_\ell = \sup_{x\in E, \|x\|_\ell \le 1} |L_n(x)|$ and if $\|L_n\|'_\ell = \infty$, then $\|y_n\|_k = 0$.

Now, we recall some functional theorems concerning (NF)-spaces with basis.

Theorem 5 (Dynin-Mitiagin [34, 35]). *Let (E, x) be a (NF)-space with basis. Then, for each $k \in \mathbb{N}$, there exist $\ell \in \mathbb{N}$ and $K_k > 0$ such that, if for $y \in E$, $y = \sum_{n=1}^{\infty} \alpha_n(y) x_n$, where $\alpha_n(y)$ is the coefficient function, then*

$$\sum_{n=1}^{\infty} |\alpha_n(y)| \|x_n\|_k < K_k \|y\|_\ell. \tag{3}$$

Corollary 6 (see [34, 35]). *Each base in a (NF)-space with basis is absolute.*

Theorem 7 (see [31]). *Let E be a (NF)-space with a complete biorthogonal system $(x_n, x'_n)_{n\in\mathbb{N}}$. Then, $x = (x_n)_{n\in\mathbb{N}}$ is a basis for E if and only if for each $k \in \mathbb{N}$, there exist $\ell \in \mathbb{N}$, and $K_k > 0$ such that for each $y \in E$,*

$$\sup_{n\in\mathbb{N}} |x'_n(y)| \|x_n\|_k \le K_k \|y\|_\ell. \tag{4}$$

Theorem 8 (Haslinger's criterion for a (NF)-space with basis [30]). *Let (E, x) be a (NF)-space with basis and let $(y_n, y'_n)_{n\in\mathbb{N}}$ be a complete biorthogonal system in E. Then, $(y_n)_{n\in\mathbb{N}}$ is a basis in E if and only if for each $k \in \mathbb{N}$, there exist $\ell \in \mathbb{N}$ and $K_k > 0$ such that for each $j \in \mathbb{N}$,*

$$\sup_{n\in\mathbb{N}} |y'_n(x_j)| \|x_n\|_k \le K_k \|x_j\|_\ell. \tag{5}$$

2.3. Nuclear Köthe Sequence Spaces. As we have seen that a Köthe sequence space $K(A)$ is an (F)-space with basis e, we have also according to [29].

Theorem 9. (see [29]). *Let $K(A)$ be a Köthe sequence space. Then, $K(A)$ is nuclear (and hence a (NF)-space with basis) if and only if for each $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that*

$$\sum_{n\in\mathbb{N}} \frac{a_n^k}{a_n^\ell} < +\infty. \tag{6}$$

Now, suppose that $K(A)$ is a Köthe sequence space and put

$$S(A) = \left\{ \xi \in \omega : \sup_{n\in\mathbb{N}} |\xi_n| a_n^k < +\infty \right\},$$
$$\|\xi\|_k = \sup_{n\in\mathbb{N}} |\xi_n| a_n^k, \quad k \in \mathbb{N}, \xi \in S(A). \tag{7}$$

Then, clearly, $Q = \{\|\bullet\|_k : k \in \mathbb{N}\}$ is a set of norms on $S(A)$. We call $S(A)$ the supremum space associated with A. Then, we have

Theorem 10 (see [29]). *$(S(A), \mathscr{P})$ is an (F)-space.*

Remark 11.

(i) Notice that, although $S(A)$ is always an (F)-space, it need not have a basis. Indeed, it is sufficient to consider $A = (a_n^k)$ with $a_n^k = 1$ for all $n, k \in \mathbb{N}$

(ii) Notice also that for this matrix A, the condition (6) needed for the nuclearity of $K(A)$ is not satisfied. This might suggest that the nuclearity of $K(A)$ could be related to the existence of a basis for $S(A)$. This is indeed the case as it was shown in [26, 30–32, 36], which can be stated in the following theorem

Theorem 12.

(i) $K(A)$ is nuclear if and only if $S(A)$ is nuclear

(ii) $K(A)$ and $S(A)$ are homeomorphic

(iii) The standard basis $e = (e_n)_{n\in\mathbb{N}}$ of $K(A)$ is also a basis for $S(A)$, and thus, $S(A)$ is a (NF)-space with basis

This theorem implies immediately that if $K(A)$ is nuclear, then $(K(A), e) \simeq (S(A), e)$.

2.4. Important Remarks. For (F)-spaces, and even for (NF)-spaces, the existence of a basis is not always guaranteed, as was shown in [34, 35]. The Haslinger's criterion mentioned previously in Theorem 8 is essential for our study, namely, to establish whether or not an infinite matrix P determines a basis transform in either (NF)-space or (NS)-space with basis. That is what we strive to achieve in the sequel. This will give a refinement of the criteria of effectiveness problem in the sense of Cannon-Whittaker theory on basis of polynomials.

Beforehand, we give a fruitful study concerning a criterion for an (F)-space to be nuclear showing by supporting examples that the provided criterion is attainable.

3. A Nuclearity Criterion for (F)-Spaces with Basis

In this section, we give a criterion stating under which conditions an (F)-space E with basis $x = (x_n)_{n\in\mathbb{N}}$ is nuclear. Let us recall that for an (F)-space E with $x = (x_n)_{n\in\mathbb{N}}$, the associated linear functional x'_n, $n \in \mathbb{N}$ is bounded according to Schauder's theorem [28] which states that "if E is an (F)-space and $(x_n)_{n\in\mathbb{N}}$ is a basis for E, then $(x_n)_{n\in\mathbb{N}}$ is a Schauder's basis."

As usual, it is tacitly understood that the topology of the (F)-space is determined by a countable sequence $(\|\bullet\|_m)_{m\in\mathbb{N}}$ of norms.

Theorem 13. *Let E be an (F)-space with basis $x = (x_n)_{n\in\mathbb{N}}$. Then, the following are equivalent:*

(i) E is nuclear

(ii) For each $t \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that

$$\sum_{n\in\mathbb{N}} \frac{\|x_n\|_t}{\|x_n\|_s} < +\infty. \tag{8}$$

Proof. Let E be a nuclear space. Then, by Dynin-Mitiagin theorem 5, it follows that for all $t \in \mathbb{N}$, there ought to exist $s \in \mathbb{N}$ such that

$$\sum_{n=0}^{\infty} \|x'_n\|_s \|x_n\|_t < \infty. \tag{9}$$

Since $x_n{}'(x_n) = 1$, we have $1 \le \|x_n{}'\|'_s \|x_n\|_s$ or $\|x_n{}'\|'_s \ge 1/\|x_n\|_s$. Consequently,

$$\sum_{n=0}^{\infty} \frac{\|x_n\|_t}{\|x_n\|_s} < +\infty. \tag{10}$$

Conversely, assume that for each $t \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that

$$\sum_{n=0}^{\infty} \frac{\|x_n\|_t}{\|x_n\|_s} < +\infty. \tag{11}$$

Let S be the following space:

$$S = \left\{ a = (a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{C} \text{ for all } n \in \mathbb{N} \text{ and } [a]_k = \sum_{n=0}^{\infty} |a_n| \|x_n\|_k < +\infty \text{for all } k \in \mathbb{N} \right\}. \tag{12}$$

Then, clearly, S is a supremum space associated with $A = (a_n^k)$ where $a_n^k = \|x_n\|_k$, hence a Fréchet space. By the condition assumed, it follows from Theorem 12 that S is nuclear.

Now, consider the map $\psi : S \to E$ such that

$$\psi : (a_n)_{n\in\mathbb{N}} \to \sum_{n=0}^{\infty} a_n x^n. \tag{13}$$

Obviously, ψ is linear, continuous, and injective. We now prove that ψ is also surjective, i.e., if $y \in E$ with $y = \sum_{n=0}^{\infty} a_n x^n$, then $a = (a_n)_{n\in\mathbb{N}} \in S$. Fix $s \in \mathbb{N}$, then as $\sum_{n=0}^{\infty} a_n x^n$ converges in F, $\lim_{n\to\infty} |a_n| \|x_n\|_s < \infty$ whence

$$b_s = \sup_{n\in\mathbb{N}} |a_n| \|x_n\|_s < +\infty. \tag{14}$$

Now, let $t \in \mathbb{N}$ be chosen arbitrarily and let $s \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} (\|x_n\|_t / \|x_n\|_s) < +\infty$. Hence,

$$\sum_{n=0}^{\infty} |a_n| \|x_n\|_t = \sum_{n=0}^{\infty} |a_n| \|x_n\|_s \frac{\|x_n\|_t}{\|x_n\|_s} \le b_s \sum_{n=0}^{\infty} \frac{\|x_n\|_t}{\|x_n\|_s} < +\infty. \tag{15}$$

Consequently, $a = (a_n)_{n\in\mathbb{N}} \in S$. By virtue of Banach's homeomorphism theorem 3 (see [28]), ψ is bicontentious and so E is homeomorphic to S or in other words E is nuclear.

Remark 14.

(1) Notice that Theorem 13 provides a relatively simple tool for determining the nuclearity of an (F)-space with basis. In what follows, for an (F)-space E having a basis $(x_n)_{n\in\mathbb{N}}$ satisfying (8), we put

$$S(A) = \left\{ a = (a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{C} \text{ and for all } k \in \mathbb{N}, [a]_k = \sum_{n=0}^{\infty} |a_n| \|x_n\|_k < +\infty \right\} \tag{16}$$

(2) As we have seen, $S(A)$ is a nuclear supremum space which is isomorphic to E. Here, $A = (a_n^k)$ with $a_n^k = \|x_n\|_k$

(3) From the isomorphism ψ constructed in the proof above, it follows that if (E, x) is a (NF)-space with basis $(x_n)_{n\in\mathbb{N}}$, then the following isomorphisms hold:

$$(E, x) \simeq (K(A), e) \simeq (S(A), e) \tag{17}$$

We thus obtain that the natural system of seminorms on E is equivalent to

(i) the system of norms

$$\|y\|_k = \sum_{n\in\mathbb{N}} |a_n| a_n^k, \quad k \in \mathbb{N} \tag{18}$$

(ii) the system of norms

$$\|y\|_k = \sup_{n\in\mathbb{N}} |a_n| a_n^k, \quad k \in \mathbb{N} \tag{19}$$

Note that in both cases, $a_n = x'_n(y)$, $n \in \mathbb{N}$.

(4) A nuclearity criterion for the more general case of locally convex spaces having an equicontinuous Schauder basis was proved by Kamthan in [32]. Our Theorem 13 is a special case of his; however, since we are working in (F)-spaces, the proof can be done in a rather easy way. For the case of Köthe spaces, we also refer to [36].

Now, we illustrate the usefulness of the criterion obtained in Theorem 13.

Example 1. Consider the space $\mathcal{O}(B(R))$ of holomorphic functions in the open disk $B(R)$ provided with the countable system $\mathcal{P}$ of seminorms p_k where

$$p_k(f) = \sup_{|z|\le r_k} |f(z)|, z \in \mathbb{C}, \tag{20}$$

$(r_k)_{k\in\mathbb{N}}$ being a strictly increasing sequence of positive numbers with $0 < r_k < R$, and $\lim_{k\to\infty} r_k = R$. One may take, for example, $r_k = R - (1/k)$ (assuming tacitly that $R > 1$, if $R \le 1$, then similar choices may, of course, be made) if R is finite and $r_k = k$ if R is infinite.

As we saw before that $(\mathcal{O}(B(R)), \xi)$ is a Fréchet space with basis $\xi = (z^n)_{n\in\mathbb{N}}$, although it is well known that $(\mathcal{O}(B(R)), \xi)$ is a (NF)-space (see [29]), its proof is not trivial.

As will be seen now, the criterion just proved yields the nuclearity of $(\mathcal{O}(B(R)), \xi)$ in an easy way. Indeed, as for each $k \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\|z^n\|_k = \sup_{|z|\le r_k} |z^n| = r_k^n. \tag{21}$$

Taking $k \in \mathbb{N}$ fixed, then for each $\ell > k$, we have

$$\sum_{n=0}^{\infty} \frac{\|z_n\|_k}{\|z_n\|_\ell} = \sum_{n=0}^{\infty} \frac{r_k^n}{r_\ell^n} = \sum_{n=0}^{\infty} \left(\frac{r_k}{r_\ell}\right)^n < +\infty. \tag{22}$$

So, the criterion applies.

Remark 15. Notice that, as we previously mentioned,

$$(\mathcal{O}(B(R)), \xi) \simeq (K(A), e) \simeq (S(A), e), \tag{23}$$

with $A = (a_n^k)$ and $a_n^k = \sup_{|z|\le r_k} |z^n|$, whence the natural system of seminorms $\mathcal{P} = \{\|\bullet\|_k : k \in \mathbb{N}\}$ and the system of seminorms $\mathcal{P}_{K(A)}$ and $\mathcal{P}_{S(A)}$ induced on $(\mathcal{O}(B(R)), \xi)$ are all equivalent. Here,

$$\mathcal{P}_{K(A)} = \{\|\bullet\|_{k,K} : k \in \mathbb{N}\},$$
$$\mathcal{P}_{S(A)} = \{[.]_k : k \in \mathbb{N}\}, \tag{24}$$

where if $f \in \mathcal{O}(B(R))$ admits the Taylor series at the origin,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{25}$$

then, for $k \in \mathbb{N}$, $[f]_k = \sup_{n\in\mathbb{N}} |c_n| a_n^k$ and $\|f\|_{k,K} = \sum_{n=0}^{\infty} |c_n| a_n^k$.

Notice that we already obtain directly, and this by using Cauchy's inequality and the triangle inequality, respectively, the following comparison between the systems of seminorms under consideration for each $k \in \mathbb{N}$ and $f \in \mathcal{O}(B(R)$,

$$[f]_k \le \|f\|_k \le \|f\|_{k,K}. \tag{26}$$

Thus, the equivalence between the systems of seminorms established above gives stronger results.

Example 2. Consider the space $\mathcal{O}(\mathrm{B(R)} \times \mathrm{B(R)})$ of holomorphic functions in two complex variables in the open polydisc $\mathrm{B(R)} \times \mathrm{B(R)}$, provided with the countable system $\mathcal{P}$ of seminorms p_k where

$$p_k(f) = \sup_{|u|\le r_k, |v|\le r_k} |f(u, v)|. \tag{27}$$

Again, $(r_k)_{k\in\mathbb{N}}$ is strictly increasing sequence of positive numbers with $0 < r_k < R$ and $\lim_{k\to\infty} r_k = R$. As is well known, $(\mathcal{O}(B(R) \times B(R)), \xi)$ is a Fréchet space. Moreover, as it is shown again by the Taylor series at the origin for any $f \in \mathcal{O}(B(R) \times B(R))$, the sequence $\xi = (u^n v^m)_{n,m\in\mathbb{N}}$ is a basis for $\mathcal{O}(B(R) \times B(R))$. Although it is known that $(\mathcal{O}(B(R) \times B(R)), \xi)$ is a (NF)-space, our criterion will yield the nuclearity of it in a very simple way. Indeed, since for each $k \in \mathbb{N}$ and $n, m \in \mathbb{N}$,

$$\|u^n v^m\|_k = \sup_{|u|\le r_k, |v|< r_k} |u^n v^m| = r_k^{n+m}, \tag{28}$$

we obtain that, taking $k \in \mathbb{N}$ fixed, for each $\ell > k$,

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\|u^n v^m\|_k}{\|u^n v^m\|_\ell} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{r_k}{r_\ell}\right)^{n+m} = \left(\frac{r_\ell}{r_\ell - r_k}\right)^2 < +\infty, \tag{29}$$

the nuclearity of this space is proved.

4. Overview on Basis Transforms in (NF)-Spaces and the General Theory of Effectiveness

4.1. Basis Transforms and Effectiveness Phenomena. In this section, we provide a general overview of the notion of basis transforms in (NF)-spaces from which a criterion is obtained. This criterion shows that under which conditions, a sequence of vectors $(x_n)_{n\in\mathbb{N}}$ in a (NF)-space E with basis is itself a basis for E. This leads to the notion of basis-preserving homeomorphisms between (NF)-spaces with basis. The given results are then applied to the case of power series spaces giving a general criterion for basis transforms which improve and refine that one of Cannon-Whittaker on the phenomena of effectiveness.

The idea behind this study is to link the theory of basis transforms in some locally convex space with the theory of Cannon-Whittaker on polynomial bases. In the meantime, we improve and refine the effectiveness phenomena which represents the core of Whittaker's theory.

It is worth mentioning that in the classical treatment of the subject of polynomial bases as was introduced by Whittaker and Cannon [17], the methods for establishing effectiveness depend on the region of effectiveness and on the class of functions for which the base is effective. The first attempt at some uniformity among the different methods was made by Newns, who gave in [4] a topological approach leading to a general theory of effectiveness. Our approach is entirely different depending on functional analytical methods and the basis transforms being performed by means of infinite matrices. In the Cannon-Whittaker theory, the main tool they used depends essentially on assuming the row-finite matrices of coefficients and operators.

In what follows, it is thus understood that E is a (NF)-space with basis $(x_n, x'_n)_{n\in\mathbb{N}}$. Moreover, $P=(P_{ij})$ and $Q=(Q_{ij})$ stand for infinite matrices over $\mathbb{C}$ and we denote the infinite identity matrix by I. In such a way, we formally write

$$y_k = \sum_{n\in\mathbb{N}} P_{kn} x_n, \quad k\in\mathbb{N}. \tag{30}$$

We start by stating the following result by Cnops and Abul-Ez [37].

Theorem 16 (uniqueness theorem). *Suppose that $(y_k)_{k\in\mathbb{N}}$ is a basis for E. Then, P has two-sided inverse Q.*

The following problem now arises: let E be a (NF)-space with basis $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be a sequence in E satisfying the relations

$$x_n = \sum_{j\in\mathbb{N}} Q_{nj} y_j \tag{31}$$

and formally,

$$y_k = \sum_{n\in\mathbb{N}} P_{kn} \left(\sum_{j\in\mathbb{N}} Q_{nj} y_j \right). \tag{32}$$

Under which condition on the matrices P and Q we conclude that $(y_k)_{k\in\mathbb{N}}$ is a basis for E?

Remark 17. In fact, (31) and (32) do not assume that $(y_k)_{k\in\mathbb{N}}$ is a basis. An example is given by Faber polynomials which are valid in a noncircular domain, see [38]. Since the nth Faber polynomials is of degree n, it is possible to write each z^k as a linear combination (even a finite one) of Faber polynomials. However, it is well known that Faber polynomials in general do not form a basis for $\mathcal{O}(B(R))$, but restricted conditions should be considered, see the work of Newns [4]. The answer of the above question is provided by the following interesting results by Cnops and Abul-Ez [37].

Theorem 18 (basis transforms in (NF)-spaces). *Let the sequence $(y_k)_{k\in\mathbb{N}}$ in E allow representations of the form (31) and (32). Then, a biorthogonal system $(y_n, y'_n)_{n\in\mathbb{N}}$ forming a basis for E may be constructed if and only if*

(B.1) $PQ = QP = I$

(B.2) For each $m\in\mathbb{N}$, there exist $\ell\in\mathbb{N}$, $K_m > 0$ such that for all $k, n \in\mathbb{N}$

$$|Q_{nk}| \|y_k\|_m \le K_m \|x_n\|_\ell. \tag{33}$$

Remark 19.

(1) From the isomorphism between $(E, (x_n)_{n\in\mathbb{N}})$ and $S(A)$ and the derived equivalent system of norms on E (see also Theorems 13 and 12), we have that (B.2) is equivalent to

(B $'$.2) For each $m\in\mathbb{N}$, there exist $\ell\in\mathbb{N}$ and $K_m>0$ such that for all $n, k, t\in\mathbb{N}$

$$|Q_{nk}| |P_{kt}| \|x_t\|_m \le K_m \|x_n\|_\ell \tag{34}$$

(2) On the other hand, using the sum-norm and applying the Dynin-Mitiagin theorem, we deduce that (B.2) and (B.2) $'$ are equivalent to

(B $''$.2) For each $m\in\mathbb{N}$, there exist $\ell\in\mathbb{N}$ and $K'_m>0$ such that for all $n\in\mathbb{N}$

$$\sum_{k\in\mathbb{N}} \sum_{t\in\mathbb{N}} |Q_{nk}| |P_{kt}| \|x_t\|_m \le K'_m \|x_n\|_\ell \tag{35}$$

(3) Some comments on the requirement that P has two-sided inverse Q. It is of course possible that P has several two-sided inverses. But, if P is taken to belong to some class of infinite matrices which forms a ring $\mathcal{R}$ under the classical rules of addition and multiplication for matrices and moreover $I\in\mathcal{R}$, then, if P has a two-sided inverses $Q\in\mathcal{R}$, Q is unique

An example of such a ring $\mathcal{R}$ is the ring $\mathcal{R}_{\mathrm{up}}$ of all infinite upper-triangular matrices or the ring $\mathcal{R}_{\mathrm{lo}}$ of all infinite lower-triangular matrices. It is clear that if, e.g., $P\in\mathcal{R}_{\mathrm{up}}$ has all diagonal elements different from zero, then P has a two-sided inverse Q in $\mathcal{R}_{\mathrm{up}}$ which is therefore unique (see [39]). Nevertheless, it may happen that although the matrix P we are considering belongs to some ring $\mathcal{R}$ of infinite matrices, P admits a two-sided inverse which does not belong to the ring $\mathscr{R}$. We address this phenomenon in the following example.

Example 3. Let $\mathscr{R}$ be the ring of all infinite row-finite matrices and $P \in \mathscr{R}$ given by $P = (P_{ij})_{i,j=1}^{\infty}$, where

$$\begin{aligned} P_{ii} &= 1, \quad i \in \mathbb{N}, \\ P_{i,i+1} &= -1, \quad i \in \mathbb{N}, \\ P_{ij} &= 0, \quad j \neq i \text{ or } j \neq i+1, \end{aligned} \tag{36}$$

that is,

$$P = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ & & & & & \end{pmatrix}. \tag{37}$$

Then, P admits the two-sided inverse $Q = (Q_{ij})_{i,j=1}^{\infty}$ with

$$Q_{ij} = \begin{cases} 0 & \text{if } j < i, \\ 1 & \text{if } j \geq i, \end{cases} \tag{38}$$

that is,

$$Q = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ & & & & & \end{pmatrix}. \tag{39}$$

Clearly, Q is not row-finite.

Remark 20. Applying P to the space $(\mathcal{O}(B(1/2)), \xi)$, $P\xi$ is given by

$$p_k(z) = z^k - z^{k+1}, \quad k = 0, 1, 2, \cdots, \tag{40}$$

and using our criterion for a basis transform, $p = P\xi$ is indeed a basis for $\mathcal{O}(B(1/2))$.

These considerations therefore suggest that, when an infinite matrix P is given and we wish to use it as a basis transform in some (NF)-space (E, x), we check the following:

(i) The existence of an infinite matrix Q which is a two-sided inverse of P, i.e., (B.1)

(ii) The criterion (B.2) or (B $'$.2) or (B $''$.2) holds

Suppose (i) is fulfilled. Then, the following situations may occur:

(i.1) Q is the unique two-sided inverse of P; then, (ii) above should be verified.

(i.2) P admits several two-sided inverses. If P defines a basis transform, there is a unique two-sided inverse Q such that Qp exists and Q satisfies (B.2). If P does not define a basis transform, the set of inverses

$$\{Q : QP = PQ = I \text{ and } QP \text{ exists}\}. \tag{41}$$

The above remark leads to the following result.

Corollary 21. *Let P be an infinite matrix having a two-sided inverse Q such that Qp converges. Then, P defines a basis transform if and only if the pair (P, Q) satisfies conditions (B.2) (equivalently (B$'$.2) or (B$''$.2)).*

4.2. Power Series Spaces. A class of Köthe sequence spaces which will be of special interest to us is that of so-called power series spaces. Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers and put for $0 < R < +\infty$ fixed,

$$a_k^n = \left(Re^{-1/k}\right)^{\alpha_n}. \tag{42}$$

Then, calling $A = (a_n^k)$ and associating with it the Köthe sequence space $K(A)$, it is well known that

$$\text{"}K(A) \text{ is nuclear} \Leftrightarrow \lim_{n\to\infty} \frac{\log n}{\alpha_n} = 0.\text{"} \tag{43}$$

In the case $R = \infty$, we put

$$a_n^k = e^{k\alpha_n} \tag{44}$$

and $A = (a_n^k)$. It is then well known that

$$\text{"}K(A) \text{ is nuclear} \Leftrightarrow \limsup_{n\to\infty} \frac{\log n}{\alpha_n} < +\infty.\text{"} \tag{45}$$

For an account of these results, we refer to [26]. In the case $0 < R < +\infty$, the space $K(A)$ is denoted by $K(A) = \Lambda_R(\alpha)$ while the case $R = \infty$, the notation $K(A) = \Lambda_\infty(\alpha)$ is used.

Notice that, putting $r_k = Re^{-1/k}$ $(0 < R < +\infty)$ or $r_k = e^{k\alpha_n}$ $(R = \infty)$, then $r_k \uparrow R$ or $r_k \uparrow \infty$.

The question now arises to determine whether or not a given (NF)-space with basis is similar to a space of the type $(\Lambda_R(\alpha), e)$ or $(\Lambda_\infty(\alpha), e)$, the latter spaces being called *power series spaces*.

Now, we illustrate the problem by means of the following.

Example 4. The space $(\mathcal{O}(B(R)),\xi)$, $0<R<+\infty$. Define for each $k\in\mathbb{N}$ and $f\in\mathcal{O}(B(R))$,

$$\|f\|_k=\sup_{|z|\le Re^{-1/k}}|f(z)|. \tag{46}$$

As $\lim_{n\to\infty}(\log n/n)=0$, we can take $\alpha=(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n=n$ for all $n\in\mathbb{N}$. Consequently, $(\mathcal{O}(B(R)),\xi)\simeq\Lambda_R(\mathbb{N},e)$.

Example 5 (the space $(\mathcal{O}(B(\mathbb{C}),\xi))$. In this case $R=\infty$ and we may describe the topology on $\mathcal{O}(\mathbb{C})$ by the system of seminorms $\mathcal{P}=\{\|\bullet\|_k : k\in\mathbb{N}\}$ where for each $k\in\mathbb{N}$ and $f\in\mathcal{O}(\mathbb{C})$,

$$\|f\|_k=\sup_{|z|\le e^k}|f(z)|. \tag{47}$$

Hence, for each $n\in\mathbb{N}$, $\|z^n\|_k=\sup_{|z|\le e^k}|z^n|=e^{nk}$.

As $\sup_{n\in\mathbb{N}}(\log n/n)<+\infty$, we can take $\alpha=(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n=n$ for all $n\in\mathbb{N}$. Consequently, $(\mathcal{O}(\mathbb{C}),\xi)\simeq\Lambda_\infty(\mathbb{N},e)$.

In the sequel, we denote for convenience $(\Lambda_R(\alpha),e)\simeq K(\alpha,R)$, the associated supremum space will be denoted by $S(\alpha,R)$.

4.3. Similarity Theorem. The following criterion will completely answer the equation we mentioned above, namely, to know under which conditions a (NF)-space with basis is similar to some power series space $K(\alpha,R)$. To reach this end, let us introduce the following notations.

Given the basis $x=(x_n)_{n\in\mathbb{N}}$ of E and the sequence $\alpha=(\alpha_n)_{n\in\mathbb{N}}$ determining $K(\alpha,R)$, we put for each $k\in\mathbb{N}$ $R(k)=\limsup_{n\to\infty}\|x_n\|_k^{1/\alpha_n}$ and $r(k)=\liminf_{n\to\infty}\|x_n\|_k^{1/\alpha_n}$. Clearly, $r(k)\le R(k)$ for each $k\in\mathbb{N}$.

Suppose that R is finite (the case $R=+\infty$ is treated in a similar way).

Theorem 22 (similarity theorem basis [37]). *Let (E,x) be a (NF)-space with basis and let $K(\alpha,R)$ be some power series space. Then, $(E,x)\simeq K(\alpha,R)$ if and only if*

(i) $r(k)\uparrow R$

(iii) $R(k)<R$ for each $k\in\mathbb{N}$

After having established which (NF)-spaces with basis are similar to a power sequence space (Theorem 22), we now aim to apply the criterion obtained for basis transforms (Theorem 18) to the case of power sequence spaces. To do this, consider the power sequence space $K(\alpha,R)$ with basis $e=(e_t)_{t\in\mathbb{N}}$. Then, we know that for each $m\in\mathbb{N}$,

$$\|e_t\|_m=\begin{cases}\left(Re^{-1/m}\right)^{\alpha_t} & \text{for } 0<R<\infty,\\ \left(e^m\right)^{\alpha_t} & \text{for } R=\infty.\end{cases} \tag{48}$$

In the sequel, we put $r_m=R^{-1/m}$ $(0<R<+\infty)$ or $r_m=e^m$. So, if P is an infinite matrix acting on e, then by a direct translation of Theorem 18, the authors of [37] established the following fundamental results.

Theorem 23 (basis criterion). *Let $K(\alpha,R)$ be a power sequence space and let P be an infinite matrix. Then, Pe is a basis for $K(\alpha,R)$ if and only if*

(i) there exists an infinite matrix Q such that $PQ=QP=I$

(ii) for each $m\in\mathbb{N}$, there exist $\ell\in\mathbb{N}$ and $K_m>0$ such that for all $n,k,t\in\mathbb{N}$,

$$|Q_{nk}||P_{kt}|r_m^{\alpha_t}\le K_m r_\ell^{\alpha_n}. \tag{49}$$

Definition 24 (see [37]). Let P be an infinite matrix which has two-sided inverse Q and let $r>0$ be fixed. Then, for $n\in\mathbb{N}$, we put

$$J_n(P,r)=\sup_{k,t}|Q_{nk}||P_{kt}|r^{\alpha_t} \tag{50}$$

and call

$$J(P,r)=\lim_{n\to\infty}\left(J_n(P,r)\right)^{1/\alpha_n} \tag{51}$$

if all $J_n(P,r)$ are finite, and $+\infty$ otherwise.

From Theorem 23, the following result was deduced in [37].

Theorem 25 (criterion for basis transforms [37]). *Let $K(\alpha,R)$ be a power sequence space and let P be an infinite matrix with two-sided inverse Q. Then, P determines a basis transform in $K(\alpha,R)$ if and only if for all $0<r<R$,*

$$J(P,r)<R. \tag{52}$$

Remark 26. We saw before that the criterion (B$'$.2) is equivalent to (B$''$.2). This of course prompts the introduction of the following entities: given $r>0$, we put

$$Z_n(P,r)=\sum_k\sum_t|Q_{nk}||P_{kt}|r^{\alpha_t}, \tag{53}$$

$$Z(P,r)=\lim_{n\to\infty}\left(Z_n(P,r)\right)^{1/\alpha_n}. \tag{54}$$

This leads to the following important result which was shown in [37].

Theorem 27 (general criterion for basis transforms [37]). *Let $K(\alpha,R)$ be a power sequence space and let P be an infinite matrix having a two-sided inverse Q. Then, P determines a basis transforms in $K(\alpha,R)$ if and only if for all $0<r<R$,*

$$Z(P,r)<R. \tag{55}$$

Remark 28. It should be noted that Cannon [15] gave a criterion of the form in (54) for certain spaces of holomorphic functions, although he did not prove that P defines a basis transform but only that every function can be represented by a series in $P\xi$.

4.4. Cannon-Whittaker's Criterion Revisited. Consider the space $(\mathcal{O}(B(R)), \xi)$. In [15], Cannon considered an infinite matrix P of the following type:

(i) P is row-finite

(ii) P possesses a row-finite two-sided inverses Q [5]

For each $0 < r < R$, the following entities were introduced [40]:

The Cannon sum $\lambda_n(P, r) \leq \sum_{k,r} |Q_{nk}| \|P_k\|_r$ for the Cannon bases. As for the non-Cannon base (general base), it introduced the Cannon sum in the form:

$$F_n(P, r) = \sup_{t_1,t_2} \sup_{|z|\leq r} \sum_{k=t_1}^{t_2} Q_{nk} \left(\sum_k P_{ks} z^s \right). \tag{56}$$

Consequently, for the corresponding Cannon functions

$$\begin{aligned} \lambda(P, r) &= \limsup_{n\to\infty} [\lambda_n(P, r)]^{1/n}, \\ \kappa(P, r) &= \limsup_{n\to\infty} [F_n(P, r)]^{1/n}. \end{aligned} \tag{57}$$

4.5. A Refinement of Cannon-Whittaker Criterion for Effectiveness. Cannon proved that the set of polynomials determined by $P\xi$ is effective in $|z| < R$ if and only if $\kappa(P, r) < R$ for all $0 < r < R$ and that for Cannon sets $\lambda(P, r) = \kappa(P, r)$. When we look at $J_n(P, r)$ in the case under consideration, it is clear that for all $n \in \mathbb{N}$ and $0 < r < R$,

$$\begin{aligned} J_n(P, r) &\leq F_n(P, r) \leq \lambda_n(P, r), \\ J(P, r) &\leq \kappa(P, r) \leq \lambda(P, r). \end{aligned} \tag{58}$$

Besides the fact that Cannon only proved the effectiveness of the set $P\xi$ under the condition $\kappa(P, r) < R$ for all $0 < r < R$ while we pointed out that $J(P, r) < R$ for all $0 < r < R$ implies that $P\xi$ is a basis, it obviously follows from $\lambda(P, r) < R$ for all $0 < r < R$ that $J(P, r) < R$ for all $0 < r < R$. Hence, the given condition here using $J(P, r)$ is weaker than the one obtained by Cannon using $\lambda(P, r)$ in the case of so-called Cannon sets (i.e., sets Pe for which $\lambda(P, r) = \kappa(P, r)$). There is even more to say. Since we can also use $Z(P, r)$ to establish whether or not $P\xi$ is a basis for $\mathcal{O}(B(R))$ and since clearly for all $n \in \mathbb{N}$ and $0 < r < R$,

$$J_n(P, r) \leq F_n(P, r) \leq \lambda_n(P, r) \leq Z_n(P, r), \tag{59}$$

whence

$$J(P, r) \leq \kappa(P, r) \leq \lambda(P, r) \leq Z(P, r). \tag{60}$$

We may conclude that $\lambda(P, r)$ may also be used in the case of non-Cannon sets. Of course, it should also be stressed that the matrices P we are considering need not be row-finite.

Let us illustrate the previous observations in the case of $(\mathcal{O}(B(1)), \xi)$, the space of holomorphic functions in the unit ball.

Example 6. Consider the function $(1 - z)^{-1} = \sum_{j=0}^{\infty} z^j \in \mathcal{O}(B(1))$. Then, we claim that set $\{1/(1 - z), z, z^2, \cdots, z^n, \cdots\}$ is a basis for $\mathcal{O}(B(1))$. Indeed, the associated infinite matrix P is given by

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ & & & & & \end{pmatrix}. \tag{61}$$

Note that P is upper-triangular and has diagonal elements 1. Hence, it possesses a two-sided inverse Q given by

$$Q = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ & & & & & \end{pmatrix}. \tag{62}$$

Taking $\alpha = (\alpha_n) = \mathbb{N}$, a straightforward calculation yields for each $0 < r < 1$ that

(i) $J_0(P, r) = \sup_{t\in\mathbb{N}, t\geq 1} r^t = 1$

(ii) $J_n(P, r) = r^n, n \geq 1$

Consequently, $J(P, r) = \limsup_{n\to\infty} [J_n(P, r)]^{1/n} = 1$, so that $J(P, r) < 1$ for each $0 < r < 1$. Hence, $P\xi$ is a basis for the space $\mathcal{O}(B(1))$.

5. Application: Chebychev Polynomials

As an actual application of the criterion of basis transforms which refines the analog one of Cannon-Whittaker, we deal with the set of Chebychev polynomials of the first kind (see [41]).

Consider the Chebychev polynomials of the first kind, namely,

$$P_n(z) = \sum_{k=0}^{[n/1]} \binom{n}{2k} z^{n-2k} (z^2 - 1)^k. \tag{63}$$

This set $\{P_n(z)\}_{n\geq 0}$ forms a basic set in the sense of Whittaker [5] and also in the sense of Theorem 16. In view of the expression (53) and (54) and after having the two matrices $|Q_{nk}|$ and $|P_{kt}|$, we may have

$$Z_n(P,r) = \liminf_{n\to\infty} \left\{ \sum_{k=0}^{n} \sum_{j=0}^{n/2} |Q_{nk}||P_{kt}|r^{k-2j}\left|r^2-1\right|^j \right\}. \quad (64)$$

Applying the well known Stirling's formula $n! \sim \sqrt{2\pi n^n}\, e^{-n}$ as $n\to\infty$ to the combinatorics coefficients

$$\binom{n}{2k} = \frac{n!}{(2k)!(n-2k)!}, \quad (65)$$

we can show after some calculations that the above series in (64) converges well only for $r\leq 1$. Consequently, $Z(P, r)\leq r$, $r\leq 1$. Hence, $\{P_n(z)\}_{n\geq 0}$ is a basis for $\mathcal{O}(B(1))$ and in the terminology of Cannon-Whittaker is effective in unit disc $B(1)$.

The material given in Sections 3 and 4 opened the door to discuss the basis transforms in further functional spaces. In the following section, we investigate some counterpart results in nuclear Silva spaces with basis.

6. Nuclear Silva Spaces with Basis

For Fréchet spaces, and even for nuclear Fréchet spaces, the existence of a basis is not always guaranteed, as shown in the fundamental paper by Mitiagin-Zobin and Mitiagin [34, 35]. This problem had been treated by Cnops and Abul-Ez [37] and has been exhaustive demonstrated in Sections 3 and 4 above, where nuclearity criterion is given for Fréchet spaces with basis, as well as basis transforms in such spaces. Related aspects had been treated for the space of holomorphic functions [10] and hyperholomorphic functions [6] to give that the set of Bessel polynomials is a basis in a functional space, and consequently, we have a (NF)-space.

It is worth mentioning that Abul-Ez [25] pointed out that the space of entire functions of finite growth order and type is a (NF)-space, as well as he studied the existence of basis in such space. In this section, we are going to show that if E is a (NF)-space and $F=E'_\beta$ is the corresponding Silva space, then F admits a basis $(x'_k)_{k\in\mathbb{N}}$ if and only if $(x'_k)_{k\in\mathbb{N}}$ is the dual basis of a basis in E. Moreover, a nuclearity criterion is given for a Silva space with basis.

By definition (see [27], p. 264), a Silva space F is the inductive limit of a sequence of Banach spaces $(F_s)_{s\in\mathbb{N}}$ such that for each $s\in\mathbb{N}$, the unit ball of F_s is contained in the unit ball of F_{s+1} and is compact in F_{s+1}. Several other characterizations of Silva spaces may be given. We mention here the following:

(i) A locally convex space F is a Silva space if and only if F is the strong dual of a Fréchet-Schwartz space E. If $(\|\cdot\|_s)_{s\in\mathbb{N}}$ is a defining sequence of seminorms on E, then F may also be considered as being the inductive limit of the sequence of Banach spaces $(E'_{\|\cdot\|'_s})_{s\in\mathbb{N}}$ whereby for each $s\in\mathbb{N}$, $E'_{\|\cdot\|'_s}$ is the linear hull of the polar of the closed $\|.\|_s$-unit ball (see again [27])

(ii) A locally convex space F is a Silva space if and only if F is a complete DF-Schwartz space in which each null sequence converges locally (see [28], Corollary 12.5.9, and [42])

Moreover, if $F=E'_\beta$ is a Silva space, E being a Fréchet-Schwartz space, then F is nuclear if and only if E is nuclear (see, e.g., [28], Theorem 21.5.3).

Now, let F be a Silva space and let $(x_n)_{n\in\mathbb{N}}$ be a basis in F. Then, by (ii) and the continuity theorem (see [28], Theorem 14.2.5), $(x_n)_{n\in\mathbb{N}}$ is already a Schauder basis in F.

Remark 29.

(1) As an inductive limit of Banach spaces $(F_s)_{s\in\mathbb{N}}$, we have the following

$$F_1\subset F_2\subset F_3\cdots\subset F_n\subset\cdots\subset F \quad (66)$$

and we can describe the topology on F by the norms $\|.\|'_s$ of the spaces F_s. This means that $\|.\|'_s$ is defined on F_s for $x\in F_s : \|x\|'_s\leq\|x\|'_{s+1}$.

In Silva spaces, we have the following criterion of convergence $x_n\to x$ if and only if

(a) there exist s such that $x_n\in F_s$ for all n and $x\in F_s$

(b) $x_n\to^{\|\cdot\|'_s} x$

(2) Every nuclear Fréchet space is a Fréchet-Schwartz space so the dual of such spaces are Silva spaces

(3) The spaces $\mathcal{O}(\Omega)$ of functions holomorphic (regular) in an arbitrary neighborhood of a compact set Ω are of this type. Since we are going to prove that having a basis in F is equivalent to having a basis in F', we can transfer the basis criterion for nuclear Fréchet spaces to nuclear Silva spaces

7. Series Representation in Silva Spaces

As a Silva space is an F_+-space in the sense of Newns, Theorem 3.2 of [4] and part of its proof may be reformulated as follows.

Theorem 30 (see [4]). *Let $F=\mathscr{L}_{s\in\mathbb{N}}F_s$ be a Silva space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in F with $x_n\neq 0$ for all $n\in\mathbb{N}$. For $s\in\mathbb{N}$ fixed, suppose that each $x\in F_s$ is represented in F by a series of the form $\sum_{n\in\mathbb{N}}\alpha_n x_n$. Then, there exists $\sigma\in\mathbb{N}$ (depending only on s) such that for each $x\in F_s$, $x=\sum_{n\in\mathbb{N}}\alpha_n x_n$ in F_σ.*

Moreover, calling U_s the space of sequences $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that the series $\sum_{n\in\mathbb{N}n}\alpha_n x_n$ converges in F_σ to some element of F_s and putting for each $(\alpha_n)_{n\in\mathbb{N}}\in U_s$,

$$\left\|(\alpha_n)_{n\in\mathbb{N}}\right\|_{U_s}=\left\|\sum_{n\in\mathbb{N}}\alpha_n x_n\right\|_s+\max_{\ell,t}\left\|\sum_{k=\ell}^{t}\alpha_k x_k\right\|_\sigma, \quad (67)$$

we have that $(U_s,\|.\|_{U_s})$ is a Banach space.

Finally, the linear mapping $u:U_s\to F_s$ given by $u((\alpha_n)_{n\in\mathbb{N}})=\sum_{n\in\mathbb{N}}\alpha_n x_n$ is a continuous surjection.

Now, suppose that $(x_n)_{n\in\mathbb{N}}$ is a basis for the Silva space, then clearly the mapping $u:U_s\to F_s$ is a continuous bijection, whence by Banach's homeomorphism theorem 3, u is bicontinuous. This leads to the following:

Theorem 31. *Let $F=\mathscr{L}_{s\in\mathbb{N}}F_s$ be a Silva space with basis $(x_n)_{n\in\mathbb{N}}$ and let for each $s\in\mathbb{N}$, U_s be defined as in Theorem 30. Then, U_s and F_s are linearly homeomorphic for all $s\in\mathbb{N}$.*

Corollary 32. *Let $F=\mathscr{L}_{s\in\mathbb{N}}F_s$ be a Silva space, and $(x_n)_{n\in\mathbb{N}}$ be a basis for F, and let for $s\in\mathbb{N},\sigma\in\mathbb{N}$ be such that $x\in F_s$ admits in F_σ the expansion $x=\sum_{n\in\mathbb{N}}\alpha_n x_n$. If the continuous linear functional α_n on F is not identically zero on F_s, i.e., ($\alpha_n|_{F_s}\neq 0$), then $x_n\in F_\sigma$.*

Proof. Suppose that for some $n\in\mathbb{N},\alpha_n|_{F_s}\neq 0$ and $x_n\notin F_\sigma$. Then, taking $y\in F_s$ such that $\alpha_n(y)\neq 0$, we have that, as in F_σ,$y=\sum_{k\in\mathbb{N}}\alpha_k x_k$, the partial sums $S_n(y)=\sum_{i=1}^{n}\alpha_i(y)x_i$ and $S_{n-1}(y)=\sum_{i=1}^{n-1}\alpha_i(y)x_i$ belong to F_σ whence $\alpha_n(y)x_n=S_n(y)-S_{n-1}(y)\in F_\sigma$, a contradiction.

8. Nuclear Silva Spaces with Basis

Let F be a nuclear Silva space. Then, as we saw in Section 6, $F=E'_\beta$ whereby E is a (NF)-space and $E\simeq F'_\beta$. In what follows, we therefore denote a basis for F (if it exists) by $(x'_k)_{k\in\mathbb{N}}$ and call $(x_k)_{k\in\mathbb{N}}$ the corresponding biorthogonal sequence in E, i.e., $x_\ell\in E$ with $x_\ell(x'_k)=\delta_{\ell k}$ for all $k,\ell\in\mathbb{N}$.

Theorem 33. *Let $F=E'_\beta$ be a (NS)-space and let $(x'_k)_{k\in\mathbb{N}}$ be a sequence of nonzero elements in F. Then, $(x'_k)_{k\in\mathbb{N}}$ is a basis in F if and only if $(x_k)_{k\in\mathbb{N}}$ is a basis in E.*

Proof. If $(x_k)_{k\in\mathbb{N}}$ is a basis for E then E being a (NF)-space, the sequence $(x'_k)_{k\in\mathbb{N}}$ is a basis in $E'_\beta=F$ (see, e.g., [28], Theorem 21.10.6). Conversely, suppose that $(x'_k)_{k\in\mathbb{N}}$ is a basis for F. Then, since $(x'_k)_{k\in\mathbb{N}}$ is a Schauder basis, the biorthogonal system $(x_k,x'_k)_{k\in\mathbb{N}}$ exists. We prove that the biorthogonal system $(x_k,x'_k)_{k\in\mathbb{N}}$ is complete, i.e., $E=\overline{\mathrm{Span}\{x_k:k\in\mathbb{N}\}}$.

Indeed, call $L=\overline{\mathrm{Span}\{x_k:k\in\mathbb{N}\}}$ and suppose that $L\neq E$. Then, if $x\in E\setminus L$, by the Hahn-Banach theorem, there ought to exist $x'\in E'$ such that $x'(x)=1$ and $x'(L)=\{0\}$. But, as $x'=\sum_{k=1}^{\infty}x_k(x')x'_k$ and $x_k(x')=0$ for all $k\in\mathbb{N},x'=0$, thus yielding a contradiction.

Now, we prove that the complete biorthogonal system $(x_k,x'_k)_{k\in\mathbb{N}}$ satisfies Haslinger's criterion [30], i.e., for all $s\in\mathbb{N}$, there exists $\sigma\in\mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}}\left\|x'_k\right\|'_\sigma\|x_k\|_s<+\infty. \quad (68)$$

To this end, take $s\in\mathbb{N}$ fixed. On the one hand, by the representation of seminorms, we have for each $k\in\mathbb{N}$ that

$$\|x_k\|_s=\|x_k\|''_s=\sup_{\|x'_k\|'_s=1}\left|x_k\left(x'\right)\right|. \quad (69)$$

On the other hand, in view of Theorem 31, there exists $\sigma\in\mathbb{N}$ and a corresponding space U_s such that U_s is linearly homeomorphic to F_s. Denoting again by u this isomorphism, we may thus find $C>0$ such that for all $x'\in F_s$,

$$\left\|x'_k\right\|'_s\geq\frac{1}{C}\left\|u^{-1}\left(x'\right)\right\|_{U_s}. \quad (70)$$

Hence, for all $x'\in F_s$ and $k\in\mathbb{N}$,

$$\left\|x'_k\right\|'_s\geq\frac{1}{C}\left\|x_k\left(x'\right)x'_k\right\|'_\sigma=\frac{1}{C}\left\|x_k\left(x'\right)\right\|\left\|x'_k\right\|'_\sigma. \quad (71)$$

Consequently,

$$\sup_{k\in\mathbb{N}}\|x_k\|_s\left\|x'_k\right\|'_\sigma\leq C. \quad (72)$$

By virtue of Haslinger's criterion (see [30]), $(x_k)_{k\in\mathbb{N}}$ is a basis for E.

Remark 34. The preceding theorem shows that if E is a (NF)-space, then each basis in $F=E'_\beta$ is the dual basis of a basis in E. In view of [28], Theorem 21.10.6, if (x_k) is absolute, so is (x'_k). But every basis in E'_β is absolute, so we obtain that each basis in a (NS)-space F is an absolute basis. This duality also leads to the following.

9. A Nuclearity Criterion for Silva Spaces with Basis

As was shown in [32], for a vector space E provided with a system of seminorms $\mathscr{P}$ and having a Schauder basis $(x_k)_{k\in\mathbb{N}}$, the following are equivalent:

(i) E is nuclear

(ii) For each $p\in\mathscr{P}$, there exists $q\in\mathscr{P}$ such that

$$\sum_{k\in\mathbb{N}}\frac{p(x_k)}{q(x_k)}<+\infty. \quad (73)$$

In this section, a criterion for the nuclearity of Silva spaces F with basis is proved whereby only the sequence of norms of the defining Banach spaces F_s is used. In such a way, a criterion for nuclearity is obtained which avoids the use of the system of seminorms defining the inductive limit topology on F.

Theorem 35. *Let $F = E'_\beta$ be a Silva with basis $(x'_k)_{k\in\mathbb{N}}$. Then, F is nuclear if and only if for each $s \in \mathbb{N}$, there exists $\sigma \in \mathbb{N}$ such that*

$$\sum_{k\in\mathbb{N}} \frac{\|x'_k\|'_\sigma}{\|x'_k\|'_s} < +\infty. \tag{74}$$

Proof. Let F be nuclear. Then, E is a (NF)-space with basis $(x_k)_{k\in\mathbb{N}}$ (see Theorem 30) whence for each $s \in \mathbb{N}$, there exists $\sigma \in \mathbb{N}$ such that (see, e.g., [32])

$$\sum_{k\in\mathbb{N}} \frac{\|x_k\|_s}{\|x_k\|_\sigma} < \infty. \tag{75}$$

Moreover, putting $A = (a_k^s)_{k,s\in\mathbb{N}}$ with $a_k^s = \|x_k\|_s$, E is linearly homeomorphic to the nuclear Köthe sequence space $K(A)$ and so the topology on E is also determined by the defining sequence of norms $[.]_s$ with

$$[x]_s = \sum_{k\in\mathbb{N}} \left|x'_k(x)\right| \|x_k\|_s, \quad x \in E \tag{76}$$

(see, e.g., [26]).

Taking duals, we thus have that

$$F = E'_\beta \simeq \mathscr{L}_{s\in\mathbb{N}}\left(E', \|\bullet\|'_s\right) \simeq \mathscr{L}_{t\in\mathbb{N}}\left(E', [\bullet]'_t\right), \tag{77}$$

whereby for each $t \in \mathbb{N}$ and $x' = (E', [\bullet]'_t)$,

$$\left[x'\right]'_t = \sup_{k\in\mathbb{N}} \frac{1}{\|x_k\|_t} \left|x'(x_k)\right|. \tag{78}$$

Notice that in particular for each $k \in \mathbb{N}$,

$$\left[x'_k\right]_t = \frac{1}{\|x_k\|_t}. \tag{79}$$

Combining (75) and (79), we thus obtain that for each $t \in \mathbb{N}$, there exists $\tau \in \mathbb{N}$ such that

$$\sum_{k\in\mathbb{N}} \frac{\left[x'_k\right]_\tau}{\left[x'_k\right]_t} < +\infty. \tag{80}$$

Now, let $s \in \mathbb{N}$ be fixed. Then, there exist $t \in \mathbb{N}$ and $K_s^* > 0$ such that for all $k \in \mathbb{N}$,

$$K_s^* \|x'_k\|_s \geq \left[x'_k\right]_t, \tag{81}$$

while for that $t \in \mathbb{N}$, there ought to exist $\tau \in \mathbb{N}$ such that (80) holds. However, for this τ, there exist $\sigma \in \mathbb{N}$ and $K_\tau > 0$ such that for all $k \in \mathbb{N}$,

$$K_\tau \left[x'_k\right]_\tau \geq \|x'_k\|'_\sigma. \tag{82}$$

Consequently, we obtain that for each $s \in \mathbb{N}$, there exists $\sigma \in \mathbb{N}$ such that (74) holds.

Conversely, suppose that for each $s \in \mathbb{N}$, there exists $\sigma \in \mathbb{N}$ such that (74) holds. Calling for each $k, s \in \mathbb{N}$,

$$a_k^s = \begin{cases} \dfrac{1}{\|x'_k\|'_s} & \text{if } x'_k \in \left(E', \|\bullet\|'_s\right) = F_s, \\ 0 & \text{if } x'_k \notin \left(E', \|\bullet\|'_s\right) = F_s, \end{cases} \tag{83}$$

and putting $A = (a_k^s)$, we have that the Köthe sequence space $K(A)$ is a (NF)-space, whence its topology is also determined by the sequence of norms $\|\bullet\|_s$, $s \in \mathbb{N}$, with

$$\begin{gathered} \|\xi\|_s = \sup_{k\in\mathbb{N}} (|\xi| a_k^s), \\ \xi = (\xi_k)_{k\in\mathbb{N}} \in K(A). \end{gathered} \tag{84}$$

Its dual $K(A)'$ is thus given by

$$K(A)' = \left\{ y = (a_k)_{k\in\mathbb{N}} \in \omega : \exists s \in \mathbb{N} \text{with } [y]_s = \sum_{k\in\mathbb{N}} \frac{|a_k|}{a_k^s} < \infty \right\}, \tag{85}$$

and of course, $K(A)'_\beta$ is a nuclear Silva space.

Now, define $B : F \to \omega$ by

$$B\left(\sum_{n\in\mathbb{N}} \alpha_n\left(x'\right) x'_n\right) = \left(\alpha_n\left(x'\right)\right)_{n\in\mathbb{N}}. \tag{86}$$

Then, we claim that $B(F) = K(A)'$.

Indeed, if $y \in K(A)'$, then there exists $s \in \mathbb{N}$ such that $\sum_{k\in\mathbb{N}} (|\alpha_k|/a_k^s)$ whence $\sum_{k\in\mathbb{N}} |\alpha_k| \|x'_k\|'_s < \infty$ or $\sum_{k\in\mathbb{N}} \alpha_k x'_k$ converges absolutely in F_s and so $\sum_{k\in\mathbb{N}} \alpha_k x'_k \in F$. This implies that $K(A)' \subset B(F)$.

Now, let $x' \in F$ admit the series representation $x' = \sum_{k\in\mathbb{N}} \alpha_k(x') x'_k$. Then, there ought to exist $s \in \mathbb{N}$ such that $x' = \sum_{k\in\mathbb{N}} \alpha_k(x') x'_k \in F_s$, the convergence being valid in F_s whence $\sup_{k\in\mathbb{N}} |\alpha_k(x')| \|x'_k\|'_s = K < \infty$. But for this $s \in \mathbb{N}$, there exists by assumption $\sigma \in \mathbb{N}$ such that (74) holds.

We claim that $[B(x')]_\sigma$ exists. Indeed,

$$\left[B\left(x'\right)\right]_\sigma = \sum_{k\in\mathbb{N}} \frac{\left|\alpha_k\left(x'\right)\right|}{a_k^\sigma} = \sum_{k\in\mathbb{N}} \frac{\left|\alpha_k\left(x'\right)\right|}{a_k^s} \frac{a_k^s}{a_k^\sigma} \leq K \sum_{k\in\mathbb{N}} \frac{a_k^s}{a_k^\sigma} < \infty. \tag{87}$$

Consequently, $B(x') \in K(A)'$ and so $B(F) \subset K(A)'$. Obviously, B is an isomorphism between F and $K(A)'$.

Now, we show that B^{-1} is continuous. Indeed, take $y \in K(A)'$. Then, there exists $s \in \mathbb{N}$ such that

$$[y]_s = \sum_{k\in\mathbb{N}} \frac{\left|\alpha_k\left(x'\right)\right|}{a_k^s} < \infty. \tag{88}$$

Putting $x' = B^{-1}y$, we have

$$[y]_s = \sum_{k\in\mathbb{N}} |\alpha_k| \|x_k'\|_s' \geq \left\|\sum_{k\in\mathbb{N}} \alpha_k x_k'\right\|_s' = \|x'\|_s'. \tag{89}$$

By virtue of the open mapping theorem, F and $K(A)'$ are linearly homeomorphic whence F is nuclear.

Remark 36. Another possible approach to the proof of Theorem 35 is one using the Grothendieck-Pietsch criterion for nuclearity for Köthe sequence spaces (see [29, 36]). Indeed, from our criterion, it follows that the basis considered is absolute and an explicit expression for the system of seminorms defining the topology of F can be given in terms of a Köthe sequence space. Applying the Grothendieck-Pietsch criterion then yields the nuclearity of the space F. On the other hand, if F is nuclear and has a basis, it follows from Theorem 33 that this basis is absolute whence F is linearly homeomorphic to some Köthe sequence space K(A), which, by assumption upon F, is nuclear. Hence, the Grothendieck-Pietsch criterion holds which can then be translated into the criterion of the above Theorem 35. For the Grothendieck-Pietsch criterion, we refer to ([28], Theorem 21.6.2).

Example 7. In [32], Kamthan introduced the following Fréchet space $(\mathcal{O}_A(\operatorname{Re} z < A), \zeta)$ of holomorphic functions. Let $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ be a fixed strictly increasing sequence of positive real numbers. With each sequence $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \sup\left(|a_n|/\lambda_n\right) \leq -A$, we associate the function

$$f(z) = \sum_{n=1}^{\infty} a_n e^{z\lambda_n}, \quad z \in \mathbb{C}. \tag{90}$$

Take $\varepsilon > 0$ arbitrary chosen and consider the half-plane

$$A_\varepsilon = \{z \in \mathbb{C} : \operatorname{Re} z \leq A - \varepsilon\}. \tag{91}$$

Then, for each $n \in \mathbb{N}$ and $z \in A_\varepsilon$,

$$|a_n|\left|e^{z\lambda_n}\right| \leq |a_n| e^{A\lambda_n} e^{-\varepsilon\lambda_n}. \tag{92}$$

But, in virtue of the assumption,

$$\lim_{n\to\infty} \sup \frac{\log |a_n|}{\lambda_n} \leq -A, \tag{93}$$

we find that

$$|a_n| e^{A\lambda_n} = e^{\log |a_n| + A\lambda}, \tag{94}$$

whence there exist $C > 0$ such that

$$\sup_{n\in\mathbb{N}} |a_n| e^{A\lambda_n} \leq C. \tag{95}$$

Consequently,

$$\sum_{n\in\mathbb{N}} \sup_{z\in A_\varepsilon} \left|a_n e^{z\lambda_n}\right| \leq C \sum_{n\in\mathbb{N}} e^{-\varepsilon\lambda_n} < +\infty, \tag{96}$$

whence the series defining f is normally convergent on each A_ε, $\varepsilon > 0$. Consequently, $f \in \mathcal{O}_A(\operatorname{Re} z < A)$.

We call $\mathcal{O}_A(\operatorname{Re} z < A)$ the subspace of $\mathcal{O}(\operatorname{Re} z < A)$ consisting of the elements f just defined and provide $\mathcal{O}_A(\operatorname{Re} z < A)$ with the system $\mathcal{P}$ of seminorms $p_k$with

$$p_k(f) = \sup_{\operatorname{Re} z \leq A-(1/k)} |f(z)|. \tag{97}$$

Then, it was proved by Kamthan that $(\mathcal{O}_A(\operatorname{Re} z < A), \mathcal{P})$ is a Fréchet space. From the definition itself of the elements f in $\mathcal{O}_A$, it follows that the sequence of functions $\zeta = (e^{z\lambda_n})_{n\in\mathbb{N}}$ is a basis for the space $(\mathcal{O}_A(\operatorname{Re} z < A), \mathcal{P})$. Then, we claim that $(\mathcal{O}_A(\operatorname{Re} z < A), \zeta)$ is a (NF)-space. Indeed, from the definition itself of $\mathcal{O}_A(\operatorname{Re} z < A)$, it follows that ζ is a basis for it.

Having that $(\mathcal{O}_A(\operatorname{Re} z < A), \zeta)$ is a (NF)-space, then using the discussion in Section 6, it can be seen that it is a (NS)-space with basis. Now, let $k \in \mathbb{N}$ be fixed and take any $l \in \mathbb{N}$ with $l > k$. Then,

$$\sum_{n=1}^{\infty} \frac{\|e^{z\lambda_n}\|_k}{\|e^{z\lambda_n}\|_l} = \sum_{n=1}^{\infty} \frac{e^{(A-(1/k))\lambda_n}}{e^{(A-(1/l))\lambda_n}} = \sum_{n=1}^{\infty} e^{((1/l)-(1/k))\lambda_n} < \infty. \tag{98}$$

Therefore, by our criterion (Theorem 35), the nuclearity is proved.

Example 8. Consider the space $(\mathcal{O}(\bar{S}_R), \xi)$ of holomorphic functions in two complex variables z, w, provided with the countable system $\mathcal{P}$ of seminorms p_k where

$$p_k(f) = \sup_{\bar{S}_{r_k}} |f(z, w)|, \tag{99}$$

and $\bar{S}_r$ is the closed hypersphere defined by

$$\sup_{\bar{S}_{r_k}} = \left\{z, w \in \mathbb{C} : |z|^2 + |w|^2 \leq r_k^2\right\}. \tag{100}$$

Again, $(r_k)_{k\in\mathbb{C}}$ is a strictly increasing sequence of positive numbers with $0 < r_k < R$ and $\lim_{k\to\infty} r_k = R$. As is well known, $(\mathcal{O}(\bar{S}_R), \xi)$ is a Fréchet, and consequently, it can be proved that it is a Silva space. Moreover, as it was shown again by the Taylor series at the origin for any f

$\in \mathcal{O}(\bar{S}_R)$, the sequence

$$\xi = (z^m w^n)_{m,n\in\mathbb{N}} \quad (101)$$

is a basis for $\mathcal{O}(\bar{S}_R)$. Although it is known that $(\mathcal{O}(\bar{S}_R), \xi)$ is a (NF)-space and then a (NS)-space, our criterion will yield the nuclearity of it in a very simple way. Indeed, as for each $k, m, n \in \mathbb{N}$, it can be proved that (see [2])

$$\|z^m w^n\|_k = \frac{r_k^{m+n}}{\sigma_{m,n}}, \quad (102)$$

where $\sigma_{m,n}$ is given by

$$\sigma_{m,n} = \begin{cases} \dfrac{(m+n)^{(1/2)(m+n)}}{m^{m/2} n^{n/2}} & \text{when } m, n > 0, \\ 1 & \text{when } m = 0 \text{ or } n = 0. \end{cases} \quad (103)$$

Then, we obtain that, taking $k \in \mathbb{N}$ fixed, for each $\ell > K$,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\|z^m w^n\|_k}{\|z^m w^n\|_\ell} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{r_k}{r_\ell}\right)^{m+n} = \left(\frac{r_\ell}{r_\ell - r_k r_\ell}\right)^2 < +\infty, \quad (104)$$

the nuclearity of this space is proved.

10. General Remarks and Comments

The properties of series of the form $\sum_{i=0}^{\infty} c_i P_i(z)$, $z \in \mathbb{C}$ where $P_i(z)$, $i = 0, 1, \cdots$ are prescribed polynomials and e_i chosen in a field $\mathbb{K}$ of scalars, widely differ according to the particular chosen polynomials. For example, the region of convergence (which is called the region of effectiveness) may be a circle (for Taylor series), an ellipse (for series of Legendre polynomials), and a half-plane (for Newton's interpolation series). Whittaker [40], in his attempt to find the common properties exhibited by all these polynomials, introduced the notion of basic sets of polynomials. In his work [5], he defined the basic sets, basic series, and effectiveness of basic sets. In [15–17], Cannon obtained the necessary and sufficient condition for the effectiveness of basic sets for classes of functions of finite radii of regularity and entire functions. In the classical treatment of the subject of basic sets [5], the methods for establishing effectiveness depend on the region of effectiveness and the class of functions for which the set is effective.

The first attempt at some uniformity among the different methods was made by Newns who gave in [4] a topological approach leading to a general theory of effectiveness. It is well known that a lot of classical function spaces are important examples of so-called nuclear Fréchet spaces, for example, spaces of null solutions of elliptic partial differential operators with constant coefficients such as the Cauchy-Riemann operator and the Laplace operator.

On an abstract level, the problem of effectiveness of basic sets of polynomials in spaces of holomorphic functions as introduced by Cannon-Whittaker may be therefore considered as being related to the problem of the change of bases in nuclear Fréchet spaces as well as in other related spaces.

In the present work, we show that general criteria for basis transforms are obtained for the nuclearity of Fréchet spaces with basis which are applied to characterize basis transforms in terms of infinite matrices in classes of nuclear Fréchet spaces. This study is considered to be a refinement of those given by Cannon, Whittaker, and Newns and all relevant generated topics.

In such a way analog results are given concerning nuclear Silva spaces with bases. This might form a starting point for further investigations regarding the basis transforms in more general locally convex spaces or higher dimensional spaces with different domains of convergence. Finally, it will be expected in the forthcoming work to study basis transforms in

(i) spaces involving product bases, inverse bases, transpose bases, derived and integrated bases, etc.

(ii) spaces of entire functions having finite growth

(iii) spaces of holomorphic functions in Faber regions

(iv) spaces of several complex variables

(v) monogenic function spaces in the framework of Clifford analysis.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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Research Article

Iterative Scheme for Split Variational Inclusion and a Fixed-Point Problem of a Finite Collection of Nonexpansive Mappings

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This article is aimed at introducing an iterative scheme to approximate the common solution of split variational inclusion and a fixed-point problem of a finite collection of nonexpansive mappings. It is proven that under some suitable assumptions, the sequences achieved by the proposed iterative scheme converge strongly to a common element of the solution sets of these problems. Some consequences of the main theorem are also given. Finally, the convergence analysis of the sequences achieved from the iterative scheme is illustrated with the help of a numerical example.

1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces endowed with inner product $\langle \cdot , \cdot \rangle$ and induced norm $\|\cdot\|$. A mapping $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is called contraction, if $\exists \kappa \in (0,1)$ such that $\|T(\varphi) - T(\psi)\| \leq \kappa\|\varphi - \psi\|$, $\forall \varphi, \psi \in \mathscr{H}_1$. If $\kappa = 1$, then T becomes nonexpansive. A mapping T is said to have a fixed point, if $\exists \varphi \in (\mathscr{H}_1)$ such that $T(\varphi) = (\varphi)$. Further, if $T_n : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$, $(n = 1,\cdots,M)$ is a finite collection of nonexpansive mappings. Then, the fixed-point problem (FPP) is defined as find $\varphi \in \mathscr{H}_1$ such that

$$\bigcap_{n=1}^{M} T_n(\varphi) = \varphi. \tag{1}$$

It is easy to show that if $\bigcap_{n=1}^{M} \mathrm{Fix}(T_n) \neq 0$, then $\bigcap_{n=1}^{M} \mathrm{Fix}(T_n)$ is closed and convex. Many iterative methods have been adopted to examine the solution of a fixed-point problem for nonexpansive mappings and its variant forms, see [1–5] and references therein.

We know that most of the techniques for solving the fixed-point problems can be acquired from Mann's iterative technique [3], namely, for arbitrary $x_0 \in \mathscr{C}$, compute

$$x_{k+1} = \alpha_k x_k + (1 - \alpha_k) T x_k, \quad k \geq 0, \tag{2}$$

where T is a nonexpansive mapping from a nonempty closed convex subset $\mathscr{C}$ of Hilbert space $\mathscr{H}_1$ to itself and α_n is a control sequence, which force $\{x_k\}$ to converge (weak) to a fixed point of T. To obtain the strong convergence result, Moudafi [4] proposed the viscosity approximation method by combining the nonexpansive mapping T with a contraction of given mapping f over $\mathscr{C}$. For an arbitrary $x_0 \in \mathscr{C}$, compute the sequence $\{x_k\}$ generated by

$$x_{k+1} = \alpha_k f(x_k) + (1 - \alpha_k) T x_k, \quad k \geq 0, \tag{3}$$

where $\alpha_n \in (0,1)$ goes slowly to zero. The sequence $\{x_{k+1}\}$ achieved from this iterative method converges strongly to a fixed point of T.

On the other hand, let us recall some work about split variational inequality/inclusion problems. A multivalued mapping $G : \mathscr{H}_1 \longrightarrow 2^{\mathscr{H}_1}$ is called maximal monotone, if its graph $\text{gph}(G) = \{(\varphi, \psi) \in \mathscr{H}_1 \times \mathscr{H}_1 : \psi \in G(\varphi)\}$ is not properly contained by the graph of any other monotone mapping. A monotone mapping G is maximal monotone if and only if for $(\varphi, \zeta) \in \mathscr{H}_1 \times \mathscr{H}_1$, $\langle \varphi - \psi, \zeta - \vartheta \rangle \geq 0$ for every $(\psi, \vartheta) \in \text{gph}(G)$ implies that $\zeta \in G(\varphi)$. If G is maximal monotone, then operator $J_\lambda^G = (I + \lambda G)^{-1}$ is well defined, nonexpansive, and known as the resolvent of G with parameter $\lambda > 0$, which is defined at every point of the domain.

