Journal of Function Spaces

Advances in Fractional Functional Analysis

Lead Guest Editor: Emanuel Guariglia
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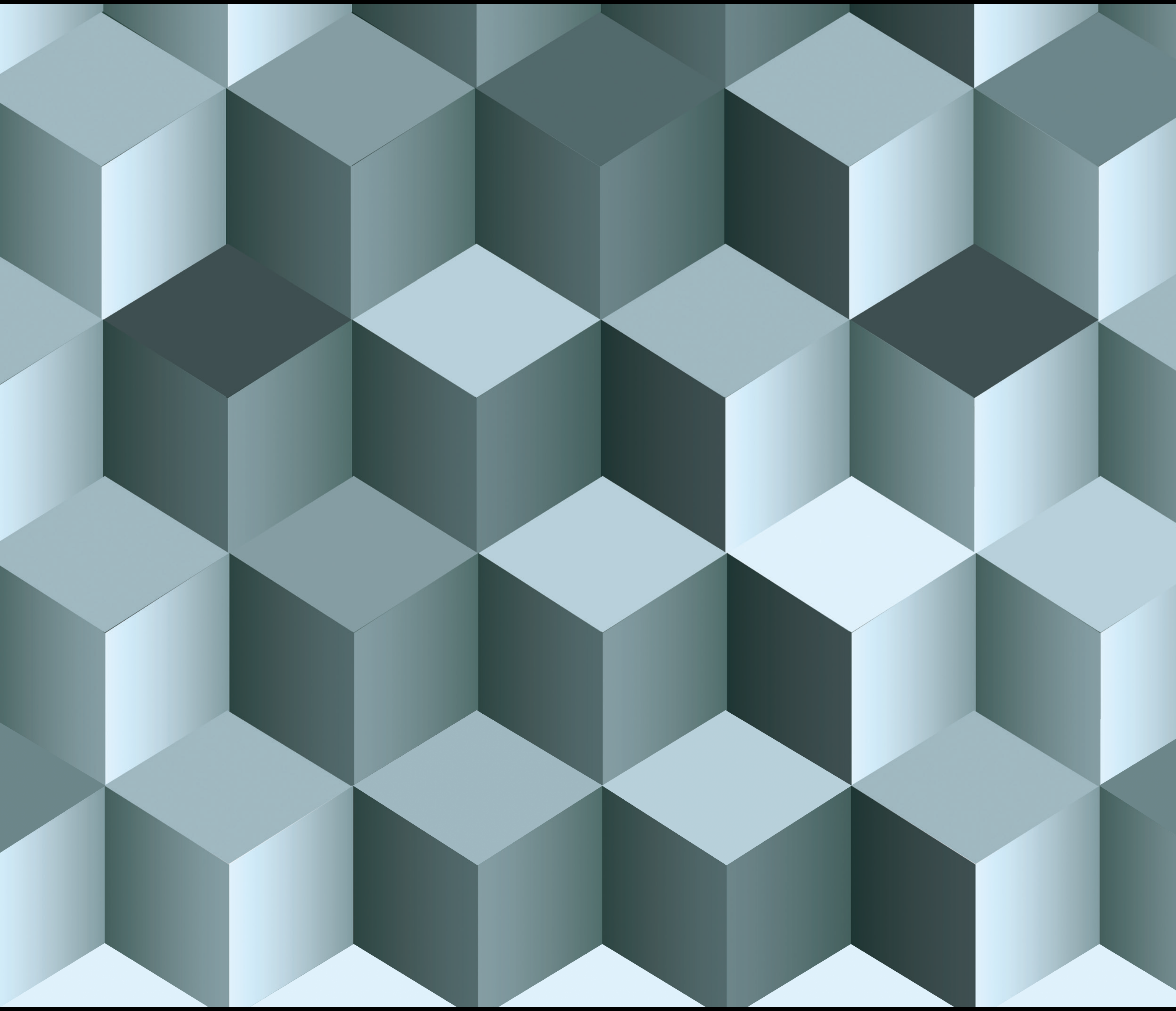

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Research Article

On a Coupled System of Fractional Differential Equations via the Generalized Proportional Fractional Derivatives

M. I. Abbas,[1] **M. Ghaderi**,[2] **Sh. Rezapour**,[2,3] **and S. T. M. Thabet**[4]

[1]*Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt*
[2]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[3]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[4]*Department of Mathematics, University of Aden, Aden, Yemen*

Correspondence should be addressed to Sh. Rezapour; rezapourshahram@yahoo.ca and S. T. M. Thabet; sabri741983@gmail.com

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This work investigates the existence and uniqueness of solutions for a coupled system of fractional differential equations with three-point generalized fractional integral boundary conditions within generalized proportional fractional derivatives of the Riemann-Liouville type. By using the Schauder and Banach fixed point theorems, we study the existence and uniqueness of solutions for the aforesaid system. Finally, we present an example to validate our theoretical outcomes.

1. Introduction

The theory of fractional calculus has become an attractive area of research for mathematicians and physicians because of its fertile aspects in many applications in natural science [1, 2], engineering [3], and many other fields. Moreover, the fractional differential equations have been employed successfully in the modeling of many biological problems, for example, human liver [4], hepatitis B [5–7], mumps virus [8] and methanol detoxification in the human body [9], and other differential models in thermodynamic and physics such as thermostat [10], pantograph [11], diffusion-wave system [12], and dynamical systems [13]. For additional specifics about the theory of fractional calculus and applications, we suggest the books of Kilbas et al. [14], Podlubny [15], and Samko et al. [16]. During the last years, there have exhibited several concepts about fractional derivatives. Here, we point out the most famous kinds including Riemann-Liouville, Liouville-Caputo, generalized Caputo [17], and Hadamard derivatives [18]. This has lead researchers to numerous research papers concerning several fractional operators which were conducted that one can see, for example, in complex plain [19, 20], extended Riemann-Liouville [21], the Mittag-Leer type function [22], the q-derivative [23], the local fractional derivative [24], and in stability result [25–27].

More recently, Jarad et al. [28] constructed a new generalized fractional derivative which is called the generalized proportional fractional derivative. This new fractional operator has the advantage of being well-behaved as it is considered to be a generalization of many of the previously known and widely used fractional operators such as Liouville-Caputo and Riemann-Liouville fractional operators. In detail, fractional differential equations with generalized proportional derivatives have seen significant contributions from an interested researcher. For instance, we refer to works of Abbas and Ragusa and Hristova and Abbas [29, 30] and Khaminsou et al. [31, 32], and the references existing therein.

At the same time, coupled systems of differential equations of fractional order with different boundary conditions have been the focus of many mathematicians. The literature on the topic involves the existence, uniqueness, and stability results. Ahmad and Luca [33] studied a system of nonlinear Caputo fractional differential equations with coupled boundary conditions involving Riemann-Liouville fractional integrals. Baitiche et al. [34] discussed the existence and uniqueness of solutions to some nonlinear fractional differential equations involving the ψ-Caputo fractional derivative with multipoint boundary conditions. Mahmudov et al. [35] investigated existence and uniqueness results for a coupled system of Caputo fractional differential equations with integral boundary conditions.

Some existing frameworks mentioned above encourage us to study the following coupled system of fractional differential equations:

$$\begin{cases} \left({}_a^R\mathscr{D}^{\alpha,\rho}u\right)(t) = \psi_1(t, u(t), v(t)), \\ \left({}_a^R\mathscr{D}^{\beta,\rho}v\right)(t) = \psi_2(t, u(t), v(t)), \\ t \in \mathscr{J} \coloneqq [a, b], \end{cases} \tag{1}$$

equipped with the generalized fractional integral boundary conditions:

$$\begin{cases} u(\delta_1) = 0, \quad u(b) = ({}_a\mathscr{I}^{\gamma_1,\rho}u)(\mu_1), \\ v(\delta_2) = 0, \quad v(b) = ({}_a\mathscr{I}^{\gamma_2,\rho}u)(\mu_2), \end{cases} \tag{2}$$

where $\rho \in (0, 1]$, ${}_a^R\mathscr{D}^{\alpha,\rho}$ and ${}_a^R\mathscr{D}^{\beta,\rho}$ denote the generalized proportional fractional derivatives of Riemann-Liouville type of order $\alpha, \beta \in (1, 2]$, ${}_a\mathscr{I}^{\gamma_1,\rho}$ and ${}_a\mathscr{I}^{\gamma_2,\rho}$ denote the generalized proportional fractional integrals of order $\gamma_1, \gamma_2 \in (0, 1)$, and $\delta_1, \delta_2, \mu_1, \mu_2 \in (a, b)$ and $\psi_1, \psi_2 : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. In the current work, we establish the existence and uniqueness of solutions of the coupled system (1) and (2) by means of Schauder's and Banach's fixed point theorems.

To the best of our knowledge, there are no contributions considering a coupled system of generalized proportional fractional differential equations with generalized fractional integral boundary conditions.

The paper structure is designed as follows: in Section 2, we collect some essential definitions and lemmas relevant to the generalized proportional fractional derivatives and integrals; in Section 3, we establish the Green function associated with the linear issue of the coupled system (1) and (2), while in Section 4, we prove the main existence and uniqueness results in the current paper; in Section 5, an example is given to validate our theoretical outcomes.

2. Preliminaries

Here, we review some definitions of the generalized proportional fractional derivatives and integrals; see [28, 30, 36].

Definition 1 (see [37]). Take $\rho \in (0, 1]$, let the functions ε_0, $\varepsilon_1 : [0, 1] \times \mathbb{R} \longrightarrow [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have $\lim_{\rho\longrightarrow 0^+}\varepsilon_1(\rho, t) = 1$, $\lim_{\rho\longrightarrow 0^+}\varepsilon_0(\rho, t) = 0$, $\lim_{\rho\longrightarrow 1^-}\varepsilon_1(\rho, t) = 0$, $\lim_{\rho\longrightarrow 1^-}\varepsilon_0(\rho, t) = 1$, $\varepsilon_1(\rho, t)$=0 for $\rho \in [0, 1)$, and $\varepsilon_0(\rho, t)$=0 for $\rho \in (0, 1]$. Then, the amended conformable derivative of order ρ is defined by

$$(\mathscr{D}^\rho v)(t) = \varepsilon_1(\rho, t)v(t) + \varepsilon_0(\rho, t)v'(t). \tag{3}$$

The above amended conformable derivative (3) is said to be a proportional derivative (see [37]). When $\varepsilon_1(\rho, t) = 1 - \rho$ and $\varepsilon_0(\rho, t) = \rho$, (3) takes the form

$$(\mathscr{D}^\rho v)(t) = (1 - \rho)v(t) + \rho v'(t). \tag{4}$$

Note that, $\lim_{\rho\longrightarrow 0^+}(\mathscr{D}^\rho v)(t) = v(t)$ and $\lim_{\rho\longrightarrow 1^-}(\mathscr{D}^\rho v)(t) = v'(t)$.

Remark 2. By using (4) for the function $v(t) = e^t$ and any arbitrary order ρ, it can be easily concluded that $(\mathscr{D}^\rho v)(t) = e^t$.

Example 1. If $v(t) = \sin(t)$, then $(\mathscr{D}^\rho v)(t) = (1 - \rho)\sin(t) + \rho\cos(t)$. One can find the graphs of $\mathscr{D}^\rho \sin(t)$ for different value of $\rho = 0.1, 0.5, 0.9, 1$, in Figure 1. As can be seen from Figure 1, in some points, the value of conformable derivative in this case is independent of ρ, and this can be one of the interesting properties of fractional calculus.

Definition 3 (see [28, 30]). Take $\rho \in (0, 1]$, $\alpha \geq 0$, we define the left generalized proportional fractional integral of the function $v \in L^1(\mathscr{J})$ by $({}_a\mathscr{I}^{0,\rho}v)(t) = v(t)$ and

$$({}_a\mathscr{I}^{\alpha,\rho}v)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{((\rho-1)/\rho)(t-s)}(t - s)^{\alpha-1}v(s)\,ds, t \in \mathscr{J}. \tag{5}$$

Definition 4 (see [28, 30]). Take $\rho \in (0, 1]$, $\alpha \geq 0$, we define the left generalized Caputo-proportional fractional derivative of the function $v \in C^{(n)}(\mathscr{J})$ by $({}_a^C\mathscr{D}^{0,\rho}v)(t) = v(t)$ and

$$\begin{aligned} \left({}_a^C\mathscr{D}^{\alpha,\rho}v\right)(t) &= {}_a\mathscr{I}^{n-\alpha,\rho}(\mathscr{D}^{n,\rho}v)(t) \\ &= \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \\ &\quad\cdot\int_a^t e^{((\rho-1)/\rho)(t-s)}(t-s)^{n-\alpha-1}(\mathscr{D}^{n,\rho}v)(s)\,ds, \end{aligned} \tag{6}$$

where $n - 1 < \alpha \leq n, n \in \mathbb{N}$, $(\mathscr{D}^{1,\rho}v)(t) = (\mathscr{D}^\rho v)(t) = (1 - \rho)v(t) + \rho v'(t)$, and $(\mathscr{D}^{n,\rho}v)(t) = (\underbrace{\mathscr{D}^\rho\mathscr{D}^\rho \cdots \mathscr{D}^\rho}_{n\,\text{times}})v(t)$.

Definition 5 (see [28, 30]). Take $\rho \in (0, 1]$, $\alpha \geq 0$, we define the left generalized proportional fractional derivative of Riemann-Liouville type of the function v by $({}_a\mathscr{D}^{0,\rho}v)(t) = v(t)$ and

$$\begin{aligned} \left({}_a^R\mathscr{D}^{\alpha,\rho}v\right)(t) &= \mathscr{D}^{n,\rho}\,{}_a\mathscr{I}^{n-\alpha,\rho}v(t) \\ &= \frac{\mathscr{D}^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \\ &\quad\cdot\int_a^t e^{((\rho-1)/\rho)(t-s)}(t-s)^{n-\alpha-1}v(s)\,ds, \end{aligned} \tag{7}$$

where $n - 1 < \alpha \leq n, n \in \mathbb{N}$.

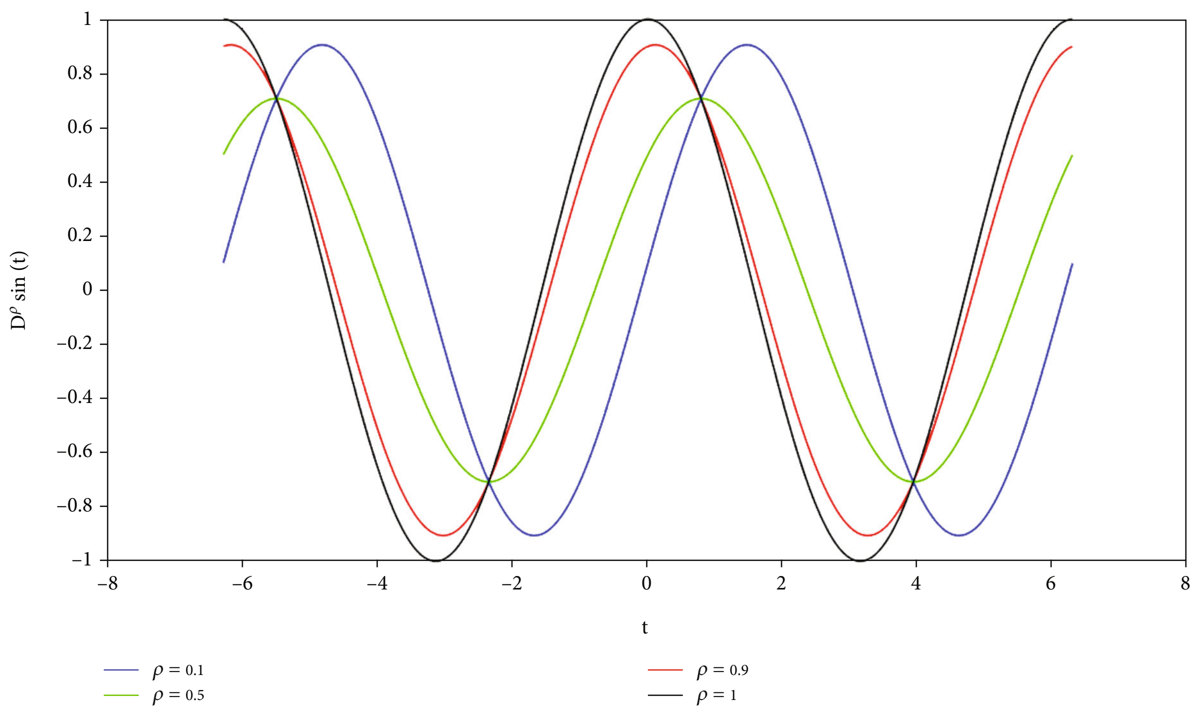

Figure 1: The graph of $\mathscr{D}^\rho \sin(t)$.

Lemma 6 (see [36]). *If $\rho \in (0, 1]$, $\beta > 0$, and $\alpha > 0$ with $n-1 < \alpha \le n$ and $v \in L^1(\mathscr{J})$, we have the following statements:*

$$\left({}_a\mathscr{J}^{\alpha,\rho} e^{\rho-1/\rho\tau}(\tau-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\rho^\alpha\Gamma(\beta+\alpha)} e^{((\rho-1)/\rho)t}(t-a)^{\alpha+\beta-1}, \tag{8}$$

$$\left({}^R_a\mathscr{D}^{\alpha,\rho} e^{\rho-1/\rho\tau}(\tau-a)^{\beta-1}\right)(t) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{((\rho-1)/\rho)t}(t-a)^{\beta-1-\alpha}, \tag{9}$$

$${}_a\mathscr{J}^{\alpha,\rho}\left({}_a\mathscr{J}^{\beta,\rho}v\right)(t) = {}_a\mathscr{J}^{\beta,\rho}\left({}_a\mathscr{J}^{\alpha,\rho}v\right)(t) = \left({}_a\mathscr{J}^{\alpha+\beta,\rho}v\right)(t), \tag{10}$$

$$\left({}^R_a\mathscr{D}^{\eta,\rho}\,{}_a\mathscr{J}^{\alpha,\rho}v\right)(t) = \left({}_a\mathscr{J}^{\alpha-\eta,\rho}v\right)(t), 0 < \eta < \alpha, \tag{11}$$

$$\left({}^R_a\mathscr{D}^{\alpha,\rho}\,{}_a\mathscr{J}^{\alpha,\rho}v\right)(t) = v(t), \tag{12}$$

$${}_a\mathscr{J}^{\alpha,\rho}\left({}^R_a\mathscr{D}^{\alpha,\rho}v\right)(t) = v(t) - \sum_{k=1}^{n} d_k e^{((\rho-1)/\rho)(t-a)}(t-a)^{\alpha-k}, \tag{13}$$

where $d_k = \left({}_a\mathscr{J}^{k-\alpha,\rho}v\right)(a)/\rho^{\alpha-k}\Gamma(\alpha-k+1)$.

Theorem 7 (Schauder's fixed point theorem) [38]. *Let $\mathscr{U}$ be a closed, convex, and nonempty subset of a Banach space C; let $\mathscr{H} : \mathscr{U} \longrightarrow \mathscr{U}$ be a continuous mapping such that $\mathscr{H}(\mathscr{U})$ is a relatively compact subset of C. Then, $\mathscr{H}$ has at least one fixed point in $\mathscr{U}$.*

3. The Equivalent Integral Equations

Let $\mathscr{C} = C(\mathscr{J}, \mathbb{R})$ be the Banach space of all continuous functions from $\mathscr{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_{\mathscr{C}} = \max_{t\in\mathscr{J}}|u(t)|, \tag{14}$$

and $\mathbf{C} = \mathscr{C} \times \mathscr{C}$ be the product Banach space with the norm

$$\|(u, v)\|_{\mathbf{C}} = \|u\|_{\mathscr{C}} + \|v\|_{\mathscr{C}}. \tag{15}$$

Definition 8. By a solution of the coupled system (1) and (2), we mean a coupled ordered pair of continuous functions $(u, v) \in C$ that satisfy (1) and (2).

Lemma 9. *Let $\rho \in (0, 1]$, $\Lambda_1\Lambda_4 = \Lambda_2\Lambda_3$, and $\omega : \mathscr{J} \longrightarrow \mathbb{R}$. Then, the solution of*

$$\begin{cases} \left({}^R_a\mathscr{D}^{\alpha,\rho}u\right)(t) = \omega(t), t \in \mathscr{J}, \alpha \in (1, 2], \\ u(\delta_1) = 0, u(b) = \left({}_a\mathscr{J}^{\gamma_1,\rho}u\right)(\mu_1), \delta_1, \mu_1 \in (a, b), \gamma_1 \in (0, 1), \end{cases} \tag{16}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) = {} & ({}_a\mathcal{I}^{\alpha,\rho}\omega)(t) + \Lambda_6 (t-a)^{\alpha-1} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot \left(\Lambda_2 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(b) - \Lambda_2 ({}_a\mathcal{I}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_4 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(\delta_1)\right) \\ & + \Lambda_5 (t-a)^{\alpha-2} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot \left(\Lambda_1 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(b) - \Lambda_1 ({}_a\mathcal{I}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_3 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(\delta_1)\right), \end{aligned} \tag{17}$$

where

$$\begin{cases} \Lambda_1 = (\delta_1 - a)^{\alpha-1} e^{((\rho-1)/\rho)(\delta_1-a)}, \quad \Lambda_2 = (\delta_1 - a)^{\alpha-2} e^{((\rho-1)/\rho)(\delta_1-a)}, \\ \Lambda_3 = (b-a)^{\alpha-1} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\alpha)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1)} (\mu_1 - a)^{\alpha+\gamma_1-1} e^{((\rho-1)/\rho)(\mu_1-a)}, \\ \Lambda_4 = (b-a)^{\alpha-2} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\alpha-1)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1-1)} (\mu_1 - a)^{\alpha+\gamma_1-2} e^{((\rho-1)/\rho)(\mu_1-a)}, \\ \Lambda_5 = (\Lambda_2\Lambda_3 - \Lambda_1\Lambda_4)^{-1}, \text{ and } \Lambda_6 = (\Lambda_1\Lambda_4 - \Lambda_2\Lambda_3)^{-1}. \end{cases} \tag{18}$$

Proof. By applying the generalized fractional proportional integral ${}_a\mathcal{I}^{\alpha,\rho}(\cdot)$ to both sides of the first equation (16) and using (13) in Lemma 6, one has

$$\begin{aligned} u(t) = {} & ({}_a\mathcal{I}^{\alpha,\rho}\omega)(t) + d_1 e^{((\rho-1)/\rho)(t-a)} (t-a)^{\alpha-1} \\ & + d_2 e^{((\rho-1)/\rho)(t-a)} (t-a)^{\alpha-2}. \end{aligned} \tag{19}$$

Using the boundary condition $u(\delta_1) = 0$ and (19), one has

$$\begin{aligned} & (\delta_1 - a)^{\alpha-1} e^{((\rho-1)/\rho)(\delta_1-a)} d_1 + (\delta_1 - a)^{\alpha-2} e^{((\rho-1)/\rho)(\delta_1-a)} d_2 \\ & = -({}_a\mathcal{I}^{\alpha,\rho}\omega)(\delta_1). \end{aligned} \tag{20}$$

Using (18), the above equation becomes

$$\Lambda_1 d_1 + \Lambda_2 d_2 = -({}_a\mathcal{I}^{\alpha,\rho}\omega)(\delta_1). \tag{21}$$

In the light of (8) and (10) in Lemma 6, the boundary condition $u(b) = ({}_a\mathcal{I}^{\gamma_1,\rho}u)(\mu_1)$ and (19) give

$$\begin{aligned} & ({}_a\mathcal{I}^{\alpha,\rho}\omega)(b) + d_1 e^{((\rho-1)/\rho)(b-a)} (b-a)^{\alpha-1} + d_2 e^{((\rho-1)/\rho)(b-a)} (b-a)^{\alpha-2} \\ & = ({}_a\mathcal{I}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) + d_1 \frac{\Gamma(\alpha)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1)} (\mu_1 - a)^{\alpha+\gamma_1-1} e^{((\rho-1)/\rho)(\mu_1-a)} \\ & \quad + d_2 \frac{\Gamma(\alpha-1)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1-1)} (\mu_1 - a)^{\alpha+\gamma_1-2} e^{((\rho-1)/\rho)(\mu_1-a)}. \end{aligned} \tag{22}$$

Again, using (18), the above equation takes the form

$$\Lambda_3 d_1 + \Lambda_4 d_2 = -({}_a\mathcal{I}^{\alpha,\rho}\omega)(b) + ({}_a\mathcal{I}^{\alpha+\gamma_1,\rho}\omega)(\mu_1). \tag{23}$$

Therefore, by merging equations (21) and (23), using (18), we get

$$\begin{aligned} d_1 &= \Lambda_6 \left(\Lambda_2 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(b) - \Lambda_2 ({}_a\mathcal{I}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_4 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(\delta_1)\right), \\ d_2 &= \Lambda_5 \left(\Lambda_1 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(b) - \Lambda_1 ({}_a\mathcal{I}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_3 ({}_a\mathcal{I}^{\alpha,\rho}\omega)(\delta_1)\right). \end{aligned} \tag{24}$$

Thus, by inserting the values of d_1 and d_2 in (19), we obtain (17). The proof is finished. □

By hint of Lemma 9, the solution $(u, v) \in \mathbf{C}$ of the system (1) and (2) is given by

$$\begin{aligned} u(t) = {} & {}_a\mathcal{I}^{\alpha,\rho}\psi_1(t, u(t), v(t)) + \Lambda_6 (t-a)^{\alpha-1} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot \left(\Lambda_{2a}\mathcal{I}^{\alpha,\rho}\psi_1(b, u(b), v(b))\right. \\ & - \Lambda_{2a}\mathcal{I}^{\alpha+\gamma_1,\rho}\psi_1(\mu_1, u(\mu_1), v(\mu_1)) \\ & \left.- \Lambda_{4a}\mathcal{I}^{\alpha,\rho}\psi_1(\delta_1, u(\delta_1), v(\delta_1))\right) \\ & + \Lambda_5 (t-a)^{\alpha-2} e^{((\rho-1)/\rho)(t-a)} \left(\Lambda_{1a}\mathcal{I}^{\alpha,\rho}\psi_1(b, u(b), v(b))\right. \\ & - \Lambda_{1a}\mathcal{I}^{\alpha+\gamma_1,\rho}\psi_1(\mu_1, u(\mu_1), v(\mu_1)) \\ & \left.- \Lambda_{3a}\mathcal{I}^{\alpha,\rho}\psi_1(\delta_1, u(\delta_1), v(\delta_1))\right), t \in \mathcal{J}, \end{aligned}$$

$$\begin{aligned} v(t) = {} & {}_a\mathcal{I}^{\beta,\rho}\psi_2(t, u(t), v(t)) + \Lambda_6' (t-a)^{\beta-1} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot \Big(\Lambda_{2a}'\mathcal{I}^{\beta,\rho}\psi_2(b, u(b), v(b)) - \Lambda_{2a}'\mathcal{I}^{\beta+\gamma_2,\rho}\psi_2 \\ & \cdot (\mu_2, u(\mu_2), v(\mu_2)) - \Lambda_{4a}'\mathcal{I}^{\beta,\rho}\psi_2(\delta_2, u(\delta_2), v(\delta_2))\Big) \\ & + \Lambda_5' (t-a)^{\beta-2} e^{((\rho-1)/\rho)(t-a)} \Big(\Lambda_{1a}'\mathcal{I}^{\beta,\rho}\psi_2(b, u(b), v(b)) \\ & - \Lambda_{1a}'\mathcal{I}^{\beta+\gamma_2,\rho}\psi_2(\mu_2, u(\mu_2), v(\mu_2)) \\ & - \Lambda_{3a}'\mathcal{I}^{\beta,\rho}\psi_2(\delta_2, u(\delta_2), v(\delta_2))\Big), t \in \mathcal{J}, \end{aligned} \tag{25}$$

where

$$\begin{cases} \Lambda_1' = (\delta_2 - a)^{\beta-1} e^{((\rho-1)/\rho)(\delta_2-a)}, \quad \Lambda_2' = (\delta_2 - a)^{\beta-2} e^{((\rho-1)/\rho)(\delta_2-a)}, \\ \Lambda_3' = (b-a)^{\beta-1} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\beta)}{\rho^{\gamma_2}\Gamma(\beta+\gamma_2)} (\mu_2 - a)^{\beta+\gamma_2-1} e^{((\rho-1)/\rho)(\mu_2-a)}, \\ \Lambda_4' = (b-a)^{\beta-2} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\beta-1)}{\rho^{\gamma_2}\Gamma(\beta+\gamma_2-1)} (\mu_2 - a)^{\beta+\gamma_2-2} e^{((\rho-1)/\rho)(\mu_2-a)}, \\ \Lambda_5' = \left(\Lambda_2'\Lambda_3' - \Lambda_1'\Lambda_4'\right)^{-1}, \text{ and } \Lambda_6' = \left(\Lambda_1'\Lambda_4' - \Lambda_2'\Lambda_3'\right)^{-1}, \Lambda_1'\Lambda_4' {=} \Lambda_2'\Lambda_3'. \end{cases} \tag{26}$$

4. Existence and Uniqueness Results

Define the operator $\mathcal{H} : \mathbf{C} \longrightarrow \mathbf{C}$ by

$$(\mathcal{H}(u, v))(t) = \begin{pmatrix} (\mathcal{H}_1(u, v))(t) \\ (\mathcal{H}_2(u, v))(t) \end{pmatrix}, \tag{27}$$

where

$$
\begin{aligned}
(\mathscr{H}_1(u,v))(t) = {}_a\mathscr{I}^{\alpha,\rho}\psi_1(t,u(t),v(t)) + \Lambda_6(t-a)^{\alpha-1}e^{((\rho-1)/\rho)(t-a)} \\
\cdot (\Lambda_{2a}\mathscr{I}^{\alpha,\rho}\psi_1(b,u(b),v(b)) - \Lambda_{2a}\mathscr{I}^{\alpha+\gamma_1,\rho}\psi_1 \\
(\mu_1,u(\mu_1),v(\mu_1)) - \Lambda_{4a}\mathscr{I}^{\alpha,\rho}\psi_1(\delta_1,u(\delta_1),v(\delta_1))) \\
+ \Lambda_5(t-a)^{\alpha-2}e^{((\rho-1)/\rho)(t-a)}(\Lambda_{1a}\mathscr{I}^{\alpha,\rho}\psi_1(b,u(b),v(b)) \\
- \Lambda_{1a}\mathscr{I}^{\alpha+\gamma_1,\rho}\psi_1(\mu_1,u(\mu_1),v(\mu_1)) \\
- \Lambda_{3a}\mathscr{I}^{\alpha,\rho}\psi_1(\delta_1,u(\delta_1),v(\delta_1))), (t \in \mathscr{J}),
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
(\mathscr{H}_2(u,v))(t) = {}_a\mathscr{I}^{\beta,\rho}\psi_2(t,u(t),v(t)) + \Lambda_6'(t-a)^{\beta-1}e^{((\rho-1)/\rho)(t-a)} \\
\cdot \Big(\Lambda_{2a}'\mathscr{I}^{\beta,\rho}\psi_2(b,u(b),v(b)) - \Lambda_{2a}'\mathscr{I}^{\beta+\gamma_2,\rho}\psi_2 \\
\cdot (\mu_2,u(\mu_2),v(\mu_2)) - \Lambda_{4a}'\mathscr{I}^{\beta,\rho}\psi_2(\delta_2,u(\delta_2),v(\delta_2))\Big) \\
+ \Lambda_5'(t-a)^{\beta-2}e^{((\rho-1)/\rho)(t-a)}\Big(\Lambda_{1a}'\mathscr{I}^{\beta,\rho}\psi_2(b,u(b),v(b)) \\
- \Lambda_{1a}'\mathscr{I}^{\beta+\gamma_2,\rho}\psi_2(\mu_2,u(\mu_2),v(\mu_2)) \\
- \Lambda_{3a}'\mathscr{I}^{\beta,\rho}\psi_2(\delta_2,u(\delta_2),v(\delta_2))\Big), \quad t \in \mathscr{J}.
\end{aligned} \tag{29}
$$

According to Lemma 9, the solution $(u,v) \in \mathbf{C}$ of the coupled system (1) and (2) conforms with the fixed point operator $\mathscr{H}$.

For fulfillment the main results, the following assumptions will be imposed.

(A1) The functions $\psi_1, \psi_2 : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous

(A2) There exist nonnegative constants L_1 and L_2 such that

$$
|\psi_i(t,u_1,v_1) - \psi_i(t,u_2,v_2)| \le L_i(|u_1-u_2| + |v_1-v_2|), \tag{30}
$$

for each $t \in \mathscr{J}$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$

Further, we set $\psi_i^* = \max_{t\in\mathscr{J}}|\psi_i(t,0,0)|, i = 1, 2.$

The following notations will be introduced:

$$
\begin{aligned}
\Delta_1 &:= \left[\frac{(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + |\Lambda_6|(b-a)^{\alpha-1}\left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\right. \\
&\quad \left. + |\Lambda_5|(b-a)^{\alpha-2}\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\right],
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\Delta_2 &:= \left[\frac{(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + |\Lambda_6'|(b-a)^{\beta-1}\left(\frac{|\Lambda_2'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_2'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_4'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right)\right. \\
&\quad \left. + |\Lambda_5'|(b-a)^{\beta-2}\left(\frac{|\Lambda_1'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_1'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_3'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right)\right].
\end{aligned} \tag{32}
$$

Theorem 10. *Assume that the assumptions (A1) and (A2) are satisfied. Then, the coupled system (1) and (2) has at least one solution on $\mathscr{J}$.*

Proof. Consider the operator $\mathscr{H} : \mathbf{C} \longrightarrow \mathbf{C}$ as defined in (27). Let us introduce the ball

$$
\zeta_\ell = \{(u,v) \in \mathbf{C} : \|(u,v)\|_{\mathbf{C}} \le \ell\}, \tag{33}
$$

where ℓ is a positive real number such that

$$
\ell \ge \frac{\Delta_1\psi_1^* + \Delta_2\psi_2^*}{1-(\Delta_1L_1+\Delta_2L_2)}, \Delta_1L_1 + \Delta_2L_2 < 1. \tag{34}
$$

It is obvious that ζ_ℓ is a closed, bounded, and convex subset of the Banach space $\mathbf{C}$. We shall show that $\mathscr{H}$ achieves the hypothesis of Schauder's fixed point theorem in four steps.

Step 1. $\mathscr{H}$ maps bounded sets into bounded sets in $\mathbf{C}$.

By virtue of (A2) and since $|e^{((\rho-1)/\rho)(t-a)}| < 1, \forall t > a$, then for each $t \in \mathscr{J}$ and $(u,v) \in \zeta_\ell$, one has

$$
|(\mathscr{H}_1(u,v))(t)| \tag{35}
$$

Thus, by using (31), we get

$$
\|(\mathscr{H}_1(u,v))\|_{\mathscr{C}} \le \Delta_1(L_1\ell + \psi_1^*). \tag{36}
$$

Similarly, we obtain that

$$
\|(\mathscr{H}_2(u,v))\|_{\mathscr{C}} \le \Delta_2(L_2\ell + \psi_2^*), \tag{37}
$$

where Δ_2 is defined in (32). Hence, we conclude that

$$
\begin{aligned}
&\|(\mathscr{H}_1(u,v))\|_{\mathscr{C}} + \|(\mathscr{H}_2(u,v))\|_{\mathscr{C}} \\
&\quad \le (\Delta_1L_1 + \Delta_2L_2)\ell + (\Delta_1\psi_1^* + \Delta_2\psi_2^*) \le \ell,
\end{aligned} \tag{38}
$$

which implies that $\mathscr{H} : \zeta_\ell \longrightarrow \zeta_\ell$

Step 2. $\mathscr{H}$ is continuous.

In view of the assumption (A1), we conclude that $\mathscr{H}_1$ and $\mathscr{H}_2$ are continuous on $\mathscr{J}$. Thus, the operator $\mathscr{H}$ is also continuous

Step 3. $\mathscr{H}(\zeta_\ell)$ is equicontinuous.

Set $\max_{t\in\mathscr{J}}|\psi_i(t,u(t),v(t))| := M_i < \infty, i = 1, 2$. For $t_1, t_2 \in \mathscr{J}$, with $t_1 < t_2$ and $(u,v) \in \zeta_\ell$, we have

$$\begin{aligned}
|(\mathscr{H}_1(u,v))(t_2) - (\mathscr{H}_1(u,v))(t_1)| \le{}& \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_a^{t_2} e^{((\rho-1)/\rho)(t_2-s)}(t_2-s)^{\alpha-1}\psi_1(s,u(s),v(s))ds - \int_a^{t_1} e^{((\rho-1)/\rho)(t_1-s)}(t_1-s)^{\alpha-1}\psi_1(s,u(s),v(s))ds\right| \\
&+ |\Lambda_6|\left|\left(e^{((\rho-1)/\rho)(t_2-a)}(t_2-a)^{\alpha-1} - e^{((\rho-1)/\rho)(t_1-a)}(t_1-a)^{\alpha-1}\right)\right| \\
&\cdot\left(\frac{|\Lambda_2|}{\rho^\alpha\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds + \frac{|\Lambda_2|}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1)}\int_a^{\mu_1}(\mu_1-s)^{\alpha+\gamma_1-1}|\psi_1(s,u(s),v(s))|ds + \frac{|\Lambda_4|}{\rho^\alpha\Gamma(\alpha)}\right. \\
&\left.\cdot\int_a^{\delta_1}(\delta_1-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds\right) + |\Lambda_5|\left|\left(e^{((\rho-1)/\rho)(t_2-a)}(t_2-a)^{\alpha-2} - e^{((\rho-1)/\rho)(t_1-a)}(t_1-a)^{\alpha-2}\right)\right| \\
&\cdot\left(\frac{|\Lambda_1|}{\rho^\alpha\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds + \frac{|\Lambda_1|}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1)}\int_a^{\mu_1}(\mu_1-s)^{\alpha+\gamma_1-1}|\psi_1(s,u(s),v(s))|ds\right. \\
&\left.+ \frac{|\Lambda_3|}{\rho^\alpha\Gamma(\alpha)}\int_a^{\delta_1}(\delta_1-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds\right) \\
\le{}& \frac{M_1}{\rho^\alpha\Gamma(\alpha)}\left(\int_a^{t_2}\left|e^{((\rho-1)/\rho)(t_2-s)}\right|\left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|ds + \int_a^{t_2}\left|e^{((\rho-1)/\rho)(t_2-s)} - e^{((\rho-1)/\rho)(t_1-s)}\right|\left|(t_1-s)^{\alpha-1}\right|ds\right. \\
&\left.+ \int_{t_1}^{t_2}\left|e^{((\rho-1)/\rho)(t_1-s)}\right|\left|(t_1-s)^{\alpha-1}\right|ds\right) + |\Lambda_6|M_1\left(\left|e^{((\rho-1)/\rho)(t_2-a)}\right|\left|(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}\right|\right. \\
&\left.+ \left|e^{((\rho-1)/\rho)(t_2-a)} - e^{((\rho-1)/\rho)(t_1-a)}\right|\left|(t_1-a)^{\alpha-1}\right|\right) \times \left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) \\
&+ |\Lambda_5|M_1\left(\left|e^{((\rho-1)/\rho)(t_2-a)}\right|\left|(t_2-a)^{\alpha-2} - (t_1-a)^{\alpha-2}\right| + \left|e^{((\rho-1)/\rho)(t_2-a)} - e^{((\rho-1)/\rho)(t_1-a)}\right|\left|(t_1-a)^{\alpha-2}\right|\right) \\
&\times\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) \\
\le{}& \frac{M_1}{\rho^\alpha\Gamma(\alpha)}\left(\int_a^{t_2}\left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|ds + \int_a^{t_2}\left|\frac{\rho-1}{\rho}(t_2-t_1)e^{((\rho-1)/\rho)(\xi_1-s)}\right|\left|(t_1-s)^{\alpha-1}\right|ds\right. \\
&\left.+ \int_{t_1}^{t_2}\left|(t_1-s)^{\alpha-1}ds\right)\right. + |\Lambda_6|M_1\left(\left((t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}\right) + \left|\frac{\rho-1}{\rho}(t_2-t_1)e^{((\rho-1)/\rho)(\xi_2-s)}\right|(t_1-a)^{\alpha-1}\right) \\
&\times\left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) + |\Lambda_5|M_1\left(\left((t_2-a)^{\alpha-2} - (t_1-a)^{\alpha-2}\right)\right. \\
&\left.+ \left|\frac{\rho-1}{\rho}(t_2-t_1)e^{((\rho-1)/\rho)(\xi_2-s)}\right|(t_1-a)^{\alpha-2}\right) \times \left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right),
\end{aligned} \tag{39}$$

where the mean value theorem is used on the function $e^{((\rho-1)/\rho)t}$ with $\xi_1, \xi_2 \in (t_1, t_2)$.

Thus, we get

$$\begin{aligned}
&|(\mathscr{H}_1(u,v))(t_2) - (\mathscr{H}_1(u,v))(t_1)| \\
&\quad\le \frac{M_1}{\rho^\alpha\Gamma(\alpha+1)}\left(\left((t_2-a)^\alpha - (t_1-a)^\alpha\right) + (t_2-t_1)(t_1-a)^\alpha\right) \\
&\qquad + |\Lambda_6|M_1\left(\left((t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}\right) + (t_2-t_1)(t_1-a)^{\alpha-1}\right) \\
&\qquad \times\left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) \\
&\qquad + |\Lambda_5|M_1\left(\left((t_2-a)^{\alpha-2} - (t_1-a)^{\alpha-2}\right) + (t_2-t_1)(t_1-a)^{\alpha-2}\right) \\
&\qquad \times\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right).
\end{aligned} \tag{40}$$

In an identical way, we obtain that

$$\begin{aligned}
&|(\mathscr{H}_2(u,v))(t_2) - (\mathscr{H}_2(u,v))(t_1)| \\
&\quad\le \frac{M_2}{\rho^\beta\Gamma(\beta+1)}\left(\left((t_2-a)^\beta - (t_1-a)^\beta\right) + (t_2-t_1)(t_1-a)^\beta\right) \\
&\qquad + |\Lambda_6'|M_2\left(\left((t_2-a)^{\beta-1} - (t_1-a)^{\beta-1}\right) + (t_2-t_1)(t_1-a)^{\beta-1}\right) \\
&\qquad \times\left(\frac{|\Lambda_2'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_2'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_4'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right) \\
&\qquad + |\Lambda_5'|M_1\left(\left((t_2-a)^{\beta-2} - (t_1-a)^{\beta-2}\right) + (t_2-t_1)(t_1-a)^{\beta-2}\right) \\
&\qquad \times\left(\frac{|\Lambda_1'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_1'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_3'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right).
\end{aligned} \tag{41}$$

As $t_2 \longrightarrow t_1$, the R.H.S. of the last two inequalities $\longrightarrow 0$ independently of $(u, v) \in \zeta_\ell$. As a consequence of Steps 1 to 3 and in the light of the Arzelá-Ascoli theorem, we conclude that the operator $\mathscr{H}(\zeta_\ell)$ is relatively compact in $\mathbf{C}$. Hence, in accordance with Schauder's fixed point theorem (Theorem 7), the operator $\mathscr{H}$ has a fixed point and so the coupled system (1) and (2) possesses at least one solution on $\mathscr{J}$. The proof is completed. □

Theorem 11. *Assume that the assumptions (A1) and (A2) are satisfied. If $\Delta_1 L_1 + \Delta_2 L_2 < 1$, then the coupled system (1) and (2) has a unique solution on $\mathscr{J}$.*

Proof. Consider the operator $\mathscr{H}$ as defined in (27). We have to show that $\mathscr{H}$ is a contraction mapping.

For each $t \in \mathscr{J}$ and $(u_1, v_1), (u_2, v_2) \in \mathbf{C}$, one has

$$\begin{aligned}
|(\mathscr{H}_1(u_1,v_1))(t)-(\mathscr{H}_1(u_2,v_2))(t)| &\le \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|\psi_1(s,u_1(s),v_1(s))-\psi_1(s,u_2(s),v_2(s))|\,ds+|\Lambda_6|(t-a)^{\alpha-1}\\
&\quad\cdot\left(\frac{|\Lambda_2|}{\rho^\alpha\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|\psi_1(s,u_1(s),v_1(s))-\psi_1(s,u_2(s),v_2(s))|ds+\frac{|\Lambda_2|}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1)}\right.\\
&\quad\cdot\int_a^{\mu_1}(\mu_1-s)^{\alpha+\gamma_1-1}|\psi_1(s,u_1(s),v_1(s))-\psi_1(s,u_2(s),v_2(s))|ds+\frac{|\Lambda_4|}{\rho^\alpha\Gamma(\alpha)}\\
&\quad\cdot\left.\int_a^{\delta_1}(\delta_1-s)^{\alpha-1}|\psi_1(s,u_1(s),v_1(s))-\psi_1(s,u_2(s),v_2(s))|ds\right)+|\Lambda_5|(t-a)^{\alpha-2}\\
&\quad\cdot\left(\frac{|\Lambda_1|}{\rho^\alpha\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|\psi_1(s,u_1(s),v_1(s))-\psi_1(s,u_2(s),v_2(s))|ds+\frac{|\Lambda_1|}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1)}\right.\\
&\quad\cdot\int_a^{\mu_1}(\mu_1-s)^{\alpha+\gamma_1-1}|\psi_1(s,u_1(s),v_1(s))-\psi_1(s,u_2(s),v_2(s))|ds+\frac{|\Lambda_3|}{\rho^\alpha\Gamma(\alpha)}\\
&\quad\cdot\left.\int_a^{\delta_1}(\delta_1-s)^{\alpha-1}|\psi_1(s,u_1(s),v_1(s))-\psi_1(s,u_2(s),v_2(s))|ds\right)\\
&\le\left[\frac{(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}+|\Lambda_6|(b-a)^{\alpha-1}\left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}+\frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)}+\frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\right.\\
&\quad\left.+|\Lambda_5|(b-a)^{\alpha-2}\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}+\frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)}+\frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\right]\times L_1\left(\|u_1-u_2\|_{\mathscr{C}}+\|v_1-v_2\|_{\mathscr{C}}\right).
\end{aligned}\tag{42}$$

Thus, by (31), we get

$$\|(\mathscr{H}_1(u_1,v_1))-(\mathscr{H}_1(u_2,v_2))\|_{\mathscr{C}} \le \Delta_1 L_1\left(\|u_1-u_2\|_{\mathscr{C}}+\|v_1-v_2\|_{\mathscr{C}}\right). \tag{43}$$

In a similar way, using (32), we get

$$\|(\mathscr{H}_2(u_1,v_1))-(\mathscr{H}_2(u_2,v_2))\|_{\mathscr{C}} \le \Delta_2 L_2\left(\|u_1-u_2\|_{\mathscr{C}}+\|v_1-v_2\|_{\mathscr{C}}\right). \tag{44}$$

From (43) and (44), we get

$$\|\mathscr{H}(u_1,v_1)-\mathscr{H}(u_2,v_2)\|_{\mathbf{C}} \le (\Delta_1 L_1+\Delta_2 L_2)\|(u_1,v_1)-(u_2,v_2)\|_{\mathbf{C}}. \tag{45}$$

By virtue of the condition $\Delta_1 L_1 + \Delta_2 L_2 < 1$, we conclude that $\mathscr{H}$ is a contraction mapping.

Hence, with the aid of Banach's fixed point theorem, we deduce that $\mathscr{H}$ has a unique fixed point, and so the coupled system (1) and (2) possesses a solution on $\mathscr{J}$ uniquely. The proof is finished. □

Example 2. Consider the following coupled system of fractional differential equations

$$\begin{cases}\left({}_{0^+}^{R}\mathscr{D}^{3/2,1/2}u\right)(t)=\psi_1(t,u(t),v(t)),\\ \left({}_{0^+}^{R}\mathscr{D}^{5/4,1/2}v\right)(t)=\psi_2(t,u(t),v(t)),\\ t\in[0,1],\end{cases}\tag{46}$$

with the generalized fractional integral boundary conditions:

$$\begin{cases}u\left(\dfrac{1}{3}\right)=0, \quad u(1)=\left({}_a\mathscr{I}^{1/7,1/2}u\right)\left(\dfrac{1}{5}\right),\\ v\left(\dfrac{1}{6}\right)=0, \quad v(1)=\left({}_a\mathscr{I}^{1/9,1/2}u\right)\left(\dfrac{1}{8}\right).\end{cases}\tag{47}$$

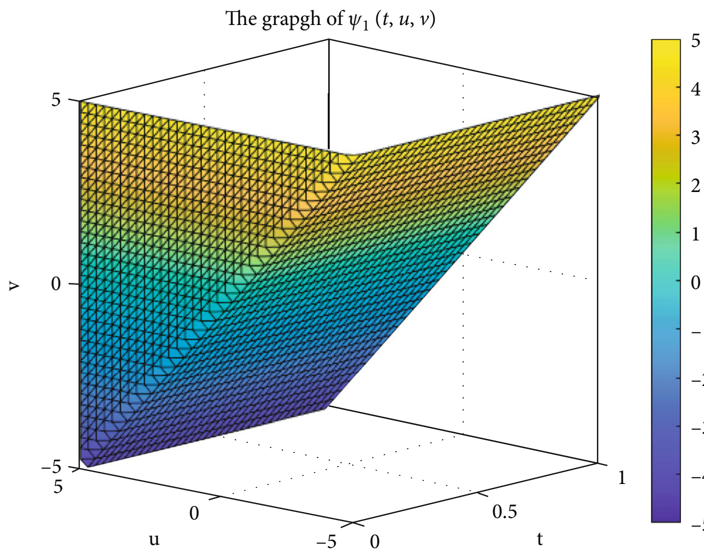

Figure 2: The graph of $\psi_1(t, u, v)$.

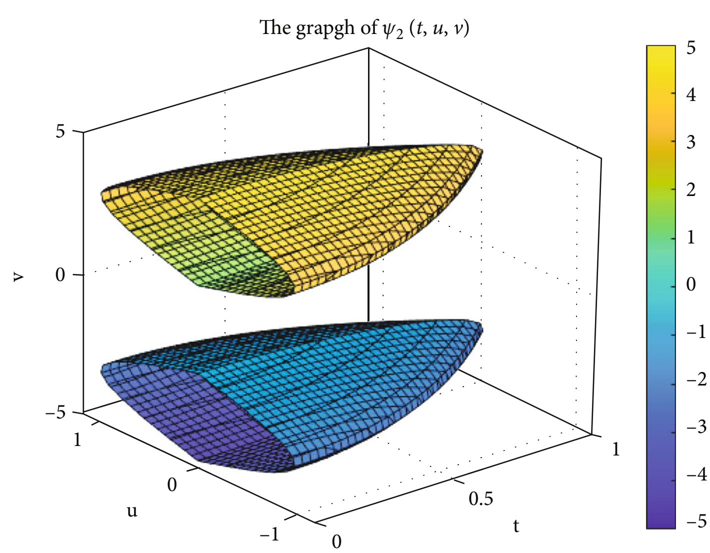

Figure 3: The graph of $\psi_2(t, u, v)$.

Here, $\alpha = 3/2, \beta = 5/4, \rho = 1/2, \gamma_1 = 1/7, \gamma_2 = 1/9, \delta_1 = 1/3$ $, \delta_2 = 1/6, \mu_1 = 1/5, \mu_2 = 1/8$, and $[a, b] = [0, 1]$. Set $\psi_1(t, u, v)$ $= (1/25(t^2 + 2))(\sin^2|u| + (|v|/1 + |v|))$ and $\psi_2(t, u, v) = (1/$ $100)(t^2 + |u| + \cos|v|)$ that their graphs show in Figures 2 and 3.

For each $t \in [0, 1]$ and $(u_1, v_1), (u_2, v_2) \in \mathbf{C}$, one has

$$|\psi_1(t, u_1, v_1) - \psi_1(t, u_2, v_2)| \leq \frac{1}{50}(|u_1 - u_2| + |v_1 - v_2|), \tag{48}$$

$$|\psi_2(t, u_1, v_1) - \psi_2(t, u_2, v_2)| \leq \frac{1}{100}(|u_1 - u_2| + |v_1 - v_2|), \tag{49}$$

which implies that the assumption (A2) holds true with $L_1 = 1/50$ and $L_2 = 1/100$. We calculate functions in (18), (26), (31), and (32) for $\rho = 1/4$, $\rho = 1/2$, and $\rho = 3/4$ and present their numerical results in Table 1. We have in all three cases:

$$L_1 \Delta_1 + L_2 \Delta_2 < 1. \tag{50}$$

By virtue of the above discussion, we infer that all the assumptions of Theorems 10 and 11 are satisfied. Consequently, we deduce that the coupled system (46) and (47) has a solution on $[0, 1]$ uniquely.

Table 1: Numerical results for some functions in Example 2.

	$\rho = \frac{1}{4}$	$\rho = \frac{1}{2}$	$\rho = \frac{3}{4}$
Λ_1	0.2124	0.4137	0.5166
Λ_2	0.6372	1.2411	1.5499
Λ_3	0.0394	0.0512	0.0865
Λ_4	-1.2107	-1.6677	-1.2144
Λ_5	0.9680	1.3272	1.6257
Λ_6	-0.9680	-1.3272	-1.6257
Λ_1'	0.3215	0.5409	0.8321
Λ_2'	2.7841	3.2451	3.9521
Λ_3'	-0.0612	-0.0902	1.5934
Λ_4'	-4.1284	-4.9254	-5.6418
Λ_5'	0.2996	0.4217	0.7294
Λ_6'	-0.2996	-0.4217	-0.7294
Δ_1	6.7451	8.0645	10.0121
Δ_2	2.7423	3.1234	4.0096
$L_1\Delta_1 + L_2\Delta_2$	0.1623	0.2225	0.2403

5. Conclusion

As you know, there are many events in nature which we know nothing about those. One of the best ways for better understanding these types of phenomena is studying new notions in the fractional calculus field. In this work, we investigated the existence and uniqueness of solutions for a coupled system of fractional differential equations with three-point generalized fractional integral boundary conditions in the frame of the generalized proportional fractional derivatives of the Riemann-Liouville type which was introduced in 2017 by Jarad et al. In this way, we provided some results under some conditions. To better explain the notion, we gave some figures of some functions. Finally, we provided an illustrated example for our main result.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

Chebyshev Wavelet Analysis

Emanuel Guariglia and Rodrigo Capobianco Guido

Institute of Biosciences, Letters and Exact Sciences, São Paulo State University (UNESP), Rua Cristóvão Colombo 2265, 15054-000 São José do Rio Preto, SP, Brazil

Correspondence should be addressed to Emanuel Guariglia; emanuel.guariglia@gmail.com

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This paper deals with Chebyshev wavelets. We analyze their properties computing their Fourier transform. Moreover, we discuss the differential properties of Chebyshev wavelets due to the connection coefficients. Uniform convergence of Chebyshev wavelets and their approximation error allow us to provide rigorous proofs. In particular, we expand the mother wavelet in Taylor series with an application both in fractional calculus and fractal geometry. Finally, we give two examples concerning the main properties proved.

1. Introduction

In the last four decades, wavelet analysis rose to the role of mathematical theory due to the introduction of multiresolution analysis [1]. In the current literature, 1909 is often recognised as the birth of wavelet analysis, when Haar introduced a complete orthonormal system for the space $L^2([0, 1])$. Nowadays, wavelet analysis is a mathematical tool widely applied in different fields. Image compression, electromagnetism, and PDE image are just three examples where wavelet methods currently play a meaningful role (see, e.g., [2–5]). In particular, Mallat produced a fast wavelet decomposition and reconstruction algorithm [6]. Over the course of time, the Mallat algorithm became the base for many wavelet applications in pure and applied science. Quite recently, wavelet analysis was also used for several techniques in image fusion, where each algorithm leads to different image decompositions. Fusing two types of information (temporal and spectral), the discrete wavelet transform (DWT) enabled the development of many DWT-based techniques. An application of the wavelet analysis in image fusion is the discrete shapelet transform (DST), which estimates the degree of similarity between the signal under analysis and a prespecified shape. This discrete transform consists of a fractal-based criterion to redefine the original Daubechies' DWTs, leading to a time-frequency-shape joint analysis. Replacing the fractal-based criterion with a correlation-based formulation, the DST can be improved significantly. More specifically, the DST of the second generation simplifies both the study of filter coefficients and the interpretation of the transformed signal [7].

The main advantage of wavelet analysis is their decomposition of mathematical entities (e.g., images and time series) into components at different scales. This property is a consequence of the multiresolution analysis. Approximation in Fourier basis can lead to unpleasant results, as in the case of Gibbs phenomenon. These approximation problems can occur in any other reconstruction, but wavelets. Likewise, fractal geometry allows us to describe irregular sets by the concepts of fractal dimension and lacunarity [8, 9]. As is well known, irregular sets provide a better representation of different natural phenomena than the classical Euclidean models. Thus, fractal-like sets are currently used for many real-world applications (e.g., antenna theory and dynamical systems). Quite recently, considerable attention has been paid to the application of hybrid methods based both on wavelet analysis and fractal geometry in nonlinear modelling. For a fuller and deeper treatment on fractal-wavelet analysis, we refer the reader to the results of Jorgensen [10, 11].

Chebyshev wavelets are generally used for numerical methods in integral equations and PDEs. In particular, Chebyshev wavelets allowed the introduction of these methods due to the operational matrices P and D defined in (12) and (13), respectively. In [12], Hyedari et al. introduced a numerical method based on Chebyshev wavelets for solution of PDEs with boundary conditions of the telegraph type. Biazar and Ebraimi proposed a method based on Chebyshev wavelets for solving nonlinear systems of Volterra integral equations [13]. Similarly, Singh and Saha Ray [14] dealt with the stochastic Itô-Volterra integral equations by Chebyshev wavelets of the second kind. Following the recent trends in nonlinear analysis, Chebyshev wavelets were also used for the numerical solutions of fractional differential equations (see, e.g., [15, 16]). Current literature showed that these methods depends on the different operational matrices in the sense of [17–19]. Moreover, Chebyshev wavelets provided sharp estimates of functions in Hölder spaces of order α [20].

In this paper, we give new results on Chebyshev wavelets. More precisely, we deal with the differentiability of Chebyshev wavelets and the possibility to use their derivatives to reconstruct a function. The differential properties of Chebyshev wavelets, expressed by the connection coefficients (also called refinable integrals), are given by finite series in terms of the Kronecker delta. Moreover, we treat the p-order derivative of Chebyshev wavelets and compute its Fourier transform. In the same spirit, we expand in Taylor series a function by Chebyshev wavelets and connection coefficients. Accordingly, Taylor expansion of the mother wavelet allows us to define the local fractional derivative of Chebyshev wavelets. More precisely, the introduction of local fractional calculus in these wavelet bases enables us to extend the local fractional derivative to nonsmooth continuous functions (e.g., fractal sets or random signals).

The rest of the paper is divided into three sections. In Section 2 we give some remarks on wavelet analysis and, particularly, on Chebyshev wavelets. Section 3 is devoted to differential properties of Chebyshev wavelets by connection coefficients. In Section 4, we deal with the Taylor expansion of Chebyshev wavelets. Finally, Section 5 extends the sought results on Chebyshev wavelets to fractal-like sets by local fractional calculus.

2. Remarks on Wavelet Analysis

This section is to devoted to recall some basic definitions and properties of wavelet analysis, which will be used throughout the paper. From now on, we refer to the set of natural numbers, denoted by $\mathbb{N}$, as the set of strictly positive integer numbers, that is $\mathbb{N} = \{1, 2, 3, \cdots\}$. Thus, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Moreover, we will use the notation $x^{\underline{n}}$ to denote the n th falling factorial of x [21].

Definition 1. The nth-order Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x), x \in [-1, 1], n \in \mathbb{N}_0, \tag{1}$$

so that

$$T_n(\cos\theta) = \cos n\theta, \theta \in [0, \pi], n \in \mathbb{N}_0.$$

Thus, $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, and so on. Definition 1 refers to the trigonometric representation of these polynomials. In literature, Chebyshev polynomials are usually defined as solutions of some Sturm-Liouville differential equations (today called Chebyshev differential equations). In particular, the definition in terms of Sturm-Liouville form leads us to prove the orthogonality of the Chebyshev polynomials with regards to the weight function:

$$w(x) = \frac{1}{\sqrt{1-x^2}}, x \in [-1, 1],$$

that is,

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}} \mathrm{d}x = \begin{cases} \dfrac{\pi}{2}\, \delta_{m,n}, & m \neq 0, n \neq 0, \\ \pi, & m = n = 0, \end{cases} \tag{2}$$

where $\delta_{m,n}$ is the Kronecker delta. Furthermore, for any $n, m \in \mathbb{N}_0$, Chebyshev polynomials can be written by the following general recurrence relation:

$$T_{n+m}(x) = 2\, T_n(x)\, T_m(x) - T_{|n-m|}(x),$$

which for $m = 1$ gives

$$T_{n+1}(x) = 2\, x\, T_n(x) - T_{n-1}(x), n \geq 1. \tag{3}$$

Chebyshev polynomials of the second, third, and fourth kinds can be defined and handled in much the same way. Moreover, all four Chebyshev polynomials admits a matrix representation (see [13] for more details). For instance, (3) can be written in matrix form as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -2x & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2x & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2x & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2x & 1 \end{pmatrix} \begin{pmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ T_3(x) \\ \vdots \\ T_n(x) \end{pmatrix} = \begin{pmatrix} 1 \\ -x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4}$$

or

$$At = c,$$

where A is the $(n+1) \times (n+1)$ matrix of the coefficients in (3) while t and c are the left-hand side and the right-hand side vectors in (4), respectively.

The properties of Chebyshev polynomials mentioned above lay the foundation for introducing a corresponding wavelet bases, termed Chebyshev wavelets. To this scope and before going ahead, let us recall the definition of wavelet orthonormal basis on $\mathbb{R}$. Wavelets are a family of functions generated by dilation and translation of one single function

ψ (called mother wavelet). In literature, all other functions of this family are usually called daughter wavelets. Thus, a family of continuous wavelets is given by

$$\psi_{a,b}(x) = |a|^{-1/p}\,\psi\left(\frac{x-b}{a}\right), p>0, a, b \in \mathbb{R}, a \neq 0, \quad (5)$$

where a and b correspond to the scale factor and time shift, respectively. In what follows, therefore, we can assume $p = 2$ in (5) which is the most common value for p. Clearly, (5) for dilation and translation parameters a^{-k} and nba^{-k} gives the following family of discrete wavelets:

$$\psi_n^k(x) = |a|^{k/2}\,\psi\left(|a|^k x - nb\right), \quad a>1, b>0, k, n \in \mathbb{Z}, \quad (6)$$

which for $a = 2$ and $b = 1$ yields

$$\psi_n^k(x) = 2^{k/2}\psi\left(2^k x - n\right), \quad k, n \in \mathbb{Z}.$$

The family of functions (6) is a wavelet basis for $L^2(\mathbb{R})$ which becomes orthonormal for $a = 2$ and $b = 1$.

2.1. Chebyshev Wavelets. Multiresolution analysis shows that Chebyshev wavelets can be built as recursive wavelets for piecewise polynomial spaces on $[0, 1]$. For this construction, we refer the reader to [22, 23], in which the problem is widely discussed.

Definition 2. Let $n = 1, 2, \cdots, 2^{k-1}$ and $m = 0, 1, \cdots, M-1$ with $(k, M) \in \mathbb{N}^2$. Chebyshev wavelets are defined as follows:

$$\psi_{n,m}^k(x) = \begin{cases} 2^{k/2}\tilde{T}_m\left(2^k x - 2n + 1\right), & \dfrac{n-1}{2^{k-1}} \leq x < \dfrac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \quad (7)$$

where

$$\tilde{T}_m(x) = \frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)T_m(x).$$

Remark 3. In Definition 2, Chebyshev wavelets depends on four parameters, that is, $\psi_{n,m}^k(x) = \psi(k, n, m, x)$. Moreover, $m \in \mathbb{N}_0$; thus,

$$\tilde{T}_m(x) = \begin{cases} \dfrac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\dfrac{2}{\pi}}\,T_m(x), & m > 0. \end{cases}$$

Remark 4. In view of (1), Definition 2 implies that Chebyshev wavelets are defined on the real interval $[0, 1)$. Note that the orthogonality of the Chebyshev polynomials on $[-1, 1]$ with regard to w implies the orthogonality of the Chebyshev wavelets on $[0, 1)$ with regard to the weight function $w_k(x) = w(2^{k-1}x - n + 1)$ with n and x as in Definition 2 (see [24] for more details).

A function $f \in L^2(\mathbb{R})$ can be expanded in terms of the wavelet basis $(\psi_n^k)_{k,n\in\mathbb{Z}}$ as follows:

$$f(x) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\beta_n^k\,\psi_n^k(x). \quad (8)$$

The coefficients ψ_n^k, usually termed wavelet coefficients, are given by $\beta_n^k = \langle f(x), \psi_n^k(x)\rangle$ where $\langle .,.\rangle$ denotes the inner product. The series representation in (8) is called a wavelet series. In the case of Chebyshev wavelets, the previous inner product is defined in $L_w^2([0, 1])$, that is,

$$\langle f(x), g(x)\rangle_w := \int_0^1 f(x)\bar{g(x)}w(x)dx, f, g \in L_w^2[0, 1].$$

Thus, any function $f \in L_w^2([0, 1])$ can be expanded in terms of Chebyshev wavelets as follows:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta_{n,m}^k\,\psi_{n,m}^k(x), \quad (9)$$

where the wavelet coefficients are given by

$$\beta_{n,m}^k = \left\langle f(x), \psi_{n,m}^k(x)\right\rangle_{w_k}. \quad (10)$$

2.2. Function Approximation and Operational Matrix. Convergence of series (9) on $L_w^2[0, 1]$ implies that f can be approximated as follows:

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\beta_{n,m}^k\,\psi_{n,m}^k(x) = B^T\mathbf{\Psi}(x), \quad (11)$$

where B and $\mathbf{\Psi}$ are $\tilde{m} = 2^{k-1}M$ column vectors. For simplicity of notation and without loss of generality, we rewrite (11) as

$$f(x) \simeq \sum_{i=1}^{\tilde{m}}\beta_i\,\psi_i(x) = \mathrm{B}^T\mathbf{\Psi}(x),$$

where $\beta_i = \beta_{n,m}^k$ and $\psi_i = \psi_{n,m}^k$. The index i is given by $i = M(n-1) + m + 1$, and thus,

$$\begin{cases} B := (\beta_1, \beta_2, \cdots, \beta_{\tilde{m}})^T, \\ \mathbf{\Psi}(x) := (\psi_1, \psi_2, \cdots, \psi_{\tilde{m}})^T. \end{cases}$$

Likewise, Chebyshev wavelets allow us to approximate every function of two variables $u = u(x, y)$ defined over $[0, 1) \times [0, 1)$ as follows:

$$u(x,y) \simeq \sum_{i=1}^{\tilde{m}} \sum_{j=1}^{\tilde{m}} u_{ij}\, \psi_i(x)\psi_i(y) = \mathbf{\Psi}^T(x) U \mathbf{\Psi}(y),$$

where $U = [u_{ij}]$ being

$$u_{ij} = \left\langle \psi_i(x), \left\langle u(x,y), \psi_j(y) \right\rangle_{w_k(y)} \right\rangle_{w_k(x)}, i, j = 1, 2, \cdots, \tilde{m}.$$

We may now integrate the vector $\mathbf{\Psi}(x)$, precisely given by

$$\int_0^x \mathbf{\Psi}(t)\, dt = P\mathbf{\Psi}(x), \tag{12}$$

where P is the $\tilde{m} \times \tilde{m}$ operational matrix of integration. It is worth noticing that, due to introduction of Chebyshev wavelets, the matrix P is sparse (see [17, 18] for more details). Furthermore, P^n allows the n-times integration of $\mathbf{\Psi}(x)$ given by

$$\underbrace{\int_0^x \cdots \int_0^x}_{n-\text{times}} \mathbf{\Psi}(t) dt_1 \cdots \mathrm{d}t_n = P^n \mathbf{\Psi}(x).$$

Similarly, we can differentiate $\mathbf{\Psi}(x)$ as follows:

$$\frac{d\mathbf{\Psi}(x)}{dx} = D\mathbf{\Psi}(x),$$

and so

$$\frac{d^n\mathbf{\Psi}(x)}{dx^n} = D^n\mathbf{\Psi}(x), \tag{13}$$

where D is the $\tilde{m} \times \tilde{m}$ operational matrix of differentiation [12, 25].

Finally, we point out that the product of two Chebyshev wavelets can be approximated as [19] follows:

$$\mathbf{\Psi}(x)\mathbf{\Psi}^T(x)X \simeq \tilde{X}\mathbf{\Psi}(x),$$

where X is a $\tilde{m}$ column vector and $\tilde{X}$ is a $\tilde{m} \times \tilde{m}$ matrix. In literature, $\tilde{X}$ is called the operational matrix of product. In particular, for $X = B$, we get

$$\mathbf{\Psi}(x)\mathbf{\Psi}^T(x)B \simeq \tilde{B}\mathbf{\Psi}(x), \tag{14}$$

where $\tilde{B}$ is a diagonal $\tilde{m} \times \tilde{m}$ matrix. In recent years, approximation (14) has been applied for solving integral equations, PDEs, and boundary valued problems (see, e.g., [17, 18]).

3. Fourier Transform, Differentiability, and Connection Coefficients

In this section, we study the differentiability of Chebyshev wavelets. More precisely, we prove our results in the weighted function space $L^2_w[0,1]$ by the introduction of connection coefficients.

3.1. Differentiability of L^2_w Functions in Chebyshev Wavelet Bases

Theorem 5. *(L^2_w convergence) A function $f \in L^2_w[0,1]$ with bounded second derivative on $[0,1]$, i.e., $|f''(x)| \le A$ for any $x \in [0,1]$, can be expanded as an infinite sum of Chebyshev wavelets and the series converges uniformly to the function f, that is*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \beta^k_{n,m} \psi^k_{n,m}(x), \tag{15}$$

where $\beta^k_{n,m} = \langle f(x), \psi^k_{n,m}(x) \rangle_{w_k}$.

Proof. We only sketch the proof. For a fuller treatment, we refer the reader to [24].

First,

$$\beta^k_{n,m} = \int_0^1 f(x)\psi^k_{n,m}(x) w_k(x) dx = \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} \frac{f(x)\, \tilde{T}_m\left(2^k x - 2n + 1\right)}{\sqrt{1 - \left(2^k x - 2n + 1\right)^2}} dx. \tag{16}$$

Now, if $m > 1$, the change of variable $2^k x - 2n + 1 = \cos\theta$ in (16) gives

$$\begin{aligned} \beta^k_{n,m} &= \frac{\sqrt{2}}{2^{3k/2} m \sqrt{\pi}} \int_0^\pi f'\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sin m\theta \sin\theta \mathrm{d}\theta \\ &= \frac{1}{2^{5k/2} m \sqrt{2\pi}} \int_0^\pi f''\left(\frac{\cos\theta + 2n - 1}{2^k}\right) h_m(\theta) \mathrm{d}\theta, \end{aligned} \tag{17}$$

where

$$h_m(\theta) = \sin\theta \left(\frac{\sin\,(m-1)\theta}{m-1} - \frac{\sin\,(m+1)\theta}{m+1}\right).$$

Since $n \le 2^{k-1}$, it follows that

$$\beta^k_{n,m} \le \frac{\sqrt{2\pi}A}{(2n)^{5/2}(m^2 - 1)}. \tag{18}$$

Similarly, if $m = 1$, (17) implies that

$$\beta^k_{n,1} \le \frac{\sqrt{2\pi}}{(2n)^{3/2}} \max_{0 \le x \le 1} |f'(x)|. \tag{19}$$

Furthermore, for $m = 0$, the series in (15) converges. In fact, $(\psi^k_{n,0})_{n=1}^{\infty}$ is an orthogonal system, which implies the convergence of $\sum_{n=1}^{\infty}\beta^k_{n,0}\,\psi^k_{n,0}(x)$. It follows that

$$\left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta^k_{n,m}\,\psi^k_{n,m}(x)\right| \le \left|\sum_{n=1}^{\infty}\beta^k_{n,0}\,\psi^k_{n,0}(x)\right| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\beta^k_{n,m}\right| < \infty.$$

Accordingly, the series $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta^k_{n,m}\psi^k_{n,m}(x)$ converges to $f(x)$ uniformly, as desired. □

Theorem 5 leads to computation of the approximation error of the wavelet expansion (30), as stated in the following proposition.

Proposition 6. *(estimation) Under the same hypotheses as in Theorem 5, we have*

$$\sigma_{k,M} \le \frac{\sqrt{\pi}}{2}\left(\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^3}\left(\max_{0\le x\le 1}{}^2\left|f'(x)\right|[\![M=1]\!] + \frac{A^2}{4n^2}\sum_{m=M}^{\infty}\frac{1}{(m^2-1)^2}M\neq 1\right)\right)^{1/2},$$

where $[\![\,]\!]$ *is the Iverson bracket notation and*

$$\sigma_{k,M} := \left(\int_0^1\left(f(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\beta^k_{n,m}\,\psi^k_{n,m}(x)\right)^2 w_k(x)dx\right)^{1/2}.$$

Proof. First, recall that the series in (30) can be approximated with the truncated series in (11). Throughout the proof, we write $\beta_{n,m}$ instead of $\beta^k_{n,m}$ to avoid confusion. Accordingly,

$$\begin{aligned}\sigma^2_{k,M} &= \int_0^1\left(f(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\beta_{n,m}\,\psi^k_{n,m}(x)\right)^2 w_k(x)\mathrm{d}x\\ &= \int_0^1\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}\left(\psi^k_{n,m}(x)\right)^2 w_k(x)dx\\ &= \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}\frac{2^{k+1}}{\pi}\int_{(n-1)/2^{k-1}}^{n/2^{k-1}}\frac{T_m^2\left(2^kx-2n+1\right)}{\sqrt{1-\left(2^kx-2n+1\right)^2}}dx.\end{aligned}\tag{20}$$

Now, the change of variable $2^kx-2n+1=x'$ in (20) and relabeling x' as x gives

$$\sigma^2_{k,M} = \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}\frac{2}{\pi}\int_{-1}^{1}\frac{T_m^2(x)}{\sqrt{1-x^2}}dx.$$

Moreover, (2) implies that

$$\int_{-1}^{1}\frac{T_m^2(x)}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{2},\quad m\ge 1,$$

therefore

$$\sigma^2_{k,M} = \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}.\tag{21}$$

From Definition 2 we see that $M\in\mathbb{N}$, thus combining (18), (19) with (21) it follows

$$\begin{aligned}\sigma^2_{k,M} &= \sum_{n=2^{k-1}+1}^{\infty}\left(\beta^2_{n,1}[\![M=1]\!] + \sum_{m=M}^{\infty}\beta^2_{n,m}[\![M\neq 1]\!]\right)\\ &\le \frac{\pi}{4}\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^3}\left(\max_{0\le x\le 1}{}^2\left|f'(x)\right|[\![M=1]\!] + \frac{A^2}{4n^2}\sum_{m=M}^{\infty}\frac{1}{(m^2-1)^2}[\![M\neq 1]\!]\right).\end{aligned}$$

The proof is complete. □

Theorem 5 and Proposition 6 show uniform convergence and accuracy estimation of Chebyshev wavelets, which lay the foundation for their wide application to the theory of integral equations (see, e.g., [12–15, 17, 18, 24]).

Our next goal is to rewrite Chebyshev wavelets as a power series. We recall [26] that

$$T_m(x) = \frac{m}{2}\sum_{r=0}^{\lfloor m/2\rfloor}\frac{(-1)^r}{m-r}\binom{m-r}{r}(2x)^{m-2r},\ m>0.$$

The change of variable $r'=m-2r$ in the previous series gives

$$\begin{aligned}T_m(x) &= m\sum_{r=0}^{m}\frac{(-1)^{(m-r)/2}}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}}(2x)^r\\ &= \sum_{r=0}^{m}\frac{2^r m\,(-1)^{\omega}}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}}x^r,\quad m+r\in E,\omega=\frac{(m-r)}{2},\end{aligned}\tag{22}$$

where E denotes the set of even numbers. Clearly, $m+r\in E$ implies that $m-r\in E$, hence $m\pm r\in E$. Moreover, the previous change of variable entails that

$$m-2\lfloor m/2\rfloor = \begin{cases}0, & m\in E,\\ 1, & m\in O,\end{cases}$$

thus, the lower index of summation in (22) is $r=0$. For simplicity of notation, we set

$$\alpha^m_r := \frac{2^r m(-1)^{\omega}}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}},\quad m+r\in E,\omega=\frac{(m-r)}{2},\tag{23}$$

thus,

$$T_m(x) = \sum_{r=0}^{m} \alpha_r^m x^r, \quad m > 0. \tag{24}$$

It is worth noticing that all contributions of the summation index r in (24) are subject to the condition $m + r \in E$ in (23), i.e., $m \pm r \in E$; thus, half of them vanish. More precisely, we have that the lower index of summation is $r = 0$ for $m \in E\backslash\{0\}$ and $r = 1$ for $m \in E$.

We see from (24) that (7) can be rewritten as follows:

$$\psi_{n,m}^k(x) = \begin{cases} 2^{k/2}\left(\frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\alpha_r^m\left(2^k x - 2n + 1\right)^r\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{25}$$

Since the parameters n and k give, respectively, a dilation and a translation of the wavelet basis (7), the wavelet mother ψ is such that $\psi = \psi(m, x)$. As a consequence, the wavelet mother ψ depends only on the its associated Chebyshev polynomial $T_m(x)$. According to the current symbology in wavelet analysis, we define the wavelet mother ψ as follows:

$$\psi(x) \coloneqq \psi_{1,m}^1(x),$$

therefore,

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\alpha_r^m(x-1)^r, & 0 \le x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

3.2. Connection Coefficients. The differential operators can be represented in wavelet bases if we compute the wavelet decomposition of the derivatives. Let f be a C^p function with $p > 0$ such that $f \in L_w^2[0,1]$ with bounded second derivative on $[0,1]$. The wavelet reconstruction (30) allows us to compute the derivatives of f as follows:

$$\frac{d^p}{dx^p} f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta_{n,m}^k \frac{d^p}{dx^p}\psi_{n,m}^k(x).$$

Thus, according to (8), the derivatives of f up to order p are uniquely determined by

$$\frac{d^p}{dx^p}\psi_{n,m}^k(x). \tag{26}$$

On the other hand, the first derivative of Chebyshev wavelets are given by

$$\frac{d}{dx}\psi_{n,m}^k(x) = \begin{cases} m\sqrt{\frac{2}{\pi}}2^{3k/2}\,\mathrm{U}_{m-1}\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{27}$$

We note that the first derivative in (27) depends on $\mathrm{U}_{m-1}(x)$, i.e., Chebyshev polynomials of the second kind. More specifically, it can be written as a Chebyshev wavelet of the second kind (see [20]). The computation of the derivatives (26) is more complicated for $p > 1$. In particular, high-order derivatives in (26) cannot be easily derived. Therefore, according to (8), we next turn to the wavelet decomposition of the derivatives (26), that is,

$$\frac{d^p}{dx^p}\psi_{n,m}^k(x) = \sum_{l=1}^{\infty}\sum_{q=0}^{\infty}\gamma_{nlmq}^{(p)kh}\,\psi_{l,q}^h(x), \tag{28}$$

with

$$\gamma_{nlmq}^{(p)kh} = \left\langle \frac{d^p}{dx^p}\psi_{n,m}^k(x), \psi_{l,q}^h(x)\right\rangle_{w_h}. \tag{29}$$

The coefficients (29) are called connection coefficients with an obvious intuitive meaning. Their computation can be obtained in the Fourier domain due to the Parseval–Plancherel identity:

$$\langle f, g\rangle \coloneqq \int_{-\infty}^{\infty} f(x)\,\bar{g}(x)\,dx = \int_{-\infty}^{\infty}\hat{f}(\xi)\,\bar{\hat{g}}(\xi)\,d\xi = \left\langle \hat{f}, \hat{g}\right\rangle, \quad f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

where $\hat{f}$ denotes the Fourier transform of f defined as follows:

$$\hat{f}(\xi) \coloneqq \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i\xi x}\,dx, \quad f \in L^1(\mathbb{R}).$$

Therefore,

$$\widehat{\frac{d^p}{dx^p}\psi_{n,m}^k(x)} = (2\pi i\xi)^p\hat{\psi}_{n,m}^k(\xi). \tag{30}$$

Let us now compute the Fourier transform of $T_m(x)$.

