Journal of Mathematics

Innovative Applications of Fractional Calculus

Lead Guest Editor: Ahmet Ocak Akdemir
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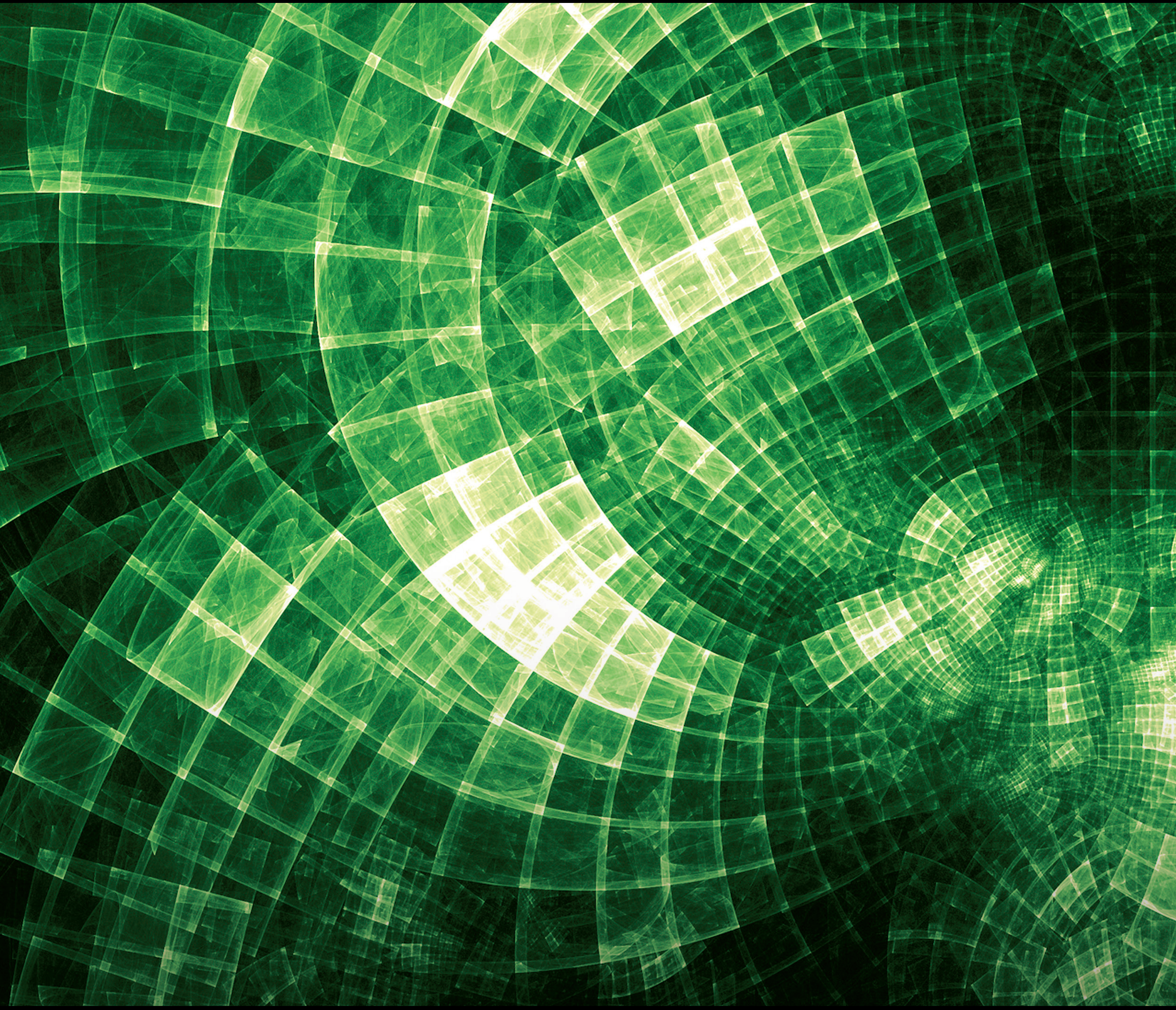

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Research Article

Shape Preserving Piecewise KNR Fractional Order Biquadratic C^2 Spline

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In a recent article, a piecewise cubic fractional spline function is developed which produces C^1 continuity to given data points. In the present paper, an interpolant continuity class C^2 is preserved which gives visually pleasing piecewise curves. The behavior of the resulting representations is analyzed intrinsically with respect to variation of the shape control parameters t and s. The data points are restricted to be strictly monotonic along real line.

1. Introduction

Among the various methods in computer aided geometric designing, piecewise spline-based techniques are the conventional methods. In many applications, one inclines interpolate or approximate univariate data by spline functions possessing certain geometric properties or shapes such as monotonicity, convexity, or nonnegativity. Due to the verity of spline algorithm, designers do not find any strain to adopt these techniques. Ample work has been done in this regard and researchers are still working on varied techniques by refining them to make it more and more diverse. The aim of spline interpolation is to get an interpolation formula that is continuous and smooth in both within the intervals and at the interpolating points. In recent past, a hatful of work have been done in the field of piecewise polynomial spline curve [1–4], rational spline [5], trigonometric spline [6], exponential spline [7], and spline-based surfaces which are used to preserve the C^2 continuity. This paper is a continuation of a previous paper [8] in which piecewise C^1 continuity is preserved. The fractional biquadratic spline is represented in terms of first and second order derivative values at the knots and provides an alternative to the ordinary spline. This paper is an attempt to embrace a novel technique on piecewise biquadratic polynomial.

Fractional calculus has been an Annex of ordinary calculus that encapsulated integrals and derivatives that are defined for arbitrary real orders. The journey of fractional calculus commenced in seventeenth century and underscored different derivatives [1] with significant pros and cons ranging from Riemann–Liouville, Hadamard, and Grünwald–Letnikov to Caputo, and so forth. Selecting apt fractional derivatives is pertinent to its considered systems; therefore, fractional operators were also a prevalent focus of various research works. Concurrently, studying generalized fractional operators is also indispensable in the field of computer graphics [9–11].

Fractional order derivatives are rapid emerging concept in different fields of mathematics, physics, and engineering in recent years [12–15]. Due to application of new approach of fractional order derivative, the computational cost is reduced. In this paper, an efficient and intuitive technique which is able to produce piecewise smooth curves in each given subinterval, $[x_i, x_{i+1}]$, $i = 0, 1, 2, 3, \ldots n$, $\forall x_i \in \mathbb{R}$, is adopted by combining both concepts of spline and Caputo–Fabrizio fractional order derivatives. With biquadratic piecewise polynomial assistance, higher accuracy is ensured.

The paper is organized in the following way. In Section 2, the formula using continuity condition is established. In Section 3, all the results are included, and in Section 4, discussion related to the novel technique is highlighted.

2. Preliminaries

There are heaps of definitions of fractional integral and derivatives; among them, few are Riemann–Liouville, Riesz, Caputo [8], Riesz–Caputo, Hadamard, Weyl, Grünwald–Letnikov, Chen, etc. Here, we are discussing Riemann–Liouville and Caputo. The proofs of results may be found in [16, 17].

Let $g\colon [a, b] \longrightarrow \mathscr{R}$ be a function, α a positive real number, n the integer satisfying $n-1 \le \alpha < n$, and Γ the Euler gamma function [11]. Then, the left and right Riemann–Liouville fractional integrals of order α are defined,

$$
\begin{aligned}
{}_aI_y^{\alpha} g(y) &= \frac{1}{\Gamma(\alpha)} \int_a^y (y-\tau)^{\alpha-1} g(\tau)\mathrm{d}\tau,\\
{}_yI_b^{\alpha} g(y) &= \frac{1}{\Gamma(\alpha)} \int_y^b (\tau-y)^{\alpha-1} g(\tau)\mathrm{d}\tau,
\end{aligned}
\tag{1}
$$

respectively.

The left and right Riemann–Liouville fractional derivatives of order α are defined by

$$
\begin{aligned}
{}_aD_y^{\alpha} g(y) &= \frac{d^m}{\mathrm{d}y^m} {}_aI_y^{m-\alpha} g(y) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{\mathrm{d}y^m} \int_a^y (y-\tau)^{m-\alpha-1} g(\tau)d\tau,\\
{}_yD_b^{\alpha} g(y) &= \frac{d^m}{\mathrm{d}y^m} {}_yI_b^{m-\alpha} g(y) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{\mathrm{d}y^m} \int_a^y (y-\tau)^{m-\alpha-1} g(\tau)d\tau.
\end{aligned}
\tag{2}
$$

Therefore, the right and left Caputo fractional derivatives of order α are defined by

$$
\begin{aligned}
{}_a^CD_y^{\alpha} g(y) &= {}_aI_y^{m-\alpha} \frac{\mathrm{d}^m}{\mathrm{d}y^m} g(y)\\
&= \frac{1}{\Gamma(m-\alpha)} \int_a^y (y-\tau)^{m-\alpha-1} g^{(m)}(\tau)\mathrm{d}\tau,\\
{}_y^CD_b^{\alpha} g(y) &= (-1)^m {}_yI_b^{m-\alpha} \frac{d^m}{\mathrm{d}y^m} g(y)\\
&= \frac{1}{\Gamma(m-\alpha)} \int_y^b (-1)^m (\tau-y)^{m-\alpha-1} g^{(m)}(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{3}
$$

Intrinsically, there exists a relation between Caputo fractional and Riemann–Liouville derivatives, and as a consequence, we have the following relations:

If $g(a) = g'(a) = \ldots = g^{(m-1)}(a) = 0$, then ${}_a^CD_y^{\alpha} g(y) = {}_aI_y^{\alpha} g(y)$;
If $g(b) = g'(b) = \ldots = g^{(m-1)}(b) = 0$, then ${}_y^CD_b^{\alpha} g(y) = {}_yD_b^{\alpha} g(y)$.

If $g \in C^m[a, b]$, then the right and left Caputo derivatives are continuous on $[a, b]$. There are some properties which are valid for integer integration and integer differentiation which are also reflected in fractional integration and differentiation [18].

3. Piecewise KNR Fractional Order Biquadratic C^2 Spline

Let $P_i(x)$, $i = 1, 2, 3, \ldots, n$, be a piecewise polynomial in a subinterval $[x_i, x_{i+1}]$ for $x \in [x_i, x_{i+1}]$:

$$
\begin{aligned}
P_i(x) &= a_i(x-x_i)^4 + b_i(x-x_i)^3 + c_i(x-x)_i^{\,2} + d_i(x-x_i)\\
&\quad + e_i, \quad i = 0, 1, 2, 3, \ldots, n,\ x \in [x_i, x_{i+1}],
\end{aligned}
\tag{4}
$$

where a_i, b_i, c_i, d_i, and e_i are unknown constants which need to be calculated by means of the given continuity and differentiability conditions:

$$
\begin{aligned}
P_i(x_{i+1}) &= P_{i+1}(x_{i+1}),\\
P_i'(x_{i+1}) &= P_{i+1}'(x_{i+1}),\\
P_i''(x_{i+1}) &= P_{i+1}''(x_{i+1}),\\
P_i^{\alpha}(x_{i+1}) &= -P_{i+1}^{\alpha}(x_{i+1}),\ 1 < \alpha < 2.
\end{aligned}
\tag{5}
$$

The parameter α that appears in the above conditions is known as fractional order derivative. It is quite evident from the given conditions that the resulting piecewise curves will be smooth in each segment and will possess C^2 continuity. The fractional order derivative of a function $f(x) \in AC^n[a, b]$ such that f is absolutely continuous of order α with $n-1 < \alpha \le n$, where n denotes the order of derivative, which is

$$
\left({}_CD_a^{\alpha} f\right)(\xi) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\xi)}{(x-\xi)^{\alpha-n+1}}\mathrm{d}\xi, \quad x > a, \tag{6}
$$

where

$$\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} \mathrm{d}u. \quad (7)$$

Let $P_i(x)$ and $P_{i+1}(x)$ be two piecewise spline polynomials with common point at $x = x_{i+1}$. The application of the above continuity and differentiability conditions will result in ten unknown constants which need to be evaluated for practical applications. Since the spline curve passes through the given data points, it will result in $e_i = y_i$ and $e_{i+1} = y_{i+1}$. The remaining eight unknowns can be calculated by applying Caputo fractional and derivative conditions.

$$\frac{1}{\Gamma(2-\alpha)} \int_t^{x_{i+1}} \frac{p_i''(\tau)}{(x_{i+1}-\tau)^{\alpha-1}} \mathrm{d}\tau = -\frac{1}{\Gamma(2-\alpha)} \int_{x_{i+1}}^{s} \frac{{p_{i+1}}''(\tau)}{(\tau - x_{i+1})^{\alpha-1}} \mathrm{d}\tau, \quad 1 < \alpha \le 2. \quad (8)$$

The given system of linear equations is of the form

$$\begin{aligned} a_i A_{\alpha_j} + b_i B_{\alpha_j} + c_i C_{\alpha_j} &= -\left(a_{i+1} E_{\alpha_j} + b_{i+1} F_{\alpha_j} + c_{i+1} G_{\alpha_j}\right), \\ A_{\alpha_j} &= -\frac{12(-t + x[i+1])^{2-\alpha_j}(K + L - M + N)}{(-4+\alpha_j)(-3+\alpha_j)(-2+\alpha_j)}, \end{aligned} \quad (9)$$

where

$$\begin{aligned} &K = 2x[i+1]^2 + 2x[i]x[i+1](-4+\alpha_j), \ L = x[i]^2(-4+\alpha_j)(-3+\alpha_j), \\ &M = 2t\left(x[i+1] + x[i](-4+\alpha_j)\right)(-2+\alpha_j), \ N = t^2(-3+\alpha_j)(-2+\alpha_j), \\ &B_{\alpha_j} = -\frac{6(-t + x[i+1])^{2-\alpha_j}\left(-x[i+1] - x[i](-3+\alpha_j) + t(-2+\alpha_j)\right)}{(-3+\alpha_j)(-2+\alpha_j)}, \\ &C_{\alpha_j} = -\frac{2(-t + x[i+1])^{2-\alpha_j}}{-2+\alpha_j}, \ E_{\alpha_j} = \frac{12(s - x[i+1])^{4-\alpha_j}}{4-\alpha_j}, \ F_{\alpha_j} = \frac{6(s - x[i+1])^{3-\alpha_j}}{3-\alpha_j}, \\ &G_{\alpha_j} = \frac{2(s - x[i+1])^{2-\alpha_j}}{2-\alpha_j}, \quad j = 1, 2, 3, \text{and } 4. \end{aligned} \quad (10)$$

We will have four linear equations.

The other four linear equations can be derived from continuity and differentiability conditions as follows:

$$\begin{aligned} a_i h_i^4 + b_i h_i^3 + c_i h_i^2 + d_i h_i &= y_{i+1} - y_i, \\ a_{i+1} h_{i+1}^4 + b_{i+1} h_{i+1}^3 + c_{i+1} h_{i+1}^2 + d_{i+1} h_{i+1} &= y_{i+2} - y_{i+1}, \\ 4a_i h_i^3 + 3b_i h_i^2 + 2c_i h_i + d_i &= d_{i+1}, \\ 12a_i h_i^2 + 6b_i h_i + 2c_i &= 2c_{i+1}, \end{aligned} \quad (11)$$

where $h_i = x_{i+1} - x_i$ and $h_{i+1} = x_{i+2} - x_{i+1}$.

The above system of linear equations will give rise to a unique solution of unknowns a_i, b_i, c_i, d_i, a_{i+1}, b_{i+1}, c_{i+1}, and d_{i+1}.

As an example, for a given set of data points, we have a piecewise biquadratic fractional spline curve. In Figures 1 and 2, we have two kinds of curves: one is concave while the other one is convex. The fractional order derivatives used in both curves are given by Table 1. These figures also indicate the potency of the technique at the bending points. We also have a liberty to control the bending due to the introduction of two parameters denoted by t and s.

$$t \in (x_i, x_{i+1}), \ s \in (x_{i+1}, x_{i+2}). \quad (12)$$

They both will serve as shape control parameters. Different choices of these parameters will cause changes in the final shapes. The piecewise curve (Figure 3) shows a C^2 KNR biquadratic fractional spline curve, whereas Figure 4 indicates the exact location of the points and Figure 5 indicates the concentration of the points.

In this method, we have the liberty to modify the path of the curve. Figures 6–9 are good examples of different

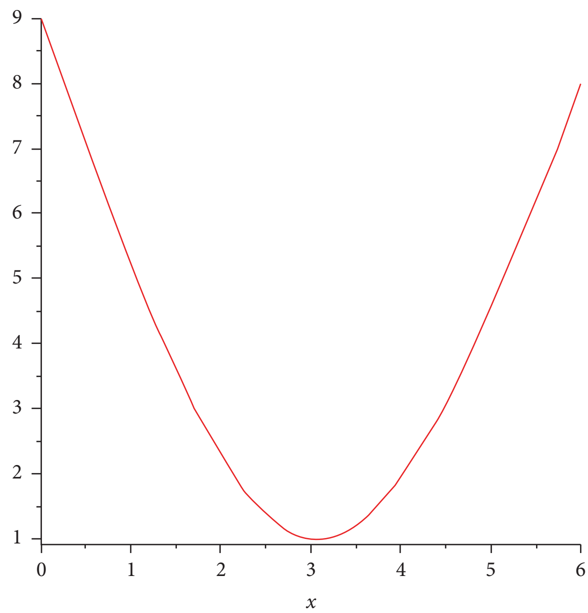

Figure 1: Convex function.

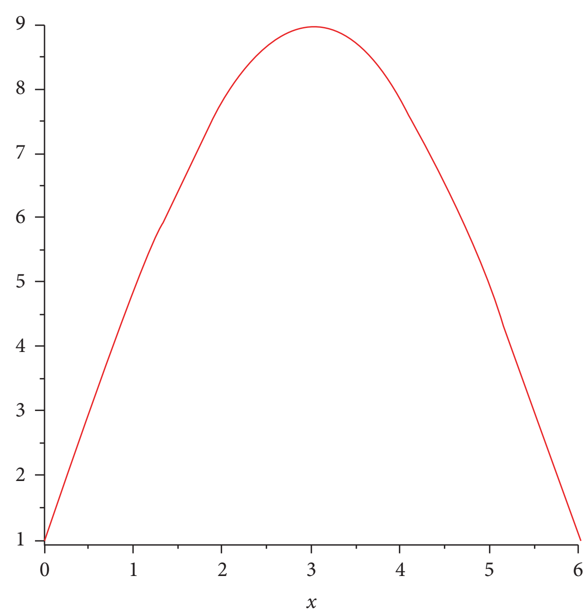

Figure 2: Concave function.

Table 1: Order of fractional derivatives used for both curves.

α_1	α_2	α_3	α_4
1.87	1.9	1.92	1.95

values of shape parameters t and s. As these parameters move away from the connecting point x_{i+1}, the curve starts to flatten at the point and will have effect on the final shape of the curve.

Figures 10–12 indicate the evidence for the effectiveness of the novel technique. The data equally reflect back after application of the newly adopted technique. The straight lines can also be graphed accordingly. Constant

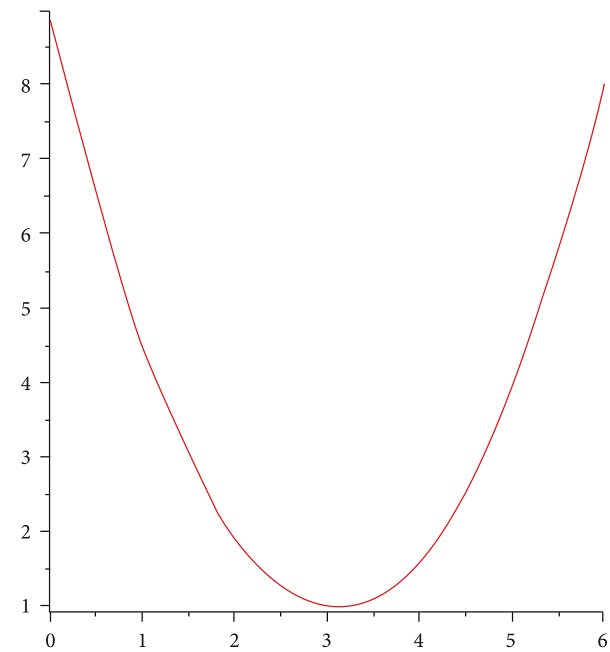

Figure 3: C^2 KNR biquadratic fractional spline curve.

function (in y-values) as shown in Figure 11 and monotone increasing data as shown in Figure 12 can also be preserved, which indicates the accuracy of the technique. In all these shapes, Table 1 is used. Effect on final shape can also be observed if the fractional order derivatives are changed.

4. Comparison of KNR Biquadratic Fractional Spline with Ordinary Cubic Spline

Since ordinary cubic spline is a conventional tool for curve generation, the given comparison indicates that the newly adopted technique coincides with the ordinary one.

For different choices of shape parameters t and s, Figures 13––15 show that the given piecewise curves can be manipulated by the choice of shape parameters. The slight adjustment of the shape parameters can give rise to different shapes. It also indicates that a small change can be made in final shape by altering these parameters.

Geometrically, we have $t \in (x_i, x_{i+1})$ and $s \in (x_{i+1}, x_{i+2})$, which gives us better control on curve's path. Different values of these parameters can change the whole geometry/pattern of the curves. Although the given fractional spline curve will pass through the given data points, but still we can have improved control on the curve.

5. Application of Fractional Spline to n Data Points

Let (x_i, y_i), $i = 0, 1, 2, \ldots, n$, be a set of n data points. Using first three data points, we can find two patches of curves as defined in this paper above. Since all the unknown constants of these two patches are already known, they can be used to find three or more patches of the curves.

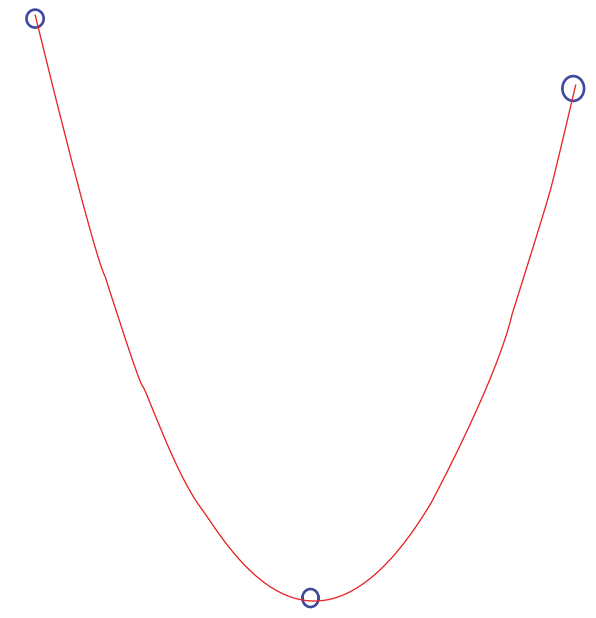

Figure 4: Location of the points.

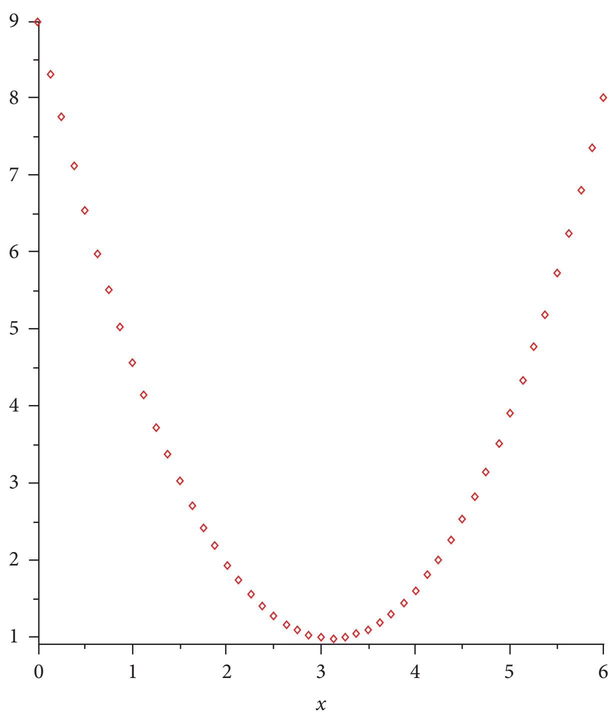

Figure 5: Concentration of the points.

By applying continuity and differentiability conditions, we have the following system of linear equations in three unknowns, namely, a_{i+1}, b_{i+1}, and c_{i+1}.

$$a_{i+1}h_{i+1}^4 + b_{i+1}h_{i+1}^3 + c_{i+1}h_{i+1}^2 = y_{i+2} - y_{i+1} - d_{i+1}h_{i+1},$$
$$a_{i+1}E_{\alpha_j} + b_{i+1}F_{\alpha_j} + c_{i+1}G_{\alpha_j} = -\left(a_iA_{\alpha_j} + b_iB_{\alpha_j} + c_iC_{\alpha_j}\right), \quad j = 1, 2, \tag{13}$$

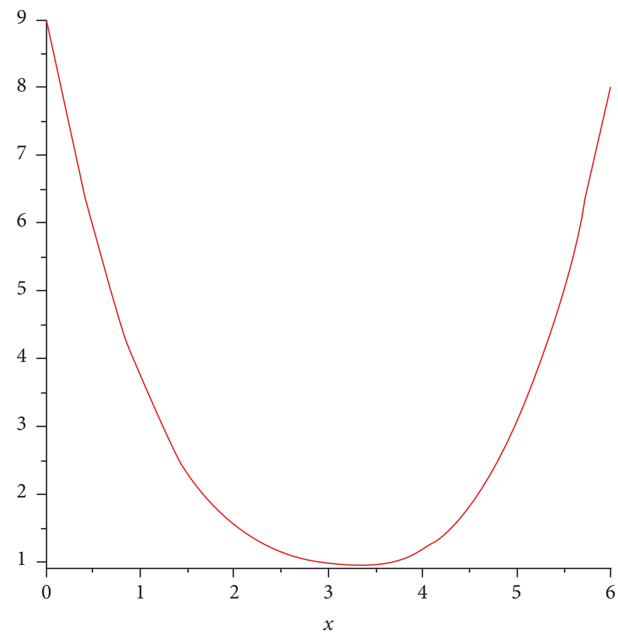

Figure 6: An example of different values of shape parameters t and s.

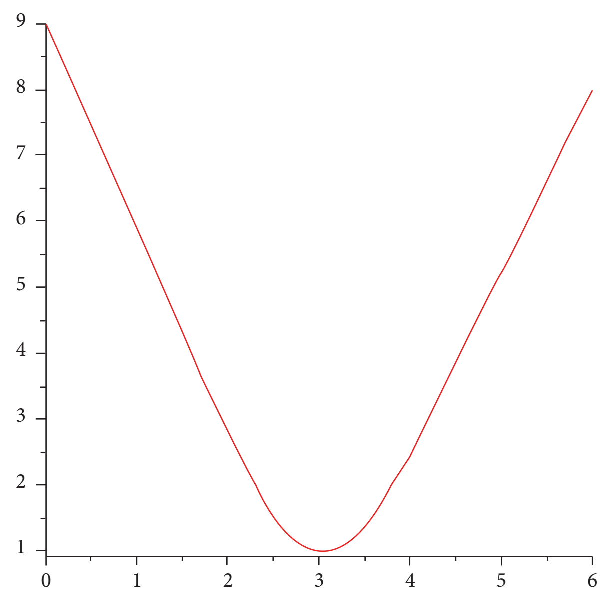

Figure 7: Impact of shape parameters t and as it moves away from connecting point.

where $h_{i+1} = x_{i+2} - x_{i+1}$, $d_{i+1} = 4a_ih_i^3 + 3b_ih_i^2 + 2c_ih_i - d_i$, A_{α_j}, B_{α_j}, C_{α_j}, E_{α_j}, F_{α_j}, and G_{α_j} are already calculated in the previous section.

The above system involves three linear equations for two values of j. In each subsequent segment of curves, we will repeatedly solve the above system for $n-1$ segments of curve. Hence, the above system is true for $i = 1, 2, \ldots, n-1$.

In Figure 16, curve segments in $[x_0, x_1]$ and $[x_1, x_2]$ intervals can easily be calculated by the algorithm as defined prior, whereas the curve segment in interval $[x_2, x_3]$, in

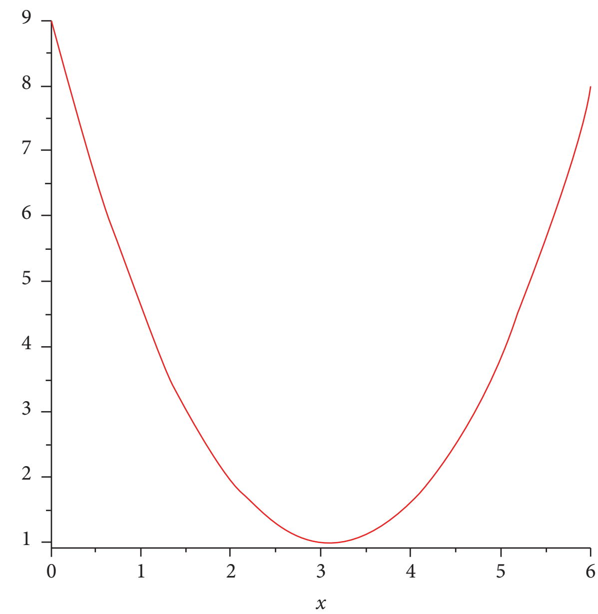

Figure 8: Impact of shape parameters t and as it moves away from connecting point.

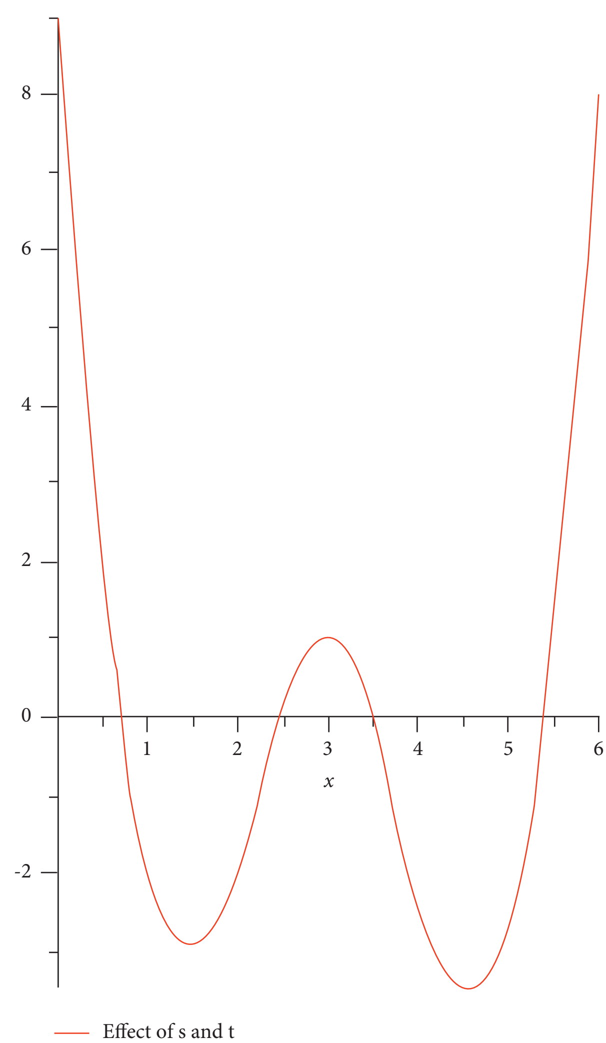

Figure 9: Impact of shape parameters t and as it moves away from connecting point.

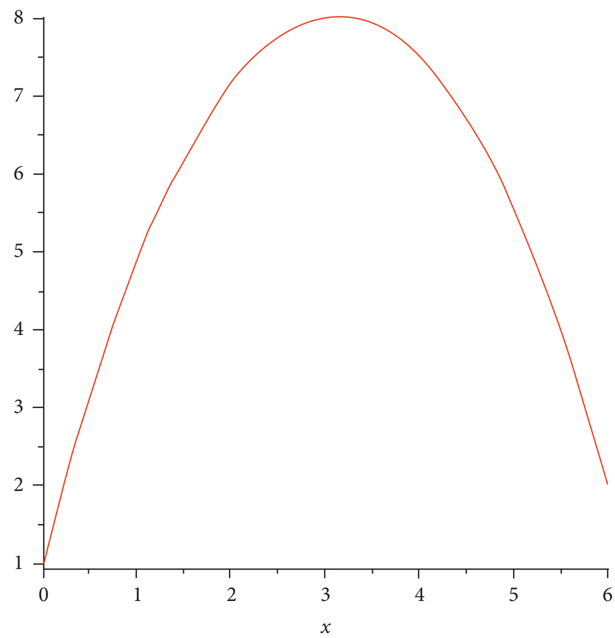

Figure 10: After application of the newly adopted technique.

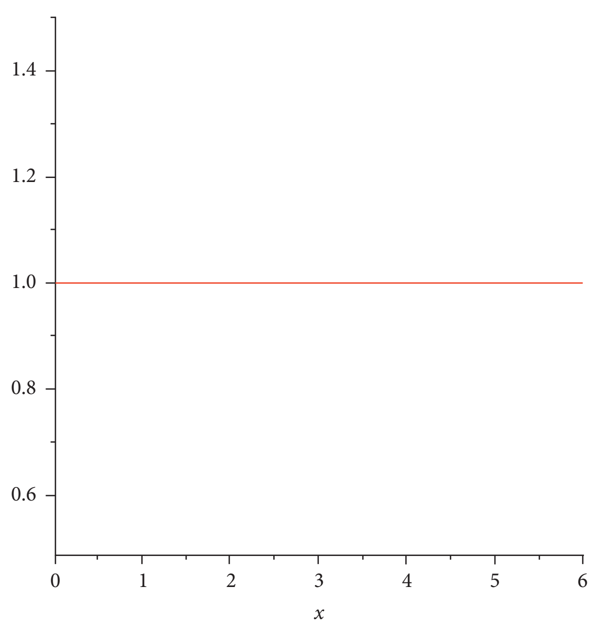

Figure 11: Constant functions are preserved.

which x_2 is the connecting point, can be evaluated by the following way:

$$\begin{aligned} P_2(x_2) &= P_3(x_2), \\ P_2'(x_2) &= P_3'(x_2), \\ P_2''(x_2) &= P_3''(x_2), \\ P_3(x_3) &= y_3, \\ P_2^{\alpha}(x_2) &= -P_3^{\alpha}(x_2). \end{aligned} \tag{14}$$

Here, in polynomial $P_3(x)$, we have five unknowns which can easily be calculated by the abovementioned conditions. Similarly, in Figure 17, one more curve segment is included by aforesaid way.

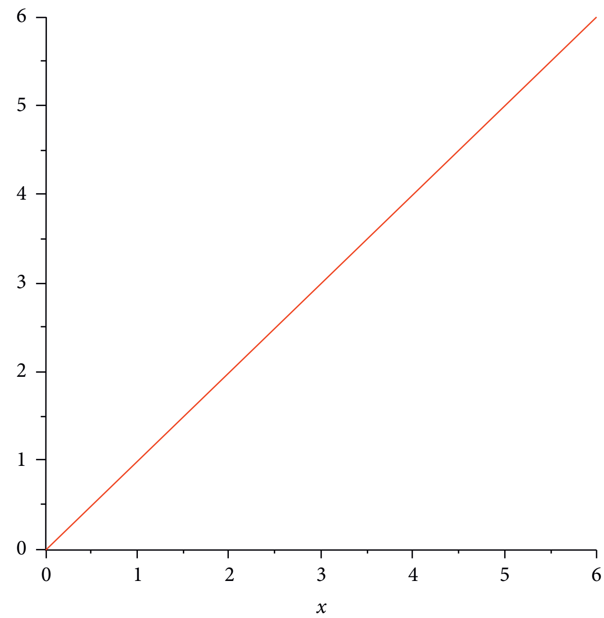

Figure 12: Monotone increasing data are preserved.

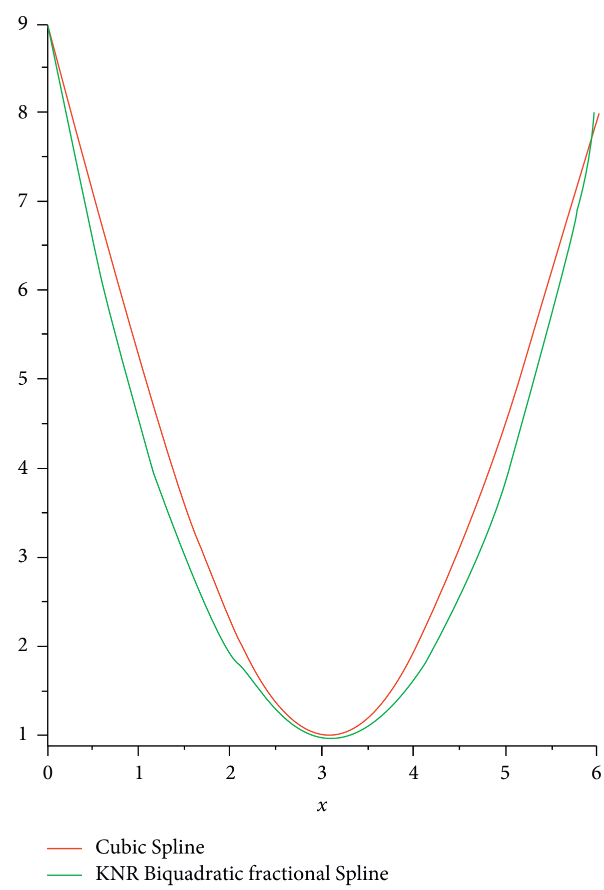

Figure 13: Piecewise curves can be manipulated by the choice of shape parameters 1.

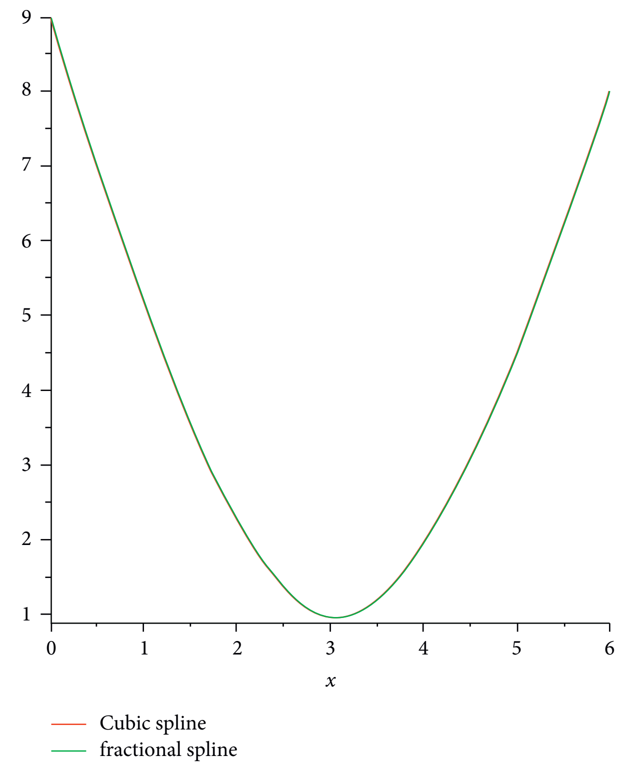

Figure 14: Piecewise curves can be manipulated by the choice of shape parameters 2.

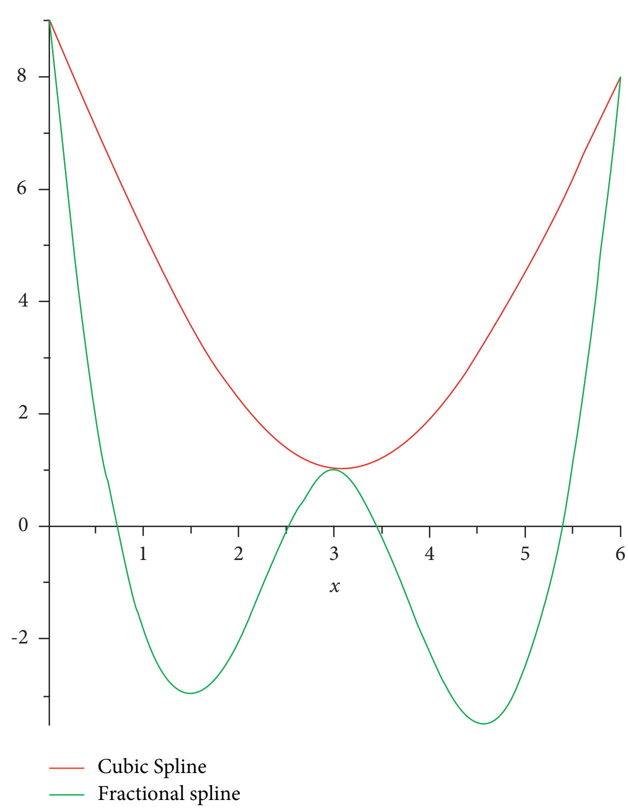

Figure 15: Piecewise curves can be manipulated by the choice of shape parameters 3.

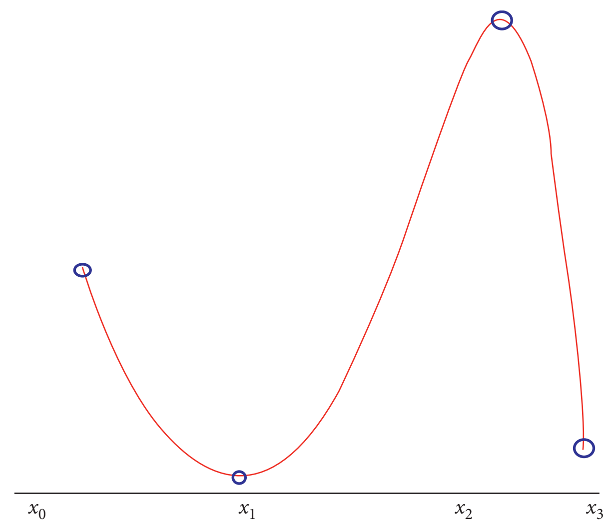

Figure 16: Curve segments.

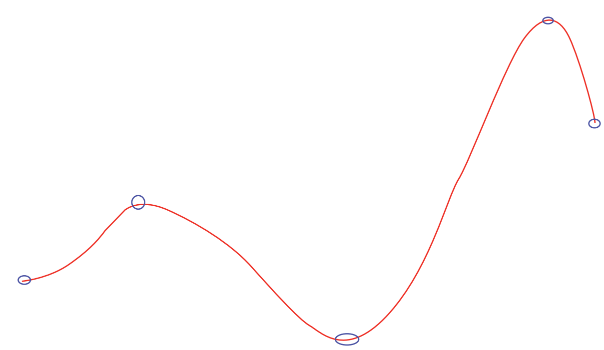

Figure 17: Another curve segment is included.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] R. Almeida, A. B. Malinowska, and T. Odzijewicz, "Fractional differential equations with dependence on the Caputo–Katugampola derivative," *Journal of Computational and Nonlinear Dynamics*, vol. 11, no. 6, p. 61017, 2016.

[2] A. Atangana and J. F. Gómez-Aguilar, "Numerical approximation of Riemann-Liouville definition of fractional derivative: from Riemann-Liouville to Atangana-Baleanu," *Numerical Methods for Partial Differential Equations*, vol. 34, no. 5, pp. 1502–1523, 2018.

[3] C. J. Howe, S. R. Cole, D. J. Westreich, S. Greenland, S. Napravnik, and J. J. Eron, "Splines for trend analysis and continuous confounder control," *Epidemiology*, vol. 22, no. 6, pp. 874-875, 2011.

[4] S. P. Yang and A. G. Xiao, "Cubic spline collocation method for fractional differential equations," *Journal of Applied Mathematics*, vol. 2013, Article ID 864025, 20 pages, 2013.

[5] L. Yan, "Cubic trigonometric nonuniform spline curves and surfaces," *Mathematical Problems in Engineering*, vol. 2016, Article ID 7067408, 9 pages, 2016.

[6] S. Yaghoobi, B. P. Moghaddam, and K. Ivaz, "An efficient cubic spline approximation for variable-order fractional differential equations with time delay," *Nonlinear Dynamics*, vol. 87, no. 2, pp. 815–826, 2017.

[7] R. Almeida and D. F. M. Torres, "Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1490–1500, 2011.

[8] O. J. J. Algahtani, "Comparing the Atangana-Baleanu and Caputo-Fabrizio derivative with fractional order: allen Cahn model," *Chaos, Solitons & Fractals*, vol. 89, pp. 552–559, 2016.

[9] A. Karim, S. Ariffin, and K. Voon Pang, "Local control of the curves using rational cubic spline," *Journal of Applied Mathematics*, vol. 2014, Article ID 872637, 12 pages, 2014.

[10] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science Limited, Amsterdam, Netherlands, 2006.

[11] S. Kirmani, N. Binti Mohd Suaib, and M. Bilal Riaz, "Shape preserving fractional order KNR C1 cubic spline," *The European Physical Journal Plus*, vol. 134, no. 7, p. 319, 2019.

[12] M. M. Khader, K. M. Saad, Z. Hammouch, and D. Baleanu, "A spectral collocation method for solving fractional KdV and KdV-Burgers equations with non-singular kernel derivatives," *Applied Numerical Mathematics*, vol. 161, pp. 137–146, 2021.

[13] T. A. Sulaiman, M. Yavuz, H. Bulut, and H. M. Baskonus, "Investigation of the fractional coupled viscous Burgers' equation involving Mittag-Leffler kernel," *Physica A: Statistical Mechanics and Its Applications*, vol. 527, p. 121126, 2019.

[14] A. Ullah, Z. Ullah, T. Abdeljawad, Z. Hammouch, and K. Shah, "A hybrid method for solving fuzzy Volterra integral equations of separable type kernels," *Journal of King Saud University Science*, vol. 33, no. 1, p. 101246, 2021.

[15] M. Al-Qurashi, S. Rashid, Y. Karaca, Z. Hammouch, D. Baleanu, and Y.-M. Chu, "Achieving more precise bounds based on double and triple integral as proposed by generalized proportional fractional operators in the hilfer sense," *Fractals*, vol. 29, no. 5, p. 2140027, 2021.

[16] W. K. Zahra and S. M. Elkholy, "The use of cubic splines in the numerical solution of fractional differential equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 638026, 16 pages, 2012.

[17] Q. Ding, "Numerical treatment of certain fractional and non-fractional differential equations," Doctoral Dissertation, Doctoral Thesis, Nanyang Technological University, Singapore, 2020.

[18] W. K. Zahra, "Finite-difference technique based on exponential splines for the solution of obstacle problems," *International Journal of Computer Mathematics*, vol. 88, no. 14, pp. 3046–3060, 2011.

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Research Article

Construction of Generalized *k*-Bessel–Maitland Function with Its Certain Properties

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The main motive of this study is to present a new class of a generalized k-Bessel–Maitland function by utilizing the k-gamma function and Pochhammer k-symbol. By this approach, we deduce a few analytical properties as usual differentiations and integral transforms (likewise, Laplace transform, Whittaker transform, beta transform, and so forth) for our presented k-Bessel–Maitland function. Also, the k-fractional integration and k-fractional differentiation of abovementioned k-Bessel–Maitland functions are also pointed out systematically.

1. Introduction and Preliminaries

The computation of fragmentary integrals of special functions is significant from the mark of perspective on the value of these outcomes in the assessment of generalized integrals, and the solution of differential and integral equations. Fractional integral formulas involving the Bessel function have been created and assume a significant part in a few physical problems. The Bessel function is significant in examining the solutions of differential equations, and they are related to a wide scope of problems in numerous regions of mathematical physics, likewise radiophysics, fluid dynamics, and material sciences. These contemplations have driven different specialists in the field of special functions to investigating the possible expansions and also applications for the Bessel function. Valuable speculation of the Bessel function called the k-Bessel function has also been presented by Diaz et al. [1–3] and Suthar et al. [4]. They have presented k-beta, k-gamma, k-zeta functions, and Pochhammer k-symbol (rising factorial). Additionally, they demonstrated some of their properties and inequalities for the above-said functions. They have likewise considered k-hypergeometric functions based on k-rising factorial.

Such functions play a discernible role in a variety of appropriate fields of science and engineering. During the past several years, several researchers have obtained various k-type function (such as k-gamma, k-beta, and k-Pochhammer). This subject has received attention of various researchers and mathematicians during the last few decades. The k symbols are well known from many references related to finite difference calculus (see, [5–11], see additionally [12–16]). Recently, k-type functions and k-type operators have been considered in the literature by various authors. For this purpose, we start with the following properties in the literature.

For our current assessment, we survey here the definition of some known functions and their generalizations. The integral representations of k-gamma and k-beta functions are as follows (see [1–3]):

$$\Gamma_k(x) = k^{(x/k)-1}\Gamma\left(\frac{x}{k}\right) = \int_0^{\infty} t^{x-1}e^{-(t^k/k)}\mathrm{d}t, \quad \Re(x) > 0, k > 0, \tag{1}$$

$$B_k(x, y) = \frac{1}{k}\int_0^1 t^{(x/k)-1}(1-t)^{(y/k)-1}\mathrm{d}t, \quad x > 0, y > 0, \tag{2}$$

where

$$\begin{aligned} B_k(x, y) &= \frac{1}{k}B\left(\frac{x}{k}, \frac{y}{k}\right), \\ B_k(x, y) &= \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \end{aligned} \tag{3}$$

The variety of the functions likewise k-Zeta function, k-Mittag–Leffler function for two and three parameters, k-Wright, and k-hypergeometric functions could be characterized by the following formulas (see also [4, 12, 13, 16–20]):

$$\begin{aligned} \xi_k(z, p) &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(z+nk)^p}, \quad k, z > 0, p > 1, \\ E_{k,\alpha,\beta}(z) &= \sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(\alpha n+\beta)}, \quad \alpha, \beta > 0, \\ E_{k,\alpha,\beta}^{\gamma}(z) &= \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)n!}z^n, \quad k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \\ W_{k,\alpha,\beta}^{\gamma}(z) &= \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)(n!)^2}z^n, \quad k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \\ F_k((\beta, k); (\gamma, k); z) &= \sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{(\gamma)_{n,k}n!}z^n, \quad k \in \mathbb{R}, \beta, \gamma \in \mathbb{C}; \Re(\beta) > 0, \Re(\gamma) > 0. \end{aligned} \tag{4}$$

Definition 1. Let f be a sufficiently well-behaved function with support in $\mathbb{R}^+$ and let α be a real number $\alpha > 0$. The k-Riemann–Liouville fractional integral of order α, $I_+^{\alpha}f$ is given by (see [21–23])

$$I_{k,a}^{\alpha}(f(z)) = \frac{1}{k\Gamma_k(\alpha)}\int_a^z (z-t)^{(\alpha/k)-1}f(t)\mathrm{d}t. \tag{5}$$

This definition unmistakably reduces the definition defined by Mubeen and Habibullah (see [14]):

$$I_k^{\alpha}(f(z)) = \frac{1}{k\Gamma_k(\alpha)}\int_0^z (z-t)^{(\alpha/k)-1}f(t)\mathrm{d}t. \tag{6}$$

It is clear that the case $k = 1$ of (6) yields the traditional Riemann–Liouville fractional integral:

$$I^{\alpha}(f(z)) = \frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}f(t)\mathrm{d}t. \tag{7}$$

Definition 2. Let β be a real number. Then, k-Riemann–Liouville fractional derivative is defined by (see [21–23])

$$D_k^{\beta}(f(t)) = \frac{d}{\mathrm{d}t}I_k^{1-\beta}f(t)\mathrm{d}t, \quad (0 < \beta \leq 1), \tag{8}$$

where

$$I_k^{1-\beta}(f(x)) = \frac{1}{k\Gamma_k(1-\beta)}\int_0^z (z-t)^{(1-\beta/k)-1}f(t)\mathrm{d}t. \tag{9}$$

Definition 3. For $u \in \varphi(R)$, the fractional Fourier transform (FFT) of order α is defined as (see [21–23])

$$\widehat{u}_a(w) = \mathsf{F}_a[u](w) = \int_R e^{iw^{1/\alpha}t}u(t)\mathrm{d}t, \quad (0 < \alpha \leq 1). \tag{10}$$

It is effectively observed that, for $\alpha = 1$, (10) reduces at the conventionally Fourier transform which is given by

$$\mathsf{F}[\varphi](z) = \int_{-\infty}^{+\infty} e^{izt}\varphi(t)\mathrm{d}t. \tag{11}$$

For $w > 0$, (10) easily recovers the FFT presented by Luchko et al. [24].

In 2018, Ghayasuddin and Khan [25] presented generalized Bessel–Maitland functions by

$$J_{\nu,\gamma,\delta}^{\mu,q,p}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}, \tag{12}$$

where $\mu,\nu,\gamma,\delta\in\mathbb{C}$, $\Re(\mu)>0, \Re(\nu)>-1, \Re(\gamma)>0, \Re(\delta)>0$; $p,q>0$, and $q<\Re(\mu)+p$.

For b_j, $j=\overline{1,q}$ different from nonpositive integers, the series (see [26, 27])

$$ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}|z\right]=\sum_{n\geq 0}\frac{(a_1)_n(a_2)_n,\ldots,(a_p)_n}{(b_1)_n(b_2)_n,\ldots,(b_q)_n}\frac{z^n}{n!}=\sum_{n\geq 0}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!} \tag{13}$$

is the generalized hypergeometric series, where the Pochhammer symbol

$$(a)_\mu:=\frac{\Gamma(a+\mu)}{\Gamma(a)}=\begin{cases}1, & \text{if } \mu=0; a\in\mathbb{C}\backslash\{0\},\\ a(a+1),\ldots,(a+n-1), & \text{if } \mu=n\in\mathbb{N}; a\in\mathbb{C},\end{cases} \tag{14}$$

and by convention $(a)_0=1$. When $p\leq q$, the generalized hypergeometric function converges for all complex values of z, that is, ${}_pF_q[z]$ is an entire function. When $p>q+1$, the series converges only for $z=0$, unless it terminates (as when one of the parameters a_j, $j=\overline{1,p}$ is a negative integer) in which case it is just a polynomial in z. When $p=q+1$, the series converges in the open unit disk $|z|<1$ and also for $|z|=1$ provided that

$$\Re\left(\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j\right)>0. \tag{15}$$

The summed up k-Wright function is addressed as follows (see details [7, 27]):

$$\begin{aligned}{}_p\Psi_q^k\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\ (\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix} z\right]&={}_p\Psi_q^k\left(\left(\alpha_j,A_j\right)_{1,p};\left(b_j,\beta_j\right)_{1,q};z\right)\\ &=\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+A_1n),\ldots,\Gamma_k(\alpha_p+A_pn)}{\Gamma_k(\beta_1+B_1n),\ldots,\Gamma_k(\beta_q+B_qn)}\frac{z^n}{n!},\end{aligned} \tag{16}$$

where $k\in\mathbb{R}^+$; $z\in\mathbb{C}$ and

$$\Re\left(\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\right)>0. \tag{17}$$

Motivated essentially by the demonstrated potential for applications of these extended generalized k-Wright hypergeometric functions, we extend the generalized k-Bessel–Maitland function (18) by means of the generalized k-Pochhammer symbol (1) and investigate certain basic properties including differentiation formulas, integral representations, Euler-Beta, Laplace, Whittaker, and fractional Fourier transforms with their several special cases and relations with the k-Bessel–Maitland function. We also derive the k-fractional integration and differentiation of k-Bessel–Maitland function.

2. Generalized k-Bessel–Maitland Function

This section deals with the new development of k-Bessel–Maitland function $J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)$ and its associated properties.

Definition 4. Let $k\in\mathbb{R}$, $\mu,\nu,\gamma,\delta\in\mathbb{C}$, $\Re(\mu)>0$, $\Re(\nu)\geq -1, \Re(\gamma)>0, \Re(\delta)>0$; $p,q>0$, and $q<\Re(\mu)+p$. The generalized k-Bessel–Maitland function is defined as

$$J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}(-z)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}. \tag{18}$$

Remark 1. We note that the case $k = 1$ in (18) leads to the generalized Bessel–Maitland function defined by Ghayasuddin and Khan [25], which further for $\delta = p = 1$ gives the Bessel–Maitland function given by Singh et al. [20].

Theorem 1. *If* $k \in \mathbb{R}$, $\mu, \nu, \gamma, \delta \in \mathbb{C}, \Re(\mu) > 0$, $\Re(\nu) \geq -1, \Re(\gamma) > 0, \Re(\delta) > 0$; $p, q > 0$ *and* $q < \Re(\mu) + p$, *then we have*

$$(\nu+1) J_{k,\nu+k,q,p}^{\mu,\gamma,\delta}(z) + \mu z \frac{d}{dz} J_{k,\nu+k,q,p}^{\mu,\gamma,\delta}(z) = J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) \quad (19)$$

and

$$J_{k,\nu,q,p}^{\mu,\gamma+k,\delta+k}(z) - J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) = z \frac{\delta q (\gamma)_{q,k}}{\gamma p (\delta)_{p,k}} J_{k,\mu+\nu,q,p}^{\mu,\gamma+qk,\delta+pk}(z). \quad (20)$$

Proof. With the help of (18) on the L.H.S of (19), we get

$$\begin{aligned}
&(\nu+1) J_{k,\nu+k,q,p}^{\mu,\gamma,\delta}(z) + \mu z \frac{d}{dz} J_{k,\nu+k,q,p}^{\mu,\gamma,\delta}(z), \\
&= (\nu+1) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn,k}} (-z)^n + \mu z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn,k}} (-z)^n \\
&= (\nu+1) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn,k}} (-z)^n + \mu z \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn,k}} n(-z)^{n-1}
\end{aligned}$$

$$(\nu+1) J_{k,\nu+k,q,p}^{\mu,\gamma,\delta}(z) + \mu z \frac{d}{dz} J_{k,\nu+k,q,p}^{\mu,\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn,k}} (-z)^n (\mu n + \nu + 1). \quad (21)$$

In view of $\Gamma_k(z+k) = z\Gamma_k(z)$, we acquire at our stated result (19).

Using Definition 3 on the L.H.S of (20), we get

$$\begin{aligned}
J_{k,\nu,q,p}^{\mu,\gamma+k,\delta+k}(z) - J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) &= \sum_{n=0}^{\infty} \frac{(\gamma+k)_{qn,k}}{\Gamma_k(\mu n + \nu + 1)(\delta+k)_{pn,k}} (-z)^n - \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} (-z)^n \\
&= \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma_k(\mu n + \nu + 1)} \left[\frac{(\gamma+k)_{qn,k} - (\gamma)_{qn,k}}{(\delta+k)_{pn,k} - (\delta)_{pn,k}} \right].
\end{aligned} \quad (22)$$

Now, by using the result given in [6], we get

$$\begin{aligned}
J_{k,\nu,q,p}^{\mu,\gamma+k,\delta+k}(z) - J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) &= \sum_{n=1}^{\infty} \frac{(-z)^n}{\Gamma_k(\mu n + \nu + 1)} \left[\frac{\delta q n k (\gamma)_{qn,k}}{\gamma p n k (\delta)_{pn,k}} \right] \\
&= \sum_{n=0}^{\infty} \frac{(-z)^{n+1}}{\Gamma_k(\mu(n+1) + \nu + 1)} \left[\frac{\delta q (\gamma)_{q(n+1),k}}{\gamma p (\delta)_{p(n+1),k}} \right].
\end{aligned} \quad (23)$$

Using the result (see [6]), we get

$$J_{k,\nu,q,p}^{\mu,\gamma+k,\delta+k}(z)-J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)=z\frac{\delta q}{\gamma p}\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma_k(\mu(n+1)+\nu+1)}\left[\frac{(\gamma)_{q,k}(\gamma+qk)_{qn,k}}{(\delta)_{p,k}(\delta+pk)_{pn,k}}\right]$$
$$=z\frac{\delta q(\gamma)_{q,k}}{\gamma p(\delta)_{p,k}}\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma_k(\mu(n+1)+\nu+1)}\left[\frac{(\gamma+qk)_{qn,k}}{(\delta+pk)_{pn,k}}\right] \tag{24}$$
$$J_{k,\nu,q,p}^{\mu,\gamma+k,\delta+k}(z)-J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)=z\frac{\delta q(\gamma)_{q,k}}{\gamma p(\delta)_{p,k}}J_{k,\mu+\nu,q,p}^{\mu,\gamma+qk,\delta+pk}(z),$$

which is our stated result (20). □

Theorem 2. *Let $k\in\mathbb{R}$, $\mu,\nu,\gamma,\delta\in\mathbb{C}$, $\Re(\mu)>0$, $\Re(\nu)\geq-1$, $\Re(\gamma)>0$, $\Re(\delta)>0$; $p,q>0$ and $q<\Re(\mu)+p$, then for $m\in\mathbb{N}$, we have*

$$\frac{d}{dz}\left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)\right]=\frac{(\gamma)_{q,k}}{(\delta)_{p,k}}(n+1)J_{k,\mu+\nu,q,p}^{\mu,\gamma+qk,\delta+pk}(z) \tag{25}$$

and

$$\left(\frac{d}{dz}\right)^m J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)=\frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}}(n+1)_m J_{k,\mu m+\nu,q,p}^{\mu,\gamma+qm,\delta+pm}(z). \tag{26}$$

Proof. With the help of (18) on the L.H.S of (25), we get

$$\frac{d}{dz}\left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)\right]=\frac{d}{dz}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}(-z)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}$$
$$=\sum_{n=1}^{\infty}\frac{(\gamma)_{qn,k}(-n)(-z)^{n-1}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{q(n+1),k}(-1)^{n+1}(n+1)(z)^n}{\Gamma_k(\mu n+\mu+\nu+1)(\delta)_{p(n+1),k}} \tag{27}$$
$$\frac{d}{dz}\left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z)\right]=\frac{(\gamma)_{q,k}}{(\delta)_{p,k}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(\gamma+qk)_{qn,k}(n+1)z^n}{\Gamma_k(\mu n+\mu+\nu+1)(\delta+pk)_{pn,k}},$$
$$=\frac{(\gamma)_{q,k}}{(\delta)_{p,k}}(n+1)J_{k,\mu+\nu,q,p}^{\mu,\gamma+qk,\delta+pk}(z),$$

which is our stated result (25).

Now, by using Definition 3 on the L.H.S of (26), we get

$$\left(\frac{d}{dz}\right)^m J^{\mu,\gamma,\delta}_{k,\nu,q,p}(z) = \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}(-z)^n$$
$$= \sum_{n=m}^{\infty} \frac{n(n-1),\ldots,(n-m+1)(\gamma)_{qn,k}(-1)^n z^{n-m}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}$$
$$= \sum_{n=0}^{\infty} \frac{(n+m)(n+m-1),\ldots,(n+1)(\gamma)_{q(n+m),k}(-1)^{(n+m)} z^n}{\Gamma_k(\mu(n+m)+\nu+1)(\delta)_{p(n+m),k}} \tag{28}$$
$$\left(\frac{d}{dz}\right)^m J^{\mu,\gamma,\delta}_{k,\nu,q,p}(z) = \frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}} \sum_{n=0}^{\infty} \frac{(n+1)_m(\gamma+qm)_{qn,k}(-z)^n}{\Gamma_k(\mu(n+m)+\nu+1)(\delta+pm)_{pn,k}}$$
$$= \frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}}(n+1)_m J^{\mu,\gamma+qm,\delta+pm}_{k,\mu m+\nu,q,p}(z),$$

which is our stated result (26). □

3. Integral Transform of a Generalized k-Bessel–Maitland Function

This section manages with some integral transforms likewise Laplace transform, Whittaker transform, beta transform, Hankel transform, K-transform, and fractional Fourier transform as follows.

Theorem 3 (k-beta transform). *Let $k \in \mathbb{R}$, $\mu,\nu,\beta,\gamma,\delta \in \mathbb{C}, \Re(\mu) > 0, \Re(\nu) \geq -1, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\beta) > 0$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have*

$$\frac{1}{\Gamma_k(\beta)} \int_0^1 t^{\nu/k}(1-t)^{(\beta/k)-1} J^{\mu,\gamma,\delta}_{k,\nu,q,p}\left(zt^{\mu/k}\right)dt = J^{\mu,\gamma,\delta}_{k,\nu+\beta,q,p}(z) \tag{29}$$

and

$$\frac{1}{\Gamma_k(\beta)} \int_t^x (x-s)^{(\beta/k)-1}(s-t)^{\nu/k} J^{\mu,\gamma,\delta}_{k,\nu,q,p}\left[z(s-t)^{\mu/k}\right]ds$$
$$= (x-t)^{\beta+\nu/k} J^{\mu,\gamma,\delta}_{k,\nu+\beta,q,p}\left[z(x-t)^{\mu/k}\right]. \tag{30}$$

Proof. By using (18) on the L.H.S of (29) and rearranging in reference to integration and summation (which is ensured under the condition), we acquire

$$\frac{1}{\Gamma_k(\beta)} \int_0^1 t^{\nu/k}(1-t)^{(\beta/k)-1} J^{\mu,\gamma,\delta}_{k,\nu,q,p}\left(zt^{\mu/k}\right)dt = \frac{1}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}(-z)^n \int_0^1 t^{\nu+\mu n/k}(1-t)^{\beta/k-1}dt$$
$$= \frac{1}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}(-z)^n B_k(\mu n+\nu+1,\beta) \tag{31}$$
$$= J^{\mu,\gamma,\delta}_{k,\nu+\beta,q,p}(z),$$

which is our stated result (29).

In the event that we set the transformation $w = s - t/x - t$ on the L.H.S of equation (30) and using Definition 3, we acquire

$$\begin{aligned}
&(x-t)^{\beta+\nu/k}\int_0^1 w^{\nu/k}(1-w)^{(\beta/k)-1}J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left[z(w(x-t))^{\mu/k}\right]\mathrm{d}w\\
&= (x-t)^{\beta+\nu/k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\left[-z(x-t)^{\mu/k}\right]^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}\int_0^1 w^{\mu n+\nu/k}(1-w)^{(\beta/k)-1}\mathrm{d}w\\
&= (x-t)^{\beta+\nu/k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\left[-z(x-t)^{\mu/k}\right]^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}B_k(\mu n+\nu+1,\beta)\\
&= (x-t)^{\beta+\nu/k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\left[-z(x-t)^{\mu/k}\right]^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}\frac{\Gamma_k(\mu n+\nu+1)\Gamma_k(\beta)}{\Gamma_k(\mu n+\beta+\nu+1)},
\end{aligned} \tag{32}$$

which is our stated result (30). □

Theorem 4 (Laplace Transform). *Let* $k\in\mathbb{R}$, $\mu,\nu,\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\Re(\mu)>0$, $\Re(\nu)\geq -1$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $p,q>0$, *and* $q<\Re(\mu)+p$, *then we have*

$$\int_0^{\infty} z^{\alpha-1}e^{-sz}J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(xz^{\beta}\right)\mathrm{d}z = \frac{\Gamma(\delta/k)}{s^{\alpha}\Gamma(\gamma/k)_3}\,\Psi_2\left[\begin{matrix}((\gamma/k),q),\ (\alpha,\beta),\ (1,1) & \\ & \dfrac{-xk^{q-p-\mu/k}}{s^{\beta}}\\ \left(\dfrac{\nu+1}{k},\dfrac{\mu}{k}\right),\left(\dfrac{\delta}{k},p\right) & \end{matrix}\right]. \tag{33}$$

Proof. By using (18) and the definition of Laplace transform

$$L[f(t)] = \int_0^{\infty} e^{-st}f(t)\mathrm{d}t, \tag{34}$$

we get

$$\begin{aligned}
\int_0^{\infty} z^{\alpha-1}e^{-sz}J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(xz^{\beta}\right)\mathrm{d}z &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}(-x)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}\int_0^{\infty}e^{-sz}z^{\alpha+\beta n-1}\mathrm{d}z\\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}(-x)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}\frac{\Gamma(\alpha+\beta n)}{s^{\alpha+\beta n}}
\end{aligned} \tag{35}$$

$$\int_0^{\infty} z^{\alpha-1}e^{-sz}J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(xz^{\beta}\right)\mathrm{d}z = \frac{\Gamma(\delta/k)}{s^{\alpha}\Gamma(\gamma/k)}\sum_{n=0}^{\infty}\frac{k^{qn}\Gamma((\gamma/k)+qn)(-x)^n}{k^{\mu n/k}\Gamma(\mu n+\nu+1/k)k^{pn}\Gamma((\delta/k)+pn)}\frac{\Gamma(\alpha+\beta n)}{s^{\beta n}}.$$

Summing up the above the series with the help of (1), we easily arrive at our stated result (33). □

Theorem 5 (Hankel transform). *If $k \in \mathbb{R}$, $\mu, \nu, \beta, \gamma, \delta, \eta, \lambda \in \mathbb{C}, \Re(\mu) > 0, \Re(\nu) \geq -1, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\lambda) > 0$; $a, b > 0$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have*

$$\int_0^\infty z^{\eta-1} J_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(bz^\beta\right)\mathrm{d}z = \frac{2^{\eta-1}\Gamma(\delta/k)}{a^\eta \Gamma(\gamma/k) k^{(\nu+1/k)-1}} \times {}_3\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k}, q\right), \left(\frac{\lambda+\eta}{2}, \frac{\beta}{2}\right), (1,1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right), \left(1 + \frac{\lambda-\eta}{2}, -\frac{\beta}{2}\right)\end{matrix} \; k^{q-p-\mu/k} - b\left(\frac{2}{a}\right)^\beta\right]. \quad (36)$$

Proof. Applying Definition 3, we have

$$\int_0^\infty z^{\eta-1} J_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(bz^\beta\right)\mathrm{d}z = \sum_{n=0}^\infty \frac{(\gamma)_{qn,k}(-b)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_0^\infty z^{\eta+\beta n-1} J_\lambda(az)\mathrm{d}z. \quad (37)$$

By following the given formula [13],

$$\int_0^\infty t^{s-1} J_\nu(\alpha t)\mathrm{d}t = \frac{2^{s-1}\alpha^{-s}\Gamma(\nu + s/2)}{\Gamma(1 + \nu - s/2)}, \Re(\nu) < \Re(s) < \frac{3}{2}, \alpha > 0, \quad (38)$$

we get

$$\int_0^\infty z^{\eta-1} J_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(bz^\beta\right)\mathrm{d}z = \frac{2^{\eta-1}\Gamma(\delta/k)}{a^\eta\Gamma(\gamma/k)k^{(\nu+1/k)-1}} \cdot \sum_{n=0}^\infty \frac{k^{qn}\Gamma(\gamma/k + qn)(-b)^n}{k^{\mu n/k}\Gamma(\mu n + \nu + 1/k)k^{pn}\Gamma(\delta/k + pn)} \times \frac{2^{\beta n}\Gamma(\lambda + \eta + \beta n/2)}{a^{\beta n}\Gamma(1 + (\lambda - \eta - \beta n/2))}. \quad (39)$$

In view of (16), we get our stated result (36). □

Theorem 6 (K-transform). *Let $k \in \mathbb{R}$, $\mu, \nu, \gamma, \delta, \eta, \lambda \in \mathbb{C}, \Re(\mu) > 0, \Re(\nu) \geq -1, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\eta \pm \lambda) > 0$; $a > 0, b > 0, w > 0$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have*

$$\int_0^\infty z^{\eta-1} K_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(bz^w\right)\mathrm{d}z = \frac{2^{\eta-2}\Gamma(\delta/k)}{a^\eta\Gamma(\gamma/k)k^{\nu+1/k-1}} \times {}_3\Psi_2\left[\begin{matrix}\left(\frac{\gamma}{k}, q\right), \left(\frac{\eta\pm\lambda}{2}, \frac{w}{2}\right), (1,1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right)\end{matrix} \; -bk^{q-p-\mu/k}\left(\frac{2}{a}\right)^w\right]. \quad (40)$$

Proof. Applying Definition 3, we have

$$\int_0^\infty z^{\eta-1} K_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(bz^w\right)\mathrm{d}z = \sum_{n=0}^\infty \frac{(\gamma)_{qn,k}(-b)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_0^\infty z^{\eta+wn-1} K_\lambda(az)\mathrm{d}z. \quad (41)$$

By using the following integral (given in [13])

$$\int_0^\infty x^{\rho-1} K_\nu(x)\mathrm{d}x = 2^{\rho-2}\Gamma\left(\frac{\rho\pm\nu}{2}\right), \quad (42)$$

in the above equation, we arrive at

$$\int_0^\infty z^{\eta-1}K_\lambda(az)J^{\mu,\gamma,\delta}_{k,\nu,q,p}(bz^w)\mathrm{d}z = \frac{2^{\eta-2}\Gamma(\delta/k)}{a^\eta\Gamma(\gamma/k)k^{(\nu+1/k)-1}} \times \sum_{n=0}^{\infty}\frac{k^{qn}\Gamma(\gamma/k+qn)(-b)^n}{k^{\mu n/k}\Gamma(\mu n+\nu+1/k)k^{pn}\Gamma(\delta/k+pn)}\frac{2^{wn}}{a^{wn}}\Gamma\left(\frac{\eta+wn\pm\lambda}{2}\right). \tag{43}$$

In view of (16), we get our stated result (40). □

Theorem 7 (Whittaker transform). *Let* $k\in\mathbb{R}$, $\mu,\nu,\alpha,\gamma,\delta,\lambda,\rho\in\mathbb{C}$, $\Re(\mu)>0$, $\Re(\nu)\geq-1$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\alpha\pm m)>-(1/2)$; $p,q>0$; *and* $q<\Re(\alpha)+p$, *then we have*

$$\int_0^\infty t^{\alpha-1}e^{-st/2}W_{\lambda,m}(st)J^{\mu,\gamma,\delta}_{k,\nu,q,p}(wt^\rho)\mathrm{d}t = \frac{\Gamma(\delta/k)}{s^\alpha\Gamma(\gamma/k)k^{\nu+1/k-1}} \times {}_4\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},q\right),\left(\frac{1}{2}\pm m+\alpha,\rho\right),(1,1)\\ \\ \left(\frac{\nu+1}{k},\frac{\mu}{k}\right),\left(\frac{\delta}{k},p\right),(1-\lambda+\alpha,\rho)\end{matrix}\ \middle|\ k^{q-p-\mu/k}\left(-\frac{w}{s^\rho}\right)\right]. \tag{44}$$

Proof. Applying Definition 3 on the L.H.S of (44) and by setting $st=z$, we get

$$\int_0^\infty e^{-(z/2)}\left(\frac{z}{s}\right)^{\alpha-1}W_{\lambda,m}(z)J^{\mu,\gamma,\delta}_{k,\nu,q,p}\left(w\left(\frac{z}{s}\right)^\rho\right)\frac{\mathrm{d}z}{s} = s^{-\alpha}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}(-w/s^\rho)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}} \times\int_0^\infty e^{-(z/2)}z^{\alpha+\rho n-1}W_{\lambda,m}(z)\mathrm{d}z. \tag{45}$$

By using the following formula (given in [11])

$$\int_0^\infty e^{-(x/2)}x^{\nu-1}W_{\lambda,m}(x)\mathrm{d}x = \frac{\Gamma(1/2\pm m+\nu)}{\Gamma(1-\lambda+\nu)}, \tag{46}$$

we get

$$\int_0^\infty e^{-(z/2)}\left(\frac{z}{s}\right)^{\alpha-1}W_{\lambda,m}(z)J^{\mu,\gamma,\delta}_{\nu,q,p}\left(w\left(\frac{z}{s}\right)^\rho\right)\frac{\mathrm{d}z}{s} = \frac{\Gamma(\delta/k)}{s^\alpha\Gamma(\gamma/k)k^{(\nu+1/k)-1}} \times\sum_{n=0}^{\infty}\frac{k^{qn}\Gamma(\gamma/k+qn)(-w/s^\rho)^n}{k^{\mu n/k}\Gamma(\mu n+\nu+1/k)k^{pn}\Gamma(\delta/k+pn}\frac{\Gamma(1/2\pm m+\alpha+\rho n)}{\Gamma(1-\lambda+\alpha+\rho n)}. \tag{47}$$

In view of (16), we get our stated result. □

Theorem 8. *Let* $k \in \mathbb{R}$, $\mu, \nu, \alpha, \gamma, \delta, \lambda, \rho \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha + m) > -(1/2)$, $p, q > 0$, *and* $q < \Re(\mu) + p$, *then we have*

$$\int_0^\infty t^{\alpha-1} e^{-st/2} M_{\lambda,m}(st) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(wt^\rho)\mathrm{d}t = \frac{\Gamma(2m+1)\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k)\Gamma(m+\lambda+1/2)k^{(\nu+1/k)-1}} \times {}_4\Psi_3\left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), \left(m+\alpha+\frac{1}{2}, \rho\right), (\lambda-\alpha, -\rho), (1,1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right), \left(m-\alpha+\frac{1}{2}, -\rho\right) \end{matrix} \; k^{q-p-\mu/k}\left(-\frac{w}{s^\rho}\right)\right]. \tag{48}$$

Proof. Applying Definition 3 on the L.H.S of (48) and by setting $st = z$, we get

$$\int_0^\infty e^{-(z/2)}\left(\frac{z}{s}\right)^{\alpha-1} M_{\lambda,m}(z) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right)\frac{\mathrm{d}z}{s} = s^{-\alpha}\sum_{n=0}^\infty \frac{(\gamma)_{qn,k}(-w/s^\rho)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}} \int_0^\infty e^{-(z/2)} z^{\alpha+\rho n-1} M_{\lambda,m}(z)\mathrm{d}z. \tag{49}$$

By using the following integral (given in [13])

$$\int_0^\infty e^{-(x/2)} x^{\nu-1} M_{\lambda,m}(x)\mathrm{d}x = \frac{\Gamma(2m+1)\Gamma(m+\nu+1/2)\Gamma(\lambda-\nu)}{\Gamma(m-\nu+1/2)\Gamma(m+\lambda+1/2)}, \tag{50}$$

we get

$$\int_0^\infty e^{-(z/2)}\left(\frac{z}{s}\right)^{\alpha-1} M_{\lambda,m}(z) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right)\frac{\mathrm{d}z}{s} = s^{-\alpha}\sum_{n=0}^\infty \frac{(\gamma)_{qn,k}(-w/s^\rho)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}} \frac{\Gamma(2m+1)\Gamma(m+\alpha+\rho n+1/2)\Gamma(\lambda-\alpha-\rho n)}{\Gamma(m-\alpha-\rho n+1/2)\Gamma(m+\lambda+1/2)}$$

$$\int_0^\infty e^{-(z/2)}\left(\frac{z}{s}\right)^{\alpha-1} M_{\lambda,m}(z) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right)\frac{\mathrm{d}z}{s} = \frac{\Gamma(2m+1)\Gamma(\delta/k)}{s^\alpha\Gamma(\gamma/k)\Gamma(m+\lambda+1/2)k^{(\nu+1/k)-1}} \sum_{n=0}^\infty \frac{k^{qn}\Gamma((\gamma/k)+qn)\left(-(w/s^\rho)\right)^n}{k^{\mu n/k}\Gamma(\mu n+\nu+1/k)k^{pn}\Gamma(\delta/k+pn)} \times \frac{\Gamma(m+\alpha+\rho n+1/2)\Gamma(\lambda-\alpha-\rho n))}{\Gamma(m-\alpha-\rho n+1/2)}. \tag{51}$$

Finally, by applying Definition 1.17, we get our stated result. □

Theorem 9. *Let* $k \in \mathbb{R}$, $\mu, \nu, \alpha, \gamma, \delta, \rho \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha) > -1$, $\Re(\alpha/2 \pm m) > -1$, $\Re(\alpha/2 \pm \lambda) > -1$; $p, q > 0$; *and* $q < \Re(\mu) + p$, *then we have*

$$\int_0^\infty t^{\alpha-1} W_{\lambda,m}(st) W_{-\lambda,m}(st) J^{\mu,\gamma,\delta}_{k,\nu,q,p}(wt^\rho)\mathrm{d}t = \frac{\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k) k^{(\nu+1/k)-1}} {}_5\Psi_4\left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), \left(\frac{\alpha}{2} \pm m + 1, \frac{\rho}{2}\right), (\alpha+1,\rho), (1,1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right), \left(1 + \frac{\alpha}{2} \pm \lambda, \frac{\rho}{2}\right)\end{matrix} \; k^{q-p-\mu/k}\left(-\frac{w}{s^\rho}\right)\right]. \quad (52)$$

Proof. Applying Definition 3 on the L.H.S of (44) and by setting $st = z$, we get

$$\int_0^\infty \left(\frac{z}{s}\right)^{\alpha-1} W_{\lambda,m}(z) W_{-\lambda,m}(z) J^{\mu,\gamma,\delta}_{k,\nu,q,p}\left(w\left(\frac{z}{s}\right)^\rho\right)\frac{\mathrm{d}z}{s} = s^{-\alpha}\sum_{n=0}^\infty \frac{(\gamma)_{qn,k}\left(-w/s^\rho\right)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \cdot \int_0^\infty z^{\alpha+\rho n-1} W_{\lambda,m}(z) W_{-\lambda,m}(z)\mathrm{d}z. \quad (53)$$

By using the integral given in [11]

$$\int_0^\infty x^{\nu-1} W_{\lambda,m}(x) W_{-\lambda,m}(x)\mathrm{d}x = \frac{\Gamma((\nu+1/2)\pm m)\Gamma(\nu+1)}{\Gamma(1+(\nu/2)\pm\lambda)}, \quad (54)$$

we get

$$= \frac{\Gamma(\delta/k)}{s^\alpha\Gamma(\gamma/k)k^{\nu+1/k-1}}\sum_{n=0}^\infty \frac{k^{qn}\Gamma(\gamma/k+qn)\left(-w/s^\rho\right)^n}{k^{\mu n/k}\Gamma(\mu n+\nu+1/k)k^{pn}\Gamma(\delta/k+pn)} \cdot \frac{\Gamma(\alpha+\rho n+1/2\pm m)\Gamma(\alpha+\rho n+1)}{\Gamma(1+(\alpha+\rho n/2)\pm\lambda)}. \quad (55)$$

Now, by summing up the above series with the help of (16), we get our stated result. □

Theorem 10 (fractional Fourier transform). *The FFT of the generalized k-Bessel–Maitland function for $t<0$ is given by*

$$\mathsf{F}_\alpha\left[J^{\mu,\gamma,\delta}_{k,\nu,q,p}(z)\right] = \sum_{n=0}^\infty \frac{n!(\gamma)_{qn,k} i^{-n-1} w^{-(n+1)/\alpha}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}. \quad (56)$$

Proof. From (11) and (18), we have

$$\mathsf{F}_\alpha\left[J^{\mu,\gamma,\delta}_{k,\nu,q,p}(z)\right] = \int_R e^{iw^{1/\alpha}z}\sum_{n=0}^\infty \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}(-z)^n\mathrm{d}z = \sum_{n=0}^\infty \frac{(\gamma)_{qn,k}(-1)^n}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}\int_{-\infty}^0 e^{iw^{1/\alpha}z} z^n\mathrm{d}z. \quad (57)$$

On changing variables $iw^{1/\alpha}z = -t$ and $iw^{1/\alpha}\mathrm{d}z = -\mathrm{d}t$, we arrive at

$$\mathsf{F}_\alpha\left[J^{\mu,\gamma,\delta}_{k,\nu,q,p}(z)\right] = \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} i^{-n-1} w^{-(n+1)/\alpha}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}\int_0^\infty e^{-t}t^n\mathrm{d}t = \sum_{n=0}^\infty \frac{n!(\gamma)_{qn,k} i^{-n-1} w^{-(n+1)/\alpha}}{\Gamma_k(\mu n+\nu+1)(\delta)_{pn,k}}, \quad (58)$$

which is our stated result (56). □

4. K-Fractional Integration and K-Fractional Differentiation

Recently, k-fractional calculus gained more attention due to its wide variety of applications in various fields [14, 17]. The k-fractional calculus of various types of special functions is used in many research papers [4, 28]. For more details about the recent works in the field of dynamic system theory, stochastic systems, non-equilibrium statistical mechanics, and quantum mechanics, we refer the interesting readers to [9, 17, 24]. In this section, we deduce the outcomes for k-fractional integration and k-fractional differentiation of the above-said function in an orderly way.

Theorem 11 (K-fractional integration). *If $k,\eta \in \mathbb{R}$; $\gamma,\delta,\mu,\nu, \in \mathbb{C}$, $\Re(\gamma)>0, \Re(\delta)>0, \Re(\mu)>0, \Re(\nu)\geq -1$, $p,q>0$, and $q<\Re(\mu)+p$, then*

$$I_k^{\eta}\left[z^{(\nu/k)-1}J_{k,\nu-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right)\right]=z^{(\nu+\eta/k)-1}J_{k,\nu+\eta-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right). \tag{59}$$

Proof. From (9) and (18), we have

$$\begin{aligned}I_k^{\eta}\left[z^{\nu/k-1}J_{k,\nu-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right)\right]&=\frac{1}{k\Gamma_k(\eta)}\int_0^z(z-t)^{\eta/k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}t^{(\mu n+\nu/k)-1}\mathrm{d}t\\&=\frac{1}{k\Gamma_k(\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}\int_0^z(z-t)^{(\eta/k)-1}t^{(\mu n+\nu/k)-1}\mathrm{d}t.\end{aligned} \tag{60}$$

Putting $t=zx$ and $\mathrm{d}t=z\mathrm{d}x$ in the above equation, we get

$$\begin{aligned}&=\frac{1}{k\Gamma_k(\eta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}z^{(\mu n+\nu+\eta/k)-1}\int_0^1(1-x)^{(\eta/k)-1}x^{(\mu n+\nu/k)-1}\mathrm{d}x\\&=\frac{1}{k\Gamma_k(\eta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}z^{\mu n+\nu+\eta/k-1}B\left(\frac{\eta}{k},\frac{\mu n+\nu}{k}\right)\\&=\frac{1}{k\Gamma_k(\eta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}z^{\mu n+\nu+\eta/k-1}\frac{\Gamma(\eta/k)\Gamma(\mu n+\nu/k)}{\Gamma(\mu n+\nu+\eta/k)}.\end{aligned} \tag{61}$$

By using Definition 2, we have

$$\begin{aligned}I_k^{\eta}\left[z^{(\nu/k)-1}J_{k,\nu-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right)\right]&=\frac{1}{k\Gamma_k(\eta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}z^{(\mu n+\nu+\eta/k)-1}\frac{k^{(\mu n+\nu+\eta/k)-1}\Gamma_k(\eta)\Gamma_k(\mu n+\nu)}{k^{(\mu n+\nu+\eta/k)-2}\Gamma_k(\mu n+\nu+\eta)}\\I_k^{\eta}\left[z^{(\nu/k)-1}J_{k,\nu-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right)\right]&=z^{(\nu+\eta/k)-1}J_{k,\nu+\eta-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right),\end{aligned} \tag{62}$$

which is our stated result. □

Theorem 12 (K-fractional differentiation). *If* $k,\eta\in\mathbb{R}$; $\gamma,\delta,\mu,\nu\in\mathbb{C}$, $\Re(\gamma)>0,\Re(\delta)>0,\Re(\mu)>0,\Re(\nu)\geq-1$, $p,q>0$, *and* $q<\Re(\mu)+p$, *then*

$$D_k^{\eta}\left[z^{(\nu/k)-1}J_{k,\nu-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right)\right]=\frac{z^{(\nu-\eta+1/k)-2}}{k}J_{k,\nu-\eta-k-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right). \tag{63}$$

Proof. From (8), (9), and (18), we have

$$\begin{aligned}D_k^{\eta}\left[z^{\nu/k-1}J_{k,\nu-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right)\right]&=\frac{1}{k\Gamma_k(1-\eta)}\frac{d}{\mathrm{d}z}\int_0^z(z-t)^{(1-\eta/k)-1}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}t^{(\mu n+\nu/k)-1}\mathrm{d}t\\&=\frac{1}{k\Gamma_k(1-\eta)}\frac{d}{\mathrm{d}z}\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}}\int_0^z(z-t)^{(1-\eta/k)-1}t^{(\mu n+\nu/k)-1}\mathrm{d}t.\end{aligned} \tag{64}$$

Putting $t=zx$ and $\mathrm{d}t=z\mathrm{d}x$ in the above equation, we get

$$= \frac{1}{k\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}} \frac{d}{dz} z^{(\mu n+\nu-\eta+1/k)-1} \int_0^1 (1-x)^{(1-\eta/k)-1} x^{(\mu n+\nu/k)-1} dx$$

$$= \frac{1}{k\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}} \frac{d}{dz} z^{\mu n+\nu-\eta+1/k-1} B\left(\frac{1-\eta}{k}, \frac{\mu n+\nu}{k}\right) \qquad (65)$$

$$= \frac{1}{k\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}} \left(\frac{\mu n+\nu-\eta+1}{k}-1\right) z^{\mu n+\nu-\eta+1/k-2} \frac{\Gamma(1-\eta/k)\Gamma(\mu n+\nu/k)}{\Gamma(\mu n+\nu-\eta+1/k-1+1)}.$$

Using Definition 2 and the result $\Gamma(n+1) = n\Gamma(n)$ in the above expression, we get

$$= \frac{1}{\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu)(\delta)_{pn,k}} \frac{(\mu n+\nu-\eta+1/k-1)}{(\mu n+\nu-\eta+1/k-1)} z^{\mu n+\nu-\eta+1/k-2}$$

$$\cdot \frac{k^{(\mu n+\nu-\eta+1-k/k)-1} \Gamma_k(1-\eta)\Gamma_k(\mu n+\nu)}{k^{(\mu n+\nu-\eta+1-k/k)-1} \Gamma_k(\mu n+\nu-\eta+1-k)} \qquad (66)$$

$$= \frac{z^{(\nu-\eta+1/k)-2}}{k} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n+\nu-\eta-k+1)(\delta)_{pn,k}} z^{\mu n/k}$$

$$D_k^{\eta}\left[z^{\nu/k-1} J_{k,\nu-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right)\right] = \frac{z^{(\nu-\eta+1/k)-2}}{k} J_{k,\nu-\eta-k-1,q,p}^{\mu,\gamma,\delta}\left(z^{\mu/k}\right).$$

This completes the proof. □

5. Concluding Remarks

In the present article, we have established generalized k-Bessel–Maitland function $J_{k,\nu,\delta,p}^{\mu,\gamma,q}(z)$ and its intriguing properties. Also, we have pointed out several integral transform such as beta transform, Laplace transform, Whittaker transform, K-transform, and fractional Fourier transform. In the last section, we deduced the outcomes for k-fractional integration and k-fractional differentiation of k-Bessel–Maitland function. Various special cases of the papers related results may be analyzed by taking appropriate values of the relevant parameters. For example, as given in Remarks 1.5, 1.6, and 1.7, we obtain the undeniable result due to Nisar et al. [15]. For several other special cases, we refer to [4, 12, 23, 24, 26, 28, 29] and leave the findings to interested readers.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] R. Diaz and C. Teruel, "q, k-Generalized gamma and beta functions," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 1, pp. 118–134, 2005.

[2] R. Diaz and E. Pariguan, "On hypergeometric functions and Pochhammer k-symbol," *Divulgaciones Matemáticas*, vol. 15, no. 2, p. 179192, 2007.

[3] R. Diaz, C. Ortiz, and E. Pariguan, "On the k-gamma, q-distribution," *Central European Journal of Mathematics*, vol. 8, no. 3, pp. 448–458, 2010.

[4] D. L. Suthar, A. M. Khan, A. Alaria, S. D. Purohit, and J. Singh, "Extended Bessel-Maitland function and its properties pertaining to integral transforms and fractional calculus," *AIMS Mathematics*, vol. 5, no. 2, pp. 1400–1410, 2020.

[5] G. A. Dorrego and R. Cerutti, "The k-Mittag Leffler function," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 15, pp. 705–716, 2012.

[6] A. Gupta and C. L. Parihar, "k-new generalized Mittag-Leffler function," *Journal of Fractional Calculus and Applications*, vol. 5, no. 1, pp. 165–176, 2014.

[7] K. S. Gehlot and J. C. Prajapati, "Fractional calculus of generalized K-Wright function," *Journal of Fractional Calculus and Applications*, vol. 4, no. 2, pp. 283–289, 2013.

[8] C. G. Kokologiannaki, "Properties and inequalities of generalized k-gamma, beta and zeta functions," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 14, pp. 653–660, 2010.
[9] V. Krasniqi, "A limit for the k-gamma and k-beta function," *International Mathematical Forum*, vol. 5, no. 33, pp. 1613–1617, 2010.
[10] W. A. Khan, K. S. Nisar, and M. Ahmad, "Euler type integral operator involving Mittag-Leffler function," *Boletim da Sociedade Paranaense de Matemática*, vol. 38, no. 5, pp. 165–174, 2020.
[11] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function Theory and Applications*, Springer, Berlin, Germany, 2010.
[12] M. Mansour, "Determining the k-generalized gamma function $k(x)$ by functional equations," *International Journal of Contemporary Mathematical Sciences*, vol. 4, no. 21, pp. 1037–1042, 2009.
[13] P. O. Mohammed, H. Aydi, A. Kashuri, Y. S. Hamed, and K. M. Abualnaja, "midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels," *Symmetry*, vol. 13, p. 550, 2021.
[14] S. Mubeen and G. M. Habibullah, "k-Fractional integrals and application," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 2, pp. 89–94, 2012.
[15] K. S. Nisar, G. Rahman, D. Baleanu, S. Mubeen, and M. Arshad, "The (k, s)-fractional calculus of the k-Mittag-Leffler function," *Advances in Difference Equations*, vol. 118, pp. 1–12, 2017.
[16] L. Romero and R. Cerutti, "Fractional calculus of a k-Wrigt type function," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 31, pp. 1547–1557, 2012.
[17] F. Merovci, "Power product inequalities for the k function," *International Journal of Mathematical Analysis*, vol. 4, no. 21, pp. 1007–1012, 2010.
[18] L. Romero, R. Cerutti, and G. Dorrego, "k-Weyl fractional integral," *International Journal of Mathematical Analysis*, vol. 6, no. 34, pp. 1685–1691, 2012.
[19] L. Romero, R. Cerutti, and G. Dorrego, "The k-Bessel function of the first kind," *International Mathematical Forum*, vol. 7, no. 38, pp. 1859–1864, 2012.
[20] M. Singh, M. A. Khan, and A. H. Khan, "On some properties of a generalization of Bessel-Maitland function," *International Journal of Mathematics Trends and Technology*, vol. 14, no. 1, pp. 46–54, 2014.
[21] L. Romero and R. Cerutti, "Fractional Fourier transform and special functions," *International Journal of Contemporary Mathematical Sciences*, vol. 17, no. 14, pp. 693–704, 2012.
[22] L. Romero, R. Cerutti, and L. Luque, "A new fractional Fourier transform and convolutions products," *International Journal of Pure and Applied Mathematics*, vol. 66, pp. 397–408, 2011.
[23] L. Romero, R. Cerutti, L. Luque, G. Dorrego, and R. Creutti, "On the k-Riemann-Liouville fractional derivative," *International Journal of Contemporary Mathematical Sciences*, vol. 8, no. 1, pp. 41–51, 2013.
[24] Y. Luchko, H. Martinez, and J. Trujillo, "Fractional Fourier transform and some of its applications," *Fractional Calculus Application and Analysis*, vol. 11, no. 4, pp. 457–470, 2008.
[25] M. Ghayasuddin and W. A. Khan, "A new generalized Bessel-Maitland function and its properties," *Matematicki Vesnik*, vol. 70, no. 4, pp. 292–302, 2018.
[26] O. I. Marichev, *"Handbook of Integral Transform and Higher Transcendental Functions," Theory and Algorithm Tables*, Ellis Horwood Chichester, New York, NY, USA, 1983.
[27] E. D. Rainville, *Special Functions*, Macmillan, New York, NY, USA, 1960.
[28] Y. Singh and R. S. Dubey, "Fractional Calculus operator with generalize k-Mittag-Leffler function," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 2, pp. 545–553, 2020.
[29] G. Rahman, K. S. Nisar, T. Kim, S. Mubeen, and M. Arshad, "The extended k-Mittag-Leffler function and its properties," *Proceedings of the Jangjeon Mathematical Society*, vol. 21, no. 3, pp. 487–495, 2018.

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Research Article

Sine Half-Logistic Inverse Rayleigh Distribution: Properties, Estimation, and Applications in Biomedical Data

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A new lifetime distribution with two parameters, known as the sine half-logistic inverse Rayleigh distribution, is proposed and studied as an extension of the half-logistic inverse Rayleigh model. The sine half-logistic inverse Rayleigh model is a new inverse Rayleigh distribution extension. In the application section, we show that the sine half-logistic inverse Rayleigh distribution is more flexible than the half-logistic inverse Rayleigh and inverse Rayleigh distributions. The statistical properties of the half-logistic inverse Rayleigh model are calculated, including the quantile function, moments, moment generating function, incomplete moment, and Lorenz and Bonferroni curves. Entropy measures such as Rényi entropy, Havrda and Charvat entropy, Arimoto entropy, and Tsallis entropy are proposed for the sine half-logistic inverse Rayleigh distribution. To estimate the sine half-logistic inverse Rayleigh distribution parameters, statistical inference using the maximum likelihood method is used. Applications of the sine half-logistic inverse Rayleigh model to real datasets demonstrate the flexibility of the sine half-logistic inverse Rayleigh distribution by comparing it to well-known models such as half-logistic inverse Rayleigh, type II Topp–Leone inverse Rayleigh, transmuted inverse Rayleigh, and inverse Rayleigh distributions.

1. Introduction

In recent years, inverse and half-inverse problems are studied in general operator theory [1–3], and many statisticians are focusing on generated families of distributions such as Kumaraswamy-G [4], T-X family [5], sine-G [6], type II half logistic-G [7], Weibull-G [8], the Burr type X-G [9], a new power Topp–Leone-G [10], truncated Cauchy power-G [11], beta generalized Marshall–Olkin–Kumaraswamy-G [12], transmuted odd Fréchet-G [13], new Kumaraswamy-G [14], Kumaraswamy Kumaraswamy-G [15], generalized Kumaraswamy-G [16], sine Topp–Leone-G [17], generalized transmuted exponentiated G [18], and Kumaraswamy transmuted-G [19].

A new generated family of distributions which is called the Sine-G (SG) family was introduced in [6]. The distribution function (CDF) of SG is

$$F(x;\varepsilon) = \sin\left[\frac{\pi}{2}G(x;\varepsilon)\right], \quad x \in R, \tag{1}$$

where $G(x;\varepsilon)$ is the CDF of baseline model with parameter vector ε and $F(x;\varepsilon)$ is the CDF derived by the T-X generator proposed in [3]. The probability density function (PDF) of the SG is

$$f(x;\ \varepsilon) = \frac{\pi}{2}g(x;\varepsilon)\cos\left[\frac{\pi}{2}G(x;\varepsilon)\right], \quad x \in R, \tag{2}$$

respectively.

The inverse Rayleigh (IR) distribution is a useful model for calculating lifetimes. Several authors have developed a number of extensions for the IR distribution in recent years, using various methods of generalization (see, for example, beta IR in [20], transmuted IR (TIR) in [21], modified IR in [22], transmuted modified IR in [23], Kumaraswamy exponentiated IR in [24], weighted IR in [25], odd Fréchet IR in [26], and half-logistic IR (HLIR) in [27]).

The CDF and PDF of HLIR distribution are given by

$$G(x;\lambda,\alpha)=\frac{1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}{1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}},\quad x,\lambda,\alpha>0, \tag{3}$$

and

$$g(x;\lambda,\alpha)=\frac{4\lambda\alpha^2x^{-3}e^{-(\alpha/x)^2}\left[1-e^{-(\alpha/x)^2}\right]^{\lambda-1}}{\left(1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)^2},\quad x,\lambda,\alpha>0, \tag{4}$$

where α is a scale parameter and λ is a shape parameter.

We now present the sine half-logistic IR (SHLIR) distribution, a new lifetime model with two parameters. Inserting (3) into (1) yields the cdf of the SHLIR distribution as

$$F(x;\lambda,\alpha)=\sin\left[\frac{\pi}{2}\left(\frac{1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}{1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}\right)\right],\quad x,\lambda,\alpha>0. \tag{5}$$

The corresponding PDF to (5) is

$$f(x;\lambda,\alpha)=\frac{2\pi\lambda\alpha^2x^{-3}e^{-(\alpha/x)^2}\left[1-e^{-(\alpha/x)^2}\right]^{\lambda-1}}{\left(1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)^2}\cos\left[\frac{\pi}{2}\left(\frac{1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}{1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}\right)\right],\quad x,\lambda,\alpha>0. \tag{6}$$

The SHLIR distribution's survival function (SF), hazard rate function (HRF), reversed HRF, and cumulative HRF are as follows:

$$R(x;\lambda,\alpha)=1-\sin\left[\frac{\pi}{2}\left(\frac{1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}{1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}\right)\right], \tag{7}$$

$$h(x;\lambda,\alpha)=\frac{2\pi\lambda\alpha^2x^{-3}e^{-(\alpha/x)^2}\left[1-e^{-(\alpha/x)^2}\right]^{\lambda-1}\cos\left[(\pi/2)\left(\left(1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)/1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)\right]}{\left(1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)^2\left[1-\sin\left[(\pi/2)\left(\left(1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)/\left(1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)\right)\right]\right]}, \tag{8}$$

$$\tau(x;\lambda,\alpha)=\frac{2\pi\lambda\alpha^2x^{-3}e^{-(\alpha/x)^2}\left[1-e^{-(\alpha/x)^2}\right]^{\lambda-1}}{\left(1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)^2}\cot\left[\frac{\pi}{2}\left(\frac{1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}{1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}\right)\right], \tag{9}$$

$$H(x;\lambda,\alpha) = -\ln\left[1-\sin\left[\frac{\pi}{2}\left(\frac{1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}{1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}}\right)\right]\right]. \tag{10}$$

Figures 1 and 2 show plots of the SHLIR PDF and HRF for various parameter values.

We can conclude from Figures 1 and 2 that the PDF of the SHLIR distribution can be unimodel and right skewed. SHLIR distribution HRF can be J-shaped and increasing.

The remainder of this paper is structured as follows. Section 2 discusses some of the structural characteristics of the SHLIR distribution, such as the quantile function, moments, incomplete moments, Lorenz and Bonferroni curves, and various measures of entropy. Section 3 discusses maximum likelihood (ML) parameter estimators for the SHLIR distribution. Section 4 implements simulation schemes. In Section 5, two sets of real-world data applications are used to demonstrate the potential of the SHLIR distribution in comparison to other distributions. The article concludes with some closing remarks.

2. Statistical Characteristics

Some statistical properties of the SHLIR distribution are obtained in this section.

2.1. Linear Representation. In this section, we will go over the most important linear PDF combinations for SHLIR distribution.

The sine function's series:

$$\sin(Z) = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}Z^{2i}. \tag{11}$$

By inserting (11) in (6), we get

$$f(x) = \sum_{i=0}^{\infty}\frac{(-1)^i\pi^{2i+1}}{(2i)!2^{2i-1}}\frac{\lambda\alpha^2x^{-3}e^{-(\alpha/x)^2}\left[1-e^{-(\alpha/x)^2}\right]^{\lambda-1}\left[1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right]^{2i}}{\left(1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)^{2(i+1)}}, \tag{12}$$

where $f(x) = f(x;\lambda,\alpha)$.

Consider the following well-known binomial expansions (for $0 < a < 1$):

$$(1+a)^{-n} = \sum_{j=0}^{\infty}(-1)^i\binom{n+j-1}{j}a^j. \tag{13}$$

Thus, inserting (13) in (12), we get

$$f(x) = \sum_{i,j=0}^{\infty}\frac{\lambda\alpha^2(-1)^{i+j}\pi^{2i+1}}{x^3(2i)!2^{2i-1}}\binom{2i+j+1}{j}e^{-(\alpha/x)^2}\left[1-e^{-(\alpha/x)^2}\right]^{\lambda(j+1)-1}\left[1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right]^{2i}. \tag{14}$$

Again, we can using the binomial expansion in the following term:

$$\left[1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right]^{2i} = \sum_{k=0}^{\infty}(-1)^k\binom{2i}{k}\left[1-e^{-(\alpha/x)^2}\right]^{k\lambda}. \tag{15}$$

Therefore, by inserting (15) in (14),

$$f(x) = \sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\pi^{2i+1}}{(2i)!2^{2i-1}}\binom{2i+j+1}{j}\binom{2i}{k}\lambda\alpha^2x^{-3}e^{-(\alpha/x)^2}\left[1-e^{-(\alpha/x)^2}\right]^{\lambda(k+j+1)-1}. \tag{16}$$

Again, we can using the binomial expansion in the following term:

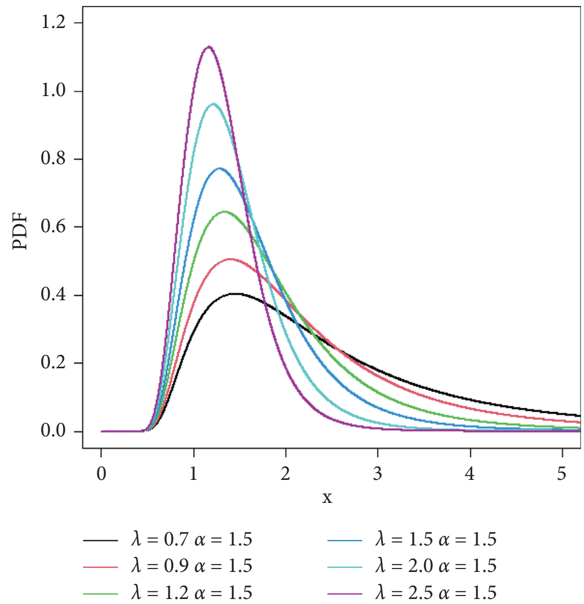

Figure 1: PDF plots for SHLIR distribution.

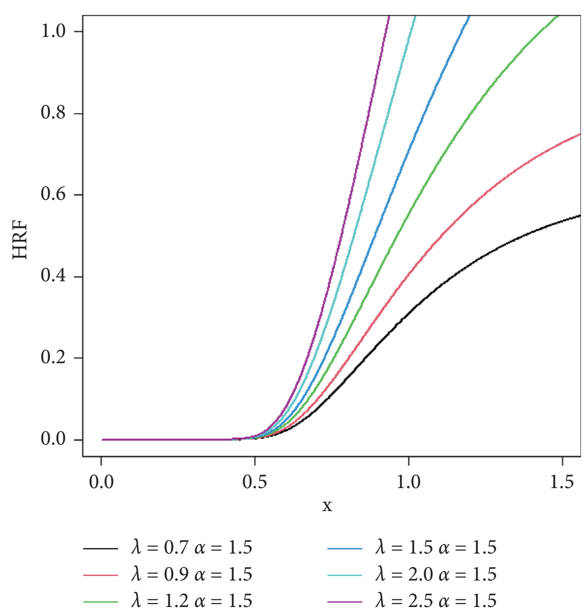

Figure 2: HRF plots for the SHLIR distribution.

$$\left[1 - e^{-(\alpha/x)^2}\right]^{\lambda(k+j+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{\lambda(k+j+1)-1}{m} e^{-m(\alpha/x)^2}. \tag{17}$$

Therefore, by inserting (17) in (16), we can write the PDF of SHLIR as

$$f(x) = 2\alpha^2 \sum_{i=0}^{\infty} \Delta_m x^{-3} e^{-(m+1)(\alpha/x)^2}, \tag{18}$$

where $\Delta_m = \sum_{i,j,k=0}^{\infty} ((-1)^i \quad +j+k+m\pi^{2i+1})/((2i) \quad !2^{2i}) \binom{2i+j+1}{j}\binom{2i}{k}\binom{\lambda(k+j+1)-1}{m}\lambda.$

2.2. Quantile Function. By inverting (5), we can obtain the quantile function of the SHLIR distribution, say $Q(u) = F^{-1}(u)$ of X, as follows:

$$Q(u) = \frac{\alpha}{\sqrt{-\ln\left(1 - \left(\left(1 - (2/\pi)\sin^{-1} u\right)/\left(1 + (2/\pi)\sin^{-1} u\right)\right)^{1/\lambda}\right)}}, \tag{19}$$

where u is thought of as a uniform random variable on (0, 1).

2.3. Moments. If X has PDF (6), then its moment can be calculated using the following relation.

$$\mu_r^{\backslash} = E(X^r) = \int_0^{\infty} x^r f(x)\mathrm{d}x. \tag{20}$$

Substituting (18) into (20) yields

$$\mu_r^{\backslash} = 2\alpha^2 \sum_{m=0}^{\infty} \Delta_m \int_0^{\infty} x^{r-3} e^{-(m+1)(\alpha/x)^2}\mathrm{d}x. \tag{21}$$

Let $y = (\alpha/x)^2$; then,

$$\mu_r^{\backslash} = \alpha^r \sum_{m=0}^{\infty} \Delta_m \int_0^{\infty} y^{-r/2} e^{-(m+1)y}\mathrm{d}y. \tag{22}$$

Then, $\mu_r^{\backslash}$ becomes

$$\mu_r^{\backslash} = \alpha^r \sum_{m=0}^{\infty} \Delta_m \frac{\Gamma(1-(r/2))}{(m+1)^{1-(r/2)}}, \quad \frac{r}{2} < 1. \tag{23}$$

The SHLIR distribution's moment generating function is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \alpha^r \sum_{r,m=0}^{\infty} \Delta_m \frac{t^r}{r!} \frac{\Gamma(1-(r/2))}{(m+1)^{1-(r/2)}}, \quad \frac{r}{2} < 1. \tag{24}$$

The incomplete moments of SHLIR are defined by

$$\phi_s(t) = \int_0^t x^s f(x)\mathrm{d}x. \tag{25}$$

Using (18), $\varphi_s(t)$ will be

$$\varphi_s(t) = 2\alpha^2 \sum_{m=0}^{\infty} \Delta_m \int_0^t x^{s-3} e^{-(m+1)(\alpha/x)^2}\mathrm{d}x = \alpha^s \sum_{m=0}^{\infty} \Delta_m \frac{\Gamma\left(1-(s/2), (m+1)(\alpha/t)^2\right)}{(m+1)^{1-(s/2)}}, \tag{26}$$

where $\Gamma(s,t) = \int_0^t x^{s-1} e^{-x}\mathrm{d}x$ is the lower incomplete gamma function.

The Lorenz and Bonferroni curves are obtained as follows:

$$L_F(x) = \frac{\varphi_1(x)}{E(X)} = \frac{\alpha \sum_{m=0}^{\infty} \Delta_m \left(\left(\Gamma\left((1/2), (m+1)(\alpha/t)^2\right)\right)/(m+1)^{1/2}\right)}{\sum_{m=0}^{\infty} \Delta_m \left((\alpha\sqrt{\pi})/(m+1)^{1/2}\right)} = \frac{\sum_{m=0}^{\infty}\left(\left(\Delta_m \Gamma\left((1/2), (m+1)(\alpha/t)^2\right)\right)/(m+1)^{1/2}\right)}{\sum_{m=0}^{\infty}\left((\Delta_m\sqrt{\pi})/(m+1)^{1/2}\right)}, \tag{27}$$

and

$$B_F(x) = \frac{L_F(x)}{F(x)} = \frac{\sum_{m=0}^{\infty}\left(\Delta_m \Gamma(1/2), (m+1)(\alpha/t)^2\right)/(i+1)^{1/2}}{\sin\left[(\pi/2)\left(\left(1-\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)/\left(1+\left[1-e^{-(\alpha/x)^2}\right]^{\lambda}\right)\right)\right]\sum_{m=0}^{\infty}(\Delta_m\sqrt{\pi})/(m+1)^{1/2}}. \tag{28}$$

2.4. Entropies. The entropy of the SHLIR model can be measured by various measures such as Rényi entropy (RE) [28], Havrda and Charvat entropy (HCE) [29], Arimoto entropy (AE) [30], and Tsallis entropy (TE) [31]. These measures of entropy are mentioned in Table 1.

$\int_0^{\infty} f^{\gamma}(x)\mathrm{d}x$ is very complicated to calculate, so it will be solved numerically.

3. Maximum Likelihood Estimation

To obtain the ML estimators (MLEs) of the SHLIR model with parameters α and λ, let $X_1, \ldots, X_n$ be observed values from this distribution. As a result, the log-likelihood function can be written as

$$\ell = n \log 2\,\pi + n \log \lambda + 2n \log \alpha - 3\sum_{i=1}^{n}\log(x_i) + (\lambda - 1)\sum_{i=1}^{n}\log(T_i) - 2\sum_{i=1}^{n}\log\left(1 + [T_i]^{\lambda}\right) + \sum_{i=1}^{n}\log\ \cos\left[\frac{\pi}{2}\left(\frac{1 - [T_i]^{\lambda}}{1 + [T_i]^{\lambda}}\right)\right]. \tag{29}$$

The ML equations of the SHLIR distribution are given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{2n}{\alpha} + 2\alpha(\lambda - 1)\sum_{i=1}^{n}\frac{e^{-(\alpha/x_i)^2}}{x_i^2(T_i)} - 4\alpha\lambda\sum_{i=1}^{n}\frac{e^{-(\alpha/x_i)^2}[T_i]^{\lambda-1}}{x_i^2\left(1 + [T_i]^{\lambda}\right)} + 2\pi\alpha\lambda\sum_{i=1}^{n}\frac{e^{-(\alpha/x_i)^2}[T_i]^{\lambda-1}}{x_i^2\left(1 + [T_i]^{\lambda}\right)^2}\tan\left[\frac{\pi}{2}\left(\frac{1 - [T_i]^{\lambda}}{1 + [T_i]^{\lambda}}\right)\right], \tag{30}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=0}^{\infty}\log(T_i) - 2\sum_{i=0}^{\infty}\frac{[T_i]^{\lambda}\ln[T_i]}{1 + [T_i]^{\lambda}} + 2\sum_{i=1}^{n}\frac{[T_i]^{\lambda}\ln\left[1 - e^{-(\alpha/x_i)^2}\right]}{\left(1 + [T_i]^{\lambda}\right)^2}\tan\left[\frac{\pi}{2}\left(\frac{1 - [T_i]^{\lambda}}{1 + [T_i]^{\lambda}}\right)\right], \tag{31}$$

where $T_i = 1 - e^{-(\alpha/x_i)^2}$. Equating $\partial l/\partial\alpha$ and $\partial l/\partial\lambda$ with zeros and solving simultaneously, we obtain the ML estimators of α and λ.

4. Numerical Results

A numerical result is evaluated and compared to evaluate and compare the behaviour of the estimates in terms of their mean square errors (MSEs). From the SHLIR model, we generate 5000 random samples $X_1, \ldots, X_n$ of sizes $n = 10$, 20, 30, 50, 100, and 200. Four distinct sets of parameters are taken into account, and their ML estimates (MLEs) are computed. The MSEs of the estimated unknown parameters are then computed. In Table 2, the simulated outcomes are listed, and the following observations are found.

For all estimates, the MSEs decrease as sample sizes increase.

5. Applications

Two data analyses are provided in this section to assess the goodness of fit of the SHLIR model in comparison to some known distributions such as type II Topp–Leone IR (TIITLIR) in [32], TIR, and IR distributions.

Maximized likelihood (A1), Akaike information criterion (A2), consistent Akaike information criterion (A3), Bayesian information criterion (A4), and Hannan–Quinn information criterion (HQIC) were used to compare the models. The model with the lowest values of A1, A2, A3, A4, and A5 is thought to be the best fit for the proposed data.

Data I: Bjerkedal [33] observed and reported the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli.

Data II: they represent the waiting times (in minutes) before service of 100 bank customers, observed and reported by Ghitany et al. [34].

Figures 3 and 4 show the fitted cumulative function (ECDF) of the SHLIR distribution, as well as the ECDFs of the compared models (HLIR, TIITLIR, TIR, and IR) for the first and second datasets.

According to Figures 3 and 4, the SHLIR distribution is the best fit when compared to the other models mentioned above for the two datasets.

Tables 3 and 4 show the ML estimates and standard errors (SEs) for the SHLIR model when compared to some known distributions such as HLIR, TIITLIR, TIR, and IR. Tables 5 and 6 also show the corresponding measures of fit statistic using A1, A2, A3, A4, and A5.

Also, Tables 5 and 6 confirm that the SHLIR distribution is the best fit among the other models for the two datasets, as the SHLIR distribution has the lowest values of A1, A2, A3, A4, and A5.

Table 1: Various measures of entropy for a distribution with PDF $f(x)$ at γ.

The measures	Formula
RE	$I_R(\gamma) = (1/1-\gamma)\mathrm{Log}[\int_0^\infty f^\gamma(x)dx], \quad \gamma \neq 1, \gamma > 0.$
HCE	$HC_R(\gamma) = (1/(2^{1-\gamma}-1))[\int_0^\infty f^\gamma(x)dx - 1], \quad \gamma \neq 1, \gamma > 0.$
AE	$A_R(\gamma) = (\gamma/1-\gamma)[(\int_0^\infty f^\gamma(x)dx)^{1/\gamma} - 1], \quad \gamma \neq 1, \gamma > 0.$
TE	$T_R(\gamma) = (1/\gamma - 1)[1 - \int_0^\infty f^\gamma(x)dx], \quad \gamma \neq 1, \gamma > 0.$

Table 2: MLEs and MSEs of SHLIR distribution.

n	$\alpha = 1.5, \lambda = 0.8$		$\alpha = 1.5, \lambda = 0.5$		$\alpha = 1.5, \lambda = 1$		$\alpha = 0.5, \lambda = 0.5$	
	MLEs	MSEs	MLEs	MSEs	MLEs	MSEs	MLEs	MSEs
10	1.716	0.305	1.825	0.617	1.699	0.266	0.609	0.072
	0.994	0.290	0.618	0.091	1.317	0.708	0.618	0.091
20	1.630	0.133	1.685	0.214	1.619	0.105	0.564	0.026
	0.907	0.082	0.557	0.024	1.139	0.127	0.557	0.024
30	1.570	0.068	1.593	0.107	1.558	0.058	0.533	0.015
	0.859	0.037	0.539	0.017	1.089	0.077	0.539	0.017
50	1.548	0.040	1.560	0.059	1.540	0.035	0.522	6.335*
	0.841	0.023	0.520	6.079*	1.046	0.036	0.520	6.076*
100	1.526	0.021	1.525	0.028	1.516	0.017	0.510	2.775*
	0.820	8.132*	0.510	2.736*	1.020	0.016	0.510	2.736*
200	1.510	8.196*	1.519	0.014	1.514	6.478*	0.508	1.225*
	0.810	4.216*	0.507	1.178*	1.012	5.964*	0.507	1.178*

* indicates that the value has multiplied by 10^{-3}.

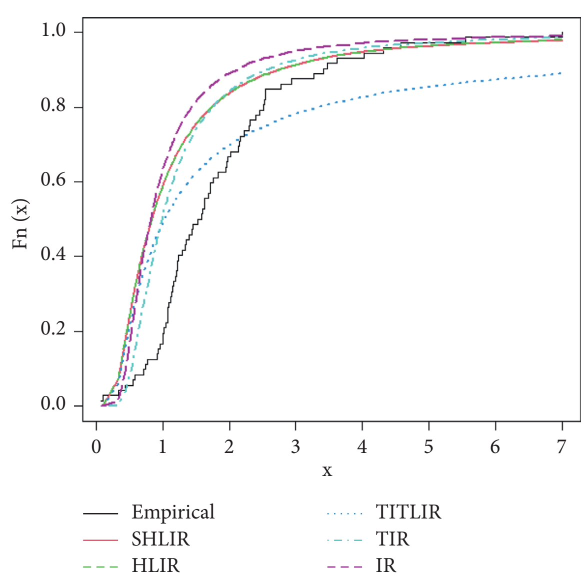

Figure 3: Fitted CDFs of models for data I.

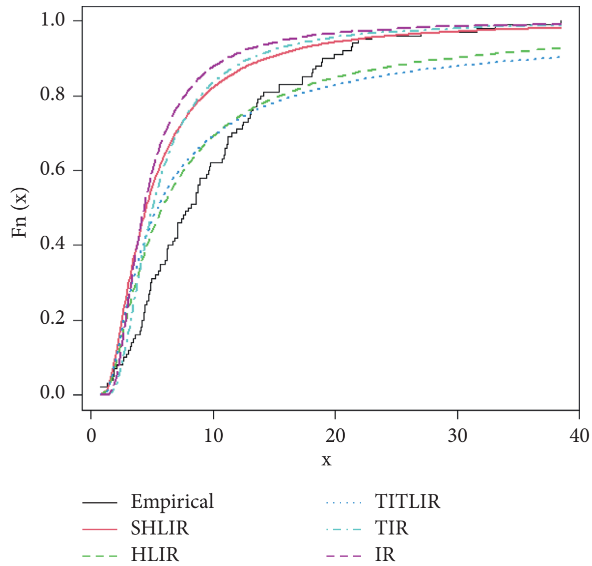

Figure 4: Fitted CDFs of models for data II.

Table 3: The MLE and SE of the model parameters for data I.

Model	MLEs and SEs	
SHLIR (α, λ)	**0.439** **(0.05)**	**0.487** **(0.061)**
HLIR (α, λ)	0.436 (0.05)	0.579 (0.07)
TIITLIR (α, λ)	0.325 (0.036)	0.404 (0.058)
TIR (α, λ)	0.352 (0.426)	−0.942 (0.351)
IR (α)	0.68 (0.04)	

Table 4: The MLE and SE of the model parameters for data I.

Model	MLEs and SEs	
SHLIR (α, λ)	2.419 (0.226)	0.499 (0.052)
HLIR (α, λ)	2.404 (0.226)	0.589 (0.06)
TIITLIR (α, λ)	1.824 (0.162)	0.43 (0.051)
TIR (α, λ)	9.978 (1.136)	−0.812 (0.085)
IR (α)	3.619 (0.181)	

Table 5: The values of A1, A2, A3, A4, and A5 for data I.

	Goodness-of-fit criteria				
Model	A1	A2	A4	A5	A3
SHLIR	**223.087**	**227.087**	**226.802**	**228.9**	**227.261**
HLIR	260.586	264.586	264.301	266.399	264.76
TIITLIR	280.492	284.492	284.207	286.305	284.666
TIR	280.538	284.538	284.253	286.351	284.712
IR	327.518	329.518	329.375	330.424	329.575

Table 6: The values of A1, A2, A3, A4, and A5 for data II.

Model	A1	A2	A4	A5	A3
	Goodness-of-fit criteria				
SHLIR	**628.185**	**632.185**	**632.185**	**634.293**	**632.308**
HLIR	680.806	684.806	684.806	686.915	684.93
TIITLIR	700.214	704.214	704.214	706.323	704.338
TIR	720.665	724.665	724.665	726.774	724.706
IR	759.629	761.629	761.629	762.683	761.67

6. Conclusion

This article investigates a new two-model distribution known as the sine half-logistic inverse Rayleigh (SHLIR). Some fundamental statistical properties of the SHLIR model are calculated and discussed, including the quantile function, moments, moment generating function, incomplete moment, and Lorenz and Bonferroni curves. Entropy measures such as Rényi entropy, Havrda and Charvat entropy, Arimoto entropy, and Tsallis entropy are investigated. The model parameter estimation is discussed using the ML method. Applications to two real datasets show that the SHLIR model outperforms other well-known competitive models such as the HLIR, TIITLIR, TIR, and IR models in terms of fit.

Data Availability

The numerical dataset used to carry out the analysis reported in this article is available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] R. Amirov, A. Ergun, and S. Durak, “Half-inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse,” *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 915–924, 2021.

[2] R. K. Amirov and A. Ergun, “Half inverse problems for the impulsive singular diffusion operator,” *Turkish Journal of Science*, vol. 5, no. 3, pp. 186–198, 2020.

[3] A. Ergün, “A half-inverse problem for the singular diffusion operator with jump conditions,” *Miskolc Mathematical Notes*, vol. 21, no. 2, pp. 805–821, 2020.

[4] G. M. Cordeiro and M. de Castro, “A new family of generalized distributions,” *Journal of Statistical Computation and Simulation*, vol. 81, no. 7, pp. 883–898, 2011.

[5] A. Alzaatreh, C. Famoye, and C. Lee, “A new method for generating families of continuous distributions,” *Metron*, vol. 71, no. 1, pp. 63–79, 2013.

[6] D. Kumar, U. Singh, and S. K. Singh, “A new distribution using sine function- its application to bladder cancer patients data,” *Journal of Statistics Applications & Probability*, vol. 4, no. 3, pp. 417–427, 2015.

[7] A. S. Hassan, M. Elgarhy, and M. Shakil, “Type II half logistic family of distributions with applications,” *Pakistan Journal of Statistics and Operation Research*, vol. 13, no. 2, pp. 245–264, 2017.

[8] M. Bourguignon, R. B. Silva, and G. M. Cordeiro, “The Weibull–G family of probability distributions,” *Journal of Data Science*, vol. 12, pp. 53–68, 2014.

[9] H. M. Yousof, A. Z. Afify, G. G. Hamedani, and G. Aryal, “The Burr X generator of distributions for lifetime data,” *Journal of Statistical Theory and Applications*, vol. 16, pp. 1–19, 2017.

[10] R. A. Bantan, F. Jamal, Ch. Chesneau, and M. Elgarhy, “A New power Topp–Leone generated family of distributions with applications,” *Entropy*, vol. 21, no. 12, pp. 1–24, 2019.

[11] M. A. Aldahlan, F. Jamal, Ch. Chesneau, M. Elgarhy, and I. Elbatal, “The truncated Cauchy power family of distributions with inference and applications,” *Entropy*, vol. 22, pp. 1–25, 2020.

[12] L. Handique and S. Chakraborty, “The beta generalized marshall-olkin kumaraswamy-g family of distributions with applications,” *International Journal of Agricultural and Statistical Sciences*, vol. 13, no. 2, pp. 721–733, 2017.

[13] M. M. Badr, I. Elbatal, F. Jamal, C. Chesneau, and M. Elgarhy, “The transmuted odd Fréchet-G family of distributions: theory and applications,” *Mathematics*, vol. 8, pp. 1–20, 2020.

[14] M. H. Tahir, M. A. Hussain, G. M. Cordeiro, M. El-Morshedy, and M. S. Eliwa, “A new kumaraswamy generalized family of distributions with properties, applications, and bivariate extension,” *Mathematics*, vol. 8, no. 11, p. 1989, 2020.

[15] A. A. Mohamed, R. M. Mahmoud, and S. A. ElSherbini, “The new kumaraswamy family of generalized distributions with application,” *Pakistan Journal of Statistics and Operation Research*, vol. 8, pp. 159–180, 2015.

[16] Z. M. Nofal, E. Altun, A. Z. Afify, and M. Ahsanullah, “The generalized kumaraswamy-g family of distributions,” *Journal of Statistical Theory and Applications*, vol. 18, no. 4, pp. 329–342, 2019.

[17] A. A. Al-Babtain, I. Elbatal, C. Chesneau, and M. Elgarhy, “Sine Topp-Leone-G family of distributions: theory and applications,” *Open Physics*, vol. 18, no. 1, pp. 574–593, 2020.

[18] H. M. Yousof, A. Z. Afify, M. Alizadeh, N. S. Butt, G. G. Hamedani, and M. M. Ali, “The transmuted exponentiated generalized-G family of distributions,” *Pakistan Journal of Statistics and Operation Research*, vol. 11, no. 4, pp. 441–464, 2015.

[19] A. Z. Afify, G. M. Cordeiro, H. M. Yousof, Z. M. Nofal, and A. Alzaatreh, “The kumaraswamy transmuted-g family of distributions: properties and applications,” *Journal of Data Science*, vol. 14, no. 2, pp. 245–270, 2016.

[20] J. Leao, H. Saulo, M. Bourguignon, J. Cintra, and L. Rego, “On some properties of the beta inverse Rayleigh distribution,” *Chilean Journal of Statistics*, vol. 4, pp. 111–131, 2013.

[21] A. Ahmad, S. P. Ahmad, and A. Ahmed, "Transmuted inverse Rayleigh distribution: a generalization of the inverse Rayleigh distribution," *Mathematical Theory and Modeling*, vol. 4, pp. 90–98, 2014.
[22] M. ShuaibKhan, "Modified inverse Rayleigh distribution," *International Journal of Computer Applications*, vol. 87, no. 13, pp. 28–33, 2014.
[23] M. S. Khan and R. King, "Transmuted modified inverse Rayleigh distribution," *Austrian Journal of Statistics*, vol. 44, no. 3, pp. 17–29, 2015.
[24] M. A. Haq, "Kumaraswamy exponentiated inverse Rayleigh distribution," *Mathematical Theory and Modeling*, vol. 6, pp. 93–104, 2016.
[25] K. Fatima and S. P. Ahmad, "Weighted inverse Rayleigh distribution," *International Journal of Statistics and Systems*, vol. 12, pp. 119–137, 2017.
[26] M. Elgarhy and S. Alrajhi, "The odd Fréchet inverse Rayleigh distribution: statistical properties and applications," *The Journal of Nonlinear Science and Applications*, vol. 12, no. 5, pp. 291–299, 2018.
[27] A. M. Almarashi, M. M. Badr, M. Elgarhy, F. Jamal, and C. Chesneau, "Statistical inference of the half-logistic inverse Rayleigh distribution," *Entropy*, vol. 22, pp. 1–23, 2020.
[28] A. Rényi, "On measures of entropy and information," *Proceedings of 4th Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1, pp. 47–561, 1960.
[29] J. Havrda and F. Charvat, "Quantification method of classification processes, concept of structural a-entropy," *Kybernetika*, vol. 3, no. 1, pp. 30–35, 1967.
[30] S. Arimoto, "Information-theoretical considerations on estimation problems," *Information and Control*, vol. 19, no. 3, pp. 181–194, 1971.
[31] C. Tsallis, "Possible generalization of Boltzmann-Gibbs statistics," *Journal of Statistical Physics*, vol. 52, no. 1-2, pp. 479–487, 1988.
[32] H. F. Mohammed and N. Yahia, "On type II Topp Leone inverse Rayleigh distribution," *Applied Mathematical Sciences*, vol. 13, no. 13, pp. 607–615, 2019.
[33] T. Bjerkedal, "Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercle Bacilli1," *American Journal of Epidemiology*, vol. 72, no. 1, pp. 130–148, 1960.
[34] M. E. Ghitany, B. Atieh, and S. Nadarajah, "Lindley distribution and its application," *Mathematics and Computers in Simulation*, vol. 78, no. 4, pp. 493–506, 2008.

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Research Article

On Multi-Index Mittag–Leffler Function of Several Variables and Fractional Differential Equations

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In this paper, we have studied a unified multi-index Mittag–Leffler function of several variables. An integral operator involving this Mittag–Leffler function is defined, and then, certain properties of the operator are established. The fractional differential equations involving the multi-index Mittag–Leffler function of several variables are also solved. Our results are very general, and these unify many known results. Some of the results are concluded at the end of the paper as special cases of our primary results.

1. Introduction

Recently, Mittag–Leffler (M-L) functions have demonstrated their special connection to fractional calculus, with a particular emphasis on fractional calculus problems arising from implementations. Several new special functions and implementations have been discovered over the last few decades. The advancement of research in the new era of special functions and their applications in mathematical modelling continues to attract many scientists from various disciplines (see recent papers; [1–13]).

The Mittag–Leffler function is extended to multi-index function in the following form [14, 15]:

$$E_{\gamma,K}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_{Kn}z^n}{\prod_{j=1}^{m}\Gamma\left(\beta_j+n\alpha_j\right)n!},\tag{1}$$

where $\alpha_j,\beta_j,\gamma\in\mathbb{C}$; $\Re(\alpha_j)>0$; $\Re(\beta_j)>0\quad(j=1,\ldots,m)$; $\Re\left(\sum_{j=1}^{m}\alpha_j\right)>0$; and K is an arbitrary complex number, i.e., $K\in\mathbb{C}$.

If we make $\gamma=K=1$ in (1) it reduces to the multi-index M-L function studied by Kiryakova [16, 17].

A multivariable extension of Mittag–Leffler function widely studied by Gautam [18], and also by Saxena et al. ([19], p. 547, Equation (7.1)), is defined and represented as follows:

$$E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1,\ldots,z_r)=\sum_{k_1,\ldots k_r=0}^{\infty}\frac{(\gamma_1)_{l_1k_1}\cdots(\gamma_r)_{l_rk_r}}{\Gamma(\lambda+k_1\rho_1+\cdots+k_r\rho_r)}\frac{z_1^{k_1}\ldots z_r^{k_r}}{k_1!\ldots k_r!},\tag{2}$$

where $\lambda,\gamma_j,l_j,\rho_j\in\mathbb{C}\,\Re(\rho_j)>0;\ \Re(l_j)>0;\ \lambda\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}$; and $j=1,2,\ldots,r$.

Motivated by the work on these functions, we consider here the subsequent multivariable and multi-index Mittag–Leffler function:

$$E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}^{(\gamma_r),(l_r)}[z_1,\ldots,z_r]=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_{l_ik_i}}{\prod_{j=1}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!},\tag{3}$$

where $\beta_j,\gamma_i,l_i,\rho_j^{(i)}\in\mathbb{C};\ \Re(\rho_j^{(i)})>0;\ \Re(l_i)>0;\ \beta_j\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\};\ (\rho_j^{(r)})\equiv\rho_j',\rho_j'',\ldots,\rho_j^{(r)};\ i=1,\ldots,r;$ and $j=1,\ldots,m$.

We have also studied here, the integral operator involving the function defined by (3), as follows:

$$\begin{aligned}&\left(E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m;\omega_r;a+}^{(\gamma_r),(l_r)}\psi\right)(x)\\&=\int_a^x(x-t)^{\beta_1-1}E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}^{(\gamma_r),(l_r)}\left[\omega_1(x-t)^{\rho_1'},\ldots,\omega_r(x-t)^{\rho_1^{(r)}}\right]\psi(t)\mathrm{d}t,\end{aligned}\tag{4}$$

with $\omega_i,\rho_j^{(i)},\gamma_i,l_i,\beta_j\in\mathbb{C};\ x>a;\ |\omega_i\ (x-t)^{\rho_1^{(i)}}|<1;\ \Re(\rho_j^{(i)})>0;\ j=1,\ldots,m;$ and $i=1,\ldots,r$.

The Riemann–Liouville fractional derivative operator D_{0+}^{α} is defined as follows [20]:

$$(D_{a+}^{\alpha}\psi)(x)=\left(\frac{d}{\mathrm{d}x}\right)^n(I_{a+}^{n-\alpha}\psi)(x),\ (\alpha\in\mathbb{C};\Re(\alpha)>0;n=|\Re(\alpha)|+1),\tag{5}$$

where $(I_{a+}^{\alpha}\psi)(x)$ is the fractional integral operator defined by

$$(I_{a+}^{\alpha}\psi)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\psi(t)}{(x-t)^{1-\alpha}}\mathrm{d}t.\tag{6}$$

The elementary definitions are also required to be mentioned as follows.

The Laplace transform of fractional derivative $(D_{0+}^{\alpha}f)(x)$ is given as

$$\mathfrak{L}(D_{0+}^{\alpha}f;s)=s^{\alpha}F(s)-\sum_{k=1}^{n}s^{k-1}D_{0+}^{\alpha-k}f(0+),\ (\Re(s)>0;\ (n-1<\alpha<n)).\tag{7}$$

Also, the formula for Laplace transform is

$$\frac{d^n}{\mathrm{d}s^n}[\mathfrak{L}\{y(x):\ (s)\}]=(-1)^n\mathfrak{L}[x^ny(x)](s).\tag{8}$$

2. Results Required

The integral for the generalized M-L function defined in (3) is given by

$$\begin{aligned}&\frac{1}{\Gamma(\sigma)}\int_0^x(x-t)^{\beta_1-1}t^{\sigma-1}E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}^{(\gamma_r),(l_r)}\left[\omega_1(x-t)^{\rho_1'},\ldots,\omega_r(x-t)^{\rho_1^{(r)}}\right]\mathrm{d}t\\&=x^{\beta_1+\sigma-1}E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+\sigma,\beta_2,\ldots,\beta_m}^{(\gamma_r),(l_r)}\left[\omega_1x^{\rho_1'},\ldots,\omega_rx^{\rho_1^{(r)}}\right].\end{aligned}\tag{9}$$

The result in (9) is established in view of definition in (3) and using the elementary beta integral.

The Laplace transform of $E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}(\omega_1 x^{\rho_1'},\ldots,\omega_r x^{\rho_1^{(r)}})$ defined in (3), easily obtained here, is as follows:

$$\mathfrak{L}\left\{x^{\beta_1-1}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}\left[\omega_1 x^{\rho_1'},\ldots,\omega_r x^{\rho_1^{(r)}}\right]\right\}(x) = s^{-\beta_1}\prod_{j=2}^{m}\left(\frac{1}{\Gamma(\beta_j)}\right) \times F^{0\,:1;\ldots;1}_{m-1:0;\ldots;0}\left[\begin{matrix} - & : & (\gamma_1,l_1);\ldots;(\gamma_r,l_r) & ; \\ \left(\beta_j;\rho_j',\ldots,\rho_j^{(r)}\right)_{2,m} & : & -;\ldots;- & ; \end{matrix}\ \omega_1 s^{-\rho_1'},\ldots,\omega_r s^{-\rho_1^{(r)}}\right], \tag{10}$$

where $\alpha,\beta_j,\rho_j^{(i)},\gamma_i,l_i,\omega_i\in\mathbb{C}$; $\Re(s)>0$; $\Re(\rho_j^{(i)})>0$; $\Re(\beta_j)$; $\Re(l_i)>0$; $j=1,\ldots,m$; $i=1,\ldots,r$; and $(\beta_j\colon\rho_j',\ldots,\rho_j^{(r)})_{2,m}\equiv(\beta_2;\rho_2',\ldots,\rho_2\ (r)),\ldots,(\beta_m;\rho_m',\ldots,\rho_m^{(r)})$.

Here, $F[\omega_1 s^{-\rho_1'},\ldots,\omega_r s^{-\rho_1^{(r)}}]$ is the generalized Lauricella function ([21], p. 37, Equations (21–23)).

3. Main Results

Theorem 1. *Let* $a\in\mathbb{R}_+$; $\alpha,\beta_j,\rho_j^{(i)},\gamma_i,l_i,\omega_i\in\mathbb{C}$; $\Re(\alpha)>0$; $\Re(\rho_j^{(i)})>0$; $\Re(\beta_j)>0$; *and* $\Re(l_i)>0$ $(j=1,\ldots,m;\ i=1,\ldots,r)$.

Then, for $x>a$, *we have*

$$D^{\alpha}_{a+}\left[(t-a)^{\beta_1-1}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}\left\{\omega_1(t-a)^{\rho_1'},\ldots,\omega_r(t-a)^{\rho_1^{(r)}}\right\}\right](x) = (x-a)^{\beta_1-\alpha-1}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1-\alpha,\beta_2,\ldots,\beta_m}\left[\omega_1(x-a)^{\rho_1'},\ldots,\omega_r(x-a)^{\rho_1^{(r)}}\right] \tag{11}$$

and

$$I^{\alpha}_{a+}\left[(t-a)^{\beta_1-1}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}\left\{\omega_1(t-a)^{\rho_1'},\ldots,\omega_r(t-a)^{\rho_1^{(r)}}\right\}\right](x) = (x-a)^{\beta_1+\alpha-1}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+\alpha,\beta_2,\ldots,\beta_m}\left[\omega_1(x-a)^{\rho_1'},\ldots,\omega_r(x-a)^{\rho_1^{(r)}}\right]. \tag{12}$$

If $\alpha,\beta_j,\gamma_i,l_i,\omega_i\in\mathbb{C}$; $\Re(\alpha)>0$; $\Re(\rho_j^{(i)})>0$; $\Re(\beta_j)>0$; $\Re(l_i)>0$; $j=1,\ldots,m$; *and* $i=1,\ldots,r$ *with the initial condition* $(I^{1-\alpha}_{0+}y)(0+)=c$ *(c is an arbitrary constant) and solution of differential equations existing in the space* $L(0,\infty)$, *then Theorems 2–4 are stated in the following form.*

Theorem 2. *If*

$$\left(D^{\alpha}_{0+}y\right)(x) = \lambda\left(E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m;\omega_r;0_+}\right)(x) + f(x), \tag{13}$$

then its solution is given by

$$y(x) = \frac{cx^{\alpha-1}}{\Gamma(\alpha)} + \lambda x^{\beta_1+\alpha}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+\alpha+1,\beta_2,\ldots,\beta_m}\left[\omega_1 x^{\rho_1'},\ldots,\omega_r x^{\rho_1^{(r)}}\right] + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\mathrm{d}t. \tag{14}$$

Theorem 3. *If*

$$\left(D_{0+}^{\alpha}y\right)(x)=\lambda\left(E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m;\omega_r;0+}^{(\gamma_r),(l_r)}\right)(x)+px^{\beta_1}E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+1,\beta_2,\ldots,\beta_m}^{(\gamma_r),(l_r)}\left[\omega_1x^{\rho_1'},\ldots,\omega_rx^{\rho_1^{(r)}}\right], \tag{15}$$

then its solution is given by

$$y(x)=\frac{cx^{\alpha-1}}{\Gamma(\alpha)}+(\lambda+p)x^{\beta_1+\alpha}E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+\alpha+1,\beta_2,\ldots,\beta_m}^{(\gamma_r),(l_r)}\left[\omega_1x^{\rho_1'},\ldots,\omega_rx^{\rho_1^{(r)}}\right]. \tag{16}$$

Theorem 4. *If*

$$x\left(D_{0+}^{\alpha}y\right)(x)=\lambda\left(E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m;\omega_r;0+}^{(\gamma_r),(l_r)}\right)(x), \tag{17}$$

then its solution is given by

$$y(x)=\frac{cx^{\alpha-1}}{\Gamma(\alpha)}+\frac{\lambda}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}(x-t)^{\beta_1-1}\times E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+1,\beta_2,\ldots,\beta_m}^{(\gamma_r),(l_r)}\left[\omega_1(x-t)^{\rho_1'},\ldots,\omega_r(x-t)^{\rho_1^{(r)}}\right]\mathrm{d}t. \tag{18}$$

Proof. In Theorem 1, let the left-hand side of result (11) be Δ_1, i.e.,

$$\Delta_1=D_{a+}^{\alpha}\left[(t-a)^{\beta_1-1}E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}^{(\gamma_r),(l_r)}\left\{\omega_1(t-a)^{\rho_1'},\ldots,\omega_r(t-a)^{\rho_1^{(r)}}\right\}\right](x). \tag{19}$$

Having used the definition of $E_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}^{(\gamma_r),(l_r)}[.]$ given in (3), we obtain the following form:

$$\Delta_1=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right]}{\prod_{j=1}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)}D_{a+}^{\alpha}\left[(t-a)^{\beta_1+\sum_{i=1}^{r}\rho_1^{(i)}k_i-1}\right](x). \tag{20}$$

On using the fractional derivative of power function $(t-a)^{\beta_1+\sum_{i=1}^{r}\rho_1^{(i)}k_i-1}$ ([20], p. 36, Equation (2.26)), we have

$$\Delta_1=(x-a)^{\beta_1-\alpha-1}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right](x-a)^{\sum_{i=1}^{r}\rho_1^{(i)}k_i}}{\prod_{j=2}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_1^{(i)}k_i\right)\Gamma\left(\beta_1-\alpha+\sum_{i=1}^{r}\rho_1^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)}. \tag{21}$$

On interpreting multiple series by the definition of $E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m}[\cdot]$, we at once arrive at (11).

The proof of (12) follows the proof of (11) using (6) and ([20], p. 40, Equation (2.44)) therein.

Theorem 2 is proved as follows.

Use the definition of operator $(E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m;\omega_r;a+}\psi)(x)$ (at $a=0$ and $\psi(x)=1$) and result (9) (at $\sigma=1$) in (13), we have

$$(D^{\alpha}_{0+}y)(x)=\lambda x^{\beta_1}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+1,\beta_2,\ldots,\beta_m}\left[\omega_1 x^{\rho_1'},\ldots,\omega_r x^{\rho_1^{(r)}}\right]+f(x). \tag{22}$$

Taking Laplace transform of (22) and then using formula (7) (for $n=1$) and (10), therein we have

$$\begin{aligned} y(s)&=cs^{-\alpha}+\lambda s^{-\beta_1-1-\alpha}\prod_{j=2}^{m}\left(\frac{1}{\Gamma(\beta_j)}\right)\\ &\quad\times F^{0\,:\,1;\ldots;1}_{m-1\,:\,0;\ldots;0}\left[\begin{matrix} - & : & (\gamma_1,l_1);\ldots;(\gamma_r,l_r) & ; \\ (\beta_j;\rho_j',\ldots,\rho_j^{(r)})_{2,m} & : & -;\ldots;- & ; \end{matrix}\;\omega_1 s^{-\rho_1'},\ldots,\omega_r s^{-\rho_1^{(r)}}\right]+f(s)s^{-\alpha}. \end{aligned} \tag{23}$$

In view of the definition of generalized Lauricella function ([21], p. 37, Equations (21–23)), we have the form

$$y(s)=cs^{-\alpha}+\lambda\frac{\sum_{k_1,\ldots,k_r=0}^{\infty}\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right]s^{\left(-\beta_1-1-\alpha-\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}}{\prod_{j=2}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)}+f(s)s^{-\alpha}. \tag{24}$$

Applying inverse Laplace transform on both sides of (24) and using convolution theorem, we find

$$\begin{aligned} y(x)&=c\frac{x^{\alpha-1}}{\Gamma(\alpha)}+\lambda\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right]}{\prod_{j=1}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)}\frac{x^{\beta_1+\alpha+\sum_{i=1}^{r}\rho_j^{(i)}k_i}}{\Gamma\left(\beta_1+1+\alpha+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t. \end{aligned} \tag{25}$$

Now, on interpreting the multiple series using (3), we at once arrive at desired result (14). □

Proof. of Theorem 3. We use (at $a=0$ and $\psi(x)=1$) and (9) (at $\sigma=1$) in (15), and it takes the following form:

$$(D^{\alpha}_{0+}y)(x)=(\lambda+p)x^{\beta_1}E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+1,\beta_2,\ldots,\beta_m}\left[\omega_1 x^{\rho_1'},\ldots,\omega_r x^{\rho_1^{(r)}}\right]. \tag{26}$$

On both sides of (26), we take Laplace transform and then using formula (7) (for $n=1$) and (10) therein, we obtain

$$y(s) = cs^{-\alpha} + (\lambda + p)s^{-\beta_1-1-\alpha} = \prod_{j=2}^{m}\left(\frac{1}{\Gamma(\beta_j)}\right) \times F^{0:1;\ldots;1}_{m-1:0;\ldots;0}\left[\begin{matrix} - & : & (\gamma_1, l_1);\ldots;(\gamma_r, l)_r & ; \\ (\beta_j;\rho_j',\ldots,\rho_j^{(r)})_{2,m} & : & -;\ldots;- & ; \end{matrix} \omega_1 s^{-\rho_1'},\ldots,\omega_r s^{-\rho_1^{(r)}}\right]. \tag{27}$$

In view of the definition of generalized Lauricella function ([21], p. 37, Equations (21–23)), we have

$$y(s) = cs^{-\alpha} + (\lambda + p)\sum_{k_1,\ldots,k_r=0}^{\infty} \frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_i k_i}(\omega_i)^{k_i}\right] s^{\left(-\beta_1-1-\alpha-\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}}{\prod_{j=1}^{m}\Gamma\left(\beta_j + \sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)}. \tag{28}$$

Applying inverse Laplace transform on (28), we have

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} + (\lambda + p)\sum_{k_1,\ldots,k_r=0}^{\infty} \frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_i k_i}(\omega_i)^{k_i}\right]}{\prod_{j=2}^{m}\Gamma\left(\beta_j + \sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)} \times \frac{x^{\left(\beta_1+\alpha+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}}{\Gamma\left(\beta_1 + 1 + \alpha + \sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}. \tag{29}$$

On interpreting the multiple series using (3), we at once arrive at result (16). □

Proof. of Theorem 4. We use operator $(E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1,\ldots,\beta_m;\omega_r;a+}\psi)(x)$ (at $a=0$ and $\psi(x)=1$) and (9) (at $\sigma=1$) in (17), and we have the following form:

$$x(D^{\alpha}_{0+}y)(x) = \lambda x^{\beta_1} E^{(\gamma_r),(l_r)}_{(\rho_1^{(r)}),\ldots,(\rho_m^{(r)});\beta_1+1,\beta_2,\ldots,\beta_m}\left[\omega_1 x^{\rho_1'},\ldots,\omega_r x^{\rho_1^{(r)}}\right]. \tag{30}$$

On both sides of (30), we take Laplace transform and use formulae (8) and (10) (for $n=1$); then, we obtain

$$\frac{d}{ds}y(s) + \frac{\alpha}{s}y(s) = -\lambda s^{-\beta_1-1-\alpha}\prod_{j=2}^{m}\left(\frac{1}{\Gamma(\beta_j)}\right) \times F^{0:1;\ldots;1}_{m-1:0;\ldots;0}\left[\begin{matrix} - & : & (\gamma_1, l_1);\ldots;(\gamma_r, l)_r & ; \\ (\beta_j;\rho_j',\ldots,\rho_j^{(r)})_{2,m} & : & -;\ldots;- & ; \end{matrix} \omega_1 s^{-\rho_1'},\ldots,\omega_r s^{-\rho_1^{(r)}}\right]. \tag{31}$$

In view of the definition of generalized Lauricella function ([21], p. 37, Equations (21–23)), we have

$$\frac{d}{ds}y(s)+\frac{\alpha}{s}y(s)=-\lambda\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right]}{\prod_{j=2}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)}s^{\left(-\beta_1-1-\alpha-\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}. \tag{32}$$

Since this is a linear differential equation of first order and first degree,

$$y(s)=\lambda\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right]}{\prod_{j=2}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)}\frac{(s)^{\left(-\beta_1-\alpha-\sum_{i=1}^{r}k_1\rho_1^{(i)}\right)}}{\left(\beta_1+\sum_{i=1}^{r}k_i\rho_1^{(i)}\right)}+cs^{-\alpha}. \tag{33}$$

Taking inverse Laplace transform of (33), we have

$$\begin{aligned}y(x)=\lambda\sum_{k_1,\ldots k_r=0}^{\infty}&\frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right]}{\prod_{j=2}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)\prod_{i=1}^{r}(k_i!)1/\left(\beta_1+\sum_{i=1}^{r}k_i\rho_1^{(i)}\right)}\\&\times\mathfrak{L}^{-1}\left[(s)^{\left(-\alpha-\beta_1-\sum_{i=1}^{r}k_1\rho_1^{(i)}\right)}\right]+c\mathfrak{L}^{-1}\left(s^{-\alpha}\right).\end{aligned} \tag{34}$$

In view of convolution theorem, we obtain

$$\begin{aligned}y(x)=\frac{\lambda}{\Gamma(\alpha)}\sum_{k_1,\ldots k_r=0}^{\infty}&\frac{\prod_{i=1}^{r}\left[(\gamma_i)_{l_ik_i}(\omega_i)^{k_i}\right]}{\prod_{j=2}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}k_i\rho_j^{(i)}\right)\prod_{i=1}^{r}(k_i!)\left(1/\Gamma\left(\beta_1+1+\sum_{i=1}^{r}k_i\rho_j^{(i)}\right)\right)}.\\&\times\int_0^x t^{\alpha-1}(x-t)^{\beta_1-1+\sum_{i=1}^{r}k_i\rho_1^{(i)}}dt+\frac{c}{\Gamma(\alpha)}x^{\alpha-1}.\end{aligned} \tag{35}$$

Now, on interpreting the multiple series in view of (3), we obtain the result in (18).

4. Conclusion

Here, we conclude further interesting known results:

(1) Our main results for $m=1$, respectively, give the known results provided by Gupta and Jaimini ([22], pp. 145–146, Equations (1–10)).

(2) If in result (10) and in Theorem 2, we take $r=1$ (i.e., $\omega_2=,\ldots,=\omega_r=0$), then result (10) reduces to the known result provided by Saxena et al. ([23], p. 10, Equation (50)) and Theorem 2 gives the correct form (at $\nu=0$) of the theorem provided by Saxena et al. ([23], p. 10, Theorem (5.1)).

(3) For $m=1$ and $r=1$, Theorems 1 to 4 reduce, respectively, to the known results (at $\nu=0$) provided by Srivastava and Tomovski [24].

(4) If in Theorem 1, we take $m=1$ and $l_1=l_2=,\ldots,=l_r=1$, then these results, respectively, reduce to the known results provided by Gautam ([18], pp. 201–202, Equations (4.64)–(4.65)).

(5) If in Theorems 1 to 4, we take $m=1$, $l_1=l_2=,\ldots,=l_r=1$, and $\rho_1'=\rho_1''=,\ldots,=\rho_1^{(r)}=1$,

then these, respectively, reduce to the results for function $\phi_2^{(r)}(\cdot)$ provided by Gupta ([25], pp. 250–253, Equations (4.9.19)–(4.9.27)).

Therefore, the results presented in the article would immediately yield a large number of results that include a wide range of special functions occurring in issues of scientific research, computer science, and applied mathematics, among others.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this article.

References

[1] H. M. Baskonus, "New acoustic wave behaviors to the Davey-Stewartson equation with power-law nonlinearity arising in fluid dynamics," *Nonlinear Dynamics*, vol. 86, no. 1, pp. 177–183, 2016.

[2] H. M. Baskonus, "Complex surfaces to the fractional (2+1)-dimensional boussinesq dynamical model with local M-derivative," *European Physical Journal Plus*, vol. 134, no. 322, pp. 1–10, 2019.

[3] S. Dave, A. M. Khan, S. D. Purohit, and D. L. Suthar, "Application of green synthesized metal nanoparticles in the photocatalytic degradation of dyes and its mathematical modelling using the Caputo–Fabrizio fractional derivative without the singular kernel," *Journal of Mathematics*, vol. 2021, Article ID 9948422, 8 pages, 2021.

[4] H. F. Ismael, H. Bulut, H. Bulut, H. M. Baskonus, and W. Gao, "Dynamical behaviors to the coupled Schrödinger-Boussinesq system with the beta derivative," *AIMS Mathematics*, vol. 6, no. 7, pp. 7909–7928, 2021.

[5] W. Gao, P. Veeresha, D. G. Prakasha, B. Senel, and H. M. Baskonus, "Iterative method applied to the fractional nonlinear systems arising in thermoelasticity with Mittag-Leffler kernel," *Fractals*, vol. 28, no. 8, 2020.

[6] W. Gao, P. Veeresha, D. G. Prakasha, and H. M. Baskonus, "New numerical simulation for fractional Benney-Lin equation arising in falling film problems using two novel techniques," *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 210–243, 2021.

[7] H. Habenom, D. L. Suthar, D. Baleanu, and S. D. Purohit, "A numerical simulation on the effect of vaccination and treatments for the fractional hepatitis B model," *ASME Journal of Computational and Nonlinear Dynamics*, vol. 16, no. 1, Article ID 11004, 2021.

[8] A. Kumar, E. Ilhan, E. Ilhan, A. Yel, and H. M. Baskonus, "Extractions of some new travelling wave solutions to the conformable Date-Jimbo-Kashiwara-Miwa equation," *AIMS Mathematics*, vol. 6, no. 5, pp. 4238–4264, 2021.

[9] G. Yel, C. Cattani, H. M. Baskonus, and W. Gao, "On the complex simulations with dark–bright to the hirota-maccari system," *Journal of Computational and Nonlinear Dynamics*, vol. 16, no. 6, p. 9, Article ID 61005, 2021.

[10] K. S. Nisar, D. L. Suthar, R. Agarwal, and S. D. Purohit, "Fractional calculus operators with Appell function kernels applied to Srivastava polynomials and extended Mittag-Leffler function," *Advances in Difference Equations*, vol. 148, 2020.

[11] D. L. Suthar, H. Amsalu, and K. Godifey, "Certain integrals involving multivariate Mittag-Leffler function," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 208, 2019.

[12] D. L. Suthar, H. Habenom, and H. Tadesse, "Generalized fractional calculus formulas for a product of Mittag-Leffler function and multivariable polynomials," *International Journal of Applied and Computational Mathematics*, vol. 4, no. 1, pp. 1–12, 2018.

[13] D. L. Suthar, M. Andualem, and B. Debalkie, "A study on generalized multivariable Mittag-Leffler function via generalized fractional calculus operators," *Journal of Mathematics*, vol. 2019, Article ID 9864737, 7 pages, 2019.

[14] R. K. Saxena and K. Nishimoto, "N-Fractional calculus of generalized Mittag-Leffler functions," *Journal of Fractional Calculus*, vol. 37, pp. 43–52, 2010.

[15] R. K. Saxena and K. Nishimoto, "Further results on the generalized Mittag-Leffler functions of fractional calculus," *Journal of Fractional Calculus*, vol. 40, pp. 29–41, 2011.

[16] V. S. Kiryakova, "Multiple (multiindex) Mittag-Leffler function related Gelfond and Leontiex operations and Laplace type integral transforms," *Fractional Calculus and Applied Analysis*, vol. 2, pp. 445–462, 1999.

[17] R. K. Saxena, S. L. Kalla, and V. S. Kiryakova, "Relation connecting multiindex Mittag-Leffler functions and Riemann-Liouville fractional calculus," *Algebras Groups and Geometries*, vol. 20, pp. 365–385, 2003.

[18] S. Gautam, *Investigations in fractional differential operators of arbitrary order and their applications to special functions of one and several variables*, Ph.D. Thesis, University of Kota, Kota, India, 2008.

[19] R. K. Saxena, S. L. Kalla, and R. Saxena, "Multivariate analogue of generalized Mittag-Leffler function," *Integral Transforms and Special Functions*, vol. 22, no. 7, pp. 533–548, 2011.

[20] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, NY, USA, 1983.

[21] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Harwood Ltd. Chichestcra, New York, NY, USA, 1985.

[22] J. Gupta and B. B. Jaimini, "On certain fractional differential equations involving generalized multivariable Mittag-Leffler function," *Note Mat*, vol. 32, no. 2, pp. 141–156, 2012.

[23] R. K. Saxena, J. P. Chauhan, R. K. Jana, and A. K. Shukla, "Further results on the generalized Mittag-Leffler function operator," *Journal of Inequalities and Applications*, vol. 75, pp. 1–12, 2015.

[24] H. M. Srivastava and Ž. Tomovski, "Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel," *Applied Mathematics and Computation*, vol. 211, no. 1, pp. 198–210, 2009.

[25] J. Gupta, *Investigations in generalized special functions and the fractional calculus with its applications to univalent and multivalent functions*, Ph.D. Thesis, University of Kota, Kota India, 2011.

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Research Article

Some Inequalities of Generalized p-Convex Functions concerning Raina's Fractional Integral Operators

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Convex functions play an important role in pure and applied mathematics specially in optimization theory. In this paper, we will deal with well-known class of convex functions named as generalized p-convex functions. We develop Hermite–Hadamard-type inequalities for this class of convex function via Raina's fractional integral operator.

1. Introduction

The subject of fractional calculus got rapid development in the last few decades. As a matter of fact, fractional calculus give more accuracy to model applied problems in engineering and other sciences then classical calculus. In order to model recent complicated problems, scientists are using fractional inequalities and fractional equations. For more on this, we refer the books [1, 2]. The models with fractional calculus have been applied successfully in ecology, aerodynamics, physics, biochemistry, environmental science, and many other branches. For more about fractional calculus and models, we refer [3–5].

Fractional integral inequalities are considered one of the important tools to study the behavior and properties of solutions of various fractional problems [6–14]. There are many interesting generalization of fractional derivatives as per need of practical problems or some theoretical approach, for example, Raina's fractional integral operator, Caputo-Fabrizio fractional integral, and extended Caputo-Fabrizio fractional integral. For recent work on it, we refer [15–20].

Convex functions also play an important role in pure and applied mathematics specially in optimization theory. Classical convexity does not fulfil needs of modern mathematics; therefore, several generalizations of convex functions are presented in literature. *s*-convex function [21], M-convex functions [22], and h-convex function [23] are some examples of generalized convex functions. It is always interesting to study properties of some generalized convex function in the setting of fractional integral operators. This paper is an effort in this direction. In this paper, we study the p-convex functions and present some of its properties in the setting of Raina's fractional integral operators.

The paper is organized as follows. In Section 2, we present some basic definition and properties of Raina's fractional integral operator. Section 3 is devoted for Hermite–Hadamard type inequalities for generalized p-convex functions in terms of Raina's fractional integral operators.

2. Preliminaries

Here, we present some basic definitions and known results.

Definition 1 (convex function). A function $\phi: I \longrightarrow R$ is said to be convex function if the following inequality holds:

$$\phi(\vartheta x + (1-\vartheta)y) \leq \vartheta\phi(x) + (1-\vartheta)\phi(y), \quad (1)$$

for $\forall x, y \in I$ and $\vartheta \in [0,1]$.

One of the novel generalization of convexity is η-convexity introduced by M. R. Delavar and S. S. Dragomir in [24].

Definition 2. A function $\phi: I \longrightarrow R$ is said to be generalized convex function with respect to $\eta: A \times A \longrightarrow B$ for appropriate $A, B \subseteq R$ if

$$\phi(\vartheta x + (1-\vartheta)y) \leq \phi(y) + \vartheta\eta(\phi(x), \phi(y)), \tag{2}$$

for $\forall x, y \in I$ and $\vartheta \in [0,1]$

In [25], Zhang and Wan gave definition of p-convex function as follows.

Definition 3. Let I be a p-convex set. A function $f: I \longrightarrow \mathbb{R}$ is said to be p-convex function if

$$\phi\left(\left[\vartheta x^p + (1-\vartheta)y^p\right]^{1/p}\right) \leq \vartheta\phi(x) + (1-\vartheta)\phi(y), \tag{3}$$

holds, for all $x, y \in I$ and $\vartheta \in [0,1]$.

In [26], the authors gave the definition of the generalized p-convex function as follows.

Definition 4. A function $\phi: I \longrightarrow R$ is said to be generalized p-convex function with respect to $\eta: A \times A \longrightarrow B$ for appropriate $A, B \subseteq R$ if

$$\phi\left(\vartheta x^p + (1-\vartheta)y^p\right)^{1/p} \leq \phi(y) + \vartheta\eta(\phi(x), \phi(y)), \tag{4}$$

for $\forall x, y \in I, p > 0$ and $\vartheta \in [0,1]$.

For some important properties and results about generalized p-convexity, see [26]. Moreover, in [26], the following Hermite–Hadamard type inequality for p-convex functions can be found.

Theorem 1. *Let $\phi: I \longrightarrow \mathbb{R}$ be generalized p-convex function for $\xi_1, \xi_2 \in I$ with condition $\xi_1 < \xi_2$; then, we obtain the inequality*

$$\begin{aligned} &\phi\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{1/p} - \frac{p}{2\left(\xi_2^p - \xi_1^p\right)} \int_{\xi_1}^{\xi_2} x^{p-1}\eta\left(\phi\left(\xi_1^p + \xi_2^p - x^p\right)^{1/p}, \phi(x)\right)\mathrm{d}x \\ &\leq \frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1}\phi(x)\mathrm{d}x \leq \frac{\phi(\xi_1) + \phi(\xi_2)}{2} + \frac{1}{4}\left[\eta(\phi(\xi_1), \phi(\xi_2)) + \eta(\phi(\xi_2), \phi(\xi_1))\right]. \end{aligned} \tag{5}$$

In [27], the author introduced a class of functions defined formally by

$$\mathscr{F}^{\sigma}_{\rho,\lambda} z = \mathscr{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda} z = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k, \tag{6}$$

where $\rho, \lambda > 0$, $|z| < \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), and $\sigma = (\sigma(1), \ldots, \sigma(k), \ldots)$ is a bounded sequence of positive real numbers.

Using (6), in [28], the authors defined the following left-sided and right-sided fractional integral operators, respectively:

$$\begin{aligned} \left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(z) &= \int_{\xi_1}^{z} (z-\vartheta)^{(\lambda-1)} \mathscr{F}^{\sigma}_{\rho,\lambda} z\left[w(z-\vartheta)^{\rho}\right]\phi(\vartheta)\mathrm{d}\vartheta, \quad (z > \xi_1), \\ \left(\mathscr{J}^{\sigma}_{\rho,\lambda,b^-;w}\phi\right)(z) &= \int_{\xi_1}^{z} (\vartheta-z)^{(\lambda-1)} \mathscr{F}^{\sigma}_{\rho,\lambda} z\left[w(\vartheta-z)^{\rho}\right]\phi(\vartheta)\mathrm{d}\vartheta, \quad (z < \xi_2), \end{aligned} \tag{7}$$

where $\rho, \lambda > 0, \omega \in \mathbb{R}$, and ϕ is such that the integral on the right side exits.

It is easy to verify that $(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi)(z)$ and $(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi)(z)$ are bounded integral operators on $L_p(\xi_1, \xi_2)$, $(1 \leq p \leq \infty)$ if

$$\mathfrak{M} := \mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(\xi_2 - \xi_1)^{\rho}\right] \leq \infty. \tag{8}$$

In fact, for $\phi \in L_p(\xi_1, \xi_2)$, we have

$$\begin{aligned} \left\|\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)\right\|_p &\leq \mathfrak{M}\|\phi\|_p, \\ \left\|\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)\right\|_p &\leq \mathfrak{M}\|\phi\|_p, \end{aligned} \tag{9}$$

where

$$\|\phi\|_p = \left(\int_{\xi_1}^{\xi_2} |\phi(z)|^p \mathrm{d}z\right)^{1/p}. \tag{10}$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Let $\phi \in L[\xi_1, \xi_2]$. The right-hand side and left-hand side Riemann–Liouville fractional integral of order $\alpha > 0$ with $\xi_2 > \xi_1 > 0$ are defined by

$$J^{\alpha}_{\xi_1+}\phi(z) = \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{z}(z-k)^{\alpha-1}\phi(k)\mathrm{d}k, \quad x>\xi_1,$$
$$J^{\alpha}_{\xi_2-}\phi(z) = \frac{1}{\Gamma(\alpha)}\int_{z}^{\xi_2}(k-z)^{\alpha-1}\phi(k)\mathrm{d}k, \quad z<\xi_2, \tag{11}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined as $\Gamma(\alpha) = \int_0^{\infty} e^{-k}k^{\alpha-1}\mathrm{d}k$.