The idea of split variational inequality problem (SVIP) given by Censor et al. [6], which amounts to saying find a solution of variational inequality whose image, under a given bounded linear operator, solves another variational inequality. Find $\varphi^* \in \mathscr{C}$ such that

$$\langle h(\varphi^*), \varphi - \varphi^* \rangle \geq 0, \quad \forall \varphi \in \mathscr{C}, \tag{4}$$

and such that

$$\psi^* = A\varphi^* \in \mathscr{D} \text{ solves} \langle g(\psi^*), \psi - \psi^* \rangle \geq 0, \quad \forall \psi \in \mathscr{D}, \tag{5}$$

where $\mathscr{C}$ and $\mathscr{D}$ are closed, convex subsets of Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively; $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ is a bounded linear operator, and $h : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ and $g : \mathscr{H}_2 \longrightarrow \mathscr{H}_2$ are two operators. They studied the weak convergent result to solve SVIP.

Moudafi [7] generalized SVIP and introduced split monotone variational inclusion problem (S_pMVIP): find $\varphi^* \in \mathscr{H}_1$ such that

$$0 \in h(\varphi^*) + G_1(\varphi^*), \tag{6}$$

and such that

$$\psi^* = A\varphi^* \in \mathscr{H}_2 \text{ solves } 0 \in g(\psi^*) + G_2(\psi^*), \tag{7}$$

where $G_1 : \mathscr{H}_1 \longrightarrow 2^{\mathscr{H}_1}$ and $G_2 : \mathscr{H}_2 \longrightarrow 2^{\mathscr{H}_2}$ are multivalued monotone mappings, $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ is a bounded linear operator, $h : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ and $g : \mathscr{H}_2 \longrightarrow \mathscr{H}_2$ are two single-valued operators. The author also composed an iterative algorithm to solve (S_pMVIP) and showed that the sequence achieved by the proposed algorithm converges weakly to the solution of (S_pMVIP). Numerous iterative methods have been investigated for split variational inequality/inclusion problems, split common fixed-point problems, split feasibility problems, and split zero problems and their generalizations, see [6, 8–16] and references therein.

If $h = g = 0$ in S_pMVIP, then we obtain the split variational inclusion problem (S_pVIP) considered in [8], stated as find $\varphi^* \in \mathscr{H}_1$ such that

$$0 \in G_1(\varphi^*) \tag{8}$$

such that

$$\psi^* = A\varphi^* \in \mathscr{H}_2 \text{ solves } 0 \in G_2(\psi^*). \tag{9}$$

Byrne et al. [8] proposed the following iterative scheme for S_pVIP and studied the strong and weak convergence. For arbitrary $x_o \in \mathscr{H}_1$, compute the iterative sequence achieved by the following scheme:

$$x_{k+1} = J_\lambda^{G_1}\left[\left(x_k + \mu A^*\left(J_\lambda^{G_2} - I\right)Ax_k\right)\right], \tag{10}$$

for $\lambda > 0$.

Recently, Kazmi and Rizvi [17] suggested and examined an iterative algorithm to estimate the common solution for S_pVIP and a fixed-point problem of a nonexpansive mapping in Hilbert spaces. Puangpee and Sauntai [18] studied the split variational inclusion problem and fixed-point problem in Banach spaces. Haitao and Li [19] investigated the split variational inclusion problem and fixed-point problem of nonexpansive semigroup without prior calculation of operator norm. Later, many authors studied the common solution of split variational inequality/inclusion problem and fixed-point problem of nonexpansive mappings in the framework of Hilbert/Banach spaces, see for example [18–24] and references therein.

Following the works in [4, 7, 8, 17] and by the current research in this flow, we propose an iterative scheme to approximate a common solution of FPP and S_pVIP. We prove that the sequences achieved by the proposed iterative scheme strongly converge to the common solution of FPP and S_pVIP. The iterative scheme and results discussed in this article are new and can be viewed as generalization and refinement of the previously published work in this area.

2. Prelude and Auxiliary Results

In this section, we assembled some underlying definitions and supporting results.

Definition 1. Let $\mathscr{C}(\mathscr{C} \subset \mathscr{H}_1)$, the metric projection $P_\mathscr{C}$ onto the set $\mathscr{C}$ is defined as $P_\mathscr{C}(\vartheta) \in \mathscr{C}$ and $\|\vartheta - P_\mathscr{C}(\vartheta)\| = \inf_{\vartheta \in \mathscr{C}} \|\vartheta - \omega\|$, $\forall \omega \in \mathscr{H}_1$.

$P_\mathscr{C}$ is also characterised by the facts that $P_\mathscr{C}(\vartheta) \in \mathscr{C}$,

$$\langle \vartheta - P_\mathscr{C}(\vartheta), \omega - P_\mathscr{C}(\vartheta) \rangle \leq 0 \tag{11}$$

and

$$\|\vartheta - \omega\|^2 \geq \|\vartheta - P_\mathscr{C}(\vartheta)\|^2 + \|\omega - P_\mathscr{C}(\vartheta)\|^2, \quad \forall \vartheta \in \mathscr{H}_1, \omega \in \mathscr{C}. \tag{12}$$

Remark 2 (see [25, 26]). For a nonexpansive mapping T and projection $P_\mathscr{C}(\vartheta)$ onto $\mathscr{C}$, the following results hold in Hilbert spaces:

(i)

$$\langle \vartheta - \omega, P_{\mathscr{C}}(\vartheta) - P_{\mathscr{C}}(\omega)\rangle \geq \|P_{\mathscr{C}}(\vartheta) - P_{\mathscr{C}}(\omega)\|^2, \quad \forall \vartheta, \omega \in \mathscr{H}_1 \tag{13}$$

(ii) For all $(\vartheta, \omega) \in \mathscr{H}_1 \times \mathscr{H}_1$,

$$\begin{aligned}&\langle(\vartheta - T(\vartheta)) - (\omega - T(\omega)), T(\omega) - T(\vartheta)\rangle \\ &\quad \leq \frac{1}{2}\|(T(\vartheta) - \vartheta) - (T(\omega) - (\omega))\|^2.\end{aligned} \tag{14}$$

Thus, for all $(\vartheta, \omega) \in \mathscr{H}_1 \times \mathrm{Fix}(T)$, we get

$$\langle \vartheta - T(\vartheta), \omega - T(\omega)\rangle \leq \frac{1}{2}\|T(\vartheta) - \vartheta\|^2 \tag{15}$$

(iii) For all $\vartheta, \omega \in \mathscr{H}_1$, $t \in (0, 1)$

$$\begin{aligned}\|t\vartheta - (1-t)\omega\|^2 &= t\|\vartheta\|^2 + (1-t)\|\omega\|^2 \\ &\quad - t(1-t)\|\vartheta - \omega\|^2\end{aligned} \tag{16}$$

Definition 3. A mapping $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is said to be

(i) monotone, if

$$\langle T\vartheta - T\omega, \vartheta - \omega\rangle \geq 0, \quad \text{for all } \vartheta, \omega \in \mathscr{H}_1 \tag{17}$$

(ii) τ-strongly monotone, if there exists a constant $\tau > 0$ such that

$$\langle T\vartheta - T\omega, \vartheta - \omega\rangle \geq \tau\|\vartheta - \omega\|^2, \quad \forall \vartheta, \omega \in \mathscr{H}_1 \tag{18}$$

(iii) γ-inverse strongly monotone, if there exists a constant $\gamma > 0$ such that

$$\langle T\vartheta - T\omega, \vartheta - \omega\rangle \geq \gamma\|T\vartheta - T\omega\|^2, \quad \forall \vartheta, \omega \in \mathscr{H}_1 \tag{19}$$

(iv) firmly nonexpansive, if

$$\langle T\vartheta - T\omega, \vartheta - \omega\rangle \geq \|T\vartheta - T\omega\|^2, \quad \forall \vartheta, \omega \in \mathscr{H}_1 \tag{20}$$

Some important characteristics of an averaged operator are mentioned below; for more details, we refer to [7, 27, 28].

Definition 4. A mapping $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is called an averaged if and only if T is the average of identity mapping and a nonexpansive mapping, that is, $T = (1-t)I + tS$, where $t \in (0, 1)$ and $S : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is nonexpansive.

Thus, firmly nonexpansive mappings are averaged. It can also be seen that averaged mappings are nonexpansive.

Proposition 5.

(i) Let $S : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ be an averaged and $V : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ be a nonexpansive mapping, then $T = (1-t)S + tV$ is averaged for $t \in (0, 1)$

(ii) If the composite $\{T_n\}_{n=1}^M$ is averaged and have a nonempty common fixed point, then

$$\bigcap_{n=1}^{M} Fix(T_n) = Fix(T_1 T_2 \cdots \cdots \cdots T_M) \tag{21}$$

(iii) If T is γ-inverse strongly monotone, then for $r > 0$, rT is r/γ-inverse strongly monotone

(iv) T is averaged if its compliment $I - T$ is γ-inverse strongly monotone for some $\gamma > (1/2)$

Lemma 6 (see [29]). *Assume that T is nonexpansive self-mapping of a closed convex subset $\mathscr{D}$ of a Hilbert space $\mathscr{H}_1$. If T has a fixed point, then $I - T$ is demiclosed, i.e., whenever $\{\omega_n\}$ is a sequence in $\mathscr{D}$ converging weakly to some $\omega \in \mathscr{D}$ and the sequence $\{(I - T)\omega_n\}$ converges strongly to some ϖ, then $(I - T)\omega = \varpi$, where I is the identity mapping on $\mathscr{H}_1$.*

Lemma 7 (see [5]). *If $\{v_k\}$ is a sequence of nonnegative real numbers such that*

$$v_{k+1} \leq (1 - \xi_n)v_k + \omega_k, \quad k = 0, 1, 2\cdots, \tag{22}$$

where $\{\xi_k\}$ is a sequence in (0,1) and $\{\omega_k\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{k=1}^{\infty}\xi_k = \infty$

(ii) $\limsup_{k\to\infty}(\omega_k/\xi_k) \leq 0$ or $\limsup_{k\to\infty}|\omega_k| < \infty$

Then, $\lim_{k\to\infty} v_k = 0$.

We denote the solution set of S_pVIP by $\Xi = \{\varphi^* \in \mathscr{H}_1 : 0 \in G_1(\varphi^*)$ and $0 \in G_2(A\varphi^*)\}$ and of FPP by $\cap_{n=1}^{M} Fix(T_n)$.

3. Iterative Scheme and Its Convergence

In this section, we present the iterative scheme and show that the sequences obtained from the proposed iterative scheme converge strongly to the common solution of FPP and S_pVIP.

For integer $K \geq 1$, we define the mapping $T_{[K]} = T_{K \bmod M}$ with the mod function, which is taking values from the set $\{1, 2, \cdots, M\}$, that is, if $K = aM + b$ for some integer $a \geq 0$ and $0 \leq b \leq M$, then $T_{[K]} = T_M$ if $b = 0$ and $T_{[K]} = T_b$ if $0 < b < M$.

Iterative Scheme 8. Step 0. Take $\{\alpha_k\} \subset (0, 1)$. Choose $u_0 \in \mathscr{H}_1$ arbitrary and let $k = 0$.

Step 1. Given $u_k \in \mathscr{H}_1$, compute $u_{k+1} \in \mathscr{H}_1$ as

$$\begin{aligned} v_k &= J_\lambda^{G_1}\left[u_k + \mu A^*\left(J_\lambda^{G_2} - I\right)Au_k\right], \\ u_{k+1} &= \alpha_k f(u_k) + (1-\alpha_k)T_{[k+1]}v_k, \end{aligned} \tag{23}$$

update $k = k+1$ and go to Step 1.

Condition C. We assume that $T_n : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$, $(n = 1, 2, \cdots, M)$, is a finite number of nonexpansive mappings such that $\cap_{n=1}^M \mathrm{Fix}(T_n) \neq 0$ and

$$\begin{aligned} \bigcap_{n=i}^{M} \mathrm{Fix}(T_n) &= \mathrm{Fix}(T_1 \circ T_2 \circ \cdots \circ T_M) = \mathrm{Fix}(T_M \circ T_1 \circ \cdots \circ T_{M-1}) \\ &= \cdots = \mathrm{Fix}(T_2 \circ T_3 \circ \cdots \circ T_M \circ T_1). \end{aligned} \tag{24}$$

Lemma 9. *$\varphi^* \in \mathscr{H}_1$ and $\psi^* = A\varphi^*$ are solutions of S_pVIP, if and only if $\varphi^* = J_\lambda^{G_1}(\varphi^*)$ and $\psi^* = A\varphi^* = J_\lambda^{G_2}(\psi^*)$, for some $\lambda > 0$.*

Proof. The proof of the lemma follows immediately from the definitions of resolvent operators.

Remark 10. If J_λ^G is the resolvent of maximal monotone mapping G, A^* is the adjoint operator of A and $\mathscr{R}$ is the spectral radius of AA^*. Then, using the properties of averaged mapping, one can easily show that the operator $[I + \mu A^*(J_\lambda^G - I)A]$ is averaged with $\lambda > 0$, $\mu \in (0, 1/\mathscr{R})$.

Now, we prove the following lemma which guarantees the contractivity of L, which is needed to prove our main result.

Lemma 11. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces and $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be a bounded linear operator. Suppose that $G_1 : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ and $G_2 : \mathscr{H}_2 \longrightarrow \mathscr{H}_2$ are maximal monotone operators and $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ be a nonexpansive mapping. Let $f : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ be a κ-contraction mapping with constant $\kappa > 0$. For any $\theta \in (0, 1]$, we define a mapping on $\mathscr{H}_1$ by*

$$L(\vartheta) = \theta f(\vartheta) + (1-\theta)T\left[J_\lambda^{G_1}\left(\vartheta + \mu A^*\left(J_\lambda^{G_2} - I\right)A\vartheta\right)\right], \quad \forall \vartheta \in \mathscr{H}_1, \tag{25}$$

where $\mu \in (0, 1/\mathscr{R})$, $\mathscr{R}$ is the spectral radius of the operator AA^*, and A^* is the adjoint operator of A. Then, the mapping L is a contraction with constant $0 < 1 - \theta(1-\kappa) < 1$; hence, L has a unique fixed point.

Proof. The operators $J_\lambda^{G_1}$ and $J_\lambda^{G_2}$ are averaged being firmly nonexpansive. For $\mu \in (0, 1/\mathscr{R})$, the operators $[I + \mu A^*(J_\lambda^{G_2} - I)A]$ and $J_\lambda^{G_1}(I + \mu A^*(J_\lambda^{G_2} - I)A)$ are averaged and hence nonexpansive. Thus, for all $u, v \in \mathscr{H}_1$, we have

$$\begin{aligned} \|L(\vartheta) - L(\omega)\| &= \left\|\theta f(\vartheta) + (1-\theta)T\left[J_\lambda^{G_1}\left(\vartheta + \mu A^*\left(J_\lambda^{G_2} - I\right)A\vartheta\right)\right]\right. \\ &\quad \left. - \theta f(\omega) - (1-\theta)T\left[J_\lambda^{G_1}\left(\omega + \mu A^*\left(J_\lambda^{G_2} - I\right)A\omega\right)\right]\right\| \\ &\le \theta\|f(\vartheta) - f(\omega)\| + (1-\theta)\|T\vartheta - T\omega\| \\ &\le \theta\kappa\|\vartheta - \omega\| + (1-\theta)\|\vartheta - \omega\| \\ &= [1 - \theta(1-\kappa)]\|\vartheta - \omega\|. \end{aligned} \tag{26}$$

Since $0 < 1 - \theta(1-\kappa) < 1$ implies that L is a contraction, hence L has a unique fixed point.

Theorem 12. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces and $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be a bounded linear operator. Assume that $G_1 : \mathscr{H}_1 \longrightarrow 2^{\mathscr{H}_1}$ and $G_2 : \mathscr{H}_2 \longrightarrow 2^{\mathscr{H}_2}$ are two maximal monotone operators and $f : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is a contraction with constant $\kappa \in (0, 1)$. Let $T_n : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$, $(n = 1, 2, \cdots, M)$, be a finite collection of nonexpansive mappings satisfying the Condition C such that $\cap_{n=1}^M Fix(T_n) \cap \Xi \neq 0$. Let $\mathscr{R}$ be a spectral radius of $A^\star A$, where $A^\star$ is the adjoint of A such that $\mu \in (0, 1/\mathscr{R})$ and $\{\alpha_k\}$ be a sequence in (0,1) such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^\infty \alpha_k = \infty$, and $\sum_{k=1}^\infty |\alpha_k - \alpha_{k-1}| < \infty$. Then, the iterative sequences $\{v_k\}$ and $\{u_k\}$ generated by Iterative Scheme 8 converge strongly to $\bar{v} \in \cap_{n=1}^M Fix(T_n) \cap \Xi$, where $\bar{v} = P_{\cap_{n=1}^M Fix(T_n)\cap\Xi} f(\bar{v})$.*

Proof. Let $u^\star \in \cap_{n=1}^M \mathrm{Fix}(T_n) \cap \Xi$, then we have $J_\lambda^{G_1}u^\star = u^\star$, $J_\lambda^{G_2}Au^\star = Au^\star$, and $T_n(u^\star) = u^\star$, $(n = 1, 2, \cdots M)$, then using Iterative Scheme 8, we evaluate

$$\begin{aligned} \|v_k - u^\star\|^2 &= \left\|J_\lambda^{G_1}\left(u_k + \mu A^*\left(J_\lambda^{G_2} - I\right)Au_k\right) - u^\star\right\|^2 \\ &= \left\|J_\lambda^{G_1}\left(u_k + \mu A^*\left(J_\lambda^{G_2} - I\right)Au_k\right) - J_\lambda^{G_1}u^\star\right\|^2 \\ &\le \|u_k - u^\star\|^2 + \mu^2\left\|A^*\left(J_\lambda^{G_2} - I\right)Au_k\right\|^2 \\ &\quad + 2\mu\left\langle u_k - u^\star, A^*\left(J_\lambda^{G_2} - I\right)Au_k\right\rangle \\ &= \|u_k - u^\star\|^2 + \mu^2\left\langle\left(J_\lambda^{G_2} - I\right)Au_k, AA^*\left(J_\lambda^{G_2} - I\right)Au_k\right\rangle \\ &\quad + 2\mu\left\langle u_k - u^\star, A^*\left(J_\lambda^{G_2} - I\right)Au_k\right\rangle \\ &\le \|u_k - u^\star\|^2 + \mu^2\mathscr{R}\left\|\left(J_\lambda^{G_2} - I\right)Au_k\right\|^2 \\ &\quad + 2\mu\left\langle u_k - u^\star, A^*\left(J_\lambda^{G_2} - I\right)Au_k\right\rangle. \end{aligned} \tag{27}$$

Denoting $\nabla = 2\mu\langle u_k - u^\star, A^*(J_\lambda^{G_2} - I)Au_k\rangle$ and using (16), we have

$$\begin{aligned}\nabla &= 2\mu\left\langle A(u_k-u^\star),\left(J_\lambda^{G_2}-I\right)Au_k\right\rangle\\&= 2\mu\left\langle A(u_k-u^\star)+\left(J_\lambda^{G_2}-I\right)Au_k\right.\\&\quad\left.-\left(J_\lambda^{G_2}-I\right)Au_k,\left(J_\lambda^{G_2}-I\right)Au_k\right\rangle\\&= 2\mu\left\{\left\langle J_\lambda^{G_2}Au_k-Au^\star,\left(J_\lambda^{G_2}-I\right)Au_k\right\rangle-\left\|\left(J_\lambda^{G_2}-I\right)Au_k\right\|^2\right\}\\&\le 2\mu\left\{\frac{1}{2}\left\|\left(J_\lambda^{G_2}-I\right)Au_k\right\|^2-\left\|\left(J_\lambda^{G_2}-I\right)Au_k\right\|^2\right\}\\&\le -\mu\left\|\left(J_\lambda^{G_2}-I\right)Au_k\right\|^2,\end{aligned}\tag{28}$$

using (28), (27) becomes

$$\|v_k-u^\star\|^2\le\|u_k-u^\star\|^2+\mu(\mathscr{R}\mu-1)\left\|\left(J_\lambda^{G_2}-I\right)Au_k\right\|^2.\tag{29}$$

Since $\mu\in(0,1/\mathscr{R})$, we obtain

$$\|v_k-u^\star\|^2\le\|u_k-u^\star\|^2.\tag{30}$$

Now, we show that $\{u_k\}$ is bounded.

$$\begin{aligned}\|u_{k+1}-u^\star\| &= \left\|\alpha_k f(u_k)+(1-\alpha_k)T_{[k+1]}v_k-u^\star\right\|\\&\le \alpha_k\|f(u_k)-u^\star\|+(1-\alpha_k)\left\|T_{[k+1]}v_k-u^\star\right\|\\&\le \alpha_k\|f(u_k)-f(u^\star)\|+\alpha_k\|f(u^\star)-u^\star\|\\&\quad+(1-\alpha_k)\left\|T_{[k+1]]}v_k-T_{[k+1]}u^\star\right\|\\&\le \alpha_k\kappa\|u_k-u^\star\|+\alpha_k\|f(u^\star)-u^\star\|\\&\quad+(1-\alpha_k)\|v_k-u^\star\|\\&= [1-\alpha_k(1-\kappa)]\|u_k-u^\star\|+\alpha_k\|f(u^\star)-u^\star\|\\&\le \max\left\{\|u_k-u^\star\|,\frac{\|f(u^\star)-u^\star\|}{1-\kappa}\right\}\\&\quad\cdots\\&\le \max\left\{\|u_0-u^\star\|,\frac{\|f(u^\star)-u^\star\|}{1-\kappa}\right\}.\end{aligned}\tag{31}$$

Hence, $\{u_k\}$ is bounded, which implies that the sequences $\{v_k\}$, $\{f(u_k)\}$, and $\{T_{[k+1]}v_k\}$ are also bounded. It follows from nonexpansiveness of T_n, $(n=1,\cdots M)$, and Lipschitz continuity of f with constant κ that

$$\begin{aligned}\|u_{k+M}-u_k\| &= \left\|\alpha_{k+M-1}f(u_{k+M-1})+(1-\alpha_{k+M-1})T_{[k+M]}(v_{k+M-1})\right.\\&\quad\left.-\alpha_{k-1}f(u_{k-1})+(1-\alpha_{k-1})T_{[k]}(v_{k-1})\right\|\\&= \left\|\alpha_{k+M-1}f(u_{k+M-1})-\alpha_{k+M-1}f(u_{k-1})\right.\\&\quad+\alpha_{k+M-1}f(u_{k-1})+(1-\alpha_{k+M-1})T_{[k]}(v_{k-1})\\&\quad-(1-\alpha_{k+M-1})T_{[k]}(v_{k-1})\\&\quad+(1-\alpha_{k+M-1})T_{[k+M]}(v_{k+M-1})-\alpha_{k-1}f(u_{k-1})\\&\quad\left.+(1-\alpha_{k-1})T_{[k]}(v_{k-1})\right\|\\&\le (\alpha_{k+M-1})\|f(u_{k+M-1})-f(u_{k-1})\|\\&\quad+\|f(u_{k-1})\||\alpha_{k+M-1}-\alpha_{k-1}|\\&\quad+(1-\alpha_{k+M-1})\left\|T_{[k+M]}(v_{k+M-1})-T_{[k]}(v_{k-1})\right\|\\&\quad+\left\|T_{[k]}(v_{k-1})\right\||\alpha_{k-1}-\alpha_{k+M-1}|,\end{aligned}\tag{32}$$

that is,

$$\begin{aligned}\|u_{k+M}-u_k\| &= \kappa(\alpha_{k+M-1})\|u_{k+M-1}-u_{k-1}\|\\&\quad+(1-\alpha_{k+M-1})\|v_{k+M-1}-v_{k-1}\|\\&\quad+2|\alpha_{k+M-1}-\alpha_{k-1}|M_1,\end{aligned}\tag{33}$$

where $M_1=\sup\{\|f(u_{k-1})\|+\|T_{[k]}(v_{k-1})\|: k\in\mathbb{N}\}$. Since, $\mu\in(0,1/\mathscr{R})$, the operator $J_\lambda^{G_1}[I+\mu A^*(J_\lambda^{G_2}-I)A]$ is average and hence nonexpansive, then we have

$$\begin{aligned}\|v_{k+M-1}-v_{k-1}\| &= \left\|J_\lambda^{G_1}\left(u_{k+M-1}+\mu A^*\left(J_\lambda^{G_2}-I\right)Au_{k+M-1}\right)\right.\\&\quad\left.-J_\lambda^{G_1}\left(u_{k-1}+\mu A^*\left(J_\lambda^{G_2}-I\right)Au_{k-1}\right)\right\|\\&\le \left\|J_\lambda^{G_1}\left[I+\mu A^*\left(J_\lambda^{G_2}-I\right)A\right]u_{k+M-1}\right.\\&\quad\left.-J_\lambda^{G_1}\left[I+\mu A^*\left(J_\lambda^{G_2}-I\right)A\right]u_{k-1}\right\|\\&\le \|u_{k+M-1}-u_{k-1}\|.\end{aligned}\tag{34}$$

From (34), (33) becomes

$$\begin{aligned}\|u_{k+M}-u_k\| &\le [1-\alpha_{k+M-1}(1-\kappa)]\|u_{k+M-1}-u_{k-1}\|\\&\quad+2|\alpha_{k+M-1}-\alpha_{k-1}|M_1,\end{aligned}\tag{35}$$

let $\xi_k=\alpha_{k+M-1}(1-\kappa)$, $\omega_k=2|\alpha_{k+M-1}-\alpha_{k-1}|M_1$, by using Lemma 7, we conclude that

$$\lim_{k\to\infty}\|u_{k+M}-u_k\|=0.\tag{36}$$

Now, we show that $\|u_k - v_k\| \longrightarrow 0$ as $k \longrightarrow \infty$. From (29), it follows that

$$\begin{aligned}\|u_{k+1} - u^\star\|^2 &= \left\|\alpha_k f(u_k) + (1-\alpha_k) T_{[k+1]} v_k - u^\star\right\|^2 \\ &\le \alpha_k \|f(u_k) - u^\star\|^2 + (1-\alpha_k)\left\|T_{[k+1]} v_k - T_{[k+1]} u^\star\right\|^2 \\ &\le \alpha_k \|f(u_k) - u^\star\|^2 + (1-\alpha_k)\|v_k - u^\star\|^2 \\ &\le \alpha_k \|f(u_k) - u^\star\|^2 + (1-\alpha_k) \\ &\quad \cdot \left[\|u_k - u^\star\|^2 + \mu(\mathcal{R}\mu - 1)\left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\|^2\right].\end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}&\mu(-\mathcal{R}\mu + 1)\left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\|^2 \\ &\quad \le \alpha_k \|f(u_k) - u^\star\|^2 + \|u_k - u^\star\|^2 - \|u_{k+1} - u^\star\|^2 \\ &\quad = \alpha_k \|f(u_k) - u^\star\|^2 + \|u_{k+1} - u_k\|[\|u_k - u^\star\| + \|u_{k+1} - u^\star\|].\end{aligned} \tag{38}$$

Since $(1 - \mathcal{R}\mu) > 0$, $\alpha_k \longrightarrow 0$, and $\|u_{k+1} - u_k\| \longrightarrow 0$, we get

$$\left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\| \longrightarrow 0, \text{as } k \longrightarrow \infty. \tag{39}$$

Since $\mu \in (0(1/\mathcal{R}))$ and using (27) and (29), we obtain

$$\begin{aligned}\|v_k - u^\star\|^2 &= \left\|J_\lambda^{G_1}\left(u_k + \mu A^*\left(J_\lambda^{G_2} - I\right) A u_k\right) - u^\star\right\|^2 \\ &= \left\|J_\lambda^{G_1}\left(u_k + \mu A^*\left(J_\lambda^{G_2} - I\right) A u_k\right) - J_\lambda^{G_1} u^\star\right\|^2 \\ &\le \left\langle v_k - u^\star, u_k + \mu A^*\left(J_\lambda^{G_2} - I\right) A u_k - u^\star\right\rangle \\ &= \frac{1}{2}\left[\|v_k - u^\star\|^2 + \left\|u_k + \mu A^*\left(J_\lambda^{G_2} - I\right) A u_k - u^\star\right\|^2\right. \\ &\quad \left. - \left\|(v_k - u^\star) - \left[u_k + \mu A^*\left(J_\lambda^{G_2} - I\right) A u_k - u^\star\right]\right\|^2\right] \\ &= \frac{1}{2}\left[\|v_k - u^\star\|^2 + \|u_k - u^\star\|^2 + \mu(\mathcal{R}\mu - 1)\right. \\ &\quad \left. \cdot \left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\|^2 - \left\|v_k - u_k - \mu A^*\left(J_\lambda^{G_2} - I\right) A u_k\right\|^2\right] \\ &\le \frac{1}{2}\left[\|v_k - u^\star\|^2 + \|u_k - u^\star\|^2 - \left[\|v_k - u_k\|^2 + \mu^2\|A^*\right.\right. \\ &\quad \left.\left. \cdot \left(J_\lambda^{G_2} - I\right) A u_k\right\|^2 - 2\mu\left\langle v_k - u_k, A^*\left(J_\lambda^{G_2} - I\right) A u_k\right\rangle\right]\right] \\ &\le \frac{1}{2}\left[\|v_k - u^\star\|^2 - \|v_k - u_k\|^2 + \|u_k - u^\star\|^2\right. \\ &\quad \left. + 2\mu\|A(v_k - u_k)\|\left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\|\right].\end{aligned} \tag{40}$$

Thus, we get

$$\begin{aligned}\|v_k - u^\star\|^2 &\le \|u_k - u^\star\|^2 + 2\mu\|A(v_k - u_k)\|\left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\| \\ &\quad - \|v_k - u_k\|^2.\end{aligned} \tag{41}$$

By (37) and (41), we have

$$\begin{aligned}\|u_{k+1} - u^\star\|^2 &\le \alpha_k \|f(u_k) - u^\star\|^2 + (1-\alpha_k) \\ &\quad \cdot \left[2\mu\|A(v_k - u_k)\|\left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\|\right. \\ &\quad \left. + \|u_k - u^\star\|^2 - \|v_k - u_k\|^2\right] \\ &\le \alpha_k \|f(u_k) - u^\star\|^2 + 2\mu\|A(v_k - u_k)\| \\ &\quad \cdot \left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\| + \|u_k - u^\star\|^2 - \|v_k - u_k\|^2,\end{aligned} \tag{42}$$

that is,

$$\begin{aligned}\|v_k - u_k\|^2 &\le \alpha_k \|f(u_k) - u^\star\|^2 + 2\mu\|A(v_k - u_k)\| \\ &\quad \cdot \left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\| + \|u_k - u^\star\|^2 - \|u_{k+1} - u^\star\|^2 \\ &\le \alpha_k \|f(u_k) - u^\star\|^2 + 2\mu\|A(v_k - u_k)\|\left\|\left(J_\lambda^{G_2} - I\right) A u_k\right\| \\ &\quad + (\|u_k - u^\star\| + \|u_{k+1} - u^\star\|)(\|u_{k+1} - u_k\|).\end{aligned} \tag{43}$$

Since $\alpha_k \longrightarrow 0$, $\|u_{k+1} - u_k\| \longrightarrow 0$, $\|(J_\lambda^{G_2} - I) A u_k\| \longrightarrow 0$ as $k \longrightarrow \infty$, we get

$$\lim_{k\to\infty} \|v_k - u_k\| = 0. \tag{44}$$

We recognized that the following relation holds:

$$\begin{aligned}u_{k+M} - u_k &= u_{k+M} - T_{[k+M]}(v_{k+M-1}) \\ &\quad + T_{[k+M]}(v_{k+M-1}) - T_{[k+M]}(u_{k+M-1}) \\ &\quad + T_{[k+M]}(u_{k+M-1}) - T_{[k+M]} T_{[k+M-1]}(v_{k+M-2}) \\ &\quad + \\ &\quad \cdots \\ &\quad + T_{[k+M]} \circ \cdots \circ T_{[k+2]}(u_{k+1}) - T_{[k+M]} \circ \cdots \circ T_{[k+2]}(v_{k+1}) \\ &\quad + T_{[k+M]} \circ \cdots \circ T_{[k+2]}(v_{k+1}) - T_{[k+M]} \circ \cdots \circ T_{[k+1]}(u_k) \\ &\quad + T_{[k+M]} \circ \cdots \circ T_{[k+1]}(u_k) - T_{[k+M]} \circ \cdots \circ T_{[k+1]}(v_k) \\ &\quad + T_{[k+M]} \circ \cdots \circ T_{[k+1]}(v_k) - u_k.\end{aligned} \tag{45}$$

By Iterative Scheme 8, we can easily see that $\|u_{k+1} - T_{[k+1]}v_k\| \longrightarrow 0$ as $k \longrightarrow \infty$. From (44) and nonexpansiveness of $T_n (n = 1, 2,\cdots,M)$, it follows that

$$\begin{aligned} &\left\|u_{k+M} - T_{[k+M]}(v_{k+M-1})\right\| \longrightarrow 0 \\ &\left\|T_{[k+M]}(v_{k+M-1}) - T_{[k+M]}(u_{k+M-1})\right\| \longrightarrow 0 \\ &\left\|T_{[k+M]}(u_{k+M-1}) - T_{[k+M]} \circ T_{[k+M-1]}(v_{k+M-2})\right\| \longrightarrow 0 \\ &\vdots \\ &\left\|T_{[k+M]}\circ\cdots\circ T_{[k+2]}(v_{k+1}) - T_{[k+M]}\circ\cdots\circ T_{[k+1]}(u_k)\right\| \longrightarrow 0 \\ &\left\|T_{[k+M]}\circ\cdots\circ T_{[k+1]}(u_k) - T_{[k+M]}\circ\cdots\circ T_{[k+1]}(v_k)\right\| \longrightarrow 0. \end{aligned} \tag{46}$$

By using (36) and (45), we conclude that

$$\lim_{k\to\infty} \left\|T_{[k+M]} \circ T_{[k+M-1]}\circ\cdots\circ T_{[k+2]} \circ T_{[k+1]}(v_k) - u_k\right\| \longrightarrow 0. \tag{47}$$

Now, using (47) and (44), we write

$$\begin{aligned} &\left\|T_{[k+M]} \circ \quad T_{[k+M-1]}\circ\cdots\circ T_{[k+1]}(v_k) - v_k\right\| \\ &\le \left\|T_{[k+M]} \circ T_{[k+M-1]}\circ\cdots\circ T_{[k+1]}(v_k) - u_k\right\| \\ &\quad + \|v_k - u_k\| \longrightarrow 0, \end{aligned} \tag{48}$$

as $k \longrightarrow \infty$, that is,

$$\left\|T_{[k+M]} \circ T_{[k+M-1]}\circ\cdots\circ T_{[k+1]}(v_k) - u_k\right\| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{49}$$

Boundedness of $\{v_k\}$ implies that there exists a subsequence $\{v_{k_i}\}$ of $\{v_k\}$, converging weakly to w. Because the collection of mappings $\{T_n : 1 \le n \le M\}$ is finite, we can say for some integer $K \in \{1, 2,\cdots,N\}$

$$T_{[k_n]} \equiv T_K, \quad \forall n \ge 1. \tag{50}$$

Thus, from (49), we have

$$\left\|T_{[n+M]} \circ T_{[n+M-1]}\circ\cdots\circ T_{[n+1]}\left(v_{k_n}\right) - v_{k_n}\right\| \longrightarrow 0. \tag{51}$$

Therefore, using Lemma 6, we conclude that

$$w \in \text{Fix}\left(T_{[n+M]}\circ\cdots T_{[n+1]}\right). \tag{52}$$

Thus, by the assumptions of Condition C, we have $w \in \cap_{n=1}^{M}\text{Fix}(T_n)$. On the other hand,

$$\begin{aligned} v_{k_i} &= J_\lambda^{G_1}\left[u_{k_i} + \mu A^*\left(J_\lambda^{G_2} - I\right)Au_{k_i}\right] \\ u_{k_i} + \mu A^*\left(J_\lambda^{G_2} - I\right)Au_{k_i} &\in (I + \lambda G_1)v_{k_i} \\ \frac{u_{k_i} - v_{k_i} + \mu A^*\left(J_\lambda^{G_2} - I\right)Au_{k_i}}{\lambda} &\in G_1\left(v_{k_i}\right). \end{aligned} \tag{53}$$

We know that the graph of a maximal monotone operator is weakly strongly closed; hence, by taking $i \longrightarrow \infty$ and using (37) and (44), we get

$$0 \in G_1(w). \tag{54}$$

Since $\{u_k\}$, $\{v_k\}$ have the same asymptotical behaviour, $\{Au_{k_i}\}$ converges weakly to Aw. Therefore, by (39), the nonexpansive property of $J_\lambda^{G_2}$ and Lemma 6, we get $0 \in G_2 (Aw)$. Thus, $w \in \cap_{n=1}^{M}\text{Fix}(T_n) \cap \Xi$.

Now, we need to show that $\limsup_{k\to\infty}\langle f(\bar{v}) - \bar{v}, u_k - \bar{v}\rangle \le 0$, where $\bar{v} = P_{\cap_{n=1}^{M}\text{Fix}(T_n)\cap\Xi} f(\bar{v})$.

We have

$$\begin{aligned} \limsup_{k\to\infty}\langle f(\bar{v}) - \bar{v}, u_k - \bar{v}\rangle &= \limsup_{k\to\infty}\left\langle f(\bar{v}) - \bar{v}, T_{[k+1]}v_k - \bar{v}\right\rangle \\ &\le \langle f(\bar{v}) - \bar{v}, w - \bar{v}\rangle \le 0, \end{aligned} \tag{55}$$

since $\bar{v} \in \cap_{n=1}^{M}\text{Fix}(T_n) \cap \Xi$.

Finally, we show that $x_k \longrightarrow \bar{v}$

$$\begin{aligned} \|u_{k+1} - \bar{v}\|^2 &= \left\|\alpha_k f(u_k) + (1 - \alpha_k)T_{[k+1]}(v_k) - \bar{v}\right\|^2 \\ &= \left\langle \alpha_k f(u_k) + (1 - \alpha_k)T_{[k+1]}(v_k) - \bar{v}, u_{k+1} - \bar{v}\right\rangle \\ &= \alpha_k\langle f(u_k) - \bar{v}, u_{k+1} - \bar{v}\rangle + (1 - \alpha_k) \\ &\quad \cdot \left\langle T_{[k+1]}v_k - \bar{v}, u_{k+1} - \bar{v}\right\rangle \le \alpha_k\langle f(u_k) - \bar{v}, u_{k+1} - \bar{v}\rangle \\ &\quad + (1 - \alpha_k)\langle v_k - \bar{v}, u_{k+1} - \bar{v}\rangle \\ &\le \alpha_k\langle f(u_k) - f(\bar{v}), u_{k+1} - \bar{v}\rangle + \alpha_k\langle f(u_k) - \bar{v}, u_{k+1} - \bar{v}\rangle \\ &\quad + (1 - \alpha_k)\langle v_k - \bar{v}, u_{k+1} - \bar{v}\rangle \\ &\le \frac{\alpha_k}{2}\left\{\|f(u_k) - f(\bar{v})\|^2 + \|u_{k+1} - \bar{v}\|^2\right\} \\ &\quad + \alpha_k\langle f(\bar{v}) - \bar{v}, u_{k+1} - \bar{v}\rangle + \frac{(1 - \alpha_k)}{2} \\ &\quad \cdot \left\{\|u_k - \bar{v}\|^2 + \|u_{k+1} - \bar{v}\|^2\right\} \\ &\le \frac{1}{2}\left[1 - \alpha_k\left(1 - \kappa^2\right)\right]\|u_{k+1} - \bar{v}\|^2 + \frac{(1 - \alpha_k)}{2} \\ &\quad \cdot \|u_k - \bar{v}\|^2 + \frac{\alpha_k}{2}\|u_{k+1} - \bar{v}\|^2 \\ &\quad + \alpha_k\langle f(\bar{v}) - \bar{v}, u_{k+1} - \bar{v}\rangle, \end{aligned} \tag{56}$$

which implies that

$$\|u_{k+1}-\bar{v}\|^2 \le \left[1-\alpha_k\left(1-\kappa^2\right)\right]\|u_{k+1}-\bar{v}\|^2 + 2\alpha_k\langle f(\bar{v})-\bar{v}, u_{k+1}-\bar{v}\rangle. \quad (57)$$

From Lemma 7 and (55), we conclude that $u_k \longrightarrow \bar{v}$ and from $\|v_k-u_k\| \longrightarrow 0$, $v_k \rightharpoonup w \in \cap_{n=1}^M \mathrm{Fix}T_n \cap \Xi$, and $u_k \longrightarrow \bar{v}$ as $k \longrightarrow \infty$, we achieve that $\bar{v}=w$. This completes the proof.

4. Consequences

Suppose $\mathscr{C}$ and $\mathscr{D}$ are closed convex subsets of Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Then, find $u \in \mathscr{H}_1$ such that

$$u \in \mathscr{C} \text{ and } Au \in \mathscr{D}, \quad (58)$$

is called the split feasibility problem (SFP), where $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ is a bounded linear operator. Byrne [9] introduced the $\mathscr{C}\mathscr{D}$ algorithm to approximate the solution of (58):

$$u_{k+1} = P_{\mathscr{C}}(u_k + \mu A^*(P_{\mathscr{D}} - I)Au_k), \quad (59)$$

where $P_{\mathscr{C}}$ and $P_{\mathscr{D}}$ are orthogonal projections onto $\mathscr{C}$ and $\mathscr{D}$, respectively.

The split common fixed-point problem (SCFPP) is an extension of Problem (58), which has been widely investigated in the present scenario. The SCFPP is the inverse problem design to search a vector in a fixed-point set so that its image under a bounded linear operator corresponds to other fixed-point set, that is, find $u \in \mathscr{H}_1$ such that

$$u = W(u) \text{ and } Au = V(Au), \quad (60)$$

where $W : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ and $W : \mathscr{H}_2 \longrightarrow \mathscr{H}_2$ are nonexpansive mappings. By putting $W = P_{\mathscr{C}}$ and $W = P_{\mathscr{D}}$, in (59), we can have an iterative scheme, which converges to the solution of SCFPP.

We denote the solution set of SFP (58) and SCFPP (60) by Ψ, and Ω, respectively. The following corollaries are given as consequences of Theorem 12.

Corollary 13. *Let $\mathscr{C}$ and $\mathscr{D}$ be two closed convex subsets of Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be a bounded linear operator and $f : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ be a contraction mapping with constant $\kappa \in (0,1)$. Let $T_n : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$, $(n = 1, 2, \cdots, M)$, be a finite collection of nonexpansive mappings satisfying the condition C such that $\cap_{n=1}^M Fix(T_n) \cap \Psi \neq 0$. Let $\mathscr{R}$ be the spectral radius of A^*A, where A^* is the adjoint of A such that $\mu \in (0, 1/\mathscr{R})$ and $\{\alpha_k\}$ be a sequence in (0,1) with $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty}\alpha_k = \infty$, and $\sum_{k=1}^{\infty}|\alpha_k - \alpha_{k-1}| < \infty$. Then, the iterative sequences $\{v_k\}$ and $\{u_k\}$ generated by Iterative Scheme 8 with $J_\lambda^{G_1} = P_{\mathscr{C}}$ and $J_\lambda^{G_2} = P_{\mathscr{D}}$ converge to $\bar{v} \in \cap_{n=1}^M Fix(T_n) \cap \Psi$, where $\bar{v} = P_{\cap_{n=1}^M Fix(T_n)\cap\Psi} f(\bar{v})$.*

Table 1: Computation of iterative sequences of Iterative Scheme 8 for the choices of parameters $\lambda = 1/4$ and $\alpha_n = 1/3k$ and different initial points $u_0 = 5$ and $u_0 = -3$.

$\lambda = 1/4$		$u_0 = 5$		$u_0 = -3$
No. iter.	v_k	$uk+1$	v_k	$uk+1$
0	3.5	-0.48502	-0.25	-1.831663
1	-1.613765	-1.034683	-1.623747	-0.726077
2	-1.26012	-0.956449	-0.794558	-0.748519
3	-0.967337	-0.926583	-0.8113989	-0.775985
4	-0.944937	-0.989916	-0.8319989	-1.037605
5	-0.992437	-0.968237	-1.028204	-0.984733
6	-0.976178	-0.948843	-0.998550	-0.964922
7	-0.961632	-0.996486	-0.973792	-0.991042
8	-0.997364	-0.980131	-0.993282	-0.977974
9	-0.985098	-0.968597	-0.983481	-0.966999
10	-0.976448	-0.995792	-0.975249	-0.996349
11	-0.986844	-0.984495	-0.997262	-0.984711
12	-0.984495	-0.975650	-0.988533	-0.975811
13	-0.981737	-0.996719	-0.981858	-0.996662
14	-0.997539	-0.997635	-0.997496	-0.987612

Corollary 14. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces and $A : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be a bounded linear operator. Assume that $G_1 : \mathscr{H}_1 \longrightarrow 2^{\mathscr{H}_1}$ and $G_2 : \mathscr{H}_2 \longrightarrow 2^{\mathscr{H}_2}$ are maximal monotone operators and $f : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is a κ-contraction mapping with constant $\kappa \in (0,1)$. Let $T_n : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$, $(n = 1, 2, \cdots, M)$, be a finite collection of nonexpansive mappings satisfying the condition C such that $\cap_{n=1}^M Fix(T_n) \cap \Omega \neq 0$. Let $\mathscr{R}$ be spectral radius of A^*A, where A^* is the adjoint of A such that $\mu \in (0, 1/\mathscr{R})$ and $\{\alpha_k\}$ be a sequence in (0,1) with $\lim_{n\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty}\alpha_k = \infty$, and $\sum_{k=1}^{\infty}|\alpha_k - \alpha_{k-1}| < \infty$. Then, the iterative sequences $\{v_k\}$ and $\{u_k\}$ obtained from Iterative Scheme 8 with $J_\lambda^{G_1} = W$ and $J_\lambda^{G_2} = V$ converge to $\bar{v} \in \cap_{n=1}^M Fix(T_n) \cap \Omega$, where $\bar{v} = P_{\cap_{n=1}^M Fix(T_n)\cap\Omega} f(\bar{v})$.*

Remark 15. If we take $T_1 = T_2 = \cdots T_M = T$, a nonexpansive mapping, then we can obtain the iterative scheme and its convergence theorem for the common solution of S_pVIP and a nonexpansive mapping T, studied in [17].

At last, we illustrate the convergence analysis of the proposed iterative scheme with the help of the following numerical example.

5. Numerical Example

Let $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$ and $G_1 : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ defined as $G_1(u) = 2(u+1)$ and $G_2 : \mathscr{H}_2 \longrightarrow \mathscr{H}_2$ defined as $G_2(u) = -(4/5)u + (12/5)$. For $\lambda = 1/4$, we compute the resolvents of G_1 and G_2 as

$$J_\lambda^{G_1}(u) = [I + \lambda G_1]^{-1}(u) = \frac{2}{3}u - \frac{1}{3}, \quad J_\lambda^{G_2}(u) = [I + \lambda G_2]^{-1}(u) = \frac{5}{4}u - \frac{3}{4}. \quad (61)$$

It can be easily seen that, here, $\Xi = \{-1\}$. Further, let T_1, T_2, and $T_3 : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ are three nonexpansive mappings, defined by

$$T_1(u) = \sin(u+1) - 1,$$
$$T_2(u) = \frac{-u-3}{2}, \tag{62}$$
$$T_3(u) = \frac{\cos(\pi u) + u}{2}$$

such that

$$\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) = \{-1\}. \tag{63}$$

Let $f : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ be a contraction mapping defined as $f(u) = u/2$ and A be a bounded linear operator defined as $Au = -3u$ with adjoint operator A^* such that $\|A\| = \|A^*\| = 3$.

Since $\mu \in (0, 1/9)$ and $\alpha_k \in (0, 1)$, so we choose $\mu = 1/18$ and $\alpha_k = 1/3k$; then, the sequences $\{v_k\}$ and $\{u_k\}$ generated by the iterative scheme are evaluated as

$$\begin{aligned} v_k &= J_\lambda^{G_1}\left[u_k + \mu A^*\left(J_\lambda^{G_2} - I\right)Au_k\right] = \frac{9}{12}u_k - \frac{3}{12}, \\ u_{k+1} &= \alpha_k f(u_k) + (1-\alpha_k)T_{[k+1]}v_k \\ &= \frac{1}{6(k+1)}u_k + \left[1 - \frac{1}{3(k+1)}\right]T_{[k+1]}v_k \end{aligned} \tag{64}$$

or for some positive integer $a \geq 0$, and $M = 3$, we can write

$$u_{k+1} = \begin{cases} \dfrac{u_k}{6(k+1)} + \left[1 - \dfrac{1}{3(k+1)}\right][\sin(v_k+1) - 1], & \text{if } k = 3a+1, \\ \dfrac{u_k}{6(k+1)} + \left[1 - \dfrac{1}{3(k+1)}\right]\dfrac{-v_k - 3}{2}, & \text{if } k = 3a+2, \\ \dfrac{u_k}{6(k+1)} + \left[1 - \dfrac{1}{3(k+1)}\right]\dfrac{\cos(\pi v_k) + v_k}{2}, & \text{if } k = 3a. \end{cases} \tag{65}$$

From Table 1, we conclude that for two arbitrary different initial points $u_0 = 5$ and $u_0 = -3$, the sequences $\{v_k\}$ and $\{u_k\}$ converge approximately to a point $u^* = -1 \in \cap_{n=1}^{M}\mathrm{Fix}(T_n) \cap \Xi$.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

No potential conflict of interest is reported by the authors.

Authors' Contributions

All authors read and approved the final manuscript.

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Research Article

Approximation by Szász-Jakimovski-Leviatan-Type Operators via Aid of Appell Polynomials

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The main purpose of the present article is to construct a newly Szász-Jakimovski-Leviatan-type positive linear operators in the Dunkl analogue by the aid of Appell polynomials. In order to investigate the approximation properties of these operators, first we estimate the moments and obtain the basic results. Further, we study the approximation by the use of modulus of continuity in the spaces of the Lipschitz functions, Peetres K-functional, and weighted modulus of continuity. Moreover, we study A -statistical convergence of operators and approximation properties of the bivariate case.

1. Introduction

In 1969, Jakimovski and Leviatan introduced a sequence of positive linear operators $\{L_n\}_{n\geq 1}$ [1], by using Appell polynomials [2] $F(v)e^{vy}=\sum_{s=0}^{\infty}P_s(y)v^s$ and defined as

$$L_n(h;y)=\frac{e^{-ny}}{F(1)}\sum_{s=0}^{\infty}P_s(ny)h\left(\frac{s}{n}\right), \quad (1)$$

where $F(1)\neq 0$, $F(v)=\sum_{s=0}^{\infty}c_s v^s$, and $P_s(y)=\sum_{i=0}^{s}c_i(y^{s-i}/(s-i)!)(s\in\mathbb{N})$. For all $n\in\mathbb{N}$ and $c_i/F(1)\geq 0$, the positive linear operators L_n are defined on $[0,1)$ given by Wood in [3]. If we take $h\in E[0,\infty)$, then an analogue of Szász operators was proved by Jakimovski and Leviatan, where $E[0,\infty)$ denotes the set of functions on $[0,\infty)\}$ such that $|h(y)|\leq ae^{\kappa y}$, where a, κ are positive constants. They established $\lim_{n\to\infty}L_n(h;y)\longrightarrow h(y)$ is uniformly on each compact subset of $[0,1)$ (see [1, 4]). Precisely, for $F(1)=1$ in (1), the well-known classical Szász operators [5] were obtained defined in 1950 such that

$$S_n(h;y)=e^{-ny}\sum_{s=0}^{\infty}\frac{(ny)^s}{s!}h\left(\frac{s}{n}\right). \quad (2)$$

Recently, Szász-Mirakyan operators have been obtained by researchers via the Dunkl generalization in approximation process; for instance, we refer the readers to [6–12]. For more details, related results relevant to the present article in different functional spaces are seen in [13–19] and [20–23]. Sucu [24] introduced Szász-Mirakyan operators by using the new exponential function given in [25] as

$$e_\lambda(y)=\sum_{s=0}^{\infty}\frac{y^s}{\gamma_\lambda(s)}.$$
$$\gamma_\lambda(2p)=\frac{2^{2p}p!\Gamma(p+\lambda+1/2)}{\Gamma(\lambda+1/2)}, \quad (3)$$
$$\gamma_\lambda(2p+1)=\frac{2^{2p+1}p!\Gamma(p+\lambda+3/2)}{\Gamma(\lambda+1/2)}.$$

For $p=0,1,2,\cdots$ a recursion of γ_λ is given as

$$\frac{\gamma_\lambda(p+1)}{(p+1+2\lambda\theta_{p+1})}=\gamma_\lambda(p),$$
$$\theta_p=\begin{cases}0 & \text{if } p=2s,\ s\in\mathbb{N},\\ 1 & \text{if } p=2s+1,\ s\in\mathbb{N}.\end{cases} \quad (4)$$

These types of generalizations gave rise to exponential function and generalization of Hermite-type polynomials, expressed in the form of the confluent hypergeometric function (see [25]).

2. Construction of Operators and Estimation of Moments

For every $h \in C_{\vartheta}[0,\infty) = \{h \in C[0,\infty): h(s) = O(s^{\vartheta})\}$ as $s \longrightarrow \infty$, and all $y \in [0,\infty)$, $\vartheta > n$, $n \in \mathbb{N}$, $F(1) \neq 0$, $\lambda \geq 0$, we define

$$\mathcal{J}^{*}_{n,\lambda}(h;y) = \frac{1}{F(1)e_{\lambda}(ny)} \sum_{s=0}^{\infty} P_s(ny) h\left(\frac{s+2\lambda\theta_s}{n}\right). \quad (5)$$

Lemma 1. *For all $y \in [0,\infty)$, $P_s(y) \geq 0$, $\lambda \geq 0$, and $F(1) \neq 0$, if we define*

$$F(\alpha)e_{\lambda}(\alpha y) = \sum_{s=0}^{\infty} P_s(y)\alpha^s. \quad (6)$$

Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
F(1)e_{\lambda}(ny) &= \sum_{s=0}^{\infty} P_s(ny),\\
\sum_{s=0}^{\infty} sP_s(ny) &= \left(F'(1) + nyF(1)\right)e_{\lambda}(ny),\\
\sum_{s=0}^{\infty} s^2P_s(ny) &= \Big(F''(1) + (2ny+1)F'(1)\\
&\quad + ny(ny+1)F(1)\Big)e_{\lambda}(ny),\\
\sum_{s=0}^{\infty} s^3P_s(ny) &= \Big(F'''(1) + 3(ny+1)F''(1)\\
&\quad + \left(3n^2y^2 + 6ny + 2\right)F'(1)\\
&\quad + ny\left(n^2y^2 + 3ny + 2\right)F(1)\Big)e_{\lambda}(ny),\\
\sum_{s=0}^{\infty} s^4P_s(ny) &= \Big(F^{iv}(1) + (4ny+6)F'''(1)\\
&\quad + \left(6n^2y^2 + 18ny + 11\right)F''(1)\\
&\quad + \left(4n^3y^3 + 18n^2y^2 + 22ny + 6\right)F'(1)\\
&\quad + ny\left(n^3y^3 + 6n^2y^2 + 11ny + 6\right)F(1)\Big)e_{\lambda}(ny).
\end{aligned} \quad (7)$$

Lemma 2. *Let $\lambda \in [0,\infty)$, $F(1) \neq 0$ and take $\phi_r = s^r$ for $r = 0, 1, 2, 3, 4$.*

Then, for operators $\mathcal{J}^{*}_{n,\lambda}(\cdot;\cdot)$ by (5), we have the following estimates:

$$\begin{aligned}
\mathcal{J}^{*}_{n,\lambda}(\phi_0;y) &= 1,\\
\mathcal{J}^{*}_{n,\lambda}(\phi_1;y) &= y + \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right),\\
\mathcal{J}^{*}_{n,\lambda}(\phi_2;y) &= y^2 + \frac{1}{n}\left(\frac{2F'(1)}{F(1)} + 4\lambda + 1\right)y\\
&\quad + \frac{1}{n^2}\left(\frac{F''(1)}{F(1)} + (1+4\lambda)\frac{F'(1)}{F(1)} + 4\lambda^2\right),\\
\mathcal{J}^{*}_{n,\lambda}(\phi_3;y) &= y^3 + \frac{3}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda + 1\right)y^2\\
&\quad + \frac{1}{n^2}\left(\frac{3F''(1)}{F(1)} + 6(1+2\lambda)\frac{F'(1)}{F(1)} + 2 + 6\lambda\right)y\\
&\quad + \frac{1}{n^3}\Bigg(\frac{3F'''(1)}{F(1)} + 3(1+2\lambda)\frac{F''(1)}{F(1)}\\
&\quad + 2\left(1 + 3\lambda + 6\lambda^2\right)\frac{F'(1)}{F(1)} + 8\lambda^3\Bigg),\\
\mathcal{J}^{*}_{n,\lambda}(\phi_4;y) &= y^4 + \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 8\lambda + 6\right)y^3 + \frac{1}{n^2}\Bigg(\frac{6F''(1)}{F(1)}\\
&\quad + 8(1+3\lambda)\frac{F'(1)}{F(1)} + 11 + 24\lambda + 24\lambda^2\Bigg)y^2\\
&\quad + \frac{1}{n^3}\Bigg(\frac{6F'''(1)}{F(1)} + 8(1+3\lambda)\frac{F''(1)}{F(1)}\\
&\quad + 2\left(11 + 24\lambda + 24\lambda^2\right)\frac{F'(1)}{F(1)}\\
&\quad + 6 + 16\lambda + 24\lambda^2 + 32\lambda^2\Bigg)y\\
&\quad + \frac{1}{n^4}\Bigg(\frac{F^{iv}(1)}{F(1)} + 2(3+4\lambda)\frac{F'''(1)}{F(1)}\\
&\quad + \left(11 + 24\lambda + 24\lambda^2\right)\frac{F''(1)}{F(1)}\\
&\quad + \left(6 + 16\lambda + 24\lambda^2 + 32\lambda^3\right)\frac{F'(1)}{F(1)} + 16\lambda^4\Bigg).
\end{aligned} \quad (8)$$

Proof.

(1) Take $h = \phi_0$, then

$$\mathcal{J}^*_{n,\lambda}(\phi_0\,;y) = \frac{1}{F(1)e_\lambda(ny)}\sum_{s=0}^{\infty} P_s(ny) = 1. \tag{9}$$

(2) For $h = \phi_1$

$$\begin{aligned}\mathcal{J}^*_{n,\lambda}(\phi_1\,;y) &= \frac{1}{F(1)e_\lambda(ny)}\sum_{s=0}^{\infty} P_s(ny)\left(\frac{s+2\lambda\theta_s}{n}\right)\\ &= \frac{1}{nF(1)e_\lambda(ny)}\sum_{s=0}^{\infty} sP_s(ny)\\ &\quad+ \frac{2\lambda}{nF(1)e_\lambda(ny)}\sum_{s=2k+1}^{\infty}\theta_s P_s(ny) \text{ for } k\\ &= 0,1,2,3,\cdots = \frac{1}{nF(1)e_\lambda(ny)}\\ &\quad\cdot\left(F'(1)+nyF(1)\right)e_\lambda(ny)+\frac{2\lambda}{n}.\end{aligned} \tag{10}$$

(3) For $h = \phi_2$

$$\begin{aligned}\mathcal{J}^*_{n,\lambda}(\phi_2\,;y) &= \frac{1}{F(1)e_\lambda(ny)}\sum_{s=0}^{\infty} P_s(ny)\left(\frac{s+2\lambda\theta_s}{n}\right)^2\\ &= \frac{1}{n^2F(1)e_\lambda(ny)}\sum_{s=0}^{\infty} s^2P_s(ny) + \frac{4\lambda}{n^2F(1)e_\lambda(ny)}\\ &\quad\cdot\sum_{s=2k+1}^{\infty} sP_s(ny)\theta_s \text{ for } k=0,1,2,3,\cdots\\ &\quad+\frac{4\lambda^2}{n^2F(1)e_\lambda(ny)}\sum_{s=2k+1}^{\infty} P_s(ny)\theta_s^2 \text{ for } k\\ &= 0,1,2,3,\cdots = \frac{1}{F(1)e_\lambda(ny)}\Big(F''(1)\\ &\quad+(2ny+1)F'(1)+ny(ny+1)F(1)\Big)e_\lambda(ny)\\ &\quad+\frac{4\lambda^2}{n^2F(1)e_\lambda(ny)}\left(F'(1)+nyF(1)\right)e_\lambda(ny)\\ &\quad+\frac{4\lambda^2}{n^2}.\end{aligned} \tag{11}$$

Similarly, we can prove easily (4) and (5).

Lemma 3. *Let $\psi_j = (\phi_1 - y)^j$ for $j = 1, 2, 3$, be the central moments, then*

$$\begin{aligned}\mathcal{J}^*_{n,\lambda}(\psi_1\,;y) &= \frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right),\\ \mathcal{J}^*_{n,\lambda}(\psi_2\,;y) &= \frac{y}{n}++\frac{1}{n^2}\left(\frac{F''(1)}{F(1)}+(1+4\lambda)\frac{F'(1)}{F(1)}+4\lambda^2\right),\\ \mathcal{J}^*_{n,\lambda}(\psi_4\,;y) &= \frac{1}{n}\left(\frac{4F'(1)}{F(1)}+8\lambda\right)y^3+\frac{1}{n^2}\left(-10\frac{F'(1)}{F(1)}+3\right)y^2\\ &\quad+\frac{1}{n^3}\left(\frac{-4F''(1)}{F(1)}+2(7+12\lambda)\frac{F'(1)}{F(1)}\right.\\ &\quad\left.+2\left(3+8\lambda+12\lambda^2\right)\right)y+\frac{1}{n^4}\left(\frac{F'^{v}(1)}{F(1)}\right.\\ &\quad+2(3+4\lambda)\frac{F'''(1)}{F(1)}+\left(11+24\lambda+24\lambda^2\right)\frac{F''(1)}{F(1)}\\ &\quad\left.+\left(6+16\lambda+24\lambda^2+32\lambda^3\right)\frac{F'(1)}{F(1)}+16\lambda^4\right)\\ &\quad+\frac{1}{n}\left(\frac{-4F'(1)}{F(1)}-8\lambda\right).\end{aligned} \tag{12}$$

3. Global Approximation

In the present section, we follow Gadžiev [11] and recall the weighted spaces of the functions on $[0,\infty)$, as well as additional conditions under which the analogous theorem of P.P. Korovkin holds for such a kind of functions. Take $y \longrightarrow \phi(y)$ be continuous and strictly increasing function with $\sigma(y) = 1 + \phi^2(y)$ and $\lim_{y\to\infty}\sigma(y) = \infty$. Let $B_\sigma[0,\infty)$ be a set of functions defined on $[0,\infty)$, verifying the results

$$B_\sigma[0,\infty) = \{h(y) : |h(y)| \le K_h\sigma(y)\}, \tag{13}$$

where K_h is a constant and depending only on function h and $B_\sigma[0,\infty)$ is space of all continuous as well as bounded functions on $[0,\infty)$. Let the set of all continuous functions on $[0,\infty)$ will be denoted by $C_\sigma[0,\infty)$ and $B_\sigma[0,\infty) \subset C_\sigma[0,\infty)$ equipped with the norm $\|h\|_\sigma = \sup_{y\in[0,\infty)} |h|/\sigma(y)$.

Let us denote

$$C^m_\sigma[0,\infty) = \left\{h \in C_\sigma : \lim_{y\to\infty}\frac{h(y)}{\sigma(y)} = k, k \text{ is positive constant}\right\}. \tag{14}$$

It is well known that (see [26]) the sequence of linear positive operators $\{L_n\}_{n\ge1}$ maps $C_\sigma[0,\infty)$ into $B_\sigma[0,\infty)$ if and only if

$$|L_n(\sigma;y)| \le C\sigma(y), \tag{15}$$

where C is a positive constant.

Definition 4. For all $h \in C[0,\infty)$, the modulus of continuity for a uniformly continuous function h defined by

$$\omega(h;\delta) = \sup_{|s_1-s_2|\le\delta} |h(s_1) - h(s_2)|, s_1, s_2 \in [0,\infty). \tag{16}$$

For every $\delta > 0$ and uniformly continuous function $h \in C[0,\infty)$, we suppose

$$|h(s_1) - h(s_2)| \le \left(1 + \frac{|s_1 - s_2|}{\delta^2}\right)\omega(h;\delta). \tag{17}$$

Theorem 5. *For all $h \in [0,\infty) \cap \{h : y \ge 0, h(y)/\sigma(y)$ is convergent as $y \longrightarrow \infty\}$, operators $\mathcal{J}^*_{n,\lambda}$ defined in (5) satisfy $\mathcal{J}^*_{n,\lambda} \Rightarrow h$ on each compact subset of $[0,\infty)$, with $\Rightarrow$ stands for uniform convergence.*

Proof. From the well-known Korovkin's theorem (see [27]), for all $r = 0, 1, 2$, it is sufficient to see that

$$\mathcal{J}^*_{n,\lambda}(\phi_r;y) \longrightarrow y^r. \tag{18}$$

In the view of Lemma 2, it is obvious that $\mathcal{J}^*_{n,\lambda}(\phi_r;y) \longrightarrow y^r$ as $n \longrightarrow \infty$, $r = 0, 1, 2$, which completes Theorem 5.

Theorem 6. *Let $\mathcal{J}^*_{n,\lambda} : C^m_\sigma[0,\infty) \longrightarrow B_\sigma[0,\infty)$. Then for every $h \in C^m_\sigma[0,\infty)$, we have*

$$\lim_{n\to\infty} \left\|\mathcal{J}^*_{n,\lambda}(h;y) - h\right\|_\sigma = 0. \tag{19}$$

Proof. We prove this theorem by applying Korovkin's theorem so it is sufficient to show that

$$\lim_{n\to\infty} \left\|\mathcal{J}^*_{n,\lambda}\left(\phi_j;y\right) - y^j\right\|_\sigma = 0, \text{ for } j = 0, 1, 2. \tag{20}$$

From Lemma 2, we easily see that

$$\left\|\mathcal{J}^*_{n,\lambda}(\phi_o;y) - y^0\right\|_\sigma = \sup_{y\in[0,\infty)} \frac{\left|\mathcal{J}^*_{n,\lambda}(1;y) - 1\right|}{\sigma(y)} = 0 \text{ for } j = 0. \tag{21}$$

Similarly, for

$$\left\|\mathcal{J}^*_{n,\lambda}(\phi_1;y) - y\right\|_\sigma = \frac{1}{n}\left(\frac{F'(1)}{F(1)} + 2\lambda\right) \sup_{y\in[0,\infty)} \frac{1}{\sigma(y)}. \tag{22}$$

which imply that $\|\mathcal{J}^*_{n,\lambda}(\phi_1;y) - y\|_\sigma \longrightarrow 0$ as $n \longrightarrow \infty$. For $j = 2$

$$\begin{aligned}
&\left\|\mathcal{J}^*_{n,\lambda}(\phi_2;y) - y^2\right\|_\sigma \\
&\quad= \sup_{y\in[0,\infty)} \frac{\left|\mathcal{J}^*_{n,\lambda}(\phi_2;y) - y^2\right|}{\sigma(y)} \\
&\quad= \frac{1}{n}\left(\frac{2F'(1)}{F(1)} + 4\lambda + 1\right) \sup_{y\in[0,\infty)} \frac{y}{\sigma(y)} \\
&\qquad+ \frac{1}{n^2}\left(\frac{F''(1)}{F(1)} + (1+4\lambda)\frac{F'(1)}{F(1)} + 4\lambda^2\right) \sup_{y\in[0,\infty)} \frac{1}{\sigma(y)},
\end{aligned} \tag{23}$$

which clearly shows that $\|\mathcal{J}^*_{n,\lambda}(\phi_2;y) - y^2\|_\sigma \longrightarrow 0$, whenever $n \longrightarrow \infty$.

Theorem 7. *For all $h \in C_B[0,\infty)$, operators given by (5) satisfy*

$$\left|\mathcal{J}^*_{n,\lambda}(h;y) - h(y)\right| \le 2\omega(h;\delta_n(y)), \tag{24}$$

where $\delta_n(y) = \sqrt{\mathcal{J}^*_{n,\lambda}(\psi_2;y)}$ and $C_B[0,\infty)$ stand for space of all continuous and bounded functions defined on $[0,\infty)$.

Proof. We prove Theorem 7 by using the well-known Cauchy-Schwarz inequality and modulus of continuity. Thus, we see that

$$\begin{aligned}
&\mathcal{J}^*_{n,\lambda}(h;y) - h(y) \\
&\le \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny);\left|h\left(\frac{s+2\lambda\theta_s}{n}\right) - h(y)\right| \\
&\le \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny)\left(1 + \frac{1}{\delta}\left|\frac{s+2\lambda\theta_s}{n} - y\right|\right)\omega(h;\delta) \\
&= \left\{1 + \frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny)\left|\frac{s+2\lambda\theta_s}{n} - y\right|\omega(h;\delta)\right. \\
&\le \left\{1 + \frac{1}{\delta}\left(\frac{1}{F(1)e_\lambda(ny)} \sum_{s=0}^{\infty} P_s(ny)\left(\frac{s+2\lambda\theta_s}{n} - y\right)^2\right)^{\frac{1}{2}} \left(\mathcal{J}^*_{n,\lambda}(\phi_0;y)\right)^{\frac{1}{2}}\right\}\omega(h;\delta) \\
&= \left(1 + \frac{1}{\delta}\left(\mathcal{J}^*_{n,\lambda}((\phi_1 - y)^2;y)\right)^{\frac{1}{2}}\right)\omega(h;\delta).
\end{aligned} \tag{25}$$

If we take $\delta = \delta_n = \sqrt{\mathcal{J}^*_{n,\lambda}(\psi_2;y)}$, we get the required result asserted by Theorem 7.