Lemma 7. *The Fourier transform of Chebyshev polynomials $T_m(x)$ is given by*

$$\hat{T}_m(\xi) = \sum_{r=0}^{m} c_r^m\delta^{(r)}(\xi),$$

where

$$c_r^m = \frac{i^m}{\pi^r}\frac{m}{m+r}\begin{pmatrix}\frac{m+r}{2}\\ \frac{m-r}{2}\end{pmatrix}, \quad m + r \in E. \tag{31}$$

Proof. The proof falls naturally into two parts ($m = 0$ and $m > 0$). For $m = 0$, it follows immediately that $c_0^0 = 1$and

$$T_0(x) = 1 \overset{\mathscr{F}}{\leftrightarrow} \delta(\xi).$$

Let us now turn to the case $m > 0$. We begin by recalling the differentiation property of Fourier transform

$$x^n f(x) \overset{\mathscr{F}}{\leftrightarrow} \left(\frac{i}{2\pi}\right)^n \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} \hat{f}(\xi), n \in \mathbb{N}_0, \tag{32}$$

which holds in the space of tempered distributions on the real line $\mathscr{S}'(\mathbb{R})$. Accordingly, from (32) for $f(x) = \delta(x)$, we get

$$x^n \overset{\mathscr{F}}{\leftrightarrow} \left(\frac{i}{2\pi}\right)^n \delta^{(n)}(\xi), n \in \mathbb{N}_0,$$

where δ is the Dirac delta distribution. Therefore,

$$T_m(\xi) \overset{(24)}{=} \sum_{r=0}^{m} \alpha_r^m \mathscr{F}\{x^r\} = \sum_{r=0}^{m} \alpha_r^m \left(\frac{i}{2\pi}\right)^r \delta^{(r)}(\xi), \quad m > 0.$$

Furthermore, since

$$\alpha_r^m \left(\frac{i}{2\pi}\right)^r = c_r^m, m > 0,$$

the desired result plainly follows. □

On the one hand, in the proof of Lemma 7, we used the fact that $\widehat{T}_0(\xi) = \delta(\xi)$. On the other hand, the principal significance of Lemma 7 is that the Fourier transform of Chebyshev polynomials $T_m(x)$ is nothing but a sum of derivatives of the Dirac delta. Condition (31) on coefficients c_r^m implies that if $m \in E$ the Fourier transform of $T_m(x)$ is the sum of all even order derivatives of the Dirac delta. Likewise, if $m \in O$, the Fourier transform of $T_m(x)$ is the sum of all odd order derivatives of the Dirac delta. We note that for $m \in E$ the Fourier transform $\widehat{T}_m(\xi)$ always contains the Dirac delta $\delta(\xi)$. Moreover, the presence of the power i^m in (31) implies that

$$\widehat{T}_m(\xi) \in \begin{cases} \mathbb{R}, & m \in E, \\ \mathbb{I}, & m \in 0, \end{cases}$$

where $\mathbb{I}$ denotes the set of imaginary numbers. These results are shown in Table 1 for the $m = 1, 2, \cdots, 10$.

Now, we are in a position to compute the Fourier transform in (30) and connection coefficients.

Theorem 8. *Let n and k be defined as in (7). Moreover, let $m, p \in \mathbb{N}$. The following statements hold:*

$$\hat{\psi}_{n,m}^k(\xi) = 2^{k/2}\left(\frac{1}{\sqrt{\pi}}\delta_m \delta(\xi) + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\sum_{t=0}^{r} c_{r,t}^{m,n}\delta^{(t)}(\xi)\right),$$

with

$$c_{r,t}^{m,n} := \left(2^{(k-1)t+r}\, i^{m-r+t}/\pi^t\right)(m/m+r)\begin{pmatrix} m+r/2 \\ m-r/2 \end{pmatrix}\begin{pmatrix} r \\ t \end{pmatrix}(1-2n)^{r-t}, m+r \in E,$$

$$\frac{d^p}{dx^p}\hat{\psi}_{n,m}^k(x) = 2^{k/2}(2\pi i\xi)^p\left(\frac{1}{\sqrt{\pi}}\delta_m\delta(\xi) + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\sum_{t=0}^{r}c_{r,t}^{m,n}\delta^{(t)}(\xi)\right),$$

$$\gamma_{nlmq}^{(p)kh} = \frac{2^{1+p+((k-h)k-h/2)}}{\sqrt{\pi}} m \left(\sum_{\substack{0\le r\le m-p: \\ m-p-r\in E}} d_{m,r}^p\left(\sqrt{2}\delta_q\lambda_{\delta_q} + (1-\delta_q)\lambda_{1-\delta_q}\right) - \frac{\sqrt{2\pi}}{4}d_m^p[\![m-p\in E]\!]\right), l \le n,$$

where

$$d_{m,r}^p := \left(\frac{m+p+r}{2}-1\right)^{\underline{p-1}}\begin{pmatrix}\frac{m+p-r}{2}-1 \\ p-1\end{pmatrix},$$

$$\lambda_{1-\delta_q} := \lambda_{\delta_q}\left(1 + \sum_{\nu=1}^{q}\alpha_\nu^q\frac{(s+\nu+2)(s+\nu+1/2)!}{(s+\nu+1)(s+\nu+2/2)!}\right),$$

with α_ν^q as in (23), $d_m^p = d_{m,0}^p$ and

$$\lambda_{\delta_q} := \sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j}\alpha_t^r\begin{pmatrix}t \\ j\end{pmatrix}\begin{pmatrix}j \\ s\end{pmatrix}2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s}\frac{((s+1)/2)!}{(s+1)\,(s/2)!}, s \in E.$$

Proof. First, from (25), it follows that

$$\hat{\psi}_{n,m}^k(\xi) = 2^{k/2}\left(\frac{1}{\sqrt{\pi}}\delta_m\delta(\xi) + \sqrt{\frac{2}{\pi}}(1-\delta_m)\mathscr{F}\left\{\sum_{r=0}^{m}\alpha_r^m\left(2^k x - 2n + 1\right)^r\right\}\right).$$

Combining the proof of Lemma 7 and binomial theorem gives

$$\mathscr{F}\left\{\sum_{r=0}^{m}\alpha_r^m\left(2^k x-2n+1\right)^r\right\} = \sum_{r=0}^{m}\sum_{t=0}^{r}\alpha_r^m 2^{kr}\begin{pmatrix}r \\ t\end{pmatrix}\left(\frac{-2n+1}{2^k}\right)^{r-t} \cdot \left(\frac{i}{2\pi}\right)^t\delta^{(t)}(\xi) \overset{(23)}{=} \sum_{r=0}^{m}\sum_{t=0}^{r}\frac{2^{(k-1)t+r}\, i^{m-r+t}}{\pi^t}\frac{m}{m+r}\begin{pmatrix}\frac{m+r}{2} \\ \frac{m-r}{2}\end{pmatrix}\begin{pmatrix}r \\ t\end{pmatrix} \cdot (1-2n)^{r-t}\delta^{(t)}(\xi), m+r \in E.$$

With the same notation as in Lemma 7, we set

$$c_{r,t}^{m,n} = \frac{2^{(k-1)t+r}\, i^{m-r+t}}{\pi^t}\frac{m}{m+r}\begin{pmatrix}\frac{m+r}{2} \\ \frac{m-r}{2}\end{pmatrix}\begin{pmatrix}r \\ t\end{pmatrix}(1-2n)^{r-t}, \quad m+r \in E,$$

hence,

$$\mathscr{F}\left\{\sum_{r=0}^{m}\alpha_r^m\left(2^k x-2n+1\right)^r\right\} = \sum_{r=0}^{m}\sum_{t=0}^{r}c_{r,t}^{m,n}\delta^{(t)}(\xi).$$

This proves (i). Furthermore, (ii) follows straightforwardly from (i) and (30).

Table 1: Fourier transform of Chebyshev polynomials $T_m(x)$ for $m = 1, 2, \cdots, 10$.

m	$T_m(x)$	$\hat{T}_m(\xi)$
1	x	$i\delta'(\xi)/(2\pi)$
2	$2x^2 - 1$	$-\delta(\xi) - \delta''(\xi)/(2\pi^2)$
3	$4x^3 - 3x$	$\left(3\pi^2\delta'(\xi) + \delta^{(3)}(\xi)\right)/(2\pi^3 i)$
4	$8x^4 - 8x^2 + 1$	$\delta(\xi) + \left(4\pi^2\delta''(\xi) + \delta^{(4)}(\xi)\right)/(2\pi^4)$
5	$16x^5 - 20x^3 + 5x$	$i\left(5\pi^4\delta'(\xi) + 5\pi^2\delta^{(3)}(\xi) + \delta^{(5)}(\xi)\right)/(2\pi^5)$
6	$32x^6 - 48x^4 + 18x^2 - 1$	$-\delta(\xi) - \left(9\pi^4\delta''(\xi) + 6\pi^2\delta^{(4)}(\xi) + \delta^{(6)}(\xi)\right)/(2\pi^6)$
7	$64x^7 - 112x^5 + 56x^3 - 7x$	$\left(7\pi^6\delta'(\xi) + 14\pi^4\delta^{(3)}(\xi) + 14\pi^2\delta^{(5)}(\xi) + \delta^{(7)}(\xi)\right)/(2\pi^7 i)$
8	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$	$\delta(\xi) + \left(16\pi^6\delta''(\xi) + 20\pi^4\delta^{(4)}(\xi) + 8\pi^2\delta^{(6)}(\xi) + \delta^{(8)}(\xi)\right)/(2\pi^8)$
9	$259x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$	$i\left(9\pi^8\delta'(\xi) + 30\pi^6\delta^{(3)}(\xi) + 27\pi^4\delta^{(5)}(\xi) + 9\pi^2\delta^{(7)}(\xi) + \delta^{(9)}(\xi)\right)/(2\pi^9)$
10	$512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$	$-\delta(\xi) - \left(25\pi^8\delta''(\xi) + 50\pi^6\delta^{(4)}(\xi) + 35\pi^4\delta^{(6)}(\xi) + 10\pi^2\delta^{(8)}(\xi) + \delta^{(10)}(\xi)\right)/(2\pi^{10})$

Finally, we can prove (iii). From (29), we have

$$\gamma_{nlmq}^{(p)kh} = \int_0^1 \frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right)\psi_{l,q}^h(x)\, w_h(x)\, dx. \tag{33}$$

By (7),

$$\begin{aligned}\frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right) &= 2^{k/2}\frac{d^p}{dx^p}\left(\frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)T_m\left(2^k x - 2n + 1\right)\right)\\ &= 2^{k/2}\sqrt{\frac{2}{\pi}}(1-\delta_m)\frac{d^p}{dx^p}\left(T_m\left(2^k x - 2n + 1\right)\right), \quad \frac{n-1}{2^{k-1}}\\ &\le x < \frac{n}{2^{k-1}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right)\psi_{l,q}^h(x) &= 2^{(h+k)/2}\,\frac{\sqrt{2}}{\pi}(1-\delta_m)\delta_q\frac{d^p}{dx^p}\left(T_m\left(2^k x - 2n + 1\right)\right)\\ &\quad + 2^{(h+k)/2}\,\frac{2}{\pi}(1-\delta_m)(1-\delta_q)\frac{d^p}{dx^p}\\ &\quad \cdot\left(T_m\left(2^k x - 2n + 1\right)\right)T_q\left(2^h x - 2l + 1\right)\\ &= 2^{(h+k)/2}\,\frac{d^p}{dx^p}\left(T_m\left(2^k x - 2n + 1\right)\right)\\ &\quad \cdot\left(\frac{\sqrt{2}}{\pi}\delta_q + \frac{2}{\pi}(1-\delta_q)T_q\left(2^h x - 2l + 1\right)\right),\end{aligned}$$

which follows from the hypothesis that $m \ge 1$. We proved in Appendix that

$$\frac{d^p}{dx^p}T_m\left(2^k x - 2n + 1\right) = 2^p m \sum_{\substack{0\le r\le m-p\,:\\ m-p-r\in E}} d_{m,r}^p\, T_r\left(2^k x - 2n + 1\right) - 2^{p-1} m\, d_m^p\; m - p \in E,$$

with

$$d_{m,r}^p := \left(\frac{m+p+r}{2} - 1\right)\underline{p-1}\begin{pmatrix}\dfrac{m+p-r}{2} - 1\\ p-1\end{pmatrix},$$

and $d_m^p = d_{m,0}^p$. Thus,

$$\begin{aligned}\frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right)\psi_{l,q}^h(x) &= 2^{p+(h+k)/2}\, m\left(\frac{\sqrt{2}}{\pi}\delta_q + \frac{2}{\pi}(1-\delta_q)T_q\left(2^h x - 2l + 1\right)\right)\\ &\quad \cdot\left(\sum_{0\le r\le m-p: m-p-r\in E} d_{m,r}^p\, T_r\left(2^k x - 2n + 1\right) - \frac{d_m^p}{2}\; m - p \in E\right),\end{aligned}$$

Now, we can proceed to compute $\gamma_{nlmq}^{(p)kh}$. Note that, expanding the integrand in (33), we have

$$\begin{cases}\dfrac{n-1}{2^{k-1}} \le x < \dfrac{n}{2^{k-1}}, & n = 1, 2, \cdots, 2^{k-1},\\[2mm] \dfrac{l-1}{2^{h-1}} \le x < \dfrac{l}{2^{h-1}}, & l = 1, 2, \cdots, 2^{h-1}.\end{cases} \tag{34}$$

The assumption $l \le n$ implies that the integrand in (33) holds for $(34)_2$. As a consequence,

$$
\begin{aligned}
\gamma_{nlmq}^{(p)kh} &= \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} 2^{p+(h+k)/2}\, m \left(\sum_{0\le r\le m-p:m-p-r\in E} d_{m,r}^{p}\, T_r\left(2^k x - 2n + 1\right) - \frac{d_m^p}{2}\, m - p \in E \right) \\
&\quad \cdot \left(\frac{\sqrt{2}}{\pi}\delta_q + \frac{2}{\pi}\left(1-\delta_q\right) T_q\left(2^h x - 2l + 1\right) \right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx \\
&= 2^{p+((h+k)/2)}\, m \sum_{0\le r\le m-p:m-p-r\in E} d_{m,r}^{p} \left(\frac{\sqrt{2}}{\pi}\delta_q \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} T_r\left(2^k x - 2n + 1\right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx \right. \\
&\quad \left. + \frac{2}{\pi}\left(1-\delta_q\right) \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} T_r\left(2^k x - 2n + 1\right) T_q\left(2^h x - 2l + 1\right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx \right) \\
&\quad - 2^{p+((h+k)h+k/2)}\, m \frac{d_m^p}{2}\, m - p \in E \left(\frac{\sqrt{2}}{\pi}\delta_q \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx + \frac{2}{\pi}\left(1-\delta_q\right) \right. \\
&\quad \left. \cdot \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} T_q\left(2^h x - 2l + 1\right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx \right).
\end{aligned}
$$

For simplicity of notation, we indicate the last four integrals with I_1,I_2, I_3, and I_4, respectively. Thus,

$$
\begin{aligned}
\gamma_{nlmq}^{(p)kh} &= 2^{p+(h+k)/2}\, m \left(\sum_{0\le r\le m-p:m-p-r\in E} d_{m,r}^{p} \left(\frac{\sqrt{2}}{\pi}\delta_q I_1 + \frac{2}{\pi}\left(1-\delta_q\right) I_2 \right) \right. \\
&\quad \left. - \frac{d_m^p}{2} \left(\frac{\sqrt{2}}{\pi}\delta_q I_3 + \frac{2}{\pi}\left(1-\delta_q\right) I_4 \right) m - p \in E \right).
\end{aligned}
\tag{35}
$$

The change of variable $x' = 2^h x - 2l + 1$ in I_1, and relabeling x' as x, gives

$$
\begin{aligned}
I_1 &= \frac{1}{2^h}\int_{-1}^{1} T_r\left(2^{k-h}(x + 2l - 1) - 2n + 1\right) \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\
&\overset{(24)}{=} \frac{1}{2^h}\int_{-1}^{1} \sum_{t=0}^{r} \alpha_t^r \left(2^{k-h}(x + 2l - 1) - 2n + 1\right)^t \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\
&= \frac{1}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j} \alpha_t^r \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s} \int_{-1}^{1} \frac{x^s}{\sqrt{1-x^2}}\mathrm{d}x.
\end{aligned}
\tag{36}
$$

We note that the last computation follows from the binomial theorem. Moreover,

$$
\int_{-1}^{1} \frac{x^s}{\sqrt{1-x^2}}\mathrm{d}x = \sqrt{\pi}\frac{\left((-1)^s + 1\right)\Gamma((s+1)/2)}{s\Gamma(s/2)}, \ \mathrm{Re}\ s > -1,
\tag{37}
$$

Since $s \ge 0$ in (35), the result above holds in the computation of I_1. Obviously, (36) vanishes for s odd. Therefore,

$$
\int_{-1}^{1} \frac{x^s}{\sqrt{1-x^2}}\mathrm{d}x = 2\sqrt{\pi}\,\frac{\Gamma((s+1)/2)}{s\Gamma(s/2)} = 2\sqrt{\pi}\frac{((s+1)/2)!}{(s+1)\,(s/2)!}, \quad s \in E,
$$

and so

$$
I_1 = \frac{\sqrt{\pi}}{2^{h-1}}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j} \alpha_t^r \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s}\frac{((s+1)/2)!}{(s+1)\,(s/2)!}, \quad s \in E.
$$

Let us now pass to the second integral I_2. We can proceed analogously to the computation of I_1. In fact, (35) implies that

$$
\begin{aligned}
I_2 &= \frac{1}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j} \alpha_t^r \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j} \\
&\quad \cdot (2l-1)^{j-s}\int_{-1}^{1}\frac{x^s}{\sqrt{1-x^2}}T_q(x)\,\mathrm{d}x \overset{(24)}{=} \frac{1}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j}\sum_{\nu=0}^{q} \alpha_t^r\alpha_\nu^q \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s}\int_{-1}^{1}\frac{x^{s+\nu}}{\sqrt{1-x^2}}\mathrm{d}x.
\end{aligned}
$$

Furthermore,

$$
\int_{-1}^{1}\frac{x^{s+\nu}}{\sqrt{1-x^2}}\mathrm{d}x = \sqrt{\pi}\frac{\left((-1)^{s+\nu}+1\right)\Gamma((s+\nu+1)/2)}{2\,\Gamma((s+\nu+2)/2)}, \qquad \mathrm{Re}\ (s+\nu) > -1.
\tag{38}
$$

As in computation of I_1, the result above holds because $s + \nu \ge 0$. We note that (37) vanishes for $s + \nu$ odd. Thus,

$$
\int_{-1}^{1}\frac{x^{s+\nu}}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\sqrt{\pi}\Gamma((s+\nu+1)/2)}{\Gamma((s+\nu+2)/2)} = \sqrt{\pi}\frac{(s+\nu+2)((s+\nu+1)/2)!}{(s+\nu+1)((s+\nu+2)/2)!}, \quad s+\nu \in E,
$$

and so

$$
I_2 = \frac{\sqrt{\pi}}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j}\sum_{\nu=0}^{q} \alpha_t^r\,\alpha_\nu^q \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s}\frac{(s+\nu+2)((s+\nu+1)/2)!}{(s+\nu+1)((s+\nu+2)/2)!}, \quad s+\nu \in E.
$$

Similarly, we can compute the integral I_3 and I_4. In fact, the same change of variable as in I_1gives

$$
\begin{aligned}
I_3 &= \frac{1}{2^h}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{2^h}, \\
I_4 &= \frac{1}{2^h}\int_{-1}^{1}T_q(x)\frac{1}{\sqrt{1-x^2}}\mathrm{d}x \overset{(2)}{=} \frac{\pi}{2^h}\delta_q.
\end{aligned}
$$

Thus, $(1-\delta_q)I_4 = 0$ in (34). This completes the proof. □

Remark 9. In the proof of Theorem 8, the hypothesis $l \le n$ played a fundamental role in the computation of connection coefficients $\gamma_{nlmq}^{(p)kh}$. Indeed, for $n < l$, the integrand in (33) holds for $(34)_1$. Thus, we get a similar computation as in the proof of Theorem 8 but the integral I_1 and I_2 will be defined on $[(n-1)/2^{k-1}, n/2^{k-1}[$. The details are left to the reader.

Remark 10. Theorem 8 allows us to define the connection coefficients of Chebyshev wavelets. Moreover, we note that the proof of Theorem 5 gives an upper bound on connection coefficients $\gamma_{nlmq}^{(p)kh}$, i.e.,

$$
\gamma_{nlmq}^{(p)kh} \le \frac{2^{1+p+(k-h)/2}}{\sqrt{\pi}} m \sum_{\substack{0\le r\le m-p:\\ m-p-r\in E}} d_{m,r}^{p}\left(\sqrt{2}\delta_q\lambda_{\delta_q} + \left(1-\delta_q\right)\lambda_{1-\delta_q}\right), \quad l \le n.
\tag{39}
$$

Theorems 5 and 8 allow us the reconstruction for $L_w^2([0,1])$ functions together with their derivative. Moreover, Theorem 8 gives the Fourier transform for any order derivative of Chebyshev wavelets, as the next example shows.

Example 1. Let us consider Chebyshev wavelets for $k=2$ re and $m=1$:

$$\psi^2_{1,1}(x)=\begin{cases}2\sqrt{\dfrac{2}{\pi}}(4x-1), & 0\le x<\dfrac{1}{2},\\ 0, & \text{otherwise,}\end{cases}\quad \psi^2_{2,1}(x)=\begin{cases}2\sqrt{\dfrac{2}{\pi}}(4x-3), & \dfrac{1}{2}\le x<1,\\ 0, & \text{otherwise.}\end{cases}$$

For the sake of simplicity and without loss of generality, we consider only $\hat{\psi}^2_{1,1}$. Thus, we leave it to the reader to deal with $\hat{\psi}^2_{2,1}$.

From Theorem 8, we have

$$\begin{aligned}\widehat{\psi}^2_{1,1}(\xi)&=\sqrt{\frac{2}{\pi}}\sum_{r=0}^{1}\sum_{t=0}^{r}c^{1,1}_{r,t}\delta^{(t)}(\xi)=\sqrt{\frac{2}{\pi}}\left(\left(c^{1,1}_{0,0}+c^{1,1}_{1,0}\right)\delta(\xi)+c^{1,1}_{1,0}\,\delta'(\xi)\right)\\&=\sqrt{\frac{2}{\pi}}\left((-1+i)\delta(\xi)+\frac{2i}{\pi}\,\delta'(\xi)\right),\end{aligned}$$

where we used that $c^{1,1}_{0,0}=i$, $c^{1,1}_{1,0}=-1$, and $c^{1,1}_{1,1}=2i/\pi$. Finally, we conclude that the sixth-order derivative of $\psi^2_{1,1}$ has the following Fourier transform:

$$\begin{aligned}\widehat{\frac{d^6}{dx^6}\psi^2_{1,1}(x)}&=(2\pi i\xi)^6\sqrt{\frac{2}{\pi}}\left((-1+i)\delta(\xi)+\frac{2i}{\pi}\delta'(\xi)\right)\\&=64\pi^5\sqrt{\frac{2}{\pi}}\xi^6\left(\pi\,(1-i)\delta(\xi)-2i\delta'(\xi)\right).\end{aligned}$$

Figures 1 and 2 show the graph of $\psi^2_{1,1}$ and $\widehat{\psi}^2_{1,1}$. We note that the Fourier transform of $\psi^2_{1,1}$ exhibits an impulsive behaviour. In particular, the imaginary part of $\widehat{\psi}^2_{1,1}$ depends also on the distributional derivative of the Dirac delta, i.e.,

$$\begin{cases}\operatorname{Re}\ \widehat{\psi}^2_{1,1}=\operatorname{Re}\ \widehat{\psi}^2_{1,1}(\delta),\\ \operatorname{Im}\ \widehat{\psi}^2_{1,1}=\operatorname{Im}\ \widehat{\psi}^2_{1,1}\left(\delta,\delta'\right).\end{cases}$$

We may approximate the Dirac delta as [27] follows:

$$\delta(x)\approx\sqrt{\frac{1}{\varepsilon\pi}}e^{-x^2/\varepsilon},\quad \varepsilon\ll 1,$$

thus,

$$\delta'(x)=-\frac{2x}{\varepsilon\sqrt{\varepsilon\pi}}e^{-x^2/\varepsilon},\quad \varepsilon\ll 1. \tag{40}$$

Approximation (39) allows us to draw $\hat{\psi}^2_{1,1}$. In Figure 1, we show the imaginary part of $\hat{\psi}^2_{1,1}$ with $\varepsilon=0.01$. Precisely, it is worth noticing that the Dirac delta $\delta(\xi)$ in Figure 1 was multiplied by a factor of 51 which takes into account the approximation error introduced by (39).

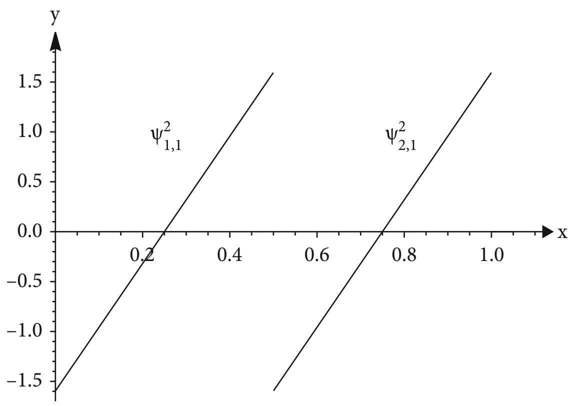

Figure 1: Chebyshev wavelets $\psi^2_{n^*,1}$ with $n^*=1,2$.

4. Taylor Series and Chebyshev Wavelets

In Section 3, we introduced the connection coefficients (29) that allows us to prove the following statement.

Theorem 11. *Let f be a C^q function such that $f\in L^2_w([0,1])$ with bounded second derivative on $[0,1]$. Then, the Taylor series of f in $x=x_0$ is given by*

$$f(x)=f(x_0)+\sum_{t=1}^{\infty}\left(\sum_{n,l=1}^{\infty}\sum_{m,q=0}^{\infty}\beta^k_{n,m}\gamma^{(t)kh}_{nlmq}\psi^h_{l,q}(x_0)\right)\frac{(x-x_0)^t}{t!}, \tag{41}$$

with $\beta^k_{n,m}$ as in (10).

Proof. The wavelet expansion (30) entails that the pth-order derivative of f (with $p\le q$) can be expanded as follows:

$$\begin{aligned}f^{(p)}(x)&=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta^k_{n,m}\frac{d^p}{dx^p}\psi^k_{n,m}(x)\overset{(28)}{=}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta^k_{n,m}\sum_{l=1}^{\infty}\sum_{q=0}^{\infty}\gamma^{(p)kh}_{nlmq}\psi^h_{l,q}(x)\\&=\sum_{n,l=1}^{\infty}\sum_{m,q=0}^{\infty}\beta^k_{n,m}\gamma^{(p)kh}_{nlmq}\psi^h_{l,q}(x),\end{aligned}$$

as desired.

It is worthy noticing that a suitable choice of the initial point x_0 allows us to simplify the Taylor expansion (40). In particular, for $x_0=0$, we have that

$$f(x)\overset{(41)}{=}f(0)+\sum_{t=1}^{\infty}\left(\sum_{n,l=1}^{\infty}\sum_{m,q=0}^{\infty}\beta^k_{n,m}\gamma^{(t)kh}_{nlmq}\psi^h_{l,q}(0)\right)\frac{x^t}{t!}.$$

Moreover, Definition 2 gives that $x_0=0\Longrightarrow l=h=1$; thus,

$$\psi^h_{l,q}(0)=\psi^1_{1,q}(0)=2^{1/2}\tilde{T}_q(-1)=\sqrt{2}\left(\frac{1}{\sqrt{\pi}}\delta_q+\sqrt{\frac{2}{\pi}}(-1)^q\left(1-\delta_q\right)\right),$$

$$\gamma^{(t)kh}_{nlmq}=\gamma^{(t)k1}_{n1mq}.$$

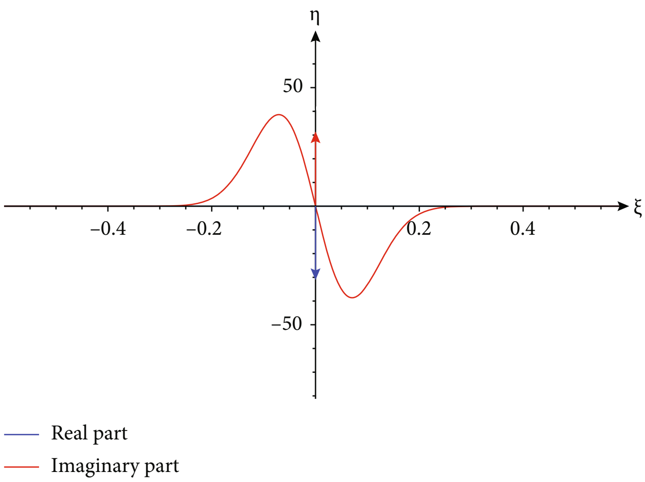

FIGURE 2: The Fourier transform of $\psi^2_{1,1}$. We drew the imaginary part of $\widehat{\psi}^2_{1,1}$ by (101) with $\varepsilon = 0.01$.

Therefore,

$$f(x) = f(0) + \sqrt{\frac{2}{\pi}}\sum_{t=1}^{\infty}\left(\sum_{n=1}^{\infty}\sum_{m,q=0}^{\infty}\beta^k_{n,m}\gamma^{(t)k1}_{n1mq}\left(\delta_q + \sqrt{2}(-1)^q(1-\delta_q)\right)\right)\frac{x^t}{t!}. \quad (42)$$

Explicitly, this means that any function $f \in L^2_w([0,1])$ can be expressed as a power series when the wavelet coefficients $\beta^k_{n,m}$ are finite. Accordingly, the Taylor series for the wavelet mother $\psi(x) = \psi^1_{1,m}(x)$ is given by

$$\psi(x) = \sqrt{\frac{2}{\pi}}\delta_m + \frac{2}{\sqrt{\pi}}(1-\delta_m)(-1)^m + \sqrt{\frac{2}{\pi}}\sum_{t=1}^{\infty}\left(\sum_{m,q=0}^{\infty}\beta^1_{1,m}\gamma^{(t)11}_{11mq}\left(\delta_q + \sqrt{2}(-1)^q(1-\delta_q)\right)\right)\frac{x^t}{t!}, \quad (43)$$

since $\psi(0) = \sqrt{2/\pi}\,\delta_m + (2/\sqrt{\pi})(1-\delta_m)(-1)^m$. We note that in the last computation the sum on the index n reduces to one term because for the wavelet mother $n = 1$.

Now, we are in a position to estimate the approximation error in (28) for a fixed scale of approximation. It is worth pointing out that the approximation depends on the upper bound in the sums. □

Theorem 12 (Approximation error of wavelet derivatives). *Let $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ be such that $p \le q+1$. Under the same hypotheses as in Definition 2, the approximation error in (28)*

$$\varepsilon_{k,M} := \left(\int_0^1\left(\frac{d^p}{dx^p}\psi^h_{l,q}(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\gamma^{(p)hk}_{lnqm}\psi^k_{n,m}(x)\right)^2 w_k(x)dx\right)^{1/2},$$

is bounded by

$$\varepsilon_{k,M} \le \frac{2^{1+p+(h-k)/2}}{\sqrt{\pi}}q(q-p+1)\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}d^2_{m,q,p,r^*}\right)^{1/2}, \quad n \le l,$$

where is the Iverson bracket notation, $d_{q,p}$ stands for d^p_q defined as in Theorem 8 and

$$d_{m,q,p,r^*} := \max_{\substack{0\le r\le q-p:\\ q-p-r\in E}} d^p_{q,r}\cdot\left(\sqrt{2}\delta_m\lambda_{\delta_m} + (1-\delta_m)\lambda_{1-\delta_m}\right),$$

with $d^p_{q,r}$ as in Theorem 8.

Proof. The proof can be handled in a similar way as in the proof of Proposition 6. In order to avoid confusion, we rewrite (28) as follows:

$$\frac{d^p}{dx^p}\psi^h_{l,q}(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\gamma^{(p)hk}_{lnqm}\psi^k_{n,m}(x),$$

with

$$\gamma^{(p)hk}_{lnqm} = \left\langle\frac{d^p}{dx^p}\psi^h_{l,q}(x), \psi^h_{n,m}(x)\right\rangle_{w_h}.$$

Moreover, throughout the proof, we denote $\gamma^{(p)hk}_{lnqm}$ briefly by γ_{lnqm}. Therefore,

$$\begin{aligned}\varepsilon^2_{k,M} &= \int_0^1\left(\frac{d^p}{dx^p}\psi^k_{n,m}(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\gamma_{lnqm}\psi^k_{n,m}(x)\right)^2 w_k(x)dx\\ &= \int_0^1\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\gamma^2_{lnqm}\left(\psi^k_{n,m}(x)\right)^2 w_k(x)dx\\ &= \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\gamma^2_{lnqm}\frac{2^{k+1}}{\pi}\int_{(n-1)/2^{k-1}}^{n/2^{k-1}}\frac{T^2_m(2^kx-2n+1)}{\sqrt{1-(2^kx-2n+1)^2}}dx.\end{aligned}$$

Now, the proof of Proposition 6 gives

$$\varepsilon^2_{k,M} = \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\gamma^2_{lnqm}.$$

We note that the coefficients $d^p_{q,r}$ defined as in Theorem 8 are nonnegative. Thus, (38) implies that

$$\begin{aligned}\gamma_{lnqm} &\le \frac{2^{1+p+(h-k)/2}}{\sqrt{\pi}}q\sum_{0\le r\le q-p:q-p-r\in E}d^p_{q,r}\left(\sqrt{2}\delta_m\lambda_{\delta_m} + (1-\delta_m)\lambda_{1-\delta_m}\right)\\ &\le \frac{2^{1+p+(h-k)/2}}{\sqrt{\pi}}q(q-p+1)\,d_{m,q,p,r^*}, \quad n \le l.\end{aligned}$$

Accordingly,

$$\gamma^2_{lnqm} \le \frac{2^{2+2p+h-k}}{\pi} q^2 (q-p+1)^2 \, d^2_{m,q,p,r^*}, \quad n \le l.$$

We conclude that

$$\varepsilon^2_{k,M} \le \frac{4}{\pi} 2^{2p+h-k} q^2 (q-p+1)^2 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} d^2_{m,q,p,r^*}, \quad n \le l,$$

which gives the desired estimate. □

Remark 13. We note that the term d^2_{m,q,p,r^*} depends on m and n by the definition of $d^p_{p,r}$ in Theorem 8. Moreover, the hypothesis $p \le q+1$ assures us that the derivative

$$\frac{d^p}{dx^p} \psi^h_{l,q}(x),$$

does not identically vanish on its domain, i.e., we exclude trivial cases in the thesis of Theorem 12.

5. Fractional Calculus of Chebyshev Wavelets

This section is devoted to the local fractional calculus over Chebyshev wavelets. Our approach follows new trends in fractional calculus. Indeed, many problems in fractional calculus are nowadays dealt with hybrid methods (e.g., [28]). In particular, we show how the local fractional calculus extends the fractional derivative to nonsmooth continuous functions (e.g., fractal functions). In fact, the introduction of a local fractional derivative on Chebyshev wavelets by connection coefficients allows us to compute the local fractional derivative of non-smooth functions.

5.1. Chebyshev Wavelets on Fractal Sets of Dimension ν. According to the recent results on fractional calculus of wavelet bases (see, e.g., [29]), we can define the Chebyshev wavelets on a fractal set of dimension ν (with $0 < \nu \le 1$) as follows:

$$\psi_\nu(x) \overset{(43)}{=} \sqrt{\frac{2}{\pi}}\delta_m + \frac{2}{\sqrt{\pi}}(1-\delta_m)(-1)^m + \sqrt{\frac{2}{\pi}} \sum_{p=1}^{\infty} \left(\sum_{m,q=0}^{\infty} \beta^1_{1,m} \gamma^{(t)11}_{11mq} \left(\delta_q + \sqrt{2}(-1)^q (1-\delta_q)\right)\right) \frac{x^{\nu p}}{\Gamma(\nu p+1)}. \quad (44)$$

Thus, if we set

$$\gamma^p_m := \begin{cases} \sqrt{\frac{2}{\pi}}\delta_m + \frac{2}{\sqrt{\pi}}(1-\delta_m)(-1)^m, & p = 0, \\ \sqrt{\frac{2}{\pi}} \sum_{m,q=0}^{\infty} \beta^1_{1,m} \gamma^{(t)11}_{11mq} \left(\delta_q + \sqrt{2}(-1)^q (1-\delta_q)\right) \frac{x^{\nu p}}{\Gamma(\nu p+1)}, & p > 0, \end{cases}$$

we have that

$$\psi_\nu(x^\nu) = \sum_{p=0}^{\infty} \frac{\gamma^p_m}{\Gamma(\nu p+1)} x^{\nu p}. \quad (45)$$

We note that power series expansion (44) for $\nu = 1$ reduces to Chebyshev wavelets on a regular domain:

$$\psi_1(x) = \psi(x) = \sum_{p=0}^{\infty} \frac{\gamma^p_m}{p!} x^p.$$

In addition to this, series expansion (44) for $\nu < 1$ gives Chebyshev wavelets in a continuous nonregular domain, which are functions of the variable x^ν. We point out that the variable x^ν can be defined in nonregular domains, such as the Cantor set shown (Figure 3). For more details, we refer the reader to [29].

In the same spirit as (42), any function $f \in L^2_w[0,1]$ can be expanded in Taylor series on a fractal set of dimension ν as follows:

$$f(x) = f(0) + \sqrt{\frac{2}{\pi}} \sum_{p=1}^{\infty} \left(\sum_{n=1}^{\infty} \sum_{m,q=0}^{\infty} \beta^k_{n,m} \gamma^{(t)k1}_{n1mq} \left(\delta_q + \sqrt{2}(-1)^q (1-\delta_q)\right)\right) \frac{x^{\nu p}}{\Gamma(\nu p+1)}. \quad (46)$$

Obviously, Taylor expansion (45) generalizes (41) on a fractal set of dimension ν. In order to clarify the generalization of power series (44) for nonregular domain, we consider one of the simplest case of nonregular set: the Cantor set. For further considerations on the Cantor set, we refer the reader to [30, 31]. It suffices for our purposes to recall that the Cantor function (or Devil's staircase) f is defined as follows:

$$f(x) \overset{u}{=} \lim_{n \longrightarrow \infty} f_n(x),$$

where $=^u$ denotes uniform convergence, $f_n[0,1] \longrightarrow \mathbb{R}$ such that $f_0(x) = x$ and

$$f_{n+1}(x) = \frac{1}{2} \begin{cases} f_n(3x) & 0 \le x \le \frac{1}{3}, \\ 1, & \frac{1}{3} \le x \le \frac{2}{3}, \\ 1 + f_n(3x-2), & \frac{2}{3} \le x \le 1. \end{cases}$$

Of course, the Cantor function maps the Cantor set C onto $[0,1]$. The family of functions $(f_n)_{n\in\mathbb{N}}$ is a set of polygonal approximations of the Cantor function f. We note that the previous definition of $(f_n)_{n\in\mathbb{N}}$ is equivalent to the following condition:

$$|f_{n+1}(x) - f_n(x)| \le 2^{-n-1}, \quad x \in [0,1], \quad n \in \mathbb{N}.$$

Moreover, Chalice [32] proved that the Cantor function f is the unique monotonic, real-valued function on $[0,1]$

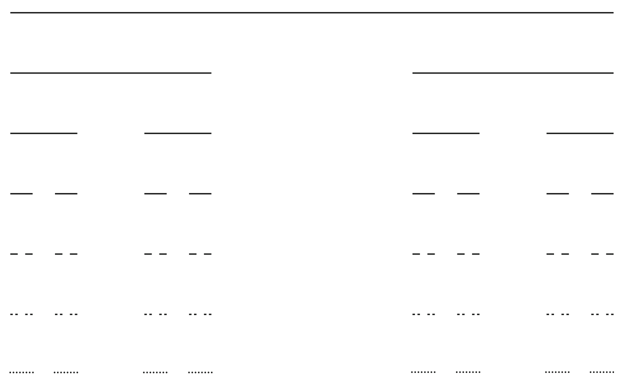

FIGURE 3: The middle third Cantor set.

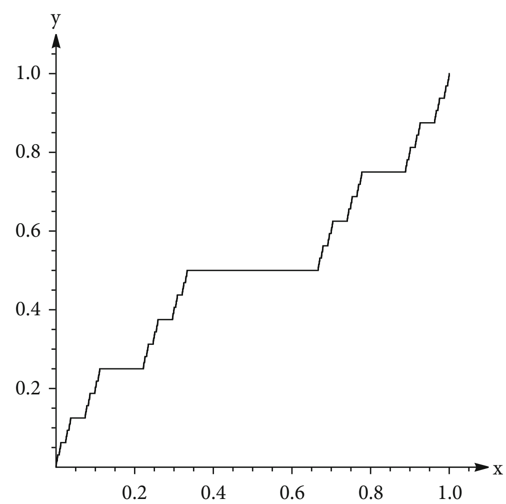

FIGURE 4: The Cantor function.

such that [label = ()]

$$f(0)=0,$$
$$f\left(\frac{x}{3}\right)=\frac{f(x)}{2},$$
$$f(1-x)=1-f(x).$$

This characterization gives us an easier way to draw the Cantor function f (Figure 4). For more details on the Cantor function, we refer the reader to [33]. We note that the method of Chalice can be used to draw the graphics of the functions $\psi_\nu(x^\nu)$ both on the compact interval $[0,1]$ and on the Cantor set.

5.2. Local fractional Calculus. We recall that if the real function $f(x)$ is such that

$$|f(x)-f(x_0)|<\varepsilon^\nu, \tag{47}$$

with $|x-x_0|<\delta$ being $\varepsilon,\delta>0$, then $f(x)$ is called fractional continuous at $x=x_0$. As in the classical case, we denote this property by

$$\lim_{x\longrightarrow x_0} f(x)=f(x_0).$$

In the same spirit, the function $f(x)$ is called local fractional continuous on the interval $]a,b[$ and denoted by $f\in C_\nu(a,b)$, if it satisfies the condition (46) for $x\in\,]a,b[$. Now, if $f\in C_\nu(a,b)$, the local fractional derivative of the real function $f(x)$ of order ν at $x=x_0$ is given [34] by

$$f^{(\nu)}(x_0)=\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}f(x)\bigg|_{x=x_0}=\Gamma(1+\nu)\lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{(x-x_0)^\nu},\quad 0<\nu\le 1. \tag{48}$$

The function f is called ν-differentiable (or differentiable of order ν) at $x=x_0$ if the limit in (47) exists and is finite. It is worth noticing that the local fractional derivative (47) does not satisfy the generalized Leibniz rule [21]. Nevertheless, if f and g are both ν-differentiable in the interval $]a,b[$ in the sense of (47), we have that

$$(fg)^{(\nu)}=f^{(\nu)}g+fg^{(\nu)}. \tag{49}$$

In fact, (47) implies that

$$\begin{aligned}\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}(f(x)\,g(x))&=\Gamma(1+\nu)\lim_{h\longrightarrow 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h^\alpha}\\&=\Gamma(1+\nu)\lim_{h\longrightarrow 0}\left(\frac{f(x+h)-f(x)}{h^\alpha}\,g(x)+f(x+h)\,\frac{g(x+h)-g(x)}{h^\alpha}\right)\\&=f^{(\nu)}(x)g(x)+f(x)g^{(\nu)}(x).\end{aligned}$$

Thus, the ν-derivative of the product of two functions is the fractional equivalent of the product rule for integer derivatives.

We note that the definition (47) works for almost all the rules of fractional calculus (except, of course, the generalized Leibniz rule). For instance, we have [35] that

$$\frac{d^\nu}{dx^\nu}\sin_\nu(x^\nu)=\cos_\nu(x^\nu),\qquad \frac{d^\nu}{dx^\nu}\cos_\nu(x^\nu)=-\sin_\nu(x^\nu).$$

Moreover,

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}x^{\nu k}=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k-1)\nu)}x^{(k-1)\nu}, \tag{50}$$

and thus, for $k=1$, it follows that

$$d^\nu x^\nu=\Gamma(1+\nu)\,dx^\nu\Rightarrow(dx)^\nu=\Gamma(1+\nu)\,dx^\nu. \tag{51}$$

In recent years, the differential operator (50) was used to define integral transformations on Cantor sets (see, e.g., [36, 37]). The results of Section 4 allow us to obtain a series expansion of the local fractional derivative (47) as stated in the following theorem.

Theorem 14. *Let $f\in L^2_w[0,1]$ such that the wavelet coefficients (10) exist and are finite. Then, the local fractional*

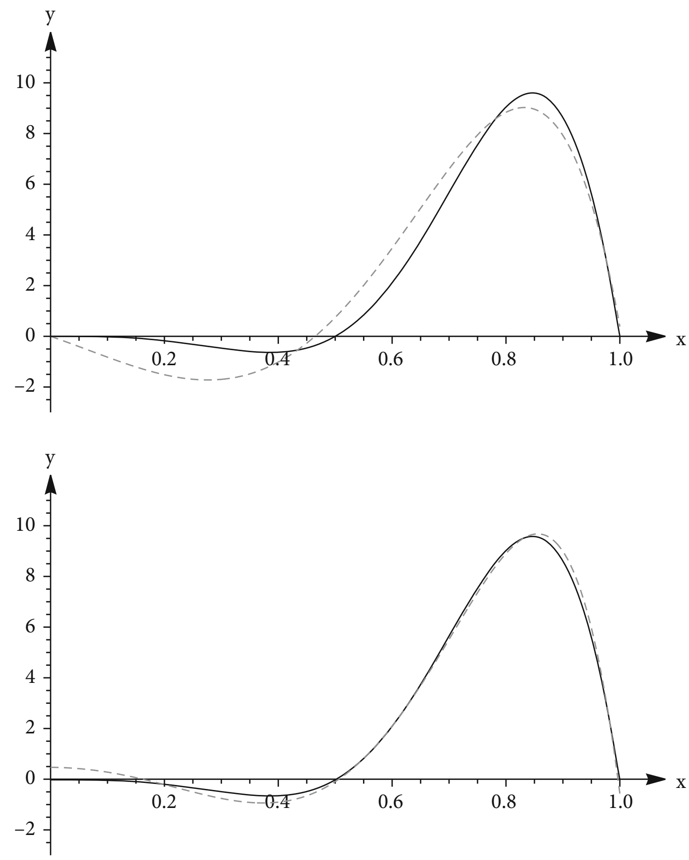

FIGURE 5: Comparison between the function $g(x) = 3x^2 e^{2x} \sin(-2\pi x)$ and its approximation (dashed line) by Chebyshev wavelets with $k^* = 6 \wedge M^* = 21$ (top) and $k^* = 9 \wedge M^* = 31$ (bottom).

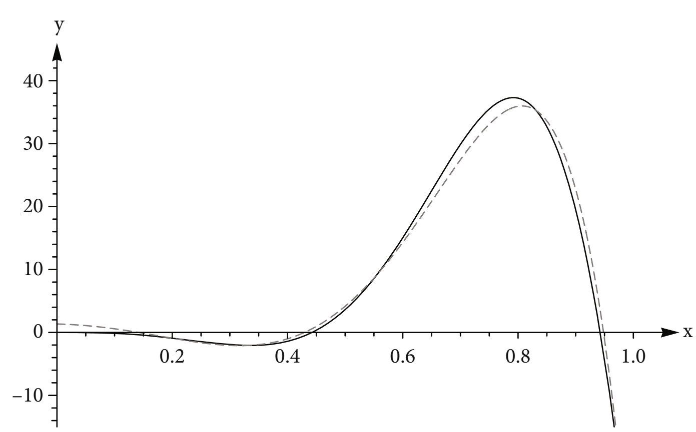

FIGURE 6: Comparison between the local fractional derivative of $g(x) = 3x^2 e^{2x} \sin(-2\pi x)$ for $\nu = 1/2$ and its approximation (dashed line) by Chebyshev wavelets with $k^* = 9 \wedge M^* = 31$.

derivative of f in the neighborhood of $x = 0$ is given by

$$\frac{d^\nu}{dx^\nu} f(x) = \sum_{p=1}^{\infty} \frac{A_p(\beta)}{\Gamma(1+(p-1)\nu)} x^{(p-1)\nu},$$

where

$$A_p(\beta) := \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \sum_{m,q=0}^{\infty} \beta_{n,m}^k \gamma_{n1mq}^{(p)k1} \left(\delta_q + \sqrt{2}(-1)^q (1-\delta_q)\right).$$

Proof. The proof follows directly by (45) and (49). In fact, we have

$$\begin{aligned}\frac{d^\nu}{dx^\nu} f(x) &= \sqrt{\frac{2}{\pi}} \sum_{p=1}^{\infty} \left(\sum_{n=1}^{\infty} \sum_{m,q=0}^{\infty} \beta_{n,m}^k \gamma_{n1mq}^{(t)k1} \left(\delta_q + \sqrt{2}(-1)^q (1-\delta_q)\right)\right) \\ &\quad \cdot \frac{1}{\Gamma(\nu p+1)} \frac{d^\nu}{dx^\nu} x^{\nu p} \\ &\overset{(50)}{=} \sqrt{\frac{2}{\pi}} \sum_{p=1}^{\infty} \left(\sum_{n=1}^{\infty} \sum_{m,q=0}^{\infty} \beta_{n,m}^k \gamma_{n1mq}^{(t)k1} \left(\delta_q + \sqrt{2}(-1)^q (1-\delta_q)\right)\right) \\ &\quad \cdot \frac{1}{\Gamma(1+(p-1)\nu)} x^{(p-1)\nu}.\end{aligned}$$

□

Example 2. Let us consider the function $g : x \in \mathbb{R} \longrightarrow 3x^2 e^{2x} \sin(-2\pi x) \in \mathbb{R}$. This example shows the efficiency of the results proved in this paper. First, we note that $g \in L_w^2[0,1]$. In fact,

$$\int_0^1 \frac{3x^2 e^{2x} \sin(-2\pi x)}{\sqrt{1-x^2}} dx \approx 5.0275.$$

Moreover, g'' is bounded on $[0, 1]$. Accordingly, we can expand the function g in Chebyshev wavelets. Now, (11) implies that g can be approximated in terms of Chebyshev wavelets as follows:

$$3x^2 e^{2x} \sin(-2\pi x) \simeq \sum_{n=1}^{2^{k^*-1}} \sum_{m=0}^{M^*-1} \beta_{n,m}^k \psi_{n,m}^k(x), \qquad (52)$$

where increasing the values of both k^* and M^* provides a better approximation in (51). Obviously, wavelet coefficients $\beta_{n,m}^k$ are computed by (10). The approximation error of g by Chebyshev wavelets is clearly shown in Figure 5.

On the one hand, Theorem 14 implies that

$$\frac{d^\nu}{dx^\nu} 3x^2 e^{2x} \sin(-2\pi x) \simeq \sum_{p=1}^{\infty} \frac{A_p^T(\beta)}{\Gamma(1+(p-1)\nu)} x^{(p-1)\nu}, \qquad (53)$$

where the coefficients $A_p^T(\beta)$ is the truncated version of $A_p(\beta)$, i.e.,

$$A_p^T(\beta) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{2^{k^*-1}} \sum_{m=0}^{M^*-1} \sum_{q=0}^{Q^*-1} \beta_{n,m}^k \gamma_{n1mq}^{(p)k1} \left(\delta_q + \sqrt{2}(-1)^q (1-\delta_q)\right).$$

On the other hand, (48) gives

$$\begin{aligned}\frac{d^\nu}{dx^\nu} g(x) &= -3 \frac{d^\nu}{dx^\nu}\left(x^2 e^{2x} \sin 2\pi x\right) \\ &= -3(2\pi)^\nu x^2 e^{2x} \left(\left(\frac{\Gamma(3)}{\Gamma(3-\nu)} \frac{1}{(2\pi x)^\nu} + \frac{1}{\pi^\nu} \right) \sin 2\pi x + \sin\left(2\pi x + \frac{\pi}{2}\nu\right)\right),\end{aligned} \qquad (54)$$

where we used the fact [35] that

$$\frac{d^\nu}{dx^\nu} e^{ax} = a^\nu e^{ax}, \quad \frac{d^\nu}{dx^\nu} \sin ax = a^\nu \sin\left(ax + \frac{\pi}{2}\nu\right), \quad a \in \mathbb{R},$$
$$\frac{d^\nu}{dx^\nu} x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} x^{n-\nu}, \quad n \geq 0.$$

Comparison between (52) and (53) shows the efficiency of this method even with rough wavelet approximations (Figure 6).

Appendix

Proposition A.1. *Let $p \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Let E be the set of even numbers. Then,*

$$\frac{d^p}{dx^p} T_m(x) = 2^p m \sum_{\substack{0 \leq r \leq m-p : \\ m-p-r \in E}} d_{m,r}^p T_r(x) - 2^{p-1} m\, d_m^p [\![m-p \in E]\!],$$

where

$$d_{m,r}^p := \left(\frac{m+p+r}{2} - 1\right)^{\underline{p-1}} \begin{pmatrix} \frac{m+p-r}{2} - 1 \\ p-1 \end{pmatrix},$$

with $d_m^p = d_{m,0}^p$.

Proof. We begin by recalling the following expansion:

$$\frac{d^p}{dx^p} T_m(x) = 2^p \sum_{0 \leq j \leq (m-p/2)} m(m-1-j)^{\underline{p-1}} \begin{pmatrix} p+j-1 \\ p-1 \end{pmatrix} T_{m-p-2j}(x) - [\![m-p \in E]\!] 2^{p-1} m \left(\frac{m+p}{2} - 1\right)^{\underline{p-1}} \begin{pmatrix} \frac{m+p}{2} - 1 \\ p-1 \end{pmatrix},$$

which is due to Prodinger [38]. By the change of variable $r = m - p - 2j$ in the previous sum, we get

$$\frac{d^p}{dx^p} T_m(x) = 2^p m \sum_{\substack{0 \le r \le m-p : \\ m-p-r \in E}} \left(\frac{m+p+r}{2} - 1\right)^{\underline{p-1}} \binom{\frac{m+p-r}{2} - 1}{p-1} T_r(x) - [\![m-p \in E]\!] 2^{p-1} m \left(\frac{m+p}{2} - 1\right)^{\underline{p-1}} \binom{\frac{m+p}{2} - 1}{p-1},$$

which completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

An Operational Matrix Technique Based on Chebyshev Polynomials for Solving Mixed Volterra-Fredholm Delay Integro-Differential Equations of Variable-Order

Kamal R. Raslan,[1] Khalid K. Ali ,[1] Emad M. H. Mohamed,[1] Jihad A. Younis ,[2] and Mohamed A. Abd El salam [1]

[1]*Department of Mathematics, Faculty of Science, Al Azhar University, Nasr City, 11884 Cairo, Egypt*
[2]*Department of Mathematics, Aden University, Aden 6014, Yemen*

Correspondence should be addressed to Jihad A. Younis; jihadalsaqqaf@gmail.com

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In this work, an algorithm for finding numerical solutions of linear fractional delay-integro-differential equations (LFDIDEs) of variable-order (VO) is introduced. The operational matrices are used as discretization technique based on shifted Chebyshev polynomials (SCPs) of the first kind with the spectral collocation method. The proposed VO-LFDIDEs have multiterms of integer, fractional-order derivatives for delayed or nondelayed and mixed Volterra-Fredholm integral terms. The introduced model is a more general form of linear fractional VO pantograph, neutral, and mixed Fredholm–Volterra integro-differential equations with delay arguments. Caputo's VO fractional derivative operator is used to generate the matrices of the derivative. Operational matrices are presented for all terms. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments. Also, some examples are included to improve the validity and applicability of the techniques. Finally, comparisons between the proposed method and other methods were used to solve this kind of equation.

1. Introduction

In recent few decades, fractional calculus has shown to be a useful implement to formulate many problems in science and engineering where the fractional derivatives and integrals can be used for the description of the properties of various real materials in different branches of science [1–7]. In 1993, Samko and Ross [8] have introduced the variable order derivative (VOD) operator just as a generalization of the fractional-order derivative, and they provided some of its main properties. In this introduced operator, the order of the derivative is a function of the independent variables such as time and space variables. With this generalization, many applications have been established in mechanics, physics, signal processing, and control [9–12]. One of the most important tools in studying the prime numbers is the Riemann zeta-function which embodies both additive and multiplicative structures in a single function, the fractional generalization of it is also found in [13–17].

Consequently, a generalized kind of differential equation appears named VO fractional differential equations (VOFDEs) [18–24]. In VOFDEs, the differential operator is VO, and the derivative changes in general concerning the independent variable or concerning an external functional behavior [8, 25]. Analytical solutions for the VOFDEs are difficult to obtain because the kernel of the VO's operator has a variable exponent, hence, there will be increasingly rapid developments in numerical approaches to VOFDEs [26–29]. The delay, neutral delay FDEs, and fractional integro-differential equations (FIDEs) are considered as a generalization and development of FDEs, and dealing with them analytically in most of the cases is difficult [30–34]. The VO fractional delay differential equations (VOFDDEs) are a kind of generalization for the fractional delay differential equations (FDDEs) [35–38]. VOFDDEs did not receive the attention that the FDDEs accomplished; however, the potential to describe complicated behavior by the VO of differentiation or integration is clear, all this was a motivation to start studying this type of equation. Soon after, a variety of definitions have been offered for variable order integral and derivative operators such as Riemann–Liouville (RL) [39], Grünwald–Letnikov [40], Caputo, and Fabrizio [41] derivatives.

In this work, we use operational matrices discretization technique based on shifted Chebyshev polynomials (SCPs) of the first kind with the spectral collocation method to solve the following type of linear fractional delay-integro-differential equations of variable-order (LFDIDEs):

$$\sum_{i=0}^{n_1} Q_i(x)\, D^{\nu_i(x)} y(x) + \sum_{j=0}^{n_2} P_j(x)\, D^{\alpha_j(x)} y(x-\tau_j) + \sum_{s=0}^{n_3} R_s(x)\, y^{(s)}(x-\varepsilon_s) = g(x) + \int_a^b \mathrm{K}(x,t)\, \mathrm{y}(t)\, dt + \int_a^x \mathrm{V}(x,t)\, \mathrm{y}(t)\, dt, x \in [a,b], \tag{1}$$

under the conditions

$$y^{(i)}(\rho_i) = \mu_i, i = 0, 1, 2 \cdots, m-1, \tag{2}$$

where $0 < \alpha_j(x) \le 1$, $0 < \nu_i(x) \le 1$, and τ_j, ε_s are positive real delay arguments, and the known functions $Q_i(x), P_j(x), R_s(x), K(x,t), V(x,t)$, and $g(x)$ are well defined, additionally, $y(x)$ is an unknown function to be determined. The symbole D denotes the variable-order derivative (VOD) operator in the Caputo sense.

Corollary 1. *The independent variable x of (1) belongs to* $[a, b]$, *which is the intersection of the intervals of the different delayed arguments and* $[0, h]$, *i.e.,* $x \in [a,b] = [\tau_j, h+\tau_j] \bigcap [\varepsilon_s, h+\varepsilon_s] \bigcap [0,h]$, *and* $\tau_j, \varepsilon_s < h$.

The introduced model (1) is a more general form of linear fractional VO pantograph, neutral, and mixed Fredholm–Volterra integro-differential equations with delay arguments. Several methods have been presented for solving VO differential equations [42–46]. For example, Ganji et al. applied the first-kind CPs to obtain the solution of variable order differential equations [47]. Doha et al. solved variable-order fractional Volterra integro-differential equations by shifted Legendre–Gauss–Lobatto collocation method [39]. A numerical method based on the Jacobi polynomials for solving variable-order differential equations has been proposed in [48]. Furthermore, Ganji et al. have introduced a numerical scheme based on the Bernstein polynomials to solve variable order diffusion-wave equations [49]. As a notation, the proposed model (1) is a generalization of our previous reports [50–54] and other work [55–57] as well.

2. Basic Definitions

In this preliminary section, the definitions and properties of Caputo's VO fractional derivative and the required mathematical tools will be introduced. Also, briefly, the SCPs of the first kind, we will present what we need from their properties.

2.1. Caputo Variable-Order's Operator

Definition 2. The Caputo's VO fractional derivative operator of order $n-1 < \alpha(x) \le n$ operates a function $g(x) \in C^n[0,h]$ in [42, 43] as

$$D^{\alpha(x)} g(x) = \begin{cases} \dfrac{1}{\Gamma[n-\alpha(x)]} \displaystyle\int_0^x \frac{g(s)^{(n)}}{(x-s)^{\alpha(x)-n+1}}\, ds, & \text{for } n-1 < \alpha(x) < n, \\ g^{(n)}(x), & \text{for } \alpha(x) = n, \end{cases} \tag{3}$$

and $n \in \aleph$.

Remark 3. In this work, we write the Variable-Order Caputo's fractional symbol $D^{\alpha(x)}$ instead of ${}_0^C D_x^{\alpha(x)}$ for short.

Remark 4. The following useful property according to Definition 2:

$$D^{\alpha(x)} x^m = \begin{cases} \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha(x))} x^{m-\alpha(x)}, & \text{for } m \ge n, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

where

$$n-1 < \alpha(x) \le n, m \in \aleph_0 \text{ and } \aleph_0 = \aleph \cup \{0\}. \tag{5}$$

Definition 5. The Caputo's VO fractional derivative operator of order $\alpha(x), 0 < \alpha(x) \le 1$; operates $g(x)$, where $g(x) \in C^n[0,h]$ can be defined as [42, 58]

$$D^{\alpha(x)} g(x) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^x \frac{g'(x)}{(x-t)^{\alpha(x)}} dt + \frac{g(0^+) - g(0^-)}{\Gamma(1-\alpha(x))} x^{-\alpha(x)}, \tag{6}$$

and $0 < \alpha(x) \le 1$. Definition 5 can be reformulated if we assume that; the starting time in a perfect situation, then:

$$D^{\alpha(x)} g(x) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^x \frac{g'(x)}{(x-t)^{\alpha(x)}} dt, 0 < \alpha(x) < 1. \tag{7}$$

Remark 6. By (7), we can get the following formulas:

(i) The Caputo's VO operator is linear

$$D^{\alpha(x)} (\mu f(x) + \lambda g(x)) = \mu D^{\alpha(x)} f(x) + \lambda D^{\alpha(x)} g(x), \tag{8}$$

where μ and λ are any two constants.

(ii) As the ordinary differentiation operator the Caputo's VO operator as well, such that $D^{\alpha(x)} K = 0$, and K is constant.

2.2. Shifted First Kind of Chebyshev Polynomials. Next, let us introduce some properties of the SCPs [59]. It is well known that the classical CPs are defined on $[-1, 1]$ by the three-term recurrence relation:

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), T_0(t) = 1, T_1(t) = t, n = 1, 2, \cdots. \tag{9}$$

Let the SCPs $T_n(2t/h - 1)$ be denoted by $T_n^h(x)$, which are defined on $[0, h]$. Then, $T_n^h(x)$ can be generated by using the following recurrence relation:

$$T_{n+1}^h(x) = 2\left(\frac{2x}{h} - 1\right)T_n^h(x) - T_{n-1}^h(x), n = 1, 2, \cdots, \tag{10}$$

where

$$T_0^h(x) = 1, T_1^h(x) = \frac{2x}{h} - 1. \tag{11}$$

The analytic form of the SCPs T_i^h of degree i is given by:

$$T_i^h(x) = \sum_{k=0}^{i} (-1)^{i-j} \frac{i(i+j-1)!2^{2j}}{(i-j)!(2j)!h^j} x^k. \tag{12}$$

The orthogonality condition of the SCPs is

$$\int_0^h T_i^h(x)T_j^h(x)w_h dx = \delta_j, \tag{13}$$

where

$$w_h = 1/\sqrt{hx - x^2}, \delta_j = \begin{cases} 0, & \text{for } i \neq j, \\ \phi_j/2\pi, & \text{for } i = j\, \phi_0 = 2, \phi_j = 1, j \geq 1. \end{cases} \tag{14}$$

A function $y(x)$, square integrable in $[0, h]$, may be expressed in terms of shifted Chebyshev polynomials as

$$y(x) = \sum_{i=0}^{\infty} a_i T_i^h(x), \tag{15}$$

where the coefficients a_i are given by

$$a_i = \frac{1}{h_i}\int_0^h y(x)T_i^h(x)w_h dx, i = 0, 1, 2, \cdots. \tag{16}$$

In practice, only the first $(1 + N)$-terms shifted Chebyshev polynomials are considered. The special values

$$T_{i,j}^h = (-1)^{i-j} \frac{i(i+j-1)!2^{2j}}{(i-j)!(2j)!h^j}, j \leq i, \tag{17}$$

will be of important use later.

In the approximation theory, the series in (15) can be approximated by taking the first $(N + 1)$ terms as follows:

$$y(x) \simeq y_N(x) = \sum_{i=0}^{N} a_i T_i^h(x) = A^T T^h(x), \tag{18}$$

where $A = [a_0, a_1, \cdots, a_N]^T$ is a vector, $T^h(x) = [T_0^h(x)\ T_1^h(x)\ \cdots\ T_N^h(x)]^T$ and if we assume that $X(x) = [x^0\quad x^1\quad \cdots\quad x^N]$.

From (12) and (17), $T^h(x)$ can be written as the following form:

$$T^h(x) = X(x)M^T, \tag{19}$$

where M is square lower triangle matrix with size $(N + 1) \times (N + 1)$ given by:

$$M = \begin{bmatrix} m_{00} & 0 & 0 & \cdots & 0 \\ m_{10} & m_{11} & 0 & \cdots & 0 \\ m_{20} & m_{21} & m_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{N+1,0} & m_{N+1,1} & m_{N+1,2} & \cdots & m_{N+1,N+1} \end{bmatrix}. \tag{20}$$

and its entries elements are given by:

$$m_{ij} = \begin{cases} 1, & \text{if } i = j = 0, \\ (-1)^{i-j} \dfrac{i(i+j-1)!2^{2j}}{(i-j)!(2j)!h^j}, & \text{if } j \leq i, \\ 0, & \text{if } j > i. \end{cases} \tag{21}$$

For example, if $N = 4$, then the square matrix M is given by:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -8 & 8 & 0 & 0 \\ -1 & 18 & -48 & 32 & 0 \\ 1 & -32 & 160 & -256 & 128 \end{pmatrix}. \tag{22}$$

Now, from (19), we can obtain the k^{th} derivative of the

Table 1: The comparisons between the absolute errors of different N for Example 1.

x	Absolute errors Our method $N=3$	Our method $N=4$	Our method $N=5$
0.2	1.66×10^{-16}	2.77×10^{-16}	1.94×10^{-16}
0.4	1.66×10^{-16}	3.33×10^{-16}	2.77×10^{-16}
0.6	1.94×10^{-16}	2.77×10^{-16}	2.22×10^{-16}
0.8	1.11×10^{-16}	3.88×10^{-16}	2.77×10^{-16}
1	1.38×10^{-16}	3.40×10^{-16}	1.75×10^{-16}

matrix $T(x)$ as

$$T^{(k)}(x)=X^{(k)}(x)M^T, k=0,1,2,\cdots. \tag{23}$$

3. Operational Matrices

In this section, we introduce the generalized operational matrices for $T^{h(k)}(x)$, $T^{h(s)}(x-\tau_s)$, $D^{\nu_i(x)}T^h(x)$, and $D^{\alpha_i(x)}T^h(x-\tau_i)$ according to fractional calculus.

Lemma 7. *The $(k)^{th}$ order derivative of the row vector $T^h(x)$ can be in the following relation form [50, 60]:*

$$T^{h(k)}(x)=X(x)B^kM^T, \tag{24}$$

where B is the $(N+1)\times(N+1)$ operational matrix of derivative for $X(x)$ and can be obtained from

$$B=(\varepsilon_{ij})=\begin{cases}0, & \text{for otherwise},\\ j+1, & \text{for } i=j+1,\end{cases}, j=0,1,\cdots,N. \tag{25}$$

Proof (see [50, 60]). The row vector $T(x-\tau)$ represents in terms of the vector $X(x)$ in the following form [50, 60]:

$$T(x-\tau)=X(x)B_{-\tau}M^T, \tag{26}$$

where $B_{-\tau}$ is square upper triangle matrix with size $(N+1)\times(N+1)$ given by:

$$B_{-\tau}=(\beta_{ij})=\begin{cases}\begin{pmatrix}j\\ i\end{pmatrix}(-\tau)^{j-i}, & \text{for } j\ge i,\\ 0, & \text{for otherwise},\end{cases}, i,j=0,1,\cdots,N. \tag{27}$$

□

Corollary 8. *The $(s)^{th}$ order derivative of the delay row vector $T^h(x-\tau_s)$ can be represented as [50, 60]*

$$T^{h(s)}(x-\tau)=X^{(s)}(x-\tau_s)M^T=X(x)B^sB_{-\tau_s}M^T. \tag{28}$$

According to the previous lemmas with the fractional calculus by using the Caputo's variable-order fractional derivative, we introduce the following theorem.

Theorem 9. *The $\nu_i(x)$th variable-order fractional derivative of the vector $T(x)$ can be written as*

$$D^{\nu_i(x)}T^h(x)=X_{\nu_i(x)}(x)B_{\nu_i(x)}(x)M^T, \tag{29}$$

where

$$X_{\nu_i(x)}(x)=\left[0,x^{1-\nu_i(x)},x^{2-\nu_i(x)},\cdots,x^{N-\nu_i(x)}\right], 0<\nu_i(x)<1, \tag{30}$$

where $B_{\nu_i(x)}(x)$ diagonal matrix with size $(N+1)\times(N+1)$, its elements λ_{ij} given by:

$$\lambda_{ij}=\begin{cases}\dfrac{\Gamma(i+1)}{\Gamma(i+1-\nu_i(x))}, & \text{for } i=j\neq 0.,\\ 0, & \text{for otherwise}.\end{cases}, i=0,1,\cdots,N. \tag{31}$$

Proof.

$$\begin{aligned}D^{\nu_i(x)}T^h(x)&=D^{\nu_i(x)}X(x)M^T=D^{\nu_i(x)}\left[1,x,x^2,\cdots,x^N\right]M^T, 0<\nu_i(x)<1,\\ &=\left[0,\frac{\Gamma(2)}{\Gamma(2-\nu_i(x))}x^{1-\nu_i(x)},\frac{\Gamma(3)}{\Gamma(3-\nu_i(x))}x^{2-\nu_i(x)},\cdots,\frac{\Gamma(N+1)}{\Gamma(N+1-\nu_i(x))}x^{N-\nu_i(x)}\right]M^T\\ &=X_{\nu_i(x)}(x)B_{\nu_i}(x)M^T.\end{aligned} \tag{32}$$

□

Corollary 10. *Relation (29) satisfies for $n-1<\nu_i(x)<n$ according to Definition 2 by induction with*

$$X_{\nu_i(x)}(x)=\left[0,0,0,\cdots,x^{n-\nu_i(x)},\cdots,x^{N-\nu_i(x)}\right], n-1<\nu_i(x)<n, \tag{33}$$

and

$$B_{\nu_i(x)}(x)=(\lambda_{ij})=\begin{cases}\dfrac{\Gamma(i+1)}{\Gamma(i+1-\nu_i(x))}, & \text{for } i=j\ge n,\\ 0, & \text{for otherwise}.\end{cases}, i=0,1,\cdots,N, n<N. \tag{34}$$

Corollary 11. *The $\alpha_i(x)$th variable-order fractional derivative of the delay vector $T(x-\tau_i)$ can be written in the following form:*

$$D^{\alpha_i(x)}T^h(x-\tau_i)=X_{\alpha_i(x)}(x)B_{\alpha_i(x)}(x)B_{-\tau_i}M^T. \tag{35}$$

Form (18), we get

$$y_N(x)=T^h(x)A, \tag{36}$$

$$y_N^{(k)}(x)=T^{h(k)}(x)A, \tag{37}$$

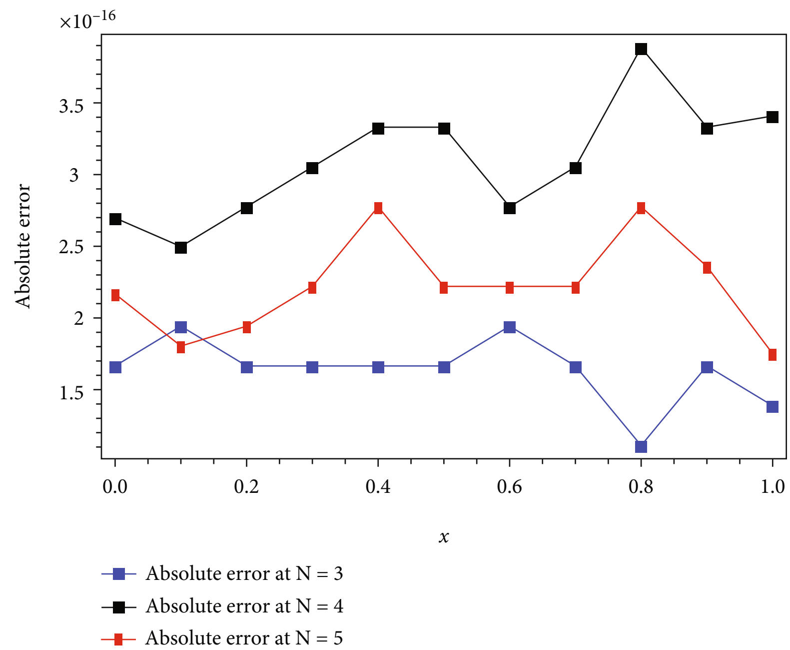
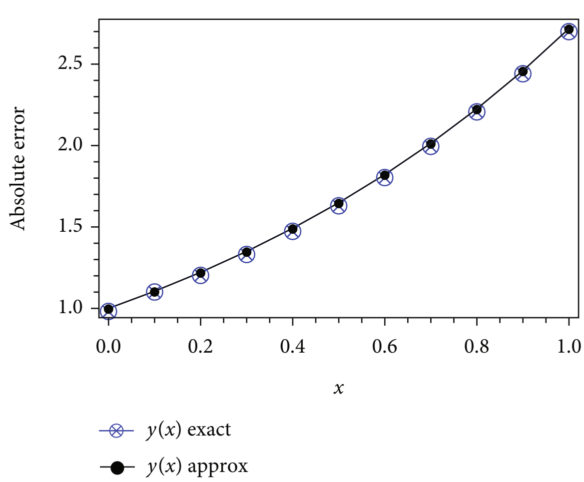
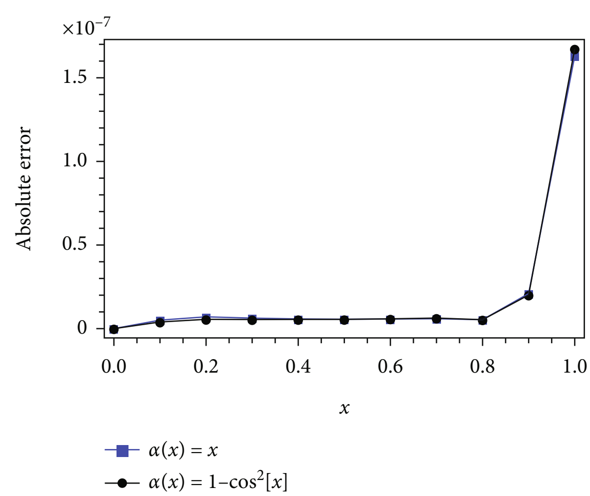


FIGURE 1: The behavior of the absolute errors for the proposed method with $N = 3, 4, 5$ for Example 1.