Lemma 1 (see [29, 30]). *Let $\lambda, \rho > 0, \omega \in \mathbb{R}$, and σ be a sequence of nonnegative real numbers. Let ϕ: $[\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be a differentiable mapping on (ξ_1, ξ_2) with $\xi_1 < \xi_2$ and $\lambda > 0$. If $\phi' \in L[\xi_1; \xi_2]$, the following equality for the fractional integral operator holds:*

$$\begin{aligned}&\frac{\phi(\xi_1)+\phi(\xi_2)}{2} - \frac{1}{2(\xi_2-\xi_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}]}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]\\ &= \frac{\xi_2-\xi_1}{2(\xi_2-\xi_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}]}\left[\int_0^1(1-\vartheta)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}(1-\vartheta)^{\rho}]\phi'(\vartheta\xi_1+(1-\vartheta)\xi_2)\mathrm{d}\vartheta\right.\\ &\left.-\int_0^1(\vartheta)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}(\vartheta)^{\rho}]\phi'(\vartheta\xi_1+(1-\vartheta)\xi_2)\mathrm{d}\vartheta.\right]\end{aligned} \tag{12}$$

3. Main Results

In this section, we establish new Hermite–Hadamard type inequalities for generalized p-convex functions in terms of Raina's fractional integral operators.

Theorem 2. *Let ϕ: $I \longrightarrow \mathbb{R}$ be generalized p-convex function and provided η (.,.) is bounded from above on $\phi(I)\times\phi(I)$ and $\phi \in L[\xi_1, \xi_2]$ with $\xi_1 < \xi_2$ and $p > 0$. Then, following fractional integral inequality holds:*

$$\begin{aligned}\phi\left(\left[\frac{\xi_1^p+\xi_2^p}{2}\right]^{1/p}\right) - N_{\eta} &\le \frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]},\\ &\le \frac{\phi(\xi_1)+\phi(\xi_2)}{2} + \frac{\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}M_{\eta},\end{aligned} \tag{13}$$

where $\sigma_1(k) = \sigma(k)(k\rho+\lambda)$, for all $k = 0, 1, 2, \ldots$ and N_{η} and M_{η} are bounds of ϕ.

Proof. From inequality (6), we have

$$\phi\left(\frac{x^p+y^p}{2}\right)^{1/p} - \frac{N_{\eta}}{2} \le \frac{\phi(x)+\phi(y)}{2} + \frac{N_{\eta}}{2}, \tag{14}$$

where N_{η} are bounds of ϕ. Substitute $x^P = \vartheta\xi_1^p + (1-\vartheta)\xi_2^p$ and $y^p = (1-\vartheta)\xi_1^p + \vartheta\xi_2^p$; then, (6) can be written as

$$2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p} - N_{\eta} \le \phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p} + \phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p} + N_{\eta}. \tag{15}$$

Multiplying both sides by $\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(\xi_2^p-\xi_1^p)^{\rho}\vartheta^{\rho}]$, we obtain

$$\begin{aligned}&\left[2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p} - N_{\eta}\right]\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right] \le \vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\\ &+\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p} + \vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]N_{\eta}.\end{aligned} \tag{16}$$

Integrate over $\vartheta \in [0,1]$, we obtain

$$\left[2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p} - N_\eta\right]\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right] \le \int_0^1 \vartheta^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\mathrm{d}\vartheta + \int_0^1 \vartheta^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p}\mathrm{d}\vartheta + \int_0^1 \vartheta^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]N_\eta\mathrm{d}\vartheta. \tag{17}$$

With the convenient change of the variable, we can observe that

$$\begin{aligned}&\int_0^1 \vartheta^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\mathrm{d}\vartheta\\
&=\frac{-p}{\xi_2^p-\xi_1^p}\int_{\xi_2}^{\xi_1}\left(\frac{\xi_2^p-x^p}{\xi_2^p-\xi_1^p}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\left(\frac{\xi_2^p-x^p}{\xi_2^p-\xi_1^p}\right)^{\rho}\right]\phi(x)x^{p-1}\mathrm{d}x\\
&=\frac{p}{\left(\xi_2^p-\xi_1^p\right)^{\lambda}}\int_{\xi_1}^{\xi_2}\left(\xi_2^p-x^p\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\left(\frac{\xi_2^p-x^p}{\xi_2^p-\xi_1^p}\right)^{\rho}\right]\phi(x)x^{p-1}\mathrm{d}x\\
&=\frac{p}{\left(\xi_2^p-\xi_1^p\right)^{\lambda}}\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2).\end{aligned} \tag{18}$$

Similarly, the second integral can be written as

$$\int_0^1 \vartheta^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p}\mathrm{d}\vartheta = \frac{p}{\left(\xi_2^p-\xi_1^p\right)^{\lambda}}\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1). \tag{19}$$

Now, equation (17) becomes

$$\begin{aligned}&\left[2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p} - N_\eta\right]\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right] \le \frac{p}{\left(\xi_2^p-\xi_1^p\right)^{\lambda}}\left[\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right] + \mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]N_\eta\\
&\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p} - N_\eta \le \frac{p}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\left[\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right],\end{aligned} \tag{20}$$

which is the left-hand side of inequality (13). To prove right-hand side of (13), using the Definition 4 of generalized p-convex function,

$$\begin{aligned}&\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\leq\phi(\xi_2)+\vartheta\eta(\phi(\xi_1),\phi(\xi_2)),\\&\phi\left(\vartheta\xi_2^p+(1-\vartheta)\xi_1^p\right)^{1/p}\leq\phi(\xi_1)+\vartheta\eta(\phi(\xi_2),\phi(\xi_1)).\end{aligned}\tag{21}$$

Multiplying both inequalities by $\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(\xi_2^p-\xi_1^p)^\rho\vartheta^\rho]$ and then adding, we obtain

$$\begin{aligned}&\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}+\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\phi\left(\vartheta\xi_2^p+(1-\vartheta)\xi_1^p\right)^{1/p}\\&\leq\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\left[\phi(\xi_2)+\vartheta\eta(\phi(\xi_1),\phi(\xi_2))\right]+\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\left[\phi(\xi_1)+\vartheta\eta(\phi(\xi_2),\phi(\xi_1))\right].\end{aligned}\tag{22}$$

Integrate over $\vartheta\in[0,1]$, we obtain

$$\begin{aligned}&\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right]}{\left(\xi_2^p-\xi_1^p\right)^\lambda}\leq[\phi(a)+\phi(\xi_2)]\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]\\&+\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right][\eta(\phi(\xi_1),\phi(\xi_2))+\eta(\phi(\xi_2),\phi(\xi_1))]\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^\lambda\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]}\\&\leq\frac{[\phi(\xi_1)+\phi(\xi_2)]}{2}+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]}{\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]}M_\eta,\end{aligned}\tag{23}$$

where $\sigma_1(k)=\sigma(k)(k\rho+\lambda)$. This completes the proof.

Corollary 1. *Let ϕ: $I\longrightarrow\mathbb{R}$ be generalized p-convex function and provided η (.,.) is bounded from above on $\phi(I)\times\phi(I)$ and $\phi\in L[a,\xi_2]$ with $a<\xi_2$ and $p>0$. Then, the following inequality holds:*

$$\begin{aligned}\phi\left(\left[\frac{\xi_1^p+\xi_2^p}{2}\right]^{1/p}\right)-N_\eta&\leq\frac{p\Gamma(\alpha+1)\left[\left(I^\alpha_{\xi_1^+}\phi\right)(\xi_2)+\left(I^\alpha_{\xi_2^-}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^\alpha},\\&\leq\frac{\phi(\xi_1)+\phi(\xi_2)}{2}+\frac{\alpha}{\alpha+1}M_\eta,\end{aligned}\tag{24}$$

where N_η and M_η are bounds of ϕ.

Proof. By taking $\lambda=\alpha$, $\sigma=(1,0,0,\ldots)$, $w=0$, and $p=1$, we obtain

$$\begin{aligned}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]&=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}=\frac{1}{\Gamma(\alpha+1)},\\\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]&=\frac{\alpha}{\Gamma(\alpha+2)},\\\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(x)&=I^\alpha_{\xi_1^+}\phi x,\\\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(x)&=I^\alpha_{\xi_2^-}\phi x.\end{aligned}\tag{25}$$

Making the substitution in (13), we obtain (26).

Remark 1. In Corollary 1, if we take $\eta(x,y)=x-y$, $\lambda=\alpha$, $\sigma=(1,0,0,\ldots)$, $w=0$, and $p=1$, then we get Theorem 1.4 of [29, 31].

Theorem 3. *Let ϕ: $[\xi_1,\xi_2]\longrightarrow\mathbb{R}$ be a differentiable function on (ξ_1,ξ_2) with $\xi_1<\xi_2$. If $|\phi'|$ is a generalized p-convex function on $[\xi_1,\xi_2]$, then the following inequality for fractional integral operator holds:*

$$\begin{aligned}&\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^\lambda\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]}\right|\\&\leq\frac{\left(\xi_2^p-\xi_1^p\right)\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]}\left[2|\phi'(\xi_2)|+\eta\left(|\phi'(\xi_1)|,|\phi'(\xi_2)|\right)\right],\end{aligned}\tag{26}$$

where $\sigma_1(k)=\sigma(k)=(1-(1/2)^{k\rho+\lambda})$.

Proof. Using Lemma 1 and definition of generalized p-convexity of $|\phi'|$, we have

$$\begin{aligned}&\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{p\left[\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\right|=\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\Bigg|\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\\&\int_0^1(1-\vartheta)^{k\rho+\lambda}\phi'\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/2}\mathrm{d}\vartheta-\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\int_0^1\vartheta^{k\rho+\lambda}\phi'\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\mathrm{d}\vartheta\Bigg|\\&=\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\times\left|\int_0^1\left((1-\vartheta)^{k\rho+\lambda}-\vartheta^{k\rho+\lambda}\right)\left(\phi'\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)\right|\right.\\&\leq\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\times\left[\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda}-\vartheta^{k\rho+\lambda}\right)\left[|\phi'(\xi_2)|+\vartheta\eta\left(|\phi'(\xi_1)|,|\phi'(\xi_2)|\right)\right]\mathrm{d}\vartheta\right]\\&\quad+\left[\int_{1/2}^1\left(\vartheta^{k\rho+\lambda}-(1-\vartheta)^{k\rho+\lambda}\right)\left[|\phi'(\xi_2)|+\vartheta\eta\left(|\phi'(\xi_1)|,|\phi'(\xi_2)|\right)\right]\mathrm{d}\vartheta\right].\end{aligned}\tag{27}$$

It is easy to verify that

$$\begin{aligned}\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda}-\vartheta^{k\rho+\lambda}\right)\mathrm{d}\vartheta&=\frac{1-(1/2)^{k\rho+\lambda}}{k\rho+\lambda+1},\\\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda}-\vartheta^{k\rho+\lambda}\right)\vartheta\mathrm{d}\vartheta&=\frac{1/2}{k\rho+\lambda+1}-\frac{1}{k\rho+\lambda+2},\\\int_{1/2}^1\left(\vartheta^{k\rho+\lambda+1}-\vartheta(1-\vartheta)^{k\rho+\lambda}\right)\mathrm{d}\vartheta&=\frac{1}{k\rho+\lambda+2}-\frac{1}{2(k\rho+\lambda+1)}.\end{aligned}\tag{28}$$

Equation (27) becomes

$$\begin{aligned}&\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{p\left[\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\right|\leq\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\\&\qquad\times\left[2|\phi'(\xi_2)|\frac{1-(1/2)^{k\rho+\lambda}}{k\rho+\lambda+1}+\eta\left(|\phi'(\xi_1)|,|\phi'(\xi_2)|\right)\frac{1-(1/2)^{k\rho+\lambda}}{k\rho+\lambda+1}\right].\end{aligned}\tag{29}$$

Finally, we can write it as

$$\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{p\left[\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\right|\leq,\frac{\left(\xi_2^p-\xi_1^p\right)\mathscr{F}^{\sigma_1}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}{2\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\left[2|\phi'(\xi_2)|+\eta\left(|\phi'(\xi_1)|,|\phi'(\xi_2)|\right)\right],\tag{30}$$

where $\sigma_1(k) = \sigma(k) = (1 - (1/2)^{k\rho+\lambda})$, which is our required result.

Corollary 2. *Let ϕ: $[\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) with $\xi_1 < \xi_2$. If $|\phi'|$ is a generalized p-convex function on $[\xi_1, \xi_2]$; then, the following inequality for fractional integral operator holds:*

$$\left|\frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{p\Gamma(\alpha+1)\left[\left(I^{\alpha}_{\xi_2^-}\phi\right)(\xi_1) + \left(I^{\alpha}_{\xi_1^+}\phi\right)(\xi_2)\right]}{2\left(\xi_2^p - \xi_1^p\right)^{\alpha}}\right| \leq \frac{\left(\xi_2^p - \xi_1^p\right)}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[2|\phi'(\xi_2)| + \eta\left(|\phi'(\xi_1)|, |\phi'(\xi_2)|\right)\right]. \quad (31)$$

Proof. By taking $\lambda = \alpha$, $\sigma = (1, 0, 0, \ldots)$, $w = 0$, and $p = 1$, we obtain

$$\begin{aligned}
\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right] &= \frac{1}{\Gamma(\alpha+1)},\\
\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right] &= \frac{1 - 1/2^{\alpha}}{\Gamma(\alpha+2)},\\
\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(x) &= I^{\alpha}_{\xi_1^+}\phi x,\\
\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w}\phi\right)(x) &= I^{\alpha}_{\xi_2^-}\phi x.
\end{aligned} \quad (32)$$

Making the substitution in (35), we obtain (31).

Remark 2. In Corollary 2, if we take $\eta(x, y) = x - y$, $\lambda = \alpha$, $\sigma = (1, 0, 0, \ldots)$, $w = 0$, and $p = 1$, then we obtain Theorem 1.5 in [29, 31].

Theorem 4. *Let ϕ: $[\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) with $\xi_1 < \xi_2$. If $|\phi'|^{q_1}$ is a generalized p-convex function on $[\xi_1, \xi_2]$ with $q_1 = p_1/p_1 + 1$ for some fixed $p_1 > 0$, then the following inequality for fractional integral operator holds:*

$$\begin{aligned}
&\left|\frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p - \xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\right|\\
&\leq \frac{\left(\xi_2^p - \xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma_1(k)}{\Gamma(k\rho_1 + \lambda_1 + 1)}\left[w^k\left(\left(\xi_2^p - \xi_1^p\right)^{1/p}\right)^{k\rho_1}\right]\\
&\times\left(\left[\frac{|\phi'(\xi_2)|^{q_1}}{2} + \vartheta\frac{\eta\left(|\phi'(\xi_1)|^{q_1}, |\phi'(\xi_2)|^{q_1}\right)}{8}\right]d\vartheta\right)^{1/q_1}\\
&+\left(\left[\frac{|\phi'(\xi_2)|^{q_1}}{2} + \vartheta\frac{\eta\left(|\phi'(\xi_1)|^{q_1}, |\phi'(\xi_2)|^{q_1}\right)}{8}\right]d\vartheta\right)^{1/q_1},
\end{aligned} \quad (33)$$

where $\rho_1 = \rho p$, $\lambda_1 = \lambda p$, and

$$\sigma_1(k) = \sigma(k)\left(\frac{1 - (1/2)^{(k\rho+\lambda)p_1+1}}{(k\rho + \lambda + 1)p_1 + 1}\right)^{1/p_1}, \quad (34)$$

for all $k = 0, 1, 2 \ldots$.

Proof. Using Lemma 1, definition of generalized p-convexity of ϕ, and Hölder inequality, we have

$$\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\right|$$

$$\begin{aligned}
&\leq\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\\
&\times\left[\left(\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda}-\vartheta^{k\rho+\lambda}\right)^{p_1}\mathrm{d}\vartheta\right)^{1/p_1}\left(\int_0^{1/2}\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right]\\
&+\left(\left[\int_{\frac{1}{2}}^{1}\left(\vartheta^{k\rho+\lambda}-(1-\vartheta)^{k\rho+\lambda}\right)^{p_1}\mathrm{d}\vartheta\right)^{1/p_1}\left(\int_{1/2}^{1}\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right]\\
&\leq\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\\
&\times\left[\left(\int_0^{1/2}\left((1-\vartheta)^{(k\rho+\lambda)p_1}-\vartheta^{(k\rho+\lambda)p_1}\right)\mathrm{d}\vartheta\right)^{1/p_1}\left(\int_0^{1/2}\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right]\\
&+\left(\left[\int_{1/2}^{1}\left(\vartheta^{(k\rho+\lambda)p_1}-(1-\vartheta)^{(k\rho+\lambda)p_1}\right)d\vartheta\right)^{1/p_1}\left(\int_{1/2}^{1}\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(a)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right]\\
&\leq\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\left(\frac{1-(1/2)^{(k\rho+\lambda)p_1+1}}{(k\rho+\lambda+1)p_1+1}\right)^{1/p_1}\\
&\times\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1}+\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1}\\
&\leq\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma_1(k)}{\Gamma(k\rho_1+\lambda_1+1)}\left[w^k\left(\left(\xi_2^p-\xi_1^p\right)^{1/p}\right)^{k\rho_1}\right]\\
&\times\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1}+\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1},
\end{aligned}\tag{35}$$

which completes the proof.

Data Availability

All data required for this research are included within the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Changyue Chen wrote the final version of this paper, verified the results, and arranged the funding for this paper, Muhammad Shoaib Saleem proposed the problem, proved the results, and supervised the work, and Muhammad Sajid Zahoor wrote the first version of the paper.

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References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.

[2] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.

[3] Y. Ouafik, "Modelling and simulation of a dynamic contact problem in thermo-piezoelectricity," *Engineering and Applied Science Letters*, vol. 4, no. 2, pp. 43–52, 2021.

[4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, Netherlands, 2006.

[5] R. I. Gweryina, F. S. Kaduna, and M. Y. Kura, "Qualitative analysis of a mathematical model of divorce epidemic with anti-divorce therapy," *Engineering and Applied Science Letters*, vol. 4, no. 2, pp. 1–11, 2021.

[6] S. Belarbi and Z. Dahmani, "On some new fractional integral inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 3, pp. 1–12, 2009.

[7] M. E. Omaba, L. O. Omenyi, and L. O. Omenyi, "Generalized fractional Hadamard type inequalities for \(Q_s\)-class functions of the second kind," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 270–278, 2021.

[8] Z. Dahmani, "On Minkowski and Hermite-Hadamard integral inequalities via fractional integration," *Annals of Functional Analysis*, vol. 1, no. 1, pp. 51–58, 2010.

[9] Z. Toghani and L. Gaggero-Sager, "Generalized fractional differential ring," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 279–287, 2021.

[10] M. A. Khan, T. Ali, S. S. Dragomir, and M. Z. Sarikaya, "Hermite-Hadamard type inequalities for conformable fractional integrals," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 112, no. 4, pp. 1033–1048, 2018.

[11] P. Fan, "New-type Hoeffding's inequalities and application in tail bounds," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 248–261, 2021.

[12] M. Kunt and İ. İşcan, "Hermite-Hadamard-Fejér type inequalities forp-convex functions," *Arab Journal of Mathematical Sciences*, vol. 23, no. 2, pp. 215–230, 2017.

[13] M. Tariq, S. I. Butt, and S. I. Butt, "Some Ostrowski type integral inequalities via generalized harmonic convex functions," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 200–208, 2021.

[14] M. A. Noor, K. I. Noor, and S. Iftikhar, "Nonconvex functions and integral inequalities," *Journal of Mathematics*, vol. 47, no. 2, p. 00, 2015.

[15] M. Gurbuz, A. O. Akdemir, S. Rashid, and E. Set, "Hermite–Hadamard inequality for fractional integrals of Caputo–Fabrizio type and related inequalities," *Journal of Inequalities and Applications*, vol. 2020, no. 1, pp. 1–10, 2020.

[16] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–13, 2015.

[17] M. A. Dokuyucu, E. Celik, H. Bulut, and H. M. Baskonus, "Cancer treatment model with the Caputo-Fabrizio fractional derivative," *The European Physical Journal Plus*, vol. 133, no. 3, pp. 1–6, 2018.

[18] D. Baleanu, A. Jajarmi, H. Mohammadi, and S. Rezapour, "A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative," *Chaos, Solitons and Fractals*, vol. 134, Article ID 109705, 2020.

[19] D. Baleanu, H. Mohammadi, and S. Rezapour, "A mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–19, 2020.

[20] K. Shah, M. Sarwar, and D. Baleanu, "Study on Krasnoselskii's fixed point theorem for Caputo–Fabrizio fractional differential equations," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–9, 2020.

[21] I. Iscan, "New estimates on generalization of some integral inequalities for s-convex functions and their applications," 2012, http://arxiv.org/abs/1207.7114.

[22] K. Murota and A. Tamura, "New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities," *Discrete Applied Mathematics*, vol. 131, no. 2, pp. 495–512, 2003.

[23] P. Burai and A. Hzy, "On approximately h-convex functions," *Journal of Convex Analysis*, vol. 18, no. 2, pp. 447–454, 2011.

[24] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of hadamard inequalities," *IEEE access*, vol. 6, pp. 64946–64953, 2018.

[25] K. S. Zhang and J. P. Wan, "p-Convex functions and their properties," *Pure Applied Mathamatics*, vol. 23, no. 1, pp. 130–133, 2007.

[26] C. Y. Jung, M. S. Saleem, W. Nazeer, M. S. Zahoor, A. Latif, and S. M. Kang, "Unification of generalized and-convexity," *Journal of Function Spaces*, vol. 2020, Article ID 4016386, 6 pages, 2020.

[27] R. K. Raina, "On generalized Wright's hypergeometric functions and fractional calculus operators," *East Asian Mathematical Journal*, vol. 21, no. 2, pp. 191–203, 2005.

[28] R. P. Agarwal, M. J. Luo, and R. K. Raina, "On Ostrowski type inequalities," *Fasciculi Mathematici*, vol. 56, no. 1, pp. 5–27, 2016.

[29] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9–10, pp. 2403–2407, 2013.

[30] E. Set, B. elik, and A. O. Akdemir, "Some new Hermite-Hadamard type inequalities for quasi-convex functions via fractional integral operator," in *AIP Conference Proceedings*vol. 1833, no. 1, AIP Publishing LLC, Article ID 020021, 2017.

[31] T. Ali, M. A. Khan, and Y. Khurshidi, "Hermite-Hadamard inequality for fractional integrals via eta-convex functions," *Acta Mathematica Universitatis Comenianae*, vol. 86, no. 1, pp. 153–164, 2017.

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Research Article

A Novel Method for Developing Efficient Probability Distributions with Applications to Engineering and Life Science Data

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In this paper, a new approach for deriving continuous probability distributions is developed by incorporating an extra parameter to the existing distributions. Frechet distribution is used as a submodel for an illustration to have a new continuous probability model, termed as modified Frechet (MF) distribution. Several important statistical properties such as moments, order statistics, quantile function, stress-strength parameter, mean residual life function, and mode have been derived for the proposed distribution. In order to estimate the parameters of MF distribution, the maximum likelihood estimation (MLE) method is used. To evaluate the performance of the proposed model, two real datasets are considered. Simulation studies have been carried out to investigate the performance of the parameters' estimates. The results based on the real datasets and simulation studies provide evidence of better performance of the suggested distribution.

1. Introduction

In the last few years, the literature of distribution theory has become rich due to the induction of additional parameters in the existing distribution. The inclusion of an extra parameter has shown greater flexibility compared to competitive models. The inclusion of a new parameter can be performed either using the available generator or by developing a new technique for generating new improved distribution compared to classical baseline distribution. Azzalini [1] proposed a modified form of the normal distribution by inserting an extra parameter, known as skew normal distribution, which indicated greater flexibility over normal distribution. Mudholkar and Srivastava [2] introduced exponentiated Weibull distribution by introducing a shape parameter in two-parameter Weibull distribution. Its cumulative distribution function is as follows:

$$G(y;\alpha,\lambda,\beta)=\left(1-e^{-\lambda x^{\alpha}}\right)^{\beta},\quad x,\alpha,\lambda,\beta>0. \tag{1}$$

This model provides greater flexibility compared to the base line distribution. Note that, for $\beta=1$, the exponentiated Weibull distribution and base line distribution coincide. Later on, various researchers have introduced different forms of exponentiated distributions; see, for example, the work of Gupta et al. [3]. Marshall and Olkin [4] introduced another technique to add an extra parameter to a probability distribution. Eugene et al. [5] suggested the beta-generated technique and applied this method to beta distribution and proposed beta-generated distribution by incorporating an extra parameter in beta distribution. Alzaatreh et al. [6] proposed a new technique and produced T-X class of continuous probability models by interchanging the probability density function of beta distribution with a probability density function, $g(t)$, of a continuous random variable and used a function $W(F(x))$ which fulfills some particular conditions. Recently, Aljarrah et al. [7] introduced T-X class of distributions using quantile functions. For more details about new techniques to produce probability distributions, see the works of Lee et al. [8] and Jones [9]. Al-Aqtash et al. [10] proposed a new class of models using the logit function as a baseline and obtained the particular case referred to as Gumbel–Weibull distribution. Alzaatreh et al. [11] studied the gamma-X class of distributions and recommended the particular case using the normal distribution as a baseline distribution. Abid and Abdulrazak [12] introduced truncated Frechet-G class of distributions. Korkmaz and Genc [13] presented a generalized two-sided class of probability distributions. Alzaghal et al. [14] worked on the T-X class of distributions. Aldeni et al. [15] used the quantile function of generalized lambda distribution and introduced a new family. For more details, see the works of Cordeiro et al. [16], Alzaatreh et al. [17], and Nasir et al. [18]. The more recent modified Weibull distributions are introduced by Abid and Abdulrazak [12], Korkmaz and Genc [13], Aldeni et al. [15], Cordeiro et al. [16], and Pe and Jurek [19].

Pearson [20] used the system of differential equation technique and produced new probability distributions. Burr [21] also proposed a new method by using the differential equation method, which may take on a wide variety of forms of the continuous distributions. Since 1980, methodologies of suggesting new models moved to the inclusion of extra parameters to an existing family of distributions to increase the level of flexibility. These include Weibull-G presented by Bourguignon et al. [22], Garhy-G proposed by Elgarhy et al. [23], Kumaraswamy (Kw-G) proposed by Cordeiro and de Castro [24], Type II half-logistic-G by Hassan et al. [25], exponentiated extended-G suggested by Elgarhy et al. [26], the Kumaraswamy–Weibull introduce by Hassan and Elgarhy [27], exponentiated Weibull by Hassan and Elgarhy [28], odd Frechet-G introduced by Haq and Elgarhy [29], and Muth-G by Almarashi and Elgarhy [30]. For a short review, one can study the work of Kotz and Vicari [31]. Recently, Mahdavi and Kundu [32] developed a new technique for proposing probability distributions which is referred as alpha power transformation (APT) technique, defined by the cumulative distribution function (CDF) as follows:

$$F_{\text{APT}}(x) = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1, \\ F(x), & \text{if } \alpha = 1. \end{cases} \tag{2}$$

The core purpose of introducing new family of distributions is to overcome the difficulties that are present in the existing probability distributions. In this study, we suggest a new method for obtaining new continuous probability distributions. Frechet distribution is used as a submodel to have a new probability distribution which is referred as modified Frechet (MF) distribution. Our proposed family of distributions also models monotonic and nonmonotonic hazard rate function and provides increased flexibility as compared to the already available probability distribution in the literature.

2. The Proposed Class of Distributions

The proposed class of probability distributions is termed as modified Frechet class (MFC) of distributions. The cumulative distribution function (CDF) and probability density function (PDF) of the suggested class of distributions are given by the following expressions:

$$G_{\text{MFC}}(x) = \frac{e^{-(F(x))^{\alpha}} - 1}{\left(e^{-1} - 1\right)}, \quad x > 0, \tag{3}$$

$$g_{\text{MFC}}(x) = \frac{\alpha f(x)(F(x))^{\alpha-1} e^{-(F(x))^{\alpha}}}{\left(1 - e^{-1}\right)}, \quad x > 0, \tag{4}$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the baseline distribution and α is the shape parameter. The method in (3) is used to produce a new model referred as modified Frechet (MF) distribution with the aim to attain more flexibility in modeling life time data. The derivation of MF distribution is given in Section 2.1.

2.1. The Proposed Distribution. The CDF of Frechet distribution is as follows:

$$F(x) = e^{-x^{-\beta}}, \quad x > 0, \tag{5}$$

where β is the shape parameter.

This portion of the manuscript is concerned with introducing a subclass of MF class of distributions using the cumulative distribution function of Frechet distribution. The resultant distribution is what we call modified Frechet (MF) distribution.

Definition 1. A random variable X is said to have MF distribution with two parameters α and β if its PDF is given as follows:

$$f_{\text{MF}}(x) = \frac{\alpha\beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{\left(1 - e^{-1}\right)}, \quad x > 0. \tag{6}$$

Its CDF is given by

$$F_{\text{MF}}(x) = \frac{e^{-\left(e^{-\alpha x^{-\beta}}\right)} - 1}{\left(e^{-1} - 1\right)}, \quad x > 0. \tag{7}$$

The hazard rate function of MF distribution is as follows:

$$h_{\text{MF}}(x) = \frac{\alpha\beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}} \left(e^{-1} - 1\right)}{\left(e^{-1} - e^{-\left(e^{-\alpha x^{-\beta}}\right)}\right)\left(1 - e^{-1}\right)}. \tag{8}$$

The survival function of the MF model is given by

$$S_{\mathrm{MF}}(x) = \frac{e^{-1} - e^{-\left(e^{-\alpha x^{-\beta}}\right)}}{e^{-1} - 1}. \tag{9}$$

These four functions have been plotted in Figures 1 and 2
.

Lemma 1. *If $f(x)$ is decreasing function, then $f_{MF}(x)$ is also decreasing function for $0 \le \alpha < 1$ and $\beta > 0$.*

Proof. If $f(x)$ is a differentiable function and if $f'(x) < 0$ or $(\mathrm{d}/\mathrm{d}x)\ln f(x) < 0$ for all X, then $f(x)$ is a decreasing function and vice versa.

Taking first derivative of $\ln f_{\mathrm{MF}}(x)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\ln f_{\mathrm{MF}}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\frac{\alpha\beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{\left(1 - e^{-1}\right)}\right], \\ \frac{\mathrm{d}}{\mathrm{d}x}\ln f_{\mathrm{MF}}(x) &= -\frac{\beta}{x} - \frac{1}{x} - \alpha\beta x^{-\beta-1}\left(e^{-\alpha x^{-\beta}} - 1\right). \end{aligned} \tag{10}$$

Thus, for $0 \le \alpha < 1$ and $\beta > 0$, $(\mathrm{d}/\mathrm{d}x)\ln f_{\mathrm{MF}}(x) < 0$. This concludes the lemma. □

Lemma 2. *For $\alpha < 1$, if $f(x)$ is log-convex and decreasing function, then $h_{MF}(x)$ is a decreasing function.*

Proof. If the second-order differential of $f(x)$ exists and $(\mathrm{d}^2/\mathrm{d}x^2)\ln f(x) > 0$, then $f(x)$ is said to be log-convex. Taking second-order derivative of equation (10), we obtain

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln f_{\mathrm{MF}}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{\beta}{x} - \frac{1}{x} - \alpha\beta x^{-\beta-1}\left(e^{-\alpha x^{-\beta}} - 1\right)\right), \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln f_{\mathrm{MF}}(x) &= \frac{\beta}{x^2} + \frac{1}{x^2} + \alpha\beta x^{-\beta-2}\left[(-\beta-1)\left(e^{-\alpha x^{-\beta}} - 1\right) + \alpha\beta x^{-\beta} e^{-\alpha x^{-\beta}}\right]. \end{aligned} \tag{11}$$

When $0 \le \alpha < 1$ and $\beta > 0$, then $(\mathrm{d}^2/\mathrm{d}x^2)\ln f_{\mathrm{MF}}(x) > 0$. Therefore, for $0 < \alpha < 1$, $f_{\mathrm{MF}}(x)$ is log-convex [33]. □

2.1.1. Quantile Function. Let $X \sim \mathrm{MF}(\alpha, \beta)$; then, the quantile function is as follows:

$$F(X) = U \Longrightarrow X = F^{-1}(U), \tag{12}$$

where U is uniformly distributed random variable. The quantile function of the MF model is given as

$$X_p = \left[-\frac{1}{\alpha}\ln\left\{-\ln\left(u\left(e^{-1} - 1\right) + 1\right)\right\}\right]^{(-1/\beta)}. \tag{13}$$

2.1.2. Median. Median of MF distribution is obtained by substituting $u = 1/2$ in equation (13), that is,

$$\text{Median} = \left[-\frac{1}{\alpha}\ln\left\{-\ln\left(\frac{1}{2}\left(e^{-1} + 1\right)\right)\right\}\right]^{(-1/\beta)}. \tag{14}$$

2.1.3. Mode. Mode of MF is obtained by solving the following equation for x.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} f_{\mathrm{MF}}(x) = 0, \quad \text{i.e. } \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\alpha\beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{\left(1 - e^{-1}\right)}\right) = 0, \\ &x^{-\beta}\left(e^{-\alpha x^{-\beta}} - 1\right) = \frac{-(\beta+1)}{\alpha\beta}. \end{aligned} \tag{15}$$

Mode of the distribution satisfies the above equation.

2.2. r^{th} Moment of MF Distribution. Let $X \sim \mathrm{MF}(\alpha, \beta)$; and the r^{th} moment of X is as follows:

$$\mu_r/ = E(X^r) = \int_0^\infty x^r \frac{\alpha\beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{\left(1 - e^{-1}\right)}\,\mathrm{d}x. \tag{16}$$

Using $x^{-\beta} = y$ and then $e^{-\alpha y} = z$ in (16), the expression will take the following form:

$$\mu_r/ = E(X^r) = \frac{(-1/\alpha)^m}{\left(1 - e^{-1}\right)} \int_0^1 (\ln z)^m e^{-z}\,\mathrm{d}z, \tag{17}$$

where $m = (-r/\beta)$.

Again substituting $\log z = u$ in (17) and after some simplification, the expression becomes

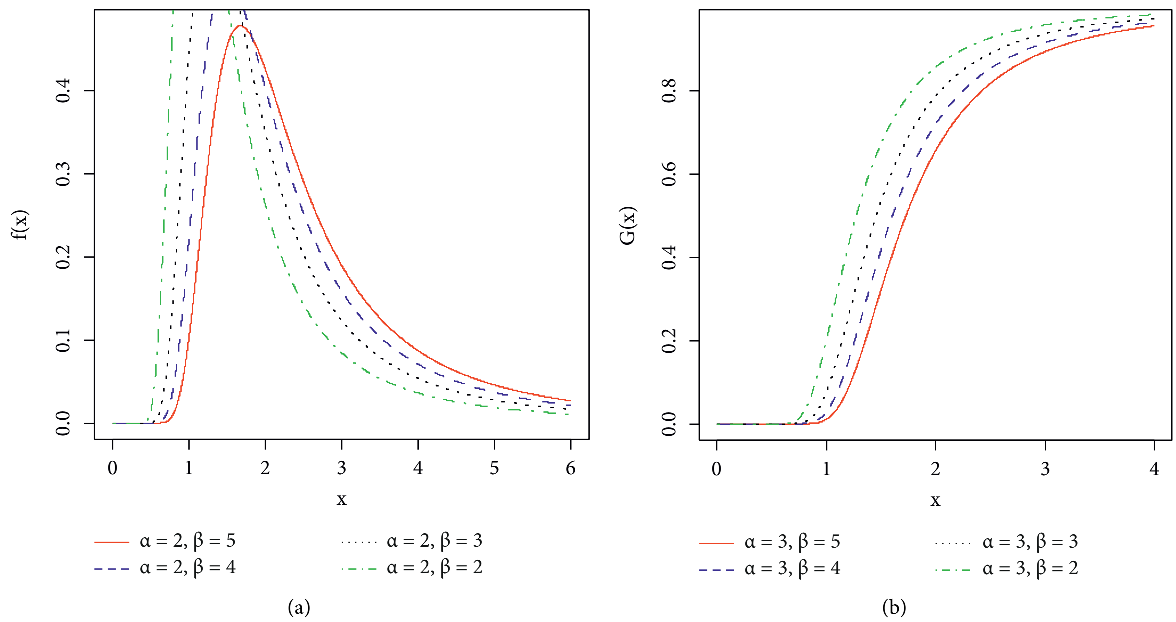

Figure 1: Graph of CDF and PDF of the MF model.

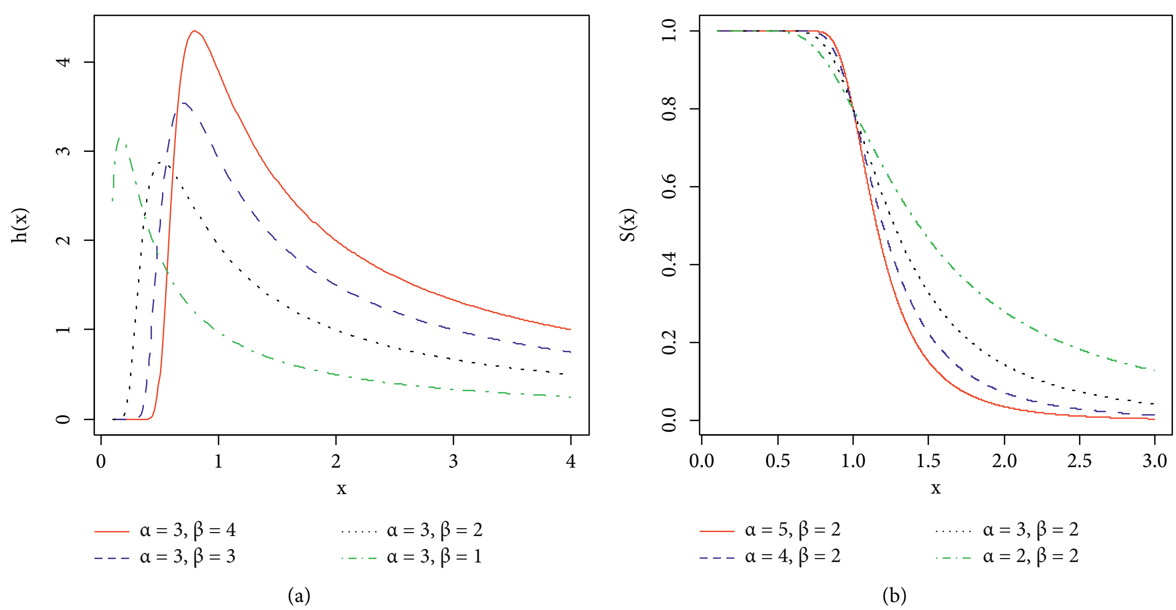

Figure 2: Graph of survival function and hazard function of MF distribution.

$$\mu_r/ = E(X^r) = \frac{(-1/\alpha)^m}{(e^{-1}-1)}\int_{-\infty}^{0} u^m e^{-e^u} e^u \mathrm{d}z. \quad (18)$$

Using series notation $e^{-e^u} = \sum_{r=0}^{\infty}((-e^u)^k/k!)$ in (18), we obtain

$$\mu_r/ = E(X^r) = \frac{(-1/\alpha)^m}{(e^{-1}-1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\lim_{b\longrightarrow-\infty} - \mathrm{it}\int_{b}^{0} u^m e^{u(k+1)}\mathrm{d}z,$$
$$\mu_r/ = E(X^r) = \frac{(-1/\alpha)^m}{(e^{-1}-1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\lim_{b\longrightarrow-\infty} - \mathrm{it}\left[\frac{\Gamma(m+1,-bk-b)-\Gamma(m+1,0)}{(-k-1)^m(k+1)}\right], \quad (19)$$

where $b > 0$.

2.3. Moment Generating Function. Let $X \sim \mathrm{MF}(\alpha,\beta)$; then, the moment generating function is given by

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\frac{\alpha\beta x^{-(\beta+1)}e^{-\alpha x^{-\beta}-e^{-\alpha x^{-\beta}}}}{(1-e^{-1})}\mathrm{d}x. \quad (20)$$

Using series $e^{tx} = \sum_{r=0}^{\infty}(t^r x^r/r!)$ in (20) and simplifying, we have

$$M_x(t) = \frac{(-1/\alpha)^m}{(e^{-1}-1)}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r(-1)^k}{r!k!}\lim_{b\longrightarrow-\infty} - \mathrm{it}\left[\frac{\Gamma(m+1,-bk-b)-\Gamma(m+1,0)}{(-k-1)^m(k+1)}\right], \quad (21)$$

where $m = (-r/\beta)$ and $b > 0$.

2.4. Order Statistics. Let $X_1, X_2, \ldots, X_n$ be a random sample taken from MF distribution, and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics. Then, the probability density function of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)[F(x)]^{i-1}[1-F(x)]^{(n-i)}. \quad (22)$$

Substitute PDF and CDF of MF in equation (22), and we obtain distribution of i^{th} order statistic as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\left(\frac{\alpha\beta x^{-(\beta+1)}e^{-\alpha x^{-\beta}-e^{-\alpha x^{-\beta}}}}{(1-e^{-1})}\right)\left[\frac{e^{-(e^{-\alpha x^{-\beta}})}-1}{e^{-1}-1}\right]^{i-1}\left[1-\left(\frac{e^{-(e^{-\alpha x^{-\beta}})}-1}{e^{-1}-1}\right)\right]^{(n-i)},$$
$$f_{i:n}(x) = \frac{n!}{(1-e^{-1})(i-1)!(n-i)!}\frac{1}{(e^{-1}-1)^{n-1}}\left[e^{-(e^{-\alpha x^{-\beta}})}-1\right]^{(i-1)}\alpha\beta x^{-(\beta+1)}e^{-\alpha x^{-\beta}-e^{-\alpha x^{-\beta}}}\left[e^{-1}-e^{-(e^{-\alpha x^{-\beta}})}\right]^{(n-i)}. \quad (23)$$

Lemma 3. *The Renyi entropy of $X \sim MF(\alpha,\beta)$ is given as*

$$\mathrm{RE}_X(\nu) = \frac{1}{(1-\nu)}\log\left[\frac{\alpha^{((1-\nu)/\beta)}\beta^{\nu-1}(-1)^l}{(1-e^{-1})^{\nu}}\sum_{k=0}^{\infty}\frac{(-\nu)^k}{k!}\lim_{b\longrightarrow-\infty} - \mathrm{it}\left[\frac{\Gamma(l+1,-b\nu-bk+b)-\Gamma(l+1,0)}{(-\nu-k+1)^l(\nu+k-1)}\right]\right], \quad (24)$$

where $l = ((\beta+1)(\nu-1)/\beta)$ and $b > 0$.

Proof. The Renyi entropy of MF is given by

$$\mathrm{RE}_X(\nu) = \frac{1}{1-\nu}\log\left\{\int_{-\infty}^{+\infty} f(x)^{\nu}\mathrm{d}x\right\} = \frac{1}{1-\nu}\log\left[\int_0^{\infty}\left(\frac{\alpha\beta x^{-(\beta+1)}e^{-\alpha x^{-\beta}-e^{-\alpha x^{-\beta}}}}{(1-e^{-1})}\right)^{\nu}\mathrm{d}x\right]. \quad (25)$$

Put $e^{-\alpha x^{-\beta}} = y$ in (25); the expression will take the form

$$\mathrm{RE}_X(\nu) = \frac{1}{(1-\nu)}\log\left[\frac{\beta^{\nu-1}}{\left(1-e^{-1}\right)^{\nu}}\alpha^{((1-\nu)/\beta)}(-1)^{l}\int_0^1 (\ln y)^{l} y^{\nu-1} e^{-\nu y}\mathrm{d}y\right], \tag{26}$$

where $l = ((\beta+1)(\nu-1)/\beta)$. Using series notation $e^{-\nu y} = \sum_{k=0}^{\infty}((\nu)^k(-y)^k/k!)$ in (26), the expression will take the following form:

$$\mathrm{RE}_X(\nu) = \frac{1}{(1-\nu)}\log\left[\frac{\alpha^{((1-\nu)/\beta)}\beta^{\nu-1}(-1)^{l}}{\left(1-e^{-1}\right)^{\nu}}\sum_{k=0}^{\infty}\frac{(-\nu)^k}{k!}\int_0^1 (\ln y)^{l} y^{\nu-1+k}\mathrm{d}y\right]. \tag{27}$$

Again substituting $\ln y = z$ in (27) and simplifying, we obtain

$$\begin{aligned}
\mathrm{RE}_X(\nu) &= \frac{1}{(1-\nu)}\log\left[\frac{\alpha^{((1-\nu)/\beta)}\beta^{\nu-1}(-1)^{l}}{\left(1-e^{-1}\right)^{\nu}}\sum_{k=0}^{\infty}\frac{(-\nu)^k}{k!}\lim_{b\longrightarrow-\infty}-\mathrm{it}\int_b^0 z^{l} e^{z(\nu-1+k)}\mathrm{d}z\right],\\
\mathrm{RE}_X(\nu) &= \frac{1}{(1-\nu)}\log\left[\frac{\alpha^{((1-\nu)/\beta)}\beta^{\nu-1}(-1)^{l}}{\left(1-e^{-1}\right)^{\nu}}\sum_{k=0}^{\infty}\frac{(-\nu)^k}{k!}\lim_{b\longrightarrow-\infty}-\mathrm{it}\left[\frac{\Gamma(l+1,-b\nu-bk+b)-\Gamma(l+1,0)}{(-\nu-k+1)^{l}(\nu+k-1)}\right]\right],
\end{aligned} \tag{28}$$

where $l = ((\beta+1)(\nu-1)/\beta)$ and. $b > 0$. □

2.5. Mean Residual Life Function. Let X be the lifetime of an object having MF distribution. The mean residual life function is the average remaining life span that a component has survived until time t. The mean residual life function, say, $\mu(t)$, has the following expression:

$$\begin{aligned}
\mu(t) &= \frac{1}{P(X>t)}\int_t^{\infty} P(X>x)\mathrm{d}x, \quad t\geq 0,\\
\mu(t) &= \frac{1}{S(t)}\left(E(t)-\int_0^t xf(x)\mathrm{d}x\right)-t, \quad t\geq 0.
\end{aligned} \tag{29}$$

Note that

$$\int_0^t xf(x)\mathrm{d}x = \frac{1}{\left(1-e^{-1}\right)}\left(\frac{-1}{\alpha}\right)^{l}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\lim_{b\longrightarrow-\infty}-\mathrm{it}\left[\frac{\Gamma\left(l+1,\left((\alpha k+\alpha)/t^{\beta}\right)\right)\left(-\alpha/t^{\beta}\right)^{l}}{(k+1)\left((\alpha k+\alpha)/t^{\beta}\right)^{l}}-\frac{b^{l}\Gamma(l+1,(-bk-b))}{(-bk-b)^{l}(k+1)}\right], \tag{30}$$

$$E(t) = \frac{\alpha^{-l}(-1)^{l}}{\left(1-e^{-1}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\lim_{b\longrightarrow-\infty}-\mathrm{it}\left[\frac{\Gamma(l+1,-bk-b)-\Gamma(l+1,0)}{(-k-1)^{l}(k+1)}\right], \tag{31}$$

where $l = (-1/\beta)$. Put equation (9), (30), and (31) in (29), and we obtain

$$\mu(t) = \frac{\left(e^{-1}-1\right)}{\left(e^{-1}-e^{-\left(e^{-\alpha x^{-\beta}}\right)}\right)} \frac{1}{\left(1-e^{-1}\right)} \left(\frac{-1}{\alpha}\right)^{l} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} \lim_{b\longrightarrow -\infty} - \text{it} \; * \\ \left[\left[\frac{\Gamma(l+1,-bk-b)-\Gamma(l+1,0)}{(-k-1)^{l}(k+1)}\right] - \left[\frac{\Gamma\left(l+1,\left((\alpha k+\alpha)/t^{\beta}\right)\right)\left(-\alpha/t^{\beta}\right)^{l}}{(k+1)\left((\alpha k+\alpha)/t^{\beta}\right)^{l}} - \frac{b^{l}\Gamma(l+1,(-bk-b))}{(-bk-b)^{l}(k+1)}\right]\right] - t. \tag{32}$$

This is the final expression of mean residual life function.

2.6. Stress-Strength Parameter. Let X_1 and X_2 be two independently and identically distributed variables such that $X_1 \sim \mathrm{MF}(\alpha_1,\beta)$ and $X_2 \sim \mathrm{MF}(\alpha_2,\beta)$. Then, the stress-strength parameter is defined by

$$R = \int_{-\infty}^{+\infty} f_1(x)F_2(x)\mathrm{d}x. \tag{33}$$

Using equation PDF and CDF of MF in the above expression, the stress-strength parameter is given as

$$R = \int_0^{\infty} \left(\frac{\alpha\beta x^{-(\beta+1)} e^{-\alpha_1 x^{-\beta} - e^{-\alpha_1 x^{-\beta}}}}{\left(1-e^{-1}\right)}\right)\left(\frac{e^{-\left(e^{-\alpha_2 x^{-\beta}}\right)}-1}{\left(e^{-1}-1\right)}\right)\mathrm{d}x, \\ R = \frac{1}{\left(e^{-1}-1\right)\left(1-e^{-1}\right)} \int_0^{\infty} \alpha_1\beta x^{-\beta-1} e^{-\alpha_1 x^{-\beta} - e^{-\alpha_1 x^{-\beta}}} e^{-\left(e^{-\alpha_2 x^{-\beta}}\right)}\mathrm{d}x - \frac{1}{\left(e^{-1}-1\right)}. \tag{34}$$

Substituting $x^{-\beta} = y$ in (34), we obtain

$$R = \frac{1}{\left(e^{-1}-1\right)\left(1-e^{-1}\right)} \int_0^{\infty} \alpha_1 e^{-\alpha_1 y} e^{-e^{-\alpha_1 y}} e^{-\left(e^{-\alpha_2 y}\right)}\mathrm{d}y - \frac{1}{\left(e^{-1}-1\right)}. \tag{35}$$

Again putting $e^{-\alpha_1 y} = z$ in (35), it will take the following form:

$$R = \frac{1}{\left(e^{-1}-1\right)^2} \int_0^1 e^{-z} e^{-\left(e^{-\alpha_2(-\ln z/\alpha_1)}\right)}\mathrm{d}z - \frac{1}{\left(e^{-1}-1\right)}. \tag{36}$$

Using series representation $e^{-z} = \sum_{k=0}^{\infty}\left((-z)^k/k!\right)$ and $e^{-\left(e^{-\alpha_2(-\ln z/\alpha_1)}\right)} = \sum_{m=0}^{\infty}\left((-e^{(\alpha_2 \ln z/\alpha_1)})^m/m!\right)$ in (36), we get the expression as follows:

$$R = \frac{1}{\left(e^{-1}-1\right)^2} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{k!m!} \int_0^1 z^k e^{n \ln z}\mathrm{d}z - \frac{1}{\left(e^{-1}-1\right)}, \tag{37}$$

where $n = m(\alpha_2/\alpha_1)$. Using series representation $e^{n \ln z} = \sum_{i=0}^{\infty}\left((n \ln z)^i/i!\right)$ in (52), after simplification, we get the following expression:

$$R = \frac{1}{\left(e^{-1}-1\right)^2} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{k+m}n^i}{k!m!i!} \int_0^1 z^k (\ln z)^i\mathrm{d}z - \frac{1}{\left(e^{-1}-1\right)}. \tag{38}$$

Again substituting $\ln z = u$ in (38) and simplifying, we obtain

$$R = \frac{1}{\left(e^{-1}-1\right)^2} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{k+m}n^i}{k!m!i!} \int_{-\infty}^{0} u^i e^{u(k+1)}\mathrm{d}u - \frac{1}{\left(e^{-1}-1\right)}, \\ R = \frac{1}{\left(e^{-1}-1\right)^2} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{k+m}n^i}{k!m!i!} \lim_{b\longrightarrow -\infty} - \text{it} \int_{b}^{0} u^i e^{u(k+1)}\mathrm{d}u - \frac{1}{\left(e^{-1}-1\right)}, \\ R = \frac{1}{\left(e^{-1}-1\right)^2} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{k+m}n^i}{k!m!i!} \lim_{b\longrightarrow -\infty} - \text{it}\left[\frac{\Gamma(i+1,-bk-b)-\Gamma(i+1)}{(-k-1)^i(k+1)}\right] - \frac{1}{\left(e^{-1}-1\right)}, \tag{39}$$

where $n = m(\alpha_2/\alpha_1)$ and $b > 0$.

Lemma 4. *The mean waiting time, say $\overline{\mu}(t)$, of MF distribution is given by*

$$\overline{\mu}(t) = t + \frac{1}{\left(e^{-\left(e^{-\alpha t^{-\beta}}\right)} - 1\right)}\left(\frac{-1}{\alpha}\right)^{l}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\lim_{b\longrightarrow-\infty} - \text{it}\left[\frac{\Gamma\left(l+1,\left((\alpha k+\alpha)/t^{\beta}\right)\right)\left(-\alpha/t^{\beta}\right)^{l}}{(k+1)\left((\alpha k+\alpha)/t^{\beta}\right)^{l}} - \frac{b^{l}\Gamma(l+1,(-bk-b))}{(-bk-b)^{l}(k+1)}\right]. \tag{40}$$

Proof. By definition, the mean waiting time of MF distribution is

$$\overline{\mu}(t) = t - \frac{1}{F(t)}\int_{0}^{t} x\, f(x)\mathrm{d}x. \tag{41}$$

The final expression of mean waiting time of MF distribution is obtained. By substituting

$$\int_{0}^{t} x f(x)\mathrm{d}x = \frac{1}{\left(1-e^{-1}\right)}\left(\frac{-1}{\alpha}\right)^{l}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\lim_{b\longrightarrow-\infty} - \text{it}\left[\frac{\Gamma\left(l+1,\left((\alpha k+\alpha)/t^{\beta}\right)\right)\left(-\alpha/t^{\beta}\right)^{l}}{(k+1)\left((\alpha k+\alpha)/t^{\beta}\right)^{l}} - \frac{b^{l}\Gamma(l+1,(-bk-b))}{(-bk-b)^{l}(k+1)}\right], \tag{42}$$

and $F(t) = \left(\left(e^{-\left(e^{-\alpha t^{-\beta}}\right)} - 1\right)/\left(e^{-1}-1\right)\right)$, in (41), we obtain

$$\overline{\mu}(t) = t + \frac{1}{\left(e^{-\left(e^{-\alpha t^{-\beta}}\right)} - 1\right)}\left(\frac{-1}{\alpha}\right)^{l}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\lim_{b\longrightarrow-\infty} - \text{it}\left[\frac{\Gamma\left(l+1,\left((\alpha k+\alpha)/t^{\beta}\right)\right)\left(-\alpha/t^{\beta}\right)^{l}}{(k+1)\left((\alpha k+\alpha)/t^{\beta}\right)^{l}} - \frac{b^{l}\Gamma(l+1,(-bk-b))}{(-bk-b)^{l}(k+1)}\right]. \tag{43}$$

□

2.7. Parameters' Estimation. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n selected from $\mathrm{MF}(\alpha,\beta)$; then, the log-likelihood function of MF distribution is given as

$$\ln l(\alpha,\beta) = n\ln(\alpha\beta) - n\ln\left(1-e^{-1}\right) - \beta\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\log x_i - \alpha\sum_{i=1}^{n}x_i^{-\beta} - e^{-\alpha\sum_{i=1}^{n}x_i^{-\beta}}. \tag{44}$$

Differentiating equation (44) with respect to α and β and equating them to 0, we obtain

$$\frac{\partial\ln l(\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}x_i^{-\beta} + e^{-\alpha\sum_{i=1}^{n}x_i^{-\beta}}\sum_{i=1}^{n}x_i^{-\beta} = 0, \tag{45}$$

$$\frac{\partial\ln l(\alpha,\beta)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log x_i - \alpha\sum_{i=1}^{n}x_i^{-\beta}\log x_i + \alpha e^{-\alpha\sum_{i=1}^{n}x_i^{-\beta}}\sum_{i=1}^{n}x_i^{-\beta}\log x_i = 0. \tag{46}$$

Solving (45) and (46) together, we get the estimates of α and β. The Newton–Raphson method or the bisection method is used to get solution of the above equations as an analytical solution which is not possible. The maximum likelihood estimators (MLE) are asymptotically normally distributed, that is, $\sqrt{n}(\widehat{\alpha}-\alpha,\widehat{\beta}-\beta) \sim N_2(0,\Sigma)$, where Σ is variance covariance matrix and can be obtained by inverting the observed Fisher information matrix F given below:

$$F = \begin{pmatrix} \dfrac{\partial^2 \log l}{\partial \alpha^2} & \dfrac{\partial^2 \log l}{\partial \alpha\, \partial \beta} \\ \\ \dfrac{\partial^2 \log l}{\partial \alpha\, \partial \beta} & \dfrac{\partial^2 \log l}{\partial \beta^2} \end{pmatrix}. \qquad (47)$$

The second derivative of equations (45) and (46) with respect to α and β yields (48)–(50) given as

$$\frac{\partial^2 \ln l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - e^{-\alpha \sum_{i=1}^{n} x_i^{-\beta}} \left(\sum_{i=1}^{n} x_i^{-\beta} \right)^2, \qquad (48)$$

$$\frac{\partial^2 \ln l}{\partial \beta^2} = -\frac{n}{\beta^2} - \alpha \sum_{i=1}^{n} x_i^{-\beta} \log x_i \left[\log x_i + \alpha e^{-\alpha \sum_{i=1}^{n} x_i^{-\beta}} - e^{-\alpha \sum_{i=1}^{n} x_i^{-\beta}} \log x_i \right], \qquad (49)$$

and

$$\frac{\partial^2 \ln l}{\partial \alpha\, \partial \beta} = \sum_{i=1}^{n} x_i^{-\beta} \log x_i^{-\beta} \left[e^{-\alpha \sum_{i=1}^{n} x_i^{-\beta}} - \alpha e^{-\alpha \sum_{i=1}^{n} x_i^{-\beta}} - 1 \right]. \qquad (50)$$

Asymptotic $(1-\zeta)100\%$ confidence intervals of the parameters of the proposed distribution can be obtained as

$$\begin{aligned} &\hat{\alpha} \pm Z_{\zeta/2} \sqrt{\Sigma_{11}}, \\ &\hat{\beta} \pm Z_{\zeta/2} \sqrt{\Sigma_{22}}, \end{aligned} \qquad (51)$$

where Z_ζ is the upper ζ^{th} percentile of the standard normal distribution.

3. Simulations' Studies

In order to measure the performance of MLE of the parameters of MF distribution, their mean square error (MSE) and bias are calculated using simulation study. We consider $W = 100$ samples of sizes $n = 50, 70,$ and 100 generated from the MF model. Bias and MSE are calculated using the expressions:

$$\begin{aligned} \text{Bias} &= \frac{1}{W} \sum_{1=1}^{w} \left(\hat{b}_i - b \right), \\ \text{MSE} &= \frac{1}{W} \sum_{1=1}^{w} \left(\hat{b}_i - b \right)^2, \quad \text{where } b = (\alpha, \beta). \end{aligned} \qquad (52)$$

Simulation results have been obtained for different values of α and β. The MSEs and bias are presented in Table 1. The consistency behavior of MLE can be easily verified from these results as the MSEs and bias of the estimates decrease for all parameter combinations with increasing sample size. Hence, we conclude that MLE procedure executes very well in estimating the parameters of MF distribution.

4. Applications

Two practical datasets are used to assess the performance of MF distribution compared to Frechet distribution (FD), exponential distribution (ED), Weibull distribution (WD), alpha power inverse Weibull distribution (APIWD) [34], alpha power Weibull distribution (APWD) [35], and Kumaraswamy inverse Weibull distribution (KIWD) [36].

4.1. Dataset 1. The performance of the suggested model is assessed using two datasets. The first dataset is taken from the work of Gross and Clark [37] which consists of 20 observations of patients receiving an analgesic and is given as follows:

$$1.1 \quad 1.4 \quad 1.3 \quad 1.7 \quad 1.9 \quad 1.8 \quad 1.6 \quad 2.2 \quad 1.7 \quad 2.7 \quad 4.1 \quad 1.8 \quad 1.5 \quad 1.2 \quad 1.4 \quad 3.0 \quad 1.7 \quad 2.3 \quad 1.6 \quad 2.0\,. \qquad (53)$$

4.2. Dataset 2. The second dataset consists of 40 wind-related catastrophes used by Hogg and klugman [38]. It includes claims of $2,000,000. The sorted values, observed in millions, are as follows:

Table 1: MSE and bias of MLE.

Parameters	N	MSE ($\hat{\alpha}$)	MSE ($\hat{\beta}$)	Bias ($\hat{\alpha}$)	Bias ($\hat{\beta}$)
$\alpha = 5.557197$ $\beta = 7.558804$	50	2.067219	0.9272915	0.5047117	0.2639728
	70	0.9708544	0.7031859	0.3088091	0.2101214
	100	0.7753223	0.372835	0.2578149	0.1197992
$\alpha = 6.457197$ $\beta = 3.558804$	50	2.026492	0.2259845	0.4415551	0.1637077
	70	1.550174	0.1505096	0.4327197	0.1085847
	100	0.8022496	0.1029645	0.1910671	0.03691258
$\alpha = 5.457197$ $\beta = 3.558804$	50	1.190428	0.2059653	0.4247818	2.342903
	70	0.8619716	0.1373483	0.2461313	2.341802
	100	0.4419579	0.08574337	0.04261996	2.333286
$\alpha = 6.457197$ $\beta = 2.558804$	50	2.518832	0.09526091	0.7185321	0.1320379
	70	1.728531	0.09429495	0.4747864	0.07025322
	100	0.8006676	0.04351014	0.1573213	0.01418112
$\alpha = 4.557197$ $\beta = 4.558804$	50	0.9564051	0.2857103	0.2557071	0.1324175
	70	0.3593258	0.2089236	0.08780059	0.04002267
	100	0.2750382	0.14968	0.02739781	0.01301724
$\alpha = 6.557197$ $\beta = 1.558804$	50	3.244032	0.04392251	0.5185741	0.03279788
	70	1.495698	0.02470381	0.3520218	0.02578572
	100	0.9790736	0.01922724	0.09188152	0.02093735
$\alpha = 5.557197$ $\beta = 7.558804$	50	2.067219	0.9272915	0.5047117	0.2639728
	70	0.9708544	0.7031859	0.3088091	0.2101214
	100	0.7753223	0.372835	0.2578149	0.1197992
$\alpha = 2.557197$ $\beta = 7.558804$	50	0.1693338	1.094077	0.1063671	0.3148875
	70	0.08930482	0.4333755	0.08538295	0.1227512
	100	0.06259663	0.3701105	0.02511625	0.03947369

Table 2: Goodness of fit results for dataset 1.

Distribution	MLE of the parameters				AIC	CAIC	BIC	HQIC	K–S	P value
MF	**6.4571**	**3.5588**			**34.88**	**35.59**	**36.87**	**35.27**	**0.107**	**0.9759**
FD	2.2255				59.16	59.39	60.16	59.36	0.473	0.0003
ED	0.5263				67.67	67.89	68.66	67.86	0.439	0.0009
WD	2.7843	2.1271			45.17	45.87	47.16	45.56	0.183	0.5104
APIWD	1.7688	4.1692	5.4473		36.79	38.29	39.77	37.37	0.124	0.9644
APWD	10.9388	2.0312	0.4230		46.58	48.08	49.57	47.16	0.162	0.6678
KIWD	1.5668	1.2318	3.7669	3.5843	38.80	41.47	42.78	39.58	0.134	0.9540

The bold values indicate that the proposed distribution is more significant as compared to other existing distributions.

Table 3: Goodness of fit results for dataset 2.

Distribution	MLE of the parameters			AIC	CAIC	BIC	HQIC	K–S	P value
MF	**6.1151**	**1.2558**		**234.65**	**234.99**	**237.98**	**235.85**	**0.190**	**0.1194**
FD	0.7779			280.98	281.09	282.64	281.58	0.558	$5.61e-11$
ED	0.1130			250.03	250.15	251.70	250.64	0.202	0.08195
WD	1.0013	8.8585		252.04	252.37	255.36	253.23	0.2017	0.0836
APIWD	2.0394	1.4913	4.9249	237.33	238.01	242.32	239.12	0.2011	0.1049
APWD	10.7622	0.69493	0.4071	255.18	255.86	260.17	256.97	0.2201	0.0458

The bold values indicate that the proposed distribution is more significant as compared to other existing distributions.

$$2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 4\ 4\ 4\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 8\ 8\ 9\ 15\ 17\ 22\ 23\ 24\ 24\ 25\ 27\ 32\ 43. \tag{54}$$

In order to compare the MF model with other models, some standard model selection criteria such as Akaike's information criteria (AIC), consistent Akaike's information criteria (CAIC), Bayesian information criterion (BIC), Hannan–Quinn information criteria (HQIC), Kolmogorov–Smirnov (K-S), and P value are used. Tables 2 and 3 demonstrate results based on dataset 1 and 2, respectively.

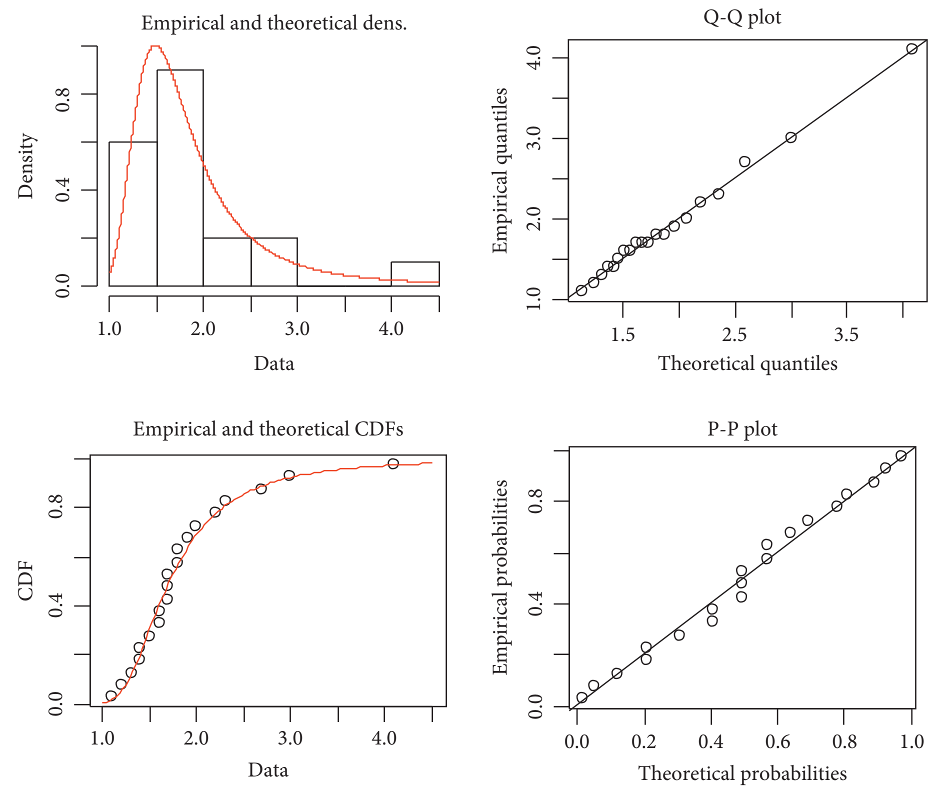

FIGURE 3: Plots of MF distribution for dataset 1.

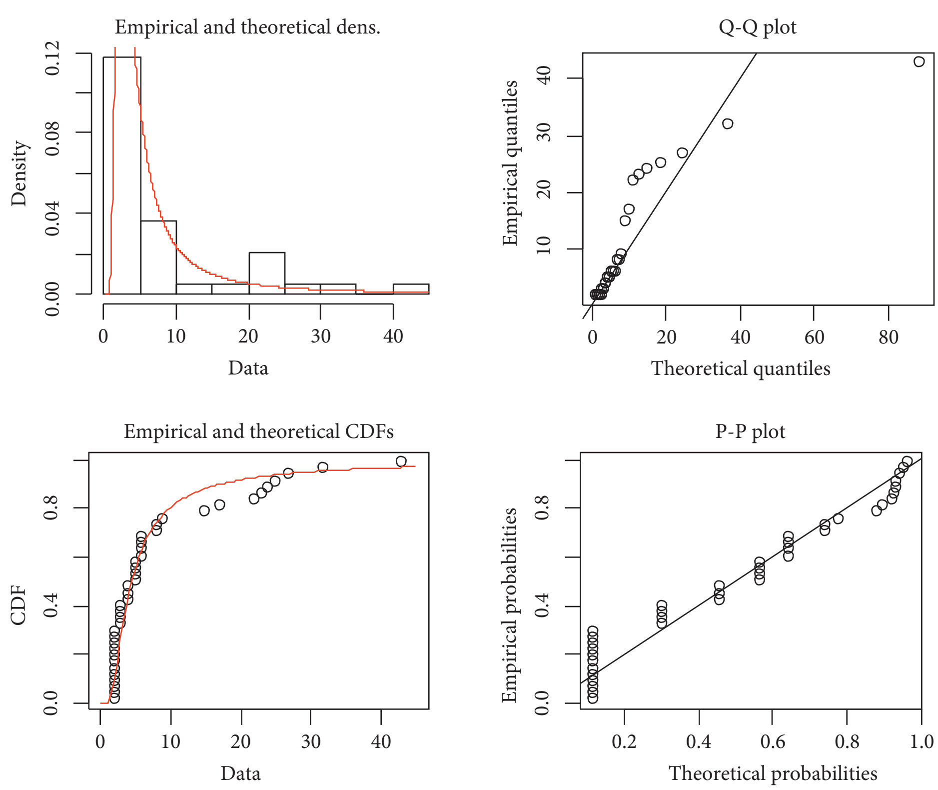

FIGURE 4: Plots of MF distribution for dataset 2.

It is evident from the results in Tables 2 and 3 that the proposed MF distribution executes well as compared to other competitive distributions.

Figures 3 and 4 represent various graphs of MF distribution for dataset 1 and dataset 2.

5. Conclusion

In this paper, a new method for deriving new continuous probability distributions has been offered which we called modified Frechet Class (MFC) of distributions. Also, a new probability model has been proposed using MFC. We called it modified Frechet (MF) distribution. Several statistical properties of the said distribution were derived and investigated for MF distribution. The MLE method was adopted to estimate the parameters of the proposed distribution. Simulation results showed that these estimates were consistent. In order to check the performance of the AFF model, two real datasets. The results based on these datasets revealed promising performance of the suggested model compared to some other distributions existing in the literature.

Data Availability

The data used in this paper are freely available upon citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

References

[1] A. Azzalini, "A class of distributions which includes the normal ones," *Scandinavian Journal of Statistics*, vol. 12, no. 2, pp. 171–178, 1985.

[2] G. S. Mudholkar and D. K. Srivastava, "Exponentiated weibull family for analyzing bathtub failure-rate data," *IEEE Transactions on Reliability*, vol. 42, no. 2, pp. 299–302, 1993.

[3] R. C. Gupta, P. L. Gupta, and R. D. Gupta, "Modeling failure time data by lehman alternatives," *Communications in Statistics-Theory and Methods*, vol. 27, no. 4, pp. 887–904, 1998.

[4] A. Marshall and I. Olkin, "A new method for adding a parameter to a family of distributions with application to the exponential and weibull families," *Biometrika*, vol. 84, no. 3, pp. 641–652, 1997.

[5] N. Eugene, C. Lee, and F. Famoye, "Beta-normal distribution and its applications," *Communications in Statistics-Theory and Methods*, vol. 31, no. 4, pp. 497–512, 2002.

[6] A. Alzaatreh, C. Lee, and F. Famoye, "A new method for generating families of continuous distributions," *Metron*, vol. 71, no. 1, pp. 63–79, 2013.

[7] M. A. Aljarrah, C. Lee, and F. Famoye, "On generating tx family of distributions using quantile functions," *Journal of Statistical Distributions and Applications*, vol. 1, no. 1, pp. 1–17, 2014.

[8] C. Lee, F. Famoye, and A. Y. Alzaatreh, "Methods for generating families of univariate continuous distributions in the recent decades," *Wiley Interdisciplinary Reviews: Computational Statistics*, vol. 5, no. 3, pp. 219–238, 2013.

[9] M. Jones, "On families of distributions with shape parameters," *International Statistical Review*, vol. 83, no. 2, 2014.

[10] R. Al-Aqtash, F. Famoye, and C. Lee, "On generating a new family of distributions using the logit function," *Journal of Probability and Statistical Science*, vol. 13, no. 1, pp. 135–152, 2015.

[11] A. Alzaatreh, F. Famoye, and C. Lee, "The gamma-normal distribution: properties and applications," *Computational Statistics & Data Analysis*, vol. 69, pp. 67–80, 2014a.

[12] S. H. Abid and R. K. Abdulrazak, "Truncated frechet-G generator of distributions," *Applied Mathematics*, vol. 3, pp. 51–66, 2017.

[13] M. Ç. Korkmaz and A. İ. Genç, "A new generalized two-sided class of distributions with an emphasis on two-sided generalized normal distribution," *Communications in Statistics-Simulation and Computation*, vol. 46, no. 2, pp. 1441–1460, 2017.

[14] A. Alzaghal, F. Famoye, and C. Lee, "Exponentiated TX family of distributions with some applications," *International Journal of Statistics and Probability*, vol. 2, no. 3, p. 31, 2013.

[15] M. Aldeni, C. Lee, and F. Famoye, "Families of distributions arising from the quantile of generalized lambda distribution," *Journal of Statistical Distributions and Applications*, vol. 4, no. 1, p. 25, 2017.

[16] G. M. Cordeiro, M. Alizadeh, T. G. Ramires, and E. M. M. Ortega, "The generalized odd half-Cauchy family of distributions: properties and applications," *Communications in Statistics-Theory and Methods*, vol. 46, no. 11, pp. 5685–5705, 2017.

[17] A. Alzaatreh, C. Lee, F. Famoye, and I. Ghosh, "The generalized Cauchy family of distributions with applications," *Journal of Statistical Distributions and Applications*, vol. 3, no. 1, p. 12, 2016.

[18] M. A. Nasir, M. Aljarrah, F. Jamal, and M. H. Tahir, "A new generalized Burr family of distributions based on quantile function," *Journal of Statistics Applications and Probability*, vol. 6, no. 3, pp. 1–14, 2017.

[19] Z. J. Pe and Z. J. Jurek, "Operator-limit distributions in probability theory," *Journal of the American Statistical Association*, vol. 89, no. 427, 1994.

[20] K. Pearson, "Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material," *Philosophical Transactions of the Royal Society of London-A*, vol. 186, pp. 343–414, 1895.

[21] I. W. Burr, "Cumulative frequency functions," *The Annals of Mathematical Statistics*, vol. 13, pp. 215–232, 1942.

[22] M. Bourguignon, R. B. Silva, and G. M. Cordeiro, "The Weibull–G family of probability distributions," *Journal of Data Science*, vol. 12, pp. 53–68, 2014.

[23] M. Elgarhy, A. S. Hassan, and M. Rashed, "Garhy-generated family of distributions with application," *Mathematical Theory and Modeling*, vol. 6, pp. 1–15, 2016.

[24] G. M. Cordeiro and M. De Castro, "A new family of generalized distributions," *Journal of Statistical Computation and Simulation*, vol. 81, pp. 883–893, 2011.

[25] A. S. Hassan, M. Elgarhy, and M. Shakil, "Type II half Logistic family of distributions with applications," *Pakistan Journal of Statistics and Operation Research*, vol. 13, pp. 245–264, 2017.

[26] M. Elgarhy, M. Haq, G. Ozel, and N. Arslan, "A new exponentiated extended family of distributions with Applications," *Gazi University Journal of Science*, vol. 30, pp. 101–115, 2017.

[27] A. S. Hassan and M. Elgarhy, "Kumaraswamy Weibull-generated family of distributions with applications," *Advances and Applications in Statistics*, vol. 48, pp. 205–239, 2016.

[28] A. S. Hassan and M. Elgarhy, "A New family of exponentiated Weibull-generated distributions," *International Journal of Mathematics and Applications*, vol. 4, pp. 135–148, 2016.
[29] M. Haq and M. Elgarhy, "The odd Frchet-G family of probability distributions," *Journal of Statistics Applications & Probability*, vol. 7, pp. 185–201, 2018.
[30] A. M. Almarashi and M. Elgarhy, "A new muth generated family of distributions with applications," *The Journal of Nonlinear Science and Applications*, vol. 11, pp. 1171–1184, 2018.
[31] S. Kotz and D. Vicari, "Survey of developments in the theory of continuous skewed distributions," *Metron*, vol. 63, pp. 225–261, 2005.
[32] A. Mahdavi and D. Kundu, "A new method for generating distributions with an application to exponential distribution," *Communications in Statistics-Theory and Methods*, vol. 46, pp. 6543–6557, 2017.
[33] S. Boyd and L. Vendenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, UK, 2004.
[34] D. A. Ramadan and A. W. Magdy, "On the alpha-power inverse weibull distribution," *International Journal of Computer Applications*, vol. 181, no. 11, pp. 6–12, 2018.
[35] M. Nassar, A. Alzaatreh, M. Mead, and A. Kasem, "Alpha power Weibull distribution :Properties and applications," *Communications in Statistics—Theory and Methods*, vol. 46, no. 20, pp. 10236–10252, 2017.
[36] M. Q. Shahbaz, S. Shahbaz, and N. S. Butt, "The kumaraswamy–inverse weibull distribution," *Pakistan Journal of Statistics and Operation Research*, vol. 8, no. 3, pp. 479–489, 2012.
[37] A. J. Gross and V. A. Clark, *Survival Distributions: Reliability Applications in the Biomedical Sciences*, John Wiley & Sons, New York, NY, USA, 1975.
[38] R. Hogg and S. A. Klugman, *Loss Distributions*, Wiley, New York, NY, USA, 1984.

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Research Article

Fractional Versions of Hadamard-Type Inequalities for Strongly Exponentially $(\alpha, h-m)$-Convex Functions

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In this article, we prove some fractional versions of Hadamard-type inequalities for strongly exponentially $(\alpha, h-m)$-convex functions via generalized Riemann–Liouville fractional integrals. The outcomes of this paper provide inequalities of strongly convex, strongly m-convex, strongly s-convex, strongly (α, m)-convex, strongly (s, m)-convex, strongly $(h-m)$-convex, strongly $(\alpha, h-m)$-convex, strongly exponentially convex, strongly exponentially m-convex, strongly exponentially s-convex, strongly exponentially (s, m)-convex, strongly exponentially $(h-m)$-convex, and exponentially $(\alpha, h-m)$-convex functions. The error estimations are also studied by applying two fractional integral identities.

1. Introduction

Fractional calculus is the study of derivatives and integrals of any arbitrary real or complex order. It is the generalization of ordinary calculus in which operations are mainly focused on integers. Its history starts when Leibniz and l'Hospital discussed the meaning of fractional order in 1695. This is the first discussion of fractional calculus. Many mathematicians devoted their efforts to make the foundation of fractional calculus. At that time, it was considered only in mathematics but now it has several applications in Science and Engineering, signal processing, mathematical biology, and rheology. In mathematics, many fractional integral operators have been introduced by researchers, see [1, 2]. Using these fractional operators, extensive inequalities are established for different types of convexity, see [3–5] and reference therein. The convex function is defined as follows:

A function $f: I \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called the convex function if the following inequality holds:

$$f(xt + (1-t)y) \leq t f(x) + (1-t) f(y), \quad t \in [0,1] \text{ and } x, y \in I. \tag{1}$$

Hadamard inequality is geometrical interpretation of the convex function, and it is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(\nu) \mathrm{d}\nu \leq \frac{f(a) + f(b)}{2}. \tag{2}$$

This inequality gives estimates of the mean value of a convex function. Recently, many mathematicians investigated different versions of Hadamard inequality and discussed its basic properties with corresponding fractional integral operators (see [6–9] and reference therein). Our aim is to establish Hadamard inequalities for the strongly exponentially $(\alpha, h-m)$-convex function via generalized Riemann–Liouville fractional integrals. Also, we have obtained error estimations for this convexity by using two fractional integral identities. Now, we recall the definition of the strongly exponentially $(\alpha, h-m)$-convex function.

Definition 1 (see [10]). Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$, and let $h\colon J \longrightarrow \mathbb{R}$ be a nonnegative function. A function $f\colon [0,b] \longrightarrow \mathbb{R}$ is called a strongly exponentially $(\alpha, h-m)$-convex function if f is nonnegative and for all x and $y \in [0,b]$, $t \in (0,1)$, $\eta \in \mathbb{R}$, and $(\alpha, m) \in (0,1]^2$, with modulus $\lambda \geq 0$, one has

$$f(tx+m(1-t)y) \leq h(t^\alpha)\frac{f(x)}{e^{\eta x}} + mh(1-t^\alpha)\frac{f(y)}{e^{\eta y}} - \frac{m\lambda}{e^{(x+y)\eta}}h(t^\alpha)h(1-t^\alpha)|y-x|^2. \tag{3}$$

The above definition provides some kinds of exponential convexities as follows:

Remark 1

(i) If we substitute $\alpha=1, \lambda=0$ and $h(t)=t^s$, then the exponentially (s,m)-convex function in the second sense introduced by Qiang et al. in [11] can be obtained

(ii) If we substitute $\alpha=1, \lambda=0$, $h(t)=t^s$, and $m=1$, then the exponentially s-convex function introduced by Mehreen et al. in [12] can be obtained

(iii) If we substitute $\alpha=1, \lambda=0$, $h(t)=t$, and $m=1$, then the exponentially convex function introduced by Awan et al. in [13] can be obtained

The classical Riemann–Liouville fractional integrals are given as follows:

Definition 2 (see [14]). Let $f \in L_1[a,b]$. Then, left-sided and right-sided Riemann–Liouville fractional integrals of a function f of the order $\xi \in \mathbb{C}$ and $\mathfrak{R}(\xi) > 0$ are given by

$$\begin{aligned} I_{a^+}^{\xi} f(x) &= \frac{1}{\Gamma(\xi)}\int_a^x (x-t)^{\xi-1} f(t)\mathrm{d}t, \quad x>a, \\ I_{b^-}^{\xi} f(x) &= \frac{1}{\Gamma(\xi)}\int_x^b (t-x)^{\xi-1} f(t)\mathrm{d}t, \quad x<b, \end{aligned} \tag{4}$$

where $\mathfrak{R}(\xi)$ denotes the real part of ξ and $\Gamma(\xi) = \int_0^\infty e^{-z}z^{\mu-1}\mathrm{d}z$.

Following two theorems are the fractional versions of Hadamard inequalities via Riemann–Liouville fractional integrals.

Theorem 1 (see [15]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following fractional integral inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\xi+1)}{2(b-a)^\xi}\left[I_{a^+}^{\xi} f(b) + I_{b^-}^{\xi} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{5}$$

with $\xi > 0$.

Theorem 2 (see [16]). *Under the assumptions of Theorem 1, the following fractional integral inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\xi-1}\Gamma(\xi+1)}{(b-a)^\xi}\left[I_{(a+b/2)^+}^{\xi} f(b) + I_{(a+b/2)^-}^{\xi} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{6}$$

with $\xi > 0$.

Following theorem is the error estimation of inequality (5).

Theorem 3 (see [15]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following fractional integral inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\xi+1)}{2(b-a)^\xi}\left[I_{a^+}^{\xi} f(b) + I_{b^-}^{\xi} f(a)\right]\right| \\ &\leq \frac{b-a}{2(\xi+1)}\left(1-\frac{1}{2^\xi}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \end{aligned} \tag{7}$$

The k-analogue of the Riemann–Liouville fractional integral is defined as follows:

Definition 3 (see [17]). Let $f \in L_1[a,b]$. Then, k-fractional Riemann–Liouville integrals of order ξ, where $\Re(\xi) > 0$ and $k > 0$, are defined as

$$
\begin{aligned}
{}_kI^{\xi}_{a^+} f(x) &= \frac{1}{k\Gamma_k(\xi)} \int_a^x (x-t)^{(\xi/k)-1} f(t)\mathrm{d}t, \quad x > a,\\
{}_kI^{\xi}_{b^-} f(x) &= \frac{1}{k\Gamma_k(\xi)} \int_x^b (t-x)^{(\xi/k)-1} f(t)\mathrm{d}t, \quad x < b,
\end{aligned}
\tag{8}
$$

where $\Gamma_k(.)$ is defined by [18]:

$$
\Gamma_k(\xi) = \int_0^{\infty} t^{\xi-1} e^{-(t^k/k)}\mathrm{d}t, \quad \Re(\xi) > 0. \tag{9}
$$

Two k-fractional versions of Hadamard inequality are given in next two theorems.

Theorem 4 (see [19]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a positive function with $0 \le a < b$. If f is a convex function on $[a,b]$, then the following inequality for k-fractional integral holds:*

$$
f\left(\frac{a+b}{2}\right) \le \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi}_{a^+} f(b) + {}_kI^{\xi}_{b^-} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{10}
$$

Theorem 5 (see [20]). *Under the assumptions of Theorem 4, the following inequality for k-fractional integral holds:*

$$
f\left(\frac{a+b}{2}\right) \le \frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(b-a)^{(\xi/k)}}\left[{}_kI^{\xi}_{(a+b/2)^+} f(b) + {}_kI^{\xi}_{(a+b/2)^-} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{11}
$$

The error estimation of inequality (10) is given in the following theorem.

Theorem 6 (see [19]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \le a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for k-fractional integral holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi}_{a^+} f(b) + {}_kI^{\xi}_{b^-} f(a)\right]\right|\\
&\le \frac{b-a}{2((\xi/k)+1)}\left(1 - \frac{1}{2^{(\xi/k)}}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right].
\end{aligned}
\tag{12}
$$

Now, we recall generalized Riemann–Liouville fractional integrals by a monotonically increasing function.

Definition 4 (see [21]). Let $f \in L_1[a,b]$. Also let ψ be an increasing and positive monotone function on $(a,b]$; further, ψ has a continuous derivative ψ' on (a,b). Therefore, left as well as right fractional integral operators of order ξ where $\Re(\xi) > 0$ of f with respect to ψ on $[a,b]$ are defined by

$$
\begin{aligned}
I^{\xi,\psi}_{a^+} f(x) &= \frac{1}{\Gamma(\xi)} \int_a^x \psi'(t)\psi(x) - \psi(t)^{\xi-1} f(t)\mathrm{d}t, \quad x > a,\\
I^{\xi,\psi}_{b^-} f(x) &= \frac{1}{\Gamma(\xi)} \int_x^b \psi'(t)\psi(t) - \psi(x)^{\xi-1} f(t)\mathrm{d}t, \quad x < b.
\end{aligned}
\tag{13}
$$

The k-analogue of generalized Riemann–Liouville fractional integrals is defined as follows.

Definition 5 (see [22]). Let $f \in L_1[a,b]$. Also let ψ be an increasing and positive monotone function on $(a,b]$; further, ψ has a continuous derivative ψ' on (a,b). Therefore, left as well as right k-fractional integral operators of order ξ where $\Re(\xi) > 0$ of f with respect to ψ on $[a,b]$ are defined by

$$
{}_kI^{\xi,\psi}_{a^+} f(x) = \frac{1}{k\Gamma_k(\xi)} \int_a^x \psi'(t)\psi(x) - \psi(t)^{(\xi/k)-1} f(t)\mathrm{d}t, \quad x > a, \tag{14}
$$

$$
{}_kI^{\xi,\psi}_{b^-} f(x) = \frac{1}{k\Gamma_k(\xi)} \int_x^b \psi'(t)\psi(t) - \psi(x)^{(\xi/k)-1} f(t)\mathrm{d}t, \quad x < b. \tag{15}
$$

For more details of fractional integrals, see [14, 23, 24]. We will utilize the following well-known hypergeometric, Beta, and incomplete Beta functions in our results [25]:

$$
\begin{aligned}
{}_2F_1[a,b;c;z] &= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\\
&\quad \cdot (1-zt)^{-a}\mathrm{d}t, \quad c > b > 0 \text{ and } |z| < 1,\\
B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\\
B(x,y;z) &= \int_0^z t^{x-1}(1-t)^{y-1}\mathrm{d}t.
\end{aligned}
\tag{16}
$$

In Section 2, we established Hadamard inequality for strongly exponentially $(\alpha, h-m)$-convex functions via generalized Riemann–Liouville fractional integrals. The special cases of these inequalities are associated with previously published papers. In Section 3, error estimations of fractional Hadamard inequality for strongly exponentially $(\alpha, h-m)$ are obtained with the help of two fractional integral identities. The outcomes of this article are connected with already established results given in [15, 16, 19, 20, 26–37].

2. Main Results

This section is concerned with two fractional versions of Hadamard inequalities for strongly exponentially $(\alpha, h-m)$-convex functions. One of them is given in the following theorem.

Theorem 7 *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a positive function with $0 \le a < mb$ and $f \in L_1[a,b]$. Also, suppose that f is the strongly exponentially $(\alpha, h-m)$-convex function on $[a,b]$ with modulus $\lambda \ge 0$. Then, for $k>0$ and $(\alpha, m) \in (0,1]^2$, the following k-fractional integral inequality holds for operators given in (14) and (15):*

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) + \frac{mg_1(\eta)\lambda h(1/2^\alpha)h(2^\alpha - 1/2^\alpha)}{(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2\right.\\
&\left.+2k\xi(b-a)\left(\frac{a}{m} - mb\right)\right] \le \frac{\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\\
&\cdot\left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right){}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right) + g_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\
&\le \frac{\xi}{k}\left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}} + \frac{mh(2^\alpha - 1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}}\right]\int_0^1 h(t^\alpha)t^{(\xi/k)-1}\mathrm{d}t\\
&+\frac{m\xi}{k}\left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}} + \frac{mh(2^\alpha - 1/2^\alpha)g_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]\int_0^1 t^{(\xi/k)-1}h(1-t^\alpha)\mathrm{d}t\\
&-\frac{m\lambda\xi}{k}\left[\frac{g_2(\eta)h(1/2^\alpha)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)h(2^\alpha - 1/2^\alpha)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right]\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{(\xi/k)-1}\mathrm{d}t,
\end{aligned} \tag{17}$$

where $\xi > 0$ and

$$\begin{aligned}
g_1(\eta) &= \begin{cases} e^{-\eta(a+b)}, & \text{if } \eta > 0,\\ e^{-\eta(mb+(a/m))}, & \text{if } \eta < 0,\end{cases}\\
g_2(\eta) &= \begin{cases} e^{-\eta mb}, & \text{if } \eta < 0,\\ e^{-\eta a}, & \text{if } \eta > 0,\end{cases}\\
g_3(\eta) &= \begin{cases} e^{-\eta b}, & \text{if } \eta < 0,\\ e^{(-\eta a/m)}, & \text{if } \eta > 0.\end{cases}
\end{aligned} \tag{18}$$

Proof. From strongly exponentially $(\alpha, h-m)$-convexity of f, we have

$$f\left(\frac{x+my}{2}\right) \le h\left(\frac{1}{2^\alpha}\right)\frac{f(x)}{e^{\eta x}} + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(y)}{e^{\eta y}} - \frac{m\lambda}{e^{\eta(x+y)}}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)|y-x|^2. \tag{19}$$

By setting $x = at + m(1-t)b$ and $y = (a/m)(1-t) + bt$, $t \in [0,1]$, in (19), multiplying the resulting inequality with $t^{(\xi/k)-1}$, and then integrating with respect to t, we get

$$\begin{aligned}\frac{k}{\xi} f\left(\frac{a+mb}{2}\right) \leq h\left(\frac{1}{2^{\alpha}}\right) \int_0^1 \frac{f(at+m(1-t)b)}{e^{\eta(at+m(1-t)b)}} t^{(\xi/k)-1} \mathrm{d}t + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) \\ \times \int_0^1 \frac{f((a/m)(1-t)+bt)}{e^{\eta((a/m)(1-t)+bt)}} t^{(\xi/k)-1} \mathrm{d}t - m\lambda h\left(\frac{1}{2^{\alpha}}\right) h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) \\ \times \int_0^1 \frac{(t(b-a)+(1-t)((a/m)-mb))^2}{e^{\eta(t(a+b)+(1-t)(mb+(a/m)))}} t^{(\xi/k)-1} \mathrm{d}t.\end{aligned} \tag{20}$$

Let

$$\begin{aligned} g_1(t) &= e^{-\eta(t(a+b)+(1-t)(mb+(a/m)))}, \\ g_1'(t) &= \eta(1-m)\left(\frac{a}{m}-b\right) e^{-\eta(t(a+b)+(1-t)(mb+(a/m)))}. \end{aligned} \tag{21}$$

Now one can see that g_1 will be increasing if $\eta < 0$ and decreasing if $\eta > 0$. Therefore, from inequality (20), we can have

$$\begin{aligned}\frac{k}{\xi} f\left(\frac{a+mb}{2}\right) \leq h\left(\frac{1}{2^{\alpha}}\right) \int_0^1 \frac{f(at+m(1-t)b)}{e^{\eta(at+m(1-t)b)}} t^{(\xi/k)-1} \mathrm{d}t + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) \\ \times \int_0^1 \frac{f((a/m)(1-t)+bt)}{e^{\eta((a/m)(1-t)+bt)}} t^{(\xi/k)-1} \mathrm{d}t - \frac{mkg_1(\eta)\lambda}{\xi(\xi+k)(\xi+2k)} h\left(\frac{1}{2^{\alpha}}\right) h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\Big[\xi(\xi+k)(b-a)^2 \\ + 2k^2\left(\frac{a}{m}-mb\right)^2 + 2k\xi(b-a)\left(\frac{a}{m}-mb\right)\Big].\end{aligned} \tag{22}$$

By setting $\psi(u) = at + m(1-t)b$ and $\psi(v) = (a/m)(1-t) + bt$ in (22), we get the following inequality:

$$\begin{aligned}\frac{k}{\xi} f\left(\frac{a+mb}{2}\right) \leq \frac{1}{(mb-a)^{(\xi/k)}} \left[h\left(\frac{1}{2^{\alpha}}\right) \int_{\psi^{-1}(a)}^{\psi^{-1}(mb)} \frac{f(\psi(u))}{e^{\eta(\psi(u))}} (mb-\psi(u))^{(\xi/k)-1} \psi'(u) \mathrm{d}u \right. \\ \left. + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) \int_{\psi^{-1}(a/m)}^{\psi^{-1}(b)} \frac{f(\psi(v))}{e^{\eta(\psi(v))}} \left(\psi(v)-\frac{a}{m}\right)^{(\xi/k)-1} \psi'(v) \mathrm{d}v \right] - \frac{mkg_1(\eta)\lambda}{\xi(\xi+k)(\xi+2k)} \\ \times h\left(\frac{1}{2^{\alpha}}\right) h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m}-mb\right)^2 + 2k\xi(b-a)\left(\frac{a}{m}-mb\right)\right].\end{aligned} \tag{23}$$

Further, multiplying by (ξ/k) and using Definition 5, we get

$$f\left(\frac{a+mb}{2}\right)\leq\frac{\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)_k I^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)\right.$$
$$\left.+g_3(\eta)m^{(\xi/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)_k I^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]-\frac{mg_1(\eta)\lambda}{(\xi+k)(\xi+2k)}$$
$$\times h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left[\xi(\xi+k)(b-a)^2+2k^2\left(\frac{a}{m}-mb\right)^2+2k\xi(b-a)\left(\frac{a}{m}-mb\right)\right]. \tag{24}$$

The above inequality leads to the first inequality of (17). Again, using strongly exponentially $(\alpha,h-m)$-convexity of f, for $t\in[0,1]$, we have

$$g_2(\eta)h\left(\frac{1}{2^\alpha}\right)f(at+m(1-t)b)+mg_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m}(1-t)+bt\right)$$
$$\leq h(t^\alpha)\left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}}\right]$$
$$+mh(1-t^\alpha)\left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]$$
$$-m\lambda h(t^\alpha)h(1-t^\alpha)\left[\frac{g_2(\eta)h(1/2^\alpha)(b-a)^2}{e^{\eta(a+b)}}+\frac{mg_3(\eta)h(2^\alpha-1/2^\alpha)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right]. \tag{25}$$

By integrating (25) over the interval $[0,1]$ after multiplying with $t^{(\xi/k)-1}$, we get

$$g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 f(ta+m(1-t)b)t^{(\xi/k)-1}\mathrm{d}t$$
$$+mg_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 f\left(\frac{a}{m}(1-t)+tb\right)t^{(\xi/k)-1}\mathrm{d}t\leq\left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}}\right]$$
$$\int_0^1 h(t^\alpha)t^{(\xi/k)-1}\mathrm{d}t+m\left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]$$
$$\int_0^1 t^{(\xi/k)-1}h(1-t^\alpha)\mathrm{d}t-m\lambda\left[\frac{g_2(\eta)h(1/2^\alpha)(b-a)^2}{e^{\eta(a+b)}}+\frac{mg_3(\eta)h(2^\alpha-1/2^\alpha)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right]$$
$$\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{(\xi/k)-1}\mathrm{d}t. \tag{26}$$

Again using substitutions as considered in (22), the above inequality leads to the second inequality of (17).

In the following remark, we give the connection of inequality (17) with already established results. □

Remark 2

(i) If we take $\eta = 0$, $\lambda = 0$, $\alpha = 1$, and ψ as the identity function in (17), then the inequality stated in Theorem 2.1 in [29] is obtained

(ii) If we take $\eta = 0$, $k = 1$, $h(t) = t$, $m = 1$, $\lambda = 0$, $\alpha = 1$, and ψ as the identity function in (17), then Theorem 1 is obtained

(iii) If we take $\eta = 0$, $h(t) = t, m = 1$, $\alpha = 1$, $\lambda = 0$, and ψ as the identity function in (17), then refinement of Theorem 1 is obtained

(iv) If we take $\alpha = 1$, $\xi = 1$, $k = 1$, $h(t) = t$, $m = 1$, $\eta = 0, \lambda = 0$, and ψ as the identity function in (17), then Hadamard inequality is obtained

(v) If we take $\eta = 0$, $m = 1$, $\alpha = 1$, $\lambda = 0$, and $h(t) = t$ in (17), then the inequality stated in Theorem 1 in [26] is obtained

(vi) If we take $\eta = 0$, $m = 1$, $\alpha = 1$, and $h(t) = t$ in (17), then the inequality stated in Theorem 10 in [32] is obtained

(vii) If we take $\eta = 0$, $k = 1$, $m = 1$, $\alpha = 1$, $\lambda = 0$, and $h(t) = t$ in (17), then the inequality stated in Theorem 2.1 in [34] is obtained

(viii) If we take $\alpha = 1, k = 1, h(t) = t, \eta = 0, \lambda = 0$, and ψ as the identity function in (17), then the inequality stated in Theorem 2.1 in [31] is obtained

(ix) If we take $\alpha = 1$, $\lambda = 0$, $h(t) = t^s$, and ψ as the identity function in (17), then the inequality stated in Theorem 2 in [36] is obtained

(x) If we take $\alpha = 1$, $\eta = 0, \lambda = 0$, and $h(t) = t^s$ in (17), then the inequality stated in Corollary 1 in [35] is obtained

(xi) If we take $\eta = 0$ and $k = 1$ in (17), then the inequality stated in Theorem 4 in [37] is obtained

(xii) If we take $\eta = 0$, $k = 1$, and $\alpha = 1$ in (17), then the inequality stated in Corollary 1 in [37] is obtained

(xiii) If we take $\lambda = 0$ in (17), then the inequality stated in Theorem 7 in [38] is obtained

Now, we give inequality (17) for strongly exponentially $(h - m)$-convex, strongly exponentially (s, m)-convex, strongly exponentially m-convex, and strongly exponentially convex functions.

Corollary 1. *If we take $\alpha = 1$ in (17), then the following inequality holds for strongly exponentially $(h - m)$-convex functions:*

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) + \frac{mg_1(\eta)\lambda h^2(1/2)}{(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2\right. \\
&\left. +2k\xi(b-a)\left(\frac{a}{m} - mb\right)\right] \leq \frac{h(1/2)\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[g_2(\eta)\,{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)\right. \\
&\left. +g_3(\eta)m^{(\xi/k)+1}\,{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right] \leq \frac{h(1/2)\xi}{k}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{mg_3(\eta)f(b)}{e^{\eta b}}\right] \\
&\times \int_0^1 h(t)t^{(\xi/k)-1}dt + \frac{mh(1/2)\xi}{k}\left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]\int_0^1 t^{(\xi/k)-1}h(1-t)dt \\
&- \frac{m\lambda h(1/2)\xi}{k}\left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right]\int_0^1 h(t)h(1-t)t^{(\xi/k)-1}dt.
\end{aligned} \tag{27}$$

Corollary 2. *If we take $\alpha = 1$ and $h(t) = t^s$ in (17), then the following inequality holds for strongly exponentially (s, m)-convex functions:*

$$f\left(\frac{a+mb}{2}\right)+\frac{mg_1(\eta)\lambda}{2^{2s}(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2+2k^2\left(\frac{a}{m}-mb\right)^2\right.$$
$$\left.+2k\xi(b-a)\left(\frac{a}{m}-mb\right)\right]\leq\frac{\Gamma_k(\xi+k)}{2^s(mb-a)^{(\xi/k)}}\left[g_2(\eta)_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)\right.$$
$$\left.+g_3(\eta)m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\leq\frac{\xi}{2^s(\xi+sk)}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}}+\frac{mg_3(\eta)f(b)}{e^{\eta b}}\right] \tag{28}$$
$$+\frac{m\xi B(1+s,(\xi/k))}{2^sk}\left[\frac{g_2(\eta)f(b)}{e^{\eta b}}+\frac{mg_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]-\frac{m\lambda\xi B(1+s,s+(\xi/k))}{2^sk}$$
$$\times\left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}}+\frac{mg_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right].$$

Corollary 3. *If we take $\alpha=1$ and $h(t)=t$ in (17), then the following inequality holds for strongly exponentially m-convex functions:*

$$f\left(\frac{a+mb}{2}\right)+\frac{mg_1(\eta)\lambda}{4(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2+2k^2\left(\frac{a}{m}-mb\right)^2\right.$$
$$\left.+2k\xi(b-a)\left(\frac{a}{m}-mb\right)\right]\leq\frac{\Gamma_k(\xi+k)}{2(mb-a)^{(\xi/k)}}\left[g_2(\eta)_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)\right.$$
$$\left.+g_3(\eta)m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\leq\frac{\xi}{2(\xi+k)}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}}+\frac{mg_3(\eta)f(b)}{e^{\eta b}}\right] \tag{29}$$
$$+\frac{mk}{2(\xi+k)}\left[\frac{g_2(\eta)f(b)}{e^{\eta b}}+\frac{mg_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]-\frac{m\lambda\xi k}{2(\xi+k)(\xi+2k)}\left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}}+\frac{mg_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right].$$

Corollary 4. *If we take $\alpha=1$, $m=1$, and $h(t)=t$ in (17), then the following inequality holds for strongly exponentially convex functions:*

$$f\left(\frac{a+b}{2}\right)+\frac{g_1(\eta)k^2\lambda}{2(\xi+k)(\xi+2k)}$$
$$\leq\frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[g_2(\eta)_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+g_3(\eta)_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right],$$
$$\leq\frac{\xi}{2(\xi+k)}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}}+\frac{mg_3(\eta)f(b)}{e^{\eta b}}\right]+\frac{k}{2(\xi+k)}\left[\frac{g_2(\eta)f(b)}{e^{\eta a}}+\frac{mg_3(\eta)f(a)}{e^{\eta b}}\right] \tag{30}$$
$$-\frac{\lambda\xi k(b-a)^2\left(g_2(\eta)+g_3(\eta)\right)}{2(\xi+k)(\xi+2k)e^{\eta(a+b)}}.$$

The next theorem is another version of Hadamard inequality for strongly exponentially $(\alpha, h-m)$-convex functions.

Theorem 8. *Under the assumptions of Theorem 7, the following k-fractional integral inequality holds:*

$$
\begin{aligned}
& f\left(\frac{a+mb}{2}\right)+h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\
& \frac{m\lambda g_1(\eta)}{4(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2+\left(\xi^2+5k\xi+8k^2\right)\left(\frac{a}{m}-mb\right)^2+2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right)\right] \\
& \leq \frac{2^{(\xi/k)}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right){}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+g_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right] \\
& \leq \frac{\xi}{k}\left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}}\right] \\
& \int_0^h h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt+\frac{\xi m}{k}\left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right] \\
& \int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt-\frac{m\lambda\xi}{k}\left[\frac{h(1/2^\alpha)g_2(\eta)(b-a)^2}{e^{\eta(a+b)}}+\frac{mh\left((2^\alpha-1)/2^\alpha\right)g_3(\eta)\left(b-a/m^2\right)^2}{e^{\eta\left(a/m^2+b\right)}}\right] \\
& \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt,
\end{aligned}
\tag{31}
$$

where $\xi>0$ *and*

$$
\begin{aligned}
g_1(\eta) &= \begin{cases} e^{(-\eta/2)(a(1+(1/m))+b(1+m))}, & \text{if } \eta>0, \\ e^{-\eta(mb+(a/m))}, & \text{if } \eta<0, \end{cases} \\
g_2(\eta) &= \begin{cases} e^{-\eta mb}, & \text{if } \eta<0, \\ e^{-\eta(a+mb/2)}, & \text{if } \eta>0, \end{cases} \\
g_3(\eta) &= \begin{cases} e^{-\eta(a+mb/2m)}, & \text{if } \eta<0, \\ e^{(-\eta a/m)}, & \text{if } \eta>0. \end{cases}
\end{aligned}
\tag{32}
$$

Proof. By setting $x=(at/2)+m(2-t/2)b$ and $y=(a/m)(2-t/2)+(bt/2)$, $t\in[0,1]$, in (19), multiplying the resulting inequality with $t^{(\xi/k)-1}$, and then integrating with respect to t, we get

$$
\begin{aligned}
\frac{k}{\xi}f\left(\frac{a+mb}{2}\right) &\leq h\left(\frac{1}{2^\alpha}\right)\int_0^1\frac{f((at/2)+m(1-(t/2))b)}{e^{\eta((at/2)+m(1-(t/2))b)}}t^{(\xi/k)-1}dt \\
&+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1\frac{f((a/m)((2-t)/2)+(bt/2))}{e^{\eta((a/m)((2-t)/2)+(bt/2))}}t^{(\xi/k)-1}dt-m\lambda g_1(\eta)h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\
&\times\int_0^1\frac{(t(b-a)+(1-t)(a/m)-mb)^2}{e^{\eta((t/2)(a+b)+(1-(t/2))(mb+(a/m)))}}t^{(\xi/k)-1}dt.
\end{aligned}
\tag{33}
$$

Let

$$g_1(t) = e^{-\eta((t/2)(a+b)+(1-(t/2))(mb+(a/m)))},$$
$$g_1'(t) = \frac{\eta}{2}(1-m)\left(\frac{a}{m}-b\right)e^{-\eta((t/2)(a+b)+(1-(t/2))(mb+(a/m)))}. \tag{34}$$

Now one can see that g_1 will be increasing if $\eta < 0$ and decreasing if $\eta > 0$. Therefore, from inequality (33), we can have

$$\begin{aligned}\frac{k}{\xi}f\left(\frac{a+mb}{2}\right) &\leq h\left(\frac{1}{2^\alpha}\right)\int_0^1 \frac{f((at/2)+m(1-(t/2))b)}{e^{\eta((at/2)+m(1-(t/2))b)}}t^{(\xi/k)-1}\mathrm{d}t \\ &+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 \frac{f((a/m)((2-t)/2)+(bt/2))}{e^{\eta((a/m)(2-t/2)+(bt/2))}}t^{((\xi/k)-1)}\mathrm{d}t - m\lambda g_1(\eta)h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\times \frac{k}{4\xi(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2+\left(\xi^2+5k\xi+8k^2\right)\left(\frac{a}{m}-mb\right)^2+2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right)\right].\end{aligned} \tag{35}$$

By setting $\psi(u) = (at/2) + m(2-t/2)b$ and $\psi(v) = (a/m)(2-t/2) + (bt/2)$ in (35), we get the following inequality:

$$\begin{aligned}&\frac{k}{\xi}f\left(\frac{a+mb}{2}\right) \leq \frac{2^{(\xi/k)}}{(mb-a)^{(\xi/k)}} \\ &\left[h\left(\frac{1}{2^\alpha}\right)\int_{\psi^{-1}(a+mb/2)}^{\psi^{-1}(mb)} \frac{f(\psi(u))}{e^{\eta(\psi(u))}}(mb-\psi(u))^{(\xi/k)-1}\psi'(u)\mathrm{d}u + m^{(\xi/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_{\psi^{-1}(a/m)}^{\psi^{-1}(a+mb/2m)} \frac{f(\psi(v))}{e^{\eta(\psi(v))}}\left(\psi(v)-\frac{a}{m}\right)^{(\xi/k)-1}\psi'(v)\mathrm{d}v\right] \\ &- m\lambda g_1(\eta)h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\frac{k}{4\xi(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2+\left(\xi^2+5k\xi+8k^2\right)\left(\frac{a}{m}-mb\right)^2+2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right)\right].\end{aligned} \tag{36}$$

Further, multiplying above inequality by (ξ/k) and using Definition 5, we get

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right) \leq \frac{2^{(\xi/k)}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \\ &\left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right){}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right) + g_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right] \\ &- h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{m\lambda g_1(\eta)}{4(\xi+k)(\xi+2k)} \\ &\left[\xi(\xi+k)(b-a)^2+\left(\xi^2+5k\xi+8k^2\right)\left(\frac{a}{m}-mb\right)^2+2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right)\right].\end{aligned} \tag{37}$$

The above inequality leads to the first inequality of (31). Again using strongly exponentially $(\alpha, h-m)$-convexity of f, for $t \in [0,1]$, we have

$$\begin{aligned}
&g_2(\eta)h\left(\frac{1}{2^\alpha}\right)f\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)+mg_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\\
&\leq h\left(\frac{t^\alpha}{2^\alpha}\right)\left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}}+\frac{mh(2^\alpha-1/2^\alpha)f(b)}{e^{\eta b}}\right]\\
&+mh\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(a/m^2)}{e^{\eta a/m^2}}\right]\\
&-m\lambda h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\left[\frac{h(1/2^\alpha)g_2(\eta)(b-a)^2}{e^{\eta(a+b)}}+\frac{h(2^\alpha-1/2^\alpha)g_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right].
\end{aligned}\tag{38}$$

By integrating (38) over $[0,1]$ after multiplying with $t^{(\xi/k)-1}$, the following inequality holds:

$$\begin{aligned}
&g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 f\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)t^{(\xi/k)-1}\mathrm{d}t\\
&+mg_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 f\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)t^{(\xi/k)-1}\mathrm{d}t\\
&\leq\left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}}+\frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}}\right]\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}\mathrm{d}t\\
&+m\left[h(1/2^\alpha)g_2(\eta)f(b)/e^{\eta b}+mh(2^\alpha-1/2^\alpha)g_2(\eta)f\left(a/m^2\right)/e^{\frac{\eta a}{m^2}}\right]\int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}\mathrm{d}t\\
&-m\lambda\left[\frac{h(1/2^\alpha)g_2(\eta)(b-a)^2}{e^{\eta(a+b)}}+\frac{h(2^\alpha-1/2^\alpha)g_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right]\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}\mathrm{d}t.
\end{aligned}\tag{39}$$

Again, using substitutions as considered in (35), the above inequality leads to the second inequality of (31).

In the following remark, we give the connection of inequality (31) with already established results. □

Remark 3

(i) If we take $\eta=0$ and $k=1$ in (31), then the inequality stated in Theorem 5 in [37] is obtained

(ii) If we take $\eta=0$ and $\alpha=1$ in (31), then the inequality stated in Corollary 3 in [37] is obtained

(iii) If we take $\alpha=1$, $h(t)=t$, $m=1$, $\lambda=0$, $k=1$, $\eta=0$, and ψ as the identity function in (31), then Theorem 2 is obtained

(iv) If we take $\alpha=1$, $m=1$, $h(t)=t$, $\lambda=0$, $\eta=0$, and ψ as the identity function in inequality (31), then refinement of Theorem 2 is obtained

(v) If we take $h(t)=t$, $m=1$, $\lambda=0$, $k=1$, $\xi=1$, $\eta=0$, $\alpha=1$, and ψ as the identity function in (31), then the Hadamard inequality is obtained

(vi) If we take $h(t)=t, m=1$, $\alpha=1$, and $\eta=0$ in (31), then the inequality stated in Theorem 11 in [32] is obtained

(vii) If we take $\alpha=1, h(t)=t, k=1, \lambda=0, \eta=0$, and ψ as the identity function in (31), then the inequality stated in Theorem 2.1 in [30] is obtained

(viii) If we take $\lambda=0$, $\alpha=1$, $h(t)=t^s$, and $\eta=0$ in (31), then the inequality stated in Corollary 3 in [35] is obtained

(ix) If we take $\lambda = 0$ in (31), then the inequality stated in Theorem 8 in [38] is obtained

Now, we give inequality (31) for strongly exponentially $(h-m)$-convex, strongly exponentially (s,m)-convex, strongly exponentially m-convex, and strongly exponentially convex functions.

Corollary 5. *If we take $\alpha = 1$ in (31), then the following inequality holds for strongly exponentially $(h-m)$-convex functions:*

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) + \frac{mh^2(1/2)\lambda g_1(\eta)}{4(\xi+k)(\xi+2k)}\left[\xi(\xi+k)(b-a)^2 + \left(\xi^2+5k\xi+8k^2\right)\left(\frac{a}{m}-mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right)\right]\\
&\leq \frac{h(1/2)2^{(\xi/k)}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[g_2(\eta)_k I^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right) + g_3(\eta)m^{(\xi/k)+1}{}_k I^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\
&\leq \frac{h(1/2)\xi}{k}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + m\frac{g_3(\eta)f(b)}{e^{\eta b}}\right]\times\int_0^1 h\left(\frac{t}{2}\right)t^{(\xi/k)-1}\mathrm{d}tr\\
&+\frac{h(1/2)\xi m}{k}\left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]\int_0^1 h\left(\frac{2-t}{2}\right)t^{(\xi/k)-1}\mathrm{d}t\\
&-m\lambda h\left(\frac{1}{2}\right)\left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{g_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right]\int_0^1 h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)t^{(\xi/k)-1}\mathrm{d}t.
\end{aligned}\tag{40}$$

Corollary 6. *If we take $\alpha = 1$ and $h(t) = t^s$ in (31), then the following inequality holds for strongly exponentially (s,m)-convex functions:*

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) + \frac{2^{-2s}m\lambda g_1(\eta)}{4(\xi+2k)(\xi+k)}\left[\xi(\xi+k)(b-a)^2 + \left(\xi^2+5k\xi+8k^2\right)\left(\frac{a}{m}-mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right)\right]\\
&\leq \frac{2^{(\xi/k)-s}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[g_2(\eta)_k I^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right) + g_3(\eta)m^{(\xi/k)+1}{}_k I^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\
&\leq \frac{\xi}{2^{2s}(\xi+k)}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{g_3(\eta)f(b)}{e^{\eta b}}\right] + m_2F_1\left(-s,\frac{\xi}{k},\frac{\xi}{k}+1;\frac{1}{2}\right)\left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]\\
&-\frac{2^{(\xi/k)-s}m\lambda\xi B(1/2, s+\xi/k, 1+s)}{k}\times\left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{g_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right].
\end{aligned}\tag{41}$$

Corollary 7. *If we take $\alpha = 1$ and $h(t) = t$ in (31), then the following inequality holds for strongly exponentially m-convex functions:*

$$
\begin{aligned}
&f\left(\frac{a+mb}{2}\right)+\frac{m\lambda g_1(\eta)}{16(\xi+2k)(\xi+k)}\left[\xi(\xi+k)(b-a)^2+\left(\xi^2+5k\xi+8k^2\right)\left(\frac{a}{m}-mb\right)^2+2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right)\right]\\
&\leq\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[g_2(\eta)_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+g_3(\eta)m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\
&\leq\frac{\xi}{4(\xi+k)}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}}+\frac{mg_3(\eta)f(b)}{e^{\eta b}}\right]+\frac{m(\xi+2k)}{4(\xi+k)}\left[\frac{g_2(\eta)f(b)}{e^{\eta b}}+\frac{mg_3(\eta)f\left(a/m^2\right)}{e^{\eta a/m^2}}\right]\\
&-\frac{m\lambda\xi(\xi+3k)}{8(\xi+k)(\xi+2k)}\left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}}+\frac{g_3(\eta)\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right].
\end{aligned}
\tag{42}
$$

Corollary 8. *If we take $\alpha=1$, $m=1$, and $h(t)=t$ in (31), then the following inequality holds for strongly exponentially convex functions:*

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)+\frac{\lambda k^2 g_1(\eta)}{2(\xi+2k)(\xi+k)}\\
&\leq\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(b-a)^{(\xi/k)}}\left[g_2(\eta)_kI^{\xi,\psi}_{\psi^{-1}(a+b/2)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+g_3(\eta)_kI^{\xi,\psi}_{\psi^{-1}(a+b/2)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right],\\
&\leq\frac{\xi}{4(\xi+k)}\left[\frac{g_2(\eta)f(a)}{e^{\eta a}}+\frac{g_3(\eta)f(b)}{e^{\eta b}}\right]+\frac{(\xi+2k)}{4(\xi+k)}\left[\frac{g_2(\eta)f(b)}{e^{\eta b}}+\frac{g_3(\eta)f(a)}{e^{\eta a}}\right]\\
&-\frac{\lambda(b-a)^2\xi(\xi+3k)\left(g_2(\eta)+g_3(\eta)\right)}{8(\xi+k)(\xi+2k)e^{\eta(a+b)}}.
\end{aligned}
\tag{43}
$$

3. Error Estimations of Hadamard Inequalities for Strongly Exponentially $(\alpha, h-m)$-Convex Functions

This section deals with error bounds of Hadamard inequalities for strongly exponentially $(\alpha, h-m)$-convex functions using generalized Riemann–Liouville fractional integrals. Estimations obtained here provide refinements of several well-known inequalities for different types of convexity. The following identity is used to prove the next theorem.

Lemma 1 (see [26]). *Let $a<b$ and $f\colon [a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b). Also, suppose that $f'\in L_1[a,b]$. Then, for $k>0$, the following identity holds for the operators given in (14) and (15):*

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]\\
&=\frac{b-a}{2}\int_0^1\left((1-t)^{(\xi/k)}-t^{(\xi/k)}\right)f'(ta+(1-t)b)\mathrm{d}t.
\end{aligned}
\tag{44}
$$

Theorem 9. *Let $f\colon [a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b) such that $f'\in L_1[a,b]$. Also, suppose that $|f'|$ is strongly exponentially $(\alpha,h-m)$-convex on $[a,b]$. Then, for $k>0$ and $\alpha,m\in(0,1]^2$, the following k-fractional integral inequality holds for the operators given in (14) and (15):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]\right|$$
$$\leq\frac{b-a}{2}\left[\begin{array}{c}\frac{|f'(a)|}{e^{\eta a}}\left(\int_0^{(1/2)}h(t^\alpha)\left((1-t)^{(\xi/k)}-t^{(\xi/k)}\right)dt+\int_{(1/2)}^1 h(t^\alpha)\left(t^{(\xi/k)}-(1-t)^{(\xi/k)}\right)dt\right)\\ +\frac{m|f'(b/m)|}{e^{\eta b/m}}\left(\int_0^{(1/2)}h(1-t^\alpha)\left((1-t)^{(\xi/k)}-t^{(\xi/k)}\right)dt+\int_{(1/2)}^1 h(1-t^\alpha)\left(t^{(\xi/k)}-(1-t)^{(\xi/k)}\right)dt\right)\\ -\frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\left(\int_0^{(1/2)}h(t^\alpha)h(1-t^\alpha)\left((1-t)^{(\xi/k)}-t^{(\xi/k)}\right)dt-\int_{\frac{1}{2}}^1 h(t^\alpha)h(1-t^\alpha)\left(t^{(\xi/k)}-(1-t)^{(\xi/k)}\right)dt\right)\end{array}\right], \tag{45}$$

where $\xi>0$.

Proof. From Lemma 1, it follows that

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]\right|$$
$$\leq\frac{b-a}{2}\int_0^1\left|(1-t)^{(\xi/k)}-t^{(\xi/k)}\right|\left|f'(ta+(1-t)b)\right|dt. \tag{46}$$

By using strongly exponentially $(\alpha,h-m)$-convexity of $|f'|$ and for $t\in[0,1]$, we have

$$\left|f'(ta+(1-t)b)\right|\leq h(t^\alpha)\frac{|f'(a)|}{e^{\eta a}}+mh(1-t^\alpha)\frac{|f'(b/m)|}{e^{\eta b/m}}-\frac{m\lambda h(t^\alpha)h(1-t^\alpha)}{e^{\eta(a+(b/m))}}\left(\frac{b}{m}-a\right)^2. \tag{47}$$

Using (47) in (46), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]\right|$$
$$\leq\frac{b-a}{2}\int_0^1\left|(1-t)^{(\xi/k)}-t^{(\xi/k)}\right|\left[h(t^\alpha)\frac{|f'(a)|}{e^{\eta a}}+mh(1-t^\alpha)\frac{|f'(b/m)|}{e^{\eta b/m}}\right.$$
$$\left.-\frac{m\lambda h(t^\alpha)h(1-t^\alpha)}{e^{\eta(a+(b/m))}}\left(\frac{b}{m}-a\right)^2\right]=\frac{b-a}{2}\left[\frac{|f'(a)|}{e^{\eta a}}\left(\int_0^{(1/2)}h(t^\alpha)\left((1-t)^{(\xi/k)}-t^{(\xi/k)}\right)dt\right.\right.$$
$$\left.+\int_{(1/2)}^1 h(t^\alpha)\left(t^{(\xi/k)}-(1-t)^{(\xi/k)}\right)dt\right)+\frac{m|f'(b/m)|}{e^{\eta b/m}}\left(\int_0^{(1/2)}h(1-t^\alpha)\left((1-t)^{(\xi/k)}-t^{(\xi/k)}\right)dt\right.$$
$$\left.+\int_{(1/2)}^1 h(1-t^\alpha)\left(t^{(\xi/k)}-(1-t)^{(\xi/k)}\right)dt\right)-\frac{m\lambda(b-a)^2}{e^{\eta(a+(b/m))}}\left(\int_0^{(1/2)}h(t^\alpha)h(1-t^\alpha)\left((1-t)^{(\xi/k)}-t^{(\xi/k)}\right)dt\right.$$
$$\left.\left.+\int_{(1/2)}^1 h(t^\alpha)h(1-t^\alpha)\left(t^{(\xi/k)}-(1-t)^{(\xi/k)}\right)dt\right)\right]. \tag{48}$$

In the following remark, we give the connection of inequality (45) with already established results. □

Remark 4

(i) If we take $\eta = 0$ and $k = 1$ in (45), then the inequality stated in Theorem 6 in [37] is obtained
(ii) If we take $\eta = 0$ and $\alpha = 1$ in (45), then the inequality stated in Corollary 7 in [37] is obtained
(iii) If we take $h(t) = t$, $m = 1$, $\alpha = 1$, and $\eta = 0$ in (45), then the inequality stated in Theorem 12 in [32] is obtained
(iv) If we take $m = 1$, $\alpha = 1$, $\eta = 0$, $\lambda = 0$, and $h(t) = t^s$ in (45), then the inequality stated in Theorem 2 in [26] is obtained
(v) If we take $m = 1$, $\alpha = 1$, $\lambda = 0$, $h(t) = t$, $\eta = 0$, and ψ as the identity function in (45), then Theorem 6 is obtained
(vi) If we take $k = 1$, $m = 1$, $\alpha = 1$, $\lambda = 0$, $h(t) = t$, $\eta = 0$, and ψ as the identity function in (45), then Theorem 3 is obtained
(vii) If we take $k = 1$, $\alpha = 1$, $\xi = 1$, $\lambda = 0$, $h(t) = t$, $m = 1$, $\eta = 0$, and ψ as the identity function in (45), then the inequality stated in Theorem 2.2 in [27] is obtained
(viii) If we take $\eta = 0$, $\alpha = 1$, $\lambda = 0$, and $h(t) = t^s$ in (45), then the inequality stated in Corollary 5 in [35] is obtained
(ix) If we take $\alpha = 1$, $h(t) = t$, $m = 1$, and $\eta = 0$ in (45), then the inequality stated in Theorem 12 in [32] is obtained
(x) If we take $\lambda = 0$ in (45), then the inequality stated in Theorem 9 in [38] is obtained

Now, we give inequality (45) for strongly exponentially $(h-m)$-convex, strongly exponentially m-convex, and strongly exponentially convex functions.

Corollary 9. *If we take $\alpha = 1$ in (45), then the following inequality holds for strongly exponentially $(h-m)$-convex functions:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right) + {}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]\right| \\
&\leq \frac{b-a}{2}\left[\frac{|f'(a)|}{e^{\eta a}}\left(\int_0^{(1/2)} h(t)\left((1-t)^{(\xi/k)} - t^{(\xi/k)}\right)\mathrm{d}t + \int_{(1/2)}^1 h(t)\left(t^{(\xi/k)} - (1-t)^{(\xi/k)}\right)\mathrm{d}t\right)\right. \\
&+ \frac{m|f'(b/m)|}{e^{\eta b/m}}\left(\int_0^{(1/2)} h(1-t)\left((1-t)^{(\xi/k)} - t^{(\xi/k)}\right)\mathrm{d}t + \int_{(1/2)}^1 h(1-t)\left(t^{(\xi/k)} - (1-t)^{(\xi/k)}\right)\mathrm{d}t\right) \\
&\left.- \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\left(\int_0^{(1/2)} h(t)h(1-t)\left((1-t)^{(\xi/k)} - t^{(\xi/k)}\right)dt - \int_{(1/2)}^1 h(t)h(1-t)\left(t^{(\xi/k)} - (1-t)^{(\xi/k)}\right)\mathrm{d}t\right)\right].
\end{aligned} \tag{49}$$

Corollary 10. *If we take $\alpha = 1$ and $h(t) = t$ in (45), then the following inequality holds for strongly exponentially m-convex functions:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right) + {}_kI^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]\right| \\
&\leq \frac{b-a}{2((\xi/k)+1)}\left(1 - \frac{1}{2^{(\xi/k)}}\right)\left(\frac{|f'(a)|}{e^{\eta a}} + \frac{m|f'(b/m)|}{e^{\eta b/m}}\right) - \frac{\lambda((b/m)-a)^3\left(1 - ((\xi/k)+4)/\left(2^{((\xi/k)+2)}\right)\right)}{e^{\eta((b/m)+a)}((\xi/k)+2)((\xi/k)+3)}.
\end{aligned} \tag{50}$$

Corollary 11. *If we take $\alpha = 1$, $m = 1$, and $h(t) = t$ in (45), then the following inequality holds for strongly exponentially convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kIk^{\xi,\psi}_{\psi^{-1}(b)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]\right|$$
$$\leq\frac{b-a}{2((\xi/k)+1)}\left(1-\frac{1}{2^{(\xi/k)}}\right)\left(\frac{|f'(a)|}{e^{\eta a}}+\frac{|f'(b)|}{e^{\eta b}}\right)-\frac{\lambda(b-a)^3\left(1-((\xi/k)+4)/\left(2^{((\xi/k)+2)}\right)\right)}{e^{\eta(b+a)}((\xi/k)+2)((\xi/k)+3)}. \tag{51}$$

The following integral identity is useful to get our next theorem.

Lemma 2 (see [35]). *Let $f\colon [a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b) such that $f'\in L_1[a,b]$. Then, for $k>0$ and $m\in(0,1]$, the following integral identity holds for operators given in (14) and (15):*

$$\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]$$
$$-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]=\frac{mb-a}{4}\left[\int_0^1 t^{(\xi/k)}f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\mathrm{d}t-\int_0^1 t^{(\xi/k)}f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\mathrm{d}t\right]. \tag{52}$$

Theorem 10. *Let $f\colon [a,b]\longrightarrow\mathbb{R}$, $[a,b]\subset[0,b]$, be a differentiable mapping on (a,b) such that $f'\in L_1[a,b]$. Also suppose that $|f'|^q$ is a strongly exponentially $(\alpha,h-m)$-convex function on $[a,b]$ for $q\geq 1$. Then, for $k>0$ and $(\alpha,m)\in(0,1]^2$, the following fractional integral inequality holds for operators given in (14) and (15):*

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4((\xi/k)+1)^{1-(1/q)}}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right.\right.$$
$$\left.+\frac{m|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t-\frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right)^{(1/q)} \tag{53}$$
$$+\left(\frac{m\left|f'\left(a/m^2\right)\right|^q}{e^{\eta a/m^2}}\int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t+\frac{m|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right.$$
$$\left.\left.-\frac{m\lambda\left(b-\left(a/m^2\right)\right)^2}{e^{\eta((a/m^2)+b)}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right)^{(1/q)}\right].$$

Proof. We divide the proof in two cases:

Case 1 (for $q=1$). Applying Lemma 2 and using strongly exponentially $(\alpha,h-m)$-convexity of $|f'|$, we have

$$\begin{aligned}&\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4}\left[\int_0^1\left|t^{(\xi/k)}f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\right|\mathrm{d}t\right.\\
&\left.+\int_0^1\left|t^{(\xi/k)}f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\right|\mathrm{d}t\right]\leq\frac{mb-a}{4}\left[\left(\frac{|f'(a)|}{e^{\eta a}}+\frac{|f'(b)|}{e^{\eta b}}\right)\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right.\\
&+m\left(\frac{|f'(b)|}{e^{\eta b}}+\frac{|f'(a/m^2)|}{e^{\eta a/m^2}}\right)\int_0^1h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}-m\lambda\left(\frac{(b-a)^2}{e^{\eta(a+b)}}+\frac{\left(b-(a/m^2)\right)^2}{e^{\eta\left((a/m^2)+b\right)}}\right)\\
&\left.\times\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t\right].\end{aligned}\tag{54}$$

Case 2 (for $q>1$). From Lemma 2 and using power mean inequality, we get

$$\begin{aligned}&\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4}\left(\int_0^1t^{(\xi/k)}\mathrm{d}t\right)^{1-(1/q)}\\
&\left[\left(\int_0^1t^{(\xi/k)}\left|f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\right|^q\mathrm{d}t\right)^{(1/q)}+\left(\int_0^1t^{(\xi/k)}\left|f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right]\\
&\leq\frac{mb-a}{4((\xi/k)+1)^{1-(1/q)}}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t+\frac{m|f'(b)|^q}{e^{\eta b}}\int_0^1h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right.\right.\\
&\left.-\frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right)^{(1/q)}+\left(\left(\frac{m\left|f'(a/m^2)\right|^q}{e^{\eta a/m^2}}\int_0^1h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right.\right.\\
&\left.\left.+\frac{|f'(b)|^q}{e^{\eta b}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t-\frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)}\mathrm{d}t\right)^{(1/q)}\right].\end{aligned}\tag{55}$$

Hence, equation (53) is obtained.

In the following remark, we give the connection of inequality (53) with already established results. □

Remark 5

(i) If we take $\lambda=0$ in (53), then the inequality stated in Theorem 10 in [38] is obtained

(ii) If we take $\eta=0$, $\alpha=1$, $\lambda=0$, and $h(t)=t^s$ in (53), then the inequality stated in Corollary 7 in [35] is obtained

(iii) If we take $h(t)=t$, $k=1$, $\eta=0$, $\alpha=1$, $\lambda=0$, and ψ as the identity function in (53), then the inequality stated in Theorem 2.4 in [30] is obtained

(iv) If we take $\eta=0$, $h(t)=t$, $m=1$, $\alpha=1$, and ψ as the identity function in (53), then the inequality stated in Theorem 3.1 in [20] is obtained

(v) If we take $h(t)=t$, $m=1$, $\eta=0$, $k=1$, $\alpha=1$, $\lambda=0$, and ψ as the identity function in (53), then the inequality stated in Theorem 5 in [16] is obtained

(vi) If we take $q=1$, $h(t)=t$, $m=1$, $\eta=0$, $k=1$, $\xi=1$, $\alpha=1$, $\lambda=0$, and ψ as the identity function in (53), then the inequality stated in Theorem 2.2 in [33] is obtained

(vii) If we take $\eta=0$, $\alpha=1$, $m=1$, and $h(t)=t$ in (53), then the inequality stated in Theorem 13 in [32] is obtained

(viii) If we take $\eta = 0$ and $k = 1$ in (53), then the inequality stated in Theorem 7 in [37] is obtained

(ix) If we take $\eta = 0$, $k = 1$, and $\alpha = 1$ in (53), then the inequality stated in Corollary 10 in [37] is obtained

Now, we give inequality (53) for strongly exponentially $(h-m)$-convex, strongly exponentially (s,m)-convex, strongly exponentially m-convex, and strongly exponentially convex functions.

Corollary 12. *If we take $\alpha = 1$ in (53), then the following inequality holds for strongly exponentially $(h-m)$-convex functions:*

$$\begin{aligned}&\Bigg|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\Bigg|\leq\frac{mb-a}{4((\xi/k)+1)^{1-(1/q)}}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}\int_0^1 h\left(\frac{t}{2}\right)t^{(\xi/k)}\mathrm{d}t\right.\right.\\&+\frac{m|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{2-t}{2}\right)t^{(\xi/k)}\mathrm{d}t-\frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\int_0^1 h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)t^{(\xi/k)}\mathrm{d}t)^{1/q}\\&+\left(\frac{m\left|f'\left(a/m^2\right)\right|^q}{e^{\eta a/m^2}}\int_0^1 h\left(\frac{2-t}{2}\right)t^{(\xi/k)}\mathrm{d}t+\frac{|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{t}{2}\right)t^{(\xi/k)}\mathrm{d}t-\frac{m\lambda\left(b-\left(a/m^2\right)\right)^2}{e^{\eta\left(\left(a/m^2\right)+b\right)}}\right.\\&\left.\left.\times\int_0^1 h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)t^{(\xi/k)}dt^{(1/q)}\right)\right].\end{aligned}\tag{56}$$

Corollary 13. *If we take $\alpha = 1$ and $h(t) = t^s$ in (53), then the following inequality holds for strongly exponentially (s,m)-convex functions:*

$$\begin{aligned}&\Bigg|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\\&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\Bigg|\leq\frac{mb-a}{4((\xi/k)+1)\left(2^s((\xi/k)+s+1)\right)^{(1/q)}}\left[\left(\frac{((\xi/k)+1)|f'(a)|^q}{e^{\eta a}}\right.\right.\\&+\frac{2^s m|f'(b)|^q((\xi/k)+s+1)_2F_1(-s,1+(\xi/k),2+(\xi/k);(1/2))}{e^{\eta b}}\\&\left.-\frac{2^{1+s+(\mu/k)}m\lambda(b-a)^2B((1/2),1+s+(\mu/k),1+s)((\xi/k)+1)((\xi/k)+s+1)}{e^{\eta(a+b)}}\right)^{(1/q)}\\&+\left(\frac{2^s m_2F_1(-s,1+(\xi/k),2+(\xi/k);(1/2))((\xi/k)+s+1)\left|f'\left(a/m^2\right)\right|^q}{e^{\eta a/m^2}}+\frac{((\xi/k)+1)|f'(b)|^q}{e^{\eta b}}\right.\\&\left.\left.-\frac{2^{1+s+(\mu/k)}m\lambda(b-a)^2B((1/2),1+s+(\mu/k),1+s)((\xi/k)+1)((\xi/k)+s+1)}{e^{\eta(a+b)}}\right)^{(1/q)}\right].\end{aligned}\tag{57}$$

Corollary 14. *If we take $\alpha = 1$, $m = 1$, and $h(t) = t$ in (53), then the following inequality holds for strongly exponentially m-convex functions:*

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(b-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+b/2)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\xi,\psi}_{\psi^{-1}(a+b/2)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{b-a}{4((\xi/k)+1)(2((\xi/k)+2))^{(1/q)}}\left[\left(\frac{((\xi/k)+1)|f'(a)|^q}{e^{\eta a}}+\frac{|f'(b)|^q((\xi/k)+3)}{e^{\eta b}}-\frac{\lambda(b-a)^2((\xi/k)+1)((\xi/k)+4)}{2e^{\eta(a+b)}((\xi/k)+3)}\right)^{(1/q)}\right. \tag{58}$$
$$\left.+\left(\frac{((\xi/k)+3)|f'((a/m^2))|^q}{e^{\eta a/m^2}}+\frac{((\xi/k)+1)|f'(b)|^q}{e^{\eta b}}-\frac{\lambda(b-a)^2((\xi/k)+1)((\xi/k)+4)}{e^{\eta(a+b)}((\xi/k)+3)}\right)^{(1/q)}\right].$$

Corollary 15. *If we take $\alpha=1$, $k=1$, $m=1$, $q=1$, $\xi=1$, $h(t)=t$, and ψ as the identity function in (53), then the following inequality is obtained:*

$$\left|\frac{1}{(b-a)}\int_0^1 f(\xi)\mathrm{d}\xi-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{8}\left[\frac{|f'(a)|}{e^{\eta a}}+\frac{|f'(b)|}{e^{\eta b}}-\frac{5\lambda(b-a)^2}{12e^{\eta(a+b)}}\right]. \tag{59}$$

Theorem 11. *Let $f\colon [a,b]\longrightarrow\mathbb{R}$, $[a,b]\subset[0,b]$, be a differentiable mapping on (a,b) with $a<b$. Also, suppose that $|f'|^q$ is the strongly exponentially $(\alpha,h-m)$-convex function on $[a,b]$ for $q>1$. Then, for $k>0$ and $(\alpha,m)\in(0,1]^2$, the following fractional integral inequality holds for the operators given in (14) and (15):*

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\le\frac{mb-a}{4((\xi p/k)+1)^{(1/p)}}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t\right.\right.$$
$$\left.+\frac{m|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t-\frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t\right)^{(1/q)} \tag{60}$$
$$+\left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}}\int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t+\frac{|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t\right.$$
$$\left.\left.-\frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}}\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t^{(1/q)}\right)\right],$$

with $(1/p)+(1/q)=1$.

Proof. Applying Lemma 2 and using the property of modulus, we get

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\le\frac{mb-a}{4}\left[\int_0^1\left|t^{(\xi/k)}f'\left(\frac{at}{2}+m\left(1-\frac{t}{2}\right)b\right)\right|\mathrm{d}t\right. \tag{61}$$
$$\left.+\int_0^1\left|t^{(\xi/k)}f'\left(\frac{a}{m}\left(1-\frac{t}{2}\right)+\frac{bt}{2}\right)\right|\mathrm{d}t\right].$$

Now applying Hölder's inequality for integrals, we get

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4((\xi p/k)+1)^{(1/p)}}\left[\left(\int_0^1\left|f'\left(\frac{at}{2}+m\left(\frac{2-t}{2}\right)b\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right.$$
$$\left.+\left(\int_0^1\left|f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right)+\frac{bt}{2}\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right].\tag{62}$$

Using strongly exponentially $(\alpha, h-m)$-convexity of $|f'|^q$, we get

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4((\xi p/k)+1)^{(1/p)}}$$
$$\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t+\frac{m|f'(b)|^q}{e^{\eta b}}\int_0^1h\left(1-\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t-\frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t\right)^{(1/q)}\right.$$
$$\left.+\left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}}\int_0^1h\left(1-\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t+\frac{|f'(b)|^q}{e^{\eta b}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t-\frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}}\int_0^1h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t\right)^{(1/q)}\right].\tag{63}$$

In the following remark, we give the connection of inequality (60) with already established results. □

Remark 6

(i) If we take $\lambda=0$ in (60), then the inequality stated in Theorem 11 in [38] is obtained
(ii) If we take $\eta=0$ and $k=1$ in (60), then the inequality stated in Theorem 8 in [37] is obtained
(iii) If we take $\eta=0$ and $\alpha=1$ in (60), then the inequality stated in Corollary 12 in [37] is obtained
(iv) If we take $h(t)=t, k=1, \eta=0, \alpha=1, \lambda=0$, and ψ as the identity function in (60), then the inequality stated in Theorem 2.7 in [30] is obtained
(v) If we take $\eta=0$, $\lambda=0$, $h(t)=t$, $\alpha=1$, $m=1$, and ψ as the identity function in (60), then the inequality stated in Theorem 3.2 in [20] is obtained
(vi) If we take $k=1, h(t)=t, m=1, \eta=0, \xi=1, \alpha=1$, $\lambda=0$, and ψ as the identity function in (60), then the inequality stated in Theorem 2.4 in [33] is obtained
(vii) If we take $\eta=0$, $\alpha=1$, and $h(t)=t^s$ in (60), then the inequality stated in Corollary 9 in [35] is obtained
(viii) If we take $\alpha=1, m=1, h(t)=t$, and $\eta=0$ in (60), then the inequality stated in Theorem 14 in [32] is obtained

Now, we give inequality (60) for strongly exponentially $(h-m)$-convex, strongly exponentially (s,m)-convex, strongly exponentially m-convex, and strongly exponentially convex functions.

Corollary 16. *If we take $\alpha=1$ in (60), then the following inequality holds for strongly exponentially $(h-m)$-convex functions:*

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4((\xi p/k)+1)^{(1/p)}}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}\int_0^1 h\left(\frac{t}{2}\right)\mathrm{d}t\right.\right.$$
$$\left.+\frac{m|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{2-t}{2}\right)dt-\frac{m\lambda(b-a)^2}{e^{\eta(a+b)}}\int_0^1 h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)\mathrm{d}t\right)^{(1/q)} \tag{64}$$
$$+\left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}}\int_0^1 h\left(\frac{2-t}{2}\right)\mathrm{d}t+\frac{|f'(b)|^q}{e^{\eta b}}\int_0^1 h\left(\frac{t}{2}\right)\mathrm{d}t\right.$$
$$\left.\left.-\frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}}\int_0^1 h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)\mathrm{d}t\right)^{(1/q)}\right].$$

Corollary 17. *If we take $\alpha=1$ and $h(t)=t^s$ in (60), then the following inequality holds for strongly exponentially (s,m)-convex functions:*

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4(1+s)^{(1/q)}((\xi p/k)+1)^{(1/p)}}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}\right.\right.$$
$$\left.+\frac{2^s m|f'(b)|^q(-1+2^{1+s})}{2^s e^{\eta b}}-\frac{2^{s+1}(1+s)mB((1/2),1+s,1+s)\lambda(b-a)^2}{e^{\eta(a+b)}}\right)^{(1/q)}+\left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}}\right. \tag{65}$$
$$\left.\left.+\frac{|f'(b)|^q(-1+2^{1+s})}{2^s e^{\eta b}}-\frac{2^{s+1}m(1+s)B((1/2),1+s,1+s)\lambda(b-(a/m^2))^2}{e^{\eta(a+b)}}\right)^{(1/q)}\right].$$

Corollary 18. *If we take $\alpha=1$ and $h(t)=t$ in (60), then the following inequality holds for strongly exponentially m-convex functions:*

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^+}(f\circ\psi)\left(\psi^{-1}(mb)\right)+m^{(\xi/k)+1}{}_kI^{\xi,\psi}_{\psi^{-1}(a+mb/2)^-}(f\circ\psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{16}\left(\frac{4}{(\xi p/k)+1}\right)^{(1/p)}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}+\frac{3m|f'(b)|^q}{e^{\eta b}}\right.\right. \tag{66}$$
$$\left.\left.-\frac{2m\lambda(b-a)^2}{3e^{\eta(a+b)}}\right)^{(1/q)}+\left(\frac{3m|f'(a/m^2)|^q}{e^{\eta a/m^2}}+\frac{|f'(b)|^q}{e^{\eta b}}-\frac{2m\lambda(b-(a/m^2))^2}{3e^{\eta(a+b)}}\right)^{(1/q)}\right].$$

Corollary 19. *If we take $\alpha=1$, $m=1$, and $h(t)=t$ in (60), then the following inequality holds for strongly exponentially convex functions:*

$$\left|\frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}}\left[{}_kI^{\xi,\psi}_{\psi^{-1}(a+b/2)^+}(f\circ\psi)\left(\psi^{-1}(b)\right)+{}_kI^{\xi,\psi}_{\psi^{-1}(a+b/2)^-}(f\circ\psi)\left(\psi^{-1}(a)\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{16}\left(\frac{4}{(\xi p/k)+1}\right)^{(1/p)}\left[\left(\frac{|f'(a)|^q}{e^{\eta a}}+\frac{3|f'(b)|^q}{e^{\eta b}}-\frac{2\lambda(b-a)^2}{3e^{\eta(a+b)}}\right)^{(1/q)}+\left(\frac{3|f'(a)|^q}{e^{\eta a}}+\frac{f'(b)}{e^{\eta b}}\right.\right. \tag{67}$$

$$\left.\left.-\frac{2\lambda\left(b-\left(a/m^2\right)\right)^2}{3e^{\eta(a+b)}}\right)^{(1/q)}\right].$$

4. Conclusion

In this paper, we have proved fractional versions of the Hadamard inequality and their estimations for strongly exponentially $(\alpha, h-m)$-convex functions via generalized Riemann–Liouville fractional integrals. The outcomes of this article give refinements and generalizations of fractional integral inequalities for different types of convex functions deducible from the definition of the exponentially $(\alpha, h-m)$-convex function. [39–41].

Data Availability

No data were used for the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

[1] U. N. Katugampola, "New approach to a generalized fractional integral," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 860–865, 2011.

[2] T. U. Khan and M. A. Khan, "Generalized conformable fractional operators," *Journal of Computational and Applied Mathematics*, vol. 346, pp. 378–389, 2019.

[3] H. Chen and U. N. Katugampola, "Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals," *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1274–1291, 2017.

[4] H. Ge-Jile, S. Rashid, S. Rashid, M. Suhail, and Y.-M. Chu, "Some unified bounds for exponentially TGS-convex functions governed by conformable fractional operators," *AIMS Mathematics*, vol. 5, no. 6, pp. 6108–6123, 2020.

[5] T. U. Khan and M. A. Khan, "Hermite-Hadamard inequality for new generalized conformable fractional operators," *AIMS Mathematics*, vol. 6, no. 1, pp. 23–38, 2021.

[6] G. Farid, A. U. Rehman, S. Bibi, and Y.-M. Chu, "Refinements of two fractional versions of Hadamard inequalities for Caputo fractional derivatives and related results," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 1–10, 2021.

[7] M. Jleli and B. Samet, "On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 3, pp. 1252–1260, 2016.

[8] S. Mehmood and G. Farid, "Fractional integrals inequalities for exponentially m-convex functions," *Open Journal of Mathematical Sciences*, vol. 4, no. 1, pp. 78–85, 2020.

[9] M. E. Özdemir, A. A. Merve, and H. Kavurmaci-Önalan, "Hermite-Hadamard type inequalities for s-convex and s-concave functions via fractional integrals," *Turkish Journal of Science*, vol. 1, no. 1, pp. 28–40, 2016.

[10] T. Yu, G. Farid, K. Mahreen, C. Y. Jung, and S. H. Shim, "On generalized strongly convex functions and unified integral operators," 2020.

[11] X. Qiang, G. Farid, J. Pečarić, and S. B. Akbar, "Generalized fractional integral inequalities for exponentially (s, m)-convex functions," *Journal of Inequalities and Applications*, vol. 2020, no. 1, p. 70, 2020.

[12] N. Mehreen and M. Anwar, "Hermite-Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 92, 2019.

[13] M. U. Awan, M. A. Noor, and K. I. Noor, "Hermite-Hadamard inequalities for exponentially convex functions," *Applied Mathematics & Information Sciences*, vol. 12, no. 2, pp. 405–409, 2018.

[14] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives-Theory and Applications Gordon and Breach*, Gordon and Breach Science Publishers, Linghorne, PA, USA, 1993.

[15] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[16] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2017.

[17] S. Mubeen and G. M. Habibullah, "k-fractional integrals and applications," *International Journal of Contemporary Mathematical Sciences*, vol. 7, pp. 89–94, 2012.

[18] S. Mubeen and A. Rehman, "A note on k-gamma function and pochhammer k-symbol," *Journal of Informatics and Mathematical Sciences*, vol. 6, no. 2, pp. 93–107, 2014.

[19] G. Farid, A. U. Rehman, and M. Zahra, "On Hadamard-type inequalities for k-fractional integrals," *Journal of Nonlinear Functional Analysis*, vol. 21, no. 3, pp. 463–478, 2016.

[20] G. Farid, A. U. Rehman, and M. Zahra, "On Hadamard inequalities for k-fractional integrals," *Konuralp Journal of Mathematics*, vol. 4, no. 2, pp. 79–86, 2016.

[21] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.

[22] A. Akkurt, M. E. Yildirim, and H. Yildirim, "On some integral inequalities for (k, h)-Riemann-Liouville fractional integral," *New Trends in Mathematical Science*, vol. 4, no. 2, pp. 138–146, 2016.

[23] M. Lazarević, M. R. Rapaic, T. B. Sekara et al., *Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modeling*, WSEAS Press, Belgrade, Serbia, 2014.

[24] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[25] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Springer Basel AG, Basel, Switzerland, 1988.

[26] M. U. Awan, S. Talib, Y.-M. Chu, M. A. Noor, and K. I. Noor, "Some new refinements of Hermite-Hadamard-type inequalities involving ψ_k-Riemann-Liouville fractional integrals and applications," *Mathematical Problems in Engineering*, vol. 2020, Article ID 3051920, 10 pages, 2020.

[27] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[28] G. Farid, S. B. Akbar, V. N. Mishra, and L. K. Mishra, "Riemann-Liouville fractional versions of Hadamard inequality for strongly m-convex functions, submitted," 2020.

[29] G. Farid, A. U. Rehman, and Q. U. Ain, "k-fractional integral inequalities of Hadamard type for $(h - m)$-convex functions," *Computational Methods for Differential Equations*, vol. 8, no. 1, pp. 119–140, 2020.

[30] G. Farid, A. U. Rehman, and B. Tariq, "On Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals," *Studia Universitatis Babes-Bolyai Matematica*, vol. 62, no. 2, pp. 141–150, 2017.

[31] G. Farid, A. U. Rehman, B. Tariq, and A. Waheed, "On Hadamard type inequalities for m-convex functions via fractional integrals," *Journal of Inequalities and Special Functions*, vol. 7, no. 4, pp. 150–167, 2016.

[32] G. Farid, H. Yasmeen, C. Y. Jung, S. H. Shim, and G. Ha, "Refinements and generalization of some fractional integral inequalities via strongly convex functions," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6667226, 18 pages, 2021.

[33] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.

[34] K. Liu, J. Wang, and D. O'Regan, "On the Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via convex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 27, 2019.

[35] C. Miao, G. Farid, H. Yasmeen, and Y. Bian, "Generalized Hadamard fractional integral inequalities for strongly (s, m)-convex functions," *Journal of Mathematics*, vol. 2021, Article ID 6642289, 19 pages, 2021.

[36] A. Ur Rehman, G. Farid, C. Yong Jung, and S. Min Kang, "k-fractional integral inequalities of Hadamard type for exponentially (s, m)-convex functions," *AIMS Mathematics*, vol. 6, no. 1, pp. 882–892, 2021.

[37] G. Farid, Y. C. Kwun, H. Yasmeen, A. Akkurt, and S. M. Kang, "Inequalities for generalized Riemann-Liouville fractional integrals of generalized strongly convex functions, submitted," 2020.

[38] U.-P. Lv, G. Farid, H. Yasmeen, W. Nazeer, and C. Y. Jung, "Generalization of some fractional versions of Hadamard inequality for exponentially $(\alpha, h\text{-}m)$-convex functions," *AIMS Mathematics*, vol. 6, no. 8, pp. 8978–8999, 2021.

[39] J. Hadamard, "Etude sur les proprietes des fonctions entieres et en particulier dune fonction considree par Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–215, 1893.

[40] E. Set and A. Gözpinar, "Hermite-Hadamard type inequalities for convex functions via generalized fractional integral operators," *Topological Algebra and Its Applications*, vol. 5, no. 1, pp. 55–62, 2016.

[41] E. Set, M. A. Noor, M. U. Awan, and A. Gözpinar, "Generalized Hermite-Hadamard type inequalities involving fractional integral operators," *Journal of Inequalities and Applications*, vol. 2017, no. 1, p. 169, 2017.

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Research Article

Properties and Bounds of Jensen-Type Functionals via Harmonic Convex Functions

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Dragomir introduced the Jensen-type inequality for harmonic convex functions (HCF) and Baloch et al. studied its different variants, such as Jensen-type inequality for harmonic h-convex functions. In this paper, we aim to establish the functional form of inequalities presented by Baloch et al. and prove the superadditivity and monotonicity properties of these functionals. Furthermore, we derive the bound for these functionals under certain conditions. Furthermore, we define more generalized functionals involving monotonic nondecreasing concave function as well as evince superadditivity and monotonicity properties of these generalized functionals.

1. Introduction

Convexity is natural and simple notion which has found applications in business, industry, and medicine. During the study of convexity, many researchers have been fascinated by generalization of this class and have tried to find out those classes of functions which have close relation with this class (but not convex in general). Harmonic convex functions (HCFs) [1], harmonic (α, m)-convex functions [2], harmonic (s, m)-convex functions [3, 4], and harmonic $(p, (s, m))$-convex functions [5] are among these classes. For a quick glance on importance of these classes and applications, see [6–9] and references therein. The class of harmonic convex functions (HCFs) and its different variants are very important classes that gained prominence in the theory of inequalities and applications as well as in other branches of mathematics. Many researchers have been working on the class of harmonic convex functions (HCFs) due to its significance and have been trying to explore about it more and more. During this study, recently different generalizations of the class of harmonic convex functions (HCFs) have been found, for example, see [10–13] and references therein.

The importance of the class of HCFs continuously encourages us and many other researchers to explore more about it, and the following paper is a link to it. For the better understanding of the results of present paper, we first recall some basic definitions.

Definition 1. Consider $I \subset \mathbb{R}\backslash\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is HCF on I if

$$f\left(\frac{w_1 w_2}{t w_1 + (1-t) w_2}\right) \leq t f(w_2) + (1-t) f(w_1), \tag{1}$$

holds, for all $w_1, w_2 \in I$ and $t \in [0, 1]$. If we reverse the above inequality, the function f becomes harmonic concave.

Remark 1 (see [14]).

(i) The function $f(w) = \ln w$ is a HCF on the interval $(0, \infty)$, but it is not a convex function.

(ii) The function

$$f(w) = \begin{cases} \frac{1-w}{w}, & 0 < w \leq 1, \\ 0, & 1 < w \leq 2, \\ \frac{w-2}{w}, & w > 2, \end{cases} \quad (2)$$

is another example of HCF, which is neither convex nor concave.

Baloch et al., in [7], observed some remarkable facts for the class of HCFs.

Proposition 1. *For $I \subseteq \mathbb{R}\backslash\{0\}$, a function $f\colon I \longrightarrow \mathbb{R}$, we have the following facts:*

(1) If $I \subset (0, \infty)$ and f is nondecreasing and convex function, then f is HCF

(2) If $I \subset (0, \infty)$ and f is nonincreasing function and harmonically convex, then f is convex function

(3) If $I \subset (-\infty, 0)$ and f is nondecreasing function and harmonically convex, then f is convex function

(4) If $I \subset (-\infty, 0)$ and f is nonincreasing function and convex, then f is HCF

Varošanec, in [15], proposed the concept of h-convexity (also, see [16–18]) to unify numerous generalized aspects of convex functions. In a similar fashion, harmonic h-convexity unifies the various types of harmonic convexities.

Definition 2 (see [19]). Consider a nonnegative function $h\colon [0,1] \longrightarrow \mathbb{R}^+$. Then, the function $f\colon I \subseteq \mathbb{R}\backslash\{0\} \longrightarrow (0, \infty)$ is said to be harmonic h-convex on I if the inequality

$$f\left(\frac{w_1 w_2}{t w_1 + (1-t) w_2}\right) \leq h(t) f(w_2) + h(1-t) f(w_1), \quad (3)$$

holds, for all $w_1, w_2 \in I$ and $t \in [0,1]$. Furthermore, if we reverse inequality (3), then f becomes harmonic h-concave.

Remark 2. We provide few examples of harmonic h-convex (concave) functions as follows:

(i) Obviously, with $h(t) = t$, the class of nonnegative harmonic convex (concave) functions on I become a particular case of the class of harmonic h-convex (concave) functions on I.

(ii) Let $t \in (0,1)$ and $h(t) = t^2$. Consider a function $f\colon [-1,0) \cup t(0,1] \longrightarrow \mathbb{R}$ defined by $f(x) = 1$, which is neither nonincreasing nor nondecreasing h-convex function. Therefore, f is a harmonic h-convex function by Proposition 1 given in [7].

(iii) Let $t \in (0,1)$ and $h\colon (0,1) \longrightarrow (0,\infty)$ be a real-valued function such that $h(t) \geq t$ on $(0,1)$. Then, the following four functions, $h_1(t) = t$, $h_2(t) = t^s$ $(s \in (0,1))$, $h_3(t) = 1/t$, and $h_4(t) = 1$, satisfy the conditions of the function h mentioned above. Therefore, f is a harmonic h_α-convex function for $\alpha = 1, 2, 3, 4$ if $f\colon I \subseteq (0,\infty) \longrightarrow (0,\infty)$ is a nondecreasing convex function, or harmonic s-convex function, or harmonic Godunova–Levin function or harmonic P-function.

(iv) Let $f\colon (0,\infty) \longrightarrow (0,\infty)$ be a nondecreasing continuous function and $h\colon [0,1] \longrightarrow (0,\infty)$ be a continuous self-concave function such that $f(tw_1 + (1-t)w_2) \leq h(t) f(w_1) + (1-t) f(w_2)$, for some $t \in (0,1)$ and all $w_1, w_2 \in (0,\infty)$. Then, f is a h-convex function by Lemma 1 of [20], and hence, f is a harmonic h-convex function by Proposition 1 of [7].

Definition 3. A function $h\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a submultiplicative function if

$$h(w_1 w_2) \leq h(w_1) h(w_2), \quad (4)$$

for all $w_1, w_2 \in I$. If inequality (4) is reversed, then h is said to be supermultiplicative function. If just equality holds in relation (4), then h is said to be multiplicative function.

Definition 4. A function $h\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a subadditive function if

$$h(w_1 + w_2) \leq h(w_1) + h(w_2), \quad (5)$$

for all $w_1, w_2 \in I$. If inequality (5) is reversed, then h is said to be superadditive function. If just the equality holds in relation (5), then h is said to be additive function.

Jensen-type inequality for HCFs is proposed by Dragomir [21].

Theorem 1. *Let $I \subseteq (0,\infty)$ be an interval. If $f\colon I \longrightarrow \mathbb{R}$ is HCF, then*

$$f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n} b_\alpha / w_\alpha}\right) \leq \left(\frac{1}{B_n}\right) \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha), \quad (6)$$

Holds, for all $w_1, \ldots, w_n \in I$ and $b_1, \ldots, b_n \geq 0$ with $\sum_{\alpha=1}^{n} b_\alpha = B_n$.

In [22], Baloch et al. derived the following results:

Theorem 2. *Let $I \subseteq \mathbb{R}\backslash\{0\}$. If $f\colon I \longrightarrow \mathbb{R}$ is HCF, then, for any finite positive sequence $(w_\alpha)_{\alpha=1}^{n} \in I$ and $b_1, \ldots, b_n \geq 0$ with $\sum_{\alpha=1}^{n} b_\alpha = B_n$, we have*

$$f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n)\sum_{\alpha=1}^{n} b_\alpha / w_\alpha}\right) \leq f(w_1) + f(w_n) - (1/B_n) \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha). \quad (7)$$

Theorem 3. *Let $b_1, \ldots, b_n$ be positive real numbers $(n \geq 2)$ and $B_n = \sum_{\alpha=1}^{n} b_\alpha$. If $h\colon I \supseteq (0,1) \longrightarrow \mathbb{R}$ is a nonnegative supermultiplicative function and if f is harmonic h-convex function, $(w_\alpha)_{\alpha=1}^{n} \in I$, then*

$$f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right) \leq \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha). \tag{8}$$

Theorem 4. *Let $h\colon I \supseteq (0,1) \longrightarrow \mathbb{R}$ be a nonnegative supermutiplicative function on I. Let $b_1, \ldots, b_n$ be positive real numbers $(n \geq 2)$ such that $B_n = \sum_{\alpha=1}^{n} b_\alpha$ and $\sum_{\alpha=1}^{n} h(b_\alpha/B_n) \leq 1$. If f is harmonic h-convex on $I \subseteq \mathbb{R}\backslash\{0\}$, then, for any finite positive increasing sequence $(w_\alpha)_{k=1}^{n} \in I$, we have*

$$f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right) \leq f(w_1) + f(w_n) - \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha). \tag{9}$$

If h is a submultiplicative function, $\sum_{\alpha=1}^{n} h(b_\alpha/B_n) \geq 1$ and f is harmonic h-concave then inequality (9) is reversed.

Remark 3. Importance of the class of HCFs can be guessed by the following applications in the field of mathematics:

(i) Harmonic convexity provides a useful analytic tool to calculate several known definite integrals such as $\int_a^b (e^w/x^n)\mathrm{d}w$, $\int_a^b e^{w^2}\mathrm{d}w$, $\int_a^b (\sin w/w^n)\mathrm{d}w$, and $\int_a^b (\cos w/w^n)\mathrm{d}x\ \forall n \in \mathbb{N}$, where $a, b \in (0, \infty)$, see [7]

(ii) Inequality (6) provides a very short proof of the discrete form of Hölder's inequality (see [22])

(iii) By inequalities (6) and (7), we can easily prove weighted HGA inequality (see [14])

Many researchers considered the functionals related to Jensen's inequality and tried to find properties and bound for these functionals (for example, see [23–30]). In the sequel, the set of all nonnegative n-tuples $\mathbf{b} = (b_1, \ldots, b_n)$, such that $B_n \coloneqq \sum_{\alpha=1}^{n} b_\alpha > 0$, will be denoted with $\mathscr{B}_n^0$.

The difference between the right-hand and the left-hand side of inequalities (6)–(9) defines the following functionals:

$$\mathscr{M}_1(f, \mathbf{w}, \mathbf{b}) \coloneqq \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha) - B_n f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right), \tag{10}$$

$$\mathscr{M}_2(f, \mathbf{w}, \mathbf{b}) \coloneqq B_n[f(w_1) + f(w_n)] - \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha) - B_n f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right), \tag{11}$$

$$\mathscr{M}_3(f, \mathbf{w}, \mathbf{b}) \coloneqq \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha) - f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right), \tag{12}$$

$$\mathscr{M}_4(f, \mathbf{w}, \mathbf{b}) \coloneqq f(w_1) + f(w_n) - \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha) - f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right). \tag{13}$$

For a fixed function f and n-tuple $\mathbf{w}$, $\mathscr{M}_1(f, \mathbf{w}, .)$, $\mathscr{M}_2(f, \mathbf{w}, .)$, $\mathscr{M}_3(f, \mathbf{w}, .)$, and $\mathscr{M}_4(f, \mathbf{w}, .)$ can be considered as functions on $\mathscr{B}_n^0$, which is a convex subset in $\mathbb{R}^n$. Furthermore, because of inequalities (6)–(9), we have $\mathscr{M}_1(f, \mathbf{w}, \mathbf{b}) \geq 0$, $\mathscr{M}_2(f, \mathbf{w}, \mathbf{b}) \geq 0$, $\mathscr{M}_3(f, \mathbf{w}, \mathbf{b}) \geq 0$, and $\mathscr{M}_4(f, \mathbf{w}, \mathbf{b}) \geq 0$, for all $\mathbf{b} \in \mathscr{B}_n^0$.

2. Main Results

In this section, we establish some properties of functionals related to Jensen-type inequalities for HCFs.

Theorem 5. *Let $\mathbf{b} = (b_1, \ldots, b_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)$ be two n-tuples from $\mathscr{B}_n^0$. Let $I \subseteq (0, \infty)$ be an interval. If $f\colon I \longrightarrow \mathbb{R}$ is a HCF, $h\colon I \supseteq (0,1) \longrightarrow \mathbb{R}$ is a nonnegative multiplicative and additive function on J, and if $\mathbf{w} = (w_1, \ldots, w_n) \in I^n$, $h(\alpha) + h(1-\alpha) \geq 2$, then $\mathscr{M}_i(f, \mathbf{w}, .)$, for $i = 1, 2, 3, 4$, defined by (10)–(12) are superadditive on $\mathscr{B}_n^0$, i.e.,*

$$\mathscr{M}_i(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) \geq \mathscr{M}_i(f, \mathbf{w}, \mathbf{b}) + \mathscr{M}_i(f, \mathbf{w}, \mathbf{c}) \geq 0, \tag{14}$$

for $i = 1, 2, 3, 4$.