4. Some Direct Results of $\mathcal{J}^*_{n,\lambda}$

The present section gives some direct approximation results in the space of K-functional and in the Lipschitz spaces. We suppose the following.

Definition 8. For every $\delta > 0$ and $h \in C[0,\infty)$, we define

$$K_2(h;\delta) = \inf\left\{\left(\|h - \psi\|_{C_B[0,\infty)} + \delta\left\|\psi''\right\|_{C_B[0,\infty)}\right) : \psi, \psi' \in \left|C^2_B[0,\infty)\right.\right\}, \tag{26}$$

where $C_B^2[0,\infty)$ is defined by

$$C_B^k[0,\infty)=\left\{h: h\in C_B[0,\infty), k\in\mathbb{N}\,; \text{ such that} \lim_{y\to\infty}\frac{h(y)}{\sigma(y)}=k_h<\infty\right\}. \quad (27)$$

Now, there exists an absolute constant $\mathscr{C}>0$ such that

$$K_2(h\,;\delta)<\mathscr{C}\left\{\omega_2\left(h\,;\sqrt{\delta}\right)+\min(1,\delta)\|h\|_{C_B[0,\infty)}\right\}, \quad (28)$$

where $\omega_2(h\,;\delta)$ is the second-order modulus of continuity given by

$$\omega_2(h\,;\delta)=\sup_{0<\eta<\delta}\sup_{y\in[0,\infty)}|h(y+2\eta)-2h(y+\eta)+h(y)|. \quad (29)$$

Moreover, the modulus of continuity of order one is

$$\omega(h\,;\delta)=\sup_{0<\eta<\delta}\sup_{y\in[0,\infty)}|h(y+\eta)-h(y)|. \quad (30)$$

Theorem 9. *Let $h\in C_B^2[0,\infty)$, we define an auxiliary operators $\mathscr{K}_{n,\lambda}^*$ such that*

$$\mathscr{K}_{n,\lambda}^*(h\,;y)=\mathscr{J}_{n,\lambda}^*(h\,;y)+h(y)-h\left\{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right\}. \quad (31)$$

Then, for every $\psi\in C_B^2[0,\infty)$, operators $\mathscr{K}_{n,\lambda}^*$ satisfy

$$\left|\mathscr{K}_{n,\lambda}^*(\psi\,;y)-\psi(y)\right|\le\{\Theta_n(y)\}\|\psi''\|, \quad (32)$$

where $\Theta_n(y)=(\delta_n(y))^2+(1/n^2)((F'(1)/F(1))+2\lambda)^2$ and $\delta_n(y)$ are defined in Theorem 7.

Proof. Take $\psi\in C_B^2[0,\infty)$; then, we easily conclude that $\mathscr{K}_{n,\lambda}^*(\phi_0\,;y)=1$ and

$$\mathscr{K}_{n,\lambda}^*(\phi_1\,;y)=\mathscr{J}_{n,\lambda}^*(\phi_1\,;y)+y-\left\{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right\}=y. \quad (33)$$

We also know easily

$$\left\|\mathscr{J}_{n,\lambda}^*(h\,;y)\right\|\le\|h\|. \quad (34)$$

Therefore,

$$\begin{aligned}\left|\mathscr{K}_{n,\lambda}^*(h\,;y)\right|&\le\left|\mathscr{J}_{n,\lambda}^*(h\,;y)\right|+|h(y)|\\&+\left|h\left\{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right\}\right|\le3\|h\|.\end{aligned} \quad (35)$$

From the Taylor series we see

$$\psi(s)=\psi(y)+(s-y)\psi'(y)+\int_y^s(s-\mu)\psi''(\mu)\mathrm{d}\mu. \quad (36)$$

Applying $\mathscr{K}_{n,\lambda}^*$, we have

$$\begin{aligned}&\mathscr{K}_{n,\lambda}^*(\psi\,;y)-\psi(y)\\&=\psi'(y)\mathscr{K}_{n,\lambda}^*(\phi_1-y\,;y)+\mathscr{K}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu\,;y\right)\\&=\mathscr{K}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu\,;y\right)\\&=\mathscr{J}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu\,;y\right)\\&\quad-\int_y^{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)}\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)-\mu\right)\psi''(\mu)\mathrm{d}\mu,\end{aligned}$$

$$\begin{aligned}&\left|\mathscr{K}_{n,\lambda}^*(\psi\,;y)-\psi(y)\right|\\&\le\left|\mathscr{J}_{n,\lambda}^*\left(\int_y^{\phi_1}(\phi_1-\mu)\psi''(\mu)\mathrm{d}\mu\,;y\right)\right|\\&\quad+\left|\int_y^{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)}\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)-\mu\right)\psi''(\mu)\mathrm{d}\mu\right|.\end{aligned} \quad (37)$$

Since we know

$$\begin{aligned}&\left|\int_y^s(s-\mu)\psi''(\mu)\mathrm{d}\mu\right|\le(s-y)^2\|\psi''\|,\\&\left|\int_y^{y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)}\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)-\mu\right)\psi''(\mu)\mathrm{d}\mu\right|\\&\quad\le\left(\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right)^2\|\psi''\|.\end{aligned} \quad (38)$$

Therefore, we get

$$\begin{aligned}&\left|\mathscr{K}_{n,\lambda}^*(\psi\,;y)-\psi(y)\right|\\&\le\left\{\mathscr{J}_{n,\lambda}^*\left((\phi_1-y)^2\,;y\right)+\frac{1}{n^2}\left(\frac{F'(1)}{F(1)}+2\lambda\right)^2\right\}\|\psi''\|.\end{aligned} \quad (39)$$

This gives the complete proof.

Theorem 10. *Let $h \in C_B[0,\infty)$ and any $\psi \in C_B^2[0,\infty)$. Then, there exists a constant $\mathscr{C} > 0$ such that*

$$\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| \leq \mathscr{C}\left\{\omega_2\left(h;\frac{\sqrt{\Theta_n(y)}}{2}\right) + \min\left(1,\frac{\Theta_n(y)}{4}\right)\|h\|_{C_{B[0,\infty)}}\right\} + \omega\left(h;\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right), \quad (40)$$

where $\Theta_n(y)$ is defined by Theorem 9.

Proof. We prove the result asserted by Theorem 10 in the light of Theorem 9. Therefore, for all $h \in C_B[0,\infty)$ and $\psi \in C_B^2[0,\infty)$, we get

$$\begin{aligned}\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| &= \left|\mathscr{K}_{n,\lambda}^*(h;y) - h(y) + h\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right) - h(y)\right| \\ &\leq \left|\mathscr{K}_{n,\lambda}^*(h-\psi;y)\right| + \left|\mathscr{K}_{n,\lambda}^*(\psi;y) - \psi(y)\right| + |\psi(y)-h(y)| \\ &\quad + \left|h\left(y+\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right) - h(y)\right| \\ &\leq 4\|h-\psi\| + \Theta_n(y)\left\|\psi''\right\| + \omega\left(h;\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|\right).\end{aligned} \quad (41)$$

Taking infimum over all $\psi \in C_B^2[0,\infty)$ and using (26), we get

$$\begin{aligned}\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| &\leq 4K_2\left(h;\frac{\Theta_n(y)}{4}\right) + \omega\left(h;\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right) \\ &\leq \mathscr{C}\left\{\omega_2\left(h;\frac{\sqrt{\Theta_n(y)}}{4}\right) + \min\left(1;\frac{\Theta_n(y)}{4}\right)\|h\|_{C_B[0,\infty)}\right\} \\ &\quad + \omega\left(h;\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right).\end{aligned} \quad (42)$$

Here, we obtain some local approximation results of $\mathscr{J}_{n,\lambda}^*$ in the Lipschitz spaces. For all the Lipschitz maximal function $h \in C[0,\infty)$, $0 < \vartheta \leq 1$ and $s, y \in [0,\infty)$, we recall that

$$\omega_\vartheta(h;y) = \sup_{s\neq y, s\in[0.\infty)} \frac{|h(s)-h(y)|}{|s-y|^\vartheta}. \quad (43)$$

Theorem 11. *Let $0 < \vartheta \leq 1$, then for all $h \in C_B[0,\infty)$, operators $\mathscr{J}_{n,\lambda}^*$ satisfy*

$$\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| \leq \omega_\vartheta(h;y)(\delta_n(y))^\vartheta, \quad (44)$$

where $\omega_\vartheta(h;y)$ is the Lipschitz maximal function defined by (43) and $\delta_n(y)$ by Theorem 7.

Proof. To prove Theorem 11, we use the well-known Hölder inequality by applying (43)

$$\begin{aligned}&\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right| \\ &\quad \leq \mathscr{J}_{n,\lambda}^*(|h(s)-h(y)|;y) \leq \omega_\vartheta(h;y)\left|\mathscr{J}_{n,\lambda}^*\left(|s-y|^\vartheta;y\right)\right. \\ &\quad \leq \omega_\vartheta(h;y)\left(\mathscr{J}_{n,\lambda}^*(\phi_0;y)\right)^{\frac{2-\vartheta}{2}}\left(\mathscr{J}_{n,\lambda}^*\left((\phi_1-y)^2;y\right)\right)^{\frac{\vartheta}{2}} \\ &\quad = \omega_\vartheta(h;y)\left(\mathscr{J}_{n,\lambda}^*(\psi_2;y)\right)^{\frac{\vartheta}{2}}.\end{aligned} \quad (45)$$

The proof is complete.

From [28] for an arbitrary $h \in C_\sigma^m[0,\infty)$, the weighted modulus of continuity is introduced such that

$$\Omega(h;\delta) = \sup_{y\in[0,\infty),|\eta|\leq\delta} \frac{|h(y+\eta)-h(y)|}{(1+\eta^2)(1+y^2)}. \quad (46)$$

The two main properties of this modulus of continuity are $\lim_{\delta\to 0}\Omega(h;\delta) = 0$ and

$$|h(s)-h(y)| \leq 2\left(1+\frac{|s-y|}{\delta}\right)\left(1+\delta^2\right)\left(1+y^2\right) \cdot \left(1+(s-y)^2\right)\Omega(h;\delta), \quad (47)$$

where $s, y \in [0,\infty)$.

Theorem 12. *Let the operators $\mathscr{J}_{n,\lambda}^*$ be defined by (5); then for every $h \in C_\sigma^m[0,\infty)$, there exists a constant $C > 0$ such that*

$$\sup_{y\in\left[0,O\left(\frac{1}{n}\right)\right)} \frac{\left|\mathscr{J}_{n,\lambda}^*(h;y) - h(y)\right|}{\sigma(y)} \leq C\left(1+O\left(\frac{1}{n}\right)\right)\Omega\left(h;\sqrt{O\left(\frac{1}{n}\right)}\right), \quad (48)$$

where $\sigma(y) = 1 + y^2$ and $C = (2 + C_1 + 2C_2)$ with $C_1 > 0, C_2 > 0$.

Proof. In light of (46), (47), and Cauchy-Schwarz inequality, we prove this theorem. Thus, we see

$$\begin{aligned}&\left|\mathcal{J}^*_{n,\lambda}(h;y)-h(y)\right|\\&\quad\le 2\left(1+\delta^2\right)\left(1+y^2\right)\Omega(h;\delta)\left(1+\mathcal{J}^*_{n,\lambda}\left((\phi_1-y)^2;y\right)\right.\\&\quad\left.+\mathcal{J}^*_{n,\lambda}\left(\left(1+(\phi_1-y)^2\right)\frac{|\phi_1-y|}{\delta};y\right)\right)\end{aligned}\tag{49}$$

$$\begin{aligned}&\mathcal{J}^*_{n,\lambda}\left(\left(1+(\phi_1-y)^2\right)\frac{|\phi_1-y|}{\delta};y\right)\\&\quad\le 1+2\left(\mathcal{J}^*_{n,\lambda}\left((\phi_1-y)^4;y\right)\right)^{\frac{1}{2}}\left(\mathcal{J}^*_{n,\lambda}\left(\frac{(\phi_1-y)^4}{\delta^2};y\right)\right)^{\frac{1}{2}}.\end{aligned}\tag{50}$$

From Lemma 3, we easily conclude that for any positive C_1 and C_1

$$\mathcal{J}^*_{n,\lambda}\left((\phi_1-y)^2;y\right)=O\left(\frac{1}{n}\right)(y+1)^2\le C_1(y+1)^2 \text{ as } n\longrightarrow\infty,\tag{51}$$

$$\mathcal{J}^*_{n,\lambda}\left((\phi_1-y)^4;y\right)=O\left(\frac{1}{n}\right)(y+1)^4\le C_2(y+1)^4 \text{ as } n\longrightarrow\infty.\tag{52}$$

Therefore,

$$\left(\mathcal{J}^*_{n,\lambda}\left(\frac{(\phi_1-y)^2}{\delta^2};y\right)\right)^{\frac{1}{2}}=\frac{1}{\delta}\sqrt{O\left(\frac{1}{n}\right)}(1+y).\tag{53}$$

$$\left(\mathcal{J}^*_{n,\lambda}\left((\phi_1-y)^4;y\right)\right)^{\frac{1}{2}}\le C_2(1+y)^2.\tag{54}$$

Hence, in light of (49), (50), (51), (52), (53) and (54) and choosing $\delta\sqrt{O(1/n)}$, if we take the supremum $y\in[0,O(1/n))$, we get the result.

5. A-Statistical Convergence

Here, we obtain the A-statistical convergence for the operators $\mathcal{J}^*_{n,\lambda}$ by (5). From [29], we recall the needed notations and notions for A-statistical convergence. Take $G=(D_{nk})$ be a nonnegative infinite summability matrix. For a given sequence $y=(y_k)$, the A-transform of y is denoted by $Gy:(Gy)_n$ where the series converges for each n and defined by

$$(Gy)_n=\sum_{k=1}^{\infty}y_kD_{nk}.\tag{55}$$

The matrix G is said to be regular if $\lim(Gy)_n=L$ whenever $\lim x=L$ and $y=(y_n)$ are said to be a A-statistically convergent to L, i.e., $st_G-\lim y=L$ if for each $\epsilon>0$, $\lim_n\sum_{k:|y_k-L|\ge\epsilon}D_{nk}=0$. For the recent work on statistical convergence and statistical approximation, we refer to [30–37].

Theorem 13. *Let operators $\mathcal{J}^*_{n,\lambda}$ be defined by 1 and a non-negative regular summability matrix be $G=(D_{nk})$; then, for every $h\in C^m_\sigma[0,\infty)$*

$$st_G-\lim_n\left\|\mathcal{J}^*_{n,\lambda}(h;y)-h\right\|_\sigma=0.\tag{56}$$

Proof. It is enough to show that

$$st_G-\lim_n\left\|\mathcal{J}^*_{n,\lambda}\left(\phi_j;y\right)-y^j\right\|_\sigma=0,\quad \text{for} j=0,1,2.\tag{57}$$

From Lemma 2, we conclude that

$$\begin{aligned}\left\|\mathcal{J}^*_{n,\lambda}(\phi_2;y)-y\right\|_\sigma&=\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|\sup_{y\in[0,\infty)}\frac{1}{\sigma(y)}\\&\le\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|.\end{aligned}\tag{58}$$

which implies that

$$st_G-\lim_n\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|=0.\tag{59}$$

Similarly for $j=2$

$$\begin{aligned}&\left\|\mathcal{J}^*_{n,\lambda}(\phi_2;y)-y^2\right\|_\sigma\\&\quad=\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)}+4\lambda+1\right)\right|\sup_{y\in[0,\infty)}\frac{y}{\sigma(y)}\\&\quad+\left|\frac{1}{n^2}\left(\frac{F''(1)}{F(1)}+(1+4\lambda)\frac{F'(1)}{F(1)}+4\lambda\right)\right|\sup_{y\in[0,\infty)}\frac{1}{\sigma(y)}\\&\quad\le\frac{1}{2}\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)}+4\lambda+1\right)\right|+\left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right|.\end{aligned}\tag{60}$$

which shows that

$$st_G-\lim_n \frac{1}{2}\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)}+4\lambda+1\right)\right| = st_G-\lim_n \left|\frac{1}{n}\left(\frac{F'(1)}{F(1)}+2\lambda\right)\right| = 0. \tag{61}$$

For a given $\varepsilon>0$, we define the sets such that

$$U_1 := \{n : \|\mathcal{J}^*_{n,\lambda}(\phi_2;y)-y^2\| \ge \epsilon\},$$

$$U_2 := \left\{n : \frac{1}{2}\left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)}+4\lambda+1\right)\right| \ge \frac{\epsilon}{2}\right\},$$

$$U_3 := \left\{n : \left|\frac{1}{n}\left(\frac{2F'(1)}{F(1)}+2\lambda\right)\right| \ge \frac{\epsilon}{2}\right\}. \tag{62}$$

Therefore, we conclude that $U_1 \subseteq U_2 \cup U_3$, and $\sum_{k_1\in U_1} D_{nk_1} \le \sum_{k_1\in U_2} D_{nk_1} + \sum_{k_1\in U_3} D_{nk_1}$. Hence, (61) implies that

$$st_G-\lim_n \|\mathcal{J}^*_{n,\lambda}(\phi_2;y)-y^2\|_\sigma = 0. \tag{63}$$

This is denumerable to complete the proof.

6. Bivariate Operators and Their Moments Estimation

Let $\mathbb{R}^2_+ = \mathbb{R}_+\times\mathbb{R}_+$ with $\mathbb{R}_+=[0,\infty)$ and $\mathbb{R}^2_+=\{(y,y_1): 0\le y<\infty, 0\le y_1<\infty\}$. Suppose $C(\mathbb{R}^2_+)$ is a set of all continuous functions on $\mathbb{R}^2_+$, endowed with the norm given by $\|g\|_{C(\mathbb{R}^2_+)} = \sup_{(y,y_1)\in\mathbb{R}^2_+}|g(y,y_1)|$. Then, for all $g\in C(\mathbb{R}^2_+)$ and $n,m\in\mathbb{N}$, we define

$$\mathcal{J}^*_{n,m}(g;y,y_1) = \frac{1}{F(1)e_\lambda(ny)G(1)e_\eta(my_1)} \cdot \sum_{s,t=0}^{\infty} P_s(ny)P_t(my_1)g\left(\frac{s+2\lambda\theta_s}{n},\frac{t+2\eta\theta_t}{m}\right), \tag{64}$$

where $\lambda,\eta\ge 0$ and $F(1),G(1)\ne 0$.

For $i,j=\{0,1,2,3,4\}$, if we take $s=s+2\lambda\theta_s/n$, $t=t+2\eta\theta_t/m$, and consider central moments as

$$g(s^i,t^j) = (s-y)^i(t-y_1)^j. \tag{65}$$

then from Lemma 2 and Lemma 3, we easily conclude Lemma 14 as follows:

Lemma 14. *For all $y,y_1\in\mathbb{R}^2_+$ and sufficiently large $n,m\in\mathbb{N}$, we have*

$$\begin{aligned}
\mathcal{J}^*_{n,m}((s-y)^2;y,y_1) &= O\left(\frac{1}{n}\right)(y+1)^2 \le M_1(y+1)^2 \text{ as } n,m\longrightarrow\infty,\\
\mathcal{J}^*_{n,m}((t-y_1)^2;y,y_1) &= O\left(\frac{1}{m}\right)(y_1+1)^2 \le M_2(y_1+1)^2 \text{ as } n,m\longrightarrow\infty,\\
\mathcal{J}^*_{n,m}((s-y)^4;y,y_1) &= O\left(\frac{1}{n}\right)(y+1)^4 \le M_3(y+1)^4 \text{ as } n,m\longrightarrow\infty,\\
\mathcal{J}^*_{n,m}((t-y_1)^4;y,y_1) &= O\left(\frac{1}{m}\right)(y_1+1)^4 \le M_4(y_1+1)^4 \text{ as } n,m\longrightarrow\infty.
\end{aligned} \tag{66}$$

Lemma 15. *If we let*

$$\mathcal{K}^*_{n,\lambda}(g;y,y_1) = \frac{1}{F(1)e_\lambda(ny)}\sum_{s=0}^{\infty}P_s(ny)g\left(\frac{s+2\lambda\theta_s}{n},y_1\right),$$

$$\mathcal{L}^*_{m,\eta}(g;y,y_1) = \frac{1}{G(1)e_\eta(my_1)}\sum_{t=0}^{\infty}P_t(my_1)g\left(y,\frac{t+2\eta\theta_t}{m}\right). \tag{67}$$

Then, it follows that

$$\mathcal{J}^*_{n,m}(g;y,y_1) = \mathcal{K}^*_{n,\lambda}\left(\mathcal{L}^*_{m,\eta}(g;y,y_1)\right) = \mathcal{L}^*_{m,\eta}\left(\mathcal{K}^*_{n,\lambda}(g;y,y_1)\right). \tag{68}$$

Proof. We easily see that

$$\begin{aligned}
\mathcal{K}^*_{n,\lambda}\left(\mathcal{L}^*_{m,\eta}(g;y,y_1)\right) &= \mathcal{K}^*_{n,\lambda}\left(\frac{1}{G(1)e_\eta(my_1)}\sum_{t=0}^{\infty}P_t(my_1)g\left(y,\frac{t+2\eta\theta_t}{m}\right)\right)\\
&= \frac{1}{G(1)e_\eta(my_1)}\sum_{t=0}^{\infty}\mathcal{K}^*_{n,\lambda}\left(g\left(y,\frac{t+2\eta\theta_t}{m}\right)\right)P_t(my_1)\\
&= \frac{1}{F(1)e_\lambda(ny_1)G(1)e_\eta(my_1)}\sum_{s,t=0}^{\infty}P_s(ny)(P_t)(my_1)g\cdot\left(\frac{s+2\lambda\theta_s}{n},\frac{t+2\eta\theta_t}{m}\right) = \mathcal{J}^*_{n,n}(g;y,y_1).
\end{aligned} \tag{69}$$

Similarly, we can see $\mathscr{L}^*_{m,\eta}(\mathscr{K}^*_{n,\lambda}(g;y,y_1)) = \mathscr{L}^*_{m,\eta}(g;y,y_1)$.

Let the weighted function ρ be $\rho(y,y_1) = 1 + y^2 + y_1^2$. Take $B_\rho(\mathbb{R}_+^2) = \{g : |g(y,y_1)| \le M_g \rho(y,y_1) \quad M_g > 0\}$. We denote the set of k-times continuously differentiable functions on $\mathbb{R}_+^2 = \{(y,y_1) \in \mathbb{R}^2 : y, y_1 \in [0,\infty)\}$ by $C^{(k)}(\mathbb{R}_+^2)$. We also denote the class of functions such that

$$C_\rho(\mathbb{R}_+^2) = \{g : g \in B_\rho \cap C_\rho(\mathbb{R}_+^2)\},$$

$$C_\rho^k(\mathbb{R}_+^2) = \left\{g : g \in C_\rho(\mathbb{R}_+^2)\,; \text{ such that } \lim_{(y,y_1)\to\infty} \frac{g(y,y_1)}{\rho(y,y_1)} = k_g < \infty\right\},$$

$$C_\rho^0(\mathbb{R}_+^2) = \left\{g : g \in C_\rho^k(\mathbb{R}_+^2)\,; \text{ such that } \lim_{(y,y_1)\to\infty} \frac{g(y,y_1)}{\rho(y,y_1)} = 0\right\}. \tag{70}$$

Let the norm on B_ρ be defined as $\|g\|_\rho = \sup_{(y,y_1)\in\mathbb{R}_+^2}(|g(y,y_1)|/\rho(y,y_1))$.

For all $g \in C_\rho^0(\mathbb{R}_+^2)$ and $\delta_1, \delta_2 > 0$, the weighted modulus of continuity is given as

$$\omega_\rho(g;\delta_1,\delta_2) = \sup_{(y,y_1)\in[0,\infty)} \sup_{0\le|\alpha|\le\delta_1, 0\le|\beta|\le\delta_2} \cdot \frac{|g(y+\alpha, y_1+\beta) - g(y,y_1)|}{\rho(y,y_1)\rho(\alpha,\beta)}, \tag{71}$$

and for any $r_1, r_2 > 0$ satisfying the inequality

$$\omega_\rho(g;r_1\delta_1, r_2\delta_2) \le 4(1+r_1)(1+r_2)(1+\delta_1^2) \cdot (1+\delta_2^2)\omega_\rho(g;\delta_1,\delta_2), \tag{72}$$

it also follows that

$$\begin{aligned} &|g(s,t) - g(y,y_1)| \\ &\quad \le \rho(y,y_1)\rho(|s-y|, |t-y_1|)\omega_\rho(g;|s-y|,|t-y_1|) \\ &\quad \le (1+y^2+y_1^2)(1+(s-y)^2) \\ &\qquad \cdot (1+(t-y_1)^2)\omega_\rho(g;|s-y|,|t-y_1|). \end{aligned} \tag{73}$$

Theorem 16. *For all $g \in C_\rho^0(\mathbb{R}_+^2)$ and sufficiently large $n, m \in \mathbb{N}$*

$$\begin{aligned} &\frac{|\mathcal{I}^*_{n,m}(g;y,y_1) - g(y,y_1)|}{(1+y^2+y_1^2)} \\ &\quad \le \xi_{y,y_1}(1+O(n^{-1}))(1+O(m^{-1}))\omega_\rho \\ &\qquad \cdot \left(g;O\left(n^{-\frac{1}{2}}\right), O\left(m^{-\frac{1}{2}}\right)\right), \end{aligned} \tag{74}$$

where $\xi_{y,y_1} = (1+(y+1)+M_1(y+1)^2+\sqrt{M_3}(y+1)^3)(1+(y_1+1)+M_2(y_1+1)^2+\sqrt{M_4}(y_1+1)^3)$, and M_1, M_2, M_3, $M_4 > 0$.

Proof. In view of the above explanation for all $\delta_n, \delta_m > 0$, we see that

$$\begin{aligned} &|g(s,t) - g(y,y_1)| \\ &\quad \le 4(1+y^2+y_1^2)(1+(s-y)^2)(1+(t-y_1)^2) \\ &\qquad \times \left(1+\frac{|s-y|}{\delta_n}\right)\left(1+\frac{|t-y_1|}{\delta_m}\right) \\ &\qquad \cdot (1+\delta_n^2)(1+\delta_m^2)\omega_\rho(g;\delta_n,\delta_m) \\ &\quad = 4(1+y^2+y_1^2)(1+\delta_n^2)(1+\delta_m^2) \\ &\qquad \times \left(1+\frac{|s-y|}{\delta_n}+(s-y)^2+\frac{|s-y|}{\delta_n}+(s-y)^2\right) \\ &\qquad \times \left(1+\frac{|t-y_1|}{\delta_m}+(t-y_1)^2+\frac{|t-y_1|}{\delta_m}(t-y_1)^2\right)\omega_\rho \\ &\qquad \cdot (g;\delta_n,\delta_m). \end{aligned} \tag{75}$$

On applying the operators $\mathcal{I}^*_{n,m}$, we get

$$\begin{aligned} &|\mathcal{I}^*_{n,m}(g;y,y_1) - g(y,y_1)| \\ &\quad \le \mathcal{I}^*_{n,m}(|g(s,t)-g(y,y_1)|;y,y_1)4(1+y^2+y_1^2) \\ &\qquad \times \mathcal{I}^*_{n,m}\left(1+\frac{|s-y|}{\delta_n}+(s-y)^2+\frac{|s-y|}{\delta_n}(s-y)^2;y,y_1\right) \\ &\qquad \times \mathcal{I}^*_{n,m}\left(1+\frac{|t-y_1|}{\delta_m}+(t-y_1)^2+\frac{|t-y_1|}{\delta_m}\right. \\ &\qquad \left.\cdot (t-y_1)^2;y,y_1\right) \times (1+\delta_n^2)(1+\delta_m^2)\omega_\rho(g;\delta_n,\delta_m) \\ &\quad = 4(1+y^2+y_1^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\rho(g;\delta_n,\delta_m) \\ &\qquad \times \left(1+\frac{1}{\delta_n}\mathcal{I}^*_{n,m}(|s-y|;y,y_1)+\mathcal{I}^*_{n,m}((s-y)^2;y,y_1)\right. \\ &\qquad \left.+\frac{1}{\delta_n}\mathcal{I}^*_{n,m}(|s-y|(s-y)^2;y,y_1)\right) \\ &\qquad \times \left(1+\frac{1}{\delta_m}\mathcal{I}^*_{n,m}(|t-y_1|;y,y_1)\right)\mathcal{I}^*_{n,m}((t-y_1)^2;y,y_1) \\ &\qquad +\frac{1}{\delta_m}\mathcal{I}^*_{n,m}(|t-y_1|(t-y_1)^2;y,y_1). \end{aligned} \tag{76}$$

From Cauchy-Schwarz inequality, we see

$$\begin{aligned}
&\left|\mathcal{J}^*_{n,m}(g;y,y_1)-g(y,y_1)\right| \\
&\quad\le 4\left(1+y^2+y_1^2\right)\left(1+\delta_n^2\right)\left(1+\delta_m^2\right)\omega_\rho(g;\delta_n,\delta_m) \\
&\qquad\times\left[1+\frac{1}{\delta_n}\sqrt{\mathcal{J}^*_{n,m}\left((s-y)^2;y,y_1\right)}+\mathcal{J}^*_{n,m}\left((s-y)^2;y,y_1\right)\right. \\
&\qquad\left.+\frac{1}{\delta_n}\sqrt{\mathcal{J}^*_{n,m}\left((s-y)^2;y,y_1\right)}\sqrt{\mathcal{J}^*_{n,m}\left((s-y)^4;y,y_1\right)}\right] \\
&\qquad\times\left[1+\frac{1}{\delta_m}\sqrt{\mathcal{J}^*_{n,m}\left((t-y_1)^2;y,y_1\right)}\right. \\
&\qquad+\mathcal{J}^*_{n,m}\left((t-y_1)^2;y,y_1\right)+1 \\
&\qquad+\frac{1}{\delta_m}\sqrt{\mathcal{J}^*_{n,m}\left((t-y_1)^2;y,y_1\right)} \\
&\qquad\left.+\sqrt{\mathcal{J}^*_{n,m}\left((t-y_1)^4;y,y_1\right)}\right].
\end{aligned} \tag{77}$$

From Lemma 14 we get

$$\begin{aligned}
&\left|\mathcal{J}^*_{n,m}(g;y,y_1)-g(y,y_1)\right| \\
&\quad\le 4\left(1+y^2+y_1^2\right)\left(1+\delta_n^2\right)\left(1+\delta_m^2\right)\omega_\rho(g;\delta_n,\delta_m) \\
&\qquad\times\left[1+\frac{1}{\delta_n}\sqrt{O\left(\frac{1}{n}\right)}(y+1)+M_1(y+1)^2\right. \\
&\qquad\left.+\frac{1}{\delta_n}\sqrt{O\left(\frac{1}{n}\right)}\sqrt{M_3}(y+1)^3\right] \\
&\qquad\times\left[1+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{m}\right)}(y_1+1)+M_2(y_1+1)^2\right. \\
&\qquad\left.+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{m}\right)}\sqrt{M_4}(y_1+1)^3\right].
\end{aligned} \tag{78}$$

By choosing $\delta_n = O(n^{-1/2})$ and $\delta_m = O(m^{-1/2})$, we arrived to our desired results.

7. Conclusion

The motivation of this present article is to provide the generalized error estimation of convergence rather than the classical Dunkl-Szász-Mirakyan operators. Here, we have defined Szász-Jakimovski-Leviatan operators by using the Appel polynomials with the aid of a new parameter $\lambda \in [0,\infty)$. These types of approximation are able to give the generalized results and error estimation in comparison to earlier study demonstrations. We have obtained the approximations via the well-known weighted Korovkin's spaces and investigated approximations in Peetre's K-functional and Lipschitz spaces with the aid of modulus of continuity. Further, we have also obtained the approximation in A-statistical convergence. Lastly, we have studied the approximation properties for the bivariate case.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors are grateful to this manuscript and declare that have no competing interest.

Authors' Contributions

All contents of this research article are checked and agreed to the integrity and accuracy of this manuscript.

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Research Article

Fixed Point Problems in Cone Rectangular Metric Spaces with Applications

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In this paper, we introduce an ordered implicit relation and investigate some new fixed point theorems in a cone rectangular metric space subject to this relation. Some examples are presented as illustrations. We obtain a homotopy result as an application. Our results generalize and extend several fixed point results in literature.

1. Introduction

Many authors generalized the classical concept of metric space, by changing the metric conditions partially. Branciari [1] introduced rectangular metric space (RMS), where the triangular inequality condition of metric space was replaced by rectangular inequality. He also proved an analog of the Banach contraction principle in rectangular metric spaces. Azam and Arshad [2] mentioned some necessary conditions to get a unique fixed point for Kannan-type mappings in this context. Later, Karapinar et al. [3] investigated some fixed points for (ψ, ϕ) contractions on rectangular metric spaces. On the other hand, Di Bari and Vetro [4] used (ψ, ϕ)-weakly contractive condition to give an extension of the results in [3]. Subsequently, a number of authors were engrossed in rectangular metric spaces and proved the existence and uniqueness of fixed point theorems for certain types of mappings [2–6].

The significance of the Banach contraction principle lies in the fact that it is a very essential tool to check the existence of solutions for differential equations, integral equations, matrix equations, and functional equations made by mathematical models of real-world problems. There has been a tendency for consistent theorists to improve both the underlying space and the contractive condition (explicit type) used by Banach [7] under the effect of one of the structures like order metric structure [8, 9], graphic metric structure [10, 11], multivalued mapping structure [12–14], α-admissible mapping structure [15], comparison functions, and auxiliary functions. The process of developing new fixed point theorems in the complete metric spaces is in progress under various new restrictions. In this regard, we can find very nice results by Debnath et al. that appeared in [10, 12, 14].

Later on, Popa [16] introduced self-mappings satisfying implicit relation and obtained fixed points, under the effect of these functions. Popa [16–18] obtained some fixed point theorems in metric spaces. However, scrutiny into the fixed points of self-mappings satisfying implicit relations in order metric structure was made by Beg and Butt [19, 20], and some common fixed point theorems were established by Berinde and Vetro [21, 22] and Sedghi et al. [23]. Huang and Zhang [24] introduced cone metric by replacing real numbers with ordering Banach spaces and established a convergence criterion for sequences in cone metric space to generalize Banach fixed point theorem. Huang and Zhang [24] considered the concept of normal cone for their drawn outcomes; however, Rezapour and Hamlbarani [25] left the normality condition in some results by Huang. Many authors have investigated fixed point theorems and common fixed point theorems of self-mappings for normal and nonnormal cones in cone metric spaces (see [26–29]).

Azam et al. [5] introduced the notion of a cone rectangular metric space and proved the Banach contraction principle in this context. In 2012, Rashwan [6] extended this idea as a continuation, which improved the results in [5]. The appealing nature of these spaces has enticed scrutiny into fixed point theorems for various contractions on cone rectangular metric spaces (see [5, 6]).

In this paper, we continue the study initiated by Azam et al. [5] subject to an ordered implicit relation. Since every cone metric space is a cone rectangular metric space but not conversely, therefore we prefer to establish results in cone rectangular metric spaces. These results are supported by some examples and an application in homotopy theory.

2. Preliminaries

Definition 1. A binary relation $\mathscr{R}$ over a set $\mathbb{Y} \neq \phi$ defines a partial order if $\mathscr{R}$ has the following axioms:

(1) reflexive

(2) antisymmetric

(3) transitive

A set having partial order $\mathscr{R}$ is known as a partially ordered set denoted by $(\mathbb{Y}, \mathscr{R})$.

In the present article, $\mathscr{E}$ stands for a real Banach space. Now, we present some definitions and relevant results, which will be required in the sequel.

Definition 2 (see [24]). *A subset $\mathscr{P}$ of $\mathscr{E}$ is called a cone if and only if the following conditions are satisfied:*

(1) $\mathscr{P}$ is closed, nonempty, and $\mathscr{P} \neq \{0\}$

(2) $\alpha\sigma + \beta\varsigma \in \mathscr{P}$, for all $\sigma, \varsigma \in \mathscr{P}$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha, \beta \geq 0$

(3) $\mathscr{P} \cap (-\mathscr{P}) = \{0\}$

Given $\mathscr{P} \subset \mathscr{E}$, define the partial order $\preceq$ with respect to $\mathscr{P}$ as follows:

$\sigma \leq \varsigma$ if and only if $\varsigma - \sigma \in \mathscr{P}$ for all $\sigma, \varsigma \in \mathscr{E}$.(5)

$\sigma \prec \varsigma$ represents that $\sigma \preceq \varsigma$ but $\sigma \neq \varsigma$, while $\sigma \ll \varsigma$ stands for $\varsigma - \sigma \in \mathscr{P}^{\circ}$ (interior of $\mathscr{P}$).

Definition 3 (see [24]). *The cone $\mathscr{P} \subseteq \mathscr{E}$ is called normal if, for all $\sigma, \varsigma \in \mathscr{P}$, there exists $\mathscr{K} > 0$ such that*

$$0 \preceq \sigma \preceq \varsigma \Rightarrow \|\sigma\| \leq \mathscr{K}\|\varsigma\|. \quad (1)$$

Throughout this paper, we assume $\mathbb{Y} = (\mathbb{Y}, \mathscr{R})$ and $\preceq$ a partial order with respect to the cone $\mathscr{P}$ defined in $\mathscr{E}$. If $\mathbb{Y} \subseteq \mathscr{E}$, then $\mathscr{R}$ and $\preceq$ are identical; otherwise, they are different.

Definition 4 (see [24]). *Let $\mathbb{Y}$ be a nonempty set, and $d_c : \mathbb{Y} \times \mathbb{Y} \mapsto \mathscr{E}$ satisfies the following:*

(d1) $d_c(\sigma, \varsigma) \geq 0$, $\forall \sigma, \varsigma \in \mathbb{Y}$ and $d_c(\sigma, \varsigma) = 0$ if and only if $\sigma = \varsigma$

(d2) $d_c(\sigma, \varsigma) = d_c(\varsigma, \sigma)$

(d3) $d_c(\sigma, \xi) \preceq d_c(\sigma, \varsigma) + d_c(\varsigma, \xi)$, $\forall \sigma, \varsigma, \xi \in \mathbb{Y}$

Then, d_c is called a cone metric on $\mathbb{Y}$, and $(\mathbb{Y}, d_c)$ is then known as a cone metric space.

Example 1 (see [5]). *Let $\mathbb{Y} = \mathbb{R}$, $\mathscr{E} = \mathbb{R}^2$, and $\mathscr{P} = \{(\sigma, \varsigma) \in \mathscr{E} : \sigma, \varsigma \geq 0\} \subset \mathbb{R}^2$. Define $d_c : \mathbb{Y} \times \mathbb{Y} \to \mathscr{E}$ by*

$$d_c(\sigma, \varsigma) = (|\sigma - \varsigma|, \alpha |\sigma - \varsigma|), \quad (2)$$

where $\alpha \geq 0$ is a constant. Then, d_c defines a cone metric on $\mathbb{Y}$.

Proposition 5 (see [5]). *Consider a cone metric space $(\mathbb{Y}, d_c)$, with cone $\mathscr{P}$. Then, for $u, v, w \in \mathscr{E}$, we have*

(1) If $u \preceq \alpha u$ and $\alpha \in [0, 1)$, then $u = 0$

(2) If $0 \preceq u \ll v$ for each $0 \ll v$, then $u = 0$

(3) If $u \preceq v$ and $v \ll w$, then $u \ll w$

Surely, cone metric space "being space" generalizes metric space, because in cone metric space, the range of a metric function is an ordered vector space instead of real numbers. Although the set of real numbers is an ordered vector space, we can find many significant ordered vector spaces in the literature (see [25, 27, 28, 30]). In Theorem 1.4 and Lemma 2.1 that appeared in [31, 32], respectively, the authors developed a metric depending on a given cone metric and proved that a complete cone metric space is always a complete metric space, and then, this relationship between metric and cone metric led them to say that every contraction mapping in a cone metric space is essentially contraction mapping in a metric space.

This paper addresses the fixed point results in the cone rectangular metric spaces. We know that every metric is a rectangular metric but rectangular metric needs not to be a metric (see [1–3, 33]). Also, we know that every cone metric is a cone rectangular metric but conversely does not hold in general (see [5, 6, 29]). In view of observations given in [5, 29], we infer that results in this paper are independent of what authors investigated in [31, 32]. The implicit relation and hence the contractive condition employed are even new and original in the rectangular metric space. The theorems in this paper are new in rectangular metric space, but we choose the cone rectangular metric space for the sake of the generality of our results.

Definition 6 (see [24]). *Let A mapping $d_{cr} : \mathbb{Y} \times \mathbb{Y} \mapsto \mathscr{E}$ is said to be a cone rectangular metric if for all $\sigma, \varsigma, \xi, \upsilon \in \mathbb{Y}$ the following conditions are satisfied:*

(dR1) $0 \preceq d_{cr}(\sigma, \varsigma)$ and $d_{cr}(\sigma, \varsigma) = 0$ if and only if $\sigma = \varsigma$

(dR2) $d_{cr}(\sigma, \varsigma) = d_{cr}(\varsigma, \sigma)$

(dR3) $d_{cr}(\sigma, \upsilon) \preceq d_{cr}(\sigma, \varsigma) + d_{cr}(\varsigma, \xi) + d_{cr}(\xi, \upsilon)$ for all distinct $\varsigma, \xi \in \mathbb{Y} \setminus \{\sigma, \upsilon\}$

The cone rectangular metric space is denoted by $(\mathbb{Y}, d_{cr})$.

Example 2 (see [5]). *Let* $\mathscr{E} = \mathbb{R}^2$, $\mathscr{P} = \{(\sigma, \varsigma) \in \mathscr{E} : \sigma, \varsigma \geq 0\} \subset \mathbb{R}^2$, $\mathbb{Y} = \mathbb{R}$. *Take* $\beta > 0$, *and define*

$$d_{cr}(\sigma,\varsigma) = \begin{cases} (0,0) & \text{if } \sigma = \varsigma, \\ (3\beta, 3) & \text{if } \sigma, \varsigma \in \{1,2\}, \sigma \neq \varsigma, \\ (\beta, 1) & \text{if } \sigma \text{ and } \varsigma \text{ cannot be both at a time in } \{1,2\}, \sigma \neq \varsigma. \end{cases} \tag{3}$$

One can easily check that $(\mathbb{Y}, d_{cr})$ is a cone rectangular metric space, but not a cone metric space, since $d_{cr}(1,2) = (3\beta, 3) \succ d_{cr}(1,3) + d_{cr}(3,2) = (2\beta, 2)$.

Definition 7 (see [1]). *Let* $\mathscr{E}$ *be a real Banach space,* $(\mathbb{Y}, d_{cr})$ *be a cone rectangular metric space and* $c \in \mathscr{E}$ *with* $0 \ll c$.

(1) A sequence $\{\sigma_n\}$ in $(\mathbb{Y}, d_{cr})$ is called a Cauchy sequence, if there exists a natural number $N \in \mathbb{N}$ such that $d_{cr}(\sigma_n, \sigma_m) \ll c$ for all $n, m > N$

(2) The sequence $\{\sigma_n\}$ is said to be convergent if there exists an $N \in \mathbb{N}$ such that $d_{cr}(\sigma_n, \sigma) \ll c$ for all $n \geq N$ and $\sigma \in \mathbb{Y}$

(3) The $(\mathbb{Y}, d_{cr})$ is called complete if every Cauchy sequence converges in $\mathbb{Y}$

3. Ordered Implicit Relations

Many authors have used implicit relations to establish fixed point results and have applied these results to solve nonlinear functional equations (see [19–22, 34, 35]).

In this section, we define a new ordered implicit relation and explain it with an example. In the next section, we use it along with some other assumptions to develop some new fixed point theorems in the cone rectangular metric space.

Let $(\mathscr{E}, \|.\|)$ be a real Banach space and $B(\mathscr{E}, \mathscr{E})$ be the space of all bounded linear operators $T : \mathscr{E} \to \mathscr{E}$ with the usual norm $\|.\|_1$ defined in $B(\mathscr{E}, \mathscr{E})$ that is $\|T\|_1 = \sup_{v \in \mathscr{E}}(\|T(v)\|/\|v\|) v \neq 0$.

In this section, generalizing the idea of [16], we define the following notion:

Let $\mathscr{L} : \mathscr{E}^6 \to \mathscr{E}$ be a continuous operator which satisfies the conditions given below:

($\mathscr{L}_1$) $v_5 \preceq \upsilon_5$ and $v_6 \preceq \upsilon_6 \Rightarrow \mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) \preceq \mathscr{L}(v_1, v_2, v_3, v_4, \upsilon_5, \upsilon_6)$

($\mathscr{L}_2$) If $\mathscr{L}(v_1, v_2, v_2, v_1, v_1 + v_2 + v_3, v_1) \preceq 0$, then there exists an order preserving operator $S \in B(\mathscr{E}, \mathscr{E})$ with $\|S\|_1 < 1$ such that $v_1 \preceq S(v_2)$ and $v_3 \preceq S(v_1)$ for all $v_1, v_2, v_3 \in \mathscr{E}$ or if $\mathscr{L}(v_1, v_2, v_1, v_2, v_1, v_1 + v_2 + v_3) \preceq 0$, then $v_2 \preceq S(v_3)$ and $v_1 \preceq S(v_2)$ for all $v_1, v_2, v_3 \in \mathscr{E}$

($\mathscr{L}_3$) $\mathscr{L}(v, 0, 0, v, v, 0) \succ 0$ whenever $\|v\| > 0$

Let $\mathscr{G} = \{\mathscr{L} : \mathscr{E}^6 \to \mathscr{E} \mid \mathscr{L}$ satisfies conditions $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3\}$.

Example 3. Let $\preceq$ *be the partial order with respect to cone* $\mathscr{P}$ *as defined in Section2and let* $(\mathscr{E}, \|.\|)$ *be a real Banach space. For all* $v_i \in \mathscr{E}(i = 1 \text{ to } 6)$, $\alpha \in (0, 1/3)$ *and* $(1 + \alpha)/2 \leq \beta \leq 1 + \alpha$, *define* $\mathscr{L} : \mathscr{E}^6 \to \mathscr{E}$, *by*

$$\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) = \alpha\{v_2 + v_3 + v_4\} - \beta(v_5 + v_6) + v_1. \tag{4}$$

Then, the operator $\mathscr{L} \in \mathscr{G}$:

($\mathscr{L}_1$) Let $v_5 \preceq \gamma_5$ and $v_6 \preceq \gamma_6$, then $\gamma_5 - v_5 \in \mathscr{P}$ and $\gamma_6 - v_6 \in \mathscr{P}$. Now, we show that $\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) - \mathscr{L}(v_1, v_2, v_3, v_4, \gamma_5, \gamma_6) \in \mathscr{P}$. Given that $\gamma_5 - v_5 \in \mathscr{P}$ and $\gamma_6 - v_6 \in \mathscr{P}$ and by Definition 2 (2), we have

$$\begin{aligned} &\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) - \mathscr{L}(v_1, v_2, v_3, v_4, \gamma_5, \gamma_6) \\ &\quad = \alpha\{v_2 + v_3 + v_4\} - \beta(v_5 + v_6) + v_1 - (\alpha\{v_2 + v_3 + v_4\} \\ &\qquad - \beta(\gamma_5 + \gamma_6) + v_1) = \beta(\gamma_5 - v_5 + \gamma_6 - v_6) \in \mathscr{P}. \end{aligned} \tag{5}$$

Thus, $\mathscr{L}(v_1, v_2, v_3, v_4, \gamma_5, \gamma_6) \preceq \mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6)$.

($\mathscr{L}_2$) Let $v_1, v_2, v_3 \in \mathscr{E}$ be such that $0 \preceq v_1, 0 \preceq v_2, 0 \preceq v_3$. If $\mathscr{L}(v_1, v_2, v_2, v_1, v_1 + v_2 + v_3, v_1) \preceq 0$ then, we have $-v_1 - \alpha\{v_2 + v_2 + v_1\} + \beta(v_1 + v_2 + v_3 + v_1) \in \mathscr{P}$.

So $(2\beta - \alpha - 1)v_1 + (\beta - 2\alpha)v_2 + \beta v_3 \in \mathscr{P}$.

By Definition 2 (2), we have for $2\beta \geq \alpha + 1$ either

$$(2\beta - \alpha - 1)v_1 + (\beta - 2\alpha)v_2 \in \mathscr{P} \tag{6}$$

or

$$(2\beta - \alpha - 1)v_1 + \beta v_3 \in \mathscr{P} \tag{7}$$

or

$$(\beta - 2\alpha)v_2 + \beta v_3 \in \mathscr{P}. \tag{8}$$

For (6), if $v_1 = 0$ and $v_2 \neq 0$, then $(\beta - 2\alpha)v_2 \in \mathscr{P}$. Thus, there exists $S : \mathscr{E} \to \mathscr{E}$ defined by $S(v_2) = \eta v_2$ ($\eta = (\beta - 2\alpha)$ is a scalar) such that $v_1 \preceq S(v_2)$. Now, if $v_2 = 0$ and $v_1 \neq 0$, then, $(2\beta - 1 - \alpha)v_1 \in \mathscr{P}$. So, there exists $S : \mathscr{E} \to \mathscr{E}$ defined by $S(v_1) = \eta v_1$ ($\eta = (2\beta - 1 - \alpha)$ is a scalar) such that $v_2 \preceq S(v_1)$, for some $\alpha \in (0, 1/3)$. For if both $v_1 \neq 0$ and $v_2 \neq 0$, then we get an absurdity.

For (7), if $v_3 = 0$ and $v_1 \neq 0$, then $(2\beta - \alpha - 1)v_1 \in \mathscr{P}$. Thus, there exists $S : \mathscr{E} \to \mathscr{E}$ defined by $S(v_1) = \eta v_1$ ($\eta = (2\beta - \alpha - 1)$ is a scalar) such that $v_3 \preceq S(v_1)$. Now, if $v_3 \neq 0$ and $v_1 = 0$, then, $\beta v_3 \in \mathscr{P}$. So, there exists $S : \mathscr{E} \to \mathscr{E}$ defined by $S(v_3) = \eta v_3$ ($\eta = \beta$ is a scalar) such that $v_1 \preceq S(v_3)$. For if both $v_1 \neq 0$ and $v_3 \neq 0$, then we get an absurdity. Similar arguments hold for (8).

($\mathscr{L}_3$) Let $v \in \mathscr{E}$ be such that $\|v\| > 0$ and consider, $0 \preceq \mathscr{L}(v, 0, 0, v, v, 0)$ then $(1 + \alpha - \beta)v \in \mathscr{P}$, which holds whenever $\|v\| > 0$.

Similarly, the operators $\mathscr{L} : \mathscr{E}^6 \to \mathscr{E}$ defined by

(1) $\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - \alpha\{v_5 + v_6\}$; $\alpha \in [1,\infty)$

(2) $\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_3 - \alpha v_5 - \beta v_6$; $\alpha + \beta < 1$, α, $\beta > 0$

(3) $\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - \alpha\{v_3 + v_4\} - (1-\alpha)\beta \max\{v_5, v_6\}$; $\alpha \in [0, 1)$ and $\beta \in [1/2, 1)$

(4) $\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - \alpha v_2$ for some $\alpha \in [0, 1)$

(5) $\mathscr{L}(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 - k(v_3 + v_4)$ for some $k \in [0, 1/2)$ are members of $\mathscr{G}$

The following remark is essential in the sequel.

Remark 8. If $S \in B(\mathscr{E}, \mathscr{E})$,the Neumann series $I + S + S^2 + \cdots + S^n + \cdots$ converges if $\|S\|_1 < 1$ and diverges otherwise. Also, if $\|S\|_1 < 1$, then there exists $\lambda > 0$ such that $\|S\|_1 < \lambda < 1$ and $\|S^n\|_1 \leq \lambda^n < 1$.

4. New Results

Recently, Popa [16] has employed implicit type contractive condition on self-mapping to obtain some fixed point theorems. Ran and Reurings [9] have presented an analog of Banach fixed point theorem for monotone self-mappings in an ordered metric space. Huang and Zhang [24] introduced the idea of cone metric spaces and obtained analogs of Banach fixed point theorem, Kannan fixed point theorem, and Chatterjea fixed point theorem in cone metric spaces. In this section, we prove some fixed point results for ordered implicit relations in a cone rectangular metric space which improves the results in [9, 16, 24]. We derive these results under two different partial orders: one defined in underlying set and the other in real Banach space.

Theorem 9. *Let $(\mathbb{Y}, d_{cr})$ be a complete cone rectangular metric space and $\mathscr{P} \subset \mathscr{E}$ be a cone. Let $f : \mathbb{Y} \to \mathbb{Y}$. If there exist $T \in B(\mathscr{E}, \mathscr{E})$, identity operator $I : \mathscr{E} \to \mathscr{E}$ and $\mathscr{L} \in \mathscr{G}$ such that, for all comparable elements $\sigma, \kappa \in \mathbb{Y}$*

$$\begin{aligned}&(I-T)^2(I+T)(d_{cr}(\sigma, f(\sigma))) \preceq d_{cr}(\sigma,\kappa) \text{ implies } \mathscr{L}\\ &\cdot (d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma,\kappa), d_{cr}(\sigma, f(\sigma)), d_{cr}(\kappa, f(\kappa)), d_{cr}\\ &\cdot (\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) \preceq 0\end{aligned} \tag{9}$$

and

(1) there exists $\sigma_0 \in \mathbb{Y}$ such that $\sigma_0 \mathscr{R} f(\sigma_0)$

(2) for all $\sigma, \kappa \in \mathbb{Y}$, $\sigma \mathscr{R} \kappa$ implies $f(\sigma)\mathscr{R} f(\kappa)$

(3) for every $\{\sigma_n\} \subseteq \mathbb{Y}$, $d_{cr}(\sigma_n, \sigma_{n+1}) \preceq T(d_{cr}(\sigma_{n-1}, \sigma_n))$

(4) for a sequence $\{\sigma_n\}$ with $\sigma_n \to x^*$ whose all sequential terms are comparable, we have $\sigma_n \mathscr{R} x^*$ for all $n \in \mathbb{N}$ and $d_{cr}(x^*, f(x^*)) \preceq d_{cr}(x^*, f^2(x^*))$

Then, $x^* = f(x^*)$.

Proof. Let $\sigma_0 \in \mathbb{Y}$ be as assumed in (1). We construct a sequence $\{\sigma_n\}$ by $f(\sigma_{n-1}) = \sigma_n$ starting with $\sigma_0 \in \mathbb{Y}$. Then, $\sigma_0 \mathscr{R} \sigma_1$. By assumption (2), we have $\sigma_1 \mathscr{R} \sigma_2$, $\sigma_2 \mathscr{R} \sigma_3 \cdots \sigma_{n-1} \mathscr{R} \sigma_n$. For $\sigma = \sigma_0$ and $\kappa = \sigma_1$, we have by (9).

$$\begin{aligned}&(I-T)^2(I+T)(d_{cr}(\sigma_0, f(\sigma_0)))\\ &\quad = (I-T)^2(I+T)(d_{cr}(\sigma_0,\sigma_1)) \preceq d_{cr}(\sigma_0,\sigma_1) \text{ implies } \mathscr{L}\\ &\quad\times (d_{cr}(f(\sigma_0), f(\sigma_1)), d_{cr}(\sigma_0,\sigma_1), d_{cr}(\sigma_0, f(\sigma_0)), d_{cr}\\ &\quad\times (\sigma_1, f(\sigma_1)), d_{cr}(\sigma_0, f^2(\sigma_1)), d_{cr}(\sigma_1, f^2(\sigma_0))) \preceq 0,\end{aligned} \tag{10}$$

that is,

$$\mathscr{L}(d_{cr}(\sigma_1,\sigma_2), d_{cr}(\sigma_0,\sigma_1), d_{cr}(\sigma_0,\sigma_1), d_{cr}(\sigma_1,\sigma_2), d_{cr}(\sigma_0,\sigma_3), d_{cr}(\sigma_1,\sigma_2)) \preceq 0. \tag{11}$$

By (dR3), we have

$$d_{cr}(\sigma_0,\sigma_3) \preceq d_{cr}(\sigma_0,\sigma_1) + d_{cr}(\sigma_1,\sigma_2) + d_{cr}(\sigma_2,\sigma_3), \tag{12}$$

and so we rewrite (11) employing condition $(\mathscr{L}_1)$ as follows:

$$\begin{aligned}&\mathscr{L}(d_{cr}(\sigma_1,\sigma_2), d_{cr}(\sigma_0,\sigma_1), d_{cr}(\sigma_0,\sigma_1), d_{cr}(\sigma_1,\sigma_2), d_{cr}(\sigma_0,\sigma_1)\\ &\quad + d_{cr}(\sigma_1,\sigma_2) + d_{cr}(\sigma_2,\sigma_3), d_{cr}(\sigma_1,\sigma_2)) \preceq 0,\end{aligned} \tag{13}$$

and thus, by $(\mathscr{L}_2)$, there exists an order preserving operator $S \in B(\mathscr{E}, \mathscr{E})$ with $\|S\|_1 < 1$ such that

$$\begin{aligned}&d_{cr}(\sigma_1,\sigma_2) \preceq S(d_{cr}(\sigma_0,\sigma_1)),\\ &d_{cr}(\sigma_2,\sigma_3) \preceq S(d_{cr}(\sigma_1,\sigma_2)).\end{aligned} \tag{14}$$

Now, put $\sigma = \sigma_1$ and $\kappa = \sigma_2$ in (9) to have

$$\begin{aligned}&(I-T)^2(I+T)(d_{cr}(\sigma_1, f(\sigma_1)))\\ &\quad = (I-T)^2(I+T)(d_{cr}(\sigma_1,\sigma_2)) \preceq d_{cr}(\sigma_1,\sigma_2) \text{ implies } \mathscr{L}\\ &\quad\times (d_{cr}(f(\sigma_1), f(\sigma_2)), d_{cr}(\sigma_1,\sigma_2), d_{cr}(\sigma_1, f(\sigma_1)), d_{cr}\\ &\quad\times (\sigma_2, f(\sigma_2)), d_{cr}(\sigma_1, f(\sigma_3)), d_{cr}(\sigma_2, f(\sigma_2))) \preceq 0,\end{aligned} \tag{15}$$

that is,

$$\mathscr{L}(d_{cr}(\sigma_2,\sigma_3), d_{cr}(\sigma_1,\sigma_2), d_{cr}(\sigma_1,\sigma_2), d_{cr}(\sigma_2,\sigma_3), d_{cr}(\sigma_1,\sigma_4), d_{cr}(\sigma_2,\sigma_3)) \preceq 0. \tag{16}$$

By (dR3), we have

$$d_{cr}(\sigma_1,\sigma_4) \preceq d_{cr}(\sigma_1,\sigma_2) + d_{cr}(\sigma_2,\sigma_3) + d_{cr}(\sigma_3,\sigma_4), \tag{17}$$

and $(\mathscr{L}_1)$ implies

$$\mathscr{L}(d_{cr}(\sigma_2,\sigma_3),d_{cr}(\sigma_1,\sigma_2),d_{cr}(\sigma_1,\sigma_2),d_{cr}(\sigma_2,\sigma_3),d_{cr}(\sigma_1,\sigma_2)+d_{cr}(\sigma_2,\sigma_3)+d_{cr}(\sigma_3,\sigma_4),d_{cr}(\sigma_2,\sigma_3))\preceq 0. \tag{18}$$

By $(\mathscr{L}_2)$, there exists $S\in B(\mathscr{E},\mathscr{E})$ with $\|S\|_1<1$ such that

$$d_{cr}(\sigma_3,\sigma_4)\preceq S(d_{cr}(\sigma_2,\sigma_3))\preceq S^2(d_{cr}(\sigma_1,\sigma_2))\preceq S^3(d_{cr}(\sigma_0,\sigma_1)). \tag{19}$$

By continuing this pattern, we can construct a sequence $\{\sigma_n\}$ such that $\sigma_n\mathscr{R}\sigma_{n+1}$ with $\sigma_{n+1}=f(\sigma_n)$, and

$$\begin{aligned}&(I-T)^2(I+T)(d_{cr}(\sigma_{n-1},f(\sigma_{n-1})))\\&\quad=(I-T)^2(I+T)(d_{cr}(\sigma_{n-1},\sigma_n))\\&\quad\preceq d_{cr}(\sigma_{n-1},\sigma_n)\text{ implies }d_{cr}(\sigma_n,\sigma_{n+1})\\&\quad\preceq S(d_{cr}(\sigma_{n-1},\sigma_n))\preceq S^2(d_{cr}(\sigma_{n-2},\sigma_{n-1}))\\&\quad\preceq\cdots\preceq S^n(d_{cr}(\sigma_0,\sigma_1)).\end{aligned} \tag{20}$$

For $m,n\in\mathbb{N}$ with $m>n$, we have by Remark 8

$$\begin{aligned}d_{cr}(\sigma_n,\sigma_m)&\le d_{cr}(\sigma_n,\sigma_{n+1})+d_{cr}(\sigma_{n+1},\sigma_{n+2})+.\cdots\cdots\\&\quad+d_{cr}(\sigma_{m-1},\sigma_m)\\&\le S^n(d_{cr}(\sigma_0,\sigma_1))+S^{n+1}(d_{cr}(\sigma_0,\sigma_1))\\&\quad+\cdots\cdots+S^{m-1}(d_{cr}(\sigma_0,\sigma_1))\\&=\left(S^n+S^{n+1}+\cdots+S^{m-1}\right)(d_{cr}(\sigma_0,\sigma_1)),\\&\le\left\{S^n\left(1+S+\cdots+S^{m-n-1}+\cdots\right)\right\}(d_{cr}(\sigma_0,\sigma_1))\\&=\left\{S^n(I-S)^{-1}\right\}(d_{cr}(\sigma_0,\sigma_1)).\end{aligned} \tag{21}$$

Since $\|S\|_1<1$, so, $S^n\to 0$ as $n\to\infty$. Thus, $\lim_{n\to\infty}d_{cr}(\sigma_n,\sigma_m)=0$ which implies that $\{\sigma_n\}$ is a Cauchy sequence in $\mathbb{Y}$. Since $(\mathbb{Y},d_{cr})$ is a complete cone rectangular metric space, so, there exists $x^*\in\mathbb{Y}$ such that $\sigma_n\to x^*$ as $n\to\infty$. Equivalently, there exists a natural number N_2 such that

$$d_{cr}(\sigma_n,x^*)\ll c\text{ for all }n\ge N_2. \tag{22}$$

We claim that

$$(I-T)^2(I+T)(d_{cr}(\sigma_n,f(\sigma_n)))\preceq d_{cr}(\sigma_n,x^*). \tag{23}$$

We assume against our claim that

$$(I-T)^2(I+T)(d_{cr}(\sigma_n,f(\sigma_n)))\succ d_{cr}(\sigma_n,x^*),(I-T)^2(I+T)\cdot(d_{cr}(\sigma_{n+2},f(\sigma_{n+2})))\succ d_{cr}(\sigma_{n+2},x^*)\text{ for some }n\in\mathbb{N}. \tag{24}$$

By (dR3), (9), and assumption (3), we have

$$\begin{aligned}&d_{cr}(\sigma_n,f(\sigma_n))\le d_{cr}(\sigma_n,x^*)+d_{cr}(x^*,\sigma_{n+2})+d_{cr}\\&\cdot(\sigma_{n+2},\sigma_{n+1})\prec(I-T)^2(I+T)(d_{cr}(\sigma_n,f(\sigma_n)))+(I-T)^2\\&\cdot(I+T)(d_{cr}(\sigma_{n+2},f(\sigma_{n+2})))+T(d_{cr}(\sigma_n,f(\sigma_n)))\prec(I-T)^2\\&\cdot(I+T)(d_{cr}(\sigma_n,f(\sigma_n)))+(I-T)^2(I+T)T^2\\&\cdot(d_{cr}(\sigma_n,f(\sigma_n)))+T(d_{cr}(\sigma_n,f(\sigma_n))),(I-T)\\&\cdot(d_{cr}(\sigma_n,f(\sigma_n)))\prec(I-T)^2(I+T)\left(I+T^2\right)\\&\cdot(d_{cr}(\sigma_n,f(\sigma_n)))d_{cr}(\sigma_n,f(\sigma_n))\prec(I-T)^{-1}(I-T)^2(I+T)\\&\cdot\left(I+T^2\right)(d_{cr}(\sigma_n,f(\sigma_n)))\prec\left(I-T^4\right)d_{cr}(\sigma_n,f(\sigma_n))).\end{aligned} \tag{25}$$

Thus, $T^4(d_{cr}(\sigma_n,f(\sigma_n)))\prec 0$, which is an absurdity. Hence, for each $n\ge 1$, we have

$$(I-T)^2(I+T)(d_{cr}(\sigma_n,f(\sigma_n)))\preceq d_{cr}(\sigma_n,x^*). \tag{26}$$

Assume that $\|d_{cr}(x^*,f(x^*))\|>0$. As $\sigma_{n-1}\mathscr{R}\sigma_n$ and by the assumption (4), we have $\sigma_n\mathscr{R}x^*$ for all $n\in\mathbb{N}$ and then by (9), we get

$$\mathscr{L}\begin{pmatrix}d_{cr}(f(\sigma_n),f(x^*)),d_{cr}(\sigma_n,x^*),d_{cr}(\sigma_n,f(\sigma_n)),d_{cr}(x^*,f(x^*)),\\d_{cr}\left(\sigma_{n-1},f^2(x^*)\right),d_{cr}\left(f^2(\sigma_n),f(\sigma_n)\right)\end{pmatrix}\preceq 0. \tag{27}$$

Letting $n\to\infty$ and in view of assumption (4) and (26), we have

$$\begin{aligned}&\mathscr{L}(d_{cr}(x^*,f(x^*)),d_{cr}(x^*,x^*),d_{cr}(x^*,x^*),d_{cr}(x^*,f(x^*)),d_{cr}\\&\cdot\left(x^*,f^2(x^*)\right),d_{cr}(x^*,x^*))\preceq 0,\mathscr{L}(d_{cr}(x^*,f(x^*)),0,0,d_{cr}\\&\cdot(x^*,f(x^*)),d_{cr}\left(x^*,f^2(x^*)\right),0)\preceq 0.\end{aligned} \tag{28}$$

By $(\mathscr{L}_1)$, we have

$$\mathscr{L}(d_{cr}(x^*,f(x^*)),0,0,d_{cr}(x^*,f(x^*)),d_{cr}(x^*,f(x^*)),0)\preceq 0. \tag{29}$$

This is a contradiction to $(\mathscr{L}_3)$. Thus, $\|d_{cr}(x^*,f(x^*))\|=0$. Hence, $d_{cr}(x^*,f(x^*))=0$. It follows from (dR1) that $x^*=f(x^*)$.

Theorem 10. *Let $(\mathbb{Y},d_{cr})$ be a complete cone rectangular metric space and f be a self-mapping on $\mathbb{Y}$. If for all comparable elements $\sigma,\kappa\in\mathbb{Y}$, there exist $T\in B(\mathscr{E},\mathscr{E})$, identity operator $I:\mathscr{E}\to\mathscr{E}$ and $\mathscr{L}\in\mathscr{G}$ such that*

$$\begin{aligned}&(I-T)^2(I+T)(d_{cr}(\sigma,f(\sigma))\preceq d_{cr}(\sigma,\kappa)\text{ implies }\mathscr{L}\\&\cdot(d_{cr}(f(\sigma),f(\kappa)),d_{cr}(\sigma,\kappa),d_{cr}(\sigma,f(\sigma)),d_{cr}\\&\cdot(\kappa,f(\kappa)),d_{cr}\left(\sigma,f^2(\kappa)\right),d_{cr}\left(\kappa,f^2(\sigma)\right))\preceq 0,\end{aligned} \tag{30}$$

and

(1) there exists $\sigma_0 \in \mathbb{Y}$ such that $f(\sigma_0)\mathscr{R}\sigma_0$

(2) for any $\sigma, \kappa \in \mathbb{Y}$, $\sigma\mathscr{R}\kappa$ implies $f(\kappa)\mathscr{R}f(\sigma)$

(3) for every $\{\sigma_n\} \subseteq \mathbb{Y}$, $d_{cr}(\sigma_n, \sigma_{n+1}) \preceq T(d_{cr}(\sigma_{n-1}, \sigma_n))$

(4) for a sequence $\{\sigma_n\}$ with $\sigma_n \to x^*$ whose all sequential terms are comparable, we have $\sigma_n\mathscr{R}x^*$ for all $n \in \mathbb{N}$ and $d_{cr}(x^*, f(x^*)) \preceq d_{cr}(x^*, f^2(x^*))$

Then, f has a fixed point in $\mathbb{Y}$.