(a)



(b)

FIGURE 2: Exact solution and numerical solutions (a), absolute error at two different cases of $\alpha(x)$ (b) for Example 2 of $N = 7$, $\tau = 0$, and $h = 1$
.

$$D^{\nu_i(x)} y_N(x) = T^{h(\nu_i(x))}(x)A, \tag{38}$$

and

$$D^{\alpha_i(x)} y_N(x - \tau_i) = T^{(\alpha_i(x))}(x - \tau_i)A,\ i = 0, 1, \cdots, N. \tag{39}$$

By putting $x \longrightarrow x - \varepsilon_s$ in the relation (37), we obtain the matrix form

$$y_N^{(s)}(x - \varepsilon_s) = T^{h(s)}(x - \varepsilon_s)A. \tag{40}$$

Consequently, by substituting the matrix form (24) into (37), we have the matrix relation

$$y_N^{(k)}(x) = X^{(k)}(x)B^k M^T A. \tag{41}$$

By substituting the matrix forms (29) into (38), we have the matrix relation

$$D^{\nu_i(x)} y_N(x) = X_{\nu_i(x)}(x) B_{\nu_i(x)}(x) M^T A. \tag{42}$$

Table 2: The comparisons between the absolute errors of the method given in [44] the presented method with different values of $\alpha(x)$, $\tau_0 = 0$ and $N = 7$ for Example 2.

x	Absolute errors Our method $\alpha(x) = x$	Method [44] $\alpha(x) = x$	Our method $\alpha(x) = 1 - \cos^2(x)$	Method [44] $\alpha(x) = 1 - \cos^2(x)$
0.1	3.97×10^{-9}	5.04×10^{-6}	5.08×10^{-9}	7.34×10^{-6}
0.3	5.43×10^{-9}	1.37×10^{-6}	6.27×10^{-9}	2.86×10^{-6}
0.5	5.53×10^{-9}	1.63×10^{-6}	5.64×10^{-9}	3.55×10^{-6}
0.7	6.32×10^{-9}	9.62×10^{-7}	5.94×10^{-9}	3.17×10^{-7}
0.9	2.01×10^{-8}	8.21×10^{-7}	2.09×10^{-8}	6.10×10^{-7}

Table 3: The comparisons between the absolute errors of the presented method for Example 2 with different values of τ and $N = 7$ in case $\alpha(x) = 1 - \cos^2(x)$.

x	Absolute errors Our method $\tau = 0.02$	Our method $\tau = 0.002$	Our method $\tau = 0.0002$
0.1	5.73×10^{-9}	5.14×10^{-9}	5.09×10^{-9}
0.3	6.86×10^{-9}	6.33×10^{-9}	$6.28 \times 10^{-}9$
0.5	6.47×10^{-9}	6.54×10^{-9}	6.70×10^{-9}
0.7	6.76×10^{-9}	5.98×10^{-9}	5.94×10^{-9}
0.9	2.22×10^{-8}	2.13×10^{-8}	2.09×10^{-8}

By using (35) and (39), we get

$$D^{\alpha_j(x)} y_N\left(x - \tau_j\right) = X_{\alpha_j(x)}(x) B_{\alpha_j(x)}(x) B_{-\tau_j} M^T A. \tag{43}$$

Similar form (28) and (40), we obtain

$$y_N^{(s)}(x - \varepsilon_s) = X(x) B^s B_{-\varepsilon_s} M^T A. \tag{44}$$

3.1. Matrix Representation for Fredholm Integral Terms. Now, we try to find the matrix form corresponding to the integral term. Assume that $K(x, t)$ can be expanded to univariate Chebyshev series concerning t, as follows

$$K(x, t) \cong \sum_{r=0}^{N} u_r(x) T_r(t). \tag{45}$$

Then the matrix representation of the kernel function $K(x, t)$ is given by

$$K(x, t) \cong U(x) T^T(t), \tag{46}$$

where

$$U(x) = [u_0(x) u_1(x) \cdots u_N(x)]. \tag{47}$$

Substituting the relations (29) and (46) in the present integral part, we obtained

$$\begin{aligned} \int_a^b K(x, t) y(t)\, dt &= \int_a^b U(x) T^T(t) \mathrm{T}(t)\, A dt = \int_a^b U(x) M X^T(t)\, \mathrm{X}(t)\, M^T A dt \\ &= \int_a^b U(x) M \left[t^0 t^1 \cdots t^N\right]^T \left[t^0 t^1 \cdots t^N\right] M^T A\, dt \\ &= U(x) M \left(\int_a^b t^p\, t^q\, dt\right) M^T A = U(x)\, M \left(\int_a^b t^{p+q}\, dt\right) M^T A \\ &= U(x)\, M Z M^T \mathrm{A}, p, q = 0, 1, \cdots, N, \end{aligned} \tag{48}$$

where

$$Z = \int_a^b t^{p+q}\, dt, p, q = 0, 1, \cdots, N, \tag{49}$$

or

$$Z = \left[z_{pq}\right] = \frac{(b)^{p+q+1} - (a)^{p+q+1}}{p + q + 1}, p, q = 0, 1, \cdots, N. \tag{50}$$

So, the present integral term can be written as

$$\int_a^b K(x, t) y(t)\, dt = U(x)\, M\, Z M^T A = U(x)\, M\, Z M^T\, A. \tag{51}$$

3.2. Matrix Representation for Volterra Integral Terms. Now, we try to find the matrix form corresponding to the integral term. By the same way, $V_c(x, t)$ can be expanded as (45)

$$V(x, t) \cong \sum_{r=0}^{N} f_r(x) T_r(t). \tag{52}$$

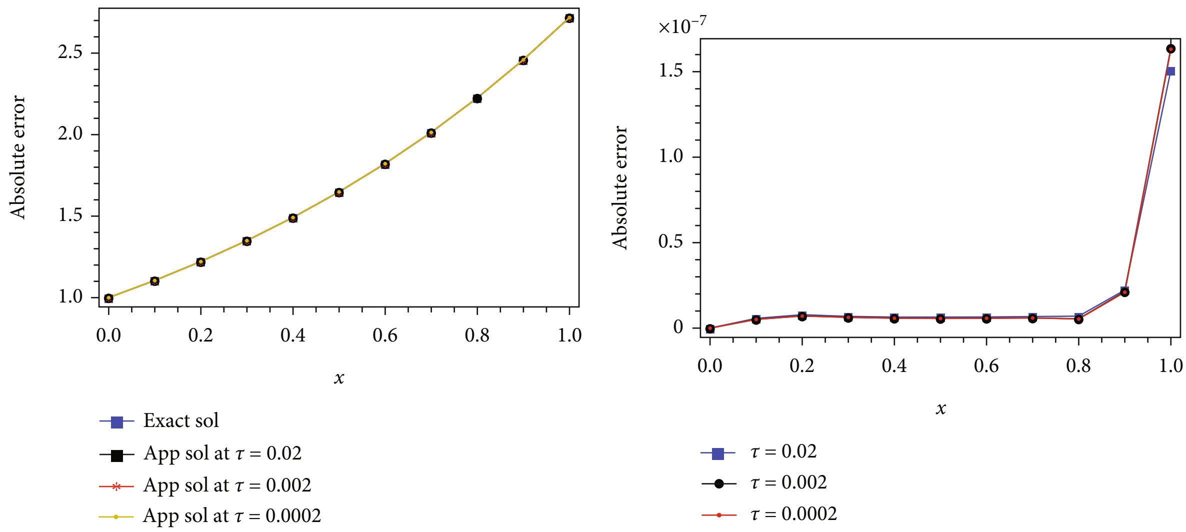
Then, the matrix representation of the kernel function $V_c(x, t)$ is given by

$$V(x, t) \cong F(x) T^T(t), \tag{53}$$

where

$$F(x) = [f_0(x) f_1(x) \cdots f_N(x)]. \tag{54}$$

Substituting the relations (29) and (53) in the present



(a) (b)

Figure 3: Exact solution and numerical solutions (a), absolute error (b) for Example 2 with three different values of τ and $N = 7$ in case $\alpha(x) = 1 - \cos^2(x)$ and $h = 1$.

Table 4: Comparison of the maximum absolute error of the method given in [45] and the presented method for Example 3 with different values of N and $h = 1$ in $\alpha(x) = x/2$ for Example 3.

x	Absolute errors			
	Our method at $N = 7$	[45] at $N = 7.$	Our method at $N = 5$	[45] at $N = 5$
0.1	9.05×10^{-16}	2.10×10^{-6}	1.56×10^{-4}	2.88×10^{-3}
0.3	1.80×10^{-15}	5.45×10^{-6}	5.58×10^{-4}	4.72×10^{-4}
0.5	3.49×10^{-15}	3.50×10^{-6}	6.27×10^{-4}	1.12×10^{-4}
0.7	3.49×10^{-15}	5.25×10^{-4}	6.94×10^{-4}	2.01×10^{-3}
0.9	9.54×10^{-15}	4.03×10^{-2}	4.18×10^{-3}	4.39×10^{-2}

integral part, we obtained

$$\begin{aligned}\int_a^x V(x,t)y(t)\,dt &= \int_a^x F(x)T^T(t)\,T(t)\,Adt = \int_a^x F(x)M\,X^T(t)X(t)M^T\,Adt \\ &= \int_a^x F(x)\,M\left[t^0t^1\cdots t^N\right]^T\left[t^0t^1\cdots t^N\right]M^T Adt \\ &= F(x)M\left(\int_a^x t^p\,t^q\,dt\right)M^T A = F(x)\,M\left(\int_a^x t^{p+q}\,dt\right)M^T A \\ &= F(x)\,\mathrm{ME}(x)M^T A, p, q = 0, 1, \cdots, N,\end{aligned} \tag{55}$$

where

$$E(x) = \int_a^x t^{p+q}\,dt, p, q = 0, 1, \cdots, N, \tag{56}$$

or

$$E(x) = \left[e_{pq}(x)\right] = \frac{(x)^{p+q+1} - (a)^{p+q+1}}{p+q+1}, p, q = 0, 1, \cdots, N. \tag{57}$$

So, the present integral term can be written as:

$$\int_a^x V(x,t)y(t)\,dt = F(x)\,M\,E(x)\,M^T\,A. \tag{58}$$

3.3. Conditions' Matrix Relation. Finally, we can obtain the matrix form for condition (2) by using (36) on the form:

$$X(b_i)B^i M^T A = \mu_i, i = 0, 1, 2\cdots, m-1, \tag{59}$$

Via, Equations (41), (42), (43), (44), (51), and (58), then, Equation (1) converted to

$$\begin{aligned}&\left[\sum_{i=0}^{n_1} Q_i(x)\,X_{v_i(x)}(x)B_{v_i(x)}(x)M^T + \sum_{j=0}^{n_2} P_j(x)\,X_{\alpha_j(x)}(x)B_{\alpha_j(x)}(x)B_{-\tau_j}M^T\right. \\ &\left. + \sum_{s=0}^{n_3} R_s(x)\,X(x)B^s B_{-\varepsilon_s}M^T - U(x)\,M\,Z\,M^T - F(x)\,M\,E(x)\,M^T\right]A = g(x).\end{aligned} \tag{60}$$

We can write (60) in the form:

$$OA = G, \text{or}[O\,;G], \tag{61}$$

where

$$\begin{aligned}O = &\sum_{i=0}^{n_1} Q_i(x)\,X_{v_i(x)}(x)B_{v_i(x)}(x)M^T + \sum_{j=0}^{n_2} P_j(x)\,X_{\alpha_j(x)}(x)B_{\alpha_j(x)}(x)B_{-\tau_j}M^T \\ &+ \sum_{s=0}^{n_3} R_s(x)\,X(x)B^s B_{-\varepsilon_s}M^T - U(x)\,M\,Z\,M^T - F(x)\,M\,E(x)\,M^T,\end{aligned} \tag{62}$$

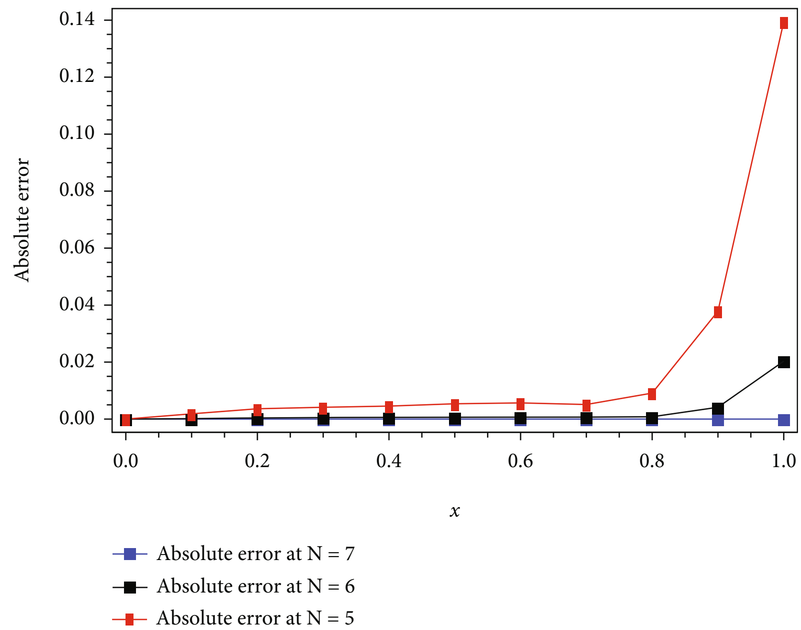


FIGURE 4: The absolute error when $h = 1$ for three different values of N for Example 3.

The collocation points are defined in this form:

$$x_l = \frac{h(2l+1)}{2N+2}, l = 0, 1, 2, \cdots, N. \tag{63}$$

By substituting (63) in (60), then (60) can be turned into the following system:

$$\Bigg[\sum_{i=0}^{n_1} Q_i(x_l)\bar{X}_{v_i(x_l)}(x_l)\bar{B}_{v_i(x_l)}(x_l)M^T + \sum_{j=0}^{n_2} P_j(x_l)\bar{X}_{\alpha_j(x_l)}(x_l)\bar{B}_{\alpha_j(x_l)}(x_l)B_{-\tau_j}M^T + \sum_{s=0}^{n_3} R_s(x_l)X(x_l)B^s B_{-\varepsilon_s}M^T - U(x_l)M\,Z\,M^T - \bar{F}\,M\,\bar{E}\,M^T\Bigg]A = G, \tag{64}$$

where $\bar{X}_{v_i(x_l)}(x_l)$, $\bar{B}_{v_i(x_l)}(x_l)$, $\bar{X}_{\alpha_j(x_l)}(x_l)$, and $\bar{B}_{\alpha_j(x_l)}(x_l)$ are block matrices, also $\bar{F}$ and $\bar{E}$ are block matrices defined in this form:

$$\bar{F} = \begin{pmatrix} f(x_0) & 0 & 0\cdots & 0 \\ 0 & f(x_1) & 0\cdots & 0 \\ 0 & 0 & f(x_2)\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots & f(x_N) \end{pmatrix}, \bar{E} = \begin{pmatrix} e(x_0) & 0 & 0\cdots & 0 \\ 0 & e(x_1) & 0\cdots & 0 \\ 0 & 0 & e(x_2)\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots & e(x_N) \end{pmatrix}.$$

$$G = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix}. \tag{65}$$

Here, the system in (64) in addition to the supplementary conditions can be solved numerically (the inverse matrix method is preferred when O be invertible) to determine the unknown vector A. Hence, the approximate analytical solution defined in series (18) can be calculated.

3.4. Convergence Analysis

Theorem 12. *Assume that a function $y(x) \in [0, h]$ be nth times continuously differentiable and $y_n(x)$ be the best square approximation of $y(x)$ defined in (15). Then, we have*

$$\|y(x) - y_N(x)\| \le \frac{MH^{(N+1)}\sqrt{l}}{(N+1)!}, \tag{66}$$

where

$$M = \max_{x\in[0,h]} y^{(n+1)}(x), H = \max\{h - x_0, x_0\} \text{ and } l = \int_0^h \frac{1}{\sqrt{x(h-x)}}dt. \tag{67}$$

Proof. Using Taylor expansion for $y(x)$ as follows:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \cdots + \frac{(x-x_0)^N}{N!}y^{(n)}(x_0) + \frac{(x-x_0)^{N+1}}{(N+1)!}y^{(N+1)}(\beta), \tag{68}$$

where $x_0 \in [0, h]$ and $\beta \in]x_0, x[$. Assume

$$\bar{y}_N(x) = y(x_0) + (x - x_0)y'(x_0) + \cdots + \frac{(x-x_0)^N}{N!}y^{(N)}(x_0) + \frac{(x-x_0)^{N+1}}{(N+1)!}y^{(N+1)}(\beta), \tag{69}$$

Table 5: Comparison of the absolute error of the method given in [45] and the presented method for Example 4 with different values of N, $h = 1$, and $\tau_0 = 0$.

x	Absolute errors			
	Our method at $N = 7$	[45] at $N = 7$	Our method at $N = 5$	[45] at $N = 5$
0.1	2.95×10^{-6}	8.24×10^{-4}	8.95×10^{-6}	1.31×10^{-3}
0.3	2.31×10^{-6}	9.11×10^{-5}	1.89×10^{-5}	1.28×10^{-4}
0.5	3.76×10^{-6}	1.25×10^{-3}	2.39×10^{-5}	3.72×10^{-3}
0.7	5.88×10^{-6}	5.25×10^{-4}	9.57×10^{-3}	2.64×10^{-2}
0.9	2.64×10^{-5}	9.57×10^{-3}	8.65×10^{-2}	1.81×10^{-1}

then

$$\|y(x) - y_N(x)\| = \left| \frac{(x - x_0)^{N+1}}{(N+1)!} y^{(N+1)}(\beta) \right|. \tag{70}$$

According to, $y_N(x)$ that given in (18), we obtain

$$\begin{aligned}
\|y(x) - y_N(x)\|^2 \le \|y(x) - \bar{y}_N(x)\|^2 &= \int_0^h \omega(x) \, [y(x) - \bar{y}_N(x)]^2 dx \\
&= \int_0^h \omega(x) \left[\frac{(x - x_0)^{N+1}}{(N+1)!} y^{(N+1)}(\beta) \right]^2 dx \\
&\le \frac{M^2}{[(N+1)!]^2} \int_0^h \omega(x) \left[(x - x_0)^{N+1} \right]^2 dx.
\end{aligned} \tag{71}$$

Now, let $H = \max \{h - x_0, x_0\}$, thus (71) rewritten as

$$\|y(x) - y_N(x)\|^2 \le \frac{M^2 \left[H^{N+1} \right]^2}{[(N+1)!]^2} \int_0^h \omega(x) dx. \tag{72}$$

Since, $\omega = 1/\sqrt{x(h - x)}$,
Then

$$\|y(x) - y_N(x)\|^2 \le \frac{M^2 \left[H^{N+1} \right]^2}{[(N+1)!]^2} \int_0^h \frac{1}{\sqrt{x(h - x)}} dx. \tag{73}$$

Hence, the proof is completed. □

4. Numerical Examples and Results Analysis

In this section, we introduce some numerical examples for VO fractional delay-differential equation to illustrate the above results. All results are coded and obtained by using Mathematica package programming. Moreover, the absolute maximum error $E_{\max}$ will be used in our computational results for the comparison between the exact and approximate solutions, where $E_{\max}$ is defined as the below formula:

$$E_{\max} = \max_{l=0,1,2,\cdots,N} |y(x_l) - y_N(x_l)|, x_l = \frac{h(2l+1)}{2N+2}, l = 0, 1, 2, \cdots, N, \tag{74}$$

where x, $y(x)$, and $y_N(x)$ are the space vectors, exact, and numerical solutions, respectively.

Example 1. Consider the following linear variable-order fractional delay integro-differential equation:

$$\begin{aligned}
&D^{\nu(x)} y(x) + D^{\alpha(x)} y(x - 0.001) + y''(x - 0.002) - y(x) \\
&\quad = g(x) + \int_0^h (x + t) \, y(t) \, dt + \int_0^x (3t - 2x) \, y(t) \, dt, x \in [0, h].
\end{aligned} \tag{75}$$

The initial conditions are $y(0) = 0$, $y'(0) = -1$, and the exact solution is $y(x) = x^2 - x$ where $h = 1$, $\nu(x) = 1 - \text{Sin}[x]/9$, $\alpha(x) = 2 - \text{Cos}[x]/7$, $Q_0(x) = 1$, $R_0(x) = -1, R_2(x) = 1$, $P_0(x) = 1$, and $g(x) = 25/12 + 7x/6 - x^2 - x^4/12 + x^{1-\alpha[x]}(1.002/-1.+\alpha[x] + 2.x/2 - 3\alpha[x] + \alpha[x]^2)/\text{Gamma}[1 - \alpha[x]] + x^{1-\upsilon[x]}(-2 + 2x + \upsilon[x])/\text{Gamma}[1 - \upsilon[x]](-2 + \upsilon[x])(-1 + \upsilon[x])$. The fundamental matrix equation of the problem (75) at $N = 3$ is defined by:

$$\begin{aligned}
&\Big[Q_0 \bar{X}_{\nu(x_l)} \bar{B}_{\nu(x_l)} M^T + P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau} M^T + R_2 X B^2 B_\varepsilon M^T \\
&\quad - R_0 X M^T - U \, M \, Z \, M^T - \bar{F} \, M \, \bar{E} \, M^T \Big] A = G,
\end{aligned} \tag{76}$$

where

$$Q_0 = R_0 = P_0 = \text{idintity}, B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 8 & 0 \\ -1 & 18 & -48 & 32 \end{pmatrix},$$

$$X = \begin{pmatrix} 1 & 0.125 & 0.015625 & 0.00195313 \\ 1 & 0.375 & 0.140625 & 0.0527344 \\ 1 & 0.625 & 0.390625 & 0.244141 \\ 1 & 0.875 & 0.765625 & 0.669922 \end{pmatrix}, G = \begin{pmatrix} 1.9211 \\ 1.85111 \\ 1.94491 \\ 2.15722 \end{pmatrix},$$

$$\bar{B}_{\nu(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.03874 & 0 & 0 \\ 0 & 0 & 1.09183 & 0 \\ 0 & 0 & 0 & 1.12841 \\ 0 & 0 & 0 & 0 \\ 0 & 1.02855 & 0 & 0 \\ 0 & 0 & 1.06608 & 0 \\ 0 & 0 & 0 & 1.0917 \\ 0 & 0 & 0 & 0 \\ 0 & 1.01898 & 0 & 0 \\ 0 & 0 & 1.04302 & 0 \\ 0 & 0 & 0 & 1.0593 \\ 0 & 0 & 0 & 0 \\ 0 & 1.01076 & 0 & 0 \\ 0 & 0 & 1.02399 & 0 \\ 0 & 0 & 0 & 1.03288 \end{pmatrix}, \bar{B}_{\alpha(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.05546 & 0 & 0 \\ 0 & 0 & 1.13733 & 0 \\ 0 & 0 & 0 & 1.19466 \\ 0 & 0 & 0 & 0 \\ 0 & 1.05846 & 0 & 0 \\ 0 & 0 & 1.14601 & 0 \\ 0 & 0 & 0 & 1.2075 \\ 0 & 0 & 0 & 0 \\ 0 & 1.06414 & 0 & 0 \\ 0 & 0 & 1.1629 & 0 \\ 0 & 0 & 0 & 1.2327 \\ 0 & 1.07187 & 0 & 0 \\ 0 & 0 & 1.1871 & 0 \\ 0 & 0 & 0 & 1.26924 \end{pmatrix},$$

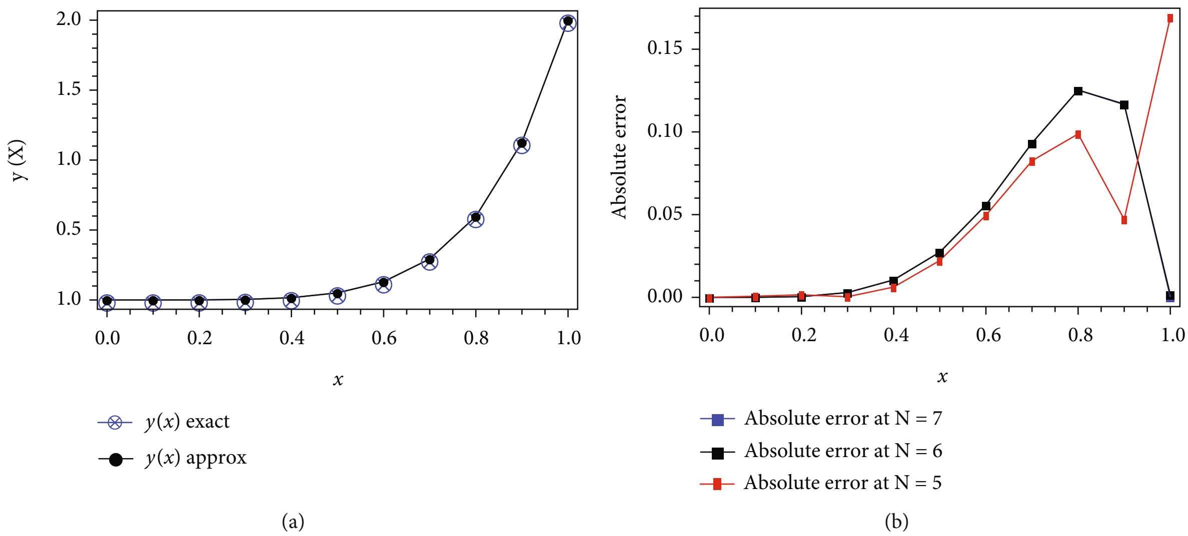


Figure 5: Exact solution and numerical solutions (a), absolute error (b) for Example 4 with different values of N and $h = 1$ in case of $h = 1$.

Table 6: The comparisons between the absolute errors of the presented method for Example 5 with value of $N = 4$.

x	Our method	Absolute errors Method [46]	Method [39]
0.1	1.11×10^{-16}	3.52×10^{-13}	4.99×10^{-16}
0.3	1.11×10^{-16}	2.33×10^{-13}	1.22×10^{-15}
0.5	0	1.35×10^{-13}	1.05×10^{-15}
0.7	0	4.86×10^{-14}	6.66×10^{-16}
0.9	0	2.99×10^{-14}	5.55×10^{-16}

$$\bar{X}_{\nu(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.153018 & 0 & 0 & 0 \\ 0.0191273 & 0 & 0 & 0 \\ 0.00239091 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.401814 & 0 & 0 \\ 0 & 0.15068 & 0 & 0 \\ 0 & 0.0565052 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.878023 & 0 \\ 0 & 0 & 0.399181 & 0 \\ 0 & 0 & 0.29992 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.931899 \\ 0 & 0 & 0 & 0.76827 \\ 0 & 0 & 0 & 0.672236 \end{pmatrix}^T,$$

$$\bar{X}_{\alpha(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.288108 & 0 & 0 & 0 \\ 0.0360135 & 0 & 0 & 0 \\ 0.00450169 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.562781 & 0 & 0 \\ 0 & 0.211043 & 0 & 0 \\ 0 & 0.079141 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.767795 & 0 \\ 0 & 0 & 0.479872 & 0 \\ 0 & 0 & 0.29992 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.931899 \\ 0 & 0 & 0 & 0.815411 \\ 0 & 0 & 0 & 0.713485 \end{pmatrix}^T,$$

$$Z = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}, B_{0.001} = \begin{pmatrix} 1 & -0.001 & 1.1 \times 10^{-6} & -1.0 \times 10^{-9} \\ 0 & 1 & -0.002 & 3.0 \times 10^{-6} \\ 0 & 0 & 1 & -0.003 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{77}$$

In Table 1, we list the absolute errors between the exact solutions and the approximate solutions using the proposed method with $\nu(x) = 1 - \mathrm{Sin}[x]/9$, $\alpha(x) = 2 - \mathrm{Cos}[x]/7$ and three choices of N. In Figure 1 is the behavior of the absolute errors for the proposed method of different N.

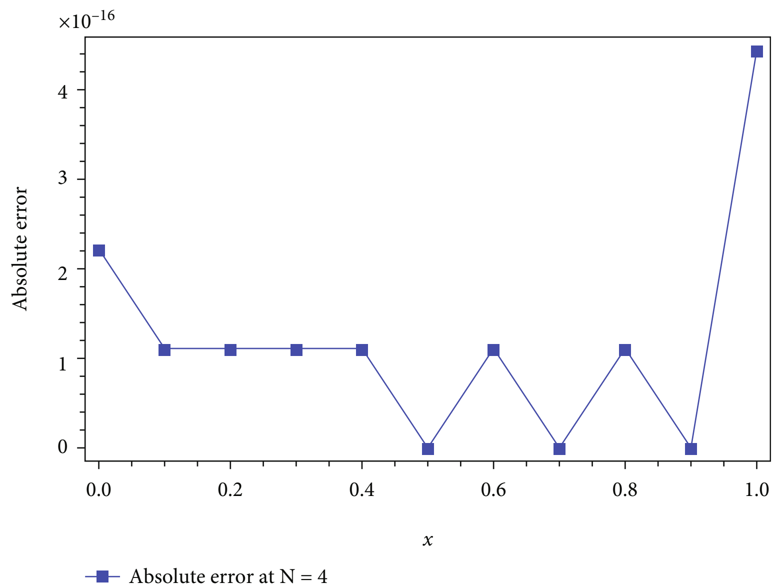


Figure 6: Exact solution and numerical solutions of Example 4-case 1 for different values of N, h.

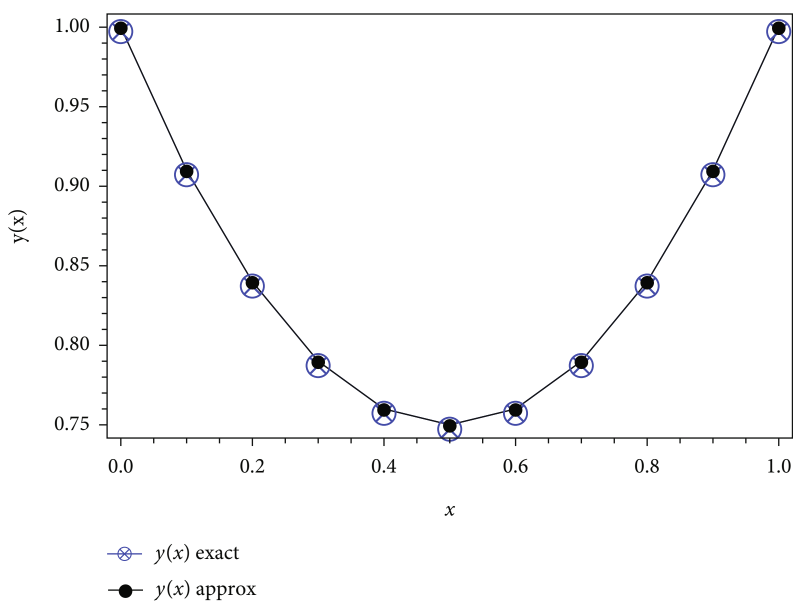


Figure 7: Comparison between the approximate solutions and the exact solutions of Example 5 for $N = 4$.

Example 2. Consider the following linear variable-order fractional delay integro-differential equation [44]:

$$D^{\alpha(x)} y(x-\tau_0) = g(x) + \int_0^h (x-t)\, y(t)\, dt + \int_0^x (t+x)\, y(t)\, dt, x \in [0,h]. \tag{78}$$

with the initial condition $y(0)=1$, $y'(0)=1$. Problem (78) found in [44] with the exact solution given as $y(x)=e^x$ at $\tau_0 = 0$. Now, by applying the suggested technique introduced in the previous section with finite terms $(N+1)$ to (78), then, we have

$$\left[P_0 X_{\alpha(x)} B_{\alpha(x)} B_{-\tau_0} M^T - U(x)\, M\, Z\, M^T - F(x)\, M\, E(x)\, M^T\right] A = g(x), \tag{79}$$

Hence, using the collocation method with the collection points x_l that is given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau_0} M^T - U\, M\, Z\, M^T - \bar{F}\, M\, \bar{E}\, M^T\right] A = G, \tag{80}$$

The numerical solutions obtained by different values of $\alpha(x)$ together with the exact solution are displayed in Figure 2. From this figure, we see that the numerical solution converges to the exact one by increasing the number of basis functions. In Table 2, the comparisons between the absolute errors of the method are given in [44] and the presented method for different values of α and $N = 7$ and choices of $\tau_0 = 0$ and $h = 1$.

In Table 3 shows the absolute errors between the exact and approximate solutions for $\nu = 1$ with $N = 6$ and various choices of τ_0. In Figure 3 is the exact solution and numerical solutions (a), absolute error (b) at $N = 6$, $\tau_0 = 0.001$, $\tau_0 = 0.0001$, and $h = 1$.

Example 3. Consider the following linear variable-order fractional delay integro-differential equation [45]:

$$D^{\alpha(x)}y(x-\tau) = g(x) + \frac{1}{3}\int_0^h (x+t)y(t)\,dt + \frac{1}{10}\int_0^x (t\,x)y(t)\,dt, x \in [0, h]. \tag{81}$$

where $\alpha(x) = x/2$, $g(x) = -17/216 - 5x/56 - 1/720x^9(9 + 8x) + 720x^{6-\alpha[x]}(7 + 7x - \alpha[x])/\mathrm{Gamma}[8 - \alpha[x]]$ and the initial condition is equal to $y(0) = 0$. The exact solution as given in [45] is given by $y(x) = x^7 + x^6$. Now, by applying the suggested technique that introduced in the previous section with finite terms $(N + 1)$ to (53) then we have

$$\left[P_0 X_{\alpha(x)} B_{\alpha(x)} B_{-\tau} M^T - U(x)\, M\, Z\, M^T - F(x)\, M\, E(x)\, M^T\right] A = g(x), \tag{82}$$

hence, using the collocation method with the collection points x_l that given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau} M^T - U\, M\, Z\, M^T - \bar{F}\, M\, \bar{E}\, M^T\right] A = G. \tag{83}$$

In Table 4, the comparison is the maximum absolute error of the method given in [45] and the presented method for different values of N and $h = 1$ in $\alpha(x) = x/2$. The absolute error when $h = 1$ for different values of N is shown in Figure 4.

Example 4. Consider the following linear variable-order fractional delay integro-differential equation [45]:

$$D^{\alpha(x)}y(x-\tau_0) = g(x) + \int_0^1 \sin(x)ty(t)\,dt + \int_0^x (t-x)y(t)\,dt, x \in [0, h], \tag{84}$$

where $P_0 = 1$, $g(x) = -16x^{27/4}/621 - 25x^{41/5}/1476 + x^{19/4-\alpha[x]}(\mathrm{Gamma}[23/4]/\mathrm{Gamma}[23/4 - \alpha[x]] + x^{29/20}\mathrm{Gamma}[36/5]/\mathrm{Gamma}[36/5 - \alpha[x]]) - 299\mathrm{Sin}[x]/1107$, $\alpha(x) = x$ and the initial condition is equal to $y(0) = 0$. The exact solution of this equation is given by $y(x) = x^{19/4} + x^{31/5}$ at $\tau_0 = 0$ (see [45]). Now, by applying the suggested technique that introduced in the previous section with finite terms $(N + 1)$ to (8), then, we have

$$\left[P_0 X_{\alpha(x)} B_{\alpha(x)} B_{-\tau_0} M^T - U(x)\, M\, Z\, M^T - F(x)\, M\, E(x)\, M^T\right] A = g(x), \tag{85}$$

Hence, using the collocation method with the collection points x_l that given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau_0} M^T - U\, M\, Z\, M^T - \bar{F}\, M\, \bar{E}\, M^T\right] A = G. \tag{86}$$

By solving the algebraic system (86), we can obtain the vector $A = [a_0, a_1, \cdots, a_n]^T$. Subsequently, numerical solution according to (18) is obtained. Table 5 gives the comparison of the absolute error at some points obtained by the present method and method of [45] with different values of N, and it is seen that our method gives more accurate results than the method of [45] at $\tau_0 = 0$. In Figure 5, the exact solution and numerical solutions given (a), where the absolute error shown (b), are all with different values of N and $h = 1$.

Example 5. Consider the following linear variable-order fractional integro-differential equation [39, 46]:

$$D^{\nu(x)}y(x) + \sin(x)y'(x) + (x-1)y(x) = g(x) - \int_0^x e^x\, y(t)\,dt, x \in [0, 1], \tag{87}$$

where $Q_0 = 1$, $R_0(x) = (x - 1)$, $R_1(x) = (\sin(x))$, and $g(x) = (-1 + x)(1 + (-1 + x)x) + 1/6e^x x(6 + x(-3 + 2x)) + (-1 + 2x)\mathrm{Sin}[x] + x^{1-\upsilon[x]}(-2 + 2x + \upsilon[x])/\mathrm{Gamma}[3 - \upsilon[x]]$ and the initial conditions is equal to $y(0) = 0, y'(0) = -1$. The exact solution of (87) according to [39, 46] is given by $y(x) = x^2 - x + 1$ at $\nu(x) = x/3$. Now, by applying the suggested technique that introduced in the previous section with finite terms $(N + 1)$ to (8), then, we have

$$\left[Q_0 X_{\nu_x} B_{\nu(x)} M^T + R_0\, X(x) M^T + R_1\, X(x) B M^T + F(x)\, M\, E(x)\, M^T\right] A = g(x), \tag{88}$$

Hence, using the present collocation method with the collection points x_l that given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[Q_0 \bar{X}_{\nu_{x_l}} \bar{B}_{\nu_{x_l}} M^T + R_0\, X(x_l) M^T + R_1\, X(x_l) B M^T + \bar{F}\, M\, \bar{E}\, M^T\right] A = G. \tag{89}$$

By solving the algebraic system (89), we can obtain the vector $A = [a_0, a_1, \cdots, a_n]^T$. Subsequently, numerical solution (18) is obtained. In Table 6, a comparison between the AEs is obtained in [39, 46], and the results are obtained in this work for $N = 4$ of for example (12). In Figure 6, we see the matching of the AEs values in this figure and that in Table 6. While Figure 7 displays the curves of exact and approximate solutions for example (12) with values of parameters listed in their caption.

5. Conclusion

In this work, the general form of mixed Fredholm–Volterra integro-differential equations with delay arguments of variable-order based on the operational matrix method with the Shifted Chebyshev polynomials (CPs) of the first kind is presented. The spectral collocation technique with the aid of CPs is used as an operational matrix method for solving the proposed model, which is reduced by the operational matrices to the matrix form. Caputo's VO fractional derivative operator is used to generate the matrices of the derivative. The accuracy of the proposed technique is obtained by many numerical examples. Finally, we used codes written with the Mathematica package to calculate our numerical results and graphs.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors carried out the proofs and conceived the study. All authors read and approved the final manuscript.

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Research Article

Higher-Order Accurate and Conservative Hybrid Numerical Scheme for Relativistic Time-Fractional Vlasov-Maxwell System

Tamour Zubair,[1] Muhammad Usman,[2] Ilyas Khan ,[3] Nawaf N. Hamadneh ,[4] Tiao Lu,[5] and Mulugeta Andualem [6]

[1]*School of Mathematical Sciences, Peking University, Beijing 100871, China*
[2]*Department of Mathematics, National University of Modern Languages (NUML), Islamabad 44000, Pakistan*
[3]*Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia*
[4]*Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia*
[5]*HEDPS and CAPT, LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, China*
[6]*Department of Mathematics, Bonga University, Bonga, Ethiopia*

Correspondence should be addressed to Ilyas Khan; i.said@mu.edu.sa, Nawaf N. Hamadneh; nhamadneh@seu.edu.sa, and Mulugeta Andualem; mulugetaandualem4@gmail.com

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The historical analysis demonstrates that plasma scientists produced a variety of numerical methods for solving “kinetic” models, i.e., the Vlasov-Maxwell (VM) system. Still, on the other hand, a significant fact or drawback of most algorithms is that they do not preserve conservation philosophies. This is a crucial fact that cannot be disregarded since the Vlasov Maxwell system is associated with conservation rules and is capable of assessing after the accomplishment of certain helpful mathematical actions. To examine the fractional-order routine of charged particles, we constructed a fractional-order plasma model and proposed a higher-order conservative numerical approach based on operational matrices theory and Shifted Gegenbauer estimations. Numerical convergence is investigated to confirm its competence and compatibility. This concept may be used in problems involving variable order and multidimensionality, such as those involving Vlasov and Boltzmann systems.

1. Introduction

The VM system is an important instrument due to its vast variety of applications [1–3]. Modern numerical plasma research is separated into two independent disciplines based on kinetic theory, notably the “particle” in cell technique (PICT) and the Vlasov equation (VE). The primary portion couple’s plasma “particle” motion equations to Maxwell equations and numerically cracks them using the particle in approach [1, 4], and [5]. The second approach employs FEM [6] and FDM [3] to discretize the VE. These are grid-constructed procedures. The main downside of debated numerical techniques is that they violate conservation commitments. Noteworthy key concerns with PICT are that in some conditions, complete energies rise dramatically in the lack of any valid source of energy. In 2010, a significant numerical study on this subject was conducted based on conservation rules. The terms “Crank” Nicolson and temporal integration are used to characterize the “conservative” arrangements of PICT [7–10].

G. Lapenta [11] published research in 2017, in which he analyzed motion equations (ME) and “Maxwell” equations (ME) using “Crank” Nicolson and leap-frog discretization methods. The technique maintains “energy” conservation, but on the other hand, it contradicts Gauss’s rule. As a result, we can conclude that formulating the PIC method, which adheres to all conservation rules, is difficult because two distinct types of the scheme are utilized, namely PICT and FDM. The forms of the “particles” are strongly retained by the integration approach used in PIC techniques, and they are also dispersion in the numerical sense, although one cannot state that FDM is dispersion-free. As a result of these mathematical conflicts, we may accomplish that it is dreadful to eloquent the PIC approach, which perfectly follows the conservation rules.

Spectral algorithms are a powerful and groundbreaking tool for tackling several kinds of mathematical models. Different modules of "polynomials" in the orthogonal form are available in the prevailing literature for spectral estimates [12–14]. Two types of structures, i.e., VM [15] and Vlasov-Poisson (VP) [15, 16], are handled (numerically) using spectral methods which further linked Crank-Nicolson. These approaches assist us in overcoming difficulties, mainly numerical dispersion [17], that the PIC method encountered.

The TFVMS is built and explored in this study utilizing a specific geometry and an improved structure of numerical scheme based on FD and SGPs approximations. Caputo sense's conservative time-fractional order finite difference approximations [18–24] has been utilized. Assuming that the "function" is constructed on multiple variables, we appropriately estimate stated "polynomials" and continue with the construction of the operational "matrices". Recent advancement in this subject is detailed in ref [19, 20]. The project tightly enforces conservation laws, which will be discussed in further detail in the next segment. The numerical framework exhibits virtuous convergence, which is also discussed in the paper. Finally, we conclude our research.

Table 1: Important symbols used in the theory.

Symbols	Name of symbols
F	Distribution function
t	Time
x, y, z	Velocity components
p_x, p_y, p_z	Momentum components
γ	Relativistic parameter
G_j^μ	Gegenbauer polynomials
G_j^μ	Shifted gegenbauer
$\wp_f(F)$	Vector norms of F,
$\wp_{E_x}(E_x)$	Vector norms of E_x,
E	Electric field
B	Magnetic field
${}_0^C D_t^\alpha$	Caputo time fractional
J	Current density
ρ	Charge density
h_i^μ	Normalizing factor
ϑ^μ	Weight function
$\wp_{B_z}(B_z)$	Vector norms of B_z,
$\wp_{E_y}(E_y)$	Vector norms of E_y

2. Mathematical Modelling and Conservative Numerical Scheme

2.1. Generalized Form. The most recent and generalized form of VMS with important laws [19–24] are explained as:

$$\begin{gathered} {}_0^C D_t^\alpha F + \frac{\partial}{\partial r} \cdot \left(\frac{p}{\gamma m} F \right) + \frac{\partial}{\partial p} \cdot \left[q \left(E + \frac{p \times B}{\gamma m c} \right) F \right], \gamma = \left(1 + \left| \frac{p}{mc} \right|^2 \right)^{1/2}, \\ \frac{1}{c} {}_0^C D_t^\alpha E - \nabla \times B + \frac{4\pi}{c} J = 0, \nabla \cdot E = 4\pi\rho, \\ -\frac{1}{c} {}_0^C D_t^\alpha B - \nabla \times E = 0, \nabla \cdot B = 0, {}_0^C D_t^\alpha \rho + \nabla \cdot J = 0. \end{gathered} \tag{1}$$

Table 1 contains an explanation of all the symbols used in the preceding system.

2.2. Mathematical Assumptions and Conversion. The significant assumptions for the problems are:

$$\begin{gathered} r = x\hat{i} + y\hat{j}, p = p_x\hat{i} + p_y\hat{j}, F = F\left(t, x, y, p_x, p_y\right), \\ E = E_x(x, y)\hat{i} + E_y(x, y)\hat{j}, J = j_x\hat{i} + j_y\hat{j}, B = B_z(x, y)\hat{k}. \end{gathered} \tag{2}$$

Therefore, we get the VMS as:

$$
\begin{aligned}
&{}_{0}^{C}D_{t}^{\alpha}F+\frac{1}{\gamma m}\left(p_{x}\frac{\partial F}{\partial x}+p_{y}\frac{\partial F}{\partial y}\right)+q\left(E_{x}\frac{\partial F}{\partial p_{x}}+E_{y}\frac{\partial F}{\partial p_{y}}\right)\\
&\quad+\frac{q}{mc}\left(\frac{p_{y}B_{z}}{\gamma}\frac{\partial F}{\partial p_{x}}-\frac{p_{x}B_{z}}{\gamma}\frac{\partial F}{\partial p_{y}}\right)=0,\\
&\frac{1}{c}{}_{0}^{C}D_{t}^{\alpha}E_{x}+\frac{4\pi}{c}j_{x}=\frac{\partial B_{z}}{\partial y},\frac{1}{c}{}_{0}^{C}D_{t}^{\alpha}E_{y}+\frac{4\pi}{c}j_{y}=-\frac{\partial B_{z}}{\partial x},\\
&\frac{1}{c}{}_{0}^{C}D_{t}^{\alpha}B_{z}=\frac{\partial E_{x}}{\partial y}-\frac{\partial E_{y}}{\partial x},\frac{\partial E_{x}}{\partial x}+\frac{\partial E_{y}}{\partial y}=4\pi\rho,\rho=q\int_{\Omega\in(-\infty,\infty)}(F)d\Omega,\\
&{}_{0}^{C}D_{t}^{\alpha}\rho+\frac{\partial}{\partial x}j_{x}+\frac{\partial}{\partial y}j_{y}=0,\ j_{x}=\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_{x}}{\gamma}F\right)d\Omega,\ j_{y}=\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_{y}}{\gamma}F\right)d\Omega.
\end{aligned}
\tag{3}
$$

2.3. Scheme Discretization. According to the defined process [19, 20], we presented our system into discretized such matrix form as follow:

$$
\begin{aligned}
&\P_{t}^{\alpha}F^{\eta+1/2}G+\frac{p_{x}}{\gamma m}C_{1}^{\eta+1}D_{x}^{1}G+\frac{p_{y}}{\gamma m}C_{1}^{\eta+1}D_{y}^{1}G+qC_{2}^{\eta}G'C_{1}^{\eta+1}D_{p_{x}}^{1}G\\
&\quad+qC_{3}^{\eta}G'C_{1}^{\eta+1}D_{p_{y}}^{1}G+\frac{q}{mc}\frac{p_{y}}{\gamma}C_{4}^{\eta}G'C_{1}^{\eta+1}D_{p_{x}}^{1}G-\frac{q}{mc}\frac{p_{x}}{\gamma}C_{4}^{\eta}G'C_{1}^{\eta+1}D_{p_{y}}^{1}G=0,\\
&\frac{1}{c}\P_{t}^{\alpha}E_{x}^{\eta+1/2}G'+\frac{4\pi}{c}\left(j_{x}^{\eta+1}\right)=C_{4}^{\eta+1}D_{y}^{1}G',\\
&\frac{1}{c}\P_{t}^{\alpha}B_{z}^{\eta+1/2}G'=C_{2}^{\eta+1}D_{y}^{1}G'-C_{3}^{\eta+1}D_{x}^{1}G',\\
&\frac{1}{c}\P_{t}^{\alpha}E_{y}^{\eta+1/2}G'+\frac{4\pi}{c}\left(j_{y}^{\eta+1}\right)=-C_{4}^{\eta+1}D_{x}^{1}G',\\
&C_{2}^{\eta+1}D_{x}^{1}G'+C_{3}^{\eta+1}D_{y}^{1}G'=4\pi\rho^{\eta+1},\\
&\P_{t}^{\alpha}\left\{\int_{\Omega\in(-\infty,\infty)}\left(F^{\eta+\frac{1}{2}}\right)d\Omega\right\}+\\
&\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_{x}}{\gamma}C_{1}^{\eta+1}D_{x}^{1}G\right)d\Omega+\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_{y}}{\gamma}C_{1}^{\eta+1}D_{y}^{1}G\right)d\Omega=0,\\
&\rho^{\eta+1}=q\int_{\Omega\in(-\infty,\infty)}\left(C_{1}^{\eta+1}G\right)d\Omega,\\
&j_{x}^{\eta+1}=\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_{x}}{\gamma}C_{1}^{\eta+1}G\right)d\Omega,\\
&j_{y}^{\eta+1}=\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_{y}}{\gamma}C_{1}^{\eta+1}G\right)d\Omega.
\end{aligned}
\tag{4}
$$

The definition of operator is [18].

$$\mathfrak{I}_t^{\alpha} F^{\eta+1/2} = \frac{\mathfrak{I}_t}{\mathfrak{I}t^{\alpha}}\left(C^{\eta+1/2} + \sum_{s=1}^{\eta}\left(C^{\eta+1/2-s}\right)\right) = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left\{\left(C^{\eta+1} - C^{\eta}\right) + \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{\eta}\left(C^{\eta+1-s} - C^{\eta-s}\right)b_s^{\alpha}\right\}. \quad (5)$$

We also have $b_s^{\alpha} = (s+1)^{1-\alpha} - s^{1-\alpha}$. We use $G_{M_1,M_2,M_3,M_4} = G$, and $G_{M_1,M_2} = G'$ for simplicity in discretization form and also $\mathfrak{I}_t^{\alpha}$ is the operator and is specified by

$$\mathfrak{I}_t F^{\eta+1/2} = \left(\delta_t\left\{C^{\eta+1/2}\right\} + \left\{\sum_{s=1}^{\eta}\delta_t\left(C^{\eta+1/2-s}\right)\right\}\right) = \left(C^{\eta+1} - C^{\eta}\right) + \sum_{s=1}^{\eta}\left(C^{\eta+1-s} - C^{\eta-s}\right). \quad (6)$$

2.4. Conservativeness of the Scheme

2.4.1. Charge Conservation. Taking $q\int_{\Omega\in(-\infty,\infty)} d\Omega$ of VE (5) and once obtained:

$$\begin{aligned}&\int_{\Omega\in(-\infty,\infty)} q\left(\mathfrak{I}_t^{\alpha} F^{\eta+1/2} G + \frac{p_x}{\gamma m} C_1^{\eta+1} D_x^1 G + \frac{p_y}{\gamma m} C_1^{\eta+1} D_y^1 G\right) d\Omega = 0,\\ &\mathfrak{I}_t^{\alpha}\left\{q\int_{\Omega\in(-\infty,\infty)} F^{\eta+1/2} G d\Omega\right\} + \left\{\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_x}{\gamma} C_1^{\eta+1} D_x^1 G\right) d\Omega\right\} + \left\{\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_y}{\gamma} C_1^{\eta+1} D_y^1 G\right) d\Omega\right\} = 0.\end{aligned} \quad (7)$$

Hence proved that charge conservation accordingly.

2.4.2. Gauss Law and Solenoidal Constrains. (1) Differential Arrangement. Using the concepts of divergence and equation (2), we obtained:

$$\frac{1}{c} div\left({}_0^C D_t^{\alpha} E\right) - div(\nabla \times B) + \frac{4\pi}{c} div(J) = 0 \Rightarrow \frac{1}{c}{}_0^C D_t^{\alpha}(div(E) - 4\pi\rho) = 0. \quad (8)$$

Hence, we can say that the above relation will satisfy if we have

$$div(E) = 4\pi\rho. \quad (9)$$

(2) Discretization Arrangement. Apply the divergence based on the assumptions, and we contract:

$$\begin{aligned}
&\frac{1}{c}\mathfrak{s}_t^{\alpha} D_x^1 E_x^{\eta+1/2} G' + \frac{4\pi}{c}\left(\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_x}{\gamma} C_1^{\eta+1} D_x^1 G\right) d\Omega\right) = C_4^{\eta+1} D_x^1 D_y^1 G',\\
&\frac{1}{c}\mathfrak{s}_t^{\alpha} D_y^1 E_y^{\eta+1/2} G' + \frac{4\pi}{c}\left(\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_y}{\gamma} C_1^{\eta+1} D_y^1 G\right) d\Omega\right) = -C_4^{\eta+1} D_x^1 D_x^1 G',\\
&\frac{1}{c}\mathfrak{s}_t^{\alpha}\left(D_x^1 E_x^{\eta+1/2} G' + D_y^1 E_y^{\eta+1/2} G'\right) + \frac{4\pi}{c}\left(\frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_x}{\gamma} C_1^{\eta+1} D_x^1 G\right) d\Omega + \frac{q}{m}\int_{\Omega\in(-\infty,\infty)}\left(\frac{p_y}{\gamma} C_1^{\eta+1} D_y^1 G\right) d\Omega\right) = 0,\\
&\frac{1}{c}\mathfrak{s}_t^{\alpha}\left(D_x^1 E_x^{\eta+1/2} G' + D_y^1 E_y^{\eta+1/2} G' - 4\pi\rho^{\eta+1}\right) = 0.
\end{aligned} \tag{10}$$

As a result, we may claim that the Gauss rule is true if and only if the following conditions are met:

$$D_x^1 E_x^0 G' + D_y^1 E_y^0 G' = -4\pi\rho^0. \tag{11}$$

It is not necessary to establish solenoidal limitations.

2.4.3. Momentum Conservation

(1) Differential Arrangement.

$$\begin{aligned}
&D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}(pF)d\Omega + \frac{\partial}{\partial r}\cdot\int_{\Omega\in(-\infty,\infty)}\left(p\frac{p}{\gamma m}F\right)d\Omega = -\int_{\Omega\in(-\infty,\infty)}\left(p\frac{\partial}{\partial p}\cdot\left\{q\left(E+\frac{p\times B}{\gamma mc}\right)F\right\}\right)d\Omega,\\
&D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}(pF)d\Omega + \frac{\partial}{\partial r}\cdot\int_{\Omega\in(-\infty,\infty)}\left(p\frac{p}{\gamma m}F\right)d\Omega = \int_{\Omega\in(-\infty,\infty)}\left(\left\{\left(qEF + qF\frac{p\times B}{\gamma mc}\right)\right\}\right)d\Omega.
\end{aligned} \tag{12}$$

For the case of x-component, we obtained

$$\begin{aligned}
&D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}(Fp_x)d\Omega + E_x\int_{\Omega\in(-\infty,\infty)}(qF)d\Omega + \frac{\partial}{\partial r}\cdot\int_{\Omega\in(-\infty,\infty)}\left(p_x\frac{p}{\gamma m}F\right)d\Omega =\\
&\frac{q}{mc}\int_{\Omega\in(-\infty,\infty)}\frac{1}{\gamma}\left(F\left\{B_z p_y\right\}\right)d\Omega,\\
&D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}(p_xF)d\Omega + \frac{\partial}{\partial r}\cdot\int_{\Omega\in(-\infty,\infty)}\left(p_x\frac{p}{\gamma m}F\right)d\Omega = \rho E_x + \frac{1}{c}\left(B_z j_y\right).
\end{aligned} \tag{13}$$

Additionally, we shall express the "Maxwell" equation in its momentum form as:

$$\begin{aligned}&\frac{B_z}{4\pi}\left(\frac{1}{c}D_t^\alpha E_y+\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}\right)+\frac{E_y}{4\pi}\left(\frac{1}{c}D_t^\alpha B_z+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)=-\frac{B_z}{4\pi}\frac{4\pi}{c}j_y,\\&-\frac{B_y}{4\pi}\left(\frac{1}{c}D_t^\alpha E_z+\frac{\partial B_x}{\partial y}-\frac{\partial B_y}{\partial x}\right)-\frac{E_z}{4\pi}\left(\frac{1}{c}D_t^\alpha B_y+\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)=\frac{B_y}{4\pi}\frac{4\pi}{c}j_z,\end{aligned}\tag{14}$$

Simplified form is

$$\frac{B_z}{4\pi}\left(\frac{1}{c}D_t^\alpha E_y+\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}\right)+\frac{E_y}{4\pi}\left(\frac{1}{c}D_t^\alpha B_z+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)=-\frac{B_z}{4\pi}\frac{4\pi}{c}j_y.\tag{15}$$

Additionally, we get

$$\frac{1}{4\pi c}D_t^\alpha\{B_zE_y\}+\frac{1}{8\pi}\left(\frac{\partial}{\partial x}\left(E^2+B^2\right)\right)-\frac{1}{4\pi}\left(div\left(E_xE+B_xB\right)\right)+\frac{1}{4\pi}\{B_x div(B)+E_x div(E)\}=-\frac{j_yB_z}{c}.\tag{16}$$

From (2), we get

$$\frac{1}{4\pi c}D_t^\alpha\{B_zE_y\}+\frac{1}{8\pi}\left(\frac{\partial}{\partial x}\left(E^2+B^2\right)\right)-\frac{1}{4\pi}\left(div\left(B_xE+B_xB\right)\right)=-E_x\rho-\frac{j_yB_z}{c}.\tag{17}$$

Further from equations (14) and (17), we get

$$D_t^\alpha\int_{\Omega\in(-\infty,\infty)}\left(p_xF+\frac{B_zE_y}{4\pi c}\right)d\Omega+\left\{\begin{array}{l}\dfrac{\partial}{\partial x}\displaystyle\int_{\Omega\in(-\infty,\infty)}\left(\begin{array}{c}p_x\dfrac{p_x}{\gamma m}F-\dfrac{E_x^2+B_x^2}{4\pi}\\ +\dfrac{E_x^2+E_y^2+B_x^2+B_y^2}{8\pi}\end{array}\right)d\Omega+\dfrac{\partial}{\partial y}\displaystyle\int_{\Omega\in(-\infty,\infty)}\left(\begin{array}{c}p_y\dfrac{p_y}{\gamma m}F\\ -\dfrac{E_yE_x+E_xE_y}{4\pi}\end{array}\right)d\Omega\\ \dfrac{\partial}{\partial x}\displaystyle\int_{\Omega\in(-\infty,\infty)}\left(\begin{array}{c}p_x\dfrac{p_x}{\gamma m}F-\dfrac{E_x^2+B_x^2}{4\pi}\\ +\dfrac{E_x^2+E_y^2+B_x^2+B_y^2}{8\pi}\end{array}\right)d\Omega+\dfrac{\partial}{\partial y}\displaystyle\int_{\Omega\in(-\infty,\infty)}\left(\begin{array}{c}p_y\dfrac{p_y}{\gamma m}F\\ -\dfrac{E_yE_x+E_xE_y}{4\pi}\end{array}\right)d\Omega\end{array}\right\}=0.\tag{18}$$

(2) Discretization Arrangement:

$$\int_{\Omega\in(-\infty,\infty)}\left\{p_x ¶_t^{\alpha} F^{\eta+\frac{1}{2}} G\right\}d\Omega+\int_{\Omega\in(-\infty,\infty)}\left\{\frac{p_x p_x}{\gamma m}C_1^{\eta+1}D_x^1 G\right\}d\Omega+\int_{\Omega\in(-\infty,\infty)}\left\{\frac{p_x p_y}{\gamma m}C_1^{\eta+1}D_y^1 G\right\}d\Omega=\left\{qC_2^{\eta}G\prime C_1^{\eta+1}G\right\}d\Omega+$$
$$\int_{\Omega\in(-\infty,\infty)}\left\{\frac{qp_y}{\gamma mc}C_4^{\eta+1}G\prime C_1^{\eta+1}G\right\}d\Omega,$$
$$\int_{\Omega\in(-\infty,\infty)}\left\{\frac{p_x p_y}{\gamma m}C_1^{\eta+1}D_y^1 G\right\}d\Omega=C_2^{\eta}G\prime\rho^{\eta+1}+\frac{1}{c}C_4^{\eta+1}G\prime j_y^{\eta+1}.\left\{p_x ¶_t^{\alpha} F^{\eta+\frac{1}{2}} G\right\}d\Omega$$
$$+\int_{\Omega\in(-\infty,\infty)}\left\{\frac{p_x p_x}{\gamma m}C_1^{\eta+1}D_x^1 G\right\}d\Omega+\int_{\Omega\in(-\infty,\infty)}\left\{\frac{p_x p_y}{\gamma m}C_1^{\eta+1}D_y^1 G\right\}d\Omega=C_2^{\eta}G\prime\rho^{\eta+1}+\frac{1}{c}C_4^{\eta+1}G\prime j_y^{\eta+1}. \tag{19}$$

Moreover; we have

$$\frac{1}{4\pi}C_4^{\eta+1}G'\left(¶_t^{\alpha}E_y^{\eta+1/2}G'+C_4^{\eta+1}D_x^1G'\right)+\frac{1}{4\pi}C_3^{\eta}G'\left(\delta_t^{\alpha}B_z^{\eta+1/2}G'-C_2^{\eta+1}D_y^1G'+C_3^{\eta+1}D_x^1G'\right)$$
$$=-\frac{1}{4\pi}C_4^{\eta+1}G'\left\{\frac{4\pi}{c}\left(j_y^{\eta+1}\right)\right\},$$
$$\frac{1}{4\pi}\left(C_4^{\eta+1}G'¶_t^{\alpha}E_y^{\eta+1/2}G'+\frac{1}{4\pi}C_3^{\eta}G'\delta_t^{\alpha}B_z^{\eta+1/2}G'\right)+\frac{1}{4\pi}\left(C_4^{\eta+1}G'¶_t^{\alpha}E_y^{\eta+1/2}G'+\frac{1}{4\pi}C_3^{\eta}G'\delta_t^{\alpha}B_z^{\eta+1/2}G'\right)$$
$$+\frac{1}{4\pi}\left(C_3^{\eta+1}G'C_3^{\eta+1}D_x^1G'+C_4^{\eta+1}G'C_4^{\eta+1}D_x^1G'-C_3^{\eta+1}G'C_2^{\eta+1}D_y^1G'\right)=-\frac{1}{c}C_4^{\eta+1}G'\left(j_y^{\eta+1}\right),$$
$$\frac{1}{4\pi}\left(C_2^{\eta+1}G'C_2^{\eta+1}D_x^1G'+C_3^{\eta+1}G'C_3^{\eta+1}D_x^1G'+C_4^{\eta+1}G'C_4^{\eta+1}D_x^1G'\right)$$
$$-\frac{1}{4\pi}C_2^{\eta+1}G'\left(C_2^{\eta+1}D_x^1G'+C_3^{\eta+1}D_y^1G'\right)=-\frac{1}{c}C_4^{\eta+1}G'\left(j_y^{\eta+1}\right).$$

From (15), we can write as:

$$\frac{1}{4\pi}\left(C_4^{\eta+1}G'¶_t^{\alpha}E_y^{\eta+1/2}G'+\frac{1}{4\pi}C_3^{\eta}G'\delta_t^{\alpha}B_z^{\eta+1/2}G'\right)$$
$$+\frac{1}{4\pi}\left(C_2^{\eta+1}G'C_2^{\eta+1}D_x^1G'+C_3^{\eta+1}G'C_3^{\eta+1}D_x^1G'+C_4^{\eta+1}G'C_4^{\eta+1}D_x^1G'\right)$$
$$-\frac{1}{4\pi}\begin{pmatrix}2C_2^{\eta+1}G'C_2^{\eta+1}D_x^1G'+C_2^{\eta+1}G'C_3^{\eta+1}D_y^1G'\\+C_3^{\eta+1}G'C_2^{\eta+1}D_y^1G'\end{pmatrix}=-\frac{1}{c}C_4^{\eta+1}G'\left(j_y^{\eta+1}\right)-\frac{1}{4\pi}C_2^{\eta+1}G'\left(4\pi\rho^{\eta+1}\right). \tag{21}$$

As a result of (20) and (21), we may conclude that the scheme follows the momentum conservation rule.

2.4.4. Energy Conservation. (1) Differential Arrangement. We shall use $\int_{\Omega\in(-\infty,\infty)} (\gamma mc^2) d\Omega$ and obtained further:

$$D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}\left(\gamma mc^2\mathbf{F}\right)d\Omega+\frac{\partial}{\partial\mathbf{r}}\cdot\int_{\Omega\in(-\infty,\infty)}\left(c^2\mathbf{pF}\right)d\Omega=\int_{\Omega\in(-\infty,\infty)}\left(\frac{\mathbf{p}}{\gamma m}\cdot\left\{q\left(E+\frac{\mathbf{p}\times B}{\gamma mc}\right)\mathbf{F}\right\}\right)d\Omega,$$

$$D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}\left(\gamma mc^2\mathbf{F}\right)d\Omega+\frac{\partial}{\partial\mathbf{r}}\cdot\int_{\Omega\in(-\infty,\infty)}\left(c^2\mathbf{pF}\right)d\Omega=\int_{\Omega\in(-\infty,\infty)}\frac{1}{\gamma m}\left(q\left(\mathbf{p}\cdot E\mathbf{F}\right)+\frac{1}{mc}\left(\mathbf{p}\cdot\frac{\mathbf{p}\times B}{\gamma}\mathbf{F}\right)\right)d\Omega \tag{22}$$

$$D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}\left(\gamma mc^2\mathbf{F}\right)d\Omega+\frac{\partial}{\partial\mathbf{r}}\cdot\int_{\Omega\in(-\infty,\infty)}\left(c^2\mathbf{pF}\right)d\Omega=\mathbf{J}\cdot E,$$

In the next we have;

$$\frac{c}{4\pi}E_x\left\{\frac{1}{c}D_t^{\alpha}E_x-\frac{\partial B_z}{\partial y}\right\}+\frac{c}{4\pi}E_y\left\{\frac{1}{c}D_t^{\alpha}E_y+\frac{\partial B_z}{\partial x}\right\}=-\frac{4\pi}{c}\frac{c}{4\pi}E_xj_x-\frac{4\pi}{c}\frac{c}{4\pi}E_yj_y,$$

$$\frac{c}{4\pi}B_z\left\{\frac{1}{c}D_t^{\alpha}B_z+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right\}=0, \tag{23}$$

$$\frac{1}{8\pi}D_t^{\alpha}\left(E_x^2+E_y^2+B_z^2\right)+\frac{c}{4\pi}\left(-E_x\frac{\partial B_z}{\partial y}+E_y\frac{\partial B_z}{\partial x}+B_z\frac{\partial E_y}{\partial x}-B_z\frac{\partial E_x}{\partial y}\right)=-j\cdot E,$$

Following a series of specific stages, we get to

$$\frac{1}{8\pi}D_t^{\alpha}\left(E^2+B^2\right)+\frac{c}{4\pi}div\left(E\times B\right)=-j\cdot E. \tag{24}$$

We have obtained from equations (23) and (24) as follow:

$$D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}\left(\gamma mc^2F+\frac{E^2+B^2}{8\pi}\right)d\Omega+div\left(\int_{\Omega\in(-\infty,\infty)}\left(c^2pf+c\frac{div\left(E\times B\right)}{4\pi}\right)d\Omega\right)=0. \tag{25}$$

(2) Discretization Arrangement:

$$D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}\left(\gamma mc^2 ¶_t^{\alpha}F^{\eta+1/2}G\right)d\Omega+\int_{\Omega\in(-\infty,\infty)}\left(c^2 p_{x,k}C_1^{\eta+1}D_x^1 G\right)d\Omega+\int_{\Omega\in(-\infty,\infty)}\left(c^2 p_{y,k}C_1^{\eta+1}D_y^1 G\right)d\Omega$$
$$=\left(\frac{qp_x}{m\gamma}C_2^{\eta}GC_1^{\eta+1}G\right)d\Omega+\int_{\Omega\in(-\infty,\infty)}\left(\frac{qp_y}{m\gamma}C_3^{\eta}GC_1^{\eta+1}G\right)d\Omega D_t^{\alpha},$$
$$\int_{\Omega\in(-\infty,\infty)}\left(\gamma mc^2 ¶_t^{\alpha}F^{\eta+1/2}G\right)d\Omega+\int_{\Omega\in(-\infty,\infty)}\left(c^2 p_{x,k}C_1^{\eta+1}D_x^1 G\right)d\Omega+\int_{\Omega\in(-\infty,\infty)}\left(c^2 p_{y,k}C_1^{\eta+1}D_y^1 G\right)d\Omega=C_2^{\eta}Gj_x+C_3^{\eta}Gj_y,$$
$$D_t^{\alpha}\int_{\Omega\in(-\infty,\infty)}\left(\gamma mc^2 ¶_t^{\alpha}F^{\eta+1/2}G\right)d\Omega+\int_{\Omega\in(-\infty,\infty)}\left(c^2 p_{x,k}C_1^{\eta+1}D_x^1 G\right)d\Omega+\int_{\Omega\in(-\infty,\infty)}\left(c^2 p_{y,k}C_1^{\eta+1}D_y^1 G\right)d\Omega=C_2^{\eta}Gj_x+C_3^{\eta}Gj_y. \tag{26}$$

The discretization procedure is elucidated as:

$$\frac{c}{4\pi}C_2^{\eta}G\left\{\frac{1}{c}¶_t^{\alpha}E_x^{\eta+1/2}G-C_4^{\eta}D_y^1G\right\}+\frac{c}{4\pi}C_3^{\eta}G\left\{\frac{1}{c}¶_t^{\alpha}E_y^{\eta+1/2}G+C_4^{\eta}D_x^1G\right\}=-\left(\frac{4\pi}{c}\frac{c}{4\pi}C_2^{\eta}Gj_{x,i,j}^{\eta}+\frac{4\pi}{c}\frac{c}{4\pi}C_3^{\eta}Gj_{y,i,j}^{\eta}\right),$$
$$\frac{c}{4\pi}C_4^{\eta}G\left\{\frac{1}{c}¶_t^{\alpha}B_z^{\eta+1/2}+C_3^{\eta}D_x^1G-C_2^{\eta}D_y^1G\right\}=0,$$
$$\frac{1}{4\pi}\left(C_2^{\eta}G\left\{¶_t^{\alpha}E_x^{\eta+1/2}G\right\}+C_3^{\eta}G\left\{¶_t^{\alpha}E_y^{\eta+1/2}G\right\}+C_4^{\eta}G\left\{¶_t^{\alpha}B_z^{\eta+1/2}\right\}\right) \tag{27}$$
$$+\frac{c}{4\pi}C_3^{\eta}G\left\{C_4^{\eta}D_x^1G\right\}+\frac{c}{4\pi}C_4^{\eta}G\left\{C_3^{\eta}D_x^1G\right\}-\frac{c}{4\pi}C_2^{\eta}G\left\{C_4^{\eta}D_y^1G\right\}$$
$$-\frac{c}{4\pi}C_4^{\eta}G\left\{C_2^{\eta}D_y^1G\right\}=-\left(C_2^{\eta}Gj_{x,i,j}^{\eta}+C_3^{\eta}Gj_{y,i,j}^{\eta}\right).$$

Hence proved.

3. Results and Discussion

Using MAPLE and Python, we generated generic code for assessing the numerical solution of the "model" using the specified numerical technique. We will explore the attitude of plasma particles for fractional concepts using the following initial perturbation.

$$F(t,x,y,p_x,p_x)=\frac{1}{\sqrt{2\pi}}e^{\frac{-\left(p_x^2+p_y^2\right)}{2}}\left(1+\varepsilon\cos(k_x x)\cos\left(k_y x\right)\right), \tag{28}$$

3.1. Numerical Convergence. To demonstrate the numerical convergence, we use concepts of norm. Therefore, we have the following relation as:

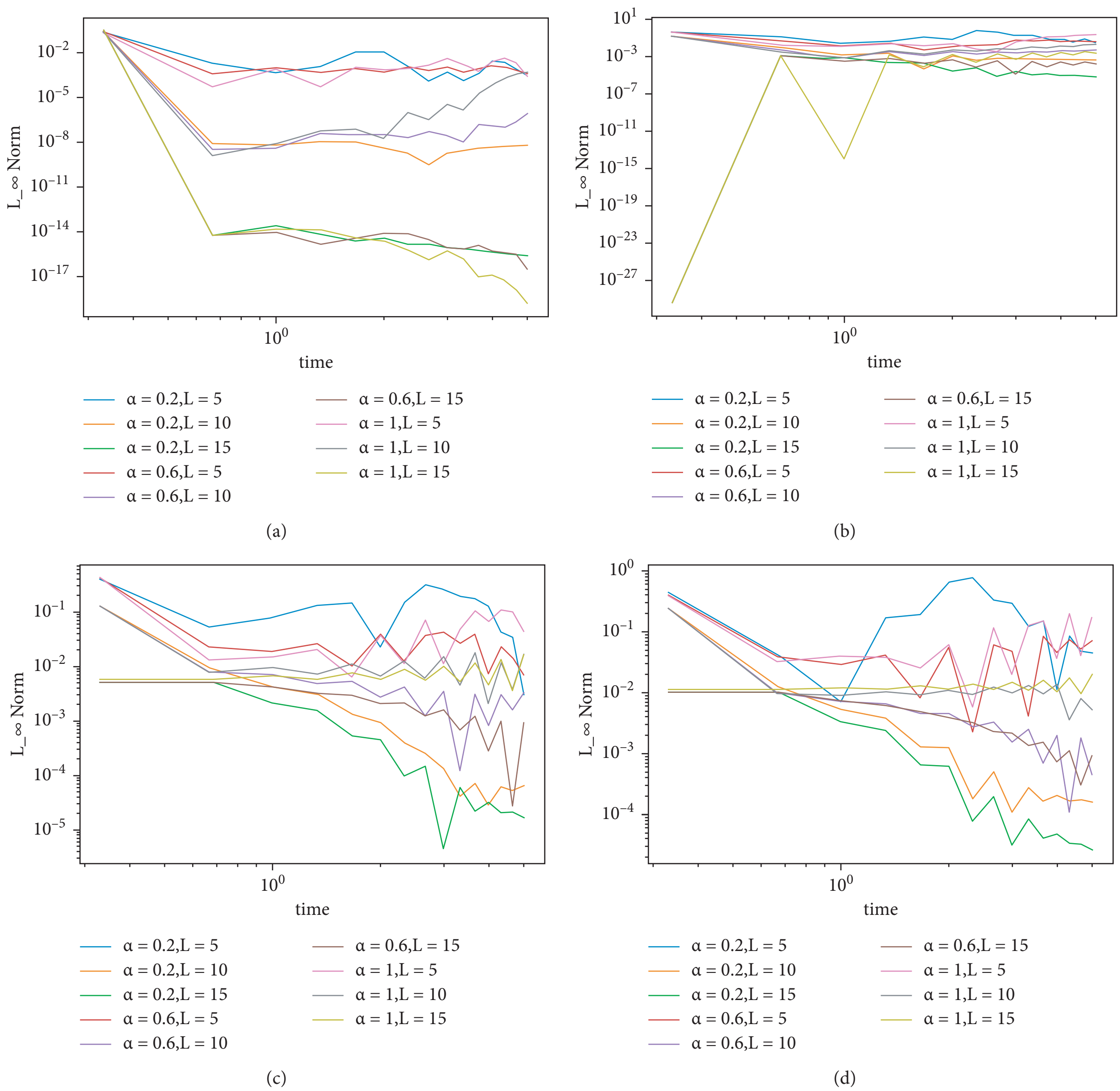

FIGURE 1: Convergence plots at $x = y = p_x = p_y = 5, 10, 15$, $N = 50$, $M_1 = M_2 = M_3 = M_4 = 5$.

$$\left\{\begin{array}{c} F_{L_\infty} = \max\left(\wp_f(F)\right), F = F^{\eta+1} - F^{n}, B_{z_{L_\infty}} = \max\left(\wp_{B_z}(B_z)\right), B_z = B_z^{\eta+1} - B_z^{\eta}, \\ E_{x_{L_\infty}} = \max\left(\wp_{E_x}(E_x)\right), E_x = E_x^{\eta+1} - E_x^{\eta}, E_{y_{L_\infty}} = \max\left(\wp_{E_y}\left(E_y\right)\right), E_y = E_y^{\eta+1} - E_y^{\eta} \end{array}\right\}. \tag{29}$$

Even under the most difficult conditions imposed by the challenge, the implemented system demonstrates efficient convergence and precision. In both cases, i.e., integer and fractional values of the fractional parameter, convergence rises steadily as the computing domain extends as shown in Figure 1(a)–1(d). Consequently, we can easily show that our technique is well-matched, competent, and appropriate for the models discussed before.

3.2. Behaviour of Charged Particles at $\alpha \in (0, 1]$. We have revealed one critical sort of graphical diagrams, namely "density". We picked two distinct time periods, $t = 1.33, 3.66$, and varied $0 < \alpha \leq 1$. While we were executing initial data, "plasma" particles were unexpectedly displaced from their initial positions. "Particles" also acquire energy as a result of this perturbation. As a result, plasma particles began moving abruptly in order to stabilize themselves.

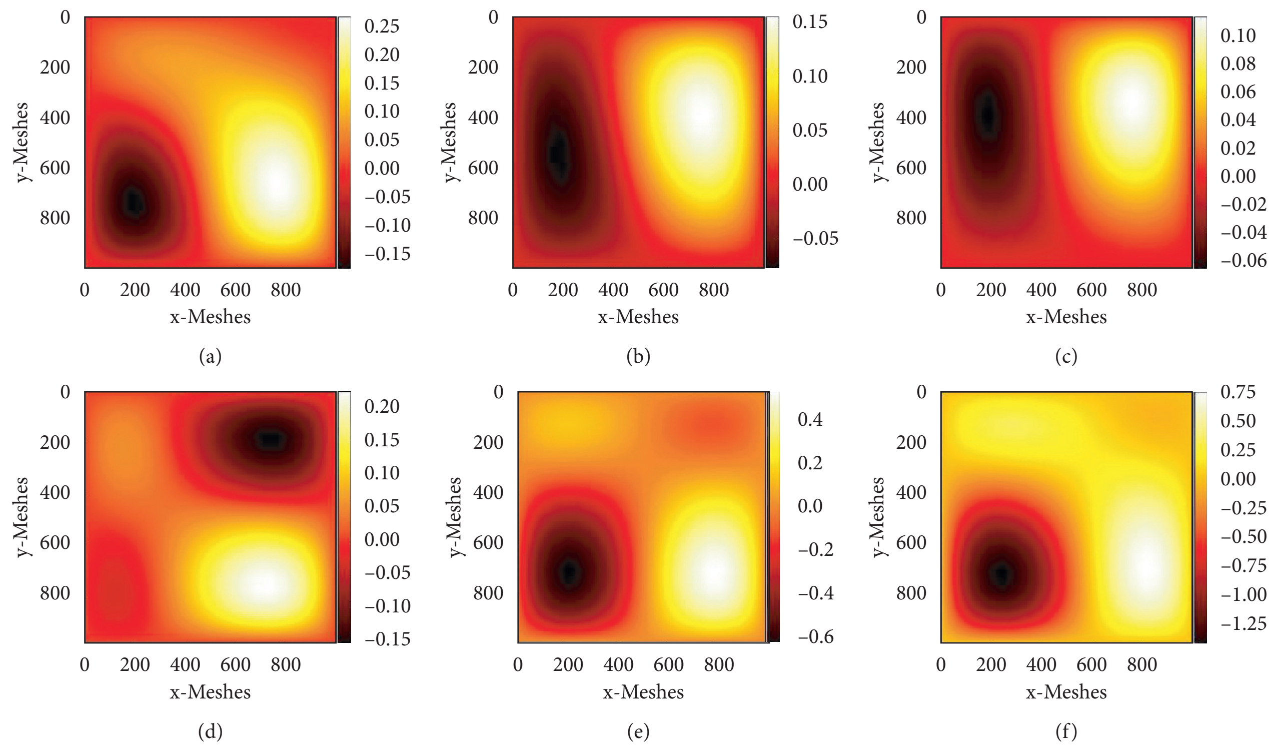

Figure 2: . Plots for Density of particles at $N = 50, M_1 = 5, M_2 = M_3 = M_4$,. (a) t = 1.33, $\alpha = 0.2$. (b) t = 1.33, $\alpha = 0.6$. (c) t = 1.33, $\alpha = 1.0$. (d) t = 3.66, $\alpha = 0.2$. (e) t = 3.66, $\alpha = 0.6$. (f) t = 3.66, $\alpha = 1.0$.

Noticed that excited "plasma" particles are accessible in cluster form in the range $400 < x, y \leq 800$. Formulation of clusters is due to its high rate of plasma particle. As we vary $t = 1.33, \alpha = 0.2$ to $t = 1.33, \alpha = 0.6$, the particles undergo modification, and the bunch magnitude and outline are repaired appropriately (see Figure 2(b)).

A thin layer is seen in Figure 2(b) around clusters of low "momentum" plasma particles. Additionally, the assortment of particles deviate position. These clusters are translated into the final location and "momentum" by selecting $t = 1.33, \alpha = 1.0$.

As a result, we can examine the particles' locus as the fractional parameter is varied. We clearly noticed significant fluctuations in the location and velocity of plasma particles during the second time period, as seen in Figures 2(d)–2(f). Some of the particles are clustered, while others are uncontrolled, covering the computing area. This procedure will continue indefinitely until the equilibrium state is reached.

When the fractional parameter is modified, we may see statistically significant variations. Therefore, fractional concepts are utilized to indicate the density of plasma particles and the path taken as a consequence of this. It dives into the complex picture of plasma particle behaviour that has remained concealed.

4. Conclusion

The current work accomplished two critical objectives: it developed a multi-order (integer and fractional) "fully relativistic" model based on several concepts, and it proposed a conservative "hybrid" numerical technique for solving the anticipated "plasma" model. To deal with the time-"fractional" derivative, the Caputo sense definition is used. The reported findings unequivocally illustrate that plasma particles exhibit significant variances over a range of fractional parameter values. By assigning specific positions, momentum, and energy to plasma particles, we may now determine their eventual state. As a consequence, the proposed model has the potential to considerably advance our understanding of plasma particles in the field of computing. The technique is adequate, well-matched, and effectual for the suggested model based on numerical convergence. It has a high rate of "convergence" for both derivatives, which grows gradually as the computing domain is enlarged.

Data Availability

No data was required to perform this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Generalized Fractional Integral Inequalities for MT-Non-Convex and pq-Convex Functions

Wei Wang,[1] **Absar Ul Haq**,[2] **Muhammad Shoaib Saleem**,[3] **and Muhammad Sajid Zahoor**[3]

[1] *School of General Education, Nantong Institute of Technology, Nantong 226002, China*
[2] *Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan*
[3] *Department of Mathematics, University of Okara, Okara, Pakistan*

Correspondence should be addressed to Wei Wang; 20200003@ntit.edu.cn

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Fractional integral inequalities have a wide range of applications in pure and applied mathematics. In the present research, we establish generalized fractional integral inequalities for MT-non-convex functions and pq-convex functions. Our results extended many inequalities already existing in the literature.