Proof. Take $i = 1$ in (28) and starting from definition, we have

$$\begin{aligned}\mathcal{M}_1(f,\mathbf{w},\mathbf{b}+\mathbf{c}) &= \sum_{\alpha=1}^{n}(b_\alpha+c_\alpha)f(w_\alpha)-(B_n+C_n)f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n}b_\alpha+c_\alpha/w_\alpha}\right)\\ &= \sum_{\alpha=1}^{n}b_\alpha f(w_\alpha)+\sum_{\alpha=1}^{n}c_\alpha f(w_\alpha)-(B_n+C_n)f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n}b_\alpha+c_\alpha/w_\alpha}\right),\end{aligned}\tag{15}$$

while, after arranging and harmonic convexity of f, yields

$$\begin{aligned}f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n}b_\alpha+c_\alpha/w_\alpha}\right) &= f\left(\frac{1}{(B_n/B_n+C_n).(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha+(C_n/B_n+C_n).(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)\\ &\leq \frac{B_n}{B_n+C_n}f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right)+\frac{C_n}{B_n+C_n}f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right).\end{aligned}\tag{16}$$

Finally, combining relation (15) and inequality (16), we obtain

$$\begin{aligned}\mathcal{M}_1(f,\mathbf{w},\mathbf{b}+\mathbf{c}) &\geq \sum_{\alpha=1}^{n}b_\alpha f(w_\alpha)+\sum_{\alpha=1}^{n}c_\alpha f(w_\alpha)-B_n f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right)\\ &\quad -C_n f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)=\mathcal{M}_1(f,\mathbf{w},\mathbf{b})+\mathcal{M}_1(f,\mathbf{w},\mathbf{c}).\end{aligned}\tag{17}$$

Now, taking $i=2$ in (28) and starting from the definition, we have

$$\begin{aligned}\mathcal{M}_2(f,\mathbf{w},\mathbf{b}+\mathbf{c}) &= (B_n+C_n)[f(w_1)+f(w_n)]-\sum_{\alpha=1}^{n}(b_\alpha+c_\alpha)f(w_\alpha)\\ &\quad -(B_n+C_n)f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/B_n+C_n)\sum_{\alpha=1}^{n}b_\alpha+c_\alpha/w_\alpha}\right)\\ &= B_n[f(w_1)+f(w_n)]+C_n[f(w_1)+f(w_n)]\\ &\quad -\sum_{\alpha=1}^{n}b_\alpha f(w_\alpha)-\sum_{\alpha=1}^{n}c_\alpha f(w_\alpha)-(B_n+C_n)\\ &\quad f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/B_n+C_n)\sum_{\alpha=1}^{n}b_\alpha+c_\alpha/w_\alpha}\right),\end{aligned}\tag{18}$$

while, after arranging and harmonic convexity of f, yields

$$
\begin{aligned}
&f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/B_n+C_n)\sum_{\alpha=1}^{n} b_\alpha+c_\alpha/w_\alpha}\right)=f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n}(b_\alpha+c_\alpha)((1/w_1)+(1/w_n)-(1/w_\alpha))}\right)\\
&=f\left(\frac{1}{(B_n/B_n+C_n)\sum_{\alpha=1}^{n}(b_\alpha/B_n)((1/w_1)+(1/w_n)-(1/w_\alpha))+(C_n/B_n+C_n)\sum_{\alpha=1}^{n}(c_\alpha/C_n)((1/w_1)+(1/w_n)-(1/w_\alpha))}\right)\\
&\leq\frac{B_n}{B_n+C_n}f\left(\frac{1}{\sum_{\alpha=1}^{n}(b_\alpha/B_n)((1/w_1)+(1/w_n)-(1/w_\alpha))}\right)+\frac{C_n}{B_n+C_n}f\left(\frac{1}{\sum_{\alpha=1}^{n}(c_\alpha/C_n)((1/w_1)+(1/w_n)-(1/w_\alpha))}\right)\\
&=\frac{B_n}{B_n+C_n}f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right)+\frac{C_n}{B_n+C_n}f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/C_n)\sum_{\alpha=1}^{n} c_\alpha/w_\alpha}\right).
\end{aligned}
\tag{19}
$$

Finally, combining relation (18) and inequality (19), we obtain

$$
\begin{aligned}
\mathcal{M}_2(f,\mathbf{w},\mathbf{b}+\mathbf{c})\geq{}& B_n[f(w_1)+f(w_n)]+C_n[f(w_1)+f(w_n)]-\sum_{\alpha=1}^{n} b_\alpha f(w_\alpha)\\
&-\sum_{\alpha=1}^{n} c_\alpha f(w_\alpha)-(B_n+C_n).\frac{B_n}{B_n+C_n}f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right)\\
&-(B_n+C_n).\frac{C_n}{B_n+C_n}f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/C_n)\sum_{\alpha=1}^{n} c_\alpha/w_\alpha}\right)=\mathcal{M}_2(f,\mathbf{w},\mathbf{b})+\mathcal{M}_2(f,\mathbf{w},\mathbf{c}).
\end{aligned}
\tag{20}
$$

Taking $i=3$ in (28) and starting from the definition, we have

$$
\begin{aligned}
\mathcal{M}_3(f,\mathbf{w},\mathbf{b}+\mathbf{c})&=\sum_{\alpha=1}^{n} h\left(\frac{b_\alpha+c_\alpha}{B_n+C_n}\right)f(w_\alpha)-f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n} b_\alpha+c_\alpha/w_\alpha}\right)\\
&=\sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n+C_n}\right)f(w_\alpha)+\sum_{\alpha=1}^{n} h\left(\frac{c_\alpha}{B_n+C_n}\right)f(w_\alpha)\\
&\quad-f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n} b_\alpha+c_\alpha/w_\alpha}\right)\\
&=\sum_{\alpha=1}^{n} h\left(\frac{B_n}{B_n+C_n}\right)h\left(\frac{b_\alpha}{B_n}\right)f(w_\alpha)+\sum_{\alpha=1}^{n} h\left(\frac{C_n}{B_n+C_n}\right)h\left(\frac{c_\alpha}{C_n}\right)f(w_\alpha)\\
&\quad-f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n} b_\alpha+c_\alpha/w_\alpha}\right),
\end{aligned}
\tag{21}
$$

while, after arranging and harmonic h-convexity of f, yields

$$f\left(\frac{1}{(1/B_n+C_n)\sum_{\alpha=1}^{n}(p_i+c_\alpha/w_\alpha)}\right)=f\left(\frac{1}{(B_n/B_n+C_n).(1/B_n)\sum_{\alpha=1}^{n}(b_\alpha/w_\alpha)+(C_n/B_n+C_n).(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)$$
$$\leq h\left(\frac{B_n}{B_n+C_n}\right)f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right)+h\left(\frac{C_n}{B_n+C_n}\right)f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)\tag{22}$$
$$=h\left(\frac{B_n}{B_n+C_n}\right)f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right)+h\left(\frac{C_n}{B_n+C_n}\right)f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right).$$

Finally, combining relation (21) and inequality (22), we obtain

$$\begin{aligned}\mathscr{M}_3(f,\mathbf{w},\mathbf{b}+\mathbf{c})&\geq h\left(\frac{B_n}{B_n+C_n}\right)\sum_{\alpha=1}^{n}h\left(\frac{b_\alpha}{B_n}\right)f(w_\alpha)+h\left(\frac{C_n}{B_n+C_n}\right)\sum_{\alpha=1}^{n}h\left(\frac{c_\alpha}{C_n}\right)f(w_\alpha)\\
&\quad-h\left(\frac{B_n}{B_n+C_n}\right)f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right)-h\left(\frac{C_n}{B_n+C_n}\right)f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)\\
&=h\left(\frac{B_n}{B_n+C_n}\right)\left[\sum_{\alpha=1}^{n}h\left(\frac{b_\alpha}{B_n}\right)f(w_\alpha)-f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right)\right]\\
&\quad+h\left(\frac{C_n}{B_n+C_n}\right)\left[\sum_{\alpha=1}^{n}h\left(\frac{c_\alpha}{C_n}\right)f(w_\alpha)-f\left(\frac{1}{(1/C_n)\sum_{i=1}^{n}c_\alpha/w_\alpha}\right)\right]\\
&\geq\left[\sum_{\alpha=1}^{n}h\left(\frac{b_\alpha}{B_n}\right)f(w_\alpha)-f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}(b_\alpha/w_\alpha)}\right)\right]+\left[\sum_{\alpha=1}^{n}h\left(\frac{c_\alpha}{C_n}\right)f(w_\alpha)-f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)\right]\\
&=\mathscr{M}_3(f,\mathbf{w},\mathbf{b})+\mathscr{M}_3(f,\mathbf{w},\mathbf{c}).\end{aligned}\tag{23}$$

Similarly, it can be proved that

$$\mathscr{M}_4(f,\mathbf{w},\mathbf{b}+\mathbf{c})\geq\mathscr{M}_4(f,\mathbf{w},\mathbf{b})+\mathscr{M}_4(f,\mathbf{w},\mathbf{c}).\tag{24}$$

□

Theorem 6. *Let* $\mathbf{b}=(b_1,\ldots,b_n)$ *and* $\mathbf{c}=(c_1,\ldots,c_n)$ *be two* n*-tuples from* $\mathscr{B}_n^0$ *such that* $\mathbf{b}\geq\mathbf{c}$*, (i.e.,* $b_\alpha\geq c_\alpha,\alpha=1,\ldots,n$*). Let* $I\subseteq(0,\infty)$ *be an interval. If* $f\colon I\longrightarrow\mathbb{R}$ *is a HCF and if* $\mathbf{w}=(w_1,\ldots,w_n)\in I^n$*, then* $\mathscr{M}_i(f,\mathbf{w},.)$*, for* $i=1,2,3,4$*, defined by (10)–(12) satisfy the following inequality:*

$$\mathscr{M}_i(f,\mathbf{w},\mathbf{b})\geq\mathscr{M}_i(f,\mathbf{w},\mathbf{c}),\tag{25}$$

on $\mathscr{B}_n^0$*.*

Proof. The monotonicity property follows directly from superadditivity. Since $\mathbf{b}\geq\mathbf{c}$, $\mathbf{b}$ can be represented as the sum of two n-tuples: $\mathbf{b}-\mathbf{c}$ and $\mathbf{c}$. Applying (28), we have

$$\mathscr{M}_i(f,\mathbf{w},\mathbf{b})=\mathscr{M}_i(f,\mathbf{w},(\mathbf{b}-\mathbf{c})+\mathbf{c})\geq\mathscr{M}_i(f,\mathbf{w},(\mathbf{b}-\mathbf{c}))+\mathscr{M}_i(f,\mathbf{w},\mathbf{c}).\tag{26}$$

Finally, $\mathscr{M}_i(f,\mathbf{w},(\mathbf{b}-\mathbf{c}))\geq 0$ by (10)–(12). So, we have that $\mathscr{M}_i(f,\mathbf{w},\mathbf{b})\geq\mathscr{M}_i(f,\mathbf{w},\mathbf{c})$, which proves the theorem. □

Theorem 7. *Let* $\mathbf{b}=(b_1,\ldots,b_n)$ *and* $\mathbf{c}=(c_1,\ldots,c_n)$ *be two* n*-tuples from* $\mathscr{B}_n^0$*. Let* m *and* M *be real constants such that*

$$\begin{aligned}&m\geq 0,\quad b_\alpha-mc_\alpha\geq 0,\\&Mc_\alpha-b_\alpha\geq 0,\quad \alpha=1,\ldots n.\end{aligned}\tag{27}$$

Let $I\subseteq(0,\infty)$ be an interval. If $f\colon I\longrightarrow\mathbb{R}$ is a HCF and if $\mathbf{w}=(w_1,\ldots,w_n)\in I^n$ be any n-tuple, then

$$M\mathscr{M}_i(f,\mathbf{w},\mathbf{c})\geq\mathscr{M}_i(f,\mathbf{w},\mathbf{b})\geq m\mathscr{M}_i(f,\mathbf{w},\mathbf{c}),\tag{28}$$

for $i=1,2,3,4$.

Proof. Since $m\geq 0$, $b_\alpha-mc_\alpha\geq 0$ and $Mc_\alpha-b_\alpha\geq 0$, $\alpha=1,\ldots n$, this implies that $\mathbf{b}-m\mathbf{c}$ and $M\mathbf{c}-\mathbf{b}$ are in $\mathscr{B}_n^0$. Then, by Theorem 5, we obtain

$$\mathcal{M}_i(f,\mathbf{w},\mathbf{b}) \geq \mathcal{M}_i(f,\mathbf{w},(\mathbf{b}-m\mathbf{c})) + \mathcal{M}_i(f,\mathbf{w},m\mathbf{c}) \geq m\mathcal{M}_i(f,\mathbf{w},\mathbf{c}). \tag{29}$$

Similarly, we obtain

$$\mathcal{M}_i(f,\mathbf{w},\mathbf{b}) \leq M\mathcal{M}_i(f,\mathbf{w},\mathbf{c}), \tag{30}$$

that is,

$$M\mathcal{M}_i(f,\mathbf{w},\mathbf{c}) \geq \mathcal{M}_i(f,\mathbf{w},\mathbf{b}) \geq m\mathcal{M}_i(f,\mathbf{w},\mathbf{c}). \tag{31}$$

□

Corollary 1. *Let* $\mathbf{b}$, $\mathbf{w}$, f, *and functional* $\mathcal{M}_i$ *be as in Theorem 5. Then,*

$$\max_{1\leq k\leq n}\{b_\alpha\}\mathcal{M}_i^{\mathcal{N}}(f,\mathbf{w}) \geq \mathcal{M}_i(f,\mathbf{w},\mathbf{b}) \geq \min_{1\leq k\leq n}\{b_\alpha\}\mathcal{M}_i^{\mathcal{N}}(f,\mathbf{w}) \quad (\text{for } i=1,2), \tag{32}$$

where

$$\begin{aligned}
\mathcal{M}_1^{\mathcal{N}}(f,\mathbf{w}) &= \sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right),\\
\mathcal{M}_2^{\mathcal{N}}(f,\mathbf{w}) &= n[f(w_1)+f(w_n)] - \sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/w_1)+(1/w_n)-(1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right).
\end{aligned} \tag{33}$$

Proof. Let $\mathbf{b}_{\min} \in \mathcal{B}_n^0$ be a constant n-tuple, i.e.,

$$\mathbf{b}_{\min} = \left(\min_{1\leq k\leq n}\{b_\alpha\},\ldots,\min_{1\leq k\leq n}\{b_\alpha\}\right). \tag{34}$$

Then, for any $\mathbf{b} \in \mathcal{B}_n^0$, we have $\mathbf{b} \geq \mathbf{b}_{\min}$. So, by applying Theorem 6, we have

$$\mathcal{M}_i(f,\mathbf{w},\mathbf{b}) \geq \mathcal{M}_i(f,\mathbf{w},\mathbf{b}_{\min}). \tag{35}$$

On the contrary,

$$\begin{aligned}
\mathcal{M}_1(f,\mathbf{w},\mathbf{b}_{\min}) &= \min_{1\leq k\leq n}\{b_\alpha\}\left\{\sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right)\right\},\\
\mathcal{M}_1(f,\mathbf{w},\mathbf{b}_{\min}) &= \min_{1\leq k\leq n}\{b_\alpha\}\left\{n[f(w_1)+f(w_n)] - \sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/w_1)+(1/w_n)-(1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right)\right\},
\end{aligned} \tag{36}$$

i.e., $\mathcal{M}_1(f,\mathbf{w},\mathbf{b}_{\min}) = \min_{1\leq k\leq n}\{b_\alpha\}\mathcal{M}_i^{\mathcal{N}}(f,\mathbf{w})$. So, it proves the right-hand side of inequality (35). The left-hand inequality is obtained similarly by exchanging the role of min and max.

To present our next results, we need to introduce the following notations:

$J(\mathbb{R}) \coloneqq \{\mathbf{b} = \{b_\alpha\}_{k\in\mathbb{N}}: b_\alpha \in \mathbb{R}$ are such that $B_K \coloneqq \sum_{\alpha\in I} b_\alpha \neq 0$, for all $K \in P_f(\mathbb{N})\}$

$F^+(C,\mathbb{R}) \coloneqq \{f \in F(C,\mathbb{R}): f(x) > 0$, for all $x \in C\}$

$J^+(\mathbb{R}) \coloneqq \{\mathbf{b} \in J(\mathbb{R}): b_\alpha \geq 0$, for all $k \in \mathbb{N}\}$

$J_*(\mathbb{R}) \coloneqq \{\mathbf{w} = \{w_\alpha\}_{\alpha\in\mathbb{N}}: w_\alpha \in C$, for all $k \in \mathbb{N}\}$

$P_f(\mathbb{N}) \coloneqq \{K \subset \mathbb{N}: K$ is finite$\}$

$H\text{Conv}(C,\mathbb{R}) \coloneqq$ the cone of all HCFs on C

$F(C,\mathbb{R}) \coloneqq$ the linear space of all real functions on C

Now, we consider more general functionals:

$$\mathscr{D}_1(f,K,\mathbf{b},\mathbf{w};\Psi) := B_K\Psi\left[\frac{1}{B_K}\sum_{\alpha\in K} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/B_K)\sum_{\alpha\in K}b_\alpha/w_\alpha}\right)\right],$$
$$\mathscr{D}_2(f,K,\mathbf{b},\mathbf{w};\Psi) := B_K\Psi\left[f(W_1)+f(X_2)\frac{1}{B_K}\sum_{\alpha\in K} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/w_1)+(1/x_2)-(1/B_K)\sum_{\alpha\in K}b_\alpha/w_\alpha}\right)\right],$$
$$\mathscr{D}_3(f,K,\mathbf{b},\mathbf{w};\Psi) := \Psi\left[\sum_{\alpha\in K} h\left(\frac{b_\alpha}{B_K}\right)f(w_\alpha) - f\left(\frac{1}{(1/B_K)\sum_{\alpha\in K}b_\alpha/w_\alpha}\right)\right],$$
$$\mathscr{D}_4(f,K,\mathbf{b},\mathbf{w};\Psi) := \Psi\left[f(W_1)+f(X_2)\frac{1}{B_K}\sum_{\alpha\in K} h\left(\frac{b_\alpha}{B_K}\right)f(w_\alpha) - f\left(\frac{1}{(1/w_1)+(1/x_2)-(1/B_K)\sum_{\alpha\in K}b_\alpha/w_\alpha}\right)\right], \tag{37}$$

where $f\in H\mathrm{Conv}(C,\mathbb{R})$, $K\in P_f(\mathbb{N})$, $\mathbf{b}\in J^+(\mathbb{R})$, $\mathbf{w}\in J_*(C)$, and $\Psi\colon(0,\infty)\longrightarrow\mathbb{R}$ is a convex function whose properties will determine the behavior of functional $\mathscr{D}_i, i=1,2,3,4$, as follows. Obviously, for $\Psi(t)=t$, we recapture, from functional $\mathscr{D}_i$, the functional $\mathscr{M}_i$ considered in Theorem 5.

First of all, we observe that, by Jensen-type inequality, the functional $\mathscr{D}_i$ is well defined and positive homogenous in the third variable, that is,

$$\mathscr{D}_i(f,K,m\mathbf{b},\mathbf{w};\Psi) = m\mathscr{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi), \tag{38}$$

for any $m>0$ and $\mathbf{b}\in J^+(\mathbb{R})$.

The following result concerning the superadditivity and the monotonicity of the functional $\mathscr{D}_i, i=1,2$, as function of weights holds. □

Theorem 8. *Let $f\in HConv(C,\mathbb{R})$, $K\in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\Psi\colon(0,\infty)\longrightarrow\mathbb{R}$ be monotonic nondecreasing and concave function.*

(i) If $\mathbf{b},\mathbf{c}\in J^+(\mathbb{R})$, then

$$\mathscr{D}_i(f,K,\mathbf{b}+\mathbf{c},\mathbf{w};\Psi)\geq\mathscr{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi)+\mathscr{D}_i(f,K,\mathbf{c},\mathbf{w};\Psi). \tag{39}$$

That is, $\mathscr{D}_i$ is superadditive as a function of weights.

(ii) If $\mathbf{b},\mathbf{c}\in J^+(\mathbb{R})$, with $b\geq c$, meaning that $b_i\geq c_i$, for each $i\in\mathbb{N}$ and $\Psi\colon(0,\infty)\longrightarrow(0,\infty)$, then

$$\mathscr{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi)\geq\mathscr{D}_i(f,K,\mathbf{c},\mathbf{w};\Psi)\geq 0. \tag{40}$$

That is, $\mathscr{D}_i$ is monotonic nondecreasing as function of weights.

Proof

(i) Let $\mathbf{b},\mathbf{c}\in J^+(\mathbb{R})$; then, by the harmonic convexity of f on C,

$$\frac{1}{B_K+C_K}\sum_{\alpha\in K}(b_\alpha+c_\alpha)f(w_\alpha) - f\left(\frac{1}{(1/B_K+B_K)\sum_{\alpha\in K}(b_\alpha+c_\alpha)/w_\alpha}\right)$$
$$=\frac{B_K((1/B_K)\sum_{\alpha\in K}b_\alpha f(w_\alpha))+C_K((1/C_K)\sum_{\alpha\in K}c_\alpha f(w_\alpha))}{B_K+C_K}$$
$$-f\left(\frac{1}{(B_K/B_K+C_K).(1/B_K)\sum_{\alpha\in K}b_\alpha/w_\alpha+(C_K/B_K+C_K).(1/C_K)\sum_{\alpha\in K}c_\alpha/w_\alpha}\right)$$
$$\geq\frac{B_K((1/B_K)\sum_{\alpha\in K}b_\alpha f(w_\alpha))+C_K((1/C_K)\sum_{\alpha\in K}c_\alpha f(w_\alpha))}{B_K+C_K}$$

$$-\frac{B_K f\left(1/(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha\right) + C_K f\left(1/(1/C_K)\sum_{\alpha\in K} c_\alpha/w_\alpha\right)}{B_K + C_K}$$
$$= \frac{B_K\left[(1/B_K)\sum_{\alpha\in K} b_\alpha f(w_\alpha) - f\left(1/(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha\right)\right]}{B_K + C_K} + \frac{C_K\left[(1/C_K)\sum_{\alpha\in K} c_\alpha f(w_\alpha) - f\left(1/(1/C_K)\sum_{\alpha\in K} c_\alpha/w_\alpha\right)\right]}{B_K + C_K}. \tag{41}$$

Since Ψ is monotonically nondecreasing and concave, then, by (40),

$$\Psi\left[\frac{1}{B_K + C_K}\sum_{\alpha\in K}(b_\alpha + c_\alpha) f(w_\alpha) - f\left(\frac{1}{(1/B_K + B_K)\sum_{\alpha\in K}(b_\alpha + c_\alpha)/w_\alpha}\right)\right]$$
$$\geq \frac{B_K\Psi\left[(1/B_K)\sum_{\alpha\in K} b_\alpha f(w_\alpha) - f\left(1/(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha\right)\right]}{B_K + C_K} + \frac{C_K\Psi\left[(1/C_K)\sum_{\alpha\in K} c_\alpha f(w_\alpha) - f\left(1/(1/C_K)\sum_{\alpha\in K} c_\alpha/w_\alpha\right)\right]}{B_K + C_K}, \tag{42}$$

which, by multiplication with $B_K + C_K > 0$, produces the desired result (39) for $i = 1$.

Similarly, we can easily verify result (39), for $i = 2$.

(ii) If $\mathbf{b} \geq \mathbf{c}$, then by (i),

$$\mathcal{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi) = \mathcal{D}_i(f, K, (\mathbf{b} - \mathbf{c}) + \mathbf{c}, \mathbf{w}; \Psi)$$
$$\geq \mathcal{D}_i(f, K, (\mathbf{b} - \mathbf{c}), \mathbf{w}; \Psi) + \mathcal{D}_i(f, K, \mathbf{c}, \mathbf{w}; \Psi) \geq \mathcal{D}_i(f, K, \mathbf{c}, \mathbf{w}; \Psi), \tag{43}$$

since $\mathcal{D}_i(f, K, (\mathbf{b} - \mathbf{c}), \mathbf{w}; \Psi) \geq 0$. □

Corollary 2. *Let $f \in HConv(C, \mathbb{R})$, $K \in P_f(\mathbb{N})$, $\mathbf{w} \in J_*(C)$, and $\Psi\colon (0, \infty) \longrightarrow \mathbb{R}$ be a monotonically nondecreasing and concave function.*

If there exist the numbers m and M with $M \geq m \geq 0$ such that $M\,\mathbf{c} \geq \mathbf{b} \geq m\mathbf{c}$, then

$$MC_K\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K} c_\alpha f(w_\alpha) - f\left(\frac{1}{(1/C_K)\sum_{\alpha\in K} c_\alpha/w_\alpha}\right)\right]$$
$$\geq B_K\Psi\left[\frac{1}{B_K}\sum_{\alpha\in K} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha}\right)\right]$$
$$\geq mC_K\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K} c_\alpha f(w_\alpha) - f\left(\frac{1}{(1/C_K)\sum_{\alpha\in K} c_\alpha/w_\alpha}\right)\right]. \tag{44}$$

In particular,

$$\frac{M}{m}\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K}c_\alpha f(w_\alpha)-f\left(\frac{1}{(1/C_K)\sum_{\alpha\in K}c_\alpha/w_\alpha}\right)\right]$$
$$\geq\Psi\left[\frac{1}{B_K}\sum_{\alpha\in K}b_\alpha f(w_\alpha)-f\left(\frac{1}{(1/B_K)\sum_{\alpha\in K}b_\alpha/w_\alpha}\right)\right] \quad (45)$$
$$\geq\frac{m}{M}\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K}c_\alpha f(w_\alpha)-f\left(\frac{1}{(1/C_K)\sum_{\alpha\in K}c_\alpha/w_\alpha}\right)\right].$$

For a function $\Phi\colon (0,\infty)\longrightarrow(0,\infty)$, we now consider the functionals

$$\mathfrak{D}_1(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi)\coloneqq\sum_{\alpha\in K}\Phi(b_\alpha)\Psi\left[\frac{1}{\sum_{\alpha\in K}\Phi(b_\alpha)}\sum_{\alpha\in K}\Phi(b_\alpha)f(w_\alpha)-f\left(\frac{1}{(1/\sum_{\alpha\in K}\Phi(b_\alpha))\sum_{\alpha\in K}\Phi(b_\alpha)/w_\alpha}\right)\right],$$
$$\mathfrak{D}_2(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi)\coloneqq\sum_{\alpha\in K}\Phi(b_\alpha)\Psi\left[f(w_1)+f(w_n)-\left(1/\sum_{\alpha\in K}\Phi(b_\alpha)\right)\sum_{\alpha\in K}\Phi(b_\alpha)f(w_\alpha)-f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/\sum_{\alpha\in K}\Phi(b_\alpha))\sum_{\alpha\in K}\Phi(b_\alpha)/w_\alpha}\right)\right], \quad (46)$$

where $f\in H\mathrm{Conv}(C,\mathbb{R})$, $K\in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\mathbf{b}\in J^+(\mathbb{R})$. Now, if we denote by $\Phi(\mathbf{b})$ the sequence $\{\Phi(b_\alpha)\}_{k\in\mathbb{N}}$, then we observe that, for $i=1,2$,

$$\mathfrak{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi)=\mathscr{D}_i(f,K,\Phi(\mathbf{b}),\mathbf{w};\Psi). \quad (47)$$

The following result may be stated.

Corollary 3. *Let $f\in HConv(C,\mathbb{R})$, $K\in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\Psi\colon (0,\infty)\longrightarrow(0,\infty)$ be monotonically nondecreasing and concave function. If $\Phi\colon (0,\infty)\longrightarrow(0,\infty)$ is concave, then $\mathfrak{D}_i(f,K,.,\mathbf{w};\Psi,\Phi)$ is also concave on $J^+(\mathbb{R})$, for $i=1,2$.*

Proof. Utilizing the properties of monotonicity, superadditivity, and positive homogeneity of functional $\mathscr{D}_i(f,K,.,\mathbf{w};\Psi)$, we have

$$\begin{aligned}\mathfrak{D}_i(f,K,t\mathbf{b}+(1-t)\mathbf{c},\mathbf{w};\Psi,\Phi)&=\mathscr{D}_i(f,K,\Phi(t\mathbf{b}+(1-t)\mathbf{c}),\mathbf{w};\Psi)\\&\geq\mathscr{D}_i(f,K,t\Phi(\mathbf{b})+(1-t)\Phi(\mathbf{c}),\mathbf{w};\Psi)\\&\geq\mathscr{D}_i(f,K,t\Phi(\mathbf{b}),\mathbf{w};\Psi)+\mathscr{D}_i(f,K,(1-t)\Phi(\mathbf{c}),\mathbf{w};\Psi)\\&=t\mathscr{D}_i(f,K,\Phi(\mathbf{b}),\mathbf{w};\Psi)+(1-t)\mathscr{D}_i(f,K,\Phi(\mathbf{c}),\mathbf{w};\Psi)\\&=t\mathfrak{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi)+(1-t)\mathfrak{D}_i(f,K,\mathbf{c},\mathbf{w};\Psi,\Phi),\end{aligned} \quad (48)$$

for any $\mathbf{b},\mathbf{c}\in J^+(\mathbb{R})$ and $t\in[0,1]$, which proves the statement.

The following result concerning the superadditivity and monotonicity of the functional $\mathscr{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi)$, for $i=1,2$, as an index set function holds. □

Theorem 9. *Let $f\in HConv(C,\mathbb{R})$, $K\in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\mathbf{b}\in J^+(\mathbb{R})$. Assume that $\Psi\colon (0,\infty)\longrightarrow(0,\infty)$ is a monotonically nondecreasing and concave function.*

(i) If $K,L\in P_f(\mathbb{N})$ with $K\cap L\neq\phi$,

$$\mathscr{D}_i(f,K\cup L,\mathbf{b},\mathbf{w};\Psi)\geq\mathscr{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi)+\mathscr{D}_i(f,L,\mathbf{b},\mathbf{w};\Psi). \quad (49)$$

That is, $\mathscr{D}_i(f,K,.,\mathbf{w};\Psi)$ is superadditive as an index set function on $P_f(\mathbb{N})$.

(ii) If $K,L\in P_f(\mathbb{N})$ with $L\subset K$ and $\Psi\colon (0,\infty)\longrightarrow(0,\infty)$, then

$$\mathcal{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi) \geq \mathcal{D}_i(f, L, \mathbf{b}, \mathbf{w}; \Psi)(\geq 0). \quad (50)$$

That is, $\mathcal{D}_i(f, K, ., \mathbf{w}; \Psi)$ is monotonically nondecreasing as an index set function on $P_f(\mathbb{N})$.

Proof

(i) Let $K, L \in P_f(\mathbb{N})$ with $K \cap L \neq \phi$. By the harmonic convexity of f on C, we have

$$\begin{aligned}
&\frac{1}{P_{K\cup L}} \sum_{j\in K\cup L} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/P_{K\cup L})\sum_{j\in K\cup L} b_\alpha/w_\alpha}\right) \\
&= \frac{B_K((1/B_K)\sum_{\alpha\in K} b_\alpha f(w_\alpha)) + P_L((1/P_L)\sum_{l\in L} q_l f(x_l))}{B_K + P_L} \\
&\quad - f\left(\frac{1}{(B_K/B_K + P_L).(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha + (P_L/B_K + P_L).(1/P_L)\sum_{l\in L} q_l/x_l}\right) \\
&\geq \frac{B_K((1/B_K)\sum_{\alpha\in K} b_\alpha f(w_\alpha)) + P_L((1/P_L)\sum_{l\in L} q_l f(x_l))}{B_K + P_L} \\
&\quad - \frac{B_K f(1/(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha) + P_L f(1/(1/P_L)\sum_{l\in L} q_l/x_l)}{B_K + P_L} \\
&= \frac{B_K[(1/B_K)\sum_{\alpha\in K} b_\alpha f(w_\alpha) - f(1/(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha)]}{B_K + P_L} \\
&\quad + \frac{P_L[(1/P_L)\sum_{l\in L} q_l f(x_l) - f(1/(1/P_L)\sum_{l\in L} q_l/x_l)]}{B_K + P_L}.
\end{aligned} \quad (51)$$

Since Ψ is monotonically nondecreasing and concave, then, by (50),

$$\begin{aligned}
&\Psi\left[\frac{1}{P_{K\cup L}} \sum_{j\in K\cup L} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/P_{K\cup L})\sum_{j\in K\cup L} b_\alpha/w_\alpha}\right)\right] \\
&\geq \frac{B_K \Psi[(1/B_K)\sum_{\alpha\in K} b_\alpha f(w_\alpha) - f(1/(1/B_K)\sum_{\alpha\in K} b_\alpha/w_\alpha)]}{B_K + P_L} \\
&\quad + \frac{P_L \Psi[(1/P_L)\sum_{l\in L} q_l f(x_l) - f(1/(1/P_L)\sum_{l\in L} q_l f(x_l))]}{B_K + P_L},
\end{aligned} \quad (52)$$

which, by multiplication with $B_K + P_L > 0$, produces the desired result (48) for $i = 1$.

Similarly, we can easily verify result (48), for $i = 2$.

(ii) Let $K, L \in P_f(\mathbb{N})$ with $L \subset K$; then,

$$\begin{aligned}
\mathcal{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi) &= \mathcal{D}_i(f, (K\backslash L) \cup L, \mathbf{b}, \mathbf{w}; \Psi) \geq \mathcal{D}_i(f, K\backslash L, \mathbf{b}, \mathbf{w}; \Psi) + \mathcal{D}_i(f, L, \mathbf{b}, \mathbf{w}; \Psi) \\
&\geq \mathcal{D}_i(f, L, \mathbf{b}, \mathbf{w}; \Psi)(\geq 0),
\end{aligned} \quad (53)$$

since $\mathcal{D}_i(f, K\backslash L, \mathbf{b}, \mathbf{w}; \Psi) \geq 0$. □

3. Conclusion

First of all, we have presented the refinement of Jensen-type inequality, and further, we have discussed several important aspect of functionals associated with Jensen-type inequalities for the HCFs. On the basis of ideas discussed in this paper along with the literature present on HCFs, we encourage the interested researcher to explore more interesting results for this class of functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally in this research article. All authors read and approved the final manuscript.

References

[1] İ. İşcan, "Hermite-Hadamard type inequaities for HCFs," *Hacettepe Journal of Mathematics and Statistic*, vol. 43, no. 6, pp. 935–942, 2014.

[2] İ. İşcan, "Hermite-Hamard type inequalities for harmonically (α, m)-convex functions," 2015, http://arxiv.org/abs/1307.5402v3.

[3] I. A. Baloch, I. Işcan, and S. S. Dragomir, "Fejer's type inequalities for harmonically (s, m)-convex functions," *International Journal of Analysis and Applications*, vol. 12, pp. 188–197, 2016.

[4] I. A. Baloch and I. Işcan, "Some ostrowski type inequalities for harmonically (s, m)-convex functions in second sense," *International Journal of Analysis*, vol. 2015, Article ID 672675, 9 pages, 2015.

[5] I. A. Baloch and I. Işcan, "Some Hermite-Hadamard type integral inequalities for Harmonically $(p, (s, m))$-convex functions," *Journal of Inequalities and Special Functions*, vol. 8, pp. 65–84, 2017.

[6] S. H. Wu, I. A. Baloch, and I. Iscan, "On harmonically (p, h, m)-preinvex functions," *Journal of Function Spaces*, vol. 2017, no. 3, Article ID 2148529, 2017.

[7] I. A. Baloch, M. De La Sen, and İ. İşcan, "Characterizations of classes of HCFs and applications," *International Journal of Analysis and Applications*, vol. 17, pp. 722–733, 2019.

[8] N. Akhtar, M. U. Awan, M. Z. Javed et al., "Ostrowski type inequalities involving harmonically convex functions and applications," *Symmetry*, vol. 13, 2021.

[9] J. Z. Xu, U. Raza, M. W. Javed, and Z. Hussain, "Hermite-Hadamard inequalities for harmonic $(s,\ m)$-convex functions," *Mathematical Problems in Engineering*, vol. 2020, Article ID 1470837, 7 pages, 2020.

[10] S. I. Butt, S. Rashid, M. Tariq, and M. K. Wang, "Novel refinements via n-polynomial harmonically s type convex functions and application in special functions," *Journal of Function Spaces*, vol. 2021, Article ID 6615948, 17 pages, 2021.

[11] E. R. Nwaeze, M. A. Khan, M. Ahmadian, M. Nazir Ahmad, and A. Kamil Mahmood, "Fractional inequalities of the Hermite-Hadamard type for $ m $-polynomial convex and harmonically convex functions," *AIMS Mathematics*, vol. 6, no. 2, pp. 1889–1904, 2021.

[12] S. I. Butt, M. Tariq, A. Aslam, H. Ahmad, and T. A. Nofal, "Hermite Hadamard type inequalities via generalized harmonic exponential convexity and applications," *Journal of Function Spaces*, vol. 2021, Article ID 5533491, 12 pages, 2021.

[13] Y. M. Chu, S. Rashid, T. Abdeljawad, A. khalid, and H. Kalsoom, "On new generalized unified bounds via generalized exponentially harmonically s-convex functions on fractal sets," *Advances in Difference Equations*, vol. 218, 2021.

[14] I. A. Baloch, A. A. Mughal, Y. M. Chu, A. U. Haq, and M. D. L. Sen, "Improvement and generalization of some results related to the class of harmonically convex functions and applications," *Journal of Mathematics and Computer Science*, vol. 22, no. 3, pp. 282–294, 2021.

[15] S. Varošanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.

[16] Z. B. Fang and R. Shi, "On the $(p,\ h)$-convex function and some integral inequalities," *Journal of Inequalities and Applications*, vol. 2014, no. 1, 45 pages, 2014.

[17] A. Hśzy, "Berstein-doetsch type results for h-convex functions," *Mathematical Inequalities & Applications,*vol. 14, no. 3, pp. 499–508, 2011.

[18] A. Olberyś, "On separation by h-convex functions," *Tatra Mountains Mathematical Publications*, vol. 62, no. 1, pp. 105–111, 2015.

[19] M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, and A. G. Khan, "Some new bounds for Simpson's rule involving special functions via harmonic h-convexity," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 4, pp. 1755–1766, 2017.

[20] M. R. Delavar, S. S. Dragomir, and M. De La Sen, "A note on characterization of h-convex functions via hermite-hadamard type inequality," *Issues of Analysis*, vol. 26, no. 2, pp. 28–36, 2019.

[21] S. S. Dragomir, "Inequalities of Jensen type for HA-convex functions," *Analele Universitati Oradea Fasc. Matematica*, vol. 27, pp. 103–124, 2020.

[22] I. Abbas Baloch, A. H. Mughal, A. Ahmad Mughal, Y.-M. Ul Haq, and M. De La Sen, "A variant of Jensen-type inequality and related results for harmonic convex functions," *Aims Mathematics*, vol. 5, no. 6, pp. 6404–6418, 2020.

[23] R. P. Agarwal and S. S. Dragomir, "The property of supermultiplicity for some classical inequalities and applications," *Computers & Mathematics with Applications*, vol. 35, no. 6, 105118 pages, 1998.

[24] S. S. Dragomir, J. E. Pećarić, and L. E. Persson, "Some inequalities of Hadamard inequality," *Soochow Journal of Mathematics*, vol. 21, no. 3, pp. 335–341, 1995.

[25] S. S. Dragomir, "Bounds for the normalized Jensen functional," *Bulletin of the Australian Mathematical Society*, vol. 74, no. 3, pp. 471–478, 2006.

[26] S. S. Dragomir, J. Pećarić, and L. E. Persson, "Properties of some functionals related to Jensens inequality," *Acta Mathematica Hungarica*, vol. 70, no. 12, pp. 129–143, 1996.

[27] P. Agarwal, M. Kadakal, İ. İşcan, and Y. M. Chu, "Better approaches for n-times differentiable convex functions," *Mathematics*, vol. 8, 2020.

[28] M. Tomar, P. Agarwal, and J. Choi, "Hermite-Hadamard type inequalities for generalized convex functions on fractal sets style," *Boletim da Sociedade Paranaense de Matemtica*, vol. 38, pp. 101–116, 2020.

[29] M. B. Khan, M. A. Noor, K. I. Noor, H. Almusawa, and K. S. Nisar, "Exponentially preinvex fuzzy mappings and fuzzy exponentially mixed variational-like inequalities," *International Journal of Analysis and Applications*, vol. 19, pp. 518–541, 2021.

[30] H. Almusawa, H. A. Hammad, and N. Sharma, "Approximation of the fixed point for unified three-step iterative algorithm with convergence analysis in busemann spaces," *Axioms*, vol. 10, no. 1, 26 pages, 2021.

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Research Article

Optical Solutions of the Date–Jimbo–Kashiwara–Miwa Equation via the Extended Direct Algebraic Method

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In this study, the solutions of $(2+1)$-dimensional nonlinear Date–Jimbo–Kashiwara–Miwa (DJKM) equation are characterized, which can be used in mathematical physics to model water waves with low surface tension and long wavelengths. The integration scheme, namely, the extended direct algebraic method, is used to extract complex trigonometric, rational and hyperbolic functions. The complex-valued solutions represent traveling waves in different structures, such as bell-, V-, and W-shaped multiwaves. The results obtained in this article are novel and more general than those contained in the literature (Wang et al., 2014, Yuan et al., 2017, Pu and Hu 2019, Singh and Gupta 2018). Furthermore, the mechanical features and dynamical characteristics of the obtained solutions are demonstrated by three-dimensional graphics.

1. Introduction

Nonlinear evolution equations (NLEEs) can represent various nonlinear problems that occur in a wide range of scientific fields such as nonlinear optics, mathematical physics, superconductivity, biophysics, optical fiber, modern optics, solid state physics, fluid mechanics, fluid dynamics, plasma physics, chemical physics, and chemical kinetics. In the literature, various effective approaches have been proposed to calculate the exact solutions for NLEEs [1–3], such as the Hereman–Nuseir method [4], inverse scattering transformation [5], Painlevé technique [6], Bäcklund transformation [7], extended modified auxiliary equation mapping method [8], Darboux transformation [9], Exp-function method [10], binary-bell-polynomial scheme [11], modified Khater method [12], ansatz method [13], sine-Gordon expansion method [14], trial equation method [15, 16], extended direct algebraic method [17], and auxiliary equation method [18].

In this study, the nonlinear DJKM equation [19] is investigated to construct various solitary wave solutions. In the integrable systems of KP hierarchy, the Jimbo–Miwa equation is the second equation used to explain such interesting $(2+1)$-dimensional waves in physics. The DJKM equation can be used in mathematical physics to model water waves with low surface tension and long wavelengths with weakly nonlinear restoring forces and frequency dispersion. Firstly, Hu and Li [19] applied bilinear Bäcklund transformations and nonlinear superposition formula for nonlinear DJKM equations, and after a gap of more than two decades, Wang et al. [20] used the bell polynomials to study the integrable properties of nonlinear DJKM equations such as Lax system, Bäcklund transformations, and infinite conservation laws along with multishock wave. Yuan et al. [21] presented Grammian- and Wronskian-type solutions by the Hirota method, and other types of solution are also obtained like auxiliary variables, the bilinear Bäcklund transformation, and N-soliton. Pu and Hu [22] employed the sine-Gordon expansion method in finding the traveling wave solutions of nonlinear DJKM equations and obtained hyperbolic, trigonometric, and complex solutions. Singh and Gupta [23] used the direct method and nonlinear self-adjointness to find the Painlevé analysis, symmetric properties, and conservation laws of the nonlinear DJKM equation. Sajid and Akram [24] utilized $\exp(-\Phi(\xi))$-expansion method and derived some exact traveling wave solutions including trigonometric, hyperbolic, and rational functions and W-shaped soliton of the DJKM equation. The proposed research analyzes some more new exact solutions such as bell-, V-, and W-shaped multiwave types of the nonlinear DJKM equation which are not yet found in the literature. To our utmost understanding, the DJKM equations were not analyzed using the extended direct algebraic method. Therefore, the benefits of this article included evaluating a wide range of advanced and contextual solutions to the considered wave equations by the use of the extended direct algebraic method. Furthermore, this beneficial and powerful approach can be used to investigate other NLEEs which frequently emerge in different scientific real-world applications.

The novelty of this paper lies in the following: (i) complex-valued solutions and solitons are in different shapes and (ii) 3-dimensional figures are first presented by the extended direct algebraic method. The limitations of this work include that the solution methods for the construction of exact solutions to the equation involve various parameters. Such parameters show up in the final precise solution expressions and create hurdles in some physical situations. These are resolved with a careful selection of appropriate parametric values which is possible through graphical interpretation and testing of the solution expressions.

The structure of this paper is organized as follows: in Section 2, detailed explanation of the extended direct algebraic method has been presented. Section 3 illustrates the method to solve the $(2+1)$-dimensional DJKM equation. In Section 3.1, the physical explanation of the solutions by mechanical features and dynamical characteristics is demonstrated. Finally, conclusion is given in Section 4.

1.1. Governing Model. Considering the governing model, $(2+1)$-dimensional nonlinear DJKM equation, as

$$\Phi_{xxxxy} + 4\Phi_{xxy}\Phi_x + 2\Phi_{xxx}\Phi_y + 6\Phi_{xy}\Phi_{xx} + \Phi_{yyy} - 2\Phi_{xxt} = 0, \tag{1}$$

where $\Phi = \Phi(x, y, t)$ is the real-valued function. The DJKM equation belongs to the well-known KP hierarchy [25, 26] which can be obtained from the first two bilinear equations using transformation $u = 2(\log \tau)_x$. The KP hierarchy is an infinite set of nonlinear PDEs.

2. Extended Direct Algebraic Method [27]

According to extended direct algebraic method, we have the following.

Step 1. Consider NLEE in three independent variables x, y, and t of the form, as

$$P\left(\Phi, \Phi_x, \Phi_y, \Phi_t, \Phi_{xx}, \Phi_{xy}, \Phi_{xt}, \Phi_{yy}, \Phi_{yt}, \Phi_{tt}, \ldots\right) = 0, \tag{2}$$

where $\Phi = \Phi(x, y, t)$ and P is the polynomial in Φ. Using the wave transformation

$$\Phi(x, y, t) = U(\xi),$$
$$\xi = \omega(x + \mu y - kt), \tag{3}$$

where ω is the wave number. After applying the transformation, equation (2) can be converted into the nonlinear ODE, as

$$Q\left(U, \omega U', \omega\mu U', -\omega k U', \omega^2 U'', \omega^2\mu U'', -\omega^2 k U'', \omega^2\mu^2 U'', -\omega^2\mu k U'', \omega^2 k^2 U'', \ldots\right) = 0, \tag{4}$$

where prime denotes the derivatives w.r.t. ξ.

Step 2. Consider that the formal solution of equation (4) has a form, as follows:

$$U(\xi) = \sum_{j=0}^{N} b_j Q^j(\xi), \quad b_N \neq 0, \tag{5}$$

where $b_0, b_1, \ldots, b_N$ are constants and $Q(\xi)$ satisfies the auxiliary equation, as

$$Q'(\xi) = Ln(A)\left(\alpha + \beta Q(\xi) + \sigma Q^2(\xi)\right), \quad A \neq 0, 1, \tag{6}$$

where σ, α, and β are constants. The solutions of equation (6) are given in the following.

Family 1. If $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then the solutions are given as

$$
\begin{aligned}
Q_1(\xi) &= -\frac{\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \tan_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\xi\right), \\
Q_2(\xi) &= -\frac{\beta}{2\sigma} - \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \cot_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\xi\right), \\
Q_3(\xi) &= -\frac{\beta}{2\sigma} + \left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\right)\left(\tan_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right) \pm \sqrt{pq}\sec_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right)\right), \\
Q_4(\xi) &- \frac{\beta}{2\sigma} - \left(\frac{-(\beta^2 - 4\alpha\sigma)}{2\sigma}\right)\left(\cot_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right) \mp \sqrt{pq}\csc_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right)\right), \\
Q_5(\xi) &= -\frac{\beta}{2\sigma} + \left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4\sigma}\right)\left(\tan_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4}\xi\right) - \cot_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4}\xi\right)\right).
\end{aligned}
\tag{7}
$$

Family 2. If $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then the solutions are given as

$$
\begin{aligned}
Q_6(\xi) &= -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma} \tanh_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\xi\right), \\
Q_7(\xi) &= -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma} \coth_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\xi\right), \\
Q_8(\xi) &= -\frac{\beta}{2\sigma} - \left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\right)\left(\tanh_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right) \mp i\sqrt{pq}\sec h_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)\right), \\
Q_9(\xi) &= -\frac{\beta}{2\sigma} - \left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\right)\left(\coth_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right) \mp \sqrt{pq}\csc h_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)\right), \\
Q_{10}(\xi) &= -\frac{\beta}{2\sigma} - \left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4\sigma}\right)\left(\tanh_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}\xi\right) + \coth_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}\xi\right)\right).
\end{aligned}
\tag{8}
$$

Family 3. If $\beta = 0$, and $\alpha\sigma > 0$, then the solutions are given as

$$Q_{11}(\xi) = \sqrt{\frac{\alpha}{\sigma}}\tan_A(\sqrt{\alpha\sigma}\,\xi),$$

$$Q_{12}(\xi) = -\sqrt{\frac{\alpha}{\sigma}}\cot_A(\sqrt{\alpha\sigma}\,\xi),$$

$$Q_{13}(\xi) = \sqrt{\frac{\alpha}{\sigma}}\left(\tan_A(2\sqrt{\alpha\sigma}\,\xi) \pm \sqrt{pq}\sec_A(2\sqrt{\alpha\sigma}\,\xi)\right), \tag{9}$$

$$Q_{14}(\xi) = -\sqrt{\frac{\alpha}{\sigma}}\left(\cot_A(2\sqrt{\alpha\sigma}\,\xi) \mp \sqrt{pq}\csc_A(2\sqrt{\alpha\sigma}\,\xi)\right),$$

$$Q_{15}(\xi) = \frac{1}{2}\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right) - \cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)\right).$$

Family 4. If $\beta = 0$ and $\alpha\sigma < 0$, then the solutions are given as

$$Q_{16}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\tanh_A(\sqrt{-\alpha\sigma}\,\xi),$$

$$Q_{17}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\coth_A(\sqrt{-\alpha\sigma}\,\xi),$$

$$Q_{18}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A(2\sqrt{-\alpha\sigma}\,\xi) \mp i\sqrt{pq}\sec h_A(2\sqrt{-\alpha\sigma}\,\xi)\right),$$

$$Q_{19}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\left(\coth_A(2\sqrt{-\alpha\sigma}\,\xi) \mp \sqrt{pq}\csc h_A(2\sqrt{-\alpha\sigma}\,\xi)\right),$$

$$Q_{20}(\xi) - \frac{1}{2}\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right) + \coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)\right). \tag{10}$$

Family 5. If $\beta = 0$ and $\sigma = \alpha$, then the solutions are given as

$$\begin{aligned}
Q_{21}(\xi) &= \tan_A(\alpha\xi),\\
Q_{22}(\xi) &= -\cot_A(\alpha\xi),\\
Q_{23}(\xi) &= \tan_A(2\alpha\xi) \pm \sqrt{pq}\sec_A(2\alpha\xi),\\
Q_{24}(\xi) &= -\cot_A(2\alpha\xi) \pm \sqrt{pq}\csc_A(2\alpha\xi),\\
Q_{25}(\xi) &= \frac{1}{2}\left(\tan_A\left(\frac{\alpha}{2}\xi\right) - \cot_A\left(\frac{\alpha}{2}\xi\right)\right).
\end{aligned} \tag{11}$$

Family 6. : If $\sigma = -\alpha$ and $\beta = 0$, then the solutions are given as

$$\begin{aligned}
Q_{26}(\xi) &= -\tanh_A(\alpha\xi),\\
Q_{27}(\xi) &= -\coth_A(\alpha\xi),\\
Q_{28}(\xi) &= -\tanh_A(2\alpha\xi) \pm i\sqrt{pq}\sec h_A(2\alpha\xi),\\
Q_{29}(\xi) &= -\coth_A(2\alpha\xi) \pm \sqrt{pq}\csc h_A(2\alpha\xi),\\
Q_{30}(\xi) &= -\frac{1}{2}\left(\tanh_A\left(\frac{\alpha}{2}\xi\right) + \coth_A\left(\frac{\alpha}{2}\xi\right)\right).
\end{aligned} \tag{12}$$

Family 7. If $\beta^2 = 4\alpha\sigma$, then the solution is given as

$$Q_{31}(\xi) = \frac{-2\alpha(\beta\xi LnA + 2)}{\beta^2\xi LnA}. \tag{13}$$

Family 8. If $\beta = l$, $\sigma = 0$, and $\alpha = ml\,(m \neq 0)$, then the solution is given as

$$Q_{32}(\xi) = A^{l\xi} - m. \tag{14}$$

Family 9. If $\beta = 0 = \sigma$, then the solution is given as

$$Q_{33}(\xi) = \alpha\xi LnA. \tag{15}$$

Family 10. If $\beta = \alpha = 0$, then the solution is given as

$$Q_{34}(\xi) = \frac{-1}{\alpha\xi LnA}. \tag{16}$$

Family 11. If $\beta \neq 0$ and $\alpha = 0$, then the solutions are given as

$$\begin{aligned}
Q_{35}(\xi) &= -\frac{p\beta}{\sigma(\cosh_A(\beta\xi) - \sinh_A(\beta\xi) + p)},\\
Q_{36}(\xi) &= -\frac{\beta(\sinh_A(\beta\xi) + \cosh_A(\beta\xi))}{\sigma(\sinh_A(\beta\xi) + \cosh_A(\beta\xi) + q)}.
\end{aligned} \tag{17}$$

Family 12. If $\beta = l$, $\alpha = 0$, and $\sigma = ml\,(m \neq 0)$, then the solution is given as

$$Q_{37}(\xi) = \frac{pA^{l\xi}}{q - mpA^{l\xi}}. \tag{18}$$

Remark 1. The generalized triangular functions and hyperbolic functions [28] are defined as follows:

$$\begin{aligned}
\sinh_A(\xi) &= \frac{pA^{\xi} - qA^{-\xi}}{2},\\
\cosh_A(\xi) &= \frac{pA^{\xi} + qA^{-\xi}}{2},\\
\tanh_A(\xi) &= \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}},\\
\coth_A(\xi) &= \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}},\\
\sec h_A(\xi) &= \frac{2}{pA^{\xi} + qA^{-\xi}},\\
\csc h_A(\xi) &= \frac{2}{pA^{\xi} - qA^{-\xi}},\\
\sin_A(\xi) &= \frac{pA^{i\xi} - qA^{-i\xi}}{2i},\\
\cos_A(\xi) &= \frac{pA^{i\xi} + qA^{-i\xi}}{2},\\
\tan_A(\xi) &= -i\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{\xi} + qA^{-\xi}},\\
\cot_A(\xi) &= i\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}},\\
\sec_A(\xi) &= \frac{2}{pA^{i\xi} + qA^{-i\xi}},\\
\csc_A(\xi) &= \frac{2i}{pA^{i\xi} - qA^{-i\xi}},
\end{aligned} \tag{19}$$

where $p, q > 0$ and ξ is an independent variable.

Step 3. Using homogeneous balancing principle in equation (4), the value of N can be determined. Substituting equation (6) along with equation (5) into equation (4), collecting the coefficients of each power $Q^j(\xi)$ $(j = 0, 1, 2, \ldots)$, and then setting each coefficient to zero give a system of equations.

Step 4. Unknowns can be found by calculating the system of equations. Putting the unknowns in equation (6), the required solutions of equation (2) are obtained.

3. Application to the DJKM Equation

The extended direct algebraic scheme is presented to obtain the optical solitons and other solutions to equation (1). After utilizing the transformation $\Phi(x, y, t) = V(\xi)$, where $\xi = \omega(x + \mu y - kt)$, to equation (1), we obtain nonlinear ODE as follows:

$$\mu\omega^2 V^{(4)} + 6\mu\omega V' V'' + \left(\mu^3 + 2k\right)V'' = 0. \tag{20}$$

Setting $U = V'$, we obtain

$$\mu\omega^2 U''' + 6\mu\omega U U' + \left(\mu^3 + 2k\right)U' = 0. \tag{21}$$

Balancing U''' with UU' in equation (21) gives $N = 2$. Thus, the solution can be written as

$$U(\xi) = b_0 + b_1 Q(\xi) + b_2 Q^2(\xi), \tag{22}$$

where b_0, b_1, and b_2 are constants to be determined. Substituting equations (22) into (21), collecting all terms with the same power of $Q(\xi)^i$ $(i = 0, 1, 2, 3, 4, 5)$, and equating the coefficients of each polynomial to zero will yield a set of algebraic equations for b_0, b_1, b_2, and ω as follows:

$$\left.\begin{aligned}
&2kb_1 Ln(A)\alpha + 2\mu\omega^2 b_1(Ln(A))^3\sigma\alpha^2 + 6\mu\omega b_0 b_1 Ln(A)\alpha\\
&+\mu\omega^2 b_1(Ln(A))^3\beta^2\alpha + 6\mu\omega^2 b_2(Ln(A))^3\beta\alpha^2 + \mu^3 b_1 Ln(A)\alpha = 0,\\
&16\mu\omega^2 b_2(Ln(A))^3\sigma\alpha^2 + 4kb_2 Ln(A)\alpha + \mu^3 b_1 Ln(A)\beta\\
&+2kb_1 Ln(A)\beta + 8\mu\omega^2 b_1(Ln(A))^3\beta\sigma\alpha + \mu\omega^2 b_1(Ln(A))^3\beta^3\\
&+2\mu^3 b_2 Ln(A)\alpha + 6\mu\omega b_0 b_1 Ln(A)\beta + 12\mu\omega b_0 b_2 Ln(A)\alpha\\
&+6\mu\omega b_1^2 Ln(A)\alpha + 14\mu\omega^2 b_2(Ln(A))^3\alpha\beta^2 = 0,\\
&\mu^3 b_1 Ln(A)\sigma + 6\mu\omega b_0 b_1 Ln(A)\sigma + 18\mu\omega b_1 b_2 Ln(A)\alpha\\
&+8\mu\omega^2 b_1(Ln(A))^3\sigma^2\alpha + 52\mu\omega^2 b_2(Ln(A))^3\alpha\beta\sigma + 4kb_2 Ln(A)\beta\\
&+2\mu^3 b_2 Ln(A)\beta + 2kb_1 Ln(A)\sigma + 7\mu\omega^2 b_1(Ln(A))^3\beta^2\sigma\\
&+8\mu\omega^2 b_2(Ln(A))^3\beta^3 + 6\mu\omega b_1^2 Ln(A)\beta + 12\mu\omega b_0 b_2 Ln(A)\beta = 0,\\
&38\mu\omega^2 b_2(Ln(A))^3\beta^2\sigma + 18\mu\omega b_1 b_2 Ln(A)\beta + 4kb_2 Ln(A)\sigma\\
&+2\mu^3 b_2 Ln(A)\sigma + 12\mu\omega^2 b_1(Ln(A))^3\beta\sigma^2 + 40\mu\omega^2 b_2(Ln(A))^3\alpha\sigma^2\\
&+12\mu\omega b_2^2 Ln(A)\alpha + 6\mu\omega b_1^2 Ln(A)\sigma + 12\mu\omega b_0 b_2 Ln(A)\sigma = 0,\\
&18\mu\omega b_1 b_2 Ln(A)\sigma + 54\mu\omega^2 b_2(Ln(A))^3\beta\sigma^2\\
&+6\mu\omega^2 b_1(Ln(A))^3\sigma^3 + 12\mu\omega b_2^2 Ln(A)\beta = 0,\\
&24\mu\omega^2 b_2(Ln(A))^3\sigma^3 + 12\mu\omega b_2^2 Ln(A)\sigma = 0.
\end{aligned}\right\} \tag{23}$$

Solving system (23) for b_0, b_1, b_2, and ω gives

$$\begin{aligned}
b_0 &= \frac{-1}{6\left(\mu\omega^2(Ln(A))^2\beta^2 + 8\mu\omega^2(Ln(A))^2\sigma\alpha + 2k + \mu^3/\mu\omega\right)},\\
b_1 &= -2\omega(Ln(A))^2\beta\sigma,\\
b_2 &= -2\omega(Ln(A))^2\sigma^2, \quad \omega = \omega.
\end{aligned} \tag{24}$$

Five families of traveling wave solutions of the DJKM equation can be obtained, as shown in the following.

Family 13. When $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, the dark, combined dark-bright, singular, combined dark-singular, and combined singular solutions are obtained, as follows:

$$\Phi_1 = \frac{-1}{6\left(\xi\left(-5\mu\omega(Ln(A))^2\beta^2 + 2k + 8\mu\omega(Ln(A))^2\sigma\alpha + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\tanh_A\left(\frac{1}{2\sqrt{\gamma}\,\xi}\right)\sqrt{\gamma} + (Ln(A))^2\xi\left(-\beta^2 + 2\alpha\sigma\right),$$

$$\Phi_2 = \frac{-1}{6\left(\xi\left(-5\mu\omega(Ln(A))^2\beta^2 + 2k + 8\mu\omega(Ln(A))^2\sigma\alpha + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\coth_A\left(\frac{1}{2\sqrt{\gamma}\,\xi}\right)\sqrt{\gamma} + 2\frac{(Ln(A))^2\left(-\beta^2 + 2\alpha\sigma\right)\arctan\, h_A\left(\coth_A\left(1/2\sqrt{\gamma}\,\xi\right)\right)}{\sqrt{\gamma}},$$

$$\Phi_3 = \frac{1}{6\left(\xi\left(-\mu\omega(Ln(A))^2\beta^2 - 2k + 4\mu\omega(Ln(A))^2\sigma\alpha - \mu^3\right)/\mu\omega\right)} + \frac{1}{2(Ln(A))^2\tanh_A\left(\sqrt{\gamma}\,\xi\right)\sqrt{\gamma}(1+pq)} - i\frac{(Ln(A))^2\sqrt{pq\gamma}}{\cosh_A\left(\sqrt{\gamma}\,\xi\right)},$$

$$\Phi_4 = \frac{1}{6\left(\xi\left(-\mu\omega(Ln(A))^2\beta^2 - 2k + 4\mu\omega(Ln(A))^2\sigma\alpha - \mu^3\right)/\mu\omega\right)} + \frac{1}{2(Ln(A))^2\sqrt{\gamma}\coth_A\left(\sqrt{\gamma}\,\xi\right)(1+pq)} - \frac{(Ln(A))^2\sqrt{pq\gamma}}{\sinh_A\left(\sqrt{\gamma}\,\xi\right)}.$$

$$\Phi_5 = \frac{1}{6\left(\xi\left(-\mu\omega(Ln(A))^2\beta^2 - 2k + 4\mu\omega(Ln(A))^2\sigma\alpha - \mu^3\right)/\mu\omega\right)} + \frac{1}{2(Ln(A))^2\tanh_A\left(1/4\sqrt{\gamma}\,\xi\right)\sqrt{\gamma}} + \frac{1}{2(Ln(A))^2\coth_A\left(1/4\sqrt{\gamma}\,\xi\right)\sqrt{\gamma}}. \tag{25}$$

Family 14. When $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, the singular, dark, and combined dark-singular solutions are obtained, as follows:

$$\Phi_6 = \frac{-1}{6\left(\xi\left(-5\mu\omega(Ln(A))^2\beta^2 + 2k + 8\mu\omega(Ln(A))^2\sigma\alpha + \mu^3\right)/\mu\omega\right)} + \sqrt{\gamma}(Ln(A))^2\tanh_A\left(\frac{1}{2\sqrt{\gamma}\,\xi}\right) - \frac{(Ln(A))^2\left(-\beta^2 + 2\alpha\sigma\right)\left(Ln\left(\tanh_A\left(1/2\sqrt{\gamma}\,\xi\right) - 1\right) - Ln\left(\tanh_A\left(1/2\sqrt{\gamma}\,\xi\right) + 1\right)\right)}{\sqrt{\gamma}},$$

$$\begin{aligned}
\Phi_7 = {} & \frac{-1}{6\left(\xi\left(-5\mu\omega(Ln(A))^2\beta^2 + 2k + 8\mu\omega(Ln(A))^2\sigma\alpha + \mu^3\right)/\mu\omega\right)} \\
& + \sqrt{\gamma}(Ln(A))^2\coth_A\left(\frac{1}{2\sqrt{\gamma}\,\xi}\right) \\
& - \frac{(Ln(A))^2\left(-\beta^2 + 2\alpha\sigma\right)\left(Ln\left(\coth_A(1/2\sqrt{\gamma}\,\xi) - 1\right) - Ln\left(\coth_A(1/2\sqrt{\gamma}\,\xi) + 1\right)\right)}{\sqrt{\gamma}}, \\
\Phi_8 = {} & \frac{1}{6\left(\xi\left(-\mu\omega(Ln(A))^2\beta^2 - 2k + 4\mu\omega(Ln(A))^2\sigma\alpha - \mu^3\right)/\mu\omega\right)} \\
& - (Ln(A))^2 i\sqrt{pq}\sqrt{\gamma}\cosh_A(\sqrt{\gamma}\,\xi) \\
& - (Ln(A))^2\beta i\sqrt{pq}\left(\arctan_A(\sinh_A(\sqrt{\gamma}\,\xi)) - 2\arctan_A\left(e^{\sqrt{\gamma}\,\xi}\right)\right) \\
& + \frac{1}{2(Ln(A))^2\tanh_A(\sqrt{\gamma}\,\xi)\sqrt{\gamma}(1+pq)} + \frac{(Ln(A))^2 i\sqrt{pq\gamma}(\sinh_A(\sqrt{\gamma}\,\xi))^2}{\cosh_A(\sqrt{\gamma}\,\xi)}, \\
\Phi_9 = {} & \frac{1}{6\left(\xi\left(-\mu\omega(Ln(A))^2\beta^2 - 2k + 4\mu\omega(Ln(A))^2\sigma\alpha - \mu^3\right)/\mu\omega\right)} \\
& + (Ln(A))^2\sqrt{pq}\sqrt{\gamma}\sinh_A(\sqrt{\gamma}\,\xi) \\
& - (Ln(A))^2\beta\sqrt{pq}\left(Ln\left(\tanh_A\left(\frac{1}{2\sqrt{\gamma}\,\xi}\right)\right) + 2\arctan h_A\left(e^{\sqrt{\gamma}\,\xi}\right)\right) \\
& + \frac{1}{2(Ln(A))^2\coth_A(\sqrt{\gamma}\,\xi)(1+pq)\sqrt{\gamma}} - \frac{(Ln(A))^2\sqrt{pq\gamma}(\cosh_A(\sqrt{\gamma}\,\xi))^2}{\sinh_A(\sqrt{\gamma}\,\xi)}, \\
\Phi_{10} = {} & \frac{1}{6\left(\xi\left(-\mu\omega(Ln(A))^2\beta^2 - 2k + 4\mu\omega(Ln(A))^2\sigma\alpha - \mu^3\right)/\mu\omega\right)} \\
& + \frac{1}{2(Ln(A))^2\tanh_A(1/4\sqrt{\gamma}\,\xi)\sqrt{\gamma}} + \frac{1}{2(Ln(A))^2\coth_A(1/4\sqrt{\gamma}\,\xi)\sqrt{\gamma}}.
\end{aligned} \tag{26}$$

Family 15. When $\alpha\sigma > 0$ and $\beta = 0$, the periodic-singular solutions are obtained as

$$\begin{aligned}
\Phi_{11} = {} & \frac{-1}{6\left(\xi\left(2k + 8\mu\omega(Ln(A))^2\sigma\alpha + \mu^3\right)/\mu\omega\right)} - 2(Ln(A))^2 \\
& \sqrt{\alpha\sigma}\left(\tan_A(\sqrt{\alpha\sigma}\,\xi) - \sqrt{\alpha\sigma}\,\xi\right), \\
\Phi_{12} = {} & \frac{-1}{6\left(\xi\left(2k + 8\mu\omega(Ln(A))^2\sigma\alpha + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\sqrt{\alpha\sigma} \\
& \left(2\cot_A(\sqrt{\alpha\sigma}\,\xi) - \pi + 2\sqrt{\alpha\sigma}\,\xi\right), \\
\Phi_{13} = {} & \frac{1}{6\left(\xi\left(-2k + 4\mu\omega(Ln(A))^2\sigma\alpha - \mu^3\right)/\mu\omega\right)} - 2\frac{(Ln(A))^2\sqrt{pq\alpha\sigma}}{\cos_A(2\sqrt{\alpha\sigma}\,\xi)} \\
& - (Ln(A))^2\sqrt{\alpha\sigma}\tan_A(2\sqrt{\alpha\sigma}\,\xi)(1+pq),
\end{aligned}$$

$$\Phi_{14}=\frac{1}{6\left(\xi\left(-2k+4\mu\omega(Ln(A))^2\sigma\alpha-\mu^3\right)/\mu\omega\right)}-2\frac{(Ln(A))^2\sqrt{pq\alpha\sigma}}{\sin_A(2\sqrt{\alpha\sigma}\,\xi)}+(Ln(A))^2\sqrt{\alpha\sigma}\cot_A(2\sqrt{\alpha\sigma}\,\xi)(1+pq),$$
$$\Phi_{15}=\frac{1}{6\left(\xi\left(-2k+4\mu\omega(Ln(A))^2\sigma\alpha-\mu^3\right)/\mu\omega\right)}-(Ln(A))^2\sqrt{\alpha\sigma}\left(\tan_A\left(\frac{1}{2\sqrt{\alpha\sigma}\,\xi}\right)-\cot_A\left(\frac{1}{2\sqrt{\alpha\sigma}\,\xi}\right)\right).\tag{27}$$

Family 16. When $\alpha\sigma<0$ and $\beta=0$, the singular, dark, combined dark-bright, combined dark-singular, and combined singular solutions are obtained, as follows:

$$\Phi_{16}=\frac{-1}{6\left(\xi\left(2k+8\mu\omega(Ln(A))^2\sigma\alpha+\mu^3\right)/\mu\omega\right)}+(Ln(A))^2\sqrt{-\alpha\sigma}\left(Ln\left(\frac{\tanh_A(\sqrt{-\alpha\sigma}\,\xi)-1}{\tanh_A(\sqrt{-\alpha\sigma}\,\xi)+1}\right)\right)+2(Ln(A))^2\sqrt{-\alpha\sigma}\tanh_A(\sqrt{-\alpha\sigma}\,\xi),$$
$$\Phi_{17}=\frac{-1}{6\left(\xi\left(2k+8\mu\omega(Ln(A))^2\sigma\alpha+\mu^3\right)/\mu\omega\right)}+(Ln(A))^2\sqrt{-\alpha\sigma}\left(Ln\left(\frac{\coth_A(\sqrt{-\alpha\sigma}\,\xi)-1}{\coth_A(\sqrt{-\alpha\sigma}\,\xi)+1}\right)\right)+2(Ln(A))^2\sqrt{-\alpha\sigma}\coth_A(\sqrt{-\alpha\sigma}\,\xi),$$
$$\Phi_{18}=\frac{-1}{6\left(\xi\left(2k-4\mu\omega(Ln(A))^2\sigma\alpha+\mu^3\right)/\mu\omega\right)}+(Ln(A))^2\sqrt{-\alpha\sigma}\tanh_A(2\sqrt{-\alpha\sigma}\,\xi)(1+pq)+2\frac{i(Ln(A))^2\sqrt{-pq\sigma\alpha}}{\cosh_A(2\sqrt{-\alpha\sigma}\,\xi)},\tag{28}$$
$$\Phi_{19}=\frac{-1}{6\left(\xi\left(2k-4\mu\omega(Ln(A))^2\sigma\alpha+\mu^3\right)/\mu\omega\right)}+(Ln(A))^2\sqrt{-\alpha\sigma}\coth_A(2\sqrt{-\alpha\sigma}\,\xi)(1+pq)+2\frac{(Ln(A))^2\sqrt{-pq\alpha\sigma}}{\sinh_A(2\sqrt{-\alpha\sigma}\,\xi)},$$
$$\Phi_{20}=\frac{-1}{6\left(\xi\left(2k-4\mu\omega(Ln(A))^2\sigma\alpha+\mu^3\right)/\mu\omega\right)}+(Ln(A))^2\sqrt{-\alpha\sigma}\tanh_A\left(\frac{1}{2\sqrt{-\alpha\sigma}\,\xi}\right)+(Ln(A))^2\sqrt{-\alpha\sigma}\coth_A\left(\frac{1}{2\sqrt{-\alpha\sigma}\,\xi}\right).$$

Family 17. When $\beta = 0$ and $\sigma = \alpha$, the periodic-singular solutions are obtained as follows:

$$\begin{aligned}
\Phi_{21} &= \frac{-1}{6\left(\xi\left(2k + 8(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} - 2(Ln(A))^2\alpha\left(\tan_A(\alpha\xi) - \alpha\xi\right),\\
\Phi_{22} &= \frac{-1}{6\left(\xi\left(2k + 8(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} + 2(Ln(A))^2\alpha\left(\cot_A(\alpha\xi) - \frac{1}{2\pi} + \alpha\xi\right),\\
\Phi_{23} &= \frac{-1}{6\left(\xi\left(2k - 4(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} - (Ln(A))^2\alpha\tan_A(2\alpha\xi)(1 + pq)\\
&\quad - 2\frac{(Ln(A))^2\alpha\sqrt{pq}}{\cos_A(2\alpha\xi)},\\
\Phi_{24} &= \frac{-1}{6\left(\xi\left(2k - 4(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\alpha\cot_A(2\alpha\xi)(1 + pq)\\
&\quad - 2\frac{(Ln(A))^2\alpha\sqrt{pq}}{\sin_A(2\alpha\xi)},\\
\Phi_{25} &= \frac{-1}{6\left(\xi\left(2k - 4(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} -\\
&\quad (Ln(A))^2\alpha\left(\tan_A\left(\frac{1}{2\alpha\xi}\right) - \cot_A\left(\frac{1}{2\alpha\xi}\right)\right).
\end{aligned} \tag{29}$$

Family 18. When $\beta = 0$ and $\sigma = -\alpha$, the singular dark, combined dark-bright, combined dark-singular, and combined singular solutions are obtained, as follows:

$$\begin{aligned}
\Phi_{26} &= \frac{-1}{6\left(\xi\left(2k - 8(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\\
&\quad \alpha\left(2\tanh_A(\alpha\xi) + Ln\left(\frac{\tanh_A(\alpha\xi) - 1}{\tanh_A(\alpha\xi) + 1}\right)\right),\\
\Phi_{27} &= \frac{-1}{6\left(\xi\left(2k - 8(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\\
&\quad \alpha\left(2\coth_A(\alpha\xi) + Ln\left(\frac{\coth_A(\alpha\xi) - 1}{\coth_A(\alpha\xi) + 1}\right)\right),\\
\Phi_{28} &= \frac{-1}{6\left(\xi\left(2k + 4(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\alpha\tanh_A(2\alpha\xi)\\
&\quad (1 + pq) - 2\frac{(Ln(A))^2\alpha i\sqrt{pq}}{\cosh_A(2\alpha\xi)},
\end{aligned}$$

$$\Phi_{29} = \frac{-1}{6\left(\xi\left(2k + 4(Ln(A))^2\alpha^2\mu\omega + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2\alpha\coth_A(2\alpha\xi)$$
$$(1 + pq) - \frac{2(Ln(A))^2\alpha\sqrt{pq}}{\sinh_A(2\alpha\xi)},$$
$$\Phi_{30} = \frac{-1}{6\left(\xi\left(2k + 4\mu\omega(Ln(A))^2\alpha^2 + \mu^3\right)/\mu\omega\right)} + (Ln(A))^2$$
$$\alpha\left(\tanh_A\left(\frac{1}{2\alpha\xi}\right) + \coth_A\left(\frac{1}{2\alpha\xi}\right)\right). \tag{30}$$

Family 19. When $\beta^2 = 4\alpha\sigma$, the rational solution is obtained, as

$$\Phi_{31} = \frac{-1}{6\left((x + \mu y - kt)\left(2k + \mu^3\right)/\mu\omega\right)} + \frac{2\omega}{(x + \mu y - kt)}. \tag{31}$$

Family 20. When $\beta = l$, $\sigma = 0$, and $\alpha = ml\,(m \neq 0)$, the rational solution is obtained, as follows:

$$\Phi_{32} = \frac{-1}{6\left((x + \mu y - kt)\left((Ln(A))^2 l^2\mu\omega + 2k + \mu^3\right)/\mu\omega\right)}. \tag{32}$$

Family 21. When $\beta = \sigma = 0$, the rational solution is obtained as

$$\Phi_{33} = \frac{-1}{6\left(\left(2k + \mu^3\right)(x + \mu y - kt)/\mu\omega\right)}. \tag{33}$$

Family 22. When $\beta = \alpha = 0$, the rational solution is obtained as

$$\Phi_{34} = \frac{-1}{6\left(2\xi^2 k + \mu^3\xi^2 - 12\mu\omega/\mu\omega\xi\right)}. \tag{34}$$

Family 23. When $\alpha = 0$ and $\beta \neq 0$, the singular and dark-singular combo solitons solutions are obtained, as follows:

$$\Phi_{35} = \frac{-1}{6\left(\xi\left(\mu\omega(Ln(A))^2\beta^2 + 2k + \mu^3\right)/\mu\omega\right)} -$$
$$4\frac{(Ln(A))^2 p\beta}{(p - 1)\left(\tanh_A(1/2\beta\xi)p - \tanh_A(1/2\beta\xi) + 1 + p\right)},$$
$$\Phi_{36} = \frac{-1}{6\left(\xi\left(\mu\omega(Ln(A))^2\beta^2 + 2k + \mu^3\right)/\mu\omega\right)} -$$
$$4\frac{(Ln(A))^2\beta}{p(-1 + p)\left(-\tanh_A(1/2\beta\xi) + \tanh_A(1/2\beta\xi)p - p - 1\right)}$$
$$+ 2(Ln(A))^2\beta Ln\left(\sinh_A(\beta\xi) + \cosh_A(\beta\xi) + p\right)$$
$$+ 2(\ln(A))^2\beta Ln\left(\tanh_A\left(\frac{1}{2\beta\xi}\right) - 1\right)$$
$$- 2(Ln(A))^2\beta Ln\left(-\tanh_A\left(\frac{1}{2\beta\xi}\right) + \tanh_A(1/2\beta\xi)p - p - 1\right). \tag{35}$$

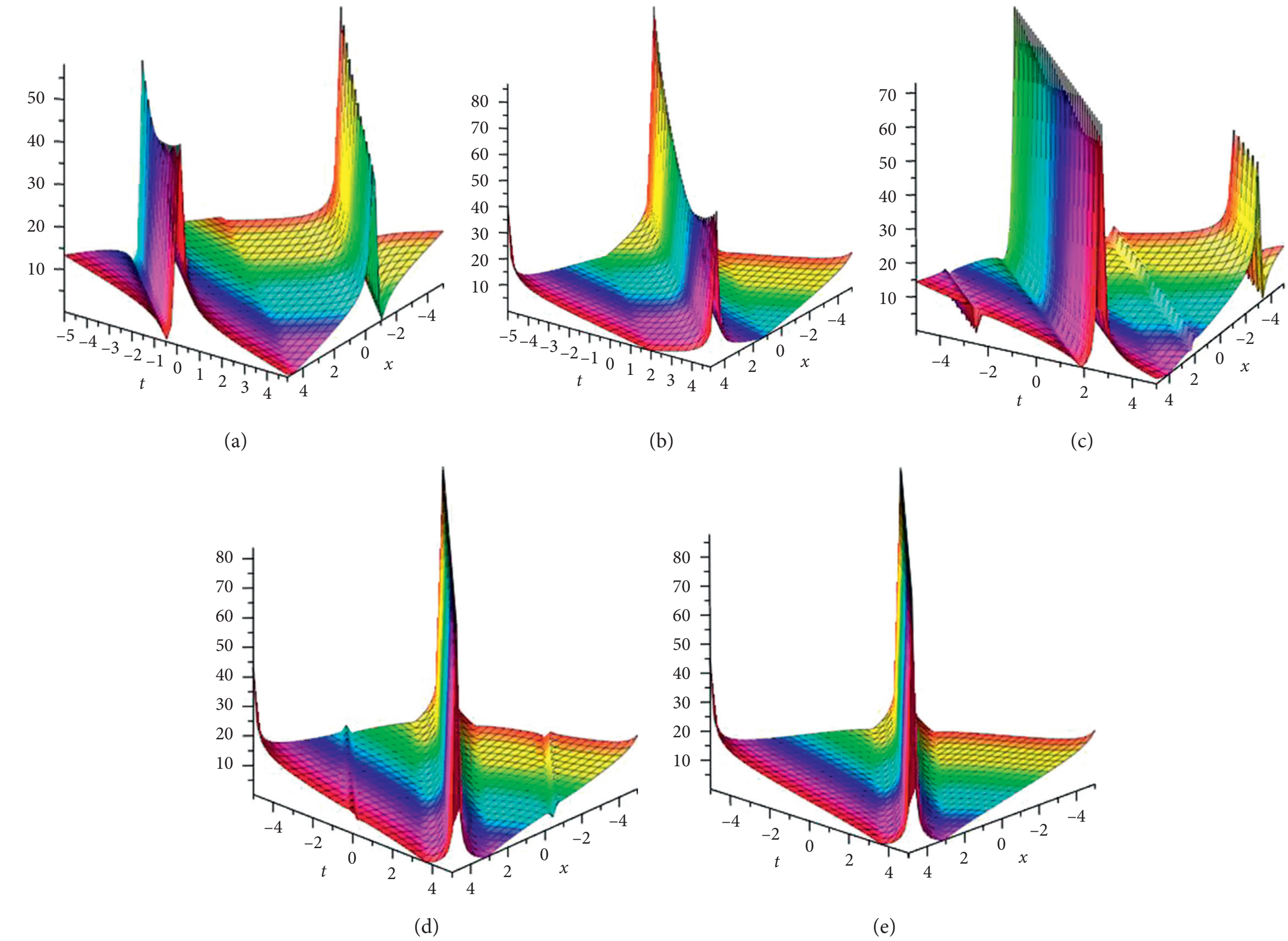


FIGURE 1: 3D profile of Family 13 with $\alpha = 1.5$, $\beta = 2$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.75$, and $A = 2.5$. (a) $|\Phi_1|$. (b) $|\Phi_2|$. (c) $|\Phi_3|$. (d) $|\Phi_4|$. (e) $|\Phi_5|$.

Family 24. When $\beta = l$, $\alpha = 0$, and $\sigma = ml\,(m \neq 0)$, the rational solution is obtained, as follows:

$$\Phi_{37} = \frac{-1}{6\left(\xi\left(\left(Ln(A)\right)^2 l^2 \mu\omega + 2k + \mu^3\right)/\mu\omega\right)} + 2Ln(A)lLn\left(\frac{q - mpA^{l\xi}}{-q + mpA^{l\xi}}\right) + 2\frac{Ln(A)lq}{-q + mpA^{l\xi}}, \tag{36}$$

where $\gamma = \beta^2 - 4\alpha\sigma$ and $\xi = \omega(x + \mu y - kt)$.

3.1. Physical Description of Solutions. A solitary wave is a restricted gravity wave that maintains its finite amplitude and propagates with consistent speed and constant shape. Solitons are the solitary wave with an elastic dispersive property. Solitons are the consequence of a delicate balance between nonlinear and dispersive impact in the medium. If the solution is in the form of tangent, secant, cotangent, and cosecant hyperbolic, then the solution is called dark, bright, singular, singular-soliton solutions, respectively. The solution of hyperbolic tangent plus hyperbolic secant form is called combined dark-bright soliton solution. The solution of hyperbolic cotangent plus hyperbolic cosecant form is called combined singular soliton solution, and the solution of hyperbolic tangent plus hyperbolic cotangent form is called dark-singular combo soliton solution.

Figure 1 demonstrates the solutions of $|\Phi_1|, |\Phi_2|, |\Phi_3|, |\Phi_4|$, and $|\Phi_5|$ for the particular parameters $\alpha = 1.5$, $\beta = 2$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.75$, and $A = 2.5$. The complex plane represents multiwaves having positive or negative jumps time to time. The modulus of this solution represents periodic long waves with positive amplitudes.

The solutions of $|\Phi_6|, |\Phi_7|, |\Phi_8|, |\Phi_9|$, and $|\Phi_{10}|$ are depicted in Figure 2 for some particular choice of the

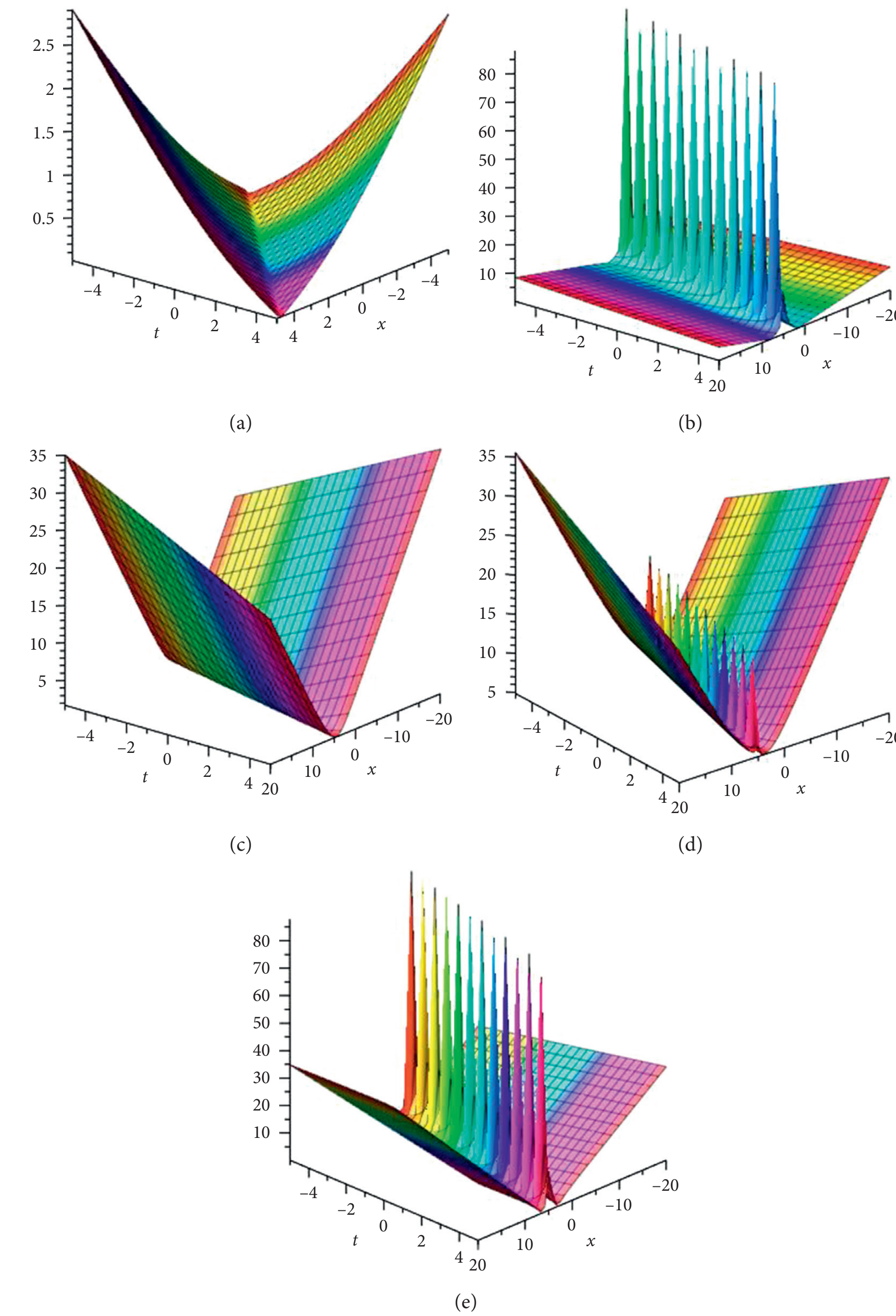


Figure 2: 3D profile of Family 14 with $\alpha = 1.5$, $\beta = 2$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.5$, and $A = 2.5$. (a) $|\Phi_6|$. (b) $|\Phi_7|$. (c) $|\Phi_8|$. (d) $|\Phi_9|$. (e) $|\Phi_{10}|$.

parameters such as $\alpha = 1.5$, $\beta = 2$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.5$, and $A = 2.5$. In Figure 3, the solutions of $|\Phi_{11}|, |\Phi_{12}|, |\Phi_{13}|, |\Phi_{14}|$, and $|\Phi_{15}|$ are plotted in the finite domain for the parameters $\alpha = 1.5$, $\beta = 0$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.75$, and $A = 2.5$. Figure 4 demonstrates the solutions of $|\Phi_{16}|, |\Phi_{17}|, |\Phi_{18}|, |\Phi_{19}|$, and $|\Phi_{20}|$ for the particular parameters $\alpha = 1.5$, $\beta = 0$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = -0.75$, and$A = 2.5$. Figure 5 demonstrates the solutions of $|\Phi_{21}|, |\Phi_{22}|, |\Phi_{23}|, |\Phi_{24}|$, and $|\Phi_{25}|$ for the particular parameters $\alpha = 1.5 = \sigma$, $\beta = 0$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, and $A = 2.5$. Figure 6 demonstrates the solutions of $|\Phi_{26}|, |\Phi_{27}|, |\Phi_{28}|, |\Phi_{29}|$, and $|\Phi_{30}|$ for the particular parameters $\alpha = 1.5$, $\beta = 0$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = -1.5$, and $A = 2.5$. Figure 7 demonstrates the 3D graphics of the solutions of $|\Phi_{31}|, |\Phi_{32}|$, and $|\Phi_{33}|$ under the particular values $\alpha = 1.5$, $k = y = 1$, $\omega = 0.5$, $\mu = 0.25$, and $A = 2.5$ for $|\Phi_{31}|$ with $\beta = 1$ and $\sigma = 0.5$, $|\Phi_{32}|$ with $\beta = 1$, $\sigma = 0$, $l = 1$, and $m = 0.5$, and $|\Phi_{33}|$ with $\beta = 0 = \sigma$. Figure 8 demonstrates the 3D graphics

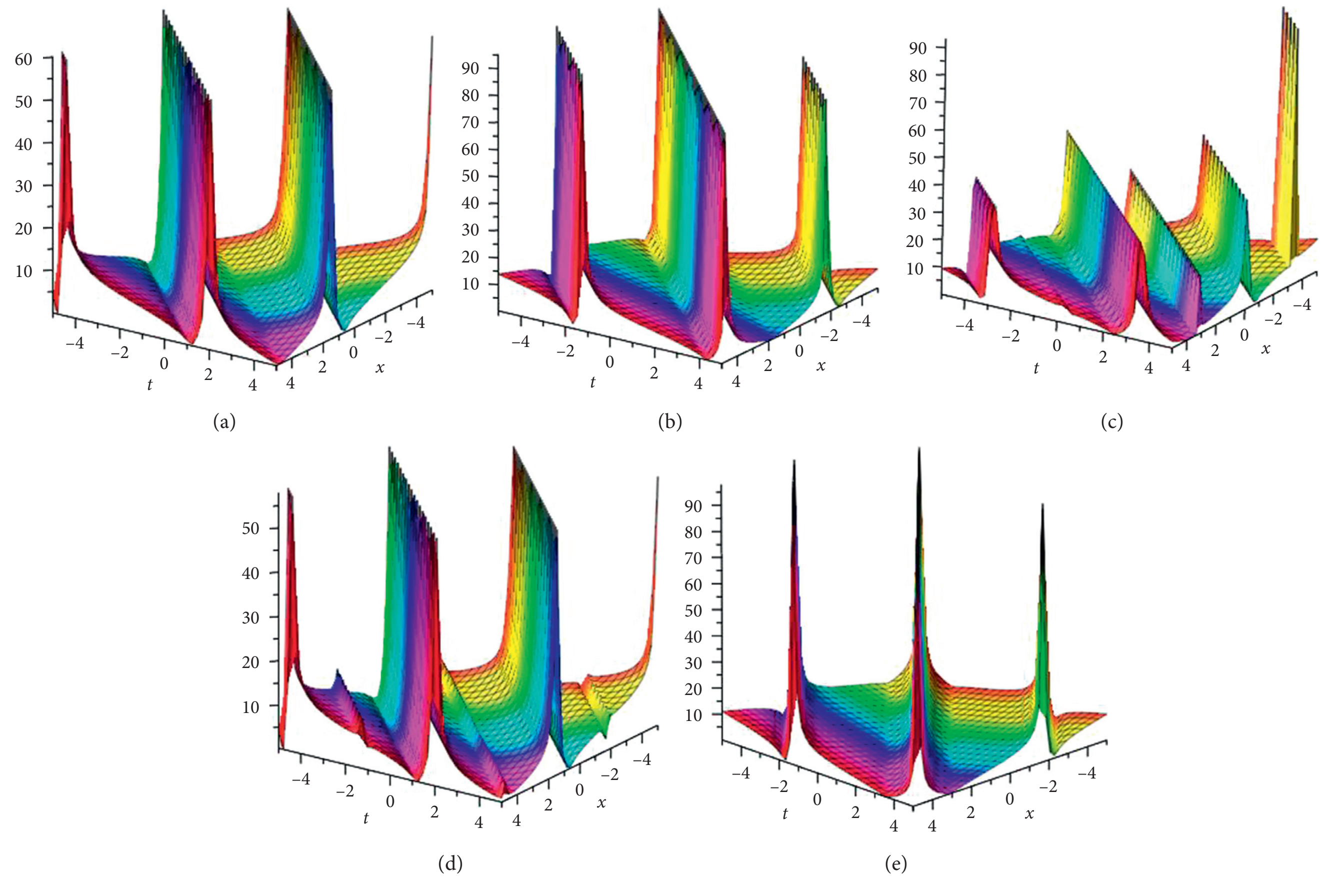


Figure 3: 3D profile of Family 15 with $\alpha = 1.5$, $\beta = 0$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.75$, and $A = 2.5$. (a) $|\Phi_{11}|$. (b) $|\Phi_{12}|$. (c) $|\Phi_{13}|$. (d) $|\Phi_{14}|$. (e) $|\Phi_{15}|$.

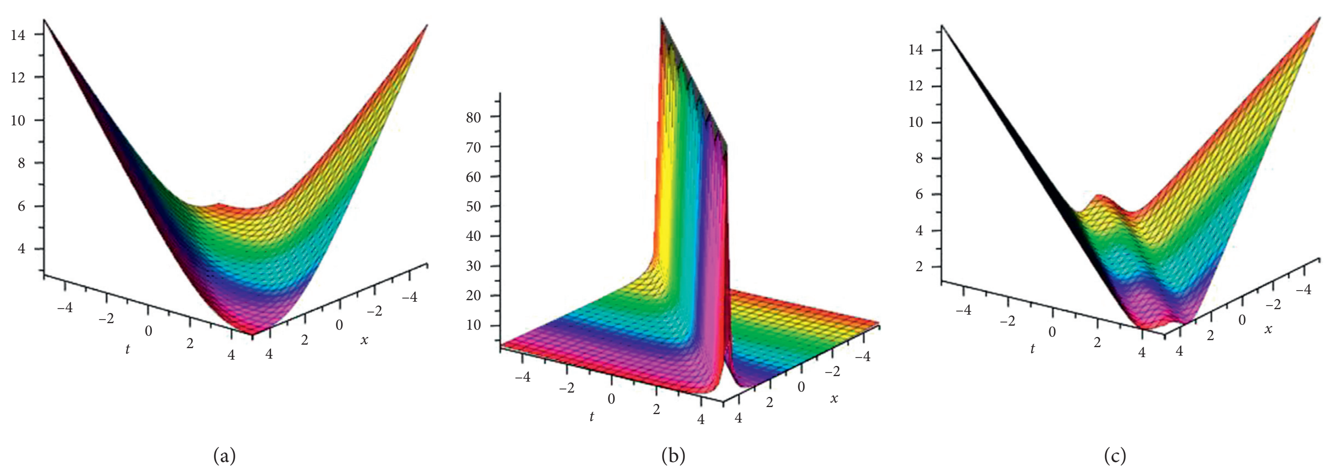
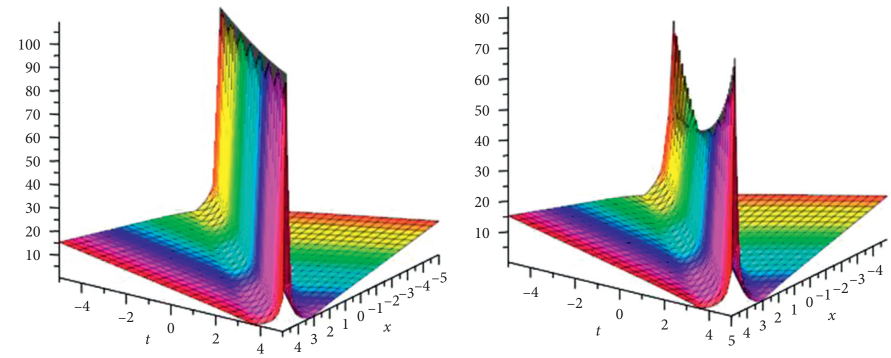
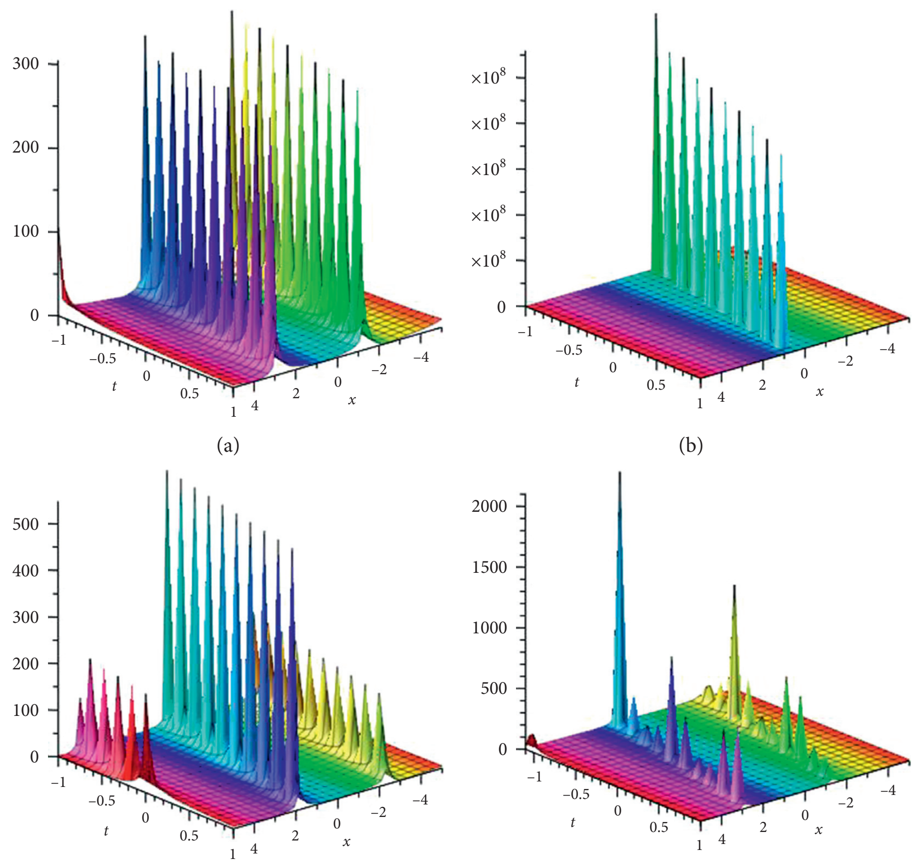


Figure 4: Continued.



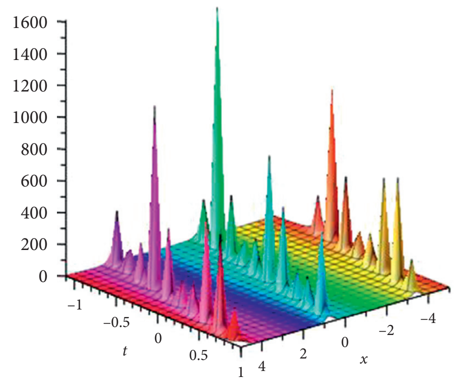
(d) (e)

Figure 4: 3D profile of Family 16 with $\alpha = 1.5$, $\beta = 0$, $k = \gamma = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = -0.75$, and $A = 2.5$. (a) $|\Phi_{16}|$. (b) $|\Phi_{17}|$. (c) $|\Phi_{18}|$. (d) $|\Phi_{19}|$. (e) $|\Phi_{20}|$.



(a) (b) (c) (d)

Figure 5: Continued.



(e)

Figure 5: 3D profile of Family 17 with $\alpha = 1.5 = \sigma, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25$, and $A = 2.5$. (a) $|\Phi_{21}|$. (b) $|\Phi_{22}|$. (c) $|\Phi_{23}|$. (d) $|\Phi_{24}|$. (e) $|\Phi_{25}|$.

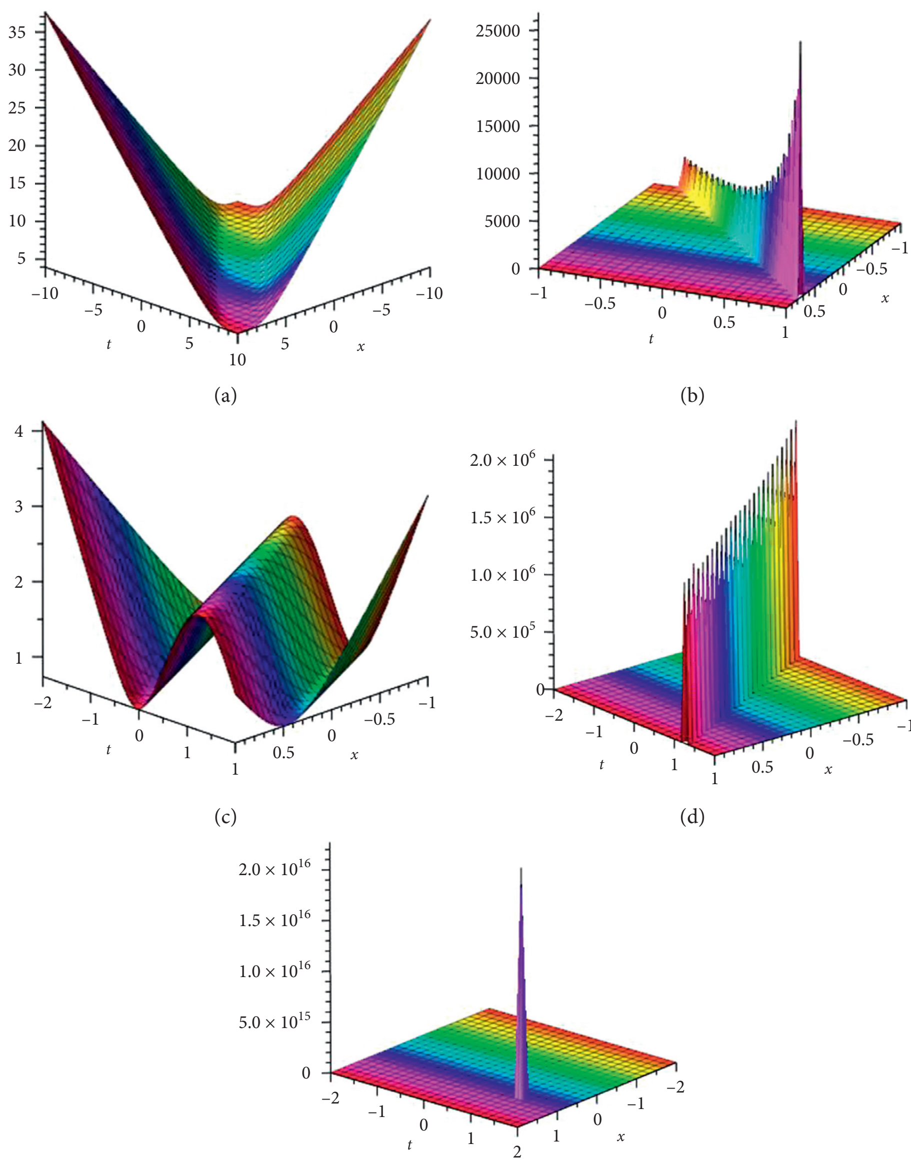
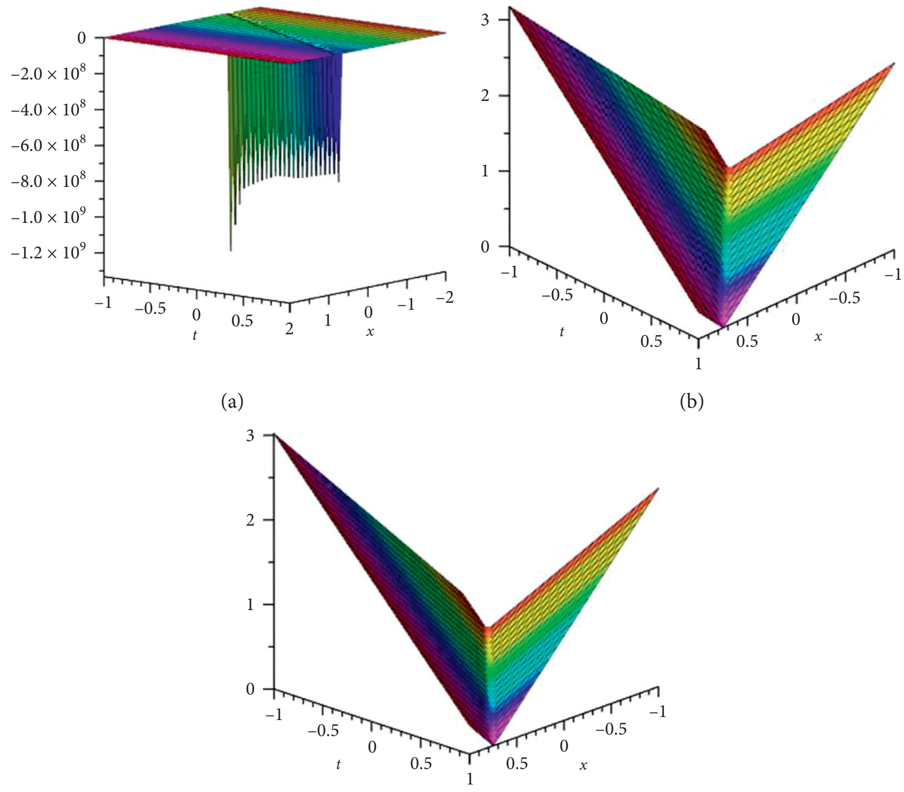
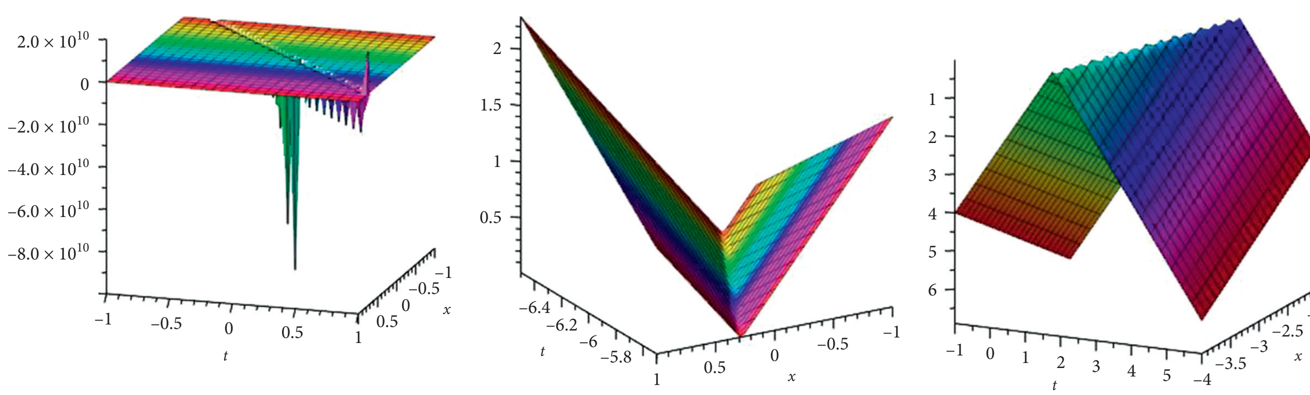


Figure 6: 3D profile of Family 18 with $\alpha = 1.5$, $\beta = 0$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = -1.5$, and $A = 2.5$. (a) $|\Phi_{26}|$. (b) $|\Phi_{27}|$. (c) $|\Phi_{28}|$. (d) $|\Phi_{29}|$. (e) $|\Phi_{30}|$.



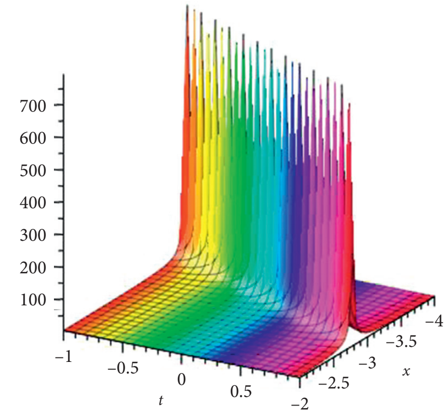
(a) (b) (c)

Figure 7: 3D graphics of Families 19, 20, and 21 under the values of $\alpha = 1.5$, $k = y = 1$, $\omega = 0.5$, $\mu = 0.25$, and $A = 2.5$ for (a) Φ_{31} with $\beta = 1$ and $\sigma = 0.5$, (b) Φ_{32} with $\beta = 1$ $\sigma = 0$, $l = 1$, and $m = 0.5$, and (c) Φ_{33} with $\beta = 0 = \sigma$.



(a) (b) (c)

Figure 8: Continued.



(d)

FIGURE 8: 3D graphics of Families 22, 23, and 24 under the values $\alpha = 0$, $k = y = 1$, $\omega = 0.5$, $\sigma = 0.5$, $\mu = 0.25$, and $A = 2.5$ for (a) Φ_{34} with $\beta = 0$, (b) Φ_{35}, (c) Φ_{36} with $\beta = 0.5$, and (d) Φ_{37} with $\beta = 0.5 = l$, $\sigma = 1$, and $m = 2$.

of the solutions $|\Phi_{34}|, |\Phi_{35}|, |\Phi_{36}|$, and $|\Phi_{37}|$ under the particular values $\alpha = 0$, $k = y = 1$, $\omega = 0.5$, $\sigma = 0.5$, $\mu = 0.25$, and $A = 2.5$ for $|\Phi_{34}|$ with $\beta = 0$, $|\Phi_{35}|$ and $|\Phi_{36}|$ with $\beta = 0.5$, and $|\Phi_{37}|$ with $\beta = 0.5 = l$, $\sigma = 1$, and $m = 2$. Families 13, 14, 15 and 16 represent the singular, dark, combined dark-bright, combined dark-singular, combined singular, and solitary wave solutions. Families 15 and 17 represent the exact periodic traveling wave solutions, whereas the Families 19, 20, 21, 22, 23, and 24 show the rational solutions.

4. Conclusion

To investigate the $(2 + 1)$-dimensional DJKM equation for exact solutions, the extended direct algebraic method is applied. By the extended direct algebraic method, many new exact solitary wave solutions are constructed including the singular, dark, combined dark-bright, periodic-singular, combined dark-singular, combined singular, and rational kinds. Such observations show that the suggested approaches are highly helpful and efficient in solving the NEEs. The complex-valued solutions represent traveling waves in different structures. Even though some are of the well-known forms such as bell-, V-, and W-shaped multiwaves, the shape of some others are completely different from them which were not found in the previous literature. The results of this investigation can be useful in illustrating the physical meaning of the studied model by 3D graphics. The performance of the method is reliable and a computerized mathematical approach to conduct other NLEEs in the field of mathematical physics and applied sciences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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References

[1] N. A. Asif, Z. Hammouch, M. B. Riaz, and H. Bulut, "Analytical solution of a Maxwell fluid with slip effects in view of the Caputo-Fabrizio derivative?" *The European Physical Journal Plus*, vol. 133, no. 7, pp. 1–13, 2018.

[2] M. B. Riaz, A. Atangana, A. Jhangeer, and S. Tahir, "Soliton solutions, soliton-type solutions and rational solutions for the coupled nonlinear Schrödinger equation in magneto-optic waveguides," *The European Physical Journal Plus*, vol. 136, no. 2, pp. 1–19, 2021.

[3] A. A. Zafar, J. Awrejcewicz, O. Mazur, and M. B. Riaz, "Study of composite fractional relaxation differential equation using fractional operators with and without singular kernels and special functions," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–16, 2021.

[4] W. Hereman and A. Nuseir, "Symbolic methods to construct exact solutions of nonlinear partial differential equations," *Mathematics and Computers in Simulation*, vol. 43, pp. 13–27, 1997.

[5] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, "The inverse scattering transform-fourier analysis for nonlinear problems," *Studies in Applied Mathematics*, vol. 53, no. 4, pp. 249–315, 1974.

[6] J. Weiss, M. Tabor, and G. Carnevale, "The painlevé property for partial differential equations," *Journal of Mathematical Physics*, vol. 24, no. 3, pp. 522–526, 1983.

[7] X. Y. Gao, "Looking at a nonlinear inhomogeneous optical fiber through the generalized higher-order variable-coefficient Hirota equation," *Applied Mathematics Letters*, vol. 73, pp. 143–149, 2017.

[8] N. Sajid and G. Akram, "Solitary dynamics of longitudinal wave equation arises in magneto-electro-elastic circular rod," *Modern Physics Letters B*, vol. 35, no. 5, 2150086 pages, 2020.

[9] T. Xu, H. Li, H. Zhang, M. Li, and S. Lan, "Darboux transformation and analytic solutions of the discrete PT-symmetric nonlocal nonlinear Schrödinger equation," *Applied Mathematics Letters*, vol. 63, pp. 88–94, 2017.

[10] M. Dehghan, J. Manafian, and A. Saadatmandi, "Application of the exp-function method for solving a partial differential equation arising in biology and population genetics," *International Journal of Numerical Methods for Heat and Fluid Flow*, vol. 21, pp. 736–753, 2011.

[11] C. Gilson, F. Lambert, J. Nimmo, and R. Willox, "On the combinatorics of the Hirota D-operators," *Proceedings of the Royal Society of London*, vol. 452, no. 1945, pp. 223–234, 1996.

[12] C. Park, M. M. Khater, A. H. Abdel-Aty et al., "Dynamical analysis of the nonlinear complex fractional emerging telecommunication model with higher–order dispersive cubic–quintic," *Alexandria Engineering Journal*, vol. 59, no. 3, pp. 1425–1433, 2020.

[13] Ö. Güner and A. Bekir, "Solving nonlinear space-time fractional differential equations via ansatz method," *Computational Methods for Differential Equations*, vol. 6, pp. 1–11, 2018.

[14] D. Kumar, K. Hosseini, and F. Samadani, "The sine-gordon expansion method to look for the traveling wave solutions of the Tzitzéica type equations in nonlinear optics," *Optik*, vol. 149, pp. 439–446, 2017.

[15] A. Biswas, Y. Yildirim, E. Yasar et al., "Optical soliton perturbation with full nonlinearity by trial equation method," *Optik*, vol. 157, pp. 1366–1375, 2018.

[16] M. Ekici, M. Mirzazadeh, A. Sonmezoglu et al., "Dispersive optical solitons with Schrödinger-Hirota equation by extended trial equation method," *Optik*, vol. 136, pp. 451–461, 2017.

[17] N. Sajid and G. Akram, "Optical solitons with full nonlinearity for the conformable space-time fractional fokas-lenells equation," *Optik*, vol. 196, Article ID 163131, 2019.

[18] H. Rezazadeh, A. Korkmaz, M. Eslami, and S. M. Mirhosseini-Alizamini, "A large family of optical solutions to Kundu-Eckhaus model by a new auxiliary equation method," *Optical and Quantum Electronics*, vol. 51, no. 3, pp. 1–12, 2019.

[19] X. Hu and Y. Li, "Bäcklund transformation and nonlinear superposition formula of DJKM equation," *Acta Mathematia Scientia*, vol. 11, pp. 164–172, 1991.

[20] Y. H. Wang, H. Wang, and C. Temuer, "Lax pair, conservation laws, and multi-shock wave solutions of the DJKM equation with Bell polynomials and symbolic computation," *Nonlinear Dynamics*, vol. 78, no. 2, pp. 1101–1107, 2014.

[21] Y. Q. Yuan, B. Tian, W. R. Sun, J. Chai, and L. Liu, "Wronskian and Grammian solutions for a $(2+1)$-dimensional date-jimbo-kashiwara-Miwa equation," *Computers and Mathematics with Applications*, vol. 74, no. 4, pp. 873–879, 2017.

[22] J. C. Pu and H. C. Hu, "Exact solitary wave solutions for two nonlinear systems," *Indian Journal of Physics*, vol. 93, no. 2, pp. 229–234, 2019.

[23] M. Singh and R. K. Gupta, "On painlevé analysis, symmetry group and conservation laws of date-Jimbo-Kashiwara-Miwa equation," *International Journal of Applied and Computational Mathematics*, vol. 4, no. 3, pp. 88–102, 2018.

[24] N. Sajid and G. Akram, "The application of the $\exp(-\Phi(\xi))$-expansion method for finding the exact solutions of two integrable equations," *Mathematical Problems in Engineering*, vol. 2018, pp. 1–10, 2018.

[25] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, "Transformation groups for soliton equations IV. A new hierarchy of soliton equations of KP-type," *Physica D*, vol. 4, pp. 343–365, 1982.

[26] T. C. Bountis, V. Papageorgiou, and P. Winternitz, "On the integrability of systems of nonlinear ordinary differential equations with superposition principles," *Journal of Mathematical Physics*, vol. 27, no. 5, pp. 1215–1224, 1986.

[27] H. Rezazadeh, "New soliton solutions of the complex Ginzburg-Landau equation with kerr law nonlinearity," *Optik*, vol. 167, pp. 218–227, 2018.

[28] Y. Ren and H. Zhang, "New generalized hyperbolic functions and auto-Bäcklund transformation to find new exact solutions of the $(2+1)$–dimensional NNV equation," *Physics Letter A*, vol. 357, no. 6, pp. 438–448, 2006.

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Research Article

Fractional Entropy-Based Test of Uniformity with Power Comparisons

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In the present paper, we use the fractional and weighted cumulative residual entropy measures to test the uniformity. The limit distribution and an approximation of the distribution of the test statistic based on the fractional cumulative residual entropy are derived. Moreover, for this test statistic, percentage points and power against seven alternatives are reported. Finally, a simulation study is carried out to compare the power of the proposed tests and other tests of uniformity.

1. Introduction

Rao et al. [1] suggested a nonnegative measure of uncertainty and called it the cumulative residual entropy (CRE). For any nonnegative continuous random variable (RV) X with a cumulative distribution function (CDF) $F(x) = P(X < x)$, the CRE is defined by

$$\mathrm{CRE}(F) = -\int_0^\infty \overline{F}(x)\ln(\overline{F}(x))\mathrm{d}x, \quad (1)$$

where $\overline{F}(x) = 1 - F(x)$ is the reliability function. Rao et al. [1] revealed many salient features of the CRE. For example, the CRE possesses more general mathematical properties than the Shannon entropy, and it can be easily computed from sample data, and these computations asymptotically converge to the true values. Moreover, the CRE deals with the quantity of information in residual life. For the standard uniform distribution, denoted by $U(0,1)$, Rao et al. [1] showed that the value of the CRE is 1/4. The literature abounds with many different results for Shannon's entropy and its modifications. Interested readers may refer to [1–17].

Xiong et al. [16] suggested the fractional cumulative residual entropy (FCRE) to extend the CRE to the case of fractional order. For any $0 \leq q \leq 1$, the FCRE for the RV X is defined by

$$\mathrm{CRE}^q(F) = \int_0^\infty \overline{F}(x)[-\ln(\overline{F}(x))]^q \mathrm{d}x. \quad (2)$$

The measure $\mathrm{CRE}^q(F)$ is a nonadditive and nonnegative. Moreover, it is a convex function of the parameter q, $\mathrm{CRE}^0(F) = \mathbb{E}(X)$, and $\mathrm{CRE}^1(F) = \mathrm{CRE}(F)$. Xiong et al. [16] derived the FCRE for some well-known distributions; for example, FCRE of the CDF $U(0,1)$ is $\Gamma(q+1)/2^{q+1}$.

Misagh et al. [15] proposed a weighted form of CRE, which is shift-dependent. This information-theoretic uncertainty measure is called the weighted cumulative residual entropy (WCRE), and it is defined by

$$\mathrm{CRE}_w(F) = -\int_0^\infty x\overline{F}(x)\ln(\overline{F}(x))\mathrm{d}x. \quad (3)$$

Later, Mirali et al. [12] and Mirali and Baratpour [13] studied several properties of this measure including its dynamic version. It is easy to observe that the WCRE of the $U(0,1)$ is 5/36.

Stephens [18] offered a practical guide to goodness-of-fit tests using statistics based on the empirical CDF. Moreover, in [18], the power comparisons of some uniformity tests were carried out. Dudewicz and Van der Meulen [9] investigated the power properties of an entropy-based test when used for testing uniformity. Moreover, via a comparison with other tests of uniformity, Dudewicz and Van der Meulen [9] showed that the entropy-based test possesses good power properties for many alternatives. Noughabi [14] constructed a test for uniformity based on the CRE and studied some of its properties. Moreover, he reported the percentage points and power comparison against seven alternative distributions. As a natural extension of the results obtained by Noughabi [14], we study the FCRE and WCRE for testing the uniformity. A result of a simulation study shows that the test based on FCRE and WCRE is competitive with the test based on CRE in terms of power. This fact gives a satisfactory motivation of our study.

Throughout this paper, we obtain the percentage points under the WCRE and FCRE by using the Monte Carlo method via the simulation and the normality asymptotic, as well as the beta approximation, respectively. Moreover, a power comparison is performed between the FCRE and WCRE and other tests. The rest of this work is systematic as follows. In Section 2, we introduce the FCRE test statistic for uniformity and discuss some of its properties. In Section 3, we propose the methods of finding the percentage points of FCRE and illustrate the WCRE test statistics for uniformity. In addition, we calculate the percentage points of FCRE and WCRE. Then, in Section 4, we use Monte Carlo simulation to perform the power comparison of uniformity of different tests against seven alternative distributions. Section 5 is devoted to the conclusions. Everywhere in what follows, the symbols $(\xrightarrow[n]{p})$, $(\xrightarrow[n]{d})$ and $(\xrightarrow[n]{\text{a.s.}})$ stand for convergence in probability, convergence in distribution, and almost surely, as $n \longrightarrow \infty$.

2. Theoretical Aspects and Test Statistic

To establish our test of the null hypothesis H_0, we need the following theorem, which shows that, for a CDF with support $[0,1]$, one always has $0 \le \mathrm{CRE}^q(F) \le e^{-q}$, and for the distribution $U(0,1)$, we have $\mathrm{FCRE} = \Gamma(q+1)/2^{q+1}$, and this value is uniquely attained by the uniform distribution, whenever q is fixed.

Theorem 1. *Let X be a nonnegative RV with an absolutely continuous CDF F with a support $[0,1]$. From (2), it holds $0 \le \mathrm{CRE}^q(F) \le e^{-q}$, and $\mathrm{CRE}^q(F) = \Gamma(q+1)/2^{q+1}$ is uniquely acquired by the distribution $U(0,1)$.*

Proof. Since $0 \le \overline{F}(x)[-\ln(\overline{F}(x))]^q \le 1$, and the function $f(x) = x(-\ln x)^q$ has a maximum at $x = e^{-q}$, $0 < x \le 1$, we get $0 \le \mathrm{CRE}^q(F) \le e^{-q}$. On the other hand, using the strict convexity of $f(x) = x(-\ln x)^q$, it is easy to see that FCRE is a concave function of distribution (with support [0, 1]). This shows that $\mathrm{CRE}^q(F) = \Gamma(q+1)/2^{q+1}$ is uniquely acquired by the distribution $U(0,1)$. This completes the proof.

Let $X_1, X_2, \ldots, X_n$ be a random sample with a continuous CDF F, with support [0, 1]. Furthermore, let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the corresponding order statistics $X_1, X_2, \ldots, X_n$. According to (2), we can obtain the empirical FCRE as an estimator of $\mathrm{FCRE}(F)$ by

$$\mathrm{CRE}^q(F_n) = \int_0^\infty \overline{F}_n(x)[-\ln(\overline{F}_n(x))]^q\mathrm{d}x, \quad (4)$$

where $\overline{F}_n(x) = 1 - F_n(x)$ and $F_n(x)$ is the empirical CDF, which is defined by

$$F_n(x) = \sum_{i=1}^{n-1}\frac{i}{n}I_{[X_{(i)},X_{(i+1)})}(x) + I_{[X_{(n)},\infty)}(x), \quad x \in \mathfrak{R}, \quad (5)$$

where $I_A(x)$ is the indicator function, i.e., $I_A(x) = 1, x \in A$; $I_A(x) = 0, x \notin A$.

To perform a consistent test of the hypothesis of uniformity, we suggest the consistent statistic test

$$R_n^q = \mathrm{CRE}^q(F_n) = \sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\left(-\ln\left(1-\frac{i}{n}\right)\right)^q\left(X_{(i+1)} - X_{(i)}\right) = \sum_{i=1}^{n-1}A_iW_i, \quad (6)$$

where $A_i = (1-(i/n))(-\ln(1-(i/n)))^q$ and $W_i = (X_{(i+1)} - X_{(i)})$, $i = 1, 2, \ldots, n-1$, $0 \le q \le 1$.

Xiong et al. [16] proved that $\mathrm{CRE}^q(F_n) \xrightarrow[n]{p} \mathrm{CRE}^q(F)$. Moreover, under the null hypothesis H_0, we get $R_n^q \xrightarrow[n]{p} \Gamma(q+1)/2^{q+1}$. On the other hand, under the alternative hypothesis (that F is any continuous CDF with support [0, 1], which is not the uniform), we have $\mathrm{CRE}^q(F_n) \xrightarrow[n]{p} r$, where r is a smaller or larger number than $\Gamma(q+1)/2^{q+1}$. □

Theorem 2. *The test based on the sample estimate R_n^q is consistent.*

Proof. From Glivenko–Cantelli theorem (see Tucker [19]), we have $\sup_t|F_n(t) - F(t)| \xrightarrow[n]{\text{a.s.}} 0$. On the other hand, Theorem 3 in Xiong et al. [16] asserts that $\mathrm{CRE}^q(F_n) \xrightarrow[n]{\text{a.s.}} \mathrm{CRE}^q(F)$, which proves the theorem. □

Theorem 3. *Suppose that the random sample $X_1, X_2, \ldots, X_n$ has been drawn from an unknown continuous CDF F defined on $[0,1]$. Then, from (6), we have $0 \le R_n^q \le e^{-q}$.*

Proof. Since the function $f(p) = p(-\ln p)^q$, $0 < p < 1$, has a maximum value at e^{-q}, $0 \le q \le 1$; therefore,

$$0 \le R_n^q = \mathrm{CRE}^q(F_n) = \sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\left(-\ln\left(1-\frac{i}{n}\right)\right)^q\left(X_{(i+1)}-X_{(i)}\right) \le \sum_{i=1}^{n-1} e^{-q}\left(X_{(i+1)}-X_{(i)}\right) = e^{-q}\left(X_{(n)}-X_{(1)}\right) \le e^{-q}. \tag{7}$$

This completes the proof of the theorem. □

Theorem 4. *Under H_0, from (6), the mean and the variance of R_n^q are, respectively,*

$$\mathbb{E}\left(R_n^q\right) = \frac{1}{n+1}\sum_{i=1}^{n-1} A_i, \tag{8}$$

$$\mathrm{Var}\left(R_n^q\right) = \frac{n}{(n+1)^2(n+2)}\sum_{i=1}^{n-1} A_i^2. \tag{9}$$

Proof. The proof directly follows by noting that, for any $i = 1, 2, \ldots, n-1$, the RV $W_i = (X_{(i+1)} - X_{(i)})$, based on the CDF $U(0,1)$, has beta distribution, i.e., $W_i \sim \mathrm{Beta}(1,n)$ (cf. [20]). This completes the proof. □

Remark 1. Under H_0, from (6), (8), and (9), we have $\lim_{n\longrightarrow\infty}\mathbb{E}(R_n^q) = \Gamma(q+1)/2^{q+1} = \mathrm{CRE}^q(U)$ and $\lim_{n\longrightarrow\infty}\mathrm{Var}(R_n^q) = 0$, where $\mathrm{CRE}^q(U)$ is the FCRE of the CDF $U(0,1)$.

The critical region, which describes the test procedure, is given by the following two inequalities:

$$\mathrm{CRE}^q(F_n) \le \mathrm{CRE}^{*q}_{\alpha/2} \coloneqq \text{lower or } \mathrm{CRE}^q(F_n) \ge \mathrm{CRE}^{*q}_{1-(\alpha/2)} \coloneqq \text{upper}, \tag{10}$$

where α is the desired level of significance, and CRE^{*q}_α is the α−quantile of the asymptotic, or approximated, CDF of the test statistic $\mathrm{CRE}^q(F_n)$, under H_0. In the next section, we derive the asymptotic and approximated CDF of the test statistic $\mathrm{CRE}^q(F_n)$. These quantiles are computed by using the Monte Carlo method.

3. Percentage Points of the Test Statistic

In this section, we obtain the asymptotic distribution of R_n^q under H_0. From (6), we can write $R_n^q = \sum_{i=1}^{n-1} T_i$, where $T_i = A_i W_i$, $i = 1, 2, \ldots, n-1$, and $W_i \sim \mathrm{Beta}(1,n)$. Thus, we can see that T_i's have the following probability density function (PDF):

$$f_{T_i}(t) = \frac{n}{A_i}\left(1-\frac{t}{A_i}\right)^{n-1}, \quad i = 1, 2, \ldots, n-1. \tag{11}$$

The mean and variance of T_i are, respectively,

$$\mu_i = \mathbb{E}(T_i) = A_i\mathbb{E}(W_i) = \frac{A_i}{n+1}, \quad \sigma_i^2 = \mathrm{Var}(T_i) = A_i^2\mathrm{Var}(W_i) = \frac{nA_i^2}{(n+1)^2(n+2)}. \tag{12}$$

According to Lyapunov central limit theorem (see Billingsley [21]), we have $\sum_{i=1}^{n-1}(T_i - \mu_i)/\sqrt{\sum_{i=1}^{n-1}\sigma_i^2} = (R_n^q - \mathbb{E}(R_n^q)/\sqrt{\mathrm{Var}(R_n^q)}) \xrightarrow[n]{d} \mathcal{N}$, where $\mathcal{N}$ is the standard normal RV (in the sequel, the standard normal distribution will be denoted by $N(0,1)$). Therefore, under H_0, the percentage point (α−quantile) CRE^{*q}_α is approximated according to the asymptotic normality of R_n^q for large n by

$$\mathrm{CRE}^{*q}_\alpha = \mathbb{E}\left(R_n^q\right) + \sqrt{\mathrm{Var}\left(R_n^q\right)}Z_\alpha, \tag{13}$$

where Z_α corresponds to the quantile $(\alpha \times 100)$ of the CDF $N(0,1)$.

Johannesson and Giri [22] proposed an approximation of the CDF of linear combination of the finite number of beta RVs. Noughabi [14] used this approximation to obtain approximately the percentage points of the CRE for finite n. By adopting the same procedure of Noughabi [14], we can obtain an approximation of R_n^q for finite n as follows:

$$R_n^q \approx \left(\sum_{i=1}^{n-1} A_i\right)Y, \tag{14}$$

where the RV Y has $\mathrm{Beta}(a,b)$ distribution,

$$a = \frac{(n+2)\left(\sum_{i=1}^{n-1} A_i\right)^2}{(n+1)\left(\sum_{i=1}^{n-1} A_i^2\right)} - \frac{1}{n+1}, \quad b = \frac{n}{n+1}\left(\frac{(n+2)\left(\sum_{i=1}^{n-1} A_i\right)^2}{\sum_{i=1}^{n-1} A_i^2} - 1\right), \tag{15}$$

and $A_i = (1-(i/n))(-\ln(1-(i/n)))^q$, $0 \le q \le 1$, $i = 1, 2, \ldots, n-1$. According to (14), the mean and variance of R_n^q are, respectively,

$$\mathbb{E}\left(R_n^q\right) = \left(\sum_{i=1}^{n-1} A_i\right)\frac{a}{a+b}, \quad \mathrm{Var}\left(R_n^q\right) = \left(\sum_{i=1}^{n-1} A_i\right)^2\frac{ab}{(a+b)^2(a+b+1)}. \tag{16}$$

Now, by using this approximation of R_n^q, the quantiles of order $\alpha/2$ and $1-(\alpha/2)$ of the approximated CDF of the test statistic $\mathrm{CRE}^q(F_n)$ under H_0 are, respectively,

$$\text{lower} \coloneqq \left(\sum_{i=1}^{n-1} A_i\right)F^{-1}\left(\frac{\alpha}{2}\right), \quad \text{upper} \coloneqq \left(\sum_{i=1}^{n-1} A_i\right)F^{-1}\left(1-\frac{\alpha}{2}\right), \tag{17}$$

where $F^{-1}(.)$ is the quantile function of the CDF F, F is the Beta(a,b) distribution, and a and b are defined in (15).

3.1. Empirical Weighted Cumulative Residual Entropy. From (3), Misagh et al. [15] proposed the empirical WCRE by

$$\begin{aligned}\mathrm{CRE}_w(F_n) &= -\sum_{i=1}^{n-1}\left(\frac{X_{(i+1)}^2 - X_{(i)}^2}{2}\right)\left(1-\frac{i}{n}\right)\ln\left(1-\frac{i}{n}\right)\\ &= \sum_{i=1}^{n-1} A_i' U_i,\end{aligned} \tag{18}$$

where $A_i' = X_{(i+1)}^2 - X_{(i)}^2/2$, $U_i = -(1-(i/n))\ln(1-(i/n))$, $i = 1, 2, \ldots, n-1$.

We suggest the following statistic of a consistent test based on (18):

$$T_n^w = \mathrm{CRE}_w(F_n) = \sum_{i=1}^{n-1} A_i' U_i. \tag{19}$$

Theorem 5. *The test based on the sample estimate T_n^w is consistent.*

Proof. From Mirali et al. [12] and by using Glivenko–Cantelli theorem, (see Tucker [19]), we have $\mathrm{CRE}_w(F_n) \xrightarrow[n]{\text{a.s.}} \mathrm{CRE}_w(F)$, which proves the theorem. □

Theorem 6. *Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from an unknown continuous CDF F defined on $[0,1]$. Then, from (18), we get $0 \le T_n^w \le 1/2e$.*

Proof. Since the function $f(p) = -p\ln p$, $0 < p < 1$, has a maximum value at $1/e$; therefore,

$$\begin{aligned}0 \le T_n^w = \mathrm{CRE}_w(F_n) &= -\sum_{i=1}^{n-1}\left(\frac{X_{(i+1)}^2 - X_{(i)}^2}{2}\right)\left(1-\frac{i}{n}\right)\ln\left(1-\frac{i}{n}\right)\\ &\le \frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{e}\left(X_{(i+1)}^2 - X_{(i)}^2\right) = \frac{1}{2e}\left(X_{(n)}^2 - X_{(1)}^2\right) \le \frac{1}{2e}.\end{aligned} \tag{20}$$

This completes the proof. □

3.2. Percentage Points. We generate 50,000 samples of size n, where $n = 10, 20, 30, 40, 50, 70, 100$, from $U(0,1)$. Using (6), the test statistic R_n^q is estimated by the empirical R_n^q for each sample and the same for T_n^w. Moreover, we can see that $\mathrm{CRE}^{0.1}(U) = 0.4438$, $\mathrm{CRE}^{0.5}(U) = 0.3133$, $\mathrm{CRE}^{0.9}(U) = 0.2576$ and $\mathrm{CRE}_w(U) = 0.1388$, where $\mathrm{CRE}^q(U)$ and $\mathrm{CRE}_w(U)$ are the FCRE and WCRE of the CDF $U(0,1)$, respectively. Consequently, for R_n^q, we present the percentage points of the Monte Carlo method, asymptotic normality, and beta approximation by using (10), (13), and (17), respectively. The result of this study is given in Table 1, where we note that the difference between the percentage points decreases when n increases. Besides, for R_n^q, the accuracy of the Monte Carlo method is more than the other two methods.

Figures 1–4 represent the empirical PDF's of the test statistics using Monte Carlo samples with $n = 10, 20, 30, 50, 100$. When n increases, it turned out that the test statistics are nearer to the exact values, which implies that the bias and the variance decrease with increasing n.

4. Power Analysis

In this section, we study the power test of Monte Carlo study under alternative distributions. The power of R_n^q is estimated by the proportion of the generated samples falling into the critical region. Under seven alternative distributions, the power of the test statistic R_n^q is calculated by the Monte Carlo study of generating 50,000 samples each of size n, where $n = 20, 30, 50$. The alternative CDFs proposed by Stephens [18] in power study of uniformity tests are as follows:

$$\begin{aligned}&A_l\colon F(y) = 1-(1-y)^l, \quad 0 \le y \le 1, l = 1.5, 2,\\ &B_l\colon F(y) = \begin{cases} 2^{l-1}y^l, & 0 \le y \le 0.5,\\ 1-2^{l-1}(1-y)^l, & 0.5 \le y \le 1, l = 1.5, 2, 3,\end{cases}\\ &C_l\colon F(y) = \begin{cases} 0.5-2^{l-1}(0.5-y)^l, & 0 \le y \le 0.5,\\ 0.5+2^{l-1}(y-0.5)^l, & 0.5 \le y \le 1, \text{ for } l = 1.5, 2.\end{cases}\end{aligned} \tag{21}$$

In Table 2, based on the Monte Carlo study, we recorded the power values of the proposed test statistics R_n^q, T_n^w, Kolmogorov–Smirnov (K-S), Kuiper (V), Cramer-von Mises (W^2), Watson (U^2), and Anderson-Darling (A^2), for $n = 10, 20, 30$ and $\alpha = 0.05$. From Table 2, we can conclude the following:

(1) If q increases and tends to 1 ($q \longrightarrow 1$), the power of *CRE*^{q} test, for alternative $A_l(B_l)(C_l)$, decreases (increases) (increases), and vice versa, if q decreases and tends to 0 ($q \longrightarrow 0$).

(2) If $q \longrightarrow 1$, the CRE^q test, for alternative $A_l(B_l)$, gives the worst (best) performance compared with the other tests.

(3) To compare the performance between CRE^q and CRE_w tests, we observe that:

   (a) For the alternative $A_l, q \longrightarrow 1$, CRE_w performs better than CRE^q and vice versa if $q \longrightarrow 0, n$ increases.

   (b) For the alternative $B_l, q \longrightarrow 1$, CRE^q performs better than CRE_w, and vice versa, if $q \longrightarrow 0, n$ increases.

   (c) For the alternative $C_l, q \longrightarrow 0$, CRE_w performs better than CRE^q, and vice versa, if $q \longrightarrow 1$.

   Stephens [18] noted that V and U^2 tests will reveal a change at variance. Therefore, we observe the following:

(1) For alternative $A_l, q \longrightarrow 0$, CRE^q performs better than V and U^2, and vice versa, if $q \longrightarrow 1$.

TABLE 1: Percentage points of the proposed test statistics R_n^q and T_n^w at level $\alpha = 0.05$.

n	q	R_n^q Monte Carlo method Upper	Lower	Normal approximation Upper	Lower	Beta approximation Upper	Lower	T_n^w Upper	Lower
	0.1	0.5131	0.2282	0.6165	0.1293	0.6495	0.1672		
10	0.5	0.3522	0.1818	0.4481	0.1065	0.47003	0.1315	0.1574	0.0669
	0.9	0.2964	0.14901	0.3732	0.0873	0.3916	0.1084		
	0.1	0.50608	0.3041	0.6012	0.2144	0.6217	0.2368		
20	0.5	0.3458	0.2341	0.4282	0.1632	0.4415	0.1777	0.1544	0.0957
	0.9	0.2892	0.1901	0.3549	0.1335	0.3661	0.1458		
	0.1	0.4995	0.3357	0.58407	0.2556	0.5987	0.2715		
30	0.5	0.3422	0.2544	0.4135	0.19007	0.423	0.2002	0.15275	0.1068
	0.9	0.2853	0.2066	0.34207	0.1555	0.35009	0.1641		
	0.1	0.4945	0.3541	0.5709	0.2809	0.5824	0.2931		
40	0.5	0.3393	0.2646	0.40309	0.2064	0.4104	0.2142	0.1515	0.1126
	0.9	0.2823	0.2153	0.3331	0.16902	0.3393	0.1756		
	0.1	0.4905	0.3653	0.5607	0.2983	0.5701	0.3082		
50	0.5	0.3374	0.2712	0.3953	0.2177	0.4013	0.2241	0.1506	0.1163
	0.9	0.2808	0.2207	0.3265	0.1783	0.3316	0.1837		
	0.1	0.4854	0.3801	0.5461	0.3213	0.553	0.3285		
70	0.5	0.3345	0.2796	0.3844	0.2326	0.3888	0.2372	0.1492	0.1206
	0.9	0.2776	0.2276	0.3172	0.1906	0.32103	0.1945		
	0.1	0.4799	0.3919	0.5317	0.3418	0.5367	0.3469		
100	0.5	0.3317	0.2859	0.37404	0.2459	0.3772	0.2492	0.14805	0.1242
	0.9	0.27508	0.2335	0.3085	0.2016	0.3112	0.2044		

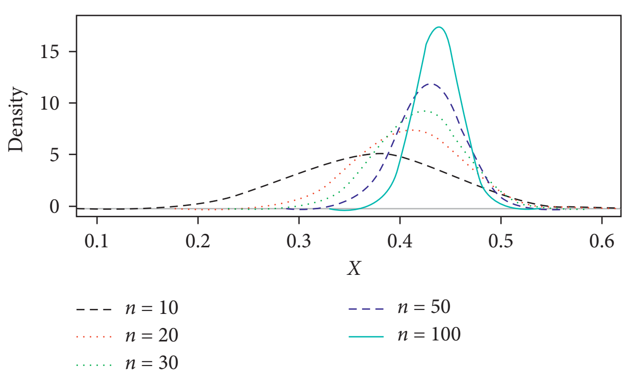

FIGURE 1: The estimated PDF's of $R_n^{0.1}$ based on U (0, 1).

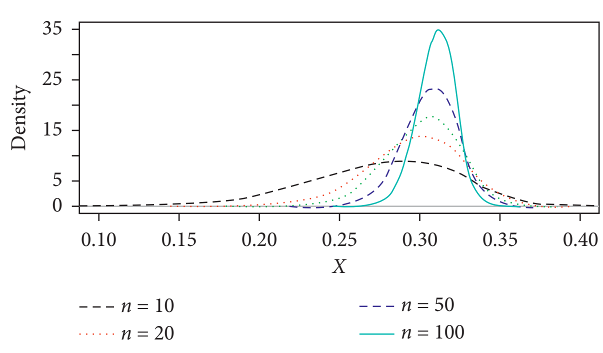

FIGURE 2: The estimated PDF's of $R_n^{0.5}$ based on U (0, 1).

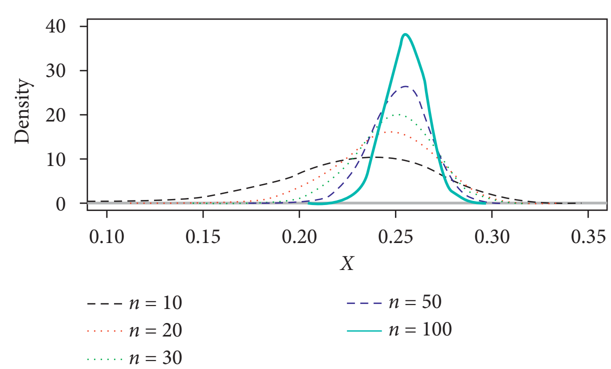

FIGURE 3: The estimated PDF's of $R_n^{0.9}$ based on U (0, 1).

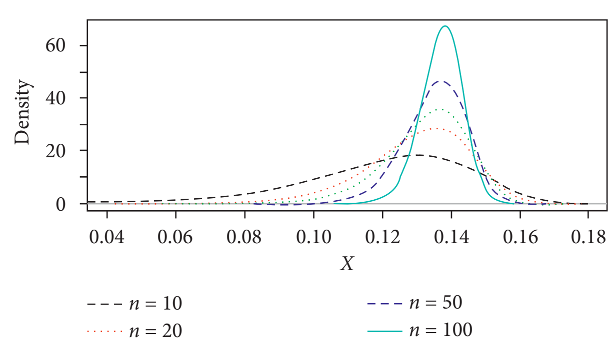

FIGURE 4: The estimated PDF's of T_n^w based on U (0, 1).

Table 2: Power estimates of the tests at level $\alpha = 0.05$.

n	Alternative	R_n^q 0.1	0.5	0.9	T_n^w	K-S	V	W^2	U^2	A^2
10	$A_{1.5}$	0.10708	0.0908	0.07208	0.14002	0.12616	0.0756	0.1456	0.07776	0.1877
	A_2	0.2771	0.2327	0.15104	0.3414	0.30298	0.1631	0.3551	0.16308	0.4761
	$B_{1.5}$	0.10406	0.1314	0.1302	0.0896	0.07352	0.0971	0.0741	0.1017	0.1349
	B_2	0.2427	0.3379	0.3357	0.21402	0.1184	0.2307	0.1104	0.2481	0.3269
	B_3	0.5763	0.7662	0.7723	0.5516	0.2424	0.5394	0.2154	0.5699	0.72308
	$C_{1.5}$	0.0843	0.119	0.1217	0.0942	0.0342	0.0974	0.0239	0.1031	0.0222
	C_2	0.1354	0.2478	0.2543	0.1723	0.0402	0.2333	0.01114	0.2475	0.00924
20	$A_{1.5}$	0.2496	0.1975	0.0909	0.2543	0.2179	0.1226	0.25208	0.1225	0.3235
	A_2	0.6679	0.5672	0.2654	0.637	0.5616	0.3486	0.6241	0.3358	0.7538
	$B_{1.5}$	0.1449	0.2828	0.2797	0.2047	0.0869	0.1634	0.0781	0.1786	0.1774
	B_2	0.38602	0.7223	0.7222	0.5511	0.1849	0.4647	0.162	0.5067	0.52802
	B_3	0.8104	0.9923	0.9931	0.954	0.4588	0.8711	0.4615	0.8978	0.93998
	$C_{1.5}$	0.0941	0.1979	0.2101	0.1516	0.0509	0.1621	0.02406	0.1791	0.0213
	C_2	0.1509	0.4551	0.4833	0.3296	0.1162	0.4633	0.0462	0.5048	0.0338
30	$A_{1.5}$	0.4045	0.3273	0.1158	0.3686	0.3144	0.18002	0.366	0.1721	0.4498
	A_2	0.8856	0.8061	0.3854	0.8285	0.7522	0.5447	0.8105	0.5071	0.8973
	$B_{1.5}$	0.1707	0.4466	0.4481	0.3331	0.1021	0.2477	0.0873	0.2667	0.2281
	B_2	0.4794	0.9148	0.9173	0.7985	0.2706	0.6695	0.25108	0.7076	0.7002
	B_3	0.9007	0.99994	0.99998	0.9983	0.6701	0.97506	0.7237	0.9819	0.99104
	$C_{1.5}$	0.1021	0.2759	0.2938	0.2084	0.07	0.2492	0.0303	0.2678	0.0271
	C_2	0.1627	0.6123	0.6513	0.4736	0.2077	0.6711	0.1258	0.7111	0.1105

(2) For the alternative $B_l, q \longrightarrow 1$, CRE^q performs better than V and U^2, and vice versa, if $q \longrightarrow 0, n$ increases.

(3) For the alternative $C_l, q \longrightarrow 0$, V and U^2 performs better than CRE^q.

(4) CRE_w performs better than V and U^2 against the alternative A_l.

(5) CRE_w performs better than V and U^2 against the alternative B_l, n increases. But, V and U^2 perform better than CRE_w against the alternative C_l.

Consequently, based on alternatives with a change toward a smaller variance, the tests CRE_w and CRE^q, $q \longrightarrow 1$, are the best. Meanwhile, under alternatives with a change toward a larger variance, the tests CRE_w and CRE^q, $q \longrightarrow 0$, are weaker.

5. Conclusion

For the CDFs with support $[0,1]$, we exhibited that the values of CRE^q and CRE_w are within $[0, e^{-q}]$ and $[0, 1/2e]$, respectively. Moreover, the test of uniformity was proposed by calculating the percentage points and power analysis of CRE^q and CRE_w. Besides, for CRE^q, we obtained the percentage points by using the Monte Carlo method via the simulation and the normality asymptotic, as well as the beta approximation. Moreover, for CRE_w the percentage points were derived by using the Monte Carlo method via the simulation. A power comparison was performed between the FCRE and WCRE and other tests, where, by changing the value of q, we indicated when the test has higher and lower power compared with the other tests.

Data Availability

The simulated data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

References

[1] M. Rao, Y. Chen, B. C. Vemuri, and F. Wang, "Cumulative residual entropy: a new measure of information," *IEEE Transactions on Information Theory*, vol. 50, no. 6, pp. 1220–1228, 2004.

[2] M. A. Abd Elgawad, M. A. Alawady, H. M. Barakat, and S. Xiong, "Concomitants of generalized order statistics from huang-kotz farlie-gumble-morgenstern bivariate distribution: some information measures," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 3, pp. 2627–2645, 2020.

[3] M. A. Abd Elgawad, H. M. Barakat, S. Xiong, and S. A. Alyami, "Information measures for generalized order statistics and their concomitants under general framework from Huang-Kotz FGM bivariate distribution," *Entropy*, vol. 23, no. 335, 2021.

[4] A. Ergun, "The multiplicity of eigenvalues of A vectorial singular diffusion equation with discontinuous conditions," *East Asian Science, Technology and Society*, vol. 6, no. 2, pp. 22–34, 2019.

[5] A. Ergün, "A half-inverse problem for the singular diffusion operator with jump conditions," *Miskolc Mathematical Notes*, vol. 21, no. 2, pp. 805–821, 2020.

[6] Z. A. Abo-Eleneen, "The entropy of progressively censored samples," *Entropy*, vol. 13, no. 2, pp. 437–449, 2011.
[7] M. A. Alawady, H. M. Barakat, and M. A. Abd Elgawad, "Concomitants of generalized order statistics from bivariate cambanis family of distributions under a general setting," *The Bulletin of the Malaysian Mathematical Society Series 2*, 2021.
[8] H. M. Barakat and I. A. Husseiny, "Some information measures in concomitants of generalized order statistics under iterated Farlie-Gumbel-Morgenstern bivariate type," *Quaestiones Mathematicae*, vol. 44, 2020.
[9] E. J. Dudewicz and E. C. Van der Meulen, "Entropy-based tests of uniformity," *Journal of the American Statistical Association*, vol. 76, no. 376, pp. 967–974, 1981.
[10] M. Mohamed, "A measure of inaccuracy in concomitants of ordered random variables under Farlie-Gumbel-Morgenstern family," *Filomat*, vol. 33, no. 15, pp. 4931–4942, 2019.
[11] M. S. Mohamed, "On cumulative residual Tsallis entropy and its dynamic version of concomitants of generalized order statistics," *Communications in Statistics-Theory and Methods*, 2020.
[12] M. Mirali, S. Baratpour, and V. Fakoor, "On weighted cumulative residual entropy," *Communications in Statistics-Theory and Methods*, vol. 46, no. 6, pp. 2857–2869, 2017.
[13] M. Mirali and S. Baratpour, "Dynamic version of weighted cumulative residual entropy," *Communications in Statistics-Theory and Methods*, vol. 46, no. 22, pp. 11047–11059, 2017.
[14] H. A. Noughabi, "Cumulative residual entropy applied to testing uniformity," *Communications in Statistics-Theory and Methods*, 2020.
[15] R. Amirov, A. Ergun, and S. Durak, "Half-inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse," *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 915–924, 2021.
[16] H. Xiong, P. Shang, and Y. Zhang, "Fractional cumulative residual entropy," *Communications in Nonlinear Science and Numerical Simulation*, vol. 78, Article ID 104879, 2019.
[17] J. Yu, W. Gui, and Y. Shan, "Statistical inference on the Shannon entropy of inverse weibull distribution under the progressive first-failure censoring," *Entropy*, vol. 21, no. 12, p. 1209, 2019.
[18] M. A. Stephens, "EDF statistics for goodness of fit and some comparisons," *Journal of the American Statistical Association*, vol. 69, no. 347, pp. 730–737, 1974.
[19] H. G. Tucker, "A generalization of the glivenko-cantelli theorem," *The Annals of Mathematical Statistics*, vol. 30, no. 3, pp. 828–830, 1959.
[20] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *A First Course in Order Statistics*, Wiley, Hoboken, NJ, USA, 1992.
[21] P. Billingsley, *Probability and Measure*, John Wiley & Sons, Hoboken, NJ, USA, 2008.
[22] B. Jóhannesson and N. Giri, "On approximations involving the beta distribution," *Communications in Statistics-Simulation and Computation*, vol. 24, no. 2, pp. 489–503, 1995.

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Research Article

Hermite-Hadamard, Jensen, and Fractional Integral Inequalities for Generalized P-Convex Stochastic Processes

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The stochastic process is one of the important branches of probability theory which deals with probabilistic models that evolve over time. It starts with probability postulates and includes a captivating arrangement of conclusions from those postulates. In probability theory, a convex function applied on the expected value of a random variable is always bounded above by the expected value of the convex function of that random variable. The purpose of this note is to introduce the class of generalized p-convex stochastic processes. Some well-known results of generalized p-convex functions such as Hermite-Hadamard, Jensen, and fractional integral inequalities are extended for generalized p-stochastic convexity.

1. Introduction

A stochastic process is a mathematical tool commonly defined as a set of random variables in various fields of probability. Verifiably, random variables were related to or listed by a lot of numbers, normally as focuses in time, giving the translation of a stochastic process, speaking to numerical estimations, some systems randomly changing over time, such as the growth of bacterial populations, fluctuations in electrical flow due to thermal noise, or the production of gas molecules. Stochastic systems are commonly used as scientific models of systems that tend to alter in an arbitrary manner. They have applications in various fields, especially in sciences, for instance, chemistry, physics, biology, neuroscience, and ecology, in addition to technology and engineering fields, for example, picture preparing, cryptography, signal processing, telecommunications, PC science, and data theory. Furthermore, apparently, arbitrary changes in money-related markets have inspired the broad utilization of stochastic processes in fund.

Convex stochastic processes and their applications have a fundamental significance in mathematics and in probability. Nikodem [1] in 1980 proposed the idea of convex stochastic processes in his article. In 1992, λ-convex and Jensen-convex stochastic processes were initiated by Skowronski [2]. More recently, Kotrys presented in [3] the results on convex stochastic processes.

For more details, refer to [4]. Many studies in the literature have been performed on some extensions of convex stochastic processes and on Hermite-Hadamard type inequalities for these extensions [5].

In the present note, we purpose to investigate the idea of generalized p-convex stochastic processes. The notion of inequality as convexity has a significant place in literature [6], as it yields a broader setting in order to investigate the mathematical programming and optimization problems. Therefore, Schur type, Hermite-Hadamard, Jensen, and fractional integral inequalities and some important results for the above said processes will be obtained in this study.

We start by definition of the stochastic process [7].

Definition 1 (see [3]). Assume a probability space $(\Omega, \mathscr{A}, P)$. A random variable is the function $\xi\colon \Omega \longrightarrow \mathbb{R}$ if ξ is $\mathscr{A}$-measurable; whereas, a stochastic process is the function $\xi\colon I \times \Omega \longrightarrow \mathbb{R}$ if $\xi(t, .)$ is a random variable for every $t \in I$.

Let us review some basic notions about stochastic processes.

Definition 2 (see [3]). The stochastic process $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$ is as follows:

(1) Continuous on I, if for all $u_\circ\in I$,

$$P-\lim_{u\longrightarrow u_\circ}\xi(u,.)=\xi(u_\circ,.),\tag{1}$$

where $P-\lim$ represents the limit in the probability;

(2) Mean-square continuous on I, if for every $u_\circ\in I$,

$$\lim_{u\longrightarrow u_\circ}\mathbb{E}(\xi(u,.)-\xi(u_\circ,.))^2=0,\tag{2}$$

Where $\mathbb{E}[\xi(u,.)]$ represents an expectation of the random variable ξ.

It is obvious in probability that if a stochastic process is mean-square continuous, then it is also continuous, but the converse is not true;

(3) Mean square is differentiable at $t\in I$, if there is a random variable $\xi'(v,.)\colon I\times\Omega\longrightarrow\mathbb{R}$, such that for every $u_\circ\in I$,

$$\xi'(u_\circ,.)=P-\lim_{u\longrightarrow u_\circ}\frac{\xi(u,.)-\xi(u_\circ,.)}{u-u_\circ}.\tag{3}$$

Definition 3 (see [3]). Consider $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$, a stochastic process with $\mathbb{E}[\xi(t,.)^2]<\infty$. We say a random variable $\nu\colon\Omega\longrightarrow\mathbb{R}$ to be the mean-square integral of the process ξ on $[a_1,a_2]$ if for each normal sequence of partitions of $[a_1,a_2]$, $a_1=u_\circ<u_1<\cdots<u_n=a_2$, and for all $\Theta_k\in[u_{k-1},u_k]$, we have

$$\lim_{n\longrightarrow\infty}\mathbb{E}\left[\left(\sum_{k=1}^{n}\xi(\Theta_k,.)(u_k-u_{k-1})-\nu(.)\right)^2\right]=0.\tag{4}$$

Then, $\nu(.)$ is written as

$$\nu(.)=\int_{a_1}^{a_2}\xi(s,.)\mathrm{d}s,\ \text{(a.e)}.\tag{5}$$

For more on mean-square integrable stochastic processes, refer [8].

Theorem 1 (see [3]). *Let us consider the Jensen-convex stochastic process $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$ that is mean-square continuous on I; then, we have*

$$\xi\left(\frac{a_1+a_2}{2},.\right)\le\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\xi(u,.)\mathrm{d}u\le\frac{\xi(a_1,.)+\xi(a_2,.)}{2},\ \text{(a.e)},\tag{6}$$

for all $a_1,a_2\in I$, $a_1<a_2$. The above inequality is Hermite-Hadamard inequality for stochastic convexity.

Let us present some important generalizations of convex stochastic processes.

Definition 4 (see [9]). A stochastic process $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$ is said to be generalized convex if for $\theta\in[0,1]$ and $u,v\in I$,

$$\xi(\theta u+(1-\theta)v,.)\le\xi(v,.)+\theta\eta(\xi(u,.),\xi(v,.)),\ \text{(a.e)}.\tag{7}$$

Definition 5 (see [10]). An interval I is a p-convex set if $[\theta u^p+(1-\theta)v^p]^{(1/p)}\in I$ for all $u,v\in I$, $\theta\in[0,1]$, and $p=2m+1$ or $p=(r/n)$, $r=2s+1$, $n=2t+1$, and $m,s,t\in N$.

Definition 6 (see [10]). A function $f\colon I\longrightarrow\mathbb{R}$ is p-convex, if for $\theta\in[0,1]$ and $u,v\in I$, we have

$$f\left([\theta u^p+(1-\theta)v^p]^{(1/p)}\right)\le\theta f(u)+(1-\theta)f(v),\tag{8}$$

where I is a p-convex set.

Remark 1 (see [11]). If $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\{0\}$, then

$$[tx^p+(1-t)y^p]^{(1/p)}\in I,\quad\text{for all } x,y\in I \text{ and } t\in[0,1].\tag{9}$$

According to Remark 1, we can give a different version of the definition of p-convex function as follows:

Definition 7 (see [11]). If $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\{0\}$. A function $f\colon I\longrightarrow\mathbb{R}$ is said to be a p-convex function if

$$f\left([tx^p+(1-t)y^p]^{(1/p)}\right)\le tf(x)+(1-t)f(y),\tag{10}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality (10) is reversed, then f is said to be p-concave.

Definition 8 (see [12]). A process $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$, where I is a p-convex set, is said to be a p-convex stochastic process if for $\theta\in[0,1]$ and $u,v\in I$, we have

$$\xi\left([\theta u^p+(1-\theta)v^p]^{(1/p)},.\right)\le\theta\xi(u,.)+(1-\theta)\xi(v,.),\ \text{(a.e)}.\tag{11}$$

In [13], the following functions are defined.

Definition 9

$$\beta(u,v)=\int_0^1\theta^{u-1}(1-\theta)^{v-1}\mathrm{d}\theta,\quad u,v>0.\tag{12}$$

Definition 10. For $w>v>0$, $|z|<1$,

$${}_2F_1(u,v;w;z)=\frac{1}{\beta(v,w-v)}\int_0^1\theta^{v-1}(1-\theta)^{w-v-1}(1-z\theta)^{-u}\mathrm{d}\theta.\tag{13}$$

Now, we are in position to define the main notion of this article.

Definition 11. A stochastic process $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$ is said to be generalized p-convex, if for $\theta\in[0,1]$ and , we have

$$\xi\left(\left[\theta u^p+(1-\theta)v^p\right]^{(1/p)},.\right)\le \xi(v,.)+\theta\eta(\xi(u,.),\xi(v,.)),\ \text{(a.e)}. \tag{14}$$

If the inequality in (14) is reversed, then ξ is the generalized p-concave.

Remark 2. It is obvious that the inequality (14) reduces to the convex stochastic process for $p=1$ and $\eta(x,y)=x-y$.

Example 1. Consider a stochastic process ξ: $(0,\infty)\times\Omega\longrightarrow\mathbb{R}$ defined by $\xi(u,.)=u^p$, $p\neq 0$, and $\eta(x,y)=x-y$; then, ξ is the generalized p-convex.

We organize our study as follows. First we derive some basic properties for this generalization. In next section, Schur type inequality is obtained. The third, fourth, and fifth sections are devoted to Hermite-Hadamard, Jensen, and fractional integral inequalities for generalized p-convex stochastic processes.

Proposition 1. *Let ξ_1,ξ_2: $I\times\Omega\longrightarrow\mathbb{R}$ be two generalized p-convex stochastic processes:*

(1) *If η is additive, then $\xi_1+\xi_2$: $I\times\Omega\longrightarrow\mathbb{R}$ is also a generalized p-convex stochastic process*

(2) *If η is nonnegatively homogeneous, then $\lambda\xi_1$: $I\times\Omega\longrightarrow\mathbb{R}$, for any $\lambda\ge 0$, is the generalized p-convex stochastic process*

The proof Proposition 1 is straightforward.

Theorem 2. *Assume a nonempty collection $\left\{\xi_j\colon I\times\Omega\longrightarrow\mathbb{R}, j\in J\right\}$ of generalized p-convex stochastic processes, such that*

(1) *There exist $\alpha\in[0,\infty]$ and $\beta\in[-1,\infty]$, such that $\eta(u,v)=\alpha u+\beta v$ for all $u,v\in\mathbb{R}$*

(2) *For each $u\in I$, $\max_{j\in J}\xi_j(u,.)$ exists in $\mathbb{R}$; then, the stochastic process defined by $\xi(u,.)=\max_{j\in J}\xi_j(u,.)$ for all $u\in I$ is the generalized p-convex.*

Proof. For any $u,v\in I$ and $\theta\in[0,1]$, we have

$$\begin{aligned}
\xi\left(\left[\theta u^p+(1-\theta)v^p\right]^{(1/p)},.\right) &= \max_{j\in J}\xi_j\left(\left[\theta u^p+(1-\theta)v^p\right]^{(1/p)},.\right)\\
&\le \max_{j\in J}\left\{\xi_j(v,.)+\theta\eta\left(\xi_j(u,.),\xi_j(v,.)\right)\right\}\\
&= \max_{j\in J}\left\{\xi_j(v,.)+\theta\left(\alpha\xi_j(u,.)+\beta\xi_j(v,.)\right)\right\}\\
&= \max_{j\in J}\left\{(1+\beta\theta)\xi_j(v,.)+\alpha\theta\xi_j(u,.)\right\}\\
&\le (1+\beta\theta)\max_{j\in J}\xi_j(v,.)+\alpha\theta\max_{j\in J}\xi_j(u,.)\\
&= (1+\beta\theta)\xi(v,.)+\alpha\theta\xi(u,.)\\
&= \xi(v,.)+\theta(\alpha\xi(u,.)+\beta\xi(v,.))\\
&= \xi(v,.)+\theta\eta(\xi(u,.),\xi(v,.)),
\end{aligned} \tag{15}$$

which is as required. □

2. Schur Type Inequality

Theorem 3. *For $I\subset(0,\infty)$ and $p>0$, let ξ: $I\times\Omega\longrightarrow\mathbb{R}$ is the generalized p-convex stochastic process. Then, $\forall u_1,u_2,u_3\in I$, such that $u_1<u_2<u_3$ and $u_3^p-u_1^p, u_3^p-u_2^p, u_2^p-u_1^p\in(0,1)$, and we have*

$$\xi(u_3,.)\left(u_3^p-u_1^p\right)-\xi(u_2,.)\left(u_3^p-u_1^p\right)+\left(u_3^p-u_2^p\right)\eta\left((\xi(u_1,.),\xi(u_3,.)\right)\ge 0,\ \text{(a.e)}. \tag{16}$$

Proof. Let $u_1,u_2,u_3\in I$ be given. Then, we can easily see that

$$\begin{aligned}
&\frac{u_3^p-u_2^p}{u_3^p-u_1^p},\frac{u_2^p-u_1^p}{u_3^p-u_1^p}\in(0,1),\\
&\frac{u_3^p-u_2^p}{u_3^p-u_1^p}+\frac{u_2^p-u_1^p}{u_3^p-u_1^p}=1.
\end{aligned} \tag{17}$$

Setting $\theta=(u_3^p-u_2^p/(u_3^p-u_1^p))$, $u_2^p=\theta u_1^p+(1-\theta)u_3^p$. As ξ is generalized p-convex, so

$$\xi(u_2,.)\le\xi(u_3,.)+\frac{u_3^p-u_2^p}{u_3^p-u_1^p}\eta\left((\xi(u_1,.),\xi(u_3,.)\right) \tag{18}$$

By assuming $u_3^p-u_1^p>0$ and multiplying the above inequality by $u_3^p-u_1^p$, we get inequality (16). □

3. Hermite-Hadamard Type Inequality

Theorem 4. *For $I\subset(0,\infty)$ and $p>0$, let a mean-square generalized p-convex stochastic process ξ: $[u_1,u_2]\times\Omega\longrightarrow\mathbb{R}$, which is integrable. Then, for any $u_1,u_2\in I$, $(u_1<u_2)$, the following inequality holds almost everywhere:*

$$\begin{aligned}
&\xi\left(\left[\frac{u_1^p+u_2^p}{2}\right]^{(1/p)},.\right)-\frac{p}{2\left(u_2^p-u_1^p\right)}\int_{u_1}^{u_2}x^{p-1}\eta\\
&\left(\left(\xi\left[u_1^p+u_2^p-x^p\right]^{(1/p)},.\right),\xi(x,.)\right)\mathrm{d}x\\
&\le\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}x^{p-1}\xi(x,.)\mathrm{d}x\\
&\le\frac{\xi(u_1,.)+\xi(u_2,.)}{2}+\frac{1}{4}\left[\eta(\xi(u_1,.),\xi(u_2,.))\right.\\
&\left.+\eta(\xi(u_2,.),\xi(u_1,.))\right].
\end{aligned} \tag{19}$$

Proof. Take $x^p=\theta u_1^p+(1-\theta)u_2^p$ and $y^p=(1-\theta)u_1^p+\theta u_2^p$; so,

$$\xi\left(\left[\frac{u_1^p+u_2^p}{2}\right]^{(1/p)},.\right)=\xi\left(\left[\frac{x^p+y^p}{2}\right]^{(1/p)},.\right). \tag{20}$$

Since ξ is the generalized p-convex, so we have

$$\begin{aligned}\xi\left(\left[\frac{u_1^p+u_2^p}{2}\right]^{(1/p)},.\right) &= \xi\left(\frac{1}{2}\left(\left[\theta u_1^p+(1-\theta)u_2^p\right]^{(1/p)},.\right)^p + \frac{1}{2}\left(\left((1-\theta)u_1^p+\theta u_1^p\right)^{(1/p)}\right)^p\right)\\ &\leq \xi\left(\left[(1-\theta)u_1^p+\theta u_2^p\right]^{(1/p)},.\right)\\ &+\frac{1}{2}\eta\left(\xi\left(\left[\theta u_1^p+(1-\theta)u_2^p\right]^{(1/p)},.\right),\right.\\ &\left.\xi\left(\left[(1-\theta)u_1^p+\theta u_2^p\right]^{(1/p)},.\right)\right).\end{aligned}\tag{21}$$

Integrating w.r.t "θ," the above inequality on [0, 1],

$$\begin{aligned}\xi\left(\left[\frac{u_1^p+u_2^p}{2}\right]^{(1/p)},.\right) &\leq \int_0^1 \xi\left(\left[(1-\theta)u_1^p+\theta u_2^p\right]^{(1/p)},.\right)\mathrm{d}\theta\\ &+\frac{1}{2}\int_0^1 \eta\left(\xi\left(\left[\theta u_1^p+(1-\theta)u_2^p\right]^{(1/p)},.\right),\right.\\ &\left.\xi\left(\left[\theta u_1^p+(1-\theta)u_2^p\right]^{(1/p)},.\right)\right)\mathrm{d}\theta,\end{aligned}\tag{22}$$

which implies

$$\begin{aligned}&\xi\left(\left[\frac{u_1^p+u_2^p}{2}\right]^{(1/p)},.\right)-\frac{p}{2\left(u_2^p-u_1^p\right)}\\ &\times\int_{u_1}^{u_2}x^{p-1}\eta\left(\xi\left(\left[u_1^p+u_2^p-x^p\right]^{(1/p)},.\right),\xi(x,.)\right)\mathrm{d}x\\ &\leq\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}x^{p-1}\xi(x,.)\mathrm{d}x.\end{aligned}\tag{23}$$

Now,

$$\begin{aligned}&\int_{u_1}^{u_2}x^{p-1}\xi(x,.)\mathrm{d}x\\ &=\frac{u_2^p-u_1^p}{p}\int_0^1\xi\left(\left[\theta u_1^p+(1-\theta)u_2^p\right]^{(1/p)},.\right)\mathrm{d}\theta\\ &\leq\frac{u_2^p-u_1^p}{p}\left(\xi(u_2,.)+\int_0^1\theta\eta(\xi(u_1,.),\xi(u_2,.))\mathrm{d}\theta\right),\end{aligned}\tag{24}$$

which implies

$$\begin{aligned}&\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}x^{p-1}\xi(x,.)\mathrm{d}x\\ &\leq\xi(u_2,.)+\int_0^1\theta\eta(\xi(u_1,.),\xi(u_2,.))\mathrm{d}\theta.\end{aligned}\tag{25}$$

Similarly,

$$\begin{aligned}&\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}x^{p-1}\xi(x,.)\mathrm{d}x\\ &\leq\xi(u_1,.)+\int_0^1\theta\eta(\xi(u_2,.),\xi(u_1,.))\mathrm{d}\theta.\end{aligned}\tag{26}$$

Adding (25) and (26),

$$\begin{aligned}&\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}x^{p-1}\xi(x,.)\mathrm{d}x\\ &\leq\frac{\xi(u_1,.)+\xi(u_2,.)}{2}\\ &+\frac{1}{4}\left[\eta(\xi(u_1,.),\xi(u_2,.))+\eta(\xi(u_1,.),\xi(u_2,.))\right].\end{aligned}\tag{27}$$

Combining (23) and (27), we obtain the inequality (22). □

Remark 3. For $p=1$ and $\eta(u,v)=u-v$ in (22), we get Hermite-Hadamard inequality (6) for the convex stochastic process.