Proof. Let σ_0 be in $\mathbb{Y}$ as assumed in (1). Define the sequence $\{\sigma_n\}$ by $\sigma_n = f(\sigma_{n-1})$ for all n. Since $\sigma_1 = f(\sigma_0)\mathscr{R}\sigma_0$ and by assumption (2) $\sigma_1 = f(\sigma_0)\mathscr{R}f(\sigma_1) = \sigma_2$, repeated application of assumption (2) gives us either $\sigma_n\mathscr{R}\sigma_{n-1}$ or $\sigma_{n-1}\mathscr{R}\sigma_n$ for each n. By (30), we have for $\sigma = \sigma_1$ and $\kappa = \sigma_0$

$$\begin{aligned}&(I-T)^2(I+T)\left(d_{cr}(f(\sigma_0), \sigma_0\right) = (I-T)^2(I+T)\\ &\quad\cdot (d_{cr}(\sigma_1, \sigma_0))) \preceq d_{cr}(\sigma_1, \sigma_0) \text{ implies } \mathscr{L}\\ &\quad\cdot (d_{cr}(f(\sigma_1), f(\sigma_0)), d_{cr}(\sigma_1, \sigma_0), d_{cr}(\sigma_1, f(\sigma_1)), d_{cr}\\ &\quad\cdot (\sigma_0, f(\sigma_0)), d_{cr}\left(\sigma_1, f^2(\sigma_0)\right), d_{cr}\left(\sigma_0, f^2(\sigma_1)\right)) \leq 0 \Rightarrow \mathscr{L}\\ &\quad\cdot (d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}\\ &\quad\cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_3)) \leq 0.\end{aligned} \tag{31}$$

By (dR3), we have

$$d_{cr}(\sigma_0, \sigma_3) \leq d_{cr}(\sigma_0, \sigma_1) + d_{cr}(\sigma_1, \sigma_2) + d_{cr}(\sigma_2, \sigma_3), \tag{32}$$

and then using $\mathscr{L}_1$, we obtain

$$\begin{aligned}&\mathscr{L}(d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}(\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1), d_{cr}\\ &\quad\cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_0, \sigma_1) + d_{cr}(\sigma_1, \sigma_2) + d_{cr}(\sigma_2, \sigma_3) \preceq 0.\end{aligned} \tag{33}$$

By $(\mathscr{L}_2)$, there exists an order preserving operator $S \in B(\mathscr{E}, \mathscr{E})$ with $\|S\|_1 < 1$ such that

$$\begin{aligned}d_{cr}(\sigma_1, \sigma_2) &\leq S(d_{cr}(\sigma_0, \sigma_1)),\\ d_{cr}(\sigma_2, \sigma_3) &\leq S(d_{cr}(\sigma_1, \sigma_2)).\end{aligned} \tag{34}$$

Using $(2) f(\sigma_0)\mathscr{R}f(\sigma_1)$, take $\sigma = \sigma_1$ and $\kappa = \sigma_2$ in (30), we have

$$\begin{aligned}&(I-T)^2(I+T)\left(d_{cr}(\sigma_1, f(\sigma_1)\right) = (I-T)^2(I+T)\\ &\quad\cdot (d_{cr}(\sigma_1, \sigma_2)) \preceq d_{cr}(\sigma_1, \sigma_2) \text{ implies } \mathscr{L}(d_{cr}(f(\sigma_1), f(\sigma_2)), d_{cr}\\ &\quad\cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_1, f(\sigma_1)), d_{cr}(\sigma_2, f(\sigma_2)), d_{cr}\left(\sigma_1, f^2(\sigma_2)\right), d_{cr}\\ &\quad\cdot \left(\sigma_2, f^2(\sigma_1)\right)) \preceq 0 \Rightarrow \mathscr{L}(d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_2), d_{cr}\\ &\quad\cdot (\sigma_1, \sigma_2), d_{cr}(\sigma_2, \sigma_3), d_{cr}(\sigma_1, \sigma_4), d_{cr}(\sigma_2, \sigma_3)) \preceq 0.\end{aligned} \tag{35}$$

By (dR3), $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$, we get

$$d_{cr}(\sigma_3, \sigma_4) \leq S(d_{cr}(\sigma_2, \sigma_3)) \preceq S^2(d_{cr}(\sigma_1, \sigma_2)) \preceq S^3(d_{cr}(\sigma_0, \sigma_1)). \tag{36}$$

By continuing the pattern, we construct a sequence $\{\sigma_n\}$ such that

$$d_{cr}(\sigma_n, \sigma_{n+1}) \preceq S(d_{cr}(\sigma_{n-1}, \sigma_n)) \preceq S^2(d_{cr}(\sigma_{n-2}, \sigma_{n-1})) \preceq \cdots \preceq S^n(d_{cr}(\sigma_0, \sigma_1)). \tag{37}$$

Hence, by the same reasoning as in the proof of Theorem 9, we have $x^* = f(x^*)$.

Theorem 11. *Let $(\mathbb{Y}, d_{cr})$ be a complete cone rectangular metric space and f be a monotone self-mapping on $\mathbb{Y}$. If for all comparable elements $\sigma, \kappa \in \mathbb{Y}$, there exist $T \in B(\mathscr{E}, \mathscr{E})$, identity operator $I : \mathscr{E} \to \mathscr{E}$, and $\mathscr{L} \in \mathscr{G}$ such that*

$$\begin{aligned}&(I-T)^2(I+T)(d_{cr}(\sigma, f(\sigma))) \leq d_{cr}(\sigma, \kappa) \textit{ implies } \mathscr{L}\\ &\quad\cdot (d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma, \kappa), d_{cr}(\sigma, f(\sigma)), d_{cr}\\ &\quad\cdot (\kappa, f(\kappa)), d_{cr}\left(\sigma, f^2(\kappa)\right), d_{cr}\left(\kappa, f^2(\sigma)\right)) \preceq 0,\end{aligned} \tag{38}$$

and

(1) there exists $\sigma_0 \in \mathbb{Y}$ such that either $\sigma_0\mathscr{R}f(\sigma_0)$ or $f(\sigma_0)\mathscr{R}\sigma_0$

(2) for every $\{\sigma_n\} \subseteq \mathbb{Y}$, $d_{cr}(\sigma_n, \sigma_{n+1}) \preceq T(d_{cr}(\sigma_{n-1}, \sigma_n))$

(3) for a sequence $\{\sigma_n\}$ with $\sigma_n \to x^*$ whose all sequential terms are comparable, we have $\sigma_n\mathscr{R}x^*$ for all $n \in \mathbb{N}$ and $d_{cr}(x^*, f(x^*))^\circ d_{cr}(x^*, f^2(x^*))$

Then, f has a fixed point in $\mathbb{Y}$.

Proof. This proof follows the same pattern as given in the previous two proofs, so, we omit it.

Remark 12.

(1) In Theorem 9, Theorem 10, and Theorem 11, uniqueness of the fixed point of f can be attained by assuming that for every pair of elements $\sigma, \kappa \in X$, there exists either an upper bound or lower bound of σ, κ

(2) The cone is taken as nonnormal in the above theorems

Theorem 13. *Let $(\mathbb{Y}, d_{cr})$ be a complete cone rectangular metric space and f be a monotone self-mapping on $\mathbb{Y}$. If for all comparable elements $\sigma, \kappa \in \mathbb{Y}$, there exist $T \in B(\mathscr{E}, \mathscr{E})$,* *identity operator $I : \mathscr{E} \to \mathscr{E}$, and $\mathscr{L} \in \mathscr{G}$ such that*

$$(I-T)^2(I+T)(d_{cr}(\sigma, f(\sigma))) \le d_{cr}(\sigma,\kappa) \text{ implies } d_{cr}(f(\sigma), f(\kappa)) \le S(d_{cr}(\sigma,\kappa)). \tag{39}$$

Moreover, if

(1) there exists $\sigma \in \mathbb{Y}$ such that $\sigma_0 \mathscr{R} f(\sigma_0)$ or $f(\sigma_0)\mathscr{R}\sigma_0$

(2) for a sequence $\{\sigma_n\}$ with $\sigma_n \to x^*$ whose all sequential terms are comparable, we have $\sigma_n \mathscr{R} x^*$ for all $n \in \mathbb{N}$ and $d_{cr}(x^*, f(x^*)) \le d_{cr}(x^*, f^2(x^*))$

Then, there exists $x^* \in \mathbb{Y}$ such that $x^* = f(x^*)$.

Proof. Define $S(\nu) = q\nu$ for all $\nu \in \mathscr{E}$ and $q \in [0,1)$, then $S \in B(\mathscr{E}, \mathscr{E})$ with $\|S\|_1 < 1$ also define implicit relation by

$$\mathscr{L}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 - \alpha\sigma_2 \text{forsome} \alpha \in [0,1). \tag{40}$$

The proof follows by the application of Theorem 9.

The following examples illustrate the main theorem.

Example 4. Let $\mathscr{E} = (\mathbb{R}^3, \|\cdot\|)$ be a real Banach space. With $\|\sigma\| = \max\{(|\sigma_1|, |\sigma_2|, |\sigma_3|)\}$, then $(\mathscr{E}, \|.\|)$ is a real Banach space. Define the partial order $\preceq$ on $\mathscr{E}$ by

$$(\sigma, \xi, \nu) \ge 0 \Leftrightarrow \sigma \ge 0, \xi \ge 0, \nu \ge 0. \tag{41}$$

Define $\mathscr{P} = \{(\sigma, \xi, \nu) \in \mathbb{R}^3 : \sigma, \xi, \nu \ge 0\}$, then, it is a cone in $\mathscr{E}$.

Let $\mathbb{Y} = \{(0,0,0), (1/4, 0, 0), (1/4, 1/4, 0), (1/4, 1/4, 1/4)\} \subset \mathscr{E}$, define order on $\mathbb{Y}$ by $\mathscr{R} =: \le$ and define $f : \mathbb{Y} \to \mathbb{Y}$ such that

$$f(\sigma) = \begin{cases} (0,0,0) & \text{if } \sigma \in \left\{(0,0,0), \left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, 0, 0\right)\right\}, \\ \left(\frac{1}{4}, \frac{1}{4}, 0\right) & \text{if } \sigma = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right). \end{cases} \tag{42}$$

Define the function $d_{cr} : \mathbb{Y} \times \mathbb{Y} \to \mathscr{E}$ by

$$d_{cr}(\sigma_1, \sigma_2) = \begin{cases} (0,0,0) & \text{if } \sigma_1 = \sigma_2, \\ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) & \text{if } \sigma_1, \sigma_2 \in \left\{(0,0,0), \left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right\}, \\ \left(\frac{1}{4}, 0, 0\right) & \text{if } \sigma_1 \ne \sigma_2 \text{ cannot be both at a time in } \left\{(0,0,0), \left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right\}. \end{cases} \tag{43}$$

It is a cone rectangular metric space but not a cone metric space. The assumptions (1)–(4) in Theorem 9 can be verified immediately. Define $T(\sigma) = \sigma/3$, then, $T \in B(\mathscr{E}, \mathscr{E})$ with $\|T\|_1 \le 1$. Now, we verify condition (9). For if $\sigma = (0,0,0)$ and $\kappa = (1/4, 0, 0)$, then

$$d_{cr}(\sigma, f(\sigma)) = d_{cr}(f(\sigma), f(\kappa)) = d_{cr}(\sigma, f^2(\kappa)) = (0,0,0), \\ d_{cr}(\kappa, f^2(\sigma)) = \left(\frac{1}{4}, 0, 0\right) = d_{cr}(\sigma, \kappa), \tag{44}$$

and thus,

$$(I-T)^2(I+T)d_{cr}(\sigma, f(\sigma)) = (0,0,0), \alpha\left[d_{cr}(\sigma, f^2(\kappa)) + d_{cr} \cdot (\kappa, f^2(\sigma)\right] = \alpha\left[(0,0,0) + \left(\frac{1}{4}, 0, 0\right)\right] = \alpha\left(\frac{1}{4}, 0, 0\right) : \alpha \ge 1. \tag{45}$$

For if $\sigma = (1/4, 0, 0)$ and $\kappa = (1/4, 1/4, 0)$, then $d_{cr}(\sigma, f(\sigma)) = d_{cr}(\sigma, \kappa) = d_{cr}(\sigma, f^2(\kappa)) = (1/4, 0, 0)$; $d_{cr}(\kappa, f^2(\sigma)) = (1/4, 1/4, 1/4)$; $d_{cr}(f(\sigma), f(\kappa)) = (0,0,0)$; $(I-T)^2(I+T)(d_{cr}(\sigma, f(\sigma))) = 16/27(1/4, 0, 0)$ and

$$\alpha\left[d_{cr}(\sigma, f^2(\kappa)) + d_{cr}(\kappa, f^2(\sigma)\right] = \alpha\left[\left(\frac{1}{4}, 0, 0\right) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right] = \alpha\left(\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\right). \tag{46}$$

Finally, for $\sigma = (1/4, 1/4, 0)$ and $\kappa = (1/4, 1/4, 1/4)$, we have $d_{cr}(\sigma, f(\sigma)) = d_{cr}(f(\sigma), f(\kappa)) = d_{cr}(\sigma, f^2(\kappa)) = d_{cr}(\sigma, \kappa) = d_{cr}(\kappa, f^2(\sigma)) = (1/4, 1/4, 1/4)$; $(I-T)^2(I+T)(d_{cr}(\sigma, f(\sigma))) = 16/27\ (1/4, 1/4, 1/4)$ and

$$\alpha\left[d_{cr}(\sigma, f^2(\kappa)) + d_{cr}(\kappa, f^2(\sigma)\right] = \alpha\left[\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\right] = \alpha\left(\frac{2}{4}, \frac{2}{4}, \frac{2}{4}\right). \tag{47}$$

Define the function $\mathscr{L}$ by

$$\mathscr{L}(d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma,\kappa), d_{cr}(\sigma, f(\sigma)), d_{cr} \cdot (\kappa, f(\kappa)), d_{cr}(\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) = d_{cr} \cdot (f(\sigma), f(\kappa)) - \alpha\left[d_{cr}(\sigma, f^2(\kappa)) + d_{cr}(\kappa, f^2(\sigma)\right]. \tag{48}$$

As a consequence, we have $(I-T)^2(I+T)(d_{cr}(\sigma, f(\sigma)) \preceq d_{cr}(\sigma,\kappa))$ implies

$$\mathscr{L}(d_{cr}(f(\sigma), f(\kappa)), d_{cr}(\sigma,\kappa), d_{cr}(\sigma, f(\sigma)), d_{cr} \cdot (\kappa, f(\kappa)), d_{cr}(\sigma, f^2(\kappa)), d_{cr}(\kappa, f^2(\sigma))) \preceq 0. \tag{49}$$

Thus, this example explains Theorem 9 well. Note that $(0,0,0)$ is a fixed point of f.

Example 5. Let $\mathscr{E} = C_R^1[1,2]$ and $\|g\|_{\mathscr{E}} = \|g\|_\infty + \|g'\|_\infty$, where $\|g\|_\infty = \sup_{t\in[1,2]} | g(t) |$ and $\mathscr{P} = \{g(t) \in \mathscr{E} : g(t) > 0, t \in [1, 2]\}$. For each $K \geq 1$, take $g(t) = t$ and $\kappa(t) = t^{2K}$. By definition $\|g\| = 2$ and $\|\kappa\| = 2K+1$. Define the partial order $\leq$ by

for any $g, \kappa \in \mathscr{E}$; $g \preceq \kappa \Leftrightarrow g(t) \leq \kappa(t)\ \forall t \in [1,2]$.

Then, $g \preceq \kappa$ implies $K\|g\| \leq \|\kappa\|$. Hence, $\mathscr{P}$ is a nonnormal cone. Define $T : \mathscr{E} \to \mathscr{E}$ such that

$$(Tg)(t) = \frac{1}{2}\int_1^t g(s)ds. \tag{50}$$

Then, $T \in B(\mathscr{E}, \mathscr{E})$. Let $\mathbb{Y} = \{1,2,3\}$ and $f : \mathbb{Y} \to \mathbb{Y}$ such that $f(1) = f(2) = 1$ and $f(3) = 2$, f is monotone with respect to usual order ($\leq$) and also assumption (1)–(2) in Theorem 13 are verified. Define d_{cr} by

$$d_{cr}(\sigma,\varsigma)(t) = \begin{cases} 0 & \text{if } \sigma = \varsigma, \\ \dfrac{e^t}{3} & \text{if } \sigma,\varsigma \in \{1,2\}, \\ e^t & \text{otherwise.} \end{cases} \tag{51}$$

The vector-valued function d_{cr} is a cone rectangular metric space but not a cone metric space. We are left to verify the contractive condition only. For this, if $\sigma = 1$ and $\varsigma = 2$, then

$$\begin{aligned} d_{cr}(\sigma, f(\sigma)) &= 0\,; \\ d_{cr}(f(\sigma), f(\varsigma)) &= 0\,; \\ d_{cr}(\varsigma, f(\varsigma)) &= \frac{e^t}{3}, \end{aligned} \tag{52}$$

and for $\sigma = 2$, $\varsigma = 3$, we have

$$\begin{aligned} d_{cr}(\sigma,\varsigma) &= e^t\,; \\ d_{cr}(\sigma, f(\sigma)) &= \frac{e^t}{3}\,; \\ d_{cr}(f(\sigma), f(\varsigma)) &= \frac{e^t}{3}\,; \\ d_{cr}(\varsigma, f(\varsigma)) &= e^t. \end{aligned} \tag{53}$$

Thus,

$$\begin{aligned} (I-T)(I-T^2)(d_{cr}(\sigma, f(\sigma))) &= \frac{e^t}{3}\left[1 - \frac{t}{2}\left(1 - \frac{t^2}{4}\right)\right], \\ (Td_{cr}(\sigma,\varsigma))(t) = \frac{te^t}{2}&,\ (Td_{cr}(\varsigma, f(\varsigma)))(t) = \frac{te^t}{2}. \end{aligned} \tag{54}$$

As $t \in [1,2]$ so $t^2 \leq 4 \Rightarrow t^2/4 \leq 1$ and $1 - t^2/4 \geq 0$. Hence,

$$(I-T)(I-T^2)(d_{cr}(\sigma, f(\sigma))) \preceq d_{cr}(\sigma,\varsigma) \text{ implies } d_{cr} \cdot (f(\sigma), f(\varsigma)) \preceq T(d_{cr}(\sigma,\varsigma)). \tag{55}$$

Note that 1 is a fixed point of f.

5. A Homotopy Result

In what follows, we derive a homotopy result by applying Theorem 13.

Theorem 14. *Let $(\mathscr{E}, \|.\|)$ be a real Banach space and $\mathscr{P} \subset \mathscr{E}$ be a cone. Let $(\mathbb{Y}, d_{cr})$ be a rectangular cone metric space and $U \subset \mathbb{Y}$ is open. Assume that there exists $T \in B(\mathscr{E}, \mathscr{E})$ and $T(\mathscr{P}) \subset \mathscr{P}$. Let the operator $h : \bar{U} \times [0,1] \to \mathbb{Y}$ satisfy ((39)) and the condition (1) of Theorem13in the first variable and*

(1) $\sigma \neq h(\sigma,\theta)$ for every $\sigma \in \partial U$ (∂U denotes the boundary of U in $\mathbb{Y}$)

(2) there exists $M \geq 0$ such that

$$\|d_{cr}(h(\sigma,\theta), h(\sigma,\mu))\| \leq M|\theta - \mu| \tag{56}$$

for every $\sigma \in \bar{U}$ and $\mu, \theta \in [0,1]$

(3) For some $\sigma \in U$, if there exists κ with $\|d_{cr}(\sigma,\kappa)\| \leq r$, then $\sigma \mathscr{R} \kappa$, where r is radius of open ball in U

If $h(\cdot, 0)$ has a fixed point in the open set U, then $h(\cdot, 1)$ also has a fixed point in the open set U.

Proof. Let

$$B = \{\theta \in [0,1] |\, \sigma = h(\sigma,\theta)\,; \text{for some } \sigma \in U\}. \tag{57}$$

Define the relation $\preceq$ in $\mathscr{E}$ by $u \preceq v$ if and only if $\|u\| \leq \|v\|$ for all $u, v \in \mathscr{E}$. Next, $0 \in B$, since $h(.,0)$ has a fixed point in the open set U. So B is nonempty. Since $d_{cr}(\sigma, h(\sigma,\theta)) = d_{cr}(\sigma,\kappa)$, $(I-T)^2(I+T)(d_{cr}(\sigma, h(\sigma,\theta))) \leq d_{cr}(\sigma,\kappa)$ for all $\sigma \mathscr{R} \kappa$, then by Theorem 13, we have

$$d_{cr}(h(\sigma,\theta), h(\kappa,\theta)) \leq S(d_{cr}(\sigma,\kappa)). \tag{58}$$

Firstly, we show that B is closed in $[0,1]$. For this, let $\{\theta_n\}_{n=1}^\infty \subseteq B$ with $\theta_n \to \theta \in [0,1]$ as $n \to \infty$. It is necessary to prove that $\theta \in B$. Since $\theta_n \in B$ for $n \in \mathbb{N}$, there exists $\sigma_n \in U$ with $\sigma_n = h(\sigma_n, \theta_n)$. Since $h(\sigma, \cdot)$ is monotone, so, for n, m

$\in \mathbb{N}$, we have $\sigma_m \mathscr{R} \sigma_n$. Since

$$(I-T)^2(I+T)(d_{cr}(\sigma_n, h(\sigma_m, \theta_m))) = (I-T)^2 \cdot (I+T)(d_{cr}(\sigma_n, \sigma_m)) \preceq d_{cr}(\sigma_n, \sigma_m), \tag{59}$$

we have

$$\begin{aligned} & d_{cr}(h(\sigma_n, \theta_m), h(\sigma_m, \theta_m)) \leq S(d_{cr}(\sigma_n, \sigma_m)), d_{cr} \\ & \cdot (\sigma_n, \sigma_{m+1}) = d_{cr}(h(\sigma_n, \theta_n), h(\sigma_{m+1}, \theta_{m+1})) \preceq d_{cr} \\ & \cdot (h(\sigma_n, \theta_n), h(\sigma_n, \theta_m)) + d_{cr}(h(\sigma_n, \theta_m), h \\ & \cdot (\sigma_n, \theta_{m+1})) + d_{cr}(h(\sigma_n, \theta_{m+1}), h(\sigma_{m+1}, \theta_{m+1})), \| d_{cr} \\ & \cdot (\sigma_n, \sigma_m) \| \leq M|\theta_n - \theta_m| + M|\theta_m - \theta_{m+1}| + \| S \\ & \cdot (d_{cr}(\sigma_n, \sigma_m)) \|, \| d_{cr}(\sigma_n, \sigma_m) \| \leq \frac{M}{1 - \|S\|} \\ & \cdot [|\theta_n - \theta_m| + |\theta_m - \theta_{m+1}|]. \end{aligned} \tag{60}$$

Since $\{\theta_n\}_{n=1}^{\infty}$ is a RMS Cauchy sequence in $[0, 1]$, we have

$$\lim_{n,m \to \infty} d_{cr}(\sigma_n, \sigma_m) = 0, \tag{61}$$

that is, $d_{cr}(\sigma_n, \sigma_m) \ll c$, whenever $n, m \to \infty$. Hence, $\{\sigma_n\}$ is a Cauchy sequence in $\mathbb{Y}$. Since $\mathbb{Y}$ is a complete cone rectangular metric space, there exists $\sigma \in \bar{U}$ with $\lim_{n\to\infty} d_{cr}(\sigma_n, \sigma) \ll c$. Hence, $\sigma_n \mathscr{R} \sigma$ for all $n \in \mathbb{N}$. Now consider

$$\begin{aligned} & d_{cr}(\sigma_n, h(\sigma, \theta)) = d_{cr}(h(\sigma_n, \theta_n), h(\sigma, \theta)) \preceq d_{cr} \\ & \cdot (h(\sigma_n, \theta_n), h(\sigma_n, \theta)) + d_{cr}(h(\sigma_n, \theta), h \\ & \cdot (\sigma_{n+1}, \theta)) + d_{cr}(h(\sigma_{n+1}, \theta), h(\sigma, \theta)), \| d_{cr} \\ & \cdot (\sigma_n, h(\sigma, \theta)) \| \leq M|\theta_n - \theta| + M|\theta_{n+1} - \theta| + \| S \\ & \cdot (d_{cr}(\sigma_{n+1}, \sigma)) \|. \end{aligned} \tag{62}$$

So we have

$$\lim_{n \to \infty} d_{cr}(\sigma_n, h(\sigma, \theta)) = 0. \tag{63}$$

Thus, $d_{cr}(\sigma, h(\sigma, \theta)) = 0$. Hence, $\theta \in B$ and so B is RMS closed in $[0, 1]$.

Next, we show that B is open in $[0, 1]$. For this, let $\theta_2 \in B$. Then, we have the existence of $\sigma_2 \in U$ with $h(\sigma_2, \theta_2) = \sigma_2$. Since U is open, there exists $r > 0$ such that $B(\sigma_2, r) \subseteq U$. Now, assume

$$l = d_{cr}(\sigma_2, \partial U) = \inf \{ d_{cr}(\sigma_2, \xi) : \xi \in \partial U \}. \tag{64}$$

Then, $r = l > 0$. Fix $\varepsilon > 0$ with $\varepsilon < ((1 - \|T\|)l)/2M$. Let $\theta \in (\theta_1 - \varepsilon, \theta_1 + \varepsilon)$ and $\theta_1 \in (\theta_2 - \varepsilon, \theta_2 + \varepsilon)$. Then,

$$\sigma \in B(\bar{\sigma_2}, r) = \{ \sigma \in X : \| d_{cr}(\sigma, \sigma_2) \| \leq r \}, \text{as}\, \sigma \mathscr{R} \sigma_2. \tag{65}$$

Consider

$$\begin{aligned} d_{cr}(h(\sigma, \theta), \sigma_2) = {} & d_{cr}(h(\sigma, \theta), h(\sigma_2, \theta_2)) \\ & \leq d_{cr}(h(\sigma, \theta), h(\sigma, \theta_1)) + d_{cr} \\ & \cdot (h(\sigma, \theta_1), h(\sigma, \theta_2)) + d_{cr} \\ & \cdot (h(\sigma, \theta_2), h(\sigma_2, \theta_2)), \| d_{cr} \\ & \cdot (h(\sigma, \theta), \sigma_2) \| \leq M|\theta_1 - \theta| + M|\theta_2 - \theta_1| + \| S \\ & \cdot (d_{cr}(\sigma_2, \sigma)) \| \leq M\varepsilon + M\varepsilon + \|S\| l = 2M\varepsilon + \|S\| l < l. \end{aligned} \tag{66}$$

Thus, for every fixed $\theta \in (\theta_2 - \varepsilon, \theta_2 + \varepsilon)$, $h(\cdot, \theta): B(\bar{\sigma}, r) \to B(\bar{\sigma}, r)$ has a fixed point in $\bar{U}$ and can be deduced by applying Theorem 13. But this fixed point should be in U as in the previous case. Hence, $\theta \in B$ for any $\theta \in (\theta_2 - \varepsilon, \theta_2 + \varepsilon)$ and so B is open in $[0, 1]$. Thus, we showed that B is RMS open as well as RMS closed in $[0, 1]$ and by connectedness, $B = [0, 1]$. Hence, $h(\cdot, 1)$ has a fixed point in U.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work.

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Research Article

Some Fixed Point Results for Perov-Ćirić-Prešić Type F-Contractions with Application

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Ćirić and Prešić developed the concept of Prešić contraction to Ćirić-Prešić type contractive mappings in the background of a metric space. On the other hand, Altun and Olgun introduced Perov type F-contractions. In this paper, we extend the concept of Ćirić-Prešić contractions to Perov-Ćirić-Prešić type F-contractions. Our results modify some known ones in the literature. To support our main result, an example and an application to nonlinear operator systems are presented.

1. Introduction

The Banach contraction principle (BCP) [1] is one of the powerful results in nonlinear analysis. It has many applications in the background of ODE and PDE.

Theorem 1 [1]. *Let (Δ, d) be a complete metric space and let $\Upsilon : \Delta \longrightarrow \Delta$ so that*

$$d(\Upsilon\iota, \Upsilon\kappa) \leq \gamma d(\iota, \kappa) \text{ for all } \iota, \kappa \in \Delta, \tag{1}$$

where $\gamma \in [0, 1)$. Then, there is a unique σ in Δ such that $\sigma = \Upsilon\sigma$. Also, for each $\zeta_0 \in \Delta$, the sequence $\zeta_{n+1} = \Upsilon\zeta_n$ converges to σ.

The BCP has been extended and generalized in many directions (see [2–4]).

Prešić [5] gave the following result.

Theorem 2 [5]. *Let (Δ, d) be a complete metric space and let $\Upsilon : \Delta^k \longrightarrow \Delta$ (k is a positive integer). Suppose that*

$$d(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1})) \leq \sum_{i=1}^{k} \lambda_i d(\zeta_i, \zeta_{i+1}), \tag{2}$$

for all $\zeta_1, \cdots, \zeta_{k+1}$ in Δ, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. Then Υ has a unique fixed point ζ^* (that is $\Upsilon(\zeta^*, \cdots, \zeta^*) = \zeta^*$). Moreover, for all arbitrary points $\zeta_1, \cdots, \zeta_{k+1}$ in Δ, the sequence $\{\zeta_n\}$ defined by $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})$, converges to ζ^*.

It is obvious that for $k = 1$, Theorem 2 coincides with the BCP.

Theorem 2 was generalized by Ćirić and Prešić [6] as follows.

Theorem 3 [6]. *Let (Δ, d) be a complete metric space and $\Upsilon : \Delta^k \longrightarrow \Delta$ (k is a positive integer). Suppose that*

$$d(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1})) \leq \lambda \max\{d(\zeta_i, \zeta_{i+1}) : 1 \leq i \leq k\}, \tag{3}$$

for all $\zeta_1, \cdots, \zeta_{k+1}$ in Δ, where $\lambda \in [0, 1)$. Then Υ has a fixed point $\zeta^* \in \Delta$. Also, for all points $\zeta_1, \cdots, \zeta_{k+1} \in \Delta$, the sequence $\{\zeta_n\}$ defined by $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})$, converges to ζ^*. The fixed point of Υ is unique if

$$d(\Upsilon(\rho,\cdots,\rho),\Upsilon(\rho,\cdots,\rho))<d(\rho,\rho), \tag{4}$$

for all $\rho,\varrho\in\Delta$ *with* $\rho\neq\varrho$.

For more details on Prešić type contractions, we refer the reader to [2, 5, 7–11].

In this paper, $\mathbb{R}_+=[0,\infty)$, $\mathbb{R}^m$ denotes the set of $m\times 1$ real matrices, $\mathbb{R}_+^m$ will be the set of $m\times 1$ real matrices with elements in $[0,\infty)$, θ denotes the zero $m\times 1$ matrix, $M_{m,m}(\mathbb{R}_+)$ denotes the set of all $m\times m$ matrices with elements in $[0,\infty)$, and Θ will be the zero $m\times m$ matrix, by I the identity $m\times m$ matrix. If $A\in M_{m,m}(\mathbb{R}_+)$, then A^T states the transpose matrix of A. Let $\pi=(\pi_i)_{i=1}^m$, $\varpi=(\varpi_i)_{i=1}^m\in\mathbb{R}^m$, then by $\pi\preceq\varpi$ (resp. $\pi\prec\varpi$), we suppose $\pi_i\le\varpi_i$ (resp. $\pi_i<\varpi_i$) for each $i\in\{1,2,\cdots,m\}$. Also, $\pi\preceq\varpi$ and $\varpi\succeq\pi$ will mean the same.

Let Δ be a nonempty set and let $V:\Delta\times\Delta\longrightarrow\mathbb{R}^m$ be a function. V is called a vector-valued metric, and (Δ,V) is called a vector-valued metric space, if

(1) $V(\iota,\kappa)=\theta$ if and only if $\iota=\kappa$,

(2) $V(\iota,\kappa)=V(\kappa,\iota)$,

(3) $V(\iota,\kappa)\preceq(\iota,\sigma)+V(\sigma,\kappa)$,

for all $\iota,\kappa,\sigma\in\Delta$.

Example 1 (Example 1.3. of [12]). Let $D_1,D_2,\cdots,D_n$ be usual metrics on X.

Then, the mapping $d:X\times X\longrightarrow\mathscr{R}^n$ defined by $d(x,y)=(D_1(x,y),D_2(x,y),\cdots,D_n(x,y))$ is a VVM on X.

From now on, we apply VVMS instead of a vector-valued metric space.

The concepts of convergence, Cauchyness, and completeness in a VVMS will be similar as in a usual metric case. Perov [13] stated the contraction mapping principle in the setting of VVMSs. Before stating this theorem, we must remember the following facts:

Let $A\in M_{m,m}(\mathbb{R}_+)$. Then A is said to converge to zero if and only if $A^n\longrightarrow\Theta$ as $n\longrightarrow\infty$ (see [14]).

Perov [13] proved the following interesting extension of BCP (see more results in [15–20]).

Theorem 4 [13]. *Let* (Δ,V) *be a VVMS and* $\Upsilon:\Delta\longrightarrow\Delta$ *be a mapping such that there exists a matrix* $A\in M_{m,m}(\mathbb{R}_+)$ *such that*

$$V(\Upsilon\iota,\Upsilon\kappa)VAV(\iota,\kappa), \tag{5}$$

for all $\iota,\kappa\in\Delta$. *If A is convergent to zero, then*

(1) Υ *has a unique fixed point* σ *in* Δ

(2) for all $x_0\in\Delta$, *the sequence* $\{x_n\}$ *defined by* $x_n=\Upsilon^n x_0$ *is convergent to* σ

(3) $V(x_n,\sigma)VA^n(I-A)^{-1}V(x_0,\Upsilon x_0)$.

In this paper, considering the recent approach of Wardowski [21], we present a generalization of Perov fixed point theorem and Ćirić-Prešić fixed point theorem. Some generalization of Wardowski results can be found in [22, 23].

As in [24], let $F:\mathbb{R}_+^m\longrightarrow\mathbb{R}^m$ be a function. Let

(F1) F be strictly increasing in each variable, i.e., $\pi\prec\varpi$; then, $F(\pi)\prec F(\varpi)$, for all $\pi=(\pi_i)_{i=1}^m$ and $\varpi=(\varpi_i)_{i=1}^m\in\mathbb{R}_+^m$,

(F2) For each sequence $\{\pi_n\}=(\pi_n^{(1)},\pi_n^{(2)},\cdots,\pi_n^{(m)})$ of $\mathbb{R}_+^m$

$$\lim_{n\to\infty}\pi_n^{(i)}=0 \text{ if and only if } \lim_{n\to\infty}\varpi_n^{(i)}=-\infty, \tag{6}$$

for each $i\in\{1,2,\cdots m\}$, where

$$F\left(\left(\pi_n^{(1)},\pi_n^{(2)},\cdots,\pi_n^{(m)}\right)\right)=\left(\varpi_n^{(1)},\varpi_n^{(2)},\cdots,\varpi_n^{(m)}\right). \tag{7}$$

(F3) There exists $k\in(0,1)$ such that $\lim_{\pi_i\to 0^+}\pi_i^k\varpi_i=0$ for each $i\in\{1,2,\cdots m\}$, where

$$F((\pi_1,\pi_2,\cdots,\pi_m))=(\varpi_1,\varpi_2,\cdots,\varpi_m). \tag{8}$$

We denote by $\mathscr{F}^m$ the set of all functions F satisfying (*F*1)–(*F*3)

Example 2 [24]. Define $F:\mathbb{R}_+^m\longrightarrow\mathbb{R}^m$ by

$$F((\pi_1,\pi_2,\cdots,\pi_m))=(\ln\pi_1,\ln\pi_2,\cdots,\ln\pi_m), \tag{9}$$

then $F\in\mathscr{F}^m$.

Note that we can define $G:\mathbb{R}^m\longrightarrow\mathbb{R}_+^m$ by

$$G((\varpi_1,\varpi_2,\cdots,\varpi_m))=\left(e^{\varpi_1},e^{\varpi_2},\cdots,e^{\varpi_m}\right), \tag{10}$$

which we can treat it as the inverse of multivariable function F.

Note that from now on, F is a continuously differentiable function from all open sets of $A\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$, and the Jacobian determinant of F at every $p\in A\subseteq\mathbb{R}^n$ is nonzero; then, according to inverse function theorem, F is invertible near p.

Example 3 [24]. Define $F:\mathbb{R}_+^2\longrightarrow\mathbb{R}^2$ by

$$F((\pi_1,\pi_2))=(\ln\pi_1,\pi_2+\ln\pi_2), \tag{11}$$

then $F\in\mathscr{F}^2$.

Example 4 [24]. Define $F : \mathbb{R}_+^3 \longrightarrow \mathbb{R}^3$ by

$$F((\pi_1, \pi_2, \pi_3)) = \left(\ln \pi_1, \pi_2 + \ln \pi_2, -\frac{1+\pi_1}{\sqrt{\pi_3}}\right), \qquad (12)$$

then $F \in \mathcal{F}^3$.

Considering the class $\mathcal{F}^m$, Altun and Olgun [24] introduced the concept of Perov type F-contraction as follows:

Definition 5 [24]. Let (Δ, V) be a VVMS and $\Upsilon : \Delta \longrightarrow \Delta$ be a map. If there exist $F \in \mathcal{F}^m$ and $\varsigma = (\varsigma_i)_{i=1}^m \in \mathbb{R}_+^m$ such that

$$\varsigma + F(V(\Upsilon \iota, \Upsilon y)) V F(V(\iota, \kappa)), \qquad (13)$$

for all $\iota, \kappa \in \Delta$ with $V(\Upsilon \iota, \Upsilon \kappa) \succ \theta$, then Υ is called a Perov type F-contraction.

If we consider $F : \mathbb{R}_+^m \longrightarrow \mathbb{R}^m$ by

$$F((\pi_1, \pi_2, \cdots, \pi_m)) = (\ln \pi_1, \ln \pi_2, \cdots, \ln \pi_m), \qquad (14)$$

then (13) turns to Perov contraction [24].

We can present new type contractions in a VVMS, via considering some function $F \in \mathcal{F}^m$ in (13).

Theorem 6 [24]. *Let (Δ, V) be a complete VVMS and let $\Upsilon : \Delta \longrightarrow \Delta$ be a Perov type F-contraction. Then Υ admits a unique fixed point.*

In this paper, we introduce the concept of Perov-Ćirić-Prešić type F-contractions. An illustrative example and an application are given to support our main result.

2. Main Results

In this section, combining the ideas of Perov, Wardowski, and Ćirić-Prešić, we obtain a new extension of BCP.

Our main result is as follows:

Theorem 7. *Let (Δ, V) be a complete VVMS and let $\Upsilon : \Delta^k \longrightarrow \Delta$ (k is a positive integer). Assume that there exist $F \in \mathcal{F}^m$ and $\varsigma = (\varsigma_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying*

$$\begin{aligned} &\varsigma + F(V(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1}))) \\ &\quad \leq F(\sup \{V(\zeta_1, \zeta_2), V(\zeta_2, \zeta_3), \cdots, V(\zeta_k, \zeta_{k+1})\}), \end{aligned} \qquad (15)$$

for all $\zeta_1, \cdots, \zeta_{k+1} \in \Delta$ with $V(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1})) \succ \theta$. Moreover, let there exists a sequence $\{\zeta_n\}$ in Δ such that $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})$ and $V(\zeta_{n+k}, \zeta_{n+k+1}) \succ \theta$, for all $n \in \mathbb{N}$. Also, if $\zeta_n \longrightarrow v$, then $V(\zeta_n, v) \succ \theta$, for all $n \in \mathbb{N}$. Then, the sequence $\{\zeta_n\}$ converges to a fixed point of Υ. Moreover, if for all $\rho, \varrho \in \Delta$ with $\rho \neq \varrho$,

$$V(\Upsilon(\rho, \cdots, \rho), \Upsilon(\rho, \cdots, \rho)) < V(\rho, \rho), \qquad (16)$$

then the fixed point of Υ is unique.

Proof. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} &F(V(\zeta_{n+k}, \zeta_{n+k+1})) \\ &\quad = F(V(\Upsilon(\zeta_n, \cdots, \zeta_{n+k-1}), \Upsilon(\zeta_{n+1}, \cdots, \zeta_{n+k}))) \\ &\quad \leq F(\sup \{V(\zeta_n, \zeta_{n+1}), V(\zeta_{n+1}, \zeta_{n+2}), \cdots, V(\zeta_{n+k-1}, \zeta_{n+k})\}) - \varsigma. \end{aligned} \qquad (17)$$

Therefore,

$$\begin{aligned} &F(V(\zeta_{k+1}, \zeta_{k+2})) \\ &\quad = F(V(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1}))) \\ &\quad \leq F(\sup \{V(\zeta_1, \zeta_2), V(\zeta_2, \zeta_3), \cdots, V(\zeta_k, \zeta_{k+1})\}) - \varsigma \\ &\quad = F(\Lambda) - \varsigma, \end{aligned} \qquad (18)$$

where $\Lambda = \sup \{V(\zeta_1, \zeta_2), V(\zeta_2, \zeta_3), \cdots, V(\zeta_k, \zeta_{k+1})\}$. Now,

$$\begin{aligned} &F(V(\zeta_{k+2}, \zeta_{k+3})) \\ &\quad = F(V(\Upsilon(\zeta_2, \cdots, \zeta_{k+1}), \Upsilon(\zeta_3, \cdots, \zeta_{k+2}))) \\ &\quad \leq F(\sup \{V(\zeta_2, \zeta_3), V(\zeta_3, \zeta_4), \cdots, V(\zeta_{k+1}, \zeta_{k+2})\}) - \varsigma \\ &\quad \leq F\left(\max \left\{\Lambda, F^{-1}(F(\Lambda) - \varsigma)\right\}\right) - \varsigma = F(\Lambda) - \varsigma. \end{aligned} \qquad (19)$$

Continuing this approach, we have

$$\begin{aligned} &F(V(\zeta_{2k}, \zeta_{2k+1})) \\ &\quad = F(V(\Upsilon(\zeta_k, \cdots, \zeta_{2k-1}), \Upsilon(\zeta_{k+1}, \cdots, \zeta_{2k}))) \\ &\quad \leq F(\sup \{V(\zeta_k, \zeta_{k+1}), V(\zeta_{k+1}, \zeta_{k+2}), \cdots, V(\zeta_{2k-1}, \zeta_{2k})\}) - \varsigma \\ &\quad \leq F\left(\max \left\{\Lambda, F^{-1}(F(\Lambda) - \varsigma)\right\}\right) - \varsigma = F(\Lambda) - \varsigma, F(V(\zeta_{2k+1}, \zeta_{2k+2})) \\ &\quad = F(V(\Upsilon(\zeta_{k+1}, \cdots, \zeta_{2k}), \Upsilon(\zeta_{k+2}, \cdots, \zeta_{2k+1}))) \\ &\quad \leq F(\sup \{V(\zeta_{k+1}, \zeta_{k+2}), V(\zeta_{k+2}, \zeta_{k+3}), \cdots, V(\zeta_{2k}, \zeta_{2k+1})\}) - \varsigma \\ &\quad \leq F\left(F^{-1}(F(\Lambda) - \varsigma)\right) - \varsigma = F(\Lambda) - 2\varsigma, F(V(\zeta_{3k}, \zeta_{3k+1})) \\ &\quad = F(V(\Upsilon(\zeta_{2k}, \cdots, \zeta_{3k-1}), \Upsilon(\zeta_{2k+1}, \cdots, \zeta_{3k}))) \\ &\quad \leq F(\sup \{V(\zeta_{2k}, \zeta_{2k+1}), V(\zeta_{2k+1}, \zeta_{2k+2}), \cdots, V(\zeta_{3k-1}, \zeta_{3k})\}) - \varsigma \\ &\quad \leq F\left(\max \left\{F^{-1}(F(\Lambda) - \varsigma), F^{-1}(F(\Lambda) - 2\varsigma)\right\}\right) - \varsigma \\ &\quad = F(\Lambda) - 2\varsigma, F(V(\zeta_{3k+1}, \zeta_{3k+2})) \\ &\quad = F(V(\Upsilon(\zeta_{2k+1}, \cdots, \zeta_{3k}), \Upsilon(\zeta_{2k+2}, \cdots, \zeta_{3k+1}))) \\ &\quad \leq F(\sup \{V(\zeta_{2k+1}, \zeta_{2k+2}), V(\zeta_{2k+2}, \zeta_{2k+3}), \cdots, V(\zeta_{3k}, \zeta_{3k+1})\}) - \varsigma \\ &\quad \leq F\left(F^{-1}(F(\Lambda) - 2\varsigma)\right) - \varsigma = F(\Lambda) - 3\varsigma. \end{aligned} \qquad (20)$$

Continuing this process, we get

$$F\left(V\left(\zeta_{pk+i}, \zeta_{pk+i+1}\right)\right) \leq F(\Lambda) - p\varsigma, \text{for all } p \in V \text{ and } i \in \{1, 2, \cdots, k\}. \qquad (21)$$

Now, taking $V(\zeta_n, \zeta_{n+1}) = (\pi_n^1, \pi_n^2, \cdots, \pi_n^m)$ and $F(\pi_n^1, \pi_n^2, \cdots, \pi_n^m) = (\varpi_n^1, \varpi_n^2, \cdots, \varpi_n^m)$, we obtain that

$$\left(\varpi^1_{pk+i}, \varpi^2_{pk+i}, \cdots, \varpi^m_{pk+i}\right) = F\left(\pi^1_{pk+i}, \pi^2_{pk+i}, \cdots, \pi^m_{pk+i}\right) \le F(\Lambda) - p\varsigma = (r_1 - p\varsigma_1, r_2 - p\varsigma_2, \cdots, r_m - p\varsigma_m), \quad (22)$$

where $\Lambda = (\Lambda_i)_{i=1}^m$ and $F((\Lambda_i)_{i=1}^m) = (r_i)_{i=1}^m$. Therefore,

$$\varpi^j_{pk+i} \le r_j - p\varsigma_j \text{ for all } j \in \{1, 2, \cdots, m\}. \quad (23)$$

Passing to the limit, we get $\lim_{p\to\infty}\varpi^j_{pk+i} = -\infty$. Therefore, $\lim_{p\to\infty}\pi^j_{pk+i} = 0$ for all $j \in \{1, 2, \cdots, m\}$. Thus, $\lim_{p\to\infty} V(\zeta_{pk+i}, \zeta_{pk+i+1}) = \theta$. From $(F3)$, there exists $\lambda \in (0, 1)$ such that

$$\lim_{p\to\infty}\left[\pi^j_{pk+i}\right]^\lambda \varpi^j_{pk+i} = 0, \text{ for all } j \in \{1, 2, \cdots, m\}. \quad (24)$$

From (23),

$$\left[\pi^j_{pk+i}\right]^\lambda \varpi^j_{pk+i} \le \left[\pi^j_{pk+i}\right]^\lambda r_j - p\left[\pi^j_{pk+i}\right]^\lambda \varsigma_j \text{ for all } j \in \{1, 2, \cdots, m\}. \quad (25)$$

Therefore,

$$\left[\pi^j_{pk+i}\right]^\lambda \varpi^j_{pk+i} - \left[\pi^j_{pk+i}\right]^\lambda r_j \le -p\left[\pi^j_{pk+i}\right]^\lambda \varsigma_j \le 0 \text{ for all } j \in \{1, 2, \cdots, m\}. \quad (26)$$

Thus, $\lim_{p\to\infty} p[\pi^j_{pk+i}]^\lambda = 0$ for all $j \in \{1, 2, \cdots, m\}$. So, for any $j \in \{1, 2, \cdots, m\}$, there exists $p_j \in \mathbb{N}$ such that $p[\pi^j_{pk+i}]^\lambda \le 1$, for all $p \ge p_j$. Thus, $\pi^j_{pk+i} \le 1/p^{1/\lambda}$, for all $p \ge p_j$. Putting $p_0 = \max\{p_j : 1 \le j \le m\}$, we have $\pi^j_{pk+i} \le 1/p^{1/\lambda}$ for all $p \ge p_0$ and all $i \in \{1, 2, \cdots, k\}$. We claim that $\{\zeta_n\}$ is a Cauchy sequence. Consider two elements $m, n \in \mathbb{N}$ so that $p_0 k \le n < m$. Then, there are $p, q \in \mathbb{N}$ and $i, j \in \{1, 2, \cdots, k\}$ such that $p_0 \le p \le q$, $n = pk + i$, and $m = qk + j$. Now, we have

$$V(\zeta_n, \zeta_m) = V\left(\zeta_{pk+i}, \zeta_{qk+j}\right) \le \sum_{r=p}^{q}\sum_{l=1}^{k} V(\zeta_{rk+l}, \zeta_{rk+l+1}) \le \sum_{r=p}^{q} k\left(\frac{1}{r^{1/\lambda}}, \cdots, \frac{1}{r^{1/\lambda}}\right) \le k\left(\sum_{r=p}^{q}\frac{1}{r^{1/\lambda}}, \cdots, \sum_{r=p}^{q}\frac{1}{r^{1/\lambda}}\right). \quad (27)$$

As $n, m \longrightarrow \infty$, we have $p, q \longrightarrow \infty$. Thus, the last term in (27) converges to θ, and so $\{\zeta_n\}$ is a Cauchy sequence in (Δ, V). Since (Δ, V) is a complete VVMS, there is $\nu \in \Delta$ so that $\lim_{n\to\infty}\zeta_n = \nu$. Now, we shall prove that ν is a fixed point of Υ. To see this, we have

$$\begin{aligned} V(\zeta_{n+k}, \Upsilon(\nu, \cdots, \nu)) &= V(\Upsilon(\zeta_n, \zeta_{n+1} \cdots, \zeta_{n+k-1}), \Upsilon(\nu, \cdots, \nu)) \\ &\le V(\Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}), \Upsilon(\zeta_{n+1}, \zeta_{n+2}, \cdots, \zeta_{n+k-1}, \nu)) \\ &\quad + V(\Upsilon(\zeta_{n+1}, \zeta_{n+2}, \cdots, \zeta_{n+k-1}, \nu), \Upsilon(\zeta_{n+2}, \zeta_{n+3}, \cdots, \zeta_{n+k-1}, \nu, \nu)) \\ &\quad + \cdots + V(\Upsilon(\zeta_{n+k-1}, \nu, \cdots, \nu), \Upsilon(\nu, \nu, \cdots, \nu)) \\ &\le F^{-1}[F(\max\{V(\zeta_n, \zeta_{n+1}), \cdots, V(\zeta_{n+k-2}, \zeta_{n+k-1}), V(\zeta_{n+k-1}, \nu)\}) - \varsigma] \\ &\quad + F^{-1}[F(\max\{V(\zeta_{n+1}, \zeta_{n+2}), \cdots, V(\zeta_{n+k-2}, \zeta_{n+k-1}), \\ &\quad \times V(\zeta_{n+k-1}, \nu)\}) - \varsigma] \cdots + F^{-1}[F(V(\zeta_{n+k-1}, \nu)) - \varsigma] \longrightarrow \theta, \end{aligned} \quad (28)$$

as $n \longrightarrow \infty$. Thus,

$$V(\nu, \Upsilon(\nu, \cdots, \nu)) = \lim_{n\to\infty} V(\zeta_{n+k}, \Upsilon(\nu, \cdots, \nu)) = \theta. \quad (29)$$

Therefore, $\nu = \Upsilon(\nu, \cdots, \nu)$. Suppose that u, ν are two distinct fixed points of Υ. From our hypothesis,

$$V(u, \nu) = V(\Upsilon(u, \cdots, u), \Upsilon(\nu, \cdots, \nu))) < V(u, \nu), \quad (30)$$

which is a contradiction. Thus, the fixed point of Υ is unique.

Note that by taking

$$F((\pi_1, \pi_2, \cdots, \pi_m)) = (\ln \pi_1, \ln \pi_2, \cdots, \ln \pi_m), \quad (31)$$

the above theorem reduces to the following theorem.

Theorem 8. *Let (Δ, V) be a complete VVMS and $\Upsilon : \Delta^k \longrightarrow \Delta$ (k is a positive integer). Suppose that there exist $F \in \mathscr{F}^m$ and $\varsigma = (\varsigma_i)_{i=1}^m \in \mathbb{R}^m_+$ satisfying*

$$V(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1})) \le A \sup\{V(\zeta_i, \zeta_{i+1}) : i = 1, \cdots, k\}, \quad (32)$$

where

$$A = \begin{pmatrix} e^{-\varsigma_1} & 0 & \cdots & 0 \\ 0 & e^{-\varsigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\varsigma_m} \end{pmatrix}_{m\times m}. \quad (33)$$

Let the sequence $\{\zeta_n\}$ in Δ be such that $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})$ and $V(\zeta_{n+k}, \zeta_{n+k+1}) \succ \theta$, for all $n \in \mathbb{N}$. Also, if $\zeta_n \longrightarrow \nu$, then $V(\zeta_n, \nu) \succ \theta$, for all $n \in \mathbb{N}$. Then, the sequence $\{\zeta_n\}$ converges to a fixed point of Υ. Also, if

$$V(\Upsilon(\rho, \cdots, \rho), \Upsilon(\rho, \cdots, \rho)) < V(\rho, \rho), \quad (34)$$

for all $\rho, \varrho \in \Delta$ with $\rho \neq \varrho$, then the fixed point of Υ is unique.

We present an example to support our main result.

Example 5. Let $\Delta = \{\varsigma_n = 1/n^2 : n = 1, 2, \cdots\} \cup \{\varsigma_0 = 0\}$, $V(\rho, \varrho) = (|\rho - \varrho|, |\rho - \varrho|)$, and define $\Upsilon : \Delta^2 \longrightarrow \Delta$ by

$$\Upsilon(\varsigma_n, \varsigma_m) = \begin{cases} \varsigma_{\max\{m,n\}+1}, & n, m \geq 1, \\ 0, & n = 0 \text{ or } m = 0. \end{cases} \quad (35)$$

Firstly, note that for all $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$, from Example 2.3 of [25], we have

$$|\varsigma_{n+1} - \varsigma_{m+1}|^{1/\sqrt{|\varsigma_{n+1}-\varsigma_{m+1}|}} |\varsigma_n - \varsigma_m|^{-1/\sqrt{|\varsigma_n-\varsigma_m|}} \leq \frac{1}{2},$$
$$\frac{1}{\sqrt{|\varsigma_{n+1} - \varsigma_{m+1}|}} - \frac{1}{\sqrt{|\varsigma_n - \varsigma_m|}} \geq 1. \quad (36)$$

As we know, $\varsigma_{n+2} = \min\{\varsigma_{n+1}, \varsigma_{n+2}\} = \Upsilon(\varsigma_n, \varsigma_{n+1})$, for all $\mathbb{N}$ and

$$\begin{aligned} V(\varsigma_{n+2}, \varsigma_{n+3}) &= (|\varsigma_{n+2} - \varsigma_{n+3}|, |\varsigma_{n+2} - \varsigma_{n+3}|) \\ &= \left(\left|\frac{1}{(n+2)^2} - \frac{1}{(n+3)^2}\right|, \left|\frac{1}{(n+2)^2} - \frac{1}{(n+3)^2}\right| \right) \\ &\succ (0,0) = \theta. \end{aligned} \quad (37)$$

Also, $\varsigma_n = 1/n^2 \longrightarrow 0$ and

$$V(\varsigma_n, 0) = (|\varsigma_n - 0|, |\varsigma_n - 0|) = \left(\frac{1}{n^2}, \frac{1}{n^2}\right) \succ (0,0) = \theta. \quad (38)$$

Define $F : \mathbb{R}_+^2 \longrightarrow \mathbb{R}^2$ by

$$F((\pi_1, \pi_2)) = \begin{cases} \left(\dfrac{\ln \pi_1}{\sqrt{\pi_1}}, \dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 \leq e \\ \left(\dfrac{\pi_1}{e\sqrt{e}}, \dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 > e. \end{cases} \quad (39)$$

Obviously, $F \in \mathscr{F}^2$. Also, take $\varsigma = (\varsigma_1, \varsigma_2) = (\ln 2, 1)$. We have

$$\begin{aligned} &\varsigma + F(V(\Upsilon(\varepsilon, \varepsilon), \Upsilon(\varepsilon, \sigma))) \\ &\quad \leq F(\max\{V(\varepsilon, \varepsilon), V(\varepsilon, \sigma)\}) \\ &\quad \Leftrightarrow (\ln 2, 1) + F(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|, |\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|) \\ &\quad \leq F(\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}) \Leftrightarrow (\ln 2, 1) \\ &\qquad + \left(\frac{\ln(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}}, \frac{-1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}}\right) \\ &\quad \leq \left(\frac{\ln(\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\})}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}}, \frac{-1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}}\right) \\ &\quad \Leftrightarrow \ln 2 + \frac{\ln(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} \leq \frac{\ln(\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\})}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}}, 1 \\ &\qquad + \frac{-1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} \leq \frac{-1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}} \\ &\quad \Leftrightarrow (|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}} \max \\ &\qquad \times \{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}} \\ &\quad \leq \frac{1}{2}, \frac{1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} - \frac{1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}} \geq 1, \end{aligned} \quad (40)$$

for any $\varepsilon, \varepsilon, \sigma \in \Delta$. Now, Let $\varepsilon = \varsigma_n$, $\varepsilon = \varsigma_m$, and $\sigma = \varsigma_p$. If $m \geq \max\{n, p\}$, then

$$\begin{aligned} V(\Upsilon(\varepsilon, \varepsilon), \Upsilon(\varepsilon, \sigma)) &= V\left(\Upsilon(\varsigma_n, \varsigma_m), \Upsilon\left(\varsigma_m, \varsigma_p\right)\right) \\ &= V(\varsigma_{m+1}, \varsigma_{m+1}) = (0,0) = \theta. \end{aligned} \quad (41)$$

So, we may assume that either $m < n$ or $m < p$. We consider the following cases:

Case 1. $n \leq m < p$. Let $n = 0$. If $n = m = 0$, then

$$V(\Upsilon(\varepsilon, \varepsilon), \Upsilon(\varepsilon, \sigma)) = (0,0) = \theta. \quad (42)$$

If $0 = n < m$, then

$$\begin{aligned} &(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}} \\ &\quad \cdot \max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}^{1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}} \\ &= \left(\varsigma_{p+1}\right)^{1/\sqrt{\varsigma_{p+1}}} (\varsigma_m)^{1/\sqrt{\varsigma_m}} = \left(\frac{1}{(p+1)^2}\right)^{p+1} \left(\frac{1}{m^2}\right)^{-m} \\ &\leq \left(\frac{1}{(p+1)^2}\right)^{p+1} \left(\frac{1}{p^2}\right)^{-p} \leq \frac{1}{(p+1)^2} \\ &\leq \frac{1}{2} \frac{1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} - \frac{1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}} \\ &= \frac{1}{\sqrt{\varsigma_{p+1}}} - \frac{1}{\sqrt{\varsigma_m}} = p + 1 - m \geq p + 1 - p = 1, \end{aligned} \quad (43)$$

and if $n > 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{m+1}-\varsigma_{p+1})^{1/\sqrt{\varsigma_{m+1}-\varsigma_{p+1}}}(\varsigma_m-\varsigma_p)^{-1/\sqrt{\varsigma_m-\varsigma_p}}\\&\le\frac{1}{2}\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_m}}=p+1-m\ge p+1-p=1.\end{aligned}\tag{44}$$

Case 2. $m < p \le n$. Here, if $m = 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=(\varsigma_{n+1})^{1/\sqrt{\varsigma_{n+1}}}(\varsigma_n)^{-1/\sqrt{\varsigma_n}}=\left(\frac{1}{(n+1)^2}\right)^{n+1}\left(\frac{1}{n^2}\right)^{-n}\le\frac{1}{(n+1)^2}\le\frac{1}{2},\end{aligned}\tag{45}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}}}-\frac{1}{\sqrt{\varsigma_n}}=n+1-n\ge n+1-n=1,\end{aligned}\tag{46}$$

and if $m > 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\\&\quad\cdot\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1}-\varsigma_{p+1})^{1/\sqrt{\varsigma_n-\varsigma_p}}(\varsigma_n-\varsigma_p)^{-1/\sqrt{\varsigma_n-\varsigma_p}}\le\frac{1}{2},\end{aligned}\tag{47}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}-\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_n-\varsigma_p}}\ge1.\end{aligned}\tag{48}$$

Case 3. $m \le n < p$. In this case, if $m = 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{p+1})^{1/\sqrt{\varsigma_{p+1}}}(\varsigma_p)^{-1/\sqrt{\varsigma_p}}=\left(\frac{1}{(p+1)^2}\right)^{p+1}\left(\frac{1}{p^2}\right)^{-p}\\&\le\frac{1}{(p+1)^2}\le\frac{1}{2},\end{aligned}\tag{49}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_p}}=p+1-p=1,\end{aligned}\tag{50}$$

and if $m > 0$, then.

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1}-\varsigma_{p+1})^{1/\sqrt{\varsigma_{n+1}-\varsigma_{p+1}}}(\varsigma_n-\varsigma_p)^{-1/\sqrt{\varsigma_n-\varsigma_p}}\le\frac{1}{2},\end{aligned}\tag{51}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}-\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_n-\varsigma_p}}\ge1.\end{aligned}\tag{52}$$

Case 4. $p \le m < n$. Here, if $p = 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1})^{1/\sqrt{\varsigma_{n+1}}}(\varsigma_n)^{-1/\sqrt{\varsigma_n}}=\left(\frac{1}{(n+1)^2}\right)^{n+1}\left(\frac{1}{n^2}\right)^{-n}\\&\le\frac{1}{(n+1)^2}\le\frac{1}{2},\end{aligned}\tag{53}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}}}-\frac{1}{\sqrt{\varsigma_n}}=n+1-n=1,\end{aligned}\tag{54}$$

and if $p > 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1}-\varsigma_{m+1})^{1/\sqrt{\varsigma_{n+1}-\varsigma_{m+1}}}(\varsigma_n-\varsigma_m)^{-1/\sqrt{\varsigma_n-\varsigma_m}}\le\frac{1}{2},\end{aligned}\tag{55}$$

and

$$\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}} = \frac{1}{\sqrt{\varsigma_{n+1}-\varsigma_{m+1}}}-\frac{1}{\sqrt{\varsigma_n-\varsigma_m}}\geq 1. \tag{56}$$

Also, let $\rho, \rho \in \Delta$ with $\rho \neq \rho$. Without loss of any generality, let $\rho = \varsigma_n$, $\rho = \varsigma_m$ with $n < m$. If $n = 0$, then

$$\begin{aligned} V(\Upsilon(\rho,\rho),\Upsilon(\rho,\rho)) &= V(\Upsilon(\varsigma_0,\varsigma_0),\Upsilon(\varsigma_m,\varsigma_m)) = V(0,\varsigma_{m+1}) \\ &= (\varsigma_{m+1},\varsigma_{m+1}) \prec (\varsigma_m,\varsigma_m) = V(0,\varsigma_m) \\ &= V(\rho,\rho), V(\varsigma_1,\varsigma_{m-1}) = \frac{m(m-1)}{2}-1 \\ &< \frac{m(m+1)}{2}-1 = V(\varsigma_1,\varsigma_m) = V(\rho,\rho), \end{aligned} \tag{57}$$

and if $n > 0$, then

$$\begin{aligned} V(\Upsilon(\rho,\rho),\Upsilon(\rho,\rho)) &= V(\Upsilon(\varsigma_n,\varsigma_n),\Upsilon(\varsigma_m,\varsigma_m)) = V(\varsigma_{n+1},\varsigma_{m+1}) \\ &= (|\varsigma_{n+1}-\varsigma_{m+1}|,|\varsigma_{n+1}-\varsigma_{m+1}|) \\ &= \left(\frac{1}{(n+1)^2}-\frac{1}{(m+1)^2},\frac{1}{(n+1)^2}-\frac{1}{(m+1)^2}\right) \\ &\prec \left(\frac{1}{n^2}-\frac{1}{m^2},\frac{1}{n^2}-\frac{1}{m^2}\right) = V(\varsigma_n,\varsigma_m) \\ &= V(\rho,\rho). \end{aligned} \tag{58}$$

We see that all of the conditions of Theorem 2 are satisfied. Thus, Υ has a unique fixed point. Here, $\Upsilon(\varsigma_0,\varsigma_0) = \varsigma_0$, and ς_0 is the unique fixed point.

We present an example in an infinite dimensional sequence space ℓ_1 which is adapted from the above example, and so, we leave the details for the reader.

Let A be the space of all convergent sequences (a_n) for which $a_i = 1/n^2$ (n is an arbitrary natural number) for exactly one i and $a_j = 0$ for other indices.

Let $\Delta = A \cup \{\varsigma_0 = (0,0,0,\cdots)\}$, $V((a_n)_{m=1}^\infty,(b_n)_{n=1}^\infty) = (\Sigma_{m,n=1}^\infty |a_n-b_n|,\Sigma_{m,n=1}^\infty |a_n-b_n|)$, and define $\Upsilon : \Delta^2 \longrightarrow \Delta$ by

$$\Upsilon\left((a_n)_{n=1}^\infty,(b_n)_{n=1}^\infty\right) = \begin{cases} \left(0,\cdots,0,a_{\max\{i,j\}+1}=\dfrac{1}{n^2},0,\cdots\right), & (a_n)\neq(0,0,0,\cdots) \text{ and } (b_n)\neq(0,0,0,\cdots), \\ (0,0,0,\cdots), & (a_n)=(0,0,0,\cdots) \text{ or } (b_n)=(0,0,0,\cdots). \end{cases} \tag{59}$$

Define $F : \mathbb{R}_+^2 \longrightarrow \mathbb{R}^2$ by

$$F((\pi_1,\pi_2)) = \begin{cases} \left(\dfrac{\ln\pi_1}{\sqrt{\pi_1}},\dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 \leq e \\ \left(\dfrac{\pi_1}{e\sqrt{e}},\dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 > e. \end{cases} \tag{60}$$

Obviously, $F \in \mathscr{F}^2$. Also, take $\varsigma = (\varsigma_1,\varsigma_2) = (\ln 2, 1)$.

Reviewing the above example, we can show that all of the conditions of Theorem 2 are satisfied. Thus, Υ has a unique fixed point. Here, $\Upsilon((0,0,0,\cdots),(0,0,0,\cdots)) = (0,0,0,\cdots)$ and $(0,0,0,\cdots)$ is the unique fixed point.

3. Application

Let $(E,\|\cdot\|_E)$ be a Banach space and $A_1,\cdots,A_k : E^k \longrightarrow E$ be k nonlinear operators. In this section, motivated by the work in [26], we will present a result on existence of a solution for the following semilinear operator system:

$$\begin{aligned} A_1(\iota_1,\iota_2,\cdots,\iota_k) &= \iota_1, \\ &\vdots \\ A_k(\iota_1,\iota_2,\cdots,\iota_k) &= \iota_k. \end{aligned} \tag{61}$$

Similar systems which appear in various branches of mathematics could be seen in [27].

Let $\Delta = E^k$ and define $V : \Delta\times\Delta \longrightarrow \mathbb{R}^k$, for $u = (\iota_1,\cdots,\iota_k)$, $v = (\varepsilon_1,\cdots,\varepsilon_k) \in \Delta$ by $V(u,v) = (\|\iota_1-\varepsilon_1\|_E,\cdots,\|\iota_k-\varepsilon_k\|_E)$. Evidently, (Δ, V) is a complete VVMS.

If we define a mapping $\Upsilon : \Delta^k \longrightarrow \Delta$ by

$$\Upsilon(u,u,\cdots,u) = (A_1(\iota_1,\iota_2,\cdots,\iota_k),\cdots,A_k(\iota_1,\iota_2,\cdots,\iota_k)), \tag{62}$$

then the system (61) can be written as a fixed point problem such as

$$\Upsilon(u,u,\cdots,u) = (\iota_1,\iota_2,\cdots,\iota_k) = u, \tag{63}$$

in the space Δ. Therefore, applying Theorem 2, we investigate the sufficient hypothesis which leads to the existence of a solution of problem (63).

Theorem 9. *Assume that there exist positive real numbers ς_i ($i = 1,\cdots,k$) such that*

$$\|A_i(\iota_1,\iota_2,\cdots,\iota_k)-A_i(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k)\|_E \leq e^{-\varsigma_i}\|\iota_i-\varepsilon_i\|_E, \tag{64}$$

for all $u = (\iota_1,\cdots,\iota_k)$, $v = (\varepsilon_1,\cdots,\varepsilon_k) \in E^k$ with $\iota_i \neq \varepsilon_i$. Then, the system (61) has a unique solution in E^k.

Proof. By the inequality (64), we have

$$\left(\varsigma_i + \ln \|A_i(\iota_1, \iota_2, \cdots, \iota_k) - A_i(\kappa_1, \kappa_2, \cdots, \kappa_k)\|_E\right) \le \ln \|\iota_i - \kappa_i\|_E, \tag{65}$$

for all $i = 1, \cdots, k$. Hence, we get

$$\begin{aligned}&\left(\varsigma_1 + \ln \|A_1(u_1) - A_1(u_2)\|_E, \cdots, \varsigma_k + \ln \|A_k(u_1) - A_k(u_2)\|_E\right)\\ &\quad \preceq (\ln \|\iota_1 - \kappa_1\|_E, \cdots, \ln \|\iota_k - \kappa_k\|_E).\end{aligned} \tag{66}$$

Taking the function $F \in \mathscr{F}^k$ as $F(\pi_1, \cdots, \pi_k) = (\ln \pi_1, \cdots, \ln \pi_k)$, the above inequality can be written as

$$\begin{aligned}&(\varsigma_1, \cdots, \varsigma_k) + F((\|A_1u_1 - A_1u_2\|_E, \cdots, \|A_ku_1 - A_ku_2\|_E))\\ &\quad \le F((\|\iota_1 - \kappa_1\|_E, \cdots, \|\iota_k - \kappa_k\|_E))\\ &\quad = F(V(u_1, u_2)), VF(\sup\{V(u_1, u_2), V(u_1, u_2), \cdots, V(u_1, u_2)\}),\end{aligned} \tag{67}$$

or, equivalently,

$$\begin{aligned}&(\varsigma_1, \cdots, \varsigma_k) + F(V(\Upsilon(u_1, \cdots, u_k), \Upsilon(u_2, \cdots, u_{k+1})))\\ &\quad \preceq F(\sup\{V(u_1, u_2), V(u_2, u_3), \cdots, V(u_k, u_{k+1})\}),\end{aligned} \tag{68}$$

where $\varsigma = (\varsigma_1, \cdots, \varsigma_k)$. Thus, applying Theorem 2, Υ possesses a unique fixed point in $\Delta = E^k$, or, equivalently, the semilinear operator system (61) has a unique solution in E^k.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the manuscript.