1. Introduction

The study of convex analysis is not modern part of mathematics, but some ancient mathematicians also used the interesting geometry of convex functions and convex sets. However, the subject of convex analysis was started in the mid of 20th century. Many remarkable facts and generalization of convex analysis have been obtained quite recently.

Convex analysis is one of the appealing subjects for the researchers of geometry and analysis. The interesting geometric, differentiability, and other facilitating properties of convex functions make it distinct from other subjects. Moreover, the convex function and convex set have diverse applications in mathematical physics, technology, economics, and optimization theory.

In the last decades, the connection of convexity got stronger due to rapid development in fractional calculus. Therefore, nowadays, it is appreciable to seek new fractional integral inequalities. Simply, we can say that convexity plays a concrete role in fractional integral inequalities and symmetry theory because of its interesting geometric features.

There are many well-known integral inequalities related to convex functions, like Jensen's inequality [1, 2], Opial-type inequality [3], Simpson inequality [4], Ostrowski inequality [5–7], Hermite–Hadamard inequality [8, 9], Olsen integral inequality [10], Fejér-type inequality [1, 11], Hardy inequality [12], and so on. One of the remarkable inequalities for convex function is the Hermite–Hadamard-type inequality.

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \varphi(\eta)\,d\eta \leq \frac{\varphi(u)+\varphi(v)}{2}, \tag{1}$$

where $\varphi\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $u, v \in I$ with $u < v$. Note that several integral inequalities can be obtained from equation (1). There are several versions of inequality equation (1) in the literature, for example, inequalities of Hermite–Hadamard type for functions whose derivatives' absolute values are quasi-convex are presented in [8]. In [13], the authors established Hermite–Hadamard-type inequalities for p-convex functions via fractional integrals. In [14], the generalized Hermite–Hadamard inequalities are presented. A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals is presented in [9]. The MT-convex functions via classical and Riemann–Liouville fractional integrals are studied in [15] by Park. He also established Hermite–Hadamard inequality for MT-convex functions. Mohammad in [16] established Hermite–Hadamard-type inequalities on differentiable coordinates for the same class of functions as studied by Park in [15], and Liua et al. [17] developed this inequality for the same class of functions for classical integrals and fractional integrals. For more details, we refer the readers to [18–20].

First of all, we recall few definitions.

Definition 1 (p-convex set) [21]. *I is said to be p-convex set, if $[\eta u^p + (1-\eta)v^p]^{(1/p)} \in I$ for all $u, v \in I$ and $\eta \in [0, 1]$.*

Definition 2 (p-convex) [13, 22]. A function $\varphi: I \longrightarrow \mathbb{R}$ is said to be p-convex function, if

$$\varphi\left(\eta u^p + (1-\eta)v^p\right)^{(1/p)} \le \eta\varphi(u) + (1-\eta)\varphi(v), \quad (2)$$

for all u, v, $\in\in I$ and $\eta \in [0, 1]$.

Definition 3 (MT-convex) [5, 15–17]. A function $\varphi: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be MT-convex on I, if

$$\varphi(\eta u + (1-\eta)v) \le \frac{\sqrt{\eta}}{2\sqrt{1-\eta}}\varphi(u) + \frac{\sqrt{1-\eta}}{2\sqrt{\eta}}\varphi(v) \quad (3)$$

holds for all u, v, $\in\in I$ and $\eta \in [0, 1]$.

Now we are ready to extend the form of convexities.

Definition 4 (MT-non-convex). A function φ: $I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be MT-non-convex. Let I be p-convex set, if

$$\left(\eta u^p + (1-\eta)v^p\right)^{(1/p)} \le \frac{\sqrt{\eta}}{2\sqrt{1-\eta}}\varphi(u) + \frac{\sqrt{1-\eta}}{2\sqrt{\eta}}\varphi(v) \quad (4)$$

holds for all u, v, $\in\in I$ and $\eta \in [0, 1]$.

Remark 1. In the above definition, for $p=1$, we get MT-convex function, and for $p=-1$, we get harmonically MT-convex function.

Definition 5 ((p-q) convex). A function φ: $I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be pq-convex. Let I be p-convex set if

$$\varphi\left(\eta u^p + (1-\eta)v^p\right)^{(1/p)} \le \left(\eta\varphi(u)^q + (1-\eta)\varphi(v)^q\right)^{(1/p)} \quad (5)$$

holds for all $u, v \in I$ and $\eta \in [0, 1]$.

Definition 6 (Riemann–Liouville fractional integral) [9]. Let $\varphi \in L^1[u, v]$ and $\gamma > 0$. The right side and left side Riemann–Liouville fractional integrals are initiated by

$$J^{\gamma}_{u+}\varphi(s) = \frac{1}{\Gamma(\gamma)}\int_u^s (s-\eta)^{\gamma-1}\varphi(\eta)\,\mathrm{d}\eta, \quad s > u,$$
$$J^{\gamma}_{v-}\varphi(s) = \frac{1}{\Gamma(\gamma)}\int_s^v (\eta-s)^{\gamma-1}\varphi(\eta)\,\mathrm{d}\eta, \quad s < v, \quad (6)$$

respectively. For details, we refer the readers to [6, 23].

Now, we define some special functions:

(1) Gamma function:

$$\Gamma(z) = \int_0^{\infty} e^{-\eta}\eta^{z-1}\,\mathrm{d}\eta, \quad z > 0. \quad (7)$$

(2) Beta function:

$$\beta(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 \eta^{u-1}(1-\eta)^{v-1}\,\mathrm{d}\eta\,(u, v > 0) \quad (8)$$

(see [24–27]).

(3) The hypergeometric function [18]:

$${}_2F_1(u, v, w, x) = \frac{1}{\beta(v, w-v)}\int_0^1 \eta^{v-1}(1-\eta)^{w-v-1} \cdot (1-d\eta)^{-u}\,\mathrm{d}\eta, \quad w > v > 0, |x| < 1. \quad (9)$$

In [28], Raina introduced a function initiated by

$$F^{\varsigma}_{\varrho,\lambda}(u) = F^{\varsigma(0),\varsigma(1),\ldots}_{\varrho,\lambda}(u) = \sum_{k=0}^{\infty}\frac{\varsigma(k)}{\Gamma(\varrho k + \lambda)}u^k, \quad (10)$$

where $\varrho, \lambda \in \mathbb{R}^+$; $|u| < \mathbb{R}$ and the coefficients $\varsigma(k) \in \mathbb{R}^+$. By using equation (10), Raina and Agarwal [28, 29] initiated the following left and right-side fractional integral operators:

$$\left(J^{\varsigma}_{\varrho,\lambda,u+;\omega}\varphi\right)(z) = \int_u^z (z-\eta)^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}\left[\omega(z-\eta)^{\varrho}\right]\varphi(\eta)\,\mathrm{d}\eta, \quad (z > u),$$
$$\left(J^{\varsigma}_{\varrho,\lambda,v-;\omega}\varphi\right)(z) = \int_v^z (\eta-z)^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}\left[\omega(\eta-z)^{\varrho}\right]\varphi(\eta)\,\mathrm{d}\eta, \quad (z < v), \quad (11)$$

where $\omega \in \mathbb{R}$ and λ, $\varrho \in \mathbb{R}^{+.}$

This paper is organized as follows. In Section 2, we will derive generalized fractional integral inequalities for MT-non-convex function. However, the last section is dedicated to establish generalized fractional integral inequalities for (p-q) convex function.

2. Fractional Integral Inequalities for MT-Non-Convex Function

The following lemma is useful to derive our main results.

Lemma 1 (see [22]). *Let λ, $\in\in \mathbb{R}^+ \varphi$: $I \subseteq \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a differentiable mapping on $I^o u$, v, $\in\in I$ such that $u < v$. If $\varphi' \in L^1[u, v]$, $p > 0$, then we obtain*

$$\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}\right]\left(v^{p}-u^{p}\right)^{\lambda}}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(v^{p}\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right)\right]$$
$$=\frac{\left(v^{p}-u^{p}\right)}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[\left(v^{p}-u^{p}\right)^{\varrho}\right]}\int_{0}^{1}\begin{array}{l}(1-\eta)^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}(1-\eta)^{\varrho}\right]\left(\eta u^{p}\right.\\ \left.+(1-\eta)v^{p}\right)^{(1/p)-1}\varphi'\left(\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)}\right)\mathrm{d}\eta\end{array} \tag{12}$$
$$+\int_{0}^{1}\eta^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}\eta^{\varrho}\right]\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}\varphi'\left(\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)}\right)\mathrm{d}\eta,$$

where $\phi(c)=c^{(1/p)}$.

Theorem 1. *Let* $\lambda,\in,\mathbb{R}^{+}\varphi\colon I\subseteq\mathbb{R}^{+}\longrightarrow\mathbb{R}^{+}$ *be a differentiable mapping on* $I^{o}u,v\in I^{o}$ *such that* $u<,v$. *If* $|\varphi'|$ *is MT-non-convex on* $[u,v]$, $p>0$, *then we obtain*

$$\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}\right]\left(v^{p}-u^{p}\right)^{\lambda}}\left[\left(J^{\sigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(v^{p}\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right)\right]$$
$$\leq\left(F^{C_1}_{\varrho,\lambda+1}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]+F^{C_3}_{\varrho,\lambda+1}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]\right)\left|\varphi'(u)\right| \tag{13}$$
$$+\left(F^{C_2}_{\varrho,\lambda+1}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]+F^{C_4}_{\varrho,\lambda+1}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]\right)\left|\varphi'(v)\right|,$$

where

$$\begin{aligned}
C_1&=\varsigma(k)v^{1-p}\frac{\varrho k+\lambda-(1/2)}{3(\varrho k+\lambda+1)}\,{}_2F_1\left(1-\frac{1}{p},\frac{3}{2};\varrho k+\lambda+2,1-\frac{u^{p}}{v^{p}}\right),\\
C_2&=\varsigma(k)v^{1-p}\frac{\varrho k+\lambda+(1/2)}{\varrho k+\lambda+1}\,{}_2F_1\left(1-\frac{1}{p},\frac{1}{2};\varrho k+\lambda+2,1-\frac{u^{p}}{v^{p}}\right),\\
C_3&=\varsigma(k)v^{1-p}\frac{(5/4)}{(\varrho k+\lambda+(3/2))(\varrho k+\lambda+1)}\,{}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{3}{2};\varrho k+\lambda+2,1-\frac{u^{p}}{v^{p}}\right),\\
C_4&=\varsigma(k)v^{1-p}\frac{1}{4(\varrho k+\lambda+(1/2))(\varrho k+\lambda+1)}\,{}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{1}{2};\varrho k+\lambda+2,1-\frac{u^{p}}{v^{p}}\right).
\end{aligned} \tag{14}$$

Proof. Employing Lemma 1 and definition of MT-non-convexity of $|\varphi'|$, we obtain

$$\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}\right]\left(v^{p}-u^{p}\right)^{\lambda}}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(v^{p}\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right)\right]$$

$$=\frac{\left(v^{p}-u^{p}\right)}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}\right]}\left[\begin{array}{l}\int_{0}^{1}\left(1-\eta^{\lambda}\right)F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}(1-\eta)^{\varrho}\right]\\ \left(\eta u^{p}+(1-\eta)v^{p}\right)\frac{1}{p}-1\varphi'\left(\eta u^{p}+(1-\eta)v^{p}\right)\frac{1}{p}\right)\mathrm{d}\eta\end{array}\right.$$

$$+\int_{0}^{1}\eta^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}\eta^{\varrho}\right]\times\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}\varphi'\left(\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)}\right)\mathrm{d}\eta]|$$

$$\leq\frac{v^{p}-u^{p}}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^{k}\left(v^{p}-u^{p}\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)}$$

$$\times\int_{0}^{1}\left[\left|(1-\eta)^{\varrho k+\lambda}+\eta^{\varrho k+\lambda}\right|\right]\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}+\left[\frac{\sqrt{\eta}}{2\sqrt{1-\eta}}\left|\varphi'(u)\right|\frac{\sqrt{1-\eta}}{2\sqrt{\eta}}\left|\varphi'(v)\right|\right]\mathrm{d}\eta$$

$$\leq\frac{v^{p}-u^{p}}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^{k}\left(v^{p}-u^{p}\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)}$$

$$\times\left[\int_{0}^{1}(1-\eta)^{\varrho k+\lambda}\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}\left[\frac{\sqrt{\eta}}{2\sqrt{1-\eta}}\left|\varphi'(u)\right|\frac{\sqrt{1-\eta}}{2\sqrt{\eta}}\left|\varphi'(v)\right|\right]\mathrm{d}\eta\right.$$

$$+\int_{0}^{1}\eta^{\varrho k+\lambda}\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1} \tag{15}$$

$$\left.\cdot\left[\frac{\sqrt{\eta}}{2\sqrt{1-\eta}}\left|\varphi'(u)\right|\left(\frac{\sqrt{1-\eta}}{2\sqrt{\eta}}\left|\varphi'(v)\right|\right)\mathrm{d}\eta\right]\right.$$

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{p}\right]\left(v^{p}-u^{p}\right)^{\lambda}}\left[J^{\varsigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(v^{p}\right)\left(J^{\varsigma}_{\varrho,\lambda,v^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right)\right]\right|$$

$$\leq\frac{v^{p}-u^{p}}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^{p}-u^{p}\right)^{p}\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^{k}\left(v^{p}-u^{p}\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)}$$

$$\times\left[\int_{0}^{1}\frac{1}{2}\eta^{(1/2)}(1-\eta)^{\varrho k+\lambda-(1/2)}\left|\varphi'(u)\right|\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}\mathrm{d}\eta\right.$$

$$+\int_{0}^{1}\frac{1}{2}\eta^{-(1/2)}(1-\eta)^{pk+\lambda+(1/2)}\left|\varphi'(v)\right|\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}\mathrm{d}\eta$$

$$+\int_{0}^{1}\frac{1}{2}\eta^{\varrho k+\lambda+(1/2)}(1-\eta)^{-(1/2)}\left|\varphi'(u)\right|\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}\mathrm{d}\eta$$

$$\left.+\int_{0}^{1}\frac{1}{2}\eta^{\varrho k+\lambda-(1/2)}(1-\eta)^{(1/2)}\left|\varphi'(v)\right|\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)-1}\mathrm{d}\eta\right].$$

So,

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^p\right]\left(v^p-u^p\right)^{\lambda}}\left[J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)\left(v^p\right)\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)\left(u^p\right)\right]\right| \leq \frac{v^p-u^p}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^p\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^k\left(v^p-u^p\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)} \times\left[v^{1-p}\frac{1}{2}\beta\left(\frac{3}{2},\varrho k+\lambda+\frac{1}{2}\right){}_2F_1\left(1-\frac{1}{p},\frac{3}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\left|\varphi'(u)\right| + v^{1-p}\frac{1}{2}\beta\left(\frac{1}{2},\varrho k+\lambda+\frac{3}{2}\right){}_2F_1\left(1-\frac{1}{p},\frac{1}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\left|\varphi'(v)\right| + v^{1-p}\frac{1}{2}\beta\left(\varrho k+\lambda+\frac{3}{2},\frac{1}{2}\right){}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{3}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\left|\varphi'(u)\right| + v^{1-p}\frac{1}{2}\beta\left(\varrho k+\lambda+\frac{1}{2},\frac{3}{2}\right){}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{1}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\left|\varphi'(v)\right|\right]. \tag{16}$$

From here,

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^p\right]\left(v^p-u^p\right)^{\lambda}}\left[J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)\left(v^p\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)\left(u^p\right)\right]\right| \leq \frac{v^p-u^p}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^p\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^k\left(v^p-u^p\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)} \times\left[v^{1-p}\frac{\varrho k+\lambda-(1/2)}{3(\varrho k+\lambda+1)}\left[{}_2F_1\left(1-\frac{1}{p},\frac{3}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\right]\left|\varphi'(u)\right| + v^{1-p}\frac{\varrho k+\lambda+(1/2)}{\varrho k+\lambda+1}\left[{}_2F_1\left(1-\frac{1}{p},\frac{1}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\right]\left|\varphi' v\right| + v^{1-p}\frac{(5/4)}{(\varrho k+\lambda+(3/2))(\varrho k+\lambda+1)}\left[{}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{3}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\right]\left|\varphi'(u)\right| + v^{1-p}\frac{1}{4(\varrho k+\lambda+(1/2))(\varrho k+\lambda+1)}\left[{}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{1}{2};\varrho k+\lambda+2,1-\frac{u^p}{v^p}\right)\right]\left|\varphi'(v)\right|\right]. \tag{17}$$

Simple calculations yield equation (13). □

Remark 2. In Theorem 1, we see the following:

(1) For $p = 1$, we have the inequality for MT-convex function:

$$\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}[|\omega|(v-u)^{\varrho}](v-u)^{\lambda}}\left[v\left(J^{\varsigma}_{\varrho,\lambda,u+;\omega}\varphi o\phi\right) + u\left(J^{\varsigma}_{\varrho,\lambda,v-;\omega}\varphi o\phi\right)\right]$$
$$\leq \left(F^{C_1}_{\varrho,\lambda+1}[|\omega|(v-u)^{\varrho}] + F^{C_3}_{\varrho,\lambda+1}[|\omega|(v-u)^{\varrho}]\right)|\varphi'(u)| \tag{18}$$
$$+\left(F^{C_2}_{\varrho,\lambda+1}[|\omega|(v-u)^{\varrho}] + F^{C_4}_{\varrho,\lambda+1}[|\omega|(v-u)^{\varrho}]\right)|\varphi'(v)|.$$

(2) For $p = -1$, we have the inequality of harmonically MT-convex function:

$$\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}[|\omega|(u-v/uv)^{\varrho}](u-v/uv)^{\lambda}}\left[\frac{1}{v}\left(J^{\varsigma}_{\varrho,\lambda,(1/u)+;\omega,}\varphi o\phi\right) + \frac{1}{u}\left(J^{\varsigma}_{\varrho,\lambda,(1/v)}\varphi o\phi\right)\right]$$
$$\leq \left(F^{C_1}_{\varrho,\lambda+1}\left[|\omega|\left(\frac{u-v}{uv}\right)^{\varrho}\right] + F^{C_3}_{\varrho,\lambda+1}\left[|\omega|\left(\frac{u-v}{uv}\right)^{\varrho}\right]\right)|\varphi'(u)| \tag{19}$$
$$+\left(F^{C_2}_{\varrho,\lambda+1}\left[|\omega|\left(\frac{u-v}{uv}\right)^{\varrho}\right] + F^{C_4}_{\varrho,\lambda+1}\left[|\omega|\left(\frac{u-v}{uv}\right)^{\varrho}\right]\right)|\varphi'(v)|.$$

Theorem 2. *Let $\lambda, \in, \mathbb{R}^+$, $\varphi: I \subset \mathbb{R} \longrightarrow \mathbb{R}$, be a MT-non-convex function on $u, v, \in\in I$ such that $u < v$. If $\varphi \in L[u, v]$, $p > 0$, then we get*

$$\varphi\left(\frac{1}{2}u^p + \frac{1}{2}v^p\right)^{(1/p)} \leq \frac{1}{2F^{\varsigma}_{\varrho,\lambda+1\left[\omega(v^p-u^p)^{\varrho}(v^p-u^p)^{\lambda}\right]}}\left[J^{\varrho,\lambda,u^p+;\omega}\varphi o\phi(v^p) + J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi(u^p)\right]$$
$$\leq \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi(u) + \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi'(v), \tag{20}$$

where $\phi(c) = c^{(1/p)}$.

Proof. Since φ is MT-non-convex on $[u, v]$, for all $c, d\in[u, v]$,

$$\varphi\left(\frac{1}{2}c^p + \frac{1}{2}d^p\right)^{(1/p)} \leq \frac{\varphi(c)+\varphi(d)}{2}, \tag{21}$$

and substituting $c = (\xi u^p + (1-\eta)v^p)^{(1/p)}$ and $d = ((1-\eta)u^p + \eta v^p)^{(1/p)}$, equation (21) yields

$$\varphi\left(\frac{1}{2}u^p + \frac{1}{2}v^p\right)^{(1/p)} \leq \varphi\left(\eta u^p + (1-\eta)v^p\right)^{(1/p)} + \varphi\left((1-\eta)u^p + \eta v^p\right)^{(1/p)}. \tag{22}$$

Multiply inequality equation (22) by $\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}[\omega(v^p-u^p)^{\varrho}\eta^{\varrho}]$, and after that, integrating it over $\eta \in[0, 1]$, then we get

$$\begin{aligned}
2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^{\varrho}\eta^{\varrho}\right]\varphi\left(\frac{1}{2}u^p+\frac{1}{2}v^p\right)^{(1/p)} &\leq \int_0^1\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}\left[\omega\left(v^p-u^p\right)^{\varrho}\eta^{\varrho}\right]\varphi\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)}\,\mathrm{d}\eta \\
&\quad+\int_0^1\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}\left[\omega\left(v^p-u^p\right)^{\varrho}\eta^{\varrho}\right]\varphi\left((1-\eta)u^p+\eta v^p\right)^{(1/p)}\,\mathrm{d}\eta \\
&=\int_0^1\eta^{\lambda-1}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k\left(v^p-u^p\right)^{\varrho k}\eta^{\varrho k}}{\Gamma(\varrho k+\lambda)}\varphi\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)}\,\mathrm{d}\eta \\
&\quad+\int_0^1\eta^{\lambda-1}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k\left(v^p-u^p\right)^{\varrho k}\eta^{\varrho k}}{\Gamma(\varrho k+\lambda)}\varphi\left((1-\eta)u^p+\eta v^p\right)^{(1/p)}\,\mathrm{d}\eta \\
&=\int_0^1\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k}{\Gamma(\varrho k+\lambda)}\eta^{\varrho k+\lambda-1}\varphi\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)}\left(v^p-u^p\right)^{\varrho k}\,\mathrm{d}\eta \\
&\quad+\int_0^1\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k}{\Gamma(\varrho k+\lambda)}\eta^{\varrho k+\lambda-1}\varphi\left((1-\eta)u^p+\eta v^p\right)^{(1/p)}\left(v^p-u^p\right)^{\varrho k}\,\mathrm{d}\eta \\
&=\int_{u^p}^{v^p}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k}{\Gamma(\varrho k+\lambda)}\frac{\left(v^p-c\right)^{\varrho k}\left(v^p-c\right)^{\lambda-1}}{\left(v^p-u^p\right)^{\varrho k}\left(v^p-u^p\right)^{\lambda-1}}\left(v^p-u^p\right)^{\varrho k}\varphi\left(c^{(1/p)}\right)\frac{dc}{v^p-u^p} \\
&\quad+\int_{u^p}^{v^p}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k}{\Gamma(\varrho k+\lambda)}\frac{\left(c-u^p\right)^{\varrho k}\left(c-u^p\right)^{\lambda-1}}{\left(v^p-u^p\right)^{\varrho k}\left(v^p-u^p\right)^{\lambda-1}}\left(v^p-u^p\right)^{\varrho k}\varphi\left(c^{(1/p)}\right)\frac{dc}{v^p-u^p} \\
&=\frac{1}{\left(v^p-u^p\right)^{\lambda}}\int_{u^p}^{v^p}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k\left(v^p-c\right)^{\varrho k}}{\Gamma(\varrho k+\lambda)}\left(v^p-c\right)^{\lambda-1}\varphi\left(c^{(1/p)}\right)\mathrm{d}c \\
&\quad+\frac{1}{\left(v^p-u^p\right)^{\lambda}}\int_{u^p}^{v^p}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^k\left(c-u^p\right)^{\varrho k}}{\Gamma(\varrho k+\lambda)}\left(c-u^p\right)^{\lambda-1}\varphi\left(c^{(1/p)}\right)\mathrm{d}c \\
&=\frac{1}{\left(v^p-u^p\right)^{\lambda}}\left(J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)\left(v^p\right) \\
&\quad+\frac{1}{\left(v^p-u^p\right)^{\lambda}}\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)\left(u^p\right)
\end{aligned}\tag{23}$$

$$2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^{p}\right]\varphi\left(\frac{1}{2}u^p+\frac{1}{2}v^p\right)^{(1/p)}\leq\frac{1}{\left(v^p-u^p\right)^{\lambda}}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)\left(v^p\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)\left(u^p\right)\right],$$

which is left side of inequality equation (23). Now we have to prove right-hand side of inequality equation (22); applying definition of MT-non-convexity of φ,

$$\varphi\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)}\leq\frac{\sqrt{\eta}}{2\sqrt{1-\eta}}\varphi(u)+\frac{\sqrt{1-\eta}}{2\sqrt{\eta}}\varphi(v),\tag{24}$$

and

$$\varphi\left((1-\eta)u^p+\eta v^p\right)^{(1/p)}\leq\frac{\sqrt{\eta}}{2\sqrt{1-\eta}}\varphi(u)+\frac{\sqrt{1-\eta}}{2\sqrt{\eta}}\varphi(v),\tag{25}$$

and by adding equations (24) and (25), we get

$$\varphi\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)}+\varphi\left((1-\eta)u^{p}+\eta v^{p}\right)^{(1/p)} \leq \frac{1}{2\sqrt{\eta}\sqrt{1-\eta}}\varphi(u)+\frac{1}{2\sqrt{\eta}\sqrt{1-\eta}}\varphi(v). \tag{26}$$

Multiply inequality equation (26) by $\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}[\omega(v^{p}-u^{p})^{\varrho}\eta^{\varrho}]$, and after that, integrating it over $\eta\in[0,1]$, then we get

$$\begin{aligned}&\frac{1}{\left(v^{p}-u^{p}\right)^{\lambda}}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(v^{p}\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right)\right]\\&\leq 2F^{\sigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\rho}\right]\frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi(u)+\frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi(v).\end{aligned} \tag{27}$$

Combining equations (23) and (27) completes equation (20). □

Remark 3. In Theorem 2, we see the following:

(1) For $p=1$, we have inequality for MT-convex function:

$$\begin{aligned}&\varphi\left(\frac{1}{2}u+\frac{1}{2}v\right),\\&\leq\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega(v-u)^{\varrho}(v-u)^{\lambda}\right]}\left[(v)J^{\varsigma}_{\varrho,\lambda,u+;\omega,}(\varphi o\phi)+(u)J^{\varsigma}_{\varrho,\lambda,v-;\omega,}(\varphi o\phi)\right],\\&\leq\frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi(u)+\frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi(v).\end{aligned} \tag{28}$$

(2) For $p=-1$, we have the inequality of harmonically MT-convex function:

$$\begin{aligned}&\varphi\left(\frac{u+v}{2uv}\right),\\&\leq\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega(u-v/uv)^{\varrho}(u-v/uv)^{\lambda}\right]}\left[\left(\frac{1}{v}\right)J^{\varsigma}_{\varrho,\lambda,(1/v)+;\omega,}(\varphi o\phi)+\left(\frac{1}{u}\right)J^{\varsigma}_{\varrho,\lambda,(1/v)-;\omega,}(\varphi o\phi)\right],\\&\leq\frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi(u)+\frac{1}{4\sqrt{\eta}\sqrt{1-\eta}}\varphi(v).\end{aligned} \tag{29}$$

3. Fractional Integral Inequalities for (p-q) Convex Functions

In this section, we will develop fractional integral inequality for (p-q) convex function.

Theorem 3. *Let $\lambda, \in, \mathbb{R}^+$, φ: $I, \subseteq \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, be a differentiable mapping on I^o $u, v \in I^o$ such that $x < y$. If $|\varphi'|$ is (p-q) convex on $[u, v]$, $p > 0$, then we obtain*

$$\begin{aligned}
&\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^{\varrho}\right]\left(v^p-u^p\right)^{\lambda}}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)\left(v^p\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)\left(u^p\right)\right],\\
&\le\left(F^{C_1}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^{\varrho}\right]+F^{C_3}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^{\varrho}\right]\right)\left|\varphi'(u)\right|,\\
&\quad+\left(F^{C_2}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^{\varrho}\right]+F^{C_4}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^{\varrho}\right]\right)\left|\varphi'(v)\right|,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
C_1 &= \varsigma(k)v^{1-p}\frac{\varrho k+\lambda}{((1/q)+1)(\varrho k+\lambda+(1/q)+1)}{}_2F_1\left(1-\frac{1}{p},\frac{1}{q}+1,\varrho k+\lambda+1,1-\frac{u^p}{v^p}\right)\\
C_2 &= \varsigma(k)v^{1-p}\frac{\varrho k+\lambda+(1/q)}{\varrho k+\lambda+(1/q)+1}{}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{1}{q}+1,\varrho k+\lambda+\frac{1}{q}+1,1-\frac{u^p}{v^p}\right)\\
C_3 &= \varsigma(k)v^{1-p}\frac{1}{\varrho k+\lambda+(1/q)+1}{}_2F_1\left(1-\frac{1}{p},\varrho k+\lambda+\frac{1}{q}+1,1,1-\frac{u^p}{v^p}\right)\\
C_4 &= \varsigma(k)v^{1-p}\frac{(1/q)}{(\varrho k+\lambda+1)(\varrho k+\lambda+(1/q)+1)}{}_2F_1\left(1-\frac{1}{p},1,\frac{1}{q}-1,1-\frac{u^p}{v^p}\right).
\end{aligned} \tag{31}$$

Proof. By making use of Lemma 1 and (p-q)-convexity of $|\varphi'|$, we obtain

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2\left(v^p-u^p\right)^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^{\varrho}\right]}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)\left(v^p\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)\left(u^p\right)\right]\right|\\
&=\left|\frac{v^p-u^p}{2pF^{\varsigma}_{\varrho,\lambda+1}\omega\left(y^p-u^p\right)^{\varrho}}\int_0^1 \begin{array}{c}(1-\eta)^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^{\varrho}(1-\eta)^{\varrho}\right]\\ \times\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\varphi'\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)}\right)d\eta\end{array}\right|\\
&\quad+\int_0^1 \begin{array}{c}\eta^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^{\varrho}\eta^{\varrho}\right]\\ \times\left(\eta u^p+1-\eta v^p\right)^{(1/p)-1}\varphi'\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)}\right)d\eta]\end{array}\\
&\le\frac{v^p-u^p}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^{\varrho}\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^k\left(v^p-u^p\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)}\\
&\quad\times\int_0^1|\left[(1-\eta)^{\varrho k+\lambda}+\eta^{\varrho k+\lambda}\right]\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\left(\eta\left|\varphi'(u)^q\right|+(1-\eta)\left|\varphi'(v)^q\right|\right)^{(1/q)}d\eta.
\end{aligned} \tag{32}$$

Moreover, we observe that

$$\begin{aligned}
&\leq \frac{v^p-u^p}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^{\varrho}\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^k\left(v^p-u^p\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)}\\
&\times\left[\int_0^1(1-\eta)^{\varrho k+\lambda}\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\left(\eta\left|\varphi'(u)^q\right|+(1-\eta)\left|\varphi'(v)^q\right|\right)^{(1/q)}\mathrm{d}\eta\right.\\
&\left.+\int_0^1\eta^{\varrho k+\lambda}\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\left(\eta\left|\varphi'(u)^q\right|+(1-\eta)\left|\varphi'(v)^q\right|\right)^{(1/q)}\mathrm{d}\eta\right],
\end{aligned} \tag{33}$$

so that

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2\left(v^p-u^p\right)^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^p-u^p\right)^{\rho}\right]}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)\left(v^p\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)\left(u^p\right)\right]\right|\\
&\leq \frac{v^p-u^p}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|\left(v^p-u^p\right)^{\varrho}\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^k\left(v^p-u^p\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)}\\
&\times\left[\int_0^1\eta^{(1/q)}(1-\eta)^{\varrho k+\lambda}\left|\varphi'(u)^q\right|^{(1/q)}\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\mathrm{d}\eta\right.\\
&+\int_0^1(1-\eta)^{\varrho k+\lambda+(1/q)}\left|\varphi'(v)^q\right|^{(1/q)}\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\mathrm{d}\eta\\
&+\int_0^1\eta^{\varrho k+\lambda+(1/q)}\left|\varphi'(u)^q\right|^{(1/q)}\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\mathrm{d}\eta\\
&\left.+\int_0^1\eta^{\varrho k+\lambda}(1-\eta)^{(1/q)}\left|\varphi'(v)^q\right|^{(1/q)}\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)-1}\mathrm{d}\eta\right].
\end{aligned} \tag{34}$$

Now

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2\left(v^{p}-u^{p}\right)^{\lambda} F_{\varrho,\lambda+1}^{\varsigma}\left[\omega\left(v^{p}-u^{p}\right)^{\rho}\right]}\left[\left(J_{\varrho,\lambda,u^{p}+;\omega}^{\varsigma}\varphi o \phi\right)\left(v^{p}\right)+\left(J_{\varrho,\lambda,v^{p}-;\omega}^{\varsigma}\varphi o \phi\right)\left(u^{p}\right)\right]\right| \\
&\quad\leq \frac{v^{p}-u^{p}}{2pF_{\varrho,\lambda+1}^{\varsigma}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]|} \sum_{k=0}^{\infty} \frac{\varsigma(k)|\omega|^{k}\left(v^{p}-u^{p}\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)} \\
&\quad\times\left[v^{1-p}\beta\left(\frac{1}{q}+1, \varrho k+\lambda+1\right){}_{2}F_{1}\left(1-\frac{1}{p}, \frac{1}{q}+1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{u^{p}}{v^{p}}\right)\left|\varphi'(u)\right|\right. \\
&\quad+v^{1-p}\beta\left(1, \varrho k+\lambda+\frac{1}{q}+1\right){}_{2}F_{1}\left(1-\frac{1}{p}, 1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{u^{p}}{v^{p}}\right)\left|\varphi'(v)\right| \\
&\quad+v^{1-p}\beta\left(\varrho k+\lambda+\frac{1}{q}+1,1\right){}_{2}F_{1}\left(1-\frac{1}{p}, \varrho k+\lambda+\frac{1}{q}+1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{1}{p}\right)\left|\varphi'(u)\right| \\
&\quad\left.+v^{1-p}\beta\left(\varrho k+\lambda+1, \frac{1}{q}+1\right){}_{2}F_{1}\left(1-\frac{1}{p}, \varrho k+\lambda+1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{u^{p}}{v^{p}}\right)\left|\varphi'(v)\right|\right].
\end{aligned} \tag{35}$$

From here,

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2\left(v^{p}-u^{p}\right)^{\lambda} F_{\varrho,\lambda+1}^{\varsigma}\left[\omega\left(v^{p}-u^{p}\right)^{\rho}\right]}\left[\left(J_{\varrho,\lambda,u^{p}+;\omega}^{\varsigma}\varphi o \phi\right)\left(v^{p}\right)+\left(J_{\varrho,\lambda,v^{p}-;\omega}^{\varsigma}\varphi o \phi\right)\left(u^{p}\right)\right]\right| \\
&\quad\leq \frac{v^{p}-u^{p}}{2pF_{\varrho,\lambda+1}^{\varsigma}\left[|\omega|\left(v^{p}-u^{p}\right)^{\varrho}\right]|} \sum_{k=0}^{\infty} \frac{\varsigma(k)|\omega|^{k}\left(v^{p}-u^{p}\right)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)} \\
&\quad\times\left[v^{1-p}\left(\frac{\varrho k+\lambda}{((1/q)+1)(\varrho k+\lambda+(1/q)+1)}\right){}_{2}F_{1}\left(1-\frac{1}{p}, \frac{1}{q}+1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{u^{p}}{v^{p}}\right)\left|\varphi'(u)\right|\right. \\
&\quad+v^{1-p}\left(\frac{\varrho k+\lambda+(1/q)}{\varrho k+\lambda+(1/q)+1}\right){}_{2}F_{1}\left(1-\frac{1}{p}, 1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{u^{p}}{v^{p}}\right)\left|\varphi'(v)\right| \\
&\quad+v^{1-p}\left(\frac{1}{\varrho k+\lambda+(1/q)+1}\right){}_{2}F_{1}\left(1-\frac{1}{p}, \varrho k+\lambda+\frac{1}{q}+1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{u^{p}}{v^{p}}\right)\left|\varphi'(u)\right| \\
&\quad\left.+v^{1-p}\left(\frac{(1/q)}{(\varrho k+\lambda+1)(\varrho k+\lambda+(1/q)+1)}\right){}_{2}F_{1}\left(1-\frac{1}{p}, \varrho k+\lambda+1, \varrho k+\lambda+\frac{1}{q}+2,1-\frac{u^{p}}{v^{p}}\right)\left|\varphi'(v)\right|\right],
\end{aligned} \tag{36}$$

and

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{2(v^p-u^p)^{\lambda}F^{\varsigma}_{\varrho,\lambda+1}\left[\omega(v^p-u^p)^{\varrho}\right]}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi\right)(v^p)+\left(J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi\right)(u^p)\right]\right|$$
$$\leq\frac{v^p-u^p}{2pF^{\varsigma}_{\varrho,\lambda+1}\left[|\omega|(v^p-u^p)^{\varrho}\right]|}\sum_{k=0}^{\infty}\frac{\varsigma(k)|\omega|^k(v^p-u^p)^{\varrho k}}{\Gamma(\varrho k+\lambda+1)}\left(C_1|\varphi'(u)|+C_2|\varphi'(v)|+C_3|\varphi'(u)|+C_4|\varphi'(v)|\right), \tag{37}$$

which is the required solution. □

Remark 4

(1) If one puts $q=1$ in equation (30), one has Theorem 5 in [22].

(2) Similarly, for $q=1$ and $p=1$ in equation (30), we get classical fractional integral of Hermite–Hadamard inequality.

Theorem 4. *Let $\lambda\in\in\mathbb{R}^+$, $\varphi: I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$, be a (p-q) convex function on u, v, $\in\in I$ with $u<v$; if $f\in L[u,v]$, $p>0$, then we obtain*

$$\varphi\left(\frac{1}{2}u^p+\frac{1}{2}v^p\right)^{(1/p)}\leq\frac{1}{2F^{\varsigma}_{\varrho,\lambda+1\left[\omega(v^p-u^p)^{\varrho}(v^p-u^p)^{\lambda}\right]}}\left[J^{\varsigma}_{\varrho,\lambda,u^p+;\omega}\varphi o\phi(v^p)+J^{\varsigma}_{\varrho,\lambda,v^p-;\omega}\varphi o\phi(u^p)\right]$$
$$\leq\frac{\left(\varphi(u)^q\right)^{(1/q)}+\left(\varphi(v)^q\right)^{(1/q)}}{2}, \tag{38}$$

where $g(c)=c^{(1/p)}$.

Proof. Since φ is $(p\text{-}q)$ convex on $[u,v]$, for all $c,d\in[u,v]$,

$$\varphi\left(\frac{1}{2}c^p+\frac{1}{2}d^p\right)^{(1/p)}\leq\frac{1}{2}\left[\varphi(c)^q+\varphi(d)^q\right]^{(1/q)}, \tag{39}$$

and substituting $c=(\eta u^p+(1-\eta)v^p)^{(1/p)}$ and $d=((1-\eta)u^p+\eta v^p)^{(1/p)}$, then equation (39) yields

$$2\varphi\left(\frac{1}{2}u^p+\frac{1}{2}v^p\right)^{(1/p)}\leq\varphi\left[\left(\left(\eta u^p+(1-\eta)v^p\right)^{(1/p)}\right)^q\right]^{(1/q)}+\left[\varphi\left(\left((1-\eta)u^p+\eta v^p\right)^{(1/p)}\right)^q\right]^{(1/q)}. \tag{40}$$

Multiply inequality equation (40) by $\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}[\omega(v^p-u^p)^{\varrho}\eta^{\varrho}]$, and after that, integrating over $\eta\in[0,1]$, we get

$$
\begin{aligned}
&2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(\nu^{p}-u^{p}\right)^{\varrho}\eta^{\varrho}\right]\varphi\Big(\frac{1}{2}u^{p}+\frac{1}{2}\nu^{p}\Big)^{(1/p)}\\
&\leq \int_{0}^{1}\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}\left[\omega\left(\nu^{p}-u^{p}\right)^{\varrho}\eta^{\varrho}\right]\Big[\varphi\Big(\left(\eta u^{p}+(1-\eta)\nu^{p}\right)^{(1/p)}\Big)^{q}\Big]^{(1/q)}\mathrm{d}\eta\\
&\quad+\int_{0}^{1}\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}\left[\omega\left(\nu^{p}-u^{p}\right)^{\varrho}\eta^{\varrho}\right]\Big[\varphi\left((1-\eta)u^{p}+\eta\nu^{p}\right)^{(1/p)})^{q}\Big]^{(1/q)}\mathrm{d}\eta\\
&=\int_{0}^{1}\eta^{\lambda-1}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}\left(\nu^{p}-u^{p}\right)^{\varrho k}\eta^{\varrho k}}{\Gamma(\varrho k+\lambda)}\Big[\varphi\Big(\left(\eta u^{p}+(1-\eta)\nu^{p}\right)^{(1/p)}\Big)^{q}\Big]^{(1/q)}\mathrm{d}\eta\\
&\quad+\int_{0}^{1}\eta^{\lambda-1}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}\left(\nu^{p}-u^{p}\right)^{\varrho k}\eta^{\varrho k}}{\Gamma(\varrho k+\lambda)}\Big[\varphi\Big(\left((1-\eta)u^{p}+\eta\nu^{p}\right)^{(1/p)}\Big)^{q}\Big]^{(1/q)}\mathrm{d}\eta\\
&=\int_{0}^{1}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}}{\Gamma(\varrho k+\lambda)}\eta^{\varrho k+\lambda-1}\Big[\varphi\Big(\left(\eta u^{p}+(1-\eta)\nu^{p}\right)^{(1/p)}\Big)^{q}\Big]^{(1/q)}\left(\nu^{p}-u^{p}\right)^{\varrho k}\mathrm{d}\eta\\
&\quad+\int_{0}^{1}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}}{\Gamma(\varrho k+\lambda)}\eta^{\varrho k+\lambda-1}\Big[\varphi\Big(\left((1-\eta)u^{p}+\eta\nu^{p}\right)^{(1/p)}\Big)^{q}\Big]^{(1/q)}\left(\nu^{p}-u^{p}\right)^{\rho k}\mathrm{d}\eta\\
&=\int_{u^{p}}^{\nu^{p}}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}}{\Gamma(\varrho k+\lambda)}\frac{\left(\nu^{p}-c\right)^{\varrho k}\left(\nu^{p}-c\right)^{\lambda-1}}{\left(\nu^{p}-u^{p}\right)^{\varrho k}\left(\nu^{p}-u^{p}\right)^{\lambda-1}}\left(\nu^{p}-u^{p}\right)^{\varrho k}\Big[\varphi\big(c^{(1/p)}\big)^{q}\Big]^{(1/q)}\frac{dc}{\nu^{p}-u^{p}}\\
&\quad+\int_{u^{p}}^{\nu^{p}}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}}{\Gamma(\varrho k+\lambda)}\frac{\left(c-u^{p}\right)^{\varrho k}\left(c-u^{p}\right)^{\lambda-1}}{\left(\nu^{p}-u^{p}\right)^{\varrho k}\left(\nu^{p}-u^{p}\right)^{\lambda-1}}\left(\nu^{p}-u^{p}\right)^{\varrho k}\Big[\varphi\big(c^{(1/p)}\big)^{q}\Big]^{(1/q)}\frac{dc}{\nu^{p}-u^{p}}\\
&=\frac{1}{\left(\nu^{p}-u^{p}\right)^{\lambda}}\int_{u^{p}}^{\nu^{p}}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}\left(\nu^{p}-c\right)^{\varrho k}}{\Gamma(\varrho k+\lambda)}\left(\nu^{p}-c\right)^{\lambda-1}\varphi\Big[\big(c^{(1/p)}\big)^{q}\Big]^{(1/q)}\mathrm{d}c\\
&\quad+\frac{1}{\left(\nu^{p}-u^{p}\right)^{\lambda}}\int_{u^{p}}^{\nu^{p}}\sum_{k=0}^{\infty}\frac{\varsigma(k)\omega^{k}\left(c-u^{p}\right)^{\varrho k}}{\Gamma(\varrho k+\lambda)}\left(c-u^{p}\right)^{\lambda-1}\Big[\varphi\big(c^{(1/p)}\big)^{q}\Big]^{(1/q)}\mathrm{d}c\\
&=\frac{1}{\left(\nu^{p}-u^{p}\right)^{\lambda}}\left(J^{\varsigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(\nu^{p}\right)\\
&\quad+\frac{1}{\left(\nu^{p}-u^{p}\right)^{\lambda}}\left(J^{\varsigma}_{\varrho,\lambda,\nu^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right).
\end{aligned}
\tag{41}
$$

So, we have left-hand side of inequality equation (38).

$$
2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(\nu^{p}-u^{p}\right)^{p}\varphi\Big(\frac{1}{2}u^{p}+\frac{1}{2}\nu^{p}\Big)^{(1/p)}\leq\frac{1}{\left(\nu^{p}-u^{p}\right)^{\lambda}}\left(J^{\varsigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(\nu^{p}\right)+\left(J^{\varsigma}_{\varrho,\lambda,\nu^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right)\right]. \tag{42}
$$

Now we have to prove other side of equation (40) from pq-convexity of φ.

$$
\varphi\left(\eta u^{p}+(1-\eta)\nu^{p}\right)^{(1/p)}\leq\left[\eta\varphi(u)^{q}+(1-\eta)\varphi(\nu)^{q}\right]^{(1/q)}, \tag{43}
$$

and

$$
\varphi\left((1-\eta)u^{p}+\eta\nu^{p}\right)^{(1/p)}\leq\left[(1-\eta)\varphi(u)^{q}+\eta\varphi(\nu)^{q}\right]^{(1/q)}, \tag{44}
$$

and by adding inequality equations (43) and (44), we get

$$\begin{aligned}&\varphi\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)}+\varphi\left((1-\eta)u^{p}+\eta v^{p}\right)^{(1/p)}\leq\left[\eta\varphi(u)^{q}+(1-\eta)\varphi(v)^{q}\right]^{(1/q)}+\left[(1-\eta)\varphi(u)^{q}+\eta\varphi(v)^{q}\right]^{(1/q)}\\&\varphi\left(\eta u^{p}+(1-\eta)v^{p}\right)^{(1/p)}+\varphi\left((1-\eta)u^{p}+\eta v^{p}\right)^{(1/p)}\leq\left[\varphi(u)^{q}\right]^{(1/q)}+\left[\varphi(v)^{q}\right]^{(1/q)}.\end{aligned}\tag{45}$$

Multiply inequality equation (45) by $\eta^{\lambda-1}F^{\varsigma}_{\varrho,\lambda}[\omega(v^{p}-u^{p})^{\varrho}\eta^{\varrho}]$, and after that, integrating inequality over $\eta\in[0,1]$, we get

$$\frac{1}{\left(v^{p}-u^{p}\right)^{\lambda}}\left[\left(J^{\varsigma}_{\varrho,\lambda,u^{p}+;\omega}\varphi o\phi\right)\left(v^{p}\right)+\left(J^{\varsigma}_{\varrho,\lambda,v^{p}-;\omega}\varphi o\phi\right)\left(u^{p}\right)\right]\leq 2F^{\varsigma}_{\varrho,\lambda+1}\left[\omega\left(v^{p}-u^{p}\right)^{\varrho}\right]\frac{\left[\varphi(u)^{q}\right]^{(1/q)}+\left[\varphi(v)^{q}\right]^{(1/q)}}{2}.\tag{46}$$

Combining equations (42) and (45) completes equation (38). □

Remark 5

(1) If one puts $q=1$ in equation (39), one has [22, Theorem 5].

(2) Similarly, for $q=1$ and $p=1$ in equation (39), we get classical fractional integral of H-H inequality.

4. Conclusion

Fractional integral inequalities are derived for MT-non-convex functions and $(p-q)$ convex functions. With the help of several lemmas, the integral inequalities are derived in generalized fractional integral operator. The remarks at the end are also given to verify the extension of results.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

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Research Article

Investigating a Class of Generalized Caputo-Type Fractional Integro-Differential Equations

Saeed M. Ali,[1] **Wasfi Shatanawi**,[2,3,4] **Mohammed D. Kassim**,[1] **Mohammed S. Abdo**,[5] **and S. Saleh**[5,6]

[1]*Department of Basic Engineering Sciences, College of Engineering, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam, Saudi Arabia*
[2]*Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia*
[3]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[4]*Department of Mathematics, Hashemite University, Zarqa, Jordan*
[5]*Department of Mathematics, Hodeidah University, Al-Hudaydah, Yemen*
[6]*Department of Computer Science, Cihan University-Erbil, Kurdistan Region, Iraq*

Correspondence should be addressed to Wasfi Shatanawi; wshatanawi@psu.edu.sa
and Mohammed S. Abdo; msabdo1977@gmail.com

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In this article, we prove some new uniqueness and Ulam-Hyers stability results of a nonlinear generalized fractional integro-differential equation in the frame of Caputo derivative involving a new kernel in terms of another function ψ. Our approach is based on Babenko's technique, Banach's fixed point theorem, and Banach's space of absolutely continuous functions. The obtained results are demonstrated by constructing numerical examples.

1. Introduction

It is notable that fractional calculus was and still is a new tool that uses fractional differential and integral equations to construct more modern mathematical models that can precisely describe complex frameworks. There are many definitions of fractional integrals (FIs) and fractional derivatives (FDs) accessible in the literature, for instance, the Riemann-Liouville and Caputo definitions that assumed a significant part in the advancement of the theory of fractional analysis. Referring to all books and papers in this field will be extremely many. In this regard, here, we refer to the most important of main references, e.g., Samko et al. [1] gave a broad comprehensive mathematical handling of fractional derivatives and integrals. Podlubny [2] and Kilbas et al. [3] have been introduced many useful results related to fractional differential equations (FDEs). Several applications have been implemented recently by a wide range of works on this subject, see [4–8].

However, the currently common operator is the generalized FD regarding another function, see [1, 3]. Agrawal [7] studied further various properties for generalized fractional derivatives and integrals. More recently, Almeida [9] inspired an idea of that generalization by projecting this generalization onto the definition of the Caputo fractional derivative with respect to another function, so-called ψ-Caputo, and introduced many interesting properties, which are more general than the classical Caputo FD. Jarad and Abdeljawad [10] provided interesting properties for generalized FDs and Laplace transform. Specifically, ψ-Caputo type FDEs with initial, boundary, and nonlocal conditions have been investigated by many researchers using fixed-point theories, see Almeida et al. [11, 12], Abdo et al. [13], and Wahash et al. [14]. A recent survey on ψ-Caputo type FDEs can be found in [15–18]. For more results in this direction, we refer to interesting works provided by Zhang et al. [19], Zhao et al. [20], Baitiche et al. [21], Benchohra et al. [22], Ravichandran et al. [23], Trujillo et al. [24], and Furati et al. [25].

Li in [17, 18] investigated some interesting results of the integral equations and the integro-differential equations involving Hadamard-type. In this work, our goal is to intend to address a general extension of these studies. Precisely, we consider the following ψ-Caputo type fractional integro-differential equation (FIDE)

$$\begin{cases} {}^C\mathbb{D}_a^{\psi,\varrho}\varphi(x) + a_1\,{}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi(x) + a_2\, I_a^{\psi,\varrho_2}\varphi(x) = \int_a^x \mathbb{G}\left(\zeta, \varphi'(\zeta)\right)d\zeta, \\ \varphi(a) = 0, \end{cases} \tag{1}$$

where

(i) $0 < \varrho_1 < \varrho < 1, \varrho_2 > 0$, and $a_1, a_2 \in \mathbb{C}$

(ii) The symbol ${}^C\mathbb{D}_a^{\psi,\sigma}$ denotes the generalized Caputo FD of order $\sigma \in \{\varrho, \varrho_1\}$

(iii) The notation I_a^{ψ,ϱ_2} means the generalized Riemann-Liouville FI of order ϱ_2

(iv) $\mathbb{G} : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and $0 \le a < b < \infty$

(v) $\varphi \in AC_0[a, b]$ such that I_a^{ψ,ϱ_2} and $\mathbb{D}_a^{\psi,\sigma}$ exist and are both continuous in $[a, b]$

Observe that the considered system (1) covers the previous standard cases of nonlinear FIDEs by defining the kernel, i.e., if $\psi(x) = x$, $\psi(x) = \log(x)$, and $\psi(x) = x^\rho$, then, the problem (1) reduces to the Caputo type FIDE, Caputo-Hadamard type FIDE, and Caputo-Katugampola type FIDE, respectively.

The aim of this work is to develop the nonlinear FIDEs. In particular, we investigate the uniqueness and Ulam-Hyers stability of solution for the problem (1) by Banach's fixed point theorem and Babenko's technique [26]. Note that the presentation and structuring of the arguments for our problem are new, and our results generalize and cover some of the known results in the literature. In addition, the obtained results here are valid when the left hand side of the considered problem (1) involves many FDs and FIs. For more details, see Remark 22.

The remainder of this paper is organized as follows: in Section 2, we present some important tools related the fractional calculus and the functional spaces, in which we aim to determine our analysis strategies. Section 3 gives the main results and their illustrative examples. Finally, our brief conclusion is included in Section 4.

2. Preliminaries

In this section, we present some properties, lemmas, definitions, and important estimations needed in the proof of our result.

Defining the Banach space as

$$AC_0[a, b] = \left\{\varphi : \varphi \in AC[a, b] \text{ with } \varphi(a) = 0 \text{ and } \|\varphi\|_0 = \int_a^b \left|\varphi'(\zeta)\right| d\zeta < \infty\right\}. \tag{2}$$

Next, we present some important definitions and properties of advanced fractional calculus.

Definition 1 [3, 9]. The ψ-Riemann-Liouville FI and ψ-Caputo FD are defined by

$$\begin{aligned} I_a^{\psi,\varrho}\omega(x) &= \frac{1}{\Gamma(\varrho)}\int_a^x (\psi(x) - \psi(\zeta))^{\varrho-1}\psi'(\zeta)\omega(\zeta)d\zeta, \varrho > 0, \\ {}^C\mathbb{D}_a^{\psi,\varrho}\omega(x) &= I_a^{\psi,n-\varrho}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\omega(x) \\ &= \frac{1}{\Gamma(n-\varrho)}\int_a^x (\psi(x) - \psi(\zeta))^{n-\varrho-1}\psi' \\ &\quad \cdot (\zeta)\omega_\psi^{[n]}(\zeta)d\zeta, \varrho > 0, \end{aligned} \tag{3}$$

respectively, where

$$n = -[-\varrho], \omega_\psi^{[n]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \omega(x). \tag{4}$$

Definition 2 [27]. The incomplete gamma function is represented by

$$\gamma(\varrho, \zeta) = \int_0^\zeta u^{\varrho-1}e^{-u}du = \zeta^\varrho\Gamma(\varrho)e^{-\zeta}\sum_{i=0}^\infty \frac{\zeta^i}{\Gamma(\varrho+i+1)}, \varrho > 0, \zeta \ge 0. \tag{5}$$

Property 3 [3, 9]. Let $\varrho \ge 0$, and $\kappa > 0$. Then

$$I_a^{\psi,\varrho}(\psi(x) - \psi(a))^\kappa = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\varrho+1)}(\psi(x) - \psi(a))^{\kappa+\varrho}, x > a, \tag{6}$$

$${}^C\mathbb{D}_a^{\psi,\varrho}(\psi(x) - \psi(a))^\kappa = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\varrho+1)}(\psi(x) - \psi(a))^{\kappa-\varrho}, x > a. \tag{7}$$

Property 4 [3, 9]. Let $\varrho, \kappa > 0$, and $\omega \in AC_0[a, b]$. Then

$${}^C\mathbb{D}_a^{\psi,\varrho}I_a^{\psi,\varrho}\omega(\zeta) = \omega(\zeta), \zeta > a, \tag{8}$$

$$I_a^{\psi,\varrho}I_a^{\psi,\kappa}\omega(\zeta) = I_a^{\psi,\varrho+\kappa}\omega(\zeta), \zeta > a, \tag{9}$$

$${}^C\mathbb{D}_a^{\psi,\kappa}I_a^{\psi,\varrho}\omega(\zeta) = I_a^{\psi,\varrho-\kappa}\omega(\zeta), \varrho > \kappa, \zeta > a, \tag{10}$$

$$I_a^{\psi,\varrho}\omega(a) = 0. \tag{11}$$

In the following, some very significant lemmas will be given.

Lemma 5. *Let $\varrho, \kappa \in [0, 1]$. If $\omega \in AC_0[a, b]$, then*

$$I_a^{\psi,\varrho}\,{}^C\mathbb{D}_a^{\psi,\varrho}\omega(\zeta) = \omega(\zeta), \zeta > a, \tag{12}$$

$$I_a^{\psi,\varrho}\,{}^C\mathbb{D}_a^{\psi,\kappa}\omega(\zeta) = I_a^{\psi,\varrho-\kappa}\omega(\zeta), \varrho > \kappa, \zeta > a. \tag{13}$$

Proof. Let $\omega \in AC_0[a,b]$. Then by Definition 1 and Property 4, we get

$$I_a^{\psi,\varrho\, C}\mathbb{D}_a^{\psi,\varrho}\omega(\zeta) = I_a^{\psi,\varrho} I_a^{\psi,1-\varrho}\omega_\psi^{[1]}(\zeta) = I_a^{\psi,1}\omega_\psi^{[1]}(\zeta) = \omega(\zeta) - \omega(a) = \omega(\zeta), \zeta > a, \tag{14}$$

$$I_a^{\psi,\varrho\, C}\mathbb{D}_a^{\psi,\kappa}\omega(\zeta) = I_a^{\psi,\varrho} I_a^{\psi,1-\kappa}\omega_\psi^{[1]}(\zeta) = I_a^{\psi,\varrho-\kappa} I_a^{\psi,1}\omega_\psi^{[1]}(\zeta) = I_a^{\psi,\varrho-\kappa}(\omega(\zeta) - \omega(a)) = I_a^{\psi,\varrho-\kappa}\omega(\zeta), \varrho > \kappa, \zeta > a. \tag{15}$$

□

Lemma 7. *Let $\varrho > 0$. Then, $I_a^{\psi,\varrho}$ is bounded from $AC_0[a,b]$ into itself, and*

$$\|I_a^{\psi,\varrho}\omega\|_0 \le \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho \|\omega\|_0. \tag{16}$$

Proof. *Let $\omega \in AC_0[a,b]$. Then*

$$\omega(\zeta) = \int_a^\zeta \omega'(s)ds = \int_a^\zeta \mathfrak{z}(s)ds, \text{ where } \mathfrak{z}(\zeta) = \omega'(\zeta) \text{ and } \omega(a) = 0. \tag{17}$$

By virtue of Definition 1, we get

$$I_a^{\psi,\varrho}\omega(x) = I_a^{\psi,\varrho}\left(\int_a^\zeta \mathfrak{z}(s)ds\right)(x) = \frac{1}{\Gamma(\varrho)}\int_a^x (\psi(x) - \psi(\zeta))^{\varrho-1}\psi'(\zeta)\int_a^\zeta \mathfrak{z}(s)dsd\zeta. \tag{18}$$

Taking advantage of the Dirichlet's formula, we have

$$\begin{aligned} I_a^{\psi,\varrho}\omega(x) &= \frac{1}{\Gamma(\varrho)}\int_a^x \mathfrak{z}(s)\int_s^x (\psi(x) - \psi(\zeta))^{\varrho-1}\psi'(\zeta)d\zeta ds \\ &= \frac{1}{\Gamma(\varrho)}\int_a^x \mathfrak{z}(s)\left[-\frac{(\psi(x) - \psi(\zeta))^\varrho}{\varrho}\right]_{\zeta=s}^x ds \\ &= \frac{1}{\Gamma(\varrho)}\int_a^x \mathfrak{z}(s)\left[\frac{(\psi(x) - \psi(s))^\varrho}{\varrho}\right] ds \\ &\le \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho \int_a^x |\mathfrak{z}(s)|ds \\ &= \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho \int_a^x |\omega'(s)|ds \\ &= \frac{1}{\Gamma(\varrho+1)}(\psi(b) - \psi(a))^\varrho \|\omega\|_0. \end{aligned} \tag{19}$$

Now, we will provide and prove the next lemma: □

Lemma 9. *If $\varrho \ge 0$, then*

$$I_a^{\psi,\varrho}e^{\psi(x)} = e^{\psi(a)}(\psi(x) - \psi(a))^\varrho \sum_{i=0}^\infty \frac{(\psi(x) - \psi(a))^i}{\Gamma(\varrho+i+1)}. \tag{20}$$

Proof. Using Definition 1, we have

$$I_a^{\psi,\varrho}e^{\psi(x)} = \frac{1}{\Gamma(\varrho)}\int_a^x (\psi(x) - \psi(\zeta))^{\varrho-1}\psi'(\zeta)e^{\psi(\zeta)}d\zeta. \tag{21}$$

Performing the substitution $s = \psi(x) - \psi(\zeta)$, we get

$$\begin{aligned} I_a^{\psi,\varrho}e^{\psi(x)} &= \frac{1}{\Gamma(\varrho)}\int_0^{\psi(x)-\psi(a)} s^{\varrho-1}e^{\psi(x)-s}ds \\ &= \frac{e^{\psi(x)}}{\Gamma(\varrho)}\int_0^{\psi(x)-\psi(a)} s^{\varrho-1}e^{-s}ds. \end{aligned} \tag{22}$$

From Definition 2, we obtain

$$I_a^{\psi,\varrho}e^{\psi(x)} = \gamma(\varrho, \psi(x) - \psi(a))\frac{e^{\psi(x)}}{\Gamma(\varrho)} = e^{\psi(a)}(\psi(x) - \psi(a))^\varrho \sum_{i=0}^\infty \frac{(\psi(x) - \psi(a))^i}{\Gamma(\varrho+i+1)}. \tag{23}$$

□

3. Main Results

Theorem 11. *Let $a_i \in \mathbb{C}(i = 1, 2)$, $0 < \varrho_1 < \varrho < 1$, and $\varrho_2 > 0$. If $h \in AC_0[a,b]$, then, the following linear problem*

$$\begin{cases} {}^C\mathbb{D}_a^{\psi,\varrho}\varphi(x) + a_1\, {}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi(x) + a_2\, I_a^{\psi,\varrho_2}\varphi(x) = h(x), \\ \varphi(a) = 0, \end{cases} \tag{24}$$

has a solution in the space $AC_0[a,b]$, that is

$$\varphi(x) = \sum_{\ell=0}^\infty (-1)^\ell \sum_{\ell_1+\ell_2=\ell} \binom{\ell}{\ell_1, \ell_2} \times a_1^{\ell_1}a_2^{\ell_2}\, I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}h(x). \tag{25}$$

Proof. Applying the operator $I_a^{\psi,\varrho}$ to both sides of Eq. (24), we obtain

$$I_a^{\psi,\varrho\, C}\mathbb{D}_a^{\psi,\varrho}\varphi(x) + a_1 I_a^{\psi,\varrho\, C}\mathbb{D}_a^{\psi,\varrho_1}\varphi(x) + a_2 I_a^{\psi,\varrho} I_a^{\psi,\varrho_2}\varphi(x) = I_a^{\psi,\varrho}\mathrm{h}(x). \tag{26}$$

According to Lemma 5, we find that

$$\varphi(x) + a_1 I_a^{\psi,\varrho-\varrho_1}\varphi(x) + a_2 I_a^{\psi,\varrho+\varrho_2}\varphi(x) = I_a^{\psi,\varrho}\mathrm{h}(x). \tag{27}$$

Observe that $\varphi(a) = 0$ and $0 < \varrho_1 < \varrho < 1$. It follows that

$$(1 + a_1 I_a^{\psi,\varrho-\varrho_1} + a_2 I_a^{\psi,\varrho+\varrho_2})\varphi(x) = I_a^{\psi,\varrho}\mathrm{h}(x). \tag{28}$$

In view of Babenko approach, we have

$$\varphi(x) = (1 + a_1 I_a^{\psi,\varrho-\varrho_1} + a_2 I_a^{\psi,\varrho+\varrho_2})^{-1}\, I_a^{\psi,\varrho}\mathrm{h}(x). \tag{29}$$

By using the multinomial theorem and Property 4, we obtain

$$\begin{aligned}\varphi(x) &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\left(a_1 I_a^{\psi,\varrho-\varrho_1} + a_2 I_a^{\psi,\varrho+\varrho_2}\right)^{\ell} I_a^{\psi,\varrho}\mathrm{h}(x)\\ &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\left(a_1 I_a^{\psi,\varrho-\varrho_1}\right)^{\ell_1}\left(a_2 I_a^{\psi,\varrho+\varrho_2}\right)^{\ell_2} I_a^{\psi,\varrho}\mathrm{h}(x)\\ &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2} a_1^{\ell_1}a_2^{\ell_2} I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x).\end{aligned} \tag{30}$$

As $x \longrightarrow a$, we get $\varphi(a)=0$. Now, we need to prove the series is absolutely continuous on $[a,b]$ and converges in the space $AC_0[a,b]$. Indeed, by Lemma 7, we have

$$\left\| I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right\|_0 \le \eta\|\mathrm{h}\|_0, \tag{31}$$

where

$$\eta = \frac{(\psi(b)-\psi(a))^{\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}}{\Gamma(\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho+1)}. \tag{32}$$

It follows that

$$\begin{aligned}\|\varphi\|_0 &\le \eta\sum_{\ell=0}^{\infty}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\left|a_1^{\ell_1}\right|\left|a_2^{\ell_2}\right|\frac{(\psi(b)-\psi(a))^{\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}}{\Gamma(\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho+1)}\|\mathrm{h}\|_0\\ &= \eta\sum_{\ell=0}^{\infty}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\\ &\quad\times\frac{\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}\right)^{\ell_1}\left(|a_2|(\psi(b)-\psi(a))^{\varrho+\varrho_2}\right)^{\ell_2}}{\Gamma(\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho+1)}\|\mathrm{h}\|_0\\ &= \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}, |a_2|(\psi(b)\right.\\ &\qquad\left.-\psi(a))^{\varrho+\varrho_2}\right)\|\mathrm{h}\|_0,\end{aligned} \tag{33}$$

where

$$E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}, |a_2|(\psi(b)-\psi(a))^{\varrho+\varrho_2}\right) < \infty, \tag{34}$$

which is the value at $\nu_1 = |a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}$, $\nu_2 = |a_2|(\psi(b)-\psi(a))^{\varrho+\varrho_2}$ of the multivariate Mittag-Leffler function $E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}(\nu_1,\nu_2)$ given in [3]. So, we conclude that the series to the right of Eq. (25) is convergent. Obviously, $\varphi(x)\in AC[a,b]$ due to $\mathrm{h}\in AC[a,b]$. To affirm that the obtained series could be a solution, we must see that it fulfills Eq. (24), i.e.,

$$\begin{aligned}&{}^{C}\mathbb{D}_a^{\psi,\varrho}\left(\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right)\\ &+a_1\,{}^{C}\mathbb{D}_a^{\psi,\varrho_1}\left(\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right)\\ &+a_2 I_a^{\psi,\varrho_2}\left(\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\mathrm{h}(x)\right)\\ &={}^{C}\mathbb{D}_a^{\psi,\varrho}\left(I_a^{\psi,\rho}\mathrm{h}(x)+\sum_{\ell=1}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\right.\\ &\left.\times a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\rho-\rho_1)+\ell_2(\rho+\rho_2)+\rho}\mathrm{h}(x)\right)\\ &+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1+1}a_2^{\ell_2}I_a^{\psi,(\ell_1+1)(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_0^{\ell_2+1}I_a^{\psi,\ell_1(\varrho-\varrho_1)+(\ell_2+1)(\varrho+\varrho_2)}\mathrm{h}(x)\\ &=\mathrm{h}(x)+\sum_{\ell=1}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\rho-\rho_1)+\ell_2(\rho+\rho_2)}\mathrm{h}(x)\\ &+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1+1}a_2^{\ell_2}I_a^{\psi,(\ell_1+1)(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2+1}I_a^{\psi,\ell_1(\varrho-\varrho_1)+(\ell_2+1)(\varrho+\varrho_2)}\mathrm{h}(x)=\mathrm{h}(x),\end{aligned} \tag{35}$$

by the cancellation. Notice that each series is absolutely convergent and also the term arrangements are possibly cancellated. In fact,

$$\begin{aligned}&-\sum_{\ell_1+\ell_2=1}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &\quad+\sum_{\ell_1+\ell_2=0}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1+1}a_2^{\ell_n}I_a^{\psi,(\ell_1+1)(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)}\mathrm{h}(x)\\ &\quad+\sum_{\ell_1+\ell_2=0}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2+1}I_a^{\psi,\ell_1(\varrho-\varrho_1)+(\ell_2+1)(\varrho+\varrho_2)}\mathrm{h}(x)=0.\end{aligned} \tag{36}$$

The remainder terms cancel each other similarly. Plainly, the uniqueness follows promptly from the fact that the FIDE

$${}^{C}\mathbb{D}_a^{\psi,\varrho}\varphi(x)+a_1\,{}^{C}\mathbb{D}_a^{\psi,\varrho_1}\varphi(x)+a_2 I_a^{\psi,\varrho_2}\varphi(x)=0, \tag{37}$$

only has solution zero due to the Babenko approach. □

Remark 13. Notice that the solution of Eq. (24) in $AC_0[a,b]$ is stable, if for all $\varepsilon>0$, there exists $\delta>0$ such that $\|\varphi\|_0<\varepsilon$

with $\|h\|_0 < \delta$. Taking advantage of the following inequality

$$\|\varphi\|_0 \le \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{\varrho-\varrho_1}, |a_2|(\psi(b) - \psi(a))^{\varrho+\varrho_2}\right)\|h\|_0, \tag{38}$$

we conclude that φ is stable.

Example 1. The following ψ-Caputo type FIDE

$$^{C}\mathbb{D}_a^{\psi,0.9}\varphi(x) + 2\,^{C}\mathbb{D}_a^{\psi,0.7}\varphi(x) - I_a^{\psi,0.4}\varphi(x) = (\psi(x)-\psi(a))^{\kappa}, \tag{39}$$

has the solution in $AC_0[a,b]$, that is

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(2)^{\ell_1}(-1)^{\ell_2} \times \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+0.2\ell_1+1.3\ell_2+1.9)}(\psi(x)-\psi(a))^{\kappa+0.2\ell_1+1.3\ell_2+0.9}. \tag{40}$$

So, according to Theorem 11, we have

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(2)^{\ell_1}(-1)^{\ell_2} \times I_a^{\psi,0.2\ell_1+1.3\ell_2+0.9}(\psi(x)-\psi(a))^{\kappa}. \tag{41}$$

From Property 3, we get

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(2)^{\ell_1}(-1)^{\ell_2} \times \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+0.2\ell_1+1.3\ell_2+1.9)}(\psi(x)-\psi(a))^{\kappa+0.2\ell_1+1.3\ell_2+0.9}. \tag{42}$$

Example 2. The following ψ-Caputo type FIDE

$$^{C}\mathbb{D}_a^{\psi,0.8}\varphi(x) + {}^{C}\mathbb{D}_a^{\psi,0.7}\varphi(x) - 3I_a^{\psi,0.2}\varphi(x) = e^{\psi(x)}, \tag{43}$$

has the solution in $AC_0[a,b]$ described as

$$\varphi(x) = e^{\psi(a)}\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(-3)^{\ell_2}(\psi(x) - \psi(a))^{0.1\ell_1+\ell_2+0.8} \times \sum_{i=0}^{\infty}\frac{(\psi(x)-\psi(a))^{i}}{\Gamma(0.1\ell_1+\ell_2+1.8+i)}. \tag{44}$$

So, as stated by Theorem 11, we obtain

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(1)^{\ell_1}(-3)^{\ell_2}I_a^{\psi,0.1\ell_1+\ell_2+0.8}e^{\psi(x)}. \tag{45}$$

By virtue of Lemma 9, we obtain

$$\varphi(x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}(-3)^{\ell_2}e^{\psi(a)}(\psi(x) - \psi(a))^{0.1\ell_1+\ell_2+0.8} \times \sum_{i=0}^{\infty}\frac{(\psi(x)-\psi(a))^{i}}{\Gamma(0.1\ell_1+\ell_2+1.8+i)}. \tag{46}$$

The uniqueness result of Eq. (1) will be proved through the following theorem.

Theorem 14. *Let $\mathbb{G} : [a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and assume that there exists a constant C such that*

$$|\mathbb{G}(x,\varphi_1) - \mathbb{G}(x,\varphi_2)| \le C|\varphi_1 - \varphi_2|, x \in [a,b], \varphi_1, \varphi_2 \in \mathbb{R}. \tag{47}$$

If

$$C\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right) < 1, \tag{48}$$

then, the problem (1) has a unique solution in $AC_0[a,b]$.

Proof. Consider the operator $\mathfrak{I}$ on $AC_0[a,b]$ defined by

$$\mathfrak{I}(\varphi) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}a_1^{\ell_1}a_2^{\ell_2} \times I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\rho}\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta. \tag{49}$$

For $\varphi \in AC_0[a,b]$, we have $\int_a^x \mathbb{G}(\zeta,\varphi'(\zeta))d\zeta \in AC_0[a,b]$, since $\varphi'(\zeta) \in L(a,b)$ and $\mathbb{G}(\zeta,\varphi'(\zeta)) \in L(a,b)$.

Hence,

$$\left\|\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right\|_0 = \int_a^b\left|\mathbb{G}\left(x,\varphi'(x)\right)\right|dx \le \int_a^b\left|\mathbb{G}\left(x,\varphi'(x)\right) - \mathbb{G}(x,0)\right|dx + \int_a^b|\mathbb{G}(x,0)|dx \le C\int_a^b\left|\varphi'(x)\right|dx + \int_a^b|\mathbb{G}(x,0)|dx < \infty. \tag{50}$$

Using the inequality (38), we obtain

$$\|\mathfrak{I}(\varphi)\|_0 < \infty \text{ and } \mathfrak{I}(\varphi)(a) = 0. \tag{51}$$

Besides, $\mathfrak{I}(\varphi)$ is absolutely continuous on $[a,b]$ via Theorem 11. So, $\mathfrak{I} : AC_0[a,b] \longrightarrow AC_0[a,b]$. Now, we just have to show that $\mathfrak{T}$ is a contraction mapping. Let $\varphi, \varphi^* \in AC_0[a, b]$. Then

$$\begin{aligned}\|\mathfrak{T}(\varphi)-\mathfrak{T}(\varphi^*)\|_0 &\leq \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)} \\ &\cdot \left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right) \\ &\times \left\|\int_a^x \mathbb{G}\left(\zeta, \varphi'(\zeta)\right) d\zeta - \int_a^x \mathbb{G}\left(\zeta, \varphi^{*'}(\zeta)\right) d\zeta\right\|_0.\end{aligned} \tag{52}$$

Since

$$\begin{aligned}&\left\|\int_a^x \mathbb{G}\left(\zeta, \varphi'(\zeta)\right) d\zeta - \int_a^x \mathbb{G}\left(\zeta, \varphi^{*'}(\zeta)\right) d\zeta\right\|_0 \\ &\quad = \int_a^b \left|\mathbb{G}\left(x, \varphi'(x)\right) - \mathbb{G}\left(x, \varphi^{*'}(x)\right)\right| dx \\ &\quad \leq C\int_a^b \left|\varphi' - \varphi^{*'}\right| dx = C\|\varphi-\varphi^*\|_0,\end{aligned} \tag{53}$$

we obtain

$$\begin{aligned}&\|\mathfrak{T}(\varphi)-\mathfrak{T}(\varphi^*)\|_0 \\ &\quad \leq C\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right) \\ &\quad \times \|\varphi-\varphi^*\|_0.\end{aligned} \tag{54}$$

Inequality (48) leads us to that $\mathfrak{T}$ is contraction mapping. □

3.1. Ulam-Hyers Stability (UHS). The first results about this type of stability emerged in 1940 by Ulam [28, 29]. From that point forward, the UHS is studied via several researchers. With the vast development of fractional calculus, the studying of stability for FDEs also attracted the numerous authors, see [30–32].

In this regard, we investigate some recent results on the UHS and generalized UHS of (1). For $\varepsilon > 0, x \in [a, b]$, and $\varphi_1 \in AC_0[a, b]$, the following inequality

$$\left|{}^C\mathbb{D}_a^{\psi,\varrho}\varphi_1(x) + a_1 {}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi_1(x) + a_2 I_a^{\psi,\varrho_2}\varphi_1(x) - \int_a^x \mathbb{G}\left(\zeta, \varphi'(\zeta)\right) d\zeta\right| \leq \varepsilon, \tag{55}$$

is satisfied.

Remark 16. Let $\varepsilon > 0$. Then, $\varphi_1 \in AC_0[a, b]$ satisfies (55) iff there exists $\xi(x) \in AC_0[a, b]$ with $\xi(0) = 0$ such that

(i) $\|\xi\|_0 = \int_a^x |\xi'(\zeta)| d\zeta \leq \varepsilon$, for $x \in [a, b]$

(ii) for $x \in [a, b]$

$$\begin{aligned}{}^C\mathbb{D}_a^{\psi,\varrho}\varphi_1(x) + a_1 {}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi_1(x) + a_2 I_a^{\psi,\varrho_2}\varphi_1(x) &= \int_a^x \mathbb{G}\left(\zeta, \varphi_1'(\zeta)\right) d\zeta \\ &+ \int_a^x |\xi'(\zeta)| d\zeta.\end{aligned} \tag{56}$$

Lemma 17. *The solution of the problem (56) with $\varphi_1(0) = 0$ satisfies the following inequality*

$$\begin{aligned}\|\varphi_1 - Z_{\mathbb{G}}\|_0 &\leq \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)} \\ &\cdot \left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon,\end{aligned} \tag{57}$$

where

$$\begin{aligned}Z_{\mathbb{G}}(x) &:= \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1, \ell_2} a_1^{\ell_1} a_2^{\ell_2} \\ &\times I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\rho}\int_a^x \mathbb{G}\left(\zeta, \varphi_1'(\zeta)\right) d\zeta,\end{aligned} \tag{58}$$

and η is defined by (32).