4. Jensen Type Inequality

The following result will be helpful in the derivation of Jensen's type inequality for the generalized p-convex stochastic process.

Lemma 1. *Let $w_1,\ldots,w_n$ be the positive real numbers $(n\geq 2)$. Assume $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$ be a generalized p-convex stochastic process and $u_1,u_2,\ldots,u_n\in I$; then, we have almost everywhere*

$$\begin{aligned}&\xi\left(\left[\frac{1}{W_n}\sum_{i=1}^n w_iu_i^p\right]^{(1/p)},.\right)\\ &=\xi\left(\left[\frac{W_{n-1}}{W_n}\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}u_i^p+\frac{w_n}{W_n}u_n^p\right]^{(1/p)},.\right)\\ &\leq\xi(u_n,.)+\frac{W_{n-1}}{W_n}\eta\left(\xi\left(\left[\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}u_i^p\right]^{(1/p)},.\right),\xi(u_n,.)\right),\end{aligned}\tag{28}$$

where $W_n=\sum_{i=1}^n w_i$.

Theorem 5 (Jensen type inequality). *Let $\xi\colon I\times\Omega\longrightarrow\mathbb{R}$ be a generalized p-convex stochastic process and $\eta\colon A\times B\longrightarrow\mathbb{R}$ be nondecreasing, nonnegatively sublinear in the first variable; then, we have almost everywhere*

$$\xi\left(\left[\frac{1}{W_n}\sum_{i=1}^n w_iu_i^p\right]^{(1/p)},.\right)\leq\xi(u_n,.)+\sum_{i=1}^n\left(\frac{W_i}{W_n}\right)\eta_\xi(u_i,u_{i+1},\ldots,u_n,.),\tag{29}$$

where $W_n = \sum_{i=1}^{n} w_i$ *and* $\eta_\xi(u_i, u_{i+1}, \ldots, u_n, .) = \eta(\eta_\xi(u_i, u_{i+1}, \ldots, u_{n-1}, .), \xi(u_n, .))$ *and* $\eta_\xi(u, .) = \xi(u, .)\forall u \in I$.

Proof. Since η is nondecreasing, nonnegatively sublinear in the first variable, so using Lemma 1, we get

$$\begin{aligned}
\xi\left(\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^p\right]^{(1/p)}, .\right) &= \xi\left(\left[\frac{w_n}{W_n}u_n^p + \sum_{i=1}^{n-1}\frac{w_i}{W_n}u_i^p\right]^{(1/p)}, .\right) \\
&= \xi\left(\left[\frac{W_{n-1}}{W_n}\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}u_i^p + \frac{w_n}{W_n}u_n^p\right]^{(1/p)}, .\right) \\
&\leq \xi(u_n, .) + \frac{W_{n-1}}{W_n}\eta\left(\xi\left(\left[\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}u_i^p\right]^{(1/p)}, .\right), \xi(u_n, .)\right) \\
&= \xi(u_n, .) + \frac{W_{n-1}}{W_n}\eta\left(\xi\left(\left[\frac{W_{n-2}}{W_{n-1}}\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}u_i^p + \frac{w_{n-1}}{W_{n-1}}u_{n-1}^p\right]^{(1/p)}, .\right), \xi(u_n, .)\right) \\
&\leq \xi(u_n, .) + \frac{W_{n-1}}{W_n}\eta\left(\xi(u_{n-1}, .) + \frac{W_{n-2}}{W_{n-1}}\eta \times \left(\xi\left(\left[\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}u_i^p\right]^{(1/p)}, .\right), \xi(u_{n-1}, .)\right), \xi(u_n, .)\right) \\
&\leq \xi(u_n, .) + \frac{W_{n-1}}{W_n}\eta(\xi(u_{n-1}, .), \xi(u_n)) + \frac{W_{n-2}}{W_n}\eta \\
&\quad \times \left(\eta\left(\xi\left(\left[\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}u_i^p\right]^{(1/p)}, .\right), \xi(u_{n-1}, .)\right), \xi(u_n, .)\right) \\
&\leq \cdots \leq \xi(u_n, .) + \frac{W_{n-1}}{W_n}\eta(\xi(u_{n-1}, .), \xi(u_n, .)) + \frac{W_{n-2}}{W_n} \\
&\quad \times \eta(\eta(\xi(u_{n-2}, .), \xi(u_n, .)), \xi(u_n, .)) \\
&\quad + \cdots + \frac{W_1}{W_n}\eta(\eta(\ldots\eta(\xi(u_1, .), \xi(u_2, .)), \xi(u_3, .))\ldots)), \xi(u_{n-1}, .), \xi(u_n, .)) \\
&= \xi(u_n, .) + \frac{W_{n-1}}{W_n}\eta_\xi(u_{n-1}, u_n, .) + \frac{W_{n-2}}{W_n}\eta_\xi(u_{n-2}, u_{n-1}, u_n, .) \\
&\quad + \cdots + \frac{W_1}{W_n}\eta_\xi(u_1, u_2, \ldots, u_{n-1}, u_n, .) \\
&= \xi(u_n, .) + \sum_{i=1}^{n-1}\left(\frac{W_i}{W_n}\right)\eta_\xi(u_i, u_{i+1}, \ldots, u_n, .).
\end{aligned} \tag{30}$$

□

5. Fractional Integral Inequalities

Lemma 2 (see [12]). *Assume a stochastic process* $\xi: I \subset (0,\infty)\times\Omega \longrightarrow \mathbb{R}$ *which is mean-square differentiable on* I^o *and* $u_1, u_2 \in I^o$ *with* $u_1 < u_2$. *If* $\xi' \in L[u_1, u_2]$, *then we have almost everywhere*

$$\frac{\xi(u_1,.)+\xi(u_2,.)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2}\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x = \frac{u_2^p - u_1^p}{2p}\int_0^1 \frac{1-2\theta}{\left[\theta u_1^p + (1-\theta)d^p\right]^{1-(1/p)}} \times \xi'\left(\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)},.\right)\mathrm{d}\theta. \quad (31)$$

Theorem 6. *For* $I \subset (0,\infty)$ *and* $p > 0$ *and under the assumptions of Lemma 2 with* $|\xi'|^q$, *a generalized p-convex stochastic process on* $[u_1, u_2]$ *for* $q \geq 1$, *then we have almost everywhere*

$$\left|\frac{\xi(u_1,.)+\xi(u_2,.)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2}\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x\right| \leq \frac{u_2^p - c^p}{2p}L_1^{1-(1/q)}\left[L_1|\xi'(u_2,.)|^q + L_2\left(\eta\left(|\xi'(u_1,.)|^q, |\xi'(u_2,.)|^q\right)\right)\right]^{(1/q)}, \quad (32)$$

where

$$L_1(u_1,u_2;p) = \frac{1}{4}\left(\frac{u_1^p - u_2^p}{2}\right)^{(1/p)-1}\left[{}_2F_1\left(1-\frac{1}{p}, 2, 3: \frac{u_1^p - u_2^p}{u_1^p + u_2^p}\right) + {}_2F_1\left(1-\frac{1}{p}, 2, 3: \frac{u_2^p - u_1^p}{c^p + u_2^p}\right)\right],$$

$$L_2(u_1,u_2;p) = \frac{1}{24}\left(\frac{u_1^p + u_2^p}{2}\right)^{(1/p)-1}\left[{}_2F_1\left(1-\frac{1}{p}, 2, 4: \frac{u_1^p - u_2^p}{u_1^p + u_2^p}\right) + {}_2F_1\left(1-\frac{1}{p}, 2, 4: \frac{u_2^p - u_1^p}{u_1^p + v^p}\right)\right]. \quad (33)$$

Proof. By making use of Lemma 2 and power mean-integral inequality, we have

$$\left|\frac{\xi(u_1,.)+\xi(u_2,.)}{2} - \frac{1}{u_2^p - u_1^p}\int_{u_1}^{u_2}\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x\right| \leq \frac{u_2^p - u_1^p}{2p}\int_0^1\left|\frac{1-2\theta}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{1-(1/p)}}\right| \times \left|\xi'\left(\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)},.\right)\right|\mathrm{d}\theta \leq \frac{u_2^p - u_1^p}{2p}\left(\int_0^1\frac{|1-2\theta|}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{1-(1/p)}}\mathrm{d}t\right)^{1-(1/q)} \times \left(\int_0^1\frac{|1-2\theta|}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{1-(1/p)}}\left|\xi'\left(\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)},.\right)\right|^q\mathrm{d}\theta\right)^{(1/q)}. \quad (34)$$

Hence, by generalized p-convexity of $|\xi'|^q$ on $[u_1, u_2]$, we have

$$\left|\frac{\xi(u_1,.)+\xi(u_2,.)}{2} - \frac{1}{u_2^p - u_1^p}\int_a^b\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x\right| \leq \frac{u_2^p - u_1^p}{2p}\left(\int_0^1\frac{|1-2\theta|}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{1-(1/p)}}\mathrm{d}t\right)^{1-(1/q)} \left(\int_0^1\frac{|1-2\theta|\left[|\xi'(u_2,.)|^q + \theta\eta\left(|\xi'(u_1,.)|^q, |\xi'(u_2,.)|^q\right)\right]}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{1-(1/p)}}\mathrm{d}t\right)^{(1/q)} \leq \frac{u_2^p - u_1^p}{2p}L_1^{1-(1/q)}\left[L_1|\xi'(u_2,.)|^q + L_2\left(\eta\left(|\xi'(u_1,.)|^q, |\xi'(u_2,.)|^q\right)\right)\right]^{(1/q)}. \quad (35)$$

It is easy to check that

$$\int_0^1\frac{|1-2\theta|}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{1-(1/p)}}\mathrm{d}\theta = L_1(u_1,u_2;p), \quad \int_0^1\frac{|1-2\theta|\theta}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{1-(1/p)}}\mathrm{d}\theta = L_2(u_1,u_2;p). \quad (36)$$

□

Remark 4. By setting $\eta(u_1,u_2) = u_1 - u_2$ in (32), we get Theorem 4 of [12].

We will get the following Corollary by taking $q = 1$ in (32).

Corollary 1. *If $|\xi'|$ is generalized p-convex on $[u_1, u_2]$, then we have almost everywhere*

$$\left|\frac{\xi(u_1,.)+\xi(u_2,.)}{2}-\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\xi(x,.)x^{1-p}\mathrm{d}x\right| \leq \frac{u_2^p-u_1^p}{2p}\left[L_1|\xi'(u_2,.)|+L_2\left(\eta\left(|\xi'(u_1,.)|,|\xi'(u_2,.)|\right)\right)\right], \tag{37}$$

where L_1 and L_2 are defined in Theorem 6.

Remark 5. If we take $\eta(u_1, u_2) = u_1 - u_2$ in (37), then we have Corollary 4 of [12].

Theorem 7. *For $I \subset (0,\infty)$ and $p > 0$ and under the assumptions of Lemma 2 with $|\xi'|^q$, a generalized p-convex stochastic process on $[u_1, u_2]$ for $1 < q$, $(1/r) + (1/q) = 1$, then we have almost everywhere*

$$\left|\frac{\xi(u_1,.)+\xi(u_2,.)}{2}-\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x\right| \leq \frac{u_2^p-u_1^p}{2p}\left(\frac{1}{r+1}\right)^{(1/r)} \left[L_4|\xi'(u_2,.)|^q+L_5\left(\eta\left(|\xi'(u_1,.)|^q,|\xi'(u_2,.)|^q\right)\right)\right]^{(1/q)}, \tag{38}$$

where

$$L_3 = L_3(u_1,u_2;p;q) = \begin{cases} \dfrac{1}{u_1^{qp-q}}\cdot {}_2F_1\left(q-\dfrac{q}{p},1;2;1-\left(\dfrac{u_2}{u_1}\right)^p\right), & \text{if } p<0,\\[2ex] \dfrac{1}{u_2^{qp-q}}\cdot {}_2F_1\left(q-\dfrac{q}{p},1;2;1-\left(\dfrac{u_1}{d}\right)^p\right), & \text{if } p>0,\end{cases}$$

$$L_4 = L_4(u_1,u_2;p;q) \begin{cases} \dfrac{1}{2u_1^{qp-q}}\cdot {}_2F_1\left(q-\dfrac{q}{p},1;3;1-\left(\dfrac{u_2}{u_1}\right)^p\right), & \text{if } p<0,\\[2ex] \dfrac{1}{2u_2^{qp-q}}\cdot {}_2F_1\left(q-\dfrac{q}{p},2;3;1-\left(\dfrac{u_1}{u_2}\right)^p\right), & \text{if } p>0. \end{cases} \tag{39}$$

Proof. By making use of Lemma 2, Holder's inequality, and generalized p-convexity of $|\xi'|^q$ on $[u_1, u_2]$, we have

$$\frac{\xi(u_1,.)+\xi(u_2,.)}{2}-\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x \leq \frac{u_2^p-u_1^p}{2p}\left(\int_0^1|1-2\theta|^r\mathrm{d}t\right)^{(1/r)} \times\left(\int_0^1\frac{1}{\left[\theta u_1^p+(1-\theta)u_2^p\right]^{q-(q/p)}}\left|\xi'\left(\left[\theta u_1^p+(1-\theta)u_2^p\right]^{(1/p)},.\right)\right|^q\mathrm{d}t\right)^{(1/q)}$$
$$\leq \frac{d^p-u_1^p}{2p}\left(\frac{1}{r+1}\right)^{(1/r)} \times\left(\int_0^1\frac{|\xi'(u_2,.)|^q+\theta\eta\left(|\xi'(u_1,.)|^q,|\xi'(u_2,.)|^q\right)}{\left[\theta u_1^p+(1-\theta)u_2^p\right]^{q-(q/p)}}\mathrm{d}\theta\right)^{(1/q)}, \tag{40}$$

where an easy calculation gives

$$\int_0^1\frac{1}{\left[\theta u_1^p+(1-\theta)u_2^p\right]^{q-(q/p)}}\mathrm{d}\theta = L_3(u_1,u_2;p;q), \tag{41}$$

$$\int_0^1\frac{\theta}{\left[\theta u_1^p+(1-\theta)u_2^p\right]^{q-(q/p)}}\mathrm{d}\theta = L_4(u_1,d;p;q). \tag{42}$$

Substituting equations (41) and (42) into (38), the proof is completed. □

Remark 6. By taking $\eta(u_1, u_2) = u_1 - u_2$ in Theorem 7, then we obtain Theorem 6 of [12].

Theorem 8. *For $I \subset (0,\infty)$ and $p > 0$ and under the assumptions of Lemma 2 with $|\xi'|^q$, a generalized p-convex on $[u_1, u_2]$ for $1 < q$, $(1/r) + (1/q) = 1$, then we have almost everywhere*

$$\left|\frac{\xi(u_1,.)+\xi(u_2,.)}{2}-\frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x\right| \leq \frac{u_2^p-u_1^p}{2p}L_5^{(1/r)}\left(\frac{1}{q+1}\right)^{(1/q)} \left(|\xi'(u_2,.)|^q+\frac{1}{2}\eta\left(|\xi'(u_1,.)|^q,|\xi'(u_2,.)|^q\right)\right)^{(1/q)}, \tag{43}$$

where

$$L_5 = L_5(u_1, u_2; p; r)$$
$$= \begin{cases} \dfrac{1}{u_1^{pr-r}} \cdot {}_2F_1\left(r - \dfrac{r}{p}, 1; 2; 1 - \left(\dfrac{u_2}{u_1}\right)^p\right), & \text{if } p < 0, \\ \dfrac{1}{u_2^{pr-r}} \cdot {}_2F_1\left(r - \dfrac{r}{p}, 1; 2; 1 - \left(\dfrac{u_1}{u_2}\right)^p\right), & \text{if } p > 0. \end{cases} \quad (44)$$

Proof. From Lemma 2, Holder's inequality, and generalized p-convexity of $|\xi'|^q$ on $[u_1, u_2]$, we have

$$\left|\frac{\xi(u_1,.) + \xi(u_2,.)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2}\frac{\xi(x,.)}{x^{1-p}}\mathrm{d}x\right|$$
$$\leq \frac{u_2^p - u_1^p}{2p}\left(\int_0^1 \frac{1}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{q-(q/p)}}\mathrm{d}\theta\right)^{(1/r)}$$
$$\times\left(\int_0^1 |1-2\theta|^q \left|\xi'\left(\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)}, .\right)\right|^q \mathrm{d}\theta\right)^{(1/q)}$$
$$\leq \frac{u_2^p - u_1^p}{2p} L_6^{(1/r)}\left(\frac{1}{q+1}\right)^{(1/q)}$$
$$\left(\left|\xi'(u_2,.)\right|^q + \frac{1}{2}\left(\eta\left(\left|\xi'(u_1,.)\right|^q, \left|\xi'(u_2,.)\right|^q\right)\right)\right)^{(1/q)}, \quad (45)$$

where an easy calculation gives

$$\int_0^1 \frac{1}{\left[\theta u_1^p + (1-\theta)u_2^p\right]^{q-(r/p)}}\mathrm{d}\theta = L_5(c, u_2; p; r), \quad (46)$$

$$\int_0^1 |1-2\theta|^q \mathrm{d}\theta = \frac{1}{(q+1)}. \quad (47)$$

Substituting (46) and (47) into (43), we obtain the required result. □

6. Conclusion

There are many applications of stochastic processes, for instance, the Kolmogorov–Smirnoff test on equality of distributions. The other application includes sequential analysis and quickest detection. In this study, we have presented a new class of convex stochastic processes which are generalized p-convex and established Jensen, Hermite-Hadamard, and fractional integral inequalities for this class. Our conclusions are applicable, since the expected value of a random variable is consistently bounded above by the expected value of the convex function of that random variable. It will be interesting to find parallel results by using the proposed definition in this study in the setting of other fractional integrals [14, 15].

Data Availability

The data used to support the findings of this study are included within this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Fangfang Ma validated the results, prepared the final draft of the manuscript, and arranged the funding for this study. Waqas Nazeer proved the main results. Mamoona Ghafoor wrote the first draft of the manuscript.

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References

[1] K. Nikodem, "On convex stochastic processes," *Aequationes Mathematicae*, vol. 20, p. 184197, 1980.
[2] A. Skowronski, "On some properties of J-convex stochastic processes," *Aequationes Mathematicae*, vol. 44, p. 249258, 1992.
[3] D. Kotrys, "Hermite-Hadamard inequality for convex stochastic processes," *Aequationes Mathematicae*, vol. 83, no. 1-2, pp. 143–151, 2012.
[4] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs*, Victoria University, Footscray, Australia, 2000.
[5] A. Skowronski, "On Wright-convex stochastic processes," *Annales Mathematicae*, vol. 9, pp. 29–32, 1995.
[6] Z. B. Fang and R. Shi, "On the (p, h)-convex function and some integral inequalities," *Journal of Inequalities and Applications*, vol. 45, 2014.
[7] H. Zhou, M. S. Saleem, M. Ghafoor, and J. Li, "Generalization of-convex stochastic processes and some classical inequalities," *Mathematical Problems in Engineering*, vol. 2020, Article ID 1583807, 9 pages, 2020.
[8] K. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer, Dordrecht, 1991.
[9] C. Yong Jung, M. S. Saleem, M. Shoaib Saleem, S. Bilal, W. Nazeer, and M. Ghafoor, "Some properties of η-convex stochastic processes," *AIMS Mathematics*, vol. 6, no. 1, pp. 726–736, 2021.
[10] K. S. Zhang and J. P. Wan, "p-convex functions and their properties," *Pure and Applied Mathematics*, vol. 23, no. 1, pp. 130–133, 2007.
[11] İ. İcan, "Ostrowski type inequalities for p-convex functions," *New Trends in Mathematical Sciences*, vol. 4, no. 3, p. 140, 2016.
[12] N. Okur, I. Işcan, and E. Yuksek Dizdar, "Hermite-Hadamard type inequalities for p-convex stochastic processes," *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, vol. 9, no. 2, pp. 148–153, 2019.
[13] İ. İcan, "Hermite-Hadamard inequalities for p-convex functions," *International Journal of Analysis and Applications*, vol. 11, no. 2, pp. 137–145, 2016.

[14] S. M. Kang, G. Farid, M. Waseem, S. Ullah, W. Nazeer, and S. Mehmood, "Generalized k-fractional integral inequalities associated with (A, m) (α, m)-convex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–14, 2019.
[15] G. Hong, G. Farid, W. Nazeer et al., "Boundedness of fractional integral operators containing mittag-leffler function via exponentially-convex functions," *Journal of Mathematics*, vol. 2020, Article ID 3584105, 7 pages, 2020.

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Research Article

Composition Formulae for the k-Fractional Calculus Operator with the S-Function

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In this study, the S-function is applied to Saigo's k-fractional order integral and derivative operators involving the k-hypergeometric function in the kernel; outcomes are described in terms of the k-Wright function, which is used to represent image formulas of integral transformations such as the beta transform. Several special cases, such as the fractional calculus operator and the S-function, are also listed.

1. Introduction and Preliminaries

Fractional calculus was first introduced in 1695, but only in the last two decades have researchers been able to use it efficiently due to the availability of computing tools. Significant uses of fractional calculus have been discovered by scholars in engineering and science. In literature, many applications of fractional calculus are available in astrophysics, biosignal processing, fluid dynamics, nonlinear control theory, and stochastic dynamical system. Furthermore, research studies in the field of applied science [1, 2], and on the application of fractional calculus in real-world problems [3, 4], have recently been published. A number of researchers [5–15] have also investigated the structure, implementations, and various directions of extensions of the fractional integration and differentiation in detail. A detailed description of such fractional calculus operators, as well as their characterization and application, can be found in research monographs [16, 17].

Recently, a series of research publications with respect to generalized classical fractional calculus operators was published. Mubeen and Habibullah [18] broughtout k-fractional order integral of the Riemann–Liouville version and its applications. Dorrego [19] introduced an alternative definition for the k-Riemann–Liouville fractional derivative.

Gupta and Parihar [20] introduced the left and right sides of Saigo k-fractional integration and differentiation operators connected with the k-Gauss hypergeometric function which are as follows:

$$\left(I_{0+,k}^{\vartheta,\varsigma,\gamma} f\right)(x) = \frac{x^{(-\vartheta-\varsigma)/k}}{k\Gamma_k(\vartheta)} \int_0^x (x-t)^{(\vartheta/k)-1} {}_2F_{1,k}\left((\vartheta+\varsigma,k), (-\gamma,k); (\vartheta,k); \left(1-\frac{t}{x}\right)\right) f(t)\mathrm{d}t; \quad (\Re(\vartheta)>0, k>0), \tag{1}$$

$$\left(I_{-,k}^{\vartheta,\varsigma,\gamma} f\right)(x) = \frac{1}{k\Gamma_k(\vartheta)} \int_x^\infty (t-x)^{(\vartheta/k)-1} t^{(-\vartheta-\varsigma)/k} {}_2F_{1,k}\left((\vartheta+\varsigma,k), (-\gamma,k); (\vartheta,k); \left(1-\frac{x}{t}\right)\right) f(t)\mathrm{d}t; \quad (\Re(\vartheta)>0, k>0). \tag{2}$$

Mubeen and Habibullah [18] defined ${}_2F_{1,k}((\vartheta,k),(\varsigma,k);(\gamma,k);x)$, i.e., the k-Gauss hypergeometric function for $x\in\mathbb{C},|x|<1,\Re(\gamma)>\Re(\varsigma)>0$:

$$ {}_2F_{1,k}((\vartheta,k),(\varsigma,k);(\gamma,k);x)=\sum_{n=0}^{\infty}\frac{(\vartheta)_{n,k}(\varsigma)_{n,k}x^n}{(\gamma)_{n,k}n!}. \quad (3) $$

Equations (1) and (2) are the left and right sides of fractional differential operators involving k-Gauss hypergeometric function, respectively:

$$ \begin{aligned}\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}f\right)(x)&=\left(\frac{d}{dx}\right)^n\left(I_{0+,k}^{-\vartheta+n,-\varsigma-n,\vartheta+\gamma-n}f\right)(x);\Re(\vartheta)>0,k>0;n=[\Re(\vartheta)+1]\\&=\left(\frac{d}{dx}\right)^n\frac{x^{\vartheta+\varsigma/k}}{k\Gamma_k(-\vartheta+n)}\int_0^x(x-t)^{-\vartheta/k+n-1}\times{}_2F_{1,k}\left((-\vartheta-\varsigma,k),(-\gamma-\vartheta+n,k);(-\vartheta+n,k);\left(1-\frac{t}{x}\right)\right)f(t)dt,\end{aligned} \quad (4) $$

$$ \begin{aligned}\left(D_{-,k}^{\vartheta,\varsigma,\gamma}f\right)(x)&=\left(-\frac{d}{dx}\right)^n\left(I_{-,k}^{-\vartheta+n,-\varsigma-n,\vartheta+\gamma}f\right)(x);\Re(\vartheta)>0,k>0;n=[\Re(\vartheta)+1]\\&=\left(-\frac{d}{dx}\right)^n\frac{1}{k\Gamma_k(-\vartheta+n)}\int_x^\infty(t-x)^{-\vartheta+n/k-1}t^{\vartheta+\varsigma/k}\times{}_2F_{1,k}\left((-\vartheta-\varsigma,k),(-\gamma-\vartheta,k);(-\vartheta+n,k);\left(1-\frac{x}{t}\right)\right)f(t)dt,\end{aligned} \quad (5) $$

where $x>0,\vartheta\in\mathbb{C},\Re(\vartheta)>0,k>0$ and $[\Re(\vartheta)]$ is the integer part of $\Re(\vartheta)$.

Remark 1. When we set $k=1$ in equations, operators (1), (2), (4), and (5) reduce into Saigo's fractional integral and derivative operators, as stated in [9], respectively.

We consider the following basic results for our study.

Lemma 1 (see p. 497, equation 4.2, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C},\Re(\varepsilon)>\max[0,\Re(\varsigma-\gamma)]$*; then,*

$$ \left(I_{0+,k}^{\vartheta,\varsigma,\gamma}t^{(\varepsilon/k)-1}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon-\varsigma+\gamma)}{\Gamma_k(\varepsilon-\varsigma)\Gamma_k(\varepsilon+\vartheta+\gamma)}x^{(\varepsilon-\varsigma/k)-1}. \quad (6) $$

Lemma 2 (see p. 497, equation 4.3, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C},\Re(\vartheta)>0,k\in\Re^+(0,\infty)$ *and* $\Re(\varepsilon)>\max[\Re(-\varsigma),\Re(-\gamma)]$*; then,*

$$ \left(I_{-,k}^{\vartheta,\varsigma,\gamma}t^{-(\varepsilon/k)}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon+\varsigma)\Gamma_k(\varepsilon+\gamma)}{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon+\vartheta+\varsigma+\gamma)}x^{-\varepsilon-\varsigma/k}. \quad (7) $$

Lemma 3 (see p. 500, equation 6.2, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C},n=[\Re(\vartheta)]+1,k\in\Re^+(0,\infty)$ *such that* $\Re(\varepsilon)>\max[0,\Re(-\vartheta-\varsigma-\gamma)]$*; then,*

$$ \left(D_{0+,k}^{\vartheta,\varsigma,\gamma}t^{(\varepsilon/k)-1}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon+\varsigma+\gamma+\vartheta)}{\Gamma_k(\varepsilon+\gamma)\Gamma_k(\varepsilon+\varsigma+n-nk)}x^{(\varepsilon+\varsigma+n/k)-n-1}. \quad (8) $$

Lemma 4 (see p. 500, equation 6.3, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C}$ *and* $n=[\Re(\vartheta)]+1,k\in\Re^+,\quad\Re(\varepsilon)>\max[\Re(-\vartheta-\gamma),\Re(\varsigma-nk+n)]$*; then,*

$$ \left(D_{-,k}^{\vartheta,\varsigma,\gamma}t^{-(\varepsilon/k)}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon-\varsigma-n+nk)\Gamma_k(\varepsilon+\vartheta+\gamma)}{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon-\varsigma+\gamma)}x^{(-\varepsilon+\varsigma+n/k)-n}. \quad (9) $$

Recent time, the S-function is defined and studied by Saxena and Daiya [21], which is generalization of k-Mittag-Leffler function, K-function, M-series, Mittag-Leffler function (see [22–25]), as well as its relationships with other special functions. These special functions have recently found essential applications in solving problems in physics, biology, engineering, and applied sciences.

The S-function is defined for $\vartheta', \delta', \gamma' \varepsilon \in \mathbb{C}$, $\mathfrak{R}(\vartheta') > 0$, $k \in \mathfrak{R}$, $\mathfrak{R}(\vartheta') > k\mathfrak{R}(\varepsilon)$, $l_i\,(i = 1, 2, \ldots, p)$, $m_j\,(j = 1, 2, \ldots, q)$, and $p < q + 1$ as

$$S_{(p,q)}^{\vartheta',\delta',\gamma',\varepsilon,k}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] = \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n (\gamma')_{n\varepsilon,k}}{(m_1)_n \ldots (m_q)_n \Gamma_k(n\vartheta' + \delta')} \frac{x^n}{n!}. \tag{10}$$

Here, Díaz and Pariguan [26] introduced the k-Pochhammer symbol and k-gamma function as follows:

$$(\gamma')_{n,k} = \begin{cases} \dfrac{\Gamma_k(\gamma' + nk)}{\Gamma_k(\gamma')}, & k \in \mathfrak{R}, \gamma' \in \mathbb{C}/\{0\}, \\ \gamma'(\gamma' + k) \ldots (\gamma' + (n-1)k), & (n \in \mathbb{N}, \gamma' \in \mathbb{C}), \end{cases} \tag{11}$$

as well as the relationship with the classic Euler's gamma function:

$$\Gamma_k(\gamma') = k^{(\gamma'/k)-1}\Gamma\left(\frac{\gamma'}{k}\right), \tag{12}$$

where $\gamma' \in \mathbb{C}, k \in \mathfrak{R}$, and $n \in \mathbb{N}$. Refer to Romero and Cerutti's papers [27] for more information on the k-Pochhammer symbol, k-special functions, and fractional Fourier transforms.

The following are some significant special cases of the S-function:

(i) For $p = q = 0$, the generalized k-Mittag-Leffler function [28]

$$E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}(x) = S_{(0,0)}^{\vartheta',\delta',\gamma',\varepsilon,k}[-;-;x] = \sum_{n=0}^{\infty} \frac{(\gamma')_{n\varepsilon,k}}{\Gamma_k(n\vartheta' + \delta')} \frac{x^n}{n!}, \mathfrak{R}\left(\left(\frac{\vartheta'}{k}\right) - \varepsilon\right) > p - q. \tag{13}$$

(ii) Again, for $k = \varepsilon = 1$, the S-function is the generalized K-function [29]:

$$\begin{aligned} K_{(p,q)}^{\vartheta',\delta',\gamma'}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] &= S_{(p,q)}^{\vartheta',\delta',\gamma',1,1}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n (\gamma')_n}{(m_1)_n \ldots (m_q)_n \Gamma(n\vartheta' + \delta')} \frac{x^n}{n!}, \quad \mathfrak{R}(\vartheta') > p - q. \end{aligned} \tag{14}$$

(iii) For $\varepsilon = k = \gamma' = 1$, the S-function reduced to generalized M-series [30]:

$$\begin{aligned} M_{(p,q)}^{\vartheta',\delta'}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] &= S_{(p,q)}^{\vartheta',\delta',1,1,1}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n}{(m_1)_n \ldots (m_q)_n \Gamma(n\vartheta' + \delta')} \frac{x^n}{n!}, \quad \mathfrak{R}(\vartheta') > p - q - 1. \end{aligned} \tag{15}$$

For our purpose, we recall the definition of generalized k-Wright function ${}_p\Psi_q^k(x)$, defined by Gehlot and Prajapati [31], for $k \in \mathbb{R}^+; x, a_i, b_j \in \mathbb{C}$, $\vartheta_i, \varsigma_j \in \mathfrak{R}\,(\vartheta_i, \varsigma_j \neq 0; i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ and $(a_i + \vartheta_i n), (b_j + \varsigma_j n) \in \mathbb{C}\backslash kz^-$, as

$${}_p\Psi_q^k(x) = {}_p\Psi_q^k\left[\begin{matrix} (a_i, \vartheta_i)_{1,p} \\ \\ (b_j, \varsigma_j)_{1,q} \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + \vartheta_i n)}{\prod_{j=1}^{q} \Gamma_k(b_j + \varsigma_j n)} \frac{(x)^n}{n!}, \tag{16}$$

which satisfies the condition

$$\sum_{j=1}^{q}\frac{\varsigma_j}{k}-\sum_{i=1}^{p}\frac{\vartheta_i}{k}>-1. \tag{17}$$

2. Saigo k-Fractional Integration in Terms of k-Wright Function

In this section, the results are displayed based on the k-fractional integrals associated with the S-function.

Theorem 1. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\gamma',\varepsilon,\varepsilon\in\mathbb{C}; k\in\Re^{+}, c\in\Re$ and $v>0$, such that $\Re(\vartheta)>0$, $\Re(\varepsilon)>\max[0,\Re(\varsigma-\gamma)]$, $\Re(\varepsilon+\gamma-\varsigma)>0$, $a_i(i=1,2,\ldots,p), b_j(j=1,2,\ldots,q)$, $\Re(\vartheta')>k\Re(\varepsilon)$; $p<q+1$. If condition (17) is satisfied and $I_{0+,k}^{\vartheta,\varsigma,\gamma}$ is the left-sided integral operator of the generalized k-fractional integration associated with S-function, then (18) holds true:*

$$\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{v/k}\right)\right)\right)(x)$$
$$=\frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}k^{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i}\times{}_{p+3}\Psi_{q+3}^{k}\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon,v),(\varepsilon+\gamma-\varsigma,v),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\varepsilon-\varsigma,v),(\varepsilon+\vartheta+\gamma,v),\end{matrix}\;kcx^{v/k}\right]. \tag{18}$$

Proof. We indicate the R.H.S. of equation (18) by I_1; invoking equation (10), we have

$$\begin{aligned}I_1&=I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma')_{n\varepsilon,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'n)}\frac{\left(ct^{v/k}\right)^n}{n!}\right)(x)\\&=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma')_{n\varepsilon,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'n)}\frac{c^n}{n!}I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+vn/k)-1}\right)(x).\end{aligned} \tag{19}$$

Now, applying equation (6) and (11), we obtain

$$I_1=\frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\Gamma_k(\gamma'+n\varepsilon k)\Gamma_k(\varepsilon+vn)\Gamma_k(\varepsilon+\gamma-\varsigma+vn)}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'n)\Gamma_k(\varepsilon-\varsigma+vn)\Gamma_k(\varepsilon+\vartheta+\gamma+vn)}\frac{\left(kcx^{v/k}\right)^n}{n!}. \tag{20}$$

Using (12) and some important simplifications on the above equation, we obtain

$$\begin{aligned}I_1&=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')}k^{(b_1+\cdots+b_q)-(a_1+\cdots+a_p)}\\&\times\sum_{n=0}^{\infty}\frac{\Gamma_k(a_1k+nk)\ldots\Gamma_k(a_pk+nk)\Gamma_k(\gamma'+n\varepsilon k)\Gamma_k(\varepsilon+vn)\Gamma_k(\varepsilon+\gamma-\varsigma+vn)}{\Gamma_k(b_1k+nk)\ldots\Gamma_k(b_qk+nk)\Gamma_k(\varsigma'+\vartheta'n)\Gamma_k(\varepsilon-\varsigma+vn)\Gamma_k(\varepsilon+\vartheta+\gamma+vn)}\frac{\left(kcx^{v/k}\right)^n}{n!}.\end{aligned} \tag{21}$$

Interpreting the definition of Wright hypergeometric function (16) on the above equation, we arrive at the desired result (18). □

Theorem 2. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\gamma',\varepsilon,\varepsilon\in\mathbb{C}; k\in\Re^{+}, c\in\Re$, and $v>0$, such that $\Re(\vartheta)>0$, $\Re(\vartheta')>0$, and $\Re(\varepsilon+\vartheta)>\max[-\Re(\varsigma),-\Re(\gamma)]$, with $\Re(\varsigma)\neq\Re(\gamma)$, $a_i(i=1,2,$*

..., p), b_j ($j = 1, 2, \ldots, q$), $\Re(\vartheta') > k\Re(\varepsilon)$, and $p < q + 1$. If condition (17) is satisfied and $I_{-,k}^{\vartheta,\varsigma,\gamma}$ is the right-sided integral operator of the generalized k-fractional integration associated with S-function, then (22) holds true:

$$\left(I_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon/k}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu/k}\right)\right)\right)(x) = k^{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\frac{x^{-\vartheta-\varepsilon-\varsigma/k}}{\Gamma_k(\gamma')} \times {}_{p+3}\Psi_{q+3}^{k}\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix}\; kcx^{-\nu/k}\right]. \quad (22)$$

Proof. The proof is parallel to that of Theorem 1. Therefore, we omit the details. □

The results given in (18) and (22), being very general, can yield a large number of special cases by assigning some suitable values to the involved parameters. Now, we demonstrate some corollaries as follows.

Corollary 1. *If we put $p = q = 0$, then (18) leads to the subsequent result of S-function:*

$$\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon/k-1}E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{\nu/k}\right)\right)\right)(x) = \frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')}\times {}_3\Psi_3^{k}\left[\begin{matrix}(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),\\(\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),\end{matrix}\; kcx^{\nu/k}\right]. \quad (23)$$

Corollary 2. *If $\varepsilon = k = 1$, in (18), we obtain the subsequent result in term of S-function as*

$$\left(I_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon-1}K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\frac{x^{\varepsilon-\varsigma-1}}{\Gamma(\gamma')} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),\end{matrix}\; cx^{\nu}\right]. \quad (24)$$

Corollary 3. *If we set $\varepsilon = 1, \gamma' = 1$, and $k = 1$, in equation (18), we obtain the following formula:*

$$\left(I_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon-1}M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}x^{\varepsilon-\varsigma-1} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),(1,1),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),\end{matrix}\; cx^{\nu}\right]. \quad (25)$$

Corollary 4. *Letting $p = q = 0$ in equation (22), then*

$$\left(I_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon/k}E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{-\nu/k}\right)\right)\right)(x) = \frac{x^{-\vartheta-\varepsilon-\varsigma/k}}{\Gamma_k(\gamma')} \times {}_3\Psi_3^k\left[\begin{matrix}(\gamma',\varepsilon k),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),\\(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} kcx^{-\nu/k}\right]. \tag{26}$$

Corollary 5. *Setting $\varepsilon = 1$, $k = 1$, then equation (22) becomes*

$$\left(I_{-}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon}K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\frac{x^{-\vartheta-\varepsilon-\varsigma}}{\Gamma_k(\gamma')} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} cx^{-\nu}\right]. \tag{27}$$

Corollary 6. *If we put $\varepsilon = 1, \gamma' = 1$, and $k = 1$ in equation (22), then equation becomes*

$$\left(I_{-}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon}M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}x^{-\vartheta-\varepsilon-\delta} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),(1,1),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} cx^{-\nu}\right]. \tag{28}$$

3. Saigo k-Fractional Differentiation in Terms of k-Wright Function

In this section, the results are displayed based on the k-fractional derivatives associated with the S-function.

Theorem 3. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\gamma',\varepsilon,\varepsilon \in \mathbb{C}$; $k \in \mathfrak{R}^+$, $c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, $\mathfrak{R}(\varepsilon) > \max[0, \mathfrak{R}(-\vartheta-\varsigma-\gamma)]$, $\mathfrak{R}(\varepsilon+\gamma+\varsigma) > 0$, $a_i\,(i = 1,2,\ldots,p)$, $b_j\,(j = 1,2,\ldots,q)$, $\mathfrak{R}(\vartheta') > k\mathfrak{R}(\varepsilon)$, and $p < q+1$. If condition (17) is satisfied and $D_{0+,k}^{\vartheta,\varsigma,\gamma}$ is the left-sided differential operator of the generalized k-fractional integration associated with S-function, then (29) holds true:*

$$\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon/k-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu/k}\right)\right)\right)(x) = \frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}k^{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i} \times {}_{p+3}\Psi_{q+3}^k\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(\varepsilon+\delta,1-k+\nu),\end{matrix} cx^{(\nu+1/k)-1}\right]. \tag{29}$$

Proof. For the sake of convenience, let the left-hand side of (29) be denoted by I_2. Using definition (10), we arrive at

$$I_2 = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma')_{n\varepsilon,k}}{(b_1)_n \ldots (b_q)_n \Gamma_k(\varsigma' + \vartheta'\nu)} \frac{c^n}{n!} D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+\nu n/k)-1}\right)(x). \tag{30}$$

Now, applying equation (8) and (11), we obtain

$$I_2 = \frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n \Gamma_k(\gamma' + \varepsilon nk)}{(b_1)_n \ldots (b_q)_n \Gamma_k(\varsigma' + \vartheta' n)} \times \frac{\Gamma_k(\varepsilon + \nu n)\Gamma_k(\varepsilon + \varsigma + \gamma + \vartheta + \nu n)}{\Gamma_k(\varepsilon + \gamma + \nu n)\Gamma_k(\varepsilon + \varsigma + n - nk + \nu n) n!}\left(cx^{(\nu+1/k)-1}\right)^n. \tag{31}$$

Using (12) and simplifications on the above equation, we obtain

$$I_2 = k^{(b_1+\cdots+b_q)-(a_1+\cdots+a_p)} \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)} \frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')} \sum_{n=0}^{\infty} \frac{\Gamma_k(\gamma' + n\varepsilon k)}{\Gamma_k(\varsigma' + \vartheta' n)} \times \frac{\Gamma_k(a_1 k + nk)\ldots\Gamma_k(a_p k + nk)\Gamma_k(\varepsilon + \nu n)\Gamma_k(\varepsilon + \varsigma + \gamma + \vartheta + \nu n)}{\Gamma_k(b_1 k + nk)\ldots\Gamma_k(b_q k + nk)\Gamma_k(\varepsilon + \gamma + \nu n)\Gamma_k(\varepsilon + \varsigma + n - nk + \nu n) n!}\left(cx^{(\nu+1/k)-1}\right)^n. \tag{32}$$

In accordance with (16), we obtain the required result (29). This completed the proof of Theorem 3. □

Theorem 4. *Let $\vartheta, \varsigma, \gamma, \vartheta', \varsigma', \gamma', \varepsilon, \varepsilon \in \mathbb{C}; k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, $\mathfrak{R}(\varepsilon) > \max[\mathfrak{R}(\vartheta + \varsigma) + n - \mathfrak{R}(\gamma)]$, and $\mathfrak{R}(\vartheta + \varsigma - \gamma) + n \neq 0$, where $n = [\mathfrak{R}(\vartheta) + 1]$, $a_i\,(i = 1, 2, \ldots, p), b_j\,(j = 1, 2, \ldots, q), \mathfrak{R}(\vartheta') > k\mathfrak{R}(\varepsilon)$, and $p < q + 1$. If condition (17) is satisfied and $D_{-,k}^{\vartheta,\varsigma,\gamma}$ is the right-sided differential operator of the generalized k-fractional integration associated with S-function, then (33) holds true:*

$$\left(D-, k^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon/k} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}(a_1, \ldots, a_p; b_1, \ldots, bq; ct - \nu/k)\right)\right)(x) = \frac{x^{\vartheta-\varepsilon+\varsigma/k}}{\Gamma_k(\gamma')} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)} k^{\sum_{j=1}^q b_j - \sum_{i=1}^p a_i} \times {}_{p+3}\Psi_{q+3}^k \left[\begin{matrix} (a_1 k, k)\ldots(a_p k, k), (\gamma', \varepsilon k), (\varepsilon - \vartheta - \delta, \nu + k - 1), (\varepsilon + \gamma, \nu), \\ (b_1 k, k)\ldots(b_q k, k), (\varsigma', \vartheta'), (\varepsilon - \vartheta, \nu), (\varepsilon - \vartheta - \varsigma + \gamma, \nu), \end{matrix} cx^{(-\nu+1/k)-1} \right]. \tag{33}$$

Proof. The proof is parallel to that of Theorem 3. Therefore, we omit the details. □

The results given in (29) and (33) are reduced as special cases by assigning some suitable values to the involved parameters. Now, we demonstrate some corollaries as follows.

Corollary 7. *If $p = q = 0$, then (29) holds the following formula:*

$$\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1} E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{-\nu/k}\right)\right)\right)(x) = \frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')} \times {}_3\Psi_3^k \left[\begin{matrix} (\gamma', \varepsilon k), (\varepsilon, \nu), (\varepsilon + \varsigma + \gamma + \vartheta, \nu), \\ (\varsigma', \vartheta'), (\varepsilon + \gamma, \nu), (\varepsilon + \delta, 1 - k + \nu), \end{matrix} cx^{(\nu+1/k)-1} \right]. \tag{34}$$

Corollary 8. *If we put $\varepsilon = 1$ and $k = 1$, then (29) gives the result in term of S-function as follows:*

$$\left(D_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x)=\frac{x^{\varepsilon+\varsigma-1}}{\Gamma(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\times{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(\varepsilon+\delta,\nu),\end{matrix}cx^{\nu}\right]. \tag{35}$$

Corollary 9. *If we put $\varepsilon=1,\gamma'=1$, and $k=1$, in equation (29), then*

$$\left(D_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x)=\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}x^{\varepsilon+\varsigma-1}\times{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),(1,1),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(\varepsilon+\delta,\nu),\end{matrix}cx^{\nu}\right]. \tag{36}$$

Corollary 10. *If we set $p=q=0$, then (33) provides the result as*

$$\left(D_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon/k}E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{-\nu/k}\right)\right)\right)(x)=\frac{x^{(\vartheta-\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')}={}_{3}\Psi_{3}^{k}\left[\begin{matrix}(\gamma',k),(\varepsilon-\vartheta-\delta,\nu+k-1),(\varepsilon+\gamma,\nu),\\(\varsigma',\vartheta'),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{matrix}cx^{(-\nu+1/k)-1}\right]. \tag{37}$$

Corollary 11. *By letting $\varepsilon=1$ and $k=1$, in equation (33), then*

$$\left(D_{-}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon}K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x)=\frac{x^{\vartheta-\varepsilon+\delta}}{\Gamma(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\times{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\varepsilon-\vartheta-\varsigma,\nu),(\varepsilon+\gamma,\nu),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{matrix}cx^{-\nu}\right]. \tag{38}$$

Corollary 12. *When $\varepsilon=1,\gamma'=1$, and $k=1$, in equation (33), then equation becomes*

$$\left(D_{-}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon}M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x)=\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}x^{\vartheta-\varepsilon+\delta}\times{}_{p+3}\Psi_{q+3}\left[\begin{matrix}\left(a_1,1\right)\ldots\left(a_p,1\right),(\varepsilon-\vartheta-\delta,\nu),(\varepsilon+\gamma,\nu),(1,1),\\ \left(b_1,1\right)\ldots\left(b_q,1\right),(\varsigma',\vartheta'),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{matrix}\; cx^{-\nu}\right]. \tag{39}$$

4. Image Formulas Associated with Integral Transforms

In this section, we establish some theorems involving the results obtained in previous sections pertaining with the integral transform. Here, we defined k-beta function as follows.

The k-beta function [32] is defined as

$$B_k(g,h)=\frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1}\mathrm{d}z,\quad g>0,h>0. \tag{40}$$

They have the following important identities:

$$B_k(g,h)=\frac{1}{k}B\left(\frac{g}{k},\frac{h}{k}\right)=\frac{\Gamma_k(g)\Gamma_k(h)}{\Gamma_k(g+h)}. \tag{41}$$

Now, we define k-beta function in the form

$$B_k(f(z);g,h)=\frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1}f(z)\mathrm{d}z,\quad g>0,h>0. \tag{42}$$

Theorem 5. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\varepsilon,\gamma',\varepsilon\in\mathbb{C};k\in\mathfrak{R}^+,c\in\mathfrak{R}$, and $\nu>0$, such that $\mathfrak{R}(\vartheta)>0$, $\mathfrak{R}(\varepsilon)>\max[0,\mathfrak{R}(\varsigma-\gamma)]$, and $\mathfrak{R}(\varepsilon+\gamma-\varsigma)>0$; then, the leading fractional order integral holds true:*

$$B_k\left(\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)\right)(x);g,h\right)=\frac{x^{(\varepsilon-\varsigma/k)-1}\Gamma_k(h)}{\Gamma_k(\gamma')}\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}k^{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i}\times{}_{p+4}\Psi_{q+4}^{k}\left[\begin{matrix}\left(a_1k,k\right)\ldots\left(a_pk,k\right),(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),(g,\nu),\\ \left(b_1k,k\right)\ldots\left(b_qk,k\right),(\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),(g+h,\nu),\end{matrix}\; kcx^{\nu/k}\right]. \tag{43}$$

Proof. Let I_3 be the left-hand side of (43), and using (42), we have

$$I_3=\frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1}\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)\right)(x)\mathrm{d}z, \tag{44}$$

which, using (10) and changing the order of integration and summation, is valid under the conditions of Theorem 1 and yields

$$I_3=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\ldots\left(a_p\right)_n\left(\gamma'\right)_{n\varepsilon,k}}{\left(b_1\right)_n\ldots\left(b_q\right)_n\Gamma_k\left(\varsigma'+\vartheta'n\right)}\frac{c^n}{n!}I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+\nu n/k)-1}\right)(x)\times\frac{1}{k}\int_0^1 z^{(g+\nu n/k)-1}(1-z)^{(h/k)-1}\mathrm{d}z. \tag{45}$$

From Lemma 1 and substituting (41) in (45), we obtain

$$I_1 = k^{(b_1+\cdots+b_q)-(a_1+\cdots+a_p)} \frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')} \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)} \sum_{n=0}^{\infty} \frac{\Gamma_k(a_1k+nk)\ldots\Gamma_k(a_pk+nk)}{(b_1k+nk)\ldots\Gamma_k(b_qk+nk)}$$
$$\times \frac{\Gamma_k(\gamma'+n\varepsilon k)\Gamma_k(\varepsilon+\nu n)\Gamma_k(\varepsilon+\gamma-\varsigma+\nu n)\Gamma_k(g+\nu n)\Gamma_k(h)}{\Gamma_k\Gamma_k(\varsigma'+\vartheta' n)\Gamma_k(\varepsilon-\varsigma+\nu n)\Gamma_k(\varepsilon+\vartheta+\gamma+\nu n)\Gamma_k(g+h+\nu n)} \frac{\left(kcx^{\nu/k}\right)^n}{n!}. \tag{46}$$

Using the definition of (16) in the right-hand side of (46), we arrive at result (43). □

Theorem 6. *Let $\vartheta, \varsigma, \gamma, \vartheta', \varsigma', \gamma', \varepsilon, \varepsilon \in \mathbb{C}; k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, and $\mathfrak{R}(\varepsilon+\vartheta) > \max[-\mathfrak{R}(\varsigma), -\mathfrak{R}(\gamma)]$, with $\mathfrak{R}(\varsigma) \neq \mathfrak{R}(\gamma)$; then, the following fractional integral holds true:*

$$B_k\left(\left(I_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon/k} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{-\nu/k}\right)\right)\right)(x);g,h\right) = k^{\sum_{j=1}^q b_j - \sum_{i=1}^p a_i} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)} \frac{\Gamma_k(h) x^{-\vartheta-\varepsilon-\varsigma/k}}{\Gamma_k(\gamma')}$$
$$\times {}_{p+4}\Psi_{q+4}^k\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(g,-\nu),(\gamma',\varepsilon k),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),\\(b_1k,k)\ldots(b_qk,k),(g+h,-\nu),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix}\; kcx^{-\nu/k}\right]. \tag{47}$$

Proof. The proof is similar of Theorem 5. Therefore, we omit the details. □

Theorem 7. *Let $\vartheta, \varsigma, \gamma, \vartheta', \varsigma', \gamma', \varepsilon, \varepsilon \in \mathbb{C}; k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, $\mathfrak{R}(\varepsilon) > \max[0, \mathfrak{R}(-\vartheta-\varsigma-\gamma)]$, and $\mathfrak{R}(\varepsilon+\gamma+\varsigma) > 0$; then, the following fractional derivative holds true:*

$$B_k\left(\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)\right)(x);g,h\right) = \frac{\Gamma_k(h)}{\Gamma_k(\gamma')} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)} x^{(\varepsilon+\varsigma/k)-1} k^{\sum_{j=1}^q b_j - \sum_{i=1}^p a_i}$$
$$\times {}_{p+4}\Psi_{q+4}^k\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),(g,\nu),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(g+h,\nu),(\varepsilon+\delta,1-k+\nu),\end{matrix}\; cx^{(\nu+1/k)-1}\right]. \tag{48}$$

Proof. Let I_4 be the left-hand side of (48), and using the definition of Beta transform, we have

$$I_4 = \frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1} D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)(x)\mathrm{d}z, \tag{49}$$

which, using (10) and changing the order of integration and summation, is reasonable under the conditions of Theorem 3 and yields

$$I_4 = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n(\gamma')_{n\varepsilon,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'\nu)} \frac{c^n}{n!} D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+\nu n/k)-1}\right)(x) \times \frac{1}{k}\int_0^1 z^{(g+\nu n/k)-1}(1-z)^{(h/k)-1}\mathrm{d}z. \tag{50}$$

From Lemma 3 and substituting equation (41) in (50), we obtain

$$I_4 = k^{(b_1 - a_1)+\cdots+(b_q - a_p)} \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)} \frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')} \sum_{n=0}^{\infty} \frac{\Gamma_k(\gamma' + n\varepsilon k)\Gamma_k(a_1 k + nk)\ldots}{\Gamma_k(\varsigma' + \vartheta' n)\Gamma_k(b_1 k + nk)\ldots} \times \frac{\Gamma_k(a_p k + nk)\Gamma_k(\varepsilon + \nu n)\Gamma_k(\varepsilon + \varsigma + \gamma + \vartheta + \nu n)\Gamma_k(g + \nu n)\Gamma_k(h)}{\Gamma_k(b_q k + nk)\Gamma_k(\varepsilon + \gamma + \nu n)\Gamma_k(\varepsilon + \varsigma + n - nk + \nu n)\Gamma_k(g + h + \nu n) n!} \left(c x^{\nu + 1/k - 1}\right)^n. \tag{51}$$

Using the definition of (16) in the above equation, we obtain the required result (48). This completed the proof of Theorem 7. □

Theorem 8. *Let $\vartheta, \varsigma, \gamma, \vartheta', \varsigma', \gamma', \varepsilon, \varepsilon \in \mathbb{C}; k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, $\mathfrak{R}(\varepsilon) > \max[\mathfrak{R}(\vartheta + \varsigma) + n - \mathfrak{R}(\gamma)]$, and $\mathfrak{R}(\vartheta + \varsigma - \gamma) + n \neq 0$, where $n = [\mathfrak{R}(\vartheta) + 1]$; then, the following fractional derivative holds true:*

$$B_k\left(D_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon/k} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p; b_1,\ldots,b_q; c(zt)^{-\nu/k}\right)\right)\right)(x) = \frac{\Gamma_k(h)}{\Gamma_k(\gamma')} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} x^{(\vartheta-\varepsilon+\varsigma/k)-1} k^{\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i} \times {}_{p+4}\Psi_{q+4}^{k}\left[\begin{matrix}(a_1 k, k)\ldots(a_p k, k), (\gamma', \varepsilon k), (\varepsilon - \vartheta - \delta, \nu + k - 1), (g, -\nu)(\varepsilon + \gamma, \nu), \\ (b_1 k, k)\ldots(b_q k, k), (\varsigma', \vartheta'), (g + h, -\nu), (\varepsilon - \vartheta, \nu), (\varepsilon - \vartheta - \varsigma + \gamma, \nu),\end{matrix} c x^{(-\nu+1/k)-1}\right]. \tag{52}$$

Proof. The proof is identical to that of Theorem 7. As a result, we exclude the specifics. □

5. Conclusion

The strength of generalized k-fractional calculus operators, also known as general operators by many scholars, is that they generalize classical Riemann-Liouville (R-L) operators and Saigo's fractional calculus operators. For $k \longrightarrow 1$, operators (1) to (5) reduce to Saigo's [9] fractional integral and differentiation operators. If we set $\delta = -\vartheta$, operators (1) to (5) reduce to k-Riemann-Liouville operators as follows:

$$\begin{aligned} \left(I_{0+,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(I_{0+,k}^{\vartheta} f\right)(x), \\ \left(I_{-,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(I_{-,k}^{\vartheta} f\right)(x), \\ \left(D_{0+,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(D_{0+,k}^{\vartheta} f\right)(x), \\ \left(D_{-,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(D_{-,k}^{\vartheta} f\right)(x). \end{aligned} \tag{53}$$

On the account of the most general character of the S-function, numerous other interesting special cases of results (18), (22), (29), 2and (33) can be obtained, but for lack of space, they are not represented here.

Data Availability

No data were used to support this study.

Conflicts of Interest

There are no conflicts of interest regarding the publication of this article.

References

[1] C. F. Cheng, Y. T. Tsay, and T. T. Wu, "Walsh operational matrices for fractional calculus and their application to distributed systems," *Journal of the Franklin Institute*, vol. 303, no. 3, pp. 267–284, 1977.

[2] W. R. Schneider and W. Wyss, "Fractional diffusion and wave equation," *Journal of Mathematical Physics*, vol. 30, pp. 134–144, 1989.

[3] A. Alaria, A. M. Khan, D. L. Suthar, and D. Kumar, "Application of fractional operators in modelling for charge carrier transport in amorphous semiconductor with multiple trapping," *International Journal of Applied and Computational Mathematics*, vol. 5, no. 6, pp. 167–10, 2019.

[4] H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Q. Chen, "A new collection of real world application of fractional calculus in science and engineering," *Communications in Nonlinear Science and Numerical Simulation*, vol. 64, pp. 213–231, 2018.

[5] W. Gao, P. Veeresha, and D. G. Prakasha, "Senel bilgin, baskonus haci mehmet: iterative method applied to the fractional nonlinear systems arising in thermoelasticity with mittag-leffler kernel," *Fractals*, vol. 28, no. 8, 2020.

[6] M. Goyal, H. M. Baskonus, and A. Prakash, "Regarding new positive, bounded and convergent numerical solution of nonlinear time fractional HIV/AIDS transmission model," *Chaos, Solitons & Fractals*, vol. 139, Article ID 110096, 2020.

[7] M. M. Khader, K. M. Saad, Z. Hammouch, and D. Baleanu, "A spectral collocation method for solving fractional KdV and

KdV-Burgers equations with non-singular kernel derivatives," *Applied Numerical Mathematics*, vol. 161, pp. 137–146, 2021.

[8] V. N. Mishra, D. L. Suthar, and S. D. Purohit, "Marichev-Saigo-Maeda fractional calculus operators, Srivastava polynomials and generalized Mittag-Leffler function," *Cogent Mathematics*, vol. 4, Article ID 1320830, 2017.

[9] M. Saigo, "A remark on integral operators involving the Gauss hypergeometric functions," *Mathematical Reports College General Eduation Kyushu University*, vol. 11, pp. 135–143, 1978.

[10] M. Saigo, "A certain boundary value problem for the Euler-Darboux equation I," *Mathematical Society of Japan*, vol. 24, pp. 377–385, 1979.

[11] M. Saigo, "A certain boundary value problem for the Euler-Darboux equation, II," *Mathematical Society of Japan*, vol. 24, pp. 211–220, 1980.

[12] R. K. Saxena, J. Ram, and D. L. Suthar, "Generalized fractional calculus of the generalized Mittag-Leffler functions," *Journal of Indian Academy of Mathematics*, vol. 31, no. 1, pp. 165–172, 2009.

[13] S. Rashid, F. Jarad, and Z. Hammouch, "Some new bounds analogous to generalized proportional fractional integral operator with respect to another function," *Discrete & Continuous Dynamical Systems-S*, 2021.

[14] D. L. Suthar, M. Andualem, and B. Debalkie, "A study on generalized multivariable Mittag-Leffler function via generalized fractional calculus operators," *Journal of Mathematics*, vol. 2019, Article ID 9864737, 7 pages, 2019.

[15] D. L. Suthar, H. Habenom, and H. Hagos, "Generalized fractional calculus formulas for a product of Mittag-Leffler function and multivariable polynomials," *International Journal of Applied and Computational Mathematics*, vol. 4, no. 1, pp. 1–12, 2018.

[16] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, Chapman and Hall/CRC Press, Boca Raton, FL, USA, 2006.

[17] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, A Wiley Intersciences Publilcation, John Wiley and Sons Inc., New York, NY, USA, 1993.

[18] S. Mubeen and G. M. Habibullah, "k-Fractional integrals and application," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 2, pp. 89–94, 2012.

[19] G. A. Dorrego, "An alternative definition for the k-Riemann-Liouville fractional derivative," *Applied Mathematical Sciences*, vol. 9, no. 10, pp. 481–491, 2015.

[20] A. Gupta and C. L. Parihar, "Saigo's k-Fractional calculus operators," *Malaya Journal of Matematik*, vol. 5, no. 3, pp. 494–504, 2017.

[21] R. K. Saxena and J. Daiya, "Integral transforms of the S-functions," *Le-Mathematiche*, vol. 70, pp. 147–159, 2015.

[22] S. D. Purohit, S. L. Kalla, and D. L. Suthar, "Fractional integral operators and the multiindex Mittag-Leffler functions," *Scientia, Series A: Mathematical Sciences*, vol. 21, pp. 87–96, 2011.

[23] S. D. Purohit, D. L. Suthar, and S. L. Kalla, "Some results on fractional calculus operators associated with the M-function," *Hadronic Journal*, vol. 33, pp. 225–236, 2010.

[24] D. L. Suthar and H. Amsalu, "Fractional integral and derivative formulas by using Marichev-Saigo-Maeda operators involving the S-function," *Abstract and Applied Analysis*, vol. 2019, Article ID 6487687, 19 pages, 2019.

[25] D. L. Suthar and H. Habenom, "Certain generalized fractional integral formulas involving the product of K-function and the general class of multivariable polynomials," *Communications in Numerical Analysis*, vol. 2017, no. 2, pp. 101–108, 2017.

[26] R. Diaz and E. Pariguan, "On hypergeometric functions and Pochhammer *ki*-symbol," *Divulgaciones Matemáticas*, vol. 15, no. 2, pp. 179–192, 2007.

[27] L. Romero and R. Cerutti, "Fractional fourier transform and special k-function," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 4, pp. 693–704, 2012.

[28] R. K. Saxena, J. Daiya, and A. Singh, "Integral transforms of the k-generalized Mittag-Leffler function," *Le-Matematiche*, vol. 69, no. 2, pp. 7–16, 2014.

[29] K. Sharma, "Application of fractional calculus operators to related areas," *General Mathematics Notes*, vol. 7, no. 1, pp. 33–40, 2011.

[30] K. Sharma and R. Jain, "A note on a generalized M-series as a special function of fractional calculus," *FCAA*, vol. 12, no. 4, pp. 449–452, 2009.

[31] K. S. Gehlot and J. C. Prajapati, "Fractional calculus of generalized k-Wright function," *Journal of Fractional Calculus and Applications*, vol. 4, no. 2, pp. 283–289, 2013.

[32] R. Diaz and C. Teruel, "q, k generalized gamma and beta functions," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 1, pp. 118–134, 2005.

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Research Article

Numerical Solution of Fractional Order Anomalous Subdiffusion Problems Using Radial Kernels and Transform

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By coupling of radial kernels and localized Laplace transform, a numerical scheme for the approximation of time fractional anomalous subdiffusion problems is presented. The fractional order operators are well suited to handle by Laplace transform and radial kernels are also built for high dimensions. The numerical computations of inverse Laplace transform are carried out by contour integration technique. The computation can be done in parallel and no time sensitivity is involved in approximating the time fractional operator as contrary to finite differences. The proposed numerical scheme is stable and accurate.

1. Introduction

In the last decades, many researchers have studied the fractional calculus [1–3]. Differential equations of fractional order have many applications in the field of science and engineering [4–7]. Analytical solution of many fractional differential equations is not possible or very hard to find, so we need a new numerical technique to find its approximate solution. Various phenomena in viscoelastic materials, economics, chemistry, finance, control theory, hydrology, physics, cosmology, solid mechanics, bioengineering, statistical mechanics, and control theory can be mathematically modeled from fractional calculus [8–17]. In literature, various numerical approaches are available for modeling anomalous diffusive behavior such as Carlo simulations [18]. An introduction of diffusion equations can be found in [19–21].

Recently, RBF-based methods were used in solving fractional partial differential equations (FPDEs) [22–24]. These methods have been employed in approximation of partial differential equations with complex domains. An implicit meshless technique based on the radial basis functions for the numerical simulation of the anomalous subdiffusion equation can be found in [25]. The convergence and stability of these mesh-free methods can be found in [26, 27]. These globally defined RBF methods cause ill-condition system matrices [28]. To overcome the problem of ill-conditioning, local RBF techniques were used in [29–31]. Unlike global RBF methods, the RBF method in local setting uses center points in each subdomain area of influence, surrounding each spatial point due to which there is reduction in the computational cost.

Recently, Laplace transform is combined with RBF method in [32, 33]. In [34–37], the authors use Laplace transform as tool in spectral method and other mesh-based methods such as finite element methods and finite difference method. To avoid the issues of computational efficiency and instability of the system matrix, we introduce a new technique Laplace transform-based local RBF method in solving the time fractional modified anomalous subdiffusion equations in irregular domain.

Here, we consider the following modified anomalous subdiffusion equation of fractional order [38]:

$$\frac{\partial w(\mathbf{x},t)}{\partial t} = \left[\nu_1 D_t^{(1-\alpha)} + \nu_2 D_t^{(1-\beta)}\right]\Delta w(\mathbf{x},t) + f(\mathbf{x},t), \quad (1)$$

where $\mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^d$, $d \geq 1$, $t > 0$, subject to the following boundary and initial conditions:

$$\mathscr{B}w(\mathbf{x}, t) = h(\mathbf{x}, t), \quad \mathbf{x} = (x, y) \in \partial\Omega, \tag{2}$$

$$w(\mathbf{x}, 0) = w_0, \quad \mathbf{x} \in \Omega, \tag{3}$$

respectively, where $\alpha, \beta \in (0,1)$, $t \in [0,T]$, ν_1, ν_2 are positive constants, Δ is the Laplace operator, and $f(\mathbf{x}, t)$ is some given function.

2. Preliminaries

Here, we introduce some fundamental definitions related to fractional calculus [39, 40].

Definition 1. Let $n-1 < \alpha < n \in Z^+$ and $\alpha > 0$, then the Caputo derivative of fractional order is defined as

$$D_t^\alpha w(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-z)^{\alpha+1-n}} \frac{\mathrm{d}^n}{\mathrm{d}z^n} w(z)\mathrm{d}z. \tag{4}$$

Definition 2. Let $w(t)$, $t \geq 0$, be a given function, then its Laplace transform is defined by

$$\widehat{w}(z) = \mathscr{L}\{w(t)\} = \int_0^\infty e^{-zt} w(t)\mathrm{d}t, \tag{5}$$

provided this integral converges.

Lemma 1. *If $w(t) \in C^p[0,\infty)$, with $\alpha \in (n-1, n) \in Z^+$, then the Laplace transform of the fractional order Caputo derivative is given by*

$$\mathscr{L}\{D_t^\alpha w(t)\}(z) = z^\alpha \widehat{w} - \sum_{i=0}^{n-1} z^{\alpha-i-1} w^{(i)}(0). \tag{6}$$

Theorem 1. *the Bromwich inversion theorem [41]). Let w(t) have a continuous derivative and let $|w(t)| < Ke^{\gamma t}$, where K and γ are positive constants. Define*

$$\widehat{w}(z) = \int_0^\infty e^{-zt} w(t)\mathrm{d}t, \quad \mathrm{Re}(z) > \gamma, \tag{7}$$

then

$$w(t) = \frac{1}{2\pi i} \int_{\xi-i\infty}^{\xi+i\infty} \widehat{w}(z) e^{zt}\mathrm{d}z. \tag{8}$$

3. Description of the Method

3.1. Time Discretization. Here, we apply Laplace transform to models (1)–(3) which gives

$$\begin{aligned} &\left[\left(zI - \left(\upsilon_1 z^{1-\alpha} + \upsilon_2 z^{1-\beta}\right)\right)\right]\widehat{w}(\mathbf{x}, z) = w(\mathbf{x}, 0) \\ &-\left(\nu_1 z^{-\alpha} + \nu_2 z^{-\beta}\right)\Delta w(\mathbf{x}, 0) + \widehat{f}(\mathbf{x}, z), \quad \mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^d, \\ &\mathscr{B}(\widehat{w}(\mathbf{x}, z)) = \widehat{h}(z), \quad \mathbf{x} = (x, y) \in \partial\Omega. \end{aligned} \tag{9}$$

In more compact form, we have

$$\mathscr{L}(\widehat{w}(\mathbf{x}, z)) = \widehat{g}(\mathbf{x}, z), \quad \mathbf{x} \in \Omega, \tag{10}$$

$$\mathscr{B}(\widehat{w}(\mathbf{x}, z)) = \widehat{h}(\mathbf{x}, z), \quad \mathbf{x} \in \partial\Omega. \tag{11}$$

The transformed problems (10) and (11) will be solved for the solution $\widehat{w}(\mathbf{x}, z)$ using local RBF method. The solution $w(\mathbf{x}, t)$ of the given models (1)–(3) will be found by using numerical inversion.

3.2. Local Radial Basis Functions Method. Here, the linear operators $\mathscr{B}$ and $\mathscr{L}$ are discretized by using local RBF [42, 43]. Consider the centers $\{x_i, i = 1\ldots, N\} \subset \Omega \subset \mathbb{R}^d, d \geq 1$, where Ω is the bounded domain. For each point $x_i, i = 1, 2, 3, \ldots, N$, we can find a subdomain Ω_j such that $n < N$. The unknown function $\widehat{w}(\mathbf{x}, t)$ can be approximated with RBF in each local subdomain Ω_i, $i = 1, 2, \ldots, N$, by the following equation:

$$w(\mathbf{x}_i, t) \approx \widehat{w}(\mathbf{x}_i, t) = \sum_{j=1}^n \lambda_{i_j}^i \phi^i\left(\left\|\mathbf{x}_i - x_j\right\|\right), \quad x_j \in \Omega_j, \tag{12}$$

where $\lambda^i = [\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i]$ are the unknown coefficients, and $r_{ij} = \|\mathbf{x}_i - x_j\|$ is the norm between nodes $\mathbf{x}_i$ and x_j, $\phi(r)$, $r \geq 0$ is a radial kernel (multiquadric radial basis function), and $\Omega_j \subset \Omega$ is a local domain for around each $\mathbf{x}_i$, containing n neighboring nodes around the node $\mathbf{x}_i$. So, we have N small size linear systems each of order $n \times n$ given by

$$\begin{pmatrix} \widehat{w}_1^i \\ \widehat{w}_2^i \\ \vdots \\ \widehat{w}_n^i \end{pmatrix} = \begin{pmatrix} \phi_{11}^i & \phi_{12}^i & \cdots & \phi_{1n}^i \\ \phi_{21}^i & \phi_{22}^i & \ldots & \phi_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1}^i & \phi_{n2}^i & \ldots & \phi_{nn}^i \end{pmatrix} \begin{pmatrix} \lambda_1^i \\ \lambda_2^i \\ \vdots \\ \lambda_n^i \end{pmatrix}, \quad i = 1, 2, \ldots, N, \tag{13}$$

which can be denoted by

$$\widehat{\mathbf{W}}^i = \mathbf{S}^i \lambda^i, \quad i = 1, 2, \ldots, N, \tag{14}$$

where $\phi_{jk}^i = \phi^i(\|x_{i_j} - x_{i_k}\|), x_{i_j}, x_{i_k} \in \Omega_i$, and matrix $\mathbf{S}^i$ is the system matrix.

Now, applying the operator $\mathscr{L}$ to (12) gives

$$\mathscr{L}\widehat{w}(x_i) = \sum_{j=1}^n \lambda_{i_j}^i \mathscr{L}\phi^i\left(\left\|x_i - x_{i_j}\right\|\right), \quad x_{i_j} \in \Omega_i. \tag{15}$$

The vector form of (15) is given by

$$\mathscr{L}\widehat{w}(x_i) = \mathbf{G}^i \cdot \lambda^i, \tag{16}$$

where $\mathbf{G}^i$ is given by

$$\mathbf{G}^i = \mathscr{L}\phi^i\left(\left\|x_i - x_{i_j}\right\|\right), \quad x_i, x_{i_j} \in \Omega_i. \tag{17}$$

From equation (14), the unknown coefficients λ^i are given by

$$\lambda^i = \left(\mathbf{S}^i\right)^{-1}\widehat{\mathbf{W}}^i, \tag{18}$$

and by inserting the values of λ^i in (16), we have

$$\mathscr{L}\widehat{w}(x_i) = \mathbf{G}^i\left(\mathbf{S}^i\right)^{-1}\widehat{\mathbf{W}}^i = \mathbf{N}^i\widehat{\mathbf{W}}^i, \tag{19}$$

where

$$\mathbf{N}^i = \mathbf{G}^i\left(\mathbf{S}^i\right)^{-1}. \tag{20}$$

Hence, the discretized form is given by

$$\mathscr{L}\widehat{w} \equiv \mathbf{H}\widehat{\mathbf{W}}, \tag{21}$$

where matrix $\mathbf{H}$ is called the sparse differentiation matrix of order $N \times N$.

4. Numerical Inversion Technique

In this section, the numerical inversion of Laplace transform for approximating the given models (1)–(3) is as follows:

$$\begin{aligned} w(\mathbf{x}, t) &= \frac{1}{2\pi\imath}\int_{\tilde{\xi}-\imath\infty}^{\tilde{\xi}+\imath\infty} \widehat{w}(\mathbf{x}, z)e^{zt}\mathrm{d}z \\ &= \frac{1}{2\pi\imath}\int_{\Psi} e^{zt}\widehat{w}(\mathbf{x}, z)\mathrm{d}z, \quad \tilde{\xi} > \tilde{\xi}_0, \end{aligned} \tag{22}$$

where Ψ is the suitable path joining $\tilde{\xi} - \imath\infty$ to $\tilde{\xi} + \imath\infty$. This Bromwich integral is numerically solved by using the following hyperbolic contour [37]:

$$z(\tilde{\eta}) = \omega + \tilde{\lambda}(1 - \sin(\tilde{\sigma} - \imath\tilde{\eta})), \quad \text{for } \tilde{\eta} \in R, \tag{23}$$

with $\tilde{\lambda} > 0$, $\omega \geq 0$, $0 < \tilde{\sigma} < \tilde{\beta}(1/2)\pi$, and $(1/2)\pi < \tilde{\beta} < \pi$.

Integral in (22) gives

$$w(\mathbf{x}, t) = \frac{1}{2\pi\imath}\int_{-\infty}^{\infty} e^{z(\tilde{\eta})t}\widehat{w}(\mathbf{x}, z(\tilde{\eta}))z'(\tilde{\eta})\mathrm{d}\tilde{\eta}. \tag{24}$$

Next applying trapezoidal rule for approximation of (24), we have

$$w_k(\mathbf{x}, t) = \frac{k}{2\pi\imath}\sum_{j=-M}^{M} \widehat{w}\left(\mathbf{x}, z_j\right)e^{z_j t}z_j', \quad z_j = z\left(\tilde{\eta}_j\right), \tilde{\eta}_j = jk, \tag{25}$$

where k is the step size.

5. Application of the Method

In this section, the proposed numerical scheme is applied to multidimensional problems. We solved four test problems and used various domain points $N \in \Omega$, stencils points $n \in \Omega_j$, and quadrature points M. Three error formulas, the error estimate, $L_{\text{est}} = e^{(-cM/\log(M))}$, L_∞, and L_2 norms are used. The radial kernel used in our computations is $\phi(r, \varepsilon) = \sqrt{1 + \varepsilon^2 r^2}$. The shape parameter ϵ is optimized by the uncertainty rule related to RBFs.

Problem 1. Consider models (1)–(3) to the following form [38]:

$$\begin{aligned} \frac{\partial w(x, t)}{\partial t} &= \left(D_{0t}^{1-\alpha} + D_{0t}^{1-\beta}\right)\left[\frac{\partial^2 w(x, t)}{\partial x^2}\right] + f(x, t), \\ f(x, t) &= \exp(x)\left[(1 + \alpha)t^\alpha - \frac{\Gamma(2 + \alpha + \beta)}{\Gamma(1 + 2\alpha + \beta)}t^{2\alpha+\beta} - \frac{\Gamma(2 + \alpha + \beta)}{\Gamma(1 + \alpha + 2\beta)}t^{\alpha+2\beta}\right], \end{aligned} \tag{26}$$

with the following boundary and initial conditions:

$$\begin{aligned} w(0, t)) &= t^{(1+\alpha+\beta)}, \\ w(1, t) &= et^{(1+\alpha+\beta)}, \quad 0 < t \leq T, \\ w(x, 0) &= 0, \quad x \in (0, 1), \end{aligned} \tag{27}$$

respectively, where the actual solution is given by

$$w(x, t) = \exp(x)t^{1+\alpha+\beta}. \tag{28}$$

In our numerical scheme, we used the hyperbolic contour (23). The optimal parameter values are taken as

$$\begin{aligned} \lambda &= \frac{\theta r_b N}{bT}, \\ b &= \cosh^{-1}\left(\frac{1}{\theta\tau\sin(\sigma)}\right), \\ \omega &= 2, \\ \sigma &= 0.3812, \\ x_k &= hk, \end{aligned} \tag{29}$$

$r_b = 2\pi r$, $r = 0.3431$, $h = b/N$, $\tau = t_0/T$, $t_0 = 0.5$, $t = 1$, and $T = 5$. This test problem is solved in the domain $(0, 1)$. Here, the number of points in domain Ω is denoted by N, the points in local subdomain $\partial\Omega_j$ are denoted by n, and the number of quadrature points relates to M. The numerical solutions are shown in Table 1 with various values of fractional order α and β and nodal points N. For comparatively smaller values of fractional order α and β, better results in terms of L_∞ and L_2 error norms are obtained. In the upper part of Table 1, condition number increases, as we increase nodal points N. Error versus various quadrature points M at $N = 21, n = 9$, and $t = 1$ and various values of α and β are shown in Figure 1. The error estimate L_{est} for $c = 1$ is well matched with L_∞ and L_2 error norms, as shown in Figure 1. Hence, our proposed method is stable and accurate.

Table 1: Numerical results using the proposed numerical scheme corresponding to Problem 1.

$M = 50,\ n = 7$	$\alpha = 0.2,\ \beta = 0.1$		
N	L_∞	L_2	κ
11	8.9603e − 004	0.0021	2.2091e + 021
21	8.5582e − 004	0.0028	9.6329e + 021
31	8.4544e − 004	0.0034	2.2821e + 022
41	8.8689e − 004	0.0041	4.4923e + 022
51	8.6646e − 004	0.0045	7.5954e + 022
71	9.2962e − 004	0.0057	1.6860e + 023
$M = 50,\ n = 7, N = 21$	L_∞	L_2	κ
$(\alpha,\beta) = (0.2, 0.6)$	0.0141	0.0462	1.0025e + 021
$(\alpha,\beta) = (0.2, 0.4)$	0.0072	0.0236	3.6776e + 020
$(\alpha,\beta) = (0.2, 0.1)$	8.5582e − 004	0.0028	9.6329e + 021
$(\alpha,\beta) = (0.6, 0.3)$	0.0079	0.0258	3.1036e + 020
$(\alpha,\beta) = (0.4, 0.3)$	0.0063	0.0207	3.1036e + 020
$(\alpha,\beta) = (0.1, 0.3)$	0.0035	0.0116	9.2023e + 021

Problem 2. Consider models (1)–(3) corresponding to the form [38]

$$\frac{\partial w(x,t)}{\partial t} = \frac{1}{2}\left(\frac{\partial^{1-\alpha} w(x,t)}{\partial t^{1-\alpha}} + \frac{\partial^{1-\beta} w(x,t)}{\partial t^{1-\beta}}\right)\left[\frac{\partial^2 w(x,t)}{\partial x^2}\right] + f(x,t),$$
$$f(x,t) = \exp(x)\left[(1+\alpha)t^{\alpha} - \frac{\Gamma(2+\alpha)}{\Gamma(1+2\alpha)}t^{2\alpha} + (1+\beta)t^{\beta} - \frac{\Gamma(2+\beta)}{\Gamma(1+2\beta)}t^{2\beta}\right], \tag{30}$$

initial and boundary conditions given by

$$\begin{aligned} &w(x,0) = 0, \quad x \in (0,1),\\ &w(0,t) = t^{(1+\alpha)} + t^{(1+\beta)},\\ &w(1,t) = \exp(1)\left(t^{(1+\alpha)} + t^{(1+\beta)}\right), \quad 0 < t \le 1. \end{aligned} \tag{31}$$

The actual solution is

$$w(x,t) = \exp(x)\left(t^{(1+\alpha)} + t^{(1+\beta)}\right). \tag{32}$$

The same domain and same parameter values as used in Problem 1 are incorporated. The numerical results are shown in Table 2 with the same as well as with various values of fractional order α and β and nodal points N. For comparatively identical values of fractional order α and β, better results in terms of L_∞ and L_2 error norms are obtained. In the upper part of Table 2, condition number of the system matrix is fixed for $11 \le N \le 71$. Error versus various quadrature points M at $N = 41, n = 9$, and $t = 1$ and various values of α and β are depicted in Figure 2. The error estimate L_{est} for $c = 0.7$ is well agreed with L_∞ and L_2error norms, as shown in Figure 1. The results obtained by our proposed numerical scheme are comparatively identical with the results in Table 2 [38].

Problem 3. Next, we consider models (1)–(3) corresponding to the form [44]

$$\frac{\partial w(x,y,t)}{\partial t} = \left(D_{0t}^{(1-\alpha)} + D_{0t}^{(1-\beta)}\right)\left[\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right] + f(x,t), \tag{33}$$

where

$$f(x,y,t) = 2t\sin(2\pi x)\sin(2\pi y)\left(1 + \frac{8\pi.^2}{\Gamma(2+\alpha)}t^{\alpha} + \frac{8\pi.^2}{\Gamma(2+\beta)}t^{\beta}\right), \tag{34}$$

initial and boundary conditions given by

$$\begin{aligned} &w(x,y,0) = 0, \quad x, y \in \Omega,\\ &w(0,t) = 0,\\ &w(1,t) = 0, \quad t > 0. \end{aligned} \tag{35}$$

The exact solution is

$$w(x,y,t) = t^2\sin(2\pi x)\sin(2\pi y). \tag{36}$$

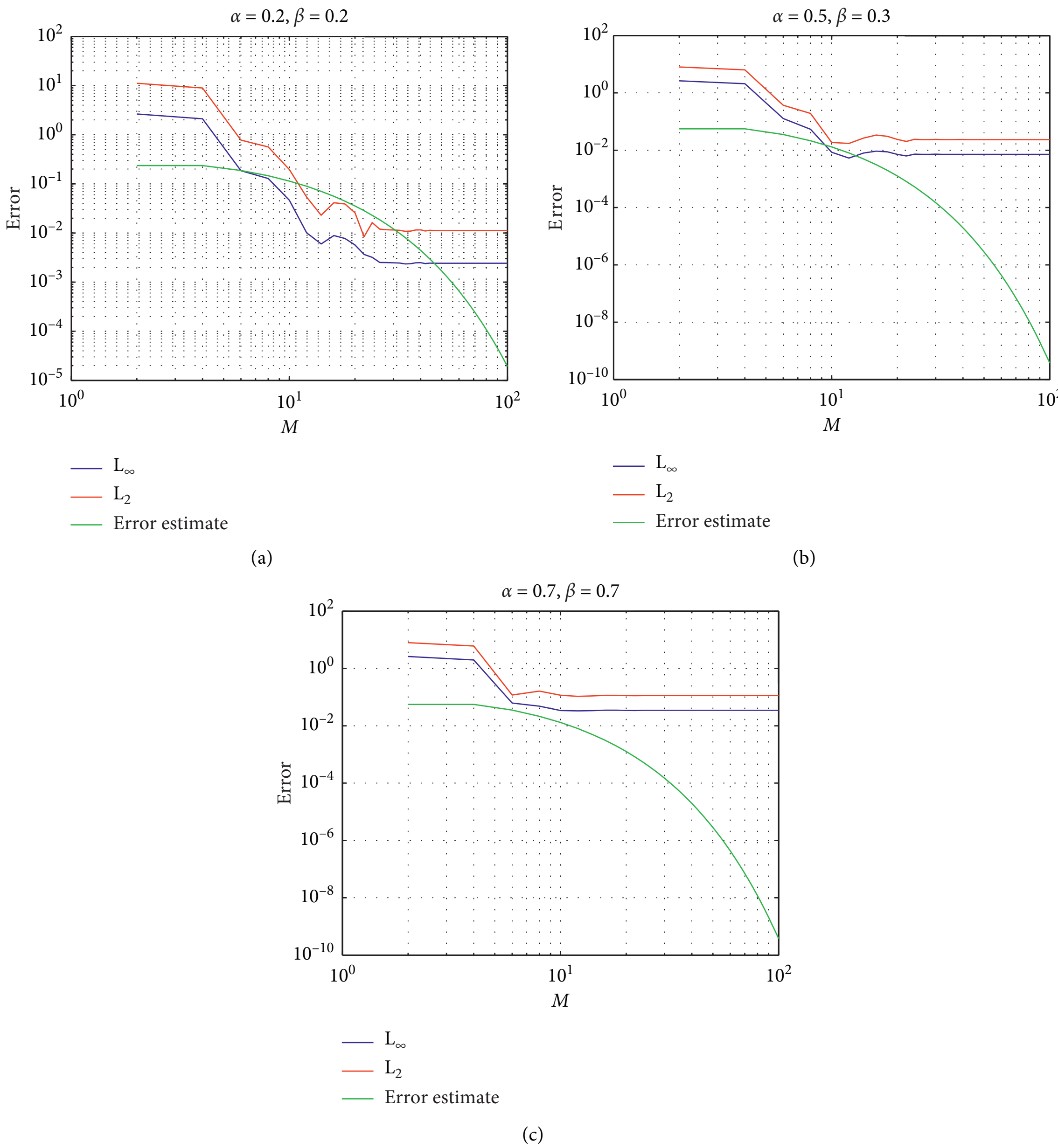


FIGURE 1: Error versus quadrature points M at $t = 1$ and various values of α, β corresponding to Problem 1.

This problem is solved over the domain $\Omega = [0, 1]$. In Table 3, for various nodal points N and stencils points $n = 11, 15$ and with various values of α and β, the L_∞ error norm is well matched with L_2 error norm. The condition number is increasing steadily as we decrease both the values of α and β at the same time.

Problem 4. Finally, we consider models (1)–(3) corresponding to the form [38]

$$\begin{aligned} \frac{\partial w(x, y, t)}{\partial t} &= \left(D_{0t}^{(1-\alpha)} + D_{0t}^{(1-\beta)}\right)\left[\frac{\partial^2 w(x, y, t)}{\partial x^2} + \frac{\partial^2 w(x, y, t)}{\partial y^2}\right] + f(x, t), \quad 0 < t \leq 1, 0 < x, y < 1, \\ f(x, y, t) &= \exp\left(-\frac{(x-0.5)^2}{\gamma} - \frac{(y-0.5)^2}{\gamma}\right)\left[f_1(x, y, t) + f_2(x, y, t)\right], \end{aligned} \tag{37}$$

Table 2: Numerical results using the proposed numerical scheme corresponding to Problem 2.

$M = 80$, $n = 9$		$\alpha = 0.5$, $\beta = 0.5$	
N	L_∞	L_2	κ
11	5.4148e − 005	1.2209e − 004	5.0986e + 033
21	1.3432e − 004	2.7509e − 004	5.0985e + 033
31	1.1372e − 004	3.2761e − 004	5.0985e + 033
41	8.6091e − 005	3.1340e − 004	5.0985e + 033
51	1.2453e − 004	5.3029e − 004	5.0985e + 033
71	4.5906e − 005	2.1446e − 004	5.0985e + 033
$M = 80$, $n = 9$, $N = 41$	L_∞	L_2	κ
$(\alpha, \beta) = (0.2, 0.6)$	0.0072	0.0331	5.1040e + 033
$(\alpha, \beta) = (0.2, 0.4)$	0.0019	0.0089	5.1040e + 033
$(\alpha, \beta) = (0.2, 0.1)$	5.1121e − 004	0.0023	2.6797e + 035
$(\alpha, \beta) = (0.6, 0.3)$	0.0039	0.0179	5.0985e + 033
$(\alpha, \beta) = (0.4, 0.3)$	4.3443e − 004	0.0019	5.0985e + 033
$(\alpha, \beta) = (0.1, 0.3)$	0.0021	0.0096	2.1983e + 034

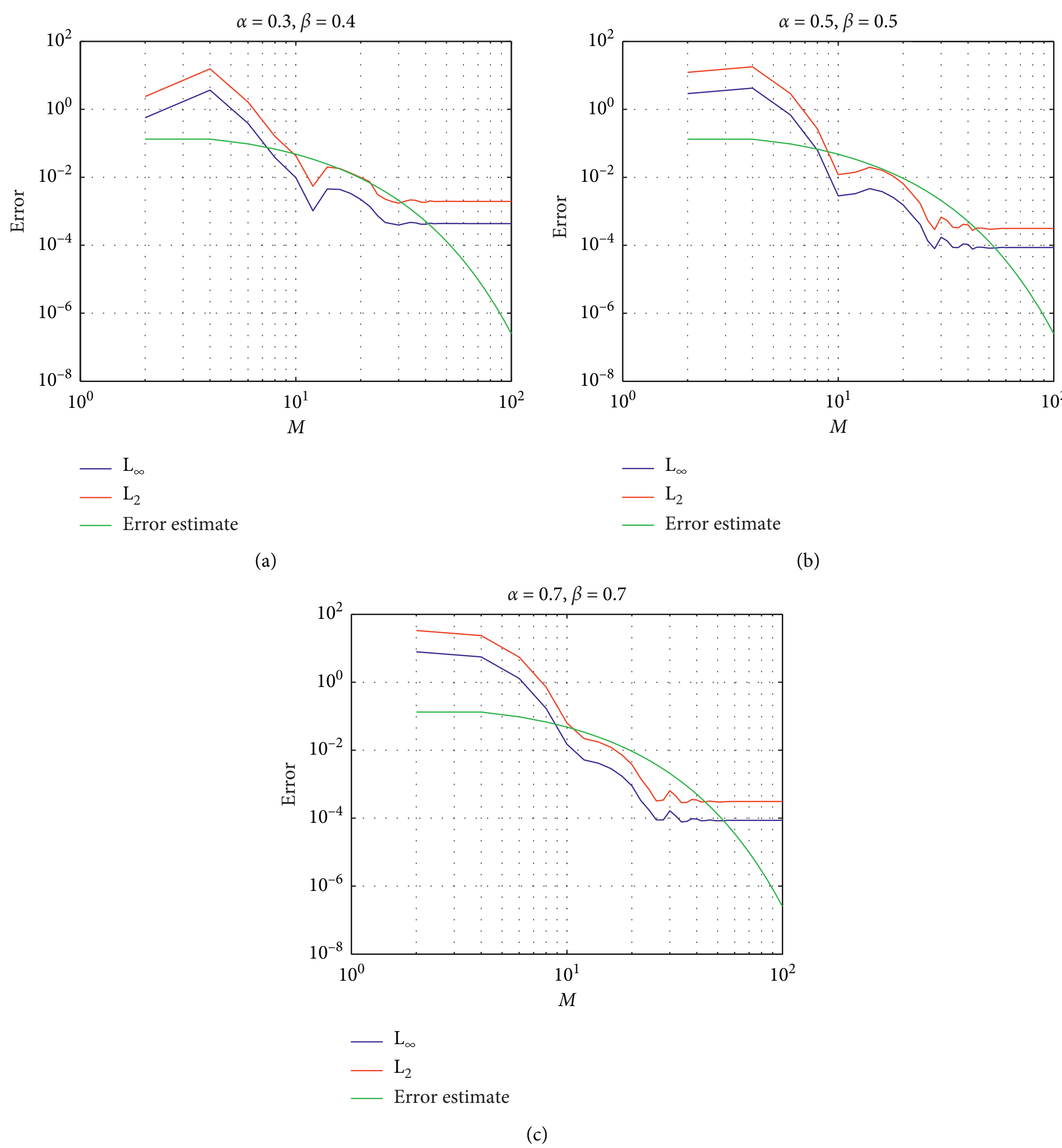


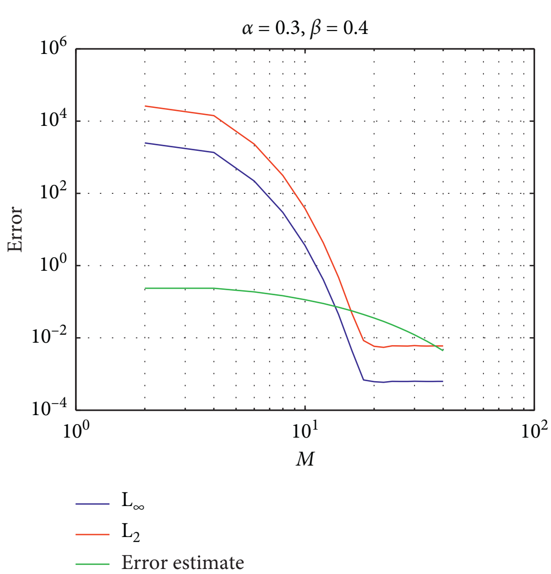
Figure 2: Error versus quadrature points M at $N = 41$, $n = 9$, and $t = 1$ and various values of α, β corresponding to Problem 2.

Table 3: Numerical results using the proposed numerical scheme corresponding to Problem 3.

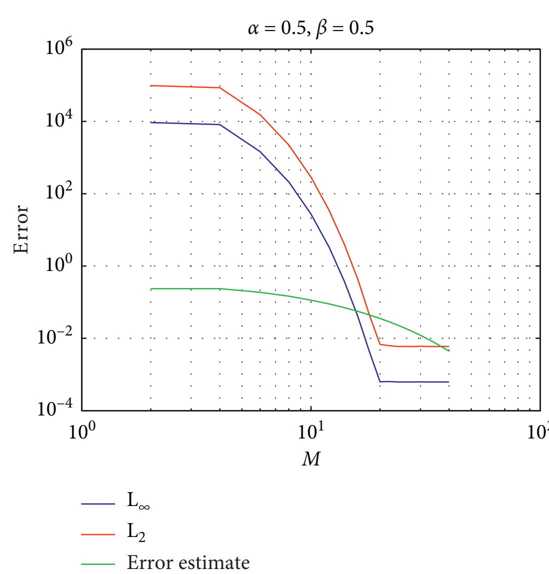
$M = 50$, $n = 11$		$\alpha = 0.5$, $\beta = 0.5$	
N	L_∞	L_2	κ
15	7.3893e − 004	0.0050	3.1023e + 020
20	0.0025	0.0232	3.1024e + 020
26	0.0029	0.0344	3.0989e + 020
41	8.6091e − 005	3.1340e − 004	5.0985e + 033
$M = 50$, $n = 15$, $N = 20$	L_∞	L_2	κ
$(\alpha, \beta) = (0.6, 0.5)$	2.2399e − 004	0.0017	3.1000e + 020
$(\alpha, \beta) = (0.5, 0.3)$	2.2409e − 004	0.0017	5.0520e + 021
$(\alpha, \beta) = (0.3, 0.2)$	2.2419e − 004	0.0017	1.1756e + 024

Table 4: Numerical results using the proposed numerical scheme corresponding to Problem 4.

$M = 50$, $n = 9$		$\alpha = 0.5$, $\beta = 0.3$, $\gamma = 0.2$	
N	L_∞	L_2	κ
11	0.0069	0.0249	1.3362e + 021
15	0.0022	0.0102	2.7478e + 021
21	4.4497e − 004	0.0057	6.7578e + 021
25	7.8109e − 004	0.0112	1.3745e + 022
$M = 50$, $N = 20$, $n = 11$	L_∞	L_2	κ
$(\alpha, \beta) = (0.2, 0.2)$	6.2477e − 004	0.0060	9.6487e + 025
$(\alpha, \beta) = (0.5, 0.5)$	6.2141e − 004	0.0060	3.1021e + 020
$(\alpha, \beta) = (0.7, 0.7)$	6.1740e − 004	0.0059	3.0989e + 020
$(\alpha, \beta) = (0.9, 0.9)$	6.1134e − 004	0.0058	3.0989e + 020



(a)



(b)

Figure 3: Error versus quadrature points M at $N = 20$, $n = 11$, and $t = 1$ and various values of α, β corresponding to Problem 4.

where

$$f_1(x,y,t) = (1+\alpha+\beta)t^{\alpha+\beta} + 2\frac{\Gamma(2+\alpha+\beta)}{\Gamma(1+2\alpha+\beta)}t^{2\alpha+\beta}\frac{2}{\gamma} + 2\frac{\Gamma(2+\alpha+\beta)}{\Gamma(1+\alpha+2\beta)}t^{\alpha+2\beta}\frac{2}{\gamma},$$
$$f_2(x,y,t) = -4\left(\frac{\Gamma(2+\alpha+\beta)}{\Gamma(1+2\alpha+\beta)}t^{2\alpha+\beta} + \frac{\Gamma(2+\alpha+\beta)}{\Gamma(1+\alpha+2\beta)}t^{\alpha+2\beta}\right)\left(\frac{(x-0.5)^2}{\gamma^2} + \frac{(y-0.5)^2}{\gamma^2}\right). \quad (38)$$

The exact solution is

$$w(x,y,t) = t^{1+\alpha+\beta}\exp\left(-\frac{(x-0.5)^2}{\gamma} - \frac{(y-0.5)^2}{\gamma}\right). \quad (39)$$

Here, the problem is solved over the domain $\Omega = [0,1]\times[0,1]$. In the upper section of Table 4, the ℓ_∞ and ℓ_2 error norms are decreasing with $\alpha = 0.5$, $\beta = 0.3$, $n = 9$, and $\gamma = 0.2$ and for nodal points $11 \le N \le 25$. In the lower section of Table 4, for same values of α and β, the ℓ_∞ and ℓ_2 error norms are decreasing steadily at $N = 20$, $n = 11$, and $M = 50$. The results are comparatively identical with the results of the paper [38]. Figure 3 shows the error with varying quadrature points M and various values of α and β at $N = 20$, $M = 50$, and $\gamma = 0.2$. The error L_∞ is well matched with estimate L_{est} for $c = 0.5$ and L_2 error norm, as shown in Figure 3. The present method is stable and accurate in multidimensional fractional order partial differential equations.

6. Conclusion

In this work, a numerical scheme is constructed which is based on Laplace transform and radial basis functions in the local setting. The proposed numerical scheme efficiently approximated time fractional anomalous subdiffusion equation. The supremacy of this method particularly for fractional order equations is its nonsensitive nature in time as contrary to finite difference approximation for fractional order operators. Since the fractional order derivative is of integral convolution type and suited to handle by Laplace transform, the spatial operators in multidimensions can be approximated by RBF in the local setting which generates small size differentiation matrices in local subdomains and these are assembled as a single sparse matrix in the global domain. So, large amount of data can be manipulated very easily and accurately.

Data Availability

The data supporting the results are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. S. Miller and B. Ross, *An Introductional the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Inc., New York, NY, USA and London, UK, 1974.

[2] A. Mohebbi and M. Dehghan, “The use of compact boundary value method for the solution of two-dimensional Schrödinger equation,” *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 124–134, 2009.

[3] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, London, UK, 1974.

[4] K. Diethelm and N. J. Ford, “Analysis of fractional differential equations,” *Journal of Mathematical Analysis and Applications*, vol. 265, no. 2, pp. 229–248, 2002.

[5] O. Keith and J. Spanier, *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Elsevier, Amsterdam, Netherlands, 1974.

[6] A. T. Patera, “A spectral element method for fluid dynamics: laminar flow in a channel expansion,” *Journal of Computational Physics*, vol. 54, no. 3, pp. 468–488, 1984.

[7] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, Amsterdam, Netherlands, 1998.

[8] R. L. Bagley and P. J. Torvik, “A theoretical basis for the application of fractional calculus to viscoelasticity,” *Journal of Rheology*, vol. 27, no. 3, pp. 201–210, 2010.

[9] F. Mainardi, “Fractional calculus,” *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, Berlin, Germany, pp. 291–348, 1997.

[10] Y. A. Rossikhin and M. V. Shitikova, “Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids,” *Applied Mechanics Reviews*, vol. 50, no. 1, pp. 15–67, 1997.

[11] R. T. Baillie, “Long memory processes and fractional integration in econometrics,” *Journal of Econometrics*, vol. 73, no. 1, pp. 5–59, 1996.

[12] S. Larsson, M. Racheva, and F. Saedpanah, “Discontinuous Galerkin method for an integro-differential equation modeling dynamic fractional order viscoelasticity,” *Computer Methods in Applied Mechanics and Engineering*, vol. 83, pp. 196–209, 2015.

[13] J. T. Machado, V. Kiryakova, and F. Mainardi, “Recent history of fractional calculus,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1140–1153, 2011.

[14] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[15] R. Amirov, A. Ergun, and S. Durak, “Half-inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse,” *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 915–924, 2021.

[16] E. R. G. Abdullah, "The multiplicity of eigenvalues of a vectorial singular diffusion equation with discontinuous conditions," *Eastern Anatolian Journal of Science*, vol. 6, no. 2, pp. 22–34, 2019.

[17] A. Rauf and E. Abdullah, "Half inverse problems for the impulsive singular diffusion operator," *Turkish Journal of Science*, vol. 5, no. 3, pp. 186–198, 2020.

[18] M. J. Saxton, "Anomalous subdiffusion in fluorescence photobleaching recovery: a Monte Carlo study," *Biophysical Journal*, vol. 81, no. 4, pp. 2226–2240, 2001.

[19] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, pp. 1–77, 2000.

[20] G. M. Zaslavsky, "Fractional kinetic equation for Hamiltonian chaos," *Physica D: Nonlinear Phenomena*, vol. 76, no. 1-3, pp. 110–122, 1994.

[21] R. Metzler and J. Klafter, "The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics," *Journal of Physics A: Mathematical and General*, vol. 37, no. 31, pp. R161–R208, 2004.

[22] M. Dehghan and A. Nikpour, "The solitary wave solution of coupled Klein-Gordon-Zakharov equations via two different numerical methods," *Computer Physics Communications*, vol. 184, no. 9, pp. 2145–2158, 2013.

[23] M. Dehghan and A. Shokri, "Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions," *Journal of Computational and Applied Mathematics*, vol. 230, no. 2, pp. 400–410, 2009.

[24] M. Tatari and M. Dehghan, "On the solution of the non-local parabolic partial differential equations via radial basis functions," *Applied Mathematical Modelling*, vol. 33, no. 3, pp. 1729–1738, 2009.

[25] Y. T. Gu, P. Zhuang, and F. Liu, "An advanced implicit meshless approach for the non-linear anomalous subdiffusion equation," *Computer Modeling in Engineering & Sciences*, vol. 56, no. 3, pp. 303–334, 2010.

[26] L. Ling and R. Schaback, "Stable and convergent unsymmetric meshless collocation methods," *SIAM Journal on Numerical Analysis*, vol. 46, no. 3, pp. 1097–1115, 2008.

[27] R. Schaback, "Convergence of unsymmetric kernel-based meshless collocation methods," *SIAM Journal on Numerical Analysis*, vol. 45, no. 1, pp. 333–351, 2007.

[28] E. J. Kansa and Y. C. Hon, "Circumventing the ill-conditioning problem with multiquadric radial basis functions: applications to elliptic partial differential equations," *Computers & Mathematics with Applications*, vol. 39, no. 7-8, pp. 123–137, 2000.

[29] B. Šarler, "A radial basis function collocation approach in computational fluid dynamics," *Computer Modelling in Engineering & Sciences*, vol. 7, pp. 185–193, 2005.

[30] B. Šarler, "Meshfree explicit local radial basis function collocation method for microscopic and macroscopic phase change simulations," PhD Thesis, Univerza v Novi Gorici, Nova Gorica, Slovenia, 2006.

[31] V. Voller, R. Vertnik, and B. Šarler, "Meshless local radial basis function collocation method for convective-diffusive solid-liquid phase change problems," *International Journal of Numerical Methods for Heat & Fluid Flow*, vol. 16, no. 5, 2006.

[32] Q. T. L. Gia and W. McLean, "Solving the heat equation on the unit sphere via laplace transforms and radial basis functions," *Advances in Computational Mathematics*, vol. 40, no. 2, pp. 353–375, 2014.

[33] G. J. Moridis and E. Kansa, *The Laplace Transform Multiquadric Method: A Highly Accurate Scheme for the Numerical Solution of Partial Differential Equations*, Lawrence Berkeley National Laboratory, Berkeley, CF, USA, 1993.

[34] M. López-Fernández and C. Palencia, "On the numerical inversion of the laplace transform of certain holomorphic mappings," *Applied Numerical Mathematics*, vol. 51, no. 2-3, pp. 289–303, 2004.

[35] B. A. Jacobs, "High-order compact finite difference and laplace transform method for the solution of time-fractional heat equations with dirchlet and neumann boundary conditions," *Numerical Methods for Partial Differential Equations*, vol. 32, no. 4, pp. 1184–1199, 2016.

[36] W. McLean and V. Thomée, "Time discretization of an evolution equation via laplace transforms," *IMA Journal of Numerical Analysis*, vol. 24, no. 3, pp. 439–463, 2004.

[37] W. McLean and V. Thomée, "Numerical solution via laplace transforms of a fractional order evolution equation," *Journal of Integral Equations Applications*, vol. 22, no. 1, pp. 57–94, 2010.

[38] M. Dehghan, M. Abbaszadeh, and A. Mohebbi, "Legendre spectral element method for solving time fractional modified anomalous sub-diffusion equation," *Applied Mathematical Modelling*, vol. 40, no. 5-6, pp. 3635–3654, 2016.

[39] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[40] I. Podlubny, "Fractional differential equations," *Mathematics in Science and Engineering*, Vol. 198, Academic Press, Cambridge, MA, USA, 1999.

[41] A. M. Cohen, *Numerical Methods for Laplace Transform Inversion*, Springer Science+Business Media, LLC, Berlin, Germany, 2007.

[42] M. Uddin, H. Ali, and A. Ali, "Kernel-based local meshless method for solving multi-dimensional wave equations in irregular domain," *CMES-Computer Modeling in Engineering & Sciences*, vol. 107, no. 6, pp. 463–479, 2015.

[43] M. Uddin, Z. Minullah, and A. Ali, "On the approximation of rapidly oscillatory hankel transform via radial kernels," *Dolomites Research Notes on Approximation*, vol. 11, no. 1, 2018.

[44] N. Zhang, W. Deng, and Y. Wu, "Finite difference/element method for a two-dimensional modified fractional diffusion equation," *Advances in Applied Mathematics and Mechanics*, vol. 4, no. 4, pp. 496–518, 2012.

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Research Article

Certain Properties of Generalized *M*-Series under Generalized Fractional Integral Operators

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The aim of this study is to introduce new (presumed) generalized fractional integral operators involving I-function as a kernel. In addition, two theorems have been developed under these operators that provide an image formula for this generalized M-series and also to study the different properties of the generalized M-series. The corresponding assertions in terms of Euler and Laplace transform methods are presented. Due to the general nature of the I-function and the generalized M-series, a number of results involving special functions can be achieved only by making appropriate values for the parameters.

1. Introduction

Recently, in a short note, Sharma and Jain [1] introduced and studied a new special function called as generalized M-series, which is a particular case of the Wright generalized hypergeometric function ${}_p\psi_q(.)$ ([2], p. 56, equation (1.11.14)) and Fox's H-function [3–5]. The generalized M-series is important because its basic cases are followed by the Mittag-Leffler function and hypergeometric function, and all these functions have actually discovered key implementations in solving problems in applied sciences, chemistry, physics, and biology. A number of researchers [6–10] have also investigated the structure, implementations, and various directions of extensions of the fractional integration and differentiation in detail. The series is defined for $z, \varphi, \varsigma \in \mathbb{C}$, $\Re(\varphi) > 0$, and $\alpha_i, \beta_j \in \mathbb{R}(-\infty, \infty)$, $(\alpha_i\colon i = 1, 2, \ldots, p; \beta_j \neq 0\colon j = 1, 2, \ldots, q)$ as

$$
{}_p\overset{\varphi,\varsigma}{M}_q(z) = {}_p\overset{\varphi,\varsigma}{M}_q\left(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{\Gamma(\varphi k + \varsigma)}, \tag{1}
$$

where $(\alpha_j)_k$, $(\beta_j)_k$ are showing the results for Pochhammer symbols. The series (1) is defined when none of the parameters β_js, $(j = 1, \ldots, q)$ is a negative integer or zero; if any numerator parameter α_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in (1) is convergent for all z if $p \leq q$, it is convergent for $|z| < \vartheta = \varphi^{\varphi}$ if , and it is divergent if $p > q + 1$. When $p = q + 1$ and $|z| < \vartheta$, the series can converge on conditions depending on the parameters ([2], for the general theory of the Wright function). The summation of the convergent series is denoted by the symbol ${}_p\overset{\varphi,\varsigma}{M}_q(.)$.

Some essential special cases of the generalized M-series are mentioned in the following:

(1) For $\varphi = \varsigma = 1$, the generalized M-series is the generalized hypergeometric function [11, 12].

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};z\right] = {}_p\overset{1,1}{M}_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z\right) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{z^k}{k!}. \tag{2} $$

(2) When $p=q=0$ and $\varsigma=1$, we have

$$ E_\varphi(z) = {}_0\overset{\varphi,1}{M}_0(-;-;z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\varphi k+1)}, \ (\varphi>0), \tag{3} $$

Where the symbol $E_\varphi(z)$ denotes the Mittag-Leffler function [13].

(3) Again, for $p=q=0$, we have

$$ E_{\varphi,\varsigma}(z) = {}_0\overset{\varphi,\varsigma}{M}_0(-;-;z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\varphi k+\varsigma)}, \ (\varphi>0,\varsigma>0), \tag{4} $$

Where the symbol $E_{\varphi,\varsigma}(z)$ denotes the two-index Mittag-Leffler function introduced by Wiman [14].

(4) Furthermore, if we put $p=q=1$, $\alpha_1=\sigma\in\mathbb{C}$, $\beta_1=1$, the generalized M-series reduces to the generalized Mittag-Leffler function [12, 15] as follows:

$$ \begin{aligned} E^\sigma_{\varphi,\varsigma}(z) &= {}_1\overset{\varphi,\varsigma}{M}_1(\sigma;1;z) = \sum_{k=0}^{\infty}\frac{(\sigma)_k}{(1)_k}\frac{z^k}{\Gamma(\varphi k+\varsigma)}\\ &= \sum_{k=0}^{\infty}\frac{(\sigma)_k}{\Gamma(\varphi k+\varsigma)}\frac{z^k}{k!}. \end{aligned} \tag{5} $$

In the present study, our aim is to study some fundamental properties of generalized M-series defined by (1), for which, we consider the two generalized fractional integral operators involving the I-function as kernel, which is described in the next section.

2. Generalized Fractional Integral Operators

In this section, we are introducing new (presumed) generalized fractional integral operators involving I-function as kernel, which are the extensions of Saxena and Kumbhat operators [16, 17]:

$$ S^{\mu,\vartheta}_{0,x;r}[f(x)] = rx^{-\mu-r\vartheta-1}\int_0^x t^\mu\left(x^r-t^r\right)^\vartheta \times I^{m,n}_{p_i,q_i:r}\left[\lambda U\left|\begin{matrix}\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m};\left(b_{ji},B_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right]f(t)\mathrm{d}t \tag{6} $$

and

$$ S^{\varepsilon,\vartheta}_{x,\infty;r}[f(x)] = rx^{\varepsilon}\int_x^\infty t^{-\varepsilon-r\vartheta-1}\left(t^r-x^r\right)^\vartheta \times I^{m,n}_{p_i,q_i:r}\left[\lambda V\left|\begin{matrix}\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m};\left(b_{ji},B_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right]f(t)\mathrm{d}t, \tag{7} $$

where

$$ U=\left(\frac{t^r}{x^r}\right)^\tau\left(1-\frac{t^r}{x^r}\right)^\nu \text{ and } V=\left(\frac{x^r}{t^r}\right)^\tau\left(1-\frac{x^r}{t^r}\right)^\nu; \quad \tau,\nu>0. \tag{8} $$

The sufficient conditions of these operators are

(i) $1\le p,q<\infty$, $p^{-1}+q^{-1}=1$;

(ii) $\mathfrak{R}(\mu+r\tau(b_j/B_j))>-q^{-1}$; $\mathfrak{R}(\vartheta+r\tau(b_j/B_j))>-q^{-1}$;

$$ \mathfrak{R}\left(\varepsilon+\vartheta+r\tau\left(\frac{b_j}{B_j}\right)\right)>-p^{-1}; \quad j=1,2,\ldots,m. \tag{9} $$

(iii) $f(x)\in L_p(0,\infty)$

(iv) $|\arg\lambda|\le\pi\Theta/2$, $\Theta>0$

$$ \Theta=\sum_{j=1}^{m}\left(A_i\right)+\sum_{j=1}^{n}\left(B_i\right)-\max_{1\le i\le r}\left[\sum_{j=n+1}^{p_i}\left(A_{ji}\right)+\sum_{j=m+1}^{q_i}\left(B_{ji}\right)\right]. \tag{10} $$

where the I-function, which is more general than Fox's H-function, is defined by Saxena [18], by means of the following Mellin-Barnes type contour integral:

$$ I[z] = I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m};\left(b_{ji},B_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_L\varphi(\zeta)z^\zeta\mathrm{d}\zeta, \tag{11} $$

where $\omega=\sqrt{-1}$ and

$$\varphi(\zeta) = \prod_{j=1}^{m}\Gamma\left(b_j - B_j\zeta\right)\frac{\prod_{j=1}^{n}\Gamma\left(1 - a_j + A_j\zeta\right)}{\sum_{j=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1 - b_{ji} + B_{ji}\zeta\right)\prod_{j=n+1}^{p_i}\left(a_{ji} - A_{ji}\zeta\right)\right\}}, \tag{12}$$

$p_i, q_i\,(i = 1, \ldots, r), m, n$ are the integers satisfying $0 \le n \le p_i, 0 \le m \le q_i$; A_j, B_j, A_{ji}, B_{ji} are the real and positive numbers, and a_j, b_j, a_{ji}, b_{ji} are the complex numbers. L is a suitable contour of the Mellin-Barnes type running from $\gamma - i\varphi$ to $\gamma + i\varphi$ (γ is real) in the complex ζ-plane. Details regarding existence conditions and various parametric restrictions of I-function are provided by Saxena [18].

For $r = 1$, (11) reduces to Fox' H-function:

$$I_{p_i,q_i;1}^{m,n}\left[z\middle|\begin{matrix}\left(a_j, A_j\right)_{1,n}; \left(a_{ji}, A_{ji}\right)_{n+1,p_i}\\ \left(b_j, B_j\right)_{1,m}; \left(b_{ji}, B_{ji}\right)_{m+1,q_i}\end{matrix}\right] = H_{p_i,q_i}^{m,n}\left[z\middle|\begin{matrix}\left(a_j, A_j\right)_{1,n}; \left(a_j, A_j\right)_{n+1,p}\\ \left(b_j, B_j\right)_{1,m}; \left(b_j, B_j\right)_{m+1,q}\end{matrix}\right]. \tag{13}$$

3. Images of Generalized M-Series under the Generalized Fractional Integral Operators

In this section, we established the image formula for the generalized M-series (1) under the generalized fractional integral operators (6) and (7) in terms of the I-function as the kernel. The results are given in Theorems 1 and 2.

Theorem 1. *Let* $a > 0, x > 0; \varphi, \varsigma, \eta, \xi \in \mathbb{C}, \Re(\varphi) > 0, \Re(\xi) > 0, \Re(\eta) > 0, \Re(\varsigma) > 0,\ 1 \le p \le 2,$ *then*

$$\begin{aligned}S_{0,x;r}^{\mu,\vartheta}\left(t^{\eta-1}\,{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(at^{\xi}\right)\right)(x) = x^{\eta-1}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots\left(\alpha_p\right)_k}{(\beta_1)_k\ldots\left(\beta_q\right)_k}\frac{\left(ax^{\xi}\right)^k}{\Gamma(\varphi k + \varsigma)}\\ \times I_{p+2,q+1;r}^{m,n+2}\left[\begin{matrix}\lambda\left|\left(a_j, A_j\right)_{1,n}; \left(a_{ji}, A_{ji}\right)_{n+1,p_i}, \left(1 - \dfrac{(\mu + \eta + \xi k)}{r}, \tau\right), (-\vartheta, \nu)\right.\\ \left(-\vartheta - \dfrac{(\mu + \eta + \xi k)}{r}, \tau + \nu\right), \left(b_j, \varsigma_j\right)_{1,m}; \left(b_{ji}, \varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right],\end{aligned} \tag{14}$$

provided the conditions, stated with operator (6), are satisfied.

Proof. We assume Ω_1 be the on the left-hand side of (14); using the definition of generalized M-series (1) and the generalized fractional integral operator (6) on the left-hand side of (14), we have

$$\Omega_1 = rx^{-\mu-r\vartheta-1}\int_0^x t^{\mu+\eta-1}\left(x^r - t^r\right)^{\vartheta}\left\{\frac{1}{2\pi\omega}\int_l \varphi(\zeta)(\lambda U)^{\zeta}\mathrm{d}\zeta\right\} \times \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots\left(\alpha_p\right)_k}{(\beta_1)_k\ldots\left(\beta_q\right)_k}\frac{\left(at^{\xi}\right)^k}{\Gamma(\varphi k + \varsigma)}\mathrm{d}t. \tag{15}$$

Now, by changing the order of the integration which is valid under the given with theorem, we get

$$\Omega_1 = rx^{-\mu-r\vartheta-1}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots\left(\alpha_p\right)_k}{(\beta_1)_k\ldots\left(\beta_q\right)_k}\frac{a^k}{\Gamma(\varphi k + \varsigma)} \times \frac{1}{2\pi\omega}\int_L \varphi(\zeta)\lambda^{\zeta}x^{r\vartheta-r\tau\zeta}\left\{\int_0^x t^{\mu+\eta+\xi k+r\tau\zeta-1}\left(1 - \frac{t^r}{x^r}\right)^{\vartheta+\nu\zeta}\mathrm{d}t\right\}\mathrm{d}\zeta. \tag{16}$$

Let the substitution $t^r/x^r = w$ and then $t = xw^{1/r}$ in (16), we get

$$\Omega_1 = x^{\eta-1}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{(ax^{\xi})^k}{\Gamma(\varphi k+\varsigma)}\times\frac{1}{2\pi\omega}\int_L\varphi(\zeta)\lambda^{\zeta}\left\{\int_0^1 w^{((\mu+\eta+\xi k)/r+\tau\zeta)-1}(1-w)^{\vartheta+\nu\zeta}\mathrm{d}w\right\}\mathrm{d}\zeta. \tag{17}$$

Using the definition of the well-known beta function in the inner integral, we have

$$\Omega_1 = x^{\eta-1}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{(ax^{\xi})^k}{\Gamma(k+\varsigma)}\frac{1}{2\pi\omega}\int_L\varphi(\zeta)\lambda^{\zeta}\frac{\Gamma(\mu+\eta+\xi k/r+\tau\zeta)\Gamma(1+\vartheta+\nu\zeta)}{\Gamma(1+\vartheta+\mu+\eta+\xi k/r+(\tau+\nu)\zeta)}\mathrm{d}\zeta. \tag{18}$$

Interpreting the right-hand side of (18), in view of the definition (11), we arrive at the result (14). □

Theorem 2. *Let* $a>0, x>0; \varphi,\varsigma,\eta,\xi\in\mathbb{C}, \mathfrak{R}(\varphi)>0, \mathfrak{R}(\xi)>0, \mathfrak{R}(\eta)>0, \mathfrak{R}(\varsigma)>0,\ 1\le p\le 2,$ *then*

$$S^{\varepsilon,\vartheta}_{x,\infty;r}\left(t^{-\eta}\,{}_pM_q^{\varphi,\varsigma}\left(\frac{a}{t^{\xi}}\right)\right)(x) = x^{-\eta}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{(a/x^{\xi})^k}{\Gamma(\varphi k+\varsigma)}\times I^{m,n+2}_{p+2,q+1;r}\left[\lambda\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},\left(1-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),(b_j,\varsigma_j)_{1,m};(b_{ji},\varsigma_{ji})_{m+1,q_i}\end{matrix}\right.\right], \tag{19}$$

provided the conditions, stated with operator (7), are satisfied.

Proof. On the left-hand side of (19), let Ω_2, using (1) and (7) on the left-hand side of (19), we have

$$\Omega_2 = rx^{\varepsilon}\int_x^{\infty}t^{-\varepsilon-\eta-r\vartheta-1}(t^r-x^r)^{\vartheta}\left\{\frac{1}{2\pi\omega}\int_L\varphi(\zeta)(\lambda V)^{\zeta}\mathrm{d}\zeta\right\}\times\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{(a/t^{\xi})^k}{\Gamma(\varphi k+\varsigma)}\mathrm{d}t. \tag{20}$$

Now, by changing the order of the integration which is valid under the given stated theorem, we get

$$\Omega_2 = rx^{\varepsilon}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{a^k}{\Gamma(\varphi k+\varsigma)}\times\frac{1}{2\pi\omega}\int_L\varphi(\zeta)\lambda^{\zeta}x^{r\tau\zeta}\left\{\int_x^{\infty}t^{-\varepsilon-\eta-\xi k-r\tau\zeta-1}\left(1-\frac{x^r}{t^r}\right)^{\vartheta+\nu\zeta}\mathrm{d}t\right\}\mathrm{d}\zeta. \tag{21}$$

Let the replacement $x^r/t^r=w$ and then $t=x/w^{1/r}$ in (21), we get

$$\Omega_2 = x^{-\eta-1}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{a^k x^{-\xi k}}{\Gamma(\varphi k+\varsigma)}\times\frac{1}{2\pi\omega}\int_L\varphi(\zeta)\lambda^{\zeta}\left\{\int_0^1 w^{(\varepsilon+\eta+\xi k/r)+\tau\zeta-1}(1-w)^{\vartheta+\nu\zeta}\mathrm{d}w\right\}\mathrm{d}\zeta. \tag{22}$$

By beta function, we have

$$\Omega_2 = x^{-\eta+r\varphi}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{\left(ax^{-\xi}\right)^k}{\Gamma(\varphi k+\varsigma)}\frac{1}{2\pi\omega}\int_L \varphi(\zeta)\lambda^{\zeta}\times\frac{\Gamma((\varepsilon+\eta+\xi k/r)+\tau\zeta)\Gamma(1+\vartheta+\nu\zeta)}{\Gamma((\varepsilon+\eta+\xi k/r)+1+\vartheta+(\tau+\nu)\zeta)}d\zeta. \tag{23}$$

Interpreting the right-hand side of (23), in view of the definition (11), we arrive at the result (19). □

4. Special Cases

(1) If we put $\varphi=\varsigma=1$ in Theorems 1 and 2, we obtain the following interesting results on the right, and it is known as the generalized hypergeometric function.

Corollary 1. *For $\varphi=\varsigma=1$, equation (14) reduces in the following form:*

$$S^{\mu,\vartheta}_{0,x;r}\left(t^{\eta-1}\,{}_p\overset{1,1}{M}_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;at^{\xi}\right)\right)(x) = x^{\eta-1}\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};ax^{\xi}\right]$$
$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{matrix}\lambda\Big|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\mu+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\mu+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right]. \tag{24}$$

Corollary 2. *For $\varphi=\varsigma=1$, equation (19) reduces in the following form:*

$$S^{\varepsilon,\vartheta}_{x,\infty;r}\left(t^{-\eta}\,{}_p\overset{1,1}{M}_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;\frac{a}{t^{\xi}}\right)\right)(x) = x^{-\eta}\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};\frac{a}{x^{\xi}}\right]$$
$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{matrix}\lambda\Big|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right]. \tag{25}$$

(2) *If we put $p=q=0$ in Theorems 1 and 2, we obtain the following interesting results on the right, and it is known as the two-index Mittag-Leffler function.*

Corollary 3. *For $p=q=0$, equation (16) reduces in the following form:*

$$S^{\mu,\vartheta}_{0,x;r}\left(t^{\eta-1}{}_0\overset{\varphi,\varsigma}{M}_0\left(-;-;at^{\xi}\right)\right)(x)=x^{\eta-1}E_{\varphi,\varsigma}\left(ax^{\xi}\right)$$
$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{array}{c}\lambda|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\mu+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\mu+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{array}\right]. \tag{26}$$

Corollary 4. *For $p=q=0$, equation (19) reduces in the following form:*

$$S^{\varepsilon,\vartheta}_{x,\infty;r}\left(t^{-\eta}{}_0\overset{\varphi,\varsigma}{M}_0\left(-;-;\frac{a}{t^{\xi}}\right)\right)(x)=x^{-\eta}E_{\varphi,\varsigma}\left(\frac{a}{x^{\xi}}\right)$$
$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{array}{c}\lambda|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{array}\right]. \tag{27}$$

(3) *If we put $p=q=1$, $\alpha_1=\sigma\in\mathbb{C}$, $\beta_1=1$ in Theorems 1 and 2, we obtain the following interesting results on the right, and it is known as the generalized Mittag-Leffler function.*

Corollary 5. *For $p=q=1$, $a_1=\xi\in\mathbb{C}$, $b_1=1$, equation (14) reduces in the following form:*

$$S^{\mu,\vartheta}_{0,x;r}\left(t^{\eta-1}{}_1\overset{\varphi,\varsigma}{M}_1\left(\sigma;1;at^{\xi}\right)\right)(x)=x^{\eta-1}E^{\sigma}_{\varphi,\varsigma}\left(ax^{\xi}\right)$$
$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{array}{c}\lambda|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\mu+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\mu+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{array}\right]. \tag{28}$$

Corollary 6. *For $p=q=1$, $\alpha_1=\sigma\in\mathbb{C}$, $\beta_1=1$, equation (19) reduces in the following form:*

$$S^{\varepsilon,\vartheta}_{x,\infty;r}\left(t^{-\eta}\,{}_1\overset{\varphi,\varsigma}{M}_1\left(\sigma;1;\frac{a}{t^{\xi}}\right)\right)(x)=x^{-\eta}E^{\sigma}_{\varphi,\varsigma}\left(\frac{a}{x^{\xi}}\right)$$

$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{matrix}\lambda|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right]. \tag{29}$$

5. Certain Integral Transforms

In this section, with the aid of the results developed in the prior segment, we will provide some very important outcomes of several theorems connected with the transforms of Euler and Laplace. To this end, we would like to define these transforms first.

Definition 1. The well-known Euler transform (e.g., [19]) of a function $f(t)$ is defined as

$$\mathfrak{B}\{f(t);c,d\}=\int_0^1 t^{c-1}(1-t)^{d-1}f(t)\mathrm{d}t;\ (c,d\in\mathbb{C},\Re(c)>0,\Re(d)>0). \tag{30}$$

Definition 2. The Laplace transform (e.g., [19]) of the function $f(t)$ is defined, as usual, by

$$\mathfrak{L}\{f(t);s\}=\int_0^{\infty}e^{-st}f(t)\mathrm{d}t;\ (\Re(s)>0). \tag{31}$$

This section would establish the following fascinating outcomes in the form of theorems. As these findings are direct implications of Definitions 1 and 2 and Theorems 1 and 2, they are provided without evidence here.

Theorem 3. *The Euler transform of the Theorem 1 gives the following result:*

$$\mathfrak{B}\left\{S^{\mu,\vartheta}_{0,x;r}\left(t^{\eta-1}\,{}_p\overset{\varphi,\varsigma}{M}_q\left(at^{\xi}\right)\right);c,d\right\}=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{a^k}{\Gamma(\varphi k+\varsigma)}B(c+\eta-1+\xi k,d)$$

$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{matrix}\lambda|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\mu+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\mu+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right], \tag{32}$$

provided that the conditions mentioned with the operator and Euler transform are satisfied.

Theorem 4. *The Euler transform of the Theorem 2 gives the following result:*

$$\mathfrak{B}\left\{S^{\varepsilon,\vartheta}_{x,\infty;r}\left(t^{-\eta}\,{}_p\overset{\varphi,\varsigma}{M}_q\left(\frac{a}{t^{\xi}}\right)\right);c,d\right\}=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{a^k}{\Gamma(\varphi k+\varsigma)}B(c-\eta-\xi k,d)$$

$$\times I^{m,n+2}_{p+2,q+1;r}\left[\begin{matrix}\lambda|\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\dfrac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right], \tag{33}$$

provided that the conditions mentioned with the operator and Euler transform are satisfied.

Theorem 5. *The Laplace transform of the Theorem 1 gives the following result:*

$$\mathfrak{L}\left\{S_{0,x;r}^{\mu,\vartheta}\left(t^{\eta-1}{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(at^{\xi}\right)\right);s\right\}=s^{-\eta}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{\Gamma(\eta+\xi k)}{\Gamma(\varphi k+\varsigma)}\left(as^{-\xi}\right)^k \times I_{p+2,q+1;r}^{m,n+2}\left[\lambda\left|\begin{matrix}\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\frac{(\mu+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\frac{(\mu+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right], \tag{34}$$

provided that the conditions mentioned with the operator and Laplace transform are satisfied.

Theorem 6. *The Laplace transform of the Theorem 2 gives the following result:*

$$\mathfrak{L}\left\{S_{x,\infty;r}^{\varepsilon,\vartheta}\left(t^{-\eta}{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(\frac{a}{t^{\xi}}\right)\right);s\right\}=s^{\eta-1}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{\Gamma(1-\eta-\xi k)}{\Gamma(\varphi k+\varsigma)}\left(as^{\xi}\right)^k \times I_{p+2,q+1;r}^{m,n+2}\left[\lambda\left|\begin{matrix}\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\frac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\frac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right], \tag{35}$$

provided that the conditions mentioned with the operator and Laplace transform are satisfied.

6. Properties of Generalized Fractional Integral Operators

Here, we establish some properties of the operators as consequences of Theorems 1 and 2. These properties show compositions of the power function.

Theorem 7. *Following all the conditions on parameters as stated in Theorem 1 with $\Re(\psi+\eta)>0$, then the following result holds true:*

$$x^{\psi}S_{0,x;r}^{\mu,\vartheta}\left(t^{\eta-1}{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(at^{\xi}\right)\right)(x)=S_{0,x;r}^{\mu-\psi,\vartheta}\left(t^{\psi+\eta-1}{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(at^{\xi}\right)\right)(x). \tag{36}$$

Proof. Let us use (14) in the left-hand side of (36), and we get

$$x^{\psi}S_{0,x;r}^{\mu,\vartheta}\left(t^{\eta-1}{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(at^{\xi}\right)\right)(x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{(a)^k}{\Gamma(\varphi n+\varsigma)}x^{\eta+\psi+\xi k-1} \times I_{p+2,q+1;r}^{m,n+2}\left[\lambda\left|\begin{matrix}\left(a_j,A_j\right)_{1,n};\left(a_{ji},A_{ji}\right)_{n+1,p_i},\left(1-\frac{(\mu+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\frac{(\mu+\eta+\xi k)}{r},\tau+\nu\right),\left(b_j,\varsigma_j\right)_{1,m};\left(b_{ji},\varsigma_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right]. \tag{37}$$

Again, using (14) in the right-hand side of (37), we get

$$S_{0,x;r}^{\mu-\psi,\vartheta}\left(t^{\psi+\eta-1}\,{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(at^{\xi}\right)\right)(x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{(a)^k}{\Gamma(\varphi k+\varsigma)}x^{\eta+\psi+\xi k-1}\times I_{p+2,q+1;r}^{m,n+2}\left[\lambda\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},\left(1-\frac{(\mu+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\frac{(\mu+\eta+\xi k)}{r},\tau+\nu\right),(b_j,\varsigma_j)_{1,m};(b_{ji},\varsigma_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \tag{38}$$

It seems that Theorem 6 readily follows due to (37) and (38). □

Theorem 8. *Follow all the conditions on parameters as stated in Theorem 2 with* $\Re(1-\psi+\eta)<1$; *then, the following result holds true:*

$$x^{-\psi}S_{x,\infty;r}^{\varepsilon,\vartheta}\left(t^{-\eta}\,{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(\frac{a}{t^{\xi}}\right)\right)(x)=S_{x,\infty;r}^{\varepsilon-\psi,\vartheta}\left(t^{-\psi-\eta}\,{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(\frac{a}{t^{\xi}}\right)\right)(x). \tag{39}$$

Proof. From (14) in the left-hand side of (39), we get

$$x^{-\psi}S_{x,\infty;r}^{\varepsilon,\vartheta}\left(t^{-\eta}\,{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(\frac{a}{t^{\xi}}\right)\right)(x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{(a)^k}{\Gamma(\varphi k+\varsigma)}x^{-\eta-\psi-\xi k}\times I_{p+2,q+1;r}^{m,n+2}\left[\lambda\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},\left(1-\frac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\frac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),(b_j,\varsigma_j)_{1,m};(b_{ji},\varsigma_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \tag{40}$$

Again, using (14) in the right-hand side of (39), we get

$$S_{x,\infty;r}^{\varepsilon-\psi,\vartheta}\left(t^{-\psi-\eta}\,{}_{p}\overset{\varphi,\varsigma}{M}_{q}\left(\frac{a}{t^{\xi}}\right)\right)(x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{(a)^k}{\Gamma(\varphi k+\varsigma)}x^{-\eta-\psi-\xi k}\times I_{p+2,q+1;r}^{m,n+2}\left[\lambda\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},\left(1-\frac{(\varepsilon+\eta+\xi k)}{r},\tau\right),(-\vartheta,\nu)\\ \left(-\vartheta-\frac{(\varepsilon+\eta+\xi k)}{r},\tau+\nu\right),(b_j,\varsigma_j)_{1,m};(b_{ji},\varsigma_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \tag{41}$$

It seems that Theorem 7 readily follows due to (40) and (41). □

7. Concluding Remark and Observations

In this study, we introduced and studied the properties of generalized M-series under the new (presumed) generalized fractional integral operators which are defined in equations (6) and (7) and also developed some new images. The results established in this study contain various special cases, such that if we take $r=1$, we recover the known results recorded in [20]. Furthermore, we can present certain very interesting results in the form of several theorems associated with Mellin, Whittaker, and K-transforms. We left this as an exercise to the interested reader.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] M. Sharma and R. Jain, "A note on a generalized M-series as a special function of fractional calculus," *Fractional Calculus and Applied Analysis*, vol. 12, no. 4, pp. 449–452, 2009.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, North-Holland Mathematics Studies, Amsterdam, Netherlands, 2006.

[3] C. Fox, "The G-and H-functions as symmetrical Fourier kernels," *Transactions of the American Mathematical Society*, vol. 98, no. 3, p. 395, 1961.

[4] A. M. Mathai and R. K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines*, Wiley Eastern Ltd., New Delhi, India, 1978.

[5] H. M. Srivastava, K. C. Gupta, and S. P. Goyal, *The H-Functions on One and Two Variables*, South Asian Publisher's, New Delhi, India, 1981.

[6] M. Kumar Bansal, D. Kumar, I. Khan, J. Singh, and K. S. Nisar, "Certain unified integrals associated with product of M-series and incomplete H-functions," *Mathematics*, vol. 7, no. 12, p. 1191, 2019.

[7] M. Kumar Bansal, D. Kumar, K. S. Nisar, and J. Singh, "Certain fractional calculus and integral transform results of incomplete $\aleph$-functions with applications," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 4, pp. 5602–5614, 2020.

[8] Y. Singh, V. Gill, J. Singh, D. Kumar, and I. Khan, "Computable generalization of fractional kinetic equation with special functions," *Journal of King Saud University-Science*, vol. 33, no. 1, p. 101221, 2021.

[9] J. Singh, D. Kumar, and D. Baleanu, "New aspects of fractional Bloch model associated with composite fractional derivative," *Mathematical Modelling of Natural Phenomena*, vol. 16, p. 10, 2021.

[10] D. L. Suthar, R. K. Parmar, and S. D. Purohit, "Fractional calculus with complex order and generalized hypergeometric functions," *Nonlinear Science Letters A*, vol. 8, no. 2, pp. 156–161, 2017.

[11] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, UK, 1999.

[12] H. Amsalu and D. L. Suthar, "Generalized fractional integral operators involving Mittag-Leffler function," *Abstract and Applied Analysis*, vol. 2018, Article ID 7034124, 8 pages, 2018.

[13] G. M. Mittag-Leffler, "Sur la nouvelle function $E_{\alpha}(x)$," *Comptes Rendus de l'Académie des Sciences Paris (Series II)*, vol. 137, pp. 554–558, 1903.

[14] A. Wiman, "Über den fundamental satz in der theorie der functionen $E_{\alpha}(x)$," *Acta Mathematica*, vol. 29, pp. 191–201, 1905.

[15] T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.

[16] R. K. Saxena and R. K. Kumbhat, "Integral operators involving H-function," *Indian Journal of Pure and Applied Mathematics*, vol. 5, pp. 1–6, 1974.

[17] D. L. Suthar, "Generalized fractional calculus operators associated with K-function," *International Journal of Mathematical Physics*, vol. 1, no. 2, pp. 1–11, 2018.

[18] V. P. Saxena, *The I-Function*, Anamaya Publisher, New Delhi, India, 2008.

[19] I. N. Sneddon, *The Use of Integral Transform*, Tata Mc Graw Hill, New Delhi, India, 1979.

[20] A. M. Khan, R. K. Kumbhat, A. Chouhan, and A. Alaria, "Generalized fractional integral operators and M-Series," *Journal of Mathematics*, vol. 2016, Article ID 2872185, 10 pages, 2016.

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Research Article

Integral-Type Fractional Equations with a Proportional Riemann–Liouville Derivative

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In this paper, we present the necessary conditions where integral-type fractional equations with a proportional Riemann–Liouville derivative have a unique solution. Also, we give an example to illustrate our work.

1. Introduction

Lately, many researchers have been focusing on the study of various types of fractional problems; we refer the reader to [1–17]. The fixed point and the monotone iterative techniques can be very useful tools to prove the existence and uniqueness of a solution to this type of problems; see [1]. In this manuscript, inspired by the work of Jankowski in [1], we investigate the existence and uniqueness of a solution to the following problem:

$$\begin{aligned} D^{\alpha,\rho}\xi(t) &= g\left(t,\xi(t),\int_0^t \mathscr{K}(t,\tau)\xi(\tau)\mathrm{d}\tau\right) \equiv \mathscr{F}\xi(t), \quad t\in J_0=(0,a];a>0,\\ \tilde{\xi}(0) &= p, \end{aligned} \tag{1}$$

where $D^{\alpha,\rho}\xi(t)$ denotes a proportional Riemann–Liouville fractional derivative for $\rho\in[0,1]$ and $0<\alpha<1$. Also, $g\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $J=[0,a]$, and $\tilde{\xi}(0)=t^{1-\alpha}e^{(\rho-1/\rho)t}\xi(t)|_{t\longrightarrow 0+}$. Now, we remind the reader of the definition of the proportional Riemann–Liouville fractional integral and derivative.

Definition 1 (see [18]). Let $\alpha\in\mathbb{C};\mathrm{Re}(\alpha)\geq 0$, $0<\rho\leq 1$, and $t>0$.

(i) The following integral is called the proportional Riemann–Liouville fractional integral:

$$I^{\alpha,\rho}f(t) := \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t e^{[\rho-1/\rho](t-\tau)}(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau. \tag{2}$$

(ii) The following derivative is called the proportional Riemann–Liouville fractional derivative:

$$\begin{aligned}\left(D^{\alpha,\rho}f\right)(t) &= D^{n,\rho}I^{n-\alpha,\rho}f(t)\\ &= \frac{D_t^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_0^t e^{[\rho-1/\rho](t-\tau)}(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau,\end{aligned} \tag{3}$$

where $n = [\mathrm{Re}(\alpha)] + 1$ and $D_t^{1,\rho} = (1-\rho) f(t) + \rho f'(t)$.

Next, we present the following proposition.

Proposition 1 (see [18]). *If $\alpha, \gamma \in \mathbb{C}$, where $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\gamma) > 0$, then for any $0 < \rho \le 1$, we have* $I^{\alpha,\rho}(t^{\gamma-1} e^{(\rho-1/\rho)t})(x) = (\Gamma(\gamma)/\rho^{\alpha}\Gamma(\alpha+\gamma)) x^{\gamma+\alpha-1} e^{(\rho-1/\rho)x}$.

In Section 2, we prove the existence and uniqueness of a solution to problem (1) using the fixed point technique. In Section 3, we prove the existence and uniqueness of a solution to problem (1) using the monotone iterative method. In the conclusion, we present an open question.

2. Fixed Point Approach

First of all, let $C_{1-\alpha}(J,\mathbb{R}) = \{ f \in C([0,a),\mathbb{R}) | t^{1-\alpha} f \in C(J,\mathbb{R})\}$. Now, define the following two weighted norms:

$$\|f\|^{*} = \max_{[0,a]} t^{1-\alpha} |f(t)|,$$
$$\|f\|_{*} = \max_{[0,a]} t^{1-\alpha} e^{-\lambda t} |f(t)| \text{ for fixed } \lambda > 0. \tag{4}$$

Theorem 1. *Let $0 < \alpha < 1, 0 < \rho \le 1$, and $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\mathcal{K} \in C(J \times J, \times \mathbb{R})$. Let $\beta: = (\rho - 1/\rho)$. Also, assume the following two hypotheses:*

(1) There exist nonnegative constants H, V, and W such that $|\mathcal{K}(t,s)| < H$ and

$$\left|g(t,u_1,u_2) - g(t,v_1,v_2)\right| \le V\left|v_1 - u_1\right| + W\left|v_2 - u_2\right|. \tag{5}$$

(2) $b \equiv (a^{\alpha}/\Gamma(2\alpha)\rho^{\alpha}) \quad [V + (HWa/2\alpha)] < 1$, for $\alpha \in (0, (1/2)]$.

Then, initial value problem (1) has a unique solution.

Proof. First, let $S\xi(t) = t^{1-\alpha} e^{(\rho-1/\rho)t} p + (1/\rho^{\alpha}\Gamma(\alpha)) \int_0^t e^{[(\rho-1/\rho)(t-\tau)]} (t-\tau)^{\alpha-1} \mathcal{F}\xi(\tau) d\tau$. Note that if S has a unique fixed point and that is $S\xi(t) = \xi(t)$, then initial value problem (1) has a unique solution, i. e., it will be enough to show that S is a contraction map. So, let $\xi, \mathcal{Y} \in C_{1-\alpha}(J,\mathbb{R})$; we have two cases:

Case 1: $\alpha \in (0, (1/2)]$.

$$\begin{aligned}
S\xi(t) &= t^{1-\alpha} e^{\beta t} p + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \mathcal{F}\xi(\tau) d\tau, \\
\|S\xi - S\mathcal{Y}\|^{*} &= \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \max_{t\in J} t^{1-\alpha} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} |\mathcal{F}\xi(\tau) - \mathcal{F}Y(\tau)| d\tau \\
&\le \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \max_{t\in J} t^{1-\alpha} \left\{ \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[\begin{matrix} V|\xi(\tau) - \mathcal{Y}(\tau)| \\ +W \int_0^{\tau} \||\mathcal{K}(t,s)\xi(s) - \mathcal{Y}(s)| ds \end{matrix} \right] d\tau \right\} \\
&\le \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \|\xi - \mathcal{Y}\|^{*} \max_{t\in J} \left\{ t^{1-\alpha} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V\tau^{\alpha-1} + HW \int_0^{\tau} s^{\alpha-1} ds \right] d\tau \right\} \\
&= \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \|\xi - \mathcal{Y}\|^{*} \max_{t\in J} t^{1-\alpha} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V\tau^{\alpha-1} + HW \frac{\tau^{\alpha}}{\alpha} \right] d\tau \\
&\le \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \|\xi - \mathcal{Y}\|^{*} \max_{t\in J} t^{1-\alpha} e^{-\beta t} \int_0^t e^{\beta(t-\tau)} (t-\tau)^{\alpha-1} \left[V e^{\beta\tau} \tau^{\alpha-1} + HW e^{\beta\tau} \frac{\tau^{\alpha}}{\alpha} \right] d\tau \\
&= \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \|\xi - \mathcal{Y}\|^{*} \max_{t\in J} t^{1-\alpha} e^{-\beta t} \left[I^{\alpha,\rho}\left(V t^{\alpha-1} e^{\beta t} \right) + I^{\alpha,\rho}\left(\frac{HW}{\alpha} t^{\alpha} e^{\beta t} \right) \right] \\
&= \frac{\Gamma(\alpha) a^{\alpha}}{\Gamma(\alpha)\Gamma(2\alpha)\rho^{\alpha}} \left[V + \frac{HWa}{2\alpha} \right] \|\xi - \mathcal{Y}\|^{*} \\
&= b\|\xi - \mathcal{Y}\|^{*}.
\end{aligned} \tag{6}$$

Hence, S is a contraction map. Therefore, S has a unique fixed point as desired.

Case 2: $\alpha \in ((1/2), 1)$; in this case, we use $\|\cdot\|_{*}$ with the positive constant $\lambda > 0$ such that

$$\sqrt{\lambda - \beta} > b_1 \equiv \frac{e^{-\beta a} (V\alpha + HWa)\Gamma(2\alpha - 1)\sqrt{a^{2\alpha-1}}}{\alpha\rho^{\alpha}\Gamma(\alpha)\sqrt{\Gamma(2(2\alpha-1))}}. \tag{7}$$

It is not difficult to see the following:

(1) $\int_0^t e^{2(\lambda-\beta)\tau}\mathrm{d}\tau \le (e^{2(\lambda-\beta)t}/2(\lambda-\beta))$

(2) $t^{1-\alpha}\sqrt{\int_0^t (t-\tau)^{2(\alpha-1)}\tau^{2(\alpha-1)}\mathrm{d}\tau} = (\Gamma(2\alpha-1)\sqrt{a^{2\alpha-1}}/\sqrt{\Gamma(2(2\alpha-1))})$

Also, recall the Schwarz inequality for integrals:

$$\int_0^t |f(\tau)\|g(\tau)|\mathrm{d}\tau \le \sqrt{\int_0^t f^2(\tau)\mathrm{d}\tau}\sqrt{\int_0^t g^2(\tau)\mathrm{d}\tau},$$

$$\begin{aligned}
\|S\xi - S\mathscr{Y}\|_* &= \frac{1}{\rho^\alpha\Gamma(\alpha)} \max_{t\in J} t^{1-\alpha}e^{-\lambda t}\int_0^t e^{\beta(t-\tau)}(t-\tau)^{\alpha-1}|\mathscr{F}\xi(\tau) - \mathscr{F}\Upsilon(\tau)|\mathrm{d}\tau \\
&\le \frac{1}{\rho^\alpha\Gamma(\alpha)}\|\xi-\mathscr{Y}\|_* \max_{t\in J}\left\{t^{1-\alpha}e^{-\lambda t}\int_0^t e^{\beta(t-\tau)}(t-\tau)^{\alpha-1}\left[Ve^{\lambda\tau}\tau^{\alpha-1}\right] + HWe^{\lambda\tau}\int_0^\tau s^{\alpha-1}\mathrm{d}s\mathrm{d}\tau\right\} \\
&\le \frac{1}{\rho^\alpha\Gamma(\alpha)}\|\xi-\mathscr{Y}\|_* \max_{t\in J}\left\{t^{1-\alpha}e^{-\lambda t}\int_0^t e^{\beta(t-\tau)}(t-\tau)^{\alpha-1}\left[V\tau^{\alpha-1}e^{\lambda\tau} + HW\frac{\tau^\alpha}{\alpha}e^{\lambda\tau}\right]\mathrm{d}\tau\right\} \\
&\le \frac{V\alpha+HWa}{\alpha\rho^\alpha\Gamma(\alpha)}\|\xi-\mathscr{Y}\|_* \max_{t\in J}\left\{t^{1-\alpha}e^{-\lambda t}\int_0^t e^{\beta(t-\tau)}e^{\lambda\tau}(t-\tau)^{(\alpha-1)}\tau^{\alpha-1}\mathrm{d}\tau\right\} \\
&= \frac{V\alpha+HWa}{\alpha\rho^\alpha\Gamma(\alpha)}\|\xi-\mathscr{Y}\|_* \max_{t\in J}\left\{t^{1-\alpha}e^{-\lambda t}\int_0^t e^{\beta(t-\tau)+\lambda\tau}(t-\tau)^{(\alpha-1)}\tau^{\alpha-1}\mathrm{d}\tau\right\} \\
&\le \frac{V\alpha+HWa}{\alpha\rho^\alpha\Gamma(\alpha)}\|\xi-\mathscr{Y}\|_* \max_{t\in J}\left\{t^{1-\alpha}e^{-\lambda t}\times\sqrt{\int_0^t (t-\tau)^{2(\alpha-1)}\tau^{2(\alpha-1)}\mathrm{d}\tau}\sqrt{\int_0^t e^{2\beta(t-\tau)+2\lambda\tau}\mathrm{d}\tau}\right\} \\
&\le \frac{V\alpha+HWa}{\alpha\rho^\alpha\Gamma(\alpha)}\|\xi-\mathscr{Y}\|_* \max_{t\in J}\left\{t^{1-\alpha}e^{-\lambda t}\times\sqrt{\int_0^t (t-\tau)^{2(\alpha-1)}\tau^{2(\alpha-1)}\mathrm{d}\tau}\sqrt{\int_0^t e^{2(\lambda-\beta)\tau}\mathrm{d}\tau}\right\} \\
&\le \frac{b_1}{\sqrt{\lambda-\beta}}\|\xi-\mathscr{Y}\|_*.
\end{aligned} \tag{8}$$

Thus, S is a contraction map. Therefore, S has a unique fixed point as required. □

As an application to Theorem 1, consider the following problem:

$$\begin{aligned} D^{\alpha,\rho}y(t) &= -L(t)y(t) + \xi(t), \\ \tilde{\xi}(0) &= r. \end{aligned} \tag{9}$$

If

$$\frac{a^\alpha}{\rho^\alpha\Gamma(2\alpha)}\max_{t\in J}|L(t)| < 1, \quad \text{for } 0 < \alpha \le \frac{1}{2}, \tag{10}$$

then it is not difficult to see that, by using Theorem 1, problem (9) has a unique solution. In closing of this section, the following linear problem is considered:

$$\begin{aligned} D^{\alpha,\rho}\xi(t) &= -L(t)\xi(t) + z(t), \quad t\in J_0, \\ \tilde{\xi}(0) &= r. \end{aligned} \tag{11}$$

Now, we introduce the following hypothesis.

Hypothesis 1 (H_1)

(1) $L(t) = L$, $t\in J$ or

(2) The function L is nonconstant on J and

$$\frac{a^\alpha}{\rho^\alpha\Gamma(2\alpha)}\max_{t\in J}|L(t)| < 1 \quad \text{only if} \quad \alpha\in\left(0,\frac{1}{2}\right). \tag{12}$$

The following lemma is a consequence of Theorem 1.

Lemma 1. *If* $\alpha\in(0,1)$, $L\in C(J,\mathbb{R})$, $z\in C_{1-\alpha}(J,\mathbb{R})$, *and hypothesis* (H_1) *holds, then problem (11) has a unique solution.*

We would like to bring to the reader's attention that, in [1], in the hypothesis ρ should be as follows: $\rho \equiv (T^q/\Gamma(2q))[K + (WLT/2q)]$ which he used to prove the case where $q\in(0,(1/2)]$. This way, his result will be stronger or he can just change the last equality to the inequality.

3. Monotone Iterative Method

First of all, we start by introducing the following hypothesis.

Hypothesis 2 (H_2)

(1) $L(t) = L$, $t \in J$ or

(2) The function L is nonconstant, and if $L(t)$ is negative, then there exists $\overline{L}$ which is nondecreasing, where $-L(t) \le \overline{L}(t)$ on J and for every $x \in J$, we have

$$\frac{e^{\beta x}}{\rho^{\alpha}\Gamma(\alpha)} \int_0^a (a-\tau)^{\alpha-1} e^{\beta(a-\tau)} \overline{L}(\tau)\mathrm{d}\tau < 1. \tag{13}$$

Now, for our purpose, we prove the following useful lemma.

Lemma 2. *Let* $\alpha \in (0,1)$ *and* $L \in C(J, [0,\infty))$ *or* $L \in C(J, (-\infty, 0])$. *Also, denote by* $\beta: = (\rho - 1/\rho)$. *Assume that* $q \in C_{1-\alpha}(J, \mathbb{R})$ *is a solution to the following problem:*

$$\begin{aligned} D^{\alpha,\rho} q(t) &\le -L(t)q(t), \quad t \in J_0, \\ \widetilde{q}(0) &< 0. \end{aligned} \tag{14}$$

If (H_2) *holds, then* $q(t) \le 0$ *for all* $t \in J$.

Proof. Assume that our lemma is false, that is, there exist $x, y \in [0, a)$ such that $q(x) = 0, q(y) > 0$, and $q(t) \le 0$ for $t \in (0, x]$; $q(t) > 0$ for $t \in (x, y]$. Let x_0 be the first maximal point of q on $[x, y]$.

Case 1: assume that $L(t) \ge 0$ for all $t \in J$. Thus, $D^{\alpha,\rho} q(t) \le 0$ for $t \in [x, y]$. Hence,

$$\int_x^{x_0} D^{\alpha,\rho} q(t) \le 0. \tag{15}$$

Therefore, $B \equiv I^{\rho,1-\alpha} q(x_0) - I^{\rho,1-\alpha} q(x) \le 0$, but

$$\begin{aligned} B &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \left[\int_0^{x_0} e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} q(\tau)\mathrm{d}\tau - \int_0^{x} e^{\beta(x-\tau)} (x-\tau)^{-\alpha} q(\tau)\mathrm{d}\tau \right] \\ &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \left\{ \int_0^{x} \left[e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} - e^{\beta(x-\tau)} (x-\tau)^{-\alpha} \right] q(\tau)\mathrm{d}\tau + \int_x^{x_0} e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} q(\tau)\mathrm{d}\tau \right\} \\ &> \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_x^{x_0} e^{\beta(x_0-\tau)} (x_0-\tau)^{-\alpha} q(\tau)\mathrm{d}\tau > 0, \end{aligned} \tag{16}$$

which leads us to a contradiction given the fact that $B \le 0$.

Case 2: assume that $L(t) \le 0$ for all $t \in J$, and consider $\widetilde{L}$ to be nondecreasing on J. Now, if we apply $I^{\alpha,\rho}$ on problem (14), we obtain

$$q(t) - \widetilde{q}(0) \frac{e^{\beta t} t^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \le -I^{\alpha,\rho}[L(t)q(t)], \quad \text{for } t \in [x, x_0]. \tag{17}$$

Notice that $\widetilde{q}(0)(e^{\beta t} t^{\alpha-1}/\rho^{\alpha-1}\Gamma(\alpha)) \le 0$ which is due to the fact that $\widetilde{q}(0) \le 0$. Thus,

$$
\begin{aligned}
q(x_0) &\le -\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{x_0}(x_0-\tau)^{\alpha-1}e^{\beta(x_0-\tau)}L(\tau)q(\tau)\mathrm{d}\tau\\
&= -\frac{1}{\rho^\alpha\Gamma(\alpha)}\left[\int_0^{x_0}(x_0-\tau)^{\alpha-1}e^{\beta(x_0-\tau)}L(\tau)q(\tau)\mathrm{d}\tau+\int_x^{x_0}(x_0-\tau)^{\alpha-1}e^{\beta(x_0-\tau)}L(\tau)q(\tau)\mathrm{d}\tau\right]\\
&< -\frac{q(x_0)}{\rho^\alpha\Gamma(\alpha)}\int_0^{x_0}(x_0-\tau)^{\alpha-1}e^{\beta(x_0-\tau)}L(\tau)\mathrm{d}\tau,\ \text{let}\ \sigma=\frac{\tau}{x_0}\\
&= -\frac{q(x_0)e^{\beta x_0}x_0^\alpha}{\rho^\alpha\Gamma(\alpha)}\int_0^1(1-\sigma)^{\alpha-1}e^{\beta(1-\sigma)}L(\sigma x_0)\mathrm{d}\sigma\\
&\le \frac{q(x_0)e^{\beta x_0}x_0^\alpha}{\rho^\alpha\Gamma(\alpha)}\int_0^1(1-\sigma)^{\alpha-1}e^{\beta(1-\sigma)}\widetilde{L}(\sigma a)\mathrm{d}\sigma\\
&= \frac{q(x_0)e^{\beta(x_0+(1/a))}x_0^\alpha}{\rho^\alpha\Gamma(\alpha)a^\alpha}\int_0^a(a-\tau)^{\alpha-1}e^{\beta(a-\tau)}\widetilde{L}(\tau)\mathrm{d}\tau\\
&\le \frac{q(x_0)e^{\beta(x_0+(1/a))}}{\rho^\alpha\Gamma(\alpha)}\int_0^a(a-\tau)^{\alpha-1}e^{\beta(a-\tau)}\widetilde{L}(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{18}
$$

Hence, $q(x_0)[1-(e^{\beta(x_0+(1/a))}/\ \rho^\alpha\Gamma(\alpha))\int_0^a(a-\tau)^{\alpha-1}e^{\beta(a-\tau)}\ \widetilde{L}(\tau)\mathrm{d}\tau]\le 0$. Using hypothesis (H_2) implies that $q(x_0)\le 0$, which leads us to a contradiction, and this concludes our proof. □

We say that y is a lower solution of problem (1) if

$$D^{\alpha,\rho}y(t)\le \mathscr{F}y(t),\quad t\in J_0, \widetilde{y}(0)\le 0, \tag{19}$$

and we say that y is an upper solution of problem (1) if

$$D^{\alpha,\rho}y(t)\ge \mathscr{F}y(t),\quad t\in J_0, \widetilde{y}(0)\le 0. \tag{20}$$

Next, the following hypothesis is defined.

Hypothesis 3. (H_3). There exists a function $L\in C(J,\mathbb{R})$ where

$$\left|g(t,u_1,u_2)-g(t,v_1,v_2)\right|\le L(t)\left|v_1-u_1\right|\ \text{whenever}\ x_0\le u_1\le v_1\le y_0\ \text{and}\ u_2\le v_2. \tag{21}$$

Theorem 2. *Assume that x_0 is a lower solution of problem (1) and y_0 is an upper solution of problem (1), where $x_0, y_0\in C_{1-\alpha}(J,\mathbb{R})$. Moreover, assume that hypotheses H_1, H_2, and H_3 hold; problem (1) has solutions in $[x_0, y_0]=\{y\in C_{1-\alpha}(J,\mathbb{R})|x_0(t)\le y(t)\le y_0(t),\ t\in J_0, \widetilde{x}_0(0)\le\ \widetilde{y}(0)\le \widetilde{y}_0(0)\}$.*

Proof. Using Lemmas 1 and 2, the proof is similar to the proof of Theorem 2 in [1]. □

Now, we present the following example.

Example 1. Let $0<\alpha<1$, $0<\rho\le 1$, $\beta=(\rho-1/\rho)$, and $\mathbb{A},\mathbb{B}\in C([0,1],(0,\infty))$ such that $\mathbb{A}(t)\le\mathbb{B}(t)$ for $t\in[0,1]$. Now, consider the following problem:

$$
\begin{aligned}
&D^{\alpha,\rho}\xi(t)\equiv\mathscr{F}\xi(t),\quad t\in J_0=(0,1],\\
&\widetilde{\xi}(0)=0,
\end{aligned}
\tag{22}
$$

where

$$\mathscr{F}\xi(t)=\frac{\rho^\alpha e^{\beta t}t^{1-\alpha}}{\Gamma(2-\alpha)}+\mathbb{A}(t)\left[te^{\beta t}-1-\xi(t)\right]^3+\frac{\beta}{e^{\beta t}-1}\mathbb{B}(t)\int_0^t[\sin(t\tau)]^4\xi(\tau)\mathrm{d}\tau. \tag{23}$$

Now, let $x_0(t) = 0$ and $y_0(t) = te^{\beta t}$; first, note that $x_0(t)$ is a lower solution of problem (22). Next, we show that $y_0(t)$ is an upper solution of problem (22):

$$\begin{aligned}\mathscr{F}y_0(t) &= \frac{\rho^{\alpha}e^{\beta t}t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \frac{\beta}{e^{\beta t}-1}\mathbb{B}(t)\int_0^t [\sin(t\tau)]^4 \tau e^{\beta\tau}d\tau \\ &\leq \frac{\rho^{\alpha}e^{\beta t}t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \frac{\beta}{e^{\beta t}-1}\mathbb{B}(t)\int_0^t e^{\beta\tau}d\tau \\ &= \frac{\rho^{\alpha}e^{\beta t}t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \frac{\beta}{e^{\beta t}-1}\mathbb{B}(t)\left[\frac{e^{\beta t}}{\beta} - \frac{1}{\beta}\right] < \frac{\rho^{\alpha}e^{\beta t}t^{1-\alpha}}{\Gamma(2-\alpha)} \\ &= D^{\alpha,\rho}y_0(t).\end{aligned} \tag{24}$$

Thus, $y_0(t)$ is an upper solution of problem (22). Now, it is not difficult to see that all the hypotheses of Theorem 2 are satisfied. Therefore, problem (22) has solutions in $[x_0, y_0]$ if $\alpha \in ((1/2), 1)$, and for $\alpha \in (0, (1/2)]$, we need to assume that $(1/\rho^{\alpha}\Gamma(2\alpha))\max_{t\in[0,1]}|\mathbb{A}(t)| < 1$.

4. Conclusion

In closing, note that the results of Jankowski [1] are a special case of our work which is by taking $\rho = 1$. Also, we would like to bring to the reader attention the following open question.