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Research Article

Some Properties of Kantorovich-Stancu-Type Generalization of Szász Operators including Brenke-Type Polynomials via Power Series Summability Method

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In this paper, we study the Kantorovich-Stancu-type generalization of Szász-Mirakyan operators including Brenke-type polynomials and prove a Korovkin-type theorem via the T-statistical convergence and power series summability method. Moreover, we determine the rate of the convergence. Furthermore, we establish the Voronovskaya- and Grüss-Voronovskaya-type theorems for T-statistical convergence.

1. Introduction and Preliminaries

Let $\mathscr{K} \subseteq \mathbb{N}$ (set of natural numbers) and $\mathscr{K}_m = \{i \leq m : i \in \mathscr{K}\}$. Then, the natural density or we can say asymptotic density of $\mathscr{K}$ is defined by $\sigma(\mathscr{K}) = \lim_m (1/m)|\mathscr{K}_m|$ whenever the limit exists, where $|\mathscr{K}_m|$ denotes the cardinality of the set $\mathscr{K}_m$. A sequence $\eta = (\eta_i)$ is statistically convergent to $\mathscr{L}$ if for every $\varepsilon > 0$

$$\lim_m \frac{1}{m} |\{i \leq m : |\eta_i - \mathscr{L}| \geq \varepsilon\}| = 0, \qquad (1)$$

and we write $st - \lim_m \eta_m = \mathscr{L}$.

Let $T = (t_{nj})$ be a matrix and $\eta = (\eta_j)$ be a sequence. The $T-$ transform of the sequence $\eta = (\eta_n)$ is defined by $T\eta = (T_n(\eta))$, $(T\eta)_n = \sum_j t_{nj}\eta_j$ if the series converges for every $n \in \mathbb{N}$. We say that η is $T-$ summable to the number $\mathscr{L}$ if $(T\eta)_n$ converges to $\mathscr{L}$. The summability matrix T is regular whenever $\lim_j \eta_j = \mathscr{L} = \lim_n (T\eta)_n$.

Let $T = (t_{nj})$ be a regular matrix. A sequence $\eta = (\eta_j)$ is said to be T -statistically convergent (see [1]) to real number $\mathscr{L}$ if for any >0, $\lim_n \sum_{j:|\eta_j - \mathscr{L}| \geq \varepsilon} t_{nj} = 0$, and we write $st_T - \lim \eta = \mathscr{L}$. If T is Cesàro matrix of order 1, then T -statistical convergence is reduced to the statistical convergence.

In this paper, we also use the power series summability method which includes several known summability methods such as Abel and Borel (see [2–9]). Note that the power method is more effective than the ordinary convergence (see [10]).

Let (p_j) be a sequence of real numbers such that $p_0 > 0$, $p_1, p_2, \cdots \geq 0$, and the corresponding power series $p(u) = \sum_{j=0}^{\infty} p_j u^j$ has radius of convergence R with $0 < R \leq \infty$. If $\lim_{u \to R^-} (1/p(u)) \sum_{j=0}^{\infty} \eta_j p_j u^j = L$ for all $t \in (0, R)$, then we say that $\eta = (\eta_j)$ is convergent in the sense of power series method (see [11, 12]). Define $A(u)B(\eta u)=\sum_{k\geq 0}p_k(\eta)u^k$, where $A(u)=\sum_{j\geq 0}a_ju^j$ and $B(u)=\sum_{j\geq 0}b_ju^j$ are analytical functions such that $a_0\neq 0$ and $b_j\neq 0$ for all $j\geq 0$ (see [13]). Clearly, $p_k(\eta)=\sum_{j=0}^{k}a_{k-j}b_j\eta^j$. Moreover, the power series method is regular if and only if $\lim_{u\to R^-}(p_ju^j/p(u))=0$ holds for each $j\in\{0,1,\cdots\}$ (see [14]).

We study a Korovkin-type theorem for the Kantorovich-Stancu-type Szász-Mirakyan operators via power series method. We determine the rate of convergence for these operators. Furthermore, we give a Voronovskaya-type theorem for $T-$ statistical convergence. Such type of operators is widely studied by several authors (see [15–19]).

We start by recalling the class of Kantorovich-Stancu-type generalization of Szász-Mirakyan operators, including Brenke-type polynomials. For every $h\in C_B[0,\infty)=C[0,\infty)\cap E$,

$$K_n^{\alpha,\beta}(h,\eta)=\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(n\eta)\int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)}h(u)du, \quad (2)$$

for $n\in\mathbb{N}$, $x\in 0,\infty)$ and $0\leq\alpha\leq\beta$, where $E=\{h: x\in[0,\infty), \lim_{x\to\infty}(h(x)/(1+x^2))<\infty\}$. In what follows, we calculate the moments and central moments for Kantorovich-Stancu of Szász-Mirakyan operators. Let us mention some properties of the functions $A(u)$ and $B(u)$ (see [13, 20]).

(1) $A(1)\neq 0$, $(a_{k-r}b_r/A(1))\geq 0$, for all $0\leq r\leq k$ and $k=0,1,2,\cdots$

(2) $B:[0,\infty)\longrightarrow(0,\infty)$

(3) Series $A(u)=\sum_{r=0}^{\infty}a_ru^r$, for $a_0\neq 0$, and $B(u)=\sum_{r=0}^{\infty}b_ru^r$, for $b_r\neq 0$ $(r\geq 0)$, are convergent for $|u|<R$ $(R>1)$ and $A(u), B(u)$ are analytic functions

The next lemma is followed immediately from the fact that $A(u)B(\eta u)=\sum_{k\geq 0}p_k(\eta)u^k$.

Lemma 1. *Let D be the operator $u(d/du)$. For all $m\geq 0$,*

$$\sum_{k\geq 0}k^mp_k(\eta)t^k=D^m(A(u)B(\eta u)). \quad (3)$$

For example, Lemma 1 for $m=0,1,2,3,4$ gives

$$\sum_{k\geq 0}p_k(\eta)=A(1)B(\eta),$$

$$\sum_{k\geq 0}kp_k(\eta)=uA^{(1)}(u)B(\eta u)+\eta uA(t)B^{(1)}(\eta u),$$

$$\begin{aligned}\sum_{k\geq 0}k^2p_k(\eta)&=u^2A^{(2)}(u)B(\eta u)+A^{(1)}(u)\left(uB(\eta u)+2\eta u^2B^{(1)}(\eta u)\right)\\&\quad+A(u)\left(\eta uB^{(1)}(\eta u)+\eta^2u^2B^{(2)}(\eta u)\right),\end{aligned}$$

$$\begin{aligned}\sum_{k\geq 0}k^3p_k(\eta)&=u^3A^{(3)}(u)B(\eta u)+A^{(2)}(u)\left(3u^2B(\eta u)+3\eta u^3B^{(1)}(\eta u)\right)\\&\quad+A^{(1)}(u)\left(uB(\eta u)+6\eta u^2B^{(1)}(\eta u)+3\eta^2u^3B^{(2)}(\eta u)\right)\\&\quad+A(u)\left(\eta uB^{(1)}(\eta u)+3\eta^2u^2B^{(2)}(\eta u)+\eta^3u^3B^{(3)}(\eta u)\right),\end{aligned}$$

$$\begin{aligned}\sum_{k\geq 0}k^4p_k(\eta)&=u^4A^{(4)}(u)B(\eta u)+A^{(3)}(u)\left(6u^3B(\eta u)+4\eta u^4B^{(1)}(\eta u)\right)\\&\quad+A^{(2)}(u)\left(7u^2B(\eta u)+18\eta u^3B^{(1)}(\eta t)+6\eta^2u^4B^{(2)}(\eta u)\right)\\&\quad+A^{(1)}(u)\left(uB(\eta u)+14\eta u^2B^{(1)}(\eta u)+18\eta^2u^3B^{(2)}(\eta u)\right.\\&\quad\left.+4\eta^3u^4B^{(3)}(\eta u)\right)+A(u)\left(\eta uB^{(1)}(\eta u)+7\eta^2u^2B^{(2)}(\eta u)\right.\\&\quad\left.+6\eta^3u^3B^{(3)}(\eta u)+\eta^4u^4B^{(4)}(\eta u)\right).\end{aligned} \quad (4)$$

Theorem 2. *Let $e_i=e_i(t)=t^i$ for all $i\geq 0$ and let D be the operator $t(d/dt)$. Then,*

$$\begin{aligned}K_n^{\alpha,\beta}(e_i,x)&=\frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{j=0}^{i+1}\binom{i+1}{j}\left((\alpha+1)^{i+1-j}\right.\\&\quad\left.-\alpha^{i+1-j}\right)D^jA(t)B(nxt)|_{t=1}.\end{aligned} \quad (5)$$

Proof. By the definition of the operators, we have

$$\begin{aligned}K_n^{\alpha,\beta}(e_i,x)&=\frac{n+\beta}{A(1)B(nx)}\sum_{k\geq 0}p_k(nx)\int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)}t^idt\\&=\frac{n+\beta}{A(1)B(nx)}\sum_{k\geq 0}p_k(nx)\\&\quad\cdot\left(\frac{(k+1+\alpha)^{i+1}}{(i+1)(n+\beta)^{i+1}}-\frac{(k+\alpha)^{i+1}}{(i+1)(n+\beta)^{i+1}}\right)\\&=\frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{k\geq 0}\sum_{j=0}^{i+1}\binom{i+1}{j}k^jp_k(nx)\\&\quad\cdot\left((\alpha+1)^{i+1-j}-\alpha^{i+1-j}\right)\\&=\frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{j=0}^{i+1}\binom{i+1}{j}\\&\quad\cdot\left((\alpha+1)^{i+1-j}-\alpha^{i+1-j}\right)\sum_{k\geq 0}k^jp_k(nx).\end{aligned} \quad (6)$$

Thus, by Lemma 1, we complete the proof.

Lemma 3 (for instance, see [21], equation (1.27)). *Let X, L be two operators on the set of functions defined by $X(f(u))=uf(u)$ and $L(f(u))=(d/du)f(u)$. Then,*

$$(XL)^m=\sum_{j=1}^{m}S(m,j)X^jL^j. \quad (7)$$

Moreover,

$$(XL)^m(f(u)g(u)) = \sum_{j=1}^{m}\sum_{i=0}^{j}\binom{j}{i}S(m,j)u^j\frac{d^i}{du^i}f(u)\cdot\frac{d^{j-i}}{du^{j-i}}g(u), \tag{8}$$

where $S(m,j)$ is the Stirling number of the second kind.

Define $a_j = (d^j/dt^j)A(t)|_{t=1}$ and $b_j = (d^j/dt^j)B(t)|_{t=nx}$, for all $j \ge 0$. Therefore, Theorem 2 with $D = XL$ and Lemma 3 imply the following theorem.

Theorem 4. *Let $e_i = e_i(t) = t^i$ for all $i \ge 0$. Then,*

$$\begin{aligned} K_n^{\alpha,\beta}(e_i,x) &= \frac{(n+\beta)^{-i}}{i+1}\left((\alpha+1)^{i+1}-\alpha^{i+1}\right) \\ &+ \frac{(n+\beta)^{-i}}{i+1}\sum_{j=1}^{i+1}\sum_{\ell=1}^{j}\sum_{s=0}^{\ell}\binom{i+1}{j}\binom{\ell}{s} \\ &\cdot S(j,\ell)\left((\alpha+1)^{i+1-j}-\alpha^{i+1-j}\right)\frac{n^{\ell-s}a_s b_{\ell-s}}{a_0 b_0}x^{2\ell-s}, \end{aligned} \tag{9}$$

where $S(m,\ell)$ is the Stirling number of the second kind.

Example 5. By applying Theorem 4 for $i = 0, 1, 2, 3, 4$ with using (2), we obtain

$$\begin{aligned} K_n^{\alpha,\beta}(e_0,x) &= 1, \\ K_n^{\alpha,\beta}(e_1,x) &= \frac{2\alpha+1}{2(n+\beta)} + \frac{a_1}{a_0(n+\beta)} + \frac{nb_1}{b_0(n+\beta)}x, \\ K_n^{\alpha,\beta}(e_2,x) &= \frac{3\alpha^2+3\alpha+1}{3(n+\beta)^2} + \frac{2a_1(\alpha+1)+a_2}{a_0(n+\beta)^2} + \frac{2nb_1(a_0(\alpha+1)+a_1)}{a_0b_0(n+\beta)^2}x + \frac{n^2b_2}{b_0(n+\beta)^2}x^2, \\ K_n^{\alpha,\beta}(e_3,x) &= \frac{4\alpha^3+6\alpha^2+4\alpha+1}{4(n+\beta)^3} + \frac{6a_1(\alpha+1)^2+6a_2(\alpha+1)+a_1+3a2+2a_3}{2a_0(n+\beta)^3} + \frac{nb_1\left(6a_0(\alpha+1)^2+12a_1(\alpha+1)+a_0+6a_1+6a_2\right)}{2a_0b_0(n+\beta)^3}x \\ &+ \frac{3n^2b_2(2a_0(\alpha+1)+a_0+2a_1)}{2a_0b_0(n+\beta)^3} + \frac{n^3b_3}{b_0(n+\beta)^3}x^3, \\ K_n^{\alpha,\beta}(e_4,x) &= \frac{5\alpha^4+10\alpha^3+10\alpha^2+5\alpha+1}{5(n+\beta)^4} + \frac{4a_1(\alpha+1)^3+6a_2(\alpha+1)^2+2(a_1+3a_2+2a_3)(\alpha+1)+3a_2+4a_3+a_4}{a_0(n+\beta)^4} \\ &+ \frac{2nb_1\left(2a_0(\alpha+1)^3+6a_1(\alpha+1)^2+(a_0+6a_1+6a_2)(\alpha+1)+3a_1+6a_2+2a_3\right)}{a_0b_0(n+\beta)^4}x \\ &+ \frac{3n^2b_2\left(2a_0(\alpha+1)^2+2(a_0+2a_1)(\alpha+1)+a_0+4a_1+2a_2\right)}{a_0b_0(n+\beta)^4}x^2 + \frac{4n^3b_3(a_0(\alpha+1)+a_0+a_1)}{a_0b_0(n+\beta)^4}x^3 \\ &+ \frac{n^4b_4}{b_0(n+\beta)^4}x^4. \end{aligned} \tag{10}$$

Theorem 6. *Let $y = \alpha - x(n+\beta)$, and let D be the operator $t(d/dt)$. Then,*

$$K_n^{\alpha,\beta}\left((t-x)^i,x\right) = \frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{j=0}^{i+1}\binom{i+1}{j}\cdot\left((1+y)^{i+1-j}-y^{i+1-j}\right)D^jA(t)B(nxt)|_{t=1}. \tag{11}$$

Proof. By the definitions, we have

$$\begin{aligned} K_n^{\alpha,\beta}\left((t-x)^i,x\right) &= \frac{n+\beta}{A(1)B(nx)}\sum_{k\ge0}p_k(nx)\int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)}(t-x)^i dt \\ &= \frac{n+\beta}{A(1)B(nx)}\sum_{k\ge0}p_k(nx)\cdot\left(\frac{(k+1+y)^{i+1}}{(i+1)(n+\beta)^{i+1}} - \frac{(k+y)^{i+1}}{(i+1)(n+\beta)^{i+1}}\right) \end{aligned}$$

$$= \frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)} \sum_{k\geq 0} \sum_{j=0}^{i+1} \binom{i+1}{j} \cdot k^j p_k(nx)\left((1+y)^{i+1-j} - y^{i+1-j}\right)$$
$$= \frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)} \sum_{j=0}^{i+1} \binom{i+1}{j} \cdot \left((1+y)^{i+1-j} - y^{i+1-j}\right) \sum_{k\geq 0} k^j p_k(nx). \quad (12)$$

Thus, Lemma 1 completes the proof.

By Theorem 6 and Lemma 3, we obtain the following result.

Theorem 7. *Let $y = \alpha - x(n+\beta)$. Then,*

$$K_n^{\alpha,\beta}\left((t-x)^i, x\right) = \frac{(n+\beta)^{-i}}{i+1}\left((y+1)^{i+1} - y^{i+1}\right) + \frac{(n+\beta)^{-i}}{i+1} \sum_{j=1}^{i+1} \sum_{\ell=1}^{j} \sum_{s=0}^{\ell} \binom{i+1}{j} \binom{\ell}{s} \cdot S(j,\ell)\left((y+1)^{i+1-j} - y^{i+1-j}\right) \frac{n^{\ell-s} a_s b_{\ell-s}}{a_0 b_0} x^{2\ell-s}, \quad (13)$$

where $S(m,\ell)$ is the Stirling number of the second kind.

Remark 8. By applying Theorem 7 for $i = 0, 1, 2, 3$, we obtain

$$K_n^{\alpha,\beta}(1,x) = 1,$$
$$K_n^{\alpha,\beta}(t-x,x) = \frac{2a_0\alpha + a_0 + 2a_1}{2a_0(n+\beta)} - \frac{(b_0 n + b_0\beta - b_1 n)}{b_0(n+\beta)} x,$$
$$K_n^{\alpha,\beta}\left((t-x)^2,x\right) = \frac{3a_0\alpha^2 + 3a_0\alpha + 6a_1\alpha + a_0 + 6a_1 + 3a_2}{3a_0(n+\beta)^2} - \frac{2a_0b_0\alpha(n+\beta) - 2a_0b_1n(\alpha+1) + a_0b_0(n+\beta) + 2a_1b_0(n+\beta) - 2a_1b_1n}{a_0b_0(n+\beta)^2} x + \frac{b_0n^2 + 2b_0n\beta + b_0\beta^2 - 2b_1n^2 - 2b_1n\beta + b_2n^2}{b_0(n+\beta)^2} x^2,$$
$$\begin{aligned} K_n^{\alpha,\beta}\left((t-x)^3,x\right) = {} & \frac{4a_0\alpha^3 + 6a_0\alpha^2 + 12a_1\alpha^2 + 4a_0\alpha + 24a_1\alpha + 12a_2\alpha + a_0 + 14a_1 + 18a_2 + 4a_3}{4a_0(n+\beta)^3} \\ & - \frac{6a_0b_0n\alpha^2 + 6a_0b_0\alpha^2\beta - 6a_0b_1n\alpha^2 + 6a_0b_0n\alpha + 6a_0b_0\alpha\beta}{2a_0b_0(n+\beta)^3} x \\ & - \frac{-12a_0b_1n\alpha + 12a_1b_0n\alpha + 12a_1b_0\alpha\beta - 12a_1b_1n\alpha + 2a_0b_0n + 2a_0b_0\beta}{2a_0b_0(n+\beta)^3} x \\ & - \frac{-7a_0b_1n + 12a_1b_0n + 12a_1b_0\beta - 18a_1b_1n + 6a_2b_0n + 6a_2b_0\beta - 6a_2b_1n}{2a_0b_0(n+\beta)^3} x \\ & + \frac{3\left(2a_0b_0n^2\alpha + 4a_0b_0n\alpha\beta + 2a_0b_0\alpha\beta^2 - 4a_0b_1n^2\alpha - 4a_0b_1n\alpha\beta + 2a_0b_2n^2\alpha\right)}{2a_0b_0(n+\beta)^3} x^2 \\ & + \frac{3\left(a_0b_0n^2 + 2a_0b_0n\beta + a_0b_0\beta^2 - 4a_0b_1n^2 - 4a_0b_1n\beta + 3a_0b_2n^2 + 2a_1b_0n^2\right)}{2a_0b_0(n+\beta)^3} x^2 \\ & - \frac{b_0n^3 + 3b_0n^2\beta + 3b_0n\beta^2 + b_0\beta^3 - 3b_1n^3 - 6b_1n^2\beta - 3b_1n\beta^2 + 3b_2n^3 + 3b_2n^2\beta - b_3n^3}{b_0(n+\beta)^3} x^3. \end{aligned} \quad (14)$$

Theorem 6 for $i = 0, 1, \cdots, 6$ (with the help of mathematical programming), we obtain the following result.

Proposition 9. *Let us consider that*

$$\lim_{n\to\infty} \frac{B^{(m)}(nx)}{B(nx)} = 1 \quad \text{for} \quad m = 0, 1, 2, \cdots, 6, \quad (15)$$

where $B^{(m)}(t)$ is the m-th derivative of $B(t)$. Then, we obtain

$$\lim_{n\to\infty} K_n^{\alpha,\beta}(1,x) = 1$$
$$\lim_{n\to\infty} n K_n^{\alpha,\beta}\left((t-x)^1,x\right) = a_1/a_0 + 1/2 + \alpha - \beta x$$
$$\lim_{n\to\infty} n K_n^{\alpha,\beta}\left((t-x)^2,x\right) = x$$

Table 1: The values of the functions $20K_{20}^{\alpha,\beta}(f;x)$ and $100K_{100}^{\alpha,\beta}(f;x)$ at $x = 0, 0.1, 0.2, \cdots, 1$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$20K_{20}^{\alpha,\beta}(f;x)$	0.01511	0.10173	0.18926	0.27770	0.36704	0.45729	0.54845
$100K_{100}^{\alpha,\beta}(f;x)$	0.00326	0.10041	0.19775	0.29529	0.39303	0.49096	0.58909

$$\lim_{n\to\infty} n^2 K_n^{\alpha,\beta}\left((t-x)^3, x\right) = 3\alpha + 5/2 + 3a_1/a_0 - 3\beta x^2$$
$$\lim_{n\to\infty} n^2 K_n^{\alpha,\beta}\left((t-x)^4, x\right) = 3x^2$$
$$\lim_{n\to\infty} n^3 K_n^{\alpha,\beta}\left((t-x)^5, x\right) = (15\alpha + 35/2 + 15a_1/a_0)x^2 - 15\beta x^3$$
$$\lim_{n\to\infty} n^3 K_n^{\alpha,\beta}\left((t-x)^6, x\right) = 15x^3. \quad (16)$$

Example 10. Let $A(t) = 1$, $B(t) = e^t$, $\alpha = 0$, $\beta = 1$, and $f(t) = (t-x)^2$. By the fact that $A(t)B(xt) = \sum_{k\geq 0} p_k(x)t^k$, we have that $p_k(x) = x^k/k!$.

Table 1 presents the values of the functions $nK_n^{\alpha,\beta}(f;x)$ and x at $x = 0, 0.1, 0.2, \cdots, 0.6$ and $n = 20$, $n = 100$, where we approximated $K_n^{\alpha,\beta}(f;x)$ as

$$K_n^{\alpha,\beta}(f,x) = \frac{n+\beta}{A(1)B(nx)} \sum_{k=0}^{3000} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)} f(t)dt. \quad (17)$$

We note that the Korovkin-type theorems are very useful tools in approximation which were studied in several function spaces [3–8, 10, 22–29]. We say that sequence of operators $K_n^{\alpha,\beta}$ converges to L in the sense of power series if

$$\lim_{u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} K_n^{\alpha,\beta}(f,x) p_n u^n = L, \quad (18)$$

for every $u \in (0, R)$.

2. Main Results

We study here $T-$ statistical convergence of the operators $K_n^{\alpha,\beta}$. Note that the Korovkin-type theorem for $T-$ statistical convergence was considered in [24] as follows:

Theorem 11. *Let (B_j) be a sequence of positive linear operators on $C[0,1]$ and let $T = (t_{nj})$ be a nonnegative regular summability matrix such that*

$$st_T - \lim_n \left\| B_j e_i - e_i \right\| = 0, i = 0, 1, 2. \quad (19)$$

Then, for any $f \in C[0,1]$

$$st_T - \lim_n \left\| B_j h - h \right\| = 0, \quad (20)$$

where $\|h\| = \max_{0\leq x\leq 1} |h(x)|$.

Based on the above theorem, we give the following result.

Theorem 12. *Let $T = (t_{nj})$ be a regular matrix and $(K_n^{\alpha,\beta})$ be as in (2) on $C[0,M] \cap E$ such that $\lim_{n\to\infty}(B^{(i)}(nx))/(B(nx)) = 1$, where $B^{(i)}(nx)$ denotes i^{th} derivative and*

$$st_A - \lim_n \left\| K_n^{\alpha,\beta} e_i - e_i \right\| = 0 \ (i = 1, 2). \quad (21)$$

Then, for any $h \in C[0,M] \cap E$

$$st_A - \lim_n \left\| K_n^{\alpha,\beta} h - h \right\| = 0, \quad (22)$$

where $\|h\| = \max_{t\in[0,M]} |h(t)|$.

Proof. From Lemma 5, we have that $st_T - \lim_n \|K_n^{\alpha,\beta} e_0 - e_0\| = 0$. Now, we will estimate the following expressions:

$$\left\| K_n^{\alpha,\beta} e_1 - e_1 \right\| \leq \left\| x\left(\frac{n}{n+\beta} \cdot \frac{B'(nx)}{B(nx)} - 1\right) \right\| + \left\| \frac{A'(1)}{(n+\beta)A(1)} \right\| + \left\| \frac{2\alpha+1}{2(n+\beta)} \right\|. \quad (23)$$

Note that $\lim_{n\to\infty}(B'(nx))/(B(nx)) = 1$. So from the last two relations we have that $\|K_n^{\alpha,\beta} e_1 - e_1\| = 0$. Moreover,

$$\left\| K_n^{\alpha,\beta} e_2 - e_2 \right\| = \left\| \left(\frac{n}{n+\beta}\right)^2 \frac{B''(nx)}{B(nx)} \cdot x^2 + \frac{nB'(nx)\left[2A'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1)B(nx)} x + \frac{1}{(n+\beta)^2 A(1)} \left\{ A''(1) + (2\alpha+2)A'(1) \cdot \left(\alpha^2 + \alpha + \frac{1}{3}\right) A(1) \right\} - x^2 \right\| \longrightarrow 0. \quad (24)$$

Now proof follows directly from Theorem 11.

This theorem is an extension of some known results for the Kantorovich-Stancu-type Szász-Mirakyan operators.

Example 13 (see [6]). Under conditions given in Theorem 12, we define the following operators

$$N_n(h,x) = (1+x_n)K_n^{\alpha,\beta}(h,x), \tag{25}$$

where the sequence (x_n) is given as follows:

$$(x_n) = \begin{cases} \frac{1}{n^3}; & m^2 - m \le n \le m^2 - 1 \\ \frac{1}{n^4}; & n = m^2;\, m \in \mathbb{N} \setminus \{1\} \\ 0; & \text{otherwise,} \end{cases} \tag{26}$$

then

$$\begin{aligned} N_n(e_0,x) &= (1+x_n), \\ N_n(e_1,x) &= (1+x_n)\left(\frac{2\alpha+1}{2(n+\beta)} + \frac{a_1}{a_0(n+\beta)} + \frac{nb_1}{b_0(n+\beta)}x\right), \\ N_n(e_2,x) &= (1+x_n)\left(\frac{3\alpha^2+3\alpha+1}{3(n+\beta)^2} + \frac{2a_1(\alpha+1)+a_2}{a_0(n+\beta)^2}\right. \\ &\quad \left. + \frac{2nb_1(a_0(\alpha+1)+a_1)}{a_0b_0(n+\beta)^2}x + \frac{n^2b_2}{b_0(n+\beta)^2}x^2\right). \end{aligned} \tag{27}$$

By Theorem 11 we obtain $st_T - \lim_n \|N_n h - h\| = 0$, but the operators $N_n(h,x)$ do not satisfy Theorem 12. Hence, the sequence (N_n) is not statistically convergent but it is $T-$ statistically convergent.

Remark 14. The sequence (x_n) is not statistically convergent and hence not convergent. As an example, consider the Cesáro matrix of order 2.

$$T = (t_{nk}) = \begin{cases} \frac{2(n+1-k)}{(n+1)(n+2)}; & 0 \le k \le n, \\ 0; & k > n, \end{cases} \tag{28}$$

where

$$\begin{aligned} 0 \le \lim_n \sum_{k:|x_k-\alpha|\ge\varepsilon} t_{nk} &= \lim_n \sum_{\substack{k=m^2-m,\cdots,m^2-1 \\ k=m^2;m\in\mathbb{N}\setminus\{1\}}} t_{nk} \\ &= \lim_n \frac{2}{(n+1)(n+2)}[1+\cdots+n] \\ &\le \lim_n \frac{2}{(n+1)(n+2)} \cdot \frac{n(n+1)}{2} = 1. \end{aligned} \tag{29}$$

This proves that $x = (x_n)$ is $T-$ statistically convergent. We have

$$\begin{aligned} N_n(e_0,x) &= (1+x_n), \\ N_n(e_1,x) &= (1+x_n)\left(\frac{2\alpha+1}{2(n+\beta)} + \frac{a_1}{a_0(n+\beta)} + \frac{nb_1}{b_0(n+\beta)}x\right), \\ N_n(e_2,x) &= (1+x_n)\left(\frac{3\alpha^2+3\alpha+1}{3(n+\beta)^2} + \frac{2a_1(\alpha+1)+a_2}{a_0(n+\beta)^2}\right. \\ &\quad \left. + \frac{2nb_1(a_0(\alpha+1)+a_1)}{a_0b_0(n+\beta)^2}x + \frac{n^2b_2}{b_0(n+\beta)^2}x^2\right). \end{aligned} \tag{30}$$

By Example 13, this shows that $N_n(h,x)$ does not satisfy Theorem 12.

In [27, 29], Korovkin-type theorems are proved by Abel summability method. Now, we discuss for power series method. Let $B[0,\infty)$ ($C[0,\infty)$) be the space of all bounded (continuous) functions on the interval $[0,\infty)$.

Theorem 15. *Let $(K_n^{\alpha,\beta})$ be a sequence of positive linear operators from $C[0,M] \cap E$ into $B[0,M] \cap E$ such that*

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} \left(K_n^{\alpha,\beta} e_i - e_i\right) p_n t^n \right\| = 0,\ i = 0, 1, 2. \tag{31}$$

Then, for $\mathfrak{h} \in C[0,M] \cap E$,

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} \left(K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\right) p_n t^n \right\| = 0. \tag{32}$$

Proof. Clearly, from (32) follows (31). Now, we show the converse that (31) implies (32). Let $\mathfrak{h} \in C[0,M] \cap E$, then there exists a constant $K > 0$ such that $|\mathfrak{h}(u)| \le K$ for all $u \in 0, M]$. Therefore,

$$|\mathfrak{h}(u) - \mathfrak{h}(x)|[\in 0, M]. \tag{33}$$

For every given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\mathfrak{h}(u) - \mathfrak{h}(x)| \le \varepsilon, \tag{34}$$

whenever $|u - x| < \delta$ for all $u \in 0, M]$. Define $\psi \equiv \psi(u,x) = (u-x)^2$. If $|u - x| \ge \delta$, then

$$|\mathfrak{h}(u) - \mathfrak{h}(x)| \le \frac{2K}{\delta^2}\psi(u,x). \tag{35}$$

From (33)–(35), we have that $|\mathfrak{h}(u) - \mathfrak{h}(x)| < \varepsilon + (2K/\delta^2)\psi(u,x)$, namely,

$$-\varepsilon - \frac{2K}{\delta^2}\psi(u,x) < \mathfrak{h}(t) - \mathfrak{h}(x) < \frac{2K}{\delta^2}\psi(u,x) + \varepsilon. \tag{36}$$

By applying the operator $K_n^{\alpha,\beta}(1,x)$, $K_n^{\alpha,\beta}(1,x)$ is a monotone and linear operator, we obtain

$$K_n^{\alpha,\beta}(1,x)\left(-\varepsilon-\frac{2K}{\delta^2}\psi\right)<K_n^{\alpha,\beta}(1,x)(\mathfrak{h}(u)-\mathfrak{h}(x))<K_n^{\alpha,\beta}(1,x)\left(\frac{2K}{\delta^2}\psi+\varepsilon\right), \quad (37)$$

which implies

$$\begin{aligned}-\varepsilon K_n^{\alpha,\beta}(1,x)-\frac{2K}{\delta^2}K_n^{\alpha,\beta}(\psi(u),x)\\ <K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)K_n^{\alpha,\beta}(1,x)\\ <\frac{2K}{\delta^2}K_n^{\alpha,\beta}(\psi(u),x)+\varepsilon K_n^{\alpha,\beta}(1,x).\end{aligned} \quad (38)$$

On the other hand,

$$\begin{aligned}K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)&=K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)K_n^{\alpha,\beta}(1,x)\\&+\mathfrak{h}(x)\left[K_n^{\alpha,\beta}(1,x)-1\right].\end{aligned} \quad (39)$$

From (38) and (39) we get

$$\begin{aligned}K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)&<\frac{2K}{\delta^2}K_n^{\alpha,\beta}(\psi(u),x)+\varepsilon K_n^{\alpha,\beta}(1,x)\\&+\mathfrak{h}(x)\left[K_n^{\alpha,\beta}(1,x)-1\right].\end{aligned} \quad (40)$$

Now, we estimate the following expression:

$$\begin{aligned}K_n^{\alpha,\beta}(\psi(u),x)&=K_n^{\alpha,\beta}\left((x-u)^2,x\right)=K_n^{\alpha,\beta}\left(\left(x^2-2xu+u^2\right),x\right)\\&=x^2K_n^{\alpha,\beta}(1,x)-2xK_n^{\alpha,\beta}(u,x)+K_n^{\alpha,\beta}\left(u^2,x\right).\end{aligned} \quad (41)$$

By (40), we obtain

$$\begin{aligned}&K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\\&<\frac{2K}{\delta^2}\left\{x^2\left[K_n^{\alpha,\beta}(1,x)-1\right]-2x\left[K_n^{\alpha,\beta}(u,x)-x\right]\right.\\&\left.+\left[K_n^{\alpha,\beta}\left(u^2,x\right)-x^2\right]\right\}+\varepsilon K_n^{\alpha,\beta}(1,x)\\&+f(x)\left[K_n^{\alpha,\beta}(1,x)-1\right]\\&=\varepsilon+\varepsilon\left[K_n^{\alpha,\beta}(1,x)-1\right]+\mathfrak{h}(x)\left[K_n^{\alpha,\beta}(1,x)-1\right]++\frac{2K}{\delta^2}\\&\cdot\left\{x^2\left[K_n^{\alpha,\beta}(1,x)-1\right]-2x\left[K_n^{\alpha,\beta}(u,x)-x\right]\right.\\&\left.+\left[K_n^{\alpha,\beta}\left(u^2,x\right)-x^2\right]\right\}.\end{aligned} \quad (42)$$

Therefore,

$$\begin{aligned}\left|K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right|&\le\varepsilon+\left(\varepsilon+K+\frac{2KM^2}{\delta^2}\right)\left|K_n^{\alpha,\beta}(1,x)-1\right|\\&+\frac{4KM}{\delta^2}\left|K_n^{\alpha,\beta}(u,x)-x\right|\\&+\frac{2K}{\delta^2}\left|K_n^{\alpha,\beta}\left(u^2,x\right)-x^2\right|.\end{aligned} \quad (43)$$

From the above relations and the linearity of $K_n^{\alpha,\beta}$, we obtain

$$\begin{aligned}&\frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(U_{n,p}(\mathfrak{h};x)-\mathfrak{h}(x)\right)p_nv^n\right\|\\&\le\varepsilon+\left(\varepsilon+K+\frac{2KM^2}{\delta^2}\right)\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}(1;x)-1\right)p_nt^n\right\|\\&+\frac{4KM}{\delta^2}\frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}(u;x)-x\right)p_nv^n\right\|\\&+\frac{2K}{\delta^2}\frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}\left(u^2;x\right)-x^2\right)p_nv^n\right\|.\end{aligned} \quad (44)$$

Hence, (32) follows from the last relation and (31).

3. Rate of Convergence

Modulus of continuity is defined by

$$\omega(\mathfrak{h},\delta)=\sup_{|h|<\delta}|\mathfrak{h}(x+h)-\mathfrak{h}(x)|,\ \mathfrak{h}(x)\in C[0,M]\cap E. \quad (45)$$

It is not hard to verify

$$|\mathfrak{h}(x)-\mathfrak{h}(y)|\le\omega(\mathfrak{h},\delta)\left(\frac{|x-y|}{\delta}+1\right). \quad (46)$$

So, we can state the following.

Theorem 16. *Let $T=(t_{ij})$ be a nonnegative regular summability matrix and $\mathfrak{h}\in C[0,M]\cap E$. If (α_n) is a sequence of positive real numbers such that $\omega(\mathfrak{h},\delta_n)=st_T-0(\alpha_n)$, then*

$$\left\|K_n^{\alpha,\beta}\mathfrak{h}-\mathfrak{h}\right\|=st_A-0(\alpha_n), \quad (47)$$

where

$$\delta_n = \Bigg\{ M^2 \left\| \left(\frac{n^2 B''(nx)}{(n+\beta)^2 B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1 \right) \right\| + M \left\| \left(\frac{nB'(nx)\left[2'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1) B(nx)} - \frac{2A'(1)}{(n+\beta)A(1)} + \frac{2(2\alpha+1)}{2(n+\beta)} \right) \right\| + \left\| \frac{1}{(n+\beta)^2 A(1)} \cdot \left\{ A''(1) + (2\alpha+2)A'(1)\left(\alpha^2+\alpha+\frac{1}{3}\right)A(1) \right\} \right\| \Bigg\}^2, \tag{48}$$

for any positive integer n.

Proof. Let $\mathfrak{h} \in C[0,M] \cap E$. By positivity and linearity of $K_n^{\alpha,\beta}$ and (46), we see

$$\begin{aligned} \left| K_n^{\alpha,\beta}(\mathfrak{h};x) - \mathfrak{h} \right| &\le K_n^{\alpha,\beta}(|\mathfrak{h}(t) - \mathfrak{h}(x)|;x) \\ &\le \frac{n+\beta}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)} \omega(\mathfrak{h},\delta) \cdot \left(1 + \frac{|t-x|}{\delta}\right) dt \\ &\le \omega(\mathfrak{h},\delta) \left[1 + \frac{1}{\delta} \frac{n+\beta}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) \cdot \int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)} (t-x) dt \right] (\text{by Lemma 1}) = \omega(\mathfrak{h},\delta) \cdot \left[1 + \frac{1}{\delta} K_n^{\alpha,\beta}(|t-x|;x) \right]. \end{aligned} \tag{49}$$

By applying the Cauchy-Schwartz inequality, we have

$$\left| K_n^{\alpha,\beta}(\mathfrak{h};x) - \mathfrak{h} \right| \le \omega(\mathfrak{h},\delta) \left[1 + \frac{1}{\delta} \left(K_n^{\alpha,\beta}\left(|t-x|^2;x\right) \right)^{1/2} \right]. \tag{50}$$

Based on Examples 5 and Remark 8, we obtain

$$\begin{aligned} K_n^{\alpha,\beta}\left((u-x)^2;x\right) &= K_n^{\alpha,\beta}(e_2;x) - 2xK_n^{\alpha,\beta}(e_1;x) + x^2 K_n^{\alpha,\beta}(e_0;x) \\ &\le M^2 \left\| \left(\frac{n^2 B''(nx)}{(n+\beta)^2 B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1 \right) \right\| \\ &+ M \left\| \left(\frac{nB'(nx)\left[2A'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1) B(nx)} - \frac{2A'(1)}{(n+\beta)A(1)} + \frac{2(2\alpha+1)}{2(n+\beta)} \right) \right\| + \left\| \frac{1}{(n+\beta)^2 A(1)} \cdot \left\{ A''(1) + (2\alpha+2)A'(1)\left(\alpha^2+\alpha+\frac{1}{3}\right)A(1) \right\} \right\|. \end{aligned} \tag{51}$$

By taking

$$\delta_n = \Bigg\{ M^2 \left\| \left(\frac{n^2 B''(nx)}{(n+\beta)^2 B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1 \right) \right\| + M \left\| \left(\frac{nB'(nx)\left[2A'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1) B(nx)} - \frac{2A'(1)}{(n+\beta)A(1)} + \frac{2(2\alpha+1)}{2(n+\beta)} \right) \right\| + \left\| \frac{1}{(n+\beta)^2 A(1)} \cdot \left\{ A''(1) + (2\alpha+2)A'(1)\left(\alpha^2+\alpha+\frac{1}{3}\right)A(1) \right\} \right\| \Bigg\}^2, \tag{52}$$

we get that $\|K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\| \le 2 \cdot \omega(\mathfrak{h},\delta_n)$. Therefore, for every $\varepsilon > 0$, we have

$$\frac{1}{\alpha_n} \sum_{\left\|K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\right\| \ge \varepsilon} t_{nj} \le \frac{1}{\alpha_n} \sum_{2\cdot\omega(f,\delta_n) \ge \varepsilon} t_{nj}. \tag{53}$$

From the conditions that are given in the theorem, we have that $\|K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\| = st_T - 0(\alpha_i)$, as claimed.

In the next result, we present the rate of convergence for the power summability method.

Theorem 17. *Let $\mathfrak{h} \in C[0,M] \cap E$ and let ϕ be a positive real function defined on $(0,M) \cap E$. If $\omega(\mathfrak{h},\psi(u)) = O(\phi(u))$, as $\nu \longrightarrow R^-$, then we have*

$$\frac{1}{p(\nu)} \left\| \sum_{n=0}^{\infty} \left(K_n^{\alpha,\beta} e_i - e_i \right) p_n \nu^n \right\| = O(\phi(\nu)), \tag{54}$$

where the function $\psi : (0,M) \cap E \longrightarrow \mathbb{R}$ is defined by relation

$$\psi(u) = \left\{ \sup_{\substack{x \in (0,M) \\ n \in \mathbb{N}}} \left\{ K_n^{\alpha,\beta}\left((u-x)^2;x\right) \right\} \right\}^{1/2}. \tag{55}$$

Proof. Let $\mathfrak{h} \in C[0, M] \cap E$. For any $u \in (0, R)$, $x \in (0, M)$, and $\delta > 0$, we have

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty}\left[K_n^{\alpha,\beta}(\mathfrak{h};x)-\mathfrak{h}(x)\right]p_n v^n\right| \\
&\quad\le \sum_{n=0}^{\infty} K_n^{\alpha,\beta}(|\mathfrak{h}(u)-\mathfrak{h}(x)|;x)p_n v^n \\
&\quad\le \sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left(\omega\left(\mathfrak{h},\frac{|u-x|}{\delta}\delta\right);x\right)p_n v^n \\
&\quad\le \sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left(\left(1+\left[\left|\frac{u-x}{\delta}\right|\right]\right)\omega(\mathfrak{h},\delta);x\right)p_n v^n \\
&\quad\le \omega(\mathfrak{h},\delta)\sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left(1+\frac{(u-x)^2}{\delta^2};x\right)p_n v^n \\
&\quad\le \omega(\mathfrak{h},\delta)\sum_{n=0}^{\infty} K_n^{\alpha,\beta}(e_0(u);x)p_n v^n \\
&\qquad+\frac{\omega(\mathfrak{h},\delta)}{\delta^2}\sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left((u-x)^2;x\right)p_n v^n \\
&\quad= p(v)\omega(\mathfrak{h},\delta)+\frac{\omega(\mathfrak{h},\delta)}{\delta^2}\sup_{\substack{0\le x\le 1\\ n\in\mathbb{N}}} \\
&\qquad\cdot\left\{K_n^{\alpha,\beta}\left((u-x)^2;x\right)\right\}\sum_{n=0}^{\infty} p_n v^n,
\end{aligned} \tag{56}$$

which leads to

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty}\left[K_n^{\alpha,\beta}(f;x)-f(x)\right]p_n v^n\right| \\
&\quad\le p(v)\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta^2}\sup_{0\le x\le 1} \\
&\qquad\cdot\left\{K_n^{\alpha,\beta}\left((u-x)^2;x\right)\right\}p(v).
\end{aligned} \tag{57}$$

If we set $\delta = \psi(u)$, then from the last inequality we have

$$0 \le \frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}\mathfrak{h}-\mathfrak{h}\right)p_n v^n\right\| \le 2\omega(\mathfrak{h},\delta), \tag{58}$$

as required.

4. Voronovskaya-Type Theorems

First, we prove a Voronovskaya-type theorem for the operators under consideration.

Theorem 18. *Let* $\mathfrak{h}, \mathfrak{h}', \mathfrak{h}'' \in C[0, M] \cap E$ *and* $\lim_{n\to\infty} B^{(i)}(nx)/B(nx) = 1$, *for* $i \in \{1, 2\}$. *Then,*

$$\begin{aligned}
&\lim_{n\to\infty}(n+\beta)\left[K_n^{\alpha,\beta}(\mathfrak{h};x)-\mathfrak{h}(x)\right] \\
&\quad=\left(\frac{A'(1)}{A(1)}+\frac{1}{2}+\alpha-\beta x\right)\mathfrak{h}'(x)+\frac{x}{2}\mathfrak{h}''(x),
\end{aligned} \tag{59}$$

for every $x \in [0, M]$.

Proof. Assume that $\mathfrak{h}', \mathfrak{h}'' \in C[0, M] \cap E$ and $x \in [0, M]$. By Taylor's formula, we have

$$\mathfrak{h}(y)=\mathfrak{h}(x)+(y-x)\mathfrak{h}'(x)+\frac{1}{2}(y-x)^2\mathfrak{h}''(x)+(y-x)^2\psi(y-x), \tag{60}$$

where $\psi(y-x) \longrightarrow 0$ and $y-x \longrightarrow 0$. Applying in both sides of the above relation operators $K_n^{\alpha,\beta}$, we obtain

$$\begin{aligned}
K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)&=\mathfrak{h}'(x)K_n^{\alpha,\beta}(y-x;x) \\
&\quad+\frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((y-x)^2;x\right) \\
&\quad+K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right),
\end{aligned} \tag{61}$$

which implies

$$\begin{aligned}
(n+\beta)\left[K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right]&=(n+\beta)\mathfrak{h}'(x)K_n^{\alpha,\beta}(y-x;x) \\
&\quad+(n+\beta)\frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((y-x)^2;x\right) \\
&\quad+(n+\beta)K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right).
\end{aligned} \tag{62}$$

Now, we will estimate this expression:

$$\lim_{n\to\infty}(n+\beta)K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right). \tag{63}$$

Let $\varepsilon > 0$ and $\delta > 0$ such that $|\psi(y-x)| < \varepsilon$, where $|y-x| < \delta$. We will split the above relation in two parts:

$$\begin{aligned}
U_1&=(n+\beta)\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\int_{|y-x|\le\delta}(y-x)^2\psi(y-x)dy \\
U_2&=(n+\beta)\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\int_{|y-x|\ge\delta}(y-x)^2\psi(y-x)dy.
\end{aligned} \tag{64}$$

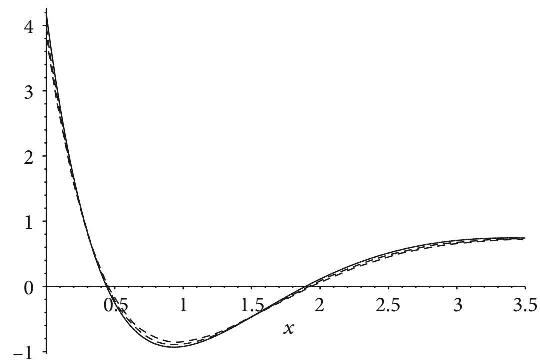

Figure 1: The graphs of $(10+\beta)[K_{10}^{\alpha,\beta}(f;x)-f(x)]$, $(20+\beta)[K_{20}^{\alpha,\beta}(f;x)-f(x)]$, and $((A'(1)/A(1))+(1/2)+\alpha-\beta x)f'(x)+(x/2)f''(x)$ when $x\in 0,3.5]$.

From the above conditions, we have

$$\begin{aligned}|U_1| &\le \varepsilon(n+\beta)^2\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\int_{k+\alpha/n+\beta}^{k+\alpha+1/n+\beta}(y-x)^2dy\\ &= \varepsilon(n+\beta)K_n^{\alpha,\beta}\left((y-x)^2;x\right).\end{aligned} \tag{65}$$

On the other hand, from Proposition 9, condition (3), we get that $|U_1|\le\varepsilon_1$.

Let us denote by $N=\sup_x\{|\psi(y-x)|;|y-x|\ge\delta\}$. Then, we obtain

$$\begin{aligned}|U_2| &\le \frac{N}{\delta^2}(n+\beta)\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\int_{|y-x|\ge\delta}(y-x)^4dy\\ &= \frac{N}{\delta^2}(n+\beta)K_n^{\alpha,\beta}\left((y-x)^4;x\right).\end{aligned} \tag{66}$$

Condition (5) in Proposition 9 tells us that $\lim_{n\to\infty}|U_2|=0$, which completes the proof.

Example 19. Let $A(t)=1$, $B(t)=e^t$, $\alpha=1/3$, $\beta=1/2$, and $\mathfrak{h}(x)=5xe^{-x}$. By $A(t)B(xt)=\sum_{k\ge0}p_k(x)t^k$, we see that $p_k(x)=x^k/k!$. Figure 1 presents the graphs of the functions $(10+\beta)[K_{10}^{\alpha,\beta}(\mathfrak{h};x)-\mathfrak{h}(x)]$, $(20+\beta)[K_{20}^{\alpha,\beta}(\mathfrak{h};x)-\mathfrak{h}(x)]$, and $((A'(1)/A(1))+(1/2)+\alpha-\beta x)\mathfrak{h}'(x)+(x/2)\mathfrak{h}''(x)$.

We extend the Voronovskaya-type theorem for $T-$ statistical method for these operators. Consider operators N_n from Example 13. We start with the following lemma.

Lemma 20. *Let $\mathfrak{h}\in C[0,M]\cap E$ such that $\mathfrak{h}',\mathfrak{h}''\in C[0,M]\cap E$, $x\in[0,M]$, and $\lim_{n\to\infty}(B^{(i)}(nx)/B(nx))=1$, for $i\in\{1,2,3,4\}$. Then, we obtain*

$$(n+\beta)^4K_n\left((y-x)^4;x\right)\sim 0(st_A)\quad on\quad [0,M]. \tag{67}$$

Proof. The proposition follows directly from Proposition 9 (5).

Theorem 21. *Let $\mathfrak{h}\in C[0,M]\cap E$ such that $\mathfrak{h}',\mathfrak{h}''\in C[0,M]\cap E$, $x\in[0,M]$, for any finite M and let $\lim_{n\to\infty}(B^{(i)}(nx)/B(nx))=1$, for $i\in\{1,2,3,4\}$. Then, for $x\in[0,M]$,*

$$\begin{aligned}&\Bigg|(n+\beta)^2\Bigg[N_n(\mathfrak{h};x)-\mathfrak{h}(x)-\mathfrak{h}'(x)\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]-\frac{\mathfrak{h}''(x)}{2}\\ &\quad\cdot\Bigg(-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\\ &\quad+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\Bigg)\Bigg]\Bigg|\\ &\sim\frac{\mathfrak{h}''(x)}{2}\cdot\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}.\end{aligned} \tag{68}$$

Proof. Taylor's formula gives

$$\mathfrak{h}(y)=\mathfrak{h}(x)+(y-x)\mathfrak{h}'(x)+\frac{1}{2}(y-x)^2\mathfrak{h}''(x)+(y-x)^2\psi(y-x), \tag{69}$$

where $\psi(y-x)\longrightarrow 0$, as $y-x\longrightarrow 0$. Taking into consideration Remark 8, after applying N_n in both sides of relation (69), we obtain

$$\begin{aligned}N_n(\mathfrak{h})&=(1+x_n)\mathfrak{h}(x)+(1+x_n)\mathfrak{h}'(x)\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)+(1+x_n)\frac{\mathfrak{h}''(x)}{2}\\&\cdot\left(\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)+(1+x_n)K_n^{\alpha,\beta}\left(\Phi^2\psi(y-x);x\right).\end{aligned} \tag{70}$$

This yields

$$\begin{aligned}(n+\beta)^2N_n(\mathfrak{h})&=(1+x_n)(n+\beta)^2\mathfrak{h}(x)+(1+x_n)\mathfrak{h}'(x)(n+\beta)^2\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)+(1+x_n)\frac{\mathfrak{h}''(x)}{2}(n+\beta)^2\\&\left(\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)+(1+x_n)(n+\beta)^2K_n^{\alpha,\beta}\left(\Phi^2\psi(y-x);x\right).\end{aligned} \tag{71}$$

Therefore,

$$\begin{aligned}&\left|(n+\beta)^2\left[N_n(\mathfrak{h};x)-\mathfrak{h}(x)-\mathfrak{h}'(x)\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]\right.\right.\\&-\frac{\mathfrak{h}''(x)}{2}\left(-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\left.\left.\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)\right]-\frac{\mathfrak{h}''(x)}{2}\cdot\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}\right|\\&\le Mx_n(n+\beta)^2+M_1x_n\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]+M_2\\&\cdot x_n\left(\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)+(n+\beta)^2K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right)\\&+x_n\left((n+\beta)^2K_n^{\alpha,\beta}(y-x)^2\psi(y-x);x\right),\end{aligned} \tag{72}$$

where $M = \sup_{x\in[0,M]}|\mathfrak{h}(x)|$, $M_1 = \sup_{x\in[0,M]}|\mathfrak{h}'(x)|$, and $M_2 = \sup_{x\in[0,M]}|\mathfrak{h}''(x)|$.

Now, we have to prove that

$$\lim_{n\to\infty} (n+\beta)^2 K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x)\,;x\right) = 0. \tag{73}$$

By applying the Cauchy-Schwartz inequality, we obtain

$$(n+\beta)^2 K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x)\,;x\right) \leq \left[(n+\beta)^4 K_n^{\alpha,\beta}\left((y-x)^4\,;x\right)\right]^{1/2} \cdot \left[K_n^{\alpha,\beta}\left(\psi^2\,;x\right)\right]^{1/2}. \tag{74}$$

Also, by setting $\eta_x(y) = (\psi(y-x))^2$, we have that $\eta_x(x) = 0$ and $\eta_x(\cdot) \in C[0,M]$. So

$$K_n^{\alpha,\beta}(\eta_x) \longrightarrow 0(st_A) \quad \text{on} \quad [0,M]. \tag{75}$$

Now, from the last relation, (74), (75), and Lemma 20, we obtain that

$$(n+\beta)^2 K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x)\,;x\right) \longrightarrow 0(st_A) \quad \text{on} \quad [0,M]. \tag{76}$$

From the construction of (x_n), it follows that $(n+\beta)^2 x_n \longrightarrow 0(st_A)$ on $[0,M]$.

For a given $\varepsilon > 0$, we define the following sets:

$$\begin{aligned}
A &= \left|\left\{n : \left|(n+\beta)^2\left[N_n(\mathfrak{h}\,;x) - \mathfrak{h}(x) - \mathfrak{h}'(x)\left[\frac{2a_0\alpha + a_0 + 2a_1}{2a_0(n+\beta)} - \frac{(b_0 n + b_0\beta - b_1 n)}{b_0(n+\beta)}x\right]\right.\right.\right.\\
&\quad - \frac{\mathfrak{h}''(x)}{2}\left(-\frac{2a_0b_0\alpha(n+\beta) - 2a_0b_1n(\alpha+1) + a_0b_0(n+\beta) + 2a_1b_0(n+\beta) - 2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\
&\quad \left.\left.\left.\left. + \frac{b_0n^2 + 2b_0n\beta + b_0\beta^2 - 2b_1n^2 - 2b_1n\beta + b_2n^2}{b_0(n+\beta)^2}x^2\right)\right] - \frac{\mathfrak{h}''(x)}{2}\cdot\frac{3a_0\alpha^2 + 3a_0\alpha + 6a_1\alpha + a_0 + 6a_1 + 3a_2}{3a_0(n+\beta)^2}\right| \geq \varepsilon\right\}\right|,\\
A_1 &= \left|\left\{n : \left|x_n(n+\beta)^2\right| \geq \frac{\varepsilon}{5M}\right\}\right|,\\
A_2 &= \left|\left\{n : \left|x_n\left[\frac{2a_0\alpha + a_0 + 2a_1}{2a_0(n+\beta)} - \frac{(b_0n + b_0\beta - b_1n)}{b_0(n+\beta)}x\right]\right| \geq \frac{\varepsilon}{5M_1}\right\}\right|,\\
A_3 &= \left|\left\{n : \left|x_n\left(\frac{3a_0\alpha^2 + 3a_0\alpha + 6a_1\alpha + a_0 + 6a_1 + 3a_2}{3a_0(n+\beta)^2} - \frac{2a_0b_0\alpha(n+\beta) - 2a_0b_1n(\alpha+1) + a_0b_0(n+\beta) + 2a_1b_0(n+\beta) - 2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\right.\right.\\
&\quad \left.\left.\left. + \frac{b_0n^2 + 2b_0n\beta + b_0\beta^2 - 2b_1n^2 - 2b_1n\beta + b_2n^2}{b_0(n+\beta)^2}x^2\right)\right| \geq \frac{\varepsilon}{5M_2}\right\}\right|,\\
A_4 &= \left|\left\{n : \left|(n+\beta)^2 K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x)\,;x\right)\right| \geq \frac{\varepsilon}{5}\right\}\right|,\\
A_5 &= \left|\left\{n : \left|x_n(n+\beta)^2 K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x)\,;x\right)\right| \geq \frac{\varepsilon}{5}\right\}\right|.
\end{aligned} \tag{77}$$

From last relations, we obtain that $A \leq A_1 + A_2 + A_3 + A_4 + A_5$. Hence, the result follows.

Remark 22. We see that the operators (N_n) (see Example 13) do not satisfy a Voronovskaya-type theorem in the usual sense.

5. Grüss-Voronovskaya-Type Theorems

This kind of result, for the first time, was shown in [30].

Theorem 23 (see [31]). *Let* $E = \{\mathfrak{h} : x \in [0,\infty), (\mathfrak{h}(x)/(1+x^2))$, *is convergent as* $x \longrightarrow \infty\}$ *and*

$$\lim_{n\to\infty} \frac{B^{(m)}(nx)}{B(nx)} = 1 \quad \text{for} \quad m = 1, 2. \tag{78}$$

If $\mathfrak{h} \in C[0,\infty) \cap E$, *then*

$$\lim_{n\to\infty} K_n^{\alpha,\beta}(f,x) = f(x), \tag{79}$$

and the operators $K_n^{\alpha,\beta}(\mathfrak{h}, x)$ converge uniformly in each compact subset of $[0,\infty)$.

Now, we are ready to prove the following result.

Theorem 24. *For $\mathfrak{h}, \mathfrak{h}', \mathfrak{h}'' \in C[0,\infty)$ and any $x \in [0,\infty)$, $\lim_{n\to\infty} B^{(i)}(nx)/B(nx) = 1$, for $i \in \{1,2,3,4,5,6\}$, where $B^{(i)}$ denotes the ith derivative of B. Then,*

$$\begin{aligned}&\Bigg|(n+\beta)\left(K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right)-\mathfrak{h}'(x)\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)-(n+\beta)\frac{\mathfrak{h}''(x)}{2}\\&\quad\cdot\left[\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\quad\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right]\Bigg|=0(1)\omega\left(\mathfrak{h}'',n^{-1/2}\right),\end{aligned} \tag{80}$$

as $n \longrightarrow \infty$.

Proof. From Taylor's theorem, we have

$$\mathfrak{h}(u)=\mathfrak{h}(x)+\mathfrak{h}'(x)(u-x)+\frac{\mathfrak{h}''(x)}{2}(u-x)^2+R(u,x), \tag{81}$$

where $R(u,x) = ((\mathfrak{h}''(\theta)-\mathfrak{h}''(x))/2)(u-x)^2$, for $\theta \in (u,x)$. Now, we obtain

$$\begin{aligned}&\Bigg|K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)-\mathfrak{h}'(x)K_n^{\alpha,\beta}((u-x);x)\\&\quad-\frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2;x\right)\Bigg|\le K_n^{\alpha,\beta}(|R(u,x)|,x).\end{aligned} \tag{82}$$

From which we get that

$$\begin{aligned}&\Bigg|(n+\beta)\left(K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right)-\mathfrak{h}'(x)\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)-(n+\beta)\frac{\mathfrak{h}''(x)}{2}\\&\quad\cdot\left[\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\quad\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right]\Bigg|\le(n+\beta)\cdot K_n^{\alpha,\beta}(|R(u,x)|,x).\end{aligned} \tag{83}$$

By the properties of modulus of continuity modulus, we have

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\le\frac{1}{2!}\left(1+\frac{|\theta-x|}{\delta}\right)\omega\left(\mathfrak{h}'',\delta\right). \tag{84}$$

On the other hand,

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\le\begin{cases}\omega\left(\mathfrak{h}'',\delta\right), & |u-x|\le\delta,\\ \dfrac{(t-x)^4}{\delta^4}\omega\left(\mathfrak{h}'',\delta\right), & |u-x|\ge\delta.\end{cases} \tag{85}$$

For $0<\delta<1$, we obtain that

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\le\omega\left(\mathfrak{h}'',\delta\right)\left(1+\frac{(u-x)^4}{\delta^4}\right), \tag{86}$$

which gives

$$\begin{aligned}|R(u,x)|&\le\omega\left(\mathfrak{h}'',\delta\right)\left(1+\frac{(u-x)^4}{\delta^4}\right)(u-x)^2\\&=\omega\left(\mathfrak{h}'',\delta\right)\left((u-x)^2+\frac{(u-x)^6}{\delta^4}\right).\end{aligned} \tag{87}$$

By the linearity of $K_n^{\alpha,\beta}$ and the above relation, we obtain

$$K_n^{\alpha,\beta}(|R(u,x)|,x) \le \omega\left(\mathfrak{h}'',\delta\right)\left(K_n^{\alpha,\beta}\left((u-x)^2,x\right)+\frac{1}{\delta^4}K_n^{\alpha,\beta}\left((u-x)^6,x\right)\right). \quad (88)$$

Taking into consideration Proposition 9, we have

$$K_n^{\alpha,\beta}(|R(u,x)|,x) \le \omega\left(\mathfrak{h}'',\delta\right)\left(0\left(\frac{1}{n}\right)+\frac{1}{\delta^4}0\left(\frac{1}{n^3}\right)\right) = 0\left(\frac{1}{n}\right)\omega\left(\mathfrak{h}'',\delta\right). \quad (89)$$

For $\delta = n^{-1/2}$, we complete the proof.

Theorem 25. *Let* $\mathfrak{h}'(x), g'(x), \mathfrak{h}''(x), g''(x), (\mathfrak{h}g)'(x), (\mathfrak{h}g)''(x) \in C[0,\infty) \cap E$, *and*

$$\lim_{n\to\infty}\frac{B^{(m)}(nx)}{B(nx)} = 1 \quad \text{for} \quad m = 0, 1, 2, \cdots, 6, \quad x \in [0, M], \quad (90)$$

where $B^{(m)}$ *is the mth derivative of B. Then,*

$$\lim_{n\to\infty}(n+\beta)\left[K_n^{\alpha,\beta}(\mathfrak{h}g,x)-K_n^{\alpha,\beta}(\mathfrak{h},x)\cdot K_n^{\alpha,\beta}(g,x)\right] = x\mathfrak{h}'(x)g'(x). \quad (91)$$

Proof. We know that

$$\begin{aligned}
&(n+\beta)\left\{K_n^{\alpha,\beta}(\mathfrak{h}g,x)-K_n^{\alpha,\beta}(\mathfrak{h},x)K_n^{\alpha,\beta}(g,x)\right\}\\
&= (n+\beta)\Bigg\{K_n^{\alpha,\beta}(\mathfrak{h}g,x)-(\mathfrak{h}g)(x)\\
&\quad -(\mathfrak{h}g)'(x)K_n^{\alpha,\beta}((u-x),x)-\frac{(\mathfrak{h}g)''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\\
&\quad -g(x)\Bigg[K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)-\mathfrak{h}'(x)K_n^{\alpha,\beta}((u-x),x)\\
&\quad -\frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\Bigg]\\
&\quad -K_n^{\alpha,\beta}(\mathfrak{h},x)\Bigg[K_n^{\alpha,\beta}(g,x)-g(x)-g'(x)K_n^{\alpha,\beta}((u-x),x)\\
&\quad -\frac{g''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\Bigg]\\
&\quad +\frac{g''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\left[\mathfrak{h}(x)-K_n^{\alpha,\beta}(\mathfrak{h},x)\right]\\
&\quad +\mathfrak{h}'(x)g'(x)K_n^{\alpha,\beta}\left((u-x)^2,x\right)-g'(x)K_n^{\alpha,\beta}((u-x),x)\\
&\quad \cdot\left[K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right]\Bigg\}.
\end{aligned} \quad (92)$$

From Proposition 9 and Theorems 23 and 24, we obtain

$$\lim_{n\to\infty}(n+\beta)\left\{K_n^{\alpha,\beta}(\mathfrak{h}g,x)-K_n^{\alpha,\beta}(\mathfrak{h},x)K_n^{\alpha,\beta}(g,x)\right\} = x\mathfrak{h}'(x)g'(x). \quad (93)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Some Trapezium-Like Inequalities Involving Functions Having Strongly n-Polynomial Preinvexity Property of Higher Order

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The main objective of this paper is to introduce a new class of preinvex functions which is called as n-polynomial preinvex functions of a higher order. As applications of this class of functions, we discuss several new variants of trapezium-like inequalities. In order to obtain the main results of the paper, we use the concepts and techniques of k-fractional calculus. We also discuss some special cases of the obtained results which show that the main results of the paper are quite unifying one.

1. Introduction

Convexity is one of the most important and natural notions in mathematics; it plays a significant role in various branches of pure and applied sciences [1–15]. For example, the set of feasible points in optimization theory is convex; the loss function used to measure the quality of solution in statistics is convex. In particular, many remarkable inequalities have been established via the convexity theory [16–32].

Recently, the classical concept of convexity has been extended and generalized in different directions. For instance, Hanson [33] introduced the notion of differentiable invex function but did not name it as invex; Craven [34] gave the term invex for this class of functions due to their property described as invariance by convexity. Mititelu [35] introduced the notion of invex set as follows.

Definition 1 (see [35]). Let $\mathscr{X} \in \mathbb{R}$ be a nonempty set and $\zeta : \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$ be a real-valued function. Then $\mathscr{X}$ is said to be invex with respect to ζ if

$$x + t\zeta(y, x) \in \mathscr{X} \tag{1}$$

for all $x, y \in \mathscr{X}$ and $t \in [0, 1]$.

We clearly see that the invexity reduces to the classical convexity if $\zeta(y, x) = y - x$. Therefore, every convex set is an invex with respect to $\zeta(y, x) = y - x$, but its converse is not true in general [35].

Weir and Mond [36] introduced the class of preinvex functions by use of the invex set.

Definition 2 (see [36]). Let $\mathscr{X} \in \mathbb{R}$ be a nonempty invex set with respect to $\zeta : \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$. Then the function

$\mathscr{F} : \mathscr{X} \mapsto \mathbb{R}$ is said to be preinvex with respect to ζ if the inequality

$$\mathscr{F}(x + t\zeta(y, x)) \le (1 - t)\mathscr{F}(x) + t\mathscr{F}(y) \tag{2}$$

holds for all $x, y \in \mathscr{X}$ and $t \in [0, 1]$.

Note that the preinvex function becomes the classical convex function if $\zeta(y, x) = y - x$; for more details regarding recent study on preinvexity property, we recommend the literature [37].

Very recently, Toplu et al. [38] introduced and investigated a new class of convexity named n-polynomial convexity, and Karamardian [39] and Polyak [40] independently introduced the class of strongly convex functions. Strong convexity is the strengthening of convexity.

Definition 3 (see [39, 40]). A function $\mathscr{F} : \mathscr{X} \mapsto \mathbb{R}$ is said to be strongly convex with respect to modulus $\mu > 0$ if the inequality

$$\mathscr{F}((1 - t)x + ty) \le (1 - t)\mathscr{F}(x) + tF(y) - \mu t(1 - t)(y - x)^2 \tag{3}$$

holds for all $x, y \in \mathscr{X}$ and $t \in [0, 1]$.

Lin et al. [41] introduced higher order strongly convex functions to simplify the study of mathematical programs with equilibrium constraints.

Definition 4 (see [41]). Let $\sigma, \mu > 0$. Then, the function $\mathscr{F} : \mathscr{X} \mapsto \mathbb{R}$ is said to be σ -order strongly convex with respect to modulus μ if

$$\mathscr{F}((1 - t)x + ty) \le (1 - t)\mathscr{F}(x) + tF(y) - \mu t(1 - t)|y - x|^\sigma \tag{4}$$

for all $x, y \in \mathscr{X}$ and $t \in [0, 1]$.

If $\sigma = 2$, then Definition 4 becomes Definition 3. Therefore, higher order strong convexity is a generalization of strong convexity. Lin et al. [41] proved that the higher order strong convexity of a function is equivalent to higher order strong monotonicity of the gradient map of the function.

It is well known that the Hermite-Hadamard inequality [42–45] is one of the most important and classical inequalities in convex function theory, which can be stated as follows.