Proof. By virtue of Lemma 8, the solution of Eq. (56) is described as

$$\begin{aligned}\varphi_1(x) &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1, \ell_2} a_1^{\ell_1} a_2^{\ell_2} \\ &\times I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\left[\int_a^x \mathbb{G}\left(\zeta, \varphi_1'(\zeta)\right) d\zeta + \int_a^x |\xi'(\zeta)| d\zeta\right].\end{aligned} \tag{59}$$

It follows from Eq. (59), Remark 16, and Eq. (38) that

$$\begin{aligned}\|\varphi_1 - Z_{\mathbb{G}}\|_0 &\leq \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)} \\ &\cdot \left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right) \\ &\times \left\|\int_a^x \mathbb{G}\left(\zeta, \varphi_1'(\zeta)\right) d\zeta + \int_a^x |\xi'(\zeta)| d\zeta - \int_a^x \mathbb{G}\left(\zeta, \varphi_1'(\zeta)\right) d\zeta\right\|_0 \\ &\leq \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\|\xi\|_0 \\ &\leq \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)}, |a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon.\end{aligned} \tag{60}$$

□

Theorem 19 (UHS). *Suppose that the hypotheses of Theorem 14 with Eq. (55) are satisfied. Then, Eq. (1) is UH stable.*

Proof. Assume that $\varepsilon > 0$ and $\varphi_1 \in AC_0[a, b]$ satisfy Eq. (55), and let $\varphi \in AC_0[a, b]$ be a unique solution of

$$\begin{cases}{}^C\mathbb{D}_a^{\psi,\varrho}\varphi_1(x) + a_1 {}^C\mathbb{D}_a^{\psi,\varrho_1}\varphi_1(x) + a_2 I_a^{\psi,\varrho_2}\varphi_1(x) = \int_a^x \mathbb{G}\left(\zeta, \varphi'(\zeta)\right) d\zeta, \\ \varphi(a) = \varphi_1(a) = 0,\end{cases} \tag{61}$$

that is

$$\varphi(x)=\varphi(a)+\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\times a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\rho}\left[\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right]. \quad (62)$$

Since $\varphi(a)=\varphi_1(a)=0$, we obtain

$$\varphi(x)=\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1,\ell_2}\times a_1^{\ell_1}a_2^{\ell_2}I_a^{\psi,\ell_1(\varrho-\varrho_1)+\ell_2(\varrho+\varrho_2)+\varrho}\left[\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right]. \quad (63)$$

According to Lemma 17 and (38), we get

$$\begin{aligned}\|\varphi_1-\varphi\|_0 &\le \|\varphi_1-Z_{\mathbb{G}}\|_0+\|Z_{\mathbb{G}}-\varphi\|_0\le \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\\ &\cdot\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon\\ &+\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\\ &\times\left\|\int_a^x \mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta-\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right\|_0.\end{aligned} \quad (64)$$

Using the assumption of Theorem 14, we have

$$\left\|\int_a^x \mathbb{G}\left(\zeta,\varphi_1'(\zeta)\right)d\zeta-\int_a^x \mathbb{G}\left(\zeta,\varphi'(\zeta)\right)d\zeta\right\|_0\le C\|\varphi_1-\varphi\|_0. \quad (65)$$

So,

$$\begin{aligned}\|\varphi_1-\varphi\|_0 &\le \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\\ &\cdot\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\varepsilon\\ &+\eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right)\\ &\times C\|\varphi_1-\varphi\|_0.\end{aligned} \quad (66)$$

From the inequality (48), we find that

$$\|\varphi_1-\varphi\|_0\le C_{\mathbb{G}}\varepsilon, \quad (67)$$

where $C_{\mathbb{G}}\coloneqq \mathfrak{R}/1-\mathfrak{R}C$ and

$$\mathfrak{R}\coloneqq \eta E_{(\varrho-\varrho_1,\varrho+\varrho_2,\varrho+1)}\left(|a_1|(\psi(b)-\psi(a))^{(\varrho-\varrho_1)},|a_2|(\psi(b)-\psi(a))^{(\varrho+\varrho_2)}\right). \quad (68)$$

□

Corollary 21. *Under assumptions of Theorem 19, if we put $\Phi(\varepsilon)=C_{\mathbb{G}}\varepsilon$ along with $\Phi(0)=0$, then Eq. (1) is a generalized UH stable.*

Example 3. Let $a=1$ and $b=\psi^{-1}(1+\psi(1))$. Then, there exists a unique solution for the following nonlinear ψ-Caputo-type FIDE

$${}^{C}\mathbb{D}_a^{\psi,0.9}\varphi(x)+I_a^{\psi,0.6}\varphi(x)=\int_a^x\left(\frac{e^{\zeta^2}}{C\left(3+e^{\zeta^2}\right)}\sin\varphi'(\zeta)+e^{\cos\zeta}+\ln\left(1+\sqrt{\zeta}\right)\right)d\zeta, \quad (69)$$

where the constant C is to be determined. It is clear that

$$\mathbb{G}(x,z)=\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}\sin z+e^{\cos x}+\ln\left(1+\sqrt{x}\right), \quad (70)$$

is continuous from $[1,\psi^{-1}(1+\psi(1))]\times\mathbb{R}$ to $\mathbb{R}$ and satisfies

$$\begin{aligned}|\mathbb{G}(x,z_1)-\mathbb{G}(x,z_2)| &=\left|\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}\sin z_1-\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}\sin z_1\right|\\ &\le\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}|\sin z_1-\sin z_1|\le\frac{e^{x^2}}{C\left(3+e^{x^2}\right)}|z_1-z_1|\\ &\le\frac{1}{C}|z_1-z_1|.\end{aligned} \quad (71)$$

Obviously, $\psi(b)-\psi(a)=1$ and

$$\sum_{\ell=0}^{\infty}\sum_{\ell_2=\ell}\binom{\ell}{\ell_2}\left(|1|(\psi(b)-\psi(a))^{1.5}\right)^{\ell_2}\frac{1}{\Gamma(1.5\ell_2+1.9)}=\sum_{\ell=0}^{\infty}\frac{1}{\Gamma(1.5\ell+1.9)}. \quad (72)$$

For $\ell\ge 1$, we have

$$\ell+1\le 1.5\ell+1.9, \quad (73)$$

$$\frac{1}{\Gamma(1.5\ell+1.9)}\le\frac{1}{\Gamma(\ell+1)}=\frac{1}{\ell!}. \quad (74)$$

Therefore,

$$\sum_{\ell=0}^{\infty}\sum_{\ell_2=\ell}\binom{\ell}{\ell_2}\frac{1}{\Gamma(1.5\ell_2+1.9)}\le\frac{1}{\Gamma(1.9)}+\sum_{\ell=1}^{\infty}\frac{1}{k!}=\frac{1}{\Gamma(1.9)}-1+\sum_{\ell=0}^{\infty}\frac{1}{k!}\le 0.04+e. \quad (75)$$

Let us choose a positive C such that

$$C < \frac{1}{0.04 + e}. \tag{76}$$

It follows from Theorem 14 that Eq. (69) has a unique solution.

Moreover, by Theorem 19, and for any solution $\varphi_1(x) \in AC_0[a, b]$ of the inequality

$$\left| {}^{C}\mathbb{D}_a^{\psi,0.9}\varphi(x) + I_a^{\psi,0.6}\varphi(x) - \int_a^x \left(\frac{e^{\zeta^2}}{C\left(3 + e^{\zeta^2}\right)} \sin \varphi_1'(\zeta) + e^{\cos \zeta} + \ln\left(1 + \sqrt{\zeta}\right) \right) d\zeta \right| \le \varepsilon, \text{for } x \in [a, b], \tag{77}$$

there exists a unique solution $\varphi(x) \in AC_0[a, b]$ of Eq. (69) such that

$$\|\varphi_1 - \varphi\|_0 \le C_{\mathbb{G}}\varepsilon, \tag{78}$$

where $C_{\mathbb{G}} \coloneqq \mathfrak{R}/1 - \mathfrak{R}C > 0$, $\mathfrak{R} = \eta/0.04 + e$ and $\eta = 1/\Gamma(\ell_1 0.9 + 1.5\ell_2 + 1.9)$. Consequently, Eq. (69) is UH stable.

Remark 22. All previous results can be generalized in which the left-hand side of (1) may contain several FDs and FIs. For example, Theorem 11 can be generalized as follows:

Theorem 23. *Suppose $a_i \in \mathbb{C}(i = 1, \cdots, n), b_{i+n} \in \mathbb{C}$ $(i = 1, \cdots, m - n)$ with $0 < \alpha_1 < \cdots < \alpha_n < \alpha < 1$ and $0 \le \kappa_{n+1} < \kappa_{n+2} < \cdots < \kappa_m \in \mathbb{R}$. If $h \in AC_0[a, b]$, then the following linear problem*

$$\begin{cases} {}^{C}\mathbb{D}_a^{\psi,\alpha}\varphi(x) + a_1 {}^{C}\mathbb{D}_a^{\psi,\alpha_1}\varphi(x) + \ldots + a_n {}^{C}\mathbb{D}_a^{\psi,\alpha_n}\varphi(x) + b_{n+1} I_a^{\psi,k_{n+1}}\varphi(x) + b_{n+2} I_a^{\psi,k_{n+2}}\varphi(x) + \ldots + b_m I_a^{\psi,k_m}\varphi(x) = \mathrm{h}(x), \\ \varphi(a) = 0, \end{cases} \tag{79}$$

has a solution

$$\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \sum_{\ell_1 + \cdots + \ell_m = \ell} \binom{\ell}{\ell_1, \ell_2, \cdots, \ell_m} \times a_1^{\ell_1} \cdots b_m^{\ell_m} I_a^{\psi,\ell_1(\alpha-\alpha_1)+\cdots+\ell_m(\alpha+\kappa_m)+\alpha} h(x). \tag{80}$$

where ${}^{C}\mathbb{D}_a^{\psi,\delta}$ is the ψ-Caputo FD of order $\delta(>0) \in \{\alpha, \alpha_i\,; i = 1, \cdots, n\}$ and $I_a^{\psi,\sigma}$ is generalized FI of order $\sigma(>0) \in \{\kappa_j\,; j = n + 1, \cdots, m\}$.

4. Conclusions

ψ-Caputo FD, a general fractional operator, is of great use because of its wide freedom to cover many classical fractional operators. In this work, we have studied the uniqueness of solution for the nonlinear ψ-Caputo type FIDE (1) by using the Banach space $AC_0[a, b]$, Banach's fixed point theorem, and Babenko's method. Moreover, the UH stability results to the proposed problem have been discussed. Also, some pertinent examples have been provided to justify the main results. The obtained results in this study extended and developed the current results introduced by [17, 18]. We have already concluded that our results are valid when the left-hand side of the considered problem (1) involves many FDs and IDs as shown in Remark 22. Furthermore, problem (1) covers previous standard cases of nonlinear FDEs and FIDEs by selecting the suitable standard kernel in the studied problem. More specifically, our results generalize some known results in literature like those that include Hadamard and Katugampola FDs.

For future research, we will consider a class of nonlinear FIDEs with the fuzzy initial conditions in a fractional case. It would also be interesting to study the same results for our current problem under the ψ-Hilfer operator [33] or Atangana-Baleanu operator [8].

Data Availability

The data of this study were used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors made equal contributions and read and supported the last manuscript.

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Research Article

Some Midpoint Inequalities for η-Convex Function via Weighted Fractional Integrals

Lei Chen,[1] Waqas Nazeer ,[2] Farman Ali ,[3] Thongchai Botmart ,[4] and Sarah Mehfooz[5]

[1]*College of Science, Qiongtai Normal University, Haikou, Hainan 571127, China*
[2]*Department of Mathematics, Government College University, Lahore 54000, Pakistan*
[3]*Department of Software, Sejong University, Seoul, Republic of Korea*
[4]*Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*
[5]*Department of Mathematics, University of Okara, Okara, Pakistan*

Correspondence should be addressed to Thongchai Botmart; thongbo@kku.ac.th

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In this research, by using a weighted fractional integral, we establish a midpoint version of Hermite-Hadamrad Fejér type inequality for η-convex function on a specific interval. To confirm the validity, we considered some special cases of our results and relate them with already existing results. It can be observed that several existing results are special cases of our presented results.

1. Introduction

In the last few decades, the classical convexity has a rapid development in fractional calculus [1]. We can say that convexity plays a vital role in fractional integral inequalities because of its geometric features [2–4].

Take a function $f{:}I \longrightarrow \mathbb{R}$ be a continuous function. Then, this function is called convex if

$$f(tm + (1-t)n) \leq tf(m) + (1-t)f(n), \tag{1}$$

$\forall m, n \in I$, and $t \in [0, 1]$.

There are many integral inequalities in the literature and one of the most common inequality is Hermite-Hadamarad or, shortly, the HH integral inequality, which is introduced by [5]:

$$f(m+n/2) \leq 1/n - m\int_m^n f(x)dx \leq f(m) + f(n)/2, m < n \in I. \tag{2}$$

In the literature, we can notice that Hermite-Hadamarad inequality (2) has been applied to distinct convexities like exponential convexity [6, 7], s-convexity [8], quasiconvexity [9, 10], GA-convexity [11], (α, m)-convexity [12], MT-convexity [13], and also, other types of convexity (see [14, 15]). Different forms of fractional integrals like Riemann-Liouville (RL), Caputo Fabrizio, Hadamrad, Riesz, Prabhakar, Ψ-RL, and weighted integrals [16–20] have been established. A lot of integer-order integral inequalities like Simpson [21], Ostrowski [22], Rozanova [23], Gagliardo-Nirenberg [24], Olsen [25], Hardy [26], Opial [27, 28], and Akdemir et al. [29, 30] have been developed and generalized from fractional point of view.

Definition 1. Let $I \subset \mathbb{R}$ be an interval and $f : I \longrightarrow \mathbb{R}$ be a continuous function. Then, the function f is called η-convex if

$$f(tm + (1-t)n) \leq f(n) + t\eta(f(m), f(n)). \tag{3}$$

Definition 2. [18] Let f is positive convex function, continuous on closed interval $[m, n]$ and $x \in [m, n]$ when $f(x) \in L^1$

$[m, n]$ with $m < n$, where left- and right-side RL fractional integrals are defined by

$$
\begin{aligned}
{}^{RL}I^{\nu}_{m+}f(x) &= \frac{1}{\Gamma(\nu)}\int_m^x (x-t)^{\nu-1}f(t)dt,\\
{}^{RL}I^{\nu}_{n-}f(x) &= \frac{1}{\Gamma(\nu)}\int_x^n (t-x)^{\nu-1}f(t)dt,
\end{aligned}
\tag{4}
$$

where Γ is famous Gamma function and for any positive integer $n, \Gamma(n) = (n-1)!$.

Definition 3 (see [19]). Let $[m, n] \subseteq \mathbb{R}$, $f : [m, n] \longrightarrow \mathbb{R}$ and $\phi : (m, n] \longrightarrow \mathbb{R}$ be monotonically increasing positive function with a continuous derivative $\phi'(x)$ on (m, n). Then, the left-sided and the right-sided weighted fractional integrals of f according to ϕ on $[m, n]$ are defined by:

$$
\begin{aligned}
\left({}_{m+}I^{\nu,\phi}_{g}f\right)(x) &= \frac{[g(x)]^{-1}}{\Gamma(\nu)}\int_m^x \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)g(t)dt,\\
\left({}_{n+}I^{\nu,\phi}_{g}f\right)(x) &= \frac{[g(x)]^{-1}}{\Gamma(\nu)}\int_x^n \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)g(t)dt, \nu > 0.
\end{aligned}
\tag{5}
$$

In this research, we denote $[g(x)]^{-1} = 1/g(x)$ and the inverse of function $\phi(x)$ by $\phi^{-1}(x)$.

Remark 4. From Definition 3, we can see some special cases:

(i) If $\phi(x) = x$ and $g(x) = 1$, then weighted fractional integrals [14] deduce to the classical RL fractional integrals [9].

(ii) If $g(x)$=1, we get fractional integrals of function f with respect to function $\phi(x)$, which is defined by [16, 17]:

$$
\begin{aligned}
\left({}_{m+}I^{\nu:\phi}\right)f(x) &= \frac{1}{\Gamma(\nu)}\int_m^x \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)dt,\\
\left({}_{n+}I^{\nu:\phi}\right)f(x) &= \frac{1}{\Gamma(\nu)}\int_x^n \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)dt, \nu > 0.
\end{aligned}
\tag{6}
$$

Lemma 5. *[31] Assume that $g : [m, n] \longrightarrow (0,\infty)$ is integrable function and symmetric with respect to $(m+n)/2, m < n$. Then,*

(i) For each $t \in [0, 1]$, we have

$$
g\left(\frac{t}{2}m + \frac{2-t}{2}n\right) = g\left(\frac{2-t}{2}m + \frac{t}{2}n\right). \tag{7}
$$

(ii) For $\nu > 0$, we have

$$
\begin{aligned}
&\left({}_{\phi^{-1}(m+n)/2+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\\
&= \left(I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-(go\phi)}\right)\left(\phi^{-1}(m)\right)\\
&= \frac{1}{2}\left[\left({}_{\phi^{-1}(m+n)/2+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\right.\\
&= \left.\left(I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-(go\phi)}\right)\left(\phi^{-1}(m)\right)\right].
\end{aligned}
\tag{8}
$$

2. Main Results

Theorem 6. *Let $0 \le m < n$ and $f : [m, n] \longrightarrow \mathbb{R}$ be an $L^1\eta$-convex function and $g : [m, n] \longrightarrow \mathbb{R}$ be an integrable, positive and weighted symmetric function with respect to $(m+n)/2$. If, in addition, ϕ is an increasing and positive function from $[m, n]$ onto itself such that its derivative $\phi'(x)$ is continues on (m, n), then for $\nu > 0$, the following inequalities are valid:*

$$
\begin{aligned}
&f\left(\frac{m+n}{2}\right) \times \left[\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\right.\\
&\quad + \left.\left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right]\\
&\le \left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\phi^{-1}(n)\right)\Big)\\
&\quad + \frac{\eta}{2}g(m)\left(f\left(\frac{t}{2}m + \frac{2-t}{2}n\right), f\left(\frac{2-t}{2}m + \frac{t}{2}n\right)\right)\\
&\quad \cdot \left.\left(\phi^{-1}(m)\right)\right].\\
&\le \left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right) + \frac{\eta}{2}g(m)\right.\\
&\quad \cdot \left.\left(f\left(\frac{t}{2}m + \frac{2-t}{2}n\right), f\left(\frac{2-t}{2}m + \frac{t}{2}n\right)\right)\left(\phi^{-1}(m)\right)\right].
\end{aligned}
\tag{9}
$$

Proof. The η-convexity of f on $[m, n]$, for all $x, y \in [m, n]$ gives

$$
f\left(\frac{x+y}{2}\right) \le \frac{f(y) + \eta(f(x), f(y))}{2}, \tag{10}
$$

setting $x = (t/2)m + ((2-t)/2)n$ and $y = ((2-t)/2)m + (t/2)n$

$$
\begin{aligned}
2f\left(\frac{m+n}{2}\right) &\le f\left(\frac{2-t}{2}m + \frac{t}{2}n\right)\\
&\quad + \eta\left(f\left(\frac{t}{2}m + \frac{2-t}{2}n\right), f\left(\frac{2-t}{2}m + \frac{t}{2}n\right)\right).
\end{aligned}
\tag{11}
$$

Multiplying both sides of inequality (11) by $t^{\nu-1}g((t/2)m+((2-t)/2)n)$ and integrating over $[0,1]$, we get

$$2f\left(\frac{m+n}{2}\right)\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt \leq \int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)dt + \int_0^1 \eta t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right) \cdot \left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right). \tag{12}$$

From the left side of inequality (12), we use

$$\begin{aligned}
&\frac{2^{\nu-1}\Gamma(\nu)}{(n-m)^{\nu}}\left[\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)+\left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right]\\
&=\frac{2^{\nu}\Gamma(\nu)}{(n-m)^{\nu}}\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\\
&=\frac{2^{\nu}}{(n-m)^{\nu}}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}(n-\phi(x))^{\nu-1}(go\phi)(x)\phi'(x)dx\\
&=\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left(\frac{2(n-\phi(x))}{n-m}\right)^{\nu-1}(go\phi)(x)\phi'(x)\frac{2dx}{n-m}\\
&=\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt,
\end{aligned} \tag{13}$$

where $t=2(n-\phi(x))/(n-m)$. It follows that

$$\begin{aligned}
&2f\left(\frac{m+n}{2}\right)\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt\\
&=\frac{2^{\nu}\Gamma(\nu)}{(n-m)^{\nu}}f\left(\frac{m+n}{2}\right)\times\left[\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)+\left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right].
\end{aligned} \tag{14}$$

By evaluating the weighted fractional operators, we see that

$$\begin{aligned}
&g(n)\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)+g(m)\left({}_{go\phi}I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(fo\phi)\right)\left(\phi^{-1}(m)\right)\\
&=g(n)\frac{(go\phi)^{-1}\left(\phi^{-1}(n)\right)}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}(n-\phi(x))^{\nu-1}\cdot(fo\phi)(x)(go\phi)(x)\phi'(x)dx\\
&+g(m)\frac{(go\phi)^{-1}\left(\phi^{-1}(n)\right)}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot(\phi(x)-m)^{\nu-1}(fo\phi)(x)(go\phi)(x)\phi'(x)dx\\
&=\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left(\frac{2(n-\phi(x))}{n-m}\right)^{\nu-1}\cdot(fo\phi)(x)(go\phi)(x)\phi'(x)\frac{2dx}{n-m}\\
&+\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left(\frac{2(\phi(x)-m)}{n-m}\right)^{\nu-1}\cdot(fo\phi)(x)(go\phi)(x)\phi'(x)\frac{2dx}{n-m},
\end{aligned} \tag{15}$$

where

$$\left[(go\phi)\left(\phi^{-1}(y)\right)\right]^{-1}=\frac{1}{(go\phi)\left(\phi^{-1}(y)\right)}=\frac{1}{g(y)}, \tag{16}$$

for $y=m,n$.

Setting $u_1=2(n-\phi(x))/(n-m)$ and $u_2=2(\phi(x)-m)/(n-m)$, one can deduce that

$$\begin{aligned}
&g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)+g(m)\left({}_{go\phi}I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-}(fo\phi)\right)\left(\phi^{-1}(m)\right)\\
&=\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\left[\int_0^1 u_1^{\nu-1}f\left(\frac{u_1}{2}m+\frac{2-u_1}{2}n\right)\cdot g\left(\frac{u_1}{2}m+\frac{2-u_1}{2}n\right)du_1+\int_0^1 u_2^{\nu-1}f\left(\frac{2-u_2}{2}m+\frac{u_2}{2}n\right)g\left(\frac{2-u_2}{2}m+\frac{u_2}{2}n\right)du_2\right]\\
&=\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\left[\int_0^1 t^{\nu-1}f\left(\frac{t}{2}m+\frac{2-t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt+\int_0^1 t^{\nu-1}f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt\right],
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\int_0^1 t^{\nu-1}f\left(\frac{t}{2}m+\frac{2-t}{2}n\right)\Bigg)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt+\int_0^1\eta t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)\cdot\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\\
&=\frac{2^{\nu}\Gamma(\nu)}{(n-m)^{\nu}}\left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right)+\eta g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\cdot\left(\phi^{-1}(m)\right)\Bigg].
\end{aligned} \tag{18}$$

By using (14) and (18) in (12), we get

$$
\begin{aligned}
& f\left(\frac{m+n}{2}\right)\times\left[\left(\phi_{\phi-1((m+n)/2)+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\right.\\
&\quad\left.+\left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right]\\
&\leq\left[g(n)\left(_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right)\\
&\quad+\eta t g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\\
&\quad\cdot\left.\left(\phi^{-1}(m)\right)\right].
\end{aligned}
\tag{19}
$$

The left side of Theorem 6 is completed.

Now, we will prove right side of inequality (9) by using η-convexity.

$$
\begin{aligned}
& f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)+t\eta\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\\
&\leq f(n)+\frac{2-t}{2}\eta(f(m),f(n))\\
&\quad+\eta\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right).
\end{aligned}
\tag{20}
$$

Multiply Equation (20) by $t^{\nu-1}g((t/2)m+((2-t)/2)n)$ and integrate over $[0,1]$ leads us to

$$
\begin{aligned}
&\left[\int_0^1 t^{\nu-1}f\left(\frac{t}{2}m+\frac{2-t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt\right.\\
&\quad\left.+\int_0^1 t^{\nu-1}f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt\right].\\
&\leq\left[f(n)+\frac{2-t}{2}\eta(f(m),f(n))\right.\\
&\quad\left.+\eta\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right).\right]\\
&\quad\times\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt.
\end{aligned}
\tag{21}
$$

By using (7) and (14) in (21), we get

$$
\begin{aligned}
& g(n)\left(_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\Big)\\
&\quad+\eta t g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\\
&\quad\cdot\left(\phi^{-1}(m)\right)\\
&\leq\left[g(n)\left(_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right)\\
&\quad+\frac{\eta}{2}g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\\
&\quad\cdot\left.\left(\phi^{-1}(m)\right)\right].\\
&\leq\left[g(n)\left(_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right.\\
&\quad+\frac{\eta}{2}g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\\
&\quad\cdot\left.\left(\phi^{-1}(m)\right)\right].
\end{aligned}
\tag{22}
$$

This completes our proof. □

Remark 7. From Theorem 6, we can get following special case:

If $\phi(x)=x$, then inequality (9) becomes

$$
\begin{aligned}
& f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n/2)+}I^{\nu}g(n)+{}^{RL}I^{\nu}_{((m+n)/2)-}g(m)\right]\\
&\leq g(n)\left({}^{RL}_{(m+n/2)+}I^{\nu}_{g}f\right)(n)+g(m)\frac{\eta}{2}\left({}^{RL}_{g}I^{\nu}_{((m+n)/2)-}f\right)(m)\\
&\quad+\frac{\eta}{2}\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\\
&\leq g(n)\left({}^{RL}_{(m+n/2)+}I^{\nu}_{g}f\right)(n)+g(m)\frac{\eta}{2}\left({}^{RL}_{g}I^{\nu}_{((m+n)/2)-}f\right)(m)\\
&\quad+\frac{\eta}{2}\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right).
\end{aligned}
\tag{23}
$$

Lemma 8. *[31] Let $0\leq m<n$ and $f:[m,n]\longrightarrow\mathbb{R}$ be a continuous with a derivative $f'\in L^1[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$ and let $g:[m,n]\longrightarrow\mathbb{R}$ be an integrable, positive, and weighted symmetric function with respect to $(m+n)/2$. If ϕ is a continuous increasing mapping form the interval $[m,n]$ onto itself with a derivative $\phi'(x)$ which is continuous on (m,n), then for $\nu>0$, the following equality is valid:*

$$
\begin{aligned}
& f\left(\frac{m+n}{2}\right)\left[\left(_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\right.\\
&\quad\left.+\left(I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-}(go\phi)\right)\left(\phi^{-1}(m)\right)\right]\\
&\quad-\left[g(n)\left(_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right.\\
&\quad\left.+g(m)\left(_{go\phi}I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-}(fo\phi)\right)\left(\phi^{-1}(m)\right)\right]\\
&=\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\\
&\quad\cdot\left[\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right]\\
&\quad\cdot\left(f'o\phi\right)(t)\phi'(t)dt-\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\\
&\quad\cdot\left[\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right]\\
&\quad\cdot\left(f'o\phi\right)(t)\phi'(t)dt.
\end{aligned}
\tag{24}
$$

Remark 9. From Lemma 8, we obtain the following special case:

If $\phi(x)=x$, then equality (24) becomes

$$f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n)/2+}I^{\nu}g(n)+{}^{Rl}I^{\nu}_{(m+n)/2-}g(m)\right]-\left[g(n)\left({}^{RL}_{(m+n)/2+}I^{\nu}_{g}f\right)(n)+g(m)\left({}^{RL}_{g}I^{\nu}_{(m+n)/2-}f\right)(m)\right]=\frac{1}{\Gamma(\nu)}\int_m^{(m+n)/2}\left[\int_m^t(x-m)^{\nu-1}g(x)dx\right]f'(t)dt-\frac{1}{\Gamma(\nu)}\int_{(m+n)/2}^{n}\left[\int_t^n(n-x)^{\nu-1}g(x)dx\right]f'(t)dt. \tag{25}$$

Theorem 10. *Let $0\le m<n$ and $f:[m,n]\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on the interval $[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$ and let $g:[m,n]\longrightarrow\mathbb{R}$ be an integrable, positive, and weighted symmetric function with respect to $(m+n)/2$. If, in addition, $|f'|$ is convex on $[m,n]$, and ϕ is an increasing and positive function from $[m,n)$ onto itself such that its derivative $\phi'(x)$ is continuous on (m,n), then for $\nu>0$, the following inequalities hold:*

$$\begin{aligned}|\sigma_1+\sigma_2|&=\left|\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left[\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt-\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left[\int_t^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt\right|\\
&\le\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+1}(\nu+1)}\cdot\left[\|g\|_{[m,(m+n)/2],\infty}f'(n)+\|g\|_{[m,(m+n)/2],\infty}f'(n)\right]+\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\|g\|_{[m,(m+n)/2],\infty}\eta\cdot\left(|f'(m)|,|f'(n)|\right)\\
&\quad+\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)}\|g\|_{[m,(m+n)/2],\infty}\eta\cdot\left(|f'(m)|,|f'(n)|\right)\\
&\le2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+1}(\nu+1)}f'(n)+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)}+\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\right]\cdot\|g\|_{[m,n],\infty}\eta\left(|f'(m)|,|f'(n)|\right).\end{aligned} \tag{26}$$

Proof. By using Lemma 8 and properties of modulus, we get

$$\begin{aligned}|\sigma_1+\sigma_2|&=\left|\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left[\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt+\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left[\int_t^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt\right|\\
&\le\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)\mathrm{d}x\right|\cdot\left|\left(f'o\phi\right)(t)\right|\phi'(t)dt+\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left|\int_t^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|\cdot\left|\left(f'o\phi\right)(t)\right|\phi'(t)dt.\end{aligned} \tag{27}$$

Since $|f'|$ is η-convex on $[m,n]$ for $t\in[\phi^{-1}(m),\phi^{-1}(n)]$, so

$$\left|\left(f'o\phi\right)(t)\right|=\left|f'\left(\frac{n-\phi(t)}{n-m}m+\frac{\phi(t)-m}{n-m}n\right)\right|\le f'(n)+\frac{n-\phi(t)}{n-m}\eta\left(f'(m),f'(n)\right). \tag{28}$$

So, using (28), we obtain

$$\begin{aligned}|\sigma_1+\sigma_1|\le&\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\times\left[|f'(n)|+\frac{n-\phi(t)}{n-m}\eta\left(|f'(m)|,|f'(n)|\right)\right]\\
&\cdot\phi'(t)dt+\frac{\|g\|_{[(m+n)/2,n],\infty}}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left|\int_t^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}dx\right|\times\left[|f'(n)|+\frac{n-\phi(t)}{n-m}\eta\left(|f'(m)|,|f'(n)|\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\cdot \phi'(t)dt \\
&\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1}} \\
&\quad \cdot \left[\|g\|_{[m,(m+n)/2],\infty} f'(n) + \|g\|_{[m,(m+n)/2],\infty} f'(n)\right] \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2}} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2}(\nu+1)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1}} f'(n) \\
&\quad + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2}(\nu+1)} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2}}\right] \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right),
\end{aligned} \tag{29}
$$

where

$$
\begin{aligned}
&\int_{\phi^{-1}(m)}^{t} \phi'(x)(\phi(x)-m)^{\nu-1}dx \\
&\quad = \frac{(\phi(t)-m)^{\nu}}{\nu}, \\
&\int_{t}^{\phi^{-1}(n)} \phi'(x)(n-\phi(x))^{\nu-1}dx \\
&\quad = \frac{(n-\phi(t))^{\nu}}{\nu}, \\
&\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)} (\phi(t)-m)^{\nu}(n-\phi(t))\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+2}(\nu+3)}{2^{\nu+2}(\nu+1)(\nu+2)}, \\
&\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)} (n-\phi(t))^{\nu}(n-\phi(t))\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+2}}{2^{\nu+2}(\nu+2)}, \\
&\int_{((m+n)/2)}^{\phi^{-1}(n)} (n-\phi(t))^{\nu}\phi'(t)dt \\
&\quad = \int_{\phi^{-1}(m)}^{\phi^{-1}m+n/2} (\phi(t)-m)^{\nu}\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+1}}{2^{\nu+1}(\nu+1)}.
\end{aligned} \tag{30}
$$

This completes our proof. □

Remark 11. From Theorem 10, we can get following inequalities:

(1) If $\phi(x)=x$, then inequality (26) becomes

$$
\begin{aligned}
&\left| f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n)/2+}I^{\nu} g(n) + {}^{RL}I^{\nu}_{m+n/2-} g(m)\right] \right. \\
&\quad - \left[g(n)\left({}^{RL}_{(m+n)/2+}I^{\nu}_{g} f\right)(n) \right. \\
&\quad \left.\left. + g(m)\left({}^{RL}_{g}I^{\nu}_{m+n/2-} f\right)(m)\right]\right| \\
&\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+1}(\nu+1)} \\
&\quad \cdot \left[\|g\|_{[m,(m+n)/2],\infty} f'(n) + \|g\|_{[m,(m+n)/2],\infty} f'(n)\right] \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+1}(\nu+1)} f'(n) \\
&\quad + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)} \right. \\
&\quad \left. + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\right]\eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right).
\end{aligned} \tag{31}
$$

(2) If $\phi(x)=x$ and $g(x)=1$, then inequality (26) becomes

$$
\begin{aligned}
&\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(n-m)^{\nu}}\right. \\
&\quad \left.\cdot \left[{}^{RL}_{(m+n)/2+}I^{\nu} f(n) + {}^{RL}I^{\nu}_{m+n/2-} f(m)\right] - f\left(\frac{m+n}{2}\right)\right| \\
&\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+1}(\nu+1)} \\
&\quad \cdot \left[\|g\|_{[m,(m+n)/2],\infty} f'(n) + \|g\|_{[m,(m+n)/2],\infty} f'(n)\right] \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)
\end{aligned}
$$

$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+1}(\nu+1)}f'(n)$$
$$+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)}+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\right]$$
$$\cdot\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right). \quad (32)$$

(3) If $\phi(x)=x$, $g(x)=1$ and $\nu=1$, then inequality (26) becomes

$$\left|\frac{1}{n-m}\int_m^n f(x)dx-f\left(\frac{m+n}{2}\right)\right|$$
$$\leq\frac{(n-m)^2}{4}f'(n)$$
$$+\left[\frac{(n-m)^2+2(n-m)^2}{48}\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right]. \quad (33)$$

Theorem 12. *Let $0\leq m\leq n$ and $f:[m,n]\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a continuously differentiable function on the interval $[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$, and let $g:[m,m]\longrightarrow\mathbb{R}$ be integrable, positive, and weighted symmetric function with respect to $(m+n)/2$. If, in addition, $|f'|^q$ is convex on $[m,n]$, $q\leq 1$, and ϕ is increasing and positive function from $[m,n]$ onto itself such that its derivative $\phi'(x)$ is continuous on $[m,m]$, then for $\nu>0$, we have:*

$$|\sigma_1+\sigma_2|\leq\frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^q,\left|f'(n)\right|^q\right)\right]^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^q,\left|f'(n)\right|^q\right)\right]^{1/q}$$
$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}}\right.$$
$$\left.+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right]\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}. \quad (34)$$

Proof. Since $|f'|^q$ is η-convex on $[m,n]$ for $t\in[\phi^{-1}(m),\phi^{-1}(n)]$, so

$$\left|\left(f'o\phi\right)(t)\right|^q=\left|f'\left(\frac{n-\phi(t)}{n-m}m+\frac{\phi(t)-m}{n-m}n\right)\right|^q$$
$$\leq\left|f'(n)\right|^q+\frac{n-\phi(t)}{n-m}\eta\left(\left|f'(m)\right|^q,\left|f'(n)\right|^q\right). \quad (35)$$

By using power mean integral, Lemma 8, and η-convexity of $|f'|^q$, we have

$$|\sigma_1+\sigma_2|\leq\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)}$$
$$\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right|$$
$$\cdot\left(f'o\phi\right)(t)\phi'(t)dt+\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)}$$
$$\cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|$$
$$\cdot\left(f'o\phi\right)(t)\phi'(t)dt$$
$$\leq\frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)}\right.$$
$$\left.\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right|\phi'(t)dt\right)^{1}$$
$$-1/q\times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)}\right.$$
$$\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right|$$
$$\left.\cdot\left|\left(f'o\phi\right)(t)\right|^q\phi'(t)dt\right)^{1/q}$$
$$+\frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)}\right.$$
$$\left.\cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|\phi'(t)dt\right)^{1-1/q}$$
$$\times\left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)}\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|\right.$$
$$\left.\cdot\left|\left(f'o\phi\right)(t)\right|^q\phi'(t)dt\right)^{1/q}$$
$$\leq\frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)}$$
$$\cdot\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q}$$
$$\times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\left|f'o\phi\right)(t)\right|^q$$

$$
\begin{aligned}
&\cdot \phi'(t)dt)^{1/q} + \frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)} \\
&\cdot \left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_{t}^{\phi^{-1}(m)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q} \\
&\times \left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_{t}^{\phi^{-1}(n)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\right. \\
&\left.\cdot \left|f'o\phi\right)(t)\right|^{q}\phi'(t)dt\Bigg)^{1/q} \\
&\leq \frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)} \\
&\cdot \left(\int_{\phi^{-1}(m)}^{\phi^{-1}m+n/2} \left|\int_{\phi^{-1}(m)}^{t} \phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q} \\
&\times \left[\int_{\phi^{-1}(m)}^{\phi^{-1}m+n/2} \left|\int_{\phi^{-1}(m)}^{t} \phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\right. \\
&\times \left(\left|f'(n)\right|^{q} + \frac{n-\phi(t)}{n-m}\eta\left(\left|f'(m)\right|^{q}, \left|f'(n)\right|^{q}\right)\right. \\
&\left.\cdot \phi'(t)dt\right]^{1/q} + \frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)} \\
&\times \left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_{t}^{\phi^{-1}(m)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q} \\
&\times \left[\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_{t}^{\phi^{-1}(n)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\right. \\
&\times \left(\left|f'(n)\right|^{q} + \frac{n-\phi(t)}{n-m}\right. \\
&\left.\cdot \eta\left(\left|f'(m)\right|^{q}, \left|f'(n)\right|^{q}\right)\phi'^{(t)}dt\right]^{1/q} \\
&= \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&+ \|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&+ \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\|g\|_{[m,m+n/2],\infty} \\
&\cdot \left[\eta\left(\left|f'(m)\right|^{q}, \left|f'(n)\right|^{q}\right)\right]^{1/q} \\
&+ \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{(1/q)}}\|g\|_{[m,m+n/2],\infty} \\
&\cdot \left[\eta\left(\left|f'(m)\right|^{q}, \left|f'(n)\right|^{q}\right)\right]^{1/q} \\
&\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&+ \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right] \\
&\cdot \left(\eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)\right)^{1/q},
\end{aligned}
\tag{36}
$$

where

$$
\begin{aligned}
&\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)} \left|\int_{\phi^{-1}(m)}^{t} (\phi(t)-m)^{\nu-1}(n-\phi(t))\right|\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+2}(\nu+3)}{2^{\nu+2}(\nu+1)(\nu+2)}, \\
&\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} |\int_{t}^{\phi^{-1}(n)} (n-\phi(t))^{\nu}(n-\phi(t))|\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+2}}{2^{\nu+2}(\nu+2)}.
\end{aligned}
\tag{37}
$$

□

Remark 13. From Theorem 12, we can get following special cases:

(1) If $\phi(x) = x$, then inequality (34) becomes

$$
\begin{aligned}
&\left|f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{m+n/2+}I^{\nu}g(n) + {}^{RL}I^{\nu}_{m+n/2-}g(m)\right]\right. \\
&\quad \left.- \left[g(n)\left({}^{RL}_{m+n/2+}I^{\nu}_{g}f\right)(n) + g(m)\left({}^{RL}_{g}I^{\nu}_{m+n/2-}f\right)(m)\right]\right| \\
&\quad \leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&\quad + \|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\|g\|_{[m,m+n/2],\infty} \\
&\quad \cdot \left[\eta\left(\left|f'(m)\right|^{q}, \left|f'(n)\right|^{q}\right)\right]^{1/q} \\
&\quad \cdot \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{(1/q)}}\|g\|_{[m,m+n/2],\infty} \\
&\quad \cdot \left[\eta\left(\left|f'(m)\right|^{q}, \left|f'(n)\right|^{q}\right)\right]^{1/q} \\
&\quad \leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&\quad \cdot \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{(1/q)}}\right. \\
&\quad \left.+ \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right] \\
&\quad \cdot \left(\eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)\right)^{1/q}.
\end{aligned}
\tag{38}
$$

(2) If $\phi(x)=x$ and $g(x)=1$, then inequality (34) becomes

$$\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(n-m)^{\nu}}\cdot\left[{}^{RL}_{m+n/2+}I^{\nu}f(n)+{}^{RL}I^{\nu}_{m+n/2-}f(m)\right]-f\left(\frac{m+n}{2}\right)\right| \leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}} \cdot\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right]\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}. \quad (39)$$

(3) If $\phi(x)=x$, $g(x)=1$ and $\nu=1$, then inequality (34) becomes

$$\left|\frac{1}{n-m}\int_m^n f(x)dx-f\left(\frac{m+n}{2}\right)\right| \leq 2\frac{(n-m)^2}{2^{2+(1/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \frac{(n-m)^2+2^{1/q}(n-m)^2}{2^{1/q}3^{3+(1/q)}}\Bigg] \cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}. \quad (40)$$

Theorem 14. *Let $0\leq m\leq n$ and $f:[m,n]\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a continuously differentiable function on the interval $[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$, and let $g:[m,m]\longrightarrow\mathbb{R}$ be integrable, positive and weighted symmetric function with respect to $(m+n)/2$. If, in addition, $|f'|^q$ is convex on $[m,n]$, $q\leq 1$, and ϕ is increasing and positive function from $[m,n]$ onto itself such that its derivative $\phi'(x)$ is continuous on $[m,m]$, then for $\nu>0$, we have*

$$|\sigma_1+\sigma_2| \leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\|g\|_{[m,(m+n)/2],\infty} \cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q} + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\|g\|_{[m,(m+n)/2],\infty} \cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$

$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\right] \cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}. \quad (41)$$

Proof. Since $|f'|^q$ is η-convex on $[m,n]$, for $t\in[\phi^{-1}(m),\phi^{-1}(n)]$, we get

$$\left|\left(f'o\phi\right)(t)\right|^{q}=\left|f'\left(\frac{n-\phi(t)}{n-m}m+\frac{\phi(t)-m}{n-m}n\right)\right|^{q} \leq\left|f'(n)\right|^{q}+\frac{n-\phi(t)}{n-m}\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right). \quad (42)$$

By using Hölder's inequality, Lemma 8, η-convexity of $|f'|^q$, and properties of modulus, we get

$$|\sigma_1+\sigma_2| \leq \frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)} \cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right| \cdot\left(f'o\phi\right)(t)\phi'(t)dt+\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)} \cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right| \cdot\left(f'o\phi\right)(t)\phi'(t)dt$$

$$\leq\frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)} \cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right|^{p}\phi'(t)dt\right)^{1/p} \times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|\left(f'o\phi\right)(t)\right|^{q}\phi'(t)dt\right)^{1/q} + \frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)} \cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|^{p}\phi'(t)dt\right)^{1/p} \times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|\left(f'o\phi\right)(t)\right|^{q}\phi'(t)dt\right)^{1/q}$$

$$\leq\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)} \times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p} \times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|f'o\phi)(t)\right|^{q}\phi'(t)dt\right)^{1/q}+\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}$$

$$\times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|\int_{t}^{\phi^{-1}(m)}\phi'(x)(n-\phi(x))^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p}$$
$$\times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|f'o\phi\right)(t)\right|^{q}\phi'(t)dt\right)^{1/q}$$
$$\leq\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}$$
$$\cdot\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n)/2}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p}$$
$$\times\left[\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n)/2}\left(\left|f'(n)\right|^{q}+\frac{n-\phi(t)}{n-m}\right.\right.$$
$$\left.\left.\cdot\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\phi'(t)dt\right]^{1/q}$$
$$+\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}\times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\right.$$
$$\left.\cdot\left|\int_{t}^{\phi^{-1}(m)}\phi'(x)(n-\phi(x))^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p}$$
$$\times\left[\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left(\left|f'(n)\right|^{q}+\frac{n-\phi(t)}{n-m}\right.\right.$$
$$\left.\left.\cdot\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\phi'(t)dt\right]^{1/q}$$
$$=\frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\right]$$
$$\cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q},\tag{43}$$

where

$$\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|\int_{\phi^{-1}(m)}^{t}(\phi(t)-m)^{\nu-1}(n-\phi(t))\right|^{p}\phi'(t)dt$$
$$=\frac{(n-m)^{p\nu+2}(p\nu+3)}{2^{p\nu+2}(p\nu+1)(p\nu+2)},$$
$$\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|\int_{t}^{\phi^{-1}(n)}(n-\phi(t))^{\nu}(n-\phi(t))\right|^{p}\phi'(t)dt$$
$$=\frac{(n-m)^{p\nu+2}}{2^{p\nu+2}(p\nu+2)}.\tag{44}$$

This completes the proof. □

Remark 15. From Theorem 14, we can obtain following special cases:

(1) If $\phi(x)=x$, then inequality (41) becomes

$$\left|f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n)/2+}I^{\nu}g(n)+{}^{RL}I^{\nu}_{(m+n)/2-}g(m)\right]\right.$$
$$-\left[g(n)\left({}^{RL}_{(m+n)/2+}I^{\nu}_{g}f\right)(n)\right.$$
$$\left.\left.+g(m)\left({}^{RL}_{g}I^{\nu}_{(m+n)/2-}f\right)(m)\right]\right|$$
$$\leq\frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\right.$$
$$\left.+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\right]$$
$$\cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}.\tag{45}$$

(2) If $\phi(x) = x$ and $g(x) = 1$, then inequality (41) becomes

$$\begin{aligned}
&\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(n-m)^{\nu}} \cdot \left[{}^{RL}_{(m+n)/2+}I^{\nu} f(n) + {}^{RL}I^{\nu}_{m+n/2-} f(m)\right] - f\left(\frac{m+n}{2}\right)\right| \\
&\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&\quad + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\right] \\
&\quad \cdot \left(\eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)\right)^{1/q}.
\end{aligned} \tag{46}$$

(3) If $\phi(x) = x$, $g(x) = 1$ and $\nu = 1$, then inequality (41) becomes

$$\begin{aligned}
&\left|\frac{1}{n-m}\int_m^n f(x)dxs - f\left(\frac{m+n}{2}\right)\right| \\
&\leq 2\frac{(n-m)^2}{2^{3+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} \\
&\quad + \frac{(n-m)^2 + (p+1)^{1/p}(n-m)^2}{(p+1)^{1/p}2^{3+(2/q)}}\Bigg] \\
&\quad \cdot \left(\eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)\right)^{1/q}.
\end{aligned} \tag{47}$$

3. Conclusion

In this paper, we established Hermite-Hadamarad Fejér type inequalities for η-convex function by using weighted fractional integrals. Our results are extensions and generalizations of many existing results in the literature.

Data Availability

All data required for this research is included within the paper.

Conflicts of Interest

The authors of this paper declare that they have no competing interests.

Authors' Contributions

Lei Chen analyzed the results, Waqas Nazeer proposed the problem, Farman Ali wrote the final version of the paper, Thongchai Botmart verified the results and arranged funding for this paper, and Sarah Mehfooz wrote the first version of the paper. Lei Chen and Farman Ali contributed equally to this work and are first co-authors.

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Research Article

Hilfer-Hadamard Nonlocal Integro-Multipoint Fractional Boundary Value Problems

Chanon Promsakon,[1] Sotiris K. Ntouyas,[2,3] and Jessada Tariboon[1]

[1] *Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*
[2] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*
[3] *Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Jessada Tariboon; jessada.t@sci.kmutnb.ac.th

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This paper is concerned with the existence and uniqueness of solutions for a new class of boundary value problems, consisting by Hilfer-Hadamard fractional differential equations, supplemented with nonlocal integro-multipoint boundary conditions. The existence of a unique solution is obtained via Banach contraction mapping principle, while the existence results are established by applying Schaefer and Krasnoselskii fixed point theorems as well as Leray-Schauder nonlinear alternative. Examples illustrating the main results are also constructed.

1. Introduction

The fractional calculus has always been an interesting research topic for many years. This is because fractional differential equations describe many real-world process related to memory and hereditary properties of various materials more accurately as compared to classical-order differential equations. Fractional differential equations arise in lots of engineering and clinical disciplines which include biology, physics, chemistry, economics, signal and image processing, and control theory (see the monographs and papers in [1–8]).

Various types of fractional derivatives were introduced among which the following Riemann-Liouville and Caputo derivatives are the most widely used ones.

(1) *Riemann-Liouville derivative*. For $n-1<\alpha<n$, the derivative of u is

$$^{RL}D_a^{\alpha}u(t) := D^n I_a^{n-\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}u(s)ds \tag{1}$$

(2) *Caputo derivative*. For $n-1<\alpha<n$, the derivative of u is

$$^{C}D_a^{\alpha}u(t) := I_a^{n-\alpha}D^n u(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\left(\frac{d}{ds}\right)^n u(s)ds, \tag{2}$$

where $D^n=(d/dt)^n$. Both Riemann-Liouville and Caputo derivatives are defined via fractional integral, the

Riemann-Liouville fractional integral, which is defined by

$$I_a^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad n-1<\alpha<n \tag{3}$$

A generalization of derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [9], known as *the Hilfer fractional derivative* of order α and a type $\beta \in [0,1]$, which interpolates between the Riemann-Liouville and Caputo derivatives, since it is reduced to the Riemann-Liouville and Caputo fractional derivatives when $\beta=0$ and $\beta=1$, respectively. The Hilfer fractional derivative of order α and parameter β of a function u is defined by

$$^H D_a^{\alpha,\beta} u(t) = I_a^{\beta(n-\alpha)} D^n I_a^{(1-\beta)(n-\alpha)} u(t), \tag{4}$$

where $n-1<\alpha<n, 0\le\beta\le1, t>a, D=d/dt$. Some properties and applications of the Hilfer derivative are given in [10, 11] and references cited therein. Initial value problems involving Hilfer fractional derivatives were studied by several authors (see, for example, [12–15] and references therein). Nonlocal boundary value problems for Hilfer fractional derivative were studied in [16, 17].

The Hadamard fractional calculus contains fractional derivative and integral with respect to the logarithmic function while someone say that it is generalized of the derivative $(t(d/dt))^\alpha$, where α is an arbitrary order. The Hadamard calculus can be obtained by replacing as $d/dt \longrightarrow td/dt$, $(t-s)^{(\cdot)} \longrightarrow (\log_e t - \log_e s)^{(\cdot)}$, and $ds \longrightarrow (1/s)d$ s in (1)–(3). In the same way, the concept of Hilfer-Hadamard derivative is arrived by the modified definition in (4). Existence and uniqueness of solutions for system of Hilfer-Hadamard sequential fractional differential equations with two point boundary conditions were studied in [18].

In this paper, we study existence and uniqueness of solutions for boundary value problems for Hilfer-Hadamard fractional differential equations with nonlocal integro-multipoint boundary conditions,

$$\begin{cases} {}^{\mathrm{HH}}D_1^{\alpha,\beta} x(t) = f(t,x(t)), & t\in[1,T], \\ x(1)=0, & \displaystyle\sum_{i=1}^m \theta_i x(\xi_i) = \lambda^H I_1^\delta x(\eta), \end{cases} \tag{5}$$

where ${}^{\mathrm{HH}}D_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha\in(1,2]$ and type $\beta\in[0,1]$, $\theta_i, \lambda \in \mathbb{R}$, $i=1,2,\cdots,$ m, are given constants, and $f:[1,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function, ${}^H I^\delta$ is the Hadamard fractional integral of order $\delta>0$, and $\eta, \xi_i \in (1,T)$, $i=1,2,\cdots,m$, are given points.

Existence and uniqueness results are established by using classical fixed point theorems. We make use of Banach's fixed point theorem to obtain the uniqueness result, while Schaefer and Krasnoselskii's fixed point theorem [19] as well as nonlinear alternative of Leray-Schauder type [20] is applied to obtain the existence results for the problem (5).

The paper is constructed as follows: in Section 2, we recall some basic facts needed in our study. The main results are proved in Section 3. Examples illustrating the main results are presented in Section 4.

2. Preliminaries

In this section, some basic definitions, lemmas, and theorems are mentioned.

Definition 1 (Hadamard fractional integral [2]). The Hadamard fractional integral of order $\alpha>0$ for a function $f:[a,\infty)\longrightarrow\mathbb{R}$ is defined as

$$^H I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \quad t>a, \tag{6}$$

provided the integral exists, where $\log(.) = \log_e(.)$.

Definition 2 (Hadamard fractional derivative [2]). The Hadamard fractional derivative of order $\alpha>0$, applied to the fuction $f:[a,\infty)\longrightarrow\mathbb{R}$, is defined as follows:

$$_H D_a^\alpha f(t) = \delta^n \left({}^H I_a^{n-\alpha} f\right)(t), n=[\alpha]+1, \tag{7}$$

where $\delta^n = (t(d/dt))^n$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 3 (Hilfer-Hadamard fractional derivative [11]). Let $n-1<\alpha<n$ and $0\le\beta\le1$, $f\in L^1(a,b)$. The Hilfer-Hadamard fractional derivative of order α and tybe β of f is defined as

$$\begin{aligned} \left({}^{\mathrm{HH}}D_a^{\alpha,\beta} f\right)(t) &= \left({}^H I_a^{\beta(n-\alpha)} \delta^{nH} I_a^{(n-\alpha)(1-\beta)} f\right)(t) \\ &= \left({}^H I_a^{\beta(n-\alpha)} \delta^{nH} I_a^{(n-\gamma)} f\right)(t) \\ &= \left({}^H I_a^{\beta(n-\alpha)} {}_H D_a^\gamma f\right)(t), \quad \gamma=\alpha+n\beta-\alpha\beta, \end{aligned} \tag{8}$$

where ${}^H I_a^{(\cdot)}$ and ${}_H D_a^{(\cdot)}$ is the Hadamard fractional integral and derivative defined by (6) and (7), respectively.

The Hilfer-Hadamard fractional derivative may be viewed as interpolating between Hadamard and Caputo-Hadamard fractional derivatives. Indeed, for $\beta=0$, this derivative reduces to the Hadamard fractional derivative while for $\beta=1$, it leads the Caputo-Hadamard derivative defined by

$${}_H^C D_a^\alpha f(t) = \left({}^H I_a^{n-\alpha} \delta^n f\right)(t), \quad n=[\alpha]+1. \tag{9}$$

We recall the following known theorem by Kilbas et al. [2] which will be used in the following.

Theorem 4 ([2]). *Let $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n = [\alpha] + 1$, and $0 < a < b < \infty$. If $f \in L^1(a, b)$ and $({}^H I_a^{n-\gamma} f)(t) \in AC_\delta^n[a, b]$, then*

$$
\begin{aligned}
{}^H I_a^{\alpha}\left({}^{HH}D_a^{\alpha,\beta} f\right)(t) &= {}^H I_a^{\gamma}\left({}^{HH}D_a^{\gamma} f\right)(t) \\
&= f(t) - \sum_{j=0}^{n-1} \frac{\left(\delta^{(n-j-1)}\left({}^H I_a^{n-\gamma} f\right)\right)(a)}{\Gamma(\gamma - j)} \left(\log \frac{t}{a}\right)^{\gamma-j-1}.
\end{aligned} \tag{10}
$$

Since $\gamma \in [\alpha, n]$, then the $\Gamma(\gamma - j)$ exists for all $j = 1, 2, \cdots, n-1$.

Finally, we will use the following well-known fixed point theorems on Banach space for proving the existence and uniqueness of the solutions to Hilfer-Hadamard fractional boundary value problem (5).

Theorem 5 (Banach's contraction principle [21]). *Let X be a Banach space, $D \subset X$ be closed, and $T : D \longrightarrow D$ be a contraction (i.e., there exists a constant $L \in (0, 1)$ such that for any $x, y \in X$, $|Tx - Ty| \le L|x - y|$). Then, T has a unique fixed point on X.*

Theorem 6 (Krasnoselskii's fixed point theorem)[19]. *Let Y be a bounded, closed, convex, and nonempty subset of a Banach space X. Let F_1 and F_2 be the operators satisfying the following conditions: (i) $F_1 y_1 + F_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$; (ii) F_1 is compact and continuous; and (iii) F_2 is a contraction mapping. Then, there exists $y \in Y$ such that $y = F_1 y + F_2 y$.*

Theorem 7 (Schaefer fixed point theorem [22]). *Let $T : E \longrightarrow E$ be a completely continuous operator (i.e., a continuous map T restricted to any bounded set in E is compact). Let $\varepsilon(T) = \{x \in E : x = \lambda T(x), 0 \le \lambda \le 1\}$. Then, either the set $\varepsilon(T)$ is unbounded or T has at least one fixed point.*

Theorem 8 (Nonlinear alternative for single-valued maps) ([20]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $\mathscr{A} : \bar{U} \longrightarrow C$ is continuous and compact (that is, $\mathscr{A}(\bar{U})$ is a relatively compact subset of C) map. Then, either*

(i) $\mathscr{A}$ has a fixed point in $\bar{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $x = \lambda \mathscr{A}(x)$

3. Main Results

In this section, we prove existence and uniqueness of solutions for nonlinear Hilfer-Hadamard fractional boundary value problem (5). Firstly, we start by proving a basic lemma concerning a linear variant of the boundary value problem (5), which will be used to transform the boundary value problem (5) into an equivalent integral equation. In this case, $n = [\alpha] + 1 = 2$; then, we have $\gamma = \alpha + (2 - \alpha)\beta$.

Lemma 9. *Let $h \in C([1, T], \mathbb{R})$ and $\Lambda \neq 0$, where*

$$
\Lambda = \sum_{i=1}^{m} \theta_i (\log \xi_i)^{\gamma-1} - \lambda \frac{\Gamma(\gamma)}{\Gamma(\gamma + \delta)} (\log \eta)^{\gamma+\delta-1}. \tag{11}
$$

Then, x is a solution of the following linear Hilfer-Hadamard fractional differential equation:

$$
{}^{HH}D_1^{\alpha,\beta} x(t) = h(t), 1 < \alpha \le 2, \quad t \in [1, T], \tag{12}
$$

supplemented with the boundary conditions in (5), if and only if

$$
\begin{aligned}
x(t) &= {}^H I_1^{\alpha} h(t) + \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \lambda \left({}^H I_1^{\alpha+\delta} h\right)(\eta) - \sum_{i=1}^{m} \theta_i \left({}^H I_1^{\alpha} h\right)(\xi_i) \right\} = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{h(\tau)}{\tau} d\tau \\
&\quad + \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \lambda \frac{1}{\Gamma(\alpha + \delta)} \int_1^{\eta} \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{h(\tau)}{\tau} d\tau - \sum_{i=1}^{m} \theta_i \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{h(\tau)}{\tau} d\tau \right\}, \quad t \in [1, T].
\end{aligned} \tag{13}
$$

Proof. By taking the Hadamard fractional integral of order α from 1 to t on both sides of (12) and using Theorem 4, it follows that

$$
x(t) - \sum_{j=0}^{1} \frac{\left(\delta^{(2-j-1)}\left({}_H I_{1^+}^{2-\gamma} x\right)\right)(1)}{\Gamma(\gamma - j)} (\log t)^{\gamma-j-1} = {}^H I_1^{\alpha} h(t). \tag{14}
$$

Then, we have

$$
\begin{aligned}
x(t) &- \frac{\delta\left({}_H I_{1^+}^{2-\gamma} x\right)(1)}{\Gamma(\gamma)} (\log t)^{\gamma-1} \\
&- \frac{\left({}_H I_{1^+}^{2-\gamma} x\right)(1)}{\Gamma(\gamma - 1)} (\log t)^{\gamma-2} = {}^H I_1^{\alpha} h(t).
\end{aligned} \tag{15}
$$

Equation (15) can be rewritten by

$$x(t) = c_0(\log t)^{\gamma-1} + c_1(\log t)^{\gamma-2} + \frac{1}{\Gamma(\alpha)}\int_1^t \frac{h(s)}{s}\left(\log \frac{t}{s}\right)^{\alpha-1} ds, \tag{16}$$

where c_0, c_1 are arbitrary constants. Now, the first boundary condition $x(1) = 0$ together with (16) yields $c_1 = 0$, since $\gamma \in [\alpha, 2]$. Putting $c_1 = 0$ in (16), we get

$$x(t) = c_0(\log\ t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds. \tag{17}$$

Next, the second boundary condition $\sum_{i=1}^{m}\theta_i x(\xi_i) = \lambda^H I_1^\delta x(\eta)$ together with (17) yields

$$c_0 = \frac{1}{\Lambda}\left\{\lambda^H I_1^{\alpha+\delta} h(\eta) - \sum_{i=1}^{m}\theta_i {}^H I_1^\alpha h(\xi_i)\right\}. \tag{18}$$

Substituting the value of c_0 in (17), we get equation (13) as desired.

The converse follows by direct computation. The proof is completed.□ □

Let us introduce the Banach space $X = C([1, T], \mathbb{R})$ endowed with the norm defined by $\|x\|{:=}\max_{t\in[1,T]}|x(t)|$.

In view of Lemma 9, we define an operator $\mathcal{F} : X \longrightarrow X$, where

$$\begin{aligned}(\mathcal{F}x)(t) = {} & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \\ & + \frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha+\delta)}\int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \right. \\ & \left. - \sum_{i=1}^{m}\theta_i \frac{1}{\Gamma(\alpha)}\int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau\right\}, \quad t \in [1, T].\end{aligned} \tag{19}$$

In the following, for convenience, we put

$$\Omega = \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^{m}|\theta_i|\frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)}\right\}. \tag{20}$$

We need the following hypotheses in the sequel:

(H_1). There exists a constant $l > 0$ such that for all $t \in [1, T]$ and $u_i \in \mathbb{R}$, $i = 1, 2$,

$$|f(t, u_1) - f(t, u_2)| \le l|u_1 - u_2|. \tag{21}$$

(H_2). There exists a continuous nonnegative function $\phi \in C([1, T], \mathbb{R}^+)$ such that

$$|f(t, u)| \le \phi(t), \quad \text{for each } (t, u) \in [1, T] \times \mathbb{R}. \tag{22}$$

(H_3). There exists a real constant $M > 0$ such that for all $t \in [1, T]$, $u \in \mathbb{R}$,

$$|f(t, u)| \le M. \tag{23}$$

(H_4). There exist $p \in C([1, T], \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$|f(t, u)| \le p(t)\psi(\|u\|) \quad \text{for each } (t, u) \in [1, T] \times \mathbb{R}. \tag{24}$$

(H_5). There exists a constant $K > 0$ such that

$$\frac{K}{\Omega\|p\|\psi(K)} > 1. \tag{25}$$

3.1. Existence and Uniqueness Result via Banach's Fixed Point Theorem. We prove an existence and uniqueness result based on Banach's contraction mapping principle.

Theorem 10. *Assume that (H_1) holds. Then, boundary value problem (5) has a unique solution on $[1, T]$, provided that $l\Omega < 1$, where Ω is defined by (20).*

Proof. We will use Banach's fixed point theorem to prove that $\mathcal{F}$, defined by (19), has a unique fixed point. Fixing $N = \max_{t\in[1,T]} | f(t, 0) | < \infty$ and using hypothesis (H_1), we obtain

$$\begin{aligned}|f(t, x(t))| &\le |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\ &\le l|x(t)| + |f(t, 0)| \le l\|x\| + N.\end{aligned} \tag{26}$$

Choose

$$r \ge \frac{N\Omega}{1 - l\Omega}. \tag{27}$$

We divide the proof into two steps.

Step I. We show that $\mathcal{F}(B_r) \subset B_r$, where $B_r = \{x \in X : \|x\| < r\}$. Let $x \in B_r$. Then, we have

$$
\begin{aligned}
|(\mathcal{F}x)(t)| \le{}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\
&+ \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\
&\left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\
\le{}& \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} (l\|x\| + N) + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \\
&\left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} (l\|x\| + N) \\
={}& \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \right. \\
&\left. \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right] (l\|x\| + N) \\
\le{}& \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \right. \\
&\left. \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right] (lr + N) = \Omega(lr + N) \le r.
\end{aligned}
\tag{28}
$$

Thus,

$$
\|(\mathcal{F}x)\| = \max_{t \in [1,T]} |(\mathcal{F}x)(t)| \le r, \tag{29}
$$

which means that $(\mathcal{F}B_r) \subset B_r$.

Step II. To show that the operator $\mathcal{F}$ is a contraction, let $x_1, x_2 \in X$. Then, for any $t \in [1, T]$, we have

$$
\begin{aligned}
|(\mathcal{F}x_2)(t) - (\mathcal{F}x_1)(t)| \le{}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x_2(\tau)) - f(s, x_1(\tau))|}{\tau} d\tau \\
&+ \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \boxtimes \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{|f(s, x_2(\tau)) - f(s, x_1(\tau))|}{\tau} d\tau \right. \\
&\left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x_2(\tau)) - f(\tau, x_1(\tau))|}{\tau} d\tau \right\} \\
\le{}& l\|x_2 - x_1\| \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right].
\end{aligned}
\tag{30}
$$

Thus,

$$
\|(\mathcal{F}x_2) - (\mathcal{F}x_1)\| = \max_{t \in [1,T]} |(\mathcal{F}x_2)(t) - (\mathcal{F}x_1)(t)| \le l\Omega \|x_2 - x_1\|, \tag{31}
$$

which, in view of $l\Omega < 1$, shows that the operator $\mathcal{F}$ is a contraction. By Theorem 5, we get that the operator $\mathcal{F}$ has a unique fixed point. Therefore, the problem (5) has a unique solution on $[1, T]$. The proof is completed. $\square$ $\square$

3.2. Existence Result via Krasnoselskii's Fixed Point Theorem. In this subsection, we prove an existence result based on Krasnoselskii's fixed point theorem.

Theorem 11. *Assume that (H_1) - (H_2) hold. Then, the problem (5) has at least one solution on $[1, T]$, provided that*

$$
\frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} l < 1. \tag{32}
$$

Proof. By assumption (H_2), we can fix $\rho \ge \Omega \|\phi\|$, where $\|\phi\| = \sup_{t \in [1,T]} |\phi(t)|$ and consider $B_\rho = \{x \in C([1, T], \mathbb{R}) : \|x\| \le \rho\}$. We split the operator $\mathcal{F} : C([1, T], \mathbb{R}) \longrightarrow C([1, T], \mathbb{R})$ defined by (19) as $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by

$$
\begin{aligned}
(\mathcal{F}_1 x)(t) ={}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau, \\
(\mathcal{F}_2 x)(t) ={}& \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \frac{\lambda}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \right. \\
&\left. - \sum_{i=1}^m \theta_i \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \right\}.
\end{aligned}
\tag{33}
$$

For any $x, y \in B_\rho$, we have

$$
\begin{aligned}
|(\mathcal{F}_1 x)(t) + (\mathcal{F}_2 y)(t)| \le{}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\
&+ \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{|f(\tau, y(\tau))|}{\tau} d\tau \right. \\
&\left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{|f(\tau, y(\tau))|}{\tau} d\tau \right\} \\
\le{}& \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right] \|\phi\| \\
={}& \Omega \|\phi\| \le r.
\end{aligned}
\tag{34}
$$

Hence, $\|(\mathcal{F}_1 x) + (\mathcal{F}_2 y)\| \le \rho$, which shows that $(\mathcal{F}_1 x) + (\mathcal{F}_2 y) \in B_\rho$. It is easy to prove, using conditions (H_1) and (32), that the operator $\mathcal{F}_2$ is a contraction mapping.

Next, the operator $\mathcal{F}_1$ is continuous by the continuity of f. Also, $\mathcal{F}_1$ is uniformly bounded on B_ρ, since

$$
\|\mathcal{F}_1 x\| \le \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|\phi\|. \tag{35}
$$

Finally, we prove the compactness of the operator $\mathscr{F}_1$. For $t_1, t_2 \in [1, T], t_1 < t_2$, we have

$$\begin{aligned} &|(\mathscr{F}_1 x)(t_2) - (\mathscr{F}_1 x)(t_1)| \\ &\quad \le \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{\tau} \right)^{\alpha-1} - \left(\log \frac{t_1}{\tau} \right)^{\alpha-1} \right] \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\quad \le \frac{\|\phi\|}{\Gamma(\alpha+1)} [2(\log t_2 - \log t_1)^\alpha + |(\log t_2)^\alpha - (\log t_1)^\alpha|], \end{aligned} \tag{36}$$

which tends to zero, independently of $x \in B_\rho$, as $t_1 \longrightarrow t_2$. Thus, $\mathscr{F}_1$ is equicontinuous. From the Arzelá-Ascoli theorem, we conclude that the operator $\mathscr{F}_1$ is compact on B_ρ. Thus, the hypotheses of Krasnoselskii fixed point theorem are satisfied, and therefore, there exists at least one solution on $[1, T]$. The proof is finished.□ □

3.3. Existence Result via Schaefer's Fixed Point Theorem. Our second existence result is based on Schaefer's fixed point theorem.

Theorem 12. *Assume that (H_3) holds. Then, the boundary value problem (5) has at least one solution on $[1, T]$.*

Proof. We will prove that the operator $\mathscr{F}$, defined by (19), has a fixed point, by using Schaefer's fixed point theorem. We divide the proof into two steps.

Step I. We show that the operator $\mathscr{F} : X \longrightarrow X$ is completely continuous.

We show first that $\mathscr{F}$ is continuous. Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in X. Then, for each $t \in [1, T]$, we have

$$\begin{aligned} &|(\mathscr{F}x_n)(t) - (\mathscr{F}x)(t)| \\ &\quad \le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{\tau} d\tau \\ &\qquad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{\tau} d\tau \right. \\ &\qquad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{\tau} d\tau \right\}. \end{aligned} \tag{37}$$

Since f is continuous, we get

$$|f(s, x_n(s)) - f(s, x(s))| \longrightarrow 0 \quad \text{as}\, x_n \longrightarrow x. \tag{38}$$

Then

$$\|(\mathscr{F}x_n) - (\mathscr{F}x)\| \longrightarrow 0 \quad \text{as}\, x_n \longrightarrow x. \tag{39}$$

Hence, $\mathscr{F}$ is continuous.

Secondly, we show that the operator $\mathscr{F}$ maps bounded sets into bounded sets in X. For a positive number R, let $B_R = \{x \in X : \|x\| \le R\}$ be a bounded ball in X. Then, for $t \in [1, T]$, we have

$$\begin{aligned} |(\mathscr{F}x)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\ &\le \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} M, \end{aligned} \tag{40}$$

and

$$\|\mathscr{F}x\| \le \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} M. \tag{41}$$

Thirdly, we show that $\mathscr{F}$ maps bounded sets into equicontinuous sets. Let $t_1, t_2 \in [1, T]$ with $t_1 < t_2$ and $u \in B_R$. Then, we have

$$\begin{aligned} &|(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| \le \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{\tau} \right)^{\alpha-1} \right. \\ &\quad \left. - \left(\log \frac{t_1}{\tau} \right)^{\alpha-1} \right] \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\quad + \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\ &\qquad \le \frac{M}{\Gamma(\alpha+1)} [2(\log t_2 - \log t_1)^\alpha + |(\log t_2)^\alpha - (\log t_1)^\alpha|] \\ &\quad + \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|\Lambda|} \left\{ \frac{|\lambda| (\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} M, \end{aligned} \tag{42}$$

which tends to zero, independently of $x \in B_R$, as $t_1 \longrightarrow t_2$.

Thus, the Arzelá-Ascoli theorem applies and hence, $\mathcal{F} : X \longrightarrow X$ is completely continuous.

Step II. We show that the set $\mathcal{E} = \{x \in X \mid x = \nu(\mathcal{F}x), 0 \le \nu \le 1\}$ is bounded. Let $x \in \mathcal{E}$, then $x = \nu(\mathcal{F}x)$. For any $t \in [1, T]$, we have $x(t) = \nu(\mathcal{F}x)(t)$. Then, in view of the hypothesis (H_3), as in Step I, we obtain

$$|x(t)| \le \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^{m} |\theta_i| \frac{(\log \xi_i)^{\alpha}}{\Gamma(\alpha+1)} \right\} M. \tag{43}$$

Thus,

$$\|x\| \le \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^{m} |\theta_i| \frac{(\log \xi_i)^{\alpha}}{\Gamma(\alpha+1)} \right\} M, \tag{44}$$

which shows that the set $\mathcal{E}$ is bounded. By Theorem 7, we get that the operator $\mathcal{F}$ has at least one fixed point. Therefore, the boundary value problem (5) has at least one solution on $[1, T]$. This completes the proof.□ □

3.4. Existence Result via Leray-Schauder Nonlinear Alternative. Our final existence result is proved via Leray-Schauder nonlinear alternative.

Theorem 13. *Assume that (H_4) and (H_5) hold. Then, the boundary value problem (5) has at least one solution on $[1, T]$.*

Proof. As in Theorem 12, we can prove that the operator $\mathcal{F}$ is completely continuous. We will prove that there exists an open set $U \subseteq C([1, T], \mathbb{R})$ with $x \ne \mu(\mathcal{F}x)$ for $\mu \in (0, 1)$ and $x \in \partial U$.

Let $x \in C([1, T], \mathbb{R})$ be such that $x = \mu(\mathcal{F}x)$ for some $0 < \mu < 1$. Then, for each $t \in [1, T]$, we have

$$\begin{aligned} |x(t)| \le{} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ & + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^{\eta} \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\ & \left. + \sum_{i=1}^{m} |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\ \le{} & \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \right. \\ & \left. \left. + \sum_{i=1}^{m} |\theta_i| \frac{(\log \xi_i)^{\alpha}}{\Gamma(\alpha+1)} \right\} \right] \|p\| \psi(\|x\|). \end{aligned} \tag{45}$$

Consequently,

$$U = \{x \in C([1, T], \mathbb{R}) : \|x\| < K\}. \tag{46}$$

In view of (H_5), there is no solution x such that $\|x\| \ne K$. Let us set

$$U = \{x \in C([1, T], \mathbb{R}) : \|x\| < K\}. \tag{47}$$

The operator $\mathcal{F} : \bar{U} \longrightarrow C([1, T], \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $x = \mu(\mathcal{F}x)$ for some $\mu \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 8), we deduce that $\mathcal{F}$ has a fixed point $x \in \bar{U}$ which is a solution of the boundary value problem (5). The proof is completed.□ □

4. Examples

Example 1. Consider the Hilfer-Hadamard nonlocal integro-multipoint fractional boundary value problem of the form

$$\begin{cases} {}^{\mathrm{HH}}D_1^{3/2,1/2} x(t) = f(t, x(t)), \quad t \in [1, 4.5], \\ x(1) = 0, \quad \dfrac{6}{29} {}^{H}I_1^{5/2} x\left(\dfrac{5}{2}\right) = \dfrac{1}{11} x\left(\dfrac{3}{2}\right) + \dfrac{2}{13} x(2) + \dfrac{3}{17} x(3) + \dfrac{4}{19} x\left(\dfrac{7}{2}\right) + \dfrac{5}{23} x(4). \end{cases} \tag{48}$$

Set constants $\alpha = 3/2$, $\beta = 1/2$, $T = 9/2$, $m = 5$, $\theta_1 = 1/11$, $\theta_2 = 2/13$, $\theta_3 = 3/17$, $\theta_4 = 4/19$, $\theta_5 = 5/23$, $\xi_1 = 3/2$, $\xi_2 = 2$, $\xi_3 = 3$, $\xi_4 = 7/2$, $\xi_5 = 4$, $\lambda = 6/29$, $\delta = 5/2$, and $\eta = 5/2$. Then, from these data, we compute that $\gamma = 7/4$, $|\Lambda| \approx 1.152315230$, and $\Omega \approx 2.250815481$,

(i) Let a nonlinear unbounded function $f : [1,4.5] \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(t, x) = \frac{e^{1-t}}{t+4} \left(\frac{x^2 + 2|x|}{1+|x|} \right) + \frac{1}{2} t^2 + 1 \tag{49}$$

Now, we can verify that function $f(t, x)$ satisfies the Lipchitz condition (H_1) with $l = 2/5$, that is, $|f(t, x) - f(t, y)| \le (2/5)|x - y|$, for all $t \in [1,4.5]$ and $x, y \in \mathbb{R}$. Then, we obtain $l\Omega \approx 0.9003261924 < 1$. The conclusion can be gotten from Theorem 10 that the problem (47), with f given by (48), has a unique solution on $[1,4.5]$.

(ii) If the term x^2 in (48) is replaced by $|x|$, that is,

$$f(t,x)=\frac{3e^{1-t}}{t+4}\left(\frac{|x|}{1+|x|}\right)+\frac{1}{2}t^2+1, \tag{50}$$

then the nonlinear function f is bounded by a function of t by

$$|f(t,x)|\le\frac{3e^{1-t}}{t+4}+\frac{1}{2}t^2+1\coloneqq\phi(t), \tag{51}$$

which is satisfied a condition (H_2). It is easy to check that function f in (49) is fulfilled the Lipchitz condition with constant $l=3/5$. Since

$$\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\frac{|\lambda|(\log\eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}+\sum_{i=1}^{m}|\theta_i|\frac{(\log\xi_i)^{\alpha}}{\Gamma(\alpha+1)}\right\}l\approx 0.5179201139<1, \tag{52}$$

we can deduce that the Hilfer-Hardamard nonlocal integro-multipoint fractional boundary value problem (47), with f given by (49), has at least one solution on $[1,4.5]$. In addition, we observe that Theorem 10 cannot be used in this case because $l\Omega\approx 1.350489289>1$

(iii) If f is a non-Lipschitzian function as

$$f(t,x)=\frac{e^{1-t}}{t+4}\sin^{12}\left(\frac{x^2+2|x|}{1+|x|}\right)+\frac{1}{2}t^2+1, \tag{53}$$

to claim the existence of boundary value problem (47), with f given by (52), we can find that

$$|f(t,x)|\le\frac{1}{5}+\frac{1}{2}\left(\frac{9}{2}\right)^2+1=\frac{453}{40} \tag{54}$$

Hence, (H_3) is satisfied with $M=453/40$. Therefore, by the benefit of Theorem 12, problem (47) with f given by (52) has at least one solution on $[1,4.5]$.

(iv) If f is a non-Lipschitzian function defined by

$$f(t,x)=\frac{e^{1-t}}{t+4}\left(\frac{x^{26}}{1+x^{24}}+\frac{1}{9}t+\frac{1}{2}\right), \tag{55}$$

we see that (H_4) holds as

$$|f(t,x)|\le\frac{e^{1-t}}{t+4}\left(x^2+1\right), \tag{56}$$

by setting $p(t)=(e^{1-t})/(t+4)$ and $\psi(x)=x^2+1$. Then, we obtain $\|p\|=1/5$ and there exists a constant $K\in($ $0.6273106970,1.594106403)$ satisfying condition (H_5). Thus, using Theorem 13, the boundary value problem (47), with f is given in (54), has at least one solution on $[1,4.5]$

5. Conclusion

In this paper, we have presented the existence and uniqueness criteria for solutions for Hilfer-Hadamard fractional differential equation complemented with nonlocal integro-multipoint boundary conditions. Firstly, we convert the given nonlinear problem into a fixed point problem. Once the fixed point operator is available, we make use of Banach contraction mapping principle to obtain the uniqueness result, while the existence results are established by applying Schaefer and Krasnoselskii fixed point theorems as well as Leray-Schauder nonlinear alternative. Our results are new in the given configuration and enrich the literature on boundary value problems involving Hilfer-Hadamard fractional differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Hermite–Hadamard-Type Inequalities for Generalized Convex Functions via the Caputo-Fabrizio Fractional Integral Operator

Dong Zhang,[1] Muhammad Shoaib Saleem ,[2] Thongchai Botmart ,[3] M. S. Zahoor,[2] and R. Bano[2]

[1]*School of Mathematics and Statistics, Cangzhou Normal University, Cangzhou 061001, China*
[2]*Department of Mathematics, University of Okara, Okara, Pakistan*
[3]*Thongchai Botmart Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*

Correspondence should be addressed to Thongchai Botmart; thongbo@kku.ac.th

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Due to applications in almost every area of mathematics, the theory of convex and nonconvex functions becomes a hot area of research for many mathematicians. In the present research, we generalize the Hermite–Hadamard-type inequalities for (p, h)-convex functions. Moreover, we establish some new inequalities via the Caputo-Fabrizio fractional integral operator for (p, h)-convex functions. Finally, the applications of our main findings are also given.

1. Introduction

In the last few decades, the subject of fractional calculus got attention of many researchers of different fields of pure and applied mathematics like mechanics, convex analysis, and relativity [1–3]. Nowadays, the researches in convex analysis cannot ignore the deep connectivity of both inequalities in convex analysis and fractional integral operator. Niels Henrik Abel gave birth to fractional calculus. The applications of fractional calculus can be seen in [4–9]. The first appearance of fractional derivative had been seen in a letter. The letter was written to Guillaume de lHopital by Gottfried Wilhelm Leibniz in 1695.