What are the necessary and sufficient conditions for problem (1) to have a unique solution if ρ is not constant , but it is a function of \$t\$ say $g(t)$, so that the problem involves $D^{\alpha,g(t)}$?

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

[1] T. Jankowski, "Fractional equations of Volterra type involving a Riemann-Liouville derivative," *Applied Mathematics Letters*, vol. 26, no. 3, pp. 344–350, 2013.

[2] T. Jankowski, "Fractional differential equations with deviating arguments," *Dynamic Systems and Applications*, vol. 17, pp. 677–684, 2008.

[3] V. Lakshmikantham, S. Leela, and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, UK, 2009.

[4] L. Lin, X. Liu, and H. Fang, "Method of upper and lower solutions for fractional differential equations," *Electronic Journal of Differential Equations*, vol. 100, pp. 1–13, 2012.

[5] Q. M. Al-Mdallal, "Monotone iterative sequences for nonlinear integro-differential equations of second order," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3665–3673, 2011.

[6] Q. M. Al-Mdallal, "Boundary value problems for nonlinear fractional integro-differential equations: theoretical and numerical results," *Advances in Difference Equations*, vol. 18, no. 1, pp. 1–13, 2012.

[7] M. Syam, Q. Al-Mdallal, and M. Anwar, "An efficient numerical algorithm for solving fractional higher-order nonlinear integrodifferential equations," *Abstract and Applied Analysis*, vol. 2015, Article ID 616438, 2015.

[8] F. A. McRae, "Monotone iterative technique and existence results for fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. 6093–6096, 2009.

[9] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, USA, 1999.

[10] J. D. Ramírez and A. S. Vatsala, "Monotone iterative technique for fractional differential equations with periodic boundary conditions," *Opuscula Mathematica*, vol. 29, no. 3, pp. 289–304, 2009.

[11] Z. Wei, Q. Li, and J. Che, "Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative," *Journal of Mathematical Analysis and Applications*, vol. 367, no. 1, pp. 260–272, 2010.

[12] G. Wang, "Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments," *Journal of Computational and Applied Mathematics*, vol. 236, no. 9, pp. 2425–2430, 2012.

[13] G. Wang, R. P. Agarwal, and A. Cabada, "Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 25, no. 6, pp. 1019–1024, 2012.

[14] S. Zhang, "Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 2087–2093, 2009.

[15] E. Deniz, A. O. Akdemir, and E. Yüksel, "New extensions of Chebyshev-Pòlya-Szegö type inequalities via conformable integrals," *AIMS Mathematics*, vol. 4, no. 6, pp. 1684–1697, 2019.

[16] M. K. Sajid, R. S. Ali, and I. Nayab, "Some results of generalized k-fractional integral operator with k-bessel function," *Turkish Journal of Science*, vol. 5, no. 3, pp. 157–169, 2020.

[17] A. Ekinci and M. E. Özdemir, "Some new integral inequalities via Riemann liouville integral operators," *Applied and Computational Mathematics*, vol. 3, no. 18, pp. 288–295, 2019.

[18] F. Jarad, M. A. Alqudah, and T. Abdeljawad, "On more general forms of proportional fractional operators," *Open Mathematics*, vol. 18, no. 1, pp. 167–176, 2020.

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Research Article

Approximate Symmetries Analysis and Conservation Laws Corresponding to Perturbed Korteweg–de Vries Equation

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The Korteweg–de Vries (KdV) equation is a weakly nonlinear third-order differential equation which models and governs the evolution of fixed wave structures. This paper presents the analysis of the approximate symmetries along with conservation laws corresponding to the perturbed KdV equation for different classes of the perturbed function. Partial Lagrange method is used to obtain the approximate symmetries and their corresponding conservation laws of the KdV equation. The purpose of this study is to find particular perturbation (function) for which the number of approximate symmetries of perturbed KdV equation is greater than the number of symmetries of KdV equation so that explore something hidden in the system.

1. Introduction

Differential equations (DEs) are ubiquitous in modeling an extensive class of physical phenomena involving variation with respect to one or more independent variables. Therefore, DEs are broadly divided into ordinary DEs (ODEs) and partial DEs (PDEs). In different sectors of science and technology, PDEs have played a significant role. PDEs have numerous applications in mathematics, physics, fluid dynamics, mechanics, and physical chemistry. Modeling of PDEs under special conditions and constraints is advantageous in different situations for an effective manipulation of the varying phenomenon. The majority of real-world problems are almost nonlinear in nature, having no analytical solutions. In order to solve nonlinear problems, various approximations and techniques are used to gain high accuracy. In this regard, the approximate symmetry methods play a significant role. We have used the method of approximate Lie symmetry [1, 2], for PDEs to deal with the dynamical system more accurately. In the 1980s, the method of approximate Lie symmetry was developed by Baikov et al. [3, 4]. In obtaining the approximate solutions to such perturbed PDEs, the approximate symmetry method is an effective one. The extension of Lie's theory was mainly the basic reason behind the development of approximate symmetry, which deals with the systems by introducing small perturbation [5]. Symmetry applications to physical problems play a pivotal role in the development of conservation laws [6, 7]. The widely recognized KdV equation is a mathematical model for the depiction of weak nonlinear long wavelength waves in various branches of engineering and physics. It explains how waves evolve due to comparable effects of weak nonlinearity and dispersion. A perturbed nonlinear wave equation is a class of approximate symmetries which is computed using two newly developed methods. For both methods, the associated invariant solution with the approximate symmetries is constructed. By discussing the advantages and disadvantages of each method, the symmetries and solutions are compared. So, the Lie group technique in finding the exact solution of a differential equation has lost its importance. But an approximate Lie group technique has been implemented and used in various methods for obtaining additional related information of differential equation. Perturbation analysis is one of the techniques which is used particularly for nonlinear systems.

This study is framed in the following manner: Section 2 is devoted to the development of exact symmetries and exact conservation laws of the KdV equation. The method to handle the approximate part of the KdV equation is developed in Section 3. The method so developed is applied to tackle the approximate part of the KdV equation for different cases and their corresponding conservation laws in Section 4. The work is concluded by describing the highlights in Section 5.

2. Exact Symmetries and Conservation Laws of the Korteweg–de Vries (KdV) Equation

The exact symmetries and conservation laws in the current study for the work considered in [8, 9] are worked out as follows:

The Korteweg–de Vries (KdV) equation which is a third-order nonlinear partial differential equation is

$$\mu_t - 6\mu\mu_x + \mu_{xxx} = 0. \tag{1}$$

The infinitesimal symmetry operator is

$$\begin{aligned}\mathbf{X}^{[3]} &= \phi\frac{\partial}{\partial x} + \varrho\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial \mu} + \varphi^t\frac{\partial}{\partial \mu_t} + \varphi^x\frac{\partial}{\partial \mu_x} + \varphi^{xx}\frac{\partial}{\partial \mu_{xx}}\\ &+ \varphi^{xt}\frac{\partial}{\partial \mu_{xt}} + \varphi^{tt}\frac{\partial}{\partial \mu_{tt}} + \varphi^{xxx}\frac{\partial}{\partial \mu_{xxx}}.\end{aligned} \tag{2}$$

Applying this symmetry operator on (1),

$$\mathbf{X}^{[3]}\left(\mu_t - 6\mu\mu_x + \mu_{xxx}\right) = 0, \tag{3}$$

we get

$$\varphi^t - 6\mu\varphi^x - 6\varphi\mu_x + \varphi^{xxx} = 0. \tag{4}$$

The expanded form of equation (4) is

$$\begin{aligned}&\left[\varphi_t - \phi_t\mu_x + \left(\varphi_\mu - \varrho_t\right)\mu_t - \phi_\mu\mu_x\mu_t - \varrho\mu\mu_t^2\right] - 6\mu\left[\varphi_x + \left(\varphi_\mu - \phi_x\right)\mu_x - \varrho_x\mu_t - \phi_\mu\mu_x^2 - \varrho_x\mu_x\mu_t\right]\\ &- 6\varphi\mu_x + \left[\varphi xxx + \left(3\varphi_{\mu xx} - \phi_{xxx}\right)\mu_x - \varrho_{xxx}\mu_t + \left(3\varphi_{\mu\mu x} - 3\phi_{xx\mu}\right)\right.\\ &\mu_x^2 - 3\varrho_{\mu xx}\mu_x\mu_t + \left(3\varphi_{\mu x} - 3\phi_{xx}\right)\mu_{xx} - 3\varrho_{xx}\mu_{xt} + \left(\varphi_{\mu\mu\mu} - \phi_{\mu\mu x}\right)\mu_x^3\\ &- 3\varrho_{\mu\mu x}\mu_x^2\mu_t + \left(3\varphi_{\mu\mu} - 9\phi_{\mu x}\right)\mu_x\mu_{xx} - 6\varrho_{x\mu}\mu_x\mu_{xt} - 3\varrho_{\mu x}\mu_t\mu_{xx} + \left(\varphi_\mu - 3\phi_x\right)\mu_{xxx}\\ &- 3\varrho_x\mu_{xxt} - \phi_{\mu\mu\mu}\mu_x^4 - \varrho_{\mu\mu\mu}\mu_x^3\mu_t - 6\phi_{\mu\mu}\mu_x^2\mu_{xx} - 3\varrho_{\mu\mu}\mu_x^2\mu_{xt} - 3\varrho_{\mu\mu}\mu_x\mu_t\mu_{xx}\\ &- 4\varphi_\mu\mu_x\mu_{xxx} - 3\varrho_\mu\mu_x\mu_{xxt} - 3\phi_\mu\mu_{xx}^2 - 3\varrho_\mu\mu_{xx}\mu_{xt} - \varrho_\mu\mu_t\mu_{xxx} = 0.\end{aligned} \tag{5}$$

Substituting equation (1) in equation (5), we get

$$\begin{aligned}&\varphi_t - \phi_t\mu_x + \left(\varphi_\mu - \varrho_t\right)\mu_t - \phi_\mu\mu_x\mu_t - \varrho_\mu\mu_t^2 - 6\varphi\mu_x - 6\mu\left[\varphi_x + \left(\varphi_\mu - \phi_x\right)\mu_x - \varrho_x\mu_t - \phi_\mu\mu_x^2 - \varrho_\mu\mu_x\mu_t\right]\\ &+ \varphi_{xxx} + \left(3\varphi_{xx\mu} - \phi_{xxx}\right)\mu_x - \varrho_{xxx}\mu_t + \left(3\varphi_{x\mu\mu} - 3\phi_{xx\mu}\right)\mu_x^2\\ &- 3\varrho_{xx\mu}\mu_x\mu_t + \left(3\varphi_{x\mu} - 3\phi_{xx}\right)\mu_{xx} - 3\varrho_{xx}\mu_{xt} + \left(\varphi_{\mu\mu\mu} - \phi_{x\mu\mu}\right)\mu_x^3 - 3\varrho_{x\mu\mu}\mu_x^2\mu_t\\ &+ \left(3\varphi_{\mu\mu} - 9\phi_{x\mu}\right)\mu_x\mu_{xx} - 6\varrho_{x\mu}\mu_x\mu_{xt} - 3\varrho_{x\mu}\mu_t\mu_{xx} + 6\mu\left(\varphi_\mu - 3\phi_x\right)\mu_x - \left(\varphi_\mu - 3\phi_x\right)\mu_t\\ &- 3\varrho_x\mu_{xxt} - \phi_{\mu\mu\mu}\mu_x^4 - \varrho_{\mu\mu\mu}\mu_x^3\mu_t - 6\phi_{\mu\mu}\mu_x^2\mu_{xx} - 3\varrho_{\mu\mu}\mu_x^2\mu_{xt} - 3\varrho_{\mu\mu}\mu_x\mu_t\mu_{xx}\\ &- 24\mu\phi_\mu\mu_x^2 + 4\phi_\mu\mu_x\mu_t - 3\varrho_\mu\mu_x\mu_{xxt} - 3\phi_\mu\mu_{xx}^2 - 3\varrho_\mu\mu_{xx}\mu_{xt} - 6\mu\varrho_\mu\mu_x\mu_{xt} + \varrho_\mu\mu_t^2 = 0.\end{aligned} \tag{6}$$

Comparing the coefficients of various terms, we get the coefficients and monomials, as shown in Table 1.

Table 1 yields the required set of PDEs as follows:

$$\varrho_\mu = 0, \tag{7}$$

$$\phi_\mu = 0, \tag{8}$$

$$\varrho_x = 0, \tag{9}$$

$$3\phi_x - \varrho_t = 0. \tag{10}$$

Form (10),

$$\begin{aligned} &\phi_{xx} = 0 \\ &\Rightarrow \varphi_{xx\mu} = \phi_{xxx} \\ &\Rightarrow \varphi_{xx\mu} = 0, \\ &\varphi = \frac{-1}{6}\phi_t - 2\mu\phi_x. \end{aligned} \tag{11}$$

As

$$\phi_x = \frac{1}{3}\varrho_t, \tag{12}$$

therefore,

$$\begin{aligned} \varphi &= -\frac{1}{6}\phi_t - \frac{2}{3}\mu\varrho_t, \\ \varphi_t &= -\frac{1}{6}\phi_{tt} - \frac{2}{3}\mu\varrho_{tt}, \\ \varphi_x &= -\frac{1}{6}\phi_{xt} - \frac{2}{3}\mu\varrho_{tx}, \end{aligned} \tag{13}$$

$$\varphi_{xxx} = 0,$$

$$\phi_{tt} = 0, \tag{14}$$

$$\varrho_{tt} = 0. \tag{15}$$

Let

$$\begin{aligned} &\varrho = A(t) \\ &\Rightarrow A_{tt}(t) = 0. \end{aligned} \tag{16}$$

Integrating twice with respect to t yields

$$\begin{aligned} &\Rightarrow A_t(t) = k_1 \\ &\Rightarrow A(t) = k_1 t + k_2 \\ &\Rightarrow \varrho = k_1 t + k_2. \end{aligned} \tag{17}$$

From (10),

$$\begin{aligned} &3\phi_x - \varrho_t = 0, \\ &\phi_\mu = 0 \Rightarrow \phi = B(x)t, \\ &\phi_x = \frac{1}{3}\varrho_t \\ &\Rightarrow \phi_x = \frac{1}{3}k_1. \end{aligned} \tag{18}$$

Integrating with respect to "x,"

$$\begin{aligned} &\Rightarrow \phi = \frac{1}{3}k_1 x + D(t) \\ &\Rightarrow \phi_{tt} = D_{tt}(t) = 0 \\ &\Rightarrow D(t) = k_3 t + k_4 \\ &\Rightarrow \phi = \frac{1}{3}k_1 x + k_3 t + k_4. \end{aligned} \tag{19}$$

From (13),

$$\begin{aligned} \varphi &= -\frac{1}{6}\phi_t - \frac{2}{3}\mu\varrho_t \\ &= -\frac{1}{6}k_3 - \frac{2}{3}k_1\mu. \end{aligned} \tag{20}$$

The general solution is

$$\begin{aligned} \varphi &= -\frac{2}{3}k_1\mu - \frac{1}{6}k_3, \\ \varrho &= k_1 t + k_2, \\ \phi &= \frac{1}{3}k_1 x + k_3 t + k_4. \end{aligned} \tag{21}$$

Hence, the Lie symmetry generators for the KdV equation are given, as shown in Table 2.

3. A New Procedure to Find the Approximate Symmetries

This section explains the development of the method for the approximate symmetries of the KdV equation. The KdV (1) is perturbed with the function $f(x, t, \mu(x, t), \mu(t, x))$ as

$$\mu_t - 6\mu\mu_x + \mu_{xxx} + \varepsilon f(x, t, \mu(x, t), \mu(t, x)) = 0, \tag{22}$$

where ε is a small parameter, causing the required perturbation in the KdV equation. The exact and approximate parts of (22) are

$$\begin{aligned} E_e &= \mu_t - 6\mu\mu_x + \mu_{xxx}, \\ E_a &= f(x, t, \mu(x, t)). \end{aligned} \tag{23}$$

Table 1: The exact symmetries of the given partial differential equation (PDE).

Coefficients	Monomials
$\varphi_t - 6\mu\varphi_x + \varphi_{xxx} = 0$	1
$-\phi_t - 6\varphi - 6\mu(\varphi_\mu - \phi_x) + 3\varphi_{xx\mu} - \phi_{xxx} + 6\mu(\varphi_\mu - 3\phi_x) = 0$	μ_x
$-\phi_\mu + 6\mu\varrho_\mu - 3\varrho_{xx\mu} + 4\phi_\mu = 0$	$\mu_x\mu_t$
$\varphi_\mu - \varrho_t + 6\mu\varrho_x - \varrho_{xxx} - (\varphi_\mu - 3\phi_x) = 0$	μ_t
$-\varrho_\mu + \varrho_\mu = 0$	μ_t^2
$6\mu\phi_\mu + 3\varphi_{x\mu\mu} - 3\phi_{xx\mu} - 24\mu\phi_\mu = 0$	μ_x^2
$3\varphi_{x\mu} - 3\phi_{xx} = 0$	μ_{xx}
$-3\varrho_{xx} = 0$	μ_{xt}
$\varphi_{\mu\mu\mu} - \phi_{x\mu\mu} = 0$	μ_x^3
$-3\varrho_{x\mu\mu} = 0$	$\mu_x^2\mu_t$
$3\varphi_{\mu\mu} - 9\phi_{x\mu} = 0$	$\mu_x\mu_{xx}$
$-6\varrho_{x\mu} - 6\mu\varrho_\mu = 0$	$\mu_x\mu_{xt}$
$-3\varrho_{x\mu} = 0$	$\mu_t\mu_{xx}$
$-3\varrho_x = 0$	μ_{xxt}
$\phi_{\mu\mu\mu} = 0$	μ_x^4
$\varrho_{\mu\mu\mu} = 0$	$\mu_x^3\mu_t$
$\phi_{\mu\mu} = 0$	$\mu_x^2\mu_{xx}$
$\varrho_{\mu\mu} = 0$	$\mu_x^2\mu_{xt}$
$\varrho_{\mu\mu} = 0$	$\mu_x\mu_t\mu_{xx}$
$\varrho_\mu = 0$	$\mu_x\mu_{xxt}$
$\phi_\mu = 0$	μ_{xx}^2
$\varrho_\mu = 0$	$\mu_{xx}\mu_{xt}$

Table 2: Lie symmetry generator of KdV equation.

Lie symmetry generators
$X_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)\mu(\partial/\partial\mu)$
$X_2 = (\partial/\partial t)$
$X_3 = (\partial/\partial)$
$X_4 = (\partial/\partial x)$

Equation (22) can now be written in a more compact form as

$$E_e + \varepsilon E_a = 0. \quad (24)$$

On similar footing, we can combine the exact and approximate Lie symmetries as

$$\mathbf{X} = \mathbf{X}_e + \varepsilon\mathbf{X}_a. \quad (25)$$

Here,

$$\mathbf{X}_e = \phi_e\frac{\partial}{\partial x} + \varrho_e\frac{\partial}{\partial t} + \varphi_e\frac{\partial}{\partial\mu}, \quad (26)$$

is the exact Lie symmetry generator, and

$$\mathbf{X}_a = \phi_a\frac{\partial}{\partial x} + \varrho_a\frac{\partial}{\partial t} + \varphi_a\frac{\partial}{\partial\mu} \quad (27)$$

is the approximate Lie symmetry generator. Furthermore, ϕ, ϱ, and φ are the unknown functions of x, t, and μ, respectively.

Now, applying the generator $\mathbf{X}$ on (24), we have

$$(\mathbf{X}_e + \varepsilon\mathbf{X}_a)(E_e + \varepsilon E_a) = 0, \quad (28)$$

which yields

$$\mathbf{X}_e E_e + \varepsilon(\mathbf{X}_a E_e + \mathbf{X}_e E_a) + O(\varepsilon^2) = 0. \quad (29)$$

The comparison of coefficients of ε^0 and ε^1, respectively, yields the exact and approximate symmetries of the corresponding PDEs as in the following:

$$\begin{aligned} \mathbf{X}_e E_e &= 0, \\ \mathbf{X}_a E_e + \mathbf{X}_e E_a &= 0. \end{aligned} \quad (30)$$

The latter equation additionally gives the approximate Lie symmetries, which will not only provide the approximate conservation laws involved in the dynamics of the KdV equation but will also give the unknown function $f(x, t, \mu(x,t), \mu_i(t,x))$ [8, 10].

4. Approximate Symmetries and Corresponding Conservation Laws of the KdV Equation

In this section, we apply the developed method to find out the approximate symmetries. This method is applied and discussed for different cases. Considering the perturbed KdV equation [6, 11, 12],

$$\mu_t - 6\mu\mu_x + \mu_{xxx} + \varepsilon f(x, y, n, t, n_t, n_x, m_t, m_x) = 0. \quad (31)$$

By employing the method developed in [13–15] for the expansion of μ,

$$\mu = m + \varepsilon n. \quad (32)$$

Using this expansion in (31),

$$\begin{aligned}
&(m_t+\varepsilon n_t)-6(m+\varepsilon n)(m_x+\varepsilon n_x)+(m_{xxx}+\varepsilon n_{xxx})=\varepsilon f(x,y,n,t,n_t,n_x,m_t,m_x),\\
&m_t+\varepsilon n_t-6mm_x-6\varepsilon mn_x-6\varepsilon nm_x-6\varepsilon^2 nn_x+m_{xxx}+\varepsilon n_{xxx}=\varepsilon f(x,y,n,t,n_t,n_x,m_t,m_x),\\
&(m_t-6mm_x+m_{xxx})+\varepsilon(n_t-6mn_x-6nm_x+n_{xxx})+\varepsilon^2(-6nn_x)=\varepsilon f(x,y,n,t,n_t,n_x,m_t,m_x).
\end{aligned} \tag{33}$$

Equation (33) in more compact form is (neglecting higher power of ϵ)

$$\Delta_e+\varepsilon\Delta_a=0. \tag{34}$$

The comparison of the coefficients of ϵ^0 and ϵ^1 in (33) gives

$$\begin{aligned}
\Delta_e &:= m_t-6mm_x+m_{xxx}=0,\\
\Delta_a &:= n_t-6mn_x-6nm_x+n_{xxx}-f(x,y,n,t,n_t,n_x,m_t,m_x)=0.
\end{aligned} \tag{35}$$

The Lie symmetry generator is

$$\mathbf{X}=\mathbf{X}_e+\varepsilon\mathbf{X}_a=0. \tag{36}$$

Here,

$$\begin{aligned}
\mathbf{X}_e &= \phi_e\frac{\partial}{\partial x}+\varrho_e\frac{\partial}{\partial t}+\varphi_e\frac{\partial}{\partial m}+\phi_e\frac{\partial}{\partial n},\\
\mathbf{X}_a &= \phi_a\frac{\partial}{\partial x}+\varrho_a\frac{\partial}{\partial t}+\varphi_a\frac{\partial}{\partial m}+\phi_a\frac{\partial}{\partial n}.
\end{aligned} \tag{37}$$

Applying the Lie generator,

$$\begin{aligned}
\mathbf{X}(\Delta_e+\varepsilon\Delta_a)&=0,\\
(\mathbf{X}_e+\varepsilon\mathbf{X}_a)(\Delta_e+\varepsilon\Delta_a)&=0,
\end{aligned} \tag{38}$$

which gives us

$$\begin{aligned}
\mathbf{X}_e\Delta_e+\varepsilon(\mathbf{X}_a\Delta_e+\mathbf{X}_e\Delta_a)+o(\varepsilon^2)&=0,\\
\mathbf{X}_e\Delta_e&=0,\\
\mathbf{X}_a\Delta_e+\mathbf{X}_e\Delta_a&=0.
\end{aligned} \tag{39}$$

We now discuss the following cases in a bit detail.

Case I. Let

$$f(x,y,n,t,n_t,n_x,m_t,m_x)=-m_t-n_t. \tag{40}$$

Then, determining the system of PDEs from (35),

$$\begin{aligned}
\varrho_{tt}&=0,\\
\varrho_m&=0,\\
\phi_t&=0,\\
\varrho_n&=0,\\
\phi_n&=0,\\
\phi_m&=0,\\
\varrho_x&=0,\\
\phi_x&=\frac{3}{\varrho_t},\\
\varphi&=\frac{-2}{3}m\varrho_t,\\
\phi&=-\frac{2}{3}\varrho_t n.
\end{aligned} \tag{41}$$

As

$$\begin{aligned}
\frac{\partial\varrho}{\partial m}&=0,\\
\frac{\partial\varrho}{\partial n}&=0,\\
\frac{\partial\varrho}{\partial x}&=0,
\end{aligned} \tag{42}$$

which implies that "ϱ" is the function of "t" alone. Therefore,

$$\varrho_{tt} = 0. \tag{43}$$

Integrating the above equation twice with respect to "t" yields

$$\varrho = c_1 t + c_2. \tag{44}$$

Also,

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= 0, \\ \frac{\partial \phi}{\partial m} &= 0, \\ \frac{\partial \phi}{\partial n} &= 0, \end{aligned} \tag{45}$$

which shows that "ϕ" is the function of "x" alone. Therefore,

$$\frac{\partial \phi}{\partial x} = \frac{1}{3}\varrho t. \tag{46}$$

Putting the value of "ϱ_t" in (46), we get

$$\frac{\partial \phi}{\partial x} = \frac{1}{3}c_1. \tag{47}$$

Integrating (47), we get

$$\phi = \frac{1}{3}c_1 x + c_3. \tag{48}$$

Now,

$$\varphi = -\frac{2}{3}m\varrho_t. \tag{49}$$

Putting the value of "ϱ_t" in (49),

$$\varphi = -\frac{2}{3}mc_1. \tag{50}$$

By taking

$$\phi = -\frac{2}{3}\varrho_t n, \tag{51}$$

and putting the value of "ϱ_t" in (51),

$$\phi = -\frac{2}{3}c_1 n. \tag{52}$$

Therefore,

$$\begin{aligned} \phi &= \frac{1}{3}c_1 x + c_3, \\ \varrho &= c_1 t + c_2, \\ \varphi &= \frac{-2}{3}mc_1, \\ \phi &= \frac{-2}{3}c_1 n. \end{aligned} \tag{53}$$

The corresponding symmetry generators are tabulated in Table 3.

4.1. Conservation Laws. The conservation laws are developed as in the following:

$$\begin{aligned} &\mathbf{X}_1\left(\psi\left(x, y, n, t, n_t, n_x, m_t, m_x\right)\right) = 0, \\ &\left(\frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{\partial}{\partial n} - \frac{2}{3}m\frac{\partial}{\partial m}\right)\psi = 0, \\ &\frac{1}{3}x\psi_x + t\psi_t - \frac{2}{3}\psi_n - \frac{2}{3}m\psi_m = 0, \\ &3\frac{dx}{x} = \frac{dt}{t} = \frac{dn}{(-2/3)} = \frac{-3}{2}\frac{dm}{m} = \frac{d\psi}{0}. \end{aligned} \tag{54}$$

Now, by taking

$$\begin{aligned} &3\frac{dx}{x} = \frac{dt}{t} \Rightarrow x^3 = c_1 t \Rightarrow c_1 = \frac{x^3}{t}, \\ &3\frac{dx}{x} = \frac{dn}{(-2/3)} \Rightarrow \ \ln x^3 = -\frac{2}{3}n + c_2 \Rightarrow c_2 = x^3 e^{(3/2)}n, \\ &3\frac{dx}{x} = \frac{-3}{2}\frac{dm}{m} \Rightarrow x^3 = c_3 m^{-}\frac{3}{2} \Rightarrow c_3 = x^3 m^{(3/2)}, \\ &\frac{dt}{t} = \frac{dn}{(-2/3)} \Rightarrow \ \ln t = -\frac{2}{3}n + c_4 \Rightarrow c_4 = te^{(3/2)}n, \\ &\frac{dt}{t} = \frac{-3}{2}\frac{dm}{m} \Rightarrow t = c_5 m^{-}\frac{3}{2} \Rightarrow c_5 = tm\frac{3}{2}, \\ &\frac{dn}{(-2/3)} = \frac{-3}{2}\frac{dm}{m} \Rightarrow n + c_6 = \ln m \Rightarrow c_6 = me^{-n}, \end{aligned} \tag{55}$$

so

$$\begin{aligned} \psi = c_1 + c_2 + c_3 &= \frac{x^3}{t} + x^3 e^{(3/2)}n + x^3 m^{(3/2)} \\ &+ te^{(3/2)}n + tm\frac{3}{2} + me^{-n}. \end{aligned} \tag{56}$$

Furthermore,

$$\begin{aligned} &X_2\left(\psi\left(x, y, n, t, n_t, n_x, m_t, m_x\right)\right) = 0, \\ &\qquad \psi_t = 0 \Rightarrow \psi = c, \\ &X_3\left(\psi\left(x, y, n, t, n_t, n_x, m_t, m_x\right)\right) = 0, \\ &\qquad \psi_x = 0 \Rightarrow \psi = c. \end{aligned} \tag{57}$$

Following are the symmetries and their corresponding conservation laws of Case 1.

Case 2. Let

$$f\left(x, y, n, t, n_t, n_x, m_t, m_x\right) = -m_x. \tag{58}$$

Table 3: Lie symmetry generators.

Lie symmetry generators
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)(\partial/\partial n) - (2/3)m(\partial/\partial m)$
$\mathbf{X}_2 = (\partial/\partial t)$
$\mathbf{X}_3 = (\partial/\partial x)$

From (35), we get after comparing the coefficients of ϵ^0 and ϵ^1,

$$\begin{aligned} m_t - 6mm_x + m_{xxx} &= 0, \\ n_t - 6mn_x - 6nm_x + n_{xxx} - m_x &= 0. \end{aligned} \tag{59}$$

Applying (36) to (59) yields the following system of PDEs:

$$\begin{aligned} &\phi_t = 0, \\ &\phi_n = 0, \\ &\phi_m = 0, \\ &\phi_m = 0, \\ &\phi_n = \frac{6\phi}{n+1}, \\ &\phi_x = \frac{1}{3}\varrho_t, \\ &\phi_x = 0, \\ &\varrho_m = 0, \\ &\phi_{tt} = 0, \\ &\varrho_n = 0, \\ &\varphi = \frac{-2}{3}\varrho_t m - \frac{1}{6}\phi_t \varrho_x = 0, \\ &\varrho_{tt} = 0. \end{aligned} \tag{60}$$

Solving the above system of PDEs, we get the following results:

$$\begin{aligned} &\varphi = \frac{-2}{3}c_1 m - \frac{1}{6}c_4, \\ &\phi = 6c_3 n + c_3, \\ &\varrho = c_1 t + c_2, \\ &\phi = \frac{1}{3}c_1 x + c_4 t + c_5. \end{aligned} \tag{61}$$

The approximate symmetries and their corresponding conservation laws in this case are given in Table 4.

Case 3. For this case, take

$$f(x, y, n, t, n_t, n_x, m_t, m_x) = -n_x. \tag{62}$$

From (35), we get after comparing the coefficients of ϵ^0 and ϵ^1,

$$\begin{aligned} m_t - 6mm_x + m_{xxx} &= 0, \\ n_t - 6mn_x - 6nm_x + n_{xxx} - n_x &= 0. \end{aligned} \tag{63}$$

This results in the following equations:

$$\begin{aligned} &\phi_t = 0, \\ &\phi_{tt} = 0, \\ &\phi_m = 0, \\ &\phi_x = 0, \\ &\phi_n = \frac{\phi}{n}, \\ &\phi_n = 0, \\ &\phi_x = 0, \\ &\varrho_t = 0, \\ &\phi_m = 0, \\ &\varrho_m = 0, \\ &\varphi = \frac{-1}{6}\phi_t \varrho_n = 0, \\ &\varrho_x = 0, \\ &\varphi = \frac{-1}{6}c_1, \\ &\phi = c_3 n, \\ &\varrho = c_4, \\ &\phi = c_1 t + c_2. \end{aligned} \tag{64}$$

Following are the symmetries and corresponding conservation laws of this Case 3.

Case 4. For this case, take

$$f(x, y, n, t, n_t, n_x, m_t, m_x) = mn, \tag{65}$$

then the system defined in (35) gives

$$\begin{aligned} m_t - 6mm_x + m_{xxx} &= 0, \\ n_t - 6mn_x - 6nm_x + n_{xxx} + mn &= 0. \end{aligned} \tag{66}$$

Applying (36) to (66), we get the following set of PDEs:

Table 4: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)m(\partial/\partial m)$	$\psi_1 = (x^3/t) + (x\sqrt{m})^3 + tm^{(3/2)}$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = f(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (6n+1)(\partial/\partial n)$	$\psi_3 = g(x, y, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_4 = t(\partial/\partial x) - (1/6)(\partial/\partial m)$	$\psi_4 = (x/t) + 6m$
$\mathbf{X}_5 = (\partial/\partial x)$	$\psi_5 = h(y, n, t, n_t, n_x, m_t, m_x)$

$$\begin{aligned}
&\varrho_t = 0,\\
&\phi_t = \frac{1}{6}n\phi_t,\\
&\phi_x = 0,\\
&\phi_m = 0,\\
&\varrho_x = 0,\\
&\phi_n = \frac{\phi}{n},\\
&\phi_{tt} = 0,\\
&\phi_x = 0,\\
&\varrho_m = 0,\\
&\phi_m = 0,\\
&\varrho_n = 0,\\
&\phi_n = 0,\\
&\varphi = -\frac{1}{6}\phi_t.
\end{aligned} \tag{67}$$

The above equations yield

$$\begin{aligned}
&\phi = c_1 t + c_2,\\
&\varrho = c_3,\\
&\varphi = \frac{-1}{6}c_1,\\
&\phi = \frac{1}{6}n\left(c_1 t + 6c_4\right).
\end{aligned} \tag{68}$$

Following are the symmetries and corresponding conservation laws of Case 4.

Case 5. Let

$$f\left(x, y, n, t, n_t, n_x, m_t, m_x\right) = -n, \tag{69}$$

then the system defined in (35) gives

$$\begin{aligned}
m_t - 6mm_x + m_{xxx} &= 0,\\
n_t - 6mn_x - 6nm_x + n_{xxx} - n &= 0.
\end{aligned} \tag{70}$$

Applying (36) to (70) produces the following set of PDEs:

$$\begin{aligned}
&\phi_t = \varrho_t n,\\
&\phi_n = 0,\\
&\phi_m = 0,\\
&\phi_m = 0,\\
&\phi_n = \frac{\phi}{n},\\
&\varrho_{tt} = 0,\\
&\phi_x = 0,\\
&\varrho_m = 0,\\
&\phi_x = \frac{1}{3}\varrho_t\varrho_n = 0,\\
&\varrho_x = 0.
\end{aligned} \tag{71}$$

Solving the above equations, we get

$$\begin{aligned}
&\phi = \frac{c_1 x}{3} + tc_3 + c_4,\\
&\varrho = c_1 t + c_2,\\
&\varphi = \frac{-2}{3}m\varrho_t c_1 - \frac{1}{6}\phi_t c_3,\\
&\phi = n\left(c_1 t + c_5\right).
\end{aligned} \tag{72}$$

Following are the symmetries and corresponding conservation laws of Case 5.

Case 6. Assume

$$f\left(x, y, n, t, n_t, n_x, m_t, m_x\right) = -nm_t, \tag{73}$$

then the system defined in (35) gives

$$\begin{aligned}
m_t - 6mm_x + m_{xxx} &= 0,\\
n_t - 6mn_x - 6nm_x + n_{xxx} - nm_t &= 0.
\end{aligned} \tag{74}$$

Applying (36) to (74) results in the following set of PDEs:

Table 5: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) + nt(\partial/\partial n) - (2/3)m(\partial/\partial m)$	$\psi_1 = (x^3/t) + x^3 + x^3 m^{(3/2)} + e^t n + tm^{(3/2)}$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = f(x, y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial x) - (1/6)(\partial/\partial m)$	$\psi_3 = x + 6m$
$\mathbf{X}_4 = (\partial/\partial x)$	$\psi_4 = g(y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_5 = n(\partial/\partial n)$	$\psi_5 = h(x, y, t, n_t, n_x, m_t, m_x)$

Table 6: Lie symmetry generator and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)(\partial/\partial n) - (2/3)m(\partial/\partial m)$	$\psi_1 = (x^3/t) + x^3 e^{(3/2)} n + x^3 m^{(3/2)} + te^{(3/2)} n + tm^{(3/2)} + me^{-n}$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = f(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial x)$	$\psi_3 = g(y, n, t, n_t, n_x, m_t, m_x)$

Table 7: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (-1/6)(\partial/\partial x) + t(\partial/\partial t)$	$\psi_1 = (e^{-6x}/t)$
$\mathbf{X}_2 = x(\partial/\partial x)$	$\psi_2 = f(y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = n(\partial/\partial n)$	$\psi_3 = g(x, y, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_4 = (\partial/\partial t)$	$\psi_4 = h(x, y, n, n_t, n_x, m_t, m_x)$

Table 8: Lie symmetry generator and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = t(\partial/\partial x) + (1/6)nt(\partial/\partial n) - (1/6)(\partial/\partial m)$	$\psi_1 = (e^{(x/t)}/n^6) + ne^m + (x/t) + 6m$
$\mathbf{X}_2 = (\partial/\partial x)$	$\psi_2 = f(y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial t)$	$\psi_3 = g(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_4 = n(\partial/\partial n)$	$\psi_4 = h(x, y, t, n_t, n_x, m_t, m_x)$

Table 9: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = n(\partial/\partial n)$	$\psi_1 = f(x, y, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = g(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial x)$	$\psi_3 = h(y, n, t, n_t, n_x, m_t, m_x)$

$$\begin{aligned}
\varrho_{tx} &= 0,\\
\phi_t &= 0,\\
\phi_x &= 0,\\
\phi_m &= 0,\\
\phi_n &= 0,\\
\phi_n &= \frac{\phi}{n},\\
\phi_m &= 0,\\
\varrho_n &= 0,\\
\phi_x &= 0,\\
\varrho_t &= 0,\\
\phi_t &= 0,\\
\varrho_m &= 0.
\end{aligned} \tag{75}$$

Solving the above set of equations, we get

$$\begin{aligned}
\phi &= c_3,\\
\varrho &= c_2,\\
\varphi &= 0,\\
\phi &= c_1 n.
\end{aligned} \tag{76}$$

Following are the symmetries and corresponding conservation laws of Case 6.

5. Conclusion

The KdV equation is a 3rd order nonlinear partial differential equation which is modeled for waves on the surface of shallow water. It admits four Lie symmetries given in Table 2. In this paper, approximate symmetry techniques are used for finding some classes of the KdV equations that admit more symmetries as compared to the exact KdV equations. We perturbed the KdV equation by different particular functions and found the corresponding Lie symmetries. We found two important classes for the perturbed KdV equation that admits five Lie symmetries. The Lie symmetries along with their conservation laws are given in Tables 2, 4 and 5. In both the tables, we have an extra symmetry which corresponds to an extra conservation law. This extra conservation law is an extra information hidden in the system, the perturbation procedure explored it. Sometimes, the symmetry does not exist for the exact equation, but perturbation enables the equation to admit a symmetry. We saw this phenomenon in this research work by comparing Tables 2–6. Table 1 contains the determining PDEs which provide the set of Lie symmetries admitted by the given PDE. We have 4 Lie symmetries given in Table 2 for exact PDE, while Tables 3 and 6 contain only three Lie symmetries; in these cases, we lose one symmetry (one conservation law). Tables 7 and 8 consist of four Lie symmetries which means that all the conservation laws are recovered in these cases. Table 9 includes the lie symmetry generators and corresponding conservation laws.

Data Availability

There are no specific data used in the study of this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] H. Liu, J. Li, and L. Liu, "Lie symmetry analysis, optimal systems and exact solutions to the fifth-order KdV types of equations," *Journal of Mathematical Analysis and Applications*, vol. 368, no. 2, pp. 551–558, 2010.

[2] T. A. Abassy, M. A. El-Tawil, and H. K. Saleh, "The solution of KdV and MKdV equations using adomian pade approximation," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 5, no. 4, pp. 327–340, 2004.

[3] V. Baikov, R. Gazizov, and N. Ibragimov, "Approximate symmetries," *Matematicheskii Sbornik*, vol. 136, no. 3, pp. 435–450, 1988.

[4] V. A. Baikov, R. K. Gazizov, and N. K. Ibragimov, "Perturbation methods in group analysis," *Journal of Soviet Mathematics*, vol. 55, no. 1, pp. 1450–1490, 1991.

[5] V. A. Baikov and N. H. Ibragimov, "Continuation of approximate transformation groups via multiple time scales method," *Nonlinear Dynamics*, vol. 22, no. 1, pp. 3–13, 2000.

[6] K. Parand, M. Dehghan, A. R. Rezaei, and S. M. Ghaderi, "An approximation algorithm for the solution of the nonlinear lane-emden type equations arising in astrophysics using hermite functions collocation method," *Computer Physics Communications*, vol. 181, no. 6, pp. 1096–1108, 2010.

[7] G. W. Bluman, A. F. Cheviakov, and S. C. Anco, *Applications Of Symmetry Methods To Partial Differential Equations*, Vol. 168, Springer, Berlin, Germany, 2010.

[8] X. R. Hu, S. Y. Lou, and Y. Chen, ""Explicit solutions from eigenfunction symmetry of the korteweg–de vries equation," *Physical Review E*, vol. 85, no. 5, Article ID 056607, 2012.

[9] A. R. Adem and C. M. Khalique, "Symmetry reductions, exact solutions and conservation laws of a new coupled KdV system," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 9, pp. 3465–3475, 2012.

[10] N. Xiaohua and P. Zuliang, "A new approach to solve perturbed nonlinear evolution equations through lie-backlund symmetry method," *Applied Mathematics-A Journal of Chinese Universities*, vol. 21, no. 1, pp. 45–51, 2006.

[11] Z. Zhi-Yong, Y. Xue-Lin, and C. Yu-Fu, "A new method to obtain approximate symmetry of nonlinear evolution equation from perturbations," *Chinese Physics B*, vol. 18, no. 7, pp. 2629–2633, 2009.

[12] X. Jiao, R. Yao, and S. Y. Lou, "Approximate similarity reduction for singularly perturbed boussinesq equation via symmetry perturbation and direct method," *Journal of Mathematical Physics*, vol. 49, no. 9, Article ID 093505, 2008.

[13] A. H. Bokhari, A. H. Kara, and F. D. Zaman, "Invariant solutions of certain nonlinear evolution type equations with

small parameters," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1075–1082, 2006.

[14] D. D. Ganji and A. Sadighi, "Application of he's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 4, pp. 411–418, 2006.

[15] J. H. He, "Addendum: new interpretation of homotopy perturbation method," *International Journal of Modern Physics B*, vol. 20, no. 18, pp. 2561–2568, 2006.

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Research Article

Qualitative Analysis of Class of Fractional-Order Chaotic System via Bifurcation and Lyapunov Exponents Notions

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This paper presents a modified chaotic system under the fractional operator with singularity. The aim of the present subject will be to focus on the influence of the new model's parameters and its fractional order using the bifurcation diagrams and the Lyapunov exponents. The new fractional model will generate chaotic behaviors. The Lyapunov exponents' theories in fractional context will be used for the characterization of the chaotic behaviors. In a fractional context, the phase portraits will be obtained with a predictor-corrector numerical scheme method. The details of the numerical scheme will be presented in this paper. The numerical scheme will be used to analyze all the properties addressed in this present paper. The Matignon criterion will also play a fundamental role in the local stability of the presented model's equilibrium points. We will find a threshold under which the stability will be removed and the chaotic and hyperchaotic behaviors will be generated. An adaptative control will be proposed to correct the instability of the equilibrium points of the model. Sensitive to the initial conditions, we will analyze the influence of the initial conditions on our fractional chaotic system. The coexisting attractors will also be provided for illustrations of the influence of the initial conditions.

1. Introduction

In the recent years, modeling chaotic and hyperchaotic systems occupy an important place in the literature and have many applications in physics, biology, electrical circuits, and many other fields [1–4]. The most used fields for the applications of chaos are modeling electrical circuits, and there exist many papers related to the implementation of the chaotic systems in this domain. Many phenomena in the real-world problems are complicated to be predicted and justify the use of chaotic models. Nowadays, there appear many tools for analyzing the chaotic systems as the phase portraits of the system using the numerical discretizations, the bifurcation diagrams to understand the influence of the models' parameters on the dynamics of the chaotic models, and the Lyapunov exponents used to determine the nature of the chaos. There exists some chaos as chaotic behaviors and hyperchaotic behaviors. As tools, we can also cite the bicoherence and the Poincare map; there is also an algorithm to focus on the initial conditions' influence. It is known that the chaos systems are sensitive to the variation of the initial conditions. Influencing initial conditions can generate a loss of chaotic behaviors or hyperchaotic behaviors. Fractional calculus has attracted much attention these years, and many fractional operators have been introduced in this new field. As operators in this field, we can cite the Riemann–Liouville derivative . We can cite the Caputo derivative [5, 6], which is the most used operator in fractional calculus due to its physical adequacy with physical problems. Other fractional operators with Mittag–Leffler kernel [7, 8] and exponential kernel exist [9] and continue to impress all the community in fractional calculus [10]. For the use of the Atangana–Baleanu derivative, see [11]. Many of them have advantages in modeling real-world problems. Modeling chaotic systems and hyperchaotic systems to capture the memories effects have constituted a new direction of research in the recent years; see [8, 12]. For the advancement of fractional calculus and its application, the readers can refer to the following papers: in [13], the authors address a new numerical scheme for solving fractional differential equations described by

Gomez–Atangana–Caputo derivative; in [14], the authors focus on the characterization of two differential fractional operators without singular kernels; and, in [15], the authors analyze an epidemic spreading model described by a fractional operator with Mittag–Leffler kernel. For recent works on numerical methods applied to partial differential equations, see [16–19].

Modeling chaotic and hyperchaotic systems are focused on the literature with integer-order derivative and the non-integer-order derivatives. We make a review of the literature in this paragraph. In [1], the authors focus on the chaotic Chua electrical circuit in the fractional version. In [20], the authors discuss and prove the algorithm to get the Lyapunov exponents in a fractional version. In [21], the authors discuss the fractional-order chaotic system and its suppression in some specific order of the fractional derivative. In [22], the authors discuss the so-called hyperchaotic chameleon system with the aid of fractional-order derivative and propose its electrical implementation. In [23], the authors propose investigations in the fractional chaotic system used in finance; the findings were interpreted financially and economically. In [24], finance and chaotic system were also interpreted in this paper. In [25], the authors investigated on fractional-order exponential jerk system and presented its electrical implementation. In [2], the authors presented a new chaotic system in integer version with multiple attractors. In [3], the authors propose a new chaotic system with a self-excited attractor. In [4], the authors proposed investigations on hidden attractors in the context of dynamical systems. In [26], the authors proposed a new simple chaotic system but with admitting a line equilibrium. In [27], the authors propose synchronization investigations using the 4D hyperchaotic jerk system. In [28], the authors propose a chaotic system with infinite equilibria located on a piecewise linear curve. In [12], the authors model the hyperchaotic system using fractional derivative with Mittag–Leffler kernel and fractional derivative with exponential derivative. In [8], the authors introduce chaos in a cancer modeling using fractional derivatives with exponential decay and Mittag–Leffler law. In [29], Baskonus et al. propose active control to stabilize a fractional-order macroeconomic model using Lyapunov direct method. See more investigations related to fractional modeling of chaotic systems using Caputo derivative, bifurcation, and Lyapunov analysis in [30], modeling class of fractional-order chaotic or hyperchaotic system with Caputo derivative in [31], analysis of a four-dimensional hyperchaotic system described by Caputo–Liouville derivative in [32], modeling Chua's electrical circuit in the fractional context in [33], and modeling chaotic processes with Caputo fractional-order derivative in [34].

In this paper, we model a chaotic system using the Caputo derivative. The main contributions of this paper are mentioned in this paragraph. First, the phase portraits are obtained using the famous predictor-corrector method valid in the fractional differential equations' discretizations. Second, the numerical scheme as the predictor-corrector method is the main contribution of this present work. Third, the small changes in our introduced fractional chaotic model have been analyzed in terms of the bifurcation diagrams. Different values of the Caputo derivative are considered; at all these values, it will be important to give the chaos's nature. In other words, we will use the Lyapunov exponents in the fractional context to decide whether we have chaotic behaviors or not. This analysis is fundamental because the Lyapunov exponents' classical theories are not valid all time in a fractional context. For example, there exist hyperchaotic systems described by the fractional operators with one positive Lyapunov exponent instead of two positive Lyapunov exponents. Fourth, we observed in our investigation that the presence of zero as the Lyapunov exponent is quasi-impossible in a fractional context. The proof of this assumption will be subject to further investigations in the future. Another contribution addressed in this paper is related to the local stability of the fractional chaotic system's equilibrium points. Due to the chaotic behaviors, all the points are not stable. Alternatively, we propose feedback control to stabilize the fractional error system after combining the slave chaotic system and the master's chaotic systems. Another contribution of the present paper is that we provide the coexisting attractors for specific values of the model's parameters at two different initial conditions.

The remainder of this paper is organized as follows. In Section 2, we recall the fractional tools used in the investigations. In Section 3, we introduce the fractional chaotic model by using the Caputo derivative. Section 4 presents the predictor corrector method proposed in the literature to discretize our fractional chaotic model. In Section 5, the phase portraits considering different fractional-order derivative values are proposed for our fractional model. In Section 6, the bifurcation diagrams for the small variation of the model's parameters are presented. In Section 7, the natures of the chaos are characterized using the Lyapunov exponents' calculation in the context of fractional calculus. In Section 8, we analyze the influence of the initial condition in the chaotic behaviors. In Section 9, the stability analysis of the fractional chaotic model's equilibrium point has been proposed, and the feedback control is presented. In Section 10, final remarks for our works are presented.

2. Fractional Operators

In this section, we make a brief recall of the fractional operators which we will use through our investigation. In this section, we will define the Caputo derivative and the Riemann–Liouville derivative.

Definition 1 (see [5, 6]). We define the Riemann–Liouville fractional integral for the function $x\colon [0,+\infty[\longrightarrow \mathbb{R}$ in the form described by

$$\left(I^{\alpha}x\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\mathrm{d}s, \tag{1}$$

with $\Gamma(.)$ representing the Gamma Euler function and we set the order as $\alpha > 0$.

Definition 2 (see [5, 6]). We define the Riemann–Liouville fractional derivative with the order $\alpha \in (0,1)$ for the function $x\colon [0,+\infty[\longrightarrow \mathbb{R}$ in the form described by

$$D^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}x(s)\mathrm{d}s, \quad (2)$$

with $\Gamma(.)$ representing the Gamma Euler function and we set the order as $\alpha\in(0,1)$.

Due to the inconveniences of the Riemann–Liouville operator, we will focus on our paper with the Caputo derivative. The description of this derivative is given in the following definition.

Definition 3 (see [5, 6]). The Caputo fractional derivative operator of order $\alpha\in(0,1)$ is symbolized as the following form when we consider a function $x\colon [0,+\infty[\longrightarrow\mathbb{R}$ in the form described by

$$D_c^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)\mathrm{d}s, \quad (3)$$

where the function $\Gamma(.)$ represents the Gamma Euler function.

For readers' interest, the Laplace transform is also fundamental for solving the fractional differential equations analytically. The following relation expresses the Laplace transform of the Caputo derivative:

$$\mathscr{L}\{(D_c^{\alpha}x)(t)\}=s^{\alpha}\mathscr{L}\{x(t)\}-s^{\alpha-1}x(0). \quad (4)$$

We set the order α with the relation $\alpha\in(0,1)$. The Caputo derivative's properties as the composition with the fractional integral, the derivative of a constant function, or the derivative of the Mittag–Leffler function can be found with more pieces of information in [5, 6]. In this paper, the fractional integral in Riemann–Liouville sense will play an important role in the discretization because the proposed method comes from the numerical scheme of this derivative; the discretization details will be found in the next sections.

3. Modeling the Fractional-Order System

In this section, we introduce our model, which we will study in the next section. An integer version of this model is a chaotic system and has been subject to investigation since 2018 in [2] and is described by the following equations [2]:

$$x'=ax-\mathrm{d}yz, \quad (5)$$

$$y'=-by+xz, \quad (6)$$

$$z'=-cz+xyz+k, \quad (7)$$

with initial conditions $x(0)=1$, $y(0)=1$, and $z(0)=1$. The strange attractor is obtained with the following values for the parameters of the previous model $a=4$, $b=9$, $c=4$, $d=1$, and $k=4$. Our new advancement here is to study the same chaotic system, but we consider its fractional version described particularly with the Caputo derivative. Therefore, in this paper, we consider the fractional differential system described as follows:

$$D_c^{\alpha}x=ax-\mathrm{d}yz, \quad (8)$$

$$D_c^{\alpha}y=-by+xz, \quad (9)$$

$$D_c^{\alpha}z=-cz+xyz+k. \quad (10)$$

We impose the initial conditions as the following forms:

$$x(0)=1,\quad y(0)=1,\quad z(0)=1. \quad (11)$$

In the above modeling, the Caputo derivative is used instead of the classical Riemann–Liouville derivative because we want to use physical initial conditions. The Riemann–Liouville derivative does not have physical initial conditions. Furthermore, the Riemann–Liouville derivative of constant value does not give zero. These inconveniences also explain in this section the use of the Caputo derivative. As it will be noticed in the phase portraits, there exist many types of chaos according to the fractional operator's order. To prove that the fractional order plays an important role in the chaos systems, notably, it allows having new attractors, contrary to the model with the integer-order derivative, where a new type of chaotic system is obtained with the variation of the models' parameters.

4. Predictor-Corrector Applied on Fractional-Order System

This section describes the numerical method that we will use to obtain the phase portraits of the fractional-order chaotic system (5)–(7). The discretizations are classical in the literature; we try to apply discretization to our model. In fractional calculus, many numerical schemes and analytical methods can be used as the homotopy methods, a domain decomposition method, the Chebyshev method, and many others. But many inconveniences of the cited methods are still to be solved due to the inconveniences in the stability and the convergences of the approximate solutions. The use of the predictor-corrector method in our system has the advantages of having Matlab codes, fundamental in chaotic and hyperchaotic systems. In the rest of this section, we use the predictor-corrector method reported by Garrappa in his review article [35]. The following can describe the solution of the fractional differential system (5)–(7):

$$x(t)=x(0)+I^{\alpha}\phi(t,x_1),\quad y(t)=y(0)+I^{\alpha}\varphi(t,x_1),\quad z(t)=z(0)+I^{\alpha}\vartheta(t,x_1). \quad (12)$$

We set the following functions from our fractional-order system (5)–(7):

$$\begin{aligned}\phi(t,x_1)&=ax-\mathrm{d}yz,\\ \varphi(t,x_1)&=-by+xz,\\ \vartheta(t,x_1)&=-cz+xyz+k,\end{aligned}\tag{13}$$

as well as point t_n; then, according to the numerical scheme named the predictor-corrector method, equations (9)–(11) can be rewritten in the following forms:

$$\begin{aligned}x(t_n)&=x(0)+h^{\alpha}\left[\overline{\kappa}_n^{(\alpha)}\phi(0)+\sum_{j=1}^{n-1}\kappa_{n-j}^{(\alpha)}\phi\left(t_j,x_{1j}\right)+\kappa_0^{(\alpha)}\phi\left(t_j,x_{1n}^P\right)\right],\\ y(t_n)&=y(0)+h^{\alpha}\left[\overline{\kappa}_n^{(\alpha)}\varphi(0)+\sum_{j=1}^{n-1}\kappa_{n-j}^{(\alpha)}\varphi\left(t_j,x_{1j}\right)+\kappa_0^{(\alpha)}\varphi\left(t_j,x_{1n}^P\right)\right],\\ z(t_n)&=z(0)+h^{\alpha}\left[\overline{\kappa}_n^{(\alpha)}\vartheta(0)+\sum_{j=1}^{n-1}\kappa_{n-j}^{(\alpha)}\vartheta\left(t_j,x_{1j}\right)+\kappa_0^{(\alpha)}\vartheta\left(t_j,x_{1n}^P\right)\right].\end{aligned}\tag{14}$$

Furthermore, the predictor has the following form in our fractional system:

$$\begin{aligned}x^P(t_n)&=x(0)+h^{\alpha}\sum_{j=1}^{n-1}\kappa_{n-j-1}^{(\alpha)}\phi\left(t_j,x_{1j}\right),\\ y^P(t_n)&=y(0)+h^{\alpha}\sum_{j=1}^{n-1}\kappa_{n-j-1}^{(\alpha)}\varphi\left(t_j,x_{1j}\right),\\ z^P(t_n)&=z(0)+h^{\alpha}\sum_{j=1}^{n-1}\kappa_{n-j-1}^{(\alpha)}\vartheta\left(t_j,x_{1j}\right).\end{aligned}\tag{15}$$

In the above formula, h denotes the step size, and the parameters of the discretization are defined in the following forms:

$$\overline{\kappa}_n^{(\alpha)}=\frac{(n-1)^{\alpha}-n^{\alpha}(n-\alpha-1)}{\Gamma(2+\alpha)},\tag{16}$$

and when the indices n describe the condition $n=1,2,\ldots$, we set the parameters as the following expressions:

$$\begin{aligned}\kappa_0^{(\alpha)}&=\frac{1}{\Gamma(2+\alpha)},\\ \kappa_n^{(\alpha)}&=\frac{(n-1)^{\alpha+1}-2n^{\alpha+1}+(n+1)^{\alpha+1}}{\Gamma(2+\alpha)}.\end{aligned}\tag{17}$$

The approximation of the functions in our model step by step is given by the expressions described in the forms

$$\begin{aligned}\phi\left(t,x_{1j}\right)&=ax_j-\mathrm{d}y_jz_j,\\ \varphi\left(t,x_{1j}\right)&=-by_j+x_jz_j,\\ \vartheta\left(t,x_{1j}\right)&=-cz_j+x_jy_jz_j+k.\end{aligned}\tag{18}$$

Before ending this section, we give a brief review concerning the method's stability and convergence; more pieces of information can be found in Garrappa's paper. We set that $x(t_n)$, $y(t_n)$, and $z(t_n)$ are the approximate solutions of the fractional system under Caputo derivative (5)–(7) and the exact solution of our model denoted by x_n, y_n, and z_n; then the residual functions as described are given in the following forms:

$$\begin{aligned}\left|x(t_n)-x_n\right|&=\mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right),\\ \left|y(t_n)-y_n\right|&=\mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right),\\ \left|z(t_n)-z_n\right|&=\mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right).\end{aligned}\tag{19}$$

It is not hard to see that when the step size converges to zero, we get the convergence of the approximate solution to the exact solutions. The predictor-corrector methods' stability can be obtained from the Lipschitz conditions of the drift functions ϕ, φ, and ϑ of our model.

The method of discretization described in this section has many advantages. Firstly, the method is stable and convergent; the Matlab implementation is useful and simple. Note that the convergence and the stability are essential in the numerical methods; comparing our method with the homotopy analysis methods, we can affirm that our method is more useful because, with the homotopy methods, we cannot determine precisely after how many iterations we have the convergence and the stability of the method. The numerical discretization described in this section is also more useful than the Laplace transform method. There are many nonlinear differential equations where the Laplace transform cannot be applied due to the complexity of the equations' forms, contrary to our described method in this section which is applicable. In the resolution of the diffusion equation, too, the use of the green function is not trivial, and here also the numerical method can be used.

5. Phase Portraits versus Fractional-Order Derivative

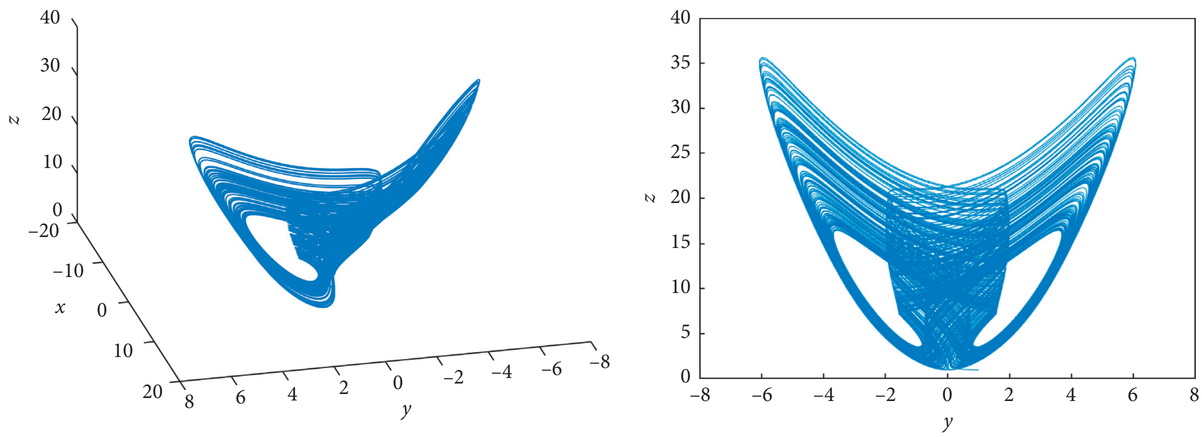
This section is devoted to observing via different phase portraits the Caputo derivative's influence in the dynamics of our fractional system (5)–(7). This section will illustrate the predictor-corrector procedure as well. To arrive at our end, we take different values of the fractional-order derivative. We consider the following values of the fractional derivative $\alpha=0.91$, $\alpha=0.93$, $\alpha=0.95$, $\alpha=0.98$, and $\alpha=0.995$. The evolution of the chaos will be observed with these different orders. We begin the representation of the phase portraits of the fractional-order system with the order $\alpha=0.95$. The graphical representations in different planes are assigned in Figure 1: $(x-y-z)$ and $(y-z)$ planes. In these first graphical representations, the considered order is $\alpha=0.95$.

The graphical representations in different planes are assigned in Figure 2: $(x-z)$ and $(x-y)$ planes.

We continue this section with the order $\alpha=0.98$. We will see the difference existing when the order of the fractional derivative varies. The graphical representations in different planes are assigned in Figure 3: $(x-y-z)$ and $(y-z)$ planes.

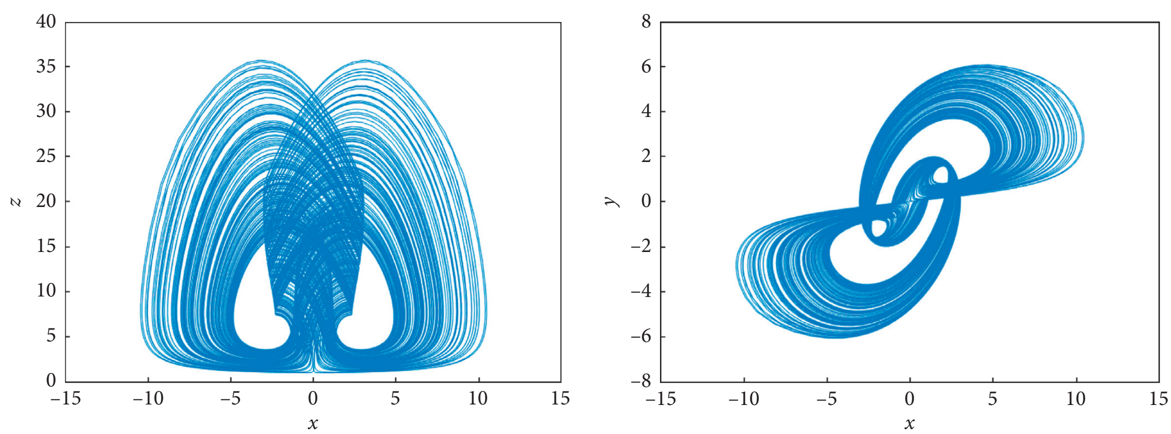
The graphical representations in different planes are assigned in Figure 4: $(x-z)$ and $(x-y)$ planes.

The first difference in the phase portrait can be observed, and we confirm the existence of new attractors. Therefore,



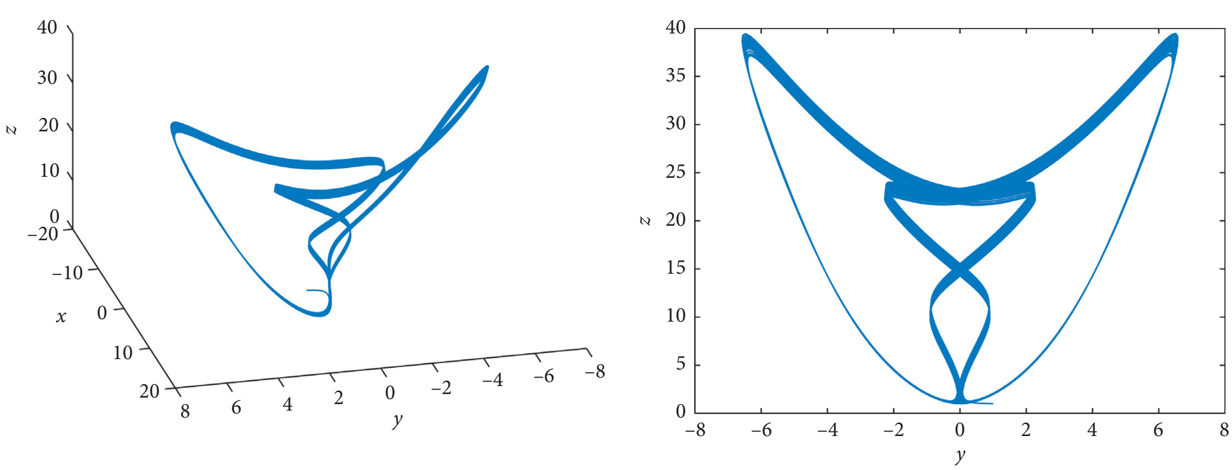
(a) (b)

Figure 1: Phase portraits of the fractional system with $\alpha = 0.95$.



(a) (b)

Figure 2: Phase portraits of the fractional system with $\alpha = 0.95$.



(a) (b)

Figure 3: Phase portraits of the fractional system with $\alpha = 0.98$.

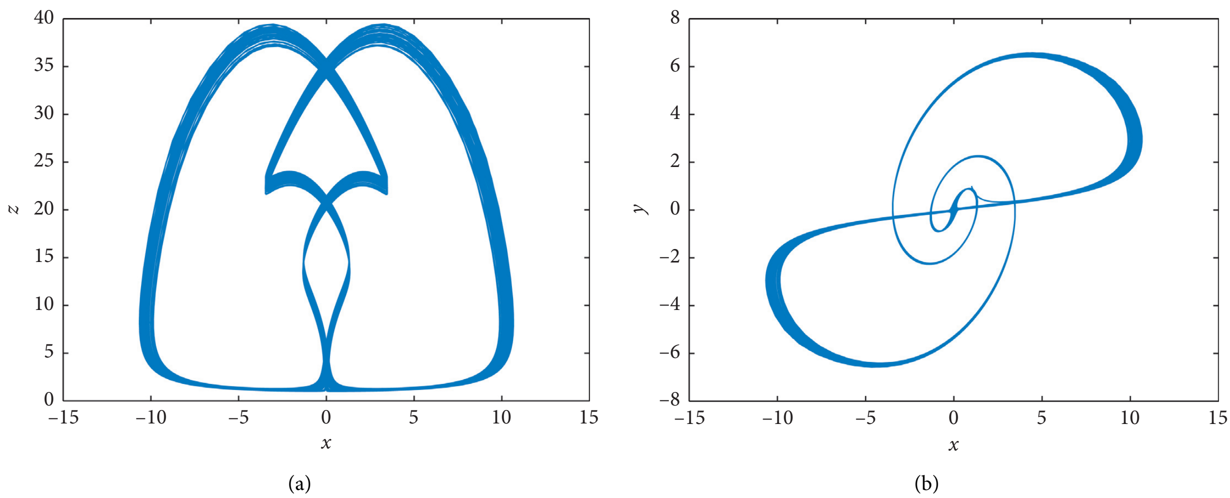


FIGURE 4: Phase portraits of the fractional system with $\alpha = 0.98$.

the fractional order can play an interesting role in the behaviors of the solutions. The graphical representations in different planes are assigned in Figure 5: $(x - y - z)$ and $(y - z)$ planes at the order $\alpha = 0.995$.

The graphical representations in different planes are assigned in Figure 6: $(x - z)$ and $(x - y)$ planes.

We finish the phase portrait section by observing the behaviors that happened behind $\alpha = 0.95$; in the next figures, we work with the order $\alpha = 0.93$. We notice in Figures 7 and 8 significant difference from the previous phase portraits, and our remark is that the chaotic behavior is not removed in the dynamics. The graphical representations in different planes are assigned in Figure 7: $(x - y - z)$ and $(y - z)$ planes.

The graphical representations in different planes are assigned in Figure 8: $(x - z)$ and $(x - y)$ planes.

The figures in this section illustrate the impact of the order of the Caputo derivative as well; this impact can be seen in the geometries of the attractors which are different in the cases considered in this section. The phase portraits represented in this section will be classified using the bifurcation maps and the Lyapunov exponents. We will notice new characterizations of chaos using the Lyapunov exponents. As will be remarked, the nature of the chaos depends on the Caputo derivative's order. In other words, the new order of the Caputo derivative generates new types of chaos.

6. Bifurcation Diagrams

In this section, we analyze the sudden qualitative changes in the nature of the solutions due to the variation of the parameters of the fractional-order system equation (5)–(7). In this section, we also illustrate the qualitative changes of the solutions of our model with the phase portraits.

In the first section, we suppose that the first parameter a of our fractional model varies in the small interval precisely in $(3, 4)$. In Figure 9 the bifurcation diagram according to the variation of parameter a is represented.

Figure 9 informs that we notice high chaotic behaviors into the interval $(3, 4)$. That is, the chaotic behaviors are not removed when the order is maintained to $\alpha = 0.95$ and the parameter has small variation. For more details, we depict phase portraits 10 of the model with parameter $a = 3.5$. The graphical representations in different planes are assigned in Figure 10: $(x - y - z)$ and $(y - z)$ planes.

The graphical representations in different planes are assigned in Figure 11: $(x - z)$ and $(x - y)$ planes.

We continue with the variation of parameter b into the interval (9.10). In Figure 12, we represent the bifurcation diagram according to the variation of parameter b.

Figure 12 informs that there exist chaotic behaviors into this interval (9.10) when parameter b has small variation. To illustrate the chaotic behaviors due to the small changes of b, we represent graphically phase portraits 13, with $b = 9.5$. The graphical representations in different planes are assigned in Figure 13: $(x - y - z)$ and $(y - z)$ planes.

The graphical representations in different planes are assigned in Figure 14: $(x - z)$ and $(x - y)$ planes.

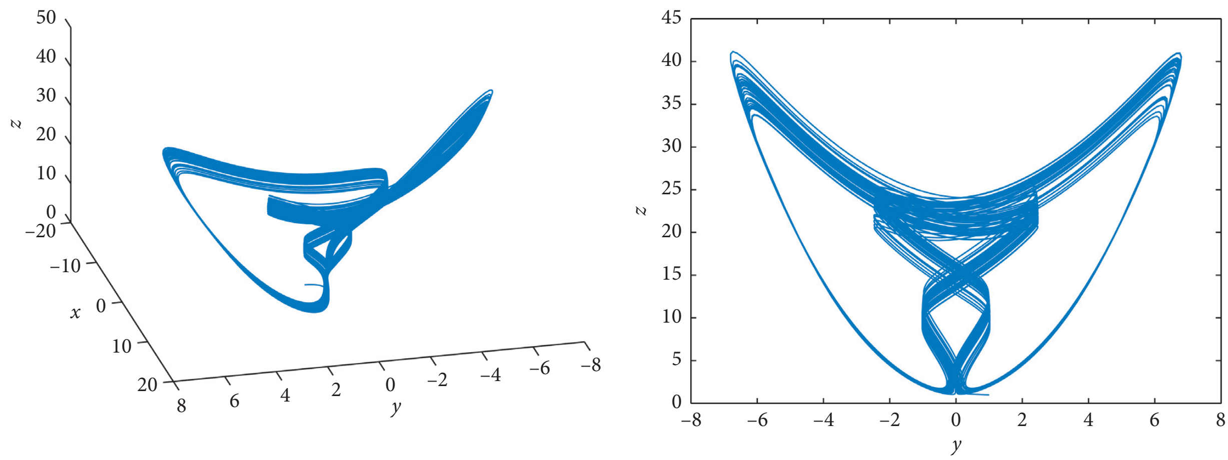
The variation of parameter c is now considered. We suppose that the parameter varies into the interval $(3, 4)$. In Figure 15, the bifurcation diagram due to the variation of parameter c is represented.

Same conclusion, the chaotic behavior is present in the considered interval and is illustrated in the following phase portraits 16, with the set $c = 3.5$. The graphical representations in different planes are assigned in Figure 16: $(x - y - z)$ and $(y - z)$ planes.

The graphical representations in different planes are assigned in Figure 17: $(x - z)$ and $(x - y)$ planes.

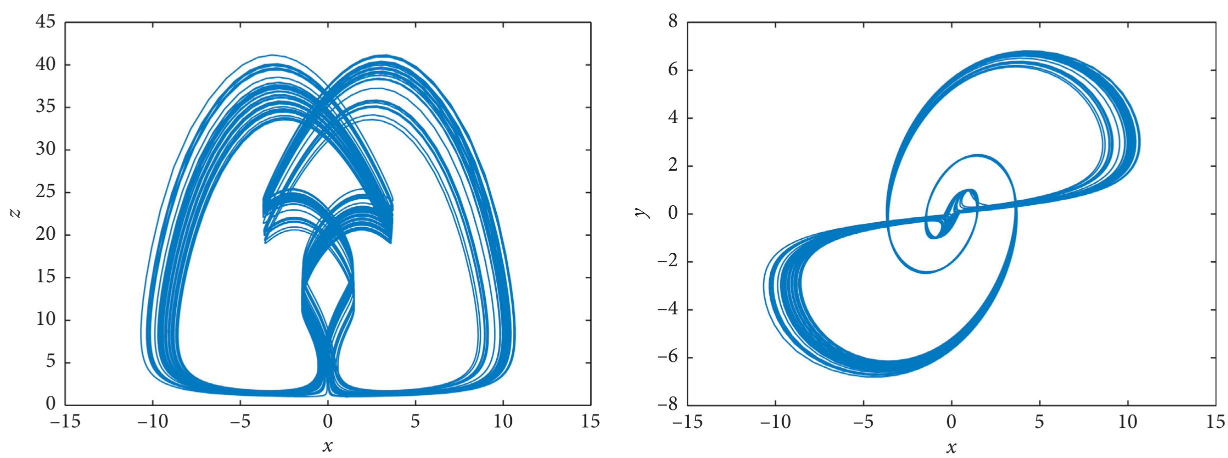
Bifurcation diagram 18 due to the variation of parameter $k = e$ into the interval $(4, 5)$ is represented in Figure 18 and we confirm the changes with phase portraits.

Phase portraits 19 are represented to illustrate the changes in the behaviors of the dynamics of our model when the parameter is into (4.5). The graphical representations in different planes are assigned in Figure 19: $(x - y - z)$ and $(y - z)$ planes.



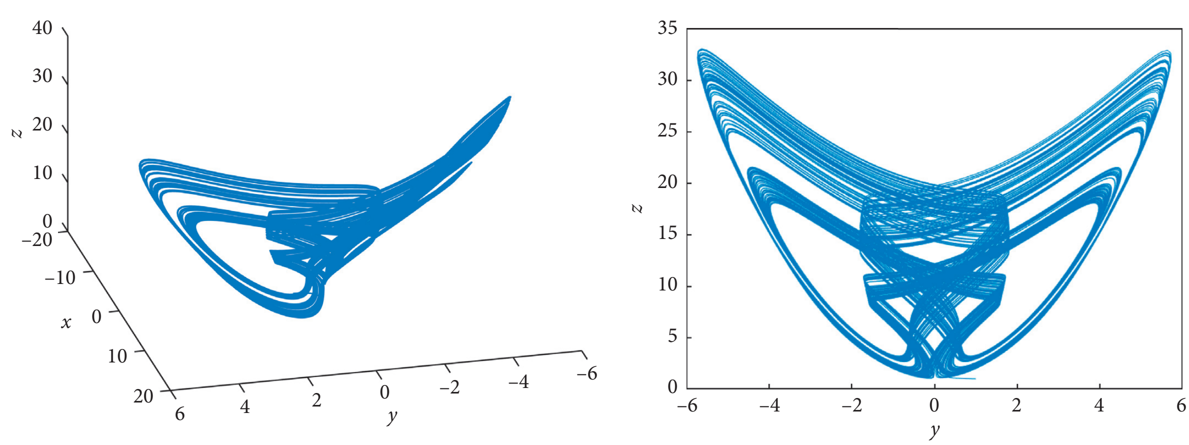
(a) (b)

Figure 5: Phase portraits of the fractional system with $\alpha = 0.995$.



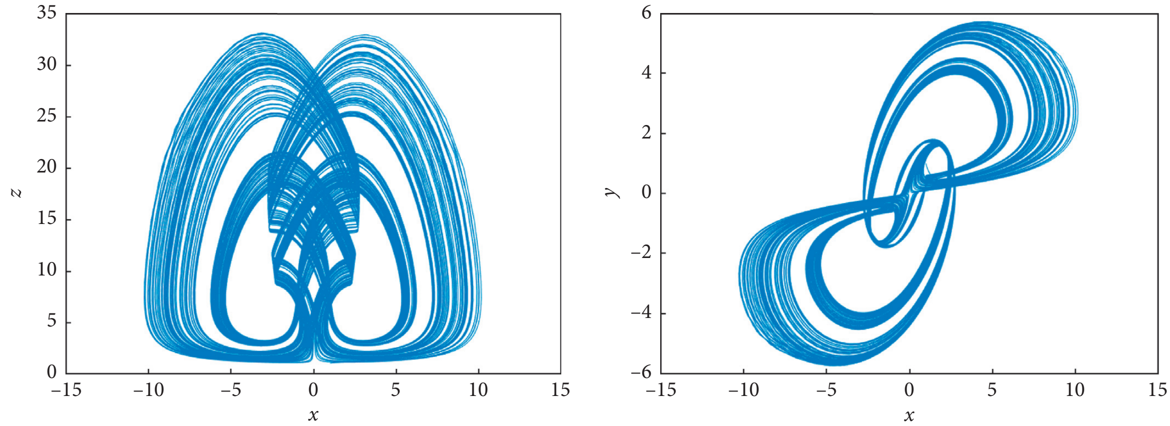
(a) (b)

Figure 6: Phase portraits of the fractional system with $\alpha = 0.995$.



(a) (b)

Figure 7: Phase portraits of the fractional system with $\alpha = 0.93$.



(a) (b)

Figure 8: Phase portraits of the fractional system with $\alpha = 0.93$.

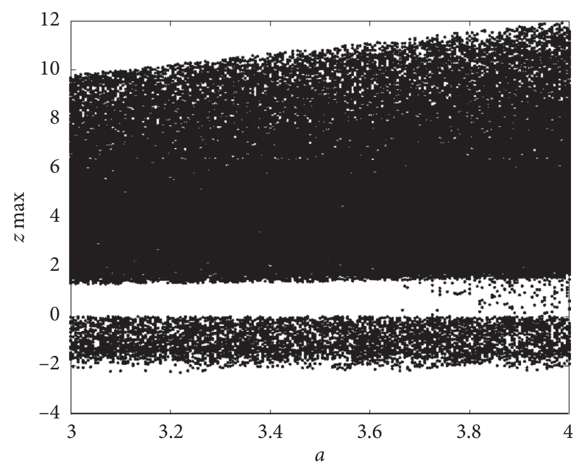
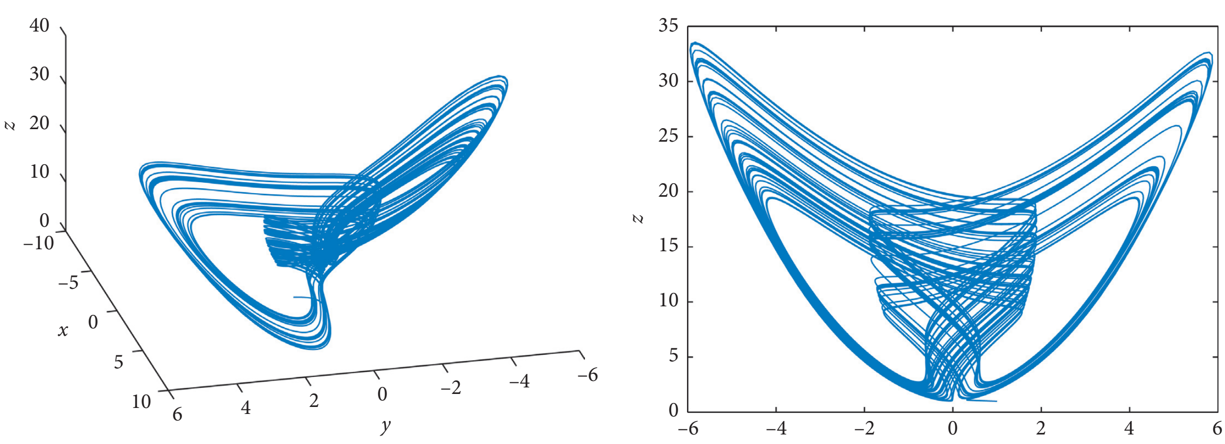


Figure 9: Bifurcation diagram according to the variation of parameter a.



(a) (b)

Figure 10: Phase portraits of the fractional system with $a = 3.5$ and $\alpha = 0.95$.

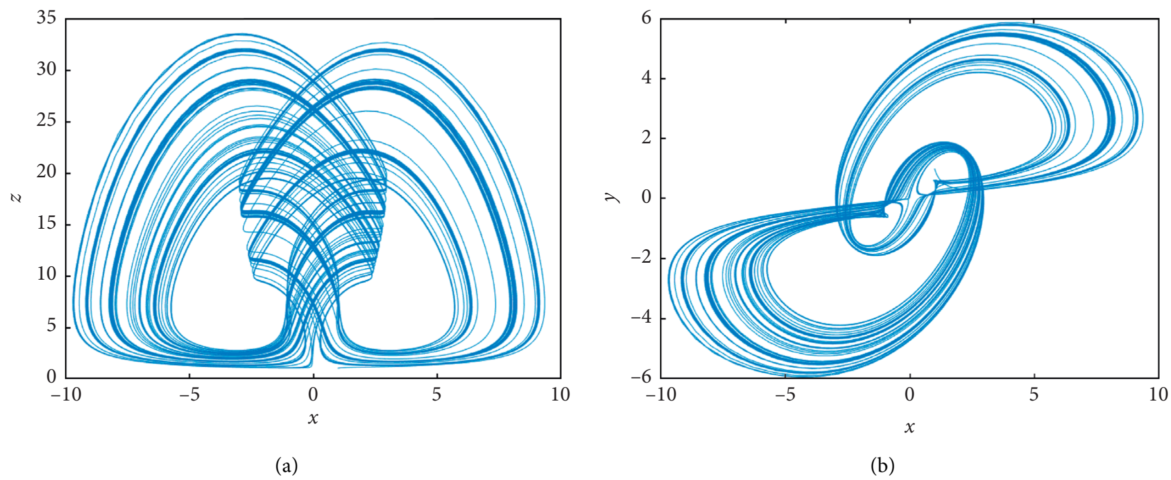


Figure 11: Phase portraits of the fractional system with $a = 3.5$ and $\alpha = 0.95$.

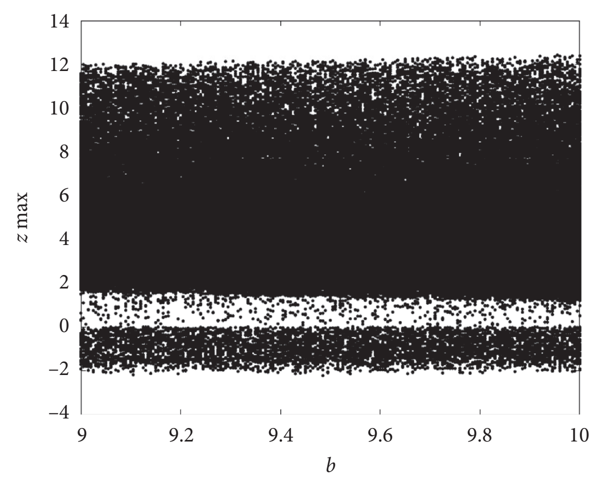


Figure 12: Bifurcation diagram according to the variation of parameter b.

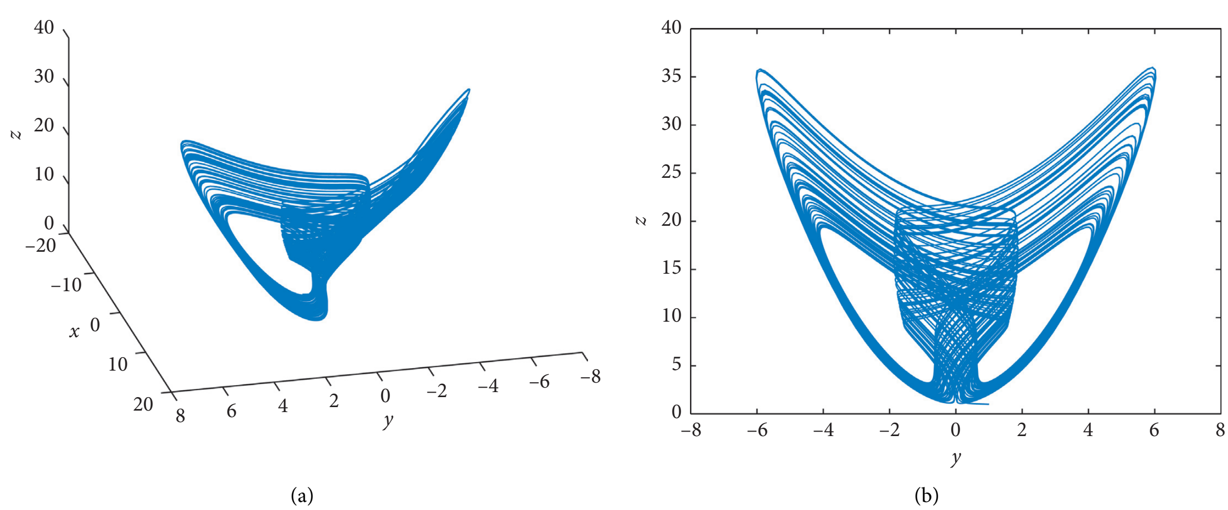
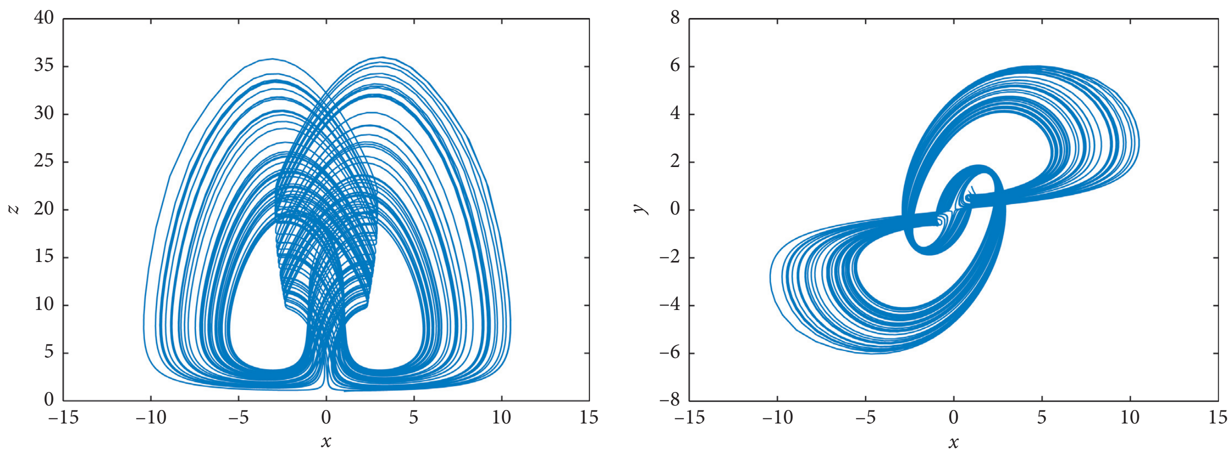


Figure 13: Phase portraits of the fractional system with $b = 9.5$ and $\alpha = 0.95$.



(a) (b)

Figure 14: Phase portraits of the fractional system with $b = 9.5$ and $\alpha = 0.95$.

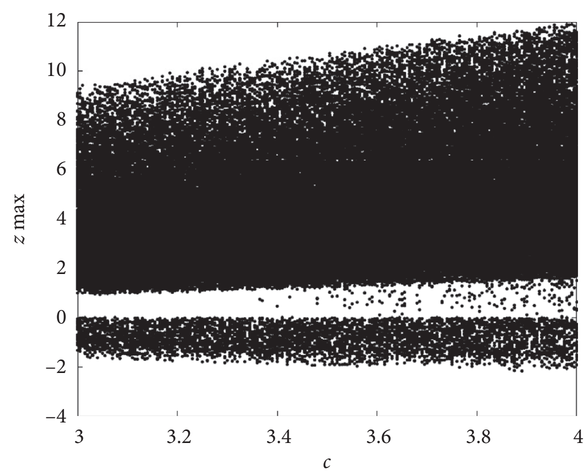
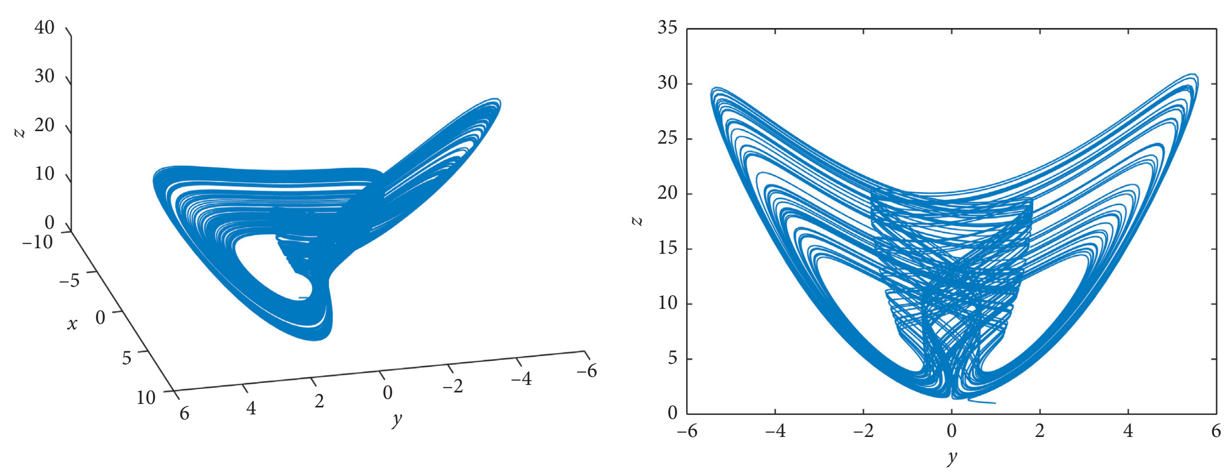


Figure 15: Bifurcation diagram according to the variation of parameter c.



(a) (b)

Figure 16: Phase portraits of the fractional system with $c = 3.5$ and $\alpha = 0.95$.

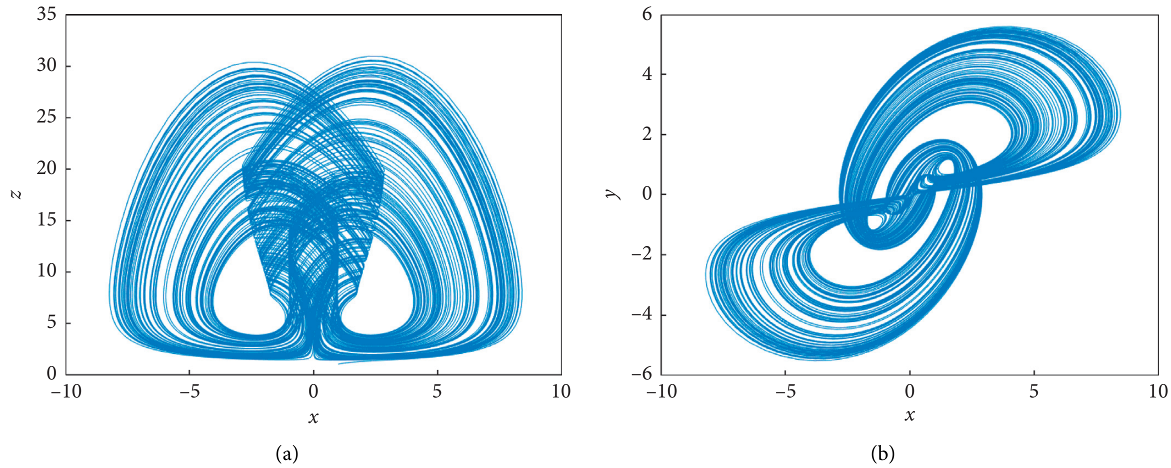


FIGURE 17: Phase portraits of the fractional system with $c = 3.5$ and $\alpha = 0.95$.

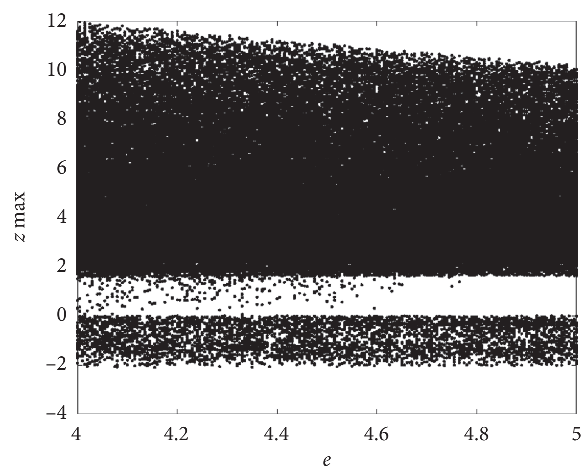


FIGURE 18: Bifurcation diagram according to the variation of parameter k.

The graphical representations in different planes are assigned in Figure 20: $(x - z)$ and $(x - y)$ planes.

The final observation using bifurcation diagrams 9, 12, 15, and 18 is that our system has chaotic behaviors. But the changes generated by parameters a, b, c, and k are approximately the same. This conclusion can be observed in the phase portraits presented in this section, which in general do not have many differences between them.

7. Chaos Detection via Lyapunov Exponents

For chaos detection, we try in this section to characterize the nature of chaos when the order of the fractional derivative varies. We calculate in particular the Lyapunov exponents at the orders $\alpha = 0.91$, $\alpha = 0.93$, $\alpha = 0.95$, $\alpha = 0.98$, and $\alpha = 0.995$. In the second part, we will localize the interval where chaotic or hyperchaotic attractors are obtained. We also calculate the dimension of the Lyapunov exponents. According to the values of the Lyapunov exponents at the order previously considered, we will calculate the sum of the Lyapunov exponents to verify whether our fractional model is dissipative or not. Before the calculations of the Lyapunov exponents, we recall the Jacobian matrix necessary for the algorithm to obtain the Lyapunov exponents; we have the following matrix:

$$J = \begin{pmatrix} a & -\mathrm{d}z & -\mathrm{d}y \\ z & -b & x \\ yz & xz & -c + xy \end{pmatrix}. \tag{20}$$

In Table 1, the Lyapunov exponents of the fractional-order system (5)–(7) are assigned according to the variations of the parameters of the fractional-order derivative.

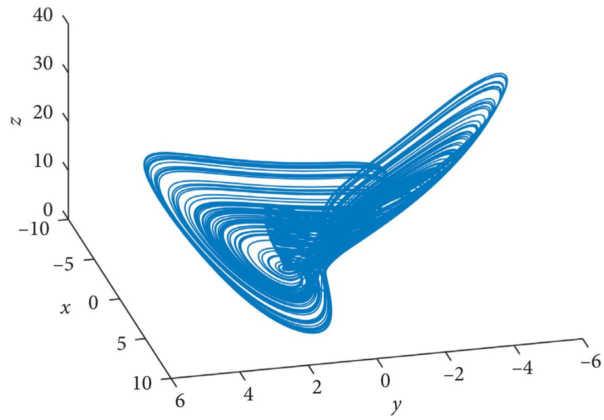
The first remark is that the Lyapunov exponents confirm the results in the bifurcation section; that is, our fractional model has chaotic behaviors. It is because there exists one positive Lyapunov exponent. The theory of Lyapunov exponents is very complex in the context of the use of the fractional operators because zero as the value of the Lyapunov exponent seems very difficult to be obtained using the algorithms to get Lyapunov exponents. This result is correct due to the complexity of the numerical scheme of the fractional operators. The second remark is that, for all considered fractional-order derivatives into $(0.9, 1)$, the sum of the Lyapunov exponents is negative, which means that the fractional-order chaotic system (5)–(7) considered in this paper is dissipative. We continue our analysis by considering the Lyapunov exponents at the order $\alpha = 0.93$. The Lyapunov exponents are given as follows:

$$\begin{aligned} \text{LE1} &= 1.6200, \\ \text{LE2} &= -0.5351, \\ \text{LE3} &= -9.4414. \end{aligned} \tag{21}$$

Their associated Kaplan–Yorke dimension is given as follows:

$$\dim(\text{LE}) = 2 + \frac{\text{LE1} + \text{LE2}}{|\text{LE3}|} = 2.1149. \tag{22}$$

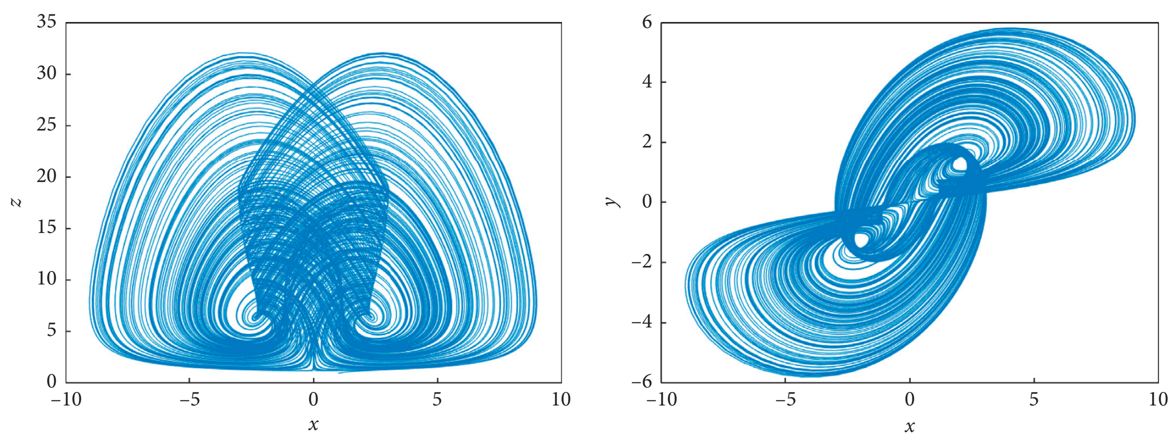
The second case is the Lyapunov exponents at the order $\alpha = 0.95$ given by the following numbers:



(a)

(b)

Figure 19: Phase portraits of the fractional system with $k = 5$ and $\alpha = 0.95$.



(a)

(b)

Figure 20: Phase portraits of the fractional system with $k = 5$ and $\alpha = 0.95$.

Table 1: Lyapunov exponents according to order α.

α	LE1	LE2	LE3
0.9	3.0972	−0.7120	−12.2030
0.91	2.2534	−0.7227	−10.4137
0.92	3.0524	−0.6030	−11.0109
0.93	1.6200	−0.5351	−9.4414
0.94	1.8800	−0.0595	−9.5865
0.95	2.9693	−0.1235	−10.2837
0.96	2.3080	−0.4528	−8.8529
0.97	2.2218	−1.1499	−7.9692
0.98	1.7822	−0.0245	−8.2439
0.99	1.8597	−0.6370	−7.0201
0.995	2.0933	−0.5649	−7.3253

$$\begin{aligned} \text{LE1} &= 2.9693, \\ \text{LE2} &= -0.1235, \\ \text{LE3} &= -10.2837. \end{aligned} \tag{23}$$

Their associated Kaplan–Yorke dimension is given as follows:

$$\dim(\text{LE}) = 2 + \frac{\text{LE1} + \text{LE2}}{|\text{LE3}|} = 2.2767. \tag{24}$$

The third case is the Lyapunov exponents at the order $\alpha = 0.98$ given by the following numbers:

$$\begin{aligned} \text{LE1} &= 1.7822, \\ \text{LE2} &= -0.0245, \\ \text{LE3} &= -8.2439. \end{aligned} \tag{25}$$

Their associated Kaplan–Yorke dimension is given as follows:

$$\dim(\text{LE}) = 2 + \frac{\text{LE1} + \text{LE2}}{|\text{LE3}|} = 2.2132. \tag{26}$$

The last case is the Lyapunov exponents at the order $\alpha = 0.995$ given by the following numbers:

$$\begin{aligned} LE1 &= 2.0933, \\ LE2 &= -0.5649, \\ LE3 &= -7.3253. \end{aligned} \tag{27}$$

Their associated Kaplan–Yorke dimension is given as follows:

$$\dim(LE) = 2 + \frac{LE1 + LE2}{|LE3|} = 2.2086. \tag{28}$$

We can notice that the chaotic attractor is more significant at the order $\alpha = 0.95$ because the positive Lyapunov exponents and the Lyapunov dimension are large. In comparison between the chaotic behaviors at $\alpha = 0.98$ and $\alpha = 0.995$, we notice by observations of the phase portraits that the chaotic behaviors are more significant when the order converges to $\alpha = 0.995$. This behavior is explained by the fact that the positive Lyapunov exponent is larger at the order $\alpha = 0.995$. The same comparison can be made for the order $\alpha = 0.95$, where the chaotic behaviors are more important than those at the order $\alpha = 0.93$. These differences can be observed with the Lyapunov exponents' values and the Lyapunov dimensions, which are larger at $\alpha = 0.95$.

8. Initial Conditions Influence and Coexistence Attractors

In this section, we analyze the impact of the initial condition. In other words, what the initial conditions give in the nature of the dynamics of our fractional system will be analyzed. The study of the changes of the initial conditions is important because chaotic and hyperchaotic systems are very sensitive to the changes in the initial conditions. We consider many cases in the initial conditions; first, we influence $x(0)$ from $x(0) = 1$ to $x(0) = 1.0001$. The illustration of this case is represented in Figure 21.

In Figure 21, the initial condition $(1, 1, 1)$ is in blue color and the initial condition $(1.0001, 1, 1)$ is in red color. We notice that significant changes can be generated by the variation of the initial condition related to $x(0)$. We continue by influencing $y(0)$ from $y(0) = 1$ to $y(0) = 1.0001$. The illustration of this case is represented in Figure 22.

In Figure 22, the initial condition $(1, 1, 1)$ is in blue color and the initial condition $(1, 1.0001, 1)$ is in red color. We notice that significant changes can be generated by the variation of the initial condition related to $y(0)$. We finish by the influence generated at the last variable $z(0)$ from $z(0) = 1$ to $z(0) = 1.0001$. The illustration of this case is represented in Figure 23.

In Figure 23, the initial condition $(1, 1, 1)$ is in blue color and the initial condition $(1, 1, 1.0001)$ is in red color. We notice that significant changes can be generated by the variation of the initial condition related to $z(0)$.

The general conclusion is that the initial conditions generate many changes in attractors. Thus, due to the fact that the Lyapunov exponents are sensitive to the initial conditions too, the values of the Lyapunov exponents will vary according to the changes in the initial conditions.

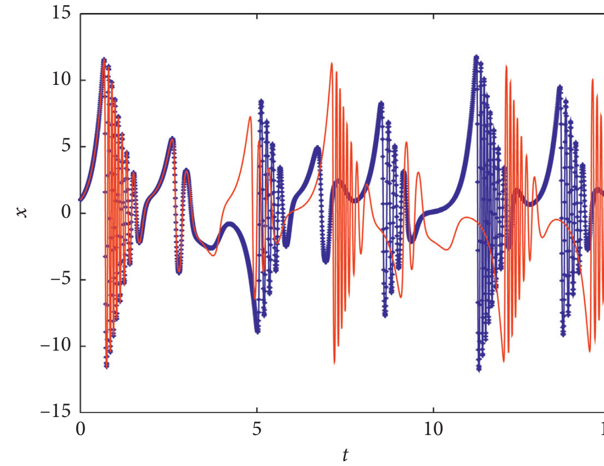


FIGURE 21: Sensitivity due to the variation of $x(0)$ from $x(0) = 1$ to $x(0) = 1.0001$.

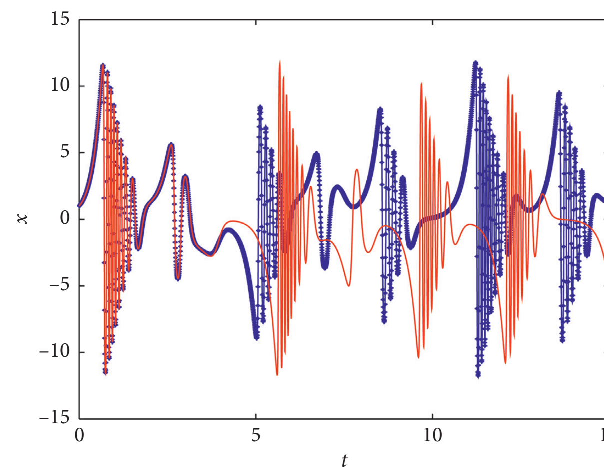


FIGURE 22: Sensitivity due to the variation of $y(0)$ from $y(0) = 1$ to $y(0) = 1.0001$.

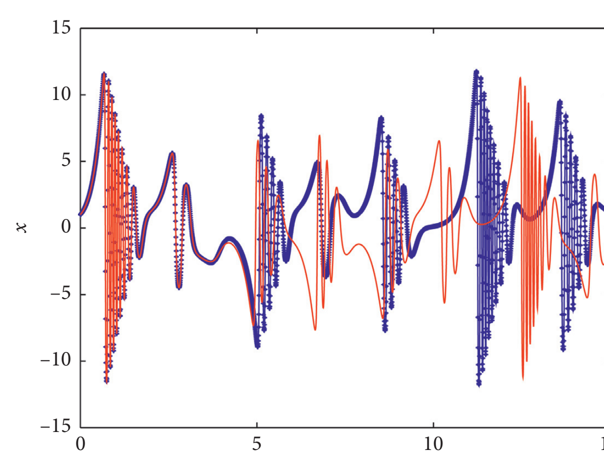


FIGURE 23: Sensitivity due to the variation of $z(0)$ from $z(0) = 1$ to $z(0) = 1.0001$.

Considering the influence of the initial conditions, we end this part by analyzing the coexisting attractors. We have, for example, the presence of two pairs of attractors when parameters $a = 3, b = 9, c = 2.9, d = 1$, and $k = 4$ at the order $\alpha = 0.995$ and two initial conditions which are given by $(1, 1, 1)$ (blue color) and $(-1, -1, -1)$ (red color). The figures of the coexisting attractors are represented in Figures 24 and 25. The graphical representations in different planes are assigned in Figure 24: $(x - y - z)$ and $(y - z)$ planes.

The graphical representations in different planes are assigned in Figure 25: $(x - z)$ and $(x - y)$ planes.

The order of the fractional operator plays an important role in the existence of pairs of attractors. To observe this influence, we maintain parameters $a = 3$, $b = 9$, $c = 2.9$, $d = 1$, and $k = 4$ and change the order to $\alpha = 0.93$; see Figures 26 and 27. The graphical representations in different planes are assigned in Figure 26: $(x - y - z)$ and $(y - z)$ planes.

The graphical representations in different planes are assigned in Figure 27: $(x - z)$ and $(x - y)$ planes.

9. Stability Analysis and Feedback Control

In this last section of our investigation, we focus on the local stability of the equilibrium points of the fractional chaotic model (5)–(7). The equilibrium points of our fractional model are given by $E_0 = (0, 0, 1)$, $E_1 = (2.236, 1.490, 6)$, and $E_2 = (-2.236, -1.490, 6)$. At the first point E_0, the Jacobian matrix in the previous section is given as follows:

$$J = \begin{pmatrix} 4 & -1 & 0 \\ 1 & -9 & 0 \\ 0 & 0 & -4 \end{pmatrix}. \tag{29}$$

The eigenvalues are given as follows: $\lambda_1 = 3.9226$, $\lambda_1 = -8.9226$, and $\lambda_3 = -4$. The second and the last eigenvalues have negative real part and thus satisfy the Matignon criterion [36], but $|\arg(\lambda_1)| = 0 < \alpha\pi/2$. Thus, the equilibrium point E_0 is not stable. At the second equilibrium point E_1, the Jacobian matrix in the previous section is given as follows:

$$J = \begin{pmatrix} 4 & -6 & -1.49 \\ 6 & -9 & 2.236 \\ 8.94 & 13.416 & -0.66836 \end{pmatrix}. \tag{30}$$

The eigenvalues are given as follows: $\lambda_1 = 2.6239 + 6.0881i$, $\lambda_2 = 2.6239 - 6.0881i$, and $\lambda_3 = -10.9161$. The last eigenvalue has negative real part and thus satisfies the Matignon criterion, but the first and the second eigenvalues do not satisfy the Matignon criterion as $\alpha > 0.9$. Thus, the equilibrium point E_1 is not stable when $\alpha > 0.9$. At the last equilibrium point E_2, the Jacobian matrix in the previous section is given as follows:

$$J = \begin{pmatrix} 4 & -6 & 1.49 \\ 6 & -9 & -2.236 \\ -8.94 & -13.416 & -0.66836 \end{pmatrix}. \tag{31}$$

We obtain the same eigenvalues as in the previous point. The eigenvalues are given as follows: $\lambda_1 = 2.6239 + 6.0881i$, $\lambda_2 = 2.6239 - 6.0881i$, and $\lambda_3 = -10.9161$. The last eigenvalue has negative real part and satisfies the Matignon criterion, but the first and the second eigenvalues do not satisfy the Matignon criterion as $\alpha > 0.9$. Thus, the equilibrium point E_2 is not stable when $\alpha > 0.9$.

In the last part, we propose a feedback control to stabilize our chaotic system because, as we observe, all the equilibrium points are not stable when the fractional-order derivative exceeds $\alpha > 0.9$. Let the slave fractional chaotic system be defined by the following equation:

$$\begin{aligned} {}_c^{}D^{\alpha} x_1 &= a x_1 - x_2 x_3, \\ {}_c^{}D^{\alpha} x_2 &= -b x_2 + x_1 x_3, \\ {}_c^{}D^{\alpha} x_3 &= -c x_3 + x_1 x_2 x_3 + k, \end{aligned} \tag{32}$$

and the master system is given by the following equation:

$$\begin{aligned} D_c^{\alpha} y_1 &= a y_1 - y_2 y_3 + u_1, \\ D_c^{\alpha} y_2 &= -b y_2 + y_1 y_3 + u_2, \\ D_c^{\alpha} y_3 &= -c y_3 + y_1 y_2 y_3 + k + u_3, \end{aligned} \tag{33}$$

where u_i represents the exogenous input, which attracts our attention. Let us define the error terms given by the following equations:

$$\begin{aligned} e_1 &= y_1 - x_1, \\ e_2 &= y_2 - x_2, \\ e_3 &= y_3 - x_3. \end{aligned} \tag{34}$$

Then, considering the slave system and the master system, we get the following fractional differential error system. Then, considering the slave system and the master system, we get the following fractional differential error system:

$$D_c^{\alpha} e_1 = a e_1 - e_2 e_3 - e_2 x_3 - e_3 x_2 + u_1, \tag{35}$$

$$D_c^{\alpha} e_2 = -b e_2 + e_1 e_3 + e_1 x_3 + e_3 x_1 + u_2, \tag{36}$$

$$D_c^{\alpha} e_3 = -c e_3 + f(e_1, e_2, e_3, x_1, x_2, x_3) + u_3. \tag{37}$$

Then, here, to stabilize the fractional error equation, we choose feedback control defined by

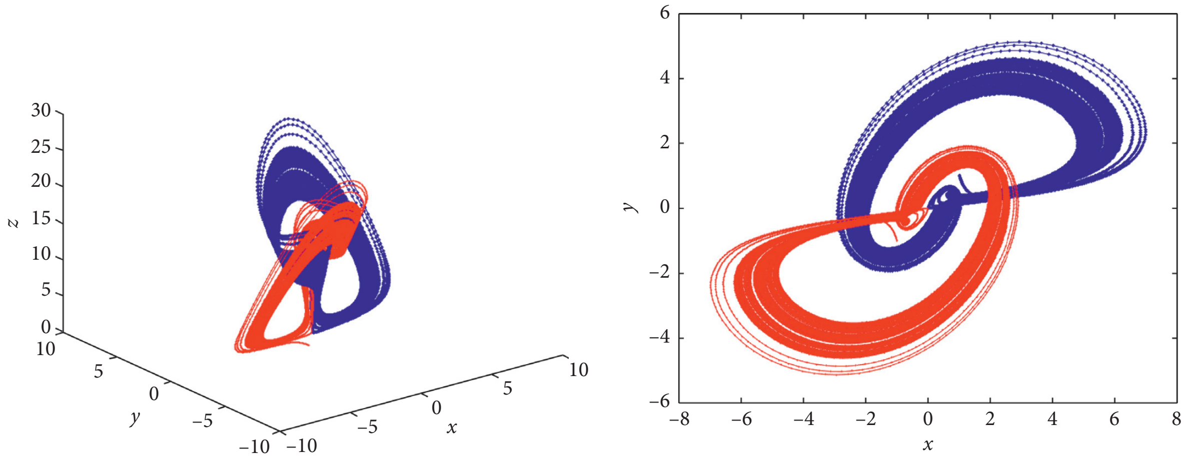
$$u_1 = -a e_1 + e_2 x_3 + e_3 x_2, \tag{38}$$

$$u_2 = -e_1 x_3 - e_3 x_1, \tag{39}$$

$$u_3 = -f(e_1, e_2, e_3, x_1, x_2, x_3). \tag{40}$$

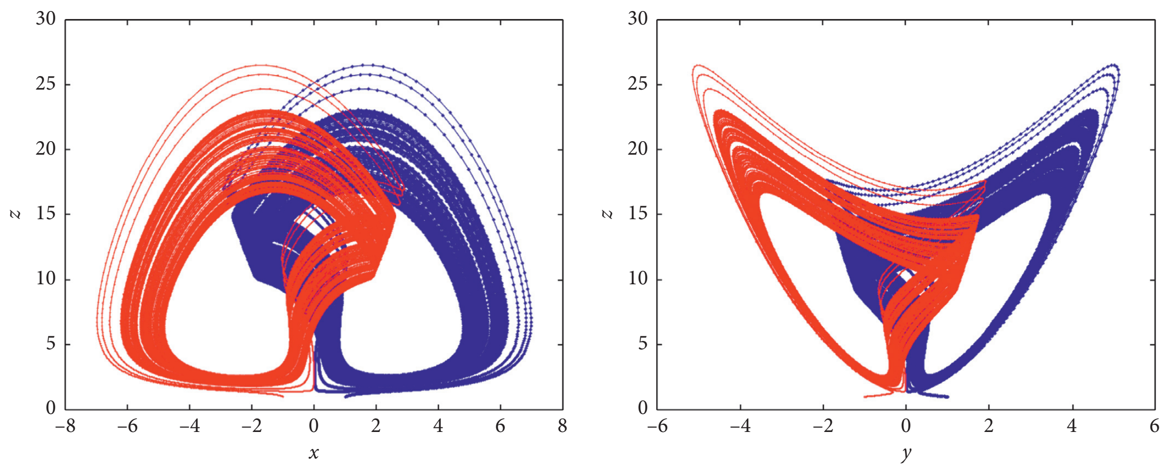
Thus, the fractional differential equation defined by equations (35)–(37) becomes as follows:

$$\begin{aligned} D_c^{\alpha} e_1 &= -e_2 e_3, \\ D_c^{\alpha} e_2 &= -b e_2 + e_1 e_3, \\ D_c^{\alpha} e_3 &= -c e_3. \end{aligned} \tag{41}$$



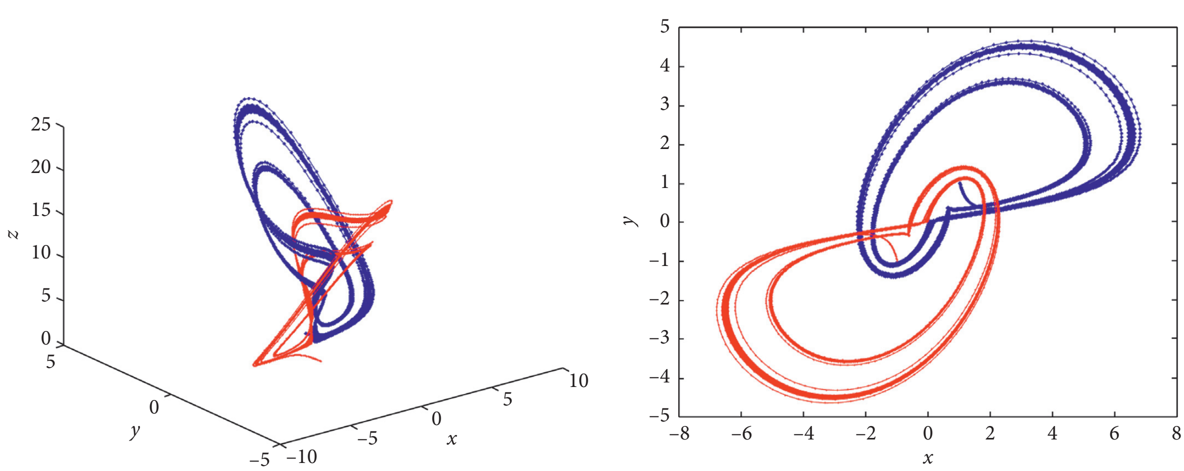
(a) (b)

Figure 24: Coexisting attractors in (x, y, z) and (x, y) planes at $\alpha = 0.995$.



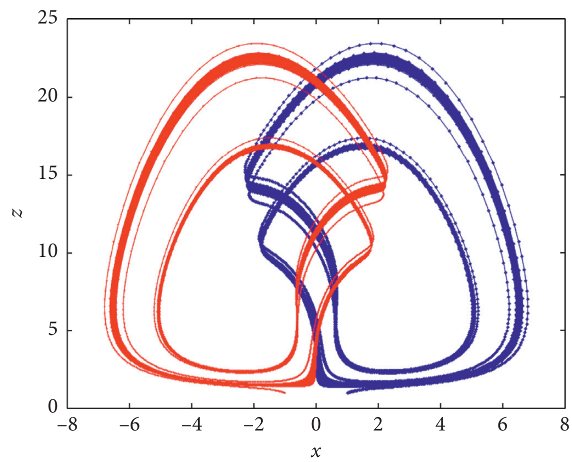
(a) (b)

Figure 25: Coexisting attractors in $(x, z)(y, z)$ planes at $\alpha = 0.995$.

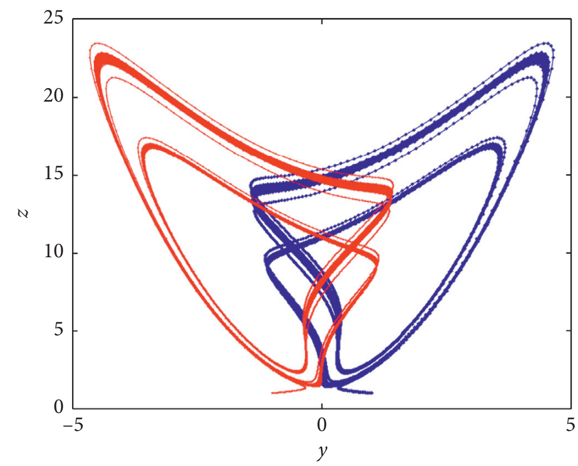


(a) (b)

Figure 26: Coexisting attractors in (x, y, z) and (x, y) planes at $\alpha = 0.93$.



(a)



(b)

FIGURE 27: Coexisting attractors in $(x, z)(y, z)$ planes at $\alpha = 0.93$.

Let the Lyapunov function be defined by $V(e_1, e_2, e_3) = 1/2(e_1^2 + e_2^2 + e_3^2)$. From the derivative of function V along the trajectories of the fractional errors equations (38)–(40) we get the following relationship:

$$D_c^\alpha V = -be_2^2 - ce_3^2,$$
$$D_c^\alpha V = -\left[be_2^2 + ce_3^2\right]. \quad (42)$$

Using Lyapunov characterization of the global asymptotic stability, we get the global asymptotic stability of the trivial equilibrium point of the fractional system (38)–(40), which in turn implies that

$$\lim_{t\longrightarrow+\infty} \|e\| = \lim_{t\longrightarrow+\infty} \|y - x\| = 0. \quad (43)$$

10. Final Remarks

This paper studies the fundamental properties of a class of fractional-order systems in terms of chaotic behaviors, Lyapunov exponents for characterizing the chaotic or hyperchaotic behaviors, the Lyapunov dimensions, and the stability of the equilibrium points of the model in the context of the Matignon criterion. We find that our system admits chaotic behaviors with all fractional-order derivatives into the interval $(0.9, 1)$ as can be observed in the figures in Section 5. The fractional chaotic system's equilibrium points are not stable due to the chaotic behaviors, but we find feedback control to stabilize the model's error term. The different figures of the dynamics of the model represented in this paper were possible with the proposed numerical discretization aid, including the Riemann–Liouville derivative discretization. The numerical method is specially called the predictor-corrector method applied in our system because it is already reported in the literature. The impact of the parameters of the introduced model is analyzed via the bifurcation concept. The main conclusions of this paper are summarized as follows: we find a region where the fractional-order system exhibits chaotic behaviors; the bifurcation diagrams in Section 6 and the Lyapunov exponents inform us that the fractional-order derivative has a significant impact on the dynamics because new attractors are generated when the order of the fractional derivative varies; the effectiveness of all the analysis in the paper is possible with the aid of the numerical scheme. The paper also informs us that the present chaotic system admits coexisting attractors when the initial conditions vary and with specific parameters; see the figures in Section 8. The paper also contributes to proposing adaptative control for global asymptotic stability. For future research directions, the Lyapunov exponents, the bifurcation diagrams, the stability analysis, the synchronization, and the electrical implementation of the model used in this paper can be focused on as regards Caputo derivative terms with different values of the fractional orders. The fractional chaotic systems with the nonsingular derivatives can also be focused on in the future. This paper addresses numerical schemes; it will be interesting in the future research to draw the circuit associated with the present chaotic model and analyze the simulations obtained in the oscilloscopes. The circuits' schematic can be done with both the integer-order version and the fractional-order version of our present system. The Poincare map of the present chaotic system in fractional version is also the perspective of new research papers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] I. Petras, "A note on the fractional-order chua's system," *Chaos, Solitons & Fractals*, vol. 38, pp. 140–147, 2008.

[2] Q. Lai 1, A. Akgul, C. Li, G. Xu, and U. Çavusoglu, "A new chaotic system with multiple attractors: dynamic analysis,

circuit realization and S-box design," *Entropy*, vol. 20, p. 12, 2018.

[3] G. Xu, Y. Shekofteh, A. Akgül, C. Li, and S. Panahi, "A new chaotic system with a self-excited attractor: entropy measurement, signal encryption, and parameter estimation," *Entropy*, vol. 20, no. 2, p. 86, 2018.

[4] D. Dudkowski, S. Jafari, T. Kapitaniak, N. V. Kuznetsov, G. A. Leonov, and A. Prasad, "Hidden attractors in dynamical systems," *Physics Reports*, vol. 637, pp. 1–50, 2016.

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, The Netherlands, 2006.

[6] I. Podlubny, "Fractional differential equations," *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1999.

[7] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model," *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.

[8] J. Gómez-Aguilar, M. López-López, V. Alvarado-Martínez, D. Baleanu, and H. Khan, "Chaos in a cancer model via fractional derivatives with exponential decay and mittag-leffler law," *Entropy*, vol. 19, no. 12, p. 681, 2017.

[9] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–15, 2015.

[10] P. Veeresha, H. M. Baskonus, D. G. Prakasha, W. Gao, and G. Yel, "Regarding new numerical solution of fractional schistosomiasis disease arising in biological phenomena," *Chaos, Solitons & Fractals*, vol. 133, Article ID 109661, 2020.

[11] S. Uçar, E. Uçar, N. Özdemir, and Z. Hammouch, "Mathematical analysis and numerical simulation for a smoking model with atangana-baleanu derivative," *Chaos, Solitons & Fractals*, vol. 118, pp. 300–306, 2019.

[12] A. Atangana and J. F. Gómez-Aguilar, "Hyperchaotic behaviour obtained via a nonlocal operator with exponential decay and mittag-leffler laws," *Chaos, Solitons & Fractals*, vol. 102, pp. 285–294, 2017.

[13] T. Mekkaoui, Z. Hammouch, D. Kumar, and J. Singh, "A new approximation scheme for solving ordinary differential equation with gomez-atangana-caputo fractional derivative," *Methods of Mathematical Modelling*, CRC Press, Boca Raton, FL, USA, 2019.

[14] M. Yavuz, "Characterizations of two different fractional operators without singular kernel," *Mathematical Modelling of Natural Phenomena*, vol. 14, no. 3, p. 302, 2019.

[15] M. Yavuz and N. Ozdemir, "Analysis of an epidemic spreading model with exponential decay law," *Mathematical Sciences and Applications E-Notes*, vol. 8, no. 1, pp. 142–154, 2020.

[16] T. Thumma, A. Wakif, and I. L. Animasaun, "Generalized differential quadrature analysis of unsteady three-dimensional MHD radiating dissipative casson fluid conveying tiny particles," *Heat Transfer*, vol. 49, no. 5, pp. 2595–2626, 2020.

[17] A. Wakif, A. Chamkha, T. Thumma, I. L. Animasaun, and R. Sehaqui, "Thermal radiation and surface roughness effects on the thermo-magneto-hydrodynamic stability of alumina-copper oxide hybrid nanofluids utilizing the generalized buongiorno's nanofluid model," *Journal of Thermal Analysis and Calorimetry*, vol. 143, no. 2, pp. 1201–1220, 2020.

[18] A. Wakif, Z. Boulahia, F. Ali, M. R. Eid, and R. Sehaqui, "Numerical analysis of the unsteady natural convection MHD couette nanofluid flow in the presence of thermal radiation using single and two-phase nanofluid models for Cu-water nanofluids," *International Journal of Applied and Computational Mathematics*, vol. 4, no. 3, 2018.

[19] W. Gao, B. Ghanbari, and H. M. Baskonus, "New numerical simulations for some real world problems with atangana-baleanu fractional derivative," *Chaos, Solitons & Fractals*, vol. 128, pp. 34–43, 2019.

[20] M. F. Danca and N. Kuznetsov, "Matlab code for Lyapunov exponents of fractional-order systems," *International Journal of Bifurcation and Chaos*, vol. 28, no. 5, Article ID 1850067, 2018.

[21] K. Rajagopal, S. Vaidyanathan, A. Karthikeyan, and P. Duraisamy, "Dynamic analysis and chaos suppression in a fractional order brushless DC motor," *Electrical Engineering*, vol. 99, no. 2, pp. 721–733, 2017.

[22] K. Rajagopal, A. Karthikeyan, and P. Duraisamy, "Hyperchaotic chameleon: fractional order FPGA implementation," *Complexity*, vol. 2017, Article ID 8979408, 16 pages, 2017.

[23] M. Diouf and N. Sene, "Analysis of the financial chaotic model with the fractional derivative operator," *Complexity*, vol. 2020, Article ID 9845031, 14 pages, 2020.

[24] W.-C. Chen, "Nonlinear dynamics and chaos in a fractional-order financial system," *Chaos, Solitons & Fractals*, vol. 36, no. 5, pp. 1305–1314, 2008.

[25] K. Rajagopal, A. Akgul, S. Jafari, A. Karthikeyan, U. Cavusoglu, and S. Kacar, "An exponential jerk system, its fractional-order form with dynamical analysis and engineering application," *Soft Computing*, vol. 24, no. 10, pp. 7469–7479, 2020.

[26] S. Jafari and J. C. Sprott, "Simple chaotic flows with a line equilibrium," *Chaos, Solitons & Fractals*, vol. 57, pp. 79–84, 2013.

[27] S. Vaidyanathan, C. Volos, V.-T. Pham, and K. Madhavan, "Analysis, adaptive control and synchronization of a novel 4-D hyperchaotic hyperjerk system and its SPICE implementation," *Archives of Control Sciences*, vol. 25, no. 1, pp. 135–158, 2015.

[28] V.-T. Pham, S. Jafari, C. Volos, S. Vaidyanathan, and T. Kapitaniak, "A chaotic system with infinite equilibria located on a piecewise linear curve," *Optik*, vol. 127, no. 20, pp. 9111–9117, 2016.

[29] H. Baskonus, T. Mekkaoui, Z. Hammouch, and H. Bulut, "Active control of a chaotic fractional order economic system," *Entropy*, vol. 17, no. 12, pp. 5771–5783, 2015.

[30] N. Sene, "Analysis of a fractional-order chaotic system in the context of the caputo fractional derivative via bifurcation and Lyapunov exponents," *Journal of King Saud University-Science*, vol. 33, no. 1, Article ID 101275, 2021.

[31] N. Sene and A. Ndiaye, "On class of fractional-order chaotic or hyperchaotic systems in the context of the caputo fractional-order derivative," *Journal of Mathematics*, vol. 2020, Article ID 8815377, 15 pages, 2020.

[32] N. Sene, "Analysis of a four-dimensional hyperchaotic system described by the caputo–liouville fractional derivative," *Complexity*, vol. 2020, Article ID 8889831, 20 pages, 2020.

[33] N. Sene, "Mathematical views of the fractional chua's electrical circuit described by the caputo-liouville derivative," *Revista Mexicana de Física*, vol. 67, no. 1, pp. 91–99, 2020.

[34] K. M. Owolabi, J. F. Gómez-Aguilar, G. Fernández-Anaya, J. E. Lavín-Delgado, and E. Hernández-Castillo, "Modelling of chaotic processes with caputo fractional order derivative," *Entropy*, vol. 22, no. 9, p. 1027, 2020.

[35] R. Garrappa, "Numerical solution of fractional differential equations: a survey and a software tutorial," *Mathematics*, vol. 6, no. 2, p. 16, 2019.

[36] D. Matignon, "Stability results on fractional differential equations to control processing," in *Proceedings of the Computational Engineering in Syatems and Application Multiconference IMACS*, pp. 963–968, IEEE-SMC, Lille, France, July 1996.

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Research Article

Application of Green Synthesized Metal Nanoparticles in the Photocatalytic Degradation of Dyes and Its Mathematical Modelling Using the Caputo–Fabrizio Fractional Derivative without the Singular Kernel

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Textile dyes are untreated discharge into the environment which results in a significant increase in water pollution levels worldwide. Due to the continuous addition of toxic organic dyes, a necessary strategic model is required for the complete degradation of dyes in textile effluent. This paper considers the possibility of biological synthesis of silver and iron nanoparticles and their use in photocatalytic degradation. The immediate change of silver nitrate solution occurring from colorless to brown is observed after the addition of the aqueous leaf extract, indicating the successive reduction of Ag+ ions to the Ag nanoparticles. These formed Ag nanoparticles were subjected to examine the photocatalytic activity under the solar radiation for the degradation of methyl orange. Green synthesized Ag nanoparticles were found to successfully degrade methyl orange up to 95% between 70 hours than the initial exposure time. The absorbance of methyl orange was measured at 465 nm. The present paper is focused on fractional mathematical modelling of dye degradation in textile effluents using the Caputo–Fabrizio fractional derivative without the singular kernel. The iterative Laplace transform method is employed to obtain an analytic solution for the absorption transport equation. The obtained experimental results showing significant removal of dyes from textile wastewater are compared using modelling results. The innovative approach is in outstanding agreement with the findings of the experiment. The mathematical modelling for the dye removal process helps to design suitable environmental management studies to reduce the adverse effect caused by toxic wastewater. Model validation has been shown by comparing analytical simulated solutions with experimental results for photocatalytic degradation using silver and iron nanoparticles as eco-friendly and low-cost agents.

1. Introduction

Dyes are the most important type of synthetic organic materials utilized in various industries such as textiles, food, and pharmaceuticals. The basic strategy for the remediation of these dye compounds from manufacturing effluents has been accompanied by the use of chemical reagents, physical aspects, and biological processes. However, these methods are laborious and inefficient and have issues with disposal as well. Recently, in [1], photocatalytic activity by metal nanoparticles sought significant attention due to the fact that it has the characteristic properties of degrading organic compounds under solar light illumination in the case of metal catalysts. Compared to traditional approaches, this process is low cost and does not produce toxic goods. Nanotechnology allows the development of nanoparticles with regulated size, design, and variance of materials at the nanometer scale length, with the aim of using them to enhance human health. Metal nanoparticles, among all nanoparticles, have a broad variety of applications in areas such as bioimaging, sensor growth, and data processing and novel applications in the biomedical research sector. The late application of metallic silver and silver nanoparticles as antimicrobial operators in various products started, for example, powder and paint, animal feed, covering of the catheter tube, wound patch dressing materials, and water purifying treatments [2], with a negligible danger of toxification in human beings. The green methodology of nanoparticles prepared from natural substances is gaining incredible popularity because it is more environmentally friendly, less harmful, and less time consuming; at present, plant materials are utilized for nanoparticles' formation because they are more perfect than the microorganism-mediated nanoparticles' procedure since they are difficult to handle.

Plant extract-based synthesis of nanoparticles is having tremendous success due to its compatibility, environmentally-friendly, and least time consuming properties [3–5]. In a recent study, silver nanoparticles were effectively fabricated using the *Cordia dichotoma* (common name: gonda) leaf extract, and the silver and iron nanoparticles synthesized were used in the degradation of dyes. A flowering plant *Cordia dichotoma* is species from the family of borage, and it is boraginaceous which is native to the regions of western Melanesia, northern Australia, and Indomalayan realm. Common vernacular names include Indian cherry, bird lime tree, pink pearl, glue berry, anonang, cumming cordia, snotty gobbles, fragrant manjack, and lasoda (gunda), respectively. *Cordia dichotoma* is a deciduous tree with a short bole and a spreading crown that grows to be small to intermediate in height. The stem bark is greyish brown in color and can be smooth or wrinkled over its base. The flowers are short stalked, whitish, and open only at night. The fruit is smooth, green-yellow, or pink-yellow globose that becomes black after ripening, and the pulp becomes viscid. Figure 1 depicts plants and their leaves found in tropical and subtropical regions. It can be found in a variety of forests, from the dry deciduous forests of Rajasthan to the wet deciduous forests of the Western Ghats and the coastal forests of Myanmar. Fabricated silver nanoparticles under exposure to sunlight have been exposed to dye degradation operation. Though a lot of work has been done to measure the performance of many adsorbents for dye degradation from industries, yet very little work has been done to model the dye degradation process to evaluate the effect of various parameters on the dye degradation process. In [6], modelling enables the future prediction and indicates the importance of various factors in the real system. The numerical iterative Laplace transform method is employed to simulate the degradation process of dyes from wastewater. The findings achieved by the proposed model could help to refine the wastewater management strategy.

2. Experimental

In order to assess the validity of the numerical modelling for the analysis of wastewater dye degradation, the simulated findings are compared with the results of the experimental studies. In the laboratory test, the following materials and methods were followed.

2.1. Preparation of the Plant Extract. Leaves of *Cordia dichotoma* (common name: gonda) were collected from the JIET campus. 10 g of fresh leaves were sliced into thin pieces and washed vigorously with double-distilled water. The leaves' content was added with 100 mL of double-distilled water and kept for boiling at 60°C for 10 min. Then, the filtrate was obtained by passing the boiled mixture through Whatman No. 1 filter paper and kept in a clean container at 4°C for further nanoparticle synthesis process.

2.2. Biosynthesis of Silver Nanoparticles. About 1 millimolar silver nitrate aqueous salt solution was prepared in double-distilled water that was procured from Sigma-Aldrich grade salt. Appropriately, 5 mL of the freshly prepared leaf extract was mixed with 45 mL of aqueous silver nitrate salt solution. For the method of reducing the silver ion to silver nanoparticles, the mixture was held for incubation at room temperature. The formation of silver NPs was identified visibly as the solution turns colorless to brown and later identified using the UV-vis spectrum analysis. The variation in pH of the leaf extract was altered to examine the effect of the production of silver nanoparticles (Figure 2). The UV-vis spectrophotometer measured the formation of silver nanoparticles at a wide range of wavelengths. The method followed is described in the earlier work published by Dave [7].

2.3. Biosynthesis of Iron Nanoparticles. Aqueous salt solution of ferrous sulphate was formulated using double-distilled water at a concentration of 1 mM of 5 mL of the newly developed leaf extract which was applied to 45 mL of aqueous salt solution of ferrous sulphate and stored at room temperature for the reduction of Fe nanoparticles (Figure 3).

2.4. Characterization of Biosynthesized Silver Nanoparticles Using UV-Vis Spectroscopy. The purified silver nanoparticles were obtained using repeated centrifugation method at 7000 rpm for 15 min followed by drying at 100°C. The successive reduction of silver nitrate into silver NPs was subjected to the double-beam UV-vis spectrophotometer for measuring the spectrum at a differential wavelength from 360 nm to 700 nm, respectively.

2.5. Photocatalytic Degradation of Dye

2.5.1. Using Silver Nanoparticles. From the biosynthesized silver nanoparticles, 5 mg Ag NPs was added to the test flask containing 50 mL of methyl orange dye solution. The control test bottle was also preserved without the inclusion of silver nanoparticles. Until exposure to sunlight irradiation, the reaction suspension was thoroughly combined with magnetic stirring for 30 min to clearly align the operating test solution. Subsequently, the dispersed solution was put under sunlight and monitored for color change significant to the

Figure 1: The plant and its leaf.

Figure 2: Synthesis of silver nanoparticles.

Figure 4: Dye degradation using silver nanoparticles.

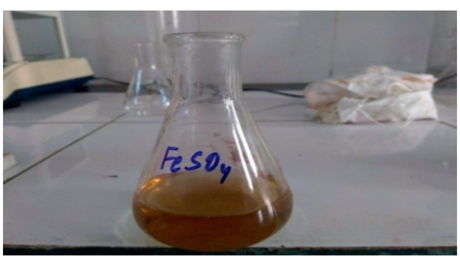

Figure 3: Synthesis of iron nanoparticles.

production of Ag NPs. At specific time intervals, aliquots of 2-3 mL from the suspension were screened and used to analyze the photocatalytic degradation activities of the dye using the UV-vis spectrophotometer at various frequencies. The degrading (Figure 4) concentration of the dye during the degradation operation was determined by observing the suspension solution value at 660 nm.

2.5.2. Using Iron Nanoparticles. For reactions similar to Fenton oxidation, iron nanoparticles were prepared using ferrous sulphate as a precursor, and 1 ml of colloidal iron nanoparticles along with 1 ml of 3% H_2O_2 was added to 9 ml of 50 ppm methyl orange in a test tube. Five replicates were prepared for each sample. A new blank (control) was also included in each round of dye degradation, containing the same volume of the dye and H_2O_2 but without colloidal water replacing the colloidal nanoparticles. The concentration was measured using a UV-vis spectrophotometer. The percentage of dye degradation was calculated using the preceding formula where the initial concentration of dye solution and concentration of dye solution are present after photocatalytic degradation (Figure 5).

3. Mathematical Modelling Using the Caputo–Fabrizio Fractional Derivative without the Singular Kernel

In this article, we based on a fractional-order mathematical model to investigate the transport of relevant textile industry effluents by using the Caputo–Fabrizio fractional derivative without the singular kernel (see also [8]). The concentration of analytical solution is obtained by the iterative Laplace transform technique, and the concentration is plotted for different input parameters. For more modern fractional-order mathematical model developments, the reader can refer to [9–16].

The transport equation due to Doulati Ardejani et al. [17] for the absorption process is given as

$$R\frac{\partial C}{\mathrm{d}t} = -KS\rho_d, \tag{1}$$

where

C: the concentration of solution

S: the quantity of absorbed mass on the surface

R: the retardation factor

K: the delay constant

ρ_d: the bulk density of the medium

The relationship between C and S due to the Langmuir isotherm [18] is given as

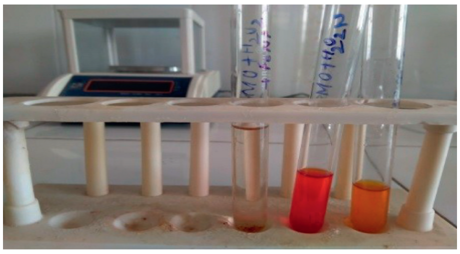

Figure 5: Dye degradation using iron nanoparticles.

$$S = \frac{Q_0 K_L C}{1 + K_L C}, \tag{2}$$

where

Q_0: the maximum absorption capacity

K_L: the Langmuir constant

Using (1) and (2), we have

$$R\frac{\partial C}{dt} = -\frac{K K_L Q_0 \rho_d C}{1 + K_L C}, \tag{3}$$

with $C(0) = C_0$.

Let $L(a,b) = \{f\colon f \in L^2(a,b) \text{ and } f' \in L^2(a,b)\}$, where $L^2(a,b)$ is the space of square-integrable functions on interval (a,b). Furthermore, $H(0,b) = \{f\colon f \in L^2(0,b) \text{ and } f' \in L^2(0,b)\}$, with $b > 0$.

Definition 1. Let $0 < \alpha < 1$; the fractional Caputo–Fabrizio [19, 20] derivative of order α for a function $f(t) \in H(0,b)$ with $b > 0$ is given by

$$^{\mathrm{CF}}\mathfrak{D}_t^{\alpha} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t f'(x) \exp\left[\frac{-\alpha}{1-\alpha}(t-x)\right] dx, \quad t \geq 0, \tag{4}$$

where $M(\alpha)$ is the normalization function.

Definition 2. Let $0 < \alpha < 1$; the fractional integral of order α for a function f is defined as

$$\begin{aligned} ^{\mathrm{CF}}\mathfrak{F}^{\alpha} f(t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) \\ &+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(x) dx, \quad t \geq 0. \end{aligned} \tag{5}$$

Remark 1. Note that, from the definition in equation (5), the fractional integral of the Caputo–Fabrizio type of function f of order $0 < \alpha \leq 1$ is a mean between the function f and its integral of order one, which means

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1, \tag{6}$$

and therefore,

$$M(\alpha) = \frac{2}{2-\alpha}, \quad 0 < \alpha \leq 1. \tag{7}$$

The advantage of the Caputo–Fabrizio operator over classical Caputo is that there is no singularity for $t = s$.

Definition 3. Due to Caputo and Mauro [19], the Laplace transform for the Caputo–Fabrizio fractional derivative operator of order $0 < \alpha \leq 1$, $M \in \mathbb{N}$, is given by

$$\begin{aligned} \mathfrak{L}\left(^{\mathrm{CF}}\mathfrak{D}_t^{M+\alpha} f(t)\right)(s) &= \frac{1}{1-\alpha} \mathfrak{L}\left(f^{(M+1)}(t)\right) \mathfrak{L}\left(\exp\left(\frac{-\alpha}{1-\alpha} t\right)\right) \\ &= \frac{s^{M+1}\mathfrak{L}(f(t)) - s^M f(0) - S^{M-1} f'(0) + \cdots + f^M(0)}{s + \alpha(1-s)}. \end{aligned} \tag{8}$$

If $M = 0$, we get

$$\mathfrak{L}\left(^{\mathrm{CF}}\mathfrak{D}_t^{\alpha} f(t)\right)(s) = \frac{s\mathfrak{L}(f(t))}{s + \alpha(1-s)}. \tag{9}$$

The fractional form of equation (3) is given as

$$^{\mathrm{CF}}\mathfrak{D}_t^{\alpha} C + \frac{K K_L Q_0 \rho_d C}{R(1 + K_L C)} = 0. \tag{10}$$

4. Iterative Laplace Transform

The nonhomogeneous Caputo–Fabrizio fractional differential equation is given as

$$\begin{aligned} &^{\mathrm{CF}}\mathfrak{D}_t^{M+\alpha} f(x,t) = u(x,t) + \phi(f(x,t)) + \psi(f(x,t)), \\ &M - 1 < \alpha \leq M, M \in \mathbb{N}, \end{aligned} \tag{11}$$

with the given condition

$$\mathfrak{D}_t^K(x,0)=\theta_K(x),\quad K=0,1,2,\ldots,M-1, \tag{12}$$

where $u(x,t)$ is a known term, ϕ is the linear operator, and ψ is the nonlinear operator.

Applying Laplace transform (8) to both sides of equation (11) yields

$$\mathfrak{L}(f(x,t))=\lambda(x,s)+\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathfrak{L}(\phi f(x,t)+\psi f(x,t)), \tag{13}$$

where

$$\begin{aligned}\lambda(x,s)&=\frac{1}{s^{n+1}}\left(s^n\theta_0(x)+s^{n-1}\theta_1(x)+\cdots+\theta_n(x)\right)+\frac{s+\alpha(1-s)}{s^{n+1}}\overline{u}(x,s),\\ f(x,t)&=\lambda(x,t)+\mathfrak{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathfrak{L}(\phi f(x,t)+\psi f(x,t))\right].\end{aligned} \tag{14}$$

Now, applying the new iterative method [21] yields the solution as an infinite series:

$$f(x,t)=\sum_{j=0}^{\infty}f_j(x,t). \tag{15}$$

Here, linear function ϕ is given as

$$\phi\left(\sum_{j=0}^{\infty}f_j(x,t)\right)=\sum_{j=0}^{\infty}\phi\big(f_j(x,t)\big). \tag{16}$$

Furthermore, nonlinear ψ is decomposed as

$$\begin{aligned}\psi\left(\sum_{j=0}^{\infty}f_j(x,t)\right)&=\psi(f_0(x,t))\\&+\sum_{j=1}^{\infty}\left\{\psi\left(\sum_{i=0}^{j}f_i(x,t)\right)-\psi\left(\sum_{i=0}^{j-1}f_i(x,t)\right)\right\}.\end{aligned} \tag{17}$$

In view of equations (15)–(17), equation (14) is equivalent to

$$\begin{aligned}\sum_{j=0}^{\infty}f_j(x,t)&=\lambda(x,t)+\mathfrak{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathfrak{L}\left(\sum_{j=0}^{\infty}\phi f_j(x,t)\right)\right]\\&+\mathfrak{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathfrak{L}\left(\psi(f_0(x,t))+\sum_{j=1}^{\infty}\left\{\psi\left(\sum_{i=0}^{j}f_i(x,t)\right)-\psi\left(\sum_{i=0}^{j-1}f_i(x,t)\right)\right\}\right)\right].\end{aligned} \tag{18}$$

The recurrence relation is given as

$$
\begin{aligned}
f_0(x,t) &= \lambda(x,t),\\
f_1(x,t) &= \mathfrak{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathfrak{L}\left(\phi\left(f_0(x,t)\right)+\psi\left(f_0(x,t)\right)\right)\right],\\
f_{p+1}(x,t) &= \mathfrak{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathfrak{L}\left(\phi\left(f_r(x,t)\right)+\left\{\psi\left(\sum_{i=0}^{p} f_i(x,t)\right)-\psi\left(\sum_{i=0}^{p-1} f_i(x,t)\right)\right\}\right)\right].
\end{aligned}
\tag{19}
$$

The p-term approximate solution is given as

$$f = f_0 + f_1 + f_2 + \cdots + f_{p-1}. \tag{20}$$

5. Results and Discussion

5.1. Optical Observation. In the beginning, by adding the leaf extract to 1 mm silver solution, color of the solution turned brown, indicating the immediate and rapid formation of silver nanoparticles. This transformation of color occurs due to the excitation of SPR (surface plasmon resonance) of the silver nanoparticles. Similar results were obtained in an experiment using the root extract of *Curculigo orchioides* by Dave and Das [6], and the color change observed was brownish yellow to dark brown.

5.2. UV-Vis Spectrophotometer. At the preliminary state, the degradation was identified by color change. The catalytic activity of silver nanoparticles on the degradation of dyes was confirmed using methyl orange as a sample dye. In solar light, silver nanoparticles were used for the degradation of methyl orange, and the amount of dye left was measured at different time intervals. Initially, color of the dye shows deep orange color which changed into light yellow after one hour of incubation along with silver nanoparticles. The absorption spectrum recorded has shown a decrease in the peak at varied time intervals. Calculation shows that the percentage of degradation efficiency of silver nanoparticles was 95.8% at 70 h (Table 1). When the exposure time of the dye and silver nanoparticle complex placed in sunlight was increased, the absorption peak also had a decrease. The absorption peak for methyl orange was canter at 660 nm in the visible region which was reduced, and at last, it disappeared when the reaction time was increased. The whole process was completed after 70 hours of incubation and was recognized by the change of reaction mixture color to colorless (Table 2).

5.3. Photocatalytic Degradation of Dye

5.3.1. Visual Observation. Photocatalytic degradation of methyl orange was carried out by using green synthesized silver nanoparticles under solar light. Dye oxidation was initially detected by a shift of hue. Initially, color of the pigment reveals deep orange color modified to light yellow after 1 h of incubation with silver nanoparticles when exposed to sunlight. Then, the hue changed from bright yellow to pale yellow, and the solution gradually became colorless. Finally, the degradation process was completed at 70 h and was recognized by the change of reaction mixture color to colorless.

6. Modelling

From equation (10), the fractional-order transport equation for the absorption process is given as

$$ {}^{\mathrm{CF}}\mathfrak{D}_t^{\alpha} C + \frac{K K_L Q_0 \rho_d C}{R\left(1+K_L C\right)} = 0, \tag{21}$$

with $C_0 = c_0 e^{-\beta t}$, β is a constant which depends on initial dye concentration.

The second-term approximate solution is given as

$$
\begin{aligned}
C(t) &= c_0 e^{-\beta t}\\
&\quad - \frac{P c_0}{\lambda}\left[(1-\alpha)\left(1+\frac{e^{\beta t}}{\lambda}\right)^{-1} + 2\alpha t + \frac{2\alpha}{\beta}\log_e\left(\frac{\lambda+1}{\lambda+e^{\beta t}}\right)\right],
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\lambda &= c_0 K_L,\\
P &= \frac{K K_L Q_0 \rho_d}{R},\\
\left|\frac{e^{\beta t}}{\lambda}\right| &< 1.
\end{aligned}
\tag{23}
$$

Figure 6 gives simulated results of solution (22), with parameters $Q_0 = 10.718$, $K_L = 0.308$, $R^2 = 0.9762$, $K = 0.1$, $R = 0.4035 \times 10^{20}$, $\rho_d = 0.001$, and $c_0 = 1.7 \times 10^4$.

We conclude that concentration decays exponentially with faster rates for initial time and takes long time to reduce for higher initial values.

Table 1: Exposure time of the amount of degradation of dye (%) using silver nanoparticles.

Exposure time (hrs.)	Amount of degradation of dye (%) using silver nanoparticles
1	2.5 ± 0.15
2	4.7 ± 0.45
3	7.3 ± 0.55
4	15.5 ± 0.47
10	19.5 ± 0.15
21	25.3 ± 0.14
22	39.5 ± 0.65
24	44.2 ± 0.34
41	47.9 ± 0.21
42	55.2 ± 0.22
44	65.2 ± 0.45
45	75.5 ± 0.65
46	83.2 ± 0.37
48	88.9 ± 0.18
65	89.1 ± 0.23
66	93.6 ± 0.88
70	95.8 ± 0.67

Table 2: Exposure time of the amount of degradation of dye (%) using iron nanoparticles and Fenton-like oxidation.

Exposure time (hrs.)	Amount of degradation of dye (%) using iron nanoparticles and Fenton-like oxidation
1	55 ± 0.35
2	78 ± 0.15
3	88.3 ± 0.25
4	95.2 ± 0.15
10	99

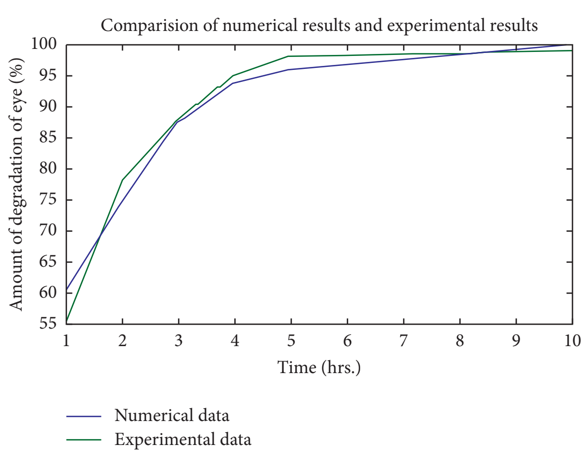

Figure 6: Comparison of numerical results with experimental results using iron nanoparticles and Fenton-like oxidation.

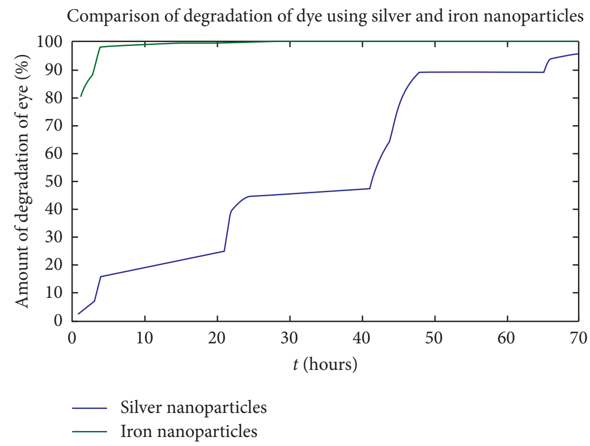

Figure 7: Comparison of the degradation of dye using silver and iron nanoparticles.

7. Conclusion

Green synthesis of silver and iron nanoparticles has been carried out using the plant extract of a locally available plant *Cordia dichotoma* (common name: gonda). Photocatalytic degradation of the dyes using these silver and iron metallic nanoparticles was successfully carried out in the laboratory. The exposure time of the amount of degradation of dye using iron nanoparticles and Fenton-like oxidation was tested, and it was found that it is much lesser with iron nanoparticles compared to the amount of degradation of dye using silver nanoparticles. A numerical fractional model for the transport equation for the concentration process involving Caputo–Fabrizio fractional-order derivatives has been developed to simulate the dye degradation from industry effluents. Iterative Laplace transform method is deployed to solve the model. Model validation has been shown by comparing the analytical simulated solution with experimental results using photocatalytic degradation using silver and iron nanoparticles as eco-friendly and low-cost adsorbents. The simulated results of the model are in good agreement with the experimental results. It is observed that the adsorption process by iron nanoparticles could be well described by the fractional model (Figures 6 and 7). Furthermore, it is clear that the rate of degradation of dye is very sensitive to the initial concentration of dye. From the present study, it is found that iron nanoparticles can be used effectively as low-cost and eco-friendly material for developing large-scale water treatment strategies to remove the toxic dyes in the effluent.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this article.

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References

[1] D. Acharya, S. Satapathy, D. Tapas Kumar Dora, S. Dave, and J. Das, "Advancement in treatment technologies of biopharmaceutical industrial effluents," *Microbial Bioremediation & Biodegradation*, Springer, Singapore, pp. 467–484, 2020.

[2] S. Dave and R. Sharma, "Use of nano-particles in water treatment: a review," *International Research Journal of Environmental Sciences*, vol. 4, no. 10, pp. 103–106, 2015.

[3] S. Bhakya, S. Muthukrishnan, M. Sukumaran, M. Muthukumar, T. K. Senthil, and M. V. Rao, "Catalytic degradation of organic dyes using synthesized silver nanoparticles: a green approach," *Journal of Bioremediation & Biodegredation*, vol. 6, no. 5, 2015.

[4] P. C. Nagajyothi, S. V. Prabhakar Vattikuti, K. C. Devarayapalli, K. Yoo, J. Shim, and T. V. M. Sreekanth, "Green synthesis: photocatalytic degradation of textile dyes using metal and metal oxide nanoparticles-latest trends and advancements," *Critical Reviews in Environmental Science and Technology*, vol. 50, no. 24, pp. 2617–2723, 2019.

[5] S. D. Purohit, A. M. Khan, D. L. Suthar, and S. Dave, "The impact on raise of environmental pollution and occurrence in biological populations pertaining to incomplete H-function," *National Academy Science Letters*, vol. 44, 2020.

[6] S. Dave and J. Das, "Technological model on advanced stages of oxidation of wastewater effluent from food industry," in *Advanced Oxidation Process for Effluent Treatment Plant*, pp. 33–49, Elsevier, Amsterdam, Netherlands, 2020.

[7] S. Dave, "Electrochemical and spectral characterization of silver nano-particles synthesized employing root extract of Curculigo orchioides," *Indian Journal of Chemical Technology*, vol. 25, no. 2, pp. 201–207, 2018.

[8] L. Mistry, A. M. Khan, and D. L. Suthar, "An epidemic Slia mathematical model with Caputo Fabrizio Fractional derivative," *Test Engineering and Management*, vol. 83, pp. 26374–26391, 2020.

[9] A. Alaria, A. M. Khan, D. L. Suthar, and D. Kumar, "Application of fractional operators in modelling for charge carrier transport in amorphous semiconductor with multiple trapping," *International Journal of Applied and Computational Mathematics*, vol. 5, no. 6, pp. 1–10, 2019.

[10] M. Ali Dokuyucu, "Caputo and atangana-baleanu-caputo fractional derivative applied to garden equation," *Turkish Journal of Science*, vol. 5, no. 1, pp. 1–7, 2020.

[11] M. Ali Dokuyucu and E. Celik, "Analyzing a novel coronavirus model (COVID-19) in the sense of caputo-fabrizio fractional operator," *Applied and Computational Mathematics an International Journal*, vol. 20, no. 1, pp. 49–69, 2021.

[12] M. Ali Dokuyucu, E. Celik, H. Bulut, and H. Mehmet Baskonus, "Cancer treatment model with the Caputo-Fabrizio fractional derivative," *The European Physical Journal Plus*, vol. 133, no. 3, p. 92, 2018.

[13] S. I. Butt, M. Nadeem, and G. Farid, "On Caputo fractional derivatives via exponential s-convex functions," *Turkish Journal of Science*, vol. 5, no. 2, pp. 140–146, 2020.

[14] I. Koca, E. Akcetin, and P. Yaprakdal, "Numerical approximation for the spread of SIQR model with Caputo fractional order derivative," *Turkish Journal of Science*, vol. 5, no. 2, pp. 124–139, 2020.

[15] J. Zhao, S. I. Butt, J. Nasir, Z. Wang, and I. Tlili, "Hermite–jensen–mercer type inequalities for Caputo fractional derivatives," *Journal of Function Spaces*, vol. 2020, Article ID 7061549, 11 pages, 2020.

[16] S. Zhao, S. I. Butt, W. Nazeer, J. Nasir, M. Umar, and Y. Liu, "Some Hermite-Jensen-Mercer type inequalities for k-Caputo-fractional derivatives and related results," *Advances in Difference Equations*, vol. 2020, no. 1, p. 262, 2020.

[17] F. Doulati Ardejani, K. Badii, N. Yousefi Limaee et al., "Numerical modelling and laboratory studies on the removal of Direct Red 23 and Direct Red 80 dyes from textile effluents using orange peel, a low-cost adsorbent," *Dyes and Pigments*, vol. 73, no. 2, pp. 178–185, 2007.

[18] E. Quiroga, P. M. Centres, N. A. Ochoa, and A. J. Ramirez-Pastor, "Fractional statistical theory of adsorption applied to protein adsorption," *Journal of Colloid and Interface Science*, vol. 390, no. 1, pp. 183–188, 2013.

[19] M. Caputo and F. Mauro, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, pp. 1–13, 2015.

[20] J. Losada and J. Nieto, "Properties of a new fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, pp. 87–92, 2015.

[21] A. Shaikh, A. Tassaddiq, K. S. Nisar, and D. Baleanu, "Analysis of differential equations involving Caputo-Fabrizio fractional operator and its applications to reaction diffusion equations," *Advances in Difference Equations*, vol. 178, pp. 1–14, 2019.

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Research Article

Weighted Estimates for Commutator of Rough p-Adic Fractional Hardy Operator on Weighted p-Adic Herz–Morrey Spaces

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The current article investigates the boundedness criteria for the commutator of rough p-adic fractional Hardy operator on weighted p-adic Lebesgue and Herz-type spaces with the symbol function from weighted p-adic bounded mean oscillations and weighted p-adic Lipschitz spaces.

1. Introduction

For a fixed prime p, it is always possible to write a nonzero rational number x in the form $x = p^{\gamma}(m/n)$, where p is not divisible by $m, n \in \mathbb{Z}$ and γ is an integer. The p-adic norm is defined as $|x|_p = \{p^{-\gamma} \cup \{0\}\colon \gamma \in \mathbb{Z}\}$. The p-adic norm $|\cdot|_p$ fulfills all the properties of a real norm along with a stronger inequality:

$$|x + y|_p \le \max\left\{|x|_p, |y|_p\right\}. \quad (1)$$

The completion of the field of rational number with respect to $|\cdot|_p$ leads to the field of p-adic numbers $\mathbb{Q}_p$. In [1], it can be seen that any $x \in \mathbb{Q}_p\backslash\{0\}$ can be represented in the formal power series form as

$$x = p^{\gamma}\sum_{j=0}^{\infty}\beta_j p^j, \quad (2)$$

where $\beta_j, \gamma \in \mathbb{Z}, \beta_j \in (\mathbb{Z}/(p\mathbb{Z}_p)), \beta_0 \ne 0$. The convergence of series (2) is followed from $|p^{\gamma}\beta_k p^j|_p = p^{-\gamma-j}$.

The n-dimensional vector space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of tuples $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{Q}_p, i = 1, 2, \ldots, n$, with the following norm:

$$|\mathbf{x}|_p = \max_{1\le i\le n}|x_i|_p. \quad (3)$$

The ball $B_{\gamma}(\mathbf{a})$ and the corresponding sphere $S_{\gamma}(\mathbf{a})$ with center at $\mathbf{a} \in \mathbb{Q}_p^n$ and radius p^{γ} in non-Archimedean geometry are given by

$$\begin{aligned}B_{\gamma}(\mathbf{a}) &= \left\{\mathbf{x} \in \mathbb{Q}_p^n\colon |\mathbf{x} - \mathbf{a}|_p \le p^{\gamma}\right\},\\ S_{\gamma}(\mathbf{a}) &= \left\{\mathbf{x} \in \mathbb{Q}_p^n\colon |\mathbf{x} - \mathbf{a}|_p = p^{\gamma}\right\}.\end{aligned} \quad (4)$$

When $\mathbf{a} = 0$, we write $B_{\gamma}(0) = B_{\gamma}, S_{\gamma}(0) = S_{\gamma}$.

Since the space $\mathbb{Q}_p^n$ is locally compact commutative group under addition, it cements the fact from the standard analysis that there exists a translation invariant Haar measure $d\mathbf{x}$. Also, the measure is normalized by

$$\int_{B_0} d\mathbf{x} = |B_0|_H = 1, \quad (5)$$

where $|E|_H$ represents the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. Furthermore, one can easily show that $|B_{\gamma}(\mathbf{a})|_H = p^{n\gamma}$, $|S_{\gamma}(\mathbf{a})|_H = p^{n\gamma}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

The last several decades have seen a growing interest in the p-adic models appearing in various branches of science. The p-adic analysis has cemented its role in the field of mathematical physics (see, for example, [2–4]). Many researchers have also paid relentless attention to harmonic analysis in the p-adic fields [5–11]. The present paper can be considered as an extension of investigation of Hardy-type operators started in [6, 7, 12–16].

The one-dimensional Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\mathrm{d}y, \quad x > 0, \tag{6}$$

was introduced by Hardy in [17] for measurable functions $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which satisfies the inequality

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty, \tag{7}$$

where the constant $q/(q-1)$ is sharp. In [18], Faris proposed an extension of an operator H on higher dimensional space $\mathbb{R}^n$ by

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{|\mathbf{y}|\leq|\mathbf{x}|} f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{8}$$

where $|\mathbf{x}| = (\sum_{i=1}^n x_i^2)^{(1/2)}$ for $\mathbf{x} = (x_1, \ldots, x_n)$. In addition, Christ and Grafakos [23] obtained the exact value of the norm of an operator H defined by (8). Over the years, Hardy operator has gained a significant amount of attention due to its boundedness properties [19–22]. For complete understanding of Hardy-type operators, we refer the interested readers to study [12, 23–29] and the references therein.

In what follows, the n-dimensional p-adic fractional Hardy operator

$$H_\alpha^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{9}$$

was defined and studied for $f \in L_1^{\mathrm{loc}}(\mathbb{Q}_p^n)$ and $0\leq\alpha < n$ in [15]. When $\alpha = 0$, the operator H_α^p transfers to the p-adic Hardy operator (see [30] for more details). Fu et al. in [30] acquired the optimal bounds of p-adic Hardy operator on $L^q(\mathbb{Q}_p^n)$. On the central Morrey spaces, the p-adic Hardy-type operators and their commutators are discussed in [16]. In this link, see also [6, 7, 14, 27].

From now on, we turn our attention towards the rough kernel version of an operator which recently received a substantial attention in analysis (see for instance [11, 31–37]). The roughness of Hardy operator was first time studied by Fu et al. in [12]. Motivated from the results of rough Hardy-type operators in Euclidean space, we define a special kind of rough fractional Hardy operator and its commutator in the p-adic field.