Theorem 5. *Let $\mathscr{F} : [a, b] \subset \mathbb{R} \mapsto \mathbb{R}$ be a convex function. Then*

$$\mathscr{F}\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \mathscr{F}(x)dx \le \frac{\mathscr{F}(a) + \mathscr{F}(b)}{2}. \tag{5}$$

In recent decades, the fractional calculus has become a powerful tool in numerous branches of mathematics, physics, and engineering. The history of fractional calculus dates back to 1695 with the work of mathematicians such as L'Hospital and Leibniz, but the logical definitions were proposed by Liouville in 1834, Riemann in 1847, and Grünwald in 1867. Fractional calculus can be considered a super set of integer-order calculus, which has the potential to accomplish what integer-order calculus cannot. The classical form of the fractional calculus is given by the Riemann-Liouville integrals as follows.

Definition 6 (see [46]). Let $\alpha > 0$, $0 \le a < b$, and $\mathscr{F} \in L_1[a, b]$. Then, the α -order Riemann-Liouville integrals $\mathfrak{J}_{a^+}^\alpha \mathscr{F}$ and $\mathfrak{J}_{b^-}^\alpha \mathscr{F}$ are defined by

$$\begin{aligned} \mathfrak{J}_{a^+}^\alpha \mathscr{F}(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x - t)^{\alpha-1}\mathscr{F}(t)\mathrm{d}t \quad (x > a), \\ \mathfrak{J}_{b^-}^\alpha \mathscr{F}(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t - x)^{\alpha-1}\mathscr{F}(t)\mathrm{d}t \quad (x < b), \end{aligned} \tag{6}$$

respectively, where

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\mathrm{d}t \tag{7}$$

is the Euler gamma function.

Sarikaya et al. [47] used the fractional calculus to obtain new variants of the Hermite-Hadamard inequality, which opened a new research area for the people who are working on the field of mathematical inequalities, particularly working on the inequalities involving convexity and its generalizations. For some more recent research work done in this direction, see [48–50].

Diaz et al. [51] introduced the generalized k-gamma function $\Gamma_k(x)$ and k-beta function $B_k(x, y)$ as follows.

$$\begin{aligned} \Gamma_k(x) &= \int_0^\infty t^{x-1}e^{-t^k/k}\mathrm{d}t, \\ B_k(x, y) &= \frac{1}{k}\int_0^1 t^{x/k-1}(1 - t)^{y/k-1}\mathrm{d}t = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)}. \end{aligned} \tag{8}$$

Making use of the generalized k-gamma function, Sarikaya et al. [52] introduced the k-Riemann-Liouville fractional integrals and discussed their properties and applications.

Definition 7 (see [52]). Let $\alpha, k > 0$, $0 \le a < b$, and $\mathscr{F} \in L_1[a, b]$. Then, the α -order k -Riemann-Liouville integrals ${}_k\mathfrak{J}_{a^+}^\alpha \mathscr{F}$ and ${}_k\mathfrak{J}_{b^-}^\alpha \mathscr{F}$ are defined by

$$\begin{aligned} {}_k\mathfrak{J}_{a^+}^\alpha \mathscr{F}(x) &= \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x - t)^{\alpha/k-1}\mathscr{F}(t)\mathrm{d}t \quad (x > a), \\ {}_k\mathfrak{J}_{b^-}^\alpha \mathscr{F}(x) &= \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t - x)^{\alpha/k-1}\mathscr{F}(t)\mathrm{d}t \quad (x < b). \end{aligned} \tag{9}$$

In recent years, several authors have used the concepts of k-fractional calculus in obtaining new variants of fractional analogues of classical inequalities. For example, Huang et al. [53] obtained k-fractional conformable analogues of Hermite-Hadamard's inequality. Rahman et al. [54] obtained fractional analogues of the Gruss type inequalities using k-conformable fractional integral operators.

The aim of the article is to obtain some new k-analogues of trapezium-like inequalities involving a new class of functions called strongly n-polynomial preinvex function of higher order. We also discuss some special cases of the obtained results. We expect that the ideas and techniques of this article will inspire interested readers working in this field. This is the main motivation of the article.

2. Results and Discussions

In this section, we discuss our main results. First of all, let us give the definition of strongly n-polynomial preinvex function of higher order.

Definition 8. Let $n \in \mathbb{N}$ and $\mathscr{X} \in \mathbb{R}$ be a nonempty invex set with respect to $\zeta : \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$. A nonnegative function $\mathscr{F} : \mathscr{X} \mapsto \mathbb{R}$ is said to be strongly n -polynomial preinvex function of higher order, if

$$\begin{aligned} &\mathscr{F}(a + t\zeta(b, a)) \\ &\quad \leq \frac{1}{n}\sum_{i=1}^{n} \left[1 - t^i\right]\mathscr{F}(a) + \frac{1}{n}\sum_{i=1}^{n} \left[1 - (1-t)^i\right]\mathscr{F}(b) \\ &\quad - \mu[t^\sigma(1-t) + t(1-t)^\sigma]\|\zeta(b, a)\|^\sigma, \\ &\quad \forall a, b \in \mathscr{X}, t \in [0, 1], \mu, \sigma > 0, \end{aligned} \tag{10}$$

Note that if $\sigma = 2$, then the class of strongly n-polynomial preinvex function of higher order reduces to the class of strongly n-polynomial preinvex function which is also new class. If we consider $\mu = 0$, then the class of strongly n-polynomial preinvex function of higher order reduces to the class of n-polynomial preinvex functions which is also new in the literature. Similarly, if we take $\zeta(b, a) = b - a$, then we have new class of strongly n-polynomial convex function of higher order. If we take $n = 1$, then the above class reduces to simple strongly preinvex function of higher order. If we take $n = 2$, then we have strongly 2-polynomial preinvex function of higher order:

$$\begin{aligned} \mathscr{F}(a + t\zeta(b, a)) &\leq \frac{2 - t - t^2}{2}\mathscr{F}(a) + \frac{3t - t^2}{2}\mathscr{F}(b) \\ &\quad - \mu[t^\sigma(1-t) + t(1-t)^\sigma]\|\zeta(b, a)\|^\sigma, \\ &\quad \forall a, b \in \mathscr{X}, t \in [0, 1], \mu, \sigma > 0, \end{aligned} \tag{11}$$

In order to obtain following result, we need the famous Condition C which was introduced by Mohan and Neogy [55].

Condition C. Let $\mathscr{X} \subset \mathbb{R}$ be an invex set with respect to bifunction $\zeta(\cdot, \cdot)$. Then, for any $x, y \in \mathscr{X}$ and $t \in [0, 1]$,

$$\begin{aligned} \zeta(y, y + t\zeta(x, y)) &= -t\zeta(x, y), \\ \zeta(x, y + t\zeta(x, y)) &= (1 - t)\zeta(x, y). \end{aligned} \tag{12}$$

Note that for every $x, y \in \mathscr{X}$, $t_1, t_2 \in [0, 1]$, and from Condition C, we have

$$\zeta(y + t_2\zeta(x, y), y + t_1\zeta(x, y)) = (t_2 - t_1)\zeta(x, y). \tag{13}$$

We now give our first main result.

Theorem 9. *Let $\mathscr{F} : [a, a + \zeta(b, a)] \mapsto \mathbb{R}$ be an strongly n-polynomial preinvex function with $\zeta(b, a) > 0$ and $\mathscr{F} \in L[a, a + \zeta(b, a)]$. If $\zeta(\cdot, \cdot)$ satisfies Condition C, then*

$$\begin{aligned} &\left(\frac{n}{n + 2^{-n} - 1}\right)\left[\mathscr{F}\left(\frac{2a + \zeta(b, a)}{2}\right) + \frac{\mu(\alpha^2 - \alpha k + 2k)}{4(\alpha + k)(\alpha + 2k)}\|\zeta(b, a)\|^\sigma\right] \\ &\leq \frac{\Gamma_k(\alpha + k)}{\zeta^{\alpha/k}(b, a)}\left[{}_k\mathfrak{J}^\alpha_{a^+}\mathscr{F}(a + \zeta(b, a)) + {}_k\mathfrak{J}^\alpha_{[a+\zeta(b,a)]^-}\mathscr{F}(a)\right] \\ &\leq \left[\frac{\mathscr{F}(a) + \mathscr{F}(b)}{n}\right]\sum_{i=1}^{n}\frac{2sk}{\alpha + sk} - \frac{2k\alpha\mu\|\zeta(b, a)\|^\sigma}{(\alpha + k)(\alpha + 2k)}. \end{aligned} \tag{14}$$

Proof. Since it is given that $\mathscr{F}$ is an an strongly n-polynomial preinvex function and $\zeta(\cdot, \cdot)$ satisfies Condition C, then

$$\begin{aligned} &\mathscr{F}\left(\frac{2a + \zeta(b, a)}{2}\right) \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\left[1 - \frac{1}{2^i}\right]\{\mathscr{F}(a + t\zeta(b, a)) + \mathscr{F}(a + (1 - t)\zeta(b, a))\} \\ &\quad - \frac{\mu}{4}\zeta^2(a + (1 - t)\zeta(b, a), a + t\zeta(b, a)) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[1 - \frac{1}{2^i}\right]\{\mathscr{F}(a + t\zeta(b, a)) + \mathscr{F}(a + (1 - t)\zeta(b, a))\} \\ &\quad - \frac{\mu}{4}(1 - 2t)^2\zeta^2(b, a). \end{aligned} \tag{15}$$

Multiplying the above inequality by $t^{\alpha/k-1}$ and then integrating the above inequality with respect to t on $[0, 1]$ yields

$$\begin{aligned} &\left(\frac{n}{n + 2^{-n} - 1}\right)\left[\frac{k}{\alpha}\mathscr{F}\left(\frac{2a + \zeta(b, a)}{2}\right) + \frac{\mu k(\alpha^2 - \alpha k + 2k)}{4\alpha(\alpha + k)(\alpha + 2k)}\zeta^2(b, a)\right] \\ &\leq \int_0^1 t^{\alpha/k-1}\mathscr{F}(a + t\zeta(b, a))\mathrm{d}t + \int_0^1 t^{\alpha/k-1}\mathscr{F}(a + (1 - t)\zeta(b, a))\mathrm{d}t \\ &= I_1 + I_2. \end{aligned} \tag{16}$$

Now

$$\begin{aligned} I_1 &= \int_a^{a+\zeta(b,a)}\left(\frac{u - a}{\zeta(b, a)}\right)^{\alpha/k-1}\mathscr{F}(u)\frac{du}{\zeta(b, a)} \\ &= \frac{k\Gamma_k(\alpha)}{\zeta^{\alpha/k}(b, a)}{}_k\mathfrak{J}^\alpha_{[a+\zeta(b,a)]^-}\mathscr{F}(a). \end{aligned} \tag{17}$$

Similarly

$$I_2 = \frac{k\Gamma_k(\alpha)}{\zeta^{\alpha/k}(b, a)}{}_k\mathfrak{J}^\alpha_{a^+}\mathscr{F}(a + \zeta(b, a)). \tag{18}$$

This imples

$$\left(\frac{n}{n+2^{-n}-1}\right)\left[\mathscr{F}\left(\frac{2a+\zeta(b,a)}{2}\right)+\frac{\mu\left(\alpha^2-\alpha k+2k\right)}{4(\alpha+k)(\alpha+2k)}\zeta^2(b,a)\right] \leq \frac{\Gamma_k(\alpha+k)}{\zeta^{\alpha/k}(b,a)}\left[{}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{[a+\zeta(b,a)]^-}\mathscr{F}(a)\right]. \tag{19}$$

For the proof of right-hand side inequality,

$$\mathscr{F}(a+t\zeta(b,a))+\mathscr{F}(a+(1-t)\zeta(b,a)) \leq [\mathscr{F}(a)+\mathscr{F}(b)]\left[\frac{1}{n}\sum_{i=1}^{n}\left[1-t^i\right]+\frac{1}{n}\sum_{i=1}^{n}\left[1-(1-t)^i\right]\right] - 2\mu t(1-t)\zeta^2(b,a). \tag{20}$$

Multiplying the above inequality by $t^{\alpha/k-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{k\Gamma_k(\alpha)}{\zeta^{\alpha/k}(b,a)}\left[{}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{[a+\zeta(b,a)]^-}\mathscr{F}(a)\right] \leq \left[\frac{\mathscr{F}(a)+\mathscr{F}(b)}{n}\right]\sum_{i=1}^{n}\int_0^1 t^{a/k-1}\left[2-t^i-(1-t)^i\right]dt - \frac{2k^2\mu\zeta^2(b,a)}{(\alpha+k)(\alpha+2k)}. \tag{21}$$

This implies that

$$\frac{\Gamma_k(\alpha+k)}{\zeta^{\alpha/k}(b,a)}\left[{}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{[a+\zeta(b,a)]^-}\mathscr{F}(a)\right] \leq \left[\frac{\mathscr{F}(a)+\mathscr{F}(b)}{n}\right]\sum_{i=1}^{n}\frac{2sk}{\alpha+sk}-\frac{2k\alpha\mu\zeta^2(b,a)}{(\alpha+k)(\alpha+2k)}. \tag{22}$$

Combining (19) and (22) completes the proof.

Note that if we take $\alpha=k=n=1$ in Theorem 9, then we get the Hermite-Hadamard like inequality involving strongly preinvex functions.

We now derive a new auxiliary result which will be helpful in obtaining the next results of the article.

Lemma 10. *Let $\mathscr{F}:[a,a+\zeta(b,a)]\mapsto\mathbb{R}$ be a differentiable function and $\mathscr{F}'\in[a,a+\zeta(b,a)]$. Then, for any $0<\alpha\leq 1$, $0<\lambda\leq 1$, the following equality for k -fractional integrals holds:*

$$\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k} - \frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right) = \frac{\zeta(b,a)}{2}\int_0^1\left(t^{a/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{a/k}\right)\mathscr{F}'(a+t\zeta(b,a))dt. \tag{23}$$

Proof. It suffices to show that

$$J=\frac{\zeta(b,a)}{2}\left[\int_0^1\left(t^{a/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{a/k}\right)\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t\right] = \frac{\zeta(b,a)}{2}\left[\int_0^1\left(t^{a/k}+\frac{\alpha}{k}(1-\lambda)\right)\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t - \int_0^1(1-t)^{a/k}\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t\right]=\frac{\zeta(b,a)}{2}[J_1+J_2]. \tag{24}$$

Integrating by parts

$$\begin{aligned}J_1&=\int_0^1\left(t^{a/k}+\frac{\alpha}{k}(1-\lambda)\right)\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t\\&=\frac{(\alpha(1-\lambda)+k)\mathscr{F}(a+\zeta(b,a))-\alpha(1-\lambda)\mathscr{F}(a)}{k\zeta(b,a)} - \frac{\alpha}{k\zeta(b,a)}\int_0^1 t^{a/k}\mathscr{F}(a+t\zeta(b,a))\mathrm{d}t\\&=\frac{(\alpha(1-\lambda)+k)\mathscr{F}(a+\zeta(b,a))-\alpha(1-\lambda)\mathscr{F}(a)}{k\zeta(b,a)} - \frac{\alpha}{k\zeta^{(\alpha/k)+1}(b,a)}\int_a^{a+\zeta(b,a)}(u-a)^{a/k-1}\mathscr{F}(u)\mathrm{d}u\\&=\frac{(\alpha(1-\lambda)+k)\mathscr{F}(a+\zeta(b,a))-\alpha(1-\lambda)\mathscr{F}(a)}{k\zeta(b,a)} - \frac{\Gamma_k(\alpha+k)}{\zeta^{a/k+1}(b,a)_k}\mathfrak{J}^{\alpha}_{(a+\zeta(b,a))^-}\mathscr{F}(a)).\end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned}J_2&=-\int_0^1(1-t)^{a/k}\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t\\&=\frac{\mathscr{F}(a)}{\zeta(b,a)}-\frac{\alpha}{k\zeta(b,a)}\int_0^1(1-t)^{a/k}\mathscr{F}(a+t\zeta(b,a))\mathrm{d}t\\&=\frac{\mathscr{F}(a)}{\zeta(b,a)}-\frac{\alpha}{k\zeta^{a/k+1}(b,a)}\int_a^{a+\zeta(b,a)}(a+\zeta(b,a)-u)^{a/k-1}\mathscr{F}(u)\mathrm{d}u\\&=\frac{\mathscr{F}(a)}{\zeta(b,a)}-\frac{\Gamma_k(\alpha+k)}{\zeta^{a/k+1}(b,a)_k}\mathfrak{J}^{\alpha}_{(a)^+}\mathscr{F}(a+\zeta(b,a))).\end{aligned} \tag{26}$$

Using (25) and (26) in (24) completes the proof.

Theorem 11. *Let $\mathscr{F}:[a,a+\zeta(b,a)]\mapsto\mathbb{R}$ be a differentiable function on $(a,a+\zeta(b,a))$ with $\zeta(b,a)>0$ and $\mathscr{F}'\in L[a,a+\zeta(b,a)]$. If $|\mathscr{F}'|$ is a strongly n-polynomial preinvex function of higher order, then*

$$\left|\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k} - \frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right)\right| \le \frac{\zeta(b,a)}{2n}\left[|\mathscr{F}'(a)|\sum_{i=1}^{n}\mathscr{M}_1+|\mathscr{F}'(b)|\sum_{i=1}^{n}\mathscr{M}_2-\mu\|\zeta(b,a)\|^{\sigma}\mathscr{M}_3\right], \tag{27}$$

where

$$\begin{aligned}\mathscr{M}_1&=\frac{\alpha i(1-\lambda)}{k(i+1)}+kB_k(\alpha+k,sk+k)-\frac{k}{\alpha+sk+k},\\ \mathscr{M}_2&=\frac{k}{\alpha+sk+k}-kB_k(sk+k,\alpha+k)+\frac{i\alpha(1-\lambda)}{k(i+1)},\\ \mathscr{M}_3&=\frac{2\alpha(1-\lambda)}{k(\sigma+1)(\sigma+2)}-kB_k(\sigma k+k,\alpha+k)\\ &\quad+kB_k(\alpha+2k,\sigma k+k)+kB_k(\sigma k+2k,\alpha+k).\end{aligned} \tag{28}$$

Proof. Using Lemma 10 and the fact that $|\mathscr{F}'|$ is a strongly n-polynomial preinvex function of higher order, we have

$$\begin{aligned}&\left|\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k} - \frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right)\right|\\
&=\frac{\zeta(b,a)}{2}\left|\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t\right|\\
&\le\frac{\zeta(b,a)}{2}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)|\mathscr{F}'(a+t\zeta(b,a))|\mathrm{d}t\\
&\le\frac{\zeta(b,a)}{2}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\left[\frac{1}{n}\sum_{i=1}^{n}\left[1-t^i\right]|\mathscr{F}'(a)|\right.\\
&\quad\left.+\frac{1}{n}\sum_{i=1}^{n}\left[1-(1-t)^i\right]|\mathscr{F}'(b)|-\mu\|\zeta(b,a)\|^{\sigma}[t^{\sigma}(1-t)+t(1-t)^{\sigma}]\right]\mathrm{d}t\\
&=\frac{\zeta(b,a)}{2n}\left[|\mathscr{F}'(a)|\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\sum_{i=1}^{n}\left[1-t^i\right]\mathrm{d}t\right.\\
&\quad+|\mathscr{F}'(b)|\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\sum_{i=1}^{n}\left[1-(1-t)^i\right]\mathrm{d}t\\
&\quad\left.-\mu\|\zeta(b,a)\|^{\sigma}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)[t^{\sigma}(1-t)+t(1-t)^{\sigma}]\mathrm{d}t\right]\\
&=\frac{\zeta(b,a)}{2n}\left[|\mathscr{F}'(a)|\sum_{i=1}^{n}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\left[1-t^i\right]\mathrm{d}t\right.\\
&\quad+|\mathscr{F}'(b)|\sum_{i=1}^{n}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\left[1-(1-t)^i\right]\mathrm{d}t\\
&\quad\left.-\mu\|\zeta(b,a)\|^{\sigma}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)[t^{\sigma}(1-t)+t(1-t)^{\sigma}]\mathrm{d}t\right]\\
&=\frac{\zeta(b,a)}{2n}\left[|\mathscr{F}'(a)|\sum_{i=1}^{n}\mathscr{M}_1+|\mathscr{F}'(b)|\sum_{i=1}^{n}\mathscr{M}_2-\mu\|\zeta(b,a)\|^{\sigma}\mathscr{M}_3\right].\end{aligned} \tag{29}$$

This completes the proof.

Theorem 12. *Let $\mathscr{F}:[a,a+\zeta(b,a)]\mapsto\mathbb{R}$ be a differentiable function on $(a,a+\zeta(b,a))$ with $\zeta(b,a)>0$, $p^{-1}+q^{-1}=1$ and $\mathscr{F}'\in L[a,a+\zeta(b,a)]$. If $|\mathscr{F}'|^q$ is strongly n -polynomial preinvex function of higher order, then*

$$\begin{aligned}&\left|\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k} - \frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right)\right|\\
&\le\frac{\zeta(b,a)}{2}\mathscr{M}_4\left(\frac{2}{n}\sum_{i=1}^{n}\frac{i}{i+1}A\left(|\mathscr{F}'(a)|^q,|\mathscr{F}'(b)|^q\right)-\frac{2\mu\|\zeta(b,a)\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{1/q},\end{aligned} \tag{30}$$

where

$$\mathscr{M}_4=\frac{\alpha(1-\lambda)}{k}+2\left(\frac{k}{\alpha p+k}\right)^{1/p}, \tag{31}$$

and $A(\cdot,\cdot)$ is the arithmetic mean.

Proof. Using Lemma 10, Hölder's integral inequality and $|\mathscr{F}'|^q$ is strongly n-polynomial preinvex function of higher order, we have

$$\begin{aligned}&\left|\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k} - \frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^{\alpha}_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right)\right|\\
&=\frac{\zeta(b,a)}{2}\left|\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t\right|\\
&=\frac{\zeta(b,a)}{2}\left(\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right)|\mathscr{F}'(a+t\zeta(b,a))|\mathrm{d}t\right.\\
&\quad\left.+\int_0^1(1-t)^{\alpha/k}|\mathscr{F}'(a+t\zeta(b,a))|\mathrm{d}t\right)\\
&\le\frac{\zeta(b,a)}{2}\left[\left(\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right)^p\mathrm{d}t\right)^{1/p}\left(\int_0^1|\mathscr{F}'(a+t\zeta(b,a))|^q\mathrm{d}t\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1(1-t)^{\alpha p/k}\mathrm{d}t\right)^{1/p}\left(\int_0^1|\mathscr{F}'(a+t\zeta(b,a))|^q\mathrm{d}t\right)^{1/q}\right]\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\zeta(b,a)}{2}\left[\left(\left(\int_0^1 t^{\alpha p/k}\mathrm{d}t\right)^{1/p}+\left(\int_0^1+\frac{\alpha^p}{k^p}(1-\lambda)^p\mathrm{d}t\right)^{1/p}\right.\right.\\
&\left.\left.+\left(\int_0^1(1-t)^{\alpha p/k}\mathrm{d}t\right)^{1/p}\right]\times\left[\frac{|\mathscr{F}'(a)|^q}{n}\sum_{i=1}^n\int_0^1\left[1-t^i\right]\mathrm{d}t\right.\\
&\left.+\frac{|\mathscr{F}'(b)|^q}{n}\sum_{i=1}^n\int_0^1\left[1-(1-t)^i\right]\mathrm{d}t-\mu\|\zeta(b,a)\|^\sigma\int_0^1\left[t^\sigma(1-t)+t(1-t)^\sigma\right]\mathrm{d}t\right]\\
&=\frac{\zeta(b,a)}{2}\mathscr{M}_4\left[\frac{|\mathscr{F}'(a)|^q}{n}\sum_{i=1}^n\frac{i}{i+1}+\frac{|\mathscr{F}'(b)|^q}{n}\sum_{i=1}^n\frac{i}{i+1}-\frac{2\mu\|\zeta(b,a)\|^\sigma}{(\sigma+1)(\sigma+2)}\right]\\
&=\frac{\zeta(b,a)}{2}\mathscr{M}_4\left(\frac{2}{n}\sum_{i=1}^n\frac{i}{i+1}A\left(|\mathscr{F}'(a)|^q,|\mathscr{F}'(b)|^q\right)-\frac{2\mu\|\zeta(b,a)\|^\sigma}{(\sigma+1)(\sigma+2)}\right)^{1/q}.
\end{aligned}\tag{32}$$

This completes the proof.

Theorem 13. *Let $\mathscr{F}:[a,a+\zeta(b,a)]\mapsto\mathbb{R}$ be a differentiable function on $(a,a+\zeta(b,a))$ with $\zeta(b,a)>0$, $q>1$ and $\mathscr{F}'\in L[a,a+\zeta(b,a)]$. If $|\mathscr{F}'|^q$ is strongly n-polynomial preinvex function of higher order, then*

$$\begin{aligned}
&\left|\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k}\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^\alpha_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^\alpha_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right)\right|\\
&\leq\frac{\zeta(b,a)}{2}\left(\frac{\alpha(1-\lambda)}{k}\right)^{1-1/q}\left[\frac{|\mathscr{F}'(a)|^q}{n}\sum_{i=1}^n\mathscr{M}_1\right.\\
&\left.+\frac{|\mathscr{F}'(b)|^q}{n}\sum_{i=1}^n\mathscr{M}_2-\mu\|\zeta(b,a)\|^\sigma\mathscr{M}_3\right],
\end{aligned}\tag{33}$$

where $\mathscr{M}_1$, $\mathscr{M}_2$, and $\mathscr{M}_3$ are given in Theorem 11.

Proof. Using Lemma 10, the power mean integral inequality and $|\mathscr{F}'|^q$ is strongly n-polynomial preinvex function of higher order, we have

$$\begin{aligned}
&\left|\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k}\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^\alpha_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^\alpha_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right)\right|\\
&=\frac{\zeta(b,a)}{2}\left|\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathscr{F}'(a+t\zeta(b,a))\mathrm{d}t\right|\\
&\leq\frac{\zeta(b,a)}{2}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)|\mathscr{F}'(a+t\zeta(b,a))|\mathrm{d}t\\
&\leq\frac{\zeta(b,a)}{2}\left(\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\right)^{1-1/q}\\
&\times\left(\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\left[\frac{1}{n}\sum_{i=1}^n\left[1-t^i\right]|\mathscr{F}'(a)|^q\right.\right.\\
&\left.\left.+\frac{1}{n}\sum_{i=1}^n\left[1-(1-t)^i\right]|\mathscr{F}'(b)|^q-\mu\|\zeta(b,a)\|^\sigma\left[t^\sigma(1-t)+t(1-t)^\sigma\right]\right]\mathrm{d}t\right)^{1/q}\\
&=\frac{\zeta(b,a)}{2}\left(\frac{\alpha(1-\lambda)}{k}\right)^{1-1/q}\left[\frac{|\mathscr{F}'(a)|^q}{n}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right.\right.\\
&\left.-(1-t)^{\alpha/k}\right)\sum_{i=1}^n\left[1-t^i\right]\mathrm{d}t+\frac{|\mathscr{F}'(b)|^q}{n}\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right.\\
&\left.-(1-t)^{\alpha/k}\right)\sum_{i=1}^n\left[1-(1-t)^i\right]\mathrm{d}t-\mu\|\zeta(b,a)\|^\sigma\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right.\\
&\left.\left.-(1-t)^{\alpha/k}\right)\left[t^\sigma(1-t)+t(1-t)^\sigma\right]\mathrm{d}t\right]^{1/q}\\
&=\frac{\zeta(b,a)}{2}\left(\frac{\alpha(1-\lambda)}{k}\right)^{1-1/q}\left[\frac{|\mathscr{F}'(a)|^q}{n}\sum_{i=1}^n\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right.\right.\\
&\left.-(1-t)^{\alpha/k}\right)\left[1-t^i\right]\mathrm{d}t+\frac{|\mathscr{F}'(b)|^q}{n}\sum_{i=1}^n\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right.\\
&\left.-(1-t)^{\alpha/k}\right)\left[1-(1-t)^i\right]\mathrm{d}t-\mu\|\zeta(b,a)\|^\sigma\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)\right.\\
&\left.\left.-(1-t)^{\alpha/k}\right)\left[t^\sigma(1-t)+t(1-t)^\sigma\right]\mathrm{d}t\right]^{1/q}\\
&=\frac{\zeta(b,a)}{2}\left(\frac{\alpha(1-\lambda)}{k}\right)^{1-\alpha/q}\left[\frac{|\mathscr{F}'(a)|^q}{n}\sum_{i=1}^n\mathscr{M}_1+\frac{|\mathscr{F}'(b)|^q}{n}\sum_{i=1}^n\mathscr{M}_2\right.\\
&\left.-\mu\|\zeta(b,a)\|^\sigma\mathscr{M}_3\right]^{1/q}.
\end{aligned}\tag{34}$$

This completes the proof.

Theorem 14. *Let $\mathscr{F}:[a,a+\zeta(b,a)]\mapsto\mathbb{R}$ be a differentiable function on $(a,a+\zeta(b,a))$ with $\zeta(b,a)>0$, $q>1$, and $\mathscr{F}'\in L[a,a+\zeta(b,a)]$. If $|\mathscr{F}'|^q$ is strongly n-polynomial preinvex function of higher order, then*

$$\begin{aligned}
&\left|\frac{(k+\alpha(1-\lambda))\mathscr{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathscr{F}(a)}{2k}\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{J}^\alpha_{a^+}\mathscr{F}(a+\zeta(b,a))+{}_k\mathfrak{J}^\alpha_{(a+\zeta(b,a))^-}\mathscr{F}(a)\right)\right|\\
&\leq\frac{\zeta(b,a)}{2}\left(\frac{\alpha}{2k}(1-\lambda)-\frac{\alpha k}{(\alpha+k)(\alpha+2k)}\right)^{1-1/q}\left(\frac{|\mathscr{F}'(a)|^q}{n}\sum_{i=1}^n\mathscr{M}_5\right.\\
&\left.+\frac{|\mathscr{F}'(b)|^q}{n}\sum_{i=1}^n\mathscr{M}_6-\mu\|\zeta(b,a)\|^\sigma\mathscr{M}_7\right)^{1/q}\\
&+\frac{\zeta(b,a)}{2}\left(\frac{\alpha k}{(\alpha+k)(\alpha+2k)}+\frac{\alpha}{2k}(1-\lambda)\right)^{1-1/q}\\
&\cdot\left(\frac{|\mathscr{F}'(a)|^q}{n}\sum_{i=1}^n\mathscr{M}_8+\frac{|\mathscr{F}'(b)|^q}{n}\sum_{i=1}^n\mathscr{M}_9-\mu\|\zeta(b,a)\|^\sigma\mathscr{M}_{10}\right)^{1/q},
\end{aligned}\tag{35}$$

where

$$\begin{aligned}
\mathscr{M}_5&=\frac{i\alpha(1-\lambda)(i+3)}{2k(i+1)(i+2)}-kB_k(sk+2k,\alpha+k)+kB_k(sk+k,\alpha+k)\\
&-\frac{k^2}{(\alpha+sk+k)(\alpha+sk+2k)}-\frac{\alpha k}{(\alpha+k)(\alpha+2k)},
\end{aligned}$$

$$\mathcal{M}_6=\frac{i\alpha(1-\lambda)}{2k(i+2)}-kB_k(\alpha+k,sk+k)+kB_k(\alpha+2k,sk+k)+\frac{k}{\alpha+sk+2k}$$

$$\begin{aligned}\mathcal{M}_7&=\frac{2k^3}{(k\sigma+\alpha+k)(k\sigma+\alpha+2k)(k\sigma+\alpha+3k)}+\frac{\alpha}{k}(1-\lambda)\left[\frac{2}{(\sigma+1)(\sigma+2)(\sigma+3)}+kB_k(2k,k\sigma+k)\right.\\&\quad\left.-kB_k(\alpha+2k,k\sigma+k)\right]+2kB_k(k\sigma+2k,\alpha+k)\\&\quad-kB_k(k\sigma+k,\alpha+k)+kB_k(\alpha+2k,k\sigma+k)\\&\quad-kB_k(2k,k\sigma+\alpha+k)-kB_k(k\sigma+3k,\alpha+k)\\&\quad-kB_k(\alpha+3k,k\sigma+k)+kB_k(3k,k\sigma+\alpha+k),\end{aligned}$$

$$\mathcal{M}_8=\frac{\alpha k}{(\alpha+k)(\alpha+2k)}+\frac{i\alpha(1-\lambda)}{2k(i+2)}+kB_k(sk+k,\alpha+k)-\frac{k}{\alpha+sk+2k}$$

$$\mathcal{M}_9=\frac{\alpha k}{(\alpha+k)(\alpha+2k)}-kB_k(sk+2k,\alpha+k)+\frac{i\alpha(1-\lambda)(i+3)}{2k(i+1)(i+2)}+\frac{k^2}{(\alpha+sk+k)(\alpha+sk+2k)},$$

$$\begin{aligned}\mathcal{M}_{10}&=\frac{k^2}{(k\sigma+\alpha+2k)(k\sigma+\alpha+3k)}+\frac{\alpha}{k}(1-\lambda)\left(\frac{1}{(\sigma+2)(\sigma+3)}+kB_k(3k,k\sigma+k)\right)\\&\quad-kB_k(k\sigma+2k,\alpha+k)+kB_k(k\sigma+3k,\alpha+k)\\&\quad+kB_k(\alpha+3k,k\sigma+k)-kB_k(3k,k\sigma+\alpha+k).\end{aligned}\tag{36}$$

Proof. Using Lemma 10. improved power mean integral inequality and $|\mathcal{F}'|^q$ is strongly n-polynomial preinvex function of higher order, we have

$$\begin{aligned}&\left|\frac{(k+\alpha(1-\lambda))\mathcal{F}(a+\zeta(b,a))+(k-\alpha(1-\lambda))\mathcal{F}(a)}{2k}-\frac{\Gamma_k(\alpha+k)}{2\zeta^{\alpha/k}(b,a)}\left({}_k\mathfrak{I}^{\alpha}_{a^+}\mathcal{F}(a+\zeta(b,a))+{}_k\mathfrak{I}^{\alpha}_{(a+\zeta(b,a))^-}\mathcal{F}(a)\right)\right|\\
&=\frac{\zeta(b,a)}{2}\left|\int_0^1\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathcal{F}'(a+t\zeta(b,a))\mathrm{d}t\right|\\
&\le\frac{\zeta(b,a)}{2}\left(\int_0^1(1-t)\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\right)^{1-1/q}\\
&\quad\times\left(\int_0^1(1-t)\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)|\mathcal{F}'(a+t\zeta(b,a))|^q\mathrm{d}t\right)^{1/q}\\
&\quad+\frac{\zeta(b,a)}{2}\left(\int_0^1 t\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\right)^{1-1/q}\\
&\quad\times\left(\int_0^1 t\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)|\mathcal{F}'(a+t\zeta(b,a))|^q\mathrm{d}t\right)^{1/q}\\
&\le\frac{\zeta(b,a)}{2}\left(\frac{\alpha}{2k}(1-\lambda)-\frac{\alpha k}{(\alpha+k)(\alpha+2k)}\right)^{1-1/q}\\
&\quad\times\left(\frac{|\mathcal{F}'(a)|^q}{n}\sum_{i=1}^n\int_0^1(1-t)\left[1-t^i\right]\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\right.\\
&\quad+\frac{|\mathcal{F}'(b)|^q}{n}\sum_{i=1}^n\int_0^1(1-t)\left[1-(1-t)^i\right]\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\\
&\quad-\mu\|\zeta(b,a)\|^\sigma\int_0^1(1-t)[t^\sigma(1-t)+t(1-t)^\sigma]\left(t^{\alpha/k}+\tfrac{\alpha}{k}(1-\lambda)\right.\\
&\quad\left.\left.-(1-t)^{\alpha/k}\right)\mathrm{d}t\right)^{1/q}+\frac{\zeta(b,a)}{2}\left(\frac{\alpha k}{(\alpha+k)(\alpha+2k)}+\frac{\alpha}{2k}(1-\lambda)\right)^{1-1/q}\\
&\quad\times\left(\frac{|\mathcal{F}'(a)|^q}{n}\sum_{i=1}^n\int_0^1 t\left[1-t^i\right]\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\right.\\
&\quad+\frac{|\mathcal{F}'(b)|^q}{n}\sum_{i=1}^n\int_0^1 t\left[1-(1-t)^i\right]\left(t^{\alpha/k}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\\
&\quad\left.-\mu\|\zeta(b,a)\|^\sigma\int_0^1 t[t^\sigma(1-t)+t(1-t)^\sigma]\left(t^{\alpha/k}+\tfrac{\alpha}{k}(1-\lambda)-(1-t)^{\alpha/k}\right)\mathrm{d}t\right)^{1/q}\\
&=\frac{\zeta(b,a)}{2}\left(\frac{\alpha}{2k}(1-\lambda)-\frac{\alpha k}{(\alpha+k)(\alpha+2k)}\right)^{1-1/q}\left(\frac{|\mathcal{F}'(a)|^q}{n}\sum_{i=1}^n\mathcal{M}_5\right.\\
&\quad\left.+\frac{|\mathcal{F}'(b)|^q}{n}\sum_{i=1}^n\mathcal{M}_6-\mu\|\zeta(b,a)\|^\sigma\mathcal{M}_7\right)^{1/q}\\
&\quad+\frac{\zeta(b,a)}{2}\left(\frac{\alpha k}{(\alpha+k)(\alpha+2k)}+\frac{\alpha}{2k}(1-\lambda)\right)^{1-1/q}\left(\frac{|\mathcal{F}'(a)|^q}{n}\sum_{i=1}^n\mathcal{M}_8\right.\\
&\quad\left.+\frac{|\mathcal{F}'(b)|^q}{n}\sum_{i=1}^n\mathcal{M}_9-\mu\|\zeta(b,a)\|^\sigma\mathcal{M}_{10}\right)^{1/q}.\end{aligned}\tag{37}$$

This completes the proof.

3. Conclusion

In this article, we have introduced the notion of strongly n-polynomial preinvex function of higher order. We have derived a new k-fractional analogue of classical Hermite-Hadamard's integral inequality utilizing the class of strongly n-polynomial preinvex functions. We established a new auxiliary result pertaining to k-fractional integrals, and utilizing this new result, we obtained several new variants of trapezium-like inequalities using the concept of strongly n-polynomial preinvex functions of higher order. We would like to emphasize here that we can recapture some other new results from the main results of this article under some suitable conditions. For example, if we take $\sigma=2$, then all the results reduce to the results for strongly n-polynomial preinvex functions. If $\zeta(b,a)=b-a$, then we have results for n-polynomial convex functions of higher order. Similarly for other suitable choices, other new and known results, we left the details to interested readers. This shows that the results obtained in this article are quite a unifying one. We hope that the ideas and techniques of this article will inspire interested readers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

M. U. Awan gave Definition 8, carried out the proof of Theorem 9 and drafted the manuscript. S. Talib carried out the proof of Theorems 10 and 11. M. A. Noor carried out the proof of Theorem 12. Y.-M. Chu provided the main idea, carried out the proof of Theorem 13, completed the final revision, and submitted the article. K. I. Noor carried out the proof of Theorem 14. All authors read and approved the final manuscript.

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Research Article

Rate of Approximation for Modified Lupaş-Jain-Beta Operators

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The main intent of this paper is to innovate a new construction of modified Lupaş-Jain operators with weights of some Beta basis functions whose construction depends on σ such that $\sigma(0)=0$ and $\inf_{x\in[0,\infty)} \sigma'(x) \geq 1$. Primarily, for the sequence of operators, the convergence is discussed for functions belong to weighted spaces. Further, to prove pointwise convergence Voronovskaya type theorem is taken into consideration. Finally, quantitative estimates for the local approximation are discussed.

1. Introduction

In 1972, Jain [1] with the help of Poisson distribution introduced a famous linear positive operators as follows:

$$P_m^{[\alpha]}(f;x)=\sum_{j=0}^{\infty}\mathscr{W}_{\alpha}(j,mx)f\left(\frac{j}{m}\right),\quad x\geq 0, \tag{1}$$

where $m\geq 1$, f defined on $[0,\infty)$, and

$$\mathscr{W}_{\alpha}(j,mx)=mx(mx+j\alpha)^{j-1}\frac{e^{-(mx+j\alpha)}}{j!},\quad 0\leq\alpha<1,$$
$$\sum_{j=0}^{\infty}\mathscr{W}_{\alpha}(j,mx)=1. \tag{2}$$

If we put $\alpha=0$ in (1), then it becomes Szász-Mirakyan-type.

In 1995, Lupaş [2] introduced a sequence of linear positive operators; later on in 1999, it was modified by Agratini [3] as follows:

$$L_m(f;x)=2^{-mx}\sum_{j=0}^{\infty}\frac{(mx)_j}{j!2^j}f\left(\frac{j}{m}\right),\quad x\geq 0, \tag{3}$$

and also discussed the Kantorovich and Durrmeyer variant of operator (1).

In 2018, Tunca et al. [4] modified operator (3) in such a way that in the construction, authors take the negative subscript -1 of the Pochhammer symbol into consideration; due to this, the calculations become simpler in a remarkable degree just as

$$L_m^{\alpha}(f;x)=2^{-(mx+j\alpha)}\sum_{j=0}^{\infty}\frac{mx(mx+1+j\alpha)_{j-1}}{j!2^j}f\left(\frac{j}{m}\right),\quad x\geq 0. \tag{4}$$

In order to approximate Lebesgue integrable functions, the most important modifications are Kantorovich and Durrmeyer integral operators. The Durrmeyer variant of operator (1) is introduced by Tarabie [5] and Mishra and Patel [6] with some beta basis functions.

In 2011, Cárdenas-Morales et al. [7] defined the Bernstein-type operators by $B_m(fo\sigma^{-1})o\sigma$ and also presents a better degree of approximation depending on σ. This type of approximation operators generalizes the Korovkin set from $\{e_0,e_1,e_2\}$ to $\{e_0,\sigma,\sigma^2\}$. The Durrmeyer variant of $B_m(fo\sigma^{-1})o\sigma$ is defined in [8]. In 2014, Aral et al. [9] defined a similar modification of Szász-Mirakyan-type operators by using a suitable function σ.

Motivated by the above mentioned work very recently, Bodur [10] introduced a new modification of operator (4) by using a suitable function σ, which satisfies the following properties:

$(\sigma_1)\sigma$ be a continuously differentiable function on $[0,\infty)$

$(\sigma_2)\sigma(0)=0$ and $\inf_{x\in[0,\infty)}\sigma'(x)\geq 1$

The new formulated operators are defined as

$$\mathscr{L}^{\alpha}_{m,\sigma}(f;x)=2^{-(m\sigma(x)+j\alpha)}\sum_{j=0}^{\infty}\frac{m\sigma(x)(m\sigma(x)+1+j\alpha)_{j-1}}{2^j j!}\cdot\left(fo\sigma^{-1}\right)\left(\frac{j}{m}\right),\quad x\geq 0, \tag{5}$$

for $m\geq 1$, $x\geq 0$, and suitable functions f defined on $[0,\infty)$.

As we know, in order to approximate Lebesgue integrable functions, the most important modifications are Kantorovich and Durrmeyer integral operators. Motivated by the above mentioned Durrmeyer type generalizations of various operators and also from [11–23], in this paper, Durrmeyer-type modification of generalized Lupaş-Jain operators (5) by taking weights of some beta basis function is defined as follows:

$$\mathscr{D}^{\alpha}_{m,\sigma}(f;x)=2^{-(m\sigma(x)+j\alpha)}\sum_{j=0}^{\infty}\frac{m\sigma(x)(m\sigma(x)+1+j\alpha)_{j-1}}{2^j j!}\int_0^{\infty}b_{m,j}(\vartheta)\left(fo\sigma^{-1}\right)(\vartheta)d\vartheta, \tag{6}$$

where $m\in\mathbb{N}$ and $b_{m,j}(\vartheta)$ is defined as

$$b_{m,j}(\vartheta)=\frac{1}{\beta(m+1,j)}\frac{\vartheta^{j-1}}{(1+\vartheta)^{m+j+1}}, \tag{7}$$

where $\beta(m+1,j)$ is the beta basis function and σ is a function satisfying the conditions (σ_1) and (σ_2) given above.

The rest of the work is organized as follows: in the second section, moments and central moments for $\mathscr{D}^{\alpha}_{m,\sigma}$ are calculated. In the third section, we study convergence properties of $\mathscr{D}^{\alpha}_{m,\sigma}$ in the light of weighted space. In the fourth section, we obtain the order of approximation of new constructed operators associated with the weighted modulus of continuity. In the fifth section, we shall prove Voronovskaya-type theorem in quantitative form. These kinds of results are very useful to describe the rate of point-wise convergence. Finally, in the last section, we obtain some local approximation results related to $\mathscr{K}$-functional.

2. Basic Results

In this section, we prove some lemmas for $\mathscr{D}^{\alpha}_{m,\sigma}$ which are required to prove our main results.

Lemma 1. *Let $\mathscr{D}^{\alpha}_{m,\sigma}$ be given by (6). Then for each $x\geq 0$, $m\in\mathbb{N}$, and $0\leq\alpha<1$, we have*

(i) $\mathscr{D}^{\alpha}_{m,\sigma}(1;x)=1$

(ii) $\mathscr{D}^{\alpha}_{m,\sigma}(\sigma;x)=\sigma(x)/1-\alpha$

(iii) $\mathscr{D}^{\alpha}_{m,\sigma}(\sigma^2;x)=(1/(m-1)(1-\alpha)^2)(m\sigma^2(x)+(\alpha^2-2\alpha+3/(1-\alpha))\sigma(x))$

(iv) $\mathscr{D}^{\alpha}_{m,\sigma}(\sigma^3;x)=(1/(m-1)(m-2)(1-\alpha)^3)(m^2\sigma^3(x)+((6m+3m(1-\alpha)^2)/1-\alpha)\sigma^2(x)+(6(\alpha+1)+6(1-\alpha)^2+2(1-\alpha)^4/(1-\alpha)^2)\sigma(x))$

(v) $\mathscr{D}^{\alpha}_{m,\sigma}(\sigma^4;x)=(1/(m-1)(m-2)(m-3)(1-\alpha)^4)(m^3\sigma^4(x)+((12m^2+6m^2(1-\alpha)^2)/1-\alpha)\sigma^3(x)+(24m\alpha+36m+36m(1-\alpha)^2+11m(1-\alpha)^4/(1-\alpha)^2)\sigma^2(x)+(26\alpha^2+68\alpha+24+(36\alpha+36)(1-\alpha)^2+22(1-\alpha)^4+6(1-\alpha)^6/(1-\alpha)^3)\sigma(x))$

By using beta function and Lemma 2.1 in [4], it can be proved. So we omit it. Now, from the linearity of the operators $\mathscr{D}^{\alpha}_{m,\sigma}$, we can state Lemma 2.

Lemma 2. *For operators $\mathscr{D}^{\alpha}_{m,\sigma}$, we have the following properties:*

(i) $\mathscr{D}^{\alpha}_{m,\sigma}(\sigma(\vartheta)-\sigma(x);x)=(\alpha/1-\alpha)\sigma(x)$

(ii) $\mathscr{D}^{\alpha}_{m,\sigma}((\sigma(\vartheta)-\sigma(x))^2;x)=(m/(m-1)(1-\alpha)^2-1+\alpha/1-\alpha)\sigma^2(x)+(\alpha^2-2\alpha+3/(m-1)(1-\alpha)^3)\sigma(x)$

(iii) $\mathscr{D}^{\alpha}_{m,\sigma}((\sigma(\vartheta)-\sigma(x))^4;x)=(1/(m-1)(m-2)(m-3)(1-\alpha)^4)(\{m^3+6m(m-2)(m-3)(1-\alpha)^2-4m^2(m-3)(1-\alpha)-4(m-1)(m-2)(m-3)(1-\alpha)^3+(m-1)(m-2)(m-3)(1-\alpha)^4\}\sigma^4(x)+\{6\alpha^2-12\alpha+18(m-2)(m-3)(1-\alpha)-(24m+12m(1-\alpha)^2)(m-3)+(12m^2+6m^2(1-\alpha)^2/(1-\alpha))\}\sigma^3(x)+\{24m\alpha+36m+36m(1-\alpha)^2+11m(1-\alpha)^4/(1-\alpha)^2-(24(\alpha+1)+24(1-\alpha)^2+8(1-\alpha)^4)(m-3)/1-\alpha\}\sigma^2(x)+(26\alpha^2+68\alpha+24+(36\alpha+36)(1-\alpha)^2+22(1-\alpha)^4+6(1-\alpha)^6/(1-\alpha)^3)\sigma(x))$

3. Convergence of $\mathscr{D}^{\alpha}_{m,\sigma}$

Here, we prove the convergence of $\mathscr{D}^{\alpha}_{m,\sigma}$ by using weight function. Let $\lambda(x)$ be a function satisfying the conditions (σ_1) and (σ_2) given above. Also, let $\lambda(x)=1+\sigma^2(x)$ be a weight function and the weighted space is defined as follows:

$$\mathscr{B}_{\lambda}[0,\infty)=\left\{f:[0,\infty)\longrightarrow\mathbb{R}\mid |f(x)|\leq\mathscr{M}_f\lambda(x),x\geq 0\right\}, \tag{8}$$

where $\mathscr{M}_f$ is a constant which depends only on f, with the norm

$$\|f\|_{\lambda}=\sup_{x\in[0,\infty)}\frac{|f(x)|}{\lambda(x)}. \tag{9}$$

Also, we mention some subspaces of $\mathscr{B}_\lambda[0,\infty)$ as

$$\begin{aligned}
\mathscr{C}_\lambda[0,\infty) &= \{f \in \mathscr{B}_\lambda[0,\infty)\colon f \text{ is continuous on } [0,\infty)\},\\
\mathscr{C}^*_\lambda[0,\infty) &= \left\{f \in \mathscr{C}_\lambda[0,\infty)\colon \lim_{x\to\infty}\frac{f(x)}{\lambda(x)} = \mathscr{M}_f = \text{constant}\right\},\\
U_\lambda[0,\infty) &= \left\{f \in \mathscr{C}_\lambda[0,\infty)\colon \frac{f(x)}{\lambda(x)} \text{ is uniformly continuous on } [0,\infty)\right\}.
\end{aligned} \tag{10}$$

It is obvious that $\mathscr{C}^*_\lambda[0,\infty) \subset U_\lambda[0,\infty) \subset \mathscr{C}_\lambda[0,\infty) \subset \mathscr{B}_\lambda[0,\infty)$.

We have the following results for the weighted Korovkin-type theorems due to Gadjiev [24]

Lemma 3. *[24].*

The positive linear operators $\mathscr{T}_m, m \geq 1$ act from $\mathscr{C}_\lambda[0,\infty)$ to $\mathscr{B}_\lambda[0,\infty)$ if and only if the inequality

$$|\mathscr{T}_m(\lambda; x)| \leq \mathscr{M}_m\lambda(x), \quad x \geq 0 \tag{11}$$

holds, where $\mathscr{M}_m > 0$ is a constant depending on m.

Theorem 4 (see [24]). *Let the sequence of positive linear operators $\mathscr{T}_m$, $m \geq 1$ acting from $\mathscr{C}_\lambda[0,\infty)$ to $\mathscr{B}_\lambda[0,\infty)$ and satisfying*

$$\lim_{m\to\infty}\|\mathscr{T}_m\sigma^r - \sigma^r\|_\lambda = 0, \quad r = 0, 1, 2. \tag{12}$$

Then, for each $f \in C^*_\lambda[0,\infty)$, we have

$$\lim_{m\to\infty}\|\mathscr{T}_m(f) - f\|_\lambda = 0. \tag{13}$$

Remark 5. Examining Lemma 1 based on the famous Korovkin theorem [25], it is clear that $(\mathscr{D}^\alpha_{m,\sigma})_{m\geq 1}$ does not form an approximation process. Now, in order to obtain convergence properties, we replace the constant α by $\alpha_m \in [0,1)$ such that $\lim_{m\to\infty}\alpha_m = 0$.

Theorem 6. *Let $0 \leq \alpha_m < 1$ such that $\lim_{m\to\infty}\alpha_m = 0$, and also, let $\mathscr{D}^\alpha_{m,\sigma}$ be the sequence of positive linear operators. Then, for each function $f \in C^*_\lambda[0,\infty)$, we have*

$$\lim_{m\to\infty}\left\|\mathscr{D}^{\alpha_m}_{m,\sigma}(f) - f\right\|_\lambda = 0. \tag{14}$$

Proof. From Lemma 1, we obtain

$$\begin{aligned}
&\left\|\mathscr{D}^{\alpha_m}_{m,\sigma}(1; x) - 1\right\|_\lambda = 0,\\
&\left\|\mathscr{D}^{\alpha_m}_{m,\sigma}(\sigma; x) - \sigma\right\|_\lambda \leq \frac{\alpha_m}{1-\alpha_m},\\
&\left\|\mathscr{D}^{\alpha}_{m,\sigma}(\sigma^2) - \sigma^2\right\|_\lambda \leq \frac{m-(m-1)(1-\alpha_m)^2}{(m-1)(1-\alpha_m)^2} + \frac{\alpha^2 - 2\alpha_m + 3}{(m-1)(1-\alpha_m)^3}.
\end{aligned} \tag{15}$$

Hence, by Theorem 4, we deduce

$$\lim_{m\to\infty}\left\|\mathscr{D}^{\alpha}_{m,\sigma}(f) - f\right\|_\lambda = 0. \tag{16}$$

4. Rate of Convergence

In this part, we would like to determine the rate of convergence for $\mathscr{D}^\alpha_{m,\sigma}$ by weighted modulus of continuity $\omega_\sigma(f;\delta)$ which was introduced by Holhos [26] in 2008, as follows:

$$\omega_\sigma(;\delta) = \sup_{x,\vartheta\in[0,\infty),|\sigma(\vartheta)-\sigma(x)|\leq\delta}\frac{|f(\vartheta) - f(x)|}{\lambda(\vartheta) + \lambda(x)}, \quad \delta > 0, \tag{17}$$

where $f \in \mathscr{C}_\lambda[0,\infty)$, with the following properties:

(i) $\omega_\sigma(f;0) = 0$

(ii) $\omega_\sigma(f;\lambda) \geq 0$, $\lambda \geq 0$, for $f \in \mathscr{C}_\lambda[0,\infty)$

(iii) $\lim_{\lambda\to 0}\omega_\sigma(f;\lambda) = 0$, for each $f \in U_\lambda[0,\infty)$

Theorem 7 (see [26]). *Let $\mathscr{T}_m: \mathscr{C}_\lambda[0,\infty) \to \mathscr{B}_\lambda[0,\infty)$ be a sequence of positive linear operators with*

$$\begin{aligned}
&\left\|\mathscr{T}_m(\sigma^0) - \sigma^0\right\|_{\lambda^0} = a_m,\\
&\left\|\mathscr{T}_m(\sigma) - \sigma\right\|_{\lambda^{1/2}} = b_m,\\
&\left\|\mathscr{T}_m(\sigma^2) - \sigma^2\right\|_\lambda = c_m,\\
&\left\|\mathscr{T}_m(\sigma^3) - \sigma^3\right\|_{\lambda^{3/2}} = d_m,
\end{aligned} \tag{18}$$

where the sequences a_m, b_m, c_m, and d_m converge to zero as $m \longrightarrow \infty$. Then

$$\|\mathscr{T}_m(f) - f\|_{\lambda^{3/2}} \leq (7 + 4a_m + 2c_m)\omega_\sigma(f;\lambda_m) + \|f\|_\lambda a_m, \tag{19}$$

for all $f \in \mathscr{C}_\lambda[0,\infty)$, where

$$\lambda_m = 2\sqrt{(a_m + 2b_m + c_m)(1 + a_m)} + a_m + 3b_m + 3c_m + d_m. \tag{20}$$

Theorem 8. *Let $0 \leq \alpha < 1$ such that $\lim_{m\to\infty}\alpha_m = 0$, and also, let $\mathscr{D}^\alpha_{m,\sigma}$ be the sequence of positive linear operators. Then for all $f \in C_\lambda[0,\infty)$, we have*

$$\begin{aligned}
\left\|\mathscr{D}^{\alpha_m}_{m,\sigma}(f) - f\right\|_{\lambda^{3/2}} \leq \Bigg(7 &+ \frac{2m - 2(m-1)(1-\alpha_m)^2}{(m-1)(1-\alpha_m)^2}\\
&+ \frac{2\alpha_m^2 - 4\alpha_m + 6}{(m-1)(1-\alpha_m)^3}\Bigg)\omega_\sigma(f;\delta_m),
\end{aligned} \tag{21}$$

where

$$\delta_m = 2\sqrt{\frac{2\alpha_m}{1-\alpha_m} + \frac{m-(m-1)(1-\alpha_m)^2}{(m-2)(1-\alpha_m)^2} + \frac{\alpha_m^2 - 2\alpha_m + 3}{(m-1)(1-\alpha_m)^3} + \frac{3\alpha_m}{1-\alpha_m} + \frac{3m-3(m-1)(1-\alpha_m)^2}{(m-1)(1-\alpha_m)^2} + \frac{3\alpha_m^2 - 6\alpha_m + 6}{(m-1)(1-\alpha_m)^3} + \frac{m^2-(m-1)(m-2)(1-\alpha_m)^3}{(m-1)(m-2)(1-\alpha_m)^3} + \frac{6m^2 - 3m(1-\alpha_m)^2}{(m-1)(m-2)(1-\alpha_m)^4} + \frac{2(1-\alpha_m)^4 + 6(1-\alpha_m)^2 + 6(\alpha_m+1)}{(m-1)(m-2)(1-\alpha_m)^5}}. \quad (22)$$

Proof. We should calculate the sequences (a_m), (b_m), (c_m), and (d_m), in order to apply Theorem 7. In light of Lemma 1, clearly, we have

$$\begin{aligned} a_m &= \left\| \mathscr{D}_{m,\sigma}^{\alpha_m}(\sigma^0) - \sigma^0 \right\|_{\lambda^0} = 0, \\ b_m &= \left\| \mathscr{D}_{m,\sigma}^{\alpha_m}(\sigma) - \sigma \right\|_{\lambda^{1/2}} \le \frac{\alpha_m}{1-\alpha_m}, \\ c_m &= \left\| \mathscr{D}_{m,\sigma}^{\alpha_m}(\sigma^2) - \sigma^2 \right\|_{\lambda} \le \frac{m-(m-1)(1-\alpha_m)^2}{(m-1)(1-\alpha_m)^2} + \frac{\alpha_m^2 - 2\alpha_m + 3}{(m-1)(1-\alpha_m)^3}. \end{aligned} \quad (23)$$

Finally,

$$d_m = \left\| \mathscr{D}_{m,\sigma}^{\alpha_m}(\sigma^3) - \sigma^3 \right\|_{\lambda^{3/2}} \le \frac{m^2-(m-1)(m-2)(1-\alpha_m)^3}{(m-1)(m-2)(1-\alpha_m)^3} + \frac{6m - 3m(1-\alpha_m)^2}{(m-1)(m-2)(1-\alpha_m)^4} + \frac{2(1-\alpha_m)^4 + 6(1-\alpha_m)^2 + 6(\alpha_m+1)}{(m-1)(m-2)(1-\alpha_m)^5}. \quad (24)$$

Thus, Theorem 7 is satisfied. Hence, we have the desired result.

Remark 9. For $\lim_{\lambda\to 0}\omega_\sigma(f;\lambda) = 0$ in Theorem 8, we obtain

$$\lim_{m\to\infty} \left\| \mathscr{D}_{m,\sigma}^{\alpha_m}(f) - f \right\|_{\lambda^{3/2}} = 0, \quad \text{for } f \in U_\lambda[0,\infty). \quad (25)$$

5. Pointwise Convergence of $\mathscr{D}_{m,\sigma}^{\alpha}$

In this section, we shall analyze pointwise convergence of $\mathscr{D}_{m,\sigma}^{\alpha}$ by obtaining the Voronovskaya theorem in a quantitative form by using the same technique in [7].

Theorem 10. *Let $0 \le \alpha < 1$ such that $\lim_{m\to\infty}\alpha_m = 0$, and also, let $f \in \mathscr{C}_\lambda[0,\infty)$, $x \in [0,\infty)$ and suppose that $(fo\sigma^{-1})'$ and $(fo\sigma^{-1})''$ exist at $\sigma(x)$. If $(fo\sigma^{-1})''$ is bounded on $[0,\infty)$, then, we have*

$$\lim_{m\to\infty} m\left[\mathscr{D}_{m,\sigma}^{\alpha}(f;x) - f(x)\right] = \frac{3\sigma(x)\left(fo\sigma^{-1}\right)''\sigma(x)}{2}. \quad (26)$$

Proof. By using Taylor expansion of $(fo\sigma^{-1})$ at $\sigma(x) \in [0,\infty)$, we have

$$\begin{aligned} f(\vartheta) &= \left(fo\sigma^{-1}\right)(\sigma(\vartheta)) \\ &= \left(fo\sigma^{-1}\right)(\sigma(x)) + \left(fo\sigma^{-1}\right)'(\sigma(x)(\sigma(\vartheta) - \sigma(x))) \\ &\quad + \frac{\left(fo\sigma^{-1}\right)''(\sigma(x))(\sigma(\vartheta)-\sigma(x))^2}{2} \\ &\quad + \lambda_x(\vartheta)(\sigma(\vartheta)-\sigma(x))^2, \end{aligned} \quad (27)$$

where

$$\lambda_x(\vartheta) = \frac{\left(fo\sigma^{-1}\right)''(\sigma(\vartheta)) - \left(fo\sigma^{-1}\right)''(\sigma(x))}{2}. \quad (28)$$

Therefore, (28) together with the assumption on f ensures that

$$|\lambda_x(\vartheta)| \le \mathscr{K}, \quad \text{for all } \vartheta \in [0,\infty) \quad (29)$$

and is convergent to zero as $\vartheta \to x$. Now applying the operators (6) to the equality (27), we obtain

$$\begin{aligned} \left[\mathscr{D}_{m,\sigma}^{\alpha}(f;x) - f(x)\right] &= \left(fo\sigma^{-1}\right)'(\sigma(x))\mathscr{D}_{m,\sigma}^{\alpha}((\sigma(\vartheta)-\sigma(x));x) \\ &\quad + \frac{\left(fo\sigma^{-1}\right)''(\sigma(x))\mathscr{D}_{m,\sigma}^{\alpha}\left((\sigma(\vartheta)-\sigma(x))^2;x\right)}{2} \\ &\quad + \mathscr{D}_{m,\sigma}^{\alpha}\left(\lambda^x(\vartheta)\left((\sigma(\vartheta)-\sigma(x))^2;x\right)\right). \end{aligned} \quad (30)$$

From Lemma 2, we get

$$\lim_{m\to\infty} m\mathscr{D}_{m,\sigma}^{\alpha}((\sigma(\vartheta)-\sigma(x));x) = 0, \quad (31)$$

$$\lim_{m\to\infty} m\mathscr{D}_{m,\sigma}^{\alpha}\left((\sigma(\vartheta)-\sigma(x))^2;x\right) \le 3\sigma(x). \quad (32)$$

By estimating the equality (30), we will get the proof.

Since from (28), for every $\varepsilon > 0$, $\lim_{\vartheta\to x}\lambda_x(\vartheta) = 0$. Let $\delta > 0$ such that $|\lambda_x(\vartheta)| < \varepsilon$ for every $\vartheta \ge 0$. By Cauchy-Schwartz inequality, we get

$$\begin{aligned} &\lim_{m\to\infty} m\mathscr{D}_{m,\sigma}^{\alpha}\left(|\lambda_x(\vartheta)|(\sigma(\vartheta)-\sigma(x))^2;x\right) \\ &\quad \le \varepsilon \lim_{m\to\infty} m\mathscr{D}_{m,\sigma}^{\alpha}\left((\sigma(\vartheta)-\sigma(x))^2;x\right) \\ &\quad + \frac{\mathscr{K}}{\delta^2}\lim_{m\to\infty}\mathscr{D}_{m,\sigma}^{\alpha}\left((\sigma(\vartheta)-\sigma(x))^4;x\right). \end{aligned} \quad (33)$$

Since

$$\lim_{m\to\infty} m\mathscr{D}_{m,\sigma}^{\alpha}\left((\sigma(\vartheta)-\sigma(x))^4;x\right) = 0, \quad (34)$$

we obtain

$$\lim_{m\to\infty} m\mathcal{D}^{\alpha}_{m,\sigma}\left(|\lambda_x(\vartheta)|\,(\sigma(\vartheta)-\sigma(x))^2\,;x\right)=0. \tag{35}$$

Thus, by taking into account equations (31), (32), and (35) to equation (30), the proof is completed.

Remark 11. If we choose $\alpha_n = 1/((m+1)^a)$ with $a>1$, then, one can easily see that $\lim_{m\to\infty}\alpha_m = 0$ and $m\lim_{m\to\infty}\alpha_m = 0$.

6. Local Approximation

In this section, for the operators $\mathcal{D}^{\alpha}_{m,\sigma}$, we shall present local approximation theorems. Let $\mathscr{C}_B[0,\infty)$ denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $\mathscr{C}_B[0,\infty)$ is defined by

$$\|f\| = \sup_{0\le x<\infty}|f(x)|. \tag{36}$$

$\mathscr{K}$-functional is defined as

$$\mathscr{K}_2(f,\delta)=\inf_{s\in W^2}\left\{\|f-s\|+\delta\left\|f''\right\|\right\}, \tag{37}$$

where $\delta>0$ and $W^2=\{s\in\mathscr{C}_B[0,\infty): s', s''\in\mathscr{C}_B[0,\infty)\}$. By Devore and Lorentz ([27], p. 177, Theorem 6.4), there exists an absolute constant $\mathscr{C}>0$ such that

$$\mathscr{K}(f,\delta)\le\mathscr{C}\omega_2\left(f,\sqrt{\delta}\right). \tag{38}$$

The second order modulus of smoothness is as follows:

$$\omega_2\left(f,\sqrt{\delta}\right)=\sup_{0<h\le\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|, \tag{39}$$

where $f\in C_B[0,\infty)$. The usual modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$\omega(f,\delta)=\sup_{0<h\le\delta}\sup_{x\in[0,\infty)}|f(x+h)-f(x)|. \tag{40}$$

Theorem 12. *Let $0\le\alpha<1$ such that $\lim_{m\to\infty}\alpha_m=0$ and for all $f\in\mathscr{C}_B[0,\infty)$. Also, let σ be a function satisfying the conditions (σ_1) and (σ_2), and $\|\sigma''\|$ is finite. Then, there exists an absolute constant $\mathscr{C}>0$ such that*

$$\left|\mathcal{D}^{\alpha}_{m,\sigma}(f;x)-f(x)\right|\le\mathscr{C}\mathscr{K}(f,\delta_m(x)), \tag{41}$$

where

$$\delta_m(x)=\left(\frac{m}{(m-1)(1-\alpha)^2}-\frac{1+\alpha}{1-\alpha}\right)\sigma^2(x)+\frac{\alpha^2-2\alpha+3}{(m-1)(1-\alpha)^3}\sigma(x). \tag{42}$$

Proof. Let $s\in W^2$ and $x,\vartheta\in[0,\infty)$. By Taylor's formula, we have

$$s(\vartheta)=s(x)+\left(so\sigma^{-1}\right)'(\sigma(x))(\sigma(\vartheta)-\sigma(x))+\int_{\sigma(x)}^{\sigma(\vartheta)}(\sigma(\vartheta)-v)\left(so\sigma^{-1}\right)''(v)dv. \tag{43}$$

By using the equality,

$$\left(so\sigma^{-1}\right)''(\sigma(x))=\frac{s''(x)}{\left(\sigma'(x)\right)^2}-s''(x)\frac{\sigma''(x)}{\left(\sigma'(x)\right)^3}. \tag{44}$$

Now, putting $v=\sigma(x)$ in the last term in equality (43), we get

$$\begin{aligned}&\int_{\sigma(x)}^{\sigma(\vartheta)}(\sigma(\vartheta)-v)\left(so\sigma^{-1}\right)''(v)dv\\&=\int_x^{\vartheta}(\sigma(\vartheta)-\sigma(x))\left[\frac{s''(x)\sigma'(x)-s'(x)\sigma''(v)}{\left(\sigma'(x)\right)^2}\right]dx\\&=\int_{\sigma(x)}^{\sigma(\vartheta)}(\sigma(\vartheta)-v)\frac{s''\left(\sigma^{-1}(v)\right)}{\left(\sigma'\left(\sigma^{-1}(v)\right)\right)^2}dv\\&\quad-\int_{\sigma(x)}^{\sigma(\vartheta)}(\sigma(\vartheta)-v)\frac{s'\left(\sigma^{-1}(v)\right)\sigma''\left(\sigma^{-1}(v)\right)}{\left(\sigma'\left(\sigma^{-1}(v)\right)\right)^3}dv.\end{aligned} \tag{45}$$

By applying operator (6) to the both sides of equality (43) and from Lemma 1, we deduce

$$\begin{aligned}\mathcal{D}^{\alpha}_{m,\sigma}(s;x)&=s(x)\\&\quad+\mathcal{D}^{\alpha}_{m,\sigma}\left(\int_{\sigma(x)}^{\sigma(\vartheta)}(\sigma(\vartheta)-v)\frac{s''\left(\sigma^{-1}(v)\right)}{\left(\sigma'\left(\sigma^{-1}(v)\right)\right)^2}dv\,;x\right)\\&\quad-\mathcal{D}^{\alpha}_{m,\sigma}\left(\int_{\sigma(x)}^{\sigma(\vartheta)}(\sigma(\vartheta)-v)\frac{s'\left(\sigma^{-1}(v)\right)\sigma''\left(\sigma^{-1}(v)\right)}{\left(\sigma'\left(\sigma^{-1}(v)\right)\right)^3}dv\,;x\right).\end{aligned} \tag{46}$$

As we know, σ is strictly increasing on $[0,\infty)$, and with condition (σ_2), we get

$$\left|\mathcal{D}^{\alpha}_{m,\sigma}(s;x)-s(x)\right|\le\mathscr{M}^{\sigma}_{m,2}(x)\left(\left\|s''\right\|+\left\|s'\right\|\left\|\sigma''\right\|\right), \tag{47}$$

where

$$\mathscr{M}^{\sigma}_{m,2}(x)=\mathcal{D}^{\alpha}_{m,\sigma}\left((\sigma(\vartheta)-\sigma(x))^2\,;x\right). \tag{48}$$

Also,

$$|\mathscr{D}^{\alpha}{}_{m,\sigma}| \leq \|f\|. \tag{49}$$

Hence, we have

$$\begin{aligned} &\left|\mathscr{D}^{\alpha}_{m,\sigma}(f;x)-f(x)\right| \\ &\quad\leq \left|\mathscr{D}^{\alpha}_{m,\sigma}(f-s;x)\right| + \left|\mathscr{D}^{\alpha}_{m,\sigma}(s;x)-s(x)\right| + |s(x)-f(x)| \\ &\quad\leq 2\|f-s\| + \left\{\left(\frac{m}{(m-1)(1-\alpha)^2}-\frac{1+\alpha}{1-\alpha}\right)\sigma^2(x)\right. \\ &\quad\left. + \frac{\alpha^2-2\alpha+3}{(m-1)(1-\alpha)^3}\sigma(x)\right\}\left(\left\|s''\right\| + \left\|s'\right\|\left\|\sigma''\right\|\right), \end{aligned} \tag{50}$$

if $\mathscr{C} = \max\{2, \|\sigma''\|\}$; then

$$\begin{aligned} &\left|\mathscr{D}^{\alpha}_{m,\sigma}(f;x)-f(x)\right| \\ &\quad\leq \mathscr{C}\left(2\|f-s\| + \left\{\left(\frac{m}{(m-1)(1-\alpha)^2}-\frac{1+\alpha}{1-\alpha}\right)\sigma^2(x)\right.\right. \\ &\quad\left.\left. + \frac{\alpha^2-2\alpha+3}{(m-1)(1-\alpha)^3}\sigma(x)\right\}\|s''\|_{W^2}\right). \end{aligned} \tag{51}$$

Taking infimum over all $s \in W^2$, we obtain

$$\left|\mathscr{D}^{\alpha}_{m,\sigma}(f;x)-f(x)\right| \leq \mathscr{C}\mathscr{K}(f,\delta_m(x)). \tag{52}$$

Theorem 12 is proved.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

We declare that there is no conflict of interest.