The fractional calculus techniques can be seen in many branches of science and engineering. Geometric and physical interpretation of fractional integration and fractional differentiation can be viewed in [10]. There are different fractional integral operators in which we use the integral inequalities (see, for example, [11–16]). The well-known inequality given by Hermite in 1881 can be stated as follows.

Theorem 1. *Let $\zeta : L \longrightarrow \mathbb{R}$ be a (p, h)-convex function defined on the interval L of real numbers and $c, d \in \mathbb{L}$ with $c < d$. Then, the following inequality holds:*

$$\frac{1}{2h(1/2)}\zeta\left[\left(\frac{c^p + d^p}{2}\right)\right]^{1/p} \leq \frac{p}{d^p - c^p}\int_c^d x^{p-1}\zeta(x)dx \leq (\zeta(c) + \zeta(d))\int_0^1 h(t)dt. \tag{1}$$

The fractional Hermite–Hadamard and Hermite–Hadamard inequalities via fractional integral can be seen in [17, 18]. For the history of Hermite–Hadamard-type inequalities, we refer to the readers [19, 20]. The outstanding applications of fractional calculus and fractional derivatives and integrals are given in [21]. Moreover, we refer the readers for a detailed study [22–27].

In the present article, we generalize the Hermite–Hadamard-type inequalities for (p, h)-convex functions. Moreover, we establish some new inequalities via the

Caputo-Fabrizio fractional integral operator for (p, h)-convex functions. Finally, the applications of our main findings are also given.

This paper is organized as follows: in Section 2, some preliminaries are given. In Section 3, we generalized Hermite–Hadamard via Caputo-Fabrizio for (p, h)-convex functions. In Section 4, we give some results related to Caputo-Fabrizio, and in Section 5, some applications to special means are given.

2. Preliminaries

We will start with some basic definitions related to our work.

Definition 2 (convex function) [28]. Let $\zeta : L \longrightarrow \mathbb{R}$ be an extended real-valued function defined on a convex set $L \subseteq \mathbb{R}^n$. Then, the function ζ is convex on L if

$$\zeta(tc + (1-t)d) \le t\zeta(c) + (1-t)\zeta(d), \tag{2}$$

for all $c, d \in L$ and $t \in (0, 1)$.

Definition 3 (p-convex function) [29]. A function $\zeta : L = [c, d] \subseteq \mathbb{R}$ is said to be a p-convex function if

$$\begin{gathered}\zeta\left(tc^p + (1-t)d^p\right)^{1/p} \le t\zeta(c) + (1-t)\zeta(d) \in L,\\ \forall c, d \in L, t \in [0, 1], p \neq 0.\end{gathered} \tag{3}$$

Definition 4 (h-convex function) [30]. Let $h : K \longrightarrow \mathbb{R}$ be a nonnegative function. We say that $\zeta : L \longrightarrow \mathbb{R}$ is an h-convex function or that $\zeta \in SX(h, L)$, if ζ is nonnegative, and for all $\forall c, d \in L, t \in [0, 1]$, we have

$$\zeta(tc + (1-t)d) \le h(t)\zeta(c) + h(1-t)\zeta(d). \tag{4}$$

If inequality (4) is reversed, then ζ is said to be h-convex, i.e., $\zeta \in SV(h, L)$.

Remark 5.

(1) If we take $h(t) = t$, then (4) reduces to (2)

(2) If the function h has the property: $h(t) \ge t$ for all $t \in [0, 1]$, then any nonnegative convex function ζ belongs to the class $SX(h, L)$

(3) If the function h has the property: $h(t) \le t$ for all $t \in [0, 1]$, then any nonnegative convex function ζ belongs to the class $SV(h, L)$

Definition 6 ((p, h)-convex function) [31]. A function $\zeta : L = [c, d] \longrightarrow \mathbb{R}$ is called a (p, h)-convex function if

$$\begin{gathered}\zeta\left(tc^p + (1-t)^p d\right)^{1/p} \le h(t)\zeta(c) + h(1-t)\zeta(d),\\ \forall c, d \in L, t \in [0, 1].\end{gathered} \tag{5}$$

Definition 7 (Caputo-Fabrizio fractional time derivative) [32]. The usual Caputo fractional time derivative (UFD_t) of order γ is given by

$$D_t^\gamma \zeta(t) = \frac{1}{\alpha(1-\gamma)} \int_c^t \frac{\zeta'(x)}{(t-x)^\gamma} dx, \tag{6}$$

with $\gamma \in [0, 1]$ and $d \in [-\infty, t), \zeta \in H^1(c, d), c < d$. By changing the Kernal $(t-x)^{-\gamma}$ with the function $\exp\left(-\gamma(t-x)^\gamma/(1-\gamma)\right)$ and $1/\alpha(1-\gamma)$ with $B(\gamma)/(1-\gamma)$, we obtain the new definition of fractional time derivative:

$$D_t^\gamma \zeta(t) = \frac{B(\gamma)}{1-\gamma} \int_c^t \zeta'(x) \exp\left(\frac{-\gamma(t-x)^\gamma}{1-\gamma}\right) dx. \tag{7}$$

Definition 8 (Caputo-Fabrizio fractional integral) [32]. Let $\zeta \in H^1(c, d), c < d, \gamma \in [0, 1]$; then, the definition of the left fractional derivative in the sense of Caputo and Fabrizio becomes

$$\left({}_c^{\mathrm{CFC}}D^\gamma \zeta\right)(t) = \frac{B(\gamma)}{1-\gamma} \int_c^t \zeta'(x) \exp\left(\frac{-\gamma(t-x)^\gamma}{1-\gamma}\right) dx, \tag{8}$$

and the associated fractional integral is

$$\left({}_c^{\mathrm{CF}}I^\gamma \zeta\right)(t) = \frac{1-\gamma}{B(\gamma)}\zeta(t) + \frac{\gamma}{B(\gamma)} \int_c^t \zeta(x) dx, \tag{9}$$

where $B(\gamma) > 0$ is a normalization function satisfying $B(\gamma) = B(1) = 1$.

In the right case, we have

$$\left({}_c^{\mathrm{CFC}}D^\gamma \zeta\right)(t) = \frac{B(\gamma)}{1-\gamma} \int_t^d \zeta'(x) \exp\left(\frac{-\gamma(t-x)^\gamma}{1-\gamma}\right) dx, \tag{10}$$

and the associated fractional integral is

$$\left({}_c^{\mathrm{CF}}I^\gamma \zeta\right)(t) = \frac{1-\gamma}{B(\gamma)}\zeta(t) + \frac{\gamma}{B(\gamma)} \int_t^d \zeta(x) dx. \tag{11}$$

Lemma 9. *Let $\zeta : L \longrightarrow \mathbb{R}$ be a differentiable mapping on L, $c, d \in L$ with $c < d$. If $\zeta' \in L_1[c, d]$, then the following inequality holds:*

$$\begin{aligned}&\frac{\zeta(c) + \zeta(d)}{2} - \frac{1}{d^p - c^p} \int_c^d \frac{\zeta(x)}{x^{1-p}} dx \\ &\quad = \frac{d^p - c^p}{2p} \int_0^1 \left(M_p^{-1}(c, d; t)\right)(1-2t)\zeta'\left(M_p(c, d; t)\right) dt,\end{aligned} \tag{12}$$

where $M_p^{-1}(c, d; t) = [tc^p + (1-t)d^p]^{(1/p)-1}$.

3. A Generalized Hermite–Hadamard-Type Inequality via the Caputo-Fabrizio Fractional Operator for a (p, h)-Convex Function

The double inequality named as Hermite–Hadamard inequality is considered one of the fundamental inequalities for convex functions.

Theorem 10. *Let a function $\zeta : [c, d] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a (p, h)-convex function on $[c, d]$ and $\zeta \in L_1[c, d]$ if $\gamma \in [0, 1]$; then, the following double inequality holds:*

$$\begin{aligned}\frac{1}{2h(1/2)}\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p} &\le \frac{2pB(\gamma)}{\gamma(d^p-c^p)}\left[\left({}_{c}^{CF}I^{\gamma}\eta\right)(k)\right.\\ &\quad\left.+\left({}^{CF}I_{d}^{\gamma}\eta\right)(k)-\frac{2(1-\gamma)}{B(\gamma)}\eta(k)\right]\\ &\le (\zeta(c)+\zeta(d))\int_0^1 h(t)dt,\end{aligned} \tag{13}$$

where $\eta(x) = x^{1-p}\zeta(x)$.

Proof. Let ζ be a (p, h)-convex function; then, Hermite–Hadamard for a (p, h)-convex function is as follows:

$$\begin{aligned}\frac{1}{2h(1/2)}\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p} &\le \frac{p}{d^p-c^p}\int_c^d x^{p-1}\zeta(x)dx\\ &\le (\zeta(c)+\zeta(d))\int_0^1 h(t)dt.\end{aligned} \tag{14}$$

Since ζ is a (p, h)-convex function on $[c, d]$, we can write

$$\begin{aligned}\frac{1}{2h(1/2)}\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p} &\le \frac{p}{d^p-c^p}\int_c^d x^{p-1}\zeta(x)dx\\ &\quad\cdot\frac{2}{2h(1/2)}\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p}\\ &\le \frac{2p}{d^p-c^p}\int_c^b x^{p-1}\zeta(x)dx\\ &\quad\cdot\frac{1}{2h(1/2)}\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p}\\ &\le \frac{2p}{d^p-c^p}\left[\int_c^k x^{p-1}\zeta(x)dx\right.\\ &\quad\left.+\int_k^d x^{p-1}\zeta(x)dx\right].\end{aligned} \tag{15}$$

Multiplying both sides of (15) with $\gamma(d^p - c^p)/2pB(\gamma)$ and adding $(k^{p-1}2(1-\gamma)/B(\gamma))\zeta(k)$, we have

$$\begin{aligned}&\frac{k^{p-1}2(1-\gamma)}{B(\gamma)}\zeta(k)+\frac{\gamma(d^p-c^p)}{2pB(\gamma)h(1/2)}\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p}\\ &\le \frac{k^{p-1}2(1-\gamma)}{B(\gamma)}\zeta(k)+\frac{\gamma}{B(\gamma)}\\ &\quad\cdot\left[\int_c^k x^{p-1}\zeta(x)dx+\int_k^d x^{p-1}\zeta(x)dx\right]\frac{2(1-\gamma)}{B(\gamma)}\eta(k)\\ &\quad+\frac{\gamma(d^p-c^p)}{2pB(\gamma)h(1/2)}\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p}\\ &\le \frac{2(1-\gamma)}{B(\gamma)}\eta(k)+\frac{\gamma}{B(\gamma)}\left[\int_c^k\eta(x)dx+\int_k^d\eta(x)dx\right]\\ &=\left[\frac{(1-\gamma)}{B(\gamma)}\eta(k)+\frac{\gamma}{B(\gamma)}\int_c^k\eta(x)dx\right.\\ &\quad\left.+\frac{(1-\gamma)}{B(\gamma)}\eta(k)+\frac{\gamma}{B(\gamma)}\int_k^d\eta(x)dx\right]\\ &=\left({}_{c}^{CF}I^{\gamma}\eta\right)(k)+\left({}^{CF}I_{d}^{\gamma}\eta\right)(k)\zeta\left[\left(\frac{c^p+d^p}{2}\right)\right]^{1/p}\\ &\le\frac{2pB(\gamma)h(1/2)}{\gamma(d^p-c^p)}\left[\left({}_{c}^{CF}I^{\gamma}\eta\right)(k)+\left({}^{CF}I_{d}^{\gamma}\eta\right)(k)\frac{2(1-\gamma)}{B(\gamma)}\eta(k)\right].\end{aligned} \tag{16}$$

For the proof of the right hand side of the Hermite–Hadamard-type inequality, we have

$$\frac{2p}{d^p-c^p}\int_c^d x^{p-1}\zeta(x)dx \le 2\left[(\zeta(c)+\zeta(d))\int_0^1 h(t)dt\right]. \tag{17}$$

Multiplying both sides of (17) with $\gamma(d^p - c^p)/2pB(\gamma)$ and adding $(k^{p-1}2(1-\gamma)/B(\gamma))\zeta(k)$, we get

$$\begin{aligned}\left({}_{c}^{CF}I^{\gamma}\eta\right)(k)+\left({}^{CF}I_{d}^{\gamma}\eta\right)(k) &\le \frac{2(1-\gamma)}{B\gamma()}\eta(k)+\frac{\gamma(d^p-c^p)}{\gamma B(\gamma)}\\ &\quad\cdot(\zeta(c)+\zeta(d))\int_0^1 h(t)dt.\end{aligned} \tag{18}$$

By recognizing (18), the proof of the right hand side of (13) is completed. This completes the proof. □

Remark 11. If we put $p = 1$ and $h(t) = t$, then we will get the Hermite-Hadamard inequality for convex function.

Theorem 12. *Let $\zeta, \theta : L \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a (p, h)-convex function. If $\zeta\theta \in L_1[c, d]$, then we have the following inequality:*

$$\begin{aligned}&\frac{2pB(\gamma)}{\gamma(d^p-c^p)}\left[\left({}_{c}^{CF}I^{\gamma}\eta\beta\right)(k)+\left({}^{CF}I_{d}^{\gamma}\eta\beta\right)(k)-\frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k)\right]\\ &\le 2\left[M(c,d)\int_0^1 h_1(t)h_2(t)dt+N(c,d)\int_0^1 h_1(t)h_2(1-t)dt\right],\end{aligned} \tag{19}$$

with $\eta(x) = x^{1-p}\zeta(x)$ and $\beta(x) = x^{1-p}\theta(x)$, where $M(c, d) = \zeta(c)\theta(c) + \zeta(d)\theta(d)$, $N(c, d) = \zeta(c)\theta(d) + \zeta(d)\theta(c)$, and $k \in [c, d]$, $B(\gamma) > 0$ is a normalization function.

Proof. Since ζ and θ are (p, h)-convex functions on $[c, d]$, we have

$$\begin{aligned}
&\zeta\left(tc^p + (1-t)d^p\right)^{1/p} \le h_1(t)\zeta(c) + h_1(1-t)\zeta(d),\\
&\theta\left(tc^p + (1-t)d^p\right)^{1/p} \le h_2(t)\theta(c) + h_2(1-t)\theta(d).
\end{aligned} \tag{20}$$

Multiplying both sides of the above inequalities, we have

$$\begin{aligned}
&\zeta\left(tc^p + (1-t)d^p\right)^{1/p}\theta\left(tc^p + (1-t)d^p\right)^{1/p}\\
&\quad\le h_1(t)h_2(t)\zeta(c)\theta(c) + h_1(1-t)h_2(1-t)\zeta(d)\theta(d)\\
&\qquad + h_1(t)h_2(1-t)\zeta(c)\theta(b) + h_1(1-t)h_2(t)\zeta(d)\theta(c).
\end{aligned} \tag{21}$$

Integrating with respect to t over $[0, 1]$ and making the change of variable, we obtain

$$\begin{aligned}
&\frac{p}{d^p - c^p}\int_c^b x^{p-1}\zeta(x)\theta(x)dx\\
&\quad\le \int_0^1 h_1(t)h_2(t)dt[\zeta(c)\theta(c) + \zeta(d)\theta(d)]\\
&\qquad + \int_0^1 h_1(t)h_2(1-t)dt[\zeta(c)\theta(d) + \zeta(d)\theta(c)],
\end{aligned} \tag{22}$$

which implies

$$\begin{aligned}
&\frac{2p}{d^p - c^p}\left[\int_c^k x^{p-1}\zeta(x)\theta(x)dx + \int_c^k x^{p-1}\zeta(x)\theta(x)dx\right]\\
&\quad\le 2\left[M(c, d)\int_0^1 h_1(t)h_2(t)dt + N(c, d)\int_0^1 h_1(t)h_2(1-t)dt\right].
\end{aligned} \tag{23}$$

By multiplying both sides with $(\gamma c^p - d^p)/2PB(\gamma)$ and adding $(k^{p-1}2(1-\gamma)/B(\gamma))\zeta(K)g(k)$, we have

$$\begin{aligned}
&\frac{\gamma}{B(\gamma)}\left[\int_c^k \eta(x)\beta(x)dx + \int_k^d \eta(x)\beta(x)dx\right] + \frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k)\\
&\quad\le \frac{\gamma c^p - d^p}{2pB(\gamma)}2\left[M(c, d)\int_0^1 h_1(t)h_2(t)dt + N(c, d)\right.\\
&\qquad\left.\cdot\int_0^1 h_1(t)h_2(1-t)dt\right] + \frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k).
\end{aligned} \tag{24}$$

Thus,

$$\begin{aligned}
&\left({}_{c}^{\mathrm{CF}}I^{\gamma}\eta\beta\right)(k) + \left({}^{\mathrm{CF}}I_d^{\gamma}\eta\beta\right)(k)\\
&\quad\le \frac{\gamma c^p - d^p}{2pB(\gamma)}\left[2M(c, d)\int_0^1 h_1(t)h_2(t)dt + 2N(c, d)\right.\\
&\qquad\left.\cdot\int_0^1 h_1(t)h_2(1-t)dt\right] + \frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k),
\end{aligned} \tag{25}$$

and with suitable rearrangements, the proof is completed. □

Remark 13. If we put $p = 1$ and $h(t) = t$ in the above theorem, we get the results for the classical convex function.

Theorem 14. *Let $\zeta, \theta : L \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a (p, h)-convex function. If $\zeta\theta \in L_1[c, d]$, then we have the following inequality:*

$$\begin{aligned}
&\frac{1}{h_1(1/2)h_2(1/2)}\zeta\left(\left[\frac{c^p + d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p + d^p}{2}\right]^{1/p}\right)\\
&\quad - \frac{2p}{\gamma\left(d^p - c^p\right)}\left[\left({}_{c}^{CF}I^{\gamma}\eta\beta\right)(k) + \left({}^{CF}I_d^{\gamma}\eta\beta\right)(k)\right]\\
&\quad + \frac{4p(1-\gamma)}{\gamma\left(d^p - c^p\right)}\eta(k)\beta(k)\\
&\quad\le 2M(c, d)\int_0^1 h_1(t)h_2(t)dt + 2N(c, d)\int_0^1 h_1(t)h_2(1-t)dt,
\end{aligned} \tag{26}$$

with $\eta(x) = x^{1-p}\zeta(x)$ and $\beta(x) = x^{1-p}\theta(x)$, where $M(c, d) = \zeta(c)\theta(c) + \zeta(d)\theta(d)$, $N(c, d) = \zeta(c)\theta(d) + \zeta(d)\theta(c)$, and $k \in [c, d]$, $B(\gamma) > 0$ is a normalization function.

Proof. Since $(c^p + d^p)/2 = ((tc^p + (1-t)d^p)/2) + (((1-t)c^p + td^p)/2)$ for $t = 1/2$, we have

$$\begin{aligned}
&\zeta\left(\left[\frac{(1-t)c^p + td^p + tc^p + (1-t)d^p}{2}\right]^{1/p}\right)\\
&\quad\le h_1\left(\frac{1}{2}\right)\zeta\left(\left((1-t)c^p + td^p\right)^{1/p}\right)\\
&\qquad + h_1\left(\frac{1}{2}\right)\zeta\left(\left(tc^p + (1-t)d^p\right)^{1/p}\right),
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\theta\left(\left[\frac{(1-t)c^p + td^p + tc^p + (1-t)d^p}{2}\right]^{1/p}\right)\\
&\quad\le h_2\left(\frac{1}{2}\right)\theta\left(\left((1-t)c^p + td^p\right)^{1/p}\right)\\
&\qquad + h_2\left(\frac{1}{2}\right)\theta\left(\left(tc^p + (1-t)d^p\right)^{1/p}\right).
\end{aligned} \tag{28}$$

Multiplying (2) and (28), we have

$$\begin{aligned}
&\zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\\
&\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\zeta\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\theta\right.\\
&\quad\left.\cdot\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\right]+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\\
&\quad\cdot\left[\zeta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\theta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[h_1(t)\zeta(c)+h_1(1-t)\zeta(d)]\\
&\quad\cdot[h_2(t)\theta(c)+h_2(1-t)\theta(d)]+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\\
&\quad\cdot[h_1(1-t)(\zeta(c)+h_1(t)\zeta(d)][h_2(1-t)\theta(c)+h_2(t)\theta(d)]\\
&=h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\zeta\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\theta\right.\\
&\quad\left.\cdot\left(\left((1-t)c^p+td^p\right)^{1/p}\right)\right]+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\\
&\quad\cdot\left[\zeta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\theta\left(\left(tc^p+(1-t)d^p\right)^{1/p}\right)\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[(h_1(t)+h_2(1-t)+h_1(1-t)h_2(t))M(c,d)]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[(h_1(t)h_2(t)+h_1(1-t)h_2(1-t))N(c,d)].
\end{aligned}\tag{29}$$

Integrating the above inequality with respect to t over $[0,1]$ and making the change of variable, one obtains

$$\begin{aligned}
&\frac{1}{h_1(1/2)h_2(1/2)}\zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\\
&\quad-\frac{p}{d^p-c^p}\int_c^d x^{p-1}\zeta(x)\theta(x)dx\\
&\le 2M(c,d)\int_0^1 h_1(t)h_2(t)dt+2N(c,d)\int_0^1 h_1(t)h_2(1-t)dt.
\end{aligned}\tag{30}$$

By multiplying both sides with $\gamma(d^p-c^p)/2pB(\gamma)$ and subtracting $(K^{p-1}2(1-\gamma)/B(\gamma))\zeta(k)\theta(k)$, we have

$$\begin{aligned}
&\frac{2\gamma\left(d^p-c^p\right)}{pB(\gamma)}\zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\\
&\quad-\frac{\gamma}{B(\gamma)}\left[\int_c^k F(x)G(x)dx+\int_k^d F(x)G(x)dx\right]\\
&\quad-\frac{2(1-\gamma)}{B(\gamma)}\zeta(k)g(k)\\
&\le\frac{\gamma\left(d^p-c^p\right)}{2pB(\gamma)}2M(c,d)\int_0^1 h_1(t)h_2(t)dt+2N(c,d)\\
&\quad\cdot\int_0^1 h_1(t)h_2(1-t)dt-\frac{2(1-\gamma)}{B(\gamma)}F(k)G(k),
\end{aligned}$$

$$\begin{aligned}
&\frac{2\gamma\left(d^p-c^p\right)}{pB(\gamma)}\zeta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\theta\left(\left[\frac{c^p+d^p}{2}\right]^{1/p}\right)\\
&\quad-\frac{\gamma}{B(\gamma)}\left[\left({}_c^{\mathrm{CF}}I^\gamma\eta\beta\right)(k)+\left({}^{\mathrm{CF}}I_d^\gamma\eta\beta\right)(k)\right]\\
&\le\frac{\gamma\left(d^p-c^p\right)}{2pB(\gamma)}2\left[M(c,d)\int_0^1 h_1(t)h_2(t)dt\right.\\
&\quad\left.+N(c,d)\int_0^1 h_1(t)h_2(1-t)dt\right]-\frac{2(1-\gamma)}{B(\gamma)}\eta(k)\beta(k).
\end{aligned}\tag{31}$$

By multiplying both sides of the above inequality by $2pB(\gamma)/\gamma(d^p-c^p)$, we get the required inequality (28). □

Remark 15. If we put $p=1$ and $h(t)=t$ in the above theorem, then we will get the result for the convex function.

4. Some New Results Related to the Caputo-Fabrizio Fractional Operator

In this section, firstly we generalize a lemma; then, we prove our main theorem with the help of this lemma.

Lemma 16. *Let $\zeta : L\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable mapping on L, $c,d\in L$ with $c<d$. If $\zeta'\in L_1[c,d]$ and $\gamma\in[0,1]$, the following equality holds:*

$$\begin{aligned}
&\frac{d^p-c^p}{2}\int_0^1 M^{-1}(c,d;t)(1-2t)\zeta'\left(M_p(c,d;t)\right)dt-\frac{2(1-\gamma)}{\gamma\left(d^p-c^p\right)}\eta(k)\\
&\quad=\frac{\zeta(c)+\zeta(d)}{2}-\frac{pB(\gamma)}{\gamma\left(d^p-c^p\right)}\left[\left({}_c^{CF}I^\gamma\eta\right)(k)+\left({}^{CF}I_d^\gamma\eta\right)(k)\right],
\end{aligned}\tag{32}$$

where $k\in[c,d]$ and $B(\gamma)>0$ is a normalization function with $\eta(x)=\zeta(x)/x^{1-p}$.

Proof. From Lemma 9, we can observe that

$$\begin{aligned}
&\int_0^1 M^{-1}(c,d;t)(1-2t)\zeta'\left(M_p(c,d;t)\right)dt\\
&\quad=\frac{\zeta(c)+\zeta(d)}{d^p-c^p}-\frac{2p}{\left(d^p-c^p\right)^2}\left(\int_c^k\frac{\zeta(x)}{x^{1-p}}dx+\int_k^d\frac{\zeta(x)}{x^{1-p}}dx\right).
\end{aligned}\tag{33}$$

By multiplying $\gamma(d^p-c^p)^2/2pB(\gamma)$ and subtracting $(2(1-\gamma)/x^{1-p}B(\gamma))\zeta(k)$, we have

$$\begin{aligned}
&\frac{\gamma\left(d^p-c^p\right)^2}{2pB(\gamma)}\int_0^1 M^{-1}(c,d;t)(1-2t)\zeta'\left(M_p(c,d;t)\right)dt-\frac{2(1-\gamma)}{x^{1-p}B(\gamma)}\zeta(k)\\
&=\frac{\gamma\left(d^p-c^p\right)\zeta(c)+\zeta(d)}{2pB(\gamma)}+\frac{2(1-\gamma)}{x^{1-p}B(\gamma)}\zeta(k)-\frac{\gamma}{B(\gamma)}\left(\int_c^k\frac{\zeta(x)}{x^{1-p}}dx+\int_k^d\frac{\zeta(x)}{x^{1-p}}dx\right)\\
&=\frac{\gamma\left(d^p-c^p\right)\zeta(c)+\zeta(d)}{2pB(\gamma)}+\frac{2(1-\gamma)}{B(\gamma)}\eta(k)-\frac{\gamma}{B(\gamma)}\left(\int_c^k\eta(x)dx+\int_k^d\eta(x)dx\right)\\
&=\frac{\gamma\left(d^p-c^p\right)\zeta(c)+\zeta(d)}{2pB(\gamma)}-\left(\frac{(1-\gamma)}{B(\gamma)}\eta(k)+\frac{\gamma}{B(\gamma)}\int_c^k\eta(x)dx\right)\\
&\quad-\left(\frac{(1-\gamma)}{B(\gamma)}\eta(k)+\frac{\gamma}{B(\gamma)}\int_c^k\eta(x)dx\right)\\
&=\frac{\gamma\left(d^p-c^p\right)\zeta(c)+\zeta(d)}{2pB(\gamma)}-\left[\left({}_c^{\mathrm{CF}}I^\gamma\eta\right)(k)+\left({}^{\mathrm{CF}}I_d^\gamma\eta\right)(k)\right].
\end{aligned}\tag{34}$$

This completes the proof. □

Theorem 17. *Let $\zeta : L\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a differential mapping on L and $|\zeta'|$ be (p,h)-convex on $[c,d]$ where $c,d\in L$ with $c<d$, $p>0$. If $\zeta'\in L_1[c,d]$ and $\gamma\in[0,1]$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\zeta(c)+\zeta(d)}{2}+\frac{2(1-\gamma)}{\gamma\left(d^p-c^p\right)}\zeta(k)-\frac{pB(\gamma)}{\gamma\left(d^p-c^p\right)}\left[\left({}_c^{\mathrm{CF}}I^\gamma\zeta\right)(k)+\left({}^{\mathrm{CF}}I_d^\gamma\zeta\right)(k)\right]\right|\\
&\le\frac{d^p-c^p}{2p}\left[d_1(c,d,p)\left|\zeta'(c)\right|+d_2(c,d,p)\left|\zeta'(d)\right|\right],
\end{aligned}\tag{35}$$

where

$$\begin{aligned}
d_1(c,d,p)&=\int_0^{1/2}\frac{(1-2t)h(t)}{\left(tc^p+(1-t)d^p\right)^{(1/p)-1}}dt+\int_{1/2}^1\frac{(2t-1)h(t)}{\left(tc^p+(1-t)d^p\right)^{(1/p)-1}}dt,\\
d_2(c,d,p)&=\int_0^{1/2}\frac{(1-2t)h(1-t)}{\left(tc^p+(1-t)d^p\right)^{(1/p)-1}}dt+\int_{1/2}^1\frac{(2t-1)h(1-t)}{\left(tc^p+(1-t)d^p\right)^{(1/p)-1}}dt,
\end{aligned}\tag{36}$$

where $k\in[c,d]$ and $B(\gamma)>0$ is a normalization function.

Proof.

$$\begin{aligned}
&\left|\frac{\zeta(c)+\zeta(d)}{2}+\frac{2(1-\gamma)}{\gamma\left(d^p-c^p\right)}\zeta(k)-\frac{pB(\gamma)}{\gamma\left(d^p-c^p\right)}\left[\left({}_c^{\mathrm{CF}}I^\gamma\zeta\right)(k)+\left({}^{\mathrm{CF}}I_d^\gamma\zeta\right)(k)\right]\right|\\
&\le\frac{d^p-c^p}{2p}\int_0^1 M^{-1}(c,d;t)|(1-2t)|\left|\zeta'\left(M_p(c,d;t)\right)\right|dt\\
&\le\frac{d^p-c^p}{2p}\int_0^1\left(\frac{|1-2t|\left|\zeta'\left(tc^p(1-t)d^p\right)^{1/p}\right|}{\left|\left(tc^p(1-t)d^p\right)^{(1/p)-1}\right|}\right)dt\\
&\le\frac{d^p-c^p}{2p}\int_0^1\left(\frac{|1-2t|\left(h(t)\left|\zeta'(c)\right|+h(1-t)\left|\zeta'(d)\right|\right)}{\left|\left(tc^p(1-t)d^p\right)^{(1/p)-1}\right|}\right)dt\\
&=\frac{d^p-c^p}{2p}\left[\int_0^{1/2}\frac{(1-2t)h(t)\left|\zeta'(c)\right|}{\left(tc^p+(1-t)d^p\right)^{(1/p)-1}}dt+\int_{1/2}^1\frac{(2t-1)h(t)\left|\zeta'(c)\right|}{\left(tc^p+(1-t)d^p\right)^{(1/p)-1}}dt\right]\\
&\quad+\frac{d^p-c^p}{2p}\left[\int_0^{1/2}\frac{(1-2t)h(1-t)\left|\zeta'(d)\right|}{\left(tc^p+(1-t)d^p\right)^{(1/p)-1}}dt+\int_{1/2}^1\frac{(2t-1)h(1-t)\left|\zeta'(d)\right|}{\left(ta^p+(1-t)d^p\right)^{(1/p)-1}}dt\right]\\
&\le\frac{d^p-a^p}{2p}\left[d_1(c,d,p)\left|\zeta'(c)\right|+d_2(c,d,p)\left|\zeta'(d)\right|\right].
\end{aligned}\tag{37}$$

This completes the proof. □

Theorem 18. *Let $\zeta : L\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a differential mapping on L and $|\zeta'|$ be (p,h)-convex on $[c,d]$ where $c,d\in L$ with $c<d$, $p>1$, and $(1/r)+(1/q)=1$. If $\zeta'\in L_1[c,d]$ and $\gamma\in[0,1]$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\zeta(c)+\zeta(d)}{2}+\frac{2(1-\gamma)}{\gamma\left(d^p-c^p\right)}\zeta(k)-\frac{pB(\gamma)}{\gamma\left(d^p-c^p\right)}\left[\left({}_c^{\mathrm{CF}}I^\gamma\zeta\right)(k)+\left({}^{\mathrm{CF}}I_d^\gamma\zeta\right)(k)\right]\right|\\
&\le\frac{d^p-a^p}{2p}\left(d_1(c,d,p)\right)^{1/r}\left(d_2(c,d,p)\left|\zeta'(c)\right|^q+d_3(c,d,p)\left|\zeta'(d)\right|^q\right)^{1/q},
\end{aligned}\tag{38}$$

where

$$\begin{aligned}
d_1(c,d,p)&=\int_0^1\left(\frac{|1-2t|^r}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt,\\
d_2(c,d,p)&=\int_0^1\left(\frac{h(t)}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt,\\
d_3(c,d,p)&=\int_0^1\left(\frac{h(1-t)}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt,
\end{aligned}\tag{39}$$

where $k\in[c,d]$ and $B(\gamma)>0$ is a normalization function.

Proof. By a similar argument to the proof of the previous theorem, by using lemma, the Hölder inequality, and convexity of $|\zeta'|$, we get

$$\begin{aligned}
&\left|\frac{\zeta(c)+\zeta(d)}{2}+\frac{2(1-\gamma)}{\gamma\left(d^p-c^p\right)}f(k)-\frac{pB(\gamma)}{\gamma\left(d^p-c^p\right)}\left[\left({}_c^{\mathrm{CF}}I^\gamma\zeta\right)(k)+\left({}^{\mathrm{CF}}I_d^\gamma\zeta\right)(k)\right]\right|\\
&\le\frac{d^p-c^p}{2p}\int_0^1 M^{-1}(c,d;t)|(1-2t)|\left|\zeta'\left(M_p(c,d;t)\right)\right|dt\\
&\le\frac{d^p-c^p}{2p}\left(\int_0^1\left(\frac{|1-2t|^r}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt\right)^{1/r}\left(\int_0^1\left(\frac{\left|\zeta'\left(tc^p(1-t)d^p\right)^{q/p}\right|}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt\right)^{1/q}\\
&\le\frac{d^p-c^p}{2p}\left(\int_0^1\left(\frac{|1-2t|^r}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt\right)^{1/r}\left(\int_0^1\left(\frac{\left(h(t)\left|\zeta'(c)\right|^q+h(1-t)\left|\zeta'(d)\right|^q\right)}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt\right)^{1/q}\\
&\le\frac{d^p-c^p}{2p}\left(\int_0^1\left(\frac{|1-2t|^r}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt\right)^{1/r}\left(\int_0^1\left(\frac{h(t)\left|\zeta'(c)\right|^q}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt\right.\\
&\quad\left.+\int_0^1\left(\frac{h(1-t)\left|\zeta'(d)\right|^q}{\left(tc^p(1-t)d^p\right)^{(1/p)-1}}\right)dt\right)^{1/q}\\
&\le\frac{d^p-c^p}{2p}\left(d_1(c,d,p)\right)^{1/r}\left(d_2(c,d,p)\left|\zeta'(a)\right|^q+d_3(c,d,p)\left|\zeta'(d)\right|^q\right)^{1/q}.
\end{aligned}\tag{40}$$

This completes the proof. □

5. Application to Special Means

The applications to special means are also used to confirm the accuracy of the findings for real numbers c,d such that $c\neq d$.

The arithmetic mean of two numbers c and d is defined as

$$A=A(c,d)=\frac{c+d}{2},\ c,d\in\mathbb{R}.\tag{41}$$

The generalized logarithmic mean is defined as

$$L = L_r^r = \frac{c^{t+1} - d^{t+1}}{(r+1)(d-c)} r \in \mathbb{R} - [-1, 0], t \in \mathbb{R}\, c, d \in \mathbb{R}, c \neq d. \quad (42)$$

Now, using the results in Section 4, we have some applications to the special means of real numbers.

Proposition 19. *Let* $c, d \in \mathbb{R}, c < d$*; then,*

$$\left|A\left(c^2, d^2\right) - pL_2^2(c, d)\right| \leq \frac{d^p - c^p}{2p}\left[d_1(2|c|) + d_2(2|d|)\right]. \quad (43)$$

Proof. In the inequality proven in Theorem 17, if we set $\zeta(z) = z^2$ with $\gamma = 1$ and $B(\gamma) = B(1) = 1$, then we obtain the result immediately. □

Remark 20. If we put $h(t) = t$ and $p = 1$ in this proposition, we will obtain this result for the convex function.

Proposition 21. *Let* $c, d \in \mathbb{R}, c < d$*; then,*

$$|A(c^n, d^n) - pL_n^n(c, d)| \leq \frac{n\left(d^p - c^p\right)}{2p}\left[d_1\left(2\left|c^{n-1}\right|\right) + d_2\left(2\left|d^{n-1}\right|\right)\right]. \quad (44)$$

Proof. In the inequality proven in Theorem 17, if we set $\zeta(z) = z^n$ where n is an even number with $\gamma = 1$ and $B(\gamma) = B(1) = 1$, then we obtain the result immediately. □

Remark 22. If we put $h(t) = t$ and $p = 1$ in this preposition, we will obtain this result for the convex function.

6. Conclusion

Convexity is very important for solving optimization problems. Fractional calculus together with convexity plays an important role in solving real-life problems. In this paper, we established several Hermite–Hadamard-type inequalities in the setting of a fractional integral operator for (p, h)-convex functions. We also presented some applications in means. Our results generalized several existing results.

Conflicts of Interest

The authors do not have any competing interests.

Authors' Contributions

Dong Zhang proved the main results, Muhammad Shoaib Saleem proposed the problem and supervised this work, Thongchai Botmart analyzed the results and arranged the funding for this paper, M. S. Zahoor proved the main results, and R. Bano wrote the first version of this paper.

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Review Article

Semilinear Fractional Evolution Inclusion Problem in the Frame of a Generalized Caputo Operator

Adel Lachouri,[1] **Abdelouaheb Ardjouni**,[2] **Fahd Jarad**,[3,4] **and Mohammed S. Abdo**[5]

[1]*Laboratory of Applied Mathematics, Department of Mathematics, Annaba University, Annaba, Algeria*
[2]*Department of Mathematics and Informatics, University of Souk Ahras, Souk Ahras, Algeria*
[3]*Department of Mathematics, Çankaya University, 06790 Etimesgut, Ankara, Turkey*
[4]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[5]*Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen*

Correspondence should be addressed to Fahd Jarad; fahd@cankaya.edu.tr

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In this paper, we study the existence of solutions to initial value problems for a nonlinear generalized Caputo fractional differential inclusion with Lipschitz set-valued functions. The applied fractional operator is given by the kernel $k(\rho, s) = \xi(\rho) - \xi(s)$ and the derivative operator $(1/\xi'(\rho))(d/d\rho)$. The existence result is obtained via fixed point theorems due to Covitz and Nadler. Moreover, we also characterize the topological properties of the set of solutions for such inclusions. The obtained results generalize previous works in the literature, where the classical Caputo fractional derivative is considered. In the end, an example demonstrating the effectiveness of the theoretical results is presented.

1. Introduction

Fractional calculus is a field of mathematical analysis that embraces the integrals and derivatives of functions of any real or complex order. For the past few decades, this field has been one of the hand-over-fist sprawling fields of mathematics by virtue of the amazing findings obtained when researchers enrolled the fractional operators in their attempts to construe some problems that arise in nature (see [1–6]). As a matter of fact, the classical fractional calculus consisted of one main integral operator, namely, the Riemann-Liouville fractional integral, and two fractional derivatives, namely, the Riemann-Liouville and Caputo derivatives. Because of the penurious number of operators, researchers were compelled to discover and develop new fractional operators that allow them to better comprehend the world around them. In the last 10 years or so, new fractional operators have been introduced and wielded on many occasions. One can touch upon the operators in [7–13]. It should be noted that some of these operators are an extension or generalization of the Riemann-Liouville integrals and Riemann-Liouville and Caputo derivatives that are nonlocal but singular kerneled. The others are brand-new ones, and they are nonlocal and contain nonsingular kernels.

One of the main applications of advanced fractional calculus is the theory of fractional evolution inclusions since they are abstract formulations for numerous problems arising in physics and engineering. Evolution equations and inclusions are commonly applied to describe the systems and models that change or evolve over time. Many studies have investigated the existence and uniqueness of solutions for fractional evolution problems based on semigroup and fixed point theories; e.g., Bedi et al. [14] studied the existence of mild solutions for Hilfer-type fractional neutral impulsive evolution problems in a Banach space. The same author, with others in [15], discussed the stability and controllability results for Hilfer-type fractional evolution equations in a Banach space. The existence of saturated mild (and global) solutions of Caputo-type fractional semilinear evolution problems with noncompact semigroups has been obtained in Banach spaces by Chen et al. [16]. The authors in [17] studied the existence of local (and global) solutions and the uniqueness of a mild solution to fractional semilinear evolution problems with compact (and noncompact) semigroups in Banach spaces. Zhou and Jiao [18] considered a class of nonlocal Cauchy problems for a Caputo-type fractional neutral evolution problem to investigate the existence and uniqueness of mild solutions. For differential equations-inclusions governed by Cauchy conditions or boundary conditions, the issue of importance is to tackle the existence, uniqueness, and stability of their solutions. These properties constitute the most essential parts of the analysis of these equations. There are heaps of papers that went about the existence, uniqueness, and stability results of differential equations-inclusions within the scope of a variety of fractional derivatives based on different types of fixed point techniques. For instance, Abdo et al. [19] have proven the existence and different types of stability of solutions for ψ-Hilfer-type fractional integro-problems. The authors in [20] investigated some existence results of Caputo-type fractional neutral inclusions by using weak topology. The existence of solutions for a generalized Caputo-type fractional differential inclusion problem with integral boundary conditions has been studied by Belmor et al. [21]. In [22], the authors discussed the existence of solutions for a Caputo-type fractional for higher-order fractional inclusion problems. Lachouri et al. [23] considered a nonlinear Hilfer-type nonlocal fractional inclusion problem to prove the existence results, taking into account the convex and nonconvex values of the nonlinear term.

Recently, Chen et al. [16] proved the existence of saturated mild solutions and global mild solutions for fractional evolution equations of the type

$$\begin{cases} {}^{C}D^{r_1}\sigma(\rho)+\mathscr{A}\sigma(\rho)=h(\rho,\sigma(\rho)),\rho\geq 0,\\ \sigma(0)=\sigma_0\in\mathscr{X}, \end{cases} \tag{1}$$

$\in(0,1)$, $\mathscr{A}:\mathfrak{D}(\mathscr{A})\subset\mathscr{X}\longrightarrow\mathscr{X}$ is a closed linear operator and $-\mathscr{A}$ generates a uniformly bounded C_0-semigroup $\{\mathscr{T}(\rho)\}_{\rho\geq 0}$ in $\mathscr{X}$, and $h:[0,\infty)\times\mathscr{X}\longrightarrow\mathscr{X}$ is the given function.

In [17], Suechoei and Ngiamsunthorn studied the following fractional evolution equations:

$$\begin{cases} {}^{C}D^{r_1;\xi}_{a+}\sigma(\rho)=\mathscr{A}\sigma(\rho)+h(\rho,\sigma(\rho)), & \rho\in(0,T],\\ \sigma(0)=\sigma_0, \quad \sigma_0\in X, \end{cases} \tag{2}$$

where ${}^{C}D^{r_1;\xi}_{\mathfrak{a}+}$ is the ξ-Caputo FD of order $r_1\in(0,1)$, $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators $\{\mathscr{T}(\rho)\}_{\rho\geq 0}$ on $\mathscr{X}$, and $h:[0,\infty)\times\mathscr{X}\longrightarrow\mathscr{X}$ is the given function.

Motivated by the aforementioned works and inspired by [17], we consider the following fractional evolution inclusion involving ξ-Caputo FD:

$$\begin{cases} {}^{C}D^{r_1;\xi}_{a+}\sigma(\rho)\in\mathscr{A}\sigma(\rho)+f(\rho,\sigma(\rho)), & \rho\in(0,T],\\ \sigma(0)=\sigma_0, \quad \sigma_0\in X, \end{cases} \tag{3}$$

where $f:[0,\mathfrak{T}]\times\mathscr{X}\longrightarrow\mathscr{O}(\mathscr{X})$ is a set-valued map (svm) from $[0,\mathfrak{T}]\times\mathscr{X}$ to the family of $\mathscr{O}(\mathscr{X})\subset\mathscr{X}$ and $\mathscr{X}$ is a Banach space with the norm $\|.\|$.

The novelty of the present work is choosing a more general operator than the classical fractional operators. More precisely, problem (3) is reduced to a Caputo-type problem, for $\xi(\rho)=\rho$; Hadamard-Caputo-type problem, for $\xi(\rho)=\log(\rho)$; and Katugampola-Caputo-type problem, for $\xi(\rho)=\rho^\delta$, $\delta>0$. In addition, we describe the topological properties of the considered solution in the present work.

In this paper, we aim to prove the existence of mild solutions for problem (3) involving the generalized Caputo derivative using the fixed point theorem (FPT) of Nadler and Covitz and to characterize the topological properties of the set of solutions for such inclusions.

The acquired results are more general and cover many of the parallel problems that contain special cases of function ξ, such as [16, 18].

2. Preliminary Notions

2.1. Fractional Calculus (FC). In this portion, we introduce several basic notions of FC and necessary lemmas that will be needed in such a study.

Let $\mathfrak{J}^*=[0,\mathfrak{T}]$. Denoted by $\mathscr{E}=C(\mathfrak{J}^*,\mathscr{X})$, we have the Banach space of continuous functions $\sigma:\mathfrak{J}^*\longrightarrow\mathscr{X}$ with

$$\|\sigma\|_{\mathscr{E}}=\sup\{\|\sigma(\rho)\|:\rho\in\mathfrak{J}^*\}. \tag{4}$$

Let $\sigma,\xi\in C^n(\mathfrak{J}^*,\mathbb{R})$ such that ξ is increasing and $\xi'(\rho)\neq 0$, $\forall\rho\in\mathfrak{J}^*$, and

$$\mathscr{M}=\sup_{\rho\in[0,\infty)}\|\mathscr{T}(\rho)\|<\infty. \tag{5}$$

Definition 1 (see [24]). The ξ-Riemann-Liouville FI of a function σ of order r_1 is defined by

$$I^{r_1;\xi}_{\mathfrak{a}+}\sigma(\rho)=\frac{1}{\Gamma(r_1)}\int_{\mathfrak{a}}^{\rho}\xi'(\upsilon)(\xi(\rho)-\xi(\upsilon))^{r_1-1}\sigma(\upsilon)d\upsilon. \tag{6}$$

Definition 2 (see [24]). The ξ-Riemann-Liouville FD of order r_1 for a function σ is given by

$$D^{r_1;\xi}_{\mathfrak{a}+}\sigma(\rho)=\left(\frac{1}{\xi'(\rho)}\frac{d}{d\rho}\right)^n I^{(n-r_1);\xi}_{\mathfrak{a}+}\sigma(\rho), \tag{7}$$

where $n=[r_1]+1$, $n\in\mathbb{N}$.

Definition 3 (see [13]). The ξ-Caputo FD of a function $\sigma\in C^n(\mathfrak{J}^*,\mathbb{R})$ of order r_1 is described by

$${}^{C}D^{r_1;\xi}_{\mathfrak{a}+}\sigma(\rho)=I^{(n-r_1);\xi}_{\mathfrak{a}+}\sigma^{[n]}(\rho), \tag{8}$$

where $\sigma^{[n]}(\rho) = ((1/\xi'(\rho))(d/d\rho))^n \sigma(\rho)$ and $n = [r_1] + 1$, $n \in \mathbb{N}$.

Lemma 4 (see [11, 25]). *Let $r_1, r_2, \mu > 0$. Then,*

$$I_{\mathfrak{a}+}^{r_1;\xi}(\xi(\rho)-\xi(\mathfrak{a}))^{r_2-1}(\rho) = \frac{\Gamma(r_2)}{\Gamma(r_1+r_2)}(\xi(\rho)-\xi(\mathfrak{a}))^{r_1+r_2-1},$$
$${}^{C}D_{\mathfrak{a}+}^{r_1;\xi}(\xi(\rho)-\xi(\mathfrak{a}))^{r_2-1}(\rho) = \frac{\Gamma(r_2)}{\Gamma(r_2-r_1)}(\xi(\rho)-\xi(\mathfrak{a}))^{r_1+r_2-1}. \tag{9}$$

Lemma 5 (see [13]). *If $\sigma \in C^n(\mathfrak{J}^*, \mathbb{R})$, $r_1 \in (n-1, n)$, and $r_2 \in (0,1)$, then*

$$I_{\mathfrak{a}+}^{r_1;\xi}\,{}^{C}D_{\mathfrak{a}+}^{r_1;\xi}\sigma(\rho) = \sigma(\rho) - \sum_{k=0}^{n-1}\frac{\sigma^{[n]}(\mathfrak{a}^+)}{k!}(\xi(\rho)-\xi(\mathfrak{a}))^k. \tag{10}$$

In particular, given $r_1 \in (0,1)$, we have

$$I_{\mathfrak{a}+}^{r_1;\xi}\,{}^{C}D_{\mathfrak{a}+}^{r_1;\xi}\sigma(\rho) = \sigma(\rho) - \sigma(\mathfrak{a}). \tag{11}$$

Definition 6 (see [11]). Let $\sigma, \xi : [\mathfrak{a}, \infty) \longrightarrow \mathbb{R}$ be real-valued functions. The generalized Laplace transform of σ is defined by

$$\mathfrak{L}\{\sigma(\rho)\}(\upsilon) = \int_{\mathfrak{a}}^{\rho} e^{-\upsilon(\xi(\rho)-\xi(\mathfrak{a}))}\sigma(\rho)\xi'(\rho)d\rho, \quad \text{for all } \upsilon. \tag{12}$$

Theorem 7 (see [11]). *Let $r_1 > 0$ and $\varkappa$ be a piecewise continuous function on each interval $[\mathfrak{a}, \rho]$ and $\xi(\rho)$-exponential order. Then,*

$$\mathfrak{L}\left\{I_{\mathfrak{a}+}^{r_1;\xi}\sigma(\rho)\right\}(\upsilon) = \frac{I_{\mathfrak{a}+}^{r_1;\xi}\sigma(\rho)}{\upsilon^{r_1}}. \tag{13}$$

Definition 8 (see [26, 27]). Let $r_1 \in (0,1)$ and $\rho \in \mathbb{C}$. The Wright-type function is defined by

$$\vartheta_{r_1}(\rho) = \sum_{\mathfrak{m}=0}^{\infty}\frac{(-\rho)^{\mathfrak{m}}}{\mathfrak{m}!\Gamma(-r_1\mathfrak{m}+1-r_1)} = \sum_{\mathfrak{m}=0}^{\infty}\frac{(-\rho)^{\mathfrak{m}}\Gamma(r_1(\mathfrak{m}+1))\sin(\pi r_1(\mathfrak{m}+1))}{\mathfrak{m}!}. \tag{14}$$

Proposition 9 (see [26, 27]). *The Wright function ϑ_{r_1} is an entire function and has the properties listed below:*

(1) $\vartheta_{r_1} \geq 0$ *for* $\rho \geq 0$ *and* $\int_0^\infty \vartheta_{r_1}(\rho)d\rho = 1$

(2) $\int_0^\infty \vartheta_{r_1}(\rho)\rho^\alpha d\rho = \Gamma(1+\alpha)/\Gamma(1+\alpha r_1)$, *for* $\alpha > -1$

(3) $\int_0^\infty \vartheta_{r_1}(\rho)e^{-k\rho}d\rho = E_{r_1}(-k)$, $k \in \mathbb{C}$

(4) $\alpha\int_0^\infty \rho\vartheta_{r_1}(\rho)e^{-k\rho}d\rho = E_{r_1,r_1}(-k)$, $k \in \mathbb{C}$

Next, we recall some conceptions concerning the semigroups of linear operators. For more details, see [28, 29].

For strongly continuous semigroups (i.e., C_0-semigroup $\{\mathcal{T}(\rho)\}_{\rho\geq 0}$ on $\mathcal{X}$), we define the generator

$$\mathcal{A}\sigma = \lim_{\rho\longrightarrow 0^+}\frac{\mathcal{T}(\rho)\sigma - \sigma}{\sigma}, \quad \sigma \in \mathcal{X}. \tag{15}$$

By $\mathfrak{D}(\mathcal{A})$, we denote the domain of $\mathcal{A}$, that is,

$$\mathfrak{D}(\mathcal{A}) = \left\{\sigma \in \mathcal{X} : \lim_{\rho\longrightarrow 0^+}\frac{\mathcal{T}(\rho)\sigma - \sigma}{\sigma} \text{ exists}\right\}. \tag{16}$$

Lemma 10 (see [28, 29]). *Let $\{\mathcal{T}(\rho)\}_{\rho\geq 0}$ be a C_0-semigroup; then, there exist constants $\mathfrak{C} > 0$ and $\mathfrak{a} \geq 0$ such that*

$$\|\mathcal{T}(\rho)\| \leq \mathfrak{C}e^{\mathfrak{a}\tau}, \quad \forall \rho \geq 0. \tag{17}$$

Lemma 11 (see [28, 29]). *A linear operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup iff*

(1) *$\mathcal{A}$ is closed, and $\bar{\mathfrak{D}}(\mathcal{A}) = \mathcal{X}$*

(2) *The resolvent set $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{R}^+$, and for every $\lambda \in \mathbb{R}^+$, we have*

$$\|R(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}, \tag{18}$$

where $R(\lambda, \mathcal{A}) = (\lambda^{r_1}I - \mathcal{A})^{-1}\sigma = \int_0^\infty e^{-\lambda\tau}\mathcal{T}(\rho)\sigma d\rho$.

In relation to problem (3), we need the following lemma which was proven in [17].

Lemma 12. *The mild solution of IVPs*

$$\begin{cases} {}^{C}D_{a+}^{r_1;\xi}\sigma(\rho) = \mathcal{A}\sigma(\rho) + q(\rho, \sigma(\rho)), & \rho \in (0, T], \\ \sigma(0) = \sigma_0, \end{cases} \tag{19}$$

is obtained as

$$\sigma(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(\upsilon)(\xi(\rho)-\xi(\upsilon))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,\upsilon)q(\upsilon,\sigma(\upsilon))d\upsilon, \tag{20}$$

where

$$\mathcal{S}^{r_1;\xi}(\rho,\upsilon)\sigma = \int_0^\infty \vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(\upsilon))^{r_1}\theta\right)\sigma d\theta,$$
$$\mathcal{T}^{r_1;\xi}(\rho,\upsilon)\sigma = r_1\int_0^\infty \theta\vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(\upsilon))^{r_1}\theta\right)\sigma d\theta, \tag{21}$$

for $0 \leq \upsilon \leq \rho \leq \mathfrak{T}$.

Lemma 13 (see [18]). *The operators $\mathcal{T}^{r_1;\xi}$ and $\mathcal{S}^{r_1;\xi}$ have the following properties:*

(1) *For any fixed $\rho \geq \upsilon \geq 0$, $\mathcal{T}^{r_1;\xi}$ and $\mathcal{S}^{r_1;\xi}$ are bounded linear operators with*

$$\left\| \mathcal{S}^{r_1;\xi}(\rho,\upsilon)(\sigma) \right\| \leq \mathcal{M}\|\sigma\|,$$
$$\left\| \mathcal{T}^{r_1;\xi}(\rho,\upsilon)(\sigma) \right\| \leq \frac{r_1\mathcal{M}}{\Gamma(1+r_1)} = \frac{\mathcal{M}}{\Gamma(r_1)}\|\sigma\|, \quad \forall \sigma \in \mathcal{X}. \tag{22}$$

(2) *$\mathcal{T}^{r_1;\xi}$ and $\mathcal{S}^{r_1;\xi}$ are strongly continuous $\forall \rho \geq \upsilon \geq 0$; that is, for every $\varpi \in \mathcal{X}$ and $0 \leq \upsilon \leq \rho_1 \leq \rho_2 \leq \mathfrak{T}$, we have*

$$\left\| \mathcal{S}^{r_1;\xi}(\rho_2,\upsilon)\sigma - \mathcal{S}^{r_1;\xi}(\rho_1,\upsilon)\sigma \right\| \longrightarrow 0,$$
$$\left\| \mathcal{T}^{r_1;\xi}(\rho_2,\upsilon)\sigma - \mathcal{T}^{r_1;\xi}(\rho_1,\upsilon)\sigma \right\| \longrightarrow 0, \tag{23}$$

as $\rho_1 \longrightarrow \rho_2$.

(3) *If $\mathcal{T}(\rho)$ is a compact operator $\forall \rho > 0$, then $\mathcal{T}^{r_1;\xi}(\rho,\upsilon)$ and $\mathcal{S}^{r_1;\xi}(\rho,\upsilon)$ are compact for all $\rho,\upsilon > 0$.*

(4) *The operators $\mathcal{T}^{r_1;\xi}(\rho,\upsilon)$ and $\mathcal{S}^{r_1;\xi}(\rho,\upsilon)$ are continuous in the uniform operator topology.*

3. Main Results

In what follows, we will utilize the notation $\mathcal{O}_{cp,c}(\mathcal{X}) = \{\mathcal{N} \in \mathcal{O}(\upsilon)$: $\mathcal{N}$ is compact and convex$\}$ for a normed space $\mathcal{X}$. For more information about the svm, we refer the reader to [30, 31].

Definition 14. A function $\sigma \in C^1(\mathfrak{J}^*, \mathcal{X})$ is a solution of (3), if $\exists \wp^\sim \in L^1(\mathfrak{J}^*, \mathcal{X})$ with $\wp^\sim(\rho) \in f(\rho,\sigma) \forall \rho \in \mathfrak{J}^*$ fulfilling the initial condition

$$\sigma(0) = \sigma_0,$$
$$\sigma(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(\upsilon)(\xi(\rho)-\xi(\upsilon))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,\upsilon)\wp^\sim(\upsilon)d\upsilon. \tag{24}$$

The first outcome treats the nonconvex f based on the theorem of Covitz and Nadler [32].

Let (σ, d) be a metric space induced from the normed space $(\sigma, \|.\|)$. Consider $\mathcal{H}_d : \mathcal{O}(\sigma) \times \mathcal{O}(\sigma) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ defined by

$$\mathcal{H}_d\left(\tilde{C}, \tilde{D}\right) = \max\left\{ \sup_{\tilde{c}\in\tilde{C}} d\left(\tilde{c}, \tilde{D}\right), \sup_{\tilde{d}\in\tilde{D}} d\left(\tilde{C}, \tilde{d}\right) \right\}, \tag{25}$$

where $d(\tilde{C}, \tilde{d}) = \inf_{\tilde{c}\in\tilde{C}} d(\tilde{c}, \tilde{d})$ and $d(\tilde{c}, \tilde{D}) = \inf_{\tilde{d}\in\tilde{D}} d(\tilde{c}, \tilde{d})$. Then, $(\mathcal{O}_{b,cl}(\sigma), \mathcal{H}_d)$ is a metric space (see [33]).

Definition 15. A svm $\mathfrak{F} : \mathcal{X} \longrightarrow \mathcal{O}_{cl}(\mathcal{X})$ is ν-Lipschitz iff $\exists \nu > 0$ such that

$$\mathcal{H}_d(\mathfrak{F}(\sigma), \mathfrak{F}(\rho)) \leq \nu d(\sigma,\rho), \quad \forall \sigma, \rho \in \mathcal{X}. \tag{26}$$

Particularly, if $\nu < 1$, then $\mathfrak{F}$ is a contraction.

Theorem 16. *Let*

$$\rho = \frac{\mathcal{M}}{\Gamma(r_1+1)}(\xi(\mathfrak{T}) - \xi(0))^{r_1}, \tag{27}$$

and assume that

(As1) $f : \mathfrak{J}^* \times \mathcal{X} \longrightarrow \mathcal{O}_{cp}(\mathcal{X})$ such that $f(.,\sigma)$: $\mathfrak{J}^* \longrightarrow \mathcal{O}_{cp}(\mathcal{X})$ is measurable for any $\sigma \in \mathcal{X}$.

(As2) $\mathcal{H}_d(f(\rho,\sigma), f(\rho,\bar{\sigma})) \leq \mathfrak{z}(\rho)\|\sigma - \bar{\sigma}\|$ for (a.e.) all $\rho \in \mathfrak{J}^*$ and $\sigma, \bar{\sigma} \in \mathbb{R}$ with $\mathfrak{z} \in C(\mathfrak{J}^*, \mathbb{R}^+)$ and $d(0, f(\rho,0)) \leq \mathfrak{z}(\rho)$ for (a.e.) all $\rho \in \mathfrak{J}^*$.

Then, (3) possesses at least one solution on $\mathfrak{J}^*$ if

$$\rho\|\mathfrak{z}\|_\infty < 1. \tag{28}$$

Proof. At first, to switch problem (3) into a FP, we formulate $\mathfrak{F} : \mathcal{E} \longrightarrow \mathcal{O}(\mathcal{E})$ as

$$\mathfrak{F}(\sigma) = \{\phi \in \mathcal{E} : \phi(\rho)$$
$$= \left\{ \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(\upsilon)(\xi(\rho)-\xi(\upsilon))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,\upsilon)\tilde{\wp}(\upsilon)d\upsilon \right\}, \tag{29}$$

for $\wp^\sim \in \mathcal{R}_{f,\sigma}$. The solution of (3) is obviously an FP of $\mathfrak{F}$. The following are the steps in the proofing process.

By virtue of assumption (As1) and [34] (Theorem III.6), f has a measurable selection, and thus, $\mathcal{R}_{f,\sigma} \neq \emptyset$. In the sequel, we demonstrate that $\mathfrak{F} : \mathcal{E} \longrightarrow \mathcal{O}(\mathcal{E})$ defined in (29) fulfills the assumptions of FPT of Covitz and Nadler. First, we show that $\mathfrak{F}(\sigma)$ is closed for every $\sigma \in \mathcal{E}$. Let $\{u_n\}_{n=0}^\infty \in \mathfrak{F}(\sigma)$ such that $\mathfrak{u}_n \longrightarrow \mathfrak{u}(n \longrightarrow \infty)$ in $\mathcal{E}$. Then, $\mathfrak{u} \in \mathcal{E}$ and there is $\wp^\sim{}_n \in \mathcal{R}_{f,\sigma_n}$ such that

$$\mathfrak{u}_n(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(\upsilon)(\xi(\rho)-\xi(\upsilon))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,\upsilon)\wp^\sim{}_n(\upsilon)d\upsilon, \quad \rho \in \mathfrak{J}^*. \tag{30}$$

Accordingly, there is a subsequence $\tilde{\wp}_n$that converges to $\tilde{\wp}$ in $L^1(\mathfrak{J}^*, \mathcal{X})$ because f has compact values. As an outcome, $\wp^\sim \in \mathcal{R}_{f,\sigma}$, and we get

$$\mathfrak{u}_n(\rho) \longrightarrow \mathfrak{u}(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)v_0 + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)\wp^{\sim}(v)dv, \quad \forall\rho\in\mathfrak{J}^*. \tag{31}$$

Hence, $u \in \mathfrak{F}(\sigma)$.

It remains to demonstrate that there is a $\theta \in (0,1)$ $(\theta = \rho\|\mathfrak{z}\|_\infty)$ such that

$$H_d(\mathfrak{F}(\sigma),\mathfrak{F}(\bar\sigma)) \le \theta\|\sigma-\bar\sigma\|, \quad \forall\sigma,\bar\sigma\in\mathscr{E}. \tag{32}$$

Let $\sigma,\bar\sigma\in\mathscr{E}$ and $\phi_1\in\mathfrak{F}(\sigma)$. Then, there exists $\wp^{\sim}{}_1(\rho)\in f(\rho,\sigma(\rho))$ such that

$$\phi_1(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)\wp^{\sim}{}_1(v)dv. \tag{33}$$

By (As2), we have

$$\mathscr{H}_d(f(\rho,\sigma),f(\rho,\bar\sigma)) \le \mathfrak{z}(\rho)\|\sigma(\rho)-\bar\sigma(\rho)\|. \tag{34}$$

So, there exists $\widehat{\varsigma}(\rho)\in f(\rho,\bar\sigma)$ such that

$$\|\wp^{\sim}{}_1(\rho)-\widehat{\varsigma}\| \le \mathfrak{z}(\rho)\|\sigma(\rho)-\bar\sigma(\rho)\|, \quad \rho\in\mathfrak{J}^*. \tag{35}$$

As follows, we build a svm $\chi : \mathfrak{J}^* \longrightarrow \mathcal{O}(\mathcal{X})$, where

$$\chi(\rho) = \{\widehat{\varsigma}\in\mathcal{X} : \|\wp^{\sim}{}_1(\rho)-\widehat{\varsigma}\| \le \mathfrak{z}(\rho)\|\sigma(\rho)-\bar\sigma(\rho)\|\}. \tag{36}$$

From [34] (Proposition III.4), the svm $\chi(\rho)\cap f(\rho,\bar\sigma)$ is measurable. We now select the function $\wp^{\sim}{}_2(\rho)\in f(\rho,\bar\sigma)$ such that

$$\|\wp^{\sim}{}_1(\rho)-\wp^{\sim}{}_2(\rho)\| \le \mathfrak{z}(\rho)\|\sigma(\rho)-\bar\sigma(\rho)\|, \quad \forall\rho\in\mathfrak{J}^*. \tag{37}$$

We define

$$\phi_2(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)\wp^{\sim}{}_2(v)dv, \quad \forall\rho\in\mathfrak{J}^*. \tag{38}$$

As a consequence, we get

$$\begin{aligned}\|\phi_1(\rho)-\phi_2(\rho)\| &\le \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)\|\wp^{\sim}{}_1(v)-\wp^{\sim}{}_2(v)\|dv \\ &\le \frac{\mathscr{M}\|\mathfrak{z}\|_\infty\|\sigma-\bar\sigma\|_{\mathscr{E}}}{\Gamma(r_1+1)}(\xi(\mathfrak{T})-\xi(0))^{r_1}.\end{aligned} \tag{39}$$

Therefore,

$$\|\phi_1-\phi_2\|_{\mathscr{E}} \le \rho\|\mathfrak{z}\|_\infty\|\sigma-\bar\sigma\|_{\mathscr{E}}. \tag{40}$$

By the analogous relation, obtained by interchanging the roles of σ and $\bar\sigma$, we get

$$\mathscr{H}_d(\mathfrak{F}(\sigma),\mathfrak{F}(\bar\sigma)) \le \rho\|\mathfrak{z}\|_\infty\|\sigma-\bar\sigma\|_{\mathscr{E}}. \tag{41}$$

Because $\mathfrak{F}$ is a contraction, we conclude that it has a FP, which is a solution of (3) based on the Covitz and Nadler theorem. □

Next, we study the topological properties of the set of solutions of problem (3).

Theorem 17. *Assume that*

(As3) $\mathcal{T}(\rho)$ is a compact operator for each $\rho > 0$.

(As4) $f : \mathfrak{J}^*\times\mathcal{X}\longrightarrow\mathcal{O}_{cp,c}(\mathcal{X})$ is a L^1-Carathéodory set-valued map.

(As5) $\exists\tilde{\ell}_1\in C(\mathfrak{J}^*,\mathbb{R}^+)$ and a nondecreasing $\tilde{\ell}_2\in C(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$\|f(\rho,\sigma)\|_{\mathcal{O}} = \sup\{\|\gamma\| : \gamma\in f(\rho,\sigma)\} \le \tilde{\ell}_1(\rho)\tilde{\ell}_2(\|\sigma\|_{\mathscr{E}}), \quad \forall(\rho,\sigma)\in\mathfrak{J}^*\times\mathcal{X}. \tag{42}$$

Then, the set of solutions of (3) is convex and relatively compact.

Proof. We consider the operator $\mathfrak{F}$ defined in (29) and demonstrate in the following steps that the set of solutions of (3) is convex and relatively compact in $\mathscr{E}$.

(Step 1) The svm $\mathfrak{F}(\sigma)$ is convex for every $\sigma\in\mathscr{E}$.

Let $\phi_1,\phi_2\in\mathfrak{F}(\sigma)$. Then, there exist $\wp^{\sim}{}_1,\wp^{\sim}{}_2\in\mathscr{R}_{f,\sigma}$ such that

$$\phi_i(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)\wp^{\sim}{}_i(v)dv, \quad i=1,2,\forall\rho\in\mathfrak{J}^*. \tag{43}$$

Let $\varkappa\in[0,1]$. Then, for every $\rho\in\mathfrak{J}^*$,

$$[\varkappa\phi_1+(1-\varkappa)\phi_2](\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)[\varkappa\wp^{\sim}{}_1(v)+(1-\varkappa)\wp^{\sim}{}_2(v)]dv. \tag{44}$$

Since f has convex values, $\mathscr{R}_{f,\sigma}$ is convex and $[\varkappa\wp^{\sim}{}_1(\rho)+(1-\varkappa)\wp^{\sim}{}_2(\rho)]\in\mathscr{R}_{\mathfrak{f},\sigma}$. Thus, $\varkappa\phi_1+(1-\varkappa)\phi_2\in\mathfrak{F}(\sigma)$.

(Step 2) $\mathfrak{F}(B_r)$ is bounded on bounded sets of $\mathcal{S}$.

For a constant $r\in\mathbb{R}^+$, let $B_r = \{\sigma\in\mathcal{S} : \|\sigma\|_{\mathscr{E}}\le r\}$ be a bounded set in $\mathcal{S}$. Then, for each $\phi\in\mathfrak{F}(\sigma)$ and $v\in B_r$, there exists $\wp^{\sim}\in\mathscr{R}_{f,\sigma}$ such that

$$\phi(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)\wp^{\sim}(v)dv. \tag{45}$$

Under hypothesis (As4) and Lemma 13 (1), for any $\rho \in \mathfrak{T}^*$, we attain

$$\begin{aligned}\|\phi(\rho)\| &\leq \left\|\mathcal{S}^{r_1;\xi}(\rho,0)\right\|\|\sigma_0\| \\ &\quad + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\left\|\mathcal{T}^{r_1;\xi}(\rho,v)\right\|\|\wp^{\sim}(v)\|dv \\ &\leq \mathcal{M}\|\sigma_0\| + \frac{\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)\mathcal{M}}{\Gamma(r_1+1)}(\xi(\mathfrak{T})-\xi(0))^{r_1}.\end{aligned} \tag{46}$$

Thus,

$$\|\phi\|_{\mathscr{E}} \leq \mathcal{M}\|\sigma_0\| + \rho\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r), \tag{47}$$

where ρ is defined in (27).

(Step 3) $\mathfrak{F}$ sends bounded sets of $\mathscr{E}$ into equicontinuous sets.

Let $\sigma \in B_r$ and $\phi \in \mathfrak{F}(\sigma)$. Then, there is a function $\wp^{\sim} \in \mathscr{R}_{f,\sigma}$ such that

$$\phi(\rho) = \mathcal{S}^{r_1;\xi}(\rho,0)\sigma_0 + \int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho,v)\wp^{\sim}(v)dv. \tag{48}$$

Let $\rho_1, \rho_2 \in \mathfrak{T}^*$, $\rho_1 < \rho_2$. Then,

$$\begin{aligned}\|\phi(\rho_2)-\phi(\rho_1)\| &\leq \left\|\mathcal{S}^{r_1;\xi}(\rho_2,0)\sigma_0 - \mathcal{S}^{r_1;\xi}(\rho_1,0)\sigma_0\right\| \\ &\quad + \left\|\int_0^{\rho_1} \xi'(v)(\xi(\rho_1)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho_2,v)\tilde{\wp}(v)dv\right. \\ &\quad + \int_{\rho_1}^{\rho_2} \xi'(v)(\xi(\rho_2)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho_2,v)\tilde{\wp}(v)dv \\ &\quad + \int_0^{\rho_1} \xi'(v)(\xi(\rho_1)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho_2,v)\tilde{\wp}(v)dv \\ &\quad - \int_0^{\rho_1} \xi'(v)(\xi(\rho_1)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho_2,v)\tilde{\wp}(v)dv \\ &\quad \left.- \int_0^{\rho_1} \xi'(v)(\xi(\rho_1)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho_1,v)\tilde{\wp}(v)dv\right\| \\ &\leq \left\|\mathcal{S}^{r_1;\xi}(\rho_2,0)\sigma_0 - \mathcal{S}^{r_1;\xi}(\rho_1,0)\sigma_0\right\| \\ &\quad + \left\|\int_{\rho_1}^{\rho_2} \xi'(v)(\xi(\rho_2)-\xi(v))^{r_1-1}\mathcal{T}^{r_1;\xi}(\rho_2,v)\tilde{\wp}(v)dv\right\| \\ &\quad + \left\|\int_0^{\rho_1} \xi'(v)\left[(\xi(\rho_2)-\xi(v))^{r_1-1}-(\xi(\rho_1)-\xi(v))^{r_1-1}\right]\mathcal{T}^{r_1;\xi}(\rho_2,v)\tilde{\wp}(v)dv\right\| \\ &\quad + \left\|\int_0^{\rho_1} \xi'(v)(\xi(\rho_1)-\xi(v))^{r_1-1}\left[\mathcal{T}^{r_1;\xi}(\rho_2,v)-\mathcal{T}^{r_1;\xi}(\rho_1,v)\right]\tilde{\wp}(v)dv\right\| \\ &=: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.\end{aligned} \tag{49}$$

From Lemma 13, it is obvious that $\mathcal{I}_1 \longrightarrow 0$ as $\rho_1 \longrightarrow \rho_2$ and we get

$$\begin{aligned}\mathcal{I}_2 &\leq \frac{\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)}{\Gamma(r_1+1)}(\xi(\rho_2)-\xi(\rho_1))^{r_1}, \\ \mathcal{I}_3 &\leq \frac{\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)}{\Gamma(r_1+1)}\left[(\xi(\rho_2))^{r_1}-(\xi(\rho_1))^{r_1}-(\xi(\rho_2)-\xi(\rho_1))^{r_1}\right],\end{aligned} \tag{50}$$

as $\rho_1 \longrightarrow \rho_2$, which yields to $\mathcal{I}_2 \longrightarrow 0$ and $\mathcal{I}_3 \longrightarrow 0$. For $\rho_1 = 0$ and $\rho_2 \in (0,\mathfrak{T}]$, it is clear that $\mathcal{I}_4 = 0$. Then, for any $0 < \varepsilon < \rho_1$, we get

$$\begin{aligned}\mathcal{I}_4 &\leq \left\|\int_0^{\rho_1-\varepsilon} \xi'(v)(\xi(\rho_1)-\xi(v))^{r_1-1}\left[\mathcal{T}^{r_1;\xi}(\rho_2,v)-\mathcal{T}^{r_1;\xi}(\rho_1,v)\right]\tilde{\wp}(v)dv\right\| \\ &\quad + \left\|\int_{\rho_1-\varepsilon}^{\rho_1} \xi'(v)(\xi(\rho_1)-\xi(v))^{r_1-1}\left[\mathcal{T}^{r_1;\xi}(\rho_2,v)-\mathcal{T}^{r_1;\xi}(\rho_1,v)\right]\tilde{\wp}(v)dv\right\| \\ &\leq \frac{\|\tilde{\ell}_1\|\tilde{\ell}_2(r)}{r_1}\left[(\xi(\rho_1)-\xi(0))^{r_1}-(\xi(\rho_1)-\xi(\rho_1-\varepsilon))^{r_1}\right] \\ &\quad \times\left(\sup_{0\leq\varepsilon<\rho_1}\left\|\mathcal{T}^{r_1;\xi}(\rho_2,v)-\mathcal{T}^{r_1;\xi}(\rho_1,v)\right\|\right) \\ &\quad + \frac{2\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)}{\Gamma(r_1+1)}\left[(\xi(\rho_1)-\xi(\rho_1-\varepsilon))^{r_1}\right],\end{aligned} \tag{51}$$

as $\rho_1 \longrightarrow \rho_2$ and $\varepsilon \longrightarrow 0$, which yields to $\mathcal{I}_4 \longrightarrow 0$; thus,

$$\|\phi(\rho_2)-\phi(\rho_1)\| \longrightarrow 0. \tag{52}$$

Hence, $\mathfrak{F}(B_r)$ is equicontinuous.

(Step 4) We show that for each $\rho \in \mathfrak{T}^*$, $\mathscr{H}(\rho) = \{(\mathfrak{F}_{\varepsilon,\delta}\sigma)(\rho) : \sigma \in B_r\}$ is relatively compact in $\mathscr{X}$.

Clearly, $\mathscr{H}(0)$ is relatively compact in $\mathscr{X}$. Let $\rho \in \mathfrak{T}^*$ be fixed, and for any $\varepsilon, \delta > 0$ and $\sigma \in B_r$, define

$$\begin{aligned}\mathfrak{F}(\sigma) &= \int_0^\infty \vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\sigma_0 d\theta \\ &\quad + r_1\int_0^{\rho-\varepsilon}\int_\delta^\infty \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta) \\ &\quad \times \mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\wp^{\sim}(v)d\theta dv \\ &= \int_0^\infty \vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\sigma_0 d\theta \\ &\quad + r_1\int_0^{\rho-\varepsilon}\int_\delta^\infty \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta) \\ &\quad \times \mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta+\varepsilon^{r_1}\delta-\varepsilon^{r_1}\delta\right)\wp^{\sim}(v)d\theta dv \\ &= \int_0^\infty \vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\sigma_0 d\theta \\ &\quad + r_1\int_0^{\rho-\varepsilon}\int_\delta^\infty \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta) \\ &\quad \times \left[\mathcal{T}(\varepsilon^{r_1}\delta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta-\varepsilon^{r_1}\delta\right)\right]\wp^{\sim}(v)d\theta dv \\ &= \int_0^\infty \vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\sigma_0 d\theta \\ &\quad + r_1\mathcal{T}(\varepsilon^{r_1}\delta)\int_0^{\rho-\varepsilon}\int_\delta^\infty \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta) \\ &\quad \times \left[\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta-\varepsilon^{r_1}\delta\right)\right]\wp^{\sim}(v)d\theta dv.\end{aligned} \tag{53}$$

Then, by the compactness of $\mathcal{T}(\varepsilon^{r_1}\delta)$ for $\varepsilon^{r_1}\delta > 0$, we see that the set $\mathcal{H}_{\varepsilon,\delta}(\rho) = \{(\mathfrak{F}_{\varepsilon,\delta}\sigma)(\rho) \colon \sigma \in B_r\}$ is relatively compact in $\mathcal{X}$ for all $\varepsilon, \delta > 0$. Moreover, for each $\sigma \in B_r$, we have

$$
\begin{aligned}
\|(\mathfrak{F}\sigma)(\rho) - (\mathfrak{F}_{\varepsilon,\delta}\sigma)(\rho)\| &= r_1 \bigg\| \int_0^\rho \int_0^\delta \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\tilde{\wp}(v)d\theta dv \\
&\quad + \int_0^\rho \int_\delta^\infty \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\tilde{\wp}(v)d\theta dv \\
&\quad - \int_0^{\rho-\varepsilon} \int_\delta^\infty \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\tilde{\wp}(v)d\theta dv \bigg\| \\
&\le r_1 \left\| \int_0^\rho \int_0^\delta \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\tilde{\wp}(v)d\theta dv \right\| \\
&\quad + r_1 \left\| \int_{\rho-\varepsilon}^\rho \int_\delta^\infty \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}\theta\vartheta_{r_1}(\theta)\mathcal{T}\left((\xi(\rho)-\xi(v))^{r_1}\theta\right)\tilde{\wp}(v)d\theta dv \right\| \\
&\le r_1\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)\left(\int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}dv\right)\left(\int_0^\delta \theta\vartheta_{r_1}(\theta)d\theta\right) \\
&\quad + r_1\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)\left(\int_{\rho-\varepsilon}^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}dv\right)\left(\int_\delta^\infty \theta\vartheta_{r_1}(\theta)d\theta\right) \\
&\le r_1\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)\left(\int_0^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}dv\right)\left(\int_0^\delta \theta\vartheta_{r_1}(\theta)d\theta\right) \\
&\quad + r_1\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)\left(\int_{\rho-\varepsilon}^\rho \xi'(v)(\xi(\rho)-\xi(v))^{r_1-1}dv\right)\left(\int_0^\infty \theta\vartheta_{r_1}(\theta)d\theta\right) \\
&\le \mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)(\xi(\rho)-\xi(0))^{r_1}\left(\int_0^\delta \theta\vartheta_{r_1}(\theta)d\theta\right) \\
&\quad + \frac{\mathcal{M}\|\tilde{\ell}_1\|_\infty \tilde{\ell}_2(r)}{\Gamma(r+1)}(\xi(\rho)-\xi(\rho-\varepsilon))^{r_1} \longrightarrow 0, \quad \text{as}\, \varepsilon, \delta \longrightarrow 0^+.
\end{aligned} \tag{54}
$$

Hence, there are relatively compact sets arbitrarily close to $\mathcal{H}(\rho)$ for $\rho > 0$, Therefore, $\mathcal{H}(\rho)$ is relatively compact in $\mathcal{X}$.