Let $f\colon \mathbb{Q}_p^n \longrightarrow \mathbb{R}$, $b\colon \mathbb{Q}_p^n \longrightarrow \mathbb{R}$ and $\Omega\colon S_0 \longrightarrow \mathbb{R}$ be measurable functions and let $0 < \alpha < n$. Then, for $\mathbf{x} \in \mathbb{Q}_p^n\setminus\{0\}$, we define a rough p-adic fractional Hardy operator $H_{\Omega,\alpha}^p$ and its commutator $H_{\Omega,\alpha}^{p,b}$ as

$$H_{\Omega,\alpha}^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \Omega\left(|\mathbf{y}|_p\mathbf{y}\right) f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{10}$$

$$H_{\Omega,\alpha}^{p,b} f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} (b(\mathbf{x}) - b(\mathbf{y}))\Omega\left(|\mathbf{y}|_p\mathbf{y}\right) f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{11}$$

whenever

$$\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \left|\Omega\left(|\mathbf{y}|_p\mathbf{y}\right) f(\mathbf{y})\right|\mathrm{d}\mathbf{y} < \infty, \tag{12}$$

$$\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \left|b(\mathbf{y})\Omega\left(|\mathbf{y}|_p\mathbf{y}\right) f(\mathbf{y})\right|\mathrm{d}\mathbf{y} < \infty. \tag{13}$$

Remark 1. Obviously

$$\left\{|\mathbf{y}|_p\colon \mathbf{y} \in \mathbb{Q}_p^n\right\} = \left\{p^\gamma\colon \gamma \in \mathbb{Z}\right\} \cup \{0\}, \tag{14}$$

holds for every integer $n \geq 1$ and prime $p \geq 2$. Since the inclusion

$$\{0\} \cup \left\{p^\gamma\colon \gamma \in \mathbb{Z}\right\} \subseteq \mathbb{Q}_p, \tag{15}$$

holds and $\mathbb{Q}_p^n$ is a linear space over field $\mathbb{Q}_p$, the product $|\mathbf{y}|_p\mathbf{y}$ is correctly defined. Moreover, if a nonzero $\mathbf{y} \in \mathbb{Q}_p^n$ has a form $\mathbf{y} = (y_1, \ldots, y_n)$ and

$$y_i = p^{\gamma_i}\left(\beta_{0,i} + \beta_{1,i}p + \beta_{2,i}p^2 + \cdots\right), \quad i = 1, \ldots, n, \tag{16}$$

(see (2)), then there is $i_0 \in \{1, \ldots, n\}$ such that

$$\left|y_{i_0}\right|_p = p^{-\gamma_{i_0}} \geq p^{-\gamma_i} = \left|y_i\right|_p, \tag{17}$$

whenever $y_i \neq 0$. Using (3), we obtain $|\mathbf{y}|_p = p^{-\gamma_{i_0}}$. Now from (16) and (17), it follows that

$$\left||\mathbf{y}|_p\mathbf{y}\right|_p = \max_{\substack{1\leq i\leq n\\ y_i\neq 0}} \left|p^{\gamma_i-\gamma_{i_0}}\right|_p = \max_{\substack{1\leq i\leq n\\ y_i\neq 0}} p^{\gamma_{i_0}-\gamma_i} = p^{\gamma_{i_0}-\gamma_{i_0}} = 1. \tag{18}$$

Thus, for every nonzero $\mathbf{y} \in \mathbb{Q}_p^n$, the vector $|\mathbf{y}|_p\mathbf{y}$ belongs to the sphere

$$S_0(0) = \left\{\mathbf{y} \in \mathbb{Q}_p^n\colon |\mathbf{y}|_p = 1\right\}. \tag{19}$$

From (12), it directly follows that $H_{\Omega,\alpha}^p \in \mathbb{R}$ for every nonzero $\mathbf{x} \in \mathbb{Q}_p^n$, and using (12) and (13), we have

$$\begin{aligned}\left|H_{\Omega,\alpha}^{p,b} f(\mathbf{x})\right| \leq{}& \frac{|b(\mathbf{x})|}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \left|\Omega\left(|\mathbf{y}|_p\mathbf{y}\right) f(\mathbf{y})\right|\mathrm{d}\mathbf{y}\\ &+ \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \left|b(\mathbf{y})\Omega\left(|\mathbf{y}|_p\mathbf{y}\right) f(\mathbf{y})\right|\mathrm{d}\mathbf{y} < \infty,\end{aligned} \tag{20}$$

for every $\mathbf{x} \in \mathbb{Q}_p^n\setminus\{0\}$. Consequently, the operators $H_{\Omega,\alpha}^p$ and $H_{\Omega,\alpha}^{p,b}$ are correctly defined.

The aim of the present paper is to study the weighted central mean oscillations (CMO) and weighted p-adic Lipschitz estimates of $H_{\Omega,\alpha}^{p,b}$ on weighted p-adic function spaces like weighted p-adic Lebesgue spaces, weighted p-adic Herz spaces and p-adic Herz–Morrey spaces. Throughout this article, the letter C represents a constant whose value may differ at all of its occurrence. Before turning to our key results, let us define and denote the relevant p-adic function spaces.

2. Notations and Definitions

Suppose $w(\mathbf{x})$ is a weight function on $\mathbb{Q}_p^n$, which is non-negative and locally integrable function on $\mathbb{Q}_p^n$. The weighted measure of E is denoted and defined as $w(E) = \int_E w(\mathbf{x})d\mathbf{x}$. Let $L^q(w, \mathbb{Q}_p^n)$, $(0<q<\infty)$ be the space of all complex-valued functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^q(w,\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{(1/q)} < \infty. \tag{21}$$

Definition 1. Suppose $1 \le q < \infty$ and w is a weight function. The p-adic space $\mathrm{CMO}^q(w, \mathbb{Q}_p^n)$ is defined as follows:

$$\|f\|_{\mathrm{CMO}^q(w,\mathbb{Q}_p^n)} = \sup_{\gamma\in Z}\left(\frac{1}{w(B_\gamma)}\int_{B_\gamma}\left|f(\mathbf{x}) - f_{B_\gamma}\right|^q w(\mathbf{x})^{1-q}d\mathbf{x}\right)^{(1/q)}, \tag{22}$$

where

$$f_{B_\gamma} = \frac{1}{|B_\gamma|}\int_{B_\gamma} f(\mathbf{x})d\mathbf{x}. \tag{23}$$

Definition 2 (see [5]). Suppose $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and w_1 and w_2 are weight functions. Then, the weighted p-adic Herz space $K_q^{\alpha,p}(w_1, w_2)$ is defined by

$$K_q^{\alpha,p}(w_1, w_2) = \left\{f \in L_{\mathrm{loc}}^q\left(w_2, \mathbb{Q}_p^n\backslash\{0\}\right): \|f\|_{K_q^{\alpha,p}(w_1,w_2)} < \infty\right\}, \tag{24}$$

where

$$\|f\|_{K_q^{\alpha,p}(w_1,w_2)} = \left(\sum_{k=-\infty}^{\infty} w_1(B_k)^{((\alpha p)/n)}\|f\chi_k\|_{L^q(w_2,\mathbb{Q}_p^n)}^p\right)^{(1/p)} \tag{25}$$

and χ_k is the characteristic function of the sphere $S_k = B_k \setminus B_{k-1}$.

Remark 2. Obviously $K_q^{0,q}(w_1, w_2) = L^q(w_2, \mathbb{Q}_p^n)$.

Definition 3 (see [5]). Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, w_1 and w_2 be weight functions and λ be a non-negative real number. Then, the weighted p-adic Herz–Morrey space $MK_{p,q}^{\alpha,\lambda}(w_1, w_2)$ is defined as follows:

$$MK_{p,q}^{\alpha,\lambda}(w_1, w_2) = \left\{f \in L_{\mathrm{loc}}^q\left(w_2, \mathbb{Q}_p^n\backslash\{0\}\right): \|f\|_{MK_{p,q}^{\alpha,\lambda}(w_1,w_2)} < \infty\right\}, \tag{26}$$

where

$$\|f\|_{MK_{p,q}^{\alpha,\lambda}(w_1,w_2)} = \sup_{k_0\in\mathbb{Z}} w_1\left(B_{k_0}\right)^{(-\lambda/n)}\left(\sum_{k=-\infty}^{k_0} w_1(B_k)^{((\alpha p)/n)}\|f\chi_k\|_{L^q(w_2,\mathbb{Q}_p^n)}^p\right)^{(1/p)}. \tag{27}$$

Remark 3. It is evident that $MK_{p,q}^{\alpha,0}(w_1, w_2) = K_q^{\alpha,p}(w_1, w_2)$.

Now, we define the weighted p-adic Lipschitz space.

Definition 4. Suppose $1 \le q < \infty$, $0 < \gamma < 1$ and w is a weight function. The p-adic space $\mathrm{Lip}_\gamma(w, \mathbb{Q}_p^n)$ is defined as

$$\|f\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)} = \sup_{B\subset\mathbb{Q}_p^n}\frac{1}{w(B)^{(\gamma/n)}}\left(\frac{1}{w(B)}\int_B\left|f(\mathbf{x}) - f_B\right|^q w(\mathbf{x})^{1-q}d\mathbf{x}\right)^{(1/q)}, \tag{28}$$

where

$$f_B = \frac{1}{|B|}\int_B f(\mathbf{x})d\mathbf{x}. \tag{29}$$

Muckenhoupt introduced the theory of A_q weights on $\mathbb{R}^n$ in [38]. Let us define the A_q weights in the p-adic field.

Definition 5. A weight function $w \in A_q(1 \le q < \infty)$, if there exists a constant C free from choice of $B \subset \mathbb{Q}_p^n$ such that

$$\left(\frac{1}{|B|}\int_B w(\mathbf{x})d\mathbf{x}\right)\left(\frac{1}{|B|}\int_B w(\mathbf{x})^{-(1/(q-1))}d\mathbf{x}\right)^{(1/q)} \le C. \tag{30}$$

For the case $q = 1, w \in A_1$, we have

$$\frac{1}{|B|}\int_B w(\mathbf{x})d\mathbf{x} \le C\operatorname*{ess\,inf}_{\mathbf{x}\in B} w(\mathbf{x}), \tag{31}$$

for every $B \subset \mathbb{Q}_p^n$.

Remark 4. A weight function $w \in A_\infty$ if it undergoes the stipulation of $A_q(1 \le q < \infty)$ weights.

3. Weighted CMO Estimates of $H_{\Omega,\alpha}^{p,b}$ on Weighted p-Adic Herz-Type Spaces

The present section discusses the boundedness of $H_{\Omega,\alpha}^{p,b}$ on weighted p-adic Lebesgue spaces as well as on the weighted p-adic Herz-type spaces. We begin the section with some useful lemmas to prove our main results.

Lemma 1 (see [39]). *Suppose $w \in A_1$; then, there exists constants C_1, C_2 and $0 < \mu < 1$ such that*

$$C_1 \frac{|A|}{|B|} \leq \frac{w(A)}{w(B)} \leq C_2 \left(\frac{|A|}{|B|}\right)^{\mu}, \tag{32}$$

for measurable subset A of a ball B.

Remark 5. If $w \in A_1$, then it follows from Lemma 1 that there exists a constant C and $\mu (0 < \mu < 1)$ such that $(w(B_k)/w(B_i)) \leq C p^{(k-i)n}$ as $i < k$ and $(w(B_k)/w(B_i)) \leq C p^{(k-i)n\mu}$ as $i \geq k$.

Lemma 2. *Suppose $w \in A_1$ and $b \in CMO^q(w, \mathbb{Q}_p^n)$; then, there is a constant C such that for i, $k \in \mathbb{Z}$,*

$$\left|b_{B_i} - b_{B_k}\right| \leq C(i-k)\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)} \frac{w(B_k)}{|B_k|}. \tag{33}$$

Proof. Firstly, we consider

$$\begin{aligned}\left|b_{2B_\gamma} - b_{B_\gamma}\right| &\leq \frac{1}{\left|B_\gamma\right|}\int_{B_\gamma}\left|b(x) - b_{2B_\gamma}\right|\mathrm{d}x \\ &\leq \frac{1}{\left|B_\gamma\right|}\int_{2B_\gamma}\left|b(x) - b_{2B_\gamma}\right|\mathrm{d}x \\ &\leq \frac{C}{\left|B_\gamma\right|}\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)} w\left(2B_\gamma\right).\end{aligned} \tag{34}$$

We assume without loss of generality that $i > k$; then, using Lemma 1, we are down to

$$\begin{aligned}\left|b_{B_i} - b_{B_k}\right| &\leq \left|b_{B_i} - b_{B_{i-1}}\right| + \cdots + \left|b_{B_{k+1}} - b_{B_k}\right| \\ &\leq \frac{1}{\left|B_{i-1}\right|}\int_{B_i}\left|b(x) - b_{B_i}\right|\mathrm{d}x + \cdots + \frac{1}{\left|B_k\right|}\int_{B_{k+1}}\left|b(x) - b_{B_{k+1}}\right|\mathrm{d}x \\ &\leq C\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)}\left(\frac{w(B_i)}{\left|B_{i-1}\right|} + \cdots + \frac{w(B_{k+1})}{\left|B_k\right|}\right) \\ &\leq C(i-k)\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)}\frac{w(B_k)}{\left|B_k\right|}.\end{aligned} \tag{35}$$

Lemma 3. *Suppose $w \in A_1$; then for $1 < q < \infty$,*

$$\int_B w(x)^{1-q'}\mathrm{d}x \leq C|B|^{q'} w(B)^{1-q'}, \tag{36}$$

where $(1/q) + (1/q') = 1$.

Proof. Since $A_1 \subset A_q\,(q > 1)$, w satisfies the A_q conditions

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B (w(x))^{-(1/(q-1))}\mathrm{d}x\right)^{q-1}\mathrm{d}x \leq C, \tag{37}$$

for every $B \subset \mathbb{Q}_p^n$.

From here, we easily get

$$\int_B w(x)^{1-q'}\mathrm{d}x \leq C|B|^{q'} w(B)^{1-q'}. \tag{38}$$

Theorem 1. *Let $1 \leq p, q < \infty$, $w \in A_1$, $(\alpha/n) + 1 = (1/s')$; then*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)} \leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{L^q\left(w,\mathbb{Q}_p^n\right)}, \tag{39}$$

holds for all $b \in CMO^{p\max\{q,q'\}}(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$ and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Now we state the results about the boundedness of commutator of rough p-adic fractional Hardy operator on weighted p-adic Herz-type spaces.

Theorem 2. *Let $0 < p_1 \leq p_2 < \infty$, $1 \leq p, q < \infty$ and let $w \in A_1$, $(\alpha/n) + 1 = 1/s'$.*

If $\beta < (n\mu/q')$, then the inequality

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{\dot{K}_q^{\beta,p_2}(w,w^{1-q})} \leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{\dot{K}_q^{\beta,p_1}(w,w)} \tag{40}$$

holds for all $b \in \mathrm{CMO}^{p\max\{q,q'\}}(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$ and $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

Remark 6. If $\beta = 0$, $p_1 = p_2 = q$, then Theorem 1 becomes a special case of Theorem 2.

Theorem 3. *Let $0 < p_1 \leq p_2 < \infty$, $1 \leq p, q < \infty$ and let $w \in A_1$, $(\alpha/n) + 1 = (1/s')$ and $\lambda > 0$. If $\beta < (n\mu/q') + \lambda$, then*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{M\dot{K}_{p_2,q}^{\beta,\lambda}(w,w^{1-q})} \leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{M\dot{K}_{p_1,q}^{\beta,\lambda}(w,w)}, \tag{41}$$

holds for all $b \in CMO^{p\max\{q,q'\}}(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$ and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Proof. of Theorem 2. By definition, we firstly have

$$\begin{aligned}
&\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^q_{L^q\left(w^{1-q},\mathbb{Q}^n_p\right)}\\
&= \int_{S_k}|\mathbf{x}|_p^{-q(n-\alpha)}\left|\int_{|\mathbf{y}|_p\le|\mathbf{x}|_p}\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})(b(\mathbf{x})-b(\mathbf{y}))d\mathbf{y}\right|^q w(\mathbf{x})^{1-q}d\mathbf{x}\\
&\le Cp^{-kq(n-\alpha)}\int_{S_k}\left(\int_{|\mathbf{y}|_p\le p^k}\left|\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})(b(\mathbf{x})-b(\mathbf{y}))\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x}\\
&\le Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{x})-b_{B_k}\right)\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x}\\
&\quad + Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y})-b_{B_k}\right)\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x}\\
&= I + II.
\end{aligned} \tag{42}$$

For $j,k\in\mathbb{Z}$ with $j\le k$, we get

$$\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^s d\mathbf{y} = \int_{|\mathbf{z}|_p=1}|\Omega(\mathbf{z})|^s p^{jn}d\mathbf{z}\le Cp^{kn}. \tag{43}$$

Also, since $w\in A_1\subset A_q$, by the application of Hölder's inequality $(((1/q)+(1/q'))=1)$ together with Lemma 3, we have

$$\begin{aligned}
\int_{S_j}f(\mathbf{y})d\mathbf{y} &\le \left(\int_{S_j}|f(\mathbf{y})|^q w(\mathbf{y})d\mathbf{y}\right)^{(1/q)}\left(\int_{S_j}w(\mathbf{y})^{(-q'/q)}d\mathbf{y}\right)^{(1/q')}\\
&\le C\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\left|B_j\right|w\left(B_j\right)^{(-1/q)}.
\end{aligned} \tag{44}$$

To estimate I, we make use of Hölder's inequality, Remark 5, and $(\alpha/n)+1=(1/s')$ along with (43) and (44) to have

$$\begin{aligned}
I &\le Cp^{-kq(n-\alpha)}\int_{B_k}\left|b(\mathbf{x})-b_{B_k}\right|^q\\
&\quad\times\left\{\sum_{j=-\infty}^{k}\left(\left(\int_{S_j}|f(\mathbf{y})|^{s'}d\mathbf{y}\right)^{1/s'}\left(\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^s d\mathbf{y}\right)^{1/s}\right)\right\}^q w(\mathbf{x})^{1-q}d\mathbf{x}\\
&\le Cp^{kq(n-\alpha)}\|b\|^q_{\mathrm{CMO}^q\left(w,\mathbb{Q}^n_p\right)}w\left(B_k\right)\left\{\sum_{j=-\infty}^{k}p^{kn/s}\int_{S_j}|f(\mathbf{y})|d\mathbf{y}\right\}^q\\
&\le Cp^{-kqn((1-\alpha)/(n-1)/s)}\|b\|^q_{\mathrm{CMO}^q\left(w,\mathbb{Q}^n_p\right)}w\left(B_k\right)\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\left|B_j\right|w\left(B_j\right)^{-1/q}\right\}^q\\
&\le Cp^{-knq}\|b\|^q_{\mathrm{CMO}^q\left(w,\mathbb{Q}^n_p\right)}\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\left|B_j\right|\left(\frac{w\left(B_k\right)}{w\left(B_j\right)}\right)^{1/q}\right\}^q\\
&\le C\|b\|^q_{\mathrm{CMO}^q\left(w,\mathbb{Q}^n_p\right)}\left(p^{(k-j)n/q'}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\right)^q.
\end{aligned} \tag{45}$$

Now, we turn our attention towards estimating II.

$$\begin{aligned} II &\leq Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y})-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\right)^q w(\mathbf{x})^{1-q}\mathrm{d}\mathbf{x} \\ &\quad + Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b_{B_k}-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\right)^q w(\mathbf{x})^{1-q}\mathrm{d}\mathbf{x} \\ &= II_1 + II_2. \end{aligned} \tag{46}$$

In order to evaluate II_1, we need the following preparation. Apply Hölder's inequality at the outset to deduce

$$\begin{aligned} &\int_{S_j}\left|f(\mathbf{y})\left(b(\mathbf{y})-b_{B_j}\right)\right|\mathrm{d}\mathbf{y} \\ &\leq\left(\int_{S_j}|f(\mathbf{y})|^q w(\mathbf{y})\mathrm{d}\mathbf{y}\right)^{(1/q)}\left(\int_{S_j}\left|b(\mathbf{y})-b_{B_j}\right|^{q'} w(\mathbf{y})^{(-q'/q)}\mathrm{d}\mathbf{y}\right)^{(1/q')} \\ &\leq w\left(B_j\right)^{(-1/q')}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\|b\|_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}. \end{aligned} \tag{47}$$

We imply Hölder's inequality, inequality (47), Lemma 3, and Remark 5 to estimate II_1.

$$\begin{aligned} II_1 &\leq Cp^{-kq(n-\alpha)}\int_{S_k}\left\{\sum_{j=-\infty}^{k}\left(\int_{S_j}\left|f(\mathbf{y})b(\mathbf{y})-b_{B_j}\right|^{s'}\mathrm{d}\mathbf{y}\right)^{1/s'}\right. \\ &\quad\left.\times\left(\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^s\mathrm{d}\mathbf{y}\right)^{(1/s)}\right\}^q w(\mathbf{x})^{1-q}\mathrm{d}\mathbf{x} \\ &\leq Cp^{-kqn((1-\alpha)/(n-1)/s)}\int_{S_k}w(\mathbf{x})^{1-q}\mathrm{d}\mathbf{x}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})b(\mathbf{y})-b_{B_j}\right|\mathrm{d}\mathbf{y}\right)^q \\ &\leq Cp^{-kqn((1-\alpha)/(n-1)/s)}\left|B_k\right|^q w\left(B_k\right)^{1-q}\|b\|^q_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}w\left(B_j\right)^{1/q'}\right)^q \\ &\leq C\|b\|^q_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left(\frac{w\left(B_j\right)}{w\left(B_k\right)}\right)^{1-(1/q)}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q \\ &\leq C\|b\|^q_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}p^{(j-k)n\mu/q'}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q. \end{aligned} \tag{48}$$

In a similar fashion, we can estimate II_2. Using Hölder's inequality, Lemmas 2 and 3, Remark 5, and inequality (44), we get

$$\begin{aligned}
II_2 &\leq C p^{-kq(n-\alpha)} \\
&\quad \times \int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)(j-k)\|b\|_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)}\frac{w\left(B_j\right)}{\left|B_j\right|}\right|\mathrm{d}\mathbf{y}\right)^q w(\mathbf{x})^{1-q}\mathrm{d}\mathbf{x} \\
&\leq C p^{-kqn((1-\alpha)/(n-1)/s)}\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)}\left|B_k\right|^q w\left(B_k\right)^{1-q} \\
&\quad \times\left(\sum_{j=-\infty}^{k}(k-j)\frac{w\left(B_j\right)}{\left|B_j\right|}\left(\int_{S_j}|f(\mathbf{y})|^{s'}\mathrm{d}\mathbf{y}\right)^{1/s'}\left(\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^{s}\mathrm{d}\mathbf{y}\right)^{1/s}\right)^q \\
&\leq C\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)} w\left(B_k\right)^{1-q} \\
&\quad \times\left(\sum_{j=-\infty}^{k}(k-j)\frac{w\left(B_j\right)}{\left|B_j\right|}\int_{S_j}|f(\mathbf{y})|\mathrm{d}\mathbf{y}\right)^q \\
&\leq C\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)} \\
&\quad \times\left(\sum_{j=-\infty}^{k}(k-j)\left(\frac{w\left(B_j\right)}{w\left(B_k\right)}\right)^{1-(1/q)}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q \\
&\leq C\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)} \\
&\quad \times\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n\mu/q'}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q.
\end{aligned} \tag{49}$$

From (45), (48), and (49) together with Jensen inequality, we have

$$\begin{aligned}
\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{\dot{K}_q^{\beta,p_2}(w,w^{1-q})} &= \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_2)/n)} \left\|\left(H_{\Omega,\alpha}^{p,b} f\right)\chi_k\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}^{p_2}\right)^{1/p_2} \\
&\leq \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left\|\left(H_{\Omega,\alpha}^{p,b} f\right)\chi_k\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}^{p_1}\right)^{1/p_1} \\
&\leq C\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)} \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left(\sum_{j=-\infty}^{k} p^{((j-k)n)/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}\right)^{1/p_1} \\
&\quad + C\|b\|_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)} \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left(\sum_{j=-\infty}^{k} p^{((j-k)n\mu)/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}\right)^{1/p_1} \\
&\quad + C\|b\|_{\mathrm{CMO}^{p}\left(w,\mathbb{Q}_p^n\right)} \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left(\sum_{j=-\infty}^{k} (k-j) p^{((j-k)n\mu)/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}\right)^{1/p_1} \\
&= S.
\end{aligned} \tag{50}$$

Consequently,

$$\begin{aligned}
S^{p_1} &\leq C\|b\|_{\mathrm{CMO}^{p\max(q,q')}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} w(B_k)^{(\beta p_1)/n} \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n\mu/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1} \\
&\leq C\|b\|_{\mathrm{CMO}^{p\max(q,q')}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n\mu/q'-\beta} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}.
\end{aligned} \tag{51}$$

From here on in the proof we consider couple of cases, $0 < p_1 \leq 1$ and $p_1 > 1$. □

Case 1. When $0 < p_1 \leq 1$, noticing that $\beta < (n\mu/q')$, we proceed as follows.

$$\begin{aligned}
S^{p_1} &\leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k} (k-j)^{p_1} w\left(B_j\right)^{(\beta p_1)/n} p^{(j-k)((n\mu/q')-\beta)p_1} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}^{p_1} = C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} w\left(B_j\right)^{((\beta p_1)/n)} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}^{p_1} \sum_{k=j}^{\infty} (k-j)^{p_1} p^{(j-k)((n\mu/q')-\beta)p_1} \\
&= C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \|f\|_{\dot{K}_q^{\beta,p_1}(w,w)}^{p_1}.
\end{aligned} \tag{52}$$

Case 2. When $p_1 > 1$, applying Hölder's inequality with $\beta < (n\mu/q')$, we get

$$
\begin{aligned}
S^{p_1} &\le C\|b\|^{p_1}_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k} w\left(B_j\right)^{((\beta p_1)/n)} \|f\chi_j\|^{p_1}_{L^q\left(0w,\mathbb{Q}_p^n\right)} p^{((j-k)((n\mu/q')-\beta)p_1)/2} \\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)^{p_1'} p^{(j-k)((n\mu/q')-\beta)p_1'/2}\right)^{p_1/p_1'} \\
&= C\|b\|^{p_1}_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sum_{k=-\infty}^{\infty} w\left(B_j\right)^{((\beta p_1)/n)} \|f\chi_j\|^{p_1}_{L^q\left(w,\mathbb{Q}_p^n\right)} \sum_{k=j}^{\infty} p^{(j-k)((n\mu/q')-\beta)p_1/2} \\
&= C\|b\|^{p_1}_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \|f\|^{p_1}_{\dot{K}^{\beta,p_1}_{q_1}(w,w)}.
\end{aligned}
\tag{53}
$$

Therefore, the proof of theorem is completed.

Proof of Theorem 3. From Theorem 2, we have

$$
\left\|\left(H^{p,b}_{\Omega,\alpha}\right)f\chi_k\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)} \le C\|b\|^{p_1}_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sum_{j=-\infty}^{k}(k-j)p^{(j-k)(n\mu/q')}\|f\chi_j\|_{L^q\left(w,\mathbb{Q}_p^n\right)}. \tag{54}
$$

By definition of weighted p-adic Herz–Morrey space and Jensen inequality together with $\beta < (n\mu/q') + \lambda$, $\lambda > 0$ and $1 < p_1 < \infty$, it follows that

$$
\begin{aligned}
\left\|H^{p,b}_{\Omega,\alpha}f\right\|_{M\dot{K}^{\beta,\lambda}_{p_2,q}(w,w^{1-q})} &= \sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n}\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{((\beta p_2)/n)}\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^{p_2}_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}\right)^{(1/p_2)} \\
&\le \sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n}\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{((\beta p_1)/n)}\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^{p_1}_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}\right)^{1/p_1} \\
&\le C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n} \\
&\quad\times\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{((\lambda p_1)/n)}\left(\sum_{j=-\infty}^{k}(k-j)p^{((j-k)n\mu/q')}\left(\frac{w\left(B_k\right)}{w\left(B_j\right)}\right)^{((\lambda p_1)/n)}\right.\right. \\
&\quad\left.\left.\times w\left(B_j\right)^{-\lambda/n}\left(\sum_{l=-\infty}^{j} w\left(B_l\right)^{\beta p_1/n}\|f\chi_j\|^{p_1}_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{1/p_1}\right)^{p_1}\right)^{1/p_1} \\
&\le C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n} \\
&\quad\times\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{((\lambda p_1)/n)}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)(((n\mu)/q')-\beta+\lambda)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}\right)^{p_1}\right)^{1/p_1} \\
&\le C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n}\times\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{((\lambda p_1)/n)}\right)^{1/p_1}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)} \\
&\le C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}.
\end{aligned}
\tag{55}
$$

4. Weighted Lipschitz Estimates for the Commutator of Rough p-Adic Fractional Hardy Operator on Herz–Morrey Spaces

In this section, we obtain the weighted p-adic Lipschitz estimates for the commutator of rough p-adic fractional Hardy operator on p-adic Lebesgue spaces and p-adic Herz-type spaces. We begin the section with a useful lemma which can be proved in the similar lines as Lemma 2.

Lemma 5. *Suppose $w \in A_1$ and $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$; then, there is a constant C such that for i, $k \in \mathbb{Z}$,*

$$\left|b_{B_i} - b_{B_k}\right| \leq C(i-k)\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)} w(B_i)^{\gamma/n} \frac{w(B_k)}{|B_k|}. \quad (56)$$

Theorem 4. *Let $1 \leq p, q < \infty$, $(1/q_1) - (1/q_2) = (\gamma/n)$, $w \in A_1$, $(\alpha/n) + 1 = (1/s')$; then,*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)} \leq C\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\|f\|_{L^q\left(w,\mathbb{Q}_p^n\right)}, \quad (57)$$

holds for all $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$, and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Now we state the results about the boundedness of commutator of rough p-adic fractional Hardy operator on weighted p-adic Herz-type spaces.

Theorem 5. *Let $0 < p_1 \leq p_2 < \infty$, $1 \leq q_1, q_2 < \infty$, $(1/q_1) - (1/q_2) = (\gamma/n)$ and let $w \in A_1$, $(\alpha/n) + 1 = (1/s')$. If $\beta < (n\mu/q_1')$, then the inequality*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{\dot{K}_{q_2}^{\beta,p_2}(w,w^{1-q_2})} \leq C\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\|f\|_{\dot{K}_{q_1}^{\beta,p_1}(w,w)}, \quad (58)$$

holds for all $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$, and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Remark 7. If $\beta = 0, p_1 = q_1 = p$ and $p_2 = q_2 = q$, then Theorem 4 can easily be obtained from Theorem 5.

Theorem 6. *Let $0 < p_1 \leq p_2 < \infty$, $1 \leq q_1, q_2 < \infty$, $(1/q_1) - (1/q_2) = (\gamma/n)$ and let $w \in A_1$, $(\alpha/n) + 1 = (1/s')$, and $\lambda > 0$. If $\beta < (n\mu/q_1') + \lambda$, then*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{M\dot{K}_{p_2,q_2}^{\beta,\lambda}(w,w^{1-q_2})} \leq C\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\|f\|_{M\dot{K}_{p_1,q_1}^{\beta,\lambda}(w,w)}, \quad (59)$$

holds for all $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$, and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Proof of Theorem 5. Following the same pattern of Theorem 2, we have

$$\begin{aligned}
&\left\|\left(H_{\Omega,\alpha}^{p,b} f\right)\chi_k\right\|_{L^{q_2}\left(w^{1-q_2},\mathbb{Q}_p^n\right)}^{q_2} \\
&\leq Cp^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}|f(\mathbf{y})|\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{x}) - b_{B_k}\right)|d\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x} \\
&\quad + Cp^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}|f(\mathbf{y})|\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y}) - b_{B_k}\right)|d\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x} \\
&= J + JJ.
\end{aligned} \quad (60)$$

To estimate J, we make use of Hölder's inequality, Remark 5, $(\alpha/n) + (1) = (1/s')$, $(\gamma/n) = (1/q_1) - (1/q_2)$, and $w \in A_1 \subset A_{q_1}$ along with (43) and (44) to have

$$\begin{aligned}
J &\leq Cp^{-kq_2n((1-\alpha)/((n-1)/s))}\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}^{q_2} w(B_k)^{((1+\gamma q_2)/n)}\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\left|B_j\right| w\left(B_j\right)^{(-1/q_1)}\right\}^{q_2} \\
&\leq Cp^{-knq_2}\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}^{q_2}\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\left|B_j\right|\left(\frac{w(B_k)}{w(B_j)}\right)^{1/q_1}\right\}^{q_2} \\
&\leq C\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}^{q_2}\left(p^{(k-j)n/q_1'}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}.
\end{aligned} \quad (61)$$

For the estimation of JJ, we need to decompose it as

$$
\begin{aligned}
JJ &\leq Cp^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y})-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}\mathrm{d}\mathbf{x}\\
&\quad + Cp^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b_{B_k}-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}\mathrm{d}\mathbf{x}\\
&= JJ_1 + JJ_2.
\end{aligned} \tag{62}
$$

We need the following preparation to estimate JJ_1. Apply Hölder's inequality to get

$$
\begin{aligned}
&\int_{S_j}\left|f(\mathbf{y})\left(b(\mathbf{y})-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\\
&\leq \left(\int_{S_j}|f(\mathbf{y})|^{q_1}w(\mathbf{y})\mathrm{d}\mathbf{y}\right)^{1/q_1}\left(\int_{S_j}\left|b(\mathbf{y})-b_{B_j}\right|^{q_1'}w(\mathbf{y})^{(-q_1'/q_1)}\mathrm{d}\mathbf{y}\right)^{(1/q_1')}\\
&\leq w\left(B_j\right)^{(-1/q_1')+(\gamma/n)}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}.
\end{aligned} \tag{63}
$$

We imply Hölder's inequality, inequality (63), Lemma 3, and Remark 5 to estimate JJ_1.

$$
\begin{aligned}
JJ_1 &\leq Cp^{-kq_2 n((1-\alpha)/(n-1)/s)}\int_{S_k}w(\mathbf{x})^{1-q_2}d\mathbf{x}\left(\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}w\left(B_j\right)^{(1/q_1'+\gamma/n)}\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}\\
&\leq Cp^{-kq_2 n((1-\alpha)/(n-1)/s)}\left|B_k\right|^{q_2}w\left(B_k\right)^{1-q_2}\|b\|^{q_2}_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}w\left(B_j\right)^{(1/q_1'+\gamma/n)}\right)^{q_2}\\
&\leq C\|b\|^{q_2}_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left(\frac{w\left(B_j\right)}{w\left(B_k\right)}\right)^{1-1/q_2}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}\\
&\leq C\|b\|^{q_2}_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}p^{(j-k)n\mu/q_2'}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}.
\end{aligned} \tag{64}
$$

Now we turn towards JJ_2. Using once again Hölder's inequality, Lemmas 5 and 3, Remark 5, and inequality (44), we get

$$\begin{aligned}
JJ_2 &\leq Cp^{-kq_2 n((1-\alpha)/(n-1)/s)}\|b\|^{q_2}_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}|B_k|^{q_2} w(B_k)^{1-q_2}\\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)w(B_k)^{\gamma/n}\frac{w(B_j)}{|B_j|}|B_j| w(B_j)^{-1/q_1}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{q_2}\\
&= C\|b\|^{q_2}_{Lip_\gamma(w,\mathbb{Q}_p^n)}\\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)\left(\frac{w(B_j)}{w(B_k)}\right)^{1-1/q_1}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{q_2}\\
&\leq C\|b\|^{q_2}_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}\\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n\mu/q_1'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{q_2}.
\end{aligned}\tag{65}$$

Rest of the proof is similar to the proof of Theorem 2. Thus, we come to an end of proof.

Proof of Theorem 6. Let $\beta < n\mu/q_1' + \lambda$. By the definition of weighted p-adic Herz–Morrey spaces along with inequalities (61), (64), and (65), we are down to

$$\begin{aligned}
\left\|H^{p,b}_{\Omega,\alpha}f\right\|_{M\dot{K}^{\beta,\lambda}_{p_2,q_2}(w,w^{1-q_2})} &\leq C\|b\|_{Lip_\gamma(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\left(\sum_{k=-\infty}^{\infty} w(B_k)^{(\beta p_2)/n}\left(\sum_{j=-\infty}^{k} p^{(j-k)n/q_1'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{p_2}\right)^{1/p_2}\\
&\quad + C\|b\|_{Lip_\gamma(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_2)/n)}\left(\sum_{j=-\infty}^{k} p^{(j-k)n\mu/q_2'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{p_2}\right)^{1/p_2}\\
&\quad + C\|b\|_{Lip_\gamma(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\left(\sum_{k=-\infty}^{\infty} w(B_k)^{(\beta p_2)/n}\left(\sum_{j=-\infty}^{k}(k-j) p^{(j-k)n\mu/q_1'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{p_2}\right)^{1/p_2}\\
&= L_1 + L_2 + L_3.
\end{aligned}\tag{66}$$

Next by applying the similar arguments as in Theorem 3, we get

$$\begin{aligned}
L_1 &\leq C\|b\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}, \quad \beta < \frac{n}{q_1'} + \lambda,\\
L_2 &\leq C\|b\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}, \quad \beta < \frac{n\mu}{q_2'} + \lambda,\\
L_3 &\leq C\|b\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}, \quad \beta < \frac{n\mu}{q_1'} + \lambda.
\end{aligned}\tag{67}$$

Therefore, we conclude the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors made equal contributions and read and supported the last original copy.

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References

[1] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1994.
[2] A. Khrennikov, *p-Adic Valued Distributions In Mathematical Physics And Its Applications*, p. 309, Kluwer Academic Publishers Group, Dordrecht, Netherlands, 1994.
[3] V. S. Varadarajan, "Path integrals for a class of p-adic Schrodiner equations," *Letters in Mathematical Physics*, vol. 39, no. 2, pp. 97–106, 1997.
[4] V. S. Vladimirov, "Tables of integrals for complex-valued functions of p-adic arguments," *Proceedings of the Steklov Institute of Mathematics*, vol. 284, no. S2, pp. 1–59, 2014.
[5] A. Hussain and N. Sarfraz, "The hausdorff operator on weighted p-adic Morrey and Herz type spaces," *p-Adic Numbers, Ultrametric Analysis and Applications*, vol. 11, no. 2, pp. 151–162, 2019.
[6] A. Hussain and N. Sarfraz, "Optimal weak type estimates for p-Adic Hardy operator. p-Adic Numb," *Ultrametric Analysis and Applications*, vol. 12, no. 1, pp. 12–21, 2020.
[7] A. Hussain, N. Sarfraz, and F. Gürbüz, "Weak bounds for p-adic Hardy operators on p-adic linear spaces," 2020, http://arxiv.org/abs/2002.08045.
[8] S. V. Kozyrev, "Methods and applications of ultrametric and p-adic analysis: from wavelet theory to biophysics," *Proceedings of the Steklov Institute of Mathematics*, vol. 274, no. S1, pp. 1–84, 2011.
[9] N. Sarfraz and F. Gürbüz, "Weak and strong boundedness for p-adic fractional Hausdorff operator and its commutators," 2019, http://arxiv.org/abs/1911.09392v1.
[10] N. Sarfraz and A. Hussain, "Estimates for the commutators of p-adic Hausdorff operator on Herz-morrey spaces," *Mathematics*, vol. 7, no. 2, p. 127, 2019.
[11] S. S. Volosivets, "Weak and strong estimates for rough Hausdorff type operator defined on p-adic linear space," *p-Adic Numbers, Ultrametric Analysis and Applications*, vol. 9, no. 3, pp. 236–241, 2017.
[12] Z. Fu, S. Lu, and F. Zhao, "Commutators of n-dimensional rough Hardy operators," *Science China Mathematics*, vol. 54, no. 1, pp. 95–104, 2011.
[13] A. Hussain, N. Sarfraz, I. Khan, and A. M. Alqahtani, "Estimates for commutators of bilinear fractional p-adic Hardy operator on Herz-type spaces," *Journal of Function Spaces*, vol. 2021, Article ID 6615604, 7 pages, 2021.
[14] R. H. Liu and J. Zhou, "Sharp estimates for the p-adic Hardy type Operator on higher-dimensional product spaces," *Journal of Inequalities and Applications*, vol. 2017, p. 13, 2017.
[15] Q. Y. Wu, "Boundedness for commutators of fractional p-adic hardy operator," *Journal of Inequalities and Applications*, vol. 2012, p. 12, 2012.
[16] Q. Y. Wu, L. Mi, and Z. W. Fu, "Boundedness of p-adic Hardy Operators and their commutators on p-adic central Morrey and BMO spaces," *Journal of Function Spaces*, vol. 2013, Article ID 359193, 10 pages, 2013.
[17] G. H. Hardy, "Note on a theorem of hilbert," *Mathematische Zeitschrift*, vol. 6, no. 3-4, pp. 314–317, 1920.
[18] W. G. Faris, "Weak Lebesgue spaces and quantum mechanical binding," *Duke Mathematical Journal*, vol. 43, pp. 365–373, 1976.
[19] Z. W. Fu, L. Grafakos, S. Z. Lu, and F. Y. Zhao, "Sharp bounds for m-linear hardy and hilbert operators," *Houston Journal of Mathematics*, vol. 38, no. 1, pp. 225–244, 2012.
[20] S. Z. Lu, D. C. Yang, and F. Y. Zhao, "Sharp bounds for Hardy type Operators on higher dimensional product spaces," *Journal of Inequalities and Applications*, vol. 148, p. 11, 2013.
[21] Y. Mizuta, A. Nekvinda, and T. Shimomura, "Optimal estimates for the fractional hardy operator," *Studia Mathematica*, vol. 227, no. 1, pp. 1–19, 2015.
[22] L.-E. Persson and S. Samko, "A note on the best constants in some Hardy inequalities," *Journal of Mathematical Inequalities*, vol. 9, no. 2, pp. 437–447, 2015.
[23] M. Christ and L. Grafakos, "Best constants for two nonconvolution inequalities," *American Mathematical Society*, vol. 123, no. 6, p. 1687, 1995.
[24] A. Ajaib and A. Hussain, "Weighted CBMO estimates for commutators of matrix Hausdorff operator on the Heisenberg group estimates for commutators of matrix Hausdorff operator on the Heisenberg group," *Open Mathematics*, vol. 18, no. 1, pp. 496–511, 2020.
[25] Z. W. Fu, Z.-G. Liu, and H. B. Wang, "Characterization for commutators of n-dimensional fractional Hardy operators," *Science in China Series A: Mathematics*, vol. 50, no. 10, pp. 1418–1426, 2007.
[26] G. Gao, "Boundedness for commutators of n-dimensional rough Hardy operators on Morrey-Herz spaces," *Computers & Mathematics with Applications*, vol. 64, no. 4, pp. 544–549, 2012.
[27] G. Gao and Y. Zhong, "Some estimates of Hardy Operators and their commutators on Morrey-Herz spaces," *Journal of Mathematical Inequalities*, vol. 11, no. 1, pp. 49–58, 2017.
[28] A. Hussain and A. Ajaib, "Some results for the commutators of generalized Hausdorff operator," *Journal of Mathematical Inequalities*, vol. 13, no. 4, pp. 1129–1146, 2019.
[29] A. Hussain and G.-L. Gao, "Some new estimates for the commutators of n-dimensional Hausdorff operator," *Applied Mathematics-A Journal of Chinese Universities*, vol. 29, no. 2, pp. 139–150, 2014.
[30] Z. W. Fu, Q. Y. Wu, and S. Z. Lu, "Sharp estimates of p-adic hardy and Hardy-Littlewood-Pólya operators," *Acta Mathematica Sinica, English Series*, vol. 29, no. 1, pp. 137–150, 2013.
[31] I. Ekincioglu, C. Keskin, and R. V. Guliyev, "Lipschitz estimates for rough fractional multilinear integral operators on local generalized Morrey spaces," *Tbilisi Mathematical Journal*, vol. 13, no. 1, pp. 47–60, 2020.

[32] G. Hu, X. Lai, and Q. Xue, "On the composition of rough singular integral operators," *Journal of Geometric Analysis Impact Factor*, vol. 31, pp. 2742–2765, 2020.

[33] K. Li, C. Pérez, I. P. Rivera-Ríos, and L. Roncal, "Weighted norm inequalities for rough singular integral operators," *The Journal of Geometric Analysis*, vol. 29, no. 3, pp. 2526–2564, 2019.

[34] M. A. Ragusa, "Regularity of solutions of divergence form elliptic equations," *American Mathematical Society*, vol. 128, no. 2, pp. 533–540, 2000.

[35] M. A. Ragusa, "Local Hölder regularity for solutions of elliptic systems," *Duke Mathematical Journal*, vol. 113, no. 2, pp. 385–397, 2002.

[36] M. A. Ragusa and A. Tachikawa, "Partial regularity of the minimizers of quadratic functionals with VMO coefficients," *Journal of the London Mathematical Society*, vol. 72, no. 3, pp. 609–620, 2005.

[37] X. Tao and G. Hu, "An estimate for the composition of rough singular integrals operators," *Canadian Mathematical Bulletin*, 2021, In press.

[38] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function," *Transactions of the American Mathematical Society*, vol. 165, p. 207, 1972.

[39] J. L. Journe, "Calderón-Zygmund operators,differential operators and the cauchy integral of Calderón," *Lecture Notes in Mathematics*, vol. 994, pp. 1–127, 1983.

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Research Article

Multivariate Dynamic Sneak-Out Inequalities on Time Scales

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In this study, we extend some "sneak-out" inequalities on time scales for a function depending on more than one parameter. The results are proved by using the induction principle and time scale version of Minkowski inequalities. In seeking applications, these inequalities are discussed in classical, discrete, and quantum calculus.

1. Introduction

Bennett and Grosse-Erdmann [1] introduce the "sneak-out" principle concerned with the equivalence of two series. Bohner and Saker [2] extended the sneak-out principle on time scales and proved some new dynamic sneak-out inequalities and their converses on time scales which, as special cases, with $\mathbb{T} = \mathbb{N}$, contain the discrete inequalities obtained by Bennett and Grosse-Erdmann (Section 6 in [1]). However, the sneak-out principle on time scales can be applied to formulate the corresponding integral inequalities by choosing $\mathbb{T} = \mathbb{R}$. The paper aims to extend the work given by Bohner and Saker in [2] for functions depending on more than one parameter. Some other inequalities, such as Hardy-type, Hardy-Copson, and Copson-Leindler-type inequalities, are also studied for functions of more than one parameter [3–5] via time scales' calculus. Some literature concerning with time scale can be seen in [6–13].

The paper is organized as follows. Section 2 provides some basics from time scales' calculus. Section 3 features two dynamic inequalities of the Copson type, which are needed to prove further results. In Section 4, we present sneak-out inequalities on time scales for functions depending on more than one parameter.

2. Preliminaries

A time scale $\mathbb{T}$ as well as close set in $\mathbb{R}$ are nonempty [14, 15]. Some examples of time scales are $\mathbb{Z}$, $\mathbb{R}$, and Cantor set. Assume that $\inf \mathbb{T} = \phi$, where ϕ is empty set and $\sup \mathbb{T} = \infty$. A time-scale interval is denoted by $[t_0, \infty)_{\mathbb{T}} \coloneqq [t_0, \infty) \cap \mathbb{T}$, for $t_0 \in \mathbb{T}$.

The operators $\sigma: \mathbb{T} \longrightarrow \mathbb{T}$ defined by $\sigma(l) \coloneqq \inf\{b \in \mathbb{T};\, b > l\}$ and $\rho: \mathbb{T} \longrightarrow \mathbb{T}$ defined by $\rho(l) \coloneqq \sup\{b \in \mathbb{T};\, b < l\}$ are forward as well as backward jump operators, respectively, for $l \in \mathbb{T}$. The point $l \in \mathbb{T}$ is right-scattered if it satisfies $\sigma(l) > l$, and left-scattered if $\rho(l) < l$. The points which are at the same time left-scattered as well as right-scattered are called *isolated*. Furthermore, the point $l \in \mathbb{T}$ is right-dense if it satisfies $l < \operatorname{Sup} \mathbb{T}$ and $\sigma(l) = l$, and left-dense if it satisfies $l > \inf \mathbb{T}$ and $\sigma(l) = l$; furthermore, the point is called dense if it is left-dense as well as right-dense at the same time. A function $\mu: \mathbb{T} \longrightarrow [0, \infty)$, defined by $\mu(l) \coloneqq \sigma(l) - l$, is called the graininess function.

If a function $g: \mathbb{T} \longrightarrow \mathbb{R}$ is continuous at all right-dense points, the left-hand limits exist and are finite at left-dense points in $\mathbb{T}$; then, it is right-dense continuous (rd-continuous) on $\mathbb{T}$. The set denoted by $C_{rd}(\mathbb{T})$ contain all rd-continuous functions on $\mathbb{T}$.

Consider a function $\beta: \mathbb{T} \longrightarrow \mathbb{R}$, and define the number $\beta^{\Delta}(\varsigma)$ if it exists with the property that, for given $\varepsilon > 0$, there is a neighborhood U of ς which satisfies

$$\left|\beta(\sigma(\varsigma)) - \beta(r) - \beta^{\Delta}(\varsigma)(\beta(\varsigma) - r)\right| \le \varepsilon|\sigma(\varsigma) - r|, \quad \forall r \in U, \tag{1}$$

then $\beta^{\Delta}(\varsigma)$ is delta derivative of function $\beta(\varsigma)$ at $\varsigma \in \mathbb{T}$.

Notation: $\zeta^{\sigma}(\varsigma) = \zeta(\sigma(\varsigma))$ for any function $\zeta \coloneqq \mathbb{T} \longrightarrow \mathbb{R}$.

(1) Product and quotient rule for delta derivative (Theorem 1.20 in [14]): assume $\zeta, \eta: \mathbb{T} \longrightarrow \mathbb{R}$ are differentiable; then,

$$\text{(i)}\ (\zeta\eta)^{\Delta} = \zeta^{\Delta}\eta + \zeta^{\sigma}\eta^{\Delta} = \zeta\eta^{\Delta} + \zeta^{\Delta}\eta^{\sigma},$$

$$\text{(ii)}\ \left(\frac{\zeta}{\eta}\right)^{\Delta}(\varsigma) = \frac{\zeta^{\Delta}(\varsigma)\eta(\varsigma) - \zeta(\varsigma)\eta^{\Delta}(\varsigma)}{\eta(\varsigma)\eta^{\sigma}(\varsigma)}, \quad \eta(\varsigma) \ne 0, \varsigma \in \mathbb{T}. \tag{2}$$

(2) Integration by parts formula (Theorem 1.77 in [14]): for two delta differentiable functions $g, h: \mathbb{T} \longrightarrow \mathbb{R}$, and $\varsigma, a, m \in \mathbb{T}$, we have

$$\int_a^m g(\varsigma)h^{\Delta}(\varsigma)\Delta\varsigma = g(\varsigma)h(\varsigma)\Big|_a^m - \int_a^m g^{\Delta}(\varsigma)h^{\sigma}(\varsigma)\Delta\varsigma. \tag{3}$$

(3) Minkowski inequality (Theorem 6.16 in [14]): for three rd-continuous functions $f: \mathbb{T} \longrightarrow \mathbb{R}$, $g: \mathbb{T} \longrightarrow \mathbb{R}$, and $h: \mathbb{T} \longrightarrow \mathbb{R}$, we have

$$\left\{\int_a^b |h(t)||f(t) + g(t)|^p\right\}^{(1/p)} \le \left\{\int_a^b |h(t)||f(t)|^p\Delta t\right\}^{(1/p)} + \left\{\int_a^b |h(t)||g(t)|^p\Delta t\right\}^{(1/p)}, \tag{4}$$

where $p > 1$ and $a, b, t \in \mathbb{T}$.

(4) Fubini's theorem [16]: let there exist two time scales' measure spaces (ν, M, ϕ_{Δ}) and $(\upsilon, N, \varphi_{\Delta})$ which have finite dimensions. If $\eta: \nu \times \upsilon \longrightarrow \mathbb{R}$ is a $\phi_{\Delta} \times \varphi_{\Delta}$ which is an integrable function and the function $\psi_1(m) = \int_{\nu}\eta(l, m)\Delta l$ exists for almost every $m \in \upsilon$ and $\psi_2(l) = \int_{\upsilon}\eta(l, m)\Delta m$ exists for almost every $l \in \nu$, then ψ_1 is φ_{Δ} integrable on υ, ψ_2 is ϕ_{Δ} integrable on ν, and

$$\int_{\nu}\Delta l\int_{\upsilon}\eta(l, m)\Delta m = \int_{\upsilon}\Delta m\int_{\nu}\eta(l, m)\Delta l. \tag{5}$$

Notation:

$$\frac{\partial}{\Delta\tau_k}g(\tau_1, \ldots, \tau_k, \ldots, \tau_n) = g^{\Delta_k}(\tau_1, \ldots, \tau_k, \ldots, \tau_n), \quad 1 \le k \le n. \tag{6}$$

Some preliminary inequalities [2]: suppose $g: \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable. Let $\beta \in \mathbb{R}$, if g^{Δ} is monotone, *i.e.*, either always negative or always positive, then

$$\beta g^{\Delta}\left(g^{\beta-1}\right)^{\sigma} \le \left(g^{\beta}\right)^{\Delta} \le \beta g^{\Delta}g^{\beta-1}, \quad \text{if } 0 \le \beta \le 1, \tag{7}$$

$$\beta g^{\Delta}g^{\beta-1} \le \left(g^{\beta}\right)^{\Delta} \le \beta g^{\Delta}\left(g^{\beta-1}\right)^{\sigma}, \quad \text{if } \beta \ge 1, \tag{8}$$

and if g^{Δ} is positive, then

$$\begin{aligned} &\left(g^{\beta}\right)^{\Delta} \le g^{\Delta}\left(g^{\beta-1}\right)^{\sigma}, \quad \text{if } 0 \le \beta \le 1, \\ &\left(g^{\beta}\right)^{\Delta} \le g^{\Delta}\left(g^{\beta-1}\right)^{\sigma}, \quad \text{if } \beta \ge 1. \end{aligned} \tag{9}$$

3. Dynamic Copson-Type Inequalities for Finite Numbers of Parameters

We assume throughout that all the functions are nonnegative and the integrals considered exist. For $h \in \mathbb{N}$, $\iota \in \{1, 2, \ldots, h\}$, let $\mathbb{T}_{\iota}$ be time scales.

Presume 1:

$$H_1 = \begin{cases} \operatorname{Sup}\mathbb{T}_{\iota} = \infty, \quad b_{\iota} \in (0, \infty)_{\mathbb{T}_{\iota}}, \\ \upsilon_{\iota}: \mathbb{T}_{\iota} \longrightarrow \mathbb{R}^{+} \text{ is rd} - \text{continuous}, \\ A_{\iota}(\tau_{\iota}) \coloneqq \int_{b_{\iota}}^{\tau_{\iota}} \upsilon_{\iota}(s_{\iota})\Delta s_{\iota}, \quad \text{for } \tau_{\iota} \in \mathbb{T}_{\iota}. \end{cases} \tag{10}$$

Theorem 1. *Assume H_1. Suppose $g: \mathbb{T}_1 \times \cdots \times \mathbb{T}_h \longrightarrow \mathbb{R}^{+}$ is such that*

$$\phi(\tau_1, \ldots, \tau_h) \coloneqq \int_{\tau_1}^{\infty} \cdots \int_{\tau_h}^{\infty} \prod_{\iota=1}^{h} \frac{\upsilon_{\iota}(s_{\iota})}{A_{\iota}(\sigma_{\iota}(s_{\iota}))} g(s_1, \ldots, s_h)\Delta s_h \cdots \Delta s_1, \tag{11}$$

is well defined and $m \ge 1$. Then,

$$\int_{b_1}^{\infty}\cdots\int_{b_h}^{\infty}\prod_{\iota=1}^{h}v_\iota(\tau_\iota)\phi^m(\tau_1,\ldots,\tau_h)\Delta\tau_h\cdots\Delta\tau_1\leqslant(m)^{hm}\int_{b_1}^{\infty}\cdots\int_{b_h}^{\infty}\prod_{\iota=1}^{h}v_\iota(\tau_\iota)g^m(\tau_1,\ldots,\tau_h)\Delta\tau_h\cdots\Delta\tau_1. \tag{12}$$

Theorem 2. *Assume H_1. Suppose $g\colon \mathbb{T}_1\times\cdots\times\mathbb{T}_h\longrightarrow\mathbb{R}$ is such that*

$$\phi(\varsigma_1,\ldots,\varsigma_h):=\int_{\varsigma_1}^{\infty}\cdots\int_{\varsigma_h}^{\infty}\prod_{\iota=1}^{h}v_\iota(s_\iota)g(s_1,\ldots,s_h)\Delta s_h\cdots\Delta s_1,\quad(\varsigma_1,\ldots,\varsigma_h)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_h, \tag{13}$$

is well defined. Let $m\geq 1$ and $0\leq c_\iota<1$. Then,

$$\int_{a_1}^{\infty}\cdots\int_{a_h}^{\infty}\prod_{\iota=1}^{h}\frac{v_\iota(\varsigma_\iota)}{A_\iota^{c_\iota}(\sigma_\iota(\varsigma_\iota))}\phi^m(\varsigma_1,\ldots,\varsigma_h)\Delta\varsigma_h\cdots\Delta\varsigma_1\leq\prod_{\iota=1}^{h}\left(\frac{m}{1-c_\iota}\right)^m\int_{a_1}^{\infty}\cdots\int_{a_h}^{\infty}\prod_{\iota=1}^{h}v_\iota(\varsigma_\iota)A_\iota^{m-c_\iota}(\sigma_\iota(\varsigma_\iota))g^m(\varsigma_1,\ldots,\varsigma_h)\Delta\varsigma_h\cdots\Delta\varsigma_1. \tag{14}$$

Proofs of Theorem 1 and Theorem 2 are after sneak-out inequalities.

4. Dynamic Sneak-Out Inequalities for Finite Numbers of Parameters

Let $i,j,r\in\{1,\ldots,h\}$ and $(i_1,\ldots,i_h)=(j_1,\ldots,j_h)=(1,\ldots,h)$.

Presume 2:

$$H_2=\begin{cases} x\colon \mathbb{T}_1\times\cdots\times\mathbb{T}_h\longrightarrow\mathbb{R}_+ \text{ is rd}-\text{continuous},\\ y\left(\tau_{i_1},\ldots,\tau_{i_h}\right):=\int_{\tau_{i_1}}^{\infty}\cdots\int_{\tau_{i_h}}^{\infty}x\left(s_{i_1},\ldots,s_{i_h}\right)\Delta s_{i_h}\cdots\Delta s_{i_1},\quad\left(\tau_{i_1},\ldots,\tau_{i_h}\right)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_h,\\ \psi\left(\tau_{i_1},\ldots,\tau_{i_h}\right):=\int_{\tau_{i_1}}^{\infty}\cdots\int_{\tau_{i_h}}^{\infty}\prod_{i_k=1}^{h}A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right)x\left(s_{i_1},\ldots,s_{i_h}\right)\Delta s_{i_h}\cdots\Delta s_{i_1}.\end{cases} \tag{15}$$

Lemma 1. *Let $\mathbb{T}_\iota$ be the time scales for $\iota\in\{1,2,\ldots,h\}$, under H_1 and H_2, and we have*

$$\psi\left(\tau_{i_1},\ldots,\tau_{i_h}\right)\leq\prod_{k=1}^{h}A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right)y\left(\tau_{i_1},\ldots,\tau_{i_h}\right)+ \sum_{1\leq j_1<\cdots<j_r\leq h}\left[\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\prod_{m=1}^{h-r}A_{i_m}^{\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\times\int_{\prod_{1\leq j_1<\cdots<j_r\leq h}\tau_{j_m}}^{\infty}\prod_{m=1}^{r}A_{j_m}^{\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right)v_{j_m}\left(s_{j_m}\right)y\left(\tau_{i_1},\ldots,\tau_{i_{h-r}},s_{j_1},\ldots,s_{j_r}\right)\underset{\prod_{1\leq j_1<\cdots<j_r\leq h}j_m}{\Delta}s_{j_m}\right]. \tag{16}$$

Proof. For $h=1$, (16) is true by Theorems 4.1 in [2], *i.e.*,

$$\psi(\tau_1)\leq A_1^{\alpha_1}(\sigma_1(\tau_1))y(\tau_1)+\alpha_1\int_{\tau_1}^{\infty}v_1(s_1)A_1^{\alpha_1-1}(\sigma_1(s_1))y(s_1)\Delta s_1. \tag{17}$$

Suppose (16) is true for $1 \le h \le p$. To prove for $h = p + 1$, by using H_2, we have defined as

$$\psi\left(\tau_{i_1}, \ldots, \tau_{i_{p+1}}\right) = \int_{\tau_{i_1}}^{\infty} \cdots \int_{\tau_{i_p}}^{\infty} \prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right) \times \left\{ \int_{\tau_{i_{p+1}}}^{\infty} A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right) y^{\Delta_1 \cdots \Delta_{p+1}}\left(s_{i_1}, \ldots, s_{i_{p+1}}\right) \Delta s_{i_{p+1}} \right\} \Delta s_{i_p} \cdots \Delta s_{i_1}. \quad (18)$$

Denote

$$Z_{i_{p+1}} = \int_{\tau_{i_{p+1}}}^{\infty} A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right) y^{\Delta_1 \cdots \Delta_{p+1}}\left(s_{i_1}, \ldots, s_{i_{p+1}}\right) \Delta s_{i_{p+1}}. \quad (19)$$

Use integration by parts' formula (3) in (19) to obtain

$$Z_{i_{p+1}} = \left| -A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(s_{p+1}\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_{p+1}}\right) \right|_{\tau_{p+1}}^{\infty} - \int_{\tau_{i_{p+1}}}^{\infty} \frac{\partial}{\Delta \tau_{i_{p+1}}}\left(A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(s_{i_{p+1}}\right)\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_{p+1}}\right) \Delta s_{i_{p+1}}. \quad (20)$$

Use the right-hand side part of inequality (8) with $A_{i_{p+1}} \le A_{i_{p+1}}^{\sigma_{i_{p+1}}}$ in (20)

$$Z_{i_{p+1}} \le A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_p}, \tau_{i_{p+1}}\right) + \alpha_{i_{p+1}} \int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right) A_{i_{p+1}}^{\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_{p+1}}\right) \Delta s_{i_{p+1}}. \quad (21)$$

Substitute (21) in (18):

$$\begin{aligned} \psi\left(\tau_{i_1}, \ldots, \tau_{i_{p+1}}\right) = {} & \int_{\tau_{i_1}}^{\infty} \cdots \int_{\tau_{i_p}}^{\infty} \prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right) \left\{ A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_p}, \tau_{i_{p+1}}\right) \right\} \Delta s_{i_p} \cdots \Delta s_{i_1} \\ & + \alpha_{i_{p+1}} \int_{\tau_{i_1}}^{\infty} \cdots \int_{\tau_{i_p}}^{\infty} \prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right) \\ & \times \left\{ \int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right) A_{i_{p+1}}^{\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_{p+1}}\right) \Delta s_{i_{p+1}} \right\} \Delta s_{i_p} \cdots \Delta s_{i_1}. \end{aligned} \quad (22)$$

Use (5) "p times" in second term of (22):

$$\begin{aligned} \psi\left(\tau_{i_1} \ldots \tau_{i_{p+1}}\right) \le {} & A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right) \int_{\tau_{i_1}}^{\infty} \cdots \int_{\tau_{i_p}}^{\infty} \prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_p}, \tau_{i_{p+1}}\right) \Delta s_{i_p} \cdots \Delta s_{i_1} \\ & + \alpha_{i_{p+1}} \int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right) A_{i_{p+1}}^{\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right) \end{aligned} \quad (23)$$

$$\times \left\{ \int_{\tau_{i_1}}^{\infty} \cdots \int_{\tau_{i_p}}^{\infty} \prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right) y^{\Delta_1 \cdots \Delta_p}\left(s_{i_1}, \ldots, s_{i_{p+1}}\right) \Delta s_{i_p} \cdots \Delta s_{i_1} \right\} \Delta s_{i_{p+1}}. \quad (24)$$

Use induction hypothesis for $\psi(\tau_{i_1},\dots,\tau_{i_{p+1}})$ with fix $\tau_{i_{p+1}}, s_{i_{p+1}} \in \mathbb{T}_{i_{p+1}}$ (instead for $\psi(\tau_{i_1},\dots,\tau_{i_p})$) in (23) and (25) to obtain

$$\begin{aligned}
\psi\left(\tau_{i_1}\dots\tau_{i_{p+1}}\right) \le\; & A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right) \\
& \times\left[\begin{array}{c}\displaystyle\prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right) y\left(\tau_{i_1},\dots,\tau_{i_{p+1}}\right) + \sum_{1\le j_1<\cdots<j_r\le p}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\prod_{i_m=1}^{p-r} A_{i_m}^{\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right) \\
\displaystyle\times\int_{\prod_{1\le j_1<\cdots<j_r\le p}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right) v_{j_m}\left(s_{j_m}\right) y\left(\tau_{i_1},\dots,\tau_{i_{p+1-r}},s_{j_1},\dots,s_{j_r}\right)\prod_{1\le j_1<\cdots<j_r\le p}\Delta_{j_m} s_{j_m}\end{array}\right] \\
& + \alpha_{i_{p+1}}\int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right) A_{i_{p+1}}^{\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right) \\
& \times\left[\begin{array}{c}\displaystyle\prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right) y\left(\tau_{i_1},\dots,\tau_{i_p},s_{i_{p+1}}\right) + \sum_{1\le j_1<\cdots<j_r\le p}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\prod_{m=1}^{p-r} A_{i_m}^{\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right) \\
\displaystyle\times\int_{\prod_{1\le j_1<\cdots<j_r\le p}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right) v_{j_m}\left(s_{j_m}\right) y\left(\tau_{i_1},\dots,\tau_{i_{p-r-1}},s_{j_1},\dots,s_{j_r}\right)\prod_{1\le j_1<\cdots<j_r\le p}\Delta_{j_m} s_{j_m}\end{array}\right]\Delta s_{i_{p+1}}.
\end{aligned} \tag{25}$$

By applying (5) "p times" on (25) and making simplification, we obtain

$$\begin{aligned}
&= \prod_{k=1}^{p} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right) \\
&\times\left[A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right) y\left(\tau_{i_1},\dots,\tau_{i_{p+1}}\right) + \alpha_{i_{p+1}}\int_{\tau_{i_{p+1}}}^{\infty} y\left(\tau_{i_1},\dots,\tau_{i_p},s_{i_{p+1}}\right)\Delta s_{i_{p+1}} y\left(\tau_{i_1},\dots,\tau_{i_p},s_{i_{p+1}}\right)\Delta s_{i_{p+1}}\right] \\
&+ \sum_{1\le j_1<\cdots<j_r\le p}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\prod_{m=1}^{i_{p-r}} A_{i_m}^{\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\int_{\prod_{1\le j_1<\cdots<j_r\le p}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right) v_{j_m}\left(s_{j_m}\right) \\
&\times\left[\begin{array}{c} A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right) y\left(\tau_{i_1},\dots,\tau_{i_{p+1-r}},s_{j_1},\dots,s_{j_r}\right) \\
\displaystyle +\alpha_{i_{p+1}}\int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right) A_{i_{p+1}}^{\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right) y\left(\tau_{i_1},\dots,\tau_{i_{p-r-1}},s_{j_1},\dots,s_{j_r}\right)\Delta s_{i_{p+1}}\end{array}\right]\prod_{1\le j_1<\cdots<j_r\le p}\Delta_{j_m} s_{j_m}.
\end{aligned} \tag{26}$$

Hence, by using (17) for $\tau_{i_{p+1}} \in \mathbb{T}_{p+1}$, we obtain

$$\psi\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right) \le \prod_{k=1}^{p+1} A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right) y\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right) + \sum_{1\le j_1<\cdots<j_r\le p+1}\left[\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\prod_{m=1}^{p+1-r} A_{i_m}^{\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\right.$$
$$\left.\times\int_{\prod_{1\le j_1<\cdots<j_r\le p+1}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right) v_{j_m}\left(s_{j_m}\right) y\left(\tau_{i_1},\ldots,\tau_{i_{p+1-r}}, s_{j_1},\ldots,s_{j_r}\right) \underset{\prod_{1\le j_1<\cdots<j_r\le p+1} j_m}{\Delta} s_{j_m}\right]. \tag{27}$$

Thus, by mathematical induction, (16) holds for all $h \in \mathbb{N}$, which completes the proof. □

Remark 1. If we chose $h = 3$ in Lemma 1, then (16) becomes the following inequality:

$$\begin{aligned}
\psi(\tau_1,\tau_2,\tau_3) &\le A_1^{\alpha_1}(\sigma_1(\tau_1))A_2^{\alpha_2}(\sigma_2(\tau_2))A_3^{\alpha_3}(\sigma_3(\tau_3))y(\tau_1,\tau_2,\tau_3)\\
&+ \alpha_1 A_2^{\alpha_2}(\sigma_2(\tau_2))A_3^{\alpha_3}(\sigma_3(\tau_3))\int_{\tau_1}^{\infty} A_1^{\alpha_1-1}(\sigma_1(s_1))a_1(s_1)y(s_1,\tau_2,\tau_3)\Delta s_1\\
&+ \alpha_2 A_1^{\alpha_1}(\sigma_1(\tau_1))A_3^{\alpha_3}(\sigma_3(\tau_3))\int_{\tau_2}^{\infty} A_2^{\alpha_2-1}(\sigma_2(s_2))a_2(s_2)y(\tau_1,s_2,\tau_3)\Delta s_2\\
&+ \alpha_3 A_1^{\alpha_1}(\sigma_1(\tau_1))A_2^{\alpha_2}(\sigma_2(\tau_2))\int_{\tau_3}^{\infty} A_3^{\alpha_3-1}(\sigma_3(s_3))a_3(s_3)y(\tau_1,\tau_2,s_3)\Delta s_3\\
&+ \alpha_1\alpha_2 A_3^{\alpha_3}(\sigma_3(\tau_3))\int_{\tau_2}^{\infty} A_2^{\alpha_2-1}(\sigma_2(s_2))a_2(s_2)\int_{\tau_1}^{\infty} A_1^{\alpha_1-1}(\sigma_1(s_1))a_1(s_1)y(s_1,s_2,\tau_3)\Delta s_1\Delta s_2\\
&+ \alpha_1\alpha_3 A_2^{\alpha_2}(\sigma_2(\tau_2))\int_{\tau_1}^{\infty} A_1^{\alpha_1-1}(\sigma_1(s_1))a_1(s_1)\int_{\tau_3}^{\infty} A_3^{\alpha_3-1}(\sigma_3(s_3))a_3(s_3)y(s_1,\tau_2,s_3)\Delta s_3\Delta s_1\\
&+ \alpha_2\alpha_3 A_1^{\alpha_1}(\sigma_1(\tau_1))\int_{\tau_2}^{\infty} A_2^{\alpha_2-1}(\sigma_2(s_2))a_2(s_2)\int_{\tau_3}^{\infty} A_3^{\alpha_3-1}(\sigma_3(s_3))a_3(s_3)y(\tau_1,s_2,s_3)\Delta s_3\Delta s_2\\
&+ \alpha_1\alpha_2\alpha_3\int_{\tau_3}^{\infty} A_3^{\alpha_3-1}(\sigma_3(s_3))a_3(s_3)\int_{\tau_2}^{\infty} A_2^{\alpha_2-1}(\sigma_2(s_2))a_2(s_2)\int_{\tau_1}^{\infty} A_1^{\alpha_1-1}(\sigma_1(s_1))a_1(s_1)y(s_1,s_2,s_3)\Delta s_1\Delta s_2\Delta s_3.
\end{aligned} \tag{28}$$

Theorem 3. *Assume H_1, H_2, and $l,\alpha_\iota \ge 1$, $\forall \iota \in \{1, 2,\ldots,h\}$, $h \in \mathbb{N}$. Then,*

$$\int_{\prod_{m=1}^{h} b_{j_m}}^{\infty}\prod_{j_m=1}^{h} v_{j_m}\left(\tau_{j_m}\right)\psi^{l}\left(\tau_{j_1},\ldots,\tau_{j_h}\right)\underset{\prod_{m=1}^{h} j_m}{\Delta}\tau_{j_m} \le \left(1+\sum_{1\le j_1<\cdots<j_r\le h} l^{r}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\right)^{l}$$
$$\times\int_{\prod_{m=1}^{h} b_{j_m}}^{\infty}\prod_{j_m=1}^{h} v_{j_m}\left(\tau_{j_m}\right)A_{j_m}^{l\alpha_{j_m}}\left(\sigma_{j_m}\left(\tau_{j_m}\right)\right)y^{l}\left(\tau_{j_1},\ldots,\tau_{j_h}\right)\underset{\prod_{m=1}^{h} j_m}{\Delta}\tau_{j_m}. \tag{29}$$

Proof. We prove the result by using mathematical induction. For $h = 1$, statement is true by Theorems 4.1 in [2]. Assume for $1 \le h \le p$, (29) holds. To prove the result for $h = p + 1$, take L.H.S of (29) in the following form:

$$\int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1} v_{j_m}\left(\tau_{j_m}\right)\psi^{l}\left(\tau_{j_1},\ldots,\tau_{j_{p+1}}\right)\underset{\prod_{m=1}^{p+1} j_m}{\Delta}\tau_{j_m}. \tag{30}$$

Using (27) in (30) for $h = p + 1$,

$$\left\{\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}\left(\tau_{j_m}\right)\psi^{l}\left(\tau_{j_1},\ldots,\tau_{j_{p+1}}\right)\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}\right\}^{(1/l)}\leq\left\{\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}\left(\tau_{j_m}\right)\left[I_{p+1}\right]^{l}\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}.\right\}^{(1/l)}, \tag{31}$$

where

$$\begin{aligned}I_{p+1}&=\prod_{k=1}^{p+1}A_{i_k}^{\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right)y\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right)+\sum_{1\leq j_1<\cdots<j_r\leq p+1}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\prod_{m=1}^{p+1-r}A_{i_m}^{\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\\&\quad\times\int_{\prod_{1\leq j_1<\cdots<j_r\leq p+1}\tau_{j_m}}^{\infty}\prod_{m=1}^{r}A_{j_m}^{\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right)v_{j_m}\left(s_{j_m}\right)y\left(\tau_{i_1},\ldots,\tau_{i_{p+1-r}},s_{j_1},\ldots,s_{j_r}\right)\Delta_{\Pi_{1\leq j_1<\cdots<j_r\leq p+1}}s.\end{aligned} \tag{32}$$

Apply Minkowski's inequality (4) on (31) to obtain

$$\begin{aligned}&\left\{\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}\left(\tau_{j_m}\right)\psi^{l}\left(\tau_{j_1},\ldots,\tau_{j_{p+1}}\right)\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}\right\}^{(1/l)}\leq\left\{\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}\left(\tau_{j_m}\right)A_{j_m}^{l\alpha_{j_m}}\left(\sigma_{j_m}\left(\tau_{j_m}\right)\right)y^{l}\left(\tau_{j_1},\ldots,\tau_{j_{p+1}}\right)\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}\right\}^{(1/l)}\\&\quad+\sum_{1\leq j_1<\cdots<j_r\leq p+1}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\left[\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}\left(\tau_{j_m}\right)\left\{\widetilde{I}_{p+1}\right\}^{l}\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}\right]^{(1/l)},\end{aligned} \tag{33}$$

where

$$\widetilde{I}_{p+1}=\prod_{m=1}^{p+1-r}A_{i_m}^{\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\times\int_{\prod_{1\leq j_1<\cdots<j_r\leq p+1}\tau_{j_m}}^{\infty}\prod_{j_m=1}^{r}A_{j_m}^{\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right)v_{j_m}\left(s_{j_m}\right)y\left(\tau_{i_1},\ldots,\tau_{i_{p+1-r}},s_{j_1},\ldots,s_{j_r}\right)\Delta_{\Pi_{1\leq j_1<\cdots<j_r\leq p+1}}s. \tag{34}$$

Denote

$$W_{p+1}=\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}\left(\tau_{j_m}\right)\left\{\widetilde{I}_{p+1}\right\}^{l}\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}, \tag{35}$$

and one has that

$$W_{p+1}=\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}(\tau_{j_m})\prod_{m=1}^{p+1-r}A_{i_m}^{l\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m}))$$
$$\times\left\{\int_{\prod_{1\le j_1<\cdots<j_r\le p+1}\tau_{j_m}}^{\infty}\prod_{j_m=1}^{r}\frac{v_{j_m}(s_{j_m})}{A_{j_m}(\sigma_{j_m}(s_{j_m}))}A_{j_m}^{\alpha_{j_m}}(\sigma_{j_m}(s_{j_m}))y\left(\tau_{i_1},\ldots,\tau_{i_{p+1-r}},s_{j_1},\ldots,s_{j_r}\right)\Delta s_{\Pi_{1\le j_1<\cdots<j_r\le p+1}}\right\}^{l}\times\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}. \tag{36}$$

Use Theorem 1 in (36) by taking $g(s_{j_1},\ldots,s_{j_r})=\prod_{j_m=1}^{r}A_{j_m}^{\alpha_{j_m}}(\sigma_{j_m}(s_{j_m}))y(\tau_{i_1},\ldots,\tau_{i_{p+1-r}},s_{j_1},\ldots,s_{j_r})$ to obtain

$$W_{p+1}\le(l)\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}(\tau_{j_m})A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m}))y^{l}\left(\tau_{j_1},\ldots,\tau_{j_{p+1}}\right)\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}. \tag{37}$$

Substitute (37) in (33) and take power l on both sides to obtain

$$\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}(\tau_{j_m})\psi^{l}\left(\tau_{j_1},\ldots,\tau_{j_{p+1}}\right)\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}\le\left(1+\sum_{1\le j_1<\cdots<j_r\le p+1}l^{r}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\right)^{l}$$
$$\times\int_{\prod_{m=1}^{p+1}b_{j_m}}^{\infty}\prod_{j_m=1}^{p+1}v_{j_m}(\tau_{j_m})A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m}))y^{l}\left(\tau_{j_1},\ldots,\tau_{j_{p+1}}\right)\Delta_{\prod_{m=1}^{p+1}j_m}\tau_{j_m}. \tag{38}$$

Thus, by mathematical induction, (29) holds for all $h\in\mathbb{N}$. □

Example 1. Let $\mathbb{T}_\iota=\mathbb{N}$ and $b_{j_m}=1$, $\forall j,m\in\{1,\ldots,h\}$ and $n_{j_m},k_{j_m},h\in\mathbb{N}\quad\forall j_m$. In this case, (29) in Theorem 3 takes the form

$$\sum_{k_{j_1}=1}^{\infty}\cdots\sum_{k_{j_h}=1}^{\infty}\prod_{m=1}^{h}v_{j_m}(k_{j_m})\left(\sum_{n_{j_1}=k_{j_1}}^{\infty}\cdots\sum_{n_{j_h}=k_{j_h}}^{\infty}\prod_{m=1}^{h}A_{j_m}^{\alpha_{j_m}}(n_{j_m}+1)x(n_{j_1},\ldots,n_{j_h})\right)^{l}$$
$$\le\left(1+\sum_{1\le j_1<\cdots<j_r\le p+1}l^{r}\left(\prod_{m=1}^{r}\alpha_{j_m}\right)\right)^{l}\sum_{k_{j_1}=1}^{\infty}\cdots\sum_{k_{j_h}=1}^{\infty}\prod_{m=1}^{h}v_{j_m}(k_{j_m})A_{j_m}^{\alpha_{j_m}l}(k_{j_m}+1)$$
$$\times\left(\sum_{n_{j_1}=k_{j_1}}^{\infty}\cdots\sum_{n_{j_h}=k_{j_h}}^{\infty}x(n_{j_1},\ldots,n_{j_h})\right)^{l}, \tag{39}$$

where

$$A_{j_m}(k_{j_m})=\sum_{n_{j_m}=1}^{k_{j_m}-1}v_{j_m}(n_{j_m}),\quad k_{j_m}\in\mathbb{N}. \tag{40}$$

Note that (39) is extension of Example 4.4 in [2].

Example 2. Let $\mathbb{T}_\iota=\mathbb{R}\,\forall j,m\in\{1,\ldots,h\}$, in Theorem 3. In this case, (29) takes the form

$$\int_{\prod_{m=1}^{h} b_{j_m}}^{\infty} \prod_{j_m=1}^{h} v_{j_m}(\tau_{j_m}) \left(\int_{\prod_{m=1}^{h} b_{j_m}}^{\infty} \prod_{m=1}^{h} A_{j_m}^{\alpha_{j_m}}(s_{j_m}) x(s_{j_1}, \dots, s_{j_h}) \prod_{m=1}^{h} d_{j_m} s_{j_m} \right)^{l} \prod_{m=1}^{h} d_{j_m} \tau_{j_m}$$
$$\leq \left(1 + \sum_{1 \leq j_1 < \cdots < j_r \leq p+1} l^r \left(\prod_{m=1}^{r} \alpha_{j_m} \right) \right)^{l} \int_{\prod_{m=1}^{h} b_{j_m}}^{\infty} \prod_{j_m=1}^{h} v_{j_m}(\tau_{j_m}) A_{j_m}^{\alpha_{j_m} l}(\tau_{j_m}) \times \left(\int_{\prod_{m=1}^{h} \tau_{j_m}}^{\infty} x(s_{j_1}, \dots, s_{j_h}) \prod_{m=1}^{h} d_{j_m} s_{j_m} \right)^{l} \prod_{m=1}^{h} d_{j_m} \tau_{j_m}, \tag{41}$$

where

$$A_{j_m}(\tau_{j_m}) = \int_{b_{j_m}}^{\tau_{j_m}} v_{j_m}(s_{j_m}) \prod_{m=1}^{h} d_{j_m} s_{j_m}, \quad \tau_{j_m} \in \mathbb{R}. \tag{42}$$

Note that (41) is extension of Example 4.3 in [2].

Example 3. Let $\mathbb{T}_\iota = q_\iota^{\mathbb{N}_0}$, $q_{j_m} > 1$, and $\forall j, m \in \{1, \dots, h\}$, in Theorem 3. In this case, (29) takes the form

$$\sum_{k_{j_1}=1}^{\infty} \cdots \sum_{k_{j_h}=1}^{\infty} \prod_{j_m=1}^{h} v_{j_m}\left(q_{j_m}^{k_{j_m}}\right)\left(q_{j_m}^{k_{j_m} l}\right)\left(\sum_{n_{j_1}=k_{j_1}}^{\infty} \cdots \sum_{n_{j_h}=k_{j_h}}^{\infty} \prod_{m=1}^{h} q_{j_m}^{n_{j_m}} A_{j_m}^{\alpha_{j_m}}\left(q_{j_m}^{n_{j_m}+1}\right) x\left(q_{j_1}^{n_{j_1}}, \dots, q_{j_h}^{n_{j_h}}\right) \right)^{l}$$
$$\leq \left(1 + \sum_{1 \leq j_1 < \cdots < j_r \leq p+1} l^r \left(\prod_{m=1}^{r} \alpha_{j_m} \right) \right)^{l}$$
$$\times \sum_{k_{j_1}=1}^{\infty} \cdots \sum_{k_{j_h}=1}^{\infty} \left[\prod_{j_m=1}^{h} v_{j_m}\left(q_{j_m}^{k_{j_m}}\right) A_{j_m}^{\alpha_{j_m} l}\left(q_{j_m}^{k_{j_m}+1}\right) q_{j_m}^{k_{j_m} l} \left(\sum_{n_{j_1}=k_{j_1}}^{\infty} \cdots \sum_{n_{j_h}=k_{j_h}}^{\infty} \prod_{m=1}^{h} q_{j_m}^{n_{j_m}} x\left(q_{j_1}^{n_{j_1}}, \dots, q_{j_h}^{n_{j_h}}\right) \right)^{l} \right], \tag{43}$$

where

$$A_{j_m}\left(q_{j_m}^{k_{j_m}}\right) = \sum_{n_{j_m}=1}^{k_{j_m}-1} v_{j_m}\left(q_{j_m}^{n_{j_m}}\right) q_{j_m}^{n_{j_m}}\left(q_{j_m} - 1\right), \quad k_{j_m} \in \mathbb{N}_0. \tag{44}$$

Lemma 2. *Let $\mathbb{T}_\iota$ be the time scales for $\iota, j, r \in \{1, 2, \dots, h\}$, under H_1 and H_2, we have*

$$y(\tau_{i_1}, \dots, \tau_{i_h}) \leq \prod_{k=1}^{h} A_{i_k}^{-\alpha_{i_k}}(\sigma_{i_k}(\tau_{i_k})) \psi(\tau_{i_1}, \dots, \tau_{i_h}) + \sum_{1 \leq j_1 < \dots < j_r \leq h} \left(\prod_{m=1}^{h-r} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) \right.$$
$$\left. \times \int_{\prod_{1 \leq j_1 < \cdots < j_r \leq h} \tau_{j_m}}^{\infty} \prod_{m=1}^{r} A_{j_m}^{-\alpha_{j_m}-1}(\sigma_{j_m}(s_{j_m})) v_{j_m}(s_{j_m}) \psi(\tau_{i_1}, \dots, \tau_{i_{h-r}}, s_{j_1}, \dots, s_{j_r}) \prod_{1 \leq j_1 < \cdots < j_r \leq h} \Delta_{j_m} s_{j_m}. \right) \tag{45}$$

Proof. For $h = 1$, (45) is true by Theorems 4.6 in [2], *i.e.*,

$$y(\tau_1) \le A_1^{-\alpha_1}(\sigma_1(\tau_1))\psi(\tau_1) + \int_{\tau_1}^{\infty} v_1(s_1)A_1^{\alpha_1-1}(\sigma_1(s_1))\psi(s_1)\Delta s_1. \tag{46}$$

Suppose (45) is true for $1 \le h \le p$. To prove for $h = p+1$, by using H_2, we have

$$y\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right) = \int_{\tau_{i_1}}^{\infty}\cdots\int_{\tau_{i_p}}^{\infty}\prod_{k=1}^{p}A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right)\times\left\{\int_{\tau_{i_{p+1}}}^{\infty}A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\psi^{\Delta_1\cdots\Delta_{p+1}}\left(s_{i_1},\ldots,s_{i_{p+1}}\right)\Delta s_{i_{p+1}}\right\}\Delta s_{i_p}\cdots\Delta s_{i_1}. \tag{47}$$

Denote

$$Z_{i_{p+1}} = \int_{\tau_{i_{p+1}}}^{\infty}A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\psi^{\Delta_1\cdots\Delta_{p+1}}\left(s_{i_1},\ldots,s_{i_{p+1}}\right)\Delta s_{i_{p+1}}. \tag{48}$$

Use integration by parts formula (3) in (48) to obtain

$$Z_{i_{p+1}} = \left|-A_{i_{p+1}}^{-\alpha_{i_{p+1}}}(s_{p+1})\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,s_{i_{p+1}}\right)\right|_{\tau_{p+1}}^{\infty} - \int_{\tau_{i_{p+1}}}^{\infty} -\frac{\partial}{\Delta\tau_{i_{p+1}}}\left(A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(s_{i_{p+1}}\right)\right)\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,s_{i_{p+1}}\right)\Delta s_{i_{p+1}}. \tag{49}$$

Use $\psi(s_{i_1},\ldots,s_{i_p},\infty) = 0$ and the right-hand side part of inequality (8) with $A_{i_{p+1}} \le A_{i_{p+1}}^{\sigma_{i_{p+1}}}$ in (49)

$$Z_{i_{p+1}} \le A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right)\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,s_{i_p},\tau_{i_{p+1}}\right) + \int_{\tau_{i_{p+1}}}^{\infty}v_{i_{p+1}}\left(s_{i_{p+1}}\right)A_{i_{p+1}}^{-\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,s_{i_{p+1}}\right)\Delta s_{i_{p+1}}. \tag{50}$$

Substitute (50) in (47)

$$\begin{aligned} y\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right) = &\int_{\tau_{i_1}}^{\infty}\cdots\int_{\tau_{i_p}}^{\infty}\prod_{k=1}^{p}A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right)\left\{A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right)\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,s_{i_p},\tau_{i_{p+1}}\right)\right\} \\ &\times\Delta s_{i_p}\cdots\Delta s_{i_1} + \int_{\tau_{i_1}}^{\infty}\cdots\int_{\tau_{i_p}}^{\infty}\prod_{k=1}^{p}A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right) \\ &\times\left\{\int_{\tau_{i_{p+1}}}^{\infty}v_{i_{p+1}}\left(s_{i_{p+1}}\right)A_{i_{p+1}}^{-\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,\Delta s_{i_{p+1}}\right)\Delta s_{i_{p+1}}\right\}\Delta s_{i_p}\cdots\Delta s_{i_1}. \end{aligned} \tag{51}$$

Use (5) "p times" on (51):

$$\begin{aligned} y\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right) \le\; &A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right)\int_{\tau_{i_1}}^{\infty}\cdots\int_{\tau_{i_p}}^{\infty}\prod_{k=1}^{p}A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right)\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,s_{i_p},\tau_{i_{p+1}}\right)\Delta s_{i_p}\cdots\Delta s_{i_1} \\ &+\int_{\tau_{i_{p+1}}}^{\infty}v_{i_{p+1}}\left(s_{i_{p+1}}\right)A_{i_{p+1}}^{-\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\left\{\int_{\tau_{i_1}}^{\infty}\cdots\int_{\tau_{i_p}}^{\infty}\prod_{k=1}^{p}A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(s_{i_k}\right)\right)\psi^{\Delta_1\cdots\Delta_p}\left(s_{i_1},\ldots,s_{i_{p+1}}\right)\Delta s_{i_p}\cdots\Delta s_{i_1}\right\}\times\Delta s_{i_{p+1}}. \end{aligned} \tag{52}$$

Use induction hypothesis for $y(\tau_{i_1},\ldots,\tau_{i_{p+1}})$ with fix $\tau_{i_{p+1}}, s_{i_{p+1}} \in \mathbb{T}_{i_{p+1}}$ (instead for $y(\tau_{i_1},\ldots,\tau_{i_p})$) in (52) to obtain

$$\begin{aligned}
y\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right) \le\ & A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right)\times\left[\prod_{k=1}^{p} A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right)+\sum_{1\le j_1<\cdots<j_r\le p}\prod_{m=1}^{p-r} A_{i_m}^{-\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\right.\\
&\left.\times\int_{\prod_{1\le j_1<\cdots<j_r\le p}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{-\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right)v_{j_m}\left(s_{j_m}\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_{p+1-r}},s_{j_1},\ldots,s_{j_r}\right)\underset{\prod_{1\le j_1<\cdots<j_r\le p} j_m}{\Delta} s_{j_m}\right]\\
&+\int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right)A_{i_{p+1}}^{-\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\times\left[\prod_{k=1}^{p} A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_p},s_{i_{p+1}}\right)\right.\\
&+\sum_{1\le j_1<\cdots<j_r\le p}\prod_{m=1}^{p-r} A_{i_m}^{-\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\\
&\times\int_{\prod_{1\le j_1<\cdots<j_r\le p}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{-\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right)v_{j_m}\left(s_{j_m}\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_{p-r-1}},s_{j_1},\ldots,s_{j_r}\right)\underset{\prod_{1\le j_1<\cdots<j_r\le p} j_m}{\Delta} s_{j_m}\Delta s_{i_{p+1}},\\
=\ & \prod_{k=1}^{p} A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right)\times\left[A_{i_{p+1}}^{-\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right)\right.\\
&\left.+\int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right)A_{i_{p+1}}^{-\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_p},s_{i_{p+1}}\right)\Delta s_{i_{p+1}}\right]\\
&+\sum_{1\le j_1<\cdots<j_r\le p}\prod_{m=1}^{p-r} A_{i_m}^{-\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\int_{\prod_{1\le j_1<\cdots<j_r\le p}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{-\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right)v_{j_m}\left(s_{j_m}\right)\\
&\times\left[A_{i_{p+1}}^{\alpha_{i_{p+1}}}\left(\sigma_{i_{p+1}}\left(\tau_{i_{p+1}}\right)\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_{p+1-r}},s_{j_1},\ldots,s_{j_p}\right)\right.\\
&\left.+\int_{\tau_{i_{p+1}}}^{\infty} v_{i_{p+1}}\left(s_{i_{p+1}}\right)A_{i_{p+1}}^{-\alpha_{i_{p+1}}-1}\left(\sigma_{i_{p+1}}\left(s_{i_{p+1}}\right)\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_{p-r-1}},s_{j_1},\ldots,s_{j_p}\right)\Delta s_{i_{p+1}}\underset{\prod_{1\le j_1<\cdots<j_r\le p} j_m}{\Delta} s_{j_m}.\right.
\end{aligned} \tag{53}$$

Hence, by using (46) for $\tau_{i_{p+1}}$, we obtain

$$\begin{aligned}
y\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right) \le\ & \prod_{k=1}^{p+1} A_{i_k}^{-\alpha_{i_k}}\left(\sigma_{i_k}\left(\tau_{i_k}\right)\right)\psi\left(\tau_{i_1},\ldots,\tau_{i_{p+1}}\right)\\
&+\sum_{1\le j_1<\cdots<j_r\le p+1}\left[\prod_{m=1}^{i_{p+1-r}} A_{i_m}^{-\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)\times\int_{\prod_{1\le j_1<\cdots<j_r\le p+1}\tau_{j_m}}^{\infty}\prod_{m=1}^{r} A_{j_m}^{-\alpha_{j_m}-1}\right.\\
&\left.\cdot\left(\sigma_{j_m}\left(s_{j_m}\right)\right)v_{j_m}\left(s_{j_m}\right)\psi\left(\tau_{i_1},\ldots,\tau_{p+1-r},s_{j_1},\ldots,s_{j_r}\right)\underset{\prod_{1\le j_1<\cdots<j_r\le p+1} j_m}{\Delta} s_{j_m}\right].
\end{aligned} \tag{54}$$

Thus, by mathematical induction, (54) holds for all $h\in\mathbb{N}$, which completes the proof. □

Theorem 4. *Assume* H_1, H_2, *and* $l,\alpha_\iota \ge 1$. *for* $\iota\in\{1,2,\ldots,h\}$, $h\in\mathbb{N}$. *Then,*

$$\int_{\prod_{m=1}^{h} b_{j_m}}^{\infty} \prod_{j_m=1}^{h} v_{j_m}(\tau_{j_m}) \psi^l(\tau_{j_1}, \ldots, \tau_{j_h}) \underset{\prod_{m=1}^{h} j_m}{\Delta} \tau_{j_m} \geq \sum_{1 \leq j_1 < \cdots < j_r \leq p+1} \left(\frac{1 + l^r \prod_{m=1}^{r} \alpha_{j_m}}{1 + l^r \prod_{m=1}^{r} \alpha_{j_m} + l^r} \right)^l \times \int_{\prod_{m=1}^{h} b_{j_m}}^{\infty} \prod_{j_m=1}^{h} v_{j_m}(\tau_{j_m}) A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) y^l(\tau_{j_1}, \ldots, \tau_{j_h}) \underset{\prod_{m=1}^{h} j_m}{\Delta} \tau_{j_m}. \tag{55}$$

Proof. We prove the result by using mathematical induction. For $h = 1$, statement is true by Theorems 4.6 in [2]. Assume for $1 \leq h \leq p$, (55) holds. To prove the result for $h = p + 1$, take L.H.S of (55) with $h = p + 1$ in the following form:

$$\int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}(\tau_{j_m}) A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) y^l(\tau_{j_1}, \ldots, \tau_{j_{p+1}}) \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m}. \tag{56}$$

Use (27) in (56) for $h = p + 1$:

$$\left\{ \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}(\tau_{j_m}) A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) y^l(\tau_{j_1}, \ldots, \tau_{j_{p+1}}) \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m}. \right\}^{(1/l)} \leq \left\{ \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}(\tau_{j_m}) A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) [I_{p+1}]^l \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m} \right\}^{(1/l)}, \tag{57}$$

where

$$I_{p+1} = \prod_{k=1}^{p+1} A_{i_k}^{-\alpha_{i_k}}(\sigma_{i_k}(\tau_{i_k})) \psi(\tau_{i_1}, \ldots, \tau_{i_{p+1}}) + \sum_{1 \leq j_1 < \cdots < j_r \leq p+1} \prod_{m=1}^{p+1-r} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) \times \int_{\prod_{1 \leq j_1 < \cdots < j_r \leq p+1} \tau_{j_m}}^{\infty} \prod_{m=1}^{r} A_{j_m}^{-\alpha_{j_m}-1}(\sigma_{j_m}(s_{j_m})) v_{j_m}(s_{j_m}) \psi(\tau_{i_1}, \ldots, \tau_{i_{p+1-r}}, s_{j_1}, \ldots, s_{j_r}) \underset{\Pi_{1 \leq j_1 < \cdots < j_r \leq p+1}}{\Delta s}. \tag{58}$$

Apply Minkowski's inequality (4) on (57) to obtain

$$\left\{ \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}(\tau_{j_m}) A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) y^l(\tau_{j_1}, \ldots, \tau_{j_{p+1}}) \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m} \right\}^{(1/l)} \leq \left\{ \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}(\tau_{j_m}) A_{j_m}^{-l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) \psi^l(\tau_{j_1}, \ldots, \tau_{j_{p+1}}) \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m} \right\}^{(1/l)} + \left[\int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}(\tau_{j_m}) A_{j_m}^{l\alpha_{j_m}}(\sigma_{j_m}(\tau_{j_m})) \{\tilde{I}_{p+1}\}^l \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m} \right]^{(1/l)}, \tag{59}$$

where

$$\widetilde{I}_{p+1} = \prod_{m=1}^{p+1-r} A_{i_m}^{-\alpha_{j_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right) \times \int_{1 \leq j_1 < \cdots < j_r \leq p+1}^{\infty} \sigma_{j_m}\left(s_{j_m}\right) v_{j_m}\left(s_{j_m}\right) \psi\left(\tau_{i_1}, \ldots, \tau_{i_{p+1-r}}, s_{j_1}, \ldots, s_{j_r}\right) \underset{\Pi_{1 \leq j_1 < \cdots < j_r \leq p+1}}{\Delta s}. \tag{60}$$

Denote

$$W_{p+1} = \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}\left(\tau_{j_m}\right) A_{j_m}^{l\alpha_{j_m}}\left(\sigma_{j_m}\left(\tau_{j_m}\right)\right)\left\{\widetilde{I}_{p+1}\right\}^{l} \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m}, \tag{61}$$

and one has that

$$\begin{aligned} W_{p+1} = {} & \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} \frac{v_{j_m}\left(\tau_{j_m}\right)}{A_{i_m}^{-l\alpha_{i_m}}\left(\sigma_{i_m}\left(\tau_{i_m}\right)\right)} \\ & \times \left\{ \int_{\prod_{1 \leq j_1 < \cdots < j_r \leq p+1} \tau_{j_m}}^{\infty} \prod_{j_m=1}^{r} v_{j_m}\left(s_{j_m}\right) A_{j_m}^{-\alpha_{j_m}-1}\left(\sigma_{j_m}\left(s_{j_m}\right)\right) \psi\left(\tau_{i_1}, \ldots, \tau_{i_{p+1-r}}, s_{j_1}, \ldots, s_{j_r}\right) \underset{\Pi_{1 \leq j_1 < \cdots < j_r \leq p+1}}{\Delta s} \right\}^{l} \times \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m}. \end{aligned} \tag{62}$$

Use Theorem 2 in (62) by taking $g(s_{j_1}, \ldots, s_{j_r}) = \prod_{j_m=1}^{r} A_{j_m}^{-\alpha_{j_m-1}}(\sigma_{j_m}(s_{j_m}))\psi(\tau_{i_1}, \ldots, \tau_{i_{p+1-r}}, s_{j_1}, \ldots, s_{j_r})$ and $c = (\prod_{m=1}^{r} \alpha_{j_m})l$:

$$w_{p+1} \leq \left(\frac{l^{rl}}{1 + l^{rl} \prod_{m=1}^{r} \alpha_{j_m}}\right)^{l} \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}\left(\tau_{j_m}\right) A_{j_m}^{\alpha_{j_m} l}\left(\sigma_{j_m}\left(\tau_{j_m}\right)\right) \psi^{l}\left(\tau_{j_1}, \ldots, \tau_{j_{p+1}}\right) \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m}. \tag{63}$$

Substitute (63) in (59) and take power l on both sides to obtain

$$\begin{aligned} \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}\left(\tau_{j_m}\right) \psi^{l}\left(\tau_{j_1}, \ldots, \tau_{j_{p+1}}\right) \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m} \geq {} & \sum_{1 \leq j_1 < \cdots < j_r \leq p+1} \left(\frac{1 + l^{r} \prod_{m=1}^{r} \alpha_{j_m}}{1 + l^{r} \prod_{m=1}^{r} \alpha_{j_m} + l^{r}}\right)^{l} \\ & \times \int_{\prod_{m=1}^{p+1} b_{j_m}}^{\infty} \prod_{j_m=1}^{p+1} v_{j_m}\left(\tau_{j_m}\right) A_{j_m}^{l\alpha_{j_m}}\left(\sigma_{j_m}\left(\tau_{j_m}\right)\right) y^{l}\left(\tau_{j_1}, \ldots, \tau_{j_{p+1}}\right) \underset{\prod_{m=1}^{p+1} j_m}{\Delta} \tau_{j_m}. \end{aligned} \tag{64}$$

Thus, by mathematical induction, (55) holds for all h, which completes the proof. □

Example 4. Let $\mathbb{T}_{\iota} = \mathbb{N}$ and $b_{j_m} = 1, \forall j, m \in \{1, \ldots, h\}$ and n_{j_m}, $k_{j_m}, h \in \mathbb{N} \forall j_m$, in Theorem 4. In this case, (55) takes the form

$$\sum_{k_{j_1}=1}^{\infty}\cdots\sum_{k_{j_h}=1}^{\infty}\prod_{m=1}^{h}v_{j_m}(k_{j_m})\left(\sum_{n_{j_1}=k_{j_1}}^{\infty}\cdots\sum_{n_{j_h}=k_{j_h}}^{\infty}\prod_{m=1}^{h}A_{j_m}^{\alpha_{j_m}}(n_{j_m}+1)x(n_{j_1},\ldots,n_{j_h})\right)^{l}$$
$$\geq\sum_{1\leq j_1<\cdots<j_r\leq p+1}\left(\frac{1+l^r\prod_{m=1}^{r}\alpha_{j_m}}{1+l^r\prod_{m=1}^{r}\alpha_{j_m}+l^r}\right)^{l}\sum_{k_{j_1}=1}^{\infty}\cdots\sum_{k_{j_h}=1}^{\infty}\prod_{m=1}^{h}v_{j_m}(k_{j_m})A_{j_m}^{\alpha_{j_m}l}(k_{j_m}+1)\times\left(\sum_{n_{j_1}=k_{j_1}}^{\infty}\cdots\sum_{n_{j_h}=k_{j_h}}^{\infty}x(n_{j_1},\ldots,n_{j_h})\right)^{l}, \tag{65}$$

where

$$A_{j_m}(k_{j_m})=\sum_{n_{j_m}=1}^{k_{j_m}-1}v_{j_m}(n_{j_m}),\quad k_{j_m}\in\mathbb{N}. \tag{66}$$

Note that (65) is extension of Example 4.7 in [2].

Example 5. Let $\mathbb{T}_\iota=\mathbb{R}\,\forall j,m\in\{1,\ldots,h\}$, in Theorem 4. In this case, (55) takes the form

$$\int_{\prod_{m=1}^{h}b_{j_m}}^{\infty}\prod_{j_m=1}^{h}v_{j_m}(\tau_{j_m})\left(\int_{\prod_{m=1}^{h}\tau_{j_m}}^{\infty}\prod_{m=1}^{h}A_{j_m}^{\alpha_{j_m}}(s_{j_m})x(s_{j_1},\ldots,s_{j_h})\underset{\prod_{m=1}^{h}j_m}{d}s_{j_m}\right)^{l}\underset{\prod_{m=1}^{h}j_m}{d}\tau_{j_m}$$
$$\geq\sum_{1\leq j_1<\cdots<j_r\leq p+1}\left(\frac{1+l^r\prod_{m=1}^{r}\alpha_{j_m}}{1+l^r\prod_{m=1}^{r}\alpha_{j_m}+l^r}\right)^{l}\int_{\prod_{m=1}^{h}b_{j_m}}^{\infty}\prod_{m=1}^{h}v_{j_m}(\tau_{j_m})A_{j_m}^{\alpha_{j_m}l}(\tau_{j_m})\times\left(\int_{\prod_{m=1}^{h}\tau_{j_m}}^{\infty}x(s_{j_1},\ldots,s_{j_h})\underset{\prod_{m=1}^{h}j_m}{d}s_{j_m}\right)^{l}\underset{\prod_{m=1}^{h}j_m}{d}\tau_{j_m}, \tag{67}$$

where

$$A_{j_m}(\tau_{j_m})=\int_{b_{j_m}}^{\tau_{j_m}}v_{j_m}(s_{j_m})\underset{\prod_{m=1}^{h}j_m}{d}s_{j_m},\quad \tau_{j_m}\in\mathbb{R}. \tag{68}$$

Note that (67) is extension of Example 4.8 in [2].

Example 6. Let $\mathbb{T}_\iota=q_\iota^{\mathbb{N}_0}$, $q_{j_m}>1$, and $\forall j,m\in\{1,\ldots,h\}$., in Theorem 4. In this case, (55) takes the form

$$\sum_{k_{j_1}=1}^{\infty}\cdots\sum_{k_{j_h}=1}^{\infty}\prod_{j_m=1}^{h}v_{j_m}\left(q_{j_m}^{k_{j_m}}\right)\left(q_{j_m}^{k_{j_m}l}\right)\left(\sum_{n_{j_1}=k_{j_1}}^{\infty}\cdots\sum_{n_{j_h}=k_{j_h}}^{\infty}\prod_{m=1}^{h}q_{j_m}^{n_{j_m}}A_{j_m}^{\alpha_{j_m}}\left(q_{j_m}^{n_{j_m}+1}\right)x\left(q_{j_1}^{n_{j_1}},\ldots,q_{j_h}^{n_{j_h}}\right)\right)^{l}$$
$$\geq\sum_{1\leq j_1<\cdots<j_r\leq p+1}\left(\frac{1+l^r\prod_{m=1}^{r}\alpha_{j_m}}{1+l^r\prod_{m=1}^{r}\alpha_{j_m}+l^r}\right)^{l}\times\sum_{k_{j_1}=1}^{\infty}\cdots\sum_{k_{j_h}=1}^{\infty} \tag{69}$$
$$\cdot\left[\prod_{j_m=1}^{h}v_{j_m}\left(q_{j_m}^{k_{j_m}}\right)A_{j_m}^{\alpha_{j_m}l}\left(q_{j_m}^{k_{j_m}+1}\right)q_{j_m}^{k_{j_m}l}\left(\sum_{n_{j_1}=k_{j_1}}^{\infty}\cdots\sum_{n_{j_h}=k_{j_h}}^{\infty}\prod_{m=1}^{h}q_{j_m}^{n_{j_m}}x\left(q_{j_1}^{n_{j_1}},\ldots,q_{j_h}^{n_{j_h}}\right)\right)^{l}\right],$$

where

$$A_{j_m}\left(q_{j_m}^{k_{j_m}}\right) = \sum_{n_{j_m}=1}^{k_{j_m}-1} v_{j_m}\left(q_{j_m}^{n_{j_m}}\right) q_{j_m}^{n_{j_m}}\left(q_{j_m}-1\right), \quad k_{j_m} \in \mathbb{N}_0. \tag{70}$$

Proof. We use mathematical induction to prove the result. For $h = 1$, (12) is true by Theorem 3.1 in [2]. Assume for $1 \le h \le p$, and (12) holds. To prove the result for $h = p + 1$, take L.H.S of (12) in the following form:

$$\int_{b_1}^{\infty} \cdots \int_{b_p}^{\infty} \prod_{\iota=1}^{p} v_\iota(\tau_\iota) \times \left\{ \int_{b_{p+1}}^{\infty} v_{p+1}(\tau_{p+1}) \phi^m(\tau_1, \ldots, \tau_{p+1}) \Delta\tau_{p+1} \right\} \Delta\tau_p \cdots \Delta\tau_1. \tag{71}$$

Denote

$$I_{p+1} = \int_{b_{p+1}}^{\infty} v_{p+1}(\tau_{p+1}) \phi^m(\tau_1, \ldots, \tau_{p+1}) \Delta\tau_{p+1}. \tag{72}$$

Use Theorem 3.1 in [2] in (72) with respect to $\tau_{p+1} \in \mathbb{T}_{p+1}$ for fix $(\tau_1, \ldots, \tau_p) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_p$ to obtain

$$\Rightarrow (I_{p+1})^m \le m^m \int_{b_{p+1}}^{\infty} v_{p+1}(\tau_{p+1}) \phi_p^m(\tau_1, \ldots, \tau_{p+1}) \Delta\tau_{p+1}, \tag{73}$$

where

$$\phi_p(\tau_1, \ldots, \tau_{p+1}) = \int_{\tau_1}^{\infty} \cdots \int_{\tau_p}^{\infty} \prod_{\iota=1}^{p} \frac{v_\iota(s_\iota)}{A_\iota(\sigma_\iota(s_\iota))} g(s_1, \ldots, s_p, \tau_{p+1}) \Delta s_p \cdots \Delta s_1. \tag{74}$$

Substitute (73) in (71) and use (5) "p times" in resultant inequality to obtain

$$\begin{aligned} \int_{b_1}^{\infty} \cdots \int_{b_{p+1}}^{\infty} \prod_{\iota=1}^{p+1} v_\iota(\tau_\iota) \phi^m(\tau_1, \ldots, \tau_{p+1}) \Delta\tau_{p+1} \cdots \Delta\tau_1 &\le m^m \int_{b_1}^{\infty} \cdots \int_{b_p}^{\infty} \prod_{\iota=1}^{p} v_\iota(\tau_\iota) \\ &\cdot \left\{ \int_{b_{p+1}}^{\infty} v_{p+1}(\tau_{p+1}) \phi_p^m(\tau_1, \ldots, \tau_{p+1}) \Delta\tau_{p+1} \right\} \Delta\tau_p \cdots \Delta\tau_1. \end{aligned} \tag{75}$$

Use induction hypothesis for $\phi_p(\tau_1 \ldots \tau_{p+1})$ with fix $\tau_{p+1} \in \mathbb{T}_{p+1}$, instead for $\phi_p(\tau_1 \ldots \tau_p)$ to obtain

$$\begin{aligned} &\int_{b_1}^{\infty} \cdots \int_{b_{p+1}}^{\infty} \prod_{\iota=1}^{p+1} v_\iota(\tau_\iota) \phi^m(\tau_1, \ldots, \tau_{p+1}) \Delta\tau_{p+1} \cdots \Delta\tau_1 \\ &\le m^{(p+1)m} \int_{b_1}^{\infty} \cdots \int_{b_{p+1}}^{\infty} \prod_{\iota=1}^{p+1} v_\iota(\tau_\iota) g^m(\tau_1, \ldots, \tau_{p+1}) \Delta\tau_{p+1} \cdots \Delta\tau_1. \end{aligned} \tag{76}$$

Thus, by mathematical induction, (12) holds for all $h \in \mathbb{N}$. □

Proof. We prove the result by using mathematical induction. For $h = 1$, statement is true by Theorem 3.3 in [2]. Assume for $1 \le h \le p$, (14) holds. To prove the result for $h = p + 1$, left-hand side of (14) can be written as

$$\int_{a_1}^{\infty} \cdots \int_{a_p}^{\infty} \prod_{\iota=1}^{p} \frac{v_\iota(\varsigma_\iota)}{A_\iota^{c_\iota}(\sigma_\iota(\varsigma_\iota))} \left\{ \int_{a_{p+1}}^{\infty} \frac{v_{p+1}(\varsigma_{p+1})}{A_{p+1}^{c_{p+1}}(\sigma_{p+1}(\varsigma_{p+1}))} \phi^m(\varsigma_1, \ldots, \varsigma_{p+1}) \Delta\varsigma_{p+1} \right\} \Delta\varsigma_p \cdots \Delta\varsigma_1. \tag{77}$$

Denote

$$I_{p+1}=\int_{a_{p+1}}^{\infty}\frac{\upsilon_{p+1}(\varsigma_{p+1})}{A_{p+1}^{c_{p+1}}(\sigma_{p+1}(\varsigma_{p+1}))}\phi^{m}(\varsigma_1,\ldots,\varsigma_{p+1})\Delta\varsigma_{p+1}. \tag{78}$$

Use Theorem 3.3 in [2] in (72) with respect to $\varsigma_{p+1}\in\mathbb{T}_{p+1}$ for fix $(\varsigma_1,\ldots,\varsigma_p)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_p$ to obtain

$$(I_{p+1})^{m}\leq\left(\frac{m}{1-c_{p+1}}\right)^{m}\int_{a_{p+1}}^{\infty}\upsilon_{p+1}(\varsigma_{p+1})A_{p+1}^{m-c_{p+1}}(\sigma_{p+1}(\varsigma_{p+1}))\phi_p^{m}(\varsigma_1,\ldots,\varsigma_{p+1})\Delta\varsigma_{p+1}, \tag{79}$$

where

$$\phi_p(\varsigma_1,\ldots,\varsigma_{p+1})\doteq\int_{\varsigma_1}^{\infty}\cdots\int_{\varsigma_p}^{\infty}\prod_{\iota=1}^{p}\upsilon_\iota(s_\iota)g(s_1,\ldots,s_p,\varsigma_{p+1})\Delta s_p\cdots\Delta s_1. \tag{80}$$

Substitute (79) in (77) and use fd5(5) "p times" in resultant inequality to obtain

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_p}^{\infty}\prod_{\iota=1}^{p}\frac{\upsilon_\iota(\varsigma_\iota)}{A_\iota^{c_\iota}(\sigma_\iota(\varsigma_\iota))}\left\{\int_{a_{p+1}}^{\infty}\frac{\upsilon_{p+1}(\varsigma_{p+1})}{A_{p+1}^{c_{p+1}}(\sigma_{p+1}(\varsigma_{p+1}))}\phi^{m}(\varsigma_1,\ldots,\varsigma_{p+1})\Delta\varsigma_{p+1}\right\}\Delta\varsigma_p\cdots\Delta\varsigma_1\\&\leq\left(\frac{m}{1-c_{p+1}}\right)^{m}\int_{a_{p+1}}^{\infty}\upsilon_{p+1}(\varsigma_{p+1})A_{p+1}^{m-c_{p+1}}(\sigma_{p+1}(\varsigma_{p+1}))\Delta\varsigma_{p+1}\times\int_{a_1}^{\infty}\cdots\int_{a_p}^{\infty}\prod_{\iota=1}^{p}\frac{\upsilon_\iota(\varsigma_\iota)}{A_\iota^{c_\iota}(\sigma_\iota(\varsigma_\iota))}\phi_p^{m}(\varsigma_1,\ldots,\varsigma_{p+1})\Delta\varsigma_p\cdots\Delta\varsigma_1.\end{aligned} \tag{81}$$

Use induction hypothesis for $\phi_p(\varsigma_1,\ldots,\varsigma_{p+1})$ with fix $\varsigma_{p+1}\in\mathbb{T}_{p+1}$, instead for $\phi_p(\varsigma_1,\ldots,\varsigma_p)$, to obtain

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_{p+1}}^{\infty}\prod_{\iota=1}^{p+1}\frac{\upsilon_\iota(\varsigma_\iota)}{A_\iota^{c_\iota}(\sigma_\iota(\varsigma_\iota))}\phi^{m}(\varsigma_1,\ldots,\varsigma_{p+1})\Delta\varsigma_{p+1}\cdots\Delta\varsigma_1\\&\leq\prod_{\iota=1}^{p+1}\left(\frac{m}{1-c_\iota}\right)^{(p+1)m}\int_{a_1}^{\infty}\cdots\int_{a_{p+1}}^{\infty}\prod_{\iota=1}^{p+1}\upsilon_\iota(\varsigma_\iota)A_\iota^{m-c_\iota}(\sigma_\iota(\varsigma_\iota))g^{m}(\varsigma_1,\ldots,\varsigma_{p+1})\Delta\varsigma_{p+1}\cdots\Delta\varsigma_1.\end{aligned} \tag{82}$$

Hence, by mathematical induction, (14) is true for all $h\in\mathbb{N}$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. G. Bennett and Grosse-Erdmann, "On series of positive terms," *Houston Journal of Mathematics*, vol. 31, no. 2, pp. 541–586, 2005.
[2] M. J. Bohner and S. H. Saker, "Sneak-out principle on time scales," *Journal of Mathematical Inequalities*, vol. 10, no. 2, pp. 393–403, 2016.
[3] M. S. Asraf, K. A. Khan, and A. Nosheen, "Hardy Copson type inequalities on time scales for the functions of n independent variables," *International Journal of Analysis and Applications*, vol. 17, no. 2, pp. 244–259, 2019.

[4] T. Donchev, A. Nosheen, and J. Pečarić, "Hardy-type inequalities on time scales via convexity in several variables," *ISRN Mathematical Analysis*, vol. 2013, Article ID 903196, 9 pages, 2013.

[5] W. Ahmad, K. A. Khan, A. Nosheen, and M. A. Sultan, "Copson-leindler type inequalities for function of several variables on time scales," *Punjab University Journal of Mathematics*, vol. 51, no. 8, pp. 157–168, 2019.

[6] A. Ekinci, "Inequalities for convex functions on time scales," *TWMS Journal of Applied and Engineering Mathematics*, vol. 9, no. 1, pp. 64–72, 2019.

[7] A. A. El-Deeb and S. Rashid, "On some new double dynamic inequalities associated with Leibniz integral rule on time scales," in *Advances in Difference Equations*, pp. 1–22, Springer, Berlin, Germany, 2021.

[8] J. M. Shen, S. Rashid, M. A. Noor, R. Ashraf, and Y. M. Chu, "Certain novel estimates within fractional calculus theory on time scales," *AIMS Mathematics*, vol. 5, no. 6, pp. 6073–6086, 2020.

[9] M. Z. Sarikaya, "New weighted Ostrowski and Čebyšev type inequalities on time scales," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1510–1514, 2010.

[10] M. Z. Sarikaya, "On weighted Iyengar type inequalities on time scales," *Applied Mathematics Letters*, vol. 22, no. 9, pp. 1340–1344, 2009.

[11] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y. M. Chu, "Hermite-Hadamard type inequalities for the class of convex functions on time scale," *Mathematics*, vol. 7, no. 10, p. 956, 2019.

[12] S. Rafeeq, H. Kalsoom, S. Hussain, S. Rashid, and Y. M. Chu, "Delay dynamic double integral inequalities on time scales with applications," in *Advances in Difference Equations*, pp. 1–32, no. 1, Springer, Berlin, Germany, 2020.

[13] U. M. Özkan, M. Z. Sarikaya, and H. Yildirim, "Extensions of certain integral inequalities on time scales," *Applied Mathematics Letters*, vol. 21, no. 10, pp. 993–1000, 2008.

[14] M. Bohner and A. Petereson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston, MA, USA, 2001.

[15] M. Bohner and A. Petereson, *Advances in Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston, MA, USA, 2003.

[16] M. J. Bohner, A. Nosheen, J. Pečarić, and A. Younus, "Some dynamic Hardy-type inequalities with general kernel," *Journal of Mathematical Inequalities*, vol. 8, no. 1, pp. 185–199, 2014.

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Research Article

Uniform Treatment of Jensen's Inequality by Montgomery Identity

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We generalize Jensen's integral inequality for real Stieltjes measure by using Montgomery identity under the effect of $\mathfrak{n}-$convex functions; also, we give different versions of Jensen's discrete inequality along with its converses for real weights. As an application, we give generalized variants of Hermite–Hadamard inequality. Montgomery identity has a great importance as many inequalities can be obtained from Montgomery identity in $q-$calculus and fractional integrals. Also, we give applications in information theory for our obtained results, especially for Zipf and Hybrid Zipf–Mandelbrot entropies.

1. Introduction

Convex functions have a great importance in mathematical inequalities, and the well-known Jensen's inequality is the characterization of convex functions. Jensen's inequality for differentiable convex functions plays a significant role in the field of inequalities as several other inequalities can be seen as special cases of it. One can find the application of Jensen's discrete inequality in discrete-time delay systems in [1].

Taking into consideration the tremendous applications of Jensen's inequality in various fields of mathematics and other applied sciences, the generalizations and improvements of Jensen's inequality have been a topic of supreme interest for the researchers during the last few decades as evident from a large number of publications on the topic (see [2–4] and the references therein).

The well-known Jensen's inequality asserts that for the function Γ it holds that

$$\Psi\left(\frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} x_{\mathcal{J}}\right) \leq \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} \Psi\left(x_{\mathcal{J}}\right), \tag{1}$$

if Ψ is a convex function on interval $I \subset \mathbb{R}$, where $p_{\mathcal{J}}$ are positive real numbers and $x_{\mathcal{J}} \in I\,(\mathcal{J} = 1, \ldots, \mathfrak{m})$, while $P_{\mathfrak{m}} = \sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}$.

However, the well-known integral analogue of Jensen's inequality is as follows.

Theorem 1. *Let $\hbar$: $[a,b] \longrightarrow [\alpha,\beta]$ be a continuous function and λ: $[a,b] \longrightarrow \mathbb{R}$ be an increasing and bounded function with $\lambda(a) \neq \lambda(b)$. Then, for every continuous convex function Ψ: $[\alpha,\beta] \longrightarrow \mathbb{R}$, the following inequality holds:*

$$\Psi(\tilde{\hbar}) \leq \frac{\int_a^b \Psi(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}, \tag{2}$$

where

$$\tilde{\hbar} = \frac{\int_a^b \hbar(\zeta)\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} \in [\alpha,\beta]. \tag{3}$$

There are several inequalities coming from Jensen's inequality both in integral and discrete cases which can be obtained by varying conditions on the function $\hbar$ and measure λ defined in Theorem 1.

Montgomery identity is used in quantum calculus or $q-$calculus. There are different identities of Montgomery, and several inequalities of Ostrowski type were formulated by using these identities. Budak and Sarikaya established the generalized Montgomery-type identities for differential mappings in [5]. Applications of Montgomery identity can be found in fractional integrals as well as in quantum integral operators. Here we utilize Montgomery's identity for the generalization of Jensen's inequality. In [6], Cerone and Dragomir developed a systematic study which produced some novel inequalities. Several interesting results related to inequalities and different types of convexity can be found in [7–21]. The class of convex functions is a very useful concept that has become a focus of interest for researchers in statistics, convex programming, and many other applied disciplines, as well as in inequality theory. The readers can find some motivated findings related to convex functions and some new integral inequalities in [22–27].

In [28], Khan et al. have mentioned about $\mathfrak{n}$-convex functions as follows.

Definition 1. A function $f\colon I \longrightarrow \mathbb{R}$ is called convex of order $\mathfrak{n}$ or $\mathfrak{n}$-convex if for all choices of $(n+1)$ distinct points $x_i, \ldots, x_{i+n}$ we have $\Delta_{(n)} f(x_i) \geq 0$.

If n-th order derivative $f^{(n)}$ exists, then f is $\mathfrak{n}-$convex if and only if $f^{(n)} \geq 0$. For $1 \leq k \leq (n-2)$, a function f is $\mathfrak{n}-$convex if and only if $f^{(k)}$ exists and is $(n-k)-$convex.

In the present paper, we will use Montgomery identity that is presented as following.

Theorem 2. *Let $\mathfrak{n} \in \mathbb{N}$, $\Psi\colon I \longrightarrow \mathbb{R}$ be such that $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ is an open interval, and $\alpha, \beta \in I$, $\alpha < \beta$. Then, the following identity holds:*

$$\Psi(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \Psi(t)\mathrm{d}t + \sum_{\ell=0}^{\mathfrak{n}-2}\frac{\Psi^{(\ell+1)}(\alpha)}{\ell!(\ell+2)}\frac{(x-\alpha)^{\ell+2}}{\beta-\alpha} - \sum_{\ell=0}^{\mathfrak{n}-2}\frac{\Psi^{(\ell+1)}(\beta)}{\ell!(\ell+2)}\frac{(x-\beta)^{\ell+2}}{\beta-\alpha} + \frac{1}{(\mathfrak{n}-1)!}\int_\alpha^\beta R_{\mathfrak{n}}(x,s)\Psi^{(\mathfrak{n})}(s)\mathrm{d}s, \tag{4}$$

where

$$R_{\mathfrak{n}}(x,s) = \begin{cases} -\dfrac{(x-s)^{\mathfrak{n}}}{\mathfrak{n}(\beta-\alpha)} + \dfrac{x-\alpha}{\beta-\alpha}(x-s)^{\mathfrak{n}-1}, & \alpha \leq s \leq x, \\[2ex] -\dfrac{(x-s)^{\mathfrak{n}}}{\mathfrak{n}(\beta-\alpha)} + \dfrac{x-\beta}{\beta-\alpha}(x-s)^{\mathfrak{n}-1}, & x < s \leq \beta. \end{cases} \tag{5}$$

2. Generalization of Jensen's Integral Inequality by Using Montgomery Identity

Before giving our main results, we consider the following assumptions that we use throughout our paper:

A_1 Let $\hbar\colon [a,b] \longrightarrow \mathbb{R}$ be continuous function.

A_2 Let $\lambda\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function or the functions of bounded variation such that $\lambda(a) \neq \lambda(b)$.

2.1. New Generalization of Jensen's Integral Inequality. In our first main result, we employ Montgomery identity to obtain the following real Stieltjes measure's theoretical representations of Jensen's inequality.

Theorem 3. *Let g, λ be as defined in A_1, A_2 such that $\hbar([a,b]) \subset [\alpha,\beta]$. Also, let $\Psi\colon [\alpha,\beta] \longrightarrow \mathbb{R}$ be such that for $\mathfrak{n} \geq 1$, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous. If Ψ is $\mathfrak{n}-$convex such that*

$$R_{\mathfrak{n}}(\tilde{\hbar}, s) \leq \frac{\int_a^b R_{\mathfrak{n}}(\hbar(\zeta), s)\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}, \quad s \in [\alpha,\beta], \tag{6}$$

with $\tilde{\hbar}$ and $R_{\mathfrak{n}}(x,s)$ as defined in (3) and (5), respectively, then we have

$$\Psi(\tilde{\hbar}) - \frac{\int_a^b \Psi(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} \leq \sum_{\ell=0}^{\mathfrak{n}-2}\frac{1}{\ell!(\ell+2)(\beta-\alpha)} \times \left[\Psi^{(\ell+1)}(\alpha)\left((\tilde{\hbar}-\alpha)^{(\ell+2)} - \frac{\int_a^b (\hbar(\zeta)-\alpha)^{(\ell+2)}\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}\right) - \Psi^{(\ell+1)}(\beta)\left((\tilde{\hbar}-\beta)^{(\ell+2)} + \frac{\int_a^b (\hbar(\zeta)-\beta)^{(\ell+2)}\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}\right)\right]. \tag{7}$$

Proof. As $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous for $(\mathfrak{n} \geq 1)$, we can use the representation of Ψ using Montgomery identity (4) and can calculate

$$\Psi(\tilde{\hbar}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \Psi(\zeta)\mathrm{d}(\zeta) + \sum_{\ell=0}^{\mathfrak{n}-2}\frac{\Psi^{(\ell+1)}(\alpha)}{\ell!(\ell+2)}\frac{(\tilde{\hbar}-\alpha)^{\ell+2}}{\beta-\alpha} - \sum_{\ell=0}^{\mathfrak{n}-2}\frac{\Psi^{(\ell+1)}(\beta)}{\ell!(\ell+2)}\frac{(\tilde{\hbar}-\beta)^{\ell+2}}{\beta-\alpha} + \frac{1}{(\mathfrak{n}-1)!}\int_\alpha^\beta R_{\mathfrak{n}}(\tilde{\hbar},s)\Psi^{(\mathfrak{n})}(s)\mathrm{d}s. \quad (8)$$

The integration of the composition of functions $\Psi \circ \hbar$ for the real measure λ on $[a,b]$ gives

$$\begin{aligned}\frac{\int_a^b \Psi(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} &= \frac{1}{\beta-\alpha}\int_\alpha^\beta \Psi(\zeta)\mathrm{d}(\zeta) + \sum_{\ell=0}^{\mathfrak{n}-2}\frac{\Psi^{(\ell+1)}(\alpha)}{\ell!(\ell+2)}\frac{\int_a^b(\hbar(\zeta)-\alpha)^{(\ell+2)}\mathrm{d}\lambda(\zeta)}{(\beta-\alpha)\int_a^b \mathrm{d}\lambda(\zeta)} \\ &\quad - \sum_{\ell=0}^{\mathfrak{n}-2}\frac{\Psi^{(\ell+1)}(\beta)}{\ell!(\ell+2)}\frac{\int_a^b(\hbar(\zeta)-\beta)^{(\ell+2)}\mathrm{d}\lambda(\zeta)}{(\beta-\alpha)\int_a^b \mathrm{d}\lambda(\zeta)} + \frac{1}{(\mathfrak{n}-1)!}\int_\alpha^\beta R_{\mathfrak{n}}(\hbar(\zeta),s)\Psi^{(\mathfrak{n})}(s)\mathrm{d}s.\end{aligned} \quad (9)$$

Now computing the difference $\Psi(\tilde{\hbar}) - \int_a^b \Psi(\hbar(\zeta))\mathrm{d}\lambda(\zeta)/\int_a^b \mathrm{d}\lambda(\zeta)$, we get the following generalized identity involving real Stieltjes measure:

$$\begin{aligned}\Psi(\tilde{\hbar}) - \frac{\int_a^b \Psi(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} &= \sum_{\ell=0}^{\mathfrak{n}-2}\frac{1}{\ell!(\ell+2)(\beta-\alpha)} \\ &\quad \times\left[\Psi^{(\ell+1)}(\alpha)\left((\tilde{\hbar}-\alpha)^{(\ell+2)} - \frac{\int_a^b(\hbar(\zeta)-\alpha)^{(\ell+2)}\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}\right) - \Psi^{(\ell+1)}(\beta)\left((\tilde{\hbar}-\beta)^{(\ell+2)} + \frac{\int_a^b(\hbar(\zeta)-\beta)^{(\ell+2)}\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}\right)\right] \\ &\quad + \frac{1}{(\mathfrak{n}-1)!}\int_\alpha^\beta\left(R_{\mathfrak{n}}(\tilde{\hbar},s) - \frac{\int_a^b R_{\mathfrak{n}}(\hbar(\zeta),s)\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}\right)\Psi^{(\mathfrak{n})}(s)\mathrm{d}s. \ (\mathrm{GI}.1).\end{aligned} \quad (10)$$

Finally, by our assumption, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous on $[\alpha,\beta]$; as a result, $\Psi^{(\mathfrak{n})}$ exists almost everywhere. Moreover, Ψ is supposed to be $\mathfrak{n}$−convex, so we have $\Psi^{(\mathfrak{n})}(x) \geq 0$ almost everywhere on $[\alpha,\beta]$. Therefore, by taking into account the last term in generalized identity (GI.1) and integral analogue of Jensen's inequality that is given in (6), we get (7). □

In the later part of this section, we will vary our conditions on functions g and Stieltjes measure $\mathrm{d}\lambda$ to obtain generalized variants of Jensen–Steffensen, Jensen–Boas, Jensen–Brunk, and Jensen-type inequalities. We start with the following generalization of Jensen–Steffensen inequality for n−convex functions.

Theorem 4. *Let Ψ defined in Theorem 3 be $\mathfrak{n}$−convex and $\hbar$ defined in M_1 be monotonic. Then, the following results hold.*

(i) If λ defined in M_2 satisfies

$$\lambda(a) \leq \lambda(x) \leq \lambda(b), \quad \forall x \in [a,b], \lambda(b) > \lambda(a), \quad (11)$$

then for even $\mathfrak{n} \geq 3$, (6) is valid.

(ii) Moreover, if (6) is valid and the function

$$H(x) := \sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\left(\Psi^{(\ell+1)}(\alpha)(x-\alpha)^{\ell+2} - \Psi^{(\ell+1)}(\beta)(x-\beta)^{\ell+2}\right) \quad (12)$$

is convex, then we get inequality (2) which is called generalized Jensen–Steffensen inequality for $\mathfrak{n}-$*convex function.*

Proof. (i) By applying second derivative test, we can show that the function $R_{\mathfrak{n}}(x,s)$ is convex for even $\mathfrak{n}>3$. Now using the assumed conditions, one can employ Jensen–Steffensen inequality given by Boas (see [29] or [30], p. 59) for convex function $R_{\mathfrak{n}}(x,s)$ to obtain (6).

(ii) Since we can rewrite the R.H.S. of (7) in the difference

$$H(\widetilde{\hbar})-\frac{\int_a^b H(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}, \tag{13}$$

for convex function H and by our assumed conditions on functions $\hbar$ and λ, this difference is non-positive by using Jensen–Steffensen inequality difference [29]. As a result, the R.H.S. of inequality (7) is non-positive and we get generalized Jensen–Steffensen inequality (2) for $\mathfrak{n}-$convex function. □

Now, we give similar results related to Jensen–Boas inequality [30], p. 59], which is a generalization of Jensen–Steffensen inequality.

Corollary 1. *Let* Ψ *defined in Theorem 3 be* $n-$*convex function. Also, let* $\hbar$ *be as defined in* M_1 *with* $a=y_0<y_1<\ldots<y_k<\ldots<y_{\mathfrak{m}-1}<y_{\mathfrak{m}}=b$ *and* $\hbar$ *be monotonic in each of the* $\mathfrak{m}$ *intervals* $((y_{k-1},y_k))$. *Then, the following results hold.*

(i) If λ *as defined in* M_2 *satisfies*

$$\lambda(a)\le\lambda(x_1)\le\lambda(y_1)\le\lambda(x_2)\le\lambda(y_2)\le\ldots\le\lambda(y_{\mathfrak{m}-1})\le\lambda(x_{\mathfrak{m}})\le\lambda(b), \tag{14}$$

$\forall x_k\in(y_{k-1},y_k)$ *and* $\lambda(b)>\lambda(a)$, *then for even* $\mathfrak{n}\ge 3$, *(6) is valid.*

(ii) Moreover, if (6) is valid and the function $H(\cdot)$ *defined in (18) is convex, then again inequality (2) holds and is called Jensen–Boas inequality for* $\mathfrak{n}-$*convex function.*

Proof. We follow the similar argument as in the proof of Theorem 4, but under the conditions of this corollary, we utilize Jensen–Boas inequality (see [29] or [24], p. 59) instead of Jensen–Steffensen inequality.

Next, we give results for Jensen–Brunk inequality. □

Corollary 2. *Let* Ψ *defined in Theorem 3 be* $\mathfrak{n}-$*convex and* $\hbar$ *defined in* M_1 *be an increasing function. Then, the following results hold.*

(i) If λ *defined in* M_2 *with* $\lambda(b)>\lambda(a)$ *and*

$$\int_a^x(\hbar(x)-\hbar(\zeta))\mathrm{d}\lambda(\zeta)\ge 0, \tag{15}$$

and

$$\int_x^b(\hbar(x)-\hbar(\zeta))\mathrm{d}\lambda(\zeta)\le 0, \tag{16}$$

$\forall x\in[a,b]$ *holds, then for even* $\mathfrak{n}\ge 3$, *(6) is valid.*

(ii) Moreover, if (6) is valid and the function $H(\cdot)$ *defined in (18) is convex, then again inequality (2) holds and is called Jensen–Brunk inequality for* $\mathfrak{n}-$*convex function.*

Proof. We proceed with the similar idea as in the proof of Theorem 4, but under the conditions of this corollary, we employ Jensen–Brunk inequality (see [31] or [30], p. 59]) instead of Jensen–Steffensen inequality. □

Remark 1. The similar result in Corollary 2 is also valid provided that the function $\hbar$ is decreasing. Also, assuming that the function $\hbar$ is monotonic, one can replace the conditions in Corollary 2(i) by

$$0\le\int_a^x|\hbar(x)-\hbar(\zeta)|\mathrm{d}\lambda(\zeta)\le\int_x^b|g(x)-\hbar(\zeta)|\mathrm{d}\lambda(\zeta). \tag{17}$$

Remark 2. It is interesting to see that by employing similar method as in Theorem 4, we can also get the generalization of classical Jensen's inequality (2) for $n-$convex functions by assuming the functions $\hbar$ and λ along with the respective conditions in Theorem 1.

Another important consequence of Theorem 3 can be given by setting the function $\hbar$ as $\hbar(\zeta)=\zeta$. This form is the generalized version of L.H.S. inequality of the Hermite-Hadamard inequality.

Corollary 3. *Let* λ: $[a,b]\longrightarrow\mathbb{R}$ *be a function of bounded variation such that* $\lambda(a)\ne\lambda(b)$ *with* $[a,b]\subset[\alpha,\beta]$ *and* $\widetilde{\zeta}=\int_a^b\zeta d\lambda(\zeta)/\int_a^b d\lambda(\zeta)\in[\alpha,\beta]$. *Under the assumptions of Theorem 3, if* Ψ *is* $\mathfrak{n}-$*convex such that*

$$R_{\mathfrak{n}}(\widetilde{\hbar},s)\le\frac{\int_a^b R_{\mathfrak{n}}(\hbar(\zeta),s)\mathrm{d}\lambda(\zeta)}{\int_a^b\mathrm{d}\lambda(\zeta)},\quad s\in[\alpha,\beta], \tag{18}$$

then we have

$$\Psi(\tilde{\zeta}) \leq \frac{\int_a^b \Psi(\zeta) \mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} + \sum_{\ell=0}^{\mathfrak{n}-2} \frac{1}{\ell!(\ell+2)(\beta-\alpha)} \times \left[\Psi^{(\ell+1)}(\alpha)\left((\tilde{\zeta}-\alpha)^{(\ell+2)} - \frac{\int_a^b (\zeta-\alpha)^{(\ell+2)} \mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}\right) - \Psi^{(\ell+1)}(\beta)\left((\tilde{\zeta}-\beta)^{(\ell+2)} + \frac{\int_a^b (\zeta-\beta)^{(\ell+2)} \mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}\right)\right]. \tag{19}$$

If the inequality (18) holds in reverse direction, then (19) also holds reversely.

The special case of above corollary can be given in the form of following remark.

Remark 3. It is interesting to see that substituting $\lambda(\zeta) = \zeta$ gives $\int_a^b \mathrm{d}\lambda(\zeta) = b - a$ and $\tilde{\zeta} = a + b/2$. Using these substitutions in (2) and by following remark (20), we get the L.H.S. inequality of renowned Hermite–Hadamard inequality for $\mathfrak{n}$−convex functions.

2.2. New Generalization of Converse of Jensen's Integral Inequality. In this section, we give the results for the converse of Jensen's inequality to hold, giving the conditions on the real Stieltjes measure $\mathrm{d}\lambda$, such that $\lambda(a) \neq \lambda(b)$, allowing that the measure can also be negative, but employing Montgomery identity.

To start with we need the following assumption for the results of this section:

A_3 Let $\mathbf{m}, \mathbf{M} \in [\alpha, \beta]\,(\mathbf{m} \neq \mathbf{M})$ be such that $\mathbf{m} \leq \hbar(\zeta) \leq \mathbf{M}$ for all $\zeta \in [a, b]$ where $\hbar$ is defined in A_1.

For a given function $\Psi\colon [\alpha, \beta] \longrightarrow \mathbb{R}$, we consider the difference

$$\mathrm{CJ}\left(\Psi, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right) = \frac{\int_a^b \Psi(\hbar(\zeta)) \mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} - \frac{\mathbf{M} - \tilde{\hbar}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{m}) - \frac{\tilde{\hbar} - \mathbf{m}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{M}), \tag{20}$$

where $\tilde{\hbar}$ is defined in (3).

Using Montgomery identity, we obtain the following representation of the converse of Jensen's inequality.

Theorem 5. *Let $\hbar, \lambda$ be as defined in A_1, A_2 and let $\Psi\colon [\alpha, \beta] \longrightarrow \mathbb{R}$ be such that for $\mathfrak{n} \geq 1$, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous. If Ψ is $\mathfrak{n}$−convex such that*

$$\mathrm{CJ}\left(R_{\mathfrak{n}}(x, s), \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right) \leq 0, \quad s \in [\alpha, \beta], \tag{21}$$

or

$$\frac{\int_a^b R_{\mathfrak{n}}(\hbar(\zeta), s) \mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} \leq \frac{\mathbf{M} - \tilde{\hbar}}{\mathbf{M} - \mathbf{m}} \left(R_{\mathfrak{n}}(\mathbf{m}, s)\right) + \frac{\tilde{\hbar} - \mathbf{m}}{\mathbf{M} - \mathbf{m}} \left(R_{\mathfrak{n}}(\mathbf{M}, s)\right), \quad s \in [\alpha, \beta], \tag{22}$$

then we get the following extension of the converse of Jensen's difference:

$$\frac{\int_a^b \Psi(\hbar(\zeta)) \mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} \leq \frac{\mathbf{M} - \tilde{\hbar}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{m}) + \frac{\tilde{\hbar} - \mathbf{m}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{M}) + \sum_{\ell=0}^{\mathfrak{n}-2} \left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right) \times \left(\Psi^{(\ell+1)}(\alpha)\mathrm{CJ}\left((x-\alpha)^{\ell+2}, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right) - \Psi^{(\ell+1)}(\beta)\mathrm{CJ}\left((x-\beta)^{\ell+2}, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right)\right), \tag{23}$$

where $R_{\mathfrak{n}}(\cdot, s)$ is defined in (5).

Proof. As $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous for $(\mathfrak{n} \geq 1)$, we can use the representation of Ψ using Montgomery identity (4) in the difference $CJ(\Psi, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda)$:

$$\begin{aligned}
CJ\left(\Psi, \hbar_{\{m,M\}}; \lambda\right) &= CJ\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \Psi(\zeta)d\zeta, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right) \\
&+ \sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\Psi^{(\ell+1)}(\alpha)CJ\left((x-\alpha)^{\ell+2}, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right) \\
&- \sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\Psi^{(\ell+1)}(\beta)CJ\left((x-\beta)^{\ell+2}, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right) + \frac{1}{(\mathfrak{n}-1)!}\int_\alpha^\beta CJ\left(R_{\mathfrak{n}}(x,s), \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right)\Psi^{(\mathfrak{n})}(s)ds.
\end{aligned} \tag{24}$$

After simplification and following the fact that $CJ(\Psi, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda)$ is zero for Ψ to be constant or linear, we get the following generalized identity:

$$\begin{aligned}
CJ\left(\Psi, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right)j &= \sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{\cdot}{\ell!(\ell+2)(\beta-\alpha)}\right) \\
&\times\left(\Psi^{(\ell+1)}(\alpha)CJ\left((x-\alpha)^{\ell+2}, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right) - \Psi^{(\ell+1)}(\beta)CJ\left((x-\beta)^{\ell+2}, \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right)\right) \\
&+ \frac{1}{(\mathfrak{n}-1)!}\int_\alpha^\beta CJ\left(R_{\mathfrak{n}}(x,s), \hbar_{\{\mathbf{m},\mathbf{M}\}}; \lambda\right)\Psi^{(\mathfrak{n})}(s)ds. \text{(CGI.1)}.
\end{aligned} \tag{25}$$

Now using characterizations of $\mathfrak{n}-$convex functions like in the proof of Theorem 3, we get (23). □

The next result gives converse of Jensen's inequality for higher-order convex functions.

Theorem 6. *Let Ψ defined in Theorem 5 be $\mathfrak{n}-$convex and $\hbar$ be as defined in A_3. Then, the following results hold.*

(i) If λ is non-negative measure on $[a,b]$, then for even $\mathfrak{n} \geq 3$, (22) is valid.

(ii) Moreover, if (22) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following inequality for $\mathfrak{n}-$convex function to be valid:

$$\frac{\int_a^b \Psi(\hbar(\zeta))d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \leq \frac{\mathbf{M}-\tilde{\hbar}}{\mathbf{M}-\mathbf{m}}\Psi(\mathbf{m}) - \frac{\tilde{\hbar}-\mathbf{m}}{\mathbf{M}-\mathbf{m}}\Psi(\mathbf{M}). \tag{26}$$

Proof. The idea of the proof is similar to that of (6), but we use converse of Jensen's inequality (see [32] or [30], p. 98). □

2.3. Applications of Jensen's Integral Inequality. In this section, we give applications of Jensen's integral inequality.

Another important consequence of Theorem 3 is by setting the function $\hbar$ as $\hbar(\zeta) = \zeta$ gives generalized version of L. H. S. inequality of the Hermite–Hadamard inequality.

Corollary 4. *Let λ: $[a,b] \longrightarrow \mathbb{R}$ be a function of bounded variation such that $\lambda(a) \neq \lambda(b)$ with $[a,b] \subset [\alpha,\beta]$ and $\tilde{\zeta} = \int_a^b \zeta d\lambda(\zeta)/\int_a^b d\lambda(\zeta) \in [\alpha,\beta]$. Under the assumptions of Theorem 5, if Ψ is $\mathfrak{n}-$convex such that*

$$\frac{\int_a^b R_{\mathfrak{n}}(\zeta,s)d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \leq \frac{b-\tilde{\zeta}}{b-a}\left(R_{\mathfrak{n}}(a,s)\right) + \frac{\tilde{\zeta}-a}{b-a}\left(R_{\mathfrak{n}}(b,s)\right), \quad s \in [\alpha,\beta], \tag{27}$$

then we have

$$\frac{\int_a^b \Psi(\zeta)\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)} \leq \frac{b-\tilde{\zeta}}{b-a}\Psi(a) + \frac{\tilde{\zeta}-a}{b-a}\Psi(b) + \sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{\cdot}{\ell!(\ell+2)(\beta-\alpha)}\right)\times$$
$$\left(\Psi^{(\ell+1)}(\alpha)\mathrm{CJ}\left((x-\alpha)^{\ell+2}, \mathrm{id}_{\{a,b\}};\lambda\right) - \Psi^{(\ell+1)}(\beta)\mathrm{CJ}\left((x-\beta)^{\ell+2}, id_{\{a,b\}};\lambda\right)\right). \tag{28}$$

If the inequality (27) holds in reverse direction, then (28) also holds reversely.

The special case of above corollary can be given in the form of following remark.

Remark 4. It is interesting to see that substituting $\lambda(\zeta) = \zeta$ and by following Theorem 6, we get the R.H.S. inequality of renowned Hermite–Hadamard inequality for $\mathfrak{n}$−convex functions.

3. Generalization of Jensen's Discrete Inequality by Using Montgomery Identity

In this section, we give generalizations for Jensen's discrete inequality by using Montgomery identity. The proofs are similar to those of continuous case as given in previous section; therefore, we give results directly.

3.1. Generalization of Jensen's Discrete Inequality for Real Weights. In discrete case, we have that $p_{\mathcal{J}} > 0$ for all $\mathcal{J} = 1, 2, \ldots, \mathfrak{m}$. Here we give generalizations of results allowing $p_{\mathcal{J}}$ to be negative real numbers. Also, with usual notations for $p_{\mathcal{J}} x_{\mathcal{J}}$ ($\mathcal{J} = 1, 2, \ldots, \mathfrak{n}$), we notate

$$\mathbf{x} = (x_1, x_2, \ldots, x_{\mathfrak{m}}) \text{ and } \mathbf{p} = (p_1, p_2, \ldots, p_{\mathfrak{m}}) \tag{29}$$

to be $\mathfrak{m}$−tuples.

$$P_\nu = \sum_{\mathcal{J}=1}^{\nu} p_{\mathcal{J}}, \overline{P}_\nu = P_{\mathfrak{m}} - P_{\nu-1} \quad (\nu = 1, 2, \ldots, \mathfrak{m}), \tag{30}$$

and

$$\overline{x} = \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} x_{\mathcal{J}}. \tag{31}$$

Using Montgomery identity (4), we obtain the following representations of Jensen's discrete inequality.

Theorem 7. *Let Ψ: $[\alpha, \beta] \longrightarrow \mathbb{R}$ be such that for $\mathfrak{n} \geq 1$, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous. Also, let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$, $p_{\mathcal{J}} \in \mathbb{R}$ ($\mathcal{J} = 1, \ldots, \mathfrak{m}$) be such that $P_{\mathfrak{m}} \neq 0$ and $\overline{x} \in [\alpha, \beta]$.*

(i) Then, the following generalized identity holds:

$$\Psi(\overline{x}) - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}\Psi(x_{\mathcal{J}}) = \sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times$$
$$\left\{\Psi^{(\ell+1)}(\alpha)\left((\overline{x}-\alpha)^{\ell+2} - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}(x_{\mathcal{J}}-\alpha)^{\ell+2}\right) - \Psi^{(\ell+1)}(\beta)\left((\overline{x}-\beta)^{\ell+2} - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}(x_{\mathcal{J}}-\beta)^{\ell+2}\right)\right\} \tag{32}$$
$$+ \frac{1}{(\mathfrak{n}-1)!}\int_\alpha^\beta\left[R_{\mathfrak{n}}(\overline{x}, s) - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} R_{\mathfrak{n}}(x_{\mathcal{J}}, s)\right]\Psi^{(\mathfrak{n})}(s)\mathrm{d}s, \text{ (DGI.1)},$$

where $R_{\mathfrak{n}}(\cdot, s)$ is defined in (5).

(ii) Moreover, if Ψ is $\mathfrak{n}$−convex and the inequality

$$R_{\mathfrak{n}}(\overline{x}, s) \leq \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} R_{\mathfrak{n}}(x_{\mathcal{J}}, s) \tag{33}$$

holds, then we have the following generalized inequality:

$$\Psi(\overline{x}) - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}\Psi(x_{\mathcal{J}}) \leq \sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times$$
$$\left\{\Psi^{(\ell+1)}(\alpha)\left((\overline{x}-\alpha)^{\ell+2} - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}(x_{\mathcal{J}}-\alpha)^{\ell+2}\right) - \Psi^{(\ell+1)}(\beta)\left((\overline{x}-\beta)^{\ell+2} - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}(x_{\mathcal{J}}-\beta)^{\ell+2}\right)\right\}. \tag{34}$$

If inequality (33) holds in reverse direction, then (34) also holds reversely.

Proof. Similar to that of Theorem 3.

In the later part of this section, we will vary our conditions on $p_{\mathcal{J}}x_{\mathcal{J}}\,(\mathcal{J}=1,2,\ldots,\mathfrak{n})$ to obtain generalized discrete variants of Jensen–Steffensen, Jensen's, and Jensen–Petrovic type inequalities. We start with the following generalization of Jensen–Steffensen discrete inequality for $\mathfrak{n}-$convex functions. □

Theorem 8. *Let Ψ be as defined in Theorem 7. Also, let $\mathbf{x}$ be monotonic $\mathfrak{n}-$tuple, $x_{\mathcal{J}}\in[a,b]\subseteq[\alpha,\beta]$, and $\mathbf{p}$ be a real $\mathfrak{n}-$tuple such that*

$$0\leq P_{\nu}\leq P_{\mathfrak{m}},\ (\nu=1,2,\ldots,\mathfrak{m}-1),\quad P_{\mathfrak{m}}>0 \tag{35}$$

is satisfied.

(i) *If Ψ is $\mathfrak{n}-$convex, then for even $\mathfrak{n}\geq 3$, (33) is valid.*

(ii) *Moreover, if (33) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized Jensen–Steffensen discrete inequality:*

$$\Psi(\overline{x})\leq\frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}\Psi\left(x_{\mathcal{J}}\right). \tag{36}$$

Proof. It is interesting to see that under the assumed conditions on tuples $\mathbf{x}$ and $\mathbf{p}$, we have that $\overline{x}\in[a,b]$. For $x_1\geq x_2\geq\ldots\geq x_{\mathfrak{m}}$,

$$P_{\mathfrak{m}}(x_1-\overline{x})=\sum_{\mathcal{J}=2}^{\mathfrak{m}}p_{\mathcal{J}}\left(x_1-x_{\mathcal{J}}\right)=\sum_{\nu=2}^{\mathfrak{m}}(x_{\nu-1}-x_{\nu})(P_{\mathfrak{m}}-P_{\nu-1})\geq 0. \tag{37}$$

This shows that $x_1\geq\overline{x}$. Also, $\overline{x}\geq x_{\mathfrak{n}}$, since we have

$$P_{\mathfrak{m}}(\overline{x}-x_{\mathfrak{m}})=\sum_{\mathcal{J}=1}^{\mathfrak{m}-1}p_{\mathcal{J}}\left(x_{\mathcal{J}}-x_{\mathfrak{m}}\right)=\sum_{\nu=1}^{\mathfrak{m}-1}(x_{\nu}-x_{\nu-1})P_{\nu}\geq 0. \tag{38}$$

For further details, see the proof of Jensen–Steffensen discrete inequality ([24], p. 57). The idea of the rest of the proof is similar to that of Theorem 3, but here we employ Theorem 7 and Jensen–Steffensen discrete inequality. □

Corollary 5. *Let Ψ be as defined in Theorem 7 and let $x_{\mathcal{J}}\in[a,b]\subseteq[\alpha,\beta]$ with $\mathbf{p}$ being a positive $\mathfrak{n}-$tuple.*

(i) *If Ψ is $\mathfrak{n}-$convex, then for even $\mathfrak{n}\geq 3$, (34) is valid.*

(ii) *Moreover, if (33) is valid and the function $H(\cdot)$ defined in (12) is convex, then again we get (36) which is called Jensen's inequality for $\mathfrak{n}-$convex functions.*

Proof. For $p_{\mathcal{J}}>0, x_{\mathcal{J}}\in[a,b]\ (\mathcal{J}=1,2,3,\ldots,\mathfrak{m})$ ensures that $\overline{x}\in[a,b]$. So, by applying classical Jensen's discrete inequality (1) and idea of Theorem 8, we will get the required results. □

Remark 5. Under the assumptions of Corollary 5, if we choose $P_{\mathfrak{m}}=1$, then Corollary 5 (*ii*) gives the following inequality for $\mathfrak{n}-$convex functions:

$$\Psi\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}x_{\mathcal{J}}\right)\leq\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}\Psi\left(x_{\mathcal{J}}\right). \tag{39}$$

Now we give following reverses of Jensen–Steffensen and Jensen-type inequalities.

Corollary 6. *Let Ψ be as defined in Theorem 7. Also, let $\mathbf{x}$ be monotonic $\mathfrak{m}-$tuple, $x_{\mathcal{J}}\in[a,b]\subseteq[\alpha,\beta]$, and $\mathbf{p}$ be a real $\mathfrak{m}-$tuple such that there exist $m\in\{1,2,\ldots,\mathfrak{m}\}$ such that*

$$0\geq P_{\nu},\quad \text{for } \nu<m \text{ and } 0\geq\overline{P}_{\nu},\ \text{for } \nu>m, \tag{40}$$

where $P_{\mathfrak{m}}>0$ and $\overline{x}\in[\alpha,\beta]$.

(i) *If is $\mathfrak{n}-$convex, then for even $\mathfrak{n}\geq 3$, then reverse of inequality (33) holds.*

(ii) *Moreover if (33) holds reversely and the function $H(\cdot)$ defined in (12) is convex, then we get reverse of generalized Jensen–Steffensen inequality (36) for $\mathfrak{n}-$convex functions.*

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ reverse of Jensen–Steffensen inequality to obtain results. □

In the next corollary, we give explicit conditions on real tuple $\mathbf{p}$ such that we get reverse of classical Jensen inequality.

Corollary 7. *Let Ψ be as defined in Theorem 7 and let $x_{\mathcal{J}}\in[a,b]\subseteq[\alpha,\beta]$ such that $\overline{x}\in[\alpha,\beta]$. Let $\mathbf{p}$ be a real $n-$tuple such that*

$$0<p_1, 0\geq p_2, p_3,\ldots,p_{\mathfrak{m}}, 0<P_{\mathfrak{m}} \tag{41}$$

is satisfied.

(i) *If Ψ is $\mathfrak{n}-$convex, then for even $\mathfrak{n}\geq 3$, the reverse of inequality (33) is valid.*

(ii) *Also, if reverse of (33) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get reverse of (36).*

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ reverse of Jensen inequality to obtain results. □

In [33] (see also [30]), one can find the result which is equivalent to the Jensen–Steffensen and the reverse Jensen–Steffensen inequality together. It is the so-called Jensen–Petrović inequality. Here, without the proof, we give the adequate corollary which uses that result. The proof goes the same way as in the previous corollaries.

Corollary 8. *Let Ψ be as defined in Theorem 7 and let $x_i \in [a,b] \subseteq [\alpha,\beta]$ be such that $x_{\mathfrak{m}} \geq x_{\mathfrak{m}-1}, \ldots, x_2 \leq x_1$. Let* $\mathbf{p}$ *be a real $\mathfrak{m}-$tuple with $P_{\mathfrak{m}} = 1$ such that*

$$0 \leq P_v, \quad \text{for } 1 \leq v < \mathfrak{m} - 1 \text{ and } 0 \leq \overline{P}_v, \quad \text{for } 2 \leq v < \mathfrak{m}, \tag{42}$$

is satisfied. Then, we get the equivalent results given in Theorem 8 (i) and (ii), respectively.

Remark 6. Under the assumptions of Corollary 8, if there exist $m \in \{1, 2, \ldots, \mathfrak{n}\}$ such that

$$0 \geq P_v, \quad \text{for } v < m \text{ and } 0 \geq \overline{P}_v, \quad \text{for } v > m, \tag{43}$$

and $\overline{x} \in [\alpha,\beta]$, then we get the equivalent results for reverse Jensen–Steffensen inequality given in Corollary 6 (i) and (ii), respectively.

Remark 7. It is interesting to see that the conditions on $p_{\mathcal{J}}, \mathcal{J} = 1, 2, \ldots, \mathfrak{m}$ given in Corollary 8 and Remark 6 are coming from Jensen–Petrović inequality which become equivalent to conditions for $p_{\mathcal{J}}, \mathcal{J} = 1, 2, \ldots, \mathfrak{m}$ for Jensen–Steffensen results given in Theorem 8 and Corollary 6, respectively, when $P_{\mathfrak{m}} = 1$.

Now we give results for Jensen and its reverses for $\mathfrak{n}-$tuples $\mathbf{x}$ and $\mathbf{p}$ when $\mathfrak{n}$ is an odd number.

Corollary 9. *Let Ψ be as defined in Theorem 7 and let $x_{\mathcal{J}} \in [a,b] \subseteq [\alpha,\beta]$ for $\mathcal{J} = 1, 2, \ldots, \mathfrak{m}$ be such that $\mathbf{x}$, $\mathbf{p}$ be real $\mathfrak{m}-$tuples, $\mathfrak{m} = 2m+1$, $m \in \mathbb{N}$ and $\widehat{x} = 1/\sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} x_{\mathcal{J}} \in [\alpha,\beta]$ for all $k = 1, 2, \ldots, m$. If for every $k = 1, 2, \ldots, m$, we have*

(i^*) $p_1 > 0, p_{2k} \leq 0, p_{2k} + p_{2k+1} \leq 0, \sum_{\mathcal{J}=1}^{2k} p_{\mathcal{J}} \geq 0, \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} > 0$

(ii^*) $x_{2k} \leq x_{2k+1}, \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} (x_{\mathcal{J}} - x_{2k+1}) \geq 0,$

then we have the following statements to be valid.

(i) If Ψ is $\mathfrak{n}-$convex, then for even $\mathfrak{n} \geq 3$, the inequality

$$R_{\mathfrak{n}}(\widehat{x}, s) \geq \frac{1}{P_{2m+1}} \sum_{\mathcal{J}=1}^{2m+1} p_{\mathcal{J}} R_{\mathfrak{n}}(x_{\mathcal{J}}, s). \tag{44}$$

(ii) Also if (44) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized inequality:

$$\Psi(\widehat{x}) \geq \frac{1}{P_{2m+1}} \sum_{\mathcal{J}=1}^{2m+1} p_{\mathcal{J}} \Psi(x_{\mathcal{J}}). \tag{45}$$

Proof. We employ the idea of the proofs of Theorems 7 and 8 for $\mathfrak{n}$ = odd along with inequality of Vasić and Janic [34]. □

Remark 8. We can also discuss the following important cases by considering the explicit conditions given in [34].

We conclude this section by giving the following important cases:

(**Case 1**)

Let the condition (i^*) hold and the reverse inequalities in condition (ii^*) hold. Then, again we can give inequalities (44) and (45), respectively, given in Corollary 9.

(**Case 2**)

If in case of conditions (i^*) and (ii^*), the following are valid:

(iii^*) $p_1 > 0, p_{2k+1} \geq 0, p_{2k} + p_{2k+1} \geq 0, \sum_{\mathcal{J}=1}^{2k} p_{\mathcal{J}} \geq 0, \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} > 0$

(iv^*) $x_{2k} \leq x_{2k+1}, \sum_{\mathcal{J}=1}^{2k-1} p_{\mathcal{J}} (x_{\mathcal{J}} - x_{2k}) \leq 0,$

then we can give reverses of inequalities (44) and (45), respectively, given in Corollary 9.

(**Case 3**)

Finally, we can also give reverses of inequalities (44) and (45), respectively, given in Corollary 9 provided that the condition (iii^*) holds and the reverse inequalities in condition (iv^*) hold.

The result given in (Case 3) is type of generalization of inequality by Szegö [35].

3.2. Generalization of Converse Jensen's Discrete Inequality for Real Weights.

In this section, we give the results for converse of Jensen's inequality in discrete case by using the Montgomery identity.

Let $x_{\mathcal{J}} \in [a,b] \subseteq [\alpha,\beta]$, $a \neq b$, $p_{\mathcal{J}} \in \mathbb{R}\,(\mathcal{J} = 1, \ldots, \mathfrak{n})$ be such that $P_{\mathfrak{m}} \neq 0$. Then, we have the following difference of converse of Jensen's inequality for Ψ: $[\alpha,\beta] \longrightarrow \mathbb{R}$:

$$\mathrm{CJ}_{\mathrm{dis}}(\Psi) = \frac{1}{P_{\mathfrak{m}}} \sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) - \frac{b - \overline{x}}{b - a} \Psi(a) - \frac{\overline{x} - a}{b - a} \Psi(b). \tag{46}$$

Similarly, we assume the Giaccardi difference [36] given as

$$G_{\mathrm{cardi}}(\Psi) = \sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) - A\Psi\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} x_{\mathcal{J}}\right) - B\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} - 1\right)\Psi(x_0), \tag{47}$$

where

$$A=\frac{\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}\left(x_{\mathcal{J}}-x_0\right)\right)}{\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}x_{\mathcal{J}}-x_0\right)},B=\frac{\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}x_{\mathcal{J}}}{\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}x_{\mathcal{J}}-x_0\right)}\text{ and }\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}x_{\mathcal{J}}\neq x_0. \tag{48}$$

Theorem 9. *Let $\Psi\colon[\alpha,\beta]\longrightarrow\mathbb{R}$ be such that for $\mathfrak{n}\geq 1$, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous. Also, let $x_0,x_{\mathcal{J}}\in[a,b]\subseteq[\alpha,\beta]$, $p_{\mathcal{J}}\in\mathbb{R}\,(\mathcal{J}=1,\ldots,\mathfrak{m})$, be such that $\sum_{\mathcal{J}=1}^{\mathfrak{m}}p_{\mathcal{J}}x_{\mathcal{J}}\neq x_0$.*

(i) Then, the following generalized identity holds:

$$\begin{aligned}\mathrm{CJ}_{\mathrm{dis}}(\Psi)=&\sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\left(\Psi^{(\ell+1)}(\alpha)\mathrm{CJ}_{\mathrm{dis}}\left(\left(x_{\mathcal{J}}-\alpha\right)^{\ell+2}\right)-\Psi^{(\ell+1)}(\beta)\mathrm{CJ}_{\mathrm{dis}}\left(\left(x_{\mathcal{J}}-\beta\right)^{\ell+2}\right)\right)\\&+\frac{1}{(\mathfrak{n}-1)!}\int_{\alpha}^{\beta}\mathrm{CJ}_{\mathrm{dis}}\left(R_{\mathfrak{n}}\left(x_{\mathcal{J}},s\right)\right)\Psi^{(\mathfrak{n})}(s)\mathrm{d}s,\ (\mathrm{DC.GI}),\end{aligned}\tag{49}$$

where $R_{\mathfrak{n}}(\cdot,s)$ is defined in (5).

(ii) Moreover, if Ψ is $\mathfrak{n}-$convex and the inequality

$$\mathrm{CJ}_{\mathrm{dis}}\left(R_{\mathfrak{n}}\left(x_{\mathcal{J}},s\right)\right)\leq 0 \tag{50}$$

holds, then we have the following generalized inequality:

$$\mathrm{CJ}_{\mathrm{dis}}(\Psi)\leq\sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\left(\Psi^{(\ell+1)}(\alpha)\mathrm{CJ}_{\mathrm{dis}}\left(\left(x_{\mathcal{J}}-\alpha\right)^{\ell+2}\right)-\Psi^{(\ell+1)}(\beta)\mathrm{CJ}_{\mathrm{dis}}\left(\left(x_{\mathcal{J}}-\beta\right)^{\ell+2}\right)\right). \tag{51}$$

If inequality (50) holds in reverse direction, then (51) also holds reversely.

Theorem 10. *Let $\Psi\colon[\alpha,\beta]\longrightarrow\mathbb{R}$ be such that for $\mathfrak{n}\geq 1$, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous. Also, let $x_{\mathcal{J}}\in[a,b]\subseteq[\alpha,\beta]$, $p_{\mathcal{J}}\in\mathbb{R}\,(\mathcal{J}=1,\ldots,\mathfrak{m})$, be such that $P_{\mathfrak{m}}\neq 0$ and $\overline{x}\in[\alpha,\beta]$.*

(i) Then, the following generalized Giaccardi identity holds:

$$\begin{aligned}G_{\mathrm{cardi}}(\Psi)=&\sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\left(\Psi^{(\ell+1)}(\alpha)G_{\mathrm{cardi}}\left(\left(x_{\mathcal{J}}-\alpha\right)^{\ell+2}\right)-\Psi^{(\ell+1)}(\beta)G_{\mathrm{cardi}}\left(\left(x_{\mathcal{J}}-\beta\right)^{\ell+2}\right)\right)\\&+\frac{1}{(\mathfrak{n}-1)!}\int_{\alpha}^{\beta}G_{\mathrm{cardi}}\left(R_{\mathfrak{n}}\left(x_{\mathcal{J}},s\right)\right)\Psi^{(\mathfrak{n})}(s)\mathrm{d}s,\ (\mathrm{GIA.GI}),\end{aligned}\tag{52}$$

where $R_{\mathfrak{n}}(\cdot,s)$ is defined in (5).

(ii) Moreover, if Ψ is $\mathfrak{n}-$convex and the inequality

$$G_{\mathrm{cardi}}\left(R_{\mathfrak{n}}\left(x_{\mathcal{J}},s\right)\right)\leq 0 \tag{53}$$

holds, then we have the following generalized Giaccardi inequality:

$$G_{\mathrm{cardi}}(\Psi)\leq\sum_{\ell=0}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\left(\Psi^{(\ell+1)}(\alpha)G_{\mathrm{cardi}}\left(\left(x_{\mathcal{J}}-\alpha\right)^{\ell+2}\right)-\Psi^{(\ell+1)}(\beta)G_{\mathrm{cardi}}\left(\left(x_{\mathcal{J}}-\beta\right)^{\ell+2}\right)\right). \tag{54}$$

If inequality (53) holds in reverse direction, then (54) also holds reversely.