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Research Article

Generalized (p, q)-Gamma-type operators

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In the present paper, the generalized (p, q)-gamma-type operators based on (p, q)-calculus are introduced. The moments and central moments are obtained, and some local approximation properties of these operators are investigated by means of modulus of continuity and Peetre $\mathscr{K}$-functional. Also, the rate of convergence, weighted approximation, and pointwise estimates of these operators are studied. Finally, a Voronovskaja-type theorem is presented.

1. Introduction

In [1], Mazhar introduced gamma operators preserving linear functions as follows:

$$\bar{G}_n(f\,;x) = \int_0^\infty g_n(x\,;u)du\int_0^\infty g_{n-1}(u\,;t)f(t)dt = \frac{(2n)!x^{n-1}}{n!(n-1)!}\int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}}f(t)\mathrm{d}t, \quad (1)$$

where $g_n(x\,;u) = (x^{n+1}/n!)e^{-xu}u^n$, $n > 1$, $x > 0$. In [2], Karsli considered new gamma operators preserving x^2 as follows:

$$\tilde{G}_n(f\,;x) = \int_0^\infty g_{n+2}(x\,;u)du\int_0^\infty g_n(u\,;t)f(t)dt = \frac{(2n+3)!x^{n+3}}{n!(n+2)!}\int_0^\infty \frac{t^n}{(x+t)^{2n+4}}f(t)dt, \quad (2)$$

where $x > 0$. In [3], Mao defined generalized gamma operators as follows:

$$G_{n,k}(f\,;x) = \int_0^\infty g_n(x\,;u)du\int_0^\infty g_{n-k}(u\,;t)f(t)dt = \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}}f(t)dt, \quad (3)$$

where $x > 0, n \geq k$. Obviously, $G_{n,1}(f\,;x) = \bar{G}_n(f\,;x)$ and $G_{n,2}(f\,;x) = \tilde{G}_{n-2}(f\,;x)$. In [4, 5], some approximation properties of operators ((1)–(3)) were discussed.

In Bernstein polynomials, some of their modifications and corresponding operators have been studied in many papers (see [6–10]). The q-analogue of well-known positive probability operators were widely studied and discussed (see books [11–13]) since Bernstein polynomials were proposed by Lupas [14] and Phillips [15]. In [16], the q-analogue of the operators (1) was defined and discussed. In [17], Cai and Zeng constructed and studied a q-analogue of the operators (2). Meantime, modifications and generalizations of the operators (2) were introduced and researched in [18–20]. In [21], Karsli constructed a q-analogue of the operators (3) and extended the works of [16, 17, 20]. Recently, many operators are constructed with two-parameter (p,q)-integers based on postquantum calculus ((p,q)-calculus) which have been used widely in many areas of sciences such as Lie group, different equations, hypergeometric series, and physical sciences. First, we recall some useful concepts and notations from (p,q)-calculus, which can be found in [11–13]. The (p,q)-integers $[m]_{p,q}$ are defined by

$$[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + \cdots + pq^{m-2} + q^{m-1} = \begin{cases} \dfrac{p^m - q^m}{p-q}, & p \neq q \neq 1, \\ mp^{m-1}, & p = q \neq 1, \\ [m]_q, & p = 1, \\ m, & p = q = 1. \end{cases} \quad (4)$$

By some simple calculation, for any $i, j \in \mathbb{N}$, we have the following relation:

$$q^i [j-i]_{p,q} = [j]_{p,q} - p^{j-i}[i]_{p,q}. \quad (5)$$

The (p,q)-factorial is defined by

$$[m]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \cdots [m]_{p,q}, & m \geq 1, \\ 1, & m = 0. \end{cases} \quad (6)$$

The (p,q)-power basis is defined by

$$\begin{aligned} (x \oplus t)^m_{p,q} &= (x+t)(px+qt)\left(p^2x + q^2t\right) \cdots \left(p^{m-1}x + q^{m-1}t\right), \\ (x!t)^m_{p,q} &= (x-t)(px-qt)\left(p^2x - q^2t\right) \cdots \left(p^{m-1}x - q^{m-1}t\right). \end{aligned} \quad (7)$$

Let f be an arbitrary function. The improper (p,q)-integral of f on $[0,\infty)$ is defined as (see [22])

$$\int_0^\infty f(x) d_{p,q}x = (p-q) \sum_{m=-\infty}^{\infty} \frac{q^m}{p^{m+1}} f\left(\frac{q^m}{p^{m+1}}\right), 0 < \frac{q}{p} < 1. \quad (8)$$

Let m be a nonnegative integer. The (p,q)-gamma function is defined as

$$\Gamma_{p,q}(m+1) = \frac{(p!q)^m_{p,q}}{(p-q)^m} = [m]_{p,q}!, 0 < q < p \leq 1. \quad (9)$$

Aral and Gupta [23] proposed a (p,q)-beta function of the second kind for $i, j \in \mathbb{N}_+ \coloneqq \{1, 2, \cdots\}$, as

$$B_{p,q}(i,j) = \int_0^\infty \frac{x^{i-1}}{(1 \oplus px)^{i+j}_{p,q}} d_{p,q}x, \quad (10)$$

and gave the relation of the (p,q)-analogues of beta and gamma functions:

$$B_{p,q}(i,j) = \frac{q\Gamma_{p,q}(i)\Gamma_{p,q}(j)}{\left(p^{i+1}q^{i-1}\right)^{i/2}\Gamma_{p,q}(i+j)}. \quad (11)$$

As a special case, if $p = q = 1$, $B(i,j) = \Gamma(i)\Gamma(j)/\Gamma(i+j)$. It is obvious that the order is important for the (p,q)-setting, which is the reason why the (p,q)-variant of beta function does not satisfy commutativity property, i.e., $B_{p,q}(i,j) \neq B_{p,q}(j,i)$.

Since Mursaleen et al. firstly introduced (p,q)-calculus in approximation theory and constructed the (p,q)-analogue of Bernstein operators [24] and (p,q)-Bernstein-Stancu operators [25], generalizations of many well-known approximation operators based on (p,q)-calculus were widely introduced and discussed by several authors (see (p,q)-Szász-Mirakjan operators [26], (p,q)-Baskakov-Durrmeyer-Stancu operators [27], (p,q)-Bernstein-Stancu-Schurer-Kantorovich operators [28], (p,q)-Baskakov-beta operators [29], (p,q)-Lorentz polynomials [30], (p,q)-Szász-Mirakjan Kantorovich operators [31], (p,q)-Bleimann-Butzer-Hahn operators [32], (p,q)-Bernstein operators [33, 34], and so on). In [35], Cheng and Zhang constructed a (p,q)-analogue of the operators (1) using the (p,q)-beta function of the second kind and studied their approximation properties. Later, Cheng et al. defined the (p,q)-analogue of the operators (2) and researched their approximation properties in [36]. All these achievements motivate us to construct the (p,q)-analogue of the gamma operator (3) and generalize the works of [35, 36]. Now, we construct generalized (p,q)-gamma-type operators as follows:

Definition 1. Let $k \in \mathbb{N}_+$, $n = k, k+1, \cdots$, $x \in (0,\infty)$, and $0 < q < p \leq 1$. *For* $f \in C(0,\infty)$, then the (p,q)-analogue of the gamma operator (3) can be defined by

$$G^{p,q}_{n,k}(f;x) = \frac{x^{n+1}p^{n^2-((k-1)n/2)+((k-1)/2)}q^{n^2+((3-k)n/2)-((k-1)/2)}}{B_{p,q}(n-k+1,n+1)} \cdot \int_0^\infty \frac{t^{n-k}}{\left((pq)^{n+((1-k)/2)}x \oplus t\right)^{2n-k+2}_{p,q}} f(t) d_{p,q}t. \quad (12)$$

In the case $p = q = 1$, $k = 1$, we obtain the operators (1); in the case $p = q = 1$, $k = 2$, we obtain the operators (2); in the case $p = q = 1$, we obtain the operators (3); in the case $p = 1$, $k = 1$, we obtain the operators [16]; in the case $p = 1$, $k = 2$, we obtain the operators [17]; in the case $p = 1$, we obtain the operators [21]; in the case $k = 1$, we obtain the operators [35]; and in the case $k = 2$, we obtain the operators [36].

The paper is organized as follows: In Section 1, we introduce the history of gamma-type operators and recall some basic notations about (p,q)-calculus; then, we construct the generalized (p,q)-gamma operators with the (p,q)-beta function. In Section 2, we obtain the auxiliary lemmas and corollaries about the moment computation formulas. And the second- and fourth-order central moment computation formula and limit equalities are also obtained. In Section 3, we discuss the local approximation about the operators by means of modulus of continuity and Peetre $\mathscr{K}$-functional. In Sections 4 and 5, the rate of convergence and weighted approximation for these operators are researched. In Section 6, two pointwise estimates are given by using the Lipschitz-type maximal function. In Section 7, the Voronovskaja-type asymptotic formula is presented.

2. Moment Estimates

In order to obtain the approximation properties of the operators $G_{n,k}^{p,q}(f;x)$, we need the following lemmas and corollaries.

Lemma 2. *For $x\in(0,\infty)$, $0<q<p\le 1$, $k\in\mathbb{N}_+$, $n=k,k+1,\cdots$, and $m=0,1,2,\cdots,n$, we have*

$$G_{n,k}^{p,q}(t^m,x)=(pq)^{\frac{m(k-m)}{2}}\frac{[n-k+m]_{p,q}![n-m]_{p,q}!}{[n-k]_{p,q}![n]_{p,q}!}x^m. \quad (13)$$

Proof. Set $C=n^2-((k-1)n/2)+(k-1)/2$, $D=n^2+((3-k)n/2)-((k-1)/2)$, and $A=B=n+((1-k)/2)$, we have

$$\begin{aligned}G_{n,k}^{p,q}(t^m;x)&=\frac{x^{n+1}p^Cq^D}{B_{p,q}(n-k+1,n+1)}\cdot\int_0^\infty\frac{t^{n-k}}{(p^Aq^Bx\oplus t)_{p,q}^{2n-k+2}}t^m d_{p,q}t\\&=\frac{x^{n+1}p^Cq^D}{B_{p,q}(n-k+1,n+1)}\frac{1}{(p^Aq^Bx)^{2n-k+2}}\cdot\int_0^\infty\frac{t^{n-k+m}}{(1\oplus(pt/p^{1+A}q^Bx))_{p,q}^{2n-k+2}}d_{p,q}t\\&=\frac{x^mp^Cq^D(p^{1+A}q^B)^{n-k+m+1}}{B_{p,q}(n-k+1,n+1)(p^Aq^B)^{2n-k+2}}\cdot\int_0^\infty\frac{(t/p^{1+A}q^Bx)^{n-k+m}}{(1\oplus(pt/p^{1+A}q^Bx))_{p,q}^{2n-k+2}}d_{p,q}\left(\frac{t}{p^{1+A}q^Bx}\right)\\&=\frac{x^mp^Cq^D(p^{1+A}q^B)^{n-k+m+1}}{(p^Aq^B)^{2n-k+2}}\frac{B_{p,q}(n-k+m+1,n-m+1)}{B_{p,q}(n-k+1,n+1)}\\&=\frac{x^mp^Cq^D(p^{1+A}q^B)^{n-k+m+1}}{(p^Aq^B)^{2n-k+2}}\frac{(p^{n-k+2}q^{n-k})^{n-k+1/2}}{(p^{n-k+m+2}q^{n-k+m})^{n-k+m+1/2}}\cdot\frac{[n-k+m]_{p,q}![n-m]_{p,q}!}{[n-k]_{p,q}![n]_{p,q}!}\\&=(pq)^{m(k-m)/2}\frac{[n-k+m]_{p,q}![n-m]_{p,q}!}{[n-k]_{p,q}![n]_{p,q}!}x^m.\end{aligned} \quad (14)$$

Then, the following corollary can be obtained immediately.

Corollary 3. *For $x\in(0,\infty)$, $0<q<p\le 1$, and $n\ge\max\{k,4\}$, $k\in\mathbb{N}$, the following equalities hold:*

$$\begin{aligned}&G_{n,k}^{p,q}(1;x)=1,\ G_{n,k}^{p,q}(t;x)=(pq)^{(k-1)/2}\frac{[n-k+1]_{p,q}}{[n]_{p,q}}x,\\&G_{n,k}^{p,q}(t^2;x)=(pq)^{k-2}\frac{[n-k+1]_{p,q}[n-k+2]_{p,q}}{[n]_{p,q}[n-1]_{p,q}}x^2,\\&G_{n,k}^{p,q}(t^3;x)=(pq)^{3(k-3)/2}\frac{[n-k+1]_{p,q}[n-k+2]_{p,q}[n-k+3]_{p,q}}{[n]_{p,q}[n-1]_{p,q}[n-2]_{p,q}}x^3,\\&G_{n,k}^{p,q}(t^4;x)=(pq)^{2(k-4)}\frac{[n-k+1]_{p,q}[n-k+2]_{p,q}[n-k+3]_{p,q}[n-k+4]_{p,q}}{[n]_{p,q}[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^4.\end{aligned} \quad (15)$$

Corollary 4. *For $x\in(0,\infty)$, $0<q<p\le 1$, and $n\ge\max\{k,4\}$, using Corollary 3, we can easily obtain the following explicit formulas for the first and second central moments:*

$$\begin{aligned}A_{n,k}^{p,q}(x)&:=G_{n,k}^{p,q}(t-x;x)=\left((pq)^{(k-1)/2}\frac{[n-k+1]_{p,q}}{[n]_{p,q}}-1\right)x,\\B_{n,k}^{p,q}(x)&:=G_{n,k}^{p,q}((t-x)^2;x)\\&=\left((pq)^{k-2}\frac{[n-k+1]_{p,q}[n-k+2]_{p,q}}{[n]_{p,q}[n-1]_{p,q}}-2(pq)^{(k-1)/2}\frac{[n-k+1]_{p,q}}{[n]_{p,q}}+1\right)x^2.\end{aligned} \quad (16)$$

Corollary 5. *The sequences $(p_n),(q_n)$ satisfy $0<q_n<p_n\le 1$ such that $p_n\longrightarrow 1$, $q_n\longrightarrow 1$ and $p_n^n\longrightarrow\alpha\in[0,1]$, $q_n^n\longrightarrow\beta\in[0,1]$, $[n]_{p_n,q_n}\longrightarrow\infty$ as $n\longrightarrow\infty$; then, for any $x\in(0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}A_{n,k}^{p_n,q_n}(x)=-\frac{k-1}{2}(\alpha+\beta)x, \quad (17)$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}B_{n,k}^{p_n,q_n}(x)=(\alpha+\beta)x^2, \quad (18)$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}G_{n,k}^{p_n,q_n}((t-x)^4;x))=0. \quad (19)$$

Proof. Using (5), we have $[n-k+1]_{p_n,q_n}=q_n^{1-k}[n]_{p_n,q_n}-p_n^n(p_nq_n)^{1-k}[k-1]_{p_n,q_n}$. Hence,

$$\begin{aligned}&\lim_{n\to\infty}[n]_{p_n,q_n}A_{n,k}^{p_n,q_n}(x)\\&=\lim_{n\to\infty}[n]_{p_n,q_n}\left((p_nq_n)^{(k-1)/2}\frac{[n-k+1]_{p_n,q_n}}{[n]_{p_n,q_n}}-1\right)\\&\cdot x=\lim_{n\to\infty}[n]_{p_n,q_n}\left(\left(\frac{p_n}{q_n}\right)^{(k-1)/2}-1-\frac{p_n^n(p_nq_n)^{(1-k)/2}[k-1]_{p_n,q_n}}{[n]_{p_n,q_n}}\right)\\&\cdot x=\lim_{n\to\infty}[n]_{p_n,q_n}\left(p_n^{(k-1)/2}-q_n^{(k-1)/2}\right)x-(k-1)\alpha x\\&=\lim_{n\to\infty}\frac{p_n^n-q_n^n}{p_n-q_n}\frac{p_n^{k-1}-q_n^{k-1}}{p_n^{(k-1)/2}x+q_n^{(k-1)/2}}-(k-1)\\&\cdot\alpha x=\frac{1}{2}\lim_{n\to\infty}(p_n^n-q_n^n)[k-1]_{p_n,q_n}-(k-1)\alpha x=-\frac{k-1}{2}(\alpha+\beta)x.\end{aligned} \quad (20)$$

Using $[n-k+1]_{p_n,q_n}/[n]_{p_n,q_n} = q_n^{1-k} - (p_n^n (p_n q_n)^{1-k} [k-1]_{p_n,q_n}/[n]_{p_n,q_n})$ and $[n-k+2]_{p_n,q_n}/[n-1]_{p_n,q_n} = q_n^{3-k} - (p_n^{n-1}(p_n q_n)^{3-k}[k-3]_{p_n,q_n}/[n-1]_{p_n,q_n})$, we have

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n} G_{n,k}^{p_n,q_n}(t(t-x);x) \\
&= \lim_{n\to\infty}[n]_{p_n,q_n}\left((p_n q_n)^{k-2}\left(q_n^{1-k} - \frac{p_n^n (p_n q_n)^{1-k}[k-1]_{p_n,q_n}}{[n]_{p_n,q_n}}\right)\right. \\
&\quad \times \left(q_n^{3-k} - \frac{p_n^{n-1}(p_n q_n)^{3-k}[k-3]_{p_n,q_n}}{[n-1]_{p_n,q_n}}\right) \\
&\quad \left. - \left(\left(\frac{p_n}{q_n}\right)^{(k-1)/2} - \frac{p_n^n (p_n q_n)^{(1-k)/2}[k-1]_{p_n,q_n}}{[n]_{p_n,q_n}}\right)\right) \\
&\quad \times x^2 = \lim_{n\to\infty}[n]_{p_n,q_n}\left(\left(\frac{p_n}{q_n}\right)^{k-2} - \left(\frac{p_n}{q_n}\right)^{(k-1)/2}\right) \\
&\quad \times x^2 - (k-3)\alpha x^2 = \lim_{n\to\infty}[n]_{p_n,q_n}\left(p_n^{(k-3)/2} - q_n^{(k-3)/2}\right) \\
&\quad - (k-3)\alpha x^2 = \lim_{n\to\infty}\left[\frac{k-3}{2}\right]_{p_n,q_n}(p_n^n - q_n^n)x^2 - (k-3)\alpha x^2 \\
&= -\frac{k-3}{2}(\alpha+\beta)x^2.
\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}[n]_{p_n,q_n} B_{n,k}^{p_n,q_n}(x) &= \lim_{n\to\infty}[n]_{p_n,q_n} G_{n,k}^{p_n,q_n}((t-x)t;x) \\
&\quad - x\lim_{n\to\infty}[n]_{p_n,q_n} G_{n,k}^{p_n,q_n}((t-x);x) = (\alpha+\beta)x^2.
\end{aligned} \tag{22}$$

Now, we prove the limit equality (19) while $k \geq 4$, $k = 1, 2, 3$ is similar. Set $r_n = p_n/q_n$ and $i \leq k$, by (5); we can obtain

$$\begin{aligned}
&(p_n q_n)^{(k-i)/2}\frac{[n-k+i]_{p_n,q_n}}{[n]_{p_n,q_n}} \\
&= r_n^{(k-i)/2}\frac{q_n^{k-i}[n-(k-i)]_{p_n,q_n}}{[n]_{p_n,q_n}} \\
&= r_n^{(k-i)/2}\frac{[n]_{p_n,q_n} - p_n^{n-(k-i)}[k-i]_{p_n,q_n}}{[n]_{p_n,q_n}} \\
&= r_n^{(k-i)/2}\left(1 - \frac{p_n^{n-(k-i)}[k-i]_{p_n,q_n}}{[n]_{p_n,q_n}}\right) \\
&= r_n^{(k-i)/2} - (p_n q_n)^{-((k-i)/2)} p_n^n \frac{[k-i]_{p_n,q_n}}{[n]_{p_n,q_n}}.
\end{aligned} \tag{23}$$

Hence, we can rewrite

$$\begin{aligned}
G_{n,k}^{p_n,q_n}(t;x) &= \left(r_n^{(k-i)/2} - (p_n q_n)^{-((k-i)/2)} p_n^n \frac{[k-1]_{p_n,q_n}}{[n]_{p_n,q_n}}\right) \\
&\quad \cdot x = \left(r_n^{(k-i)/2} - \frac{(k-1)\alpha}{[n]_{p_n,q_n}} + o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x, \\
G_{n,k}^{p_n,q_n}(t^2;x) &= \left(\prod_{l=0}^{1}\left(r_n^{(k-2)/2} - (p_n q_n)^{-((k-2)/2)} p_n^{n-l}\frac{[k-2]_{p_n,q_n}}{[n-l]_{p_n,q_n}}\right)\right) \\
&\quad \cdot x^2 = \left(r_n^{k-2} - \frac{2(k-2)\alpha}{[n]_{p_n,q_n}} + o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^2, \\
G_{n,k}^{p_n,q_n}(t^3;x) &= \left(\prod_{l=0}^{2}\left(r_n^{(k-3)/2} - (p_n q_n)^{-((k-3)/2)} p_n^{n-l}\frac{[k-3]_{p_n,q_n}}{[n-l]_{p_n,q_n}}\right)\right) \\
&\quad \cdot x^3 = \left(r_n^{3(k-3)/2} - \frac{3(k-3)\alpha}{[n]_{p_n,q_n}} + o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^3, \\
G_{n,k}^{p_n,q_n}(t^4;x) &= \left(\prod_{l=0}^{3}\left(r_n^{(k-4)/2} - (p_n q_n)^{-((k-4)/2)} p_n^{n-l}\frac{[k-4]_{p_n,q_n}}{[n-l]_{p_n,q_n}}\right)\right) \\
&\quad \cdot x^4 = \left(r_n^{2(k-4)} - \frac{4(k-4)\alpha}{[n]_{p_n,q_n}} + o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^4.
\end{aligned} \tag{24}$$

Further, we can easily get

$$\begin{aligned}
G_{n,k}^{p_n,q_n}((t-x)^4;x) &= \left(r_n^{2(k-4)} - 4r_n^{3(k-3)/2} + 6r_n^{k-2}\right. \\
&\quad \left. - 4r_n^{(k-1)/2} + 1 + o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^4.
\end{aligned} \tag{25}$$

For any $l \in \mathbb{N}_+$, $[n]_{p_n,q_n}(r_n^l - 1) \sim [n]_{p_n,q_n}(p_n^l - q_n^l) \sim (\alpha - \beta)l$. We can obtain

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n} G_{n,k}^{p_n,q_n}((t-x)^4;x)) \\
&= \lim_{n\to\infty}[n]_{p_n,q_n}\left(r_n^{2(k-4)} - 4r_n^{3(k-3)/2} + 6r_n^{k-2}\right. \\
&\quad \left. - 4r_n^{(k-1)/2} + 1 + o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^4 \\
&= \lim_{n\to\infty}[n]_{p_n,q_n}\left(\left(r_n^{2(k-4)} - 1\right) - 4\left(r_n^{3(k-3)/2} - 1\right)\right. \\
&\quad \left. + 6\left(r_n^{k-2} - 1\right) - 4\left(r_n^{(k-1)/2} - 1\right)\right)x^4 \\
&= (\alpha-\beta)[2(k-4) - 6(k-3) + 6(k-2) - 2(k-1)]x^4 = 0,
\end{aligned} \tag{26}$$

we obtain the required result.

Corollary 6. *Let us denote the norm $\|f\| = \sup_{x\in(0,\infty)}|f(x)|$ on $C_B(0,\infty)$ (the class of real valued continuous bounded functions on $(0,\infty)$). For any $f \in C_B(0,\infty)$, we have*

$$\left\|G_{n,k}^{p,q}(f;x)\right\| \leq \|f\|. \tag{27}$$

Proof. In view of (12) and Corollary 3, the proof of this corollary can be obtained easily.

3. Local Approximation

For any $f \in C_B(0,\infty)$, let us consider the following $\mathscr{K}$-functional:

$$\mathscr{K}(f;\delta) = \inf_{g\in S^2}\left\{\|f-g\| + \delta\left\|g''\right\|\right\}, \tag{28}$$

where $\delta \in (0,\infty)$ and $S^2 = \{g \in C_B(0,\infty): g', g'' \in C_B(0,\infty)\}$. The usual modulus of continuity and the second-order modulus of smoothness of f can be defined by

$$\begin{aligned}\omega(f;\delta) &= \sup_{0<|t|\le\delta}\sup_{x\in(0,\infty)}|f(x+t)-f(x)|,\\ \omega_2(f;\delta) &= \sup_{0<|t|\le\delta}\sup_{x\in(0,\infty)}|f(x+2t)-2f(x+t)+f(x)|.\end{aligned} \tag{29}$$

By [37] (p.177, Theorem 2.4), there exists an absolute constant $C>0$ such that

$$\mathscr{K}\left(f;\delta^2\right) \le C\omega_2(f;\delta), \delta>0. \tag{30}$$

Theorem 7. *Let $(p_n), (q_n)$ be the sequences defined in Corollary 5 and $f \in C_B(0,\infty)$. Then, for all $k = 1, 2, 3, \cdots, n \ge \max\{k, 2\}$, there exists an absolute positive $C_1 = 4C$ such that*

$$\begin{aligned}\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right| &\le C_1\omega_2\left(f;\sqrt{\left(A_{n,k}^{p_n,q_n}(x)\right)^2 + B_{n,k}^{p_n,q_n}(x)}\right)\\ &\quad + \omega\left(f;\left|A_{n,k}^{p_n,q_n}(x)\right|\right).\end{aligned} \tag{31}$$

Proof. Define the following new operators:

$$\mathbf{G}_{n,k}^{p_n,q_n}(f;x) = G_{n,k}^{p_n,q_n}(f;x) - f\left(A_{n,k}^{p_n,q_n}(x)+x\right) + f(x), \quad x\in(0,\infty). \tag{32}$$

Let $x, t \in (0,\infty)$ and $g \in S^2$. By Taylor's expansion formula, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)du. \tag{33}$$

Applying $\mathbf{G}_{n,k}^{p_n,q_n}$ to the above equality and $\mathbf{G}_{n,k}^{p_n,q_n}(t-x;x) = 0$, we can obtain

$$\begin{aligned}\mathbf{G}_{n,k}^{p_n,q_n}(g;x) - g(x) &= \mathbf{G}_{n,k}^{p_n,q_n}\left(\int_x^t g''(u)(t-u)du;x\right)\\ &\le G_{n,k}^{p_n,q_n}\left(\left|\int_x^t |t-u|\, g''(u)du\right|;x\right)\\ &\quad + \left|\int_x^{A_{n,k}^{p_n,q_n}(x)+x} g''(u)\left(A_{n,k}^{p_n,q_n}(x)+x-u\right)du\right|\\ &\le G_{n,k}^{p_n,q_n}\left((t-x)^2;x\right)\left\|g''\right\| + \left(A_{n,k}^{p_n,q_n}(x)\right)^2\left\|g''\right\|\\ &= \left(A_{n,k}^{p_n,q_n}(x) + B_{n,k}^{p_n,q_n}(x)\right)^2\left\|g''\right\|.\end{aligned} \tag{34}$$

By Corollary 6 and (32), we easily know $|\mathbf{G}_{n,k}^{p_n,q_n}(f;x)| \le 3\|f\|$. Hence,

$$\begin{aligned}\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right| &= \left|\mathbf{G}_{n,k}^{p_n,q_n}(f;x) + f\left(A_{n,k}^{p_n,q_n}(x)+x\right)\right.\\ &\quad \left.- 2f(x)\right| \le \left|\mathbf{G}_{n,k}^{p_n,q_n}(f-g;x) - (f-g)(x)\right| + \left|\mathbf{G}_{n,k}^{p_n,q_n}(g;x)\right.\\ &\quad \left.- g(x)\right| + \left|f\left(A_{n,k}^{p_n,q_n}(x)+x\right) - f(x)\right| \le 4\|f-g\|\\ &\quad + \left(\left(A_{n,k}^{p_n,q_n}(x) + B_{n,k}^{p_n,q_n}(x)\right)^2 + \right)\left\|g''\right\| + \omega\left(f;\left|A_{n,k}^{p_n,q_n}(x)\right|\right).\end{aligned} \tag{35}$$

Taking the infimum on the right-hand side over all $g \in S^2$ and using (30), we obtain the desired assertion.

Corollary 8. *Let $(p_n), (q_n)$ be the sequences defined in Corollary 5 and $f \in C_B(0,\infty)$. Then, for any finite interval $I \subset (0,\infty)$, the sequence $\{G_{n,k}^{p_n,q_n}(f;x)\}$ converges to f uniformly on I.*

4. Rate of Convergence

Let

$$\begin{aligned}B_w(0,\infty) &= \left\{f : |f(x)| \le M_f w(x)\right\},\\ C_w(0,\infty) &= \{f : f\in B_w(0,\infty)\cap C(0,\infty)\},\\ C_w^0(0,\infty) &= \left\{f : f\in C_w(0,\infty), \lim_{x\to\infty}\frac{|f(x)|}{w(x)}<\infty\right\},\end{aligned} \tag{36}$$

where $w(x)$ is the weighted function given by $w(x) = 1 + x^2$ and M_f is an absolute constant depending only on f. $C_w^0(0,\infty)$ is equipped with the norm $\|f\|_w = \sup_{x\in(0,\infty)}|f(x)|/w(x)$. As is known, if $f \in C(0,\infty)$ is not uniform, we cannot obtain $\lim_{\delta\to0^+}\omega(f;\delta) = 0$. In [38], Ispir defined the following weighted modulus of continuity:

$$\Omega(f;\delta) = \sup_{0<t\le\delta, x\in(0,\infty)}\frac{|f(x+t)-f(x)|}{w(x)w(t)}, \tag{37}$$

and proved the properties of monotone increasing about Ω

$(f;\delta)$ as $\delta>0$, $\lim_{\delta\to 0^+}\Omega(f;\delta)=0$, and the inequality

$$\Omega(f;\lambda\delta)\le 2(1+\lambda)\left(1+\delta^2\right)\Omega(f;\delta), \tag{38}$$

while $\lambda>0$ and $f\in C_w^0(0,\infty)$. Meantime, we recall the modulus of continuity of f on the interval $(0,\kappa]\subset(0,\infty)$ by

$$\omega_\kappa(f;\delta)=\sup_{x,t\in(0,\kappa],|x-t|\le\delta}|f(t)-f(x)|, \quad \delta>0. \tag{39}$$

Theorem 9. *Let $f\in C_w(0,\infty)$, $0<q<p\le 1$, and $\kappa>0$, we have*

$$\begin{aligned}\left\|G_{n,k}^{p,q}(f;x)-f(x)\right\|_{C(0,\kappa)}&\le 4M_f\left(3+2\kappa^2\right)B_{n,k}^{p,q}(\kappa)\\&+2\omega_{\kappa+1}\left(f;\sqrt{B_{n,k}^{p,q}(\kappa)}\right).\end{aligned} \tag{40}$$

Proof. For any $x\in(0,\kappa)$ and $t>\kappa+1$, we easily have $1\le(t-\kappa)^2\le(t-x)^2$; thus,

$$\begin{aligned}|f(t)-f(x)|&\le|f(t)|+|f(x)|\le M_f\left(2+t^2+x^2\right)\\&=M_f\left(2+x^2+(t-x+x)^2\right)\le M_f\left(2+2x^2+(t-x)^2\right)\\&\le M_f\left(3+2x^2\right)(t-x)^2\le M_f\left(3+2\kappa^2\right)(t-x)^2,\end{aligned} \tag{41}$$

and for any $x\in(0,\kappa)$, $t\in(0,\kappa+1)$, and $\delta>0$, we have

$$|f(t)-f(x)|\le\omega_{\kappa+1}(|t-x|;x)\le\left(1+\frac{|t-x|}{\delta}\right)\omega_{\kappa+1}(f;\delta). \tag{42}$$

For (41) and (42), we can get

$$|f(t)-f(x)|\le M_f\left(3+2\kappa^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{\kappa+1}(f;\delta). \tag{43}$$

By Schwarz's inequality, for any $x\in(0,\kappa)$, we can get

$$\begin{aligned}\left|G_{n,k}^{p,q}(f;x)-f(x)\right|&\le G_{n,k}^{p,q}(|f(t)-f(x)|;x)\\&\le M_f\left(3+2\kappa^2\right)G_{n,k}^{p,q}\left((t-x)^2;x\right)\\&+G_{n,k}^{p,q}\left(\left(1+\frac{|t-x|}{\delta}\right);x\right)\omega_{\kappa+1}(f;\delta)\\&\le M_f\left(3+2\kappa^2\right)G_{n,k}^{p,q}\left((t-x)^2;x\right)\\&+\omega_{\kappa+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{G_{n,k}^{p,q}\left((t-x)^2;x\right)}\right)\\&\le M_f\left(3+2\kappa^2\right)B_{n,k}^{p,q}+\omega_{\kappa+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{B_{n,k}^{p,q}(x)}\right)\\&\le M_f\left(3+2\kappa^2\right)B_{n,k}^{p,q}(\kappa)+\omega_{\kappa+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{B_{n,k}^{p,q}(\kappa)}\right).\end{aligned} \tag{44}$$

By taking $\delta=\sqrt{B_{n,k}^{p,q}(\kappa)}$ and the supremum over all $x\in(0,\kappa)$, we accomplish the proof of Theorem 9.

5. Weighted Approximation

In this section, we will discuss the following three theorems about weighted approximation for the operators $G_{n,k}^{p_n,q_n}(f;x)$:

Theorem 10. *Let $f\in C_w^0(0,\infty)$ and the sequences (p_n), (q_n) satisfy $0<q_n<p_n\le 1$ such that $p_n\longrightarrow 1$, $q_n\longrightarrow 1$, $p_n^n\longrightarrow\alpha\in[0,1]$, $q_n^n\longrightarrow\beta\in[0,1]$, $[n]_{p_n,q_n}\longrightarrow\infty$ as $n\longrightarrow\infty$; then, there exists $N\in\mathbb{N}_+$ such that for all $n>N$ and $\rho>0$, the inequality*

$$\sup_{x\in(0,\infty)}\frac{\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{(5/2)+\rho}}\le 8\left(2+\sqrt{2}\right)\Omega\left(f;\frac{1}{\sqrt{[n]_{p_n,q_n}}}\right) \tag{45}$$

holds.

Proof. Using (37) and (38), we can write

$$\begin{aligned}|f(t)-f(x)|&\le\left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;|t-x|)\\&\le 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1+(t-x)^2\right)\left(1+x^2\right)\\&\le\begin{cases}4\left(1+\delta^2\right)^2\left(1+x^2\right)\Omega(f;\delta), & |t-x|\le\delta,\\ 4\left(1+\delta^2\right)\left(1+x^2\right)\Omega(f;\delta)\dfrac{|t-x|+|t-x|^3}{\delta}, & |t-x|>\delta.\end{cases}\end{aligned} \tag{46}$$

For any $\delta\in(0,1)$ and $x,t\in(0,\infty)$, (46) can be rewritten:

$$|f(t)-f(x)|\le 8\left(1+x^2\right)\Omega(f;\delta)\left(2+\frac{|t-x|+|t-x|^3}{\delta}\right). \tag{47}$$

Using (18) and (19), there exists $N\in\mathbb{N}_+$ such that for any $n>N$,

$$\begin{aligned}G_{n,k}^{p_n,q_n}\left((t-x)^2;x\right)&\le\frac{2}{[n]_{p_n,q_n}}x^2,\\G_{n,k}^{p_n,q_n}\left((t-x)^4;x\right)&\le x^4.\end{aligned} \tag{48}$$

By Schwarz's inequality, we can obtain

$$G_{n,k}^{p_n,q_n}(|t-x|;x)\le\sqrt{G_{n,k}^{p_n,q_n}\left((t-x)^2;x\right)}\le\sqrt{2}\frac{x}{\sqrt{[n]_{p_n,q_n}}}, \tag{49}$$

$$G_{n,k}^{p_n,q_n}\left(|t-x|^3;x\right) \le \sqrt{G_{n,k}^{p_n,q_n}\left((t-x)^2;x\right)} \times \sqrt{G_{n,k}^{p_n,q_n}\left((t-x)^4;x\right)} \le \sqrt{2}\frac{x^3}{\sqrt{[n]_{p_n,q_n}}}. \quad (50)$$

Since $G_{n,k}^{p_n,q_n}$ is linear and positive, using (47), (49), and (50), we can obtain

$$\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right| \le 8\left(1+x^2\right)\Omega(f;\delta) \cdot \left(2+\frac{G_{n,k}^{p_n,q_n}\left(|t-x|+|t-x|^3;x\right)}{\delta}\right) \le 8\left(1+x^2\right)\left(2+\frac{\sqrt{2}\left(x+x^3\right)}{\delta\sqrt{[n]_{p_n,q_n}}}\right)\Omega(f;\delta). \quad (51)$$

By choosing $\delta = 1/\sqrt{[n]_{p_n,q_n}}$, the conclusion holds.

Theorem 11. *Let $(p_n),(q_n)$ be the sequences defined in Theorem 10. Then, for any $f \in C_w^0(0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|G_{n,k}^{p_n,q_n}(f;x)-f\right\|_w = 0. \quad (52)$$

Proof. By the weighted Korovkin theorem in [39], we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|G_{n,k}^{p_n,q_n}\left(t^k;x\right)-x^k\right\|_w = 0, \quad k=0,1,2. \quad (53)$$

Since $G_{n,k}^{p_n,q_n}(1;x)=1$, then (53) holds true for $k=0$. By Corollary 3, we can obtain

$$\begin{aligned}\left\|G_{n,k}^{p_n,q_n}(t;x)-x\right\|_w &= \sup_{x\in(0,\infty)}\frac{\left|G_{n,k}^{p_n,q_n}(t;x)-x\right|}{1+x^2} \\ &\le \left|\left(p_nq_n\right)^{(k-1)/2}\frac{[n-k+1]_{p_n,q_n}}{[n]_{p_n,q_n}}-1\right|\sup_{x\in(0,\infty)}\frac{x}{1+x^2} \\ &= \frac{1}{2}\left|\left(p_nq_n\right)^{(k-1)/2}\frac{[n-k+1]_{p_n,q_n}}{[n]_{p_n,q_n}}-1\right| \to 0, \quad n\to\infty, \\ \left\|G_{n,k}^{p_n,q_n}\left(t^2;x\right)-x^2\right\|_w &= \sup_{x\in(0,\infty)}\frac{\left|G_{n,k}^{p_n,q_n}\left(t^2;x\right)-x^2\right|}{1+x^2} \\ &\le \sup_{x\in(0,\infty)}\frac{x^2}{1+x^2}\left|\left(p_nq_n\right)^{k-2}\frac{[n-k+1]_{p_n,q_n}[n-k+2]_{p_n,q_n}}{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}-1\right| \\ &= \left|\left(p_nq_n\right)^{k-2}\frac{[n-k+1]_{p_n,q_n}[n-k+2]_{p_n,q_n}}{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}-1\right| \to 0, \quad n\to\infty.\end{aligned} \quad (54)$$

Thus, the proof of Theorem 11 is completed.

Theorem 12. *Let $(p_n),(q_n)$ be the sequences defined in Theorem 10. Then, for any $f \in C_w^0(0,\infty)$ and $\lambda > 0$, we have*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}_+}\frac{\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\lambda}} = 0. \quad (55)$$

Proof. Let $x_0 \in (0,\infty)$ be arbitrary but fixed. Then,

$$\begin{aligned}&\sup_{x\in(0,\infty)}\frac{\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\lambda}} \\ &\le \sup_{x\in(0,x_0]}\frac{\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\lambda}} + \sup_{x\in(x_0,\infty)}\frac{\left|G_{n,k}^{p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\lambda}} \\ &\le \left\|G_{n,k}^{p_n,q_n}(f;x)-f\right\|_{C(0,x_0]} + \|f\|_w\sup_{x\in(x_0,\infty)}\frac{\left|G_{n,k}^{p_n,q_n}\left(1+t^2;x\right)\right|}{\left(1+x^2\right)^{1+\lambda}} \\ &\quad + \sup_{x\in(x_0,\infty)}\frac{|f(x)|}{\left(1+x^2\right)^{1+\lambda}}.\end{aligned} \quad (56)$$

Since $|f(x)| \le M_f(1+x^2)$, we have $\sup_{x\in(x_0,\infty)}(|f(x)|/(1+x^2)^{1+\lambda}) \le (M_f\|f\|_w/(1+x_0^2)^\lambda)$. Let $\varepsilon > 0$ be arbitrary; we can choose x_0 to be so large such that

$$\frac{M_f\|f\|_w}{\left(1+x_0^2\right)^\lambda} < \frac{\varepsilon}{3}. \quad (57)$$

In view of Corollary 3, while $x \in (x_0,\infty)$, we can obtain

$$\|f\|_w\lim_{x\to\infty}\lim_{n\to\infty}\frac{\left|G_{n,k}^{p_n,q_n}\left(1+t^2;x\right)\right|}{\left(1+x^2\right)^{1+\lambda}} = \|f\|_w\lim_{x\to\infty}\frac{1}{\left(1+x^2\right)^\lambda} = 0. \quad (58)$$

Hence, we can choose N and x_0 to be so large such that for any $n > N$, the inequality

$$\sup_{x\in[x_0,\infty)}\|f\|_w\frac{\left|G_{n,k}^{p_n,q_n}\left(1+t^2;x\right)\right|}{\left(1+x^2\right)^{1+\lambda}} < \frac{\varepsilon}{3} \quad (59)$$

holds. Also, the first term of the above inequality tends to zero by Theorem 9, that is,

$$\left\|G_{n,k}^{p_n,q_n}(f;x)-f\right\|_{C(0,x_0]} < \frac{\varepsilon}{3}. \quad (60)$$

Thus, combining (57), (59), and (60), we obtain the desired result.

6. Pointwise Estimates

In this section, we establish two pointwise estimates of the operators (12). First, we give the relation between the local smoothness of f and local approximation. We denote that $f \in C(0,\infty)$ is in $\mathrm{Lip}_M(\gamma, E), \gamma \in (0,1], E \subset (0,\infty)$ if it satisfies the following condition:

$$|f(t)| - f(x) \leq M|t-x|^{\gamma}, \quad t \in (0,\infty), x \in E, \tag{61}$$

where M is a constant depending only on γ and f.

Theorem 13. *Let $0 < q < p \leq 1$, $\gamma \in (0,1]$, and E be any bounded subset on $(0,\infty)$. If $f \in C_B(0,\infty) \bigcap Lip_M(\gamma, E)$, then, for all $x \in (0,\infty)$, we have*

$$\left|G_{n,k}^{p,q}(f;x) - f(x)\right| \leq M\left(\left(B_{n,k}^{p,q}(x)\right)^{\gamma/2} + 2d^{\gamma}(x;E)\right), \tag{62}$$

where $d(x;E)$ denotes the distance between x and E defined by

$$d(x;E) = \inf\{|t-x| : t \in E\}. \tag{63}$$

Proof. Let $\bar{E}$ be the closure of E. Using the properties of the infimum, there is at least a point $t_0 \in \bar{E}$ such that $d(x;E) = |x - t_0|$. By the triangle inequality

$$|f(t) - f(x)| \leq |f(t) - f(t_0)| + |f(x) - f(t_0)|, \tag{64}$$

we can obtain

$$\begin{aligned} \left|G_{n,k}^{p,q}(f;x) - f(x)\right| &\leq G_{n,k}^{p,q}(|f(t) - f(t_0)|;x) \\ &+ G_{n,k}^{p,q}(|f(x) - f(t_0)|;x) \leq M\left\{G_{n,k}^{p,q}(|t-t_0|^{\gamma};x)\right. \\ &\left.+ |x-t_0|^{\gamma}\right\} \leq M\left\{G_{n,k}^{p,q}(|t-x|^{\gamma} + |x-t_0|^{\gamma};x) + |x-t_0|^{\gamma}\right\} \\ &= M\left\{G_{n,k}^{p,q}(|t-x|^{\gamma};x) + 2|x-t_0|^{\gamma}\right\}. \end{aligned} \tag{65}$$

Choosing $p_1 = 2/\gamma$ and $p_2 = 2/(2-\gamma)$ and using the well-known Hölder inequality, we have

$$\begin{aligned} \left|G_{n,k}^{p,q}(f;x) - f(x)\right| &\leq M\left\{\left(G_{n,k}^{p,q}(|t-x|^{p_1\gamma};x)\right)^{1/p_1}\right. \\ &\left.\times \left(G_{n,k}^{p,q}(1^{p_2};x)\right)^{1/p_2} + 2d^{\gamma}(x;E)\right\} \\ &\leq M\left\{\left(G_{n,k}^{p,q}((t-x)^2;x)\right)^{\gamma/2} + 2d^{\gamma}(x;E)\right\} \\ &\leq M\left(\left(B_{n,k}^{p,q}(x)\right)^{\gamma/2} + 2d^{\gamma}(x;E)\right). \end{aligned} \tag{66}$$

Next, we obtain the local direct estimate of the operators $G_{n,k}^{p,q}$, using the Lipschitz-type maximal function of the order γ introduced by Lenze [40] as

$$\tilde{\omega}_{\gamma}(f;x) = \sup_{x,t \in (0,\infty), x \neq t} \frac{|f(t) - f(x)|}{|t-x|^{\gamma}}, \quad \gamma \in (0,1]. \tag{67}$$

Theorem 14. *Let $f \in C_B(0,\infty)$ and $\gamma \in (0,1]$. Then, for all $x \in (0,\infty)$, we have*

$$\left|G_{n,k}^{p,q}(f;x) - f(x)\right| \leq \tilde{\omega}_{\gamma}(f;x)\left(B_{n,k}^{p,q}(x)\right)^{\gamma/2}. \tag{68}$$

Proof. From equation (67), we have

$$\left|G_{n,k}^{p,q}(f;x) - f(x)\right| \leq \tilde{\omega}_{\gamma}(f;x) G_{n,k}^{p,q}(|t-x|^{\gamma};x). \tag{69}$$

Applying the well-known Hölder inequality, we can get

$$\begin{aligned} \left|G_{n,k}^{p,q}(f;x) - f(x)\right| &\leq \tilde{\omega}_{\gamma}(f;x)\left(G_{n,k}^{p,q}((t-x)^2;x)\right)^{\gamma/2} \\ &\leq \tilde{\omega}_{\gamma}(f;x)\left(B_{n,k}^{p,q}(x)\right)^{\gamma/2}. \end{aligned} \tag{70}$$

7. Voronovskaja-Type Theorem

In this section, we give a Voronovskaja-type asymptotic formula for the operators (12) by means of the second and fourth central moments.

Theorem 15. *Let (p_n), (q_n) be the sequences defined in Corollary 5 and $f \in C_B(0,\infty)$. Suppose that $f''(x)$ exists at a point $x \in (0,\infty)$, then we can obtain*

$$\begin{aligned} &\lim_{n\to\infty} [n]_{p_n,q_n}\left(G_{n,k}^{p_n,q_n}(f;x) - f(x)\right) \\ &= \frac{\alpha+\beta}{2}\left(x^2 f''(x) - (k-1)x f'(x)\right). \end{aligned} \tag{71}$$

Proof. Using Taylor's expansion formula, we can obtain

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + R(t,x)(t-x)^2, \tag{72}$$

where $R(t,x)$ is the Peano form of the remainder and $\lim_{t\to x} R(t,x) = 0$. Applying $G_{n,k}^{p_n,q_n}$ to the both sides of (72), we have

$$\begin{aligned} [n]_{p_n,q_n}\left(G_{n,k}^{p_n,q_n}(\zeta;x) - \zeta(x)\right) &= [n]_{p_n,q_n} f'(x) G_{n,k}^{p_n,q_n}(t-x;x) \\ &+ [n]_{p_n,q_n}\frac{f''(x)}{2} G_{n,k}^{p_n,q_n}((t-x)^2;x) \\ &+ [n]_{p_n,q_n} G_{n,k}^{p_n,q_n}(R(t,x)(t-x)^2;x). \end{aligned} \tag{73}$$

By Schwarz's inequality, we have

$$\begin{aligned} &G_{n,k}^{p_n,q_n}(R(t,x)(t-x)^2;x) \\ &\leq \sqrt{G_{n,k}^{p_n,q_n}(R^2(t,x);x)}\sqrt{G_{n,k}^{p_n,q_n}((t-x)^4;x)}. \end{aligned} \tag{74}$$

We observe that $R^2(x,x) = 0$ and $R^2(\cdot,x) \in C_B(0,\infty)$. Then, it follows from Corollary 8 that

$$\lim_{n\to\infty} G_{n,k}^{p_n,q_n}(R^2(t,x);x) = R^2(x,x) = 0. \tag{75}$$

Hence, from (19), (74), and (75), we can obtain

$$\lim_{n\to\infty} [n]_{p_n,q_n} G_{n,k}^{p_n,q_n}(R(t,x)(t-x)^2;x) = 0. \tag{76}$$

Combining (17), (18), and (76), we obtain the required result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Eigenvalues of s-Type Operators on Prequasi Normed $C(t, p)$

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We investigate some new topological properties of the multiplication operator on $C(t, p)$ defined by Bilgin (The Punjab University Journal of Mathematics, vol. 30, pp. 67–77, 1997) equipped with the prequasi norm and the prequasi operator ideal formed by this sequence space and s-numbers.

1. Introduction

Summability, multiplication, and ideal operator theorems are very important in mathematical models and have numerous implementations, such as normal series theory, ideal transformations, geometry of Banach spaces, approximation theory, and fixed point theory. For more details, see [1–11]. By $\mathbb{C}^{\mathbb{N}}$, ℓ_∞, ℓ_r, and c_0, we denote the spaces of every, bounded, r-absolutely summable, and convergent to zero sequences of complex numbers. $\mathbb{N}$ indicates the set of nonnegative integers. For a sequence (p_r) with $\inf_r p_r > 0$, Bilgin [12] defined and studied the sequence space

$$C(t, p) = \left\{x = (x_n) \in \mathbb{C}^{\mathbb{N}} : h(\beta x) < \infty \quad \text{for some } \beta > 0\right\}, \tag{1}$$

where $h(x) = \sum_{n=0}^{\infty} \left(\sum_{k=2^n-1}^{2^{n+1}-2} (k+1)^{-t} \mid x_k \mid / 2^n\right)^{p_n}$ and $t \geq 0$.

The space $C(t, p)$ is a Banach space with the Luxemburg norm

$$\|x\| = \inf\left\{\beta > 0 : h\left(\frac{x}{\beta}\right) \leq 1\right\}. \tag{2}$$

If $(p_n) \in \ell_\infty$, we have

$$C(t, p) = \left\{(x_i) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} \left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2} (k+1)^{-t} \mid x_k \mid}{2^n}\right)^{p_n} < \infty\right\}. \tag{3}$$

Here and after, we denote $p = (p_n)$, if $t = 0$, then $C(t, p) = ces(p)$, where $\inf_n p_n > 0$ was defined and investigated by Lim [13]. If $t = 0$ and $p_n = p$, for all $n \in \mathbb{N}$, then $C(t, p) = ces_p$, Lim [14] defined and determined its dual spaces and characterized some matrix classes. For any pair of Banach spaces$T$ and U, the following notations will be used throughout the article: $\mathscr{B}(T, U)$ (the space of all bounded linear operators from T into U), and if$T = U$, we write $\mathscr{B}(T)$, $(s_r(A))_{r=0}^{\infty}$ [15] (the$s$-number sequence of operator $A \in \mathscr{B}(T, U)$), $(\alpha_r(A))_{r=0}^{\infty}$ (the approximation number sequence of operator $A \in \mathscr{B}(T, U)$), $\mathscr{F}(T)$ (the space of all finite rank operators on T), $\mathscr{K}(T)$ (the space of all compact operators on T), and $\mathscr{A}(T)$ (the space of all approximable operators on T). For any sequence space E, we will use the notations [16]

$$\begin{aligned} E^{(s)} &:= \left\{E^{(s)}(T,U)\right\}, \text{ where } E^{(s)}(T,U) \\ &:= \left\{A \in \mathscr{B}(T,U):((s_i(A))_{i=0}^{\infty} \in E\right\}, \\ E^{(\alpha)} &:= \left\{E^{(\alpha)}(T,U)\right\}, \text{ where } E^{(\alpha)}(T,U) \\ &:= \left\{A \in \mathscr{B}(T,U):((\alpha_i(A))_{i=0}^{\infty} \in E\right\}, \\ E^{(\lambda)} &:= \left\{E^{(\lambda)}(T,U)\right\}, \text{ where} \\ E^{(\lambda)}(T,U) &:= \{A \in \mathscr{B}(T,U):((\lambda_n(A))_{n=0}^{\infty} \\ &\in E \text{ and } \|A - \lambda_r(A)I\| = 0, \text{ for all } r \in \mathbb{N}\}. \end{aligned} \tag{4}$$

A few of operator ideals in the class of Hilbert spaces or Banach spaces are defined by distinct scalar sequence spaces, such as the ideal of compact operators $\mathscr{K}$ formed by $(d_r(V))$ and c_0. Pietsch [17] studied the quasi-ideals $(\ell_r)^{(\alpha)}$ for $r \in (0,\infty)$, the ideals of Hilbert Schmidt operators between Hilbert spaces constructed by ℓ_2, and the ideals of nuclear operators generated by ℓ_1. He examined that $\bar{\mathscr{F}} = (\ell_r)^{(\alpha)}$ for $r \in [1,\infty)$, where $\bar{\mathscr{F}}$ is the closed class of all finite rank operators, and the class $(\ell_r)^{(\alpha)}$ is a simple Banach and small [18]. The strict inclusion $(\ell_r)^{(\alpha)}(T,U) \subsetneq (\ell_j)^{(\alpha)}(T,U) \subsetneq \mathscr{B}(T,U)$, whenever $j > r > 0$, T and U are infinite dimensional Banach spaces investigated through Makarov and Faried [19]. Faried and Bakery [16] investigated a generalization of the class of operator ideal which is the prequasi operator ideal; they studied several geometric and topological structures of $(\ell_M)^{(s)}$ and $(\mathrm{ces}(r))^{(s)}$. On sequence spaces, Mursaleen and Noman [20, 21] investigated the compact operators on some difference sequence spaces. The multiplication operators on $(\mathrm{ces}(r), \|\bullet\|)$ with the Luxemburg norm $\|\bullet\|$ investigated by Komal et al. [22]. The point of this article is to explain some results of $(C(t,p))_h$ equipped with a prequasi norm h. Firstly, we give the conditions on $(C(t,p))_h$ to be a Banach space. Secondly, we investigate the multiplication operator defined on $(C(t,p))_h$. Finally, some geometric and topological structures of $(C(t,p))_h^{(s)}$ have been studied, such as closed, small, simple Banach, and $(C(t,p))_h^{(s)} = (C(t,p))_h^{(\lambda)}$. A strict inclusion relation of $(C(t,p))_h^{(s)}$ has been studied for different p and t.

2. Definitions and Preliminaries

We will use $e_r = (0,0,\cdots,1,0,0,\cdots)$, where 1 shows at the r^{th} place, for all $r \in \mathbb{N}$.

Lemma 1 [17]. *If $E \in \mathscr{B}(T,U)$ and $E \notin \mathscr{A}(T,U)$, then there are $G \in \mathscr{B}(T)$ and $D \in \mathscr{B}(U)$ such that $DEGe_r = e_r$, for all $r \in \mathbb{N}$.*

Definition 2 [17]. A Banach space T is called simple if the algebra $\mathscr{B}(T)$ has one and only one nontrivial closed ideal.

Theorem 3 [17]. *If T is a Banach space with* $\dim(T) = \infty$, *then*

$$\mathscr{F}(T) \subsetneq \mathscr{A}(T) \subsetneq \mathscr{K}(T) \subsetneq \mathscr{B}(T). \tag{5}$$

Definition 4 [23]. An operator $A \in \mathscr{B}(T)$ is called Fredholm if $\dim((R(A))^c) < \infty$, $\dim(\ker(A)) < \infty$, and $R(\bar{A}) = R(A)$, where $(R(A))^c$ is the complement of range A.

Definition 5 [24]. A class of linear sequence spaces T is called a special space of sequences (sss) if

(1) $e_r \in T$, for every $r \in \mathbb{N}$

(2) if $t = (t_r) \in \mathbb{C}^{\mathbb{N}}$, $z = (z_r) \in T$, and $|t_r| \leq |z_r|$, for all $r \in \mathbb{N}$, then $t \in T$, i.e., "T is solid"

(3) if $(t_r)_{r=0}^{\infty} \in T$, then $(t_{[r/2]})_{r=0}^{\infty} \in T$, where $[r/2]$ denotes the integral part of $r/2$

Definition 6 [24]. A subclass T_h is called a premodular (sss) if there is a function $h : T \longrightarrow [0,\infty)$ verifying the conditions

(i) $h(t) \geq 0$ for each $t \in T$ and $h(t) = 0 \Leftrightarrow t = \theta$, where θ is the zero vector of T

(ii) there is $d \geq 1$ such that $h(\eta t) \leq d|\eta| h(t)$, for each $t \in T$ and $\eta \in \mathbb{C}$

(iii) there is $M \geq 1$, $h(t+z) \leq M(h(t) + h(z))$, for all $t, z \in T$

(iv) if $|t_r| \leq |z_r|$, for every $r \in \mathbb{N}$, then $h((t_r)) \leq h((z_r))$

(v) there is $M_0 \geq 1$, $h((t_r)) \leq h((t_{[r/2]})) \leq M_0 h((t_r))$

(vi) for all $t = (t_r)_{r=0}^{\infty} \in T$, and $\varepsilon > 0$ there is $r_0 \in \mathbb{N}$ such that $h((t_r)_{r=r_0}^{\infty}) < \varepsilon$

(vii) there is $\alpha > 0$ such that $h(\eta, 0, 0, 0, \cdots) \geq \alpha |\eta| h(1, 0, 0, 0, \cdots)$, for all $\eta \in \mathbb{C}$

Definition 7 [24]. Let T be a (sss). If there is a function $h : T \longrightarrow [0,\infty)$ verifying the parts (i), (ii), and (iii) of Definition 6. The space T with h is called prequasi normed (sss) and denoted by T_h, which gives a class more general than the quasi normed (sss). If the space T is complete with h, then T_h, is called a prequasi Banach (sss).

Theorem 8 [24]. *Every quasinorm (sss) is prequasi norm (sss).*

Theorem 9 [24]. *Every premodular (sss) is prequasi normed (sss).*

A generalization of the usual classes of operator ideal is the class prequasi operator ideal.

Definition 10 [24]. A function $V : \Psi \longrightarrow [0,\infty)$ is called a prequasi norm on the class Ψ if it satisfies

(1) if $A \in \Psi(T, U)$, then $V(A) \geq 0$ and $V(A) = 0 \Leftrightarrow A = 0$

(2) there is $d \geq 1$ such that $V(\eta A) \leq d\,|\,\eta\,|\,V(A)$, for every $A \in \Psi(T, U)$ and $\eta \in \mathbb{C}$

(3) there is $M \geq 1$ such that $V(A_1 + A_2) \leq M[V(A_1) + V(A_2)]$, for every $A_1, A_2 \in \Psi(T, U)$

(4) there is $\nu \geq 1$ such that if $X \in \mathscr{B}(T_0, T)$, $Y \in \Psi(T, U)$, and $Z \in \mathscr{B}(U, U_0)$, then $V(ZYX) \leq \nu \|Z\| V(Y) \|X\|$, where T_0 and U_0 are normed spaces

Theorem 11 [25]. *If E_h is a premodular (sss), then $\mathscr{F}(\bar{T}, U) = E_h^{(s)}(T, U)$, where $V(A) = h(s_r(A)_{r=0}^{\infty})$.*

Theorem 12 [16]. *If E_h is a premodular (sss), then $V(B) = h(s_r(B))_{r=0}^{\infty}$ is a prequasi norm on $E_h^{(s)}$.*

The following inequality [26] will be used in the sequel: $|a_r + b_r|^{p_r} \leq H(|a_r|^{p_r} + |b_r|^{p_r})$, where $H = \max\{1, 2^{\sup_r p_r - 1}\}$, $0 \leq p_r \leq \sup_r p_r < \infty$, and $a_r, b_r \in \mathbb{C}$ for each $r \in \mathbb{N}$.

3. Main Results

3.1. Premodular Banach (sss). The conditions on t and p are such that the space $(C(t,p))_h$ is a premodular Banach (sss), where $h(z) = \sum_{r=0}^{\infty} (\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, z_k | / 2^r)^{p_r}$, for all $z \in C(t, p)$.

Theorem 13. *Let $(p_r) \in \ell_\infty$ be an increasing with $p_0 > 0$ and $t \geq 0$, then the space $(C(t,p))_h$ is a premodular Banach (sss).*

Proof. (1-i) Assume $z, w \in C(t, p)$. From $(p_r) \in \ell_\infty$, we get

$$\begin{aligned} h(z+w) &= \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, z_k + w_k \,|}{2^r} \right)^{p_r} \\ &\leq H \left(\sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, z_k \,|}{2^r} \right)^{p_r} + \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, w_k \,|}{2^r} \right)^{p_r} \right) \\ &= H(h(z) + h(w)) < \infty, \end{aligned} \tag{6}$$

hence $z + w \in C(t,p)$.

(1-ii) If $\eta \in \mathbb{C}$, $z \in C(t,p)$, and from $(p_r) \in \ell_\infty$, one has

$$\begin{aligned} h(\eta z) &= \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, \eta z_k \,|}{2^r} \right)^{p_r} \\ &\leq \sup_r |\eta|^{p_r} \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, z_k \,|}{2^r} \right)^{p_r} \\ &= \sup_r |\eta|^{p_r} h(z) < \infty. \end{aligned} \tag{7}$$

Therefore, $\eta z \in C(t,p)$, from (1-i) and (1-ii), the space $C(t,p)$ is linear. Since $(p_r) \in \ell_\infty$ with $p_0 > 0$, we obtain

$$\begin{aligned} h(e_m) &= \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} (e_m)_k}{2^r} \right)^{p_r} \\ &= \left(\frac{(m+1)^{-t}}{2^{r_0}} \right)^{p_{r_0}} < \infty, \end{aligned} \tag{8}$$

where $r_0 \in \mathbb{N}$ be such that $2^{r_0} - 1 \leq m \leq 2^{r_0+1} - 2$. Hence $e_m \in C(t,p)$, for all $m \in \mathbb{N}$.

(2) Let $|z_r| \leq |w_r|$, for all $r \in \mathbb{N}$ and $w \in C(t,p)$. Hence,

$$\begin{aligned} h(z) &= \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, z_k \,|}{2^r} \right)^{p_r} \\ &\leq \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, w_k \,|}{2^r} \right)^{p_r} = h(w) < \infty, \end{aligned} \tag{9}$$

hence, $z \in C(t,p)$.

(3) Let $(z_r) \in C(t,p)$, we have

$$\begin{aligned} h\left(z_{[r/2]}\right) &= \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} ([k/2]+1)^{-t} \,|\, z_{[k/2]} \,|}{2^r} \right)^{p_r} \\ &= \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^{2r}-1}^{2^{2r+1}-2} ([k/2]+1)^{-t} \,|\, z_{[k/2]} \,|}{2^{2r}} \right)^{p_{2r}} + \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^{2r+1}-1}^{2^{2r+2}-2} ([k/2]+1)^{-t} \,|\, z_{[k/2]} \,|}{2^{2r+1}} \right)^{p_{2r+1}} \\ &\leq \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^{2r}-2}^{2^{2r+1}-2} ([k/2]+1)^{-t} \,|\, z_{[k/2]} \,|}{2^{2r}} \right)^{p_r} + \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^{2r+1}-2}^{2^{2r+2}-2} ([k/2]+1)^{-t} \,|\, z_{[k/2]} \,|}{2^{2r+1}} \right)^{p_r} \\ &\leq \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} 3(2^r)(k+1)^{-t} \,|\, z_k \,|}{2^{2r}} \right)^{p_r} + \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} 3(2^r)(k+1)^{-t} \,|\, z_k \,|}{2^{2r+1}} \right)^{p_r} \\ &\leq 2(3)^{\sup p_r} \sum_{r=0}^{\infty} \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} \,|\, z_k \,|}{2^r} \right)^{p_r}, \end{aligned} \tag{10}$$

then $(z_{[r/2]}) \in C(t,p)$.

(i) Evidently, $h(z) \geq 0$ and $h(z) = 0 \Leftrightarrow z = \theta$

(ii) There is $d = \max\{1, \sup_r |\eta|^{p_r - 1}\} \geq 1$ with $h(\eta z) \leq d\,|\,\eta\,|\,h(z)$, for each $z \in C(t,p)$ and $\eta \in \mathbb{C}$

(iii) From condition (1), there is $M \geq 1$ with $h(z+w) \leq M(h(z)+h(w))$ for each $z, w \in C(t,p)$

(iv) From condition (2), we have $(C(t,p))_h$ which is solid

(v) From condition (3), we find $M_0 \geq 2(3)^{\sup p_r} \geq 1$

(vi) Clearly, $\bar{F} = C(t,p)$

(vii) There is $0 < \alpha \leq |\eta|^{p_0 - 1}$ with $h(\eta, 0, 0, 0, \cdots) \geq \alpha \mid \eta \mid h(1, 0, 0, 0, \cdots)$, for each $\eta \neq 0$ and $\alpha > 0$, when $\eta = 0$

Therefore, the space $(C(t,p))_h$ is premodular (sss). To prove that $(C(t,p))_h$ is a premodular Banach (sss), suppose $z^r = (z_i^r)_{i=0}^{\infty}$ is a Cauchy sequence in $(C(t,p))_h$; hence, for every $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that for every $r, v \geq r_0$, one has

$$h(z^r - z^v) = \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} \mid z_j^r - z_j^v \mid}{2^i} \right)^{p_i} < \varepsilon^{\sup_i p_i}. \tag{11}$$

Hence, for $r, v \geq r_0$ and $i \in \mathbb{N}$, we get $|z_i^r - z_i^v| < \varepsilon$. So (z_i^v) is a Cauchy sequence in $\mathbb{C}$ for fixed $i \in \mathbb{N}$; this gives $\lim_{v \to \infty} z_i^v = z_i^0$ for fixed $i \in \mathbb{N}$. Hence, $h(z^r - z^0) < \varepsilon^{\sup_i p_i}$, for all $r \geq r_0$. Finally, to prove that $z^0 \in C(t,p)$, we have

$$h(z^0) = h(z^0 - z^r + z^r) \leq H(h(z^r - z^0) + h(z^n)) < \infty, \tag{12}$$

so $z^0 \in C(t,p)$. This means that $(C(t,p))_h$ is a premodular Banach (sss).