From the Arzela-Ascoli theorem, $\mathfrak{F}(B_r)$ is relatively compact in $\mathcal{E}$. Consequently, $\mathfrak{F}(\sigma)$ is relatively compact in $\forall \sigma \in \mathcal{E}$. □

4. An Example

Let $\mathcal{X} = L^2([0,\pi])$. Consider the following fractional partial differential inclusions with Caputo derivative $(\xi(\rho) = \rho)$:

$$
\begin{cases}
{}^C D_{a+}^{1/2;\rho}\sigma(\rho,\nu) \in \dfrac{\partial^2}{\partial \nu^2}\sigma(\rho,\nu) + f(\rho,\sigma(\rho,\nu)), & \rho \in (0,1), \sigma \in (0,\pi], \\
\sigma(\rho,0) = \sigma(\rho,\pi) = 0, \quad \sigma(0,\nu) = \sigma_0, &
\end{cases} \tag{55}
$$

where $r_1 = 1/2$ and $\mathcal{T} = 1$, and we define an operator $\mathcal{A}$ by $\mathcal{A}\sigma = \sigma''$ for $\sigma \in \mathfrak{D}(\mathcal{A})$, where $\mathfrak{D}(\mathcal{A}) = \{\sigma \in \mathcal{X} : \sigma, \sigma'$ are absolutely continuous, and $\sigma(0) = \sigma(\pi) = 0\}$. Then, $\mathcal{A}$ generates a strongly continuous semigroup $\{\mathcal{T}(\rho)\}_{\rho \ge 0}$ which is compact. Furthermore, $\|\mathcal{T}(\rho)\| \le e^{-\rho} \le 1 = \mathcal{M}$, $\rho \ge 0$. Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathcal{O}(\mathbb{R})$ defined by

$$
\sigma(\rho,\nu) \longrightarrow f(\rho,\sigma(\rho,\nu)) = \left[0, \frac{\sin(\sigma(\rho,\nu))}{(\exp(\rho^2)+2)} + \frac{1}{15}\right]. \tag{56}
$$

Clearly, $H_d(f(\rho,\sigma), f(\rho,\bar{\sigma})) \le \mathfrak{z}(\rho)\|\sigma - \bar{\sigma}\|$, where $\mathfrak{z}(\rho) = 1/(\exp(\rho^2)+2)$ and $d(0, f(\rho,0)) = 1/15 \le \mathfrak{z}(\rho)$ for (a.e.) all $\rho \in [0,1]$. Besides, we obtain $\|\mathfrak{z}\|_\infty = 1/3$ which implies $\rho\|\mathfrak{z}\| \simeq 0.38 < 1$. Consequently, all items of Theorem 16 are satisfied. Then, there exists at least one solution of (10) on $[0,1]$.

Remark 18. Our current results for problem (3) remain true for the following cases:

(i) Caputo-type inclusion problem: $\xi(\rho) = \rho$.

(ii) Caputo-Hadamard-type inclusion problem: $\xi(\rho) = \log(\rho)$.

(iii) Generalized Caputo-type inclusion problem: $\xi(\rho) = \rho^\rho$, $\rho > 0$.

Remark 19. The obtained results for problem (3) include the results of Suechoei and Ngiamsunthorn [17]; i.e., problem (3) is reduced to the considered problem in [17] when $\{f(\rho, \sigma(\rho))\} = h(\rho, \sigma(\rho))$.

5. Concluding Remarks

We have studied the class of IVPs for generalized Caputo FDIs with Lipschitz set-valued functions. We obtained the existence result by using the FPT of Covitz and Nadler. Also, we characterized the topological properties of the set of solutions. We confirm that our acquired outcomes are new in the frame of generalized Caputo FDIs with initial conditions and it greatly contributes to enriching the existing literature on this theme.

In future works, many cases can be established when one takes a more generalized operator, for example, ψ-Hilfer fractional operator [13]. Further, it will be of interest to study the existing problem in this article for the Mittag-Leffler power law [8] and for fractal fractional operators [35].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors made equal contributions and read and supported the last original copy.

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Research Article

Numerical Analysis of the Fractional-Order Nonlinear System of Volterra Integro-Differential Equations

Pongsakorn Sunthrayuth,[1] Roman Ullah,[2] Adnan Khan,[3] Rasool Shah,[3] Jeevan Kafle,[4] Ibrahim Mahariq,[5] and Fahd Jarad[6,7]

[1]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Thanyaburi, Pathum Thani, Thailand*
[2]*Department of Computing, Muscat College, Muscat, Oman*
[3]*Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan*
[4]*Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal*
[5]*College of Engineering and Technology, American University of the Middle East, Kuwait*
[6]*Department of Mathematics, Cankaya University, Etimesgut, Ankara, Turkey*
[7]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*

Correspondence should be addressed to Jeevan Kafle; jeevan.kafle@cdmath.tu.edu.np and Fahd Jarad; fahd@cankaya.edu.tr

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This paper presents the nonlinear systems of Volterra-type fractional integro-differential equation solutions through a Chebyshev pseudospectral method. The proposed method is based on the Caputo fractional derivative. The results that we get show the accuracy and reliability of the present method. Different nonlinear systems have been solved; the solutions that we get are compared with other methods and the exact solution. Also, from the presented figures, it is easy to conclude that the CPM error converges quickly as compared to other methods. Comparing the exact solution and other techniques reveals that the Chebyshev pseudospectral method has a higher degree of accuracy and converges quickly towards the exact solution. Moreover, it is easy to implement the suggested method for solving fractional-order linear and nonlinear physical problems related to science and engineering.

1. Introduction

Fractional calculus has a long history as classical calculus. The concept of fractional calculus arouse when Leibnitz used a proper representation $d^n f/dx^n$ for the nth derivative in his publications. L'hopital raises a question on the particular notation on what happens if "n" is a noninteger. It was the beginning of fractional calculus [1]. Recently, mathematicians focused on fractional calculus due to its numerous applications in every field of science: viscoelastic materials [2], economics [3], continuum and statistical mechanics [4], dynamics of interfaces between soft nanoparticles and rough substrates [5], solid mechanics [6], and much more [7–14].

Mathematical formulations solve many problems of nature with the help of converting the physical phenomena to the equation form. Differential equations (DEs) are among those that play the main role in modeling various phenomena. However, some problems are complex and cannot be handled with the help of a differential equation. In this regard, the researchers utilized fractional differential equations (FDEs) that model the phenomenon more accurately than differential equations having order integers. Nowadays, FDEs got the importance of real-world modeling problems: such as electrode-electrolyte polarization [15], electrochemistry of corrosion [16], circuit systems [17], optics and signal processing [18], heat conduction [19], diffusion wave [20], control theory of dynamical systems [21], fluid flow [22], probability and statistics [23, 24], and so on (see [25–28]).

The role of fractional integral and integrodifferential equations is found in every field of engineering and science. When a physical phenomenon is modeled under the differential equation, it finally gives a differential equation, an integral equation, or an integrodifferential equation. Some applications of these types of equations are nanohydrodynamics [29], glass-forming process [30], wind ripple in the desert [31], and drop-wise condensation [32]. The analytical solution of integral and integrodifferential equations does not exist in most cases. Even if it exists in certain cases, it is hard to find. Different numerical methods have been developed for finding an approximate solution of integral and integro-differential equations. The most common among these methods are the Chebyshev polynomials [33], Haar wavelet [34], triangular function method [35], collocation method [36], Legendre wavelet operational method [37], Taylor series expansion method [38], homotopy perturbation method [39], reproducing kernel Hilbert space method [40], Adomian decomposition method [41], Euler wavelet method [42], variational iteration method [43], spectral collocation method [44], least square method [45], homotopy analysis method [46], and differential transform method [47].

We apply Chebyshev pseudospectral method (CPM) to solve nonlinear Volterra integro-differential equation systems in the present work. CPM is a powerful technique for solving linear and nonlinear problems. The obtained results show the higher convergence rate of the present technique. The solution that we get shows that CPM has good agreement with the exact solution. Error analysis reveals the efficiency of the proposed technique that CPM has greater accuracy than other methods.

2. Definitions and Preliminary Concept

This unit shows the preliminary concept and some essential definitions taken from fractional calculus and used in our present research work.

2.1. Definition. The definition for fractional derivative by Caputo of order α is showed by the following mathematical expression [48]:

$$D^{\alpha} j(s) = \frac{1}{\Gamma(n-\alpha)} \int_0^s (s-t)^{n-\alpha-1} j^{(n)}(t) dt, \tag{1}$$

for $n-1<\alpha \leq n$, $n \in \mathbb{N}$, $s>0$, $j \in \mathbb{C}_{-1}^m$.

2.2. Definition. The fractional derivatives by Jin-Hunan He are described as [48]

$$\frac{D^{\alpha} j(s)}{D s^{\alpha}} = \Gamma(1+\alpha) \lim_{\Delta s=s_1-s_2 \longrightarrow L} \frac{f(s_1)-f(s_2)}{(s_1-s_2)^{\alpha}}, \tag{2}$$

where Δs does not approach zero.

2.3. Definition. Xiao-Jun explains derivatives having fractional order as [48]

$$D_s^{\alpha} j(s_0) = j^{\alpha}(s_0) = \left.\frac{d^{\alpha} j(s)}{d s^{\alpha}}\right|_{s=s_0} = \lim_{s \longrightarrow s_0} \frac{\Delta^{\alpha}(j(s)-j(s_0))}{(s-s_0)^{\alpha}}, \tag{3}$$

where

$$\Delta^{\alpha}(j(s)-j(s_0)) \cong \Gamma(1+\alpha) \Delta(j(s)-j(s_0)). \tag{4}$$

2.4. Definition. The integral operator by Riemann-Liouville for order α is [48]

$$I^{\alpha} j(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} j(t) dt. \tag{5}$$

The Caputo derivative operator and Riemann-Liouville integral operator have the following properties

$$\begin{aligned} & D^{\alpha} I^{\alpha} j(s) = j(s), \\ & I^{\alpha} D^{\alpha} j(s) = j(s) - \sum_{k=0}^{n-1} \frac{j^{(k)}(0^+)}{k!} s^k, s \geq 0 \; n-1<\alpha<n. \end{aligned} \tag{6}$$

3. Chebyshev Pseudospectral Method (CPM)

The Chebyshev polynomials are defined in the $[-1,1]$ interval and can be described by the following recurrence formula:

$$R_{n+1}(t) = 2u R_n(s) - R_{n-1}(s), \quad n=1,2,\cdots, \tag{7}$$

where

$$\begin{aligned} & R_0(s) = 1, \\ & R_1(s) = s. \end{aligned} \tag{8}$$

To apply the Chebyshev polynomials in the $[0,1]$ interval, we define the Chebyshev shifted polynomials $\widehat{R}_n(s)$ which are defined in the manner of Chebyshev polynomials $R_n(s)$ by relation

$$\widehat{R}_n(s) = R_n(2s-1). \tag{9}$$

And the recurrence formula is as follows:

$$\widehat{R}_{n+1}(s) = 2(2s-1) \widehat{R}_n(s) - \widehat{R}_{n-1}(s), \quad n=1,2,\cdots, \tag{10}$$

where

$$\begin{aligned} & \widehat{R}_0(s) = 1, \\ & \widehat{R}_1(s) = 2s-1. \end{aligned} \tag{11}$$

Table 1: Exact versus CPM solution of problem 1 at $m = 10$.

s	Exact $j(s)$	Exact $k(s)$	CPM solution $j(s)$	CPM solution $k(s)$
0	0.0000000000	1.0000000000	0.0000000000	1.0000000000
0.1	0.1001667500	1.0050041680	0.1001667500	1.0050041680
0.2	0.2013360025	1.0200667556	0.2013360025	1.0200667556
0.3	0.3045202934	1.0453385141	0.3045202934	1.0453385141
0.4	0.4107523258	1.0810723718	0.4107523258	1.0810723718
0.5	0.5210953054	1.1276259652	0.5210953054	1.1276259652
0.6	0.6366535821	1.1854652182	0.6366535821	1.1854652182
0.7	0.7585837018	1.2551690056	0.7585837018	1.2551690056
0.8	0.8881059821	1.3374349463	0.8881059821	1.3374349462
0.9	1.0265167257	1.4330863854	1.0265167253	1.4330863853
1.0	1.1752011936	1.5430806348	1.1752011918	1.5430806343

Table 2: Error comparison of CPM versus other methods of Section 4.1 at $m = 10$.

s	Error(j_{CPM})	Error(k_{CPM})	Error(j_{OTM})	Error(k_{OTM})
0	$0.0000000000E+00$	$0.0000000000E+00$	$0.00E+00$	$0.00E+00$
0.1	$8.2692537238E-17$	$1.4541999963E-17$	$1.39E-17$	$0.00E+00$
0.2	$9.1822482293E-16$	$1.6662883612E-16$	$5.27E-16$	$0.00E+00$
0.3	$1.2443994903E-15$	$2.2699378388E-16$	$4.45E-14$	$1.11E-15$
0.4	$1.2607906760E-15$	$2.3113111594E-16$	$1.05E-12$	$3.51E-14$
0.5	$7.2897283903E-16$	$1.2454060032E-16$	$1.23E-11$	$5.10E-13$
0.6	$2.5175064758E-13$	$5.5241090380E-14$	$9.11E-11$	$4.55E-12$
0.7	$5.8189005235E-12$	$1.3209327312E-12$	$4.97E-10$	$2.90E-11$
0.8	$5.8725732247E-11$	$1.3786485036E-11$	$2.16E-9$	$1.44E-10$
0.9	$3.7796170567E-10$	$9.1669608328E-11$	$7.90E-9$	$5.92E-10$
1.0	$1.8157658168E-09$	$4.5450903393E-10$	$2.52E-8$	$2.10E-9$

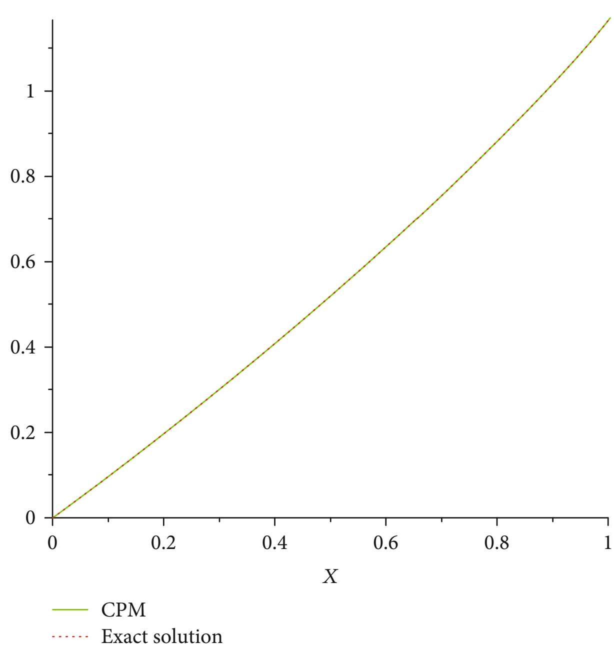

Figure 1: The solution graph of example 1. (a) Exact solution and (b) CPM solution.

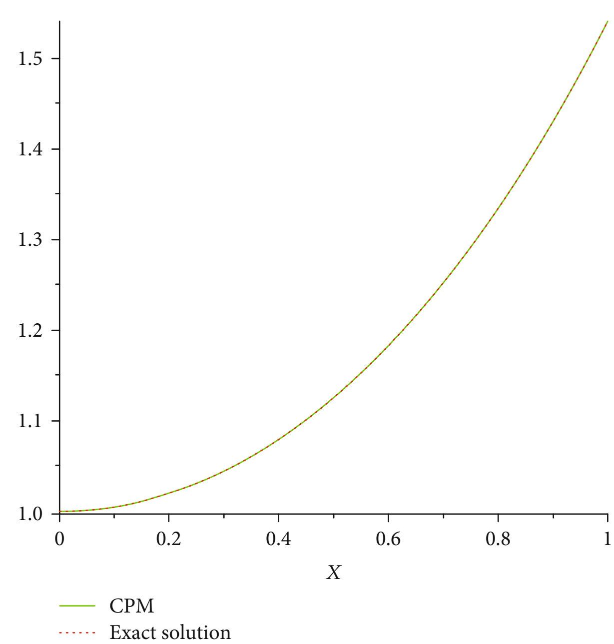

Figure 2: The solution graph of example 1. (a) Exact solution and (b) CPM solution.

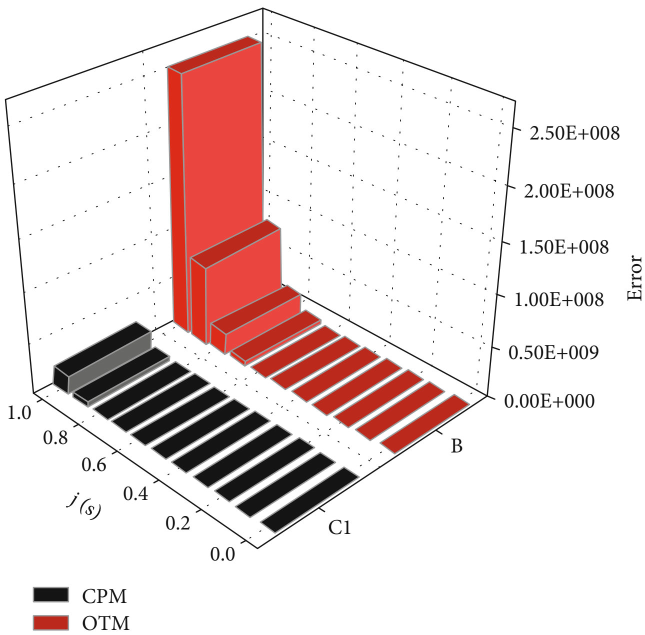

Figure 3: CPM and OTM error graph of Section 4.1.

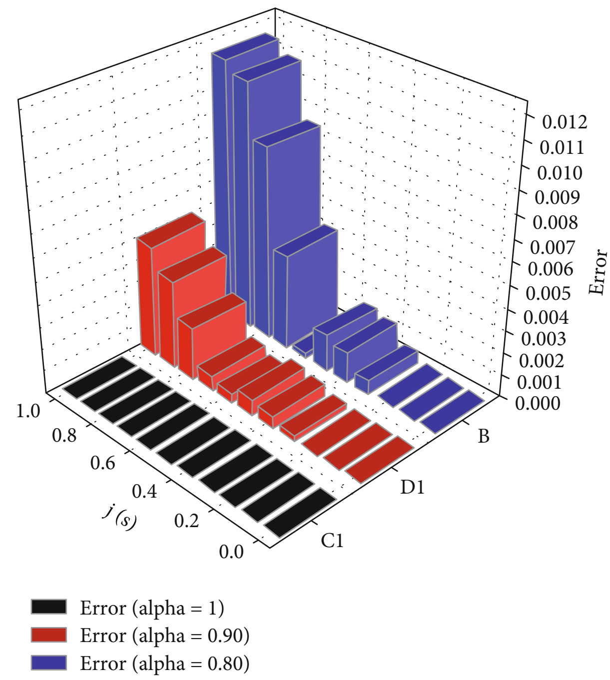

Figure 5: The absolute error graph of Section 4.1 at a different fractional order.

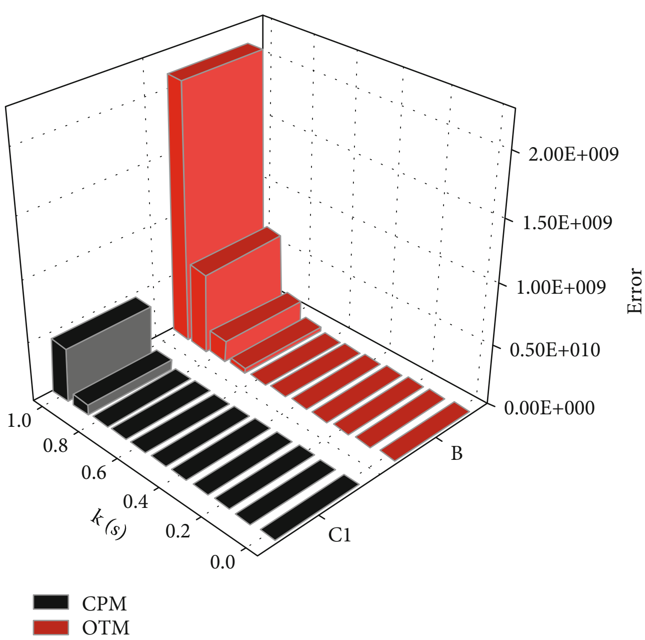

Figure 4: CPM and OTM error graph of Section 4.1.

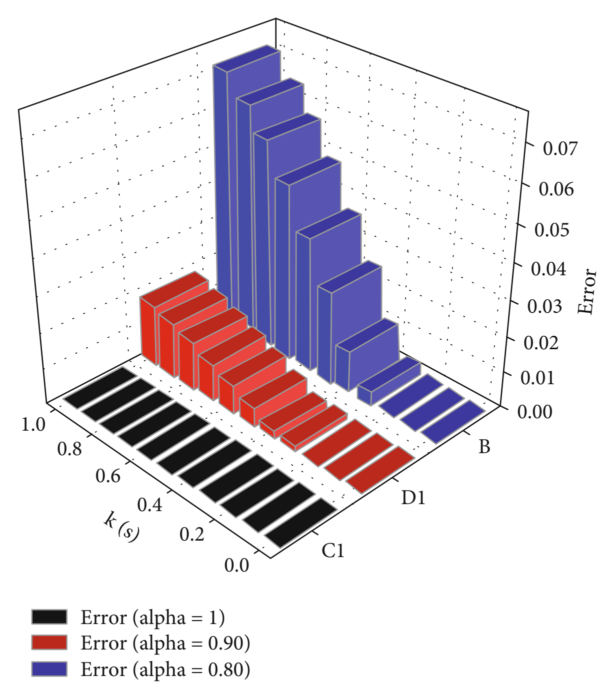

Figure 6: The absolute error graph of Section 4.1 at a different fractional order.

A function $j(s) \in L_2[0,1]$, in terms of Chebyshev shifted polynomials described as

$$j(s) = \sum_{n=1}^{\infty} c_n \widehat{R}_n(s). \tag{12}$$

Table 3: Exact and CPM solutions of Section 4.2 at $m = 3$.

s	Exact $j(s)$	Exact $k(s)$	CPM solution $j(s)$	CPM solution $k(s)$
0.05	0.047500000000000	−0.002375000000000	0.047500000050000	−0.002375000000000
0.15	0.127500000000000	−0.019125000000000	0.127500000100000	−0.019124999970000
0.25	0.187500000000000	−0.046875000000000	0.187500000100000	−0.046874999940000
0.35	0.227500000000000	−0.079625000000000	0.227500000100000	−0.079624999950000
0.45	0.247500000000000	−0.111375000000000	0.247500000000000	−0.111374999900000
0.55	0.247500000000000	−0.136125000000000	0.247500000000000	−0.136125000000000
0.65	0.227500000000000	−0.147875000000000	0.227500000000000	−0.147875000000000
0.75	0.187500000000000	−0.140625000000000	0.187500000000000	−0.140625000000000
0.85	0.127500000000000	−0.108375000000000	0.127500000000000	−0.108375000000000
0.95	0.047500000000000	−0.045125000000000	0.047499999999999	−0.045125000100000

Table 4: Error comparison of CPM versus other methods of Section 4.2.

s	Error(j_{CPM})	Error(k_{CPM})	Error(j_{SCM})	Error(k_{SCM})
0.05	$5.0000000000E-11$	$0.0000000000E+00$	$8.17886E-8$	$1.71222E-7$
0.15	$1.0000000000E-10$	$3.0000000000E-11$	$4.14502E-8$	$5.20012E-8$
0.25	$1.0000000000E-10$	$6.0000000000E-11$	$3.00945E-9$	$1.53079E-7$
0.35	$1.0000000000E-10$	$5.0000000000E-11$	$5.66834E-8$	$1.82626E-7$
0.45	$0.0000000000E+00$	$1.0000000000E-10$	$3.81977E-8$	$6.42170E-7$
0.55	$0.0000000000E+00$	$0.0000000000E+00$	$3.16220E-8$	$6.19236E-7$
0.65	$0.0000000000E+00$	$0.0000000000E+00$	$6.05974E-8$	$1.37882E-7$
0.75	$0.0000000000E+00$	$0.0000000000E+00$	$9.63834E-9$	$1.53242E-7$
0.85	$0.0000000000E+00$	$0.0000000000E+00$	$4.55344E-8$	$1.18939E-8$
0.95	$5.0000000000E-11$	$1.0000000000E-10$	$8.32363E-8$	$9.10621E-8$

The Chebyshev shifted polynomials first $(m+1)$ terms are considered as

$$j_m(s) = \sum_{n=0}^{m} c_n \widehat{R}_n(s),$$
$$\frac{d^\alpha}{ds^\alpha}\left(\sum_{n=0}^{m} c_n \widehat{R}_n(s)\right) + \sum_{n=0}^{m} c_n \widehat{R}_n(s) + \int_{u_0}^{s}\left(\sum_{n=0}^{m} c_n \widehat{R}_n(s)\right) ds = g(s, j). \tag{13}$$

For finding the system of equations, we have

$$\frac{d^\alpha}{ds^\alpha}\left(\sum_{n=0}^{m} c_n \widehat{R}_n(s_i)\right) + \sum_{n=0}^{m} c_n \widehat{R}_n(s_i) + \int_{s_0}^{s}\left(\sum_{n=0}^{m} c_n \widehat{R}_n(s_i)\right) du = g(s_i, j). \tag{14}$$

whereas

$$s_i = \frac{i-0.5}{2^{k-1}M}. \tag{15}$$

I solved the resultant system using maple software, which provide CPM solution for the given problem.

4. Numerical Representation

4.1. Problem. Consider the nonlinear FIDE system having B.Cs $j(0) = 0$, $k(0) = 1$

$$D^\alpha j(s) + \frac{1}{2}\left(\frac{dk}{ds}\right)^2 - \int_0^s [(s-t)k(t) + k(t)j(t)]dt = 1,$$
$$D^\alpha k(s) + sj(s) - \int_0^s \left[(s-t)j(t) + k^2(t)\right]dt = 2s, \tag{16}$$

having $j(s) = \sinh(s)$, $k(s) = \cosh(s)$ as the exact solution at $\alpha = 1$.

The exact solution and numerical results obtained by means of the proposed method are shown in Table 1. The absolute error comparison of our method and those obtained from OTM is given in Table 2. The behavior of the exact solution and approximate solution (our method) of this example when $\alpha = 1$ is presented in Figures 1 and 2 whereas the error comparison of CPM and OTM can be observed in Figures 3 and 4. The graphical representation for different fractional order of α is seen in Figures 5 and 6 which confirm that the solution converge to the exact solution as the value of α converges from the fractional order to the integer order.

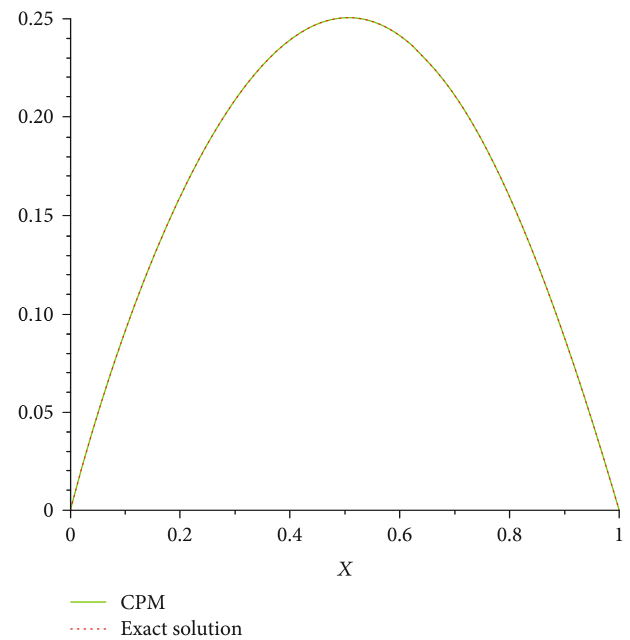

Figure 7: The solution graph of example 2. (a) Exact solution and (b) CPM solution.

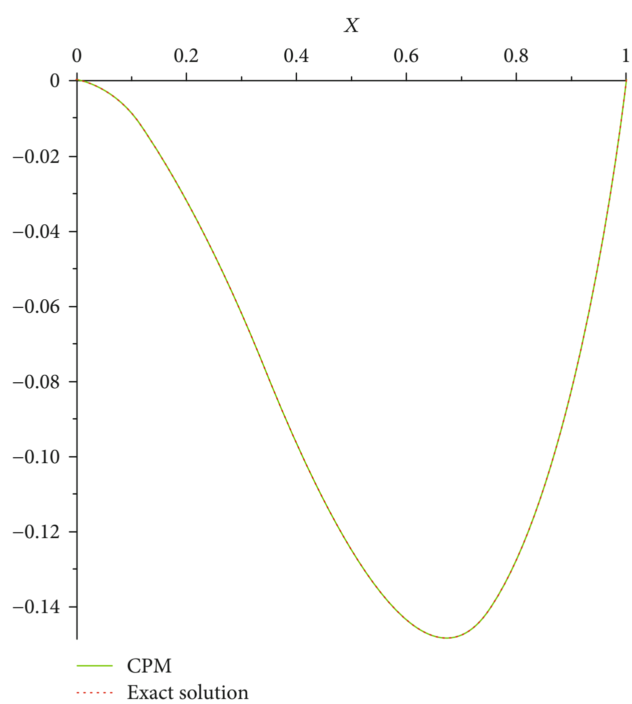

Figure 8: The solution graph of example 2. (a) Exact solution and (b) CPM solution.

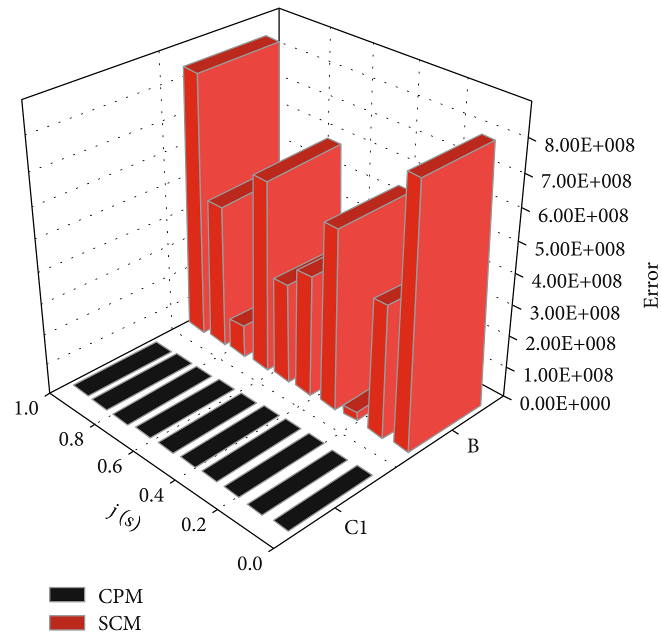

Figure 9: CPM and SCM error graph of Section 4.2.

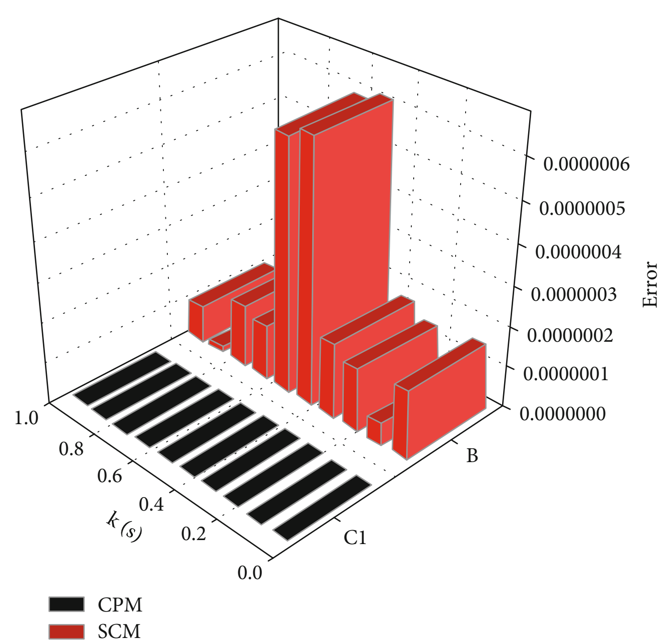

Figure 10: CPM and SCM error graph of Section 4.2.

4.2. Problem. Consider the FIDE system with B.Cs $j(0) = j(1) = 0,\ k(0) = k(0) = 0$

$$D^{\alpha} j(s) + k^2(s) + \frac{s}{2}\frac{dk}{ds} - \int_0^s ((s-t)k(t) + j(t)k(t))dt = g_2(s),$$
$$D^{\alpha} k(s) + j^2(s) - \int_0^s \left((s-t)j(t) - k^2(t) + j^2(t)\right)dt = g_1(s), \tag{17}$$

Table 5: Exact versus CPM solution of Section 4.3 at $m = 10$.

s	Exact $j(s)$	Exact $k(s)$	CPM solution $j(s)$	CPM solution $k(s)$
0	1.0000000000	−1.0000000000	1.0000000000	−1.0000000000
0.1	1.2051709180	−1.0051709180	1.2051709180	−1.0051709180
0.2	1.4214027580	−1.0214027580	1.4214027580	−1.0214027580
0.3	1.6498588080	−1.0498588075	1.6498588080	−1.0498588075
0.4	1.8918246980	−1.0918246976	1.8918246980	−1.0918246976
0.5	2.1487212710	−1.1487212707	2.1487212710	−1.1487212707
0.6	2.4221188000	−1.2221188003	2.4221188000	−1.2221188003
0.7	2.7137527070	−1.3137527074	2.7137527070	−1.3137527074
0.8	3.0255409280	−1.4255409284	3.0255409280	−1.4255409284
0.9	3.3596031110	−1.5596031111	3.3596031111	−1.5596031109
1.0	3.7182818280	−1.7182818284	3.7182818284	−1.7182818275

Table 6: Error comparison of CPM versus other methods of Section 4.3 at $m = 10$.

s	Error(j_{CPM})	Error(k_{CPM})	Error(j_{OTM})	Error(k_{OTM})
0	$0.0000000000E+00$	$0.0000000000E+00$	$0.00E+00$	$0.00E+00$
0.1	$8.1999412724E-17$	$8.2624090006E-17$	$2.22E-16$	$0.00E+00$
0.2	$3.8043301351E-15$	$3.8490016915E-15$	$4.44E-16$	$6.66E-16$
0.3	$1.9076326419E-14$	$1.9470658817E-14$	$4.55E-14$	$4.55E-14$
0.4	$3.8309289099E-14$	$3.9767165526E-14$	$1.09E-12$	$1.09E-12$
0.5	$5.6578756405E-14$	$6.0124928716E-14$	$1.28E-11$	$1.28E-11$
0.6	$3.2264909603E-13$	$3.2822178282E-13$	$9.57E-11$	$9.57E-11$
0.7	$6.8521655232E-12$	$6.8380840814E-12$	$5.26E-10$	$5.26E-10$
0.8	$7.2715580869E-11$	$7.2515818424E-11$	$2.30E-9$	$2.30E-9$
0.9	$4.8186768333E-10$	$4.8069973992E-10$	$8.49E-9$	$8.49E-9$
1.0	$2.3551879379E-09$	$2.3502526547E-9$	$2.73E-8$	$2.73E-8$

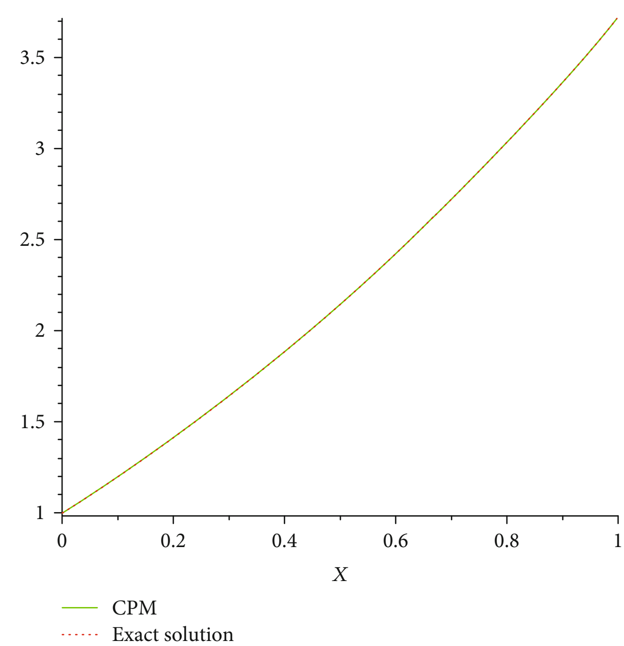

Figure 11: The solution graph of example 3. (a) Exact solution and (b) CPM solution.

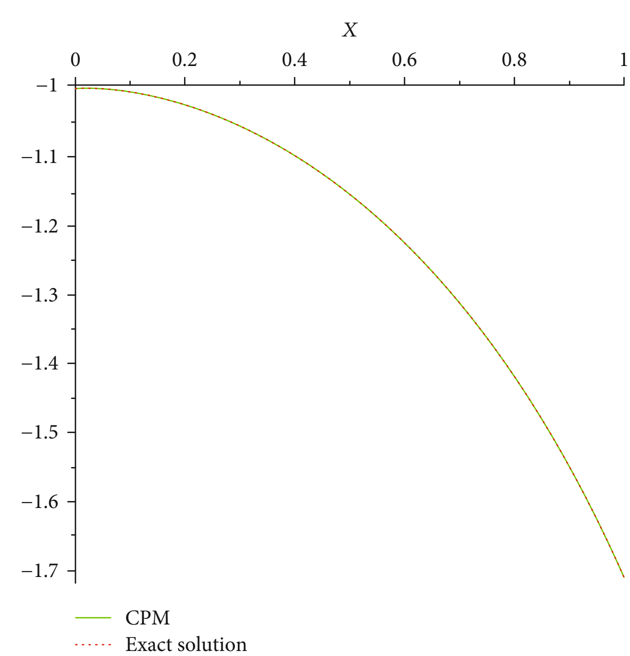

Figure 12: The solution graph of example 3. (a) Exact solution and (b) CPM solution.

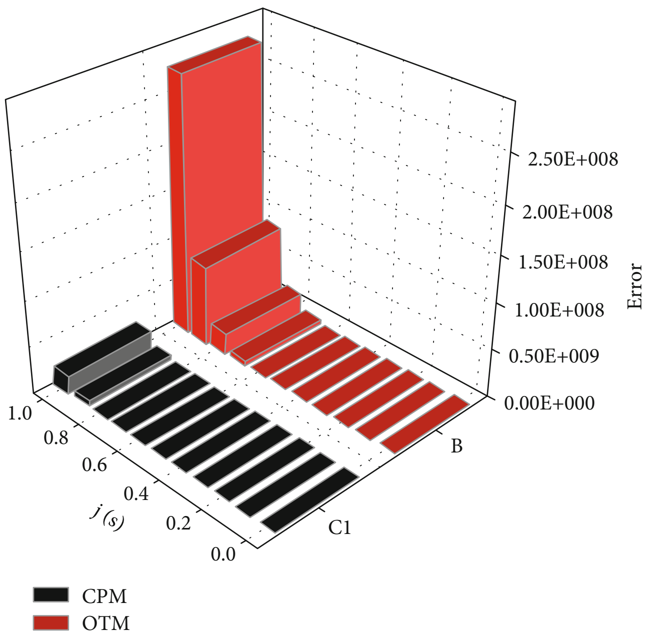

FIGURE 13: CPM and OTM error graph of Section 4.3.

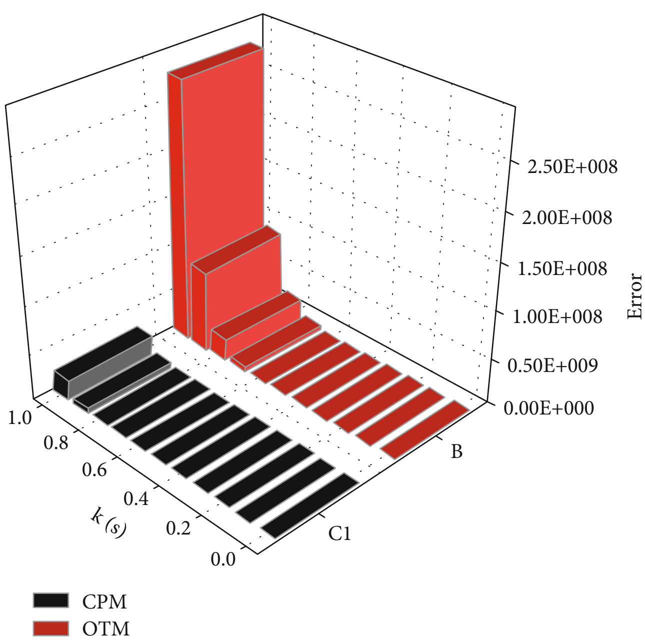

FIGURE 14: CPM and OTM error graph of Section 4.3.

with the exact solution $(j(s) = s - s^2, k(s) = s^3 - s^2)$ at $\alpha = 2$, where

$$g_1(s) = \frac{7}{6}s^6 - \frac{49}{20}s^5 + \frac{4}{3}s^4 + \frac{3}{2}s^3 - s^2 - 2,$$
$$g_2(s) = \frac{s^7}{7} - \frac{s^6}{3} + \frac{19}{12}s^4 - \frac{5}{2}s^3 + s^2 + 6s - 2. \quad (18)$$

In Table 3, we give the numerical values of the exact solution and CPM solution for $m = 3$. The absolute errors

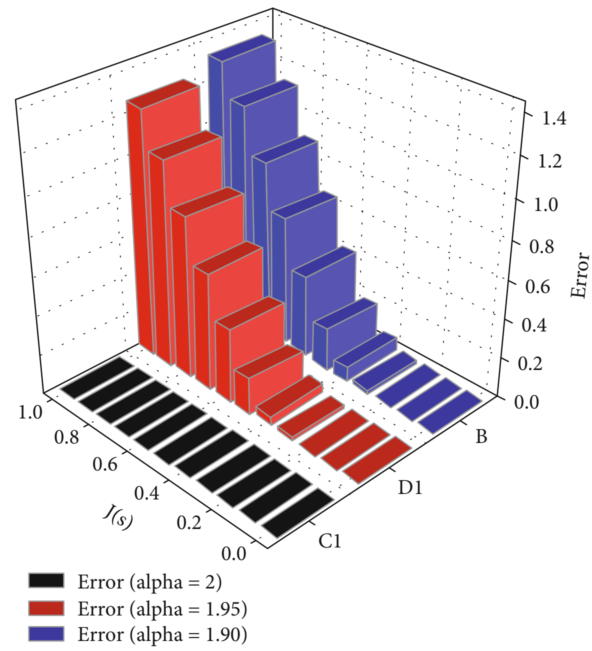

FIGURE 15: The absolute error graph of Section 4.3 at a different fractional order.

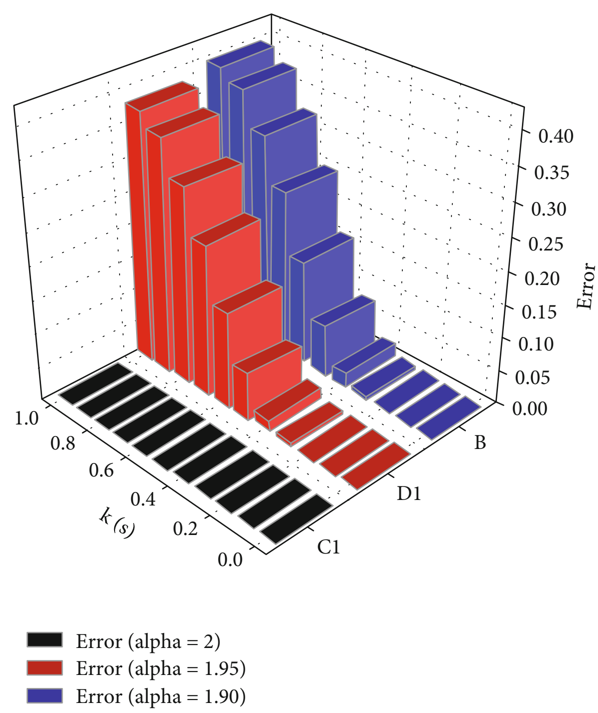

FIGURE 16: The absolute error graph of Section 4.3 at a different fractional order.

obtained by the present method are compared with SCM in Table 4. We compare the actual and estimated solution in Figures 7 and 8 which tells us that both the solutions are quite close to each other. Also, Figures 9 and 10 display the error comparison of CPM and SCM which verify that our method is in good agreement with the exact solution.

4.3. Problem. Consider the nonlinear FIDE system having B.Cs $j(0)=1$, $j'(0)=2$, $k(0)=-1$, and $k'(0)=0$

$$D^{\alpha}j(s)+\frac{1}{2}\left(\frac{dk}{ds}\right)^2-\frac{1}{2}\int_0^s\left[j^2(t)+k^2(t)\right]dt=1-\frac{1}{3}s^3,$$
$$D^{\alpha}k(s)+sj(s)-\frac{1}{4}\int_0^s\left[j^2(t)-k^2(t)\right]dt=s^2-1, \tag{19}$$

having $j(s)=s+e^s$, $k(s)=s-e^s$ as the exact solution.

To solve this example, we implement the method suggested in Section 4 for $\alpha=2$ with $m=10$. The exact solution and estimated solution by CPM are presented in Table 5. The absolute error of our method and those obtained from OTM are given in Table 6. In Figures 11 and 12, it is clear that the numerical solution of the proposed method is in good contact with the exact solution. In order to illustrate the effectiveness of CPM, the error comparison with OTM is shown in Figures 13 and 14. Also, in Figures 15 and 16, we can obtain that as $\alpha\longrightarrow 2$ the estimated solution approach to the exact solutions.

5. Conclusion

In this work, we implemented the Chebyshev pseudospectral method for solving nonlinear fractional integral and integro-differential equation systems. The proposed technique reduces this type of systems to the solution of the system of linear and nonlinear algebraic equations. Special attention is given to study the convergence of the proposed method. The results that we get by implementing the suggested technique are in excellent agreement with the exact solution and show more accuracy than the solution obtained using other methods. Also, from the presented figures, it is easy to conclude that the CPM error converges quickly as compared to other methods. The computation work in this article is done using Maple.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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Research Article

Fractional Integral Inequalities via Atangana-Baleanu Operators for Convex and Concave Functions

Ahmet Ocak Akdemir,[1] **Ali Karaoğlan**,[2] **Maria Alessandra Ragusa**,[3,4] **and Erhan Set**[1]

[1]*Department of Mathematics, Faculty of Sciences and Arts, Ağri University, Ağri, Turkey*
[2]*Department of Mathematics, Faculty of Sciences and Arts, Ordu University, Ordu, Turkey*
[3]*Dipartimento di Matematica e Informatica, Universitá di Catania, Catania, Italy*
[4]*Rudn University, 6 Miklukho, Maklay St., Moscow 117198, Russia*

Correspondence should be addressed to Maria Alessandra Ragusa; mariaalessandra.ragusa@unict.it

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Recently, many fractional integral operators were introduced by different mathematicians. One of these fractional operators, Atangana-Baleanu fractional integral operator, was defined by Atangana and Baleanu (Atangana and Baleanu, 2016). In this study, firstly, a new identity by using Atangana-Baleanu fractional integral operators is proved. Then, new fractional integral inequalities have been obtained for convex and concave functions with the help of this identity and some certain integral inequalities.

1. Introduction

Mathematics is a tool that serves pure and applied sciences with its deep-rooted history as old as human history and sheds light on how to express and then solve problems. Mathematics uses various concepts and their relations with each other while performing this task. By defining spaces and algebraic structures built on spaces, mathematics creates structures that contribute to human life and nature. The concept of function is one of the basic structures of mathematics, and many researchers have focused on new function classes and made efforts to classify the space of functions. One of the types of functions defined as a product of this intense effort is the convex function, which has applications in statistics, inequality theory, convex programming, and numerical analysis. This interesting class of functions is defined as follows.

Definition 1. The mapping $f : [\theta_1, \theta_2] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{1}$$

is valid for all $x, y \in [\theta_1, \theta_2]$ and $\lambda \in [0, 1]$.

Many inequalities have been obtained by using this unique function type and varieties in inequality theory, which is one of the most used areas of convex functions. We will continue by introducing the Hermite-Hadamard inequality that generate limits on the mean value of a convex function and the famous Bullen inequality as follows.

Assume that $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$, where $\theta_1 < b$. The following statement:

$$f\left(\frac{\theta_1+\theta_2}{2}\right) \le \frac{1}{\kappa_2-\theta_1}\int_{\theta_1}^{\theta_2} f(x)dx \le \frac{f(\theta_1)+f(\theta_2)}{2}, \quad (2)$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

Bullen's integral inequality can be presented as

$$\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} f(x)dx \le \frac{1}{2}\left[f\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{f(\theta_1)+f(\theta_2)}{2}\right], \quad (3)$$

where $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a convex mapping on the interval I of $\mathbb{R}$ where $\kappa_1, \theta_2 \in I$ with $\theta_1 < \theta_2$.

To provide detail information on convexity, let us consider some earlier studies that have been performed by many researchers. In [1], Jensen introduced the concept of convex function to the literature for the first time and drew attention to the fact that it seems to be the basis of the concept of incremental function. In [2], Beckenbach has mentioned about the concept of convexity and emphasized several features of this useful function class. In [3], the authors have focused the relations between convexity and Hermite-Hadamard's inequality. This study has led many researchers to the link between convexity and integral inequalities, which has guided studies in this field. Based on these studies, many papers have been produced for different kinds of convex functions. In [4], Akdemir et al. have proved several new integral inequalities for geometric-arithmetic convex functions via a new integral identity. Several new Hadamard's type integral inequalities have been established with applications to special means by Kavurmaci et al. in [5]. Therefore, a similar argument has been carried out by Zhang et al. but now for $s-$geometrically convex functions in [6]. On all of these, Xi et al. have extended the challenge to $m-$ and $(\alpha, m)-$convex functions by providing Hadamard type inequalities in [7].

Although fractional analysis has been known since ancient times, it has recently become a more popular subject in mathematical analysis and applied mathematics. The adventure that started with the question of whether the solution will exist if the order is fractional in a differential equation has developed with many derivative and integral operators. By defining the derivative and integral operators in fractional order, the researchers who aimed to propose more effective solutions to the solution of physical phenomena have turned to new operators with general and strong kernels over time. This orientation has provided mathematics and applied sciences several operators with kernel structures that differ in terms of locality and singularity, as well as generalized operators with memory effect properties. The struggle that started with the question of how the order in the differential equation being a fraction would have consequences has now evolved into the problem of how to explain physical phenomena and find the most effective fractional operators that will provide effective solutions to real-world problems. Let us introduce some fractional derivative and integral operators that have broken ground in fractional analysis and have proven their effectiveness in different fields by using by many researchers.

We will remember the Caputo-Fabrizio derivative operators. Also, we would like to note that the functions belong to Hilbert spaces denoted by $H^1(0,\theta_2)$.

Definition 2. [8]. Let $f \in H^1(0,\theta_2)$, $\theta_2 > \theta_1$, $\alpha \in [0,1]$, then the definition of the new Caputo fractional derivative is

$${}^{CF}D^{\alpha}f(\tau_1) = \frac{M(\alpha)}{1-\alpha}\int_{\kappa_1}^{\tau_1} f'(s)\exp\left[-\frac{\alpha}{(1-\alpha)}(\tau_1-s)\right]ds, \quad (4)$$

where $M(\alpha)$ is normalization function.

Depending on this interesting fractional derivative operator, the authors have defined the Caputo-Fabrizio fractional integral operator as follows.

Definition 3. [9] Let $f \in H^1(0,\theta_2)$, $\theta_2 > \theta_1$, $\alpha \in [0,1]$, then the definition of the left and right side of Caputo-Fabrizio fractional integral is

$$\left({}^{CF}_{\theta_1}I^{\alpha}\right)(\tau_1) = \frac{1-\alpha}{B(\alpha)}f(\tau_1) + \frac{\alpha}{B(\alpha)}\int_{\theta_1}^{\tau_1} f(y)dy, \quad (5)$$

and

$$\left({}^{CF}I^{\alpha}_{\theta_2}\right)(\tau_1) = \frac{1-\alpha}{B(\alpha)}f(\tau_1) + \frac{\alpha}{B(\alpha)}\int_{\tau_1}^{\kappa_2} f(y)dy, \quad (6)$$

where $B(\alpha)$ is the normalization function.

The Caputo-Fabrizio fractional derivative, which is used in dynamical systems, physical phenomena, disease models, and many other fields, is a highly functional operator by definition, but has a deficiency in terms of not meeting the initial conditions in the special case $\alpha = 1$. The improvement to eliminate this deficiency has been provided by the new derivative operator developed by Atangana-Baleanu, which has versions in the sense of Caputo and Riemann. In the sequel of this paper, we will denote the normalization function with $B(\alpha)$ with the same properties with the $M(\alpha)$ which is defined in Caputo-Fabrizio definition.

Definition 4. [10] Let $f \in H^1(\theta_1,\theta_2)$, $\theta_2 > \kappa_1$, $\alpha \in [0,1]$, then the definition of the new fractional derivative is given:

$${}^{ABC}_{\theta_1}D^{\alpha}_{\tau_1}[f(\tau_1)] = \frac{B(\alpha)}{1-\alpha}\int_{a}^{\tau_1} f'(x)E_{\alpha}\left[-\alpha\frac{(\tau_1-x)^{\alpha}}{(1-\alpha)}\right]dx. \quad (7)$$

Definition 5. [10] Let $f \in H^1(\theta_1,\theta_2)$, $\theta_2 > \kappa_1$, $\alpha \in [0,1]$, then the definition of the new fractional derivative is given:

$${}^{ABR}_{\theta_1}D^{\alpha}_{\tau_1}[f(\tau_1)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\tau_1}\int_{\theta_1}^{\tau_1} f(x)E_{\alpha}\left[-\alpha\frac{(\tau_1-x)^{\alpha}}{(1-\alpha)}\right]dx. \quad (8)$$

Equations (7) and (8) have a nonlocal kernel. Also, in Equation (8), when the function is constant, we get zero.

The associated fractional integral operator has been defined by Atangana-Baleanu as follows.

Definition 6. [10] The fractional integral associate to the new fractional derivative with nonlocal kernel of a function $f \in H^1(\kappa_1, \theta_2)$ as defined:

$$ {}_{\theta_1}^{AB}I^{\alpha}\{f(\tau_1)\} = \frac{1-\alpha}{B(\alpha)} f(\tau_1) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\theta_1}^{\tau_1} f(y)(\tau_1 - y)^{\alpha-1} dy, \tag{9} $$

where $\theta_2 > \theta_1$ and $\alpha \in [0, 1]$.

In [11], Abdeljawad and Baleanu introduced the right hand side of integral operator as follows: the right fractional new integral with ML kernel of order $\alpha \in [0, 1]$ is defined by

$$ \left({}^{AB}I_{\theta_2}^{\alpha}\right)\{f(\tau_1)\} = \frac{1-\alpha}{B(\alpha)} f(\tau_1) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \cdot \int_{\tau_1}^{\theta_2} f(y)(y - \tau_1)^{\alpha-1} dy. \tag{10} $$

In [9], Abdeljawad and Baleanu has presented some new results based on fractional-order derivatives and their discrete versions. Conformable integral operators have been defined by Abdeljawad in [12]. This useful operator has been used to prove some new integral inequalities in [13]. Another important fractional operator—Riemann-Liouville fractional integral operators—have been used to provide some new Simpson type integral inequalities in [14]. Ekinci and Ozdemir have proved several generalizations by using Riemann-Liouville fractional integral operators in [15], and the authors have established some similar results with this operator in [16]. In [17], Akdemir et al. have presented some new variants of celebrated Chebyshev inequality via generalized fractional integral operators. The argument has been proceed with the study of Rashid et al. (see [18]) that involves new investigations related to generalized k-fractional integral operators. In [19], Rashid et al. have presented some motivated findings that extend the argument to the Hilbert spaces. For more information related to different kinds of fractional operators, we recommend to consider [20]. The applications of fractional operators have been demonstrated by several researchers; we suggest to see the papers [21–23].

The main motivation of this paper is to prove an integral identity that includes the Atangana-Baleanu integral operator and to provide some new Bullen type integral inequalities for differentiable convex and concave functions with the help of this integral identity. Some special cases are also considered.

2. Main Results

We will start with a new integral identity that will be used as proofs of our main findings.

Lemma 7. *Let $f : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ be differentiable function on (θ_1, θ_2) with $\kappa_1 < \theta_2$. Then, we have the following identity for Atangana-Baleanu fractional integral operators:*

$$ \begin{aligned} &\frac{2(\theta_2-\theta_1)^{\alpha} + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}} \left[f(\kappa_1) + f(\theta_2) + 2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] \\ &\quad - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}} \left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2} f(\theta_1) + {}_{\theta_1}^{AB}I^{\alpha} f\left(\frac{\kappa_1+\theta_2}{2}\right) \right. \\ &\quad \left. + {}_{((\theta_1+\theta_2)/2)}^{AB}I^{\alpha} f(\theta_2) + {}^{AB}I^{\alpha}_{\theta_2} f\left(\frac{\theta_1+\theta_2}{2}\right)\right] \\ &= \int_0^1 ((1-\tau_1)^{\alpha} - \tau_1^{\alpha}) f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right) d\tau_1 \\ &\quad + \int_0^1 (\tau_1^{\alpha} - (1-\tau_1)^{\alpha}) f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\kappa_1\right) d\tau_1, \end{aligned} \tag{11} $$

where $\alpha, \tau_1 \in [0, 1]$, $\Gamma(.)$ is the gamma function, and $B(\alpha)$ is the normalization function.

Proof. By adding I_1 and I_2, we have

$$ \begin{aligned} I_1 + I_2 &= \int_0^1 ((1-\tau_1)^{\alpha} - \tau_1^{\alpha}) f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\kappa_2\right) d\tau_1 \\ &\quad + \int_0^1 (\tau_1^{\alpha} - (1-\tau_1)^{\alpha}) f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right) d\tau_1. \end{aligned} \tag{12} $$

By using integration, we have

$$ \begin{aligned} I_1 &= \int_0^1 ((1-\tau_1)^{\alpha} - \tau_1^{\alpha}) f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\kappa_2\right) d\tau_1 \\ &= \left.\frac{((1-\tau_1)^{\alpha} - \tau_1^{\alpha}) f(((1+\tau_1)/2)\theta_1 + ((1-\tau_1)/2)\theta_2) d\tau_1}{(\theta_1-\theta_2)/2}\right|_1^0 \\ &\quad - \frac{2\alpha}{\kappa_2-\theta_1}\int_0^1 ((1-\tau_1)^{\alpha-1} + \tau_1^{\alpha-1}) f\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\kappa_2\right) d\tau_1 \\ &= -\frac{2}{\theta_1-\theta_2} f(\theta_1) - \frac{2}{\kappa_1-\theta_2} f\left(\frac{\theta_1+\theta_2}{2}\right) \\ &\quad - \frac{2\alpha}{\kappa_2-\theta_1}\int_0^1 (1-\tau_1)^{\alpha-1} f\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right) d\tau_1 \\ &\quad - \frac{2\alpha}{\theta_2-\theta_1}\int_0^1 \tau_1^{\alpha-1} f\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right) d\tau_1 \\ &= \frac{2}{\theta_2-\theta_1}\left(f(\theta_1) + f\left(\frac{\kappa_1+\theta_2}{2}\right)\right) \\ &\quad - \frac{2^{\alpha+1}\alpha}{(\kappa_2-\theta_1)^{\alpha+1}} \int_{\theta_1}^{(\theta_1+\kappa_2)/2} (x-\theta_1)^{\alpha-1} f(x) dx \\ &\quad - \frac{2^{\alpha+1}\alpha}{(\theta_2-\theta_1)^{\alpha+1}} \int_{\theta_1}^{(\theta_1+\theta_2)/2} \left(\frac{\kappa_1+\theta_2}{2} - x\right)^{\alpha-1} f(x) dx. \end{aligned} \tag{13} $$

Multiplying both sides of (13) identity by $(\kappa_2-\theta_1)^{\alpha+1}/(2^{\alpha+1}B(\alpha)\Gamma(\alpha))$, we have

$$\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}I_1 = \frac{(\theta_2-\theta_1)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(f(\theta_1)+f\left(\frac{\theta_1+\kappa_2}{2}\right)\right) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\kappa_1}^{(\theta_1+\theta_2)/2}(x-\theta_1)^{\alpha-1}f(x)dx - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\theta_1}^{(\theta_1+\theta_2)/2}\left(\frac{\theta_1+\theta_2}{2}-x\right)^{\alpha-1} \cdot f(x)dx. \tag{14}$$

Similarly, by using integration, we get

$$\begin{aligned} I_2 &= \int_0^1 (\tau_1^\alpha-(1-\tau_1)^\alpha)f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\kappa_1\right)d\tau_1 \\ &= \frac{(\tau_1^\alpha-(1-\tau_1)^\alpha)f(((1+\tau_1)/2)\theta_2+((1-\tau_1)/2)\theta_1)d\tau_1}{(\theta_2-\theta_1)/2}\Bigg|_1^0 \\ &\quad - \frac{2\alpha}{\kappa_2-\theta_1}\int_0^1\left(\tau_1^{\alpha-1}+(1-\tau_1)^{\alpha-1}\right)f\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\kappa_1\right)d\tau_1 \\ &= \frac{2}{\theta_2-\theta_1}\left(f(\theta_2)+f\left(\frac{\kappa_1+\theta_2}{2}\right)\right) \\ &\quad - \frac{2^{\alpha+1}\alpha}{(\kappa_2-\theta_1)^{\alpha+1}}\int_{(\theta_1+\theta_2)/2}^{\theta_2}\left(x-\frac{\theta_1+\theta_2}{2}\right)^{\alpha-1}f(x)dx \\ &\quad - \frac{2^{\alpha+1}\alpha}{(\theta_2-\theta_1)^{\alpha+1}}\int_{(\theta_1+\theta_2)/2}^{\theta_2}(\theta_2-x)^{\alpha-1}f(x)dx. \end{aligned} \tag{15}$$

Multiplying both sides of (13) identity by $(\theta_2-\kappa_1)^{\alpha+1}/(2^{\alpha+1}B(\alpha)\Gamma(\alpha))$, we get

$$\begin{aligned} &\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}I_2 \\ &\quad = \frac{(\theta_2-\theta_1)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(f(\theta_2)+f\left(\frac{\theta_1+\kappa_2}{2}\right)\right) \\ &\qquad - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\theta_1+\theta_2)/2}^{\theta_2}\left(x-\frac{\theta_1+\kappa_2}{2}\right)^{\alpha-1}f(x)dx \\ &\qquad - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\theta_1+\kappa_2)/2}^{\theta_2}(\theta_2-x)^{\alpha-1}f(x)dx. \end{aligned} \tag{16}$$

By adding identities (14) and (16), we obtain

$$\begin{aligned} &\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}[I_1+I_2] \\ &\quad = \frac{(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^\alpha\Gamma(\alpha)}{2^\alpha B(\alpha)\Gamma(\alpha)}\left[f(\theta_1)+f\left(\frac{\theta_1+\theta_2}{2}\right)\right] \\ &\qquad - \frac{1-\alpha}{B(\alpha)}f(\theta_1) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\theta_1}^{(\theta_1+\theta_2)/2}(x-\theta_1)^{\alpha-1}f(x)dx \\ &\qquad - \frac{1-\alpha}{B(\alpha)}f\left(\frac{\theta_1+\theta_2}{2}\right) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\theta_1}^{(\theta_1+\theta_2)/2} \\ &\qquad \cdot \left(\frac{\theta_1+\theta_2}{2}-x\right)^{\alpha-1}f(x)dx + \frac{(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^\alpha\Gamma(\alpha)}{2^\alpha B(\alpha)\Gamma(\alpha)} \\ &\qquad \cdot \left[f(\theta_2)+f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{1-\alpha}{B(\alpha)}f(\theta_2) \\ &\qquad - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\kappa_1+\theta_2)/2}^{\theta_2}(\theta_2-x)^{\alpha-1}f(x)dx \\ &\qquad - \frac{1-\alpha}{B(\alpha)}f\left(\frac{\theta_1+\theta_2}{2}\right) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\kappa_1+\theta_2)/2}^{\theta_2} \\ &\qquad \cdot \left(x-\frac{\theta_1+\theta_2}{2}\right)^{\alpha-1}f(x)dx. \end{aligned} \tag{17}$$

Using the definition of Atangana-Baleanu fractional integral operators, we get

$$\begin{aligned} &\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[\int_0^1((1-\tau_1)^\alpha-\tau_1^\alpha)f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)d\tau_1 \right.\\ &\qquad \left. + \int_0^1(\tau_1^\alpha-(1-\tau_1)^\alpha)f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)d\tau_1\right] \\ &\quad = \frac{(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^\alpha\Gamma(\alpha)}{2^\alpha B(\alpha)\Gamma(\alpha)}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] \\ &\qquad - {}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\kappa_1) + {}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\theta_1+\theta_2}{2}\right) \\ &\qquad + {}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2) + {}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\Bigg]. \end{aligned} \tag{18}$$

□□ □

Theorem 8. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} &\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right. \\ &\quad - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right. \\ &\quad \left.\left. + {}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2) + {}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right| \\ &\leq \frac{2\left[|f'(\theta_1)|+|f'(\theta_2)|\right]}{\alpha+1}, \end{aligned} \tag{19}$$

where $\alpha\in[0,1]$ and $B(\alpha)$ is the normalization function.

Proof. By using Lemma 7, we can write

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1$$
$$+\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1. \tag{20}$$

By using convexity of $|f'|$, we get

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta 1+\theta 2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|+\frac{1-\tau_1}{2}|f'(\theta_2)|\right]d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|+\frac{1-\tau_1}{2}|f'(\theta_2)|\right]d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|+\frac{1-\tau_1}{2}|f'(\kappa_1)|\right]d\tau_1$$
$$+\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|+\frac{1-\tau_1}{2}|f'(\theta_1)|\right]d\tau_1. \tag{21}$$

By computing the above integral, we obtain

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta 1+\theta 2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\frac{2\left[|f'(\theta_1)|+|f'(\theta_2)|\right]}{\alpha+1}, \tag{22}$$

and the proof is completed.□□ □

Corollary 9. *In Theorem 8, if we choose $\alpha=1$, we obtain*

$$\left|\frac{f(\theta_1)+f(\theta_2)+2f((\theta_1+\kappa_2)/2)}{\theta_2-\theta_1}-\frac{4}{(\theta_2-\kappa_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right|$$
$$\le\frac{|f'(\theta_1)|+|f'(\kappa_2)|}{2}. \tag{23}$$

Theorem 10. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta 1+\theta 2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\frac{2}{(\alpha p+1)^{1/p}}\left[\left(\frac{3|f'(\theta_1)|^q+|f'(\kappa_2)|^q}{4}\right)^{1/q}\right.$$
$$\left.+\left(\frac{3|f'(\theta_2)|^q+|f'(\kappa_1)|^q}{4}\right)^{1/q}\right], \tag{24}$$

where $p^{-1}+q^{-1}=1$, $\alpha\in[0,1]$, $q>1$, and $B(\alpha)$ is the normalization function.

Proof. By using the identity that is given in Lemma 7, we have

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta 1+\theta 2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1$$
$$+\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1. \tag{25}$$

By applying Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&\quad-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\quad\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\leq\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\kappa_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}\\
&\quad+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}\\
&\quad+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}\\
&\quad+\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}.
\end{aligned}\tag{26}$$

By using convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&\quad-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\quad\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\leq\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_1)|^q\right.\right.\\
&\quad\left.\left.+\frac{1-\tau_1}{2}|f'(\kappa_2)|^q\right]dv\right)^{1/q}+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\\
&\quad\cdot\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_1)|^q+\frac{1-\tau_1}{2}|f'(\kappa_2)|^q\right]d\tau_1\right)^{1/q}\\
&\quad+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_2)|^q\right.\right.\\
&\quad\left.\left.+\frac{1-\tau_1}{2}|f'(\kappa_1)|^q\right]d\tau_1\right)^{1/q}+\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\\
&\quad\cdot\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_2)|^q+\frac{1-\tau_1}{2}|f'(\kappa_1)|^q\right]d\tau_1\right)^{1/q}.
\end{aligned}\tag{27}$$

By calculating the integrals that is in the above inequalities, we get desired result.□□ □

Corollary 11. *In Theorem 10, if we choose $\alpha=1$, we obtain*

$$\begin{aligned}
&\left|\frac{f(\theta_1)+f(\theta_2)+2f((\kappa_1+\theta_2)/2)}{\theta_2-\theta_1}-\frac{4}{(\kappa_2-\theta_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right|\\
&\leq\frac{1}{(p+1)^{1/p}}\left[\left(\frac{3|f'(\theta_1)|^q+|f'(\kappa_2)|^q}{4}\right)^{1/q}\right.\\
&\quad\left.+\left(\frac{3|f'(\theta_2)|^q+|f'(\theta_1)|^q}{4}\right)^{1/q}\right].
\end{aligned}\tag{28}$$

Theorem 12. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&\quad-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\quad\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\leq\left(\frac{1}{\alpha+1}\right)^{1-(1/q)}\left[\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\kappa_1)|^q\right.\right.\\
&\quad\left.+\frac{1}{2(\alpha+2)}|f'(\kappa_2)|^q\right)^{1/q}+\left(\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\theta_1)|^q\right.\\
&\quad\left.+\frac{1}{2(\alpha+1)(\alpha+2)}|f'(\theta_2)|^q\right)^{1/q}\\
&\quad+\left(\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\theta_2)|^q+\frac{1}{2(\alpha+1)(\alpha+2)}|f'(\theta_1)|^q\right)^{1/q}\\
&\quad\left.+\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\theta_2)|^q+\frac{1}{2(\alpha+2)}|f'(\theta_1)|^q\right)^{1/q}\right],
\end{aligned}\tag{29}$$

where $\alpha\in[0,1]$, $q\geq1$, and $B(\alpha)$ is the normalization function.

Proof. By Lemma 7, we get

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&\quad-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\quad\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\leq\int_0^1(1-\tau_1)^\alpha\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1\\
&\quad+\int_0^1 t^\alpha\left|f'\left(\frac{1+\tau_1}{2}\kappa_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1\\
&\quad+\int_0^1\tau_1^\alpha\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1\\
&\quad+\int_0^1(1-\tau_1)^\alpha\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1.
\end{aligned}\tag{30}$$

By applying power mean inequality, we get

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\le\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1\right.\right.\right.\\
&\left.\left.\left.+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\\
&\cdot\left(\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}\\
&+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}\\
&+\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2\right.\right.\right.\\
&\left.\left.\left.+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}.
\end{aligned}\tag{31}$$

By using convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\le\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|^q\right.\right.\\
&\left.\left.+\frac{1-\tau_1}{2}|f'(\theta_2)|^q\right]d\tau_1\right)^{1/q}+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\\
&\cdot\left(\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|^q+\frac{1-\tau_1}{2}|f'(\theta_2)|^q\right]d\tau_1\right)^{1/q}\\
&+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|^q\right.\right.\\
&\left.\left.+\frac{1-\tau_1}{2}|f'(\theta_1)|^q\right]d\tau_1\right)^{1/q}+\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\\
&\cdot\left(\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|^q+\frac{1-\tau_1}{2}|f'(\theta_1)|^q\right]d\tau_1\right)^{1/q}.
\end{aligned}\tag{32}$$

By computing the above integrals, the proof is completed.□□ □

Corollary 13. *In Theorem 12, if we choose $\alpha=1$, we obtain*

$$\begin{aligned}
&\left|\frac{2[f(\theta_1)+f(\theta_2)+2f((\kappa_1+\theta_2)/2)]}{\theta_2-\theta_1}-\frac{8}{(\theta_2-\theta_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right|\\
&\le\left(\frac{1}{2}\right)^{1-(1/q)}\left[\left(\frac{2|f'(\theta_1)|^q+|f'(\theta_2)|^q}{6}\right)^{1/q}\right.\\
&+\left(\frac{5|f'(\theta_1)|^q+|f'(\theta_2)|^q}{12}\right)^{1/q}\\
&+\left(\frac{5|f'(\theta_2)|^q+|f'(\theta_1)|^q}{12}\right)^{1/q}\\
&\left.+\left(\frac{2|f'(\theta_2)|^q+|f'(\theta_1)|^q}{6}\right)^{1/q}\right].
\end{aligned}\tag{33}$$

Theorem 14. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\le\frac{4}{p(\alpha p+1)}+\frac{2\left[|f'(\kappa_1)|^q+|f'(\theta_2)|^q\right]}{q},
\end{aligned}\tag{34}$$

where $p^{-1}+q^{-1}=1$, $\alpha\in[0,1]$, $q>1$, and $B(\alpha)$ is the normalization function.

Proof. By using identity that is given in Lemma 7, we get

$$\begin{aligned}
&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\
&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\
&\le\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1
\end{aligned}$$

$$+\int_0^1 \tau_1^\alpha \left| f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)\right| d\tau_1$$
$$+\int_0^1 \tau_1^\alpha \left| f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)\right| d\tau_1$$
$$+\int_0^1 (1-\tau_1)^\alpha \left| f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)\right| d\tau_1. \quad (35)$$

By using the Young inequality as $xy \leq (1/p)x^p + (1/q)y^q$

$$\left|\frac{2(\theta_2-\theta_1)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^\alpha_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^\alpha f\left(\frac{\kappa_1+\theta_2}{2}\right) + {}^{AB}_{::(\theta1+\theta2)/2}I^\alpha f(\theta_2) + {}^{AB}I^\alpha_{\theta_2} f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq \frac{1}{p}\int_0^1 (1-\tau_1)^{\alpha p} d\tau_1 + \frac{1}{q}\int_0^1 \left| f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1$$
$$+\frac{1}{p}\int_0^1 \tau_1^{\alpha p} d\tau_1 + \frac{1}{q}\int_0^1 \left| f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1$$
$$+\frac{1}{p}\int_0^1 \tau_1^{\alpha p} d\tau_1 + \frac{1}{q}\int_0^1 \left| f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1$$
$$+\frac{1}{p}\int_0^1 (1-\tau_1)^{\alpha p} d\tau_1 + \frac{1}{q}\int_0^1 \left| f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1. \quad (36)$$

By using convexity of $|f'|^q$ and by a simple computation, we have the desired result.□□ □

Corollary 15. *In Theorem 14, if we choose $\alpha = 1$, we obtain*

$$\left|\frac{f(\theta_1)+f(\theta_2)+2f((\theta_1+\kappa_2)/2)}{\theta_2-\theta_1} - \frac{4}{(\theta_2-\kappa_1)^2}\int_{\theta_1}^{\theta_2} f(x)dx\right| \leq \frac{2}{p^2+p} + \frac{|f'(\theta_1)|^q + |f'(\theta_2)|^q}{q}. \quad (37)$$

Theorem 16. *Let $f : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ be differentiable function on (θ_1, θ_2) with $\kappa_1 < \theta_2$ and $f' \in L_1[\theta_1, \theta_2]$. If $|f'|$ is a concave for $q > 1$, then we have*

$$\left|\frac{2(\theta_2-\theta_1)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^\alpha_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^\alpha f\left(\frac{\kappa_1+\theta_2}{2}\right) + {}^{AB}_{::(\theta1+\theta2)/2}I^\alpha f(\theta_2) + {}^{AB}I^\alpha_{\theta_2} f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq \left(\frac{1}{\alpha+1}\right)\left[\left| f'\left(\frac{\theta_1(\alpha+3)+\theta_2(\alpha+1)}{2(\alpha+2)}\right)\right| + \left| f'\left(\frac{\theta_1(2\alpha+3)+\theta_2}{2(\alpha+2)}\right)\right| + \left| f'\left(\frac{\theta_2(2\alpha+3)+\theta_1}{2(\alpha+2)}\right)\right| + \left| f'\left(\frac{\kappa_2(\alpha+3)+\theta_1(\alpha+1)}{2(\alpha+2)}\right)\right|\right], \quad (38)$$

where $\alpha \in [0, 1]$ and $B(\alpha)$ is the normalization function.

Proof. From Lemma 7 and the Jensen integral inequality, we have

$$\left|\frac{2(\theta_2-\theta_1)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^\alpha_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^\alpha f\left(\frac{\kappa_1+\theta_2}{2}\right) + {}^{AB}_{::(\theta1+\theta2)/2}I^\alpha f(\theta_2) + {}^{AB}I^\alpha_{\theta_2} f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq \left(\int_0^1 (1-\tau_1)^\alpha d\tau_1\right) \cdot \left| f'\left(\frac{\int_0^1 (1-\tau_1)^\alpha(((1+\tau_1)/2)\theta_1 + ((1-\tau_1)/2)\theta_2) d\tau_1}{\int_0^1 (1-\tau_1)^\alpha d\tau_1}\right)\right|$$
$$+\left(\int_0^1 \tau_1^\alpha d\tau_1\right)\left| f'\left(\frac{\int_0^1 \tau_1^\alpha(((1+\tau_1)/2)\theta_1 + ((1-\tau_1)/2)\theta_2) d\tau_1}{\int_0^1 \tau_1^\alpha d\tau_1}\right)\right|$$
$$+\left(\int_0^1 \tau_1^\alpha d\tau_1\right)\left| f'\left(\frac{\int_0^1 \tau_1^\alpha(((1+\tau_1)/2)\theta_2 + ((1-\tau_1)/2)\theta_1) d\tau_1}{\int_0^1 \tau_1^\alpha d\tau_1}\right)\right|$$
$$+\left(\int_0^1 (1-\tau_1)^\alpha d\tau_1\right) \cdot \left| f'\left(\frac{\int_0^1 (1-\tau_1)^\alpha(((1+\tau_1)/2)\theta_2 + ((1-\tau_1)/2)\theta_1) d\tau_1}{\int_0^1 (1-\tau_1)^\alpha d\tau_1}\right)\right|. \quad (39)$$

By computing the above integrals, we have

$$\left|\frac{2(\theta_2-\theta_1)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^\alpha_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^\alpha f\left(\frac{\kappa_1+\theta_2}{2}\right) + {}^{AB}_{::(\theta1+\theta2)/2}I^\alpha f(\theta_2) + {}^{AB}I^\alpha_{\theta_2} f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq \left(\frac{1}{\alpha+1}\right)\left[\left| f'\left(\frac{\theta_1(\alpha+3)+\theta_2(\alpha+1)}{2(\alpha+2)}\right)\right| + \left| f'\left(\frac{\theta_1(2\alpha+3)+\theta_2}{2(\alpha+2)}\right)\right| + \left| f'\left(\frac{\theta_2(2\alpha+3)+\theta_1}{2(\alpha+2)}\right)\right| + \left| f'\left(\frac{\kappa_2(\alpha+3)+\theta_1(\alpha+1)}{2(\alpha+2)}\right)\right|\right]. \quad (40)$$

So, the proof is completed.□□ □

Corollary 17. *In Theorem 16, if we choose $\alpha = 1$, we obtain*

$$\left|\frac{f(\theta_1) + f(\theta_2) + 2f((\kappa_1 + \theta_2)/2)}{\theta_2 - \theta_1} - \frac{4}{(\kappa_2 - \theta_1)^2}\int_{\theta_1}^{\theta_2} f(x)dx\right| \le \left(\frac{1}{4}\right)\left[\left|f'\left(\frac{2\theta_1 + \theta_2}{3}\right)\right| + \left|f'\left(\frac{5\theta_1 + \theta_2}{6}\right)\right| + \left|f'\left(\frac{5\theta_2 + \theta_1}{6}\right)\right| + \left|f'\left(\frac{2\theta_2 + \theta_1}{3}\right)\right|\right]. \tag{41}$$

Theorem 18. *Let $f : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ be differentiable function on (θ_1, θ_2) with $\kappa_1 < \theta_2$ and $f' \in L_1[\theta_1, \theta_2]$. If $|f'|^q$ is a concave function, we have*

$$\begin{aligned}&\left|\frac{2(\theta_2 - \theta_1)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2 - \theta_1)^{\alpha+1}}\left[f(\kappa_1) + f(\theta_2) + 2f\left(\frac{\theta_1 + \theta_2}{2}\right)\right]\right. \\ &\quad - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2 - \kappa_1)^{\alpha+1}}\left[{}^{AB}I^\alpha_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^\alpha f\left(\frac{\kappa_1 + \theta_2}{2}\right)\right. \\ &\quad \left.\left. + {}^{AB}_{::(\theta1+\theta2)/2}I^\alpha f(\theta_2) + {}^{AB}I^\alpha_{\theta_2}f\left(\frac{\theta_1 + \theta_2}{2}\right)\right]\right| \\ &\le \frac{2}{(\alpha p + 1)^{1/p}}\left[\left|f'\left(\frac{3\theta_1 + \theta_2}{4}\right)\right| + \left|f'\left(\frac{3\theta_2 + \theta_1}{4}\right)\right|\right],\end{aligned} \tag{42}$$

where $p^{-1} + q^{-1} = 1$, $\alpha \in 0, 1]$, and $q > 1$.