In the later part of this section, we will vary our conditions on $p_{\mathcal{J}}x_{\mathcal{J}}$ $(\mathcal{J}=1,2,\ldots,\mathfrak{m})$ to obtain generalized converse discrete variants of Jensen's inequality and Giaccardi inequality for $\mathfrak{n}-$convex functions.

Theorem 11. *Let Ψ be as defined in Theorem 9. Also, let $x_{\mathcal{J}} \in [a,b] \subseteq [\alpha,\beta]$ and $\mathbf{p}$ be a positive $\mathfrak{m}-$tuple.*

(i) If Ψ is $\mathfrak{n}-$convex, then for even $\mathfrak{n} \geq 3$, (50) is valid.

(ii) Moreover, if (50) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized converse of Jensen's inequality:

$$\frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}\Psi\left(x_{\mathcal{J}}\right) \leq \frac{b-\overline{x}}{b-a}\Psi(a) + \frac{\overline{x}-a}{b-a}\Psi(b). \tag{55}$$

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ converse of Jensen's inequality (see [32] or [30], p. 98) to obtain results. □

Finally, in this section, we give Giaccardi inequality for higher-order convex functions.

Theorem 12. *Let Ψ be as defined in Theorem 9. Also, let $x_0, x_{\mathcal{J}} \in [a,b]\subseteq[\alpha,\beta]$ and $\mathbf{p}$ be a positive $\mathfrak{m}-$tuple such that*

$$\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}} \neq x_0 \text{ and } \left(x_v - x_0\right)\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}} - x_v\right) \geq 0, \quad (v = 1,\ldots,\mathfrak{m}). \tag{56}$$

(i) If Ψ is $\mathfrak{n}-$convex, then for even $\mathfrak{n} \geq 3$, (53) is valid.

(ii) Moreover, if (53) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized Giaccardi inequality:

$$\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}\Psi\left(x_{\mathcal{J}}\right) \leq A\Psi\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}}\right) + B\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}} - 1\right)\Psi\left(x_0\right), \tag{57}$$

where A and B are defined in (47).

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ Giaccardi inequality (see [36] or [37], p. 11) to obtain results. □

3.3. Applications in Information Theory for Jensen's Discrete Inequality. Jensen's inequality plays a key role in information theory to construct lower bounds for some notable inequalities, but here we will use it to make connections between inequalities in information theory.

Let $\Psi: \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ be a convex function and let $p \coloneqq (p_1,\ldots,p_{\mathfrak{m}})$ and $q \coloneqq (q_1,\ldots,q_{\mathfrak{m}})$ be positive probability distributions; then, Ψ-divergence functional is defined (in [38]) as follows:

$$I_{\Psi}(\mathbf{p},\mathbf{q}) = \sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}}\Psi\left(\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}\right). \tag{58}$$

Horváth et al. in [39] defined the generalized Csiszár divergence functional as follows.

Definition 2. Let I be an interval in $\mathbb{R}$ and $\Psi: I \longrightarrow \mathbb{R}$ be a function. Also, let $\mathbf{p} \coloneqq (p_1,\ldots,p_{\mathfrak{m}}) \in \mathbb{R}^{\mathfrak{m}}$ and $\mathbf{q} \coloneqq (q_1,\ldots,q_{\mathfrak{m}}) \in (0,\infty)^{\mathfrak{m}}$ such that

$$\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}} \in I, \quad \mathcal{J} = 1,\ldots,\mathfrak{m}. \tag{59}$$

Then, let

$$\widetilde{I}_{\Psi}(\mathbf{p},\mathbf{q}) = \sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}}\Psi\left(\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}\right). \tag{60}$$

In this section, we write Jensen's difference here that we use in upcoming results:

$$F\left(\mathbf{p}, x_{\mathcal{J}}, \Psi\right) = \Psi(\overline{x}) - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}\Psi\left(x_{\mathcal{J}}\right). \tag{61}$$

Theorem 13. *Under the assumptions of Theorem 9 (ii), let (51) hold and Ψ be $\mathfrak{n}-$convex. Also, let $\mathbf{p} \coloneqq (p_1,\ldots,p_{\mathfrak{m}})$ in $\mathbb{R}^{\mathfrak{m}}$ and $\mathbf{q} \coloneqq (q_1,\ldots,q_{\mathfrak{m}}) \in (0,\infty)^{\mathfrak{m}}$; then, we have the following results:*

$$\widetilde{I}_{\Psi}(\mathbf{p},\mathbf{q}) \geq P_{\mathfrak{m}}\Psi(1) - P_{\mathfrak{m}}\sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times\left\{\Psi^{(\ell+1)}(\alpha)\left[F\left(\mathbf{q},\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}},(x-\alpha)^{\ell+2}\right)\right] - \Psi^{(\ell+1)}(\beta)\left[F\left(\mathbf{q},\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}},(x-\beta)^{\ell+2}\right)\right]\right\}. \tag{62}$$

Proof. From Theorem 9 by following Jensen's difference (61), we can rearrange (34) as

$$\Psi(\overline{x}) - \frac{1}{P_{\mathfrak{m}}}\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}\Psi(x_{\mathcal{J}}) \leq \sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times\left\{\Psi^{(\ell+1)}(\alpha)\left(F\left(\mathbf{p},x_{\mathcal{J}},(x-\alpha)^{\ell+2}\right)\right) - \Psi^{(\ell+1)}(\beta)\left(F\left(\mathbf{p},x_{\mathcal{J}},(x-\beta)^{\ell+2}\right)\right)\right\}. \tag{63}$$

Now replace $p_{\mathcal{J}}$ with $q_{\mathcal{J}}$ and $x_{\mathcal{J}}$ with $p_{\mathcal{J}}/q_{\mathcal{J}}$, and we get (62). □

For positive $\mathfrak{n}$-tuple $\mathbf{q} = (q_1,\ldots,q_{\mathfrak{m}})$ such that $\sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}} = 1$, the *Shannon entropy* is defined by

$$S(q) = -\sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}}\ln q_{\mathcal{J}}. \tag{64}$$

Corollary 10. *Under the assumptions of Theorem 9 (ii), let (51) hold and Ψ be $\mathfrak{n}$−convex.*

(i) If $\mathbf{q} := (q_1,\ldots,q_{\mathfrak{m}}) \in (0,\infty)^{\mathfrak{m}}$, then

$$\sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}}\ln q_{\mathcal{J}} \leq P_{\mathfrak{m}}\sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times\left\{\frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}}F\left(q,\frac{1}{q_{\mathcal{J}}},-\ln(\cdot)\right) - \frac{(-1)^{\ell+1}\ell!}{\beta^{(\ell+1)}}F\left(q,\frac{1}{q_{\mathcal{J}}},-\ln(\cdot)\right)\right\}. \tag{65}$$

(ii) We can get bounds for the Shannon entropy of $\mathbf{q}$, if we choose $\mathbf{q} := (q_1,\ldots,q_{\mathfrak{n}})$ to be a positive probability distribution.

$$S(\mathbf{q}) \leq P_{\mathfrak{m}}\sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times\left\{\frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}}F\left(\mathbf{q},\frac{1}{q_{\mathcal{J}}},-\ln(\cdot)\right) - \frac{(-1)^{\ell+1}\ell!}{\beta^{(\ell+1)}}F\left(\mathbf{q},\frac{1}{q_{\mathcal{J}}},-\ln(\cdot)\right)\right\}. \tag{66}$$

Proof. (i) Substituting $\Psi(x) := -\ln x$ and using $\mathbf{p} := (1,1,\ldots,1)$ in Theorem 13, we get (65).

(ii) Since we have $\sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}} = 1$, by multiplying −1 on both sides of (65) and taking into account (64), we get (66).

The Kullback–Leibler distance [40] between the positive probability distributions $\mathbf{p} = (p_1,\ldots,p_{\mathfrak{m}})$ and $\mathbf{q} = (q_1,\ldots,q_{\mathfrak{m}})$ is defined by

$$D(\mathbf{q}\,\|\,\mathbf{p}) = \sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}}\ln\left(\frac{q_{\mathcal{J}}}{p_{\mathcal{J}}}\right). \tag{67}$$

□

Corollary 11. *Under the assumptions of Corollary 10,*

(i) If $\mathbf{q} := (q_1,\ldots,q_{\mathfrak{m}}), \mathbf{p} := (p_1,\ldots,p_{\mathfrak{m}}) \in (0,\infty)^{\mathfrak{m}}$, then

$$\sum_{\mathcal{J}=1}^{\mathfrak{m}} q_{\mathcal{J}}\ln\left(\frac{q_{\mathcal{J}}}{p_{\mathcal{J}}}\right) \leq P_{\mathfrak{m}}\sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times\left\{\frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}}F\left(q,\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}},-\ln(\cdot)\right) - \frac{(-1)^{\ell+1}\ell!}{\beta^{(\ell+1)}}F\left(q,\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}},-\ln(\cdot)\right)\right\}. \tag{68}$$

(ii) If $\mathbf{q} := (q_1,\ldots,q_{\mathfrak{m}}), \mathbf{p} := (p_1,\ldots,p_{\mathfrak{m}})$ are positive probability distributions, then we have

$$D(\mathbf{q}\,\|\,\mathbf{p}) \le P_{\mathfrak{m}} \sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times\left\{\frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}}F\left(q,\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}},-\ln(\cdot)\right)-\frac{(-1)^{\ell+1}\ell!}{\beta^{\ell+1}}F\left(q,\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}},-\ln(\cdot)\right)\right\}. \tag{69}$$

Proof.

(i) Using $\Psi(x) := -\ln x$ (which is $\mathfrak{n}$-convex for even $\mathfrak{n}$) in Theorem 13, we get (68) after simplification.

(ii) It is a special case of (i). □

3.4. Results for Zipf and Hybrid Zipf–Mandelbrot Entropy. One of the basic laws in information science is Zipf's law [41,42] which is highly applied in linguistics. Let $c \ge 0$, $d > 0$, and $N \in \{1, 2, \ldots\}$; Zipf–Mandelbrot entropy can be given as

$$Z_M(H,c,d) = \frac{d}{H_{c,d}^N}\sum_{\mathcal{J}=1}^{N}\frac{\ln(\mathcal{J}+c)}{(\mathcal{J}+c)^d} + \ln\left(H_{c,d}^N\right), \tag{70}$$

where

$$H_{c,d}^N = \sum_{\sigma=1}^{N}\frac{1}{(\sigma+c)^d}. \tag{71}$$

Consider

$$q_{\mathcal{J}} = \Psi(\mathcal{J}; N, c, d) = \frac{1}{\left((\mathcal{J}+c)^d H_{c,d}^N\right)}, \tag{72}$$

where $\Psi(\mathcal{J}; m, c, d)$ is discrete probability distribution known as Zipf–Mandelbrot law. Zipf–Mandelbrot law has many application in linguistics and information sciences. Some of the recent study about Zipf–Mandelbrot law can be seen in the listed references (see [39, 43]). Now we state our results involving entropy introduced by Mandelbrot law by establishing the relationship with Shannon and relative entropies.

Theorem 14. *Let* $\mathbf{q}$ *be Zipf–Mandelbrot law as defined in (72) with parameters* $c \ge 0$, $d > 0$, *and* $N \in \{1, 2, \ldots\}$, *and we have*

$$Z_M(H,c,d) = S(\mathbf{q}) \le N \times \sum_{\ell=2}^{\mathfrak{n}-2}\left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)}\right)\times \left\{\frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}}F\left(\mathbf{q},\left((\mathcal{J}+c)^d H_{c,d}^N\right),-\ln(\cdot)\right)-\frac{(-1)^{\ell+1}\ell!}{\beta^{(\ell+1)}}F\left(\mathbf{q},\left((\mathcal{J}+c)^d H_{c,d}^N\right),-\ln(\cdot)\right)\right\}. \tag{73}$$

Proof. It is interesting to see that for $q_{\mathcal{J}}$ defined in (72), $\sum_{\mathcal{J}=1}^N q_{\mathcal{J}} = 1$. Therefore, using above $q_{\mathcal{J}}$ in Shannon entropy (64), we get Mandelbrot entropy (70):

$$S(\mathbf{q}) = -\sum_{\mathcal{J}=1}^{N} q_{\mathcal{J}}\ln q_{\mathcal{J}} = -\sum_{\mathcal{J}=1}^{N}\frac{1}{\left((\mathcal{J}+c)^d H_{c,d}^N\right)}\ln\frac{1}{\left((\mathcal{J}+c)^d H_{c,d}^N\right)} = \frac{d}{\left(H_{c,d}^N\right)}\sum_{\mathcal{J}=1}^{N}\frac{\ln(\mathcal{J}+c)}{(\mathcal{J}+c)^d} + \ln\left(H_{c,d}^N\right). \tag{74}$$

Finally, substituting this $q_{\mathcal{J}} = 1/((\mathcal{J}+c)^d H_{c,d}^N)$ in Corollary 10 (ii), we get the desired result. □

Corollary 12. *Let* $\mathbf{q}$ *and* $\mathbf{p}$ *be Zipf–Mandelbrot law with parameters* $c_1, c_2 \in [0,\infty), d_1, d_2 > 0$, *and let*

$H^N_{c_1,d_1} = \sum_{\sigma=1}^N 1/(\sigma + c_1)^{d_1}$ *and* $H^N_{c_2,d_2} = \sum_{\sigma=1}^N 1/(\sigma + c_2)^{d_2}$. *Now using* $q_{\mathcal{J}} = 1/(\mathcal{J} + c_1)^{d_1} H^N_{c_1,d_1}$ *and* $p_{\mathcal{J}} = 1/(\mathcal{J} + c_2)^{d_2} H^N_{c_2,d_2}$ *in Corollary 11 (ii), the following holds:*

$$\begin{aligned} D(\mathbf{q}\,\|\,\mathbf{p}) &= \sum_{\mathcal{J}=1}^{N} \frac{1}{(\mathcal{J}+c_1)^{d_1} H^N_{c_1,d_1}} \ln\left(\frac{(\mathcal{J}+c_2)^{d_2} H^N_{c_2,d_2}}{(\mathcal{J}+c_1)^{d_1} H^m_{c_1,d_1}}\right) \\ &= -Z(H, c_1, d_1) + \frac{d_2}{H^N_{c_1,d_1}} \sum_{\mathcal{J}=1}^{N} \frac{\ln(\mathcal{J}+c_2)}{(\mathcal{J}+c_1)^{d_1}} + \ln\left(H^m_{c_2,d_2}\right) \\ &\leq N \sum_{\ell=2}^{\mathfrak{n}-2} \left(\frac{1}{\ell!\,(\ell+2)(\beta-\alpha)}\right) \times \left\{ \frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}} F\left(\mathbf{q}, \frac{(\mathcal{J}+c_1)^{d_1} H^N_{c_1,d_1}}{(\mathcal{J}+c_2)^{d_2} H^N_{c_2,d_2}}, -\ln(\cdot)\right) - \frac{(-1)^{\ell+1}\ell!}{\beta^{(\ell+1)}} F\left(\mathbf{q}, \frac{(\mathcal{J}+c_1)^{d_1} H^N_{c_1,d_1}}{(\mathcal{J}+c_2)^{d_2} H^N_{c_2,d_2}}, -\ln(\cdot)\right) \right\}. \end{aligned} \tag{75}$$

The Next Result for Hybrid Zipf–Mandelbrot Entropy. Further generalization of Zipf–Mandelbrot entropy is Hybrid Zipf–Mandelbrot entropy. Let $N \in \{1, 2, \ldots\}$, $c \geq 0\ \omega > 0$; *then, Hybrid Zipf–Mandelbrot entropy can be given as*

$$\widehat{Z}_M(H^*, c, d, \omega) = \frac{1}{H^*_{c,d,\omega}} \sum_{\mathcal{J}=1}^{N} \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^d} \ln\left(\frac{(\mathcal{J}+c)^d}{\omega^{\mathcal{J}}}\right) + \ln\left(H^*_{c,d,\omega}\right), \tag{76}$$

where

$$H^*_{c,d,\omega} = \sum_{\mathcal{J}=1}^{N} \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^d}. \tag{77}$$

Consider

$$q_{\mathcal{J}} = \Psi(\mathcal{J}; N, c, d, \omega) = \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^d H^*_{c,d,\omega}}, \tag{78}$$

which is called Hybrid Zipf–Mandelbrot law. There is a unified approach, maximization of Shannon entropy [44], that naturally follows the path of generalization from Zipf's to Hybrid Zipf's law. Extending this idea, Jakšetic et al. in [45] presented a transition from Zipf–Mandelbrot to Hybrid Zipf–Mandelbrot law by employing maximum entropy technique with one additional constraint. It is interesting that examination of its densities provides some new insights of Lerch's transcendent.

Theorem 15. *Let* $\mathbf{q}$ *be* ***Hybrid Zipf–Mandelbrot*** *law as defined in (78) with parameters* $c \geq 0$, $d, \omega > 0$, *and* $N \in \{1, 2, \ldots\}$, *and we have*

$$\widehat{Z}_M(H^*, c, d, \omega) = S(\mathbf{q}) \leq N \sum_{\ell=2}^{\mathfrak{n}-2} \left(\frac{1}{\ell!\,(\ell+2)(\beta-\alpha)}\right) \times \left\{ \frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}} F\left(\mathbf{q}, \frac{(\mathcal{J}+c)^d H^*_{c,d,\omega}}{\omega^{\mathcal{J}}}, -\ln(\cdot)\right) - \frac{(-1)^{\ell+1}\ell!}{\beta^{(\ell+1)}} F\left(\mathbf{q}, \frac{(\mathcal{J}+c)^d H^*_{c,d,\omega}}{\omega^{\mathcal{J}}}, -\ln(\cdot)\right) \right\}. \tag{79}$$

Proof. It is interesting to see that for $q_{\mathcal{J}}$ defined in (78), $\sum_{\mathcal{J}=1}^{N} q_{\mathcal{J}} = 1$. Therefore, using above $q_{\mathcal{J}}$ in Shannon entropy (64), we get Hybrid Zipf–Mandelbrot entropy (76):

$$\begin{aligned} S(\mathbf{q}) &= -\sum_{\mathcal{J}=1}^{N} q_{\mathcal{J}} \ln q_{\mathcal{J}} = -\sum_{\mathcal{J}=1}^{N} \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^{d} H^{*}_{c,d,\omega}} \ln \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^{d} H^{*}_{c,d,\omega}} \\ &= \frac{-1}{H^{*}_{c,d,\omega}} \sum_{\mathcal{J}=1}^{N} \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^{d}} \left[\ln\left(\frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^{d}} \right) + \ln\left(\frac{1}{H^{*}_{c,d,\omega}} \right) \right] \\ &= \frac{1}{H^{*}_{c,d,\omega}} \sum_{\mathcal{J}=1}^{N} \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^{d}} \left[\ln\left(\frac{(\mathcal{J}+c)^{d}}{\omega^{\mathcal{J}}} \right) + \ln\left(H^{*}_{c,d,\omega} \right) \right] \\ &= \frac{1}{H^{*}_{c,d,\omega}} \sum_{\mathcal{J}=1}^{N} \frac{\omega^{\mathcal{J}}}{(\mathcal{J}+c)^{d}} \ln\left(\frac{(\mathcal{J}+c)^{d}}{\omega^{\mathcal{J}}} \right) + \ln\left(H^{*}_{c,d,\omega} \right). \end{aligned} \tag{80}$$

Finally, substituting this $q_{\mathcal{J}} = \omega^{\mathcal{J}}/(\mathcal{J}+c)^{d} H^{*}_{c,d,\omega}$ in Corollary 10 (ii), we get the desired result. □

Corollary 13. *Let* $\mathbf{q}$ *and* $\mathbf{p}$ *be Hybrid Zipf–Mandelbrot law with parameters* $c_1, c_2 \in [0,\infty), \omega_1, \omega_2, d_1, d_2 > 0$. *Now using* $q_{\mathcal{J}} = \omega_1^{\mathcal{J}}/(\mathcal{J}+c_1)^{d_1} H^{*}_{c_1,d_1,\omega_1}$ *and* $p_{\mathcal{J}} = \omega_2^{\mathcal{J}}/(\mathcal{J}+c_2)^{d_2} H^{*}_{c_2,d_2,\omega_2}$ *in Corollary 11 (ii), the following holds:*

$$\begin{aligned} D(\mathbf{q} \,\|\, \mathbf{p}) &= \sum_{\mathcal{J}=1}^{N} \frac{\omega_1^{\mathcal{J}}}{(\mathcal{J}+c_1)^{d_1} H^{*}_{c_1,d_1,\omega_1}} \ln\left(\frac{\omega_1^{\mathcal{J}} (\mathcal{J}+c_2)^{d_2} H^{*}_{c_2,d_2,\omega_2}}{\omega_2^{\mathcal{J}} (\mathcal{J}+c_1)^{d_1} H^{*}_{c_1,d_1,\omega_1}} \right) \\ &= -\widehat{Z}_M (H^{*}, c_1, d_1, \omega_1) + \frac{1}{H^{*}_{c_1,d_1,\omega_1}} \sum_{\mathcal{J}=1}^{N} \frac{\omega_1^{\mathcal{J}}}{(\mathcal{J}+c_1)^{d_1}} \ln\left(\frac{(\mathcal{J}+c_2)^{d_2}}{\omega_2^{\mathcal{J}}} \right) + \ln\left(H^{*}_{c_2,d_2,\omega_2} \right) \\ &\leq N \sum_{\ell=2}^{\mathfrak{n}-2} \left(\frac{1}{\ell!(\ell+2)(\beta-\alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1}\ell!}{\alpha^{(\ell+1)}} F\left(\mathbf{q}, \frac{\omega_2^{\mathcal{J}} (\mathcal{J}+c_1)^{d_1} H^{*}_{c_1,d_1,\omega_1}}{\omega_1^{\mathcal{J}} (\mathcal{J}+c_2)^{d_2} H^{*}_{c_2,d_2,\omega_2}}, -\ln(\cdot) \right) \right. \\ &\quad \left. - \frac{(-1)^{\ell+1}\ell!}{\beta^{(\ell+1)}} F\left(\mathbf{q}, \frac{\omega_2^{\mathcal{J}} (\mathcal{J}+c_1)^{d_1} H^{*}_{c_1,d_1,\omega_1}}{\omega_1^{\mathcal{J}} (\mathcal{J}+c_2)^{d_2} H^{*}_{c_2,d_2,\omega_2}}, -\ln(\cdot) \right) \right\}. \end{aligned} \tag{81}$$

Remark 9. Similarly, we can give results for Shannon entropy, Kullback–Leibler distance, Zipf–Mandelbrot entropy, and Hybrid Zipf–Mandelbrot entropy by using generalized Giaccardi inequality defined in (54) on the same steps.

4. Concluding Remarks

In this paper, we gave generalization of Jensen's inequality as well as converse of Jensen's inequality by using Montgomery identity. We also formulate results for other inequalities like Jensen–Steffensen inequality, Jensen–Boas inequality, and Jensen–Brunk inequality. We can obtain Jensen–Steffensen inequality, Jensen–Boas inequality, and Jensen–Brunk inequality by changing the assumption of Jensen's inequality. At the end, we gave applications in information theory for our obtained results, especially we gave results for Hybrid Zipf–Mandelbrot entropy for our obtained results [46].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] X. L. Zhu and G. H. Yang, "Jensen inequality approach to stability analysis of discrete-time systems with time-varying delay," in *Proceedings of the American Control Conference*, pp. 1644–1649, Seattle, WA, USA, June 2008.

[2] S. I. Butt, M. Klaričić Bakula, Đ. Pečarić, and J. Pečarić, "Jensen-grüss inequality and its applications for the zipf-mandelbrot law," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 2, pp. 1664–1673, 2021.

[3] M. A. Khan, D. Pečarić, and J. Pečarić, "New refinement of the Jensen inequality associated to certain functions with applications," *Journal of Inequalities and Applications*, vol. 76, 2020.

[4] S. Khan, M. Adil Khan, S. I. Butt, and Y.-M. Chu, "A new bound for the Jensen gap pertaining twice differentiable functions with applications," *Advances in Difference Equations*, vol. 1, 2020.

[5] H. Budak and M. Z. Sarikaya, "On generalized Ostrowski-type inequalities for functions whose first derivatives absolute values are convex," *Turkish Journal of Mathematics*, vol. 40, pp. 1193–1210, 2016.

[6] P. Cerone and S. S. Dragomir, "On some inequalities arising from Montgomery's identity," *Journal of Computational Analysis and Applications*, vol. 5, no. 4, pp. 341–367, 2003.

[7] S. Bernstein, "Sur les fonctions absolument monotones," *Acta Mathematica*, vol. 52, pp. 1–66, 1929.

[8] V. Čuljak and J. Pečarić, "Interpolation polynomials and inequalities for convex functions of higher order," *Acta Mathematica*, vol. 5, pp. 369–386, 2012.

[9] A. M. Fink, "Bounds of the deviation of a function from its averages," *Czechoslovak Mathematical Journal*, vol. 42, no. 117, pp. 289–310, 1992.

[10] J. Pečarić and V. Čuljak, "Interpolation polinomials and inequalities for convex functions of higher order," *Mathematical Inequalities & Applications*, vol. 5, no. 3, pp. 369–386, 2002.

[11] J. Pečarić, V. Čuljak, and A. M. Fink, "On some inequalities for convex function of higher order," *Advanced Nonlinear Studies*, vol. 6, no. 2, pp. 131–140, 1999.

[12] J. Pečarić, V. Čuljak, and M. Rogina, "On some inequalities for convex function of higher order II," *Advanced Nonlinear Studies*, vol. 45, pp. 281–294, 2001.

[13] J. Pečarić, I. Perić, and M. Rodić-Lipanović, "Uniform treatment of Jensen type inequalities," *Mathematical Reports*, vol. 16, no. 66, pp. 183–205, 2014.

[14] J. Pečarić and J. Perić, "Improvement of the Giaccardi and the petrović inequality and related stolarsky type means," *Annals of the University of Craiova - Mathematics and Computer Science*, vol. 39, no. 1, pp. 65–75, 2012.

[15] J. Pečarić, M. Praljak, and A. Witkowski, "Linear operator inequality for n−convex functions at a point," *Mathematical Inequalities & Applications Appl*, vol. 18, pp. 1201–1217, 2015.

[16] Z. Pavić, "Refinements of Jensen's inequality for infinite convex combinations," *Turkish Journal of Inequalities*, vol. 2, no. 2, pp. 44–53, 2018.

[17] R. Bibi, A. Nosheen, and J. Pečarić, "Extended Jensen's type inequalities for diamond integrals via Taylors formula," *Turkish Journal of Inequalities*, vol. 3, no. 1, pp. 7–18, 2019.

[18] T. Niaz, K. A. Khan, and J. Pečarić, "On refinement of Jensen's inequality for 3−convex function at a point," *Turkish Journal of Inequalities*, vol. 4, no. 1, pp. 70–80, 2020.

[19] A. Ekinci, A. O. Akdemir, and M. E. Ozdemir, "Integral inequalities for different kinds of convexity via classical inequalities," *Turkish Journal of Science*, vol. 5, no. 3, pp. 305–313, 2020.

[20] Y. M. Chu, S. Talib, E. Set, M. U. Awan, and M. A. Noor, "(p, q)− Analysis of Montgomery identity and estimates of (p, q)−bounds with applications," *Journal of Inequalities and Applications*, vol. 2021, 9 pages, 2021.

[21] M. Z. Sarikaya, H. Yaldrz, and E. Set, "On fractional inequalities via Montgomery identities," *International Journal of Open Problems in Complex Analysis*, vol. 6, no. 2, pp. 36–43, 2014.

[22] A. Bnouhachem, "A descent SQP alternating direction method for minimizing the sum of three convex functions," *Journal of Nonlinear Functional Analysis*, vol. 4, pp. 469–482, 2020.

[23] A. O. Akdemir, S. I. Butt, M. Nadeem, and M. A. Ragusa, "New general variants of Chebyshev type inequalities via generalized fractional integral operators," *Mathematics*, vol. 92, 2021.

[24] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir, and Y. Ming Chu, "New Hermite Jensen Mercer type inequalities via k−fractional integrals," *Advances in Difference Equations*, vol. 635, 2020.

[25] S. I. Butt, M. Nadeem, S. Qaisar, A. O. Akdemir, and T. Abdeljawad, "Hermite Jensen Mercer type inequalities for conformable integrals and related results," *Advances in Difference Equations*, vol. 501, 2020.

[26] A. Bnouhachem and X. Qin, "An inertial proximal Peaceman-Rachford splitting method with SQP regularization for convex programming," *Journal of Nonlinear Functional Analysis*, vol. 50, 2020.

[27] S. Hamann, "Minimality conditions for convex composite functions and efficiency conditions in vector optimization," *Applied Set-Valued Analysis and Optimization*, vol. 1, pp. 221–229, 2019.

[28] A. R. Khan, J. Pečarić, and M. Praljak, "Popoviciu type inequalities for n-convex functions via extension of Montgomery identity$n$−convex functions via extension of Montgomery identity," *Analele Universitatii "Ovidius" Constanta-Seria Matematica*, vol. 24, no. 3, pp. 161–188, 2016.

[29] R. P. Boas, "The Jensen-Steffensen inequality," *Elektrotehn. Fak. Ser. Mat. Fiz*, vol. 302–319, pp. 1–8, 1970.

[30] J. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, NY, USA, 1992.

[31] H. D. Brunk, "On an inequality for convex functions," *Proceedings of the American Mathematical Society*, vol. 7, no. 5, p. 817, 1956.

[32] P. R. Beesack and J. E. Pečarić, "On Jessen's inequality for convex functions," *Journal of Mathematical Analysis and Applications*, vol. 110, no. 2, pp. 536–552, 1985.

[33] R. Barlow, A. Marshall, and F. Proschan, "Some inequalities for starshaped and convex functions," *Pacific Journal of Mathematics*, vol. 29, no. 1, pp. 19–42, 1969.

[34] J. E. Pečarić, "On an inequality of PMV Janić," *Publications de l'Institut Mathématique*, vol. 28, no. 42, pp. 145–149, 1980.
[35] G. Szegö, "Über eine Verallgemeinerung des Dirichletschen Integrals," *Mathematische Zeitschrift*, vol. 52, no. 1, pp. 676–685, 1950.
[36] F. Giaccardi, "Su alcune disuguaglianie," *Journal of Research of the National Bureau of Standards*, vol. 1, no. 4, pp. 139–153, 1955.
[37] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and new inequalities in analysis. Mathematics and its Applications (East European Series)*, Vol. 61, Kluwer Academic Publishers Group, Dordrecht, Netherlands, 1993.
[38] I. Csiszár, "Information-type measures of diference of probability distributions and indirect observations," *Studia Science Math Hungar*, vol. 2, pp. 299–318, 1967.
[39] L. Horváth, D. Pečarić, and J. Pečarić, "Estimations of f - and rényi divergences by using a cyclic refinement of the Jensen's inequality," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, 2017.
[40] S. Kullback, *Information Theory and Statistics*, Wiley, New York, NY, USA, 1959.
[41] S. T. Piantadosi, "Zipf's word frequency law in natural language: a critical review and future directions," *Psychonomic Bulletin and Review*, vol. 21, no. 5, pp. 1112–1130, 2014.
[42] Z. K. Silagadze, "Citations and the zipf–mandelbrot law," *Complex Systems*, no. 11, pp. 487–499, 1997.
[43] J. Jakšetic, D. Pečarić, and J. Pečarić, "Some properties of zipf-mandelbrot law and hurwitz ζ-function," *Mathematical Inequalities & Applications*, vol. 21, no. 2, pp. 575–584, 2018.
[44] M. Visser, "Zipf's law, power laws and maximum entropy," *Mathematical Inequalities & Applications*, vol. 15, Article ID 043021, 2013.
[45] J. Jakšetic, D. Pečarić, and J. Pečarić, "Hybrid zipf-mandelbrot law," *Mathematical Inequalities & Applications*, vol. 13, no. 1, pp. 275–286, 2019.
[46] D. V. Widder, "Completely convex functions and Lidstone series," *Transactions of the American Mathematical Society*, vol. 51, p. 387, 1942.

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Research Article

The Hermite–Hadamard–Jensen–Mercer Type Inequalities for Riemann–Liouville Fractional Integral

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In this paper, we give Hermite–Hadamard type inequalities of the Jensen–Mercer type for Riemann–Liouville fractional integrals. We prove integral identities, and with the help of these identities and some other eminent inequalities, such as Jensen, Hölder, and power mean inequalities, we obtain bounds for the difference of the newly obtained inequalities.

1. Introduction

The concept of convex functions plays a vital role in both pure and applied mathematics. Convex functions also have many applications in other branches of science such as finance, economics, and engineering.

Definition 1 (see [1]). A function $\psi\colon [m, M] \longrightarrow \mathbb{R}$ is convex if

$$\psi(sx + (1-s)y) \leq s\psi(x) + (1-s)\psi(y), \tag{1}$$

for all $x, y \in [m, M]$ and $s \in [0, 1]$.

If the inequality in (1) is strict for $x \neq y$, then ψ is said to be a strictly convex function, and if $-\psi$ is convex, then ψ is said to be a concave function [2, 3].

Many important inequalities such as Jensen, Jensen–Mercer, Hermite–Hadamard, and support line inequalities hold for convex functions. The classical Jensen's inequality is among the most prominent inequalities stated as follows [4, 5].

If $\psi\colon [m, M] \longrightarrow \mathbb{R}$ is convex, then

$$\psi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \psi(x_i), \tag{2}$$

for all $x_i \in [m, M]$ and $w_i \in [0, 1]\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$.

In [6], Mercer presented a type of Jensen's inequality called Jensen–Mercer inequality.

Theorem 1. *If $\psi\colon [m, M] \longrightarrow \mathbb{R}$ is convex, then*

$$\psi\left(m + M - \sum_{i=1}^{n} w_i x_i\right) \leq \psi(m) + \psi(M) - \sum_{i=1}^{n} w_i \psi(x_i), \tag{3}$$

for each $x_i \in [m, M]$ and $w_i \in [0, 1]\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$.

For a convex function, there exist at least one line lies on or below the graph of the function.

Definition 2 (see [7]). A function $\psi\colon I \longrightarrow \mathbb{R}$ has a support at $x_0 \in I$ if

$$\psi(x_0) + c(u - x_0) \leq \psi(u), \tag{4}$$

for all $x_0 \in I$ and for each $u \in [m, M] \subset I$. Inequality (4) is said to be the support line inequality.

The following theorem connects the support line inequality with convex functions.

Theorem 2 (see [7]). *ie function ψ: $[m, M] \longrightarrow \mathbb{R}$ is convex if and only if ψ has at least one line of support at each $x_0 \in [m, M]$.*

The Hermite–Hadamard inequality is one of the most investigated inequality in the theory of convex functions due to its geometrical significance and applications. Because of the importance of Hermite–Hadamard inequality, there is an ample amount of research work dedicated to the extensions, generalizations, refinements, and applications of the Hermite–Hadamard inequality. The Hermite–Hadamard inequality is given below [8].

Let ψ: $I \longrightarrow \mathbb{R}$ be a convex function, where I is an interval and $m, M \in I$ such that $m < M$. Then,

$$\psi\left(\frac{m+M}{2}\right) \leq \frac{1}{M-m}\int_m^M \psi(x)\mathrm{d}x \leq \frac{\psi(m)+\psi(M)}{2}. \tag{5}$$

If ψ is concave, then (5) holds in the reversed direction. For more results associated with Hermite–Hadamard inequality, see [9–18].

The Hermite–Hadamard inequality has been extended by means of fractional integral operators. Most popular of them is the Riemann–Liouville fractional operator given in the following definition [19–22].

Definition 3 (see [23, 24]). Let ψ be an integrable function defined on $[m, M]$. Then, the integrals $J^{\alpha}_{m^+}\psi(x)$ and $J^{\alpha}_{M^-}\psi(x)$ defined by

$$J^{\alpha}_{m^+}\psi(x) = \frac{1}{\Gamma(\alpha)}\int_m^x (x-s)^{\alpha-1}\psi(s)\mathrm{d}s, \quad x > m, \tag{6}$$

$$J^{\alpha}_{M^-}\psi(x) = \frac{1}{\Gamma(\alpha)}\int_x^M (s-x)^{\alpha-1}\psi(s)\mathrm{d}s, \quad x < M, \tag{7}$$

are called the left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ respectively. Here, Γ represents gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-s}s^{\alpha-1}\mathrm{d}s$.

In [25, 26], authors used the following lemmas to obtain trapezoidal and midpoint type inequalities.

Lemma 1 (see [25]). *Let ψ: $I^{\circ} \longrightarrow \mathbb{R}$ (where I° is the interior of I) be a differentiable function and $m, M \in I^{\circ}$ such that $m < M$. If $\psi' \in L[m, M]$, then*

$$\frac{\psi(m)+\psi(M)}{2} - \frac{1}{M-m}\int_m^M \psi(u)\mathrm{d}u = \frac{M-m}{2}\int_0^1 (1-2s)\psi'(sm+(1-s)M)\mathrm{d}s. \tag{8}$$

Lemma 2 (see [26]). *Let all the assumptions of Lemma 1 hold. Then,*

$$\frac{1}{M-m}\int_m^M \psi(u)\mathrm{d}u - \psi\left(\frac{m+M}{2}\right) = (M-m)\left(\int_0^{(1/2)} s\psi'(sm+(1-s)M)\mathrm{d}s + \int_{(1/2)}^1 (s-1)\psi'(sm+(1-s)M)\mathrm{d}s\right). \tag{9}$$

In this article, we establish fractional Hermite–Hadamard–Jensen–Mercer type inequalities. We give identities involving fractional integrals, and from these identities, we derive trapezoidal and midpoint type inequalities.

Throughout this article, α represents a positive real number.

2. Main Results

We begin this section with our first main result.

Theorem 3. *Suppose ψ: $[m, M] \longrightarrow \mathbb{R}$ is a convex function and $x, y \in [m, M]$ such that $x < y$. Then,*

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\leq\psi(m)+\psi(M)-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{x^{+}}\psi(y)$$
$$\leq\psi(m)+\psi(M)-\psi\left(\frac{\alpha x+y}{\alpha+1}\right), \tag{10}$$

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)$$
$$\leq\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}$$
$$\leq\psi(m)+\psi(M)-\frac{\alpha\psi(x)+\psi(y)}{\alpha+1}. \tag{11}$$

Proof. Since ψ is convex, it has support line at each point $x_0\in[m,M]$, that is,

$$\psi(x_0)+c(u-x_0)\leq\psi(u), \tag{12}$$

for each $u\in[m,M]$. Substituting $x_0=m+M-(\alpha x+y)/(\alpha+1)$ and $u=m+M-sx-(1-s)y$, where $s\in[0,1]$, in inequality (12), we obtain

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)+c\left[-sx-(1-s)y+\frac{\alpha x+y}{\alpha+1}\right]$$
$$\leq\psi(m+M-sx-(1-s)y). \tag{13}$$

Multiplying (13) with $\alpha s^{\alpha-1}$ and integrate with respect to s, we obtain

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)+c\left[-\frac{\alpha x+y}{\alpha+1}+\frac{\alpha x+y}{\alpha+1}\right]$$
$$\leq\alpha\int_o^1 s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s$$
$$\Longrightarrow\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\leq\alpha\int_o^1 s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s. \tag{14}$$

Using Mercer's inequality, we obtain

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\leq\alpha\int_o^1 s^{\alpha-1}(\psi(m)+\psi(M)-(s\psi(x)+(1-s)\psi(y)))\mathrm{d}s$$
$$\Longrightarrow\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\leq\psi(m)+\psi(M)-\alpha\int_o^1 s^{\alpha-1}(s\psi(x)+(1-s)\psi(y))\mathrm{d}s. \tag{15}$$

Since ψ is convex, we have $-(s\psi(x)+(1-s)\psi(y))\leq-\psi(sx+y(1-s))$ and (15) becomes

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\leq\psi(m)+\psi(M)-\alpha\int_o^1 s^{\alpha-1}\psi(sx+(1-s)y)\mathrm{d}s. \tag{16}$$

Substituting $sx + (1-s)y = w$ in (16), we obtain

$$\psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq \psi(m) + \psi(M) - \frac{\alpha}{(y-x)^{\alpha}} \int_x^y (y-w)^{\alpha-1}\psi(w)\mathrm{d}w$$
$$\Longrightarrow \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq \psi(m) + \psi(M) - \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{x^+}^{\alpha}\psi(y). \tag{17}$$

Now, we prove the second inequality of (10). Put $x_0 = (\alpha x + y)/(\alpha + 1)$ and $u = sx + (1-s)y$ in (12), we obtain

$$\psi\left(\frac{\alpha x + y}{\alpha + 1}\right) + c\left[sx + (1-s)y - \frac{\alpha x + y}{\alpha + 1}\right] \leq \psi(sx + (1-s)y). \tag{18}$$

Multiplying the above inequality with $\alpha s^{\alpha-1}$ and integrating and using $\int_0^1 s^{\alpha-1}\mathrm{d}s = 1/\alpha$ and $\int_0^1 s^{\alpha} ds = 1/(\alpha+1)$, we obtain

$$\psi\left(\frac{\alpha x + y}{\alpha + 1}\right) \leq \alpha \int_0^1 s^{\alpha-1}\psi(sx + (1-s)y)\mathrm{d}s. \tag{19}$$

Put $sx + (1-s)y = w$, and we obtain

$$\psi\left(\frac{\alpha x + y}{\alpha + 1}\right) \leq \frac{\alpha}{(y-x)^{\alpha}} \int_x^y (y-w)^{\alpha-1}\psi(w)\mathrm{d}w$$
$$\Longrightarrow \psi\left(\frac{\alpha x + y}{\alpha + 1}\right) \leq \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{x^+}^{\alpha}\psi(y)$$
$$\Longrightarrow -\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{x^+}^{\alpha}\psi(y) \leq -\psi\left(\frac{\alpha x + y}{\alpha + 1}\right). \tag{20}$$

Adding $\psi(m) + \psi(M)$ on both sides of (20), we obtain

$$\psi(m) + \psi(M) - \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{x^+}^{\alpha}\psi(y) \leq \psi(m) + \psi(M) - \psi\left(\frac{\alpha x + y}{\alpha + 1}\right), \tag{21}$$

and on combining (17) and (21), we obtain (10).

Now, we prove the inequalities in (11). Let $u = m + M - sx - (1-s)y \Longrightarrow s = (u - m - M + y)/(y-x)$ and (14) become

$$\psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq \alpha \int_{m+M-y}^{m+M-x} \left(\frac{u - m - M + y}{y-x}\right)^{\alpha-1} \psi(u) \frac{\mathrm{d}u}{y-x}$$
$$= \frac{\alpha\Gamma(\alpha)}{(y-x)^{\alpha}} \frac{1}{\Gamma(\alpha)} \int_{m+M-y}^{m+M-x} (u - (m+M-y))^{\alpha-1}\psi(u)\mathrm{d}u$$
$$\Longrightarrow \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{(m+M-x)^-}^{\alpha}\psi(m + M - y). \tag{22}$$

Now, we prove the other two inequalities of (11). As ψ is a convex function, we have

$$\psi(m + M - sx - (1-s)y) = \psi(s(m+M-x) + (1-s)(m+M-y))$$
$$\Longrightarrow \psi(m + M - sx - (1-s)y) \leq s\psi(m+M-x) + (1-s)\psi(m+M-y)$$
$$\leq \psi(m) + \psi(M) - s\psi(x) - (1-s)\psi(y). \tag{23}$$

Multiplying with $\alpha s^{\alpha-1}$ and integrating, we obtain

$$
\begin{aligned}
&\alpha \int_0^1 s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s \\
&\le \alpha\psi(m+M-x)\int_0^1 s^{\alpha}\mathrm{d}s + \alpha\psi(m+M-y)\int_0^1\left(s^{\alpha-1}-s^{\alpha}\right)\mathrm{d}s \\
&\le \psi(m)+\psi(M)-\alpha\psi(x)\int_0^1 s^{\alpha}\mathrm{d}s - \alpha\psi(y)\int_0^1\left(s^{\alpha-1}-s^{\alpha}\right)\mathrm{d}s \\
&\Longrightarrow \alpha\int_0^1 s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s \\
&\le \frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1} \\
&\le \psi(m)+\psi(M)-\frac{\alpha\psi(x)+\psi(y)}{\alpha+1}.
\end{aligned}
\tag{24}
$$

By changing of variable, (24) becomes

$$
\begin{aligned}
&\alpha\int_{m+M-y}^{m+M-x}\left(\frac{u-m-M+y}{y-x}\right)^{\alpha-1}\psi(u)\frac{\mathrm{d}u}{y-x} \\
&\le \frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1} \le \psi(m)+\psi(M)-\frac{\alpha\psi(x)+\psi(y)}{\alpha+1} \\
&\Longrightarrow \frac{\alpha\Gamma(\alpha)}{(y-x)^{\alpha}}\frac{1}{\Gamma(\alpha)}\int_{m+M-y}^{m+M-x}(u-(m+M-y))^{\alpha-1}\psi(u)\mathrm{d}u \\
&\le \frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1} \le \psi(m)+\psi(M)-\frac{\alpha\psi(x)+\psi(y)}{\alpha+1} \\
&\Longrightarrow \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y) \\
&\le \frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1} \le \psi(m)+\psi(M)-\frac{\alpha\psi(x)+\psi(y)}{\alpha+1},
\end{aligned}
\tag{25}
$$

and on combining (22) and (25), we obtain (11). □

Remark 1. If we put $\alpha = 1$ in Theorem 3 and in the obtained expressions substitute $u = sx + (1-s)y$ and $u = m + M - v$, respectively, we obtain

$$\psi\left(m + M - \frac{x+y}{2}\right) \leq \psi(m) + \psi(M) - \int_0^1 \psi(sx + (1-s)y)\mathrm{d}s \leq \psi(m) + \psi(M) - \psi\left(\frac{x+y}{2}\right), \tag{26}$$

$$\begin{aligned}\psi\left(m + M - \frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_x^y \psi(m+M-v)dv\\ &\leq \frac{\psi(m+M-x) + \psi(m+M-y)}{2}\\ &\leq \psi(m) + \psi(M) - \frac{\psi(x)+\psi(y)}{2},\end{aligned} \tag{27}$$

respectively. Inequalities (26) and (27) have been proved in [27].

Remark 2. Substituting $\alpha = 1$, $x = m$, and $y = M$ in (11), one can obtain Hermite–Hadamard inequality.

3. Bounds for the Difference of Hermite–Hadamard–Jensen–Mercer Type Inequalities

Throughout this section, we consider $\psi\colon [m, M] \longrightarrow \mathbb{R}$ is a differentiable function. To give the bounds for the difference of Hermite–Hadamard–Jensen–Mercer type inequalities, first, we present the following lemmas.

Lemma 3. *Let $x, y \in [m, M]$ such that $x < y$ and let $\psi' \in L[m, M]$. Then,*

$$\begin{aligned}&\frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\\ &= \frac{y-x}{\alpha+1}\int_0^1 \left((\alpha+1)s^{\alpha} - 1\right)\psi'(m+M-sx-(1-s)y)\mathrm{d}s.\end{aligned} \tag{28}$$

Proof. Using the techniques of integration, we have

$$\begin{aligned}
&\frac{y-x}{\alpha+1}\int_0^1\left[(\alpha+1)s^\alpha-1\right]\psi'(m+M-sx-(1-s)y)\mathrm{d}s\\
&=(y-x)\int_0^1 s^\alpha\psi'(m+M-sx-(1-s)y)\mathrm{d}s\\
&\quad-\frac{y-x}{\alpha+1}\int_0^1\psi'(m+M-sx-(1-s)y)\mathrm{d}s\\
&=s^\alpha\psi(m+M-sx-(1-s)y)\big|_0^1-\alpha\int_0^1 s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s\\
&\quad-\frac{1}{\alpha+1}\psi(m+M-sx-(1-s)y)\big|_0^1,
\end{aligned}\tag{29}$$

$$\begin{aligned}
&=\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\alpha\int_0^1 s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s\\
&=\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}\\
&\quad-\frac{\alpha}{(y-x)^\alpha}\int_{m+M-y}^{m+M-x}(u-(m+M-y))^{\alpha-1}\psi(u)\mathrm{d}u\\
&=\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^\alpha}J^\alpha_{(m+M-x)^-}\psi(m+M-y).
\end{aligned}\tag{30}$$

□

Remark 3. Substituting $\alpha=1$, $x=m$, and $y=M$ in (28), we obtain (8).

Lemma 4. *Let all the assumptions of Lemma 3 hold. Then,*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{(y-x)^\alpha}J^\alpha_{(m+M-x)^-}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\\
&=(y-x)\left(\int_{\alpha/(\alpha+1)}^1(1-s^\alpha)\psi'(m+M-sx-(1-s)y)\mathrm{d}s-\int_0^{\alpha/(\alpha+1)}s^\alpha\psi'(m+M-sx-(1-s)y)\mathrm{d}s\right).
\end{aligned}\tag{31}$$

Proof. Using techniques of integration, we have

$$\begin{aligned}
&(y-x)\int_{\alpha/(\alpha+1)}^{1}(1-s^{\alpha})\psi'(m+M-sx-(1-s)y)\mathrm{d}s\\
&\quad-(y-x)\int_{0}^{\alpha/(\alpha+1)}s^{\alpha}\psi'(m+M-sx-(1-s)y)\mathrm{d}s\\
&=(y-x)\left[\int_{\alpha/(\alpha+1)}^{1}\psi'(m+M-sx-(1-s)y)\mathrm{d}s\right]\\
&\quad-(y-x)\left[\int_{0}^{1}s^{\alpha}\psi'(m+M-sx-(1-s)y)\mathrm{d}s\right]\\
&=\psi(m+M-sx-(1-s)y)|_{\alpha/(\alpha+1)}^{1}-s^{\alpha}\psi(m+M-sx-(1-s)y)|_{0}^{1}\\
&\quad+\alpha\int_{0}^{1}s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s\\
&=-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)+\alpha\int_{0}^{1}s^{\alpha-1}\psi(m+M-sx-(1-s)y)\mathrm{d}s\\
&=\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right).
\end{aligned} \tag{32}$$

□

Remark 4. If we put $\alpha=1$, $x=m$, and $y=M$ in (31), we obtain (9).

We use Lemmas 3 and 4 and obtain bounds for the difference of the inequalities in (11).

Theorem 4. *Let $|\psi'|$ be a convex function defined on $[m,M]$ and let $x,y\in[m,M]$ such that $x<y$. Then,*

$$\begin{aligned}
&\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)\right|\\
&\leq\frac{y-x}{\alpha+1}\left[P_1(\alpha)|\psi'(m)|+P_1(\alpha)|\psi'(M)|-P_2(\alpha)|\psi'(x)|-P_3(\alpha)|\psi'(y)|\right],
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
P_1(\alpha)&=\frac{2\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}},\\
P_2(\alpha)&=\frac{\alpha\left[2+(\alpha+1)^{2/\alpha}\right]}{2(\alpha+2)(\alpha+1)^{2/\alpha}},\\
P_3(\alpha)&=\frac{\alpha\left[4(\alpha+2)(\alpha+1)^{(1/\alpha)-1}-(\alpha+1)^{2/\alpha}-2\right]}{2(\alpha+2)(\alpha+1)^{2/\alpha}}.
\end{aligned} \tag{34}$$

Proof. From Lemma 3, we have

$$\begin{aligned}&\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|\\&\leq\frac{y-x}{\alpha+1}\int_0^1\left|(\alpha+1)s^{\alpha}-1\right|\left|\psi'(m+M-sx-(1-s)y)\right|\mathrm{d}s.\end{aligned}\tag{35}$$

Since $|\psi'|$ is convex, using Mercer's inequality, we obtain

$$\begin{aligned}&\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|\\&\leq\frac{y-x}{\alpha+1}\int_0^1\left|((\alpha+1)s^{\alpha}-1)\right|\left(\left|\psi'(m)\right|+\left|\psi'(M)\right|-s\left|\psi'(x)\right|-(1-s)\left|\psi'(y)\right|\right)\mathrm{d}s\\&=\frac{y-x}{\alpha+1}\left(\int_0^{1/(\alpha+1)^{1/\alpha}}(1-(\alpha+1)s^{\alpha})\left(\left|\psi'(m)\right|+\left|\psi'(M)\right|-s\left|\psi'(x)\right|-(1-s)\left|\psi'(y)\right|\right)\mathrm{d}s\right.\\&\left.+\int_{1/(\alpha+1)^{1/\alpha}}^1((\alpha+1)s^{\alpha}-1)\left(\left|\psi'(m)\right|+\left|\psi'(M)\right|-s\left|\psi'(x)\right|-(1-s)\left|\psi'(y)\right|\right)\mathrm{d}s\right),\end{aligned}\tag{36}$$

equivalent to

$$\begin{aligned}&\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|\\&\leq\frac{y-x}{\alpha+1}\Big(\left[L_1(\alpha)+N_1(\alpha)\right]\left|\psi'(m)\right|+\left[L_1(\alpha)+N_1(\alpha)\right]\left|\psi'(M)\right|\\&-\left[L_2(\alpha)+N_2(\alpha)\right]\left|\psi'(x)\right|-\left[L_3(\alpha)+N_3(\alpha)\right]\left|\psi'(y)\right|\Big),\end{aligned}\tag{37}$$

where

$$L_1(\alpha)=\int_0^{1/(\alpha+1)^{1/\alpha}}\left[1-(\alpha+1)s^{\alpha}\right]\mathrm{d}s=\frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}},$$

$$L_2(\alpha)=\int_0^{1/(\alpha+1)^{1/\alpha}}\left[1-(\alpha+1)s^{\alpha}\right]s\mathrm{d}s=\frac{\alpha}{2(\alpha+2)(\alpha+1)^{2/\alpha}},$$

$$L_3(\alpha)=\int_0^{1/(\alpha+1)^{1/\alpha}}\left(1-(\alpha+1)s^{\alpha}\right)(1-s)\mathrm{d}s=\frac{\alpha\left(2(\alpha+2)(\alpha+1)^{(1/\alpha)-1}-1\right)}{2(\alpha+2)(\alpha+1)^{2/\alpha}},$$

$$N_1(\alpha)=\int_{1/(\alpha+1)^{1/\alpha}}^1\left[(\alpha+1)s^{\alpha}-1\right]\mathrm{d}s=\frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{38}$$

$$N_2(\alpha)=\int_{1/(\alpha+1)^{1/\alpha}}^1\left[(\alpha+1)s^{\alpha}-1\right]s\mathrm{d}s=\frac{\alpha\left((\alpha+1)^{2/\alpha}+1\right)}{2(\alpha+2)(\alpha+1)^{(2/\alpha)}},$$

$$N_3(\alpha)=\int_{1/(\alpha+1)^{1/\alpha}}^1\left[(\alpha+1)s^{\alpha}-1\right](1-s)\mathrm{d}s$$

$$=\frac{\alpha\left(2(\alpha+2)(\alpha+1)^{(1/\alpha)-1}-(\alpha+1)^{2/\alpha}-1\right)}{2(\alpha+2)(\alpha+1)^{2/\alpha}}.$$

Substituting these values in (37), we get (33). □

Remark 5. If we put $\alpha=1$, $x=m$, and $y=M$ in (33), we get the inequality given in Theorem 2.2 of [25].

Theorem 5. *Let $|\psi'|^q$ be a convex function for $q\geq 1$ and let $x,y\in[m,M]$ such that $x<y$. Then,*

$$\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|$$
$$\leq\frac{y-x}{\alpha+1}\left(P_1(\alpha)\right)^{1-1/q} \tag{39}$$
$$\times\left(P_1(\alpha)|\psi'(m)|^q+P_1(\alpha)|\psi'(M)|^q-P_2(\alpha)|\psi'(x)|^q-P_3(\alpha)|\psi'(y)|^q\right)^{1/q},$$

where $P_1(\alpha)$, $P_2(\alpha)$, and $P_3(\alpha)$ are the same as defined in Theorem 4.

Proof. From Lemma 3, we have (35). Applying power mean inequality, we obtain

$$\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|$$
$$\leq\frac{y-x}{\alpha+1}\left(\int_0^1\left|(\alpha+1)s^{\alpha}-1\right|\mathrm{d}s\right)^{1-1/q} \tag{40}$$
$$\times\left(\int_0^1\left|((\alpha+1)s^{\alpha}-1)\right|\left|\psi'(m+M-sx-(1-s)y)\right|^q\mathrm{d}s\right)^{1/q}.$$

Since $|\psi'|^q$ is convex, using Mercer's inequality, we have

$$\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|$$
$$\leq\frac{y-x}{\alpha+1}\left(\int_0^{1/(\alpha+1)^{1/\alpha}}\left[1-(\alpha+1)s^{\alpha}\right]ds+\int_{1/(\alpha+1)^{1/\alpha}}^1\left[(\alpha+1)s^{\alpha}-1\right]ds\right)^{1-1/q}$$
$$\times\left(\int_0^1|((\alpha+1)s^{\alpha}-1)|\left(|\psi'(m)|^q+|\psi'(M)|^q-s|\psi'(x)|^q-(1-s)|\psi'(y)|^q\right)ds\right)^{1/q}$$
$$=\frac{y-x}{\alpha+1}\left(\frac{2\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}\right)^{1-(1/q)}$$
$$\times\left(\int_0^{1/(\alpha+1)^{(1/\alpha)}}\left(1-(\alpha+1)s^{\alpha}\right)\left(|\psi'(m)|^q+|\psi'(M)|^q-s|\psi'(x)|^q-(1-s)|\psi'(y)|^q\right)ds\right.$$
$$\left.+\int_{1/(\alpha+1)^{(1/\alpha)}}^1\left((\alpha+1)s^{\alpha}-1\right)\left(|\psi'(m)|^q+|\psi'(M)|^q-s|\psi'(x)|^q-(1-s)|\psi'(y)|^q\right)ds\right)^{1/q}. \tag{41}$$

This implies that

$$\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|$$
$$\leq\frac{y-x}{\alpha+1}\left(\frac{2\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}\right)^{1-(1/q)}\left((L_1(\alpha)+N_1(\alpha))|\psi'(m)|^q+(L_1(\alpha)+N_1(\alpha))|\psi'(M)|^q\right.$$
$$\left.-(L_2(\alpha)+N_2(\alpha))|\psi'(x)|^q-(L_3(\alpha)+N_3(\alpha))|\psi'(y)|^q\right)^{1/q} \tag{42}$$

Substituting the values of L_1, L_2, L_3, N_1, N_2, and N_3 as given in the proof of Theorem 4 in (42), we get (39). □

Remark 6. If we put $\alpha=1$, $x=m$, and $y=M$ in (39), we obtain the inequality proved in Theorem 1 of [28].

In the following theorem, we derive trapezoidal type inequality.

Theorem 6. *Let $p,q>1$ and $|\psi'|^q$ be a convex function, and let $x,y\in[m,M]$ such that $x<y$. Then,*

$$\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|$$
$$\leq\frac{y-x}{\alpha+1}(L_4(\alpha,p))^{1/p}\left(|\psi'(m)|^q+|\psi'(M)|^q-\frac{|\psi'(x)|^q+|\psi'(y)|^q}{2}\right)^{1/q}, \tag{43}$$

where $L_4(\alpha,p)=\int_0^1|(\alpha+1)s^{\alpha}-1|^p ds$ such that $(1/p)+(1/q)=1$.

Proof. Using Lemma 3, we have (35). Applying Hölder's inequality, we obtain

$$\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)\right|$$
$$\le\frac{y-x}{\alpha+1}\left(\int_0^1\left|(\alpha+1)s^{\alpha}-1\right|^{p}ds\right)^{1/p}\left(\int_0^1\left|\psi'(m+M-sx-(1-s)y)\right|^{q}ds\right)^{1/q}$$
$$\le\frac{y-x}{\alpha+1}(L_4(\alpha,p))^{1/p}\left(\int_0^1\left(\left|\psi'(m)\right|^{q}+\left|\psi'(M)\right|^{q}-s\left|\psi'(x)\right|^{q}-(1-s)\left|\psi'(y)\right|^{q}\right)ds\right)^{1/q}$$
$$=\frac{y-x}{\alpha+1}(L_4(\alpha,p))^{1/p}\left(\left|\psi'(m)\right|^{q}+\left|\psi'(M)\right|^{q}-\frac{\left|\psi'(x)\right|^{q}+\left|\psi'(y)\right|^{q}}{2}\right)^{1/q}. \tag{44}$$

□

Remark 7. Substituting $\alpha=1$, $x=m$, and $y=M$ in Theorem 6, we obtain Theorem 2.3 of [25].

Theorem 7. *Let $|\psi'|$ be a convex function and let $x,y\in[m,M]$ such that $x<y$. Then,*

$$\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|$$
$$\le(y-x)\left(P_5(\alpha)\left|\psi'(x)\right|+P_6(\alpha)\left|\psi'(y)\right|\right), \tag{45}$$

where

$$P_5(\alpha)=\frac{-\alpha}{2(\alpha+2)(\alpha+1)^{2}},$$
$$P_6(\alpha)=\frac{\alpha}{2(\alpha+2)(\alpha+1)^{2}}. \tag{46}$$

Proof. Using Lemma 4, we have

$$\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|$$
$$\le(y-x)\int_{\alpha/(\alpha+1)}^{1}(1-s^{\alpha})\left|\psi'(m+M-sx-(1-s)y)\right|ds$$
$$-(y-x)\int_0^{\alpha/(\alpha+1)}s^{\alpha}\left|\psi'(m+M-sx-(1-s)y)\right|ds. \tag{47}$$

Since $|\psi'|$ is convex, using Mercer's inequality, we obtain

$$\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|$$
$$\le(y-x)\int_{\alpha/(\alpha+1)}^{1}(1-s^{\alpha})\left(\left|\psi'(m)\right|+\left|\psi'(M)\right|-s\left|\psi'(x)\right|-(1-s)\left|\psi'(y)\right|\right)ds$$
$$-(y-x)\int_0^{\alpha/(\alpha+1)}s^{\alpha}\left(\left|\psi'(m)\right|+\left|\psi'(M)\right|-s\left|\psi'(x)\right|-(1-s)\left|\psi'(y)\right|\right)ds, \tag{48}$$

which implies that

$$\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^{-}}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|$$
$$\le(y-x)\left((N_5(\alpha)-L_5(\alpha))\left|\psi'(x)\right|+(N_6(\alpha)-L_6(\alpha))\left|\psi'(y)\right|\right), \tag{49}$$

where

$$\begin{aligned}
L_5(\alpha) &= \int_{\alpha/(\alpha+1)}^{1} (1-s^\alpha) s\mathrm{d}s = \frac{\alpha(\alpha+1)^\alpha + 2\alpha^{\alpha+2}}{2(\alpha+2)(\alpha+1)^{\alpha+2}},\\
L_6(\alpha) &= \int_{\alpha/(\alpha+1)}^{1} (1-s^\alpha)(1-s)\mathrm{d}s = \frac{4\alpha^{\alpha+1} - \alpha(\alpha+1)^\alpha}{2(\alpha+2)(\alpha+1)^{\alpha+2}},\\
N_5(\alpha) &= \int_{0}^{\alpha/(\alpha+1)} s^{\alpha+1}\mathrm{d}s = \frac{\alpha^{\alpha+2}}{(\alpha+2)(\alpha+1)^{\alpha+2}},\\
N_6(\alpha) &= \int_{0}^{\alpha/(\alpha+1)} s^{\alpha}(1-s)\mathrm{d}s = \frac{2\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}}.
\end{aligned} \tag{50}$$

Substituting these values in (49), we get (45).

In next theorem, we use power mean inequality and derive midpoint type inequality. □

Theorem 8. *Let $|\psi'|^q$ be a convex function for $q \geq 1$ and let $x, y \in (m, M)$ such that $x < y$. Then,*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J^{\alpha}_{(m+M-x)^-}\psi(m+M-y) - \psi\left(m+M-\frac{\alpha x + y}{\alpha+1}\right)\right|\\
&\leq (y-x)\left(L_8(\alpha)\right)^{1-(1/q)}\\
&\times\Big(\left(L_8(\alpha)|\psi'(m)|^q + L_8(\alpha)|\psi'(M)|^q - L_5(\alpha)|\psi'(x)|^q - L_6(\alpha)|\psi'(y)|^q\right)^{1/q}\\
&-\left(L_8(\alpha)|\psi'(m)|^q + L_8(\alpha)|\psi'(M)|^q - N_5(\alpha)|\psi'(x)|^q - N_6(\alpha)|\psi'(y)|^q\right)^{1/q}\Big),
\end{aligned} \tag{51}$$

where

$$L_8(\alpha) = \int_{\alpha/(\alpha+1)}^{1} (1-s^\alpha)\mathrm{d}s = \int_{0}^{\alpha/(\alpha+1)} s^\alpha \mathrm{d}s = \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}, \tag{52}$$

and $L_5(\alpha)$, $L_6(\alpha)$, $N_5(\alpha)$, and $N_6(\alpha)$ are given in the proof of Theorem 7.

Proof. Using power mean inequality in (47), we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J^{\alpha}_{(m+M-x)^-}\psi(m+M-y) - \psi\left(m+M-\frac{\alpha x + y}{\alpha+1}\right)\right|\\
&\leq (y-x)\left(\int_{\alpha/(\alpha+1)}^{1}(1-s^\alpha)\mathrm{d}s\right)^{1-(1/q)}\left(\int_{\alpha/\alpha+1}^{1}(1-s^\alpha)|\psi'(m+M-sx-(1-s)y)|^q\mathrm{d}s\right)^{1/q}\\
&-(y-x)\left(\int_{0}^{\alpha/(\alpha+1)} s^\alpha\mathrm{d}s\right)^{1-(1/q)}\left(\int_{0}^{\alpha/(\alpha+1)} s^\alpha|\psi'(m+M-sx-(1-s)y)|^q ds\right)^{1/q}.
\end{aligned} \tag{53}$$

Since $|\psi'|^q$ is convex, using Mercer's inequality, we obtain

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|\\&\leq(y-x)\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)^{1-(1/q)}\times\left(\left(\int_{\alpha/(\alpha+1)}^{1}(1-s^{\alpha})\left(|\psi'(m)|^q+|\psi'(M)|^q-s|\psi'(x)|^q-(1-s)|\psi'(y)|^q\right)ds\right)^{1/q}\right.\\&\left.-\left(\int_{0}^{\alpha/(\alpha+1)}s^{\alpha}\left(|\psi'(m)|^q+|\psi'(M)|^q-s|\psi'(x)|^q-(1-s)|\psi'(y)|^q\right)ds\right)^{1/q}\right)\\&=(y-x)(L_8(\alpha))^{1-(1/q)}\times\left(L_8(\alpha)|\psi'(m)|^q+L_8(\alpha)|\psi'(M)|^q-L_5(\alpha)|\psi'(x)|^q-L_6(\alpha)|\psi'(y)|^q\right)^{1/q}\\&-\left(L_8(\alpha)|\psi'(m)|^q+L_8(\alpha)|\psi'(M)|^q-N_5(\alpha)|\psi'(x)|^q-N_6(\alpha)|\psi'(y)|^q\right)^{1/q}\Big).\end{aligned}\tag{54}$$

□

Remark 8. Substituting $\alpha=1$, $x=m$, and $y=M$ in Theorem 8, we obtain the following midpoint type inequality:

$$\begin{aligned}&\left|\frac{1}{M-m}\int_m^M f(u)du-\psi\left(\frac{m+M}{2}\right)\right|\\&\leq\frac{M-m}{8}\left(\left(\frac{|\psi'(m)|^q+2|\psi'(M)|^q}{3}\right)^{1/q}-\left(\frac{2|\psi'(m)|^q+|\psi'(M)|^q}{3}\right)^{1/q}\right).\end{aligned}\tag{55}$$

Another midpoint type inequality is presented in the following theorem.

Theorem 9. *Let $p, q>1$ and $|\psi'|^q$ be convex, and let $x, y\in[m, M]$ such that $x<y$. Then,*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|\\&\leq(y-x)(L_9(\alpha,p))^{1/p}\\&\times\left(\frac{1}{\alpha+1}|\psi'(m)|^q+\frac{1}{\alpha+1}|\psi'(M)|^q-\frac{2\alpha+1}{2(\alpha+1)^2}|\psi'(x)|^q-\frac{1}{2(\alpha+1)^2}|\psi'(y)|^q\right)^{1/q}\\&-(y-x)(L_{10}(\alpha,p))^{1/p}\\&\times\left(\frac{\alpha}{\alpha+1}|\psi'(m)|^q+\frac{\alpha}{\alpha+1}|\psi'(M)|^q-\frac{\alpha^2}{2(\alpha+1)^2}|\psi'(x)|^q-\frac{\alpha(\alpha+2)}{2(\alpha+1)^2}|\psi'(y)|^q\right)^{1/q},\end{aligned}\tag{56}$$

where $L_9(\alpha, p) = \int_{\alpha/(\alpha+1)}^{1} (1-s^{\alpha})^p ds$ *and* $L_{10}(\alpha, p) = \int_0^{\alpha/(\alpha+1)} s^{\alpha p} ds$ *such that* $(1/p) + (1/q) = 1$.

Proof. Applying Hölder's inequality in (47), we have

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J^{\alpha}_{(m+M-x)^-} \psi(m+M-y) - \psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right| \\ &\leq (y-x)\left(\int_{\alpha/(\alpha+1)}^{1} (1-s^{\alpha})^p \mathrm{d}s\right)^{1/p} \left(\int_{\alpha/(\alpha+1)}^{1} |\psi'(m+M-sx-(1-s)y)|^q \mathrm{d}s\right)^{1/q} \\ &\quad - (y-x)\left(\int_0^{\alpha/(\alpha+1)} s^{\alpha p} \mathrm{d}t\right)^{1/p} \left(\int_0^{\alpha/(\alpha+1)} |\psi'(m+M-sx-(1-s)y)|^q \mathrm{d}s\right)^{1/q}.\end{aligned} \tag{57}$$

As $|\psi'|^q$ is convex, applying Mercer's inequality, we obtain

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J^{\alpha}_{(m+M-x)^-} \psi(m+M-y) - \psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right| \\ &\leq (y-x)\left(L_9(\alpha,p)\right)^{1/p} \times \left(\int_{\alpha/(\alpha+1)}^{1} \left(|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q\right)\mathrm{d}s\right)^{1/q} \\ &\quad - (y-x)\left(L_{10}(\alpha,p)\right)^{1/p} \\ &\quad \times \left(\int_0^{\alpha/(\alpha+1)} \left(|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q\right)\mathrm{d}s\right)^{1/q} \\ &= (y-x)\left(L_9(\alpha,p)\right)^{1/p} \times \left(\frac{1}{\alpha+1}|\psi'(m)|^q + \frac{1}{\alpha+1}|\psi'(M)|^q - \frac{2\alpha+1}{2(\alpha+1)^2}|\psi'(x)|^q - \frac{1}{2(\alpha+1)^2}|\psi'(y)|^q\right)^{1/q} \\ &\quad - (y-x)\left(L_{10}(\alpha,p)\right)^{1/p} \\ &\quad \times \left(\frac{\alpha}{\alpha+1}|\psi'(m)|^q + \frac{\alpha}{\alpha+1}|\psi'(M)|^q - \frac{\alpha^2}{2(\alpha+1)^2}|\psi'(x)|^q - \frac{\alpha(\alpha+2)}{2(\alpha+1)^2}|\psi'(y)|^q\right)^{1/q}.\end{aligned} \tag{58}$$

Remark 9. Substituting $\alpha = 1$, $x = m$, and $y = M$ in Theorem 9, we obtain

$$\begin{aligned}&\left|\frac{1}{M-m}\int_m^M f(u)\mathrm{d}u - \psi\left(\frac{m+M}{2}\right)\right| \\ &\leq \frac{M-m}{4(p+1)^{1/p}}\left(\left(\frac{|\psi'(m)|^q + 3|\psi'(M)|^q}{4}\right)^{1/q} - \left(\frac{3|\psi'(m)|^q + |\psi'(M)|^q}{4}\right)^{1/q}\right).\end{aligned} \tag{59}$$

Theorem 10. *Let $x, y \in [m, M]$ such that $x < y$. If $|\psi'|$ is concave on $[m, M]$, then*

$$\begin{aligned}&\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|\\&\leq\frac{y-x}{\alpha+1}\Big(L_1(\alpha)\left|\psi'(m+M-L_{11}(\alpha)x-L_{12}(\alpha)y)\right|\\&\quad+N_1(\alpha)\left|\psi'(m+M-L_{13}(\alpha)x-L_{14}(\alpha)y)\right|\Big),\end{aligned}\tag{60}$$

where L_1 and N_1 are given in the proof of Theorem 4 and

$$\begin{aligned}L_{11}(\alpha)&=\frac{(\alpha+1)^{1-(1/\alpha)}}{2(\alpha+2)},\\L_{12}(\alpha)&=\frac{(\alpha+1)^{1-(1/\alpha)}\left[2(\alpha+2)(\alpha+1)^{(1/\alpha)-1}-1\right]}{2(\alpha+2)},\\L_{13}(\alpha)&=\frac{(\alpha+1)^{1-(1/\alpha)}\left[(\alpha+1)^{2/\alpha}+1\right]}{2(\alpha+2)},\\L_{14}(\alpha)&=\frac{(\alpha+1)^{1-(1/\alpha)}\left[2(\alpha+2)(\alpha+1)^{(1/\alpha)-1}-(\alpha+1)^{2/\alpha}-1\right]}{2(\alpha+2)}.\end{aligned}\tag{61}$$

Proof. Lemma 3 implies that

$$\begin{aligned}&\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|\\&\leq\frac{y-x}{\alpha+1}\int_0^1\left|(\alpha+1)s^{\alpha}-1\right|\left|\psi'(m+M-sx-(1-s)y)\right|ds\\&=\frac{y-x}{\alpha+1}\left(\int_0^{1/(\alpha+1)^{1/\alpha}}(1-(\alpha+1)s^{\alpha})\left|\psi'(m+M-sx-(1-s)y)\right|ds\right.\\&\quad\left.+\int_{1/(\alpha+1)^{1/\alpha}}^1((\alpha+1)s^{\alpha}-1)\left|\psi'(m+M-sx-(1-s)y)\right|ds\right).\end{aligned}\tag{62}$$

Since $|\psi'|$ is concave, using Jensen's inequality, we obtain

$$\begin{aligned}
&\left|\frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)\right|\\
&\leq\frac{y-x}{\alpha+1}\left(\int_0^{1/(\alpha+1)^{1/\alpha}}(1-(\alpha+1)s^{\alpha})\mathrm{d}s\times\left|\psi'\left(\frac{\int_0^{1/(\alpha+1)^{1/\alpha}}(1-(\alpha+1)s^{\alpha})(m+M-sx-(1-s)y)\mathrm{d}s}{\int_0^{1/(\alpha+1)^{1/\alpha}}(1-(\alpha+1)s^{\alpha})\mathrm{d}s}\right)\right|\right.\\
&\left.+\int_{1/(\alpha+1)^{1/\alpha}}^{1}((\alpha+1)s^{\alpha}-1)\mathrm{d}s\times\left|\psi'\left(\frac{\int_{1/(\alpha+1)^{1/\alpha}}^{1}((\alpha+1)s^{\alpha}-1)(m+M-sx-(1-s)y)\mathrm{d}s}{\int_{1/(\alpha+1)^{1/\alpha}}^{1}((\alpha+1)s^{\alpha}-1)\mathrm{d}s}\right)\right|\right).\\
&=\frac{y-x}{\alpha+1}\left(L_1(\alpha)\left|\psi'(m+M-L_{11}(\alpha)x-L_{12}(\alpha)y)\right|\right.\\
&\left.+N_1(\alpha)\left|\psi'(m+M-L_{13}(\alpha)x-L_{14}(\alpha)y)\right|\right).
\end{aligned}\tag{63}$$

□

Theorem 11. *Let $|\psi'|$ be a concave function, and let $x, y\in[m, M]$ such that $x<y$. Then,*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|\\
&\leq(y-x)\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)\left(\left|\psi'(m+M-L_{15}(\alpha)x-L_{16}(\alpha)y)\right|\right.\\
&\left.-\left|\psi'(m+M-L_{17}(\alpha)x-L_{18}(\alpha)y)\right|\right),
\end{aligned}\tag{64}$$

where

$$\begin{aligned}
L_{15}(\alpha)&=\frac{(\alpha+1)^{\alpha}+2\alpha^{\alpha+1}}{2\alpha^{\alpha}(\alpha+2)},\\
L_{16}(\alpha)&=\frac{4\alpha^{\alpha}-(\alpha+1)^{\alpha}}{2\alpha^{\alpha}(\alpha+2)},\\
L_{17}(\alpha)&=\frac{\alpha}{\alpha+2},\\
L_{18}(\alpha)&=\frac{2}{\alpha+2}.
\end{aligned}\tag{65}$$

Proof. From Lemma 4, we have (47). As $|\psi'|$ is concave, using Jensen's inequality, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J^{\alpha}_{(m+M-x)^-}\psi(m+M-y)-\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right)\right|\\
&\leq(y-x)\int_{\alpha/(\alpha+1)}^{1}(1-s^{\alpha})\mathrm{d}s\left|\psi'\left(\frac{\int_{\alpha/(\alpha+1)}^{1}(1-s^{\alpha})(m+M-sx-(1-s)y)\mathrm{d}s}{\int_{\alpha/(\alpha+1)}^{1}(1-s^{\alpha})\mathrm{d}s}\right)\right|\\
&-(y-x)\int_0^{\alpha/(\alpha+1)}s^{\alpha}\mathrm{d}s\left|\psi'\left(\frac{\int_0^{\alpha/(\alpha+1)}s^{\alpha}(m+M-sx-(1-s)y)\mathrm{d}s}{\int_0^{\alpha/(\alpha+1)}s^{\alpha}\mathrm{d}s}\right)\right|\\
&=(y-x)\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)\left(\left|\psi'(m+M-L_{15}(\alpha)x-L_{16}(\alpha)y)\right|\right.\\
&\left.-\left|\psi'(m+M-L_{17}(\alpha)x-L_{18}(\alpha)y)\right|\right).
\end{aligned}\tag{66}$$

□

4. Conclusion

In this paper, we establish the fractional Hermite–Hadamard type inequalities of Mercer type by using support line inequality. We expect that this work will lead to the new fractional integral studies for Hermite–Hadamard inequality. It is an open problem to prove inequalities (10) and (11) by any other method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to this paper. All authors read and approved the final manuscript.

References

[1] J. Pečarić, F. Prochan, and Y. L. Tong, *Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, New York, NY, USA, 1991.

[2] A. Ekinci, M. E. Özdemir, and E. Set, "New integral inequalities of Ostrowski type for quasi-convex functions with applications," *Turkish Journal of Science*, vol. 5, no. 3, pp. 290–304, 2020.

[3] M. Gürbüz and M. E. Özdemir, "On some inequalities for product of different kinds of convex functions," *Turkish Journal of Science*, vol. 5, no. 1, pp. 23–27, 2020.

[4] M. Adil Khan, J. Khan, and J. Pečarić, "Generalization of Jensen's and Jensen-Steffensen's inequalities by generalized majorization theorem," *Journal of Mathematical Inequalities*, vol. 11, no. 4, pp. 1049–1074, 2017.

[5] J. Khan, M. Adil Khan, and J. Pecaric, "On Jensen's type inequalities via generalized majorization inequalities," *FILOMAT*, vol. 32, no. 16, pp. 5719–5733, 2018.

[6] A. Mercer, "A variant of Jensen's inequality," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, p. 73, 2003.

[7] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, NY, USA, 1973.

[8] J. Hadamard, "Étude sur les propriétés des fonctions entières et en particulier d̀une fonction considérée par Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–215, 1893.

[9] M. Adil Khan, A. Iqbal, M. Suleman, and Y. M. Chu, "Hermite-Hadamard type inequalities for fractional integrals via Green's function," *Journal of Inequalities and Applications*, vol. 2018, p. 161, 2018.

[10] M. Adil Khan, T. U. Khan, and Y. M. Chu, "Generalized Hermite-Hadamard type inequalities for quasi-convex functions with applications," *Journal of Inequalities and Special Functions*, vol. 11, no. 1, pp. 24–42, 2020.

[11] M. Adil Khan, Y. Khurshid, T. S. Du, and Y. M. Chu, "Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals," *Journal of Function Spaces*, vol. 2018, Article ID 5357463, , 2018.

[12] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir, and Y. M. Chu, "New Hermite-Jensen-Mercer-type inequalities via k-fractional integrals," *Advances in Difference Equations*, vol. 635, pp. 1–24, 2020.

[13] S. S. Dragomir and C. E. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Footscray, Australia, 2000.

[14] M. Gürbüz, A. O. Akdemir, S. Rashid, and E. Set, "Hermite-Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities," *Journal of Inequalities and Applications*, vol. 172, pp. 1–10, 2020.

[15] A. Iqbal, M. Adil Khan, N. Mohammed, E. R. Nwaeze, and Y. M. Chu, "Revisting the Hermite-Hadamard fractional integrals via Green's function," *AIMS Mathematics*, vol. 5, pp. 6087–6107, 2020.

[16] M. E. Özdemir, M. Avci, and H. Kavurmaci, "Hermite-Hadamard type inequalities for s-convex and s-concave functions via fractional integrals," *Turkish Journal of Science*, vol. 1, no. 1, pp. 28–40, 2016.

[17] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[18] E. Set, İ. İşcan, M. Zeki Sarikaya, and M. Emin Özdemir, "On new inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals," *Applied Mathematics and Computation*, vol. 259, pp. 875–881, 2015.

[19] S. I. Butt, M. Nadeem, and G. Farid, "On Caputo fractional derivatives via exponential s-convex functions," *Turkish Journal of Science*, vol. 5, no. 2, pp. 140–146, 2020.

[20] A. Ekinci and M. E. Özdemir, "Some new integral inequalities via Riemann-Liouville integral operators," *Applied and Computational Mathematics*, vol. 3, no. 18, pp. 288–295, 2019.

[21] M. K. Sajid, R. S. Ali, and I. Nayab, "Some results of generalized k-fractional integral operator with k-Bessel function," *Turkish Journal of Science*, vol. 5, no. 3, pp. 157–169, 2020.

[22] E. Set, A. O. Akdemir, and F. Demirci, "Grüss type inequalities for fractional integral operator involving the extended generalized Mittag-Leffler function," *Applied and Computational Mathematics*, vol. 19, no. 3, pp. 402–414, 2020.

[23] M. Kunt, D. Karapinar, S. Turhan, and İ. İşcan, "The left Riemann-Liouville fractional Hermite-Hadamard type inequalities for convex functions," *Mathematica Slovaca*, vol. 69, no. 4, pp. 773–784, 2019.

[24] S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Hoboken, NJ, USA, 1993.

[25] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[26] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.

[27] M. Kian and M. S. Moslehian, "Refinement of the operator Jensen-Mercer inequality," *The Electronic Journal of Linear Algebra*, vol. 26, p. 50, 2013.

[28] C. E. M. Pearce and J. Pečarić, "Inequalities for differentiable mappings with application to special means and quadrature formulæ," *Applied Mathematics Letters*, vol. 13, no. 2, pp. 51–55, 2000.

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Research Article

Efficient Exponential Time-Differencing Methods for the Optical Soliton Solutions to the Space-Time Fractional Coupled Nonlinear Schrödinger Equation

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The coupled nonlinear Schrödinger equation is used in simulating the propagation of the optical soliton in a birefringent fiber. Hereditary properties and memory of various materials can be depicted more precisely using the temporal fractional derivatives, and the anomalous dispersion or diffusion effects are better described by the spatial fractional derivatives. In this paper, one-step and two-step exponential time-differencing methods are proposed as time integrators to solve the space-time fractional coupled nonlinear Schrödinger equation numerically to obtain the optical soliton solutions. During this procedure, we take advantage of the global Padé approximation to evaluate the Mittag-Leffler function more efficiently. The approximation error of the Padé approximation is analyzed. A centered difference method is used for the discretization of the space-fractional derivative. Extensive numerical examples are provided to demonstrate the efficiency and effectiveness of the modified exponential time-differencing methods.

1. Introduction

The coupled nonlinear Schrödinger equation (CNLSE) can be employed in simulating the propagation of the optical soliton in a birefringent fiber [1–3]. A soliton is a solitary pulse which can travel at a constant speed and keep a stationary shape due to the balancing of the self-phase modulation and the group velocity dispersion effect in fiber optics [4]. According to Agrawal [5], in a fiber communication system, the input pulse may be orthogonally polarized in a birefringent fiber. The polarized components can form solitary waves, which are named as vector solitons. Because of the nonlinear coupling effect, the vector solitons can propagate undistorted even when the components have different widths or peak powers.

During the last few decades, researchers have found that hereditary properties and memory of various materials can be depicted more precisely using the temporal fractional derivatives [6–8]. It is also shown in [9, 10] that the anomalous dispersion or diffusion effects are better described by the spatial fractional derivatives. The anomalous effects reflect the Lévy-type particle movement, different from Brownian motion, which depicts the classical random movement of particles. Therefore, the space-time fractional coupled nonlinear Schrödinger equation (FCNLSE) is useful in modeling solitons in fractional fiber optics.

In this article, we consider the FCNLSE given as follows [11]:

$$\begin{aligned} &iD_t^{\alpha}u + D_x^{\beta}u + \delta\left(|u|^2 + \gamma|v|^2\right)u = 0, \quad x \in \mathbb{R}, 0 < t \leq T,\\ &iD_t^{\alpha}v + D_x^{\beta}v + \delta\left(|v|^2 + \gamma|u|^2\right)v = 0, \quad x \in \mathbb{R}, 0 < t \leq T, \end{aligned} \tag{1}$$

with the initial conditions

$$\begin{aligned} u(0,x) &= u_0(x),\\ v(0,x) &= v_0(x), \end{aligned} \tag{2}$$

and homogeneous Dirichlet boundary conditions on $[x_L, x_R]$, where $i = \sqrt{-1}$ and complex functions u and v represent the amplitudes of orthogonally polarized waves in a birefringent optical fiber. $D_t^{\alpha} u = \partial^{\alpha} u / \partial t^{\alpha}$ is the Caputo derivative of u in time, $D_x^{\beta} u = \partial^{\beta} u / \partial x^{\beta}$ is the Riesz derivative of u in space, $0 < \alpha \leq 1$, $0 < \beta \leq 2$, and the parameters δ and ρ are some real constants.

Both analytical treatments and numerical methods have been investigated for the fractional Schrödinger equations and some novel types of nonlinear Schrödinger equations. In [12], an extended sinh-Gordon equation expansion method is adopted to solve the space-time fractional Schrödinger equation analytically. In [13], a modified residual power series method is implemented on the fractional Schrödinger equation. In [14], the L1 scheme together with the Fourier–Galerkin spectral method is employed to discretize the time-fractional Schrödinger model. In [15], a Fourier spectral exponential splitting scheme is constructed to solve the space-fractional initial boundary value problems. In [16], a generalized exponential rational function method is applied to a new extension of the nonlinear Schrödinger equation. In [17], a cubic-quartic nonlinear Schrödinger equation is solved analytically for the dark, singular, and bright-singular soliton solutions. In [18], a modified expansion function method and an extended sinh-Gordon method are proposed for the M-fractional paraxial nonlinear Schrödinger equation to obtain soliton solutions.

However, to the best of our current knowledge, numerical methods for the coupled space-time fractional Schrödinger equations are rarely considered. In this paper, we modify the exponential time-differencing (ETD) method for the time-fractional nonlinear PDEs, introduced in [19], by applying the Padé approximation. Then, we combine the modified ETD scheme with a fourth-order fractional compact scheme in space. During this procedure, the nonlinear term of the equation is computed explicitly, and the calculation of the fractional exponential time integral is undertaken more efficiently.

2. Discretization in Space

The spatial Riesz derivative is defined in [10] as

$$D_x^{\beta} u(t,x) = \frac{\partial^{\beta}}{\partial x^{\beta}} u(t,x) - \frac{1}{2\cos \pi\beta/2}\left[{}_{-\infty}D_x^{\beta} u(t,x) + {}_xD_{+\infty}^{\beta} u(t,x)\right], \tag{3}$$

where $1 < \beta < 2$. ${}_{-\infty}D_x^{\beta} u(t, x)$ and ${}_xD_{+\infty}^{\beta} u(t, x)$ are the left and right Riemann–Liouville derivatives:

$$\begin{aligned} {}_{-\infty}D_x^{\beta} u(t,x) &= \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} \frac{u(t,\xi)}{(x-\xi)^{\beta-1}} \mathrm{d}\xi, \\ {}_xD_{+\infty}^{\beta} u(t,x) &= \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_{x}^{+\infty} \frac{u(t,\xi)}{(\xi-x)^{\beta-1}} \mathrm{d}\xi, \end{aligned} \tag{4}$$

in which $\Gamma(\cdot)$ is the gamma function.

It is stated in [20] that the approximation of the left derivative ${}_{-\infty}D_x^{\beta} v(t, x)$ is calculated using matrix $B_M^{(\beta)}$:

$$\left[v_M^{(\beta)} \;\; v_{M-1}^{(\beta)} \;\; \ldots \;\; v_1^{(\beta)} \;\; v_0^{(\beta)} \right]^{\top} = B_M^{(\beta)} \left[v_M \;\; v_{M-1} \;\; \ldots \;\; v_1 \;\; v_0 \right]^{\top}, \tag{5}$$

where

$$B_M^{(\beta)} = \frac{1}{h^{\beta}} \begin{bmatrix} \omega_0^{(\beta)} & \omega_1^{(\beta)} & \ddots & \ddots & \omega_{M-1}^{(\beta)} & \omega_M^{(\beta)} \\ 0 & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \ddots & \ddots & \omega_{M-1}^{(\beta)} \\ 0 & 0 & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & \omega_0^{(\beta)} & \omega_1^{(\beta)} \\ 0 & 0 & \cdots & 0 & 0 & \omega_0^{(\beta)} \end{bmatrix}, \tag{6}$$

$$\omega_j^{(\beta)} = (-1)^j \begin{pmatrix} \beta \\ j \end{pmatrix},$$

with $x = jh$ $(j = 0, 1, \ldots, M)$, where h is a single spatial step.

Similarly, the approximation of the right derivative ${}_xD_{+\infty}^{\beta} v(t, x)$ is calculated using matrix $L_M^{(\beta)}$:

$$\left[v_M^{(\beta)} \;\; v_{M-1}^{(\beta)} \;\; \cdots \;\; v_1^{(\beta)} \;\; v_0^{(\beta)} \right]^{\top} = L_M^{(\beta)} \left[v_M \;\; v_{M-1} \;\; \cdots \;\; v_1 \;\; v_0 \right]^{\top}. \tag{7}$$

Matrices $L_M^{(\beta)}$ and $B_M^{(\beta)}$ are transposes to each other in (5) and (6).

Furthermore, we use the centered difference method for the fractional derivative to approximate the Riesz derivative in the following way [21]:

$$\left[v_M^{(\beta)} \;\; v_{M-1}^{(\beta)} \;\; \cdots \;\; v_1^{(\beta)} \;\; v_0^{(\beta)} \right]^{\top} = H_M^{(\beta)} \left[v_M \;\; v_{M-1} \;\; \cdots \;\; v_1 \;\; v_0 \right]^{\top}, \tag{8}$$

where

$$H_M^{(\beta)} = \frac{1}{h^{\beta}} \begin{bmatrix} \omega_0^{(\beta)} & \omega_1^{(\beta)} & \omega_2^{(\beta)} & \omega_3^{(\beta)} & \cdots & \omega_M^{(\beta)} \\ \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \omega_2^{(\beta)} & \cdots & \omega_{M-1}^{(\beta)} \\ \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \cdots & \omega_{M-2}^{(\beta)} \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{M-1}^{(\beta)} & \ddots & \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} \\ \omega_M^{(\beta)} & \omega_{M-1}^{(\beta)} & \ddots & \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} \end{bmatrix},$$

$$\omega_j^{(\beta)} = \frac{(-1)^j \Gamma(\beta+1) \cos(\beta\pi/2)}{\Gamma(\beta/2 - j + 1)\Gamma(\beta/2 + j + 1)}, \quad j = 0, 1, \ldots, M. \tag{9}$$

Noticed that scheme (8) is second-order convergent in space, Ding et al. [22] generated a compact scheme to improve the order of convergence:

$$\begin{aligned}\frac{\partial^\beta v(t,x)}{\partial x^\beta} &= \frac{1}{h^\beta}\left[\frac{\beta}{24}\Delta_h^\beta v(t,x-h) - \left(1+\frac{\beta}{12}\right)\Delta_h^\beta v(t,x)\right.\\ &\quad \left.+ \frac{\beta}{24}\Delta_h^\beta v(t,x+h)\right] + \mathcal{O}\left(h^4\right)\\ &= -\frac{1}{h^\beta}\left(1-\frac{\beta}{24}\delta_x^2\right)\Delta_h^\beta v(t,x) + \mathcal{O}\left(h^4\right)\\ &= -\frac{1}{h^\beta}\left(1+\frac{\beta}{24}\delta_x^2\right)^{-1}\Delta_h^\beta v(t,x) + \mathcal{O}\left(h^4\right),\end{aligned} \tag{10}$$

where $\delta_x^2 v(t,x) = v(t,x-h) - 2v(t,x) + v(t,x+h)$ and $-\Delta_h^\beta v(t,x)/h^\beta$ is the second-order approximation (8). As been proved by Theorem 11 in [22], compact scheme (10) is fourth-order convergent spatially.

3. The Exponential Time Integrator

We obtain a system of time-fractional equations after discretizing FCNLSE (1) in space:

$$\frac{\partial^\alpha}{\partial t^\alpha}U(t) + \mathrm{A}U(t) = F(U(t)), \tag{11}$$

where $\partial^\alpha/\partial t^\alpha$ denotes the Caputo derivative, A is the $M_x \times M_x$ matrix in the Riesz derivative approximation, $F\colon \mathbb{R}^{M_x} \longrightarrow \mathbb{R}^{M_x}$ contains the nonlinear function and the boundary conditions, and $U(t) = (U_1(t), U_2(t), \ldots, U_{M_x}(t))^{\mathrm{T}}$ with $U_j(t) = u(x_j, t)$, $j = 1, \ldots, M_x$, and the initial condition is $U(0) = U_0$.

As been computed using the variation of constant formula in [23], system (11) has an analytical solution:

$$U(t) = e_{\alpha,1}(t; A)U_0 + \int_0^t e_{\alpha,\alpha}(t-s; A)F(U(s))\mathrm{d}s, \tag{12}$$

where $e_{\alpha,\beta}(t;\lambda)$ denotes the inverse function of the Laplace transform $s^{\alpha-\beta}/(s^\alpha+\lambda)$ to A, and $e_{\alpha,\beta}(t;\lambda)$ can be calculated taking advantage of the Mittag-Leffler (ML) function $E_{\alpha,\beta}(z)$:

$$e_{\alpha,\beta}(t;\lambda) = t^{\beta-1}E_{\alpha,\beta}(-t^\alpha\lambda), E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{13}$$

Formula (12) can be written in a discrete form after discretization on $[0,T]$ with an equal-spaced mesh-grid $t_n = n\tau$, $n = 0, 1, \ldots$:

$$U(t_n) = e_{\alpha,1}(t_n; A)U_0 + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} e_{\alpha,\alpha}(t_n - s; A)F(U(s))\mathrm{d}s. \tag{14}$$

Then, the ETD scheme can be denoted as [19, 23]

$$U_n = e_{\alpha,1}(t_n; A)U_0 + \sum_{j=0}^{n-1} W_{n,j}F\left(U_j\right), \tag{15}$$

where U_j is the numerical approximation to $U(t_j)$, and the convolution weights $W_{n,j}$ can be computed as

$$W_{n,j} = e_{\alpha,\alpha+1}\left(t_n - t_j; A\right) - e_{\alpha,\alpha+1}\left(t_n - t_{j+1}; A\right). \tag{16}$$

Scheme (15) is called the one-step ETD scheme.

Garrappa and Popolizio proved in [19] that the one-step ETD scheme (15) has the absolute approximation error $\mathrm{Err}_n = \|U(t_n) - U_n\|$ satisfying

$$\left\|U(t_n) - U_n\right\| \le C\tau, \quad n = 1, \ldots, M, \tag{17}$$

where M is the temporal step number and C is a constant relating to T and α. This inequality tells us that ETD scheme (15) is first-order convergent temporally.

The two-step ETD scheme is also constructed in [19]:

$$\begin{aligned}U_n &= e_{\alpha,1}(t_n; A)U_0 + W_n^{(1)}F(U_0) + \sum_{j=0}^{n-1} W_{n,j}^{(2)}F\left(U_j\right)\\ &\quad - W_{n,n}^{(2)}F(U_{n-2}) + 2W_{n,n}^{(2)}F(U_{n-1}),\end{aligned} \tag{18}$$

where

$$\begin{aligned}W_n^{(1)} &= e_{\alpha,\alpha+2}(t_{n-1}; A) + e_{\alpha,\alpha+1}(t_n; A) - e_{\alpha,\alpha+2}(t_n; A),\\ W_{n,j}^{(2)} &= \begin{cases} e_{\alpha,\alpha+2}(t_1; A), & n = j,\\ e_{\alpha,\alpha+2}\left(t_n - t_{j+1}; A\right) - 2e_{\alpha,\alpha+2}\left(t_n - t_j; A\right) + e_{\alpha,\alpha+2}\left(t_n - t_{j-1}; A\right), & n > j.\end{cases}\end{aligned} \tag{19}$$

Garrappa and Popolizio also proved in [19] that the two-step ETD scheme (18) has the absolute approximation error $\mathrm{Err}_n = \|U(t_n) - U_n\|$ satisfying

$$\left\|U(t_n) - U_n\right\| \le C\tau^{1+\alpha}, \quad n = 1, \ldots, M, \tag{20}$$

where M is the temporal step number and C is a constant relating to T and α. This inequality tells us that ETD scheme (18) is $\{1+\alpha\}$-order convergent temporally.

To relief the burden of computing the function $e_{\alpha,\beta}(t; A)$, we transform it using the multiplication of eigenvectors and functions of eigenvalues [23]:

$$f(A) = Zf(D)Z^{-1} = Z\begin{bmatrix} f(\lambda_1) & & & \\ & f(\lambda_1) & & \\ & & \ddots & \\ & & & f(\lambda_1) \end{bmatrix}Z^{-1}, \tag{21}$$

where A is diagonalizable, with λ_k's to be its eigenvalues, Z is the composition of A's eigenvectors, and $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$. Using this decomposition, we avoid the computation of the ML function of matrices, which is really time consuming. We only need to calculate the ML function with inputs of numbers and multiply the matrices, which reduces the time of computation significantly.

Moreover, we use the Padé approximation $R^{3,2}_{\alpha,\beta}$ to compute the value of the ML function [24, 25]:

$$E_{\alpha,\beta}(-x) \approx R^{3,2}_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta-\alpha)x} \cdot \frac{p_1 + x}{q_0 + q_1 x + x^2}, \tag{22}$$

with coefficients

$$\begin{cases} p_1 = c_{\alpha,\beta}\left[\Gamma(\beta)\Gamma(\beta+\alpha) - \dfrac{\Gamma(\beta+\alpha)\Gamma^2(\beta-\alpha)}{\Gamma(\beta-2\alpha)}\right], \\ q_0 = c_{\alpha,\beta}\left[\dfrac{\Gamma^2(\beta)\Gamma(\beta+\alpha)}{\Gamma(\beta-\alpha)} - \dfrac{\Gamma(\beta)\Gamma(\beta+\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta-2\alpha)}\right], \\ q_1 = c_{\alpha,\beta}\left[\Gamma(\beta)\Gamma(\beta+\alpha) - \dfrac{\Gamma(\beta-\alpha)\Gamma^2(\beta)}{\Gamma(\beta-2\alpha)}\right], \\ c_{\alpha,\beta} = \dfrac{1}{\Gamma(\beta+\alpha)\Gamma(\beta-\alpha) - \Gamma^2(\beta)}. \end{cases} \tag{23}$$

After simplification, formula (22) becomes

$$E_{\alpha,\beta}(-x) \approx R^{3,2}_{\alpha,\beta}(x) = \frac{\alpha}{\Gamma(1+\alpha) + 2\Gamma(1-\alpha)^2/\Gamma(1-2\alpha)x + \Gamma(1-\alpha)x^2}. \tag{24}$$

The Padé approximation (24) to the ML function can be applied to the one-step and two-step ETD schemes (15) and (18) to enhance the efficiency.

4. Approximation Error Analysis

The approximation error of formula (24) is defined as [24]

$$e^{3,2}_{\alpha,\beta}(x) := E_{\alpha,\beta}(-x) - R^{3,2}_{\alpha,\beta}(x), \quad x > 0. \tag{25}$$

Then, we have

$$E_{\alpha,\beta}(-x) = \frac{1}{s_{\alpha,\beta}(x)}\mathscr{E}_{\alpha,\beta}(x), \tag{26}$$

where

$$\begin{gathered} s_{\alpha,\beta}(x) = \begin{cases} \Gamma(\beta-\alpha)x, & \beta > \alpha, \\ -\Gamma(-\alpha)x^2, & \beta = \alpha, \end{cases} \\ \mathscr{E}_{\alpha,\beta}(x) = \begin{cases} \mathscr{E}^0_{\alpha,\beta}(x) + \mathcal{O}(x^m), & \text{as } x \longrightarrow 0, \quad m \geq \begin{cases} 2, \ \beta > \alpha, \\ 3, \ \beta = \alpha, \end{cases} \\ \mathscr{E}^\infty_{\alpha,\beta}(x^{-1}) + \mathcal{O}(x^{-n}), & \text{as } x \longrightarrow \infty, \quad n \geq \begin{cases} 1, \ \beta > \alpha, \\ 2, \ \beta = \alpha, \end{cases} \end{cases} \end{gathered} \tag{27}$$

in which

$$\mathscr{E}^{0}_{\alpha,\beta}(x) = \begin{cases} \Gamma(\beta-\alpha)x \displaystyle\sum_{k=0}^{m-2} \frac{(-x)^k}{\Gamma(\beta+\alpha k)}, & \beta > \alpha, \\ -\Gamma(-\alpha)x^2 \displaystyle\sum_{k=0}^{m-3} \frac{(-x)^k}{\Gamma(\alpha+\alpha k)}, & \beta = \alpha, \end{cases}$$
$$\mathscr{E}^{\infty}_{\alpha,\beta}\left(x^{-1}\right) = \begin{cases} -\Gamma(\beta-\alpha)x \displaystyle\sum_{k=1}^{n} \frac{(-x)^k}{\Gamma(\beta-\alpha k)}, & \beta > \alpha, \\ \Gamma(-\alpha)x^2 \displaystyle\sum_{k=1}^{n} \frac{(-x)^{-(k+1)}}{\Gamma(-\alpha k)}, & \beta = \alpha. \end{cases} \tag{28}$$

Then, we compute the error of approximation as

$$e^{3,2}_{\alpha,\beta}(x) = E_{\alpha,\beta}(-x) - R^{3,2}_{\alpha,\beta}(x) = \frac{1}{s_{\alpha,\beta}(x)}\left\{\mathscr{E}_{\alpha,\beta}(x) - \frac{p(x)}{q(x)}\right\} = \frac{1}{s_{\alpha,\beta}(x)}\left\{\mathscr{O}\left(x^3\right) + \mathscr{O}(x)\right\} = \begin{cases} \mathscr{O}(1), & \beta > \alpha, \\ \mathscr{O}\left(x^{-1}\right), & \beta = \alpha, \end{cases} \text{ as } x \longrightarrow 0,$$
$$e^{3,2}_{\alpha,\beta}(x) = E_{\alpha,\beta}(-x) - R^{3,2}_{\alpha,\beta}(x) = \frac{1}{s_{\alpha,\beta}(x)}\left\{\mathscr{E}_{\alpha,\beta}(x) - \frac{p(x)}{q(x)}\right\} = \frac{1}{s_{\alpha,\beta}(x)}\mathscr{O}\left(x^{-2}\right) = \begin{cases} \mathscr{O}\left(x^{-3}\right), & \beta > \alpha, \\ \mathscr{O}\left(x^{-4}\right), & \beta = \alpha, \end{cases} \text{ as } x \longrightarrow \infty. \tag{29}$$

As stated by Sarumi et al. [24], to make the approximation of $R^{m,n}_{\alpha,\beta}$ reliable for $\beta \neq \alpha$, we need to have $m \geq n+1$. This is why we use $R^{3,2}_{\alpha,\beta}$ to approximate the Mittag-Leffler function.

5. Numerical Experiments

We tested the ETD schemes with Padé approximation on an initial boundary value problem with analytical solutions. The numerical errors in this section are computed as

$$\mathrm{err}(\tau) = \left\| U(t_n) - U_n \right\|_{L_2}. \tag{30}$$

The rate of convergence in time is computed as

$$\mathrm{p} = \frac{\log\left(\mathrm{err}(\tau_k)/\mathrm{err}(\tau_{k+1})\right)}{\log\left(\tau_k/\tau_{k+1}\right)}. \tag{31}$$

The experiments were compiled on an Intel Core i5-6200U 2.30 GHz workstation, and MATLAB R2016b was chosen as computation software.

Firstly, we consider the following FCNLSE as suggested by Esen et al. [12]:

$$\begin{aligned} iD^{\alpha}_{t}u + D^{\beta}_{x}u + \delta\left(|u|^2 + \gamma|v|^2\right)u &= 0, \\ iD^{\alpha}_{t}v + D^{\beta}_{x}v + \delta\left(|v|^2 + \gamma|u|^2\right)v &= 0, \end{aligned} \tag{32}$$

with initial conditions

$$u(x,0) = \mu\sqrt{-\frac{2}{\delta(1+\gamma)}}\operatorname{sech}\left(\mu\left(\frac{x^{\beta/2}}{\beta/2}\right)\right)e^{i\left(-k\frac{x^{\beta/2}}{\beta/2}+p\right)},$$
$$v(x,0) = -\mu\sqrt{-\frac{2}{\delta(1+\gamma)}}\operatorname{sech}\left(\mu\left(\frac{x^{\beta/2}}{\beta/2}\right)\right)e^{i\left(-k\frac{x^{\beta/2}}{\beta/2}+p\right)}, \tag{33}$$

and homogeneous Dirichlet boundary conditions on $[-20, 20]$, where the parameters can be chosen as $\gamma = 0.25$, $\mu = 0.45$, $\delta = -0.35$, $p = 1.5$, $k = -\sqrt{\mu^2 - \omega}$, and $\omega = -3$.

The analytical solutions to FCNLSE (32) are given in [12] as

$$u(x,t) = \mu\sqrt{-\frac{2}{\delta(1+\gamma)}}\operatorname{sech}\left(\mu\left(\frac{x^{\beta/2}}{\beta/2} + 2k\frac{t^{\alpha}}{\alpha}\right)\right)e^{i\left(-kx^{\beta/2}/\beta/2+\omega t^{\alpha}/\alpha+p\right)},$$
$$v(x,t) = -\mu\sqrt{-\frac{2}{\delta(1+\gamma)}}\operatorname{sech}\left(\mu\left(\frac{x^{\beta/2}}{\beta/2} + 2k\frac{t^{\alpha}}{\alpha}\right)\right)e^{i\left(-kx^{\beta/2}/\beta/2+\omega t^{\alpha}/\alpha+p\right)}, \tag{34}$$

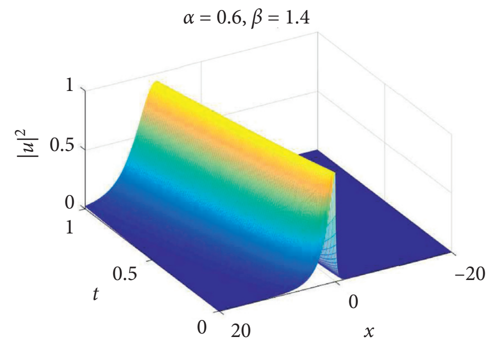

Figure 1: The trace of the solution to $|u|^2$ of FCNLSE (32) with $\alpha = 0.6$ and $\beta = 1.4$.

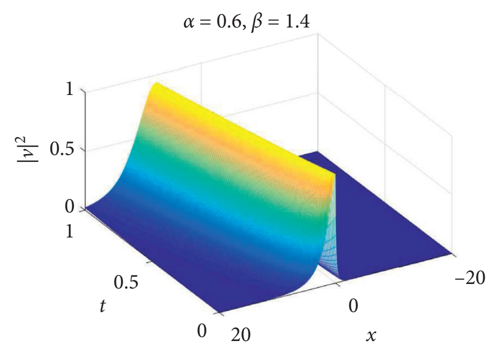

Figure 2: The trace of the solution to $|v|^2$ of FCNLSE (32) with $\alpha = 0.6$ and $\beta = 1.4$.

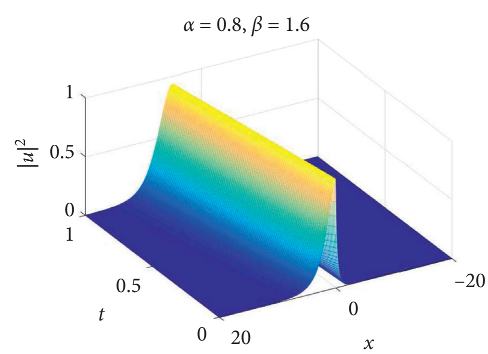

Figure 3: The trace of the solution to $|u|^2$ of FCNLSE (32) with $\alpha = 0.8$ and $\beta = 1.6$.

where $k = -\sqrt{\mu^2 - \omega}$ and $\mu^2 - \omega > 0$ for valid solitons.

We plot the traces of numerical solutions to FCNLSE (32) with initial conditions (33) using the one-step ETD scheme (15) and the central difference method (8) for different α and β values in Figures 1–4. It can be seen from the plots that $|u|^2$ and $|v|^2$ travel in the same pace and direction. This is due to the fact that u and v model vector solitons in a birefringent fiber. Because of the nonlinear coupling effect, the vector solitons can propagate undistorted even when the components have different widths or peak powers.

In Tables 1 and 2, the temporal convergence rates of the two-step ETD scheme (18) are computed according to formulas (30) and (31). The spatial step size is chosen as $h = 0.001$ which is relatively small. The experiments are performed for both $\alpha = 0.6$ and $\alpha = 0.8$. It can be noticed from the convergence rates that the order of convergence for $\alpha = 0.6$ is around 1.6, and the order of convergence for $\alpha = 0.8$ is around 1.8, which means the two-step ETD scheme (18) has a temporal order of $\{1 + \alpha\}$.

Secondly, we solve FCNLSE (32) with initial conditions

$$\begin{aligned} u(x,0) &= \operatorname{sech}(x), \\ v(x,0) &= -\operatorname{sech}(x), \end{aligned} \tag{35}$$

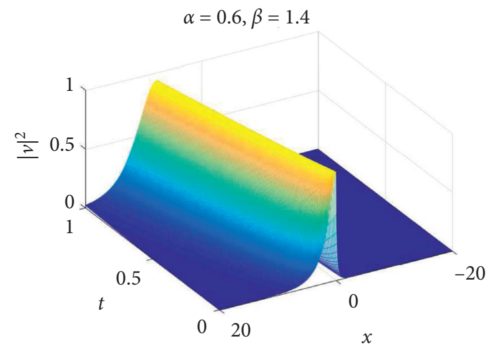

Figure 4: The trace of the solution to $|v|^2$ of FCNLSE (32) with $\alpha = 0.8$ and $\beta = 1.6$.

Table 1: Temporal convergence rates of the two-step ETD scheme with Padé approximation for FCNLSE (32) with $\alpha = 0.6$.

τ	$\mathrm{err}(\tau)$ for $\beta = 1.4$	Order p	$\mathrm{err}(\tau)$ for $\alpha = 1.8$	Order p
0.1	$2.3309e-3$	—	$2.2265e-3$	—
0.05	$7.6231e-4$	1.6124	$7.2183e-4$	1.6250
0.025	$2.4983e-4$	1.6094	$2.3503e-4$	1.6188
0.0125	$8.1755e-5$	1.6116	$7.6287e-5$	1.6233

Table 2: Temporal convergence rates of the two-step ETD scheme with Padé approximation for FCNLSE (32) with $\alpha = 0.8$.

τ	$\mathrm{err}(\tau)$ for $\beta = 1.4$	Order p	$\mathrm{err}(\tau)$ for $\alpha = 1.8$	Order p
0.1	$1.8527e-3$	—	$1.7532e-3$	—
0.05	$5.3631e-4$	1.7885	$4.9810e-4$	1.8155
0.025	$1.5159e-4$	1.8229	$1.3967e-4$	1.8344
0.0125	$4.2812e-5$	1.8241	$3.9562e-5$	1.8198

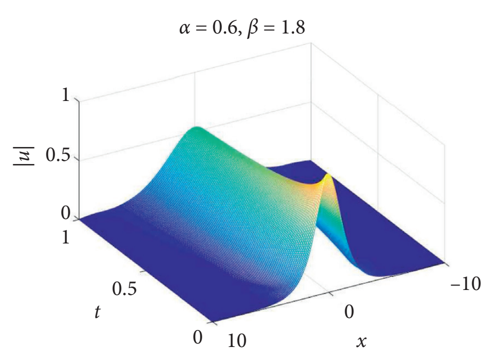

Figure 5: The trace of the solution to $|u|$ of FCNLSE (32) with $\alpha = 0.6$ and $\beta = 1.8$.

and homogeneous Dirichlet boundary conditions on $[-10, 10]$, where the parameters are chosen as $\delta = -1$ and $\gamma = 1$.

In Figures 5–12, the evolution traces of solutions to FCNLSE (32) with initial conditions (35) are depicted with different values of α and β, using the two-step ETD scheme (18) in time and the compact scheme (10) in space. It can be observed from the mesh plots that the absolute values of u and v remain the same, which means the magnitudes of the pulses remain identical, while the real parts of u and v remain opposite to each other. It can also be seen from the evolution profiles that different α and β values result in different diffusion effects and time delay effects.

In Table 3, the computation time is recorded solving FCNLSE (32) using the two-step ETD scheme (18) taking advantage of the Padé approximation (24) for different α and β values by counting the CPU time used in MATLAB. As we

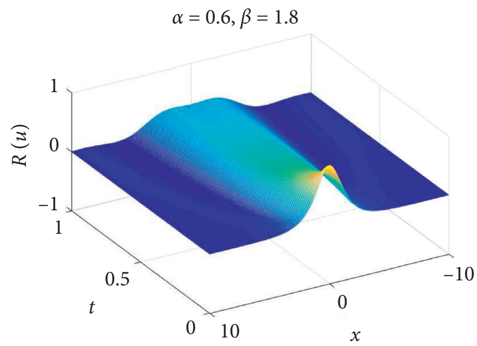

FIGURE 6: The trace of the solution to the real part of u with $\alpha = 0.6$ and $\beta = 1.8$.

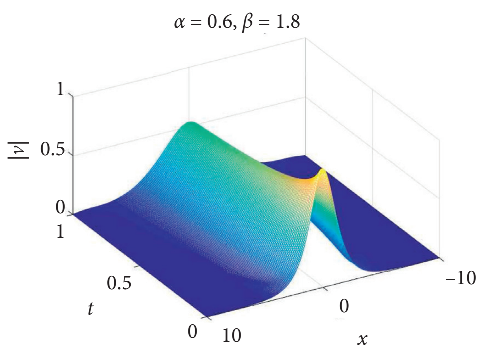

FIGURE 7: The trace of the solution to $|v|$ of FCNLSE (32) with $\alpha = 0.6$ and $\beta = 1.8$.

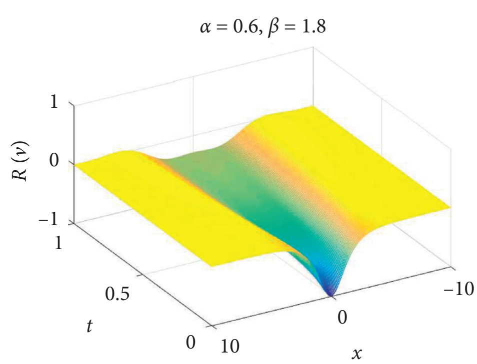

FIGURE 8: The trace of the solution to the real part of v with $\alpha = 0.6$ and $\beta = 1.8$.

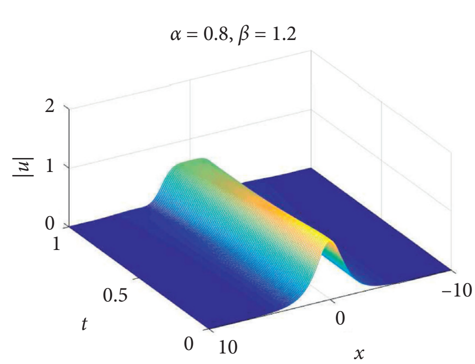

FIGURE 9: The trace of the solution to $|u|$ of FCNLSE (32) with $\alpha = 0.8$ and $\beta = 1.2$.

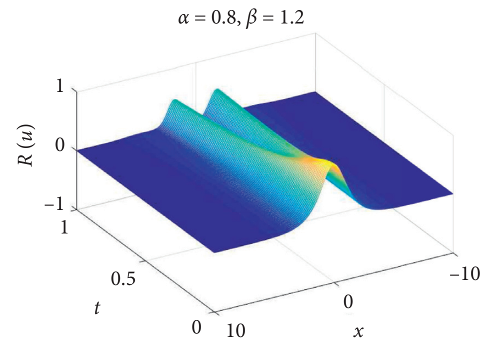

FIGURE 10: The trace of the solution to the real part of u with $\alpha = 0.8$ and $\beta = 1.2$.

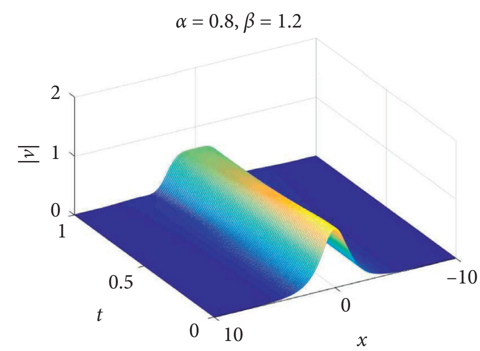

FIGURE 11: The trace of the solution to $|v|$ of FCNLSE (32) with $\alpha = 0.8$ and $\beta = 1.2$.

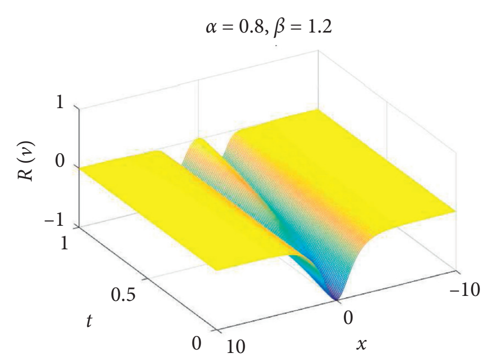

FIGURE 12: The trace of the solution to the real part of v with $\alpha = 0.8$ and $\beta = 1.2$.

TABLE 3: Computation time (CPU time in s) needed for solving FCNLSE (32) with initial conditions (35) via the two-step ETD scheme (18) utilizing the Padé approximation (24), taking $h = 0.05$ and $\tau = 0.01$.

α	0.3	0.3	0.3	0.6	0.6	0.6
β	1.2	1.5	1.8	1.2	1.5	1.8
$t = 0.25$	2.452	2.445	2.360	2.313	2.388	2.426
$t = 0.5$	4.736	4.823	4.633	4.577	4.579	4.752
$t = 0.75$	7.305	7.537	7.345	7.292	7.334	7.436
$t = 1$	9.238	9.426	9.332	9.443	9.563	9.573

took the similar experiments without using the Padé approximation, the CPU time needed for the computation is about 15 times longer. This indicates the efficiency and necessity of the Padé approximation for the ETD schemes.

6. Conclusion

To solve the space-time fractional coupled nonlinear Schrödinger equation efficiently, we employed exponential time-differencing schemes for the fractional derivative in time. During this process, the Mittag-Leffler function is computed using the Padé approximation. It has been shown in the numerical experiments that the Padé approximation reduces the computational time markedly compared to the original exponential time-differencing scheme. The error of the Padé approximation to the Mittag-Leffler function has been analyzed, and the convergence rates of the schemes have been computed and demonstrated in the Numerical Experiments section. Figures 1–4 express the bright soliton solutions, and Figures 5–12 depict orthogonally polarized optical waves in a birefringent fiber. The main contribution of this paper is the modification of the exponential time-differencing methods by applying the Padé approximation, as well as obtaining the soliton solutions to the fractional coupled nonlinear Schrödinger equation, which might be applicable in the industry of fiber optics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors contributed equally.

Acknowledgments

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References

[1] D. Anderson, "Variational approach to nonlinear pulse propagation in optical fibers," *Physical Review A*, vol. 27, no. 6, pp. 3135–3145, 1983.

[2] A. Aydin, "Multisymplectic integration of N-coupled nonlinear Schrödinger equation with destabilized periodic wave solutions," *Chaos Solitons and Fractals*, vol. 41, pp. 735–751, 2009.

[3] H. Bulut, T. A. Sulaiman, and B. Demirdag, "Dynamics of soliton solutions in the chiral nonlinear Schrödinger equations," *Nonlinear Dynamics*, vol. 91, no. 3, pp. 1985–1991, 2018.

[4] M. Eslami, "New exact solutions for higher order nonlinear Schrödinger equation in optical fibers," *Optical and Quantum Electronics*, vol. 50, Article ID 47, 2018.

[5] G. P. Agrawal, *Nonlinear Fiber Optics, Optics and Photonics Series*, Academic Press, San Diego, CA, USA, 2013.

[6] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 1999.

[7] X. Liang, "Efficient exponential time differencing methods with Padé approximations for the semilinear space-time-fractional Schrödinger equation," *Modern Physics Letters B*, vol. 34, Article ID 2050428, 2020.

[8] S. Longhi, "Fractional Schrödinger equation in optics," *Optics Letters*, vol. 40, no. 6, pp. 1117–1120, 2015.

[9] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, pp. 1–77, 2000.

[10] Q. Yang, F. Liu, and I. Turner, "Numerical methods for fractional partial differential equations with Riesz space fractional derivatives," *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 200–218, 2010.

[11] M. Lakestani and J. Manafian, "Analytical treatments of the space-time fractional coupled nonlinear Schrödinger equations," *Optical and Quantum Electronics*, vol. 50, Article ID 396, 2018.

[12] A. Esen, T. A. Sulaiman, H. Bulut, and H. M. Baskonus, "Optical solitons to the space-time fractional $(1+1)$-dimensional coupled nonlinear Schrödinger equation," *Optik*, vol. 167, pp. 150–156, 2018.

[13] A. Demir, M. A. Bayrak, and E. Ozbilge, "New approaches for the solution of space-time fractional Schrödinger equation," *Advances in Difference Equations*, vol. 23, Article ID 133, 2020.

[14] J. Zhang, J. Wang, and Y. Zhou, "Numerical analysis for time-fractional Schrödinger equation on two space dimensions," *Advances in Difference Equations*, vol. 23, Article ID 53, 2020.

[15] X. Liang and A. Q. M. Khaliq, "An efficient Fourier spectral exponential time differencing method for the space-fractional nonlinear Schrödinger equations," *Computers & Mathematics with Applications*, vol. 75, no. 12, pp. 4438–4457, 2018.

[16] B. Ghanbari, H. Günerhan, O. A. Ilhan, and H. M. Baskonus, "Some new families of exact solutions to a new extension of nonlinear Schrödinger equation," *Physica Scripta*, vol. 95, no. 7, pp. 1–12, 2020.

[17] W. Gao, H. F. Ismael, A. M. Husien, H. Bulut, and H. M. Baskonus, "Optical soliton solutions of the cubic-quartic nonlinear schrödinger and resonant nonlinear schrödinger equation with the parabolic law," *Applied Sciences*, vol. 10, no. 1, pp. 1–20, 2020.

[18] W. Gao, H. F. Ismael, S. A. Mohammed, H. M. Baskonus, and H. Bulut, "Complex and real optical soliton properties of the paraxial nonlinear Schrödinger equation in Kerr media with M-fractional," *Frontiers in Physics*, vol. 7, no. 197, pp. 1–8, 2019.

[19] R. Garrappa and M. Popolizio, "Generalized exponential time differencing methods for fractional order problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 876–890, 2011.

[20] I. Podlubny, A. Chechkin, T. Skovranek, Y. Chen, and B. M. Vinagre Jara, "Matrix approach to discrete fractional calculus II: partial fractional differential equations," *Journal of Computational Physics*, vol. 228, no. 8, pp. 3137–3153, 2009.

[21] M. D. Ortigueira, "Riesz potential operators and inverses via fractional centred derivatives," *International Journal of*

Mathematics and Mathematical Sciences, vol. 2006, pp. 1–12, 2006.

[22] H. F. Ding, C. P. Li, and Y. Q. Chen, "High-order algorithms for riesz derivative and their applications," *Abstract and Applied Analysis*, vol. 23, Article ID 653797, 2014.

[23] R. Garrappa, "Exponential integrators for time-fractional partial differential equations," *The European Physical Journal Special Topics*, vol. 222, no. 8, pp. 1915–1927, 2013.

[24] I. O. Sarumi, K. M. Furati, and A. Q. M. Khaliq, "Highly accurate global Padé approximations of generalized Mittag-CLeffler function and its inverse," *Journal of Scientific Computing*, vol. 82, Article ID 46, 2020.

[25] C. Zeng and Y. Chen, "Global Padé approximations of the generalized Mittag-Leffler function and its inverse," *Fractional Calculus and Applied Analysis*, vol. 18, pp. 1492–1506, 2015.

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Research Article

q-Hermite–Hadamard Inequalities for Generalized Exponentially $(s, m; \eta)$-Preinvex Functions

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In this article, we introduce a new extension of classical convexity which is called generalized exponentially $(s, m; \eta)$-preinvex functions. Also, it is seen that the new definition of generalized exponentially $(s, m; \eta)$-preinvex functions describes different new classes as special cases. To prove our main results, we derive a new ${}^{m\kappa_2}q$-integral identity for the twice ${}^{m\kappa_2}q$-differentiable function. By using this identity, we show essential new results for Hermite–Hadamard-type inequalities for the ${}^{m\kappa_2}q$-integral by utilizing differentiable exponentially $(s, m; \eta)$-preinvex functions. The results presented in this article are unification and generalization of the comparable results in the literature.

1. Introduction and Preliminaries

In mathematics, quantum calculus is equivalent to usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word "quantus," and literally, it means how much, in Swedish "Kvant"). It has two major branches: q-calculus and h-calculus. And both of them were worked out by Cheung and Kac [1] in the early twentieth century. In the same era, Jackson started to work on quantum calculus or q-calculus, but Euler and Jacobi had already figured out this type of calculus. A number of studies have recently been widely used in the field of q-analysis, beginning with Euler, due to the vast necessity for mathematics that models of quantum computing q-calculus exist in the framework between physics and mathematics. In 2013, Tariboon and Ntouyas introduced the ${}_{\kappa_1}Dq$-difference operator [2, 3]. This inspired other researchers, and as a consequence, numerous novel results concerning quantum analogues of classical mathematical results have already been launched in the literature. In various mathematical fields, it has many applications, such as theory of numbers, combinations, orthogonal polynomials, basic hypergeometric functions and other subjects, quantum mechanics, physics, and the principle of relativity. Many important aspects of quantum calculus are covered in the articles by Humaira et al. [4–7]. The quantum calculus is currently a subfield of the more general scientific field of time-scale calculus. New developments have recently been made in the research and methodology of dynamic derivatives on time scales. The research offers a consolidation and application of traditional differential and difference equations. Moreover, it is a unification of the discrete theory with the continuous theory, from the theoretical perspective. Recently, in 2020, Bermudo et al. introduced the notion of the ${}^{\kappa_2}Dq$-derivative and integral [8]. For more details, see [9–15] and references cited therein.

The discussion and application of convex functions has become a very rich source of motivational material in pure and applied science. This vision not only promoted new and profound results in many branches of mathematical and engineering sciences but also provided a comprehensive framework for the study of many problems. Many scholars have studied various classes of convex sets and convex functions; see [16, 17]. The concept of convexity has been extended in several directions, since these generalized versions have significant applications in different fields of pure and applied sciences. One of the convincing examples on extensions of convexity is the introduction of invex function, which was introduced by Hanson [18] Weir and Mond [19] explored the idea of preinvex functions and actualized it to the foundation of adequate optimality conditions and duality in nonlinear programming.

The Hermite–Hadamard inequality was introduced by Hermite and Hadamard; see [20]. It is one of the most recognized inequalities in the theory of convex functional analysis, which is stated as follows.

Let $F\colon \mathcal{Q}\subseteq\mathcal{R} \longrightarrow \mathcal{R}$ be a convex mapping and $\kappa_1, \kappa_2 \in \mathcal{Q}$ with $\kappa_1 < \kappa_2$. Then,

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathrm{d}\varkappa \le \frac{F(\kappa_1)+F(\kappa_2)}{2}. \quad (1)$$

If F is concave, both inequalities hold in the reverse direction.

The important objective of this paper is to introduce an exponentially generalized definition of $(s, m; \eta)$-preinvex functions. Furthermore, the new ${}^{m\kappa 2}q$-integral identity is determined. By using this new identity, we proved many new estimates of bounds for it, essentially based on the concept of quantum calculus.

2. Preliminaries

In this section, we derive a new definition of the generalized exponentially $(s, m; \eta)$-preinvex function. Also, we present all necessary concepts related to quantum calculus.

First of all, let $\mathcal{Q} \subset \mathcal{R}^n$ be a nonempty set, $F\colon \mathcal{Q} \longrightarrow \mathcal{R}$ be a continuous function, and $\eta\colon \mathcal{Q}\times\mathcal{Q}\times(0,1] \longrightarrow \mathcal{R}\setminus\{0\}$ and $\vartheta\colon \mathcal{Q}\times\mathcal{Q}\times(0,1] \longrightarrow \mathcal{R}^n$ be two continuous functions.

Definition 1. A set $\mathcal{Q}\subseteq\mathcal{R}^n$ is supposed to be η-invex concerning $\eta(\cdot,\cdot,\cdot)$ and $\vartheta(\cdot,\cdot,\cdot)$ with some fixed $m \in (0,1]$ if

$$m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m) \in \mathcal{Q}, \quad (2)$$

for all $\kappa_1, \kappa_2 \in \mathcal{Q}$ and $k \in [0,1]$.

If $\eta(\kappa_1,\kappa_2,m) = 1$, the above equation is called the convex set, and $\vartheta(\kappa_1,\kappa_2,m) = \kappa_1 - m\kappa_2$ is an invex set; however, the reverse is not possible.

Example 1. Consider $\mathcal{Q} = [-3,-2] \cup [-1,2]$ and

$$\vartheta(\kappa_1,\kappa_2,m) = \begin{cases} \kappa_1 - m\kappa_2 & \text{if } 2 \ge \kappa_2 \ge -1, 2 \ge \kappa_1 \ge -1, \\ \kappa_1 - m\kappa_2 & \text{if } -3 \le \kappa_2 \le -2, -3 \le \kappa_1 \le -2, \\ -1 - m\kappa_2 & \text{if } -3 \le \kappa_2 \le -2, -1 \le \kappa_1 \le 2, \\ -3 - m\kappa_2 & \text{if } -1 \le \kappa_2 \le 2, -3 \le \kappa_1 \le -2. \end{cases} \quad (3)$$

As one can see, $\mathcal{Q}$ is also an invex set for ϑ, but not a convex set.

Definition 2. A function $F\colon \mathcal{Q} \longrightarrow \mathcal{R}$ is said to be a generalized exponentially $(s, m; \eta)$-preinvex function if there exist $\eta(\cdot,\cdot,\cdot)$ and $\vartheta(\cdot,\cdot,\cdot)$, $\chi \ge 1$, and nonpositive α such that

$$F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) \le k^s\frac{F(\kappa_1)}{\chi^{\alpha\kappa_1}} + (1-k)^s\frac{F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}, \quad (4)$$

for all $\kappa_1, \kappa_2 \in \mathcal{Q}$ and $k \in [0,1]$ and for some fixed $s, m \in (0,1]$.

Remark 1. In Definition 2,

(1) If we choose $\alpha = 0$ or $\chi = 1$, then the definition of the generalized exponentially $(s, m; \eta)$-preinvex function is converted into the definition of the generalized $(s, m; \eta)$-preinvex function

(2) If we choose $\alpha = 0$ and $\eta(\kappa_1,\kappa_2,m) = 1$, then we get the definition of (s, m)-preinvexity

(3) If we choose $\alpha = 0$, $\eta(\kappa_1,\kappa_2,m) = 1$, and $\vartheta(\kappa_1,\kappa_2,m) = \kappa_1 - m\kappa_2$, then we get the definition of (s, m)-convexity

(4) If we choose $m = 1$ and $\vartheta(\kappa_1,\kappa_2,m) = 1$, we get the definition in [21]

(4) If we choose $\chi = e$, then we have the definition of exponentially $(s, m; \eta)$-preinvex functions, stated as follows

Definition 3. A function $F\colon \mathcal{Q} \longrightarrow \mathcal{R}$ is called exponentially $(s, m; \eta)$-preinvex if there exist $\eta(\cdot,\cdot,\cdot)$, $\vartheta(\cdot,\cdot,\cdot)$, and nonpositive α such that

$$F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) \le k^s\frac{F(\kappa_1)}{e^{\alpha\kappa_1}} + (1-k)^s\frac{F(m\kappa_2)}{e^{\alpha m\kappa_2}}, \quad (5)$$

for all $\kappa_1, \kappa_2 \in \mathcal{Q}$ and $k \in [0,1]$ and for some fixed $s, m \in (0,1]$.

Many researchers proved several results about the importance and development in the theory of exponentially convex functions and their applications. For more details, see [22–25] and references cited therein.

Jackson derived the q-Jackson integral in [12] from 0 to κ_2 for $0 < q < 1$ as follows:

$$\int_0^{\kappa_2} F(\varkappa)\,\mathrm{d}_q\varkappa = (1-q)\kappa_2\sum_{n=0}^{\infty} q^n F(\kappa_2 q^n), \quad (6)$$

provided the sum converges absolutely.

The q-Jackson integral in a generic interval $[\kappa_1, \kappa_2]$ was given by Jackson in [12] and defined as follows:

$$\int_{\kappa_1}^{\kappa_2} F(\varkappa)\,d_q\varkappa = \int_0^{\kappa_2} F(\varkappa)\,d_q\varkappa - \int_0^{\kappa_1} F(\varkappa)\,d_q\varkappa. \quad (7)$$

Definition 4 (see [3]). We suppose that $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathcal{R}$ is an arbitrary function. Then, the q_{κ_1}-derivative of F at $\varkappa \in [\kappa_1, \kappa_2]$ is defined as follows:

$$ {}_{\kappa_1}D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa + (1-q)\kappa_1)}{(1-q)(\varkappa - \kappa_1)}, \quad \varkappa \neq \kappa_1. \tag{8}$$

Since F is an arbitrary function from $[\kappa_1, \kappa_2]$ to $\mathscr{R}$, ${}_{\kappa_1}D_q F(\kappa_1) = \lim_{\varkappa \longrightarrow \kappa_1} {}_{\kappa_1}D_q F(\varkappa)$. The function F is said to be q-differentiable on $[\kappa_1, \kappa_2]$ if ${}_{\kappa_1}D_q F(t)$ exists for all $\varkappa \in [\kappa_1, \kappa_2]$. If $\kappa_1 = 0$ in (3), then ${}_0D_q F(\varkappa) = D_q F(\varkappa)$, where $D_q F(\varkappa)$ is a familiar q-derivative of F at $\varkappa \in [\kappa_1, \kappa_2]$ defined by the following expression (see [1]):

$$ D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa)}{(1-q)\varkappa}, \quad \varkappa \neq 0. \tag{9}$$

Definition 5 (see [8]). We suppose that $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ is an arbitrary function; then, the q^{κ_2}-derivative of F at $\varkappa \in [\kappa_1, \kappa_2]$ is defined as follows:

$$ {}^{\kappa_2}D_q F(\varkappa) = \frac{F(q\varkappa + (1-q)\kappa_2) - F(\varkappa)}{(1-q)(\kappa_2 - \varkappa)}, \quad \varkappa \neq \kappa_2. \tag{10}$$

Definition 6 (see [3]). We suppose that $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ is an arbitrary function; then, the q_{κ_1}-definite integral on $[\kappa_1, \kappa_2]$ is defined as follows:

$$ \begin{aligned} \int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q \varkappa &= (1-q)(\kappa_2 - \kappa_1) \sum_{n=0}^{\infty} q^n F(q^n \kappa_2 + (1-q^n)\kappa_1) \\ &= (\kappa_2 - \kappa_1) \int_0^1 F((1-t)\kappa_1 + t\kappa_2) d_q t. \end{aligned} \tag{11}$$

In [10], Alp et al. established the q_{κ_1}-Hermite–Hadamard inequalities for convexity, which are defined as follows.

Theorem 1. *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite–Hadamard inequalities are as follows:*

$$ F\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q \varkappa \leq \frac{qF(\kappa_1) + F(\kappa_2)}{[2]_q}. \tag{12}$$

On the contrary, the following new description and related Hermite–Hadamard-form inequalities were given by Bermudo et al.

Definition 7 (see [8]). Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be an arbitrary function. Then, the q^{κ_2}-definite integral on $[\kappa_1, \kappa_2]$ is defined as

$$ \begin{aligned} \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q \varkappa &= (1-q)(\kappa_2 - \kappa_1) \sum_{n=0}^{\infty} q^n F(q^n \kappa_1 + (1-q^n)\kappa_2) \\ &= (\kappa_2 - \kappa_1) \int_0^1 F(t\kappa_1 + (1-t)\kappa_2) d_q t. \end{aligned} \tag{13}$$

Theorem 2 (see [8]). *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite–Hadamard inequalities are as follows:*

$$ F\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q \varkappa \leq \frac{F(\kappa_1) + qF(\kappa_2)}{[2]_q}. \tag{14}$$

From Theorems 1 and 2, one can achieve the following inequalities.

Corollary 1 (see [8]). *For any convex function $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ and $0 < q < 1$, we have*

$$ F\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) + F\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \left\{ \int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q \varkappa + \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q \varkappa \right\} \leq F(\kappa_1) + F(\kappa_2), \tag{15}$$

and

$$ \begin{aligned} F\left(\frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{1}{2(\kappa_2 - \kappa_1)} \left\{ \int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q \varkappa + \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q \varkappa \right\} \\ &\leq \frac{F(\kappa_1) + F(\kappa_2)}{2}. \end{aligned} \tag{16}$$

Alp and Sarikaya, by using the area of trapezoids, introduced the following generalized quantum integral which we will call as $\kappa_1 T_q$-integral.

Definition 8 (see [11]). Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be an arbitrary function. For $\varkappa \in [\kappa_1, \kappa_2]$,

$$ \begin{aligned} \int_{\kappa_1}^{\kappa_2} F(\xi)_{\kappa_1} d_q^T &= \frac{(1-q)(\kappa_2 - \kappa_1)}{2q} \\ &\cdot \left[[2]_q \sum_{n=0}^{\infty} q^n F(q^n \kappa_2 + (1-q^n)\kappa_1) - F(\kappa_2) \right], \end{aligned} \tag{17}$$

where $0 < q < 1$.

Theorem 3 ($\overline{q}$-Hermite–Hadamard; see [11]). *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be a convex continuous function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, we have*

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q^T \varkappa \leq \frac{F(\kappa_1)+F(\kappa_2)}{2}. \tag{18}$$

Definition 9 (see [11]). For any real number n, the q analogue of n is defined as

$$[n]_q = \frac{1-q^n}{1-q}. \tag{19}$$

Definition 10 (see [11]). Let $k, p > 0$. Then, $\mathbb{B}_q(k,p)$ is defined by

$$\mathbb{B}_q(k,p) = \int_0^1 x^{k-1}(1-qx)_q^{p-1} d_q x. \tag{20}$$

3. A New ${}^{m\kappa_2}q$-Integral Identity

In this section, we present a new ${}^{m\kappa_2}q$-integral identity.

Lemma 1. *For $m \in (0,1]$ with $0<q<1$, let there be an arbitrary function $F\colon \mathcal{Q} \longrightarrow \mathcal{R}$ such that ${}^{m\kappa_2}D_2^q F$ is ${}^{m\kappa_2}q$-integrable on $\mathcal{Q}$. Then, one has*

$$\begin{aligned}{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x) &= \frac{F(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) + qF(m\kappa_2)}{[2]_q} \\ &\quad - \frac{1}{\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}\int_{m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}^{m\kappa_2} F(x)^{m\kappa_2} d_q x \\ &= \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2} D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) d_q k.\end{aligned} \tag{21}$$

Proof. We suppose that

$$\begin{aligned}&\int_0^1 k(1-qk)^{m\kappa_2} D_q^2 F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) d_q k \\
&= \int_0^1 k(1-qk)\left\{\frac{qF(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{k^2 q(1-q)^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} - \frac{[2]_q F(m\kappa_2+qk\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) + F\left(m\kappa_2+q^2k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{k^2 q(1-q)^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}\right\} d_q k \\
&= \frac{q\sum_{n=0}^{\infty} F(m\kappa_2+q^n\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) - [2]_q\sum_{n=0}^{\infty} F\left(m\kappa_2+q^{n+1}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} + \frac{\sum_{n=0}^{\infty} F\left(m\kappa_2+q^{n+2}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} \\
&\quad - q\left\{\frac{q(1-q)\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\sum_{n=0}^{\infty} q^n F(m\kappa_2+q^n\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q(1-q)^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\right. \\
&\quad - \frac{[2]_q(1-q)\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\sum_{n=0}^{\infty} q^{n+1} F\left(m\kappa_2+q^{n+1}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q^2(1-q)^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)} \\
&\quad + \frac{(1-q)\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\sum_{n=0}^{\infty} q^{n+2} F\left(m\kappa_2+q^{n+2}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q^3(1-q)^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)} \\
&= \frac{q(F(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) - F(m\kappa_2)) - F(m\kappa_2+q\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} \\
&\quad + \frac{F(m\kappa_2)}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} - \frac{[2]_q}{q^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\int_{m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}^{m\kappa_2} F(x)^{m\kappa_2} d_q x \\
&\quad - \frac{\left(q^2+q-1\right)F(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q^2(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} + \frac{F(m\kappa_2+q\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} \\
&= \frac{F(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) + q\Lambda(m\kappa_2)}{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)} - \frac{[2]_q}{q^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\int_{m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}^{m\kappa_2} F(x)^{m\kappa_2} d_q x.\end{aligned} \tag{22}$$

Multiplying both sides of the above equality by $q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)/[2]_q$, we get the required result. □

4. Hermite–Hadamard Inequalities for Generalized Exponentially $(s, m; \eta)$-Preinvex Functions

Theorem 4. *We assume that the conditions of Lemma 1 with $\chi \geq 1$ and $\alpha \in \mathscr{R}$ hold. If $|^{m\kappa_2}D_q^2 F|^{\mathfrak{u}}$ is a generalized exponentially $(s, m; \eta)$-preinvex function and $\mathfrak{u} \geq 1$, then for some fixed $s, m \in (0, 1]$, we have*

$$\left|^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}} \cdot \left(\Omega_1\left|\frac{^{m\kappa_2}D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \Omega_2\left|\frac{^{m\kappa_2}D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}, \tag{23}$$

where

$$\Omega_1 = \mathbb{B}_q(s+2, \mathfrak{u}+1), \tag{24}$$

$$\Omega_2 = 2^{1-s}\mathbb{B}_q(2, \mathfrak{u}+1) - \mathbb{B}_q(s+2, \mathfrak{u}+1). \tag{25}$$

Proof. By utilizing conditions of Lemma 1 and the famous power mean inequality, we obtain

$$\begin{aligned}
&\left|^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \\
&= \left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right| \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|d_qk \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 k d_qk\right)^{1-1/\mathfrak{u}} \times \left(\int_0^1 k(1-qk)^{\mathfrak{u}}\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}}\left(\int_0^1 k(1-qk)^{\mathfrak{u}}\left[k^s\left|\frac{^{m\kappa_2}D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{^{m\kappa_2}D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}}\left(\int_0^1 k(1-qk)_q^{\mathfrak{u}}\left[k^s\left|\frac{^{m\kappa_2}D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{^{m\kappa_2}D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}} \\
&= \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}}\left(\Omega_1\left|\frac{^{m\kappa_2}D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \Omega_2\left|\frac{^{m\kappa_2}D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}},
\end{aligned} \tag{26}$$

where

$$\Omega_1 = \int_0^1 k^{s+1}(1-qk)_q^{\mathfrak{u}} d_qk = \mathbb{B}_q(s+2, \mathfrak{u}+1), \tag{27}$$

and

$$\begin{aligned}\Omega_1 &= \int_0^1 k\left(2^{1-s} - k^s\right)(1-qk)_q^{\mathfrak{u}} d_qk \\ &= 2^{1-s}\mathbb{B}_q(2, \mathfrak{u}+1) - \mathbb{B}_q(s+2, \mathfrak{u}+1) \geq 0\end{aligned} \tag{28}$$

due to $2^{1-s} - k^s \geq 0$ for all $k \in [0, 1]$ and $s \in (0, 1]$.

We proved our result. □

Theorem 5. *We assume that the conditions of Lemma 1 with* $\chi \geq 1$ *and* $\alpha \in \mathscr{R}$ *hold. If* $|^{m\kappa_2}D_q^2 F|^{\mathfrak{u}}$ *is a generalized exponentially* $(s,m;\eta)$*-preinvex function and* $\mathfrak{u} > 1$ *with* $\mathfrak{p}^{-1} + \mathfrak{u}^{-1} = 1$*, then for some fixed* $s, m \in (0,1]$*, we obtain*

$$\left|^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)^{1/\mathfrak{p}}\mathbb{B}q(2,\mathfrak{p}+1)}{[2]_q} \times \left(\frac{[2]_q\left|^{m\kappa_2}D_q^2 F(\kappa_1)/\chi^{\alpha\kappa_1}\right|^{\mathfrak{u}} + \left(2^{1-s}[s+2]_q - [2]_q\right)\left|m\kappa_2 D_q^2 F(m\kappa_2)/\chi^{\alpha m\kappa_2}\right|^{\mathfrak{u}}}{[2]_q[s+2]_q}\right)^{1/\mathfrak{u}}. \tag{29}$$

Proof. By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}
&\left|^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \\
&= \left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_q k\right| \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|d_q k \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 k(1-qk)^{\mathfrak{p}}d_q k\right)^{1/\mathfrak{p}} \times \left(\int_0^1 k\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_q k\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 k(1-qk)_q^{\mathfrak{p}}d_q k\right)^{1/\mathfrak{p}} \times \left(\int_0^1 k\left[k^s\left|\frac{^{m\kappa_2}D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{^{m\kappa_2}D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_q k\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)\mathbb{B}_q^{1/\mathfrak{p}}(2,\mathfrak{p}+1)}{[2]_q} \times \left(\frac{[2]_q\left|^{m\kappa_2}D_q^2 F(\kappa_1)/\chi^{\alpha\kappa_1}\right|^{\mathfrak{u}} + \left(2^{1-s}[s+2]_q - [2]_q\right)\left|^{m\kappa_2}D_q^2 F(m\kappa_2)/\chi^{\alpha m\kappa_2}\right|^{\mathfrak{u}}}{[2]_q[s+2]_q}\right)^{1/\mathfrak{u}}.
\end{aligned} \tag{30}$$

This completes the proof. □

Theorem 6. *We assume that the conditions of Lemma 1 with* $\chi \geq 1$ *and* $\alpha \in \mathscr{R}$ *hold. If* $|^{m\kappa_2}D_q^2 F|^{\mathfrak{u}}$ *is a generalized exponentially* $(s,m;\eta)$*-preinvex function and* $\mathfrak{u} > 1$ *with* $\mathfrak{p}^{-1} + \mathfrak{u}^{-1} = 1$*, then for some fixed* $s, m \in (0,1]$*, we obtain*

$$\left|^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q} \cdot \left(\mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\left|\frac{m\kappa_2 D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \left(\left(2^{1-s}\mathbb{B}_q(\mathfrak{u}+1,\mathfrak{u}+1) - \mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\right)\left|\frac{m\kappa_2 D_q^2 F(\kappa_1)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}. \tag{31}$$

Proof. By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}
&\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \\
&=\left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right| \\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk \\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 1d_qk\right)^{1/\mathfrak{p}}\times\left(\int_0^1 k^{\mathfrak{u}}(1-qk)^{\mathfrak{u}}\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{\frac{1}{\mathfrak{u}}} \\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\times\left(\int_0^1 k^{\mathfrak{u}}(1-qk)^{\mathfrak{u}}\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}+(1-k)^s\left|\frac{m\kappa_2D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}} \\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\left|\frac{m\kappa_2D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}\right. \\
&\quad\left.+\left(\left(2^{1-s}\mathbb{B}_q(\mathfrak{u}+1,\mathfrak{u}+1)-\mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\right)\left|\frac{m\kappa_2D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)\right)^{1/\mathfrak{u}},
\end{aligned}\tag{32}$$

where

$$2^{1-s}\mathbb{B}_q(\mathfrak{u}+1,\mathfrak{u}+1)-\mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\geq 0. \tag{33}$$

We proved our result. □

Theorem 7. *We assume that the conditions of Lemma 1 with $\chi\geq 1$ and $\alpha\in\mathscr{R}$ hold. If $|{}^{m\kappa_2}D_q^2F|^{\mathfrak{u}}$ is a generalized exponentially $(s,m;\eta)$-preinvex function and $\mathfrak{u}>1$ with $\mathfrak{p}^{-1}+\mathfrak{u}^{-1}=1$, then for some fixed $s,m\in(0,1]$, we obtain*

$$\begin{aligned}
\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right|&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)\mathbb{B}_q^{1/\mathfrak{p}}(\mathfrak{p}+1,\mathfrak{p}+1)}{[2]_q} \\
&\quad\times\left(\vartheta_1\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}+\vartheta_2\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}},
\end{aligned}\tag{34}$$

where

$$\begin{aligned}
\vartheta_1&=\frac{1-q}{1-q^{s+1}}=\frac{1}{[s+1]_q}, \\
\vartheta_2&=(1-q)\sum_{n=0}^{\infty}q^n(1-q^n)^s.
\end{aligned}\tag{35}$$

Proof. By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}
&\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \\
&=\left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right| \\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|d_qk \\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 k^p(1-qk)^pd_qk\right)^{1/p}\times\left(\int_0^1\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{1/\mathfrak{u}} \\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)\mathbb{B}_q^{1/\mathfrak{p}}(\mathfrak{p}+1,\mathfrak{p}+1)}{[2]_q}\times\left(\int_0^1\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}+(1-k)^s\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}}.
\end{aligned}\tag{36}$$

Applying the definition of quantum integral, we get

$$\vartheta_1 = \int_0^1 k^s\, {}_0d_qk = \frac{1-q}{1-q^{s+1}} = \frac{1}{[s+1]_q},$$
$$\vartheta_2 = \int_0^1 (1-k)^s\, {}_0d_qk = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^n\right)^s. \tag{37}$$

This completes the proof. □

Theorem 8. *We assume that the conditions of Lemma 1 with $\chi \geq 1$ and $\alpha \in \mathscr{R}$ hold. If $|^{m\kappa_2}D_q^2F|^{\mathfrak{u}}$ is a generalized exponentially $(s,m;\eta)$-preinvex function and $\mathfrak{u}>1$ with $\mathfrak{p}^{-1}+\mathfrak{u}^{-1}=1$, then for some fixed $s,m\in(0,1]$, we obtain*

$$\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q[\mathfrak{p}+1]_q^{1/\mathfrak{p}}} \times \left(\omega_1\left|\frac{m\kappa_2 D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \omega_2\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}, \tag{38}$$

where

$$\omega_1 = (1-q)\sum_{n=0}^{\infty} q^{n(s+1)}\left(1-q^{n+1}\right)^{\mathfrak{u}}, \omega_2 = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^n\right)^s\left(1-q^{n+1}\right)^{\mathfrak{u}}. \tag{39}$$

Proof. By utilizing conditions of Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \\
&= \left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right| \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|d_qk \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 k^{\mathfrak{p}}d_qk\right)^{1/\mathfrak{p}} \times\left(\int_0^1 (1-qk)^{\mathfrak{u}}\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q[\mathfrak{p}+1]_q^{1/\mathfrak{p}}} \times\left(\int_0^1 (1-qk)^{\mathfrak{u}}\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}}.
\end{aligned} \tag{40}$$

Applying the definition of quantum integral, we get

$$\omega_1 = \int_0^1 k^s(1-qk)^{\mathfrak{u}}\,{}_0d_qk = (1-q)\sum_{n=0}^{\infty} q^{n(s+1)}\left(1-q^{n+1}\right)^{\mathfrak{u}}, \omega_2 = \int_0^1 (1-k)^s(1-qk)^{\mathfrak{u}}\,{}_0d_qk = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^n\right)^s\left(1-q^{n+1}\right)^{\mathfrak{u}}. \tag{41}$$

This completes the proof. □

Theorem 9. *We assume that the conditions of Lemma 1 with $\chi \geq 1$ and $\alpha \in \mathscr{R}$ hold. If $|^{m\kappa_2}D_q^2 F|^{\mathfrak{u}}$ is a generalized exponentially $(s,m;\eta)$-preinvex function and $\mathfrak{u} > 1$ with $\mathfrak{p}^{-1} + \mathfrak{u}^{-1} = 1$, then for some fixed $s, m \in (0,1]$, we obtain*

$$\left|^{m\kappa_2} L_q(\kappa_1,\kappa_2,m,x)\right| \leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)\mathbb{B}_q^{1/\mathfrak{p}}(1,\mathfrak{p}+1)}{[2]_q} \times \left(\sigma_1\left|\frac{m\kappa_2 D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \sigma_2\left|\frac{m\kappa_2 D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}, \quad (42)$$

where

$$\sigma_1 = \frac{1}{[s+\mathfrak{u}+1]_q}, \qquad \sigma_2 = (1-q)\sum_{n=0}^{\infty} q^{n(1+\mathfrak{u})}(1-q^n)^s. \quad (43)$$

Proof. By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}
&\left|^{m\kappa_2} L_q(\kappa_1,\kappa_2,m,x)\right| \\
&= \left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\mathrm{d}_q k\right| \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right| d_q k \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 (1-qk)^{\mathfrak{p}} d_q k\right)^{1/\mathfrak{p}} \times \left(\int_0^1 k^{\mathfrak{u}}\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}} d_q k\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)\mathbb{B}_q^{1/\mathfrak{p}}(1,\mathfrak{p}+1)}{[2]_q} \times \left(\int_0^1 k^{\mathfrak{u}}\left[k^s\left|\frac{^{m\kappa_2}D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{^{m\kappa_2}D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right] d_q k\right)^{1/\mathfrak{u}}.
\end{aligned} \quad (44)$$

Applying the definition of quantum integral, we get

$$\begin{aligned}
\sigma_1 &= \int_0^1 k^{s+\mathfrak{u}}\,{}_0d_q k = \frac{1}{[s+\mathfrak{u}+1]_q}, \\
\sigma_2 &= \int_0^1 k^{\mathfrak{u}}(1-k)^s\,{}_0d_q k = (1-q)\sum_{n=0}^{\infty} q^{n(1+\mathfrak{u})}(1-q^n)^s.
\end{aligned} \quad (45)$$

This completes the proof. □

5. Conclusion

In this article, we established the new definition of generalized exponentially $(s,m;\eta)$-preinvex functions and proved a new modified $^{m\kappa_2}q$-integral identity. Using this new identity, we have been able to obtain new estimates of the quantum bounds applying the concept of generalized exponentially $(s,m;\eta)$-preinvex functions. It is worth to mention here that if we take $\chi = e$, then all of the main results reduce to the results for exponentially $(s,m;\eta)$-preinvex functions. For further research, we could expand the inequality-based analysis to other fields, including the inequality-based theory, quantum calculus, machine learning, robotics, weather forecasting, and optimizations.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

This study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

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References

[1] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, NY, USA, 2002.

[2] J. Tariboon and S. K. Ntouyas, "Quantum integral inequalities on finite intervals," *Journal of Inequalities and Applications*, vol. 2014, p. 121, 2014.

[3] J. Tariboon and S. K. Ntouyas, "Quantum calculus on finite intervals and applications to impulsive difference equations," *Advances in Difference Equations*, vol. 2013, p. 282, 2013.

[4] H. Kalsoom, M. Idrees, M. Idrees, A. Uzair Awan, and Y.-M. Chu, "Some new $ (p_1p_2,q_1q_2) $-estimates of ostrowski-type integral inequalities via n-polynomials s-type convexity," *AIMS Mathematics*, vol. 5, no. 6, pp. 7122–7144, 2020.

[5] H. Kalsoom, M. A. Latif, S. Rashid, D. Baleanu, and Y. M. Chu, "New $(p, q)\alpha$-estimates for different types of integral inequalities via (m)-convex mappings," *Open Mathematics*, vol. 18, no. 1, pp. 1830–1854, 2020.

[6] H. Kalsoom, M. Amer, M. U. D. Junjua, S. Hussain, and G. Shahzadi, "Some (p, q)-estimates of hermite-hadamard-type inequalities for coordinated convex and quasi-convex functions," *Mathematics*, vol. 7, no. 8, p. 683, 2019.

[7] H. Kalsoom, J.-D. Wu, S. Hussain, and M. A. Latif, "Simpson's type inequalities for Co-ordinated convex functions on quantum calculus," *Symmetry*, vol. 11, no. 6, p. 768, 2019.

[8] S. Bermudo, P. Kórus, and J. N. Valdés, "On q-Hermite–Hadamard inequalities for general convex functions," *Acta Mathematica Hungarica*, vol. 162, pp. 364–374, 2020.

[9] T. Ernst, *A Comprehensive Treatment of Q-Calculus*, Springer, Basel, Switzerland, 2012.

[10] N. Alp, M. Z. Sarıkaya, M. Kunt, and İ. İşcan, "q-hermite hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions," *Journal of King Saud University-Science*, vol. 30, no. 2, pp. 193–203, 2018.

[11] N. Alp and M. Z. Sariakaya, "A new definition and properties of quantum integral which calls $\overline{q}$-integral," *Konuralp Journal of Mathematics*, vol. 5, no. 2, pp. 146–159, 2017.

[12] F. H. Jackson, "On a q-definite integrals," *Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[13] W. Sudsutad, S. K. Ntouyas, and J. Tariboon, "Quantum integral inequalities for convex functions," *Journal of Mathematical Inequalities*, vol. 9, no. 3, pp. 781–793, 2015.

[14] H. Zhuang, W. Liu, and J. Park, "Some quantum estimates of hermite-hadamard inequalities for quasi-convex functions," *Mathematics*, vol. 7, no. 2, p. 152, 2019.

[15] N. Alp and M. Z. Sarikaya, "Hermite hadamard's type inequalities for co-ordinated convex functions on quantum integral," 2013.

[16] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, UK, 2013.

[17] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization*, Springer, New York, NY, USA, 2000.

[18] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[19] T. Weir and B. Mond, "Pre-invex functions in multiple objective optimization," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 29–38, 1988.

[20] J. Hadamard, "Etude sur les propri et ´ es des fonctions enti ´ eres et en particulier dune fonction consider ´ ee par Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–215, 1893.

[21] M. U. Awan, S. Talib, A. Kashuri, M. A. Noor, K. I. Noor, and Y. M. Chu, "A new q-integral identity and estimation of its bounds involving generalized exponentially μ-preinvex functions," *Advances in Difference Equations*, vol. 2020, no. 575, pp. 1–12, 2020.

[22] S. I. Butt, A. Kashuri, M. Tariq, J. Nasir, A. Aslam, and W. Gao, "Hermite–hadamard-type inequalities via n-polynomial exponential-type convexity and their applications," *Advances in Difference Equations*, vol. 2020, p. 508, 2020.

[23] S. I. Butt, M. Tariq, A. Aslam, H. Ahmad, and T. A. Nofal, "Hermite–hadamard type inequalities via generalized harmonic exponential convexity and applications," *Journal of Function Spaces*, vol. 2021, Article ID 5533491, 12 pages, 2021.

[24] S. I. Butt, R. Jakšic, L. Kvesic, and J. Pecaric, "n-exponential convexity of weighted hermite-hadamard's inequality," *Journal of Mathematical Inequalities*, vol. 8, no. 2, pp. 299–311, 2014.

[25] S. I. Butt, J. Pečarić, and A. U. Rehman, "Exponential convexity of petrović and related functional," *Journal of Inequalities and Applications*, vol. 2011, no. 1, pp. 1–16, 2011.

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Research Article

On Some Classes with Norms of Meromorphic Function Spaces Defined by General Spherical Derivatives

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The main concerned target of this article is to define and study some concerned classes of meromorphic function spaces using the general spherical derivatives. The general Besov-type classes of meromorphic functions as well as the general normal functions are considered intensively and both are compared deeply with each other. Specifically, multiple results concerning general meromorphic-type classes as well as non-normal classes are obtained by the help of general spherical derivatives. The concerned results are proved by constructing some specific mild conditions on the sequences of points belonging to the concerned meromorphic-type classes. The obtained results generalize and improve the corresponding previous results in some concerned respects. The concerned proofs and methods are simply presented.

1. Introduction

The area of complex function spaces is fundamental and essential in many branches of pure and applied mathematics. Some decades ago, there have been obvious interests on meromorphic function classes, from concerned point of view of their singularities. For various studies on meromorphic function spaces, we may refer to all citations therein. As a concerned result, some new general classes of meromorphic functions shall be introduced by using the general spherical derivatives, which will be associated to obtain the new classes of meromorphic function spaces. Fundamental concerned properties of these concerned aforementioned meromorphic-type classes which include generalizations of meromorphic Besov spaces as well as normal function classes shall be studied and intensively discussed. As a concerned consequence of our investigation, some relevant special cases can be pointed out. Furthermore, to capture some new generalized results under the current concerned proofs, some new concepts and definitions are introduced. Let $U = \{w\colon |w| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $dm(w)$ be the usual Euclidean area element on U. The symbol $M(U)$ stands for the concerned class of all meromorphic functions in U. The pseudohyperbolic metric between the points w and z is defined by $d(z,w) = |\varphi_z(w)|$. For $0 < R < 1$, assume that $U(a,R) = \{w \in U\colon d(w,a) < r\}$ defines the concerned pseudohyperbolic disc which is centered $a \in U$ with the specific radius R. For $0 < q < \infty$ and $0 < s < \infty$, the classes $M^{\#}(p,q,s)$ are defined by (see [1] pp.10)

$$M^{\#}(p,q,s) = \left\{h \in M(U)\colon \sup_{a\in U} \iint_U \left(h^{\#}(w)\right)^p \left(1-|w|^2\right)^q \left(1-|\varphi_a(w)|^2\right)^s dm(w) < \infty\right\}, \quad (1)$$

where $h^{\#}(w) = (|h\prime(w)|/(1+|h(w)|^2))$ is the usual spherical derivative of h. The meromorphic $M^{\#}(q, q-2, 0)$ classes are called the meromorphic Besov classes and denoted by $B_q^{\#}$, for which

$$B_q^{\#} = \left\{ h \in M(U)\colon \sup_{a\in U} \iint_U \left(h^{\#}(w)\right)^q \left(1-|w|^2\right)^{q-2} dm(w) < \infty \right\}. \tag{2}$$

For the analytic corresponding classes of Besov spaces, we cite [2–7]. In this article, the general meromorphic Besov-type classes always refer to the concerned classes $B^{\#}(q, q-2, s; n)$. Using the general spherical derivative $(|h^{(n)}(w)|/(1+|h(w)|^{n+1}))$ (see [8]), we give the following general meromorphic spaces.

Let $n \in \mathbb{N}$, $0 < q < \infty$, $0 < s < \infty$. Then, the general meromorphic Besov-type spaces are defined by

$$B^{\#}(q, q-2, s; n) = \left\{ h \in M(U)\colon \sup_{a\in U} \int_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \left(1-|w|^2\right)^{q-2} \left(1-|\varphi_a(w)|^2\right)^s dm(w) < \infty \right\}, \tag{3}$$

where the concerned weight function is $(1-|w|^2)^{q-2}(1-|\varphi_a(w)|^2)^s$ and $w \in U$. Here, $\varphi_a(w)$ denotes the usual Möbius transformation $\varphi_a(w) = ((a-w)/(1-\bar{a}w))$. Also,

$$h^{(n)}(w) = \frac{d^n h(w)}{dw^n}, \text{ that is, we have “}n\text{” times derivatives, } n \in \mathbb{N}. \tag{4}$$

The concerned meromorphic counterpart of the Bloch-type space is the class of all concerned normal functions $\mathcal{N}$ (see [1, 9]); this class of meromorphic functions can be extended to the following concerned class.

Definition 1. Assume that h is a meromorphic function in U. When

$$\|h\|_{\mathcal{N}_n} = \sup_{w\in U} \left(1-|w|^2\right) \frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}} < \infty, \tag{5}$$

$h \in \mathcal{N}_n$ of concerned normal functions.

Definition 2. Suppose that the function h stands for a concerned meromorphic function in U. The concerned sequence of points $\{a_m\}(|a_m| \longrightarrow 1)$ in U is called a $q_{(N,n)}$-sequence if

$$\lim_{m\longrightarrow\infty} \frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}} \left(1-|a_m|^2\right) = +\infty. \tag{6}$$

In Definitions 2, by letting $n = 1$, we obtain the class of all usual normal functions $\mathcal{N}$ (see [1, 9]). For more interesting various studies on different meromorphic function classes, we refer to [10–15] and others. The following definitions can be introduced.

Definition 3. Assume that h is a meromorphic function in U. For $2 < q < \infty$ and $0 < s < \infty$, the concerned sequence of points $\{a_m\}(|a_m| \longrightarrow 1)$ in U is called a $b_{(q,n)}$-sequence if

$$\lim_{m\longrightarrow\infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \left(1-|w|^2\right)^{q-2} \left(1-|\varphi_{a_m}(w)|^2\right)^s dm(w) = +\infty. \tag{7}$$

2. Families of $\mathbf{b}_{(q,n)}$- and $\mathbf{q}_{(N,n)}$-Type Sequences

Theorem 1. *Let $h \in M(U)$. Suppose that $\{a_m\}$ defines the $q_{(N,n)}$–type sequence, thus any sequence of points $\{c_m\}$ in U, such that $d(a_m, c_m) \longrightarrow 0$ is a $b_{(q,n)}$-type sequence for all values of q with $2 < q < \infty$.*

Proof. In view of [16], we can find two concerned sequences $\{c_m\} \subset U$ and $\{d_m\} \subset \mathbb{R}^+$, with $d(a_m, c_m) \longrightarrow 0$ with

$$\frac{d_m}{\left(1-|c_m|^2\right)} \longrightarrow 0, \tag{8}$$

where the concerned sequence of functions $\{h_m(t)\} = \{h(b_m + d_m t)\}$ has uniformly converging type on each concerned compact subset of $\mathbb{C}$ to a concerned nonconstant meromorphic function $G(t)$. Thus,

$$\begin{aligned}
&\sup_{c_m \in U} \iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(w)\right|^2\right)^s dm(w)\\
&\geq \iint_{U(c_m,(1/e))} \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(w)\right|^2\right)^s dm(w)\\
&\geq \iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1+|h_m(t)|^{n+1}}\right]^q \left(1-\left|c_m+d_m t\right|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(c_m+d_m t)\right|^2\right)^s d_m^{2-q}\, dm(t)\\
&= \iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1+|h_m(t)|^{n+1}}\right]^q \left(\frac{1-\left|c_m+d_m t\right|^2}{d_m}\right)^{q-2} \times\left(\left|\frac{c_m-(c_m+d_m t)}{1-\bar{c}_m(c_m+d_m t)}\right|^2\right)^s dm(t)\\
&= \iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1+|h_m(t)|^{n+1}}\right]^q \left(\frac{1-\left|c_m+d_m t\right|^2}{d_m}\right)^{q-2} \times\left(1-\left|\frac{1}{\left(\left(1-|c_m|^2\right)/d_m t\right)-\bar{c}_m}\right|^2\right)^s dm(t).
\end{aligned} \tag{9}$$

Using the uniform convergence techniques, we deduce that

$$\iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1+|h_m(t)|^{n+1}}\right]^q dm(t) \longrightarrow \iint_{U(0,r)} \left(G_n^{\#}(t)\right)^q dm(t), \tag{10}$$

where the last defined integral is positive, since $G(t)$ is a concerned nonconstant meromorphic function. Further, by (6), when $m \longrightarrow \infty$, we conclude that

$$1-\left|\frac{1}{\left(\left(1-|c_m|^2\right)/d_m t\right)-\bar{c}_m}\right|^2 \longrightarrow 1. \tag{11}$$

Therefore, we can obtain that

$$\iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1+|h_m(t)|^{n+1}}\right]^q \left(\frac{1-\left|a_m+d_m t\right|^2}{d_m}\right)^{q-2} \left(1-\left|\frac{1}{\left(\left(1-|a_m|^2\right)/d_m t\right)-\bar{a}_m}\right|^2\right)^s dm(t) \longrightarrow \infty. \tag{12}$$

Hence, when $2<q<\infty$, we have that

$$\iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(w)\right|^2\right)^s dm(w) \longrightarrow \infty, \tag{13}$$

Thus, $\{c_m\} \in U$ is a $b_{(q,n)}$-type sequence for all q, with $2<q<\infty$. The proof of Theorem 1 is completely finished. □

Theorem 2. *We can find a concerned non-normal function h and a concerned sequence $\{a_m\}$ in U which is a $b_{(q,n)}$-sequence for all q, with $2<q<\infty$, whereas $\{a_m\}$ is not a $q_{(N,n)}$-sequence.*

Proof. Assume that the function $h(w)=\exp(i/(1-w))$ is a non-normal function where $i=\sqrt{-1}$. Considering the concerned sequence $\{c_m\}=\{m^2/(1+m^2)\}$, after simple computation, we deduce that

$$\lim_{m\longrightarrow\infty}\left(1-|c_m|^2\right)\frac{\left|h^{(n)}(c_m)\right|}{1+\left|h(c_m)\right|^{n+1}} = +\infty. \tag{14}$$

Applying Theorem 1 for any concerned sequence of specific points $\{a_m\}$ in U, with $d(a_m,c_m)\longrightarrow 0$, we get

$$\lim_{m\longrightarrow\infty}\iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \left(1-|w|^2\right)^{q-2} \cdot\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s dm(w) = +\infty, \tag{15}$$

for all q, with $2<q<\infty$. Let $\{a_m\}=\{(m^2/(1+m^2))-(i/(m+m^3))\}$, and note that $d(a_m,c_m)\longrightarrow 0$. But

$$\lim_{m\longrightarrow\infty}\left(1-|a_m|^2\right)\frac{\left|h^{(n)}(a_m)\right|}{1+\left|h(m_m)\right|^{n+1}} = 0. \tag{16}$$

Hence, the concerned sequence $\{a_m\}$ is our needed sequence of points. □

Theorem 3. *Let $h \in M(U)$ and suppose that $2 < q_1 < q < \infty$ and $0 < s_1 < s < \infty$. For a concerned sequence of points $\{a_m\}$ in the disc U, when*

$$\lim_{m \longrightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s dm(w) = +\infty, \tag{17}$$

$$\lim_{m \longrightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q_1} \left(1-|w|^2\right)^{q_1-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^{s_1} dm(w) = +\infty. \tag{18}$$

Proof. When condition (17) holds, then for $2 < q_1 < q < \infty$ and $0 < s_1 < s < \infty$, using the known inequality of Hölder, we conclude that

$$\begin{aligned}
&\iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q_1} \left(1-|w|^2\right)^{q_1-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^{s_1} dm(w) \\
&\leq \left(\iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q} \left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^{s} dm(w)\right)^{(q_1/q)} \\
&\times\left(\iint_U \left(1-\left|\varphi_{a_m}(w)\right|^2\right)^{(s_1-(sq_1/q))(q/(q-q_1))}\left(1-|w|^2\right)^{-2} dm(w)\right)^{(1-(q_1/q))} \\
&= \left(\iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q} \left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^{s} dm(w)\right)^{(q_1/q)} \\
&\times\left(\iint_U \left(1-|w|^2\right)^{((s_1q-sq_1)/(q-q_1)-2)} dm(w)\right)^{(1-(q_1/q))}.
\end{aligned} \tag{19}$$

It is obvious to see that $((s_1q-sq_1)/(q-q_1)-2) = (\eta-2) > -1$, for $\eta > 1$, and we obtain

$$\iint_U \left(1-|w|^2\right)^{((s_1q-sq_1)/(q-q_1)-2)} dm(w) = \iint_U \left(1-|w|^2\right)^{(\eta-2)} dm(w) < C_1 < \infty, \tag{20}$$

where $C_1 > 0$. Therefore, $B^{\#}(q, q-2, s; n) \subset B^{\#}(q_1, q_1-2, s_1; n)$.

Thus, the following inequality can be followed:

$$\begin{aligned}
&\iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q_1} \left(1-|w|^2\right)^{q_1-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^{s_1} dm(w) \\
&\geq \iint_U \left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q} \left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^{s} dm(w) = +\infty.
\end{aligned} \tag{21}$$

Then, condition (18) must be verified. Thus, the proof is established completely. □

Remark 1. Using the specific condition (17), we deduce that the function h not in the classes $B^{\#}(q,q-2,s;n)$, this because the concerned meromorphic classes $B^{\#}(q,q-2,s;n)$ have a specific nesting property and the meromorphic function h is not belong to the meromorphic classes $B^{\#}(q_1,q_1-2,s_1)$ when $2<q_1<q<\infty$ and $0<s_1<s<\infty$. Nevertheless, Theorem 3 shows further details on this case which clearing that the similar concerned sequence of points $\{a_m\}$, for which $B^{\#}(q,q-2,s;n)$-condition can be excluded, also it excludes the $B^{\#}(q_1,q_1-2,s_1;n)$–condition.

Remark 2. From the concerned proof of Theorem 3, we can clearly show that for a fixed ρ_0, $0<\rho_0<1$ and $\rho>0$, when

$$\lim_{m\longrightarrow\infty}\iint_{U(a_m,\rho_0)}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w)=+\infty. \tag{22}$$

Thus, we can find a concerned sequence of points $\{c_m\}$ in $\Delta_\rho^n=\left\{w\colon\ (1-|\varphi_a(w)|^2)>\rho\right\}$, for which

$$\lim_{m\longrightarrow\infty}\left(1-\left|c_m\right|^2\right)\frac{\left|h^{(n)}(c_m)\right|}{1+\left|h(c_m)\right|^{n+1}}=+\infty. \tag{23}$$

Theorem 4. *Let $h\in M(U)$. For a concerned sequence of points $\{a_m\}\subset U$, when*

$$\lim_{m\longrightarrow\infty}\left(1-\left|a_m\right|^2\right)\frac{\left|h^{(n)}(a_m)\right|}{1+\left|h(a_m)\right|^{n+1}}=+\infty, \tag{24}$$

for the same concerned sequence $\{a_m\}$, we have

$$\lim_{m\longrightarrow\infty}\iint_{U(a_m,\rho)}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w)=+\infty, \tag{25}$$

for all values of q, s where $2<q<\infty$ and $0<s<\infty$ as well as ρ, with $0<\rho<1$.

Proof. Assume that condition (25) holds. Then, we have ρ_0, $0<\rho_0<1$, such that

$$\lim_{m\longrightarrow\infty}\sup\iint_{U(a_m,\rho_0)}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w)=K<+\infty. \tag{26}$$

Thus, we can find a concerned subsequence $\left\{a_{m_k}\right\}$ of $\{a_m\}$, for which

$$\iint_{U\left(a_{m_k},\rho_0\right)}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_{m_k}}(w)\right|^2\right)^s \mathrm{d}m(w)\leq K+1, \tag{27}$$

this can be verfied for sufficiently large k. Let ρ_1, $0<\rho_1<\rho_0$, $U(a_{m_k},\rho_1)=\left\{w\in U\colon\ |\varphi_{a_{m_k}}(w)|<\rho_1\right\}$, which verifies that

$$\frac{K+1}{\left(1-\rho^2\right)^{s+q-2}}<\frac{\pi}{2}. \tag{28}$$

This implies that

$$\iint_{U\left(a_{m_k},\rho_1\right)}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q \mathrm{d}m(w)\leq\frac{K+1}{\left(1-\rho_1^2\right)^{s+q-2}}<\frac{\pi}{2}, \tag{29}$$

where $(1-|\varphi_{a_{m_k}}(w)|^2)\geq(1-\rho_1^2)$. Applying the theorem of Dufresngy (see [15]), we deduce that

$$\left(1-\left|a_{m_k}\right|^2\right)\frac{\left|h^{(n)}\left(a_{m_k}\right)\right|}{1+\left|h\left(a_{m_k}\right)\right|^{n+1}}\leq\frac{1}{\rho_1}, \tag{30}$$

and this is a contradiction of the concerned assumption. Therefore, the concerned proof of Theorem 4 is finished. □

Theorem 5. *Suppose that $h\in M(U)$. For $q,s\in(0,\infty)$, we can find a concerned sequence of points $\{a_m\}\subset U$, for which*

$$\lim_{m\longrightarrow\infty}\iint_U\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s dm(w)=+\infty. \tag{31}$$

Hence, for any concerned sequence of points $\{c_m\}$ in U such that $d(a_m,c_m)\longrightarrow 0$, we have

$$\lim_{m\longrightarrow\infty}\iint_U\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(w)\right|^2\right)dm(w)=+\infty. \tag{32}$$

Proof. Let K_1 and K_2 be two specific positive constants satisfying $K_2<K_1$. Suppose that

$$U^m_{K_1}=\left\{w:\left(1-\left|\varphi_{a_m}(w)\right|^2\right)>K_1\right\}\text{ and }$$
$$U^m_{K_2}=\left\{w:\left(1-\left|\varphi_{a_m}(w)\right|^2\right)>K_2\right\}. \tag{33}$$

Thus, when $w\in U^m_{K_1}$ and $w\in U\backslash U^m_{K_2}$, we have that $C(1-|\varphi_{a_n}(w)|^2)\leq(1-|\varphi_{a_n}(w)|^2$ for some specific constant $C>0$. Thus, for all m, we conclude that

$$\begin{aligned}&\iint_{U\backslash U^m_{M_2}}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(z)\right|^2\right)^s dm(w)\\&\geq C^s\iint_{U\backslash U^m_{K_2}}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s dm(w).\end{aligned} \tag{34}$$

This inequality holds for any concerned sequence of points $\{c_m\}$ in U with $d(a_m,c_m)\longrightarrow 0$. When

$$\lim_{m\longrightarrow\infty}\sup\iint_{U\backslash U^m_{K_2}}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s dm(w)=+\infty. \tag{35}$$

Therefore, using (10), we obtain

$$\lim_{m\longrightarrow\infty}\sup\iint_{U\backslash U^m_{K_2}}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(w)\right|^2\right)^s dm(w)=+\infty, \tag{36}$$

If

$$\lim_{m\longrightarrow\infty}\sup\iint_{U^m_{K_2}}\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s dm(w)=+\infty. \tag{37}$$

Thus, we can consider two cases. □

Case 1. In this case, we can find a concerned sequence of points $\{c_m\}$ in $U_{K_2}^m$, such that $\mathrm{d}(a_m, c_m) \longrightarrow 0$, for which

$$\lim_{m\longrightarrow\infty}\left(1-\left|c_m\right|^2\right)\frac{\left|h^{(n)}(c_m)\right|}{1+\left|h(c_m)\right|^{n+1}} = +\infty, \tag{38}$$

or we can consider the following case.

Case 2. We can find R_0, $0 < R_0 < e^{-K_2}$; also, there exists $\lambda > 0$, for which

$$\left(1-|w|^2\right)\frac{\left|h^{(n)}(w)\right|}{1+|h(w)|^{n+1}} \leq \lambda, \tag{39}$$

where we consider all $w \in U(a_m, R_0)$. If Case 1 is verified, by Theorem 1, for the aforementioned sequence $\{c_m\}$ such that $\mathrm{d}(a_m, c_m) \longrightarrow 0$, we deduce that

$$\lim_{m\longrightarrow\infty}\sup\iint_U\left[\frac{\left|h^{(n)}(w)\right|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(w)\right|^2\right)^s \mathrm{d}m(w) = +\infty. \tag{40}$$

This is because $\mathrm{d}(a_m, c_m) \longrightarrow 0$. Also, when Case 2 holds, using the same concerned conclusions for the concerned weight functions, we obtain that necessarily condition for any concerned sequence of points $\{c_n\}$ such that $\mathrm{d}(a_m, c_m) \longrightarrow 0$,

$$\lim_{m\longrightarrow\infty}\sup\iint_U\left[\frac{\left|h^{(n)}(w)\right|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{c_m}(w)\right|^2\right)^s \mathrm{d}m(w) = +\infty. \tag{41}$$

This is the end of the concerned proof.

Now we are dealing with the following interesting question:

Assume that $2 < q < \infty$ for any concerned sequence $\{a_m\}$ and assume also that

$$\lim_{m\longrightarrow\infty}\iint_U\left[\frac{\left|h^{(n)}(w)\right|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w) = +\infty. \tag{42}$$

Is the following equation correct?

$$\lim_{m\longrightarrow\infty}\sup\iint_U\left[\frac{\left|h^{(n)}(w)\right|}{1+|h(w)|^{n+1}}\right]^{q_1}\left(1-|w|^2\right)^{q_1-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w) = +\infty, \tag{43}$$

for q_1 with $q < q_1$.

We give the answer of this important question by introducing Theorem 6 with its concerned proof.

Definition 4. Let $\{a_m\}$ be any concerned sequence of points in U; then, $\{a_m\}$ is said to be a $m_{(q,n)}$−sequence, when

$$\lim_{m\longrightarrow\infty}\sup\iint_U\left[\frac{\left|h^{(n)}(w)\right|}{1+|h(w)|^{n+1}}\right]^q\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^q \mathrm{d}m(w) = +\infty. \tag{44}$$

Theorem 6. *Let $q \in (2, \infty)$ and assume that*

$$\lim_{m\longrightarrow\infty}\iint_U\left[\frac{\left|h^{(n)}(w)\right|}{1+|h(w)|^{n+1}}\right]^q\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w) = +\infty. \tag{45}$$

When the concerned sequence of points $\{a_m\}$ in U is not a concerned $m_{(q,n)}$−sequence, for any q_1 with $q<q_1$, we conclude that

$$\lim_{m\longrightarrow\infty}\iint_U\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q_1}\left(1-|w|^2\right)^{q_1-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w)=+\infty. \tag{46}$$

Proof. As in [1], we can deduce that $\mathcal{N}_n\cap B^{\#}_{q,q-2,;n}$. Also, we have

(i) $B^{\#}(q,q-2,s;n)\subset\mathcal{N}_n$ for all q, with $2<q<\infty$ and $0<s<1$.

(ii) For all values of q, q_1, $2<q<\infty$, $0<s<1$ with $q_1+s>1$, we have that

$$\cup_{2<q<q_1}B^{\#}(q,q-2,s;n)\subsetneq B^{\#}(q_1,q_1-2,s;n). \tag{47}$$

□

Therefore, it is obvious to get that

$$\lim_{m\longrightarrow\infty}\iint_U\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q_1}\left(1-|w|^2\right)^{q_1-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w)=+\infty. \tag{48}$$

Remark 3. The recent developments of fractional calculus as well as its applications are more essential to complex function spaces with the specific arbitrary fractional order derivatives. For some recent interesting studies on the subject of fractional calculus, we can here refer to [17–19] and others. To the best of our knowledge, a few number of manuscripts researched some certain classes of analytic function spaces by the help of general fractional derivatives (see [20]). For further research work, the following specific interesting question can be considered:

How one can define and study the Besov spaces of general meromorphic functions by using the general fractional derivatives?

3. Conclusions

Certain concerned weighted classes of meromorphic function spaces using the general spherical derivatives are studied and discussed in this article. The general Besov-type classes of meromorphic functions as well as the general normal functions are considered intensively and both are compared deeply with each other. For a concerned non-normal function h, the concerned families of points $\{a_m\}$ and $\{c_m\}$, for which

$$\begin{aligned}&\lim_{m\longrightarrow\infty}\left(1-|a_m|^2\right)\frac{\left|h^{(n)}(a_m)\right|}{1+\left|h(a_m)\right|^{n+1}}=+\infty,\quad n\in\mathbb{N},\\&\lim_{m\longrightarrow\infty}\iint_U\left[\frac{|h^{(n)}(w)|}{1+|h(w)|^{n+1}}\right]^{q}\left(1-|w|^2\right)^{q-2}\left(1-\left|\varphi_{a_m}(w)\right|^2\right)^s \mathrm{d}m(w)=+\infty\end{aligned} \tag{49}$$

are introduced and discussed. Several connections between families (sequences) of $b_{(q,n)}$ and $q_{(N,n)}$ type are established. The obtained results improve, extend, and generalize numerous results in [21–23].

Remark 4. Quite recently there are some important enjoyable research studies on hyperbolic function classes (see [24, 25]). For more interesting research, how we can construct some workable conditions on some hyperbolic-type sequences of points that make guarantee to belong to some specific hyperbolic-type classes?

Data Availability

No data were applied or considered to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] J. Rättyä, "On some Complex function spaces and classes," *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica. Dissertationes*, vol. 124, pp. 1–73, 2001.
[2] C. Batty, A. Gomilko, and Y. Tomilov, "A Besov algebra calculus for generators of operator semigroups and related norm-estimates," *Mathematische Annalen*, vol. 379, no. 1-2, pp. 23–93, 2021.
[3] S. Dominguez and D. Girela, "Radial growth of the derivatives of analytic functions in Besov spaces," *Concrete Operators*, vol. 8, no. 1, pp. 1–12, 2021.
[4] S. Dominguez and D. Girela, "A radial integrability result concerning bounded functions in analytic Besov spaces with applications," *Results in Mathematics*, vol. 75, 2020.
[5] K. Stroethoff, "Besov-type characterisations for the Bloch space," *Bulletin of the Australian Mathematical Society*, vol. 39, no. 3, pp. 405–420, 1989.
[6] M. Tjani, "Compact composition operators on some mobius invariant banach spaces," Thesis, Michigan State University, Michigan, USA, 1996.
[7] X. Zhu and N. Hu, "Weighted composition operators from Besov Zygmund-type spaces into Zygmund-type spaces," *Journal of Function Spaces*, vol. 2020, Article ID 2384971, 7 pages, 2020.
[8] A. El-Sayed Ahmed and M. Al-Bogami, "Generalizations on some meromorphic function spaces in the unit disc," *Journal of Computational Analysis and Applications*, vol. 22, no. 1, pp. 148–160, 2017.
[9] R. Aulaskari and P. Lappan, "Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, Complex analysis and its applications," *Longman Scientific and Technical Harlow*, vol. 305, pp. 136–146, 1994.
[10] A. El-Sayed Ahmed and A. Kamal, "-type spaces of analytic and meromorphic functions," *Mathematica*, vol. 54, no. 77, pp. 26–37, 2012.
[11] L. Bernal-Gonzälez, A. Jung, and J. Müller, "Universality vs. non-normality of families of meromorphic functions," *Proceedings of the American Mathematical Society*, vol. 149, no. 2, pp. 761–771, 2021.
[12] R. Aulaskari and P. Lappan, "Additive automorphic functions and Bloch functions," *Canadian Journal of Mathematics*, vol. 46, no. 3, pp. 474–484, 1994.
[13] C. Fang and Y. Xu, "Normal family of meromorphic functions concerning fixed-points," *Analysis and Mathematical Physics*, vol. 9, no. 1, pp. 197–207, 2019.
[14] S. Makhmutov, "Integral characterizations of Bloch functions," *New Zealand Journal of Mathematics*, vol. 26, pp. 201–212, 1997.
[15] J. L. Schiff, *Normal Families*, Springer-Verlay, New York, NY, USA, 1993.
[16] H. Wulan, "On some classes of meromorphic functions, Annales Academiae Scientiarum Fennicae. Series A I. Mathematica," *Doctoral dissertations*, vol. 116, pp. 1–57, Suomalainen Tiedeakatemia, Helsinki, Finland, 1998.
[17] S. Rashid, M. A. Latif, Z. Hammouch, and Y.-M. Chu, "Fractional integral inequalities for strongly h-preinvex functions for a kth order differentiable functions," *Symmetry*, vol. 11, no. 12, p. 1448, 2019.
[18] S. Rashid, H. Kalsoom, Z. Hammouch, R. Ashraf, D. Baleanu, and Y.-M. Chu, "New multi-parametrized estimates having pth-order differentiability in fractional calculus for predominating $\hbar$-convex functions in hilbert space," *Symmetry*, vol. 12, no. 2, p. 222, 2020.
[19] S. S. Zhou, S. Rashid, S. Parveen, A. O. Akdemir, and Z. Hammouch, "New computations for extended weighted functionals within the Hilfer generalized proportional fractional integral operators," *AIMS Mathematics*, vol. 6, no. 5, pp. 4507–4525, 2021.
[20] A. Perälä, "General fractional derivatives and the Bergman projection," *Annales Academiae Scientiarum Fennicae Mathematica*, vol. 45, no. 2, pp. 903–913, 2020.
[21] A. El-Sayed Ahmed and M. A. Bakhit, "Sequences of some meromorphic function spaces," *Bulletin of the Belgian Mathematical Society—Simon Stevin*, vol. 16, no. 3, pp. 395–408, 2009.
[22] R. Aulaskari, S. Makhmutov, and H. Wulan, "On Q p Sequences," *Finite or Infinite Dimensional Complex Analysis and Applications*, vol. 2, pp. 117–125, 2004.
[23] R. Aulaskari, S. Makhmutov, and J. Rättyä, "Hyperbolically close to $Q_p^{\#}$-sequences," *Bulletin of the Korean Mathematical Society*, vol. 57, no. 1, pp. 133–138, 2020.
[24] A. El-Sayed Ahmed, "Refinements on some classes of complex function spaces," *Symmetry*, vol. 13, no. 2, p. 339, 2021.
[25] A. El-Sayed Ahmed and A. E. Shammaky, "Certain classes of operators on some weighted hyperbolic function spaces," *Journal of Mathematics*, vol. 2021, Article ID 6664398, 6 pages, 2021.

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Research Article

Image Denoising of Adaptive Fractional Operator Based on Atangana–Baleanu Derivatives

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A fractional integral operator can preserve an image edge and texture details as a denoising filter. Recently, a newly defined fractional-order integral, Atangana–Baleanu derivatives (ABC), has been used successfully in image denoising. However, determining the appropriate order requires numerous experiments, and different image regions using the same order may cause too much smoothing or insufficient denoising. Thus, we propose an adaptive fractional integral operator based on the Atangana–Baleanu derivatives. Edge intensity, global entropy, local entropy, and local variance weights are used to construct an adaptive order function that can adapt to changes in different regions of an image. Then, we use the adaptive order function to improve the masks based on the Grumwald–Letnikov scheme (GL_ABC) and Toufik–Atangana scheme (TA_ABC), namely, Ada_GL_ABC and Ada_TA_ABC, respectively. Finally, multiple evaluation indicators are used to assess the proposed masks. The experimental results demonstrate that the proposed adaptive operator can better preserve texture details when denoising than other similar operators. Furthermore, the image processed by the Ada_TA_ABC operator has less noise and more detail, which means the proposed adaptive function has universality.

1. Introduction

The theory of fractional-order derivatives has been applied in many fields, such as physics, fluid mechanics, physiology, medical science, and epidemic diseases [1–4]. With the development of information science, fractional operators have gained incomparable advantages over integral operators in many fields. A fractional derivative recurrent neural network can effectively improve estimation accuracy in parameter identification [5]. Complex behaviors in fractional-order financial systems can provide theoretical basis for the government [6]. Fractional-order control systems perform more accurately and elegantly than traditional systems [7]. In signal processing, the characteristics of fractional differential operators, such as "nonlocality," "memorability," and "weak derivatives," are also applied [8–10]. These properties can improve the high frequency of an image while preserving the performance of the low and medium frequencies. In other words, methods based on fractional calculus for enhanced images can enhance the texture details while preserving the texture details of the smooth region in images [11, 12]. Therefore, many scholars are engaged in research on the application of fractional operators in image enhancement and denoising. The most representative scholar is Y.F. Pu, who, with his team, constructed image enhancement and denoising operators by fractional calculus [10, 13]. Based on the Grunwald–Letnikov (GL) approximation, a medical image enhancement method was proposed by Guan et al. [14]. An adaptive image enhancement operator based on fractional-differential and image gradient feature was proposed by Lan [15]. Arian Azarang inferred different structure mask to image fusion [16]. An adaptive fractional-order integral filter was presented for echocardiographic image denoising [17].

The basic theory of abovementioned fractional operators is mainly the definitions of GL and Riemann–Liouville (RL). The Caputo derivative is another definition of fractional order that is widely studied and applied; it includes the numerical solutions of fractional equations and the properties of systems [18, 19]. New fractional derivatives and applications based on the frame of the Caputo derivative have received much attention from experts. The existence and stability of Belouso–Zhabotinskii reaction systems with Atangana–Baleanu fractional-order derivatives are discussed in [20]. In [21], the locally and globally asymptotically stable of symbiosis system modelling by the Atangana–Baleanu derivative are analyzed. With the development of research, fractional-differential operators with nonlocal and nonsingular kernels are used to image filters [22–24]. Furthermore, an AB-fractional differential mask based on the Gaussian kernel has been introduced to detect blood vessels in retinal images [25]. Behzad Ghanbari and Abdon Atangana designed an ABC-fractional derivatives mask that is used for image denoising. The ABC-fractional derivatives mask is computationally efficient and has excellent performance in the denoising of nosy images [26]. In the process of denoising, many experiments are required to determine the order of the mask. Moreover, because using a fixed order may lead to excess or deficiency for denoising effect, an adaptive fractional operator based on Atangana–Baleanu derivatives is proposed in this paper, which is called Ada_GL_ABC. We consider the gradient of the image, local entropy, global entropy, and local variance weights to construct a function for solving the adaptive order. The starting point of this idea is removing the image noise while preserving the edge and texture details of the image as much as possible. The adaptive function proposed by us is different from that of other studies. We consider both global and local information, the adaptive function contains more comprehensive information when determining the order, and the order used for denoising is more appropriate. And, the adaptive function designed by us has a certain generality. The function can be applied not only to GL_ABC mask but also to TA_ABC mask, which is rarely seen in the previous literature.

The remainder of this paper is organized as follows: the basic definitions of fractional derivatives and the structure of fractional-masks are introduced in Section 2. In Section 3, the function of the adaptive fractional-order integral operator based on Atangana–Baleanu derivatives is described. The performance of the proposed adaptive operator is discussed in Section 4. In Section 5, the conclusions are elaborated.

2. Preliminaries

2.1. Definitions of the Fractional Derivatives. Many basic definitions of fractional derivatives exist [27]. Recently, the Mittag–Leffler function was introduced to compute fractional derivatives. This new definition is named the Atangana–Baleanu fractional derivative; it is based on the definition of Liouville–Caputo (ABC) and can be defined as follows [28]:

$$ {}_{0}^{ABC}D_t^{\gamma} f(t) = \frac{A(\gamma)}{1-\gamma}\int_0^t E_{\gamma}\left[-\beta\frac{(t-\tau)^{\gamma}}{1-\gamma}\right] \dot{f}(\tau)\mathrm{d}\tau, \quad 0<\gamma\leq 1. \tag{1} $$

The Atangana–Baleanu fractional integral with order β can be depicted as

$$ {}_{0}^{ABC}J_t^{\gamma} f(t) = \frac{1-\gamma}{A(\gamma)} f(t) + \frac{\gamma}{\Gamma(\gamma)A(\gamma)}\int_0^T f(\tau)(t-\tau)^{\gamma-1}\mathrm{d}\tau, \quad 0<\gamma\leq 1, \tag{2} $$

where $A(*)$ is a normalization function, and this function satisfies $A(0)=A(1)=1$. It can be described by

$$ A(\gamma) = 1-\gamma+\frac{\gamma}{\Gamma(\gamma)}. \tag{3} $$

The ABC derivative inherits the memory of the Mittag–Leffler function, which, with index γ, is denoted as

$$ E_{\gamma}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\gamma k+1)}, \quad \gamma>0. \tag{4} $$

The GL definition is one of the best-known definitions of discrete fractional calculus and is widely applied to image processing. Details of the GL definition are expounded in Definition 1.

Definition 1. The GL definition of fractional calculus formula with α-order of [29] is described as

$$ {}_{0}D_b^{\alpha} f(x) = \lim_{h\longrightarrow 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{[(b-a)/h]}(-1)^j\binom{\beta}{j} f(x-jh), \tag{5} $$

where h is the step, $[*]$ represents the rounded operation, $\binom{\beta}{j} = ((\Gamma(\beta+1))/(j!\Gamma(\beta-j+1)))$, and $\Gamma(\beta)$ is Gamma function.

Equation (5) can be further decomposed as follows:

$$ D_{GL}^{\alpha} f(x) = \lim_{h\longrightarrow 0}\frac{1}{h^{\alpha}}\left(f(t)+(-\alpha)f(t-h)+\frac{(\alpha)(\alpha+1)}{2}f(t-h)+\cdots+\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-N+1)}f(t-jh)\right). \tag{6} $$

We know that equation (6) is the fractional derivative operator with $\alpha>0$, and it takes a part of the fractional integral operator with $\alpha<0$. When $\alpha>0$, we set $\gamma=-\alpha$, and the integral GL of order γ using equation (6) is described as

$$J_{GL}^{\gamma} f(x) \approx \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} \mathrm{d}\tau = \lim_{h \longrightarrow 0} \frac{1}{h^{-\gamma}} \left(f(t) + \gamma f(t-h) + \frac{(-\gamma)(-\gamma+1)}{2} f(t-h) + \cdots + \frac{\Gamma(-\gamma+1)}{k!\Gamma(-\gamma-N+1)} f(t-jh) \right). \tag{7}$$

From the above discussion, the ABC-fractional integral can be described by equation (7):

$$\begin{aligned} {}_0^{ABC}J_t^{\beta}(t) &= \frac{1-\gamma}{A(\gamma)} f(t) + \frac{\gamma}{\Gamma(\gamma) A(\gamma)} \int_0^T f(\tau)(t-\tau)^{\gamma-1} \mathrm{d}\tau \\ &\approx \frac{1-\gamma}{A(\gamma)} f(t) + \frac{\gamma}{A(\gamma)} h^{-\gamma} \left(f(t) + \gamma f(t-h) + \frac{(-\gamma)(-\gamma+1)}{2} f(t-h) + \cdots \right). \end{aligned} \tag{8}$$

In [26], the newly defined fractional order integral is mentioned. It can be approximated to the following as $t = t_n$:

$$\begin{aligned} {}_0^{ABC}J_t^{\gamma}(t_n) &= \frac{1-\gamma}{A(\gamma)} f(t) + \frac{\gamma}{\Gamma(\gamma) A(\gamma)} \int_0^{t_n} f(\tau)(t-\tau)^{\gamma-1} \mathrm{d}\tau \\ &= \frac{1-\gamma}{A(\gamma)} f(t) + \frac{\gamma}{\Gamma(\gamma) A(\gamma)} \sum_{k=0}^{N} \int_{t_k}^{t_{k+1}} f(\tau)(t-\tau)^{\gamma-1} \mathrm{d}\tau. \end{aligned} \tag{9}$$

The function $f(\tau)$ can be described by a two-step Lagrange polynomial interpolation as follows:

$$f(\tau) = \frac{f(t_k)}{h} (\tau - t_{k-1}) + \frac{f(t_{k-1})}{h} (\tau - t_k). \tag{10}$$

Using equation (10), equation (9) can be discretized as follows:

$$\begin{aligned} {}_0^{ABC}J_t^{\gamma}(t_n) &= \left[\frac{(1-\gamma)\Gamma(\gamma+2) + \gamma h^{\gamma}(\beta+2)}{A(\gamma)\Gamma(\gamma+2)} \right] f(t_n) + \left[\frac{\gamma h^{\gamma}}{A(\gamma)} \left(\frac{-2 * \gamma + (\gamma+3)2^{\gamma} - 4}{\Gamma(\gamma+2)} \right) \right] f(t_{n-1}) \\ &+ \left[\frac{\gamma h^{\gamma} \left(-2^{\gamma+1}\gamma + (\gamma+4)3^{\gamma} + a - 6 * 2^{\gamma} + 2 \right)}{A(\gamma)\Gamma(\gamma+2)} \right] f(t_{n-2}) + \cdots. \end{aligned} \tag{11}$$

2.2. The Mask Based on Grunwald–Letnikov (GL_ABC) and Toufik–Atangana (TA-ABC). In an image, the distance between two pixels can be assumed to be 1. This distance is the same as h in equation (7). Therefore, the GL integral with fractional-order in the x and y directions [30] are described by equations (12) and (13):

$${}^{x}J_{GL}^{\gamma} f(x,y) \approx \frac{1}{A(\gamma)} f(x,y) + \frac{\gamma^2}{A(\gamma)} f(x-1,y) + \frac{\gamma^3 - \gamma^2}{2A(\gamma)} f(x-2,y) + \cdots, \quad (0 < \gamma \leq 1), \tag{12}$$

$${}^{y}J_{GL}^{\gamma} f(x,y) \approx \frac{1}{A(\gamma)} f(x,y) + \frac{\gamma^2}{A(\gamma)} f(x,y-1) + \frac{\gamma^3 - \gamma^2}{2A(\gamma)} f(x,y-2) + \cdots, \quad (0 < \gamma \leq 1). \tag{13}$$

The TA_ABC integral with fractional-order [31] in the x and y directions is presented as equations (14) and (15):

$$ {}^{x}J^{\gamma}_{\text{TA-ABC}} f(x,y) \approx \left[\frac{(1-\gamma)\Gamma(\gamma+2)+\gamma(\gamma+2)}{A(\gamma)\Gamma(\gamma+2)}\right] f(x,y) + \left[\frac{\gamma h^{\gamma}}{A(\gamma)}\left(\frac{-2*\gamma+(\gamma+3)2^{\gamma}-4}{\Gamma(\gamma+2)}\right)\right] f(x-1,y) + \left[\frac{\gamma\left(-2^{\gamma+1}\gamma+(\gamma+4)3^{\gamma}+a-6*2^{\gamma}+2\right)}{A(\gamma)\Gamma(\gamma+2)}\right] f(x-2,y), \tag{14} $$

$$ {}^{y}J^{\gamma}_{\text{TA-ABC}} f(x,y) \approx \left[\frac{(1-\gamma)\Gamma(\gamma+2)+\gamma(\gamma+2)}{A(\gamma)\Gamma(\gamma+2)}\right] f(x,y) + \left[\frac{\gamma h^{\gamma}}{A(\gamma)}\left(\frac{-2*\gamma+(\gamma+3)2^{\gamma}-4}{\Gamma(\gamma+2)}\right)\right] f(x,y-1) + \left[\frac{\gamma\left(-2^{\gamma+1}\gamma+(\gamma+4)3^{\gamma}+a-6*2^{\gamma}+2\right)}{A(\gamma)\Gamma(\gamma+2)}\right] f(x,y-2). \tag{15} $$

with equations (12)–(15), the $5*5$ fractional integral mask can be constructed as follows:

This $5*5$ mask is used in image denoising as the filter. The mask is rotation-invariant mainly because it is obtained by superimposition of fractional integral in eight directions. Thus, we can use different fractional-order integrals for airspace filtering to denoise images. Therefore, the coefficients of GL_ABC and TA_ABC mask are described by equations (16) and (17), respectively (Table 1).

$$ \begin{aligned} \mathrm{H}_0 &= \frac{1}{A(\gamma)}, \\ \mathrm{H}_1 &= \frac{\gamma^2}{A(\gamma)}, \\ \mathrm{H}_2 &= \frac{\gamma^3-\gamma^2}{2A(\beta)}, \end{aligned} \tag{16} $$

$$ \begin{aligned} \mathrm{H}_0 &= \frac{(1-\gamma)\Gamma(\gamma+2)+\gamma(\gamma+2)}{A(\gamma)\Gamma(\gamma+2)}, \\ \mathrm{H}_1 &= \frac{\gamma}{A(\gamma)}\left(\frac{-2\gamma+(\gamma+3)2^{\gamma}-4}{\Gamma(\gamma+2)}\right), \\ \mathrm{H}_2 &= \frac{\gamma\left(-2^{\gamma+1}\gamma+(\gamma+4)3^{\gamma}+a-6*2^{\gamma}+2\right)}{A(\gamma)\Gamma(\gamma+2)}. \end{aligned} \tag{17} $$

3. Adaptive Fractional Operators Based on Atangana–Baleanu Derivatives

For an image with noise of different intensities and in different regions, one fixed order in the fractional integral operator is insufficient to achieve a good denoising effect. Therefore, this paper proposes an adaptive fractional operator for image denoising. The edge intensity coefficient, image entropy, local entropy, and local variance weight are used to construct the expression of the adaptive fractional order. The image gradient represents the image edge intensity information. In this paper, The Kirsch algorithm is applied to calculate the image edge intensity. However, the Kirsch algorithm can suppress image noise [32].

$$ G[I(x,y)] = \max\left\{1, \max\left\{\left|5r_k - 3s_k\right|, \quad k = 0,1,\ldots,7\right\}\right\}. \tag{18} $$

Here, $r_k = W_k + W_{k-(\pi/4)} + W_{k-(\pi/2)}$; $s_k = W_{k-(3\pi/4)} + W_{k-\pi} + W_{k-(5\pi/4)} + W_{k-(3\pi/2)} + W_{k-(7\pi/4)}$. Moreover, when $k = (0, (\pi/4), (\pi/2), (3\pi/4), \pi, (5\pi/4), (3\pi/2), (7\pi/4))$, the eight directions of masks are depicted as follows:

$$ W_0 = \begin{bmatrix} -3 & -3 & 5 \\ -3 & 0 & 5 \\ -3 & -3 & 5 \end{bmatrix}, $$

$$ W_{(\pi/4)} = \begin{bmatrix} -3 & 5 & 5 \\ -3 & 0 & 5 \\ -3 & -3 & -3 \end{bmatrix}, $$

$$ W_{(\pi/2)} = \begin{bmatrix} 5 & 5 & 5 \\ -3 & 0 & -3 \\ -3 & -3 & -3 \end{bmatrix}, $$

$$ W_{(3\pi/4)} = \begin{bmatrix} 5 & 5 & -3 \\ 5 & 0 & -3 \\ -3 & -3 & -3 \end{bmatrix}, $$

$$ W_{\pi} = \begin{bmatrix} 5 & -3 & -3 \\ 5 & 0 & -3 \\ 5 & -3 & -3 \end{bmatrix}, $$

$$ W_{(5\pi/4)} = \begin{bmatrix} -3 & -3 & -3 \\ 5 & 0 & -3 \\ 5 & 5 & -3 \end{bmatrix}, $$

Table 1: 5*5 mask.

H_2	0	H_2	0	H_2
0	H_1	H_1	H_1	0
H_2	H_1	8 H_0	H_1	H_2
0	H_1	H_1	H_1	0
H_2	0	H_2	0	H_2

$$W_{(3\pi/2)} = \begin{bmatrix} -3 & -3 & -3 \\ -3 & 0 & -3 \\ 5 & 5 & 5 \end{bmatrix},$$

$$W_{(7\pi/4)} = \begin{bmatrix} -3 & -3 & -3 \\ -3 & 0 & 5 \\ -3 & 5 & 5 \end{bmatrix}. \tag{19}$$

Image entropy determines how much information an image contains. The smaller the entropy is, the more information it contains [31].

$$E_l = -\sum_{L=1}^{255} P\left(I_{j,k}\right)\log_2 P\left(I_{j,k}\right), \tag{20}$$

P is the probability that an image pixel will appear. The local variance weight can not only measure the local gray change of an image but also reflect the importance of the image local change rate in the whole image. The larger the difference in partial pixel values is, the greater the local variance weight. Conversely, the smaller the changes are, the smaller the value of the local variance weight [33].

$$\mathrm{St}(h) = \frac{1}{\mathrm{Num}} \sum_{i=1}^{\mathrm{Num}} \frac{\sigma_I^2(h\prime)}{\sigma_I^2(h)}, \tag{21}$$

where Num represents the number of image pixels, I is the image to be processed, h is the local pixel, h' is the local pixel of the current window, and $\sigma_I^2(h\prime)$ is the variance in the pixel value in the current window.

The function established in this paper takes the global entropy of the image as a measure of the overall image characteristics. The order value should be small to maintain the texture details. We consider taking the product of three measures of frequency information, to ensure that the fractional-order is inversely proportional to high frequency information such as edges and texture details. The adaptive order function is as follows:

$$a\ da_v = E_t * \varepsilon - G * St * E_l, \tag{22}$$

where E_t represents the entropy of the whole image, ε is the coefficient of E_t and take 0.22 in the experiment, and $G * St * E_l$ is the product of the local information entropy, local gradient, and local variance. Then, the entropy of the global image is subtracted from the product. The results are small in the region of the edge and texture and high in the region of smoothness. According to this equation, the orders of different local textures of the image vary. As shown in Figure 1, the order used in the edge and texture details is relatively small, while the order used in the smooth area is larger. In this way, the obtained orders are reduced in the edge and texture detail region and enhanced in the smooth region so that the edge and texture detail information can be preserved as much as possible while denoising.

4. Numerical Examples

In this paper, the peak signal-to-noise ratio (PSNR), entropy, and structural similarity index measurement (SSIM) are used to assess the performance of the proposed operator. The PSNR is the most popular assessment criterion to evaluate the performance of denoising algorithms. In general, the value of the PSNR is higher when the image quality is better. The PSNR is defined as follows [34]:

$$\mathrm{PSNR} = 10\lg\frac{255^2}{\mathrm{MSE}}, \tag{23}$$

$$\mathrm{MSE} = \frac{1}{M*N}\sum_{j=1}^{M}\sum_{k=1}^{N}\left[I\prime(j,k) - I(j,k)\right]^2, \tag{24}$$

where M and N are the size dimensions of the original image. $I(j,k)$ and $I'(j,k)$ are the original and denoised images, respectively. The SSIM is also a well-known criterion among image quality assessment metrics [35] defined as

$$\mathrm{SSIM} = \frac{\left(2\varphi_p\varphi_q + \rho_1\right)\left(2\sigma_{pq} + \rho_2\right)}{\left(\varphi_p^2 + \varphi_q^2 + \rho_1\right)\left(\sigma_p^2 + \sigma_b^2 + \rho_2\right)}, \tag{25}$$

where p and q represent different images; φ_p and φ_q represent the mean of images p and q; σ_p^2 and σ_b^2 represent the variances of p and q, respectively; σ_{pq} is the covariance of p and q; and ρ_1 and ρ_2 are constants added to maintain stability. The value of the SSIM represents how similar two images are. When the SSIM value is higher, the pixel values of the two images are closer. The range of this index is [0, 1]. If the value of this index is closer to one, the two images are more approximate.

In the experiments, we employed five grayscale images to test the proposed mask: "Lena," "Elaine," "Goldhill," "Peppers," and "Cameraman," with $512 * 512$ pixels each. We use the proposed adaptive function to improve the TA_ABC and GL_ABC mask. The improved mask "Ada_-TA_ABC" and "Ada_GL_ABC" compare the "GL_ABC mask" [31], "TA_ABC mask" [30], and the method proposed in [36]. The orders γ in "GL_ABC mask" and "TA_ABC mask" are from literature [30]. In the experiments, we added noise with different variances $\sigma \in \{15, 20, 25\}$ to the test images, respectively. Figures 2–16 show the results for image denoising by the different methods. Tables 2–7 show the PSNR, SSIM and entropy of these test images for the different algorithms. From Figures 1–16, we find that the test images lost image details when TA_ABC mask was applied. The method proposed by [36] has a poor denoising ability. The Ada_TA_ABC mask performs better than the TA_ABC mask, the order of which is determined by our proposed

(a) (b)

Figure 1: A map of the adaptive order. (a) Lena. (b) Peppers.

(a) (b) (c)

(d) (e) (f)

Figure 2: Comparison of different operators on “Lena” under Gaussian noise with variance $\sigma = 15$.

method, as depicted by equation (15). The test images processed by the GL_ABC mask and Ada_GL_ABC mask contain less noise and more details. The Ada_GL_ABC mask is better than the GL_ABC mask. From another perspective, this result shows that the adaptive function proposed by us has certain universality. This result is verified in Tables 2–6. The PNSR of images proposed by the Ada_GL_ABC mask is higher than that of the other methods. This outcome means that the quality of images processed by the Ada_GL_ABC mask is better than that delivered by other methods. Meanwhile, the images processed by the Ada_GL_ABC mask are closer to the original images. This conclusion can be confirmed by the higher SSIM, which indicates the similarity between two images. Additionally, we calculated

Figure 3: Comparison of different operators on “Lena” under Gaussian noise with variance $\sigma = 20$.

Figure 4: Comparison of different operators on “Lena” under Gaussian noise with variance $\sigma = 25$.

Figure 5: Comparison of different operators on "Elaine" under Gaussian noise with variance $\sigma = 15$.

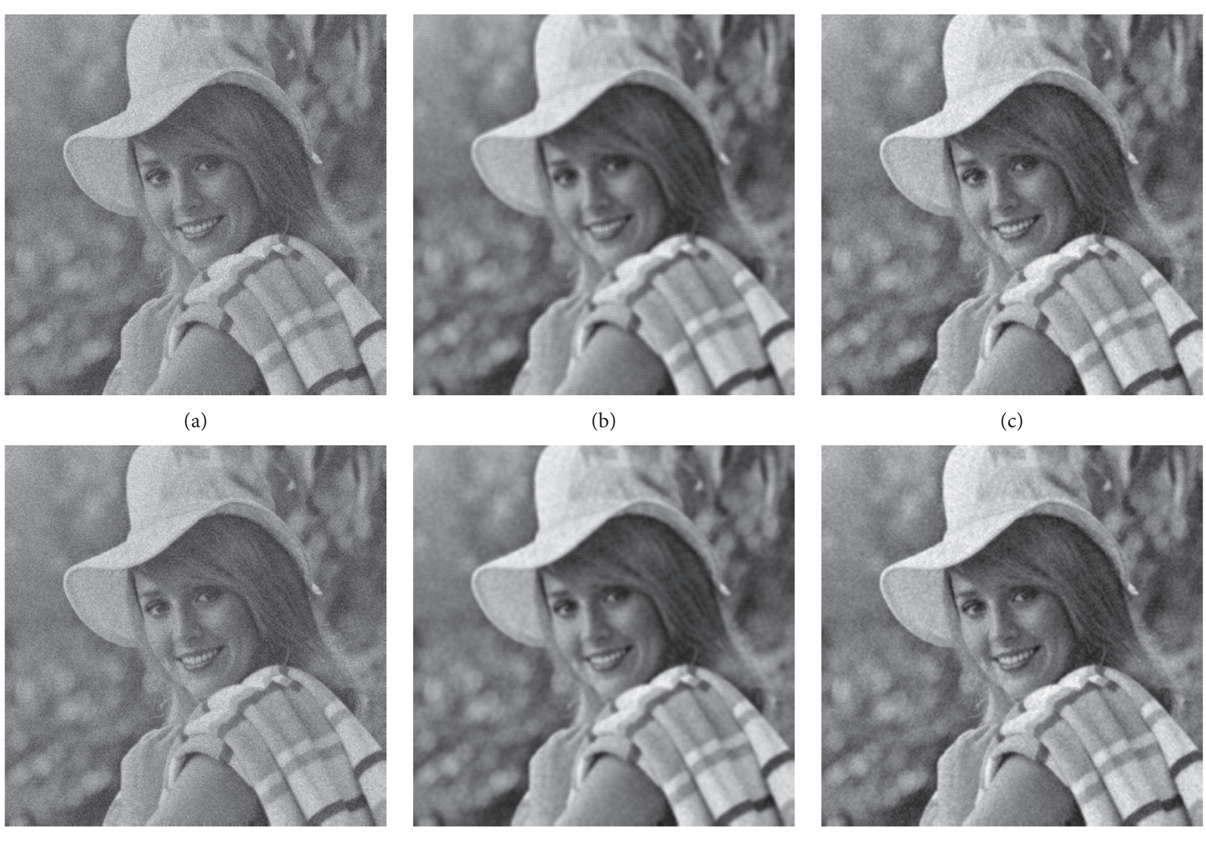
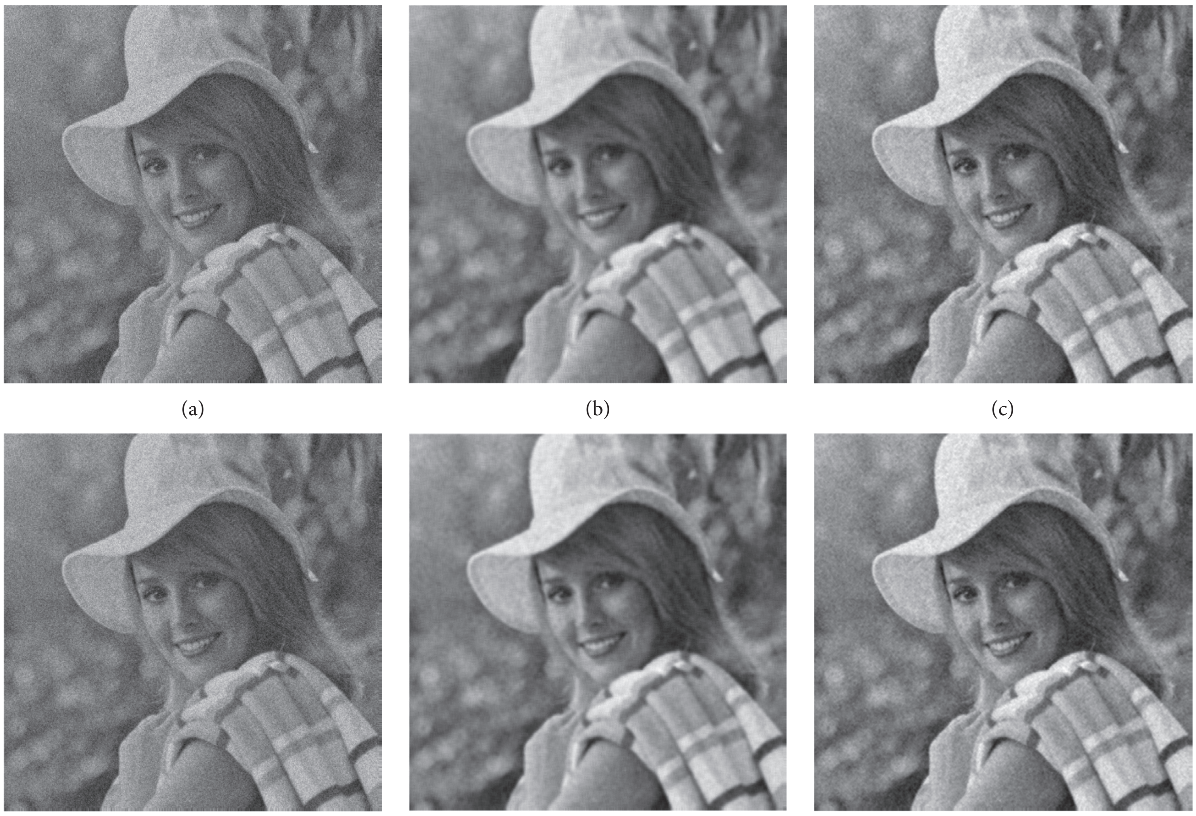


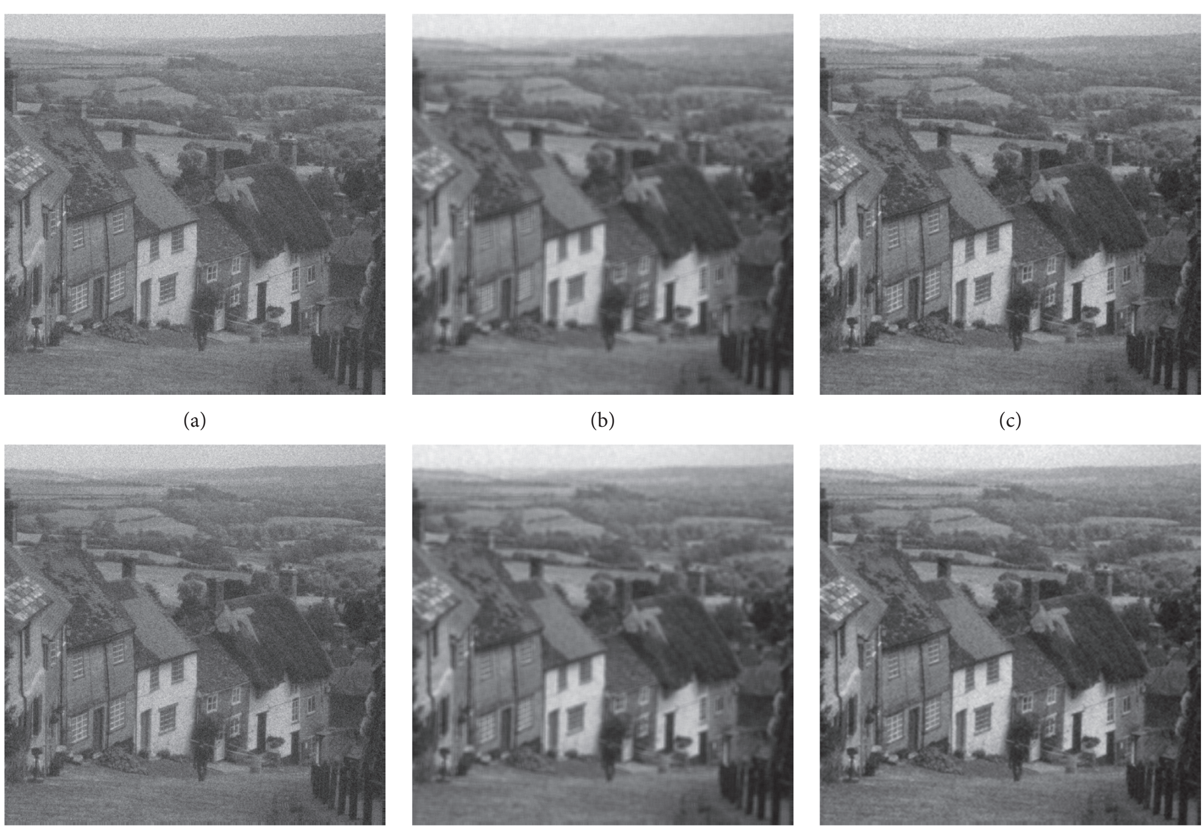
Figure 6: Comparison of different operators on "Elaine" under Gaussian noise with variance $\sigma = 20$.



(a) (b) (c)

(d) (e) (f)

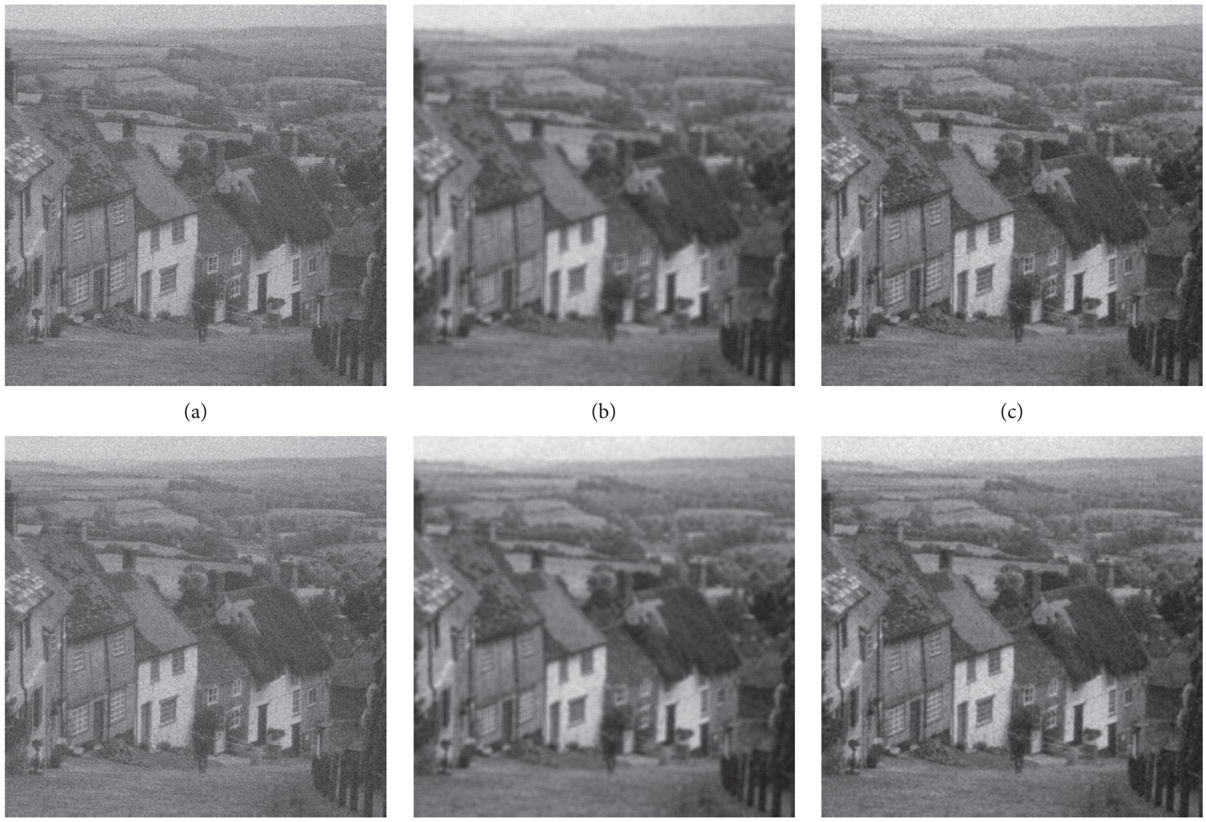
Figure 7: Comparison of different operators on "Elaine" under Gaussian noise with variance $\sigma = 25$.



(a) (b) (c)

(d) (e) (f)

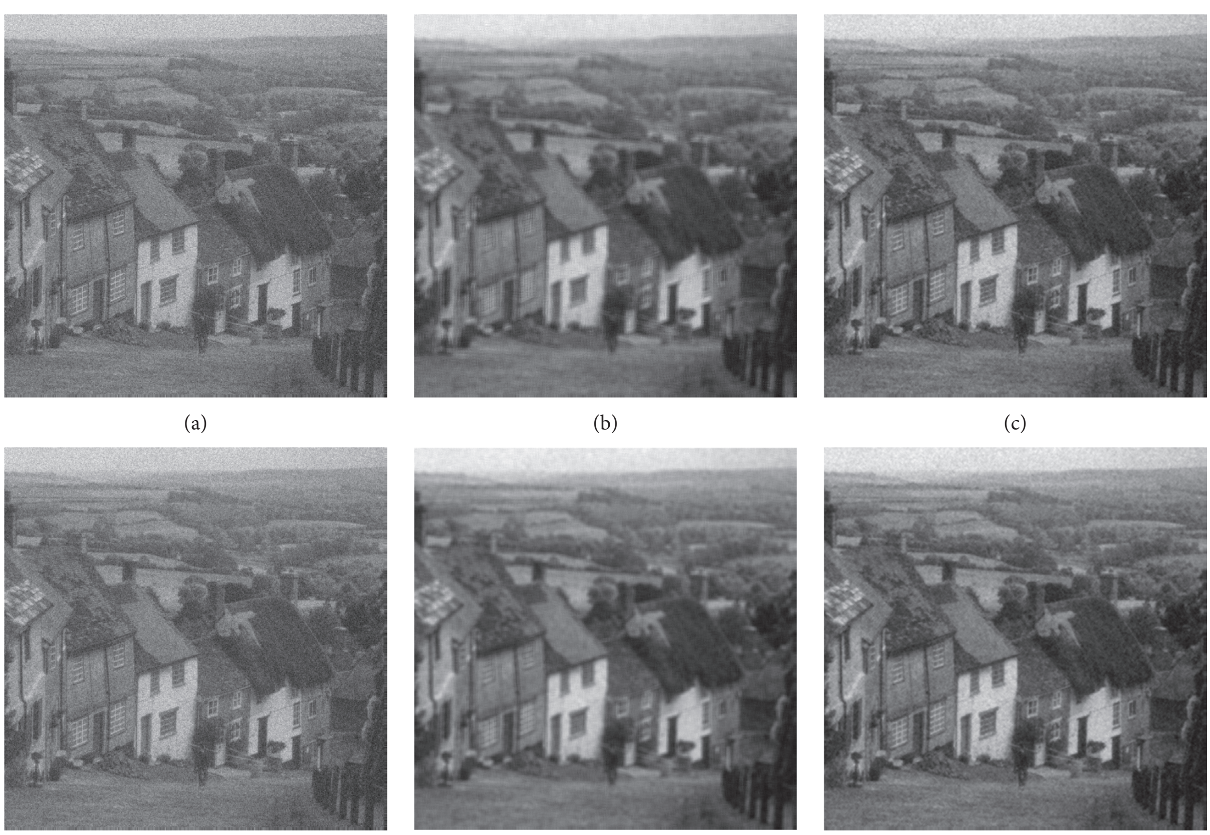
Figure 8: Comparison of different operators on "Goldhill" under Gaussian noise with variance $\sigma = 15$.



(a) (b) (c)

(d) (e) (f)

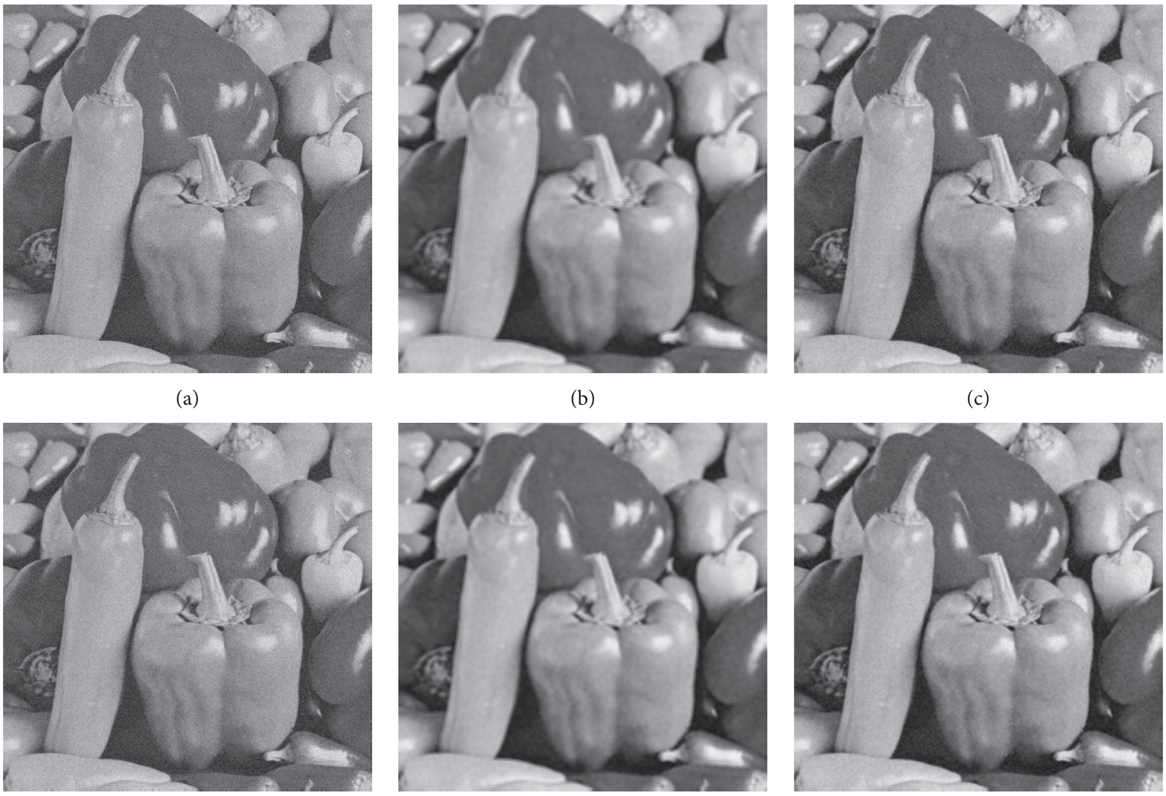
Figure 9: Comparison of different operators on “Goldhill” under Gaussian noise with variance $\sigma = 20$.



(a) (b) (c)

(d) (e) (f)

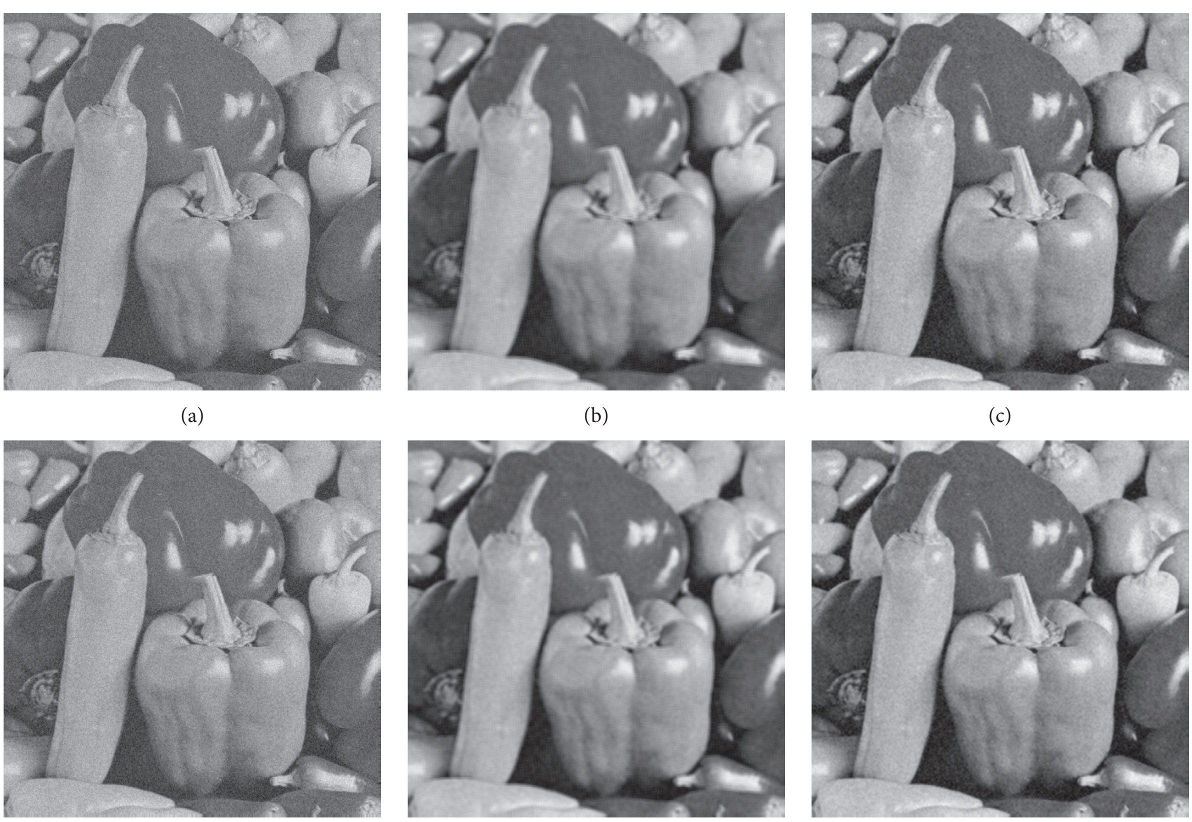
Figure 10: Comparison of different operators on “Goldhill” under Gaussian noise with variance $\sigma = 25$.



(a) (b) (c)

(d) (e) (f)

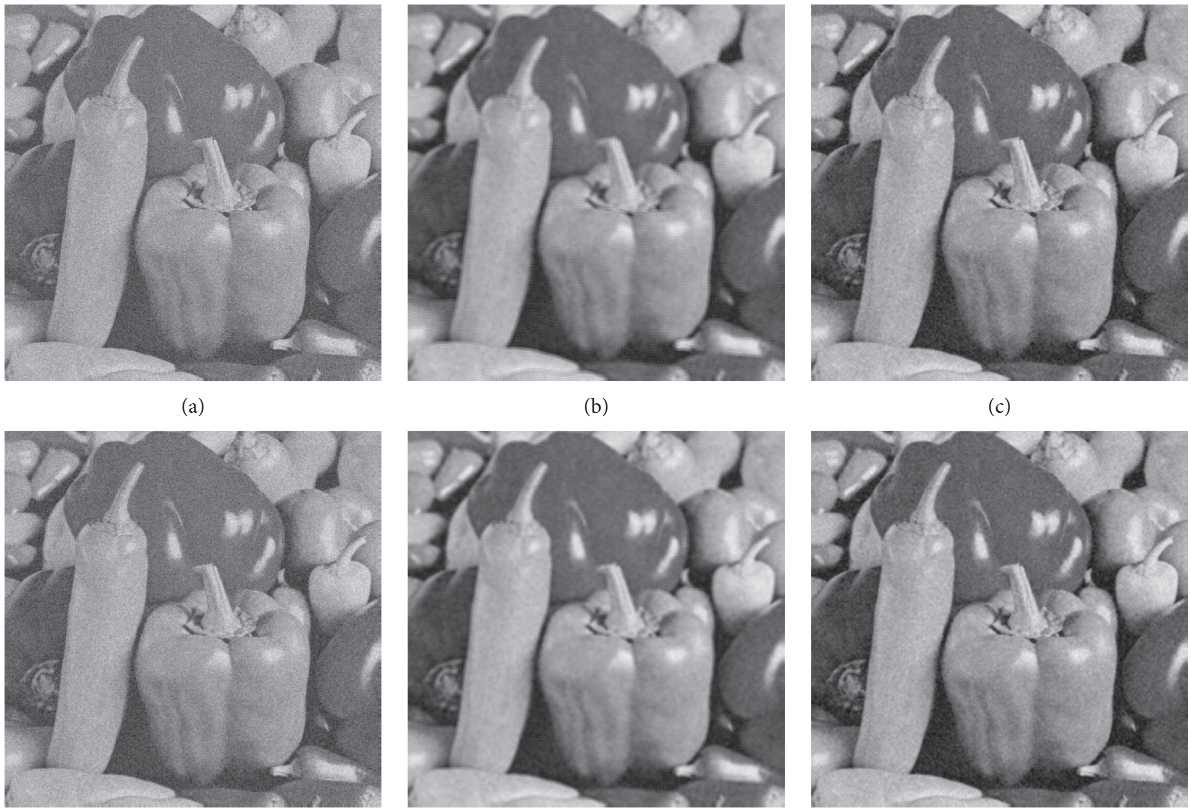
Figure 11: Comparison of different operators on “Pepper” under Gaussian noise with variance $\sigma = 15$.



(a) (b) (c)

(d) (e) (f)

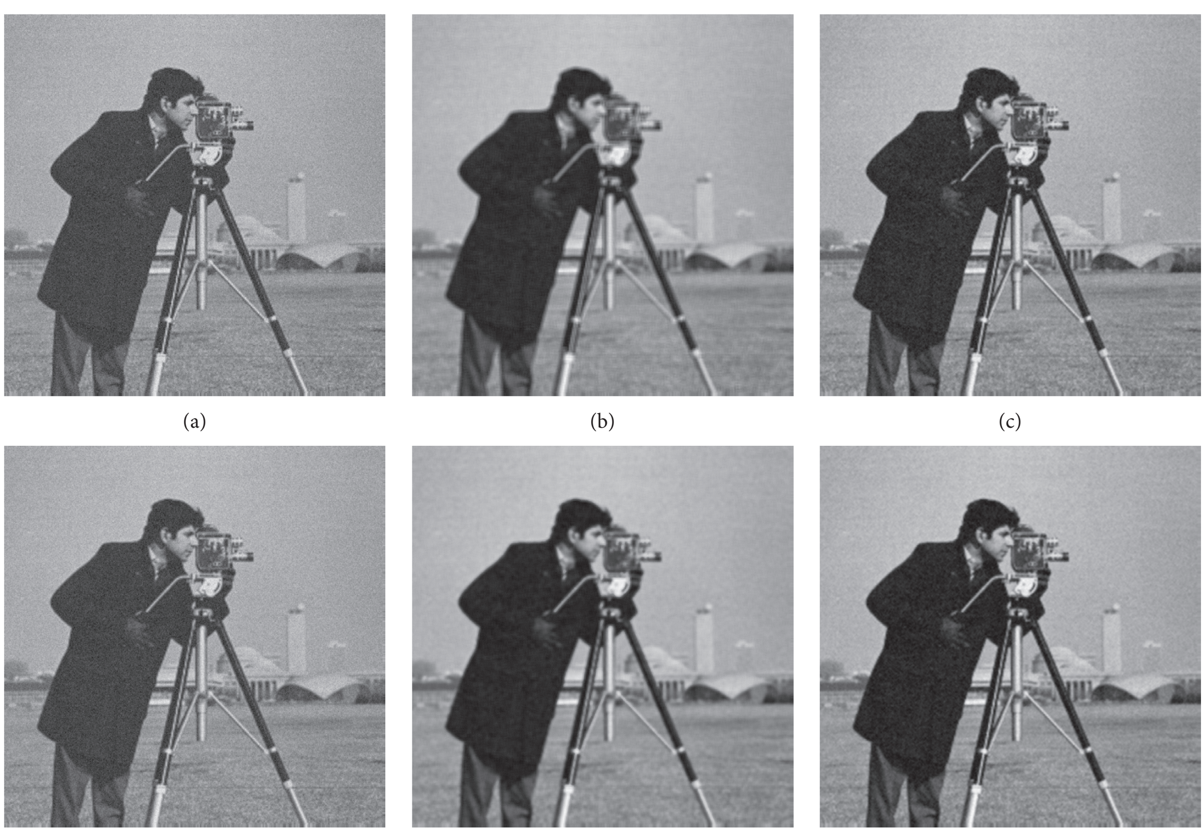
Figure 12: Comparison of different operators on “Pepper” under Gaussian noise with variance $\sigma = 20$.



(a) (b) (c)

(d) (e) (f)

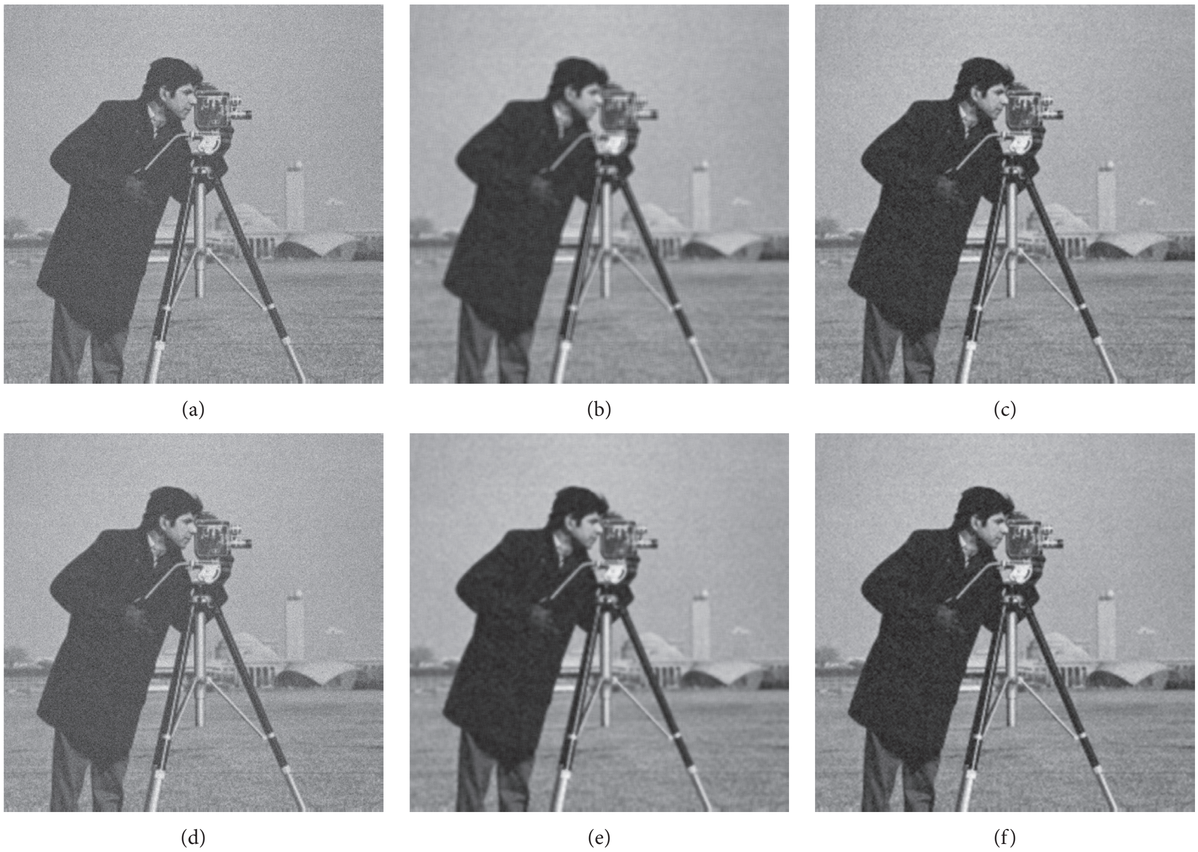
Figure 13: Comparison of different operators on "Pepper" under Gaussian noise with variance $\sigma = 25$.



(a) (b) (c)

(d) (e) (f)

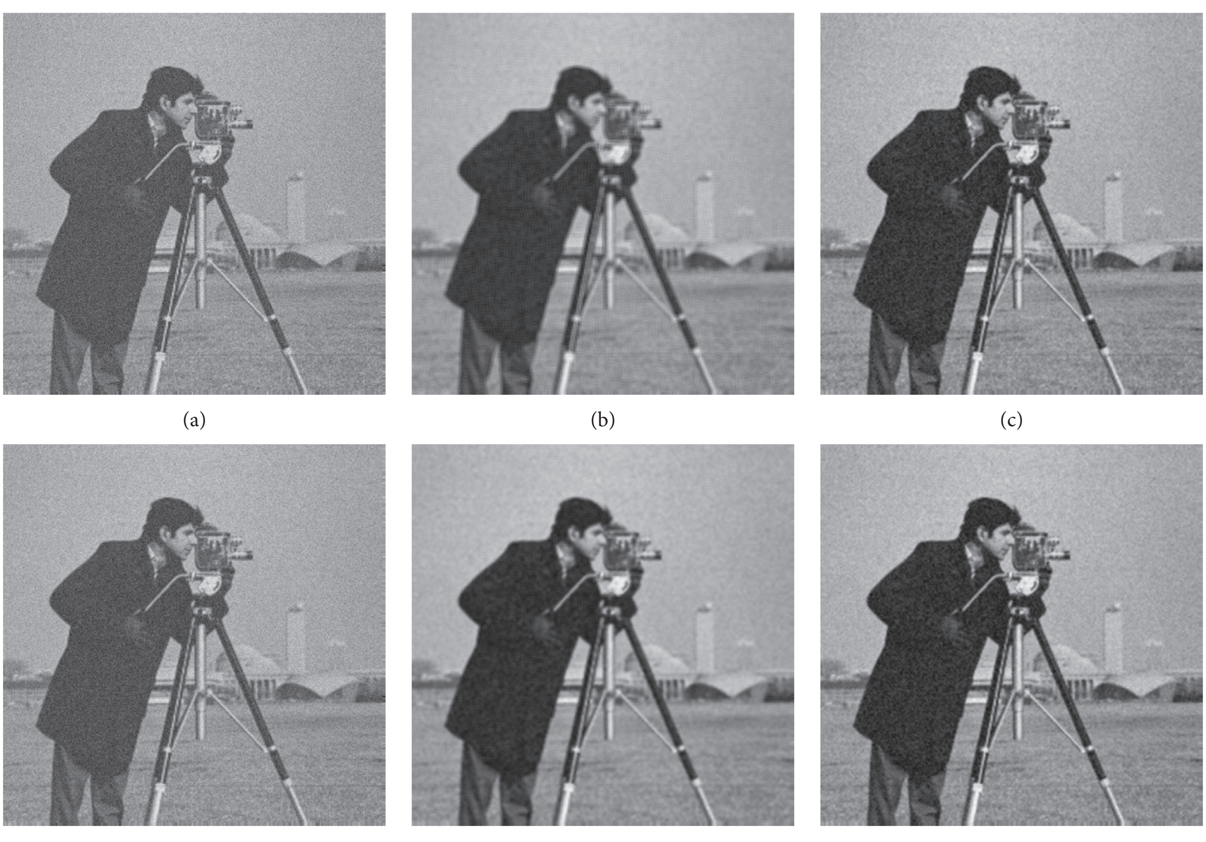
Figure 14: Comparison of different operators on "Cameraman" under Gaussian noise with variance $\sigma = 15$.



(a) (b) (c)

(d) (e) (f)

Figure 15: Comparison of different operators on “Cameraman” under Gaussian noise with variance $\sigma = 20$.



(a) (b) (c)

(d) (e) (f)

Figure 16: Comparison of different operators on “Cameraman” under Gaussian noise with variance $\sigma = 25$. (a) Noisy image, (b) TA_ABC, (c) GL_ABC, (d) Method in [38], (e) Ada_TA_ABC, (f) Ada_GL_ABC.

Table 2: Comparison of the effectiveness of different fractional operators for the Lena image.

		Noisy image	Method in [36]	TA_ABC	Ada_TA_ABC	GL_ABC	Ada_GL_ABC
$\sigma = 15$	PSNR	24.6047	25.8887	26.6857	28.3095	30.6827	31.2764
	SSIM	0.4466	0.498	0.6325	0.7147	0.7498	0.8126
$\sigma = 20$	PSNR	22.1006	23.6471	26.0032	27.6077	29.2995	29.9716
	SSIM	0.3392	0.3953	0.556	0.6524	0.6924	0.7482
$\sigma = 25$	PSNR	20.1839	22.0718	25.3131	27.0041	28.3211	28.7544
	SSIM	0.2682	0.3296	0.4895	0.5979	0.6516	0.6845

Table 3: Comparison of the effectiveness of different fractional operators for the Elaine image.

		Noisy image	Method in [38]	TA_ABC	Ada_TA	GL_ABC	Ada_GL_ABC
$\sigma = 15$	PSNR	24.602	25.8	29.1284	30.5185	31.5693	32.5138
	SSIM	0.4239	0.4754	0.6484	0.7174	0.7575	0.8033
$\sigma = 20$	PSNR	22.1297	23.4999	28.0504	29.5162	30.4062	30.8692
	SSIM	0.3101	0.363	0.574	0.66	0.714	0.739
$\sigma = 25$	PSNR	20.1507	21.7676	26.9874	28.542	28.8114	29.3865
	SSIM	0.233	0.2867	0.5048	0.6024	0.6385	0.6719

Table 4: Comparison of the effectiveness of different fractional operators for the Goldhill image.

		Noisy image	Method in [38]	TA_ABC	Ada_TA_ABC	GL_ABC	Ada_GL_ABC
$\sigma = 15$	PSNR	24.6053	25.5984	25.8959	26.7892	29.265	29.8402
	SSIM	0.5246	0.562	0.5339	0.6021	0.7312	0.7831
$\sigma = 20$	PSNR	22.0996	23.2623	25.2557	26.3092	28.5353	28.9227
	SSIM	0.4058	0.449	0.4725	0.5579	0.7035	0.7343
$\sigma = 25$	PSNR	20.1693	21.6197	24.6607	25.8326	27.6622	27.9387
	SSIM	0.3205	0.3707	0.4226	0.5176	0.6634	0.6823

Table 5: Comparison of the effectiveness of different fractional operators for the Pepper image.

		Noisy image	Method in [38]	TA_ABC	Ada_TA_ ABC	GL_ABC	Ada_GL_ABC
$\sigma = 15$	PSNR	24.6089	25.5672	26.6575	27.508	29.9746	**30.3144**
	SSIM	0.4525	0.4913	0.6472	0.7039	0.7407	**0.7979**
$\sigma = 20$	PSNR	22.1164	23.3792	26.0054	27.0188	28.8524	**29.3238**
	SSIM	0.3451	0.3904	0.576	0.6504	0.6927	**0.7383**
$\sigma = 25$	PSNR	20.1598	21.6034	25.2919	26.4364	27.72	**28.2121**
	SSIM	0.2723	0.3178	0.5095	0.5944	0.6326	**0.6733**

Table 6: Comparison of the effectiveness of different fractional operators for the Cameraman image.

		Noisy image	Method in [38]	TA_ABC	Ada_TA_ ABC	GL_ABC	Ada_GL_ABC
$\sigma = 15$	PSNR	24.6171	25.7532	25.724	28.3601	30.9423	31.5346
	SSIM	0.4084	0.4524	0.6394	0.7668	0.7398	0.8214
$\sigma = 20$	PSNR	22.1178	23.6034	25.1439	27.7603	29.5537	30.0694
	SSIM	0.3116	0.3615	0.5502	0.7002	0.6902	0.7429
$\sigma = 25$	PSNR	20.1543	21.7478	24.5397	26.8904	28.3857	28.7435
	SSIM	0.2465	0.2923	0.4724	0.6241	0.635	0.6674

Table 7: Comparison of the entropy of different fractional operators for all images.

	Lena			Elaine			Goldhill			Peppers			Cameraman		
Σ	15	20	25	15	20	25	15	20	25	15	20	25	15	20	25
Original image		7.4456			7.5001			7.4778			7.5715			7.0480	
Method in [36]	7.6008	7.6512	7.6891	7.5873	7.6181	7.6525	7.6079	7.6478	7.6744	7.6767	7.6973	7.7121	7.3948	7.4416	7.4843
TA_ABC	7.44	7.4675	7.4989	7.4760	7.4874	7.5084	7.4494	7.4734	7.4956	7.6141	7.6268	7.6412	7.2339	7.2925	7.3342
Ada_TA_ ABC	7.4298	7.4515	7.4788	7.4734	7.4808	7.4979	7.4427	7.4621	7.4800	7.6071	7.6183	7.6302	7.1802	7.2381	7.2886
GL_ABC	7.4878	7.5028	7.5189	7.5118	7.5125	7.5303	7.5285	7.5197	7.5248	7.6269	7.6329	7.6444	7.2693	7.2965	7.3226
Ada_GL_ABC	7.4508	7.4744	7.5038	7.4944	7.5042	7.5205	7.7405	7.4901	7.5101	7.6101	7.6224	7.6365	7.2111	7.2674	7.3085

the entropy of test images, as shown in Table 7. The entropy of images processed by the Ada_GL_ABC mask is closer to that of the original images than that got by other masks, but it is also higher than that of the original images. The results demonstrate that the detailed information of images is preserved while denoising. For Figures 2–16, (e) and (f) are clearer than the others by visual evaluation. TA_ABC and GL_ABC have been improved by the proposed adaptive function. According to the quantitative indicators shown in the tables, the Ada_GL_ABC mask has better denoising and detail-preserving ability than other masks. Furthermore, Ada_GL_ABC and Ada_TA_ABC masks are robust for different intensity noise by the analysis of the results. The effectiveness of our proposed adaptive operator can be proved from the two aspects of vision and evaluation index.

5. Conclusions

In this paper, the adaptive denoising mask is proposed based on Atangana–Baleanu derivatives. The key to this method is the calculation of order. The order is determined by the intensity of the gradient, global entropy, local entropy, and local variance. These variables represent the whole and local information of the image. To protect the texture details, we design the adaptive order integral operator considering global and local information. This operator can produce smaller orders in the image edge and texture details while larger orders in the smooth region. The proposed function is used to improve the GL_ABC mask and TA_ABC mask operator. We test the effectiveness of our proposed algorithm on multiple images. From a visual point of view, the denoising ability of Ada_TA_ABC and Ada_GL_ABC are reliable. Compared with other operators by the evaluation indicators, the Ada_GL_ABC operator works better. And, the PSNR and the SSIM are all higher under different intensities of noise. The information entropy index of the image processed by Ada_TA_ABC and Ada_GL_ABC operators is closer to the original images. The entropy of image filtered by Ada_GL_ABC mask is slightly larger, which indicates that Ada_GL_ABC mask can preserve texture details. These experimental results confirm that GL_ABC and TA_ABC have all been improved. And, the proposed adaptive function has a certain degree of universality.

Data Availability

The test images used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

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References

[1] S. Zhou, X. Lin, H. Li et al., "Chaotic synchronization of a fractional-order system based on washout filter control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1533–1540, 2011.

[2] M. A. Khan and A. Atangana, "Modeling the dynamics of novel coronavirus (2019-nCov) with fractional derivative," *Alexandria Engineering Journal*, vol. 59, no. 4, pp. 2379–2389, 2020.

[3] G. Wang, Y. Liu, Y. Wu et al., "Symmetry analysis for a seventh-order generalized KdV equation and its fractional version in fluid mechanics," *Fractals*, vol. 28, no. 3, pp. 127–134, 2020.

[4] T. J. Freeborn, "A survey of fractional-order circuit models for biology and biomedicine," *IEEE Journal on Emerging and Selected Topics in Circuits and Systems*, vol. 3, no. 3, pp. 416–424, 2013.

[5] B. N. Lundstrom, M. H. Higgs, W. J. Spain, and A. L. Fairhall, "Fractional differentiation by neocortical pyramidal neurons," *Nature Neuroscience*, vol. 11, no. 11, pp. 1335–1342, 2008.

[6] Z. Wang, X. Huang, and G. Shi, "Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1531–1539, 2011.

[7] X. Hu, Q. Song, M. Ge, and R. Li, "Fractional-order adaptive fault-tolerant control for a class of general nonlinear systems," *Nonlinear Dynamics*, vol. 101, no. 1, pp. 379–392, 2020.

[8] H. A. Jalab, R. W. Ibrahim, A. Ahmed et al., "Image denoising algorithm based on the convolution of fractional Tsallis entropy with the Riesz fractional derivative," *Neural Computing and Applications*, vol. 28, no. S1, pp. 217–223, 2017.

[9] F. Hu, S. Si, H. S. Wong, B. Fu, M. Si, and H. Luo, "An adaptive approach for texture enhancement based on a fractional differential operator with non-integer step and order," *Neurocomputing*, vol. 158, no. 22, pp. 295–306, 2015.

[10] J. Bai and X.-C. Feng, "Fractional-order anisotropic diffusion for image denoising," *IEEE Transactions on Image Processing*, vol. 16, no. 10, pp. 2492–2502, 2007.

[11] Y. Pu, P. Siarry, A. Chatterjee et al., "A fractional-order variational framework for retinex: fractional-order partial differential equation-based formulation for multi-scale nonlocal contrast enhancement with texture preserving," *IEEE Transactions on Image Processing*, vol. 27, pp. 1214–1229, 2017.

[12] Y. Pu, "Application of fractional differential approach to digital image processing," *Advanced Engineering Sciences*, vol. 39, no. 3, pp. 124–132, 2007.

[13] Y. Pu, J. Zhou, and X. Yuan, "Fractional differential mask: a fractional differential-based approach for multiscale texture enhancement," *IEEE Transactions on Image Processing*, vol. 19, no. 2, pp. 491–511, 2010.

[14] J. Guan, J. Ou, Z. Lai et al., "The medical image enhancement method based on the fractional order derivative and the directional derivative," *International Journal of Pattern Recognition and Artificial Intelligence*, vol. 32, Article ID 1857001, 2018.

[15] C. Wang, L. Lan, and S. Zhou, "Grunwald-Letnikov based adaptive fractional differential algorithm on image texture enhancing," *Journal of Computational Information Systems*, vol. 9, no. 2, pp. 445–454, 2013.

[16] A. Azarang and H. Ghassemian, "Application of fractional-order differentiation in multispectral image fusion," *Remote Sensing Letters*, vol. 9, no. 1, pp. 91–100, 2017.

[17] A. Saadia and A. Rashdi, "Incorporating fractional calculus in echo-cardiographic image denoising," *Computers & Electrical Engineering*, vol. 67, pp. 134–144, 2018.

[18] W. Gao, P. Veeresha, D. G. Prakasha et al., "New numerical results for the time-fractional phi-four equation using a novel analytical approach," *Symmetry*, vol. 12, Article ID 478, 2020.

[19] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "New numerical simulation for fractional Benney–Lin equation arising in falling film problems using two novel techniques," *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 210–243, 2021.

[20] K. M. Owolabi and Z. Hammouch, "Spatiotemporal patterns in the Belousov-Zhabotinskii reaction systems with Atangana-Baleanu fractional order derivative," *Physica A: Statistical Mechanics and Its Applications*, vol. 523, pp. 1072–1090, 2019.

[21] K. M. Owolabi and Z. Hammouch, "Mathematical modeling and analysis of two-variable system with noninteger-order derivative," *Chaos*, vol. 29, no. 1, Article ID 013145, 2019.

[22] J. D. Djida, A. Atangana, and I. Area, "Numerical computation of a fractional derivative with non-local and non-singular kernel," *Mathematical Modelling of Natural Phenomena*, vol. 12, no. 3, pp. 4–13, 2017.

[23] D. Baleanu and A. Fernandez, "On some new properties of fractional derivatives with Mittag-Leffler kernel," *Communications in Nonlinear Science and Numerical Simulation*, vol. 59, pp. 444–462, 2017.

[24] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 51 pages, 2011.

[25] J. E. Solís-Pérez, J. F. Gómez-Aguilar, R. F. Escobar-Jiménez, and J. Reyes-Reyes, "Blood vessel detection based on fractional Hessian matrix with non-singular Mittag-Leffler Gaussian kernel," *Biomedical Signal Processing and Control*, vol. 54, Article ID 101584, 2019.

[26] B. Ghanbari and A. Atangana, "A new application of fractional Atangana-Baleanu derivatives: designing ABC-fractional masks in image processing," *Physica A Statistical Mechanics & Its Applications*, vol. 542, Article ID 123516, 2019.

[27] G. Huang, L. Xu, and Y. Pu, "Summary of research on image processing using fractional calculus," *Application Research of Computers*, vol. 29, no. 2, pp. 414–426, 2012.

[28] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model," *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.

[29] K. B. Oldham and J. Spanier, "The fractional calculus," *Mathematical Gazette*, vol. 56, no. 247, pp. 396–400, 1974.

[30] J. Yu, L. Tan, S. Zhou et al., "Image denoising algorithm based on entropy and adaptive fractional order calculus operator," *IEEE Access*, vol. 5, pp. 12275–12285, 2017.

[31] J. Prehl, F. Boldt, K. H. Hoffmann et al., "Symmetric fractional diffusion and entropy production," *Entropy*, vol. 18, no. 8, p. 275, 2016.

[32] B. Qiao, L. Jin, and Y. Yang, "An adaptive algorithm for grey image edge detection based on grey correlation analysis," in *Proceedings of the International Conference on Computational Intelligence & Security*, pp. 470–474, Wuxi, China, December 2017.

[33] W. Xie, M. You, and Y. Zhou, "Guided image filter and application based on adaptive fractional-order differential," *Application Research of Computers*, vol. 34, no. 001, pp. 283–286, 2017.

[34] N. He, J.-B. Wang, L.-L. Zhang, and K. Lu, "An improved fractional-order differentiation model for image denoising," *Signal Processing*, vol. 112, pp. 180–188, 2015.

[35] W. Zhou, A. C. Bovik, H. R. Sheikn et al., "Image quality assessment: from error visibility to structural similarity," *IEEE Transactions on Image Processing*, vol. 13, no. 4, pp. 600–612, 2004.

[36] B. Li and W. Xie, "Image enhancement and denoising algotithms based on adaptive fractional differential and integral," *Systems Engineering and Electronics*, vol. 38, pp. 185–192, 2016.

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Research Article

On Strongly Convex Functions via Caputo–Fabrizio-Type Fractional Integral and Some Applications

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The theory of convex functions plays an important role in the study of optimization problems. The fractional calculus has been found the best to model physical and engineering processes. The aim of this paper is to study some properties of strongly convex functions via the Caputo–Fabrizio fractional integral operator. In this paper, we present Hermite–Hadamard-type inequalities for strongly convex functions via the Caputo–Fabrizio fractional integral operator. Some new inequalities of strongly convex functions involving the Caputo–Fabrizio fractional integral operator are also presented. Moreover, we present some applications of the proposed inequalities to special means.

1. Introduction

The theory of fractional calculus got rapid development, and it has brought the attention of many researchers from various disciplines [1–3]. In the last few years, it was observed that fractional calculus is very useful for modeling complicated problems of engineering, chemistry, mechanics, and many other branches. Various interesting notations of fractional calculus exist in the literature, for example, the Riemann–Liouville fractional integral and Caputo–Fabrizio fractional integral [4–14].

Among these notions, Riemann–Liouville and Caputo involve the following singular kernal [11]:

$$K(\zeta, x) = \frac{(\zeta - x)^{-\varsigma}}{\Gamma(1-\varsigma)}, \quad 0 < \varsigma < 1. \tag{1}$$

However, it was observed by Caputo and Fabrizio in [8] that certain phenomena cannot be modelled by the already existing definition in the literature. That is why, they proposed a more general fractional derivative in [8] and named it as the Caputo–Fabrizio fractional integral operator. It mainly involves the following nonsingular kernal:

$$K(\zeta, x) = e^{-\varsigma(\zeta - x)/(1-\varsigma)}, \quad 0 < \varsigma < 1. \tag{2}$$

Nowadays, many researchers of applied sciences are using the Caputo–Fabrizio fractional integral operator to model their problem. For more details about the fractional integral with a nonsingular kernal, we refer [15–19] to the readers.

The theory of inequalities also plays an important role in applied as well as in pure mathematics. The Hermite–Hadamard inequality is the most important inequality in the literature, and this inequality has been studied for different classes of convex functions, see [20–24]. The classical version of the Hermite–Hadamard inequality for convex functions is stated as follows:

If $\varrho\colon I = [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is an integrable and continuous convex function, then its mean value remains between the value of ϱ at $(a+b)/2$ of interval $I = [a,b]$ and arithmetic mean value of ϱ at the endpoints $a, b \in I = [a,b]$. In other words, it means that

$$\varrho\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varrho(x)x \leq \frac{\varrho(a)+\varrho(b)}{2}. \tag{3}$$

Inequality (3), in the literature, is generalized by several fractional integral operators to meet the desired results, see, for instance, [25–28]. In this paper, we present the Hermite–Hadamard inequality for a strongly convex function in the setting of the Caputo–Fabrizio fractional integral operator. We also present some new inequalities for strongly convex functions in the setting of the Caputo–Fabrizio fractional integral operator. We also give some applications of the presented inequalities in special mean.

2. Preliminaries

In this section, we present some definitions from the literature.

Definition 1. A function is convex if

$$\varrho(\zeta x+(1-\zeta)y)\leq \zeta\varrho(x)+(1-\zeta)\varrho(y), \tag{4}$$

for every $x, y \in I$ and $\zeta \in [0,1]$.

Definition 2 (see [29]). Assume $\lambda \geq 0$. A function $\varrho\colon I \longrightarrow \mathbb{R}$ is strongly convex if

$$\varrho(\zeta x+(1-\zeta)y)\leq \zeta\varrho(x)+(1-\zeta)\varrho(y)-\lambda\zeta(1-\zeta)(x-y)^2, \tag{5}$$

for every $x, y \in I$ and $\zeta \in [0,1]$.

Remark 1. Setting $\lambda = 0$ in inequality (5), we obtain convex function (4).

Definition 3 (see [8]). Let $\varrho \in H^1(a,b)$, $a<b, \varsigma \in [0,1]$; then, the left Caputo–Fabrizio fractional derivative is defined by

$$\left({}_{a}^{\mathrm{CF}}D^{\varsigma}\varrho\right)(\zeta)=\frac{B(\varsigma)}{(1-\varsigma)}\in \zeta_a^{\zeta}\varrho'(x)e^{-\varsigma(\zeta-x)^{\varsigma}/(1-\varsigma)}\mathrm{d}x, \tag{6}$$

and the left Caputo–Fabrizio fractional integral is defined by

$$\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta)=\frac{(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)+\frac{\varsigma}{B(\varsigma)}\in \zeta_a^{\zeta}\varrho(x)\mathrm{d}x, \tag{7}$$

where $B(\varsigma)>0$ is a normalization of function with $B(0)=B(1)=1$.

Definition 4 (see [8]). Let $\varrho \in H^1(a,b)$, $a<b, \varsigma \in [0,1]$; then, the right Caputo–Fabrizio fractional derivative is defined by

$$\left({}^{\mathrm{CF}}D_b^{\varsigma}\varrho\right)(\zeta)=\frac{-B(\varsigma)}{(1-\varsigma)}\in \zeta_{\zeta}^{b}\varrho'(x)e^{-\varsigma(x-\zeta)^{\varsigma}/(1-\varsigma)}\mathrm{d}x, \tag{8}$$

and the right Caputo–Fabrizio fractional integral is defined by

$$\left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho\right)(\zeta)=\frac{(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)+\frac{\varsigma}{B(\varsigma)}\in \zeta_{\zeta}^{b}\varrho(x)\mathrm{d}x, \tag{9}$$

where $B(\varsigma)>0$ is a normalization of function with $B(0)=B(1)=1$.

3. Hermite–Hadamard-Type Inequalities via Caputo–Fabrizio Fractional Integrals for Strongly Convex Functions

Theorem 1. *Assume $\varrho\colon I \longrightarrow \mathbb{R}$ to be a strongly convex function with modulus $\lambda \geq 0$ and $\varrho \in L_1[a,b]$; then, the inequality*

$$\begin{aligned}
&\varrho\left(\frac{a+b}{2}\right)+\frac{\lambda}{12}(b-a)^2\\
&\leq \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho\right)(\zeta)-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)\right]\\
&\leq \frac{\varrho(a)+\varrho(b)}{2}-\frac{\lambda}{6}(b-a)^2,
\end{aligned} \tag{10}$$

holds, where $B(\varsigma)>0$ is a normalization function, $\varsigma \in [0,1]$, and $\zeta \in [0,1]$.

Proof. Since ϱ is strongly convex function, we have

$$\begin{aligned}
&\varrho\left(\frac{a+b}{2}\right)-\frac{\lambda}{12}(b-a)^2\leq \frac{1}{b-a}\int_a^b\varrho(x)\mathrm{d}x\\
&\leq \frac{\varrho(a)+\varrho(b)}{2}-\frac{\lambda}{6}(b-a)^2.
\end{aligned} \tag{11}$$

The left side of inequality (11) yields

$$\begin{aligned}
&2\varrho\left(\frac{a+b}{2}\right)-\frac{\lambda}{6}(b-a)^2\leq \frac{2}{b-a}\int_a^b\varrho(x)\mathrm{d}x,\\
&=\frac{2}{b-a}\left[\int_a^{\zeta}\varrho(x)g(x)\mathrm{d}x+\int_{\zeta}^{b}\varrho(x)g(x)\mathrm{d}x\right].
\end{aligned} \tag{12}$$

Multiplying $\varsigma(b-a)/2B(\varsigma)$ on both sides of the abovementioned inequality, adding $(2(1-\varsigma)/B(\varsigma))\varrho(\zeta)$ $g(\zeta)$ and rearranging the terms, we obtain

$$\begin{aligned}
&\varrho\left(\frac{a+b}{2}\right)+\frac{\lambda}{12}(b-a)^2\\
&\leq \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho\right)(\zeta)-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)\right],
\end{aligned} \tag{13}$$

which is the left side of Theorem 1.

Now, to prove the right side of Theorem 1, we use the right side of (11), which is

$$\frac{2}{b-a}\int_a^b\varrho(x)\mathrm{d}x\leq \varrho(a)+\varrho(b)-\frac{\lambda}{3}(b-a)^2. \tag{14}$$

Applying the same operations on the abovementioned inequality as on (12) yields the right side of Theorem 1, which is

$$\leq \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta)+\left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho\right)(\zeta)-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)\right]$$
$$\leq \frac{\varrho(a)+\varrho(b)}{2}-\frac{\lambda}{6}(b-a)^{2}. \tag{15}$$

The combination of (13) and (15) completes the proof. □

Theorem 2. *Assume that* $\varrho, g\colon I \longrightarrow \mathbb{R}$ *are two strongly convex functions with modulus* $\lambda \geq 0$ *and* $f, g \in L_1[a,b]$; *then, the inequality*

$$\frac{2B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho g\right)(\zeta)-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta)\right]$$
$$\leq \frac{2}{3}P(a,b)+\frac{1}{3}Q(a,b)-\frac{\lambda}{3}(b-a)^{2}\left[R(a,b)-\frac{\lambda}{5}(b-a)^{2}\right], \tag{16}$$

holds with normalization function $B(\varsigma)>0$, $\varsigma \in [0,1]$, *and* $\zeta \in [0,1]$, *where* $P(a,b)=\varrho(a)g(a)+\varrho(b)g(b)$, $Q(a,b)=\varrho(a)\ g(b)+\varrho(b)g(a)$, *and* $R(a,b)=\varrho(a)+g(a)+\varrho(b)+g(b)$.

Proof. Since ϱ and g are strongly convex functions defined on I, by definition, we have

$$\varrho(\zeta a+(1-\zeta)b)\leq \zeta\varrho(a)+(1-\zeta)\varrho(b)-\lambda\zeta(1-\zeta)(b-a)^{2}, \tag{17}$$

$$g(\zeta a+(1-\zeta)b)\leq \zeta g(a)+(1-\zeta)g(b)-\lambda\zeta(1-\zeta)(b-a)^{2}, \tag{18}$$

for all $a,b \in I$ and $\zeta \in [0,1]$.

Multiplying (17) and (18), we have

$$\begin{aligned}
&\varrho(\zeta a+(1-\zeta)b)g(\zeta a+(1-\zeta)b)\\
&\leq \zeta^{2}\varrho(a)g(a)+(1-\zeta)^{2}\varrho(b)g(b)\\
&\quad+\zeta(1-\zeta)[\varrho(a)g(b)+\varrho(b)g(a)]\\
&\quad-\lambda\zeta(1-\zeta)^{2}(b-a)^{2}[\varrho(b)+g(b)]\\
&\quad-\lambda\zeta^{2}(1-\zeta)g(a)(b-a)^{2}\\
&\quad-\lambda\zeta^{2}(1-\zeta)(b-a)^{2}[\varrho(a)+g(a)]\\
&\quad+\lambda^{2}\zeta^{2}(1-\zeta)^{2}(b-a)^{4}.
\end{aligned} \tag{19}$$

Integrating the abovementioned inequality w.r.t "ζ" over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{b-a}\int_{a}^{b}\varrho(x)g(x)\mathrm{d}x \leq \frac{2}{3}[\varrho(a)g(a)+\varrho(b)g(b)]+\frac{1}{3}[\varrho(a)g(b)+\varrho(b)g(a)]\\
&-\frac{\lambda}{3}(b-a)^{2}[\varrho(a)+\varrho(b)+g(a)+g(b)]-\frac{\lambda}{5}(b-a)^{2},\\
&\frac{2}{b-a}\int_{a}^{b}\varrho(x)g(x)\mathrm{d}x\\
&\leq \frac{2}{3}[P(a,b)]+\frac{1}{3}[Q(a,b)]-\frac{\lambda}{3}(b-a)^{2}[R(a,b)]-\frac{\lambda}{5}(b-a)^{2}.
\end{aligned} \tag{20}$$

Multiplying $\varsigma(b-a)/2B(\varsigma)$ on both sides and adding $(2(1-\varsigma)/B(\varsigma))\varrho(\zeta)g(\zeta)$, we obtain

$$\begin{aligned}
&\frac{\varsigma}{B(\varsigma)}\left[\int_{a}^{\zeta}\varrho(x)g(x)\mathrm{d}x+\int_{\zeta}^{b}\varrho(x)g(x)\mathrm{d}x\right]+\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta)\\
&\leq \frac{\varsigma(b-a)}{2B(\varsigma)}\left[\frac{2}{3}[P(a,b)]+\frac{1}{3}[Q(a,b)]-\frac{\lambda}{3}(b-a)^{2}[R(a,b)]-\frac{\lambda}{5}(b-a)^{2}\right]\\
&\quad+\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta).
\end{aligned} \tag{21}$$

Now, the use of (7) and (9) and rearrangements of the terms of abovementioned inequality complete the proof. □

Theorem 3. *Assume that* $f, g\colon I \longrightarrow \mathbb{R}$ *are two strongly convex functions with modulus* $\lambda \geq 0$ *and* $f, g \in L_1[a,b]$; *then, the inequality*

$$\frac{2\varsigma}{B(\varsigma)}\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho g\right)(\zeta)\right]$$
$$+\frac{2(1-\varsigma)}{B(\varsigma)(b-a)}\varrho(\zeta)g(\zeta) \tag{22}$$
$$\leq\frac{\varsigma}{2B(\varsigma)}\left[\frac{2}{3}Q(a,b)+\frac{1}{3}P(a,b)-\frac{\lambda}{3}(b-a)^{2}\left[R(a,b)-\frac{\lambda}{5}(b-a)^{2}-\frac{(b-a)^{2}}{2}-\frac{1}{5}\right]\right]$$

holds with normalization function $B(\varsigma)>0$, $\varsigma\in[0,1]$, *and* $\zeta\in[0,1]$, *where* $P(a,b)=\varrho(a)g(a)+\varrho(b)g(b)$, $Q(a,b)=\varrho(a)g(b)+\varrho(b)\,g(a)$, *and* $R(a,b)=\varrho(a)+g(a)+\varrho(b)+g(b)$.

Proof. Since ϱ and g be the two strongly convex functions, so for $\zeta=1/2$, we have

$$\varrho\left(\frac{a+b}{2}\right)\leq\frac{\varrho(\zeta a+(1-\zeta)b)+\varrho(\zeta a+(1-\zeta)a)}{2}-\frac{\lambda}{4}(2\zeta-1)(b-a)^{2}, \tag{23}$$

$$g\left(\frac{a+b}{2}\right)\leq\frac{g(\zeta a+(1-\zeta)b)+g(\zeta a+(1-\zeta)a)}{2}-\frac{\lambda}{4}(2\zeta-1)(b-a)^{2}, \tag{24}$$

for all $a,b\in I$ and $\zeta\in[0,1]$.

Multiplying (23) and (24), we obtain

$$\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leq\frac{1}{4}[\varrho(\zeta a+(1-\zeta)b)g(\zeta a+(1-\zeta)b)+\varrho(\zeta a+(1-\zeta)a)g(\zeta a+(1-\zeta)a)$$
$$+\left(\zeta^{2}+(1-\zeta)^{2}\right)[\varrho(a)g(b)+(b)g(a)]+2\zeta(1-\zeta)[\varrho(a)g(a)+\varrho(b)g(b)]$$
$$-\lambda(b-a)^{2}\left(\zeta^{2}(1-\zeta)+\zeta(1-\zeta)^{2}\right)[\varrho(a)+g(b)+\varrho(b)+g(a)] \tag{25}$$
$$+2\lambda^{2}\zeta^{2}(1-\zeta)^{2}(b-a)^{2}+2\lambda^{2}\zeta(1-\zeta)(2\zeta-1)^{2}(b-a)^{4}+\frac{\lambda}{2}(2\zeta-1)^{2}(b-a)^{4}$$
$$-\frac{\lambda}{2}(2\zeta-1)^{2}(b-a)^{2}R(a,b).$$

Integrating the abovementioned inequality w.r.t "ζ" over $[0,1]$ and using the technique of change of variable, we obtain

$$4\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{2}{b-a}\int_{a}^{b}\varrho(x)g(x)\mathrm{d}x+\frac{1}{3}P(a,b)+\frac{2}{3}Q(a,b)$$
$$-\frac{\lambda}{3}(b-a)^{2}\left[R(a,b)-\frac{\lambda}{5}-\frac{(b-a)^{2}}{2}-\frac{1}{5}\right]. \tag{26}$$

Multiplying $\varsigma(b-a)/2B(\varsigma)$ on both sides and subtracting $(2(1-\varsigma)/B(\varsigma))\varrho(\zeta)g(\zeta)$, we obtain

$$\begin{aligned}&\frac{2\varsigma(b-a)}{2B(\varsigma)}\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\leq\frac{\varsigma}{B(\varsigma)}\left[\int_a^\zeta\varrho(x)g(x)\mathrm{d}x+\int_\zeta^b\varrho(x)g(x)\mathrm{d}x\right]-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta)\\&+\frac{\varsigma(b-a)}{2B(\varsigma)}\left[\frac{1}{3}P(a,b)+\frac{2}{3}Q(a,b)-\frac{\lambda}{3}(b-a)^2\left[R(a,b)-\frac{\lambda}{5}-\frac{(b-a)^2}{2}-\frac{1}{5}\right]\right]\\&-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta).\end{aligned}\tag{27}$$

Now, the use of (7) and (9) and rearrangements of the terms of the abovementioned inequality complete the proof. □

4. Some New Caputo–Fabrizio Fractional Integral Inequalities for Strongly Convex Functions

Lemma 1 (see [28, 30]). *Assume that $\varrho\colon I\longrightarrow\mathbb{R}$ is a differentiable mapping on I°, where $a,b\in I$ with $a<b$. If $\varrho'\in L_1[a,b]$, then the inequality*

$$\begin{aligned}&\frac{b-a}{2}\in\zeta_1^0(1-2\zeta)\varrho'(\zeta a+(1-\zeta)b)\mathrm{d}\zeta-\frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta)\\&=\frac{\varrho(a)+\varrho(b)}{2}-\frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_a^{\mathrm{CF}}I^\varsigma\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^\varsigma\varrho g\right)(\zeta)\right]\end{aligned}\tag{28}$$

holds, where $B(\varsigma)>0$ is a normalization function, $\varsigma\in[0,1]$, and $\zeta\in[0,1]$.

Theorem 4. *Assume that $\varrho\colon I\longrightarrow\mathbb{R}$ is a differentiable positive mapping on I°, where $a,b\in I$ with $a<b$. If $\varrho'\in L_1[a,b]$ and $|\varrho'|$ are two strongly convex functions, then the inequality*

$$\begin{aligned}&\left|\frac{\varrho(a)+\varrho(b)}{2}+\frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta)-\frac{B(\varsigma)}{\varsigma(b-a)}\right.\\&\left.\left[\left({}_a^{\mathrm{CF}}I^\varsigma\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^\varsigma\varrho g\right)(\zeta)\right]\right|\\&\leq\frac{(b-a)\left(\left|\varrho'(a)\right|+\left|\varrho'(b)\right|\right.}{8}-\frac{\lambda}{32}(b-a)^3\end{aligned}\tag{29}$$

holds, where $B(\varsigma)>0$ is a normalization function, $\varsigma\in[0,1]$, and $\zeta\in[0,1]$.

Proof. By using Lemma 1, convexity of $|\varrho'|$, and the property of absolute value, we get

$$\begin{aligned}&\left|\frac{\varrho(a)+\varrho(b)}{2}+\frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta)-\frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_a^{\mathrm{CF}}I^\varsigma\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^\varsigma\varrho g\right)(\zeta)\right]\right|,\\&=\left|\frac{b-a}{2}\in\zeta_0^1(1-2\zeta)\varrho'(\zeta a+(1-\zeta)b)\mathrm{d}\zeta\right|\\&\leq\frac{b-a}{2}\in\zeta_0^1(1-2\zeta)\left|\varrho'(\zeta a+(1-\zeta)b)\right|\mathrm{d}\zeta\\&\leq\frac{b-a}{2}\in\zeta_0^1|(1-2\zeta)|\left[t\left|\varrho'(a)\right|+(1-\zeta)\left|\varrho'(b)\right|-\lambda\zeta(1-\zeta)(b-a)^2\right]\mathrm{d}\zeta,\\&=\frac{(b-a)\left|\varrho'(a)\right|+\left|\varrho'(b)\right|}{8}-\frac{\lambda}{32}(b-a)^3.\end{aligned}\tag{30}$$

This completes the proof. □

Theorem 5. *Assume $\varrho: I \longrightarrow \mathbb{R}$ to be a differentiable positive mapping on I°, $a, b \in I$ with $a < b$, and $(1/p) + (1/q) = 1$. If $\varrho' \in L_1[a,b]$ and $|\varrho'|^q$ is a strongly convex function, then the inequality*

$$\left|\frac{\varrho(a)+\varrho(b)}{2} + \frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta) - \frac{B(\varsigma)}{\varsigma(b-a)}\right.$$
$$\left.\left[\left({}_a^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta) + \left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho g\right)(\zeta)\right]\right|$$
$$\leq \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|\varrho'(a)|^{p/(p-1)} + |\varrho'(b)|^{p/(p-1)}}{2} - \frac{\lambda}{6}(b-a)^2\right]^{(p-1)/p} \tag{31}$$

holds, where $B(\varsigma) > 0$ is a normalization function, $\varsigma \in [0,1]$, and $\zeta \in [0,1]$.

Proof. We start the proof by using Lemma 1, convexity of $|\varrho'|^q$, the property of absolute value, where $(1/p) + (1/q) = 1$, and Holder's inequality to obtain

$$\begin{aligned}
&\left|\frac{\varrho(a)+\varrho(b)}{2} + \frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta) - \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_a^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta) + \left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho g\right)(\zeta)\right]\right|,\\
&= \left|\frac{b-a}{2} \in \zeta_0^1(1-2\zeta)\varrho'(\zeta a + (1-\zeta)b)\mathrm{d}\zeta\right|\\
&\leq \frac{b-a}{2} \in \zeta_0^1|(1-2\zeta)||\varrho'(\zeta a + (1-\zeta)b)|\mathrm{d}\zeta\\
&\leq \frac{b-a}{2}\left(\in \zeta_0^1|(1-2\zeta)|^p\mathrm{d}\zeta\right)^{1/p}\left(\int_0^1|\varrho'(\zeta a + (1-\zeta)b)|^q\mathrm{d}\zeta\right)^{1/q}\\
&\leq \frac{b-a}{2}\left(\in \zeta_0^1|(1-2\zeta)|^p\mathrm{d}\zeta\right)^{1/p}\left(\int_0^1\left[t|\varrho'(a)| + (1-\zeta)|\varrho'(b)| - \lambda\zeta(1-\zeta)(b-a)^2\right]\mathrm{d}\zeta\right)^{1/q},\\
&= \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|\varrho'(a)|^q + |\varrho'(b)|^q}{2} - \frac{\lambda}{6}(b-a)^3\right)^{1/q},\\
&= \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|\varrho'(a)|^{p/(p-1)} + |\varrho'(b)|^{p/(p-1)}}{2} - \frac{\lambda}{6}(b-a)^3\right)^{(p-1)/p},
\end{aligned} \tag{32}$$

where $\int_0^1|1-2\zeta|^p\mathrm{d}\zeta = \int_0^{1/2}(1-2\zeta)\ {}^p\mathrm{d}\zeta + \int_{1/2}^1(1-2\zeta)^p\mathrm{d}\zeta = 2\int_0^{1/2}(1-2\zeta)^p\mathrm{d}\zeta = 1/(p+1)$. This completes the proof. □

5. Some Applications of Caputo–Fabrizio Fractional Integral Inequalities to Special Means

Means are important in applied and pure mathematics; especially, they are used frequently in numerical approximation. In the literature, they are ordered in the following way:

$$H \leq G \leq L \leq I \leq A. \tag{33}$$

The special means of two numbers a and b in the order of $b > a$ are known as arithmetic mean, geometric mean, harmonic mean, power mean, logarithmic mean, p-logarithmic mean, and identric mean. They are listed below from (34)–(40), respectively.

$$A(a,b) = \frac{a+b}{2}, \tag{34}$$

$$G(a,b) = \sqrt{ab}, \tag{35}$$

$$H(a,b) = \frac{2ab}{a+b}, \tag{36}$$

$$M_p(a,b) = \left[\left(\frac{a^p+b^p}{2}\right)^{1/p}\right], \quad p \neq 0, \tag{37}$$

$$L(a,b) = \frac{b-a}{\ln(b) - \ln(a)}, \tag{38}$$

$$L_p(a,b) = \left[\left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}\right], \quad p \in \mathbb{R} \setminus -1, 0, \tag{39}$$

$$I(a,b) = \frac{1}{e}\left(\frac{b^p}{a^p}\right)^{1/(b-a)}. \tag{40}$$

There are several results connecting these means, see [31] for some new relations; however, very few results are known for arbitrary real numbers. For this, it is clear that we can extend some of the abovementioned means as follows:

$$A(a,b) = \frac{a+b}{2}, a, b \in \mathbb{R},$$

$$\overline{L}(a,b) = \frac{b-a}{\ln|b| - \ln|a|}, \quad a, b \in \mathbb{R} \setminus \{0\},$$

$$L_n(a,b) = \left[\left(\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right)^{1/n}\right]. \quad n \in \mathbb{N}, n \geq 1, a, b \in \mathbb{R}, a < b. \tag{41}$$

Now, we shall use the results of Sections 3 and 4 to prove the following new inequalities connecting the abovementioned means for arbitrary real numbers.

Proposition 1. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, the following inequality holds:*

$$\left|A(a^n, b^n) - L_n^n(a,b)\right| \leq \frac{b-a}{8}\left[nA\left(|a|^{n-1}, |b|^{n-1}\right) - \frac{\lambda}{4}(b-a)^2\right]. \tag{42}$$

Proof. Insertion of $\varrho(x) = x^n$, where $n \in \mathbb{N}, n \geq 2$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 4 completes the proof. □

Proposition 2. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &\left|A(a^n, b^n) - L_n^n(a,b)\right| \\ &\leq \frac{n(b-a)}{2(p+1)^{1/p}}\left[nA\left(|a|^{(n-1)(p/(p-1))}, |b|^{(n-1)(p/(p-1))}\right)\right. \\ &\left. -\frac{\lambda(b-a)^2}{6n^{p/(p-1)}}\right]^{(p-1)/p}. \end{aligned} \tag{43}$$

Proof. Insertion of $\varrho(x) = x^n$, where $n \in \mathbb{N}, n \geq 2$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 5 completes the proof. □

Proposition 3. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &A^{-1}(a,b) + \frac{\lambda}{12}(b-a)^2 \leq \overline{L}^{-1}(a,b) \\ &\leq A\left(a^{-1}, b^{-1}\right) + \frac{\lambda}{6}(b-a)^2. \end{aligned} \tag{44}$$

Proof. Insertion of $\varrho(x) = x^n$, where $n \in \mathbb{N}, n \geq 2$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 1 completes the proof. □

Proposition 4. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &\left|A\left(a^{-1}, b^{-1}\right) - \overline{L}^{-1}(a,b)\right| \\ &\leq \frac{b-a}{4}A\left(|a|^{-2}, |b|^{-2}\right) - \frac{\lambda}{32}(b-a)^3. \end{aligned} \tag{45}$$

Proof. Insertion of $\varrho(x) = x^{-1}$, where $x \in [a,b]$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 4 completes the proof. □

Proposition 5. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &\left|A\left(a^{-1}, b^{-1}\right) - \overline{L}^{-1}(a,b)\right| \\ &\leq \frac{b-a}{2(p+1)^{1/p}}\left[A\left(|a|^{-2p/(p-1)}, |b|^{-2p/(p-1)}\right) - \frac{\lambda}{6}(b-a)^2\right]^{(p-1)/p}. \end{aligned} \tag{46}$$

Proof. Insertion of $\varrho(x) = x^{-1}$, where $x \in [a,b]$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 4 completes the proof. □

6. Some Applications of Caputo–Fabrizio Fractional Integral Inequalities to the Trapezoidal Formula

Suppose d is the division of interval $[a,b]$, d: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and consider the trapezoidal formula

$$T(\varrho, d) = \sum_{i=0}^{i=1} \frac{\varrho(x_i) + \varrho(x_{i+1})}{2}(x_{i+1} - x_i). \tag{47}$$

It is well known that if the mapping $\varrho: I \longrightarrow \mathbb{R}$ is twice differentiable on (a,b) and $M = \max_{x \in (a,b)}[\varrho''(x)] < \infty$, then

$$\int_a^b \varrho(x)\mathrm{d}x = T(\varrho, d) + E(\varrho, d), \tag{48}$$

where the approximation error $E(\varrho, d)$ of the integral $\int_a^b \varrho(x)\mathrm{d}x$ by the trapezoidal formula $T(\varrho, d)$ satisfies

$$|E(\varrho, d)| \leq \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^3. \tag{49}$$

It is clear that if the mapping f is not twice differentiable or the second derivative is not bounded on (a,b), then (49) cannot be applied. In recent studies [30, 32–35], Dragomir and Wang showed that the remainder term $E(\varrho, d)$ can be estimated in terms of the first derivative only. These estimates have a wider range of applications. Here, we shall propose some new estimates of the remainder term $E(\varrho, d)$ which supplement, in a sense, those established in [30, 32–35].

Proposition 6. *Assume that $\varrho: I \longrightarrow \mathbb{R}$ is a differentiable positive mapping on I°, $a, b \in I$ with $a < b$. If $\varrho' \in L_1[a, b]$ and $|\varrho'|$ is a strongly convex function, then for every division d of $[a, b]$, the following inequality holds:*

$$\begin{aligned} |E(\varrho, d)| &\leq \frac{1}{8}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\left(|\varrho'(x_i)| + |\varrho'(x_{i+1})|\right) - \frac{\lambda}{4}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right] \\ &\leq \frac{1}{4}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\max\left\{\left(|\varrho'(a)|, |\varrho'(b)|\right)\right\} - \frac{\lambda}{8}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right]. \end{aligned} \tag{50}$$

Proof. Applying subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, of the division d from Theorem 4, we obtain

$$\begin{aligned} &\left|\frac{\varrho(x_i) + \varrho(x_{i+1})}{2}(x_{i+1}-x_i) - \int_{x_i}^{x_{i+1}}\varrho(x)\mathrm{d}x\right| \\ &\leq \frac{(x_{i+1}-x_i)^2}{4}\left[\frac{\left(|\varrho'(x_i)| + |\varrho'(x_{i+1})|\right)}{2} - \frac{\lambda}{16}(x_{i+1}-x_i)^2\right]. \end{aligned} \tag{51}$$

Summing over $i = 0, \ldots, n-1$ and taking that $|\varrho'|$ is a strongly convex function, then by using (47), (48), and triangular inequality, we complete the proof. □

Proposition 7. *Assume that $\varrho: I \longrightarrow \mathbb{R}$ is a differentiable positive mapping on I°, $a, b \in I$ with $a < b$ and $(1/p) + (1/q) = 1$. If $\varrho' \in L_1[a, b]$ and $|\varrho'|^q$ is a strongly convex function, then for every division d of $[a, b]$, the following inequality holds:*

$$\begin{aligned} |E(\varrho, d)| &\leq \frac{1}{2(p+1)^{1/p}}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\left(\frac{|\varrho'(x_i)|^{p/(p-1)} + |\varrho'(x_{i+1})|^{p/(p-1)}}{2}\right) - \frac{\lambda}{6}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right]^{p/(p-1)} \\ &\leq \frac{1}{2(p+1)^{1/p}}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\max\left\{\left(|\varrho'(a)|, |\varrho'(b)|\right)\right\} - \frac{\lambda}{6}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right]. \end{aligned} \tag{52}$$

Proof. Applying subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, of the division d, we obtain from Theorem 5

$$\begin{aligned} &\left|\frac{\varrho(x_i) + \varrho(x_{i+1})}{2}(x_{i+1}-x_i) - \int_{x_i}^{x_{i+1}}\varrho(x)\mathrm{d}x\right| \\ &\leq \frac{(x_{i+1}-x_i)^2}{2(p+1)^{1/p}}\left[\frac{\left(|\varrho'(x_i)|^{p/(p-1)} + |\varrho'(x_{i+1})|^{p/(p-1)}\right)}{2} - \frac{\lambda}{6}(x_{i+1}-x_i)^2\right]^{(p-1)/p}. \end{aligned} \tag{53}$$

Summing over $i = 0, \ldots, n-1$ and taking that $|\varrho'|^q$, where $(1/p) + (1/q) = 1$, is a strongly convex function, then by using (47), (48), and triangular inequality, we complete the proof. □

7. Conclusions

The convex functions play an important role in approximation theory, and the fractional calculus has been found the best to model physical and engineering processes. Some properties of strongly convex functions via the Caputo–Fabrizio fractional integral operator have been studied in this paper. Precisely speaking, Hermite–Hadamard-type and some new inequalities for strongly convex functions via the Caputo–Fabrizio fractional integral operator are proved, and applications of the proposed inequalities to special means are also presented in this paper.

Data Availability

All data required for this research are available within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Qi Li added the application section to justify the novelty of the paper, Muhammad Shoaib Saleem designed the problem and supervised the work, Peiyu Yan wrote the literature review and arranged the funding for this paper, Muhammad Sajid Zahoor proved the main results, and Muhammad Imran wrote the first draft of the paper. All authors read and approved the final version of this paper.

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References

[1] A. Fernandez and P. Mohammed, "Hermite-Hadamard inequalities in fractional calculus defined using mittag-leffler kernels," *Mathematical Methods in the Applied Sciences*, vol. 2020, pp. 1–18, 2020.

[2] X. Yang, G. Farid, G. Farid, W. Nazeer, Y.-M. Chu, and C. Dong, "Fractional generalized hadamard and fejér-hadamard inequalities for m-convex functions," *AIMS Mathematics*, vol. 5, no. 6, pp. 6325–6340, 2020.

[3] P. O. Mohammed, M. Z. Sarikaya, and D. Baleanu, "On the generalized hermite-hadamard inequalities via the tempered fractional integrals," *Symmetry*, vol. 12, no. 4, p. 595, 2020.

[4] M. E. Gordji, M. R. Delavar, and M. D. L. Sen, "On φ-convex functions," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 173–183, 2016.

[5] Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and S. M. Kang, "Inequalities for a unified integral operator and associated results in fractional calculus," *IEEE Access*, vol. 7, pp. 126283–126292, 2019.

[6] İ. İşcan, "Generalization of different type integral inequalities for s-convex functions via fractional integrals," *Applicable Analysis*, vol. 93, no. 9, pp. 1846–1862, 2014.

[7] Y. Y. Gambo, F. Jarad, D. Baleanu, and T. Abdeljawad, "On Caputo modification of the hadamard fractional derivatives," *Advances in Difference Equations*, vol. 2014, no. 1, p. 10, 2014.

[8] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–13, 2015.

[9] F. Song, Z. Yu, and H. Yang, "Modeling and analysis of fractional neutral disturbance waves in arterial vessels," *Mathematical Modelling of Natural Phenomena*, vol. 14, no. 3, p. 301, 2019.

[10] X.-J. Yang, M. Abdel-Aty, and C. Cattani, "A new general fractional-order derivataive with Rabotnov fractional-exponential kernel applied to model the anomalous heat transfer," *Thermal Science*, vol. 23, no. 3 Part A, pp. 1677–1681, 2019.

[11] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1140–1153, 2011.

[12] F. Ali, M. Saqib, I. Khan, and N. A. Sheikh, "Application of caputo-fabrizio derivatives to MHD free convection flow of generalized walters'-B fluid model," *The European Physical Journal Plus*, vol. 131, no. 10, p. 377, 2016.

[13] A. Atangana, "On the new fractional derivative and application to nonlinear fisher's reaction-diffusion equation," *Applied Mathematics and Computation*, vol. 273, pp. 948–956, 2016.

[14] M. Caputo and M. Fabrizio, "Applications of new time and spatial fractional derivatives with exponential kernels," *Progress in Fractional Differentiation and Applications*, vol. 2, no. 1, pp. 1–11, 2016.

[15] G. Farid, A. Ur Rehman, S. Bibi, A. U. Rehman, S. Bibi, and Y.-M. Chu, "Refinements of two fractional versions of hadamard inequalities for caputo fractional derivatives and related results," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 1–10, 2021.

[16] B. Meftah, A. Souahi, and A. Souahi, "Cebyšev inequalities for co-ordinated \(QC\)-convex and \((s, QC)\)-convex," *Engineering and Applied Science Letters*, vol. 4, no. 1, pp. 14–20, 2021.

[17] T. Abdeljawad and D. Baleanu, "On fractional derivatives with exponential kernel and their discrete versions," *Reports on Mathematical Physics*, vol. 80, no. 1, pp. 11–27, 2017.

[18] T. Abdeljawad, "Fractional operators with exponential kernels and a Lyapunov type inequality," *Advances in Difference Equations*, vol. 2017, no. 1, pp. 1–11, 2017.

[19] S. E. Fadugba, "Development of a new numerical scheme for the solution of exponential growth and decay models," *Open Journal of Mathematical Sciences*, vol. 4, no. 1, pp. 18–26, 2021.

[20] N. Merentes and K. Nikodem, "Remarks on strongly convex functions," *Aequationes Mathematicae*, vol. 80, no. 1-2, pp. 193–199, 2010.

[21] M. Eshaghi, S. S. Dragomir, and M. Rostamian Delavar, "An inequality related to η-convex functions (II)," *International Journal of Nonlinear Analysis and Applications*, vol. 6, no. 2, pp. 27–33, 2015.

[22] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "On some Hadamard-type inequalities for h-convex functions," *Journal of Mathematical Inequalities*, vol. 2, no. 3, pp. 335–341, 2008.

[23] H. Angulo, J. Giménez, A. Milena Moros, and K. Nikodem, "On strongly \$h\$-convex functions h-convex functions," *Annals of Functional Analysis*, vol. 2, no. 2, pp. 85–91, 2011.

[24] M. A. Latif, S. Rashid, S. S. Dragomir, and Y. M. Chu, "Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications," *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–33, 2019.

[25] J. Wang, X. Li, M. Fekan, and Y. Zhou, "Hermite-hadamard-type inequalities for Riemann-liouville fractional integrals via two kinds of convexity," *Applicable Analysis*, vol. 92, no. 11, pp. 2241–2253, 2013.

[26] A. A. Al-Gonah and W. K. Mohammed, "A new forms of extended hypergeometric functions and their properties,"

Engineering and Applied Science Letters, vol. 4, no. 1, pp. 30–41, 2021.
[27] G. Farid, K. Mahreen, and Y.-M. Chu, “Study of inequalities for unified integral operators of generalized convex functions,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 80–93, 2021.
[28] S. Foschi and D. Ritelli, “The lambert function, the quintic equation and the proactive discovery of the implicit function theorem,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 94–114, 2021.
[29] B. T. Polyak, “Existence theorems and convergence of minimizing sequences in extremum problems with restrictions,” *Soviet Mathematics-Doklady*, vol. 7, pp. 72–75, 1966.
[30] S. S. Dragomir and R. P. Agarwal, “Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,” *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.
[31] R. P. Agarwal and S. S. Dragomir, “An application of Hayashi’s inequality for differentiable functions,” *Computers & Mathematics with Applications*, vol. 32, no. 6, pp. 95–99, 1996.
[32] S. S. Dragomir and S. Wang, “Applications of Ostrowski’s inequality to the estimation of error bounds for some special means and for some numerical quadrature rules,” *Applied Mathematics Letters*, vol. 11, no. 1, pp. 105–109, 1998.
[33] G. Hong, G. Farid, W. Nazeer et al., “Boundedness of fractional integral operators containing mittag-leffler function via exponentially-convex functions,” *Journal of Mathematics*, vol. 2020, Article ID 3584105, 7 pages, 2020.
[34] Y. C. Kwun, M. S. Saleem, M. Ghafoor, W. Nazeer, and S. M. Kang, “Hermitehadamard-type inequalities for functions whose derivatives are ?-convex via fractional integrals,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–16, 2019.
[35] J. Zhao, S. I. Butt, J. Nasir, Z. Wang, I. Tlili, and Y. Liu, “Hermite-jensen-mercer type inequalities for caputo fractional derivatives,” *Journal of Function Spaces*, vol. 2020, no. 1, 11 pages, Article ID 7061549, 2020.

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Research Article

Generalized Conformable Mean Value Theorems with Applications to Multivariable Calculus

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The conformable derivative and its properties have been recently introduced. In this research work, we propose and prove some new results on the conformable calculus. By using the definitions and results on conformable derivatives of higher order, we generalize the theorems of the mean value which follow the same argument as in the classical calculus. The value of conformable Taylor remainder is obtained through the generalized conformable theorem of the mean value. Finally, we introduce the conformable version of two interesting results of classical multivariable calculus via the conformable formula of finite increments.

1. Introduction

The history of fractional calculus goes back to the late seventeenth century when L'Hospital proposed the fractional-order derivative. With the introduction of fractional calculus, various newly proposed definitions have been introduced. Some of the common definitions are the Caputo, Riesz, Riesz-Caputo, and Riemann-Liouville fractional ones (refer to [1, 2] for more information about fractional definitions, and see [3, 4] for research studies on the mathematical analysis of fractional calculus). A new local-type fractional definition [5] of derivative and integral has been recently proposed by Khalil et al. in [6]. Conformable derivative is basically considered as a natural extension of the classical derivative that satisfies the properties of usual derivative. In addition, conformable derivative is a generalized version of q-derivative or fractal derivative (refer to the introduction section in [7] for discussion about this relationship). Almeida et al. (2016) discussed in [8] that conformable derivative is an interesting topic of research that deserves to be studied further. In addition, both Zhao & Luo (2017) and Khalil et al. (2019) presented the physical and geometrical meaning of conformable derivative in [9, 10], respectively. Tuan et al. [11] investigated the mild solutions' existence and regularity of the proposed initial value problem for time diffusion equation in the sense of conformable derivative. This main goal of this newly introduced definition is to overcome the difficulties associated with obtaining the solutions for the equations formulated in the sense of nonlocal fractional definitions [12]. Motivated by the introduction of this definition, several research works have been conducted on the mathematical analysis of functions of a real variable formulated in the sense of conformable definition such as chain rule, mean value theorem, Rolle's theorem, power series expansion, and integration by parts formulas [6, 12–14]. The conformable partial derivative of the order $\alpha \in (0, 1]$ of the real-valued functions of several variables and the conformable gradient vector has been defined as well as the conformable Clairaut's theorem for partial derivative has also been studied in [15]. The conformable Jacobian matrix has been proposed in [16], and the chain rule for multivariable conformable derivative has also been proposed. The conformable Euler's theorem on homogeneous has been successfully defined in [17].

Furthermore, many research studies have been conducted on the theoretical and practical elements of conformable differential equations shortly after the proposition of this new definition [5, 7, 12, 18–35]. Conformable derivative has also been applied in modeling and investigating phenomena in applied sciences and engineering [12] such as the nonlinear Boussinesq equation's travelling wave solutions [36], the coupled nonlinear Schrödinger equations [34] and regularized long wave Burgers equation [35] deterministic and stochastics forms, the approximate long water wave equation's exact solutions [37], the $(1+3)$-Zakharov-Kuznetsov equation with power-law nonlinearity analytical and numerical solutions [38], the $(2+1)$-dimensional Zoomeron equation [39, 40] and 3[rd]-order modified KdV equation analytical solutions [39], and the exact solutions for Whitham-Broer-Kaup equation's three various models in shallow water [41].

The paper is organized as follows: The main concepts of the conformable calculus are presented in the next section. After that, with the help of the definitions and results on conformable derivatives of higher order, the theorems of the mean value are generalized which follow the same argument as in the classical calculus. We also introduce the value of conformable Taylor remainder via the generalized conformable theorems of the mean value. Finally, we characterize the functions of several variables in which one of their conformable partial derivatives is null, and we also obtain the first conformable formula of finite increments.

2. Basic Definitions and Tools

Definition 1. Given a function $f\colon [0,\infty) \longrightarrow R$. Then, the conformable derivative of order α [6] is defined by

$$(T_\alpha f)(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{1}$$

for all $t > 0$, $0 < \alpha \le 1$. If f is α differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\longrightarrow 0^+}(T_\alpha f)(t)$ exists, then it is defined as

$$(T_\alpha f)(0) = \lim_{t \longrightarrow 0^+} (T_\alpha f)(t). \tag{2}$$

Theorem 1 (see [6]). *If a function $f\colon [0,\infty) \longrightarrow R$ is α-differentiable at $t_0 > 0$, $0 < \alpha \le 1$, then f is continuous at t_0.*

Theorem 2 (see [6]). *Let $0 < \alpha \le 1$, and let f, g be α differentiable at a point $t > 0$. Then, we have*

(i) $T_\alpha(af + bg) = a(T_\alpha f) + b(T_\alpha g)$, $\forall a, b \in R$.
(ii) $T_\alpha(t^p) = pt^{p-\alpha}$, $\forall p \in R$.
(iii) $T_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
(iv) $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$.
(v) $T_\alpha(f/g) = (g(T_\alpha f) - f(T_\alpha g)/g^2)$.
(vi) If, in addition, f is differentiable, then $(T_\alpha f)(t) = t^{1-\alpha}(df/dt)(t)$.

The conformable derivative of certain functions using the above definition is given as follows:

(i) $T_\alpha(1) = 0$.
(ii) $T_\alpha(\sin(at)) = at^{1-\alpha}\cos(at)$.
(iii) $T_\alpha(\cos(at)) = -at^{1-\alpha}\sin(at)$.
(iv) $T_\alpha(e^{at}) = ae^{at}$, $a \in R$.

Definition 2. The (left) conformable derivative starting from a of a given function $f\colon [a,\infty) \longrightarrow R$ of order $0 < \alpha \le 1$ [13] is defined by

$$(T_\alpha^a f)(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{3}$$

When $a = 0$, it is expressed as $(T_\alpha f)(t)$. If f is α differentiable in some (a, b), then the following can be defined as

$$(T_\alpha^a f)(a) = \lim_{t \longrightarrow a^+} (T_\alpha^a f)(t). \tag{4}$$

Theorem 3 Chain Rule (see [13]). *Assume $f, g\colon (a,\infty) \longrightarrow R$ be (left) α differentiable functions, where $0 < \alpha \le 1$. By letting $h(t) = f(g(t))$, $h(t)$ isα differentiable for all $t \ne a$ and $g(t) \ne 0$; therefore, we have the following:*

$$(T_\alpha^a h)(t) = (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot (g(t))^{\alpha-1}. \tag{5}$$

If $t = a$, then we obtain

$$(T_\alpha^a h)(a) = \lim_{t \longrightarrow a^+} (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot (g(t))^{\alpha-1}. \tag{6}$$

Theorem 4 Rolle's Theorem (see [6]). *Let $a > 0$, $\alpha \in (0,1]$, and $f\colon [a,b] \longrightarrow R$ be a given function that satisfies*

(i)- f is continuous on $[a,b]$.
(ii) f is α-differentiable on (a,b).
(iii)- $f(a) = f(b)$.

Then, there exists $c \in (a,b)$ such that $(T_\alpha f)(c) = 0$.

Corollary 1 (see [14]). *Let $I \subset [o,\infty)$, $\alpha \in (0,1]$, and $f\colon I \longrightarrow R$ be a given function that satisfies*

(i)- f is α differentiable on I.
(ii)- $f(a) = f(b) = 0$ for certain $a, b \in I$.

Then, there exists $c \in (a,b)$, such that $(T_\alpha f)(c) = 0$.

Theorem 5 *Mean Value Theorem* (see [6]). *Let $a > 0$, $\alpha \in (0,1]$, and $f\colon [a,b] \longrightarrow R$ be a given function that satisfies*

(i) f is continuous on $[a,b]$.
(ii)- f is α differentiable on (a,b).

Then, there exists $c \in (a,b)$, such that

$$(T_\alpha f)(c) = \frac{f(b) - f(a)}{(b^\alpha/\alpha) - (a^\alpha/\alpha)}. \tag{7}$$

Theorem 6 (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and f: $[a, b] \longrightarrow R$ be a given function that satisfies*

(i) f is continuous on $[a, b]$.

(ii)- f is α differentiable on (a, b).

If $(T_\alpha g)(t) = 0$ for all $t \in (a, b)$, then f is a constant on $[a, b]$.

Corollary 2 (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and F, G: $[a, b] \longrightarrow R$ be functions such that $(T_\alpha F)(t) = (T_\alpha G)(t)$ for all $t \in (a, b)$. Then, there exists a constant C such that*

$$F(t) = G(t) + C. \tag{8}$$

Theorem 7 Extended Mean Value Theorem (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and f, g: $[a, b] \longrightarrow R$ be functions that satisfy*

(i) f, g are continuous on $[a, b]$.

(ii) f, g are α differentiable on (a, b).

(iii) $(T_\alpha g)(t) \neq 0$ for all $t \in (a, b)$.

(iv) $g(b) \neq g(a)$.

(v) $(T_\alpha f)(t)$ and $(T_\alpha g)(t)$ not annulled simultaneously on $[a, b]$.

Then, there exists $c \in (a, b)$, such that

$$\frac{(T_\alpha f)(c)}{(T_\alpha g)(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{9}$$

Remark 1. Observe that Theorem 5 is a special case of this theorem for $g(t) = (t^\alpha/\alpha)$.

Theorem 8 (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and f: $[a, b] \longrightarrow R$ be a given function that satisfies*

(i) f is continuous on $[a, b]$.

(ii) f is α-differentiable on (a, b).

Then, we have the following:

(i) If $(T_\alpha f)(t) > 0$ for all $t \in (a, b)$, then f is increasing on $[a, b]$.

(ii) If $(T_\alpha f)(t) < 0$ for all $t \in (a, b)$, then f is decreasing on $[a, b]$.

Theorem 9 (see [13]). *Assume f is infinitely α differentiable function, for some $0 < \alpha \leq 1$ at the neighborhood of a point t_0. Then, f has the following fractional power series expansion:*

$$f(t) = \sum_{k=0}^{\infty} \frac{\left({}^{(k)}T_\alpha^{t_0}\right)(t_0)}{\alpha^k k!}(t - t_0)^{k\alpha}, \quad t_0 < t < t_0 + R^{(1/\alpha)}. \tag{10}$$

Here, $({}^{(k)}T_\alpha^{t_0})(t_0)$ means the application of the conformable derivative k times.

Finally, the conformable partial derivative of a real-valued function with several variables is defined as follows.

Definition 3 (see [15, 16]). Let f be a real-valued function with n variables and $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$ be a point whose i^{th} component is positive. Then, the limit can be expressed as follows

$$\lim_{\varepsilon \longrightarrow 0} \frac{f\left(a_1, \ldots, a_i + \varepsilon a_i^{1-\alpha}, \ldots, a_n\right) - f(a_1, \ldots, a_n)}{\varepsilon}, \tag{11}$$

if the above limit exists, then we have the i^{th} conformable partial derivative of f of the order $\alpha \in (0, 1]$ at $\mathbf{a}$, denoted by $(\partial^\alpha/\partial x_i^\alpha) f(\mathbf{a})$.

3. Main Results

From the definitions and results on conformable derivatives of higher order, the theorems of the mean value are easily generalized which follow the same argument as in the classical calculus [42].

Theorem 10. *Let $a > 0$, $\alpha \in (0, 1]$, and f, g: $[a, b] \longrightarrow R$ be functions that satisfy*

(i)- $f, g \in C^{(n-1)\alpha}([a, b])$.

(ii)- $({}^n T_\alpha f)(t)$ and $({}^n T_\alpha g)(t)$ exist for all t in $[a, b)$.

In addition, the following $n - 1$ equations are assumed:

$$\left({}^k T_\alpha f\right)(a)[g(b) - g(a)] = \left({}^k T_\alpha g\right)(a)[f(b) - f(a)], \quad \text{for } k = 1, 2, \ldots, n - 1. \tag{12}$$

Then, there exists $c \in (a, b)$, such that

$$({}^n T_\alpha f)(c)[g(b) - g(a)] = ({}^n T_\alpha g)(c)[f(b) - f(a)]. \tag{13}$$

Proof. Consider the following function:

$$F(t) = f(t)[g(b) - g(a)] - g(t)[f(b) - f(a)], \quad \forall t \in [a, b]. \tag{14}$$

Since F is continuous on $[a, b]$, α differentiable on (a, b), and $F(a) = F(b)$, then by Theorem 4, there exists $c_1 \in (a, b)$ such that

$$(T_\alpha f)(c_1)[g(b)-g(a)]-(T_\alpha g)(c_1)[f(b)-f(a)]=0. \tag{15}$$

Let us now consider the following function:

$$(T_\alpha F)(t)=(T_\alpha f)(t)[g(b)-g(a)] - (T_\alpha g)(t)[f(b)-f(a)], \quad \forall t\in[a,c_1], \tag{16}$$

which is continuous on $[a,c_1]$, α differentiable on (a,c_1), and it is null at the extremes of interval $[a,c_1]$, by virtue of the above equation and hypothesis. Then, by Theorem 4, there exists $c_2\in(a,c_1)$ such that

$$\left({}^2T_\alpha f\right)(c_2)[g(b)-g(a)]-\left({}^2T_\alpha g\right)(c_2)[f(b)-f(a)]=0. \tag{17}$$

So, we reiterate this process until we obtain the following equality:

$$\left({}^{n-1}T_\alpha f\right)(c_{n-1})[g(b)-g(a)]-\left({}^{n-1}T_\alpha h\right)(c_{n-1})[f(b)-f(a)]=0. \tag{18}$$

Then, we consider functions: ${}^{n-1}T_\alpha f$ and ${}^{n-1}T_\alpha g$, that are continuous on $[a,c_{n-1}]$, and α differentiable on (a,c_{n-1}). So, by Theorem 7, there exists $c\in(a,c_{n-1})\subset(a,b)$ with

$$({}^nT_\alpha f)(c)[g(b)-g(a)]=({}^nT_\alpha g)(c)[f(b)-f(a)]. \tag{19}$$

This completes the proof of the theorem. □

Remark 2. The generalized conformable formula of extended mean value theorem is derived from previous theorem by taking $g(t)=(t^\alpha-a^\alpha)^n$.

Theorem 11. *Let $a_0>0$, $a\in(a_0,b)$ $\alpha\in(0,1]$, and $f\colon(a_0,b]\longrightarrow R$ be a function that satisfies*

(i) f is continuous on $[a,b]$.
(ii)- f is $n-1$ times α differentiable on (a,b).
(iii)- $({}^nT_\alpha f)(t)$ exist for all t in $[a,b)$.

In addition, the following $n-1$ equations are assumed:

$$(T_\alpha f)(a)=\left({}^2T_\alpha f\right)(a)=\cdots=\left({}^{n-1}T_\alpha f\right)(a)=0. \tag{20}$$

Then, there exists $c\in(a,b)$, such that

$$f(b)-f(a)=\frac{({}^nT_\alpha f)(c)}{\alpha^n\cdot n!}(t^\alpha-a^\alpha)^n. \tag{21}$$

Remark 3. A generalization of the conformable formula of mean value of Cauchy is also obtained.

Theorem 12. *Let $a_0>0$, $a\in(a_0,b)$, $\alpha\in(0,1]$, and $f,g\colon(a_0,b]\longrightarrow R$ be functions that satisfy*

(i) f,g are continuous on $[a,b]$.
(ii) f,g are $n-1$ times α differentiable on (a,b).
(iii) $({}^nT_\alpha f)(t)$ and $({}^nT_\alpha g)(t)$ exist for all t in $[a,b)$.
(iv)- $({}^nT_\alpha g)(t)\neq0$ $\forall t\in(a,b)$.

In addition, the following $n-1$ equations are assumed:

$$\left({}^kT_\alpha f\right)(a)[g(b)-g(a)]=\left({}^kT_\alpha g\right)(a)[f(b)-f(a)], \quad \text{for } k=1,2,\ldots,n-1. \tag{22}$$

Then, there exists $c\in(a,b)$, such that

$$\frac{({}^nT_\alpha f)(c)}{({}^nT_\alpha g)(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}. \tag{23}$$

Proof. Using the formula (21) and the fact that $({}^nT_\alpha g)(t)\neq0$ $\forall t\in(a,b)$, it follows that $g(b)-g(a)\neq0$.

Dividing the two members of equality

$$({}^nT_\alpha f)(c)[g(b)-g(a)]=({}^nT_\alpha g)(c)[f(b)-f(a)], \tag{24}$$

by the product $({}^kT_\alpha g)(c)[g(b)-g(a)]$, the desired result is obtained.

We end this section by obtaining the value of conformable Taylor remainder through the generalized conformable theorems of the mean value. □

Definition 4. Let an open set $X\subset R$, $a\in X$, $\alpha\in(0,1]$, and $f\colon X\longrightarrow R$ be a function that satisfies

(i)- f is $n-1$ times α differentiable on a neighborhood of a point a.
(ii)- $({}^nT_\alpha f)(a)$ exists.

Then, the conformable Taylor remainder is defined by

$$R(t)=f(t)-p_n(x)=f(t)-\sum_{k=0}^{n}\left({}^kT_\alpha f\right)(a)\cdot\frac{(t^\alpha-a^\alpha)^k}{\alpha^k\cdot k!}, \quad \forall t\in X. \tag{25}$$

Theorem 13. *Let an open set $X\subset R$, $a\in X$, $\alpha\in(0,1]$, and $f\colon X\longrightarrow R$. If f is $n+1$ times α differentiable on $[a,t]\subset X$, Then, there exists $c\in(a,t)$, such that*

$$R(t)=\left({}^{n+1}T_\alpha f\right)(c)\cdot\frac{(t^\alpha-a^\alpha)^{n+1}}{\alpha^{n+1}\cdot(n+1)!}, \tag{26}$$

where R is called the conformable Lagrange form of the remainder.

Proof. By applying Theorem 12 to a function $R = f - p_n$ and using the fact that $({}^{n+1}T_\alpha R)(t) = ({}^{n+1}T_\alpha f)(t)$ and $({}^{k}T_\alpha R)(a) = 0$ for $k = 1, 2, \ldots, n$, our result is followed. □

4. Applications to Multivariable Calculus

In this section, we will introduce the conformable version of two interesting classical results on functions of several variables [42]. Using the conformable formula of finite increments [16], these results will be proven.

Theorem 14. *Let $\alpha \in (0, 1]$, $f\colon X \longrightarrow R$ be a real-valued function defined in an open and convex set $X \subset R^n$, such that for all $\mathbf{x} = (x_1, \ldots, x_n) \in X$, each $x_i > 0$. If the conformable partial derivative of f with respect to x_i exists and is null on X, then $f(\mathbf{x}) = f(x')$ for any points $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n), x' = (x_1, \ldots, x_i', \ldots, x_n) \in X$, for $i = 1, 2, \ldots, n$.*

Proof. Since x' is a convex set and $\mathbf{x} = (x_1, \ldots x_i, \ldots, x_n), x' = (x_1, \ldots, x_i', \ldots, x_n) \in X$, all points of the line segment $[\mathbf{x}, x']$ are also in X, so the function g is defined in the interval of endpoints x_i and x_i':

$$t \mapsto g(t) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n). \tag{27}$$

This function is α differentiable on the above interval, and its derivative at a point t is given by

$$(T_\alpha g)(t) = \frac{\partial^\alpha f(x_1, \ldots, t, \ldots, x_n)}{\partial x_i^\alpha}. \tag{28}$$

Therefore, by applying Theorem 5, there is a point c_i between x_i and x_i', such that

$$g(x_i') - g(x_i) = \left(\frac{x_i'^\alpha}{\alpha} - \frac{x_i^\alpha}{\alpha}\right) \cdot (T_\alpha g)(c_i). \tag{29}$$

Since point $\mathbf{c} = (x_1, \ldots, c_i, \ldots, x_n) \in X$; therefore, $(\partial^\alpha f(\mathbf{c})/\partial x_i^\alpha) = 0$, and the above equality leads to

$$f(x') - f(\mathbf{x}) = (x_i' - x_i) \cdot \frac{\partial^\alpha f(c)}{\partial x_i^\alpha}, \tag{30}$$

then $f(\mathbf{x}) = f(x')$, for $i = 1, 2, \ldots, n$, as we wanted to prove.

Finally, we introduce the first formula of finite increments for functions of several variables, involving conformable partial derivatives. □

Theorem 15. *Let $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in R^n$, $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x_n}$ be points $\mathbf{x_i} = (b_1, \ldots, b_i, a_{i+1}, \ldots, a_n)$ (note that $\mathbf{x}_0 = \mathbf{a}$ and $\mathbf{x_n} = \mathbf{b}$), and line segment $S_i = [\mathbf{x_{i-1}}, \mathbf{x_i}]$, for $i = 1, 2, \ldots, n$. Let $\alpha \in (0, 1]$, and $f\colon X \longrightarrow R$ be a real-valued function defined in an open set $X \subset R^n$ containing line segments $S_1, S_2, \ldots, S_n$, such that for all $\mathbf{x} = (x_1, \ldots, x_n) \in X$, each $x_i > 0$. If the conformable partial derivative of f with respect to x_i exists on X, then there is a point c_i between a_i and b_i, for $i = 1, 2, \ldots, n$, such that*

$$f(b_1, b_2, \ldots b_n) - f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \left(\frac{b_i^\alpha}{\alpha} - \frac{a_i^\alpha}{\alpha}\right) \cdot \frac{\partial^\alpha f(b_1, \ldots, b_{i-1}, c_i, a_{i+1} \ldots, a_n)}{\partial x_i^\alpha}. \tag{31}$$

Proof. First, we will express the difference $f(\mathbf{b}) - f(\mathbf{a})$ as follows:

$$f(\mathbf{b}) - f(\mathbf{a}) = f(\mathbf{x_n}) - f(\mathbf{x}_0) = \sum_{i=1}^{n} [f(\mathbf{x_i}) - f(\mathbf{x_{i-1}})]. \tag{32}$$

Let us now consider, for $i = 1, 2, \ldots, n$, the real function g_i of the real variable t, defined on the closed interval of endpoints a_i and b_i, by

$$t \longmapsto g_i(t) = f(b_1, \ldots, b_{i-1}, t, a_{i+1}, \ldots, a_n). \tag{33}$$

Since the conformable partial derivative of f with respect to x_i exists on X and $S_i \subset X$, then g_i is α differentiable on the above interval, and its derivative at a point t, is given by

$$(T_\alpha g_i)(t) = \frac{\partial^\alpha f(b_1, \ldots, b_{i-1}, t, a_{i+1}, \ldots, a_n)}{\partial x_i^\alpha}. \tag{34}$$

Therefore, by applying Theorem 5, there is a point c_i between a_i and b_i, such that

$$g_i(b_i) - g_i(a_i) = \left(\frac{b_i^\alpha}{\alpha} - \frac{a_i^\alpha}{\alpha}\right) \cdot (T_\alpha g_i)(c_i). \tag{35}$$

Then, it is verified that

$$f(\mathbf{x_i}) - f(\mathbf{x_{i-1}}) = \left(\frac{b_i^\alpha}{\alpha} - \frac{a_i^\alpha}{\alpha}\right) \cdot \frac{\partial^\alpha f(b_1, \ldots, b_{i-1}, t, a_{i+1}, \ldots, a_n)}{\partial x_i^\alpha}. \tag{36}$$

By taking the above expression to equation (32), our result is followed. □

5. Conclusion

In this research work, some new results regarding the conformable mean value theorems have been proposed. As in classical calculus, higher-order derivatives have been applied to generalize the mean value theorems. Likewise, the Lagrange expression has been established for the Taylor conformable remainder. In the context of the calculus of functions of several variables, according to the conformable mean value theorem, the functions in which one of its conformable partial derivatives is null have been characterized, and the first conformable formula of finite increments has been obtained. The findings of this investigation indicate that the results obtained in the sense of the conformable derivative coincide with the results obtained in the classical case of integer order. Finally, our obtained results, in addition to a theoretical interest, show great potential to be applied in a future research work concerning various applications in the field of natural sciences and engineering.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. Kilbas, H. Srivastava, and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland, Amsterdam, Netherlands, 2006.

[2] K. S. Miller, *An Introduction to Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Hoboken, NJ, USA, 1993.

[3] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, and S. Rezapour, "Investigation of the p-laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives," *Advances in Difference Equations*, vol. 2021, no. 68, pp. 1–18, 2021.

[4] S. Rezapour, A. Imran, A. Hussain, F. Martínez, S. Etemad, and M. K. A. Kaabar, "Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs," *Symmetry*, vol. 13, no. 3, p. 469, 2021.

[5] F. Martínez, I. Martínez, M. K. A. Kaabar, R. Ortíz-Munuera, and S. Paredes, "Note on the conformable fractional derivatives and integrals of complex-valued functions of a real variable," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 3, pp. 609–615, 2020.

[6] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[7] M. Kaabar, "Novel methods for solving the conformable wave equation," *Journal of New Theory*, vol. 31, pp. 56–85, 2020.

[8] R. Almeida, M. Guzowska, and T. Odzijewicz, "A remark on local fractional calculus and ordinary derivatives," *Open Mathematics*, vol. 14, no. 1, pp. 1122–1124, 2016.

[9] D. Zhao and M. Luo, "General conformable fractional derivative and its physical interpretation," *Calcolo*, vol. 54, no. 3, pp. 903–917, 2017.

[10] R. Khalil, M. Al Horani, and M. Abu Hammad, "Geometric meaning of conformable derivative via fractional cords," *Journal of Mathematics and Computer Science*, vol. 19, no. 4, pp. 241–245, 2019.

[11] N. H. Tuan, T. B. Ngoc, D. Baleanu, and D. O'Regan, "On well-posedness of the sub-diffusion equation with conformable derivative model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 89, Article ID 105332, 2020.

[12] F. Martínez, I. Martínez, M. K. A. Kaabar, and S. Paredes, "On conformable laplace's equation," *Mathematical Problems in Engineering*, vol. 2021, Article ID 5514535, 10 pages, 2021.

[13] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015.

[14] O. S. Iyiola and E. R. Nwaeze, "Some new results on the new conformable fractional calculus with application using D ´Alambert approach," *Progress in Fractional Differentiation and Applications*, vol. 2, no. 2, pp. 115–122, 2016.

[15] A. Atangana, D. Baleanu, and A. Alsaedi, "New properties of conformable derivative," *Open Mathematics*, vol. 13, pp. 57–63, 2015.

[16] N. Yazici and U. Gözütok, "Multivariable conformable fractional calculus," *Filomat*, vol. 32, no. 2, pp. 45–53, 2018.

[17] F. Martínez, I. Martínez, and S. Paredes, "Conformable Euler ´s theorem on homogeneous functions," *Computational Mathematical Methods*, vol. 1, no. 5, pp. 1–11, 2018.

[18] E. Ünal, A. An, and E. Çelik, "Solutions of sequential conformable fractional differential equations around an ordinary point and conformable fractional hermite differential equation," *British Journal of Applied Science & Technology*, vol. 10, no. 2, pp. 1–11, 2015.

[19] M. Al Masalmeh, "Series method to solve conformable fractional ric-cati differential equations," *International Journal of Applied Mathematical Research*, vol. 6, no. 1, pp. 30–33, 2017.

[20] M. Al Horani and R. Khalil, "Total fractional differential with applications to exact fractional differential equations," *International Journal of Computer Mathematics*, vol. 95, no. 6-7, pp. 1444–1452, 2018.

[21] M. Al Horani, M. Abu Hammad, and R. Khalil, "Variation of parameters for local fractional nonhomogenous linear differential equations," *Journal of Mathematics and Computer Science*, vol. 16, no. 2, pp. 147–153, 2016.

[22] R. Khalil, M. Al Horani, and D. Anderson, "Undetermined coefficients for local fractional differential equations," *Journal of Mathematics and Computer Science*, vol. 16, no. 2, pp. 140–146, 2016.

[23] M. A. Hammad and R. Khalil, "Abel´s formula and wronskian for conformable fractional differential equations," *International Journal of Differential Equations and Applications*, vol. 13, no. 2, pp. 177–183, 2014.

[24] M. A. Hammad and R. Khalil, "Legendre fractional differential equation and Legendre fractional polynomials," *International Journal of Applied Mathematical Research*, vol. 3, no. 3, pp. 214–219, 2014.

[25] F. S. Silva, M. D. Moreira, and M. A. Moret, "Conformable laplace transform of fractional differential equations," *Axioms*, vol. 7, no. 55, 2018.

[26] Z. Al-Zhour, N. Al-Mutairi, F. Alrawajeh, and R. Alkhasawneh, "Series solutions for the laguerre and lane-emden fractional differential equations in