Corollary 14. *If $0 < p < \infty$, then $(ces_p)_h$ is a premodular Banach (sss), where $h(z) = \sum_{i=0}^{\infty} (\sum_{j=2^i-1}^{2^{i+1}-2} \mid z_j \mid / 2^i)^p$ for every $z \in ces_p$.*

4. Some Properties of Multiplication Operators on $(C(t,p))_h$

We investigate the necessity and sufficient conditions on t and p such that the multiplication operator defined on the prequasi normed (sss), $(C(t,p))_h$, becomes an element of $\mathscr{B}((C(t,p))_h)$, $\mathscr{A}((C(t,p))_h)$, $\mathscr{K}((C(t,p))_h)$, closed range, and Fredholm operator.

Definition 15. Pick up $\zeta \in \mathbb{C}^{\mathbb{N}} \cap \ell_\infty$, and T_h is a prequasi normed (sss). An operator $A_\zeta : T_h \longrightarrow T_h$ is called a multiplication operator, if $A_\zeta z = \zeta z = (\zeta_k z_k)_{k=0}^{\infty}$, for all $z \in T$. It is called a multiplication operator generated by ζ, if $A_\zeta \in \mathscr{B}(T_h)$ is a multiplication operator.

Theorem 16. *If $\zeta \in \mathbb{C}^{\mathbb{N}}$, $(p_n) \in \ell_\infty$ is increasing with $p_0 > 0$ and $t \geq 0$, then*

$$\zeta \in \ell_\infty \Leftrightarrow A_\zeta \in \mathscr{B}((C(t,p))_h). \tag{13}$$

Proof. Let $\zeta \in \ell_\infty$. Therefore, there is $D > 0$ such that $|\zeta_r| \leq D$, for every $r \in \mathbb{N}$. Since $(p_r) \in \ell_\infty$ and for $z \in (C(t,p))_h$, we have

$$\begin{aligned} h(A_\zeta z) = h(\zeta z) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| |z_j|}{2^i} \right)^{p_i} \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} D \mid z_j \mid}{2^i} \right)^{p_i} \leq \sup_i (D^{p_i})\, h(z), \end{aligned} \tag{14}$$

this implies that $A_\zeta \in \mathscr{B}((C(t,p))_h)$. Conversely, let $A_\zeta \in \mathscr{B}((C(t,p))_h)$, to show that $\zeta \in \ell_\infty$. For, if $\zeta \notin \ell_\infty$, then for all $r \in \mathbb{N}$, there are $i_r \in \mathbb{N}$ such that $\zeta_{i_r} > r$. Hence,

$$\begin{aligned} h(A_\zeta e_{i_r}) = h(\zeta e_{i_r}) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} \mid \zeta_j \mid (e_{i_r})_j}{2^i} \right)^{p_i} \\ &= \left(\frac{(i_r+1)^{-t} \mid \zeta_{i_r} \mid}{2^{i_0}} \right)^{p_{i_0}} > \left(\frac{(i_r+1)^{-t} r}{2^{i_0}} \right)^{p_{i_0}} \\ &= r^{p_{i_0}} h(e_{i_r}), \end{aligned} \tag{15}$$

where $i_0 \in \mathbb{N}$ be such that $2^{i_0} - 1 \leq i_r \leq 2^{i_0+1} - 2$. This shows that $A_\zeta \notin \mathscr{B}((C(t,p))_h)$. So, $\zeta \in \ell_\infty$.

Theorem 17. *Assume $\zeta \in \mathbb{C}^{\mathbb{N}}$ and $(C(t,p))_h$ is a prequasi normed (sss). $|\zeta_r| = 1$, for every $r \in \mathbb{N}$ if and only if A_ζ is an isometry.*

Proof. If $|\zeta_r| = 1$, for every $r \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} h(A_\zeta z) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| |z_j|}{2^i} \right)^{p_i} \\ &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} \mid z_j \mid}{2^i} \right)^{p_i} = h(z), \end{aligned} \tag{16}$$

for all $z \in (C(t,p))_h$. Hence, A_ζ is an isometry. Conversely, let A_ζ be an isometry and $|\zeta_r| < 1$, for all $r \in \mathbb{N}$. Then,

$$\begin{aligned} h(A_\zeta z) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| |z_j|}{2^i} \right)^{p_i} \\ &< \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} \mid z_j \mid}{2^i} \right)^{p_i} = h(z). \end{aligned} \tag{17}$$

Also if $|\zeta_r|>1$, then we get $h(\zeta z)>h(z)$. We have a contradiction in the two cases. Therefore, $|\zeta_r|=1$, for every $r\in\mathbb{N}$.

By $N(D)$, we indicate the cardinality of the set D.

Theorem 18. *For $\zeta\in\mathbb{C}^{\mathbb{N}}$, $(p_r)\in\ell_\infty$ is increasing with $p_0>0$, $t\geq 0$, and $A_\zeta\in\mathscr{B}((C(t,p))_h)$. Then,*

$$A_\zeta\in\mathscr{A}\left((C(t,p))_h\right)\Leftrightarrow(\zeta_r)_{r=0}^{\infty}\in c_0. \tag{18}$$

Proof. Let $A_\zeta\in\mathscr{A}((C(t,p))_h)$; hence $A_\zeta\in\mathscr{K}((C(t,p))_h)$. To show that $(\zeta_r)_{r=0}^{\infty}\in c_0$. Assume $(\zeta_r)_{r=0}^{\infty}\notin c_0$. Therefore, the set $B_\delta=\{r\in\mathbb{N}:|\zeta_r|\geq\delta\}$ has $N(B_\delta)=\infty$, for $\delta>0$. Suppose $d_r\in B_\delta$, for all $r\in\mathbb{N}$. Therefore, $E=\{e_{d_r}:d_r\in B_\delta\}\subseteq\ell_\infty\cap(C(t,p))_h$ with $N(E)=\infty$. We have

$$\begin{aligned}
h\left(A_\zeta e_{d_r}-A_\zeta e_{d_m}\right)&=h\left(\zeta e_{d_r}-\zeta e_{d_m}\right)\\
&=\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\left|\zeta_j\left(\left(e_{d_r}\right)_j-\left(e_{d_m}\right)_j\right)\right|}{2^i}\right)^{p_i}\\
&\geq\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\left|\delta\left(\left(e_{d_r}\right)_j-\left(e_{d_m}\right)_j\right)\right|}{2^i}\right)^{p_i}\\
&\geq\left(\inf_i\delta^{p_i}\right)\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\left|\left(\left(e_{d_r}\right)_j-\left(e_{d_m}\right)_j\right)\right|}{2^i}\right)^{p_i}\\
&=\left(\inf_i\delta^{p_i}\right)h\left(e_{d_r}-e_{d_m}\right),
\end{aligned} \tag{19}$$

for all $d_r,d_m\in B_\delta$. This proves $A_\zeta(E)$ is not a convergent subsequence. This shows that $A_\zeta\notin\mathscr{K}((C(t,p))_h)$; hence, $A_\zeta\notin\mathscr{A}((C(t,p))_h)$. This gives a contradiction. So, $\lim_{r\to\infty}\zeta_r=0$. Conversely, suppose $\lim_{r\to\infty}\zeta_r=0$. Hence, for all $\delta>0$, we have $N(B_\delta)<\infty$. Therefore, for all $\delta>0$ the space

$$\left((C(t,p))_h\right)_{B_\delta}=\left\{z=(z_r)\in\mathbb{C}^{B_\delta}:h(z)<\infty\right\}, \tag{20}$$

is a finite dimensional. Hence, $A_\zeta\,|\,((C(t,p))_h)_{B_\delta}\in\mathscr{F}((C(t,p))_h)_{B_\delta}$. Let $r\in\mathbb{N}$ and define $\zeta_r\in\mathbb{C}^{\mathbb{N}}$ as

$$(\zeta_r)_m=\begin{cases}\zeta_m, & m\in B_{1/r},\\ 0, & \text{otherwise.}\end{cases} \tag{21}$$

Obviously, $A_{\zeta_r}\in\mathscr{F}((C(t,p))_h)_{B_{1/r}}$ since $\dim((C(t,p))_h)_{B_{1/r}}<\infty$, for all $r\in\mathbb{N}$. Since $(p_i)\in\ell_\infty$ is increasing with $p_0>0$, we have

$$\begin{aligned}
h\left(\left(A_\zeta-A_{\zeta_r}\right)z\right)&=\sum_{i=0,i\in B_{1/r}}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\left|\left(\zeta_j-(\zeta_r)_j\right)z_j\right|}{2^i}\right)^{p_i}\\
&\quad+\sum_{i=0,i\notin B_{1/r}}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\left|\left(\zeta_j-(\zeta_n)_j\right)z_j\right|}{2^i}\right)^{p_i}\\
&=\sum_{i=0,i\in B_{1/r}}^{\infty}\left(\frac{\sum_{j=2^i-1,j\notin B_{1/r}}^{2^{i+1}-2}(j+1)^{-t}\left|\zeta_j z_j\right|}{2^i}\right)^{p_i}\\
&\quad+\sum_{i=0,i\notin B_{1/r}}^{\infty}\left(\frac{\sum_{j=2^i-1,j\notin B_{1/r}}^{2^{i+1}-2}(j+1)^{-t}\left|\zeta_j z_j\right|}{2^i}\right)^{p_i}\\
&<\frac{1}{r^{p_0}}\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\left|z_j\right|}{2^i}\right)^{p_i}=\frac{1}{r^{p_0}}h(z).
\end{aligned} \tag{22}$$

This proves that $\|A_\zeta-A_{\zeta_r}\|\leq 1/r^{p_0}$ and $A_\zeta=\lim_{r\to\infty}A_{\zeta_r}$. Therefore, $A_\zeta\in\mathscr{A}((C(t,p))_h)$.

According to Theorem 3, it is easy to prove the following theorem.

Theorem 19. *For $\zeta\in\mathbb{C}^{\mathbb{N}}$, $(p_r)\in\ell_\infty$ is increasing with $p_0>0$, $t\geq 0$, and $A_\zeta\in\mathscr{B}((C(t,p))_h)$. Then,*

$$A_\zeta\in\mathscr{K}\left((C(t,p))_h\right)\Leftrightarrow(\zeta_r)_{r=0}^{\infty}\in c_0. \tag{23}$$

Corollary 20. *If $(p_i)\in\ell_\infty$ is increasing with $p_0>0$ and $t\geq 0$, we have*

$$\mathscr{K}\left((C(t,p))_h\right)\subsetneqq\mathscr{B}\left((C(t,p))_h\right). \tag{24}$$

Proof. The operator $I=A_\zeta$ on $(C(t,p))_h$ is a multiplication operator generated by $\zeta=(1,1,\cdots)$. Therefore, $I\notin\mathscr{K}((C(t,p))_h)$ and $I\in\mathscr{B}((C(t,p))_h)$.

Theorem 21. *Suppose $(C(t,p))_h$ is a prequasi Banach (sss) and $A_\zeta\in\mathscr{B}((C(t,p))_h)$. Then, there is $\delta>0$ such that $|\zeta_r|\geq\delta$, for all $r\in(\ker(\zeta))^c$, if and only if, A_ζ has a closed range.*

Proof. Let the sufficient condition be verified, to show that $R(\bar{A}_\zeta)=R(A_\zeta)$. If f is a limit point of $R(A_\zeta)$, then there is $A_\zeta x_r\in(C(t,p))_h$, for each $r\in\mathbb{N}$ with $\lim_{r\to\infty}A_\zeta x_r=f$. Clearly, the sequence $A_\zeta x_r$ is a Cauchy sequence. Since h is nondecreasing, one can see

$$h(A_\zeta x_r - A_\zeta x_m) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\mid\zeta_j(x_r)_j-\zeta_j(x_m)_j\mid}{2^i}\right)^{p_i}$$
$$= \sum_{i=0,i\in(\ker(\zeta))^c}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\mid\zeta_j(x_r)_j-\zeta_j(x_m)_j\mid}{2^i}\right)^{p_i}$$
$$+ \sum_{i=0,i\notin(\ker(\zeta))^c}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\mid\zeta_j(x_r)_j-\zeta_j(x_m)_j\mid}{2^i}\right)^{p_i}$$
$$\geq \sum_{i=0,i\in(\ker(\zeta))^c}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\mid\zeta_j(x_r)_j-\zeta_j(x_m)_j\mid}{2^i}\right)^{p_i}$$
$$= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\mid\zeta_j(y_r)_j-\zeta_j(y_m)_j\mid}{2^i}\right)^{p_i}$$
$$> \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\mid\varepsilon(y_r)_j-\varepsilon(y_m)_j\mid}{2^i}\right)^{p_i}$$
$$= h(\varepsilon(y_r - y_m)), \tag{25}$$

where

$$(y_r)_k = \begin{cases}(x_r)_k, & k\in(\ker(\zeta))^c,\\ 0, & k\notin(\ker(\zeta))^c.\end{cases} \tag{26}$$

This proves that (y_r) is a Cauchy sequence in $(C(t,p))_h$. But $(C(t,p))_h$ is complete. Therefore, there exists $x\in(C(t,p))_h$ such that $\lim_{r\to\infty}y_r = x$. In view of the continuity of A_ζ, hence $\lim_{r\to\infty}A_\zeta y_r = A_\zeta x$. But $\lim_{n\to\infty}A_\zeta x_n = \lim_{n\to\infty}A_\zeta y_n = f$. Therefore, $A_\zeta x = f$. Hence $f\in R(A_\zeta)$. Therefore, $R(\bar{A}_\zeta) = R(A_\zeta)$. Let the necessary condition be satisfied. Therefore, there is $\varepsilon>0$ such that $h(A_\zeta x)\geq h(\varepsilon x)$, for all $x\in((C(t,p))_h)_{(\ker(\zeta))^c}$. Let $D=\{k\in(\ker(\zeta))^c : |\zeta_k|<\varepsilon\}$. If $D\neq\phi$, then for $n_0\in D$, since h is nondecreasing, we have

$$h(A_\zeta e_{n_0}) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}|\zeta_j|\left|(e_{n_0})_j\right|}{2^i}\right)^{p_i}$$
$$< \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\varepsilon\mid(e_{n_0})_j\mid}{2^i}\right)^{p_i} = h(\varepsilon e_{n_0}), \tag{27}$$

which gives a contradiction. So, $D=\phi$ with $|\zeta_r|\geq\varepsilon$, for every $r\in(\ker(\zeta))^c$.

Theorem 22. *Pick up $\zeta\in\mathbb{C}^{\mathbb{N}}$, and $(C(t,p))_h$ is a prequasi Banach (sss). There is $d>0$ and $D>0$ with $d<\zeta_r<D$, for every $r\in\mathbb{N}$, if and only if, $A_\zeta\in\mathscr{B}((C(t,p))_h)$ is invertible.*

Proof. Let the condition be true. Define $\eta\in\mathbb{C}^{\mathbb{N}}$ by $\eta_n = 1/\zeta_n$. Then, by Theorem 16, A_ζ and T_η are bounded linear operators. Also $A_\zeta\cdot A_\eta = A_\eta\cdot A_\zeta = I$. Hence, A_ζ is the inverse of A_η. Conversely, let A_ζ be invertible. Hence, $R(A_\zeta) = ((C(t,p))_h)_{\mathbb{N}}$. Therefore, $R(\bar{A}_\zeta) = R(A_\zeta)$. Hence, by Theorem 21, there is $d>0$ with $|\zeta_r|\geq d$, for every $r\in(\ker(\zeta))^c$. Now, $\ker(\zeta)=\phi$; otherwise, $\zeta_{r_0}=0$; for some $r_0\in\mathbb{N}$, we have $e_{r_0}\in\ker(A_\zeta)$; this implies a contradiction, since $\ker(A_\zeta)=\phi$. So, $|\zeta_r|\geq d$, for every $r\in\mathbb{N}$. Since $A_\zeta\in\mathscr{B}((C(t,p))_h)$, from Theorem 16, there is $D>0$ with $|\zeta_r|\leq D$, for every $r\in\mathbb{N}$. Therefore, we get that $d\leq|\zeta_r|\leq D$, for each $r\in\mathbb{N}$.

Theorem 23. *For a prequasi Banach (sss) $(C(t,p))_h$. The operator A_ζ is Fredholm, if and only if, (i) $N(\ker(\zeta))<\infty$ and (ii) $|\zeta_r|\geq\varepsilon$, for each $r\in(\ker(\zeta))^c$.*

Proof. Let the sufficient condition be true and $N(\ker(\zeta))=\infty$, so $e_r\in\ker(A_\zeta)$, for every $r\in\ker(\zeta)$. We have dim $(\ker(A_\zeta))=\infty$, since e_n's are linearly independent. This implies a contradiction. Hence, $N(\ker(\zeta))<\infty$. From Theorem 21, the condition (ii) follows. Next, if the necessary conditions are true, from Theorem 21, the condition (ii) gives that $R(\bar{A}_\zeta) = R(A_\zeta)$. By condition (i), we have dim $(\ker(A_\zeta))<\infty$ and dim $(R((A_\zeta))^c)<\infty$. This implies that A_ζ is Fredholm.

5. Banach and Closed Prequasi Ideal

We explain the sufficient conditions on t and p such that $((C(t,p))_h^{(s)}, V)$ is Banach and a closed prequasi ideal, where $h(x)=\sum_{i=0}^{\infty}(\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\mid x_j|/2^i)^{p_i}$, for all $x\in C(t,p)$ and $V(A)=h((s_n(A))_{n=0}^{\infty})$.

Theorem 24. *Let T, U be Banach spaces, $(p_r)\in\ell_\infty$ be increasing with $p_0>0$ and $t\geq 0$, then $((C(t,p))_h^{(s)}, V)$ is a prequasi Banach ideal.*

Proof. Let the sufficient conditions be satisfied. From Theorem 13, the space $(C(t,p))_h$ is a premodular (sss). Therefore, by Theorem 12, the function $V(A)=h((s_n(A))_{n=0}^{\infty})$ is a prequasi norm on $(C(t,p))_h^{(s)}$. Let (A_i) be a Cauchy sequence in $(C(t,p))_h^{(s)}(T,U)$, since $\mathscr{B}(T,U)\supseteq(C(t,p))_h^{(s)}(T,U)$; we get

$$V(A_i - A_j) = \sum_{n=0}^{\infty}\left(\frac{\sum_{m=2^n-1}^{2^{n+1}-2}(m+1)^{-t}s_m(A_i-A_j)}{2^n}\right)^{p_n} \geq \left\|A_i - A_j\right\|^{p_0}. \tag{28}$$

Hence, $(A_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathscr{B}(T,U)$. Therefore, there is $A\in\mathscr{B}(T,U)$ such that $\lim_{i\to\infty}\|A_i - A\|=0$, since $(s_r(A_i))_{r=0}^{\infty}\in(C(t,p))_h$ for every $i\in\mathbb{N}$.

By using parts (ii), (iii), and (iv) of Definition (6), one can see

$$\begin{aligned} V(A) &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_j(A)}{2^i}\right)^{p_i} \\ &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_j(A-A_m+A_m)}{2^i}\right)^{p_i} \\ &\le M\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_{[j/2]}(A-A_m)}{2^i}\right)^{p_i} \\ &\quad + M\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_{[j/2]}(A_m)}{2^i}\right)^{p_i} \\ &\le M\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\|A_m-A\|}{2^i}\right)^{p_i} \\ &\quad + M M_0\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_j(A_m)}{2^i}\right)^{p_i} < \varepsilon. \end{aligned} \tag{29}$$

Therefore, $(s_r(A))_{r=0}^{\infty} \in (C(t,p))_h$; hence, $A \in (C(t,p))_h^{(s)}(T,U)$.

Theorem 25. *Let T, U be normed spaces and $(p_r) \in \ell_\infty$ be increasing with $p_0 > 0$ and $t \ge 0$, then $((C(t,p))_h^{(s)}, V)$ is a pre-quasi closed ideal.*

Proof. Let the sufficient conditions be satisfied. From Theorem 13, the space $(C(t,p))_h$ is a premodular (sss). Therefore, by Theorem 12, the function $V(A) = h((s_n(A))_{n=0}^{\infty})$ is a pre-quasi norm on $(C(t,p))_h^{(s)}$. Assume $A_m \in (C(t,p))_h^{(s)}(T,U)$ for every $m \in \mathbb{N}$ and $\lim_{m\to\infty} V(A_m - A) = 0$, since $\mathscr{B}(T,U) \supseteq (C(t,p))_h^{(s)}(T,U)$, we have

$$\begin{aligned} V(A-A_j) &= \sum_{n=0}^{\infty}\left(\frac{\sum_{m=2^n-1}^{2^{n+1}-2}(m+1)^{-t}s_m(A-A_j)}{2^n}\right)^{p_n} \\ &\ge \left\|A-A_j\right\|^{p_0}. \end{aligned} \tag{30}$$

Hence, $(A_j)_{j\in\mathbb{N}}$ is a convergent sequence in $\mathscr{B}(T,U)$. So, $(s_r(A_j))_{r=0}^{\infty} \in (C(t,p))_h$ for each $j \in \mathbb{N}$. By using parts (ii), (iii), and (iv) of Definition (6), we have

$$\begin{aligned} V(A) &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_j(A)}{2^i}\right)^{p_i} \\ &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_j(A-A_m+A_m)}{2^i}\right)^{p_i} \\ &\le M\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_{[j/2]}(A-A_m)}{2^i}\right)^{p_i} \\ &\quad + M\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_{[j/2]}(A_m)}{2^i}\right)^{p_i} \\ &\le M\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}\|A_m-A\|}{2^i}\right)^{p_i} \\ &\quad + M M_0\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t}s_j(A_m)}{2^i}\right)^{p_i} < \varepsilon. \end{aligned} \tag{31}$$

Therefore, $(s_r(A))_{r=0}^{\infty} \in (C(t,p))_h$; hence, $A \in (C(t,p))_h^{(s)}(T,U)$.

6. Smallness and Simplicity of the Prequasi Ideal

We examine here some inclusion relations and topological structures of $(C(t,p))_h^{(s)}$.

Theorem 26. *Let T or U be finite dimensional Banach spaces, $t_q > t_p > 0$ and increasing sequences $(p_n) \in \ell_\infty$, $(q_n) \in \ell_\infty$ with $0 < p_r < q_r$, for all $r \in \mathbb{N}$, then*

$$\left(C(t_p,p)\right)_h^{(s)}(T,U)Þ\left(C(t_q,q)\right)_h^{(s)}(T,U)\mathscr{U}\mathscr{B}(T,U). \tag{32}$$

Proof. Let the conditions be satisfied, if $A \in (C(t_p,p))_h^{(s)}(T,U)$, then $(s_j(A)) \in C(t_p,p)$. We have

$$\begin{aligned} &\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t_q}s_j(A)}{2^i}\right)^{q_i} \\ &\quad < \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t_p}s_j(A)}{2^i}\right)^{p_i} < \infty, \end{aligned} \tag{33}$$

hence, $A \in (C(t_q,q))_h^{(s)}(T,U)$. Next, if we take $(s_j(A))_{j=0}^{\infty}$ such that $\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t_p}s_j(A) = 2^i(i+1)^{-(1/p_i)}$, one can find $A \in \mathscr{B}(T,U)$ with

$$\begin{aligned} &\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t_p}s_j(A)}{2^i}\right)^{p_i} = \sum_{i=0}^{\infty}\frac{1}{i+1} = \infty, \\ &\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t_q}s_j(A)}{2^i}\right)^{q_i} \le \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t_p}s_j(A)}{2^i}\right)^{q_i} \\ &\qquad = \sum_{i=0}^{\infty}\left(\frac{1}{i+1}\right)^{q_i/p_i} < \infty. \end{aligned} \tag{34}$$

Hence, $A \notin (C(t_p,p))_h^{(s)}(T,U)$ and $A \in (C(t_q,q))_h^{(s)}(T,U)$. Clearly, $(C(t_q,q))_h^{(s)}(T,U) \subset \mathscr{B}(T,U)$. Secondly, by choosing $(s_j(A))_{j=0}^{\infty}$ such that $\sum_{j=2^i-1}^{2^{i+1}-2}(j+1)^{-t_q}s_j(A) = 2^i(i+1)^{-(1/q_i)}$.

Hence, there is $A \in \mathscr{B}(T, U)$ with $A \notin (C(t_q, q))_h^{(s)}(T, U)$. This completes the proof.

We study in the next theorem that $(C(t,p))_h^{(s)}$ is not small.

Theorem 27. *If T and U are infinite dimensional Banach spaces, $(p_r) \in \ell_\infty$ is increasing with $p_0 > 0$ and $t > 0$, then $(C(t,p))_h^{(s)}$ is not small.*

Proof. Let the conditions be satisfied. Therefore, $(C(t,p))_h^{(s)}$ is a prequasi operator ideal. For all $A \in \mathscr{B}(T, U)$, there is $C > 0$ such that $\|A\| \leq C$; we have

$$h(A) = \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} s_j(IA)}{2^i} \right)^{p_i} \leq \sup_n C^{p_n} \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} s_j(I)}{2^i} \right)^{p_i} \leq \sup_n C^{p_n} \sum_{i=0}^{\infty} \frac{1}{2^{itp_0}} < \infty. \quad (35)$$

Hence, $A \in (C(t,p))_h^{(s)}(T, U)$ which gives $(C(t,p))_h^{(s)}(T, U) = \mathscr{B}(T, U)$. This finishes the proof.

We investigate here the simplicity of $(C(t,p))_h^{(s)}$.

Theorem 28. *For any finite dimensional Banach spaces T or U, $t_q > t_p > 0$ and increasing sequences $(p_r) \in \ell_\infty$, $(q_r) \in \ell_\infty$ with $0 < p_r < q_r$, for every $r \in \mathbb{N}$, then*

$$\mathscr{B}\left((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)} \right) = \mathscr{A}\left((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)} \right). \quad (36)$$

Proof. Let there is $D \in \mathscr{B}((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)})$ with $D \notin \mathscr{A}((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)})$. By Lemma 1, we have $A \in \mathscr{B}((C(t_q, q))_h^{(s)})$ and $B \in \mathscr{B}((C(t_p, p))_h^{(s)})$ with $\mathrm{BDA}I_k = I_k$. Then, it follows, for all $k \in \mathbb{N}$ that

$$\|I_k\|_{(C(t_p,p))_h^{(s)}} = \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t_p} s_j(I_k)}{2^i} \right)^{p_i} \leq \|\mathrm{BDA}\| \|I_k\|_{(C(t_q,q))_h^{(s)}} \leq \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t_q} s_j(I_k)}{2^i} \right)^{q_i}. \quad (37)$$

By Theorem 26, we get a contradiction. Therefore, $D \in \mathscr{A}((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)})$.

Theorem 29. *For any finite dimensional Banach spaces T or U. If $(p_r) \in \ell_\infty$ is increasing with $p_0 > 1$, then $(C(t,p))_h^{(s)}$ is simple.*

Proof. Assume there is $D \in \mathscr{K}((C(t,p))_h^{(s)})$ with $D \notin \mathscr{A}((C(t,p))_h^{(s)})$. By Lemma 1, we have $A, B \in \mathscr{B}((C(t,p))_h^{(s)})$ with $\mathrm{BDA}I_k = I_k$. This means that $I_{(C(t,p))_h^{(s)}} \in \mathscr{K}((C(t,p))^{(s)})$. Consequently, $\mathscr{B}(S_{(C(t,p))_h^{(s)}}) = \mathscr{K}((C(t,p))_h^{(s)})$. Therefore, $\mathscr{A}((C(t,p))_h^{(s)})$ is the only nontrivial closed ideal in $\mathscr{B}((C(t,p))_h^{(s)})$.

We study the approximation of the prequasi ideal $(C(t,p))_h^{(s)}$.

Theorem 30. *For any finite dimensional Banach spaces T or U. If $(p_r) \in \ell_\infty$ is increasing with $p_0 > 0$, then $\mathscr{F}(\bar{T}, U) = (C(t,p))_h^{(s)}(T, U)$.*

Proof. Since $(C(t,p))_h$ is a premodular (sss), then from Theorem 11, we have $\mathscr{F}(\bar{T}, U) = (C(t,p))_h^{(s)}(T, U)$. Conversely, since $I_4 \in S_{(C(1,1))_h}$ but $p_0 = 0$. This gives a counter example.

7. Spectrum of s-Type Operators

We explain the spectrum of the prequasi ideal $(C(t,p))_h^{(s)}$.

Theorem 31. *For any finite dimensional Banach spaces T or U. If $(p_r) \in \ell_\infty$ is increasing with $p_0 > 0$, then*

$$(C(t,p))_h^{(\lambda)}(T, U) = (C(t,p))_h^{(s)}(T, U). \quad (38)$$

Proof. Let $A \in (C(t,p))_h^{(\lambda)}(T, U)$, then $(\lambda_r(A))_{r=0}^\infty \in C(t,p)$ and $\|A - \lambda_n(A)I\| = 0$, for every $r \in \mathbb{N}$. Therefore, $A = \lambda_r(A)I$, for every $r \in \mathbb{N}$, then $s_r(A) = s_r(\lambda_r(A)I) = |\lambda_r(A)|$, for every $r \in \mathbb{N}$. Hence $(s_r(A))_{r=0}^\infty \in C(t,p)$, so $A \in (C(t,p))_h^{(s)}(T, U)$. Therefore, the sequence $(s_r(A))_{r=0}^\infty$ is the eigenvalues of A. Secondly, let the sufficient conditions be verified and $A \in (C(t,p))_h^{(s)}(T, U)$. Therefore, $(s_r(A))_{r=0}^\infty \in C(t,p)$. Hence, we have

$$\sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} (j+1)^{-t} s_j(A)}{2^i} \right)^{p_i} \geq \sum_{i=0}^{\infty} [s_i(A)]^{p_i}. \quad (39)$$

Hence, $(s_i(A))_{i=0}^\infty \in \ell_{(p_i)}$, so $\lim_{i\to\infty} s_i(A) = 0$. Assume $\|A - s_i(A)I\|^{-1}$ exists, for every $i \in \mathbb{N}$. Therefore, $\|A - s_i(A)I\|^{-1}$ exists and is bounded, for every $i \in \mathbb{N}$. So, $\lim_{i\to\infty} \|A - s_i(A)I\|^{-1} = \|A\|^{-1}$ exists and is bounded. From the prequasi operator ideal of $((C(t,p))^{(s)}, V)$, we obtain

$$I = AA^{-1} \in (C(t,p))_h^{(s)}(T,U) \Rightarrow (s_i(I))_{i=0}^{\infty} \in C(t,p) \Rightarrow \lim_{i\to\infty} s_i(I) = 0. \quad (40)$$

We have a contradiction, since $\lim_{i\to\infty} s_i(I) = 1$. Therefore, $\|A - s_i(A)I\| = 0$, for every $i \in \mathbb{N}$. This gives $A \in (C(t,p))_h^{(\lambda)}(T,U)$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Generalized Weighted Statistical Convergence for Double Sequences of Fuzzy Numbers and Associated Korovkin-Type Approximation Theorem

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We define the notions of weighted (λ, μ)-statistical convergence of order (γ_1, γ_2) and strongly weighted (λ, μ)-summability of (γ_1, γ_2) for fuzzy double sequences, where $0 < \gamma_1, \gamma_2 \leq 1$. We establish an inclusion result and a theorem presenting a connection between these concepts. Moreover, we apply our new concept of weighted (λ, μ)-statistical convergence of order (γ_1, γ_2) to prove Korovkin-type approximation theorem for functions of two variables in a fuzzy sense. Finally, an illustrative example is provided with the help of q-analogue of fuzzy Bernstein operators for bivariate functions which shows the significance of our approximation theorem.

1. Introduction and Preliminaries

The notion of weighted statistical convergence for sequences of real numbers has been studied by Karakaya and Chishti [1] as a generalization of the concept of statistical convergence which is related to the idea of asymptotic density (or natural density) of a subset of $\mathbb{N}$, the set of natural numbers, according to Fast [2]. The weighted statistical convergence was further improved by Mursaleen et al. [3] and later generalized by Belen and Mohiuddine [4] with a view of nondecreasing sequence of positive numbers. Ghosal [5] added constraints to these ideas. For some other related work, we refer the interested reader to [6–10]. Recently, Çolak [11] defined the notion of statistical convergence of order $\alpha (0 < \alpha \leq 1)$ and strong p-Cesàro summability of order α while the order of statistical was also given in [12], and it reduces to strong p-Cesàro summability for $\alpha = 1$ according to Connor [13].

The idea convergence for double sequence $y = (y_{kl})$ was presented by Pringsheim [14]; that is, (y_{kl}) is convergent to ξ in the Pringsheim sense if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|y_{kl} - \xi| < \varepsilon$ whenever $k, l > N$. The idea of statistical convergence for $y = (y_{kl})_{(k,l) \in \mathbb{N} \times \mathbb{N}}$ was introduced by Mursaleen and Edely [15] as follows: (y_{kl}) is statistically convergent to ξ if for every $\varepsilon > 0$,

$$\lim_{k_1, l_1 \to \infty} \frac{1}{k_1 l_1} \{(k, l), k \leq k_1, l \leq l_1 : |y_{kl} - \xi| \geq \epsilon\} = 0, \qquad (1)$$

where the limit is taken in Pringsheim's sense while the order of notion was studied in [16]. This idea was further generalized by Mursaleen et al. [17] and called it (λ, μ)-statistical convergence, and for single sequence in [18] and its weighted variant were presented by Cinar and Et [19], which were later on studied by various authors in various setups [20–22]. Throughout this paper, limit of the double sequences means limit in Pringsheim's sense.

In the very recent past, for sequence of fuzzy numbers, Mohiuddine et al. [23] weighted statistical convergence and strong weighted summability by using the idea of difference operator and established a connection between these notions while under certain conditions, these notions reduce to statistical convergence and strong p-Cesáro summability, respectively, in [24, 25]. Savas and Mursaleen [26] defined statistical convergence and statistically Cauchy for fuzzy double sequences and obtained that these concepts are equivalent. The statistical analogue of pointwise and uniformly convergent double sequences for fuzzy-valued functions were discussed by Mohiuddine et al. [21]. These concepts were studied and extended by several authors; for instance, see [27–31].

Recall that, in [32], a fuzzy number is a fuzzy set on the real axis; that is, $u : \mathbb{R} \longrightarrow [0,1]$ which is normal, fuzzy convex, upper semicontinuous and the closure of the set $\{y \in \mathbb{R} : u(y) > 0\}$ is compact. By using the symbol $\mathscr{L}(\mathbb{R})$, we denote the set of all fuzzy numbers on $\mathbb{R}$. α-Level set $[u]_\alpha$ of u is given as follows:

$$[u]_a = \begin{cases} \{y \in \mathbb{R} : u(y) \geq \alpha\} & \text{if } 0 < \alpha \leq 1, \\ \overline{\{y \in \mathbb{R} : u(y) > 0\}} & \text{if } \alpha = 0. \end{cases} \tag{2}$$

For each $\alpha \in [0,1]$, $[u]_\alpha$ is closed and bounded and nonempty interval for each $\alpha \in [0,1]$ defined by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, where $u_\alpha^- \leq u_\alpha^+$ and $u_\alpha^-, u_\alpha^+ \in \mathbb{R}$. For any $u_1, u_2 \in \mathscr{L}(\mathbb{R})$, the partial ordering is defined by

$$\begin{aligned} &u_1 \preceq u_2 \Leftrightarrow u_{1\alpha}^- \leq u_{2\alpha}^-, \\ &u_{1\alpha}^+ \leq u_{2\alpha}^+, \end{aligned} \tag{3}$$

for all $0 \leq \alpha \leq 1$. For any $u_1, u_2 \in \mathscr{L}(\mathbb{R})$ and $\alpha \in [0,1]$, define the following:

$$\begin{aligned} &[u_1 \oplus u_2]_\alpha = [u_1]_\alpha \oplus [u_2]_\alpha, \\ &[\mu \odot \mu_1]_\alpha = \mu [u_1]_\alpha \quad (\mu \in \mathbb{R}). \end{aligned} \tag{4}$$

By means of the Hausdorff metric D, the metric $d : \mathscr{L}(\mathbb{R}) \times \mathscr{L}(\mathbb{R}) \longrightarrow \mathbb{R}_+$ is defined by

$$\begin{aligned} d(u_1, u_2) &= \sup_{\alpha \in [0,1]} D([u_1]_\alpha, [u_2]_\alpha) \\ &= \sup_{\alpha \in [0,1]} \max \{|u_{1\alpha}^- - u_{2\alpha}^-|, |u_{1\alpha}^+ - u_{2\alpha}^+|\}. \end{aligned} \tag{5}$$

It is clear from [33] that $(\mathscr{L}(\mathbb{R}), d)$ is a complete metric space.

Suppose $g_1, g_2 : [a,b] \times [a,b] \longrightarrow \mathscr{L}(\mathbb{R})$ are two fuzzy number-valued functions. Then, the distance between g_1 and g_2 is defined by

$$\begin{aligned} d^*(g_1, g_2) = &\sup_{y \in [a,b] \times [a,b]} \sup_{\alpha \in [0,1]} \max \\ &\cdot \{|g_{1\alpha}^- - g_{2\alpha}^-|, |g_{1\alpha}^+ - g_{2\alpha}^+|\}. \end{aligned} \tag{6}$$

2. Generalized Weighted Statistical Convergence of Order (γ_1, γ_2)

Suppose two nondecreasing sequence $\lambda = (\lambda_k)$ and $\mu = (\mu_l)$ of positive integers such that both λ and μ tending to infinity; that is, $\lambda_k \longrightarrow \infty$ and $\mu_l \longrightarrow \infty$ as $k \longrightarrow \infty$ and $l \longrightarrow \infty$, respectively, such that

$$\begin{aligned} &\lambda_{k+1} \leq \lambda_k + 1, \quad \lambda_k = 1, \\ &\mu_{l+1} \leq \mu_l + 1, \quad \mu_l = 1. \end{aligned} \tag{7}$$

Consider two sequences $u = (u_m)$ and $v = (v_n)$ of nonnegative numbers such that $\liminf_m u_m > 0$, $\liminf_n v_n > 0$ and

$$\begin{aligned} U_{\lambda_k} &= \sum_{m \in [k - \lambda_k + 1, k]} u_m \longrightarrow \infty (k \longrightarrow \infty), \\ V_{\mu 1} &\sum_{n \in [l - \mu l + 1, l]} v_n \longrightarrow \infty (l \longrightarrow \infty). \end{aligned} \tag{8}$$

Assume that $M \subseteq \mathbb{N} \times \mathbb{N}$ and let $0 < \gamma_1, \gamma_2 \leq 1$. We define the weighted (λ, μ) density of order (γ_1, γ_2), denoted by $\delta_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$, of M by

$$\delta_{2,\gamma 1,\gamma 2}^{\bar{N}(\lambda,\mu)}(M) = \lim_{k,l \to \infty} \frac{\left| A_{U_{\lambda_k}, V_{\mu_l}} \right|}{U_{\lambda k}^{\gamma 1} V_{\mu l}^{\gamma 2}}, \tag{9}$$

provided the limit in the last relation (9) exists, where

$$A_{U_{\lambda_k}, V_{\mu l}} = \{m \leq U_{\lambda_k}, n \leq V_{\mu l} : (m,n) \in M\}. \tag{10}$$

Definition 1. A fuzzy double sequence (s_{mn}) is said to be weighted (λ, μ) -statistically convergent of order (γ_1, γ_2) , where $0 < \gamma_1, \gamma_2 \leq 1$; in short, we shall write $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$ -convergent, to a fuzzy number s_0 if for every $\varepsilon > 0$, the set

$$B = \{(m,n) \in \mathbb{N} \times \mathbb{N} :: u_m v_n d(s_{mn}, s_0) \geq \epsilon\} \tag{11}$$

has $\delta_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$ density zero; that is,

$$S_{2,\gamma 1,\gamma 2}^{\bar{N}(\lambda,\mu)}(B) = 0. \tag{12}$$

Equivalently, we can write

$$\begin{aligned} \lim_{k,l \to \infty} \frac{1}{U_{\lambda k}^{\gamma_1} V_{\mu l}^{\gamma_2}} &|\{(m,n), m \leq U_{\lambda_k}, n \\ &\leq V_{\mu l} : u_m v_n d(s_{mn}, s_0) \geq \epsilon\}| = 0. \end{aligned} \tag{13}$$

We denote this convergence by $s_{mn} \xrightarrow{S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}} s_0$ or $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} - \lim s_{mn} = s_0$. We denote the set of all $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$-convergence by $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$.

The choice of $\gamma_1 = \gamma_2 = 1$ in the notion of $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$-convergence gives weighted (λ, μ)-statistically convergence for fuzzy double sequences. In addition, if we choose $\lambda_k = k$, $\mu_l = l$, $u_m = 1$, and $v_n = 1$ for all m, n, then $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$-convergence gives the notion of (λ, μ)-statistically convergence in a fuzzy sense [26].

For $0 < \gamma_1, \gamma_2 \leq 1$, the notion of weighted (λ, μ)-statistically convergent of order (γ_1, γ_2) is well but not well defined for $\gamma_1, \gamma_2 > 1$. To show this assertion, we consider the following example.

Example 1. Assume that $\gamma_1, \gamma_2 > 1$, and define the fuzzy double sequence (s_{mn}) by

$$s_{mn} = \begin{cases} \bar{1} & \text{if } m+n \text{ even}, \\ \bar{0} & \text{if } m+n \text{ odd}. \end{cases} \quad (14)$$

Let $\varepsilon > 0$ be given, and let $u_m = 1$ and $v_n = 1 \forall m, n \in \mathbb{N}$. Then,

$$\lim_{k,l\to\infty} \frac{1}{U_{\lambda_k}^{\gamma1} V_{\mu l}^{\gamma2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu l} : u_m v_n d(s_{mn}, \bar{1}) \ge \epsilon\right\}\right| \le \lim_{k,l\to\infty} \frac{\lambda_k \mu_l}{2\lambda_k{}^{\gamma1} \mu l^{\gamma_2}} = 0. \quad (15)$$

On the other hand,

$$\lim_{k,l\to\infty} \frac{1}{U_{\lambda_k}^{\gamma1} V_{\mu l}^{\gamma2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu l} : u_m v_n d(s_{mn}, \bar{0}) \ge \epsilon\right\}\right| \le \lim_{k,l\to\infty} \frac{\lambda_k \mu_l}{2\lambda_k{}^{\gamma1} \mu l^{\gamma_2}} = 0. \quad (16)$$

We conclude from (15) and (16) that

$$S_{2,\gamma1,\gamma2}^{\bar{N}(\lambda,\mu)} - \lim s_{mn} = \bar{1}, \quad S_{2,\gamma1,\gamma2}^{\bar{N}(\lambda,\mu)} - \lim s_{mn} = \bar{0}, \quad (17)$$

which is impossible.

Example 2. Define (s_{mn}) as

$$s_{mn} = \begin{cases} b & \text{if } m = k^2, n = l^2, \\ \bar{0} & \text{otherwise}, \end{cases} \quad (18)$$

for $i, j \in \mathbb{N}$, where b stands for a fixed fuzzy number. Let $\gamma_1 = \gamma_2 = 1$, and let $u_m = 1$ and $v_n = 1 \forall m, n$. Let $\lambda_k = k$ and $\mu_l = l$. Suppose b_α^+ is the upper bound of the α-cut. Then, for $\varepsilon > 0$, $0 \le \alpha \le 1$ and $i, j \in \mathbb{N}$, one writes

$$d(s_{mn}, \bar{0}) = \begin{cases} \sup_{0\le\alpha\le1} \left|b_\alpha^+\right| & \text{if } m = k^2, n = l^2, \\ \bar{0} & \text{otherwise}. \end{cases} \quad (19)$$

Thus,

$$\lim_{k,l\to\infty} \frac{1}{U_{\lambda_k}^{\gamma_1} V_{\mu_l}^{\gamma_2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu_l} : u_m v_n d(s_{mn}, \bar{0}) \ge \varepsilon\right\}\right| \le \lim_{k,l\to\infty} \frac{\sqrt[2]{k}\sqrt[2]{l}}{kl} = 0. \quad (20)$$

Hence, $s_{mn} \xrightarrow{S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}} \bar{0}$ but (s_{mn}) is not convergent.

Theorem 2. *Suppose $0 < \gamma_1, \gamma_2, \eta_1, \eta_2 \le 1$ such that $\gamma_1 \le \eta_1$ and $\gamma_2 \le \eta_2$. Then, $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} \subseteq S_{2,\eta_1,\eta_2}^{\bar{N}(\lambda,\mu)}$.*

Proof. Let $\varepsilon > 0$ be given, and let $\gamma_1, \gamma_2, \eta_1, \eta_2 \in (0, 1]$. If $\gamma_1 \le \eta_1$ and $\gamma_2 \le \eta_2$, then, for every $\varepsilon > 0$, we can write

$$\frac{1}{U_{\lambda_k}^{\gamma_1} V_{\mu_l}^{\gamma_2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu_l} : u_m v_n d(s_{mn}, s_0) \ge \varepsilon\right\}\right|, \le \frac{1}{U_{\lambda_k}^{\eta_1} V_{\mu_l}^{\eta_2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu_l} : u_m v_n d(s_{mn}, s_0) \ge \varepsilon\right\}\right|, \quad (21)$$

which yields that $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} \subseteq S_{2,\eta_1,\eta_2}^{\bar{N}(\lambda,\mu)}$.

Theorem 3. *Suppose two fuzzy double sequence (s_{mn}) and (t_{mn}) such that $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} - \lim s_{mn} = s_0$ and $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} - \lim t_{mn} = t_0$. Then,*

(i) $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} - \lim c s_{mn} = c s_0 (c \in \mathbb{R})$

(ii) $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} - \lim(s_{mn} + t_{mn}) = s_0 + t_0$

Proof.

(i) Let $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} - \lim s_{mn} = s_0$. It is clear for $c = 0$. Let $c \ne 0$. For $\varepsilon > 0$, we write

$$\begin{aligned} &\frac{1}{U_{\lambda_k}^{\gamma_1} V_{\mu_l}^{\gamma_2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu_l} : u_m v_n d(c s_{mn}, c s_0) \ge \varepsilon\right\}\right| \\ &= \frac{1}{U_{\lambda_k}^{\gamma_1} V_{\mu_l}^{\gamma_2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu_l} : u_m v_n |c| d(s_{mn}, s_0) \ge \varepsilon\right\}\right| \\ &= \frac{1}{U_{\lambda_k}^{\gamma_1} V_{\mu_l}^{\gamma_2}} \left|\left\{(m,n), m \le U_{\lambda_k}, n \le V_{\mu_l} : u_m v_n d(s_{mn}, s_0) \ge \frac{\varepsilon}{|c|}\right\}\right| \end{aligned} \quad (22)$$

Thus, (i) holds.

Suppose that $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}-\lim s_{mn}=s_0$ and $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}-\lim t_{mn}=t_0$. Let $\varepsilon>0$ be given. Then,

$$\begin{aligned}&\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\left|\left\{(m,n), m\le U_{\lambda_k}, n\le V_{\mu_l}: u_m v_n d(s_{mn}+t_{mn}, s_0+t_0)\right.\right.\\&\left.\left.\ge\varepsilon\right\}\right|\le\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\left|\left\{(m,n), m\le U_{\lambda_k}, n\le V_{\mu_l}: u_m v_n d(s_{mn},s_0)\right.\right.\\&\left.\left.\ge\frac{\varepsilon}{2}\right\}\right|+\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\left|\left\{(m,n), m\le U_{\lambda_k}, n\right.\right.\\&\left.\left.\le V_{\mu_l}: u_m v_n d(t_{mn},t_0)\ge\frac{\varepsilon}{2}\right\}\right|\end{aligned}\tag{23}$$

Hence, (ii) holds.

Definition 4. Assume that $\gamma_1,\gamma_2\in(0,1]$. Then, a fuzzy double sequence (s_{mn}) is said to be strongly weighted (λ,μ)-summable of order (γ_1,γ_2); in short, we shall write $S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}$-summable, to fuzzy number s_0, denoted by

$$s_{mn}\xrightarrow{M_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0,\tag{24}$$

such that

$$\lim_{k,l\to\infty}\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{m\in[k-\lambda_k+1,k]}\sum_{n\in[l-\mu_l+1,l]}u_m v_n d(s_{mn},s_0)=0.\tag{25}$$

Theorem 5. *Consider a fuzzy double sequence $s=(s_{mn})$, and let $0<\gamma_1,\gamma_2\le 1$. Then,*

(i) $s_{mn}\xrightarrow{M_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$ implies $s_{mn}\xrightarrow{S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$

(ii) (s_{mn}) is a bounded double sequence and $s_{mn}\xrightarrow{S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$ imply $s_{mn}\xrightarrow{M_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$

Proof.

(i) Let $s_{mn}\xrightarrow{M_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$, and let $\varepsilon>0$ be given. Then,

$$\begin{aligned}&\sum_{m\in[k-\lambda_k+1,k]}\sum_{n\in[l-\mu_l+1,l]}u_m v_n d(s_{mn},s_0)\\&=\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}u_m v_n d(s_{mn},s_0)\\&+\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}u_m v_n d(s_{mn},s_0)\\&\ge\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}u_m v_n d(s_{mn},s_0)\end{aligned}\tag{26}$$

Therefore, we get

$$\begin{aligned}&\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{m\in[k-\lambda_k+1,k]}\sum_{n\in[l-\mu_l+1,l]}u_m v_n d(s_{mn},s_0)\\&\ge\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}\varepsilon\\&\ge\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\left|\left\{(m,n), m\le U_{\lambda_k}, n\right.\right.\\&\left.\left.\le V_{\mu_l}: u_m v_n d(s_{mn},s_0)\ge\varepsilon\right\}\right|\varepsilon.\end{aligned}\tag{27}$$

Letting $k,l\longrightarrow\infty$ on both sides of the above relation, we get $s_{mn}\xrightarrow{S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$.

Let $s_{mn}\xrightarrow{S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$. Since $s=(s_{mn})\in\mathscr{L}_{2,\infty}$, $d(_{mn},s_0)\le C$ for all $m,n\in\mathbb{N}$. For a given $\varepsilon>0$, one writes

$$\begin{aligned}&\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{m\in[k-\lambda_k+1,k]}\sum_{n\in[l-\mu_l+1,l]}u_m v_n d(s_{mn},s_0)\\&=\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)\ge\varepsilon}}u_m v_n d(s_{mn},s_0)\\&+\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}u_m v_n d(s_{mn},s_0)\\&\le\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}C\\&+\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\sum_{\substack{m\in[k-\lambda_k+1,k]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}\sum_{\substack{n\in[l-\mu_l+1,l]\\u_m v_n d(s_{mn},s_0)<\varepsilon}}\varepsilon\\&\le\frac{1}{U_{\lambda_k}^{\gamma_1}V_{\mu_l}^{\gamma_2}}\left|\left\{(m,n), m\le U_{\lambda_k}, n\right.\right.\\&\left.\left.\le V_{\mu_l}: u_m v_n d(s_{mn},s_0)\ge\varepsilon\right\}\right|C+\varepsilon.\end{aligned}\tag{28}$$

It follows that $s_{mn}\xrightarrow{M_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}}s_0$.

3. Application to Fuzzy Korovkin-Type Theorems

The fuzzy version of Korovkin theorem has been obtained by Anastassiou [34] while the classification of this result has been established by Korovkin in [35] (also see [36–42]), and then Anastassiou and Duman [43] and Karaisa and Kadak [44] studied this result in a statistical sense. In the recent past, Mohiuddine et al. [23] investigated the fuzzy Korovkin-type approximation theorem through weighted statistical convergence of fuzzy sequence based on difference operators. The fuzzy Korovkin theorem for function of two variables was discussed by Demirci and Karakus [45] with the help of A-statistical convergence for a sequence of fuzzy numbers. Here, we prove the fuzzy Korovkin-type theorem for the function of two variables with the help of $S^{\bar{N}(\lambda,\mu)}_{2,\gamma_1,\gamma_2}$ -convergence.

Assume that $I=[a,b](a,b\in\mathbb{R})$ and $g: I\times I(=I^2)\longrightarrow \mathscr{L}(\mathbb{R})$ is a fuzzy number-valued function. Then, g is fuzzy continuous at a point S of $I\times I$ if $D(g(s),g(S))\longrightarrow 0$ whenever $s=(s_{mn})$ is Pringsheim's convergent to S in a fuzzy sense. We denote the set of fuzzy continuous function on $I^2=[a,b]\times[a,b]$by $C_{\mathscr{L}}(I^2)$. Here, $C_{\mathscr{L}}(I^2)$ is only a cone but not a vector space and let $T: C_{\mathscr{L}}(I^2)\longrightarrow C_{\mathscr{L}}(I^2)$ be an operator. Further, T is said to be a fuzzy linear operator if

$$T(\beta_1\cdot g_1\oplus\beta_2\cdot g_2;x,y)=\beta_1\cdot T(g_1;x,y)\oplus\beta_2\cdot T(g_2;x,y), \tag{29}$$

for all $\beta_1,\beta_2\in\mathbb{R}$, $g_1,g_2\in C_{\mathscr{L}}(I^2)$, and $(x,y)\in I^2$. In addition, if $T(g_1;x,y)^\circ T(g_2;x,y)$ holds for all $g_1,g_2\in C_{\mathscr{L}}(I^2)$ and $(x,y)\in I^2$ with $g_1(x,y)^\circ g_2(x,y)$, then T is called fuzzy position linear operator.

Theorem 6. *Let $(A_{nr})_{n,r\in\mathbb{N}}$ be a double sequence such that $A_{nr}: C_{\mathscr{L}}(I^2)\longrightarrow C_{\mathscr{L}}(I^2)$. Let $(\tilde{A}_{nr})_{n,r\in\mathbb{N}}$ be corresponding positive linear operators acting from $C(I^2)$ into itself having the relation*

$$\{A_{nr}(g;x,y)\}^{\pm}_{\alpha}=\tilde{A}_{nr}\left(g^{\pm}_{\alpha};x,y\right)\left(\forall\alpha\in[0,1],(x,y)\in I^2,g\in C_{\mathscr{L}}\left(I^2\right),n,r\in\mathbb{N}\right). \tag{30}$$

Assume further that

$$S^{\bar{N}(\lambda,\mu)}_{2,\gamma_1,\gamma_2}-\lim\left\|\tilde{A}_{nr}(g_i)-g_i\right\|=0(i=0,1,2,3), \tag{31}$$

where $g_0(t_1,t_2)=1$, $g_1(t_1,t_2)=t_1$, $g_2(t_1,t_2)=t_2$, and $g_3(t_1,t_2)=t_1+t_2$. Then,

$$S^{\bar{N}(\lambda,\mu)}_{2,\gamma_1,\gamma_2}-\lim d^*(A_{nr}(g),g)=0, \tag{32}$$

for all $n,r\in\mathbb{N}$ and $g\in C_{\mathscr{L}}(I^2)$.

Proof. Suppose that (31) holds and $(n,r)\in\mathbb{N}\times\mathbb{N}$. Suppose also that $(x,y)\in I^2$, $g\in C_{\mathscr{L}}(I^2)$, and $\alpha\in[0,1]$. Since $g^{\pm}_{\alpha}\in C(I^2)$, it follows that, for a given $\varepsilon>0$, there exists $\delta>0$ such that for all $(t_1,t_2)\in I^2$, we have $|g^{\pm}_{\alpha}(t_1,t_2)-g^{\pm}_{\alpha}(x,y)|<\varepsilon$ whenever $|t_1-x|<\delta$ and $|t_2-y|<\delta$.

By the fuzzy boundedness of g, we may write $|g^{\pm}_{\alpha}(x,y)|\le C^{\pm}_{\alpha}$ for all $(x,y)\in I^2$, where $C^{\pm}_{\alpha}=\|g^{\pm}_{\alpha}\|$ for all $a<y<b$. We then obtain for all $(t_1,t_2),(x,y)\in I^2$ that

$$\left|g^{\pm}_{\alpha}(t_1,t_2)-g^{\pm}_{\alpha}(x,y)\right|\le 2C^{\pm}_{\alpha}. \tag{33}$$

With the help of last relations, for all $|t_1-x|<\delta$ and $|t_2-y|<\delta$, we obtain

$$\left|g^{\pm}_{\alpha}(t_1,t_2)-g^{\pm}_{\alpha}(x,y)\right|<\varepsilon+\frac{2C^{\pm}_{\alpha}}{\delta^2}\left\{(t_1-x)^2+(t_2-y)^2\right\}. \tag{34}$$

Since $\tilde{A}_{nr}$ is a positive linear operator, by applying this operator to (34), we get

$$\begin{aligned}&\left|\tilde{A}_{nr}\left(g^{\pm}_{\alpha}(t_1,t_2);x,y\right)-g^{\pm}_{\alpha}(x,y)\right|\\&=\left|\tilde{A}_{nr}\left(g^{\pm}_{\alpha}(t_1,t_2)-g^{\pm}_{\alpha}(x,y);x,y\right)+g^{\pm}_{\alpha}(x,y)\left[\tilde{A}_{nr}(g_0;x,y)-g_0(x,y)\right]\right|\\&\le\tilde{A}_{nr}\left(\left|g^{\pm}_{\alpha}(t_1,t_2)-g^{\pm}_{\alpha}(x,y)\right|;x,y\right)+C^{\pm}_{\alpha}\left|\tilde{A}_{nr}(g_0;x,y)-g_0(x,y)\right|\\&\le\tilde{A}_{nr}\left(\varepsilon+\frac{2C^{\pm}_{\alpha}}{\delta^2}\left[(t_1-x)^2+(t_2-y)^2\right];x,y\right)+C^{\pm}_{\alpha}\left|\tilde{A}_{nr}(g_0;x,y)-g_0(x,y)\right|\\&\le\varepsilon+\left(\varepsilon+C^{\pm}_{\alpha}+\frac{4C^{\pm}_{\alpha}E^2}{\delta^2}\right)\left|\tilde{A}_{nr}(g_0;x,y)-g_0(x,y)\right|\\&+\frac{4C^{\pm}_{\alpha}E}{\delta^2}\left|\tilde{A}_{nr}(g_1;x,y)-g_1(x,y)\right|\\&+\frac{4C^{\pm}_{\alpha}E}{\delta^2}\left|\tilde{A}_{nr}(g_2;x,y)-g_2(x,y)\right|\\&+\frac{2C^{\pm}_{\alpha}}{\delta^2}\left|\tilde{A}_{nr}(g_3;x,y)-g_3(x,y)\right|,\end{aligned} \tag{35}$$

where $E=\max\{|x|,|y|\}$. We thus obtain by letting

$$\Omega^{\pm}_{\alpha}(\varepsilon)=\max\left\{\varepsilon+C^{\pm}_{\alpha}+\frac{4C^{\pm}_{\alpha}E^2}{\delta^2},\frac{4C^{\pm}_{\alpha}E}{\delta^2},\frac{2C^{\pm}_{\alpha}E}{\delta^2}\right\} \tag{36}$$

and taking $\sup_{(x,y)\in I^2}$ that

$$\begin{aligned}\left\|\tilde{A}_{nr}\left(g^{\pm}_{\alpha}\right)-g^{\pm}_{\alpha}\right\|&\le\varepsilon+\Omega^{\pm}_{\alpha}(\varepsilon)\left\{\left\|\tilde{A}_{nr}(g_0)-g_0\right\|\right.\\&+\left\|\tilde{A}_{nr}(g_1)-g_1\right\|+\left\|\tilde{A}_{nr}(g_2)-g_2\right\|\\&\left.+\left\|\tilde{A}_{nr}(g_3)-g_3\right\|\right\}.\end{aligned} \tag{37}$$

From (30), we can write

$$\begin{aligned}d^*(A_{mr}(g),g)&=\sup_{(x,y)\in I^2}d(A_{mr}(g;x,y),g(x,y))\\&=\sup_{(x,y)\in I^2}\sup_{\alpha\in[0,1]}\max\left\{\left|\tilde{A}_{mr}\left(g^{-}_{\alpha};x,y\right)-g^{-}_{\alpha}(x,y)\right|,\right.\\&\left.\left|\tilde{A}_{mr}\left(g^{+}_{\alpha};x,y\right)-g^{+}_{\alpha}(x,y)\right|\right\},\end{aligned} \tag{38}$$

and so

$$d^*(A_{nr}(g), g) = \sup_{\alpha\in[0,1]} \max\left\{\left\|\tilde{A}_{nr}(g_\alpha^-) - g_\alpha^-\right\|, \left\|\tilde{A}_{nr}(g_\alpha^+) - g_\alpha^+\right\|\right\}. \quad (39)$$

With the help of (37) and (39), we get

$$d^*(A_{nr}(g), g) \le \varepsilon + \Omega(\varepsilon)\left\{\left\|\tilde{A}_{nr}(g_0) - g_0\right\| + \left\|\tilde{A}_{nr}(g_1) - g_1\right\| + \left\|\tilde{A}_{nr}(g_2) - g_2\right\| + \left\|\tilde{A}_{nr}(g_3) - g_3\right\|\right\}, \quad (40)$$

where

$$\Omega(\varepsilon) = \sup_{\alpha\in[0,1]} \max\left\{\Omega_\alpha^-(\varepsilon), \Omega_\alpha^+(\varepsilon)\right\}. \quad (41)$$

Multiplying by the product of two nonnegative sequences (u_n) and (v_r) as details given in the previous section, one gets

$$\begin{aligned} u_n v_r d^*(A_{nr}(g), g) \le u_n v_r \varepsilon + \Omega(\varepsilon)\Big\{ & u_n v_r\left\|\tilde{A}_{nr}(g_0) - g_0\right\| \\ & + u_n v_r\left\|\tilde{A}_{nr}(g_1) - g_1\right\| \\ & + u_n v_r\left\|\tilde{A}_{nr}(g_2) - g_2\right\| \\ & + u_n v_r\left\|\tilde{A}_{nr}(g_3) - g_3\right\|\Big\}. \end{aligned} \quad (42)$$

For a given $\varrho > 0$, choose $\varepsilon > 0$ such that $\varepsilon u_n v_r < \varrho$. Then, upon setting,

$$\begin{aligned} F &= \left\{(n, r): u_n v_r d^*(A_{nr}(g), g) \ge \varrho\right\}, \\ F_0 &= \left\{(n, r): u_n v_r\left\|\tilde{A}_{nr}(g_0) - g_0\right\| \ge \frac{\varrho - \varepsilon u_n v_r}{4\Omega(\varepsilon)}\right\}, \\ F_1 &= \left\{(n, r): u_n v_r\left\|\tilde{A}_{nr}(g_1) - g_1\right\| \ge \frac{\varrho - \varepsilon u_n v_r}{4\Omega(\varepsilon)}\right\}, \\ F_2 &= \left\{(n, r): u_n v_r\left\|\tilde{A}_{nr}(g_2) - g_2\right\| \ge \frac{\varrho - \varepsilon u_n v_r}{4\Omega(\varepsilon)}\right\}, \\ F_3 &= \left\{(n, r): u_n v_r\left\|\tilde{A}_{nr}(g_3) - g_3\right\| \ge \frac{\varrho - \varepsilon u_n v_r}{4\Omega(\varepsilon)}\right\}. \end{aligned} \quad (43)$$

It follows that

$$F \subset F_0 \cup F_1 \cup F_2 \cup F_3. \quad (44)$$

Consequently, by taking weighted (λ, μ) density of order (γ_1, γ_2), we obtain

$$\begin{aligned} \delta_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}(F) \le\ & \delta_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}(F_0) + \delta_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}(F_1) \\ & + \delta_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}(F_2) + \delta_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}(F_3). \end{aligned} \quad (45)$$

Using the hypothesis (31), we conclude that

$$S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)} - \lim d^*(A_{nr}(g), g) = 0 \quad (46)$$

holds for all $(n, r) \in \mathbb{N} \times \mathbb{N}$ and $g \in C_{\mathscr{L}}(I^2)$.

Recall that, for any nonnegative integer j, the q-integer $[j]_q (q > 0)$ is given by

$$[j]_q = \begin{cases} \dfrac{1 - q^j}{1 - q} & \text{if } q \ne 1, \\ j & \text{if } q = 1, \end{cases} \quad (47)$$

and the q-factorial $[j]_q!$ by

$$\begin{aligned} &[j]_q! = [j]_q![j-1]_q!\cdots[1]_q!, \\ &[0]_q! = 1. \end{aligned} \quad (48)$$

The q-binomial coefficients, for $0 \le m \le j$, is defined by

$$\begin{bmatrix} j \\ m \end{bmatrix}_q = \frac{[j]_q!}{[m]_q![j-m]_q!}. \quad (49)$$

The bivariate case of classical q-Bernstein operators were introduced and studied by Barbosu [46]. We are now considering the fuzzy analogue of these operators to construct an example to illustrate our last theorem.

Example 3. Consider the fuzzy analogue of bivariate q -Bernstein operators, defined by

$$B_{nr}^{\mathscr{L},q_1,q_2}(g; x, y) = \bigoplus_{k=0}^{n}\bigoplus_{m=0}^{r} g\left(\frac{[k]_{q_1}}{[n]_{q_1}}, \frac{[m]_{q_2}}{[r]_{q_2}}\right) \odot b_{k,m,n,r}^{q_1,q_2}(x, y) (n, r \in \mathbb{N}), \quad (50)$$

for all $g \in C_{\mathscr{L}}(I^2) (I^2 = [0, 1] \times [0, 1])$, $(x, y) \in [0, 1] \times [0, 1]$, and $0 < q_1, q_2 < 1$, where

$$b_{k,m,n,r}^{q_1,q_2}(x, y) = \begin{bmatrix} n \\ k \end{bmatrix}_{q_1} \begin{bmatrix} r \\ m \end{bmatrix}_{q_2} x^k (1 - x)_{q_1}^{n-k} y^m (1 - y)_{q_2}^{r-m}. \quad (51)$$

Note that

$$(1 - x)_q^n = \prod_{k=0}^{n-1}\left(1 - q^k x\right). \quad (52)$$

The above operators $B_{nr}^{\mathscr{L},q_1,q_2}(g;x,y)$ are positive and linear, and one writes

$$\begin{aligned}&\left\{B_{nr}^{\mathscr{L},q_1,q_2}(g;x,y)\right\}_{\alpha}^{\pm}\\&\quad=B_{nr}^{q_1,q_2}\left(g_{\alpha}^{\pm};x,y\right)\\&\quad=\sum_{k=0}^{n}\sum_{m=0}^{r}b_{k,m,n,r}^{q_1,q_2}(x,y)\,g_{\alpha}^{\pm}\left(\frac{[k]_{q_1}}{[n]_{q_1}},\frac{[m]_{q_2}}{[r]_{q_2}}\right),\end{aligned}\tag{53}$$

for all $0\leq\alpha\leq 1$, $g_{\alpha}^{\pm}\in C[0,1]$). Suppose that $q_1=q_{1n}$ and $q_2=q_{2r}$ such that $q_{1n}\longrightarrow 1$ as $s\longrightarrow\infty$ and $q_{2r}\longrightarrow 1$ as $r\longrightarrow\infty$. In view of this assumption, one can easily find that

$$\begin{aligned}&[n]_{q_1}\longrightarrow\infty(n\longrightarrow\infty),\\&[r]_{q_2}\longrightarrow\infty(r\longrightarrow\infty).\end{aligned}\tag{54}$$

We define the sequence of positive linear operators by

$$\tilde{\Phi}_{nr}\left(g_{\alpha}^{\pm};x,y\right)=(1+s_{nr})B_{nr}^{q_1,q_2}\left(g_{\alpha}^{\pm};x,y\right).\tag{55}$$

In this case, (s_{nr}) is same as defined in Example 2 with $\gamma_1=\gamma_2=1$, $\lambda_k=k$, $\mu_l=l$, $u_n=1$, and $v_r=1$ for all $n,r\in\mathbb{N}$. Therefore, we get

$$\begin{aligned}&\tilde{\Phi}_{nr}(g_0;x,y)=1+s_{nr},\\&\tilde{\Phi}_{nr}(g_1;x,y)=(1+s_{nr})x,\\&\tilde{\Phi}_{nr}(g_2;x,y)=(1+s_{nr})y,\\&\tilde{\Phi}_{nr}(g_3;x,y)=(1+s_{nr})\left(x^2+y^2+\frac{x-x^2}{[n]_{q_1}}+\frac{y-y^2}{[r]_{q_2}}\right).\end{aligned}\tag{56}$$

Since

$$S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}-\lim s_{nr}=0,\tag{57}$$

we observe that

$$S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}-\lim\left\|\tilde{\Phi}_{nr}(g_i)-g_i\right\|=0\ (i=0,1,2,3).\tag{58}$$

Consequently, by Theorem 6, we obtain

$$S_{2,\gamma_1,\gamma_2}^{\bar{N}(\lambda,\mu)}-\lim d^{*}(\Phi_{nr}(g),g)=0,\tag{59}$$

for all $g\in C_{\mathscr{L}}(I^2)$. Hence, all the conditions of Theorem 6 hold true for our operators $\tilde{\Phi}_{nr}$ but Theorem 2.2 obtained in [45] for convergence in Pringsheim's sense does not work for $\tilde{\Phi}_{nr}$ since the fuzzy double sequence (s_{nr}) is not convergent in Pringsheim's sense. Thus, we conclude that Theorem 6 is stronger than the one proved for convergence in Pringsheim's sense.

Data Availability

Not applicable.

Conflicts of Interest

The author declares there are no conflicts of interest.

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