Proof. By using the Lemma 7 and Hölder integral inequality, we can write

$$\begin{aligned}&\left|\frac{2(\theta_2 - \theta_1)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2 - \theta_1)^{\alpha+1}}\left[f(\kappa_1) + f(\theta_2) + 2f\left(\frac{\theta_1 + \theta_2}{2}\right)\right]\right. \\ &\quad - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2 - \kappa_1)^{\alpha+1}}\left[{}^{AB}I^\alpha_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^\alpha f\left(\frac{\kappa_1 + \theta_2}{2}\right)\right. \\ &\quad \left.\left. + {}^{AB}_{::(\theta1+\theta2)/2}I^\alpha f(\theta_2) + {}^{AB}I^\alpha_{\theta_2}f\left(\frac{\theta_1 + \theta_2}{2}\right)\right]\right| \\ &\le \left(\int_0^1 (1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\kappa_1 + \frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q} \\ &\quad + \left(\int_0^1 \tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q} \\ &\quad + \left(\int_0^1 \tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q} \\ &\quad + \left(\int_0^1 (1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}.\end{aligned} \tag{43}$$

By using concavity of $|f'|^q$ and Jensen integral inequality, we get

$$\begin{aligned}&\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1 \\ &= \int_0^1 \tau_1^0\left|f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1 \\ &\le \left(\int_0^1 \tau_1^0 d\tau_1\right)\left|f'\left(\frac{\int_0^1\tau_1^0(((1+\tau_1)/2)\theta_1 + ((1-\tau_1)/2)\theta_2)d\tau_1}{\int_0^1\tau_1^0 d\tau_1}\right)\right|^q \\ &= \left|f'\left(\frac{3\theta_1 + \theta_2}{4}\right)\right|^q.\end{aligned} \tag{44}$$

Similarly,

$$\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1 \le \left|f'\left(\frac{3\theta_2 + \theta_1}{4}\right)\right|^q, \tag{45}$$

so we obtain

$$\begin{aligned}&\left|\frac{2(\theta_2 - \theta_1)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2 - \theta_1)^{\alpha+1}}\left[f(\kappa_1) + f(\theta_2) + 2f\left(\frac{\theta_1 + \theta_2}{2}\right)\right]\right. \\ &\quad - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2 - \kappa_1)^{\alpha+1}}\left[{}^{AB}I^\alpha_{\theta_1+\theta_2/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^\alpha f\left(\frac{\kappa_1 + \theta_2}{2}\right)\right. \\ &\quad \left.\left. + {}^{AB}_{::\theta_1+\theta_2/2}I^\alpha f(\theta_2) + {}^{AB}I^\alpha_{\theta_2}f\left(\frac{\theta_1 + \theta_2}{2}\right)\right]\right| \\ &\le \frac{2}{(\alpha p + 1)^{1/p}}\left[\left|f'\left(\frac{3\theta_1 + \theta_2}{4}\right)\right| + \left|f'\left(\frac{3\theta_2 + \theta_1}{4}\right)\right|\right].\end{aligned} \tag{46}$$

□□ □

Corollary 19. *In Theorem 18, if we choose $\alpha = 1$, we obtain*

$$\left|\frac{f(\theta_1) + f(\theta_2) + 2f((\kappa_1 + \theta_2)/2)}{\theta_2 - \theta_1} - \frac{4}{(\kappa_2 - \theta_1)^2}\int_{\theta_1}^{\theta_2} f(x)dx\right| \le \frac{1}{(p+1)^{1/p}}\left[\left|f'\left(\frac{3\theta_1 + \theta_2}{4}\right)\right| + \left|f'\left(\frac{3\theta_2 + \theta_1}{4}\right)\right|\right]. \tag{47}$$

3. Conclusion

In this study, an integral identity including Atangana-Baleanu integral operators has been proved. Some integral inequalities are established by using Hölder inequality, power-mean inequality, Young inequality, and convex functions with the help of Lemma 7 which has the potential to produce Bullen type inequalities. Some special cases of the results in this general form have been pointed out. Researchers can establish new equations such as the integral identity in the study and reach similar inequalities of these equality-based inequalities.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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Research Article

Fixed Point and Endpoint Theories for Two Hybrid Fractional Differential Inclusions with Operators Depending on an Increasing Function

Sh. Rezapour[1,2] M. Q. Iqbal,[3] A. Hussain[3] A. Zada[4] and S. Etemad[2]

[1]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[2]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[3]*Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*
[4]*Department of Mathematics, University of Peshawar, Peshawar, Khyber Pakhtunkhwa, Pakistan*

Correspondence should be addressed to S. Etemad; sina.etemad@azaruniv.ac.ir

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The main concentration of the present research is to explore several theoretical criteria for proving the existence results for the suggested boundary problem. In fact, for the first time, we formulate a new hybrid fractional differential inclusion in the φ-Caputo settings depending on an increasing function φ subject to separated mixed φ-hybrid-integro-derivative boundary conditions. In addition to this, we discuss a special case of the proposed φ-inclusion problem in the non-φ-hybrid structure with the help of the endpoint notion. To confirm the consistency of our findings, two specific numerical examples are provided which simulate both φ-hybrid and non-φ-hybrid cases.

1. Introduction

Arbitrary order calculus theory is considered as an important topic of research for all mathematicians, researchers, engineers, and scientists due to the applicability of mentioned theory in several contexts in engineering and applied science and its flexibility to model different systems and phenomena having memory effects (see, e.g., [1–3] and reference therein). Several arbitrary order derivatives have been introduced in the past decade, and the most common of them are Riemann-Liouville, Caputo, and Hadamard derivatives. Hence, arbitrary order boundary value problems (BVPs) can be formulated in the framework of different operators. In the meantime, some recent research investigations have been conducted with the aid of these operators to establish the relevant analytical results for proposed BVPs. For instance, Alzabut et al. [4] investigated the oscillatory behavior of a kind of fractional differential equations (FDEs) supplemented with damping and forcing terms by terms of generalized proportional operators. In [5], Baleanu et al. modeled an applied instrument in engineering in the context of a hybrid Caputo FBVP and studied its existence theory. Also, the same authors [6] established similar results by means of Caputo and Riemann-Liouville conformable derivation and integration operators. In 2019, Matar et al. [7] devoted their focus on solvability of nonlinear systems of FDEs via nonlocal initial value problems by terms of fixed point methods and after that, Mohammadi et al. [8] utilized another fractional operator entitled Caputo–Hadamard for modeling a hybrid FBVP with Hadamard integral boundary conditions. Zhou et al. [9] presented a fractional antiperiodic model of Langevin equation and investigated qualitative aspects of its solutions with the aid of techniques appeared in functional analysis. Similarly, one can find some papers on applications of fractional operators [10–13].

In 2017, a generalization of the Caputo fractional operator known as φ-Caputo derivative (φ-CF) was presented by Almeida [14] in which its kernel is with respect to a given increasing function φ. One of the most important advantages of the φ-CF derivative operator is its ability to produce all previous fractional derivatives, and also, it involves the semigroup property. As a result, φ-CF derivative is known as an extended structure of arbitrary order derivative operators.

To get acquainted with some previous research works done based on φ-CF operators so far, we refer to a paper published by Wahash et al. [15]. In that paper, Wahash et al. designed a generalized φ-fractional differential equation with a simple integral condition as

$$\begin{cases} {}^{\mathscr{C}}\mathfrak{D}_{0^+}^{\sigma^*;\varphi} w(z) = \hbar_*(z, w(z)), \\ w(0) = \nu + d\displaystyle\int_0^1 \xi(q)u(q)dq, \end{cases} \tag{1}$$

where $z \in [0,1]$, $\sigma^* \in (0,1)$, $\nu \in \mathbb{R}_+$, and $d \in \mathbb{R}^{\geq 0}$, and also, $\hbar_* : [0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ stands for a continuous function along with $\xi \in L^1_{\mathbb{R}_+}([0,1])$. The lower-upper solution is a technique implemented in that article by Wahash et al. in which they utilized a fixed point method on cones. Further, lower-upper control maps are provided with respect to the nonlinear term without a certain monotonicity criterion [15]. Similarly, by using the newly introduced φ-CF operator and its generalizations, several articles have been published like as [16] in which Almeida et al. considered a FDE via a Caputo derivative with respect to a kernel function and reviewed some applications of them. Derbazi et al. [17] used such a generalized operator to investigate a nonlinear initial value problem via monotone iterative method. Samet et al. derived some Lyapunov-type inequalities in relation to an antiperiodic FBVP involving φ-Caputo operator [18]. The analysis of the stability to an φ-Hilfer impulsive FDE is another instance of applications of such generalized operators which was studied by Sousa et al. in [19]. In 2020, Tariboon et al. [20] turned to establishment of existence theorems to sequential generalized inclusion FBVP, and then, Thabet et al. [21] achieved to similar findings on a new structure of the pantograph inclusion FBVP. In a higher level, Vivek et al. [22] defined generalized φ-operators in the context of partial operators and analyzed a PDE in the φ-Caputo settings.

With regard to ideas of aforesaid research works, we consider the following φ-hybrid fractional differential inclusion in the sense of Caputo represented as

$${}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right) \in \tilde{\mathfrak{O}}(z,\varpi^*(z)), \tag{2}$$

supplemented with separated mixed φ-hybrid-integro-derivative boundary conditions

$$\begin{cases} \tilde{m}_1\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a} = s_1^* + \tilde{m}_2\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a}, \\ \tilde{m}_1 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi}{}^{\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T} = s_2^* + \tilde{m}_2 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi}{}^{\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T}, \end{cases} \tag{3}$$

where $z \in [a, T]$ with $a \geq 0$, $\sigma^* \in (1,2)$, $\tilde{\mu} \in (0,1)$, $\mu^* > 0$, $\tilde{m}_1, \tilde{m}_2 \in \mathbb{R}^{\neq 0}$, and $s_1^*, s_2^* \in \mathbb{R}^+$. Two notations ${}^{\mathscr{C}}\mathfrak{D}_a^{(\cdot);\varphi}$ and ${}^{\mathscr{RL}}\mathscr{I}_a^{(\cdot);\varphi}$ stand for the φ-CF derivative and the φ-Riemann-Liouville integral (φ-RLF), respectively. Also, notice that ${}^{\mathscr{C}}\mathfrak{D}_a^{1;\varphi} = (1/\varphi'(z))(d/dz)$. Besides, $\mathfrak{N}_* : [a,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be a nonzero continuous single-valued operator, and $\tilde{\mathfrak{O}} : [0,1] \times \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ is assumed to be a set-valued operator equipped with some required properties. Notice that by putting $\mathfrak{N}_*(z,\varpi^*(z)) = 1$, the given φ-hybrid Caputo fractional differential inclusion BVP (2) and (3) is transformed into a non-φ-hybrid separated inclusion BVP presented by

$$\begin{cases} {}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*(z) \in \tilde{\mathfrak{O}}(z,\varpi^*(z)), \quad (z \in [a,T]), \\ \tilde{m}_1\varpi^*(a) = s_1^* + \tilde{m}_2\varpi^*(a), \\ \tilde{m}_1 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi}{}^{\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\varpi^*(T) = s_2^* + \tilde{m}_2 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi}{}^{\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\varpi^*(T). \end{cases} \tag{4}$$

Note that by taking into account the authors' knowledge, there are no research manuscripts on φ-CF operators involving mixed φ-hybrid-integro-derivative boundary conditions simultaneously. In addition, this given structure is formulated in a unique and general form in which we can consider some standard special cases studied before. Here, we derive some analytical criteria to prove the existence results for the proposed novel φ-hybrid fractional differential inclusion in the φ-Caputo settings (2) equipped with separated mixed φ-hybrid-integro-derivative boundary conditions (3). The applied approach to achieve desired purposes is based on Dhage's fixed point result. In addition, we discuss the special case of the proposed φ-inclusion problem in the non-φ-hybrid version with the aid of the endpoint notion. We organize the present manuscript as the following construction. In Section 2, we briefly collect auxiliary preliminaries on the φ-fractional operators and some required notions on the multifunctions and related properties. In Section 3, the existence criteria of solutions for both proposed φ-hybrid and non-φ-hybrid BVPs (2)–(4) are derived by two different analytical methods. To confirm the applicability of our analytical findings, two simulative numerical examples are formulated in Section 4 which cover both φ-hybrid and non-φ-hybrid cases.

2. Auxiliary Preliminaries

By continuing the path ahead, we assemble and recall several auxiliary and fundamental notions in the direction of our theoretical methods implemented in this paper. The concept of RLF integral for $\varpi^* : [0,+\infty) \longrightarrow \mathbb{R}$ of order $\sigma^* > 0$ is defined as

$${}^{\mathscr{RL}}\mathscr{I}_0^{\sigma^*}\varpi^*(z) = \int_0^z \frac{(z-q)^{\sigma^*-1}}{\Gamma(\sigma^*)}\varpi^*(q)dq, \tag{5}$$

provided that the integral has finite value [23, 24]. In this position, let us take $n-1<\sigma^*<n$ in which $n=[\sigma^*]+1$. Regarding a continuous function $\varpi^*:[0,+\infty)\longrightarrow\mathbb{R}$, the RLF derivative of order ρ^* is defined as

$$ {}^{\mathscr{RL}}\mathfrak{D}_0^{\sigma^*}\varpi^*(z)=\left(\frac{d}{dz}\right)^n\int_0^z\frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)}\varpi^*(q)\,dq, \tag{6}$$

provided that the integral has finite value [23, 24]. In the next step, for an absolutely continuous n-times differentiable real-valued function ϖ^* on $[0,+\infty)$, the derivative in the Caputo settings of order σ^* is defined as

$$ {}^{\mathscr{C}}\mathfrak{D}_0^{\sigma^*}\varpi^*(z)=\int_0^z\frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)}\varpi^{*(n)}(q)\,dq, \tag{7}$$

such that it is finite-valued [23, 24]. Now, let $\varphi\in\mathscr{C}^n([a,b])$ be increasing with $\varphi'(z)>0, \forall z\in[a,b]$. Then, an integral in the sense of φ-Riemann-Liouville for an integrable $\varpi^*:[a,b]\longrightarrow\mathbb{R}$ of order σ^* depending on increasing function φ is defined as

$$ {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}\varpi^*(z)=\frac{1}{\Gamma(\sigma^*)}\int_a^z\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\varpi^*(q)dq, \tag{8}$$

provided that the RHS of above equality involves the finite value [25, 26]. It is clear that if we take $\varphi(z)=z$, then $\varphi'(z)=1$, and thus by inserting them into (8), we see that the φ-RLF integral is converted to the standard RLF integral given by (5). For a continuous function $\varpi^*:[0,+\infty)\longrightarrow\mathbb{R}$, a derivative in the sense of φ-RL of order σ^* is given by

$$ {}^{\mathscr{RL}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*(z)=\frac{1}{\Gamma(n-\sigma^*)}\left(\frac{1}{\varphi'(z)}\frac{d}{dz}\right)^n \cdot\int_a^z\varphi'(q)(\varphi(z)-\varphi(q))^{n-\sigma^*-1}\varpi^*(q)\,dq, \tag{9}$$

provided that the RHS of above equality exists [25, 26]. If $\varphi(z)=z$, then the φ-RLF derivative (9) is converted to the standard RLF derivative (6). Motivated by such operators, Almeida gave a φ-version of the CF derivative as follows:

$$ {}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*(z)=\frac{1}{\Gamma(n-\sigma^*)}\int_a^z\varphi'(q)(\varphi(z)-\varphi(q))^{n-\sigma^*-1}\cdot\left(\frac{1}{\varphi'(q)}\frac{d}{dq}\right)^n\varpi^*(q)\,dq, \tag{10}$$

provided that the RHS of above equality exists [14]. If $\varphi(z)=z$, then the φ-CF derivative (10) is converted to the standard CF derivative (7). Some useful properties of the φ-CF and φ-RLF operators can be seen in the following.

Lemma 1 [14, 24]. *Let $\sigma^*,\varrho^*,\beta^*>0$ and $\varphi\in\mathscr{C}^n([a,b])$ be increasing with $\varphi'(z)>0,\forall z\in[a,b]$. Then,*

(i1) $ {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}({}^{\mathscr{RL}}\mathscr{I}_a^{\varrho^*;\varphi}\varpi^*)(z)=({}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*+\varrho^*;\varphi}\varpi^*)(z)$

(i2) $ {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}(\varphi(z)-\varphi(a))^{\beta^*}(y)=(\Gamma(\beta^*+1)/\Gamma(\sigma^*+\beta^*+1))(\varphi(y)-\varphi(a))^{\sigma^*+\beta^*}$

(i3) $ {}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}(\varphi(z)-\varphi(a))^{\beta^*}(y)=(\Gamma(\beta^*+1)/\Gamma(\beta^*-\sigma^*+1))(\varphi(y)-\varphi(a))^{\beta^*-\sigma^*}, (\beta^*>\sigma^*)$

(i4) $ {}^{\mathscr{RL}}\mathfrak{D}_a^{\sigma^*;\varphi}({}^{\mathscr{RL}}\mathscr{I}_a^{\rho^*;\varphi}\varpi^*)(z)=({}^{\mathscr{RL}}\mathscr{I}_a^{\rho^*-\sigma^*;\varphi}\varpi^*)(z), (\sigma^*<\rho^*)$

For instance, we plot the graph of φ-RLF integral and φ-CF derivative of $\varpi(z)=(z-1)^{6.5}$ for $\varphi(z)=2z+3/2$ in Figure 1.

Lemma 2 [14]. *Let $n-1<\sigma^*<n$. Then, for each $\varpi^*\in\mathscr{C}^{n-1}([a,b])$,*

$$ {}^{\mathscr{RL}}\mathscr{I}_a^{(\sigma^*;\varphi)}\left({}^{\mathscr{C}}\mathfrak{D}_a^{(\sigma^*;\varphi)}\varpi^*\right)(z)=\varpi^*(z)-\sum_{j=0}^{n-1}\frac{(\delta_\varphi)^j\varpi^*(a)}{j!}(\varphi(z)-\varphi(a))^j, \cdot\left(\delta_\varphi=\frac{1}{\varphi'(z)}\frac{d}{dz}\right). \tag{11}$$

In accordance with above lemma, the authors proved that the series solution for given homogeneous differential equation $({}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*)(z)=0$ has such a form

$$\varpi^*(z)=\sum_{j=0}^{n-1}\tilde{k}_j^*(\varphi(z)-\varphi(a))^j=\tilde{k}_0^*+\tilde{k}_1^*(\varphi(z)-\varphi(a))+\tilde{k}_2^*(\varphi(z)-\varphi(a))^2+\cdots+\tilde{k}_{n-1}^*(\varphi(z)-\varphi(a))^{n-1}, \tag{12}$$

where $n-1<\sigma^*<n$ and $\tilde{k}_0^*,\tilde{k}_1^*,\cdots,\tilde{k}_{n-1}^*\in\mathbb{R}$ [14].

We consider the normed space by notation $(\mathfrak{W},\|\cdot\|_{\mathfrak{W}})$. Also, we introduce the notations $\mathscr{P}(\mathfrak{W})$, $\mathscr{P}_{\text{bnd}}(\mathfrak{W})$, $\mathscr{P}_{\text{cls}}(\mathfrak{W})$, $\mathscr{P}_{\text{cmp}}(\mathfrak{W})$, and $\mathscr{P}_{\text{cvx}}(\mathfrak{W})$ for the category of all nonempty subsets, all bounded subsets, all closed subsets, all compact subsets, and all convex subsets of $\mathfrak{W}$, respectively. In the subsequent path, a metric function attributed to Pompeiu-Hausdorff $\mathbb{PH}_{d_{\mathfrak{W}}}:\mathscr{P}(\mathfrak{W})\times\mathscr{P}(\mathfrak{W})\longrightarrow\mathbb{R}\cup\{\infty\}$ is defined by

$$\mathbb{PH}_{d_{\mathfrak{W}}}(\mathscr{E}_1,\mathscr{E}_2)=\max\left\{\sup_{e_1\in\mathscr{E}_1}d_{\mathfrak{W}}(e_1,\mathscr{E}_2),\sup_{e_2\in\mathscr{E}_2}d_{\mathfrak{W}}(\mathscr{E}_1,e_2)\right\}, \tag{13}$$

so that $d_{\mathfrak{W}}(\mathscr{E}_1,e_2)=\inf_{e_1\in\mathscr{E}_1}d_{\mathfrak{W}}(e_1,e_2)$ and $d_{\mathfrak{W}}(e_1,\mathscr{E}_2)=\inf_{e_2\in\mathscr{E}_2}d_{\mathfrak{W}}(e_1,e_2)$ [27]. We say that $\tilde{\mathfrak{O}}:\mathfrak{W}\longrightarrow\mathscr{P}_{cls}(\mathfrak{W})$ is Lipschitzian with constant $\hat{c}>0$ if $\mathbb{PH}_{d_{\mathfrak{W}}}(\tilde{\mathfrak{O}}(\varpi_1^*),\tilde{\mathfrak{O}}(\varpi_2^*))\le\hat{c}\,d_{\mathfrak{W}}(\varpi_1^*,\varpi_2^*),\forall\varpi_1^*,\varpi_2^*\in\mathfrak{W}$. Also, $\tilde{\mathfrak{O}}$ is a contraction if $\hat{c}\in[0,1)$ [27].

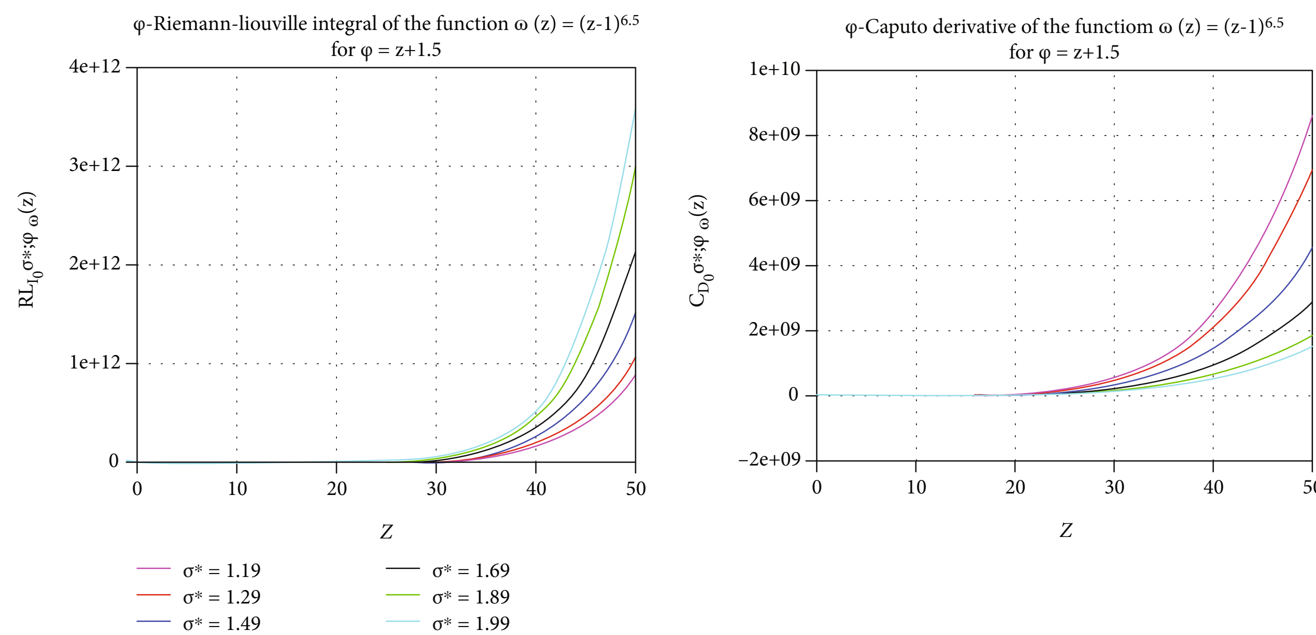

Figure 1: The RLF-integral and CF-derivative of $\varpi(z) = (z-1)^{6.5}$ for $\varphi(z) = 2z + 3/2$.

We represent the collection of all existing selections of $\tilde{\mathfrak{D}}$ at point $\varpi^* \in \mathscr{C}_{\mathbb{R}}([0,1])$ by

$$(\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*} := \left\{\widehat{\kappa} \in \mathscr{L}^1_{\mathbb{R}}([0,1]) \colon \widehat{\kappa}(z) \in \tilde{\mathfrak{D}}(z, \varpi^*(z))\right\}, \quad (14)$$

for almost all $z \in [0,1]$ [27, 28]. We note that $\varpi^* \in \mathfrak{W}$ is an endpoint for given set-valued operator $\tilde{\mathfrak{D}} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ whenever we have $\tilde{\mathfrak{D}}(\varpi^*) = \{\varpi^*\}$ [29]. Also, the mapping $\tilde{\mathfrak{D}}$ possesses an approximate endpoint property (APXEndP-property) whenever

$$\inf_{\varpi_1^* \in \mathfrak{W}} \sup_{\varpi_2^* \in \tilde{\mathfrak{D}}(\varpi_1^*)} d_{\mathfrak{W}}(\varpi_1^*, \varpi_2^*) = 0, \quad (15)$$

[29]. We need next results.

Theorem 3 (Closed graph theorem [30]). *Let $\mathfrak{W}$ be a separable Banach space, $\tilde{\mathfrak{D}} : [0,1] \times \mathfrak{W} \longrightarrow \mathscr{P}_{cmp,cvx}(\mathfrak{W})$ be $\mathscr{L}^1$-Carathéodory and $\coprod : \mathscr{L}^1_{\mathfrak{W}}([0,1]) \longrightarrow \mathscr{C}_{\mathfrak{W}}([0,1])$ be a linear continuous map. Then, $\coprod \circ (\mathbb{SEL})_{\tilde{\mathfrak{D}}} : \mathscr{C}_{\mathfrak{W}}([0,1]) \longrightarrow \mathscr{P}_{cmp,cvx}(\mathscr{C}_{\mathfrak{W}}([0,1]))$ is another operator in $\mathscr{C}_{\mathfrak{W}}([0,1]) \times \mathscr{C}_{\mathfrak{W}}([0,1])$ with action $\varpi^* \mapsto (\coprod \circ (\mathbb{SEL})_{\tilde{\mathfrak{D}}})(\varpi^*) = \coprod((\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*})$ having closed graph property.*

Theorem 4 (Dhage's theorem [31]). *Consider the Banach algebra $\mathfrak{W}$, and the operators $\mathbb{A}_1^* : \mathfrak{W} \longrightarrow \mathfrak{W}$ and $\mathbb{A}_2^* : \mathfrak{W} \longrightarrow \mathscr{P}_{cmp,cvx}(\mathfrak{W})$ satisfying the following:*

(i) $\mathbb{A}_1^*$ is Lipschitzian (with $l^* > 0$)

(ii) $\mathbb{A}_2^*$ is compact upper semicontinuous

(iii) $2l^*\widehat{\mathbb{O}} < 1$ with $\widehat{\mathbb{O}} = \|\mathbb{A}_2^*(\mathfrak{W})\|$

Then, either (1i) $\mathscr{O}^* = \{\varpi^* \in \mathfrak{W} \mid \alpha_0\varpi^* \in \mathbb{A}_1^*\varpi^*\mathbb{A}_2^*\varpi^*, \alpha_0 > 1\}$ is unbounded, or (2i) a solution, belonging to $\mathfrak{W}$, exists for which $\varpi^* \in \mathbb{A}_1^*\varpi^*\mathbb{A}_2^*\varpi^*$.

Theorem 5 (Endpoint theorem [29]). *Suppose that $(\mathfrak{W}, d_{\mathfrak{W}})$ be complete and $\psi : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}^{\geq 0}$ admits the upper semicontinuity via $\psi(z) < z$ and $\liminf_{z\longrightarrow\infty}(z - \psi(z)) > 0, \forall z > 0$. Besides, we assume that $\tilde{\mathfrak{D}} : \mathfrak{W} \longrightarrow \mathscr{P}_{cls,bnd}(\mathfrak{W})$ is such that $\mathbb{PH}_{d_{\mathfrak{W}}}(\tilde{\mathfrak{D}}\varpi_1^*, \tilde{\mathfrak{D}}\varpi_2^*) \leq \psi(d_{\mathfrak{W}}(\varpi_1^*, \varpi_2^*))$ for each $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$. Then, an endpoint (uniquely) exists for $\tilde{\mathfrak{D}}$ iff $\tilde{\mathfrak{D}}$ involves the APXEndP-property.*

3. New Existence Criteria

In two previous sections, we assembled some auxiliary and useful notions to achieve our main goals. Now in the following, we first establish a required lemma to derive the main existence results. To do this, we need to consider a sup–norm given by $\|\varpi^*\|_{\mathfrak{W}} = \sup_{z\in[0,1]}|\varpi^*(z)|$ on the space $\mathfrak{W} = \{\varpi^*(z) \colon \varpi^*(z) \in \mathscr{C}_{\mathbb{R}}([0,1])\}$. In this case, the Banach space $(\mathfrak{W}, \|\cdot\|_{\mathfrak{W}})$ along with the multiplication action defined as $(\varpi_1^* \cdot \varpi_2^*)(z) = \varpi_1^*(z)\varpi_2^*(z)$ is a Banach algebra for all $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$.

Lemma 6. *Let $h_* \in \mathfrak{W}$, $a \geq 0$, $\sigma^* \in (1,2)$, $\tilde{\mu} \in (0,1)$, $\mu^* > 0$, $\tilde{m}_1, \tilde{m}_2 \in \mathbb{R}^{\neq 0}$, and $s_1^*, s_2^* \in \mathbb{R}^+$. An element $\varpi_0^* \in \mathfrak{W}$ is a solution for given φ-hybrid fractional equation*

$${}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z, \varpi^*(z))}\right) = h_*(z), (z \in [a, T], \sigma^* \in (1,2)), \quad (16)$$

supplemented with separated mixed φ-integro-derivative boundary conditions

$$\begin{cases} \tilde{m}_1\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a} = s_1^* + \tilde{m}_2\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a}, \\ \tilde{m}_1{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T} = s_2^* + \tilde{m}_2{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T}, \end{cases} \tag{17}$$

which is given by the following:

$$\begin{aligned}\varpi^*(z) = \mathfrak{N}_*(z,\varpi^*(z))\Bigg[&\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}h_*(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}h_*(q)dq \\ &+ \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}h_*(q)dq\Bigg],\end{aligned} \tag{18}$$

so that $\tilde{m}^*$ is a positive real constant given as

$$\tilde{m}^* := \frac{\tilde{m}_1(\varphi(T)-\varphi(a))^{1+\mu^*-\tilde{\mu}}}{\Gamma(2+\mu^*-\tilde{\mu})} - \frac{\tilde{m}_2(\varphi(T)-\varphi(a))^{\mu^*}}{\Gamma(1+\mu^*)} \neq 0. \tag{19}$$

Proof. At first, the element ϖ_0^* is assumed to be a solution for the hybrid φ-Caputo differential Equation (16). Then, there exist $\tilde{k}_0^*, \tilde{k}_1^* \in \mathbb{R}$ such that

$$\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))} = {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}h_*(z) + \tilde{k}_0^* + \tilde{k}_1^*(\varphi(z)-\varphi(a)), \tag{20}$$

or more precisely, we have

$$\begin{aligned}\varpi_0^*(z) = \mathfrak{N}_*(z,\varpi_0^*(z))\Bigg[&\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}}{\Gamma(\sigma^*)}h_*(q)\,\mathrm{d}q \\ &+ \tilde{k}_0^* + \tilde{k}_1^*(\varphi(z)-\varphi(a))\Bigg].\end{aligned} \tag{21}$$

In view of the notion of fractional derivative in the φ-Caputo framework, we get the following relations for $\tilde{\mu} \in (0,1)$:

$${}^{\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = \int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-2}}{\Gamma(\sigma^*-1)}h_*(q)\,dq + \tilde{k}_1^*, \tag{22}$$

$$\begin{aligned}{}^{\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = &\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-\tilde{\mu}-1}}{\Gamma(\sigma^*-\tilde{\mu})}h_*(q)dq \\ &+ \tilde{k}_1^*\frac{(\varphi(z)-\varphi(a))^{1-\tilde{\mu}}}{\Gamma(2-\tilde{\mu})}.\end{aligned} \tag{23}$$

In the following, by taking integral of order $\mu^* > 0$ in the φ-Riemann-Liouville settings on both sides of (22) and (23), we obtain

$$\begin{aligned}{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = &\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*+\mu^*-2}}{\Gamma(\sigma^*+\mu^*-1)}h_*(q)\,dq \\ &+ \tilde{k}_1^*\frac{(\varphi(z)-\varphi(a))^{\mu^*}}{\Gamma(\mu^*+1)}, \\ {}^{\mathscr{RL}}\mathscr{I}_a^{(\mu^*;\varphi)C}\mathfrak{D}_a^{(\tilde{\mu};\varphi)}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = &\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}}{\Gamma(\sigma^*+\mu^*-\tilde{\mu})}h_*(q)dq \\ &+ \tilde{k}_1^*\frac{(\varphi(z)-\varphi-(a))^{1+\mu^*-\tilde{\mu}}}{\Gamma(\mu^*-\tilde{\mu}+2)}.\end{aligned} \tag{24}$$

In this step, by considering the first boundary condition in (17), we find that $(\tilde{m}_1-\tilde{m}_2)\tilde{k}_0^* = s_1^*$ and so

$$\tilde{k}_0^* = \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2}. \tag{25}$$

In addition, the second integro-derivative boundary condition given in (17) yields

$$\begin{aligned}\tilde{k}_1^* = \frac{s_2^*}{\tilde{m}^*} - &\frac{\tilde{m}_1}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}h_*(q)dq \\ + &\frac{\tilde{m}_2}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}h_*(q)dq.\end{aligned} \tag{26}$$

In the last step, if we insert the values $\tilde{k}_0^*$ and $\tilde{k}_1^*$ obtained in (25) and (26) into (21), then we get

$$\begin{aligned}\varpi_0^*(z) = \mathfrak{N}_*(z,\varpi_0^*(z))\Bigg[&\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}h_*(q)\,dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}h_*(q)dq + \frac{\tilde{m}_2(\varphi(z)-o(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}h_*(q)dq\Bigg].\end{aligned} \tag{27}$$

The resultant integral equation confirms that ϖ_0^* satisfies the mentioned φ-integral Equation (18), and the proof is completed. □

Now, by considering Lemma 6, we can present the following definition.

Definition 7. An absolutely continuous function $\varpi^* : [a,T] \longrightarrow \mathbb{R}$ is called a solution function for the φ-hybrid inclusion BVP in the sense of φ-Caputo (2) and (3) if there is $\widehat{\kappa} \in \mathscr{L}^1([a,T],\mathbb{R})$ with $\widehat{\kappa}(z) \in \tilde{\mathfrak{O}}(z,\varpi^*(z))$ for almost all $z \in [a,T]$ which satisfies separated mixed φ-integro-derivative boundary conditions

$$\begin{cases} \tilde{m}_1\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a} = s_1^* + \tilde{m}_2\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a}, \\ \tilde{m}_1{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T} = s_2^* + \tilde{m}_2{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T}, \end{cases} \tag{28}$$

and also

$$\begin{aligned} \varpi^*(z) = \mathfrak{N}_*(z,\varpi^*(z))\Bigg[&\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq\Bigg], \end{aligned} \tag{29}$$

for each $z \in [a, T]$.

Now, we are in a position that we can prove the first existence result about the hybrid φ-Caputo inclusion BVP (2) and (3).

Theorem 8. *Assume that $\tilde{\mathfrak{D}} : [a, T] \times \mathfrak{W} \longrightarrow \mathscr{P}_{cmp,cvx}(\mathfrak{W})$ is a set-valued operator and a function $\mathfrak{N}_* : [a, T] \times \mathfrak{W} \longrightarrow \mathfrak{W} \setminus \{0\}$ is continuous. In addition, let*

($\mathfrak{C}_1$) a bounded function $\chi : [a, T] \longrightarrow \mathbb{R}^+$ exists such that for each $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$ and $z \in [a, T]$

$$|\mathfrak{N}_*(z,\varpi_1^*(z)) - \mathfrak{N}_*(z,\varpi_2^*(z))| \le \chi(z)|\varpi_1^*(z) - \varpi_2^*(z)| \tag{30}$$

($\mathfrak{C}_2$) $\tilde{\mathfrak{D}}$ is $\mathscr{L}^1$-Caratheodory

($\mathfrak{C}_3$) a function $Y(z) \in \mathscr{L}^1([a, T], \mathbb{R}^+)$ exists such that

$$\left\|\tilde{\mathfrak{D}}(z,\varpi^*)\right\| = \sup\left\{|\widehat{\kappa}|: \widehat{\kappa} \in \tilde{\mathfrak{D}}(z,\varpi^*(z))\right\} \le Y(z), \tag{31}$$

for all $\varpi^* \in \mathfrak{W}$ and for almost all $z \in [a, T]$

($\mathfrak{C}_4$) a real number $\widehat{\varepsilon} \in \mathbb{R}$ exists so that

$$\widehat{\varepsilon} > \frac{\mathscr{F}^*\Pi^*\|Y\|_{\mathscr{L}^1}}{1-\chi^*\Pi^*\|Y\|_{\mathscr{L}^1}}, \tag{32}$$

where $\mathscr{F}^* = \sup_{z\in[a,T]}|\tilde{\mathfrak{N}}_*(z,0)|$, $\|Y\|_{\mathscr{L}^1} = \int_a^T |Y(q)|dq$, $\chi^* = \sup_{z\in[a,T]}|\chi(z)|$, and

$$\begin{aligned} \Pi^* = &\frac{s_1^*}{|\tilde{m}_1-\tilde{m}_2|} + \frac{(\varphi(T)-\varphi(a))^{\sigma^*}}{\Gamma(\sigma^*+1)} + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(T)-\varphi(a)) \\ &+ \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)} + \frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)} \end{aligned} \tag{33}$$

If $\chi^*\Pi^*\|Y\|_{\mathscr{L}^1} < 1/2$, then the φ-hybrid inclusion BVP (2) and (3) has at least a solution.

Proof. For each $\varpi^* \in \mathfrak{W}$, the collection of all existing selections of $\tilde{\mathfrak{D}}$ is defined as

$$(\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*} := \left\{\widehat{\kappa} \in \mathscr{L}^1_{\mathbb{R}}([a, T]): \widehat{\kappa}(z) \in \tilde{\mathfrak{D}}(z,\varpi^*(z))\right\}, \tag{34}$$

for every $\varpi^* \in \mathfrak{W}$ and for almost all $z \in [a, T]$. Define a set-valued map $\mathfrak{K} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$\mathfrak{K}(\varpi^*) = \left\{\zeta^* \in \mathfrak{W} : \zeta^*(z) = \psi^*(z)\right\}, \tag{35}$$

where

$$\begin{aligned} \psi^*(z) = \mathfrak{N}_*(z,\varpi^*(z))\Bigg[&\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq\Bigg], \end{aligned} \tag{36}$$

for some $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*}$ and for almost all $z \in [a, T]$. It is obvious that the function ψ_0^* is a solution to the φ-hybrid BVP (2) and (3) if ψ_0^* is a fixed point of $\mathfrak{K}$. Now, define $\mathbb{A}_1^* : \mathfrak{W} \longrightarrow \mathfrak{W}$ by $(\mathbb{A}_1^*\varpi^*)(z) = \mathfrak{N}_*(z,\varpi^*(z))$ and $\mathbb{A}_2^* : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$(\mathbb{A}_2^*\varpi^*)(z) = \left\{h^* \in \mathfrak{W} : h^*(z) = \xi^*(z)\right\}, \tag{37}$$

where

$$\begin{aligned} \xi^*(z) = &\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ &+ \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq, \end{aligned} \tag{38}$$

for some $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*}$ and for almost all $z \in [a, T]$. This implies $\mathfrak{K}(\varpi^*) = (\mathbb{A}_1^*\varpi^*)(\mathbb{A}_2^*\varpi^*)$. We show that both operators $\mathbb{A}_1^*$ and $\mathbb{A}_2^*$ satisfy Theorem 4. We at first prove that $\mathbb{A}_1^*$ is Lipschitzian. Let $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$. We have

$$\begin{aligned} |(\mathbb{A}_1^*\varpi_1^*(z)) - (\mathbb{A}_1^*\varpi_2^*(z))| &= |\mathfrak{N}_*(z,\varpi_1^*(z)) - \mathfrak{N}_*(z,\varpi_2^*(z))| \\ &\le \chi(z)|\varpi_1^*(z) - \varpi_2^*(z)|, \end{aligned} \tag{39}$$

for all $z \in [a, T]$. Therefore, for all $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$, we get

$$\|\mathbb{A}_1^*\varpi_1^* - \mathbb{A}_1^*\varpi_2^*\|_{\mathfrak{W}} \le \chi^*\|\varpi_1^* - \varpi_2^*\|\mathfrak{W}. \tag{40}$$

Hence, $\mathbb{A}_1^*$ is Lipschitz with constant $l^* = \chi^* > 0$. In the current moment, we check the convexity of $\mathbb{A}_2^*$. For this, let $\varpi_1^*, \varpi_2^* \in \mathbb{A}_2^*\varpi^*$. Choose $\widehat{\kappa}_1, \widehat{\kappa}_2 \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*}$ such that

$$\begin{aligned}\varpi_i^*(z) &= \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}_i(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_i(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_i(q)dq, (i=1,2),\end{aligned} \tag{41}$$

for almost all $z \in [a, T]$. Let $0 < \eta < 1$. Then,

$$\begin{aligned}\eta\varpi_1^*(z) + (1-\eta)\varpi_2^*(z) &= \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) + \frac{1}{\Gamma(\sigma^*)} \\ &\cdot \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}[\eta\widehat{\kappa}_1(q) + (1-\eta)\widehat{\kappa}_2(q)]dq \\ &- \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \\ &\cdot [\eta\widehat{\kappa}_1(q) + (1-\eta)\widehat{\kappa}_2(q)]dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}[\eta\widehat{\kappa}_1(q) + (1-\eta)\widehat{\kappa}_2(q)]dq,\end{aligned} \tag{42}$$

for almost all $z \in [a, T]$. With due attention to the convexity of $\tilde{\mathfrak{D}}$, $(\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*}$ has convex values. This implies that

$$\eta\widehat{\kappa}_1(z) + (1-\eta)\widehat{\kappa}_2(z) \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*}, \tag{43}$$

for almost all $z \in [a, T]$. Therefore, $\mathbb{A}_2^*\varpi^*$ is convex for each $\varpi^* \in \mathfrak{W}$. Next, we claim that $\mathbb{A}_2^*$ is completely continuous. To confirm this claim, we verify that the set $\mathbb{A}_2^*(\mathfrak{W})$ is equicontinuous and uniformly bounded. Firstly, we prove that $\mathbb{A}_2^*$ corresponds bounded sets to bounded sets contained in $\mathfrak{W}$. For $\alpha^* \in \mathbb{R}^+$, define the bounded ball $\mathfrak{B}_{\alpha^*} = \{\varpi^* \in \mathfrak{W} : \|\varpi^*\|_{\mathfrak{W}} \le \alpha^*\}$. For every $\varpi^* \in \mathfrak{B}_{\alpha^*}$ and $\xi^* \in \mathbb{A}_2^*\varpi^*$, there exists a function $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*}$ such that

$$\begin{aligned}\xi^*(z) &= \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ &+ \left.\frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq\right],\end{aligned} \tag{44}$$

for almost all $z \in [a, T]$. Then,

$$\begin{aligned}|\xi^*(z)| &\le \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) \\ &+ \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq \\ &+ \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq \\ &+ \frac{s_1^*}{|\tilde{m}_1 - \tilde{m}_2|} \le \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}Y(q)dq \\ &+ \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\,\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}Y(q)dq + \frac{|\tilde{m}_2|\,(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\,\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}Y(q)dq + \frac{s_1^*}{|\tilde{m}_1 - \tilde{m}_2|} \\ &\le \left[\frac{(\varphi(T)-\varphi(a))^{\sigma^*}}{\Gamma(\sigma^*+1)} + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(T)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)}\right. \\ &+ \left.\frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)} + \frac{s_1^*}{|\tilde{m}_1 - \tilde{m}_2|}\right]\|Y\|_{\mathscr{L}^1} = \Pi^*\|Y\|_{\mathscr{L}^1},\end{aligned} \tag{45}$$

where Π^* is given in (33). Thus, $\|\xi^*\| \le \Pi^*\|Y\|_{\mathscr{L}^1}$, and so the set $\mathbb{A}_2^*(\mathfrak{W})$ is uniformly bounded. Now, we want to prove that $\mathbb{A}_2^*$ corresponds bounded sets to equicontinuous sets. Take $\varpi^* \in \mathfrak{B}_{\alpha^*}$, $\xi^* \in \mathbb{A}_2^*\varpi^*$ and choose $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^*}$ so that

$$\begin{aligned}\xi^*(z) &= \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\,dq + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) \\ &- \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ &+ \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq \\ &+ \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2},\end{aligned} \tag{46}$$

for almost all $z \in [a, T]$. Let $z_1, z_2 \in [a, T]$ with $z_1 < z_2$, Then,

$$\begin{aligned}|\xi^*(z_2) - \xi^*(z_1)| &\le \frac{1}{\Gamma(\sigma^*)}\int_a^{z_1} \varphi'(q)\left[(\varphi(z_2)-\varphi(q))^{\sigma^*-1} - (\varphi(z_1)-\varphi(q))^{\sigma^*-1}\right] \\ &\cdot |\widehat{\kappa}(q)|dq + \frac{1}{\Gamma(\sigma^*)}\int_{z_1}^{z_2} \varphi'(q)(\varphi(z_2)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq \\ &+ \frac{s_2^*}{|\tilde{m}^*|}[\varphi(z_2)-\varphi(z_1)] + \frac{|\tilde{m}_1|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq + \frac{|\tilde{m}_2|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq \le \frac{1}{\Gamma(\sigma^*)}\int_a^{z_1}\varphi'(q) \\ &\cdot \left[(\varphi(z_2)-\varphi(q))^{\sigma^*-1} - (\varphi(z_1)-\varphi(q))^{\sigma^*-1}\right]Y(q)dq + \frac{1}{\Gamma(\sigma^*)} \\ &\cdot \int_{z_1}^{z_2} \varphi'(q)(\varphi(z_2)-\varphi(q))^{\sigma^*-1}Y(q)dq + \frac{s_2^*}{|\tilde{m}^*|}[\varphi(z_2)-\varphi(z_1)] \\ &+ \frac{|\tilde{m}_1|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}Y(q)dq \\ &+ \frac{|\tilde{m}_2|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}Y(q)dq.\end{aligned} \tag{47}$$

The right-hand side of the latter inequalities tends to zero (independent of $\varpi^* \in \mathfrak{B}_{\alpha^*}$) as z_1 tends to z_2. Application of Arzela–Ascoli theorem gives the complete continuity of $\mathbb{A}_2^*$. We here discuss that $\mathbb{A}_2^*$ has a closed graph, and this finding implies that $\mathbb{A}_2^*$ is upper semicontinuous. To achieve this aim, let $\varpi_n^* \in \mathfrak{B}_{\alpha^*}$ and $\xi_n^* \in (\mathbb{A}_2^* \varpi_n^*)$ with $\varpi_n^* \longrightarrow \varpi^{**}$ and $\xi_n^* \longrightarrow \tilde{\xi}^*$. We claim that $\tilde{\xi}^* \in (\mathbb{A}_2^* \varpi^{**})$. For every $n \geq 1$ and $\xi_n^* \in (\mathbb{A}_2^* \varpi_n^*)$, choose $\widehat{\kappa}_n \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi_n^*}$ such that

$$\begin{aligned}
\xi_n^*(z) &= \frac{1}{\Gamma(\sigma^*)} \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1} \widehat{\kappa}_n(q) \mathrm{d}q \\
&\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\
&\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \widehat{\kappa}_n(q) dq \\
&\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-1)} \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2} \widehat{\kappa}_n(q) dq \\
&\quad + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},
\end{aligned} \tag{48}$$

for almost all $z \in [a, T]$. It is suffices to find that there is a member $\kappa\wedge^* \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^{**}}$ so that

$$\begin{aligned}
\tilde{\xi}^*(z) &= \frac{1}{\Gamma(\sigma^*)} \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1} \kappa\wedge^*(q) dq \\
&\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\
&\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \kappa\wedge^*(q) dq \\
&\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-1)} \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2} \kappa\wedge^*(q) dq \\
&\quad + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},
\end{aligned} \tag{49}$$

for almost all $z \in [a, T]$. Define a linear continuous operator $\coprod : \mathscr{L}^1([a, T], \mathbb{R}) \longrightarrow \mathfrak{W} = \mathscr{C}([a, T], \mathbb{R})$ as

$$\begin{aligned}
\coprod \widehat{\kappa}(z) = \varpi^*(z) &= \frac{1}{\Gamma(\sigma^*)} \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1} \widehat{\kappa}(q) dq \\
&\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\
&\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \widehat{\kappa}(q) dq \\
&\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-1)} \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2} \widehat{\kappa}(q) dq \\
&\quad + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},
\end{aligned} \tag{50}$$

for almost all $z \in [a, T]$. Hence,

$$\begin{aligned}
\left\| \xi_n^*(z) - \tilde{\xi}^*(z) \right\| &= \left\| \frac{1}{\Gamma(\sigma^*)} \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1} \right. \\
&\quad \cdot (\widehat{\kappa}_n(q) - \kappa\wedge^*(q)) dq + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) \\
&\quad - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-\tilde{\mu})} \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \\
&\quad \cdot (\widehat{\kappa}_n(q) - \kappa\wedge^*(q)) dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-1)} \int_a^T \varphi' \\
&\quad \left. \cdot (q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2} (\widehat{\kappa}_n(q) - \kappa\wedge^*(q)) dq \right\| \longrightarrow 0.
\end{aligned} \tag{51}$$

Application of Theorem 3 shows that $\coprod \circ (\mathbb{SEL})_{\tilde{\mathfrak{D}}}$ has a closed graph. Besides, since $\xi_n^* \in \coprod((\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi_n})$ and $\varpi_n \longrightarrow \varpi^{**}$, so there exists $\kappa\wedge^* \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^{**}}$ such that

$$\begin{aligned}
\tilde{\xi}^*(z) &= \frac{1}{\Gamma(\sigma^*)} \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1} \kappa\wedge^*(q) dq \\
&\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\
&\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \kappa\wedge^*(q) dq \\
&\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^* \Gamma(\sigma^*+\mu^*-1)} \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2} \kappa\wedge^*(q) dq \\
&\quad + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},
\end{aligned} \tag{52}$$

for almost all $z \in [a, T]$. Hence, $\tilde{\xi}^* \in (\mathbb{A}_2^* \varpi^{**})$, and so $\mathbb{A}_2^*$ possesses closed graph which implies that $\mathbb{A}_2^*$ is upper semicontinuous. On the other hand, because of the compactness of values of $\mathbb{A}_2^*$, it is immediately deduced that $\mathbb{A}_2^*$ is compact and upper semicontinuous. Utilizing $(\mathfrak{C}_3)$, we get

$$\begin{aligned}
\widehat{\mathbb{O}} &= \|\mathbb{A}_2^*(\mathfrak{W})\| = \sup \{|\mathbb{A}_2^* \varpi^*| : \varpi^* \in \mathfrak{W}\} \\
&= \left[\frac{s_1^*}{|\tilde{m}_1-\tilde{m}_2|} + \frac{(\varphi(T)-\varphi(a))^{\sigma^*}}{\Gamma(\sigma^*+1)} + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(T)-\varphi(a)) \right. \\
&\quad \left. + \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)} + \frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)} \right] \\
&\quad \cdot \|Y\|_{\mathscr{L}^1} = \Pi^* \|Y\|_{\mathscr{L}^1}.
\end{aligned} \tag{53}$$

Put $l^* = \chi^*$. We have $\widehat{\mathbb{O}} l^* < 1/2$. Utilizing Theorem 4, we prove that one of the items (i) or (ii) is possible. First, we check that the item (i) is not the case. From Theorem 4 and the assumption $(\mathfrak{C}_4)$, consider an arbitrary member ϖ_0^* of Σ^* with $\|\varpi_0^*\| = \widehat{\varepsilon}$. Then, $\alpha_0 \varpi_0^*(z) \in (\mathbb{A}_1^* \varpi_0^*)(\mathbb{A}_2^* \varpi_0^*)(z)$ for all $\alpha_0 > 1$.

Choosing a function $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi_0^*}$, for each $\alpha_0 > 1$, we have

$$\begin{aligned}\varpi_0^*(z) = \frac{1}{\alpha_0}\mathfrak{N}_*(z,\varpi_0^*(z))\Bigg[\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}\\ \cdot (q)dq + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq\Bigg],\end{aligned} \tag{54}$$

for almost all $z \in [a, T]$. Thus, one can write

$$\begin{aligned}|\varpi_0^*(z)| = \frac{1}{\alpha_0}|\mathfrak{N}_*(z,\varpi_0^*(z))|\Bigg[\frac{s_1^*}{|\tilde{m}_1-\tilde{m}_2|} + \frac{1}{\Gamma(\sigma^*)}\\ \cdot \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq\\ + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq\\ + \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq\Bigg]\\ \le \frac{1}{\alpha_0}[|\mathfrak{N}_*(z,\varpi_0^*(z)) - \mathfrak{N}_*(z,0)| + |\mathfrak{N}_*(z,0)|]\\ \times\Bigg[\frac{s_1^*}{|\tilde{m}_1-\tilde{m}_2|} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq\\ + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\\ \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq + \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)}\\ \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq\Bigg] \le [\chi^*\widehat{\varepsilon} + \mathscr{F}^*]\Pi^*\|Y\|_{\mathscr{L}^1}.\end{aligned} \tag{55}$$

Hence, we get

$$\widehat{\varepsilon} \le \frac{\|Y\|_{\mathscr{L}^1}\mathscr{F}^*\Pi^*}{1-\Pi^*\chi^*\|Y\|_{\mathscr{L}^1}}, \tag{56}$$

which is a contradiction. Hence, the item (ii) indicated in Theorem 4 is valid. Thus, $\varpi^* \in \mathfrak{W}$ exists so that $\varpi^* \in (\mathbb{A}_1^*\varpi^*)(\mathbb{A}_2^*\varpi^*)$. In consequence, the operator $\mathfrak{K}$ has a fixed point. So the φ-hybrid inclusion BVP (2) and (3) has a solution, and this completes the proof. □

Definition 9. An absolutely continuous function $\varpi^* : [a, T] \longrightarrow \mathbb{R}$ is called a solution for the non-φ-hybrid inclusion BVP (4) in the sense of Caputo if there is $\widehat{\kappa} \in \mathscr{L}^1([a, T], \mathbb{R})$ with $\widehat{\kappa}(z) \in \tilde{\mathfrak{D}}(z, \varpi^*(z))$ for almost all $z \in [a, T]$ which satisfies separated mixed φ-integro-derivative boundary conditions

$$\begin{cases}\tilde{m}_1\varpi^*(a) = s_1^* + \tilde{m}_2\varpi^*(a),\\ \tilde{m}_1{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\varpi^*(T) = s_2^* + \tilde{m}_2{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\varpi^*(T),\end{cases} \tag{57}$$

$$\begin{aligned}\varpi^*(z) = \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\mathrm{d}q\\ + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq\\ + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq,\end{aligned} \tag{58}$$

for almost all $z \in [a, T]$.

For each $\varpi^* \in \mathfrak{W}$, the collection of all existing selections of $\mathfrak{D}$ is defined as

$$(\mathbb{SEL})_{\mathfrak{D},\varpi^*} = \left\{\widehat{\kappa} \in \mathscr{L}^1([a, T]) : \widehat{\kappa}(z) \in \mathfrak{D}(z, \varpi^z)\right\}, \tag{59}$$

for almost all $z \in [a, T]$. Define $\mathfrak{F} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ as

$$\mathfrak{F}(\varpi^*) = \{\vartheta \in \mathfrak{W} : \vartheta(z) = \rho(z)\}, \tag{60}$$

where

$$\begin{aligned}\varrho(z) = \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\mathrm{d}q\\ + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq\\ + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}\\ \cdot (q)dq, \widehat{\kappa} \in (\mathbb{SEL})_{\mathfrak{D},\varpi^*}.\end{aligned} \tag{61}$$

By making use of endpoints for the multifunction $\mathfrak{F}$, we prove the following theorem.

Theorem 10. *Consider $\mathfrak{D} : [a, T] \times \mathfrak{W} \longrightarrow \mathscr{P}_{cp}(\mathfrak{W})$ as a set-valued operator. Let*

($\mathfrak{C}_5$) $\psi : [0,\infty) \longrightarrow [0,\infty)$ be increasing and upper semi-continuous with $\liminf_{z\longrightarrow\infty}(z - \psi(z)) > 0$ and $z > \psi(z), \forall z > 0$

($\mathfrak{C}_6$) the multifunction $\mathfrak{D} : [a, T] \times \mathfrak{W} \longrightarrow \mathscr{P}_{cp}(\mathfrak{W})$ be integrable and bounded so that $\mathfrak{D}(\cdot, \varpi^*) : [a, T] \longrightarrow \mathscr{P}_{cp}(\mathfrak{W})$ be measurable for each $\varpi^* \in \mathfrak{W}$

($\mathfrak{C}_7$) a function $\varrho \in C([a, T], [0,\infty))$ exists such that

$$\mathbb{PH}_{d\mathfrak{W}}\left(\mathfrak{D}(z,\varpi_1^*), \mathfrak{D}\left(z,\varpi'^*_1\right)\right) \le \varrho(z)\psi\left(\left|\varpi_1^*(z) - \varpi'^*_1(z)\right|\right)\frac{1}{\Omega^{**}}, \tag{62}$$

for almost all $z \in [a, T]$ *and all* $\varpi^*_1, {\varpi'}^*_1 \in \mathfrak{W}$, *where* $\sup_{z\in[a,T]} |\varrho(z)| = \|\varrho\|$ *and*

$$\Omega^{**} = \left[\frac{1}{\Gamma(\sigma^* + 1)} - \frac{|\tilde{m}_1| (\varphi(T) - \varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^* - \tilde{\mu} + 1)} + \frac{|\tilde{m}_2| (\varphi(T) - \varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^*)}\right] \|\varrho\| \tag{63}$$

($\mathfrak{C}_8$) *the operator* $\mathfrak{F}$ *given by (60) possesses APXEndP-property*

Then, a solution exists to the non-φ-hybrid inclusion FBVP (4).

Proof. In such an argument, we try to prove the existence of endpoint to the set-valued operator $\mathfrak{F} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ defined by (60). To proceed this, we first investigate that $\mathfrak{F}(\varpi^*)$ is closed for each $\varpi^* \in \mathfrak{W}$. By taking into account the hypothesis ($\mathfrak{C}_6$), $z \mapsto \mathfrak{O}(z, \varpi^*(z))$ is a closed-valued measurable multifunction for each $\varpi^* \in \mathfrak{W}$. In consequence, $\mathfrak{O}$ has a measurable selection $(\mathbb{SEL})_{\mathfrak{O},\varpi^*} \neq \varnothing$. Now, we show that $\mathfrak{F}(\varpi^*) \subseteq \mathfrak{W}$ is closed for all $\varpi^* \in \mathfrak{W}$. Consider the sequence $(\varpi^*_n)_{n\geq 1}$ contained in $\mathfrak{F}(\varpi^*)$ with $\varpi^*_n \longrightarrow \mu$. For each n, there exists $\widehat{\kappa}_n \in (\mathbb{SEL})_{\mathfrak{O},\varpi^*}$ such that

$$\begin{aligned}\varpi^*_n(z) = {} & \frac{s^*_1}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z) - \varphi(q))^{\sigma^*-1}\widehat{\kappa}_n(q)dq \\ & + \frac{s^*_2}{\tilde{m}^*}(\varphi(z) - \varphi(a)) - \frac{\tilde{m}_1(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - \tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_n(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - 1)}\int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_n(q)dq,\end{aligned} \tag{64}$$

for almost all $z \in [a, T]$. Since $\mathfrak{O}$ is compact multifunction, we acquire a subsequence $(\widehat{\kappa}_n)_{n\geq 1}$ tending to $\widehat{\kappa} \in \mathscr{L}^1([a, T])$. Hence, we have $\widehat{\kappa} \in (\mathbb{SEL})_{\mathfrak{O},\varpi^*}$ and

$$\begin{aligned}\lim_{n\longrightarrow\infty} \varpi^*_n(z) = {} & \frac{s^*_1}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z) - \varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\mathrm{d}q \\ & + \frac{s^*_2}{\tilde{m}^*}(\varphi(z) - \varphi(a)) - \frac{\tilde{m}_1(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - \tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - 1)}\int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa} \\ & \cdot (q)dq = \varpi^*(z),\end{aligned} \tag{65}$$

for almost all $z \in [a, T]$. Hence, $\varpi^* \in \mathfrak{F}$ which indicates that $\mathfrak{F}$ is closed-valued. In addition, $\mathfrak{F}(\varpi^*)$ is bounded for each $\varpi^* \in \mathfrak{W}$ since $\mathfrak{O}$ is compact. Finally, we investigate if $\mathbb{P}\mathbb{H}_{d_{\mathfrak{W}}}(\mathfrak{F}(\varpi^*), \mathfrak{F}(\tilde{\varpi}^*)) \leq \psi(\|\varpi^* - \tilde{\varpi}^*\|)$ holds. Let $\varpi^*, \tilde{\varpi}^* \in \mathfrak{W}$, and $x_1 \in \mathfrak{F}(\tilde{\varpi}^*)$. Choose $\widehat{\kappa}_1 \in (\mathbb{SEL})_{\mathfrak{O},\tilde{\varpi}^*}$ such that

$$\begin{aligned}x_1(z) = {} & \frac{s^*_1}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z) - \varphi(q))^{\sigma^*-1}\widehat{\kappa}_1(q)dq \\ & + \frac{s^*_2}{\tilde{m}^*}(\varphi(z) - \varphi(a)) - \frac{\tilde{m}_1(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - \tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_1(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - 1)}\int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_1(q)dq,\end{aligned} \tag{66}$$

for almost all $z \in [a, T]$. Since

$$\mathbb{P}\mathbb{H}_{d_{\mathfrak{W}}}\left(\mathfrak{O}(z, \varpi^*), \mathfrak{O}\left(z, \tilde{\varpi}^*\right)\right) \leq \varrho(z)\left(\psi\left(\varpi^*(z) - \varpi^{\sim *}(z)\right)\right)\frac{1}{\Omega^{**}}, \tag{67}$$

for almost all $z \in [a, T]$, so there exists $k^* \in \mathfrak{O}(z, \varpi^*)$ such that

$$|\widehat{\kappa}_1(z) - k^*| \leq \left(\psi\left(\varpi^*(z) - \varpi^{\sim *}(z)\right)\right)\frac{\varrho(z)}{\Omega^{**}}, \tag{68}$$

for almost all $z \in [a, T]$. Define the multifunction $\mathfrak{A} : [a, T] \longrightarrow \mathscr{P}(\mathfrak{W})$ given by

$$\mathfrak{A}(z) = \left\{k^* \in \mathfrak{W} : |\widehat{\kappa}_1(z) - k^*| \leq \varrho(z)\left(\psi\left(\varpi^*(z) - \varpi^{\sim *}(z)\right)\right)\frac{1}{\Omega^{**}}\right\}. \tag{69}$$

Since $\widehat{\kappa}$ and $\sigma = \varrho(\psi(\varpi^* - \tilde{\varpi}^*))1/\Omega^{**}$ are measurable, thus we choose $\widehat{\kappa}_2(z) \in \mathfrak{O}(z, \varpi^*(z))$ such that

$$|\widehat{\kappa}_1(z) - \widehat{\kappa}_2(z)| \leq \left(\psi\left(\varpi^*(z) - \varpi^{\sim *}(z)\right)\right)\frac{\varrho(z)}{\Omega^{**}}, \tag{70}$$

for almost all $z \in [a, T]$. Select $x_2 \in \mathfrak{F}(\varpi^*)$ such that

$$\begin{aligned}x_2(z) = {} & \frac{s^*_1}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z) - \varphi(q))^{\sigma^*-1}\widehat{\kappa}_2(q)\mathrm{d}q \\ & + \frac{s^*_2}{\tilde{m}^*}(\varphi(z) - \varphi(a)) - \frac{\tilde{m}_1(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - \tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_2(q)\, dq \\ & + \frac{\tilde{m}_2(\varphi(z) - \varphi(a))}{\tilde{m}^*\Gamma(\sigma^* + \mu^* - 1)}\int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_2(q)dq,\end{aligned} \tag{71}$$

for almost all $z \in [a, T]$. Hence, we get

$$\begin{aligned} |x_1(z) - x_2(z)| &\le \frac{1}{\Gamma(\sigma^*)} \int_a^z \varphi'(q)(\varphi(z) - \varphi(q))^{\sigma^*-1} |\widehat{\kappa}_1(q) - \widehat{\kappa}_2(q)| dq \\ &+ \frac{|\tilde{m}_1|(\varphi(z) - \varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^* - \tilde{\mu})} \int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \\ &\cdot |\widehat{\kappa}_1(q) - \widehat{\kappa}_2(q)| dq + \frac{|\tilde{m}_2|(\varphi(z) - \varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^* - 1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T) - \varphi(q))^{\sigma^*+\mu^*-2} |\widehat{\kappa}_1(q) - \widehat{\kappa}_2(q)| dq \\ &\le \frac{1}{\Gamma(\sigma^* + 1)} \|\varrho\| \psi(\|\varpi^* - \tilde{\varpi}^*\|) \frac{1}{\Omega^{**}} - \frac{|\tilde{m}_1|(\varphi(T) - \varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^* - \tilde{\mu} + 1)} \\ &\cdot \|\varrho\| \psi(\|\varpi^* - \tilde{\varpi}^*\|) \frac{1}{\Omega^{**}} + \frac{|\tilde{m}_2|(\varphi(T) - \varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^*)} \|\varrho\| \psi \\ &\cdot (\|\varpi^* - \tilde{\varpi}^*\|) \frac{1}{\Omega^{**}} = \left[\frac{1}{\Gamma(\sigma^* + 1)} - \frac{|\tilde{m}_1|(\varphi(T) - \varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^* - \tilde{\mu} + 1)} \right. \\ &\left. + \frac{|\tilde{m}_2|(\varphi(T) - \varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^* + \mu^*)} \right] \|\varrho\| \psi(\|\varpi^* - \tilde{\varpi}^*\|) \frac{1}{\Omega^{**}} \\ &= \Omega^{**} \psi(\|\varpi^* - \tilde{\varpi}^*\|) \frac{1}{\Omega^{**}} = \psi(\|\varpi^* - \tilde{\varpi}^*\|). \end{aligned} \tag{72}$$

This gives $\|x_1 - x_2\| \le \psi(\|\varpi^* - \tilde{\varpi}^*\|)$ and shows that $\mathbb{PH}_{d\mathfrak{W}}(\mathfrak{F}(\varpi^*), \mathfrak{F}(\tilde{\varpi}^*)) \le \psi(\|\varpi^* - \tilde{\varpi}^*\|)$ for all $\varpi^*, \tilde{\varpi}^* \in \mathfrak{W}$. Also from $(\mathfrak{C}_8)$, we realize that $\mathfrak{F}$ has approximate endpoint property. Application of Theorem 5 gives that $\mathfrak{K}$ has a unique endpoint, i.e., there exists $\varpi^{**} \in \mathfrak{W}$ such that $\mathfrak{F}(\varpi^{**}) = \{\varpi^{**}\}$. In conclusion, ϖ^{**} is a solution to the non-φ-hybrid inclusion BVP (4). □

4. Some Examples

This section involves two different numerical simulation examples corresponding to the relevant φ-hybrid and non-φ-hybrid fractional inclusion boundary problems to guarantee the applicability of proved theorems.

Example 1. With due attention to (2) and (3), we design the Caputo φ-hybrid differential inclusion BVP as

$$\begin{aligned} {}^{\mathscr{C}}\mathfrak{D}_0^{1.62;z+2} \left(\frac{\varpi^*(z)}{z \sin \varpi^*(z)/42 + 1/4} \right) &\in \\ \cdot \left[\frac{\sin \varpi^*(z)}{z(2 + |\sin z|)}, \frac{|\cos \varpi^*(z)|}{4(1 + |\cos \varpi^*(z)|)} + \frac{3}{5} \right], \end{aligned} \tag{73}$$

supplemented with separated mixed φ-integro-derivative boundary conditions

$$\begin{cases} 0.8 \left(\dfrac{\varpi^*(z)}{z \sin \varpi^*(z)/42 + 1/4} \right) \Bigg|_{z=0} = 1.2 + (0.4) \left(\dfrac{\varpi^*(z)}{z \sin \varpi^*(z)/42 + 1/4} \right) \Bigg|_{z=0}, \\ 0.8 \, {}^{\mathscr{RL}}\mathscr{I}_0^{1.4;z+2} \, {}^{\mathscr{C}}\mathfrak{D}_0^{0.4;z+2} \left(\dfrac{\varpi^*(z)}{z \sin \varpi^*(z)/42 + 1/4} \right) \Bigg|_{z=1} = 0.9 + 0.4 \, {}^{\mathscr{RL}}\mathscr{I}_0^{1.4;z+2} \, {}^{\mathscr{C}}\mathfrak{D}_0^{1;z+2} \left(\dfrac{\varpi^*(z)}{z \sin \varpi^*(z)/42 + 1/4} \right) \Bigg|_{z=1}, \end{cases} \tag{74}$$

where $z \in [0, 1]$, $\sigma^* = 1.62$, $\tilde{\mu} = 0.4$, $\mu^* = 1.4$, $\tilde{m}_1 = 0.8$, $\tilde{m}_2 = 0.4$, $s_1^* = 1.2$, $s_2^* = 0.9$, and $\varphi(z) = z + 2$. We define $\mathfrak{N}_* : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}\{0\}$ by $\mathfrak{N}_*(z, \varpi^*(z)) = z \sin \varpi^*(z)/42 + 1/4$ which is nonzero and continuous. Notice that $\mathscr{F}^* = \sup_{z\in[0,1]} |\mathfrak{N}_*(z, 0)| = 1/4$. Moreover, the function $\mathfrak{N}_*$ is Lipschitz, that is, for each $\varphi_1^*, \varphi_2^* \in \mathbb{R}$, we have

$$\begin{aligned} |\mathfrak{N}_*(z, \varpi_1^*(z)) - \mathfrak{N}_*(z, \varpi_2^*(z))| &= \left| z \sin \frac{\varpi_1^*(z)}{42} - z \sin \frac{\varpi_2^*(z)}{42} \right| \\ &\le \frac{z}{42} |\varpi_1^*(z) - \varpi_2^*(z)|. \end{aligned} \tag{75}$$

If $\chi(z) = z/42$, then $\chi^* = \sup_{z\in[0,1]} |\chi(z)| = 1/42$. Now, define $\tilde{\mathfrak{O}} : [0, 1] \times \mathbb{R} \longrightarrow P(\mathbb{R})$ by

$$\tilde{\mathfrak{O}}(z, \varpi^*(z)) = \left[\frac{\sin \varpi^*(z)}{z|\sin z| + 2z}, \frac{|\cos \varpi^*(z)|}{4(|\cos \varpi^*(z)| + 1)} + \frac{3}{5} \right]. \tag{76}$$

Since

$$|\zeta| \le \max \left[\frac{\sin \varpi^*(z)}{z|\sin z| + 2z}, \frac{|\cos \varpi^*(z)|}{4(|\cos \varpi^*(z)| + 1)} + \frac{3}{5} \right] \le 1, \tag{77}$$

therefore, we have

$$\left\| \tilde{\mathfrak{O}}(z, \varpi^*(z)) \right\| = \sup \left\{ |\widehat{\kappa}| : \widehat{\kappa} \in \tilde{\mathfrak{O}}(z, \varpi^*(z)) \right\} \le Y(z) = 1. \tag{78}$$

Then, $\|Y\|_{\mathscr{L}_1} = \int_0^1 |Y(q)| dq = \int_0^1 1 \cdot dq = 1$. By using above values, we have $\Pi^* = 20.1356266$. Also, we can find $\widehat{\varepsilon}$ with $\widehat{\varepsilon} > 9.67254$. Finally, we have $\chi^* \Pi^* \|Y\|_{\mathscr{L}^1} = 0.47949 < 1/2$. Thus, all assertions of Theorem 8 are verified. Hence, the φ-hybrid Caputo differential inclusion BVP (73) supplemented with separated mixed φ-integro-derivative boundary conditions (74) has a solution.

Example 2. With due attention to (4), we design the Caputo non-φ-hybrid differential inclusion BVP as

$$ {}^{\mathscr{C}}\mathfrak{D}_0^{1.4;2z+3/2}\varpi^*(z) \in \left[0, \frac{2|\varpi^*(z)|}{3(1/2+z)(2+|\varpi^*(z)|)}\right], \tag{79} $$

supplemented with separated mixed φ-integro-derivative boundary

$$ \begin{cases} (0.73)\varpi^*(0) = 0.7 + 0.3\varpi^*(0), \\ 0.73^{\mathscr{RL}}\mathscr{I}_0^{1.6;2z+3/2\,\mathscr{C}}\mathfrak{D}_0^{0.73;2z+3/2}\varpi^*(1) = 0.6 + 0.3^{\mathscr{RL}}\mathscr{I}_0^{1.6;2z+3/2\,\mathscr{C}}\mathfrak{D}_0^{1;2z+3/2}\varpi^*(1). \end{cases} \tag{80} $$

where ${}^{\mathscr{C}}\mathfrak{D}_0^{1.4;2z+3/2}$ denotes the φ-CF derivative of order $\sigma^* = 1.4$, $z \in [0, 1]$, $\tilde{\mu} = 0.6$, $\mu^* = 1.6$, $\tilde{m}_1 = 0.73$, $\tilde{m}_2 = 0.3$, $s_1^* = 0.7$, $s_2^* = 0.6$, and $\varphi(z) = 2z + 3/2$. Using these values, we have $\tilde{m}^* = 0.15516$. We consider the Banach space $\mathfrak{W} = \{\varpi^*(z) : \varpi^*(z) \in \mathscr{C}([0, 1], \mathbb{R})\}$ equipped with $\|\varpi^*\|_{\mathfrak{W}} = \sup_{z\in[0,1]}|\varpi^*(z)|$. Now, we define a multivalued map $\tilde{\mathfrak{O}} : [0, 1] \times \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$ \tilde{\mathfrak{O}}(z, \varpi^*(z)) = \left[0, \frac{2|\varpi^*(z)|}{36(1/2+z)(2+|\varpi^*(z)|)}\right], \tag{81} $$

for almost all $z \in [0, 1]$. We define $\psi : ;[0,\infty) \longrightarrow [0,\infty)$ by $\psi(z) = z/3, \forall z > 0$. Obviously, $\lim\inf_{z\longrightarrow\infty}(z - \psi(z)) > 0$ and $\psi(z) < z$ for all $z > 0$. Now, for each $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$, we have

$$ \begin{aligned} &\mathbb{PH}_{d_{\mathfrak{W}}}\left(\tilde{\mathfrak{O}}(z, \varpi_1^*(z)), \tilde{\mathfrak{O}}(z, \varpi_2^*(z))\right) \\ &\quad \le \frac{2}{36(1/2+z)}(|\varpi_1^*(z) - \varpi_2^*(z)|) \le \psi(|\varpi_1^*(z) - \varpi_2^*(z)|)\frac{\rho(z)}{\Omega^{**}}, \end{aligned} \tag{82} $$

where $\Omega^{**} = 0.21377$ and $\varrho \in \mathscr{C}([0, 1], [0,\infty))$ is defined as $\varrho(z) = 2/12(1/2 + z)$ for all z. Then, $\|\varrho\| = \sup_{z\in[0,1]} = 1/3$. Lastly, we introduce $\mathfrak{F} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$ \mathfrak{F}(\varpi^*) = \left\{\vartheta \in \mathfrak{W} : \text{there exists } \tilde{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*} \text{ s.t.} \vartheta(z) = \rho(z), \forall z \in [0, 1]\right\}, \tag{83} $$

in which

$$ \begin{aligned} \rho(z) = {}& \frac{0.7}{0.43} + \frac{1}{\Gamma(1.4)}\int_0^z \varphi'(q)(\varphi(z) - \varphi(q))^{0.4}\widehat{\kappa}(q)dq \\ & + \frac{0.6}{0.15516}(\varphi(z) - \varphi(0)) - \frac{0.73(\varphi(z) - \varphi(0))}{0.15516\Gamma(2.4)} \\ & \cdot \int_0^T \varphi'(q)(\varphi(1) - \varphi(q))^{1.4}\widehat{\kappa}(q)dq + \frac{0.3(\varphi(z) - \varphi(0))}{0.15516\Gamma(2)} \\ & \cdot \int_0^T \varphi'(q)(\varphi(1) - \varphi(q))\widehat{\kappa}(q)dq. \end{aligned} \tag{84} $$

Thus, all assertions of Theorem 10 are verified. Hence, the non-φ-hybrid Caputo differential inclusion BVP (79) with separated mixed φ-integro-derivative boundary (80) has a solution.

5. Conclusion

In the current research study, we derived some theoretical criteria to prove the existence results to a new φ-hybrid fractional differential inclusion in the Caputo settings depending on the increasing function φ with separated mixed φ-hybrid-integro-derivative boundary conditions. The applied method to achieve desired purposes is based on Dhage's fixed point result. In addition, we discussed a special case of the proposed φ-inclusion problem in the non-φ-hybrid structure with the help of the endpoint notion. To confirm the applicability of our theoretical findings, two specific numerical examples are provided which simulate both φ-hybrid and non-φ-hybrid cases. Hence, this research work can motivate other researchers in this field to concentrate on various investigations of different φ-hybrid structures formulated by other fractional operators.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors declare that the present study was realized in collaboration with equal responsibility. All authors read and approved the final version of the current manuscript.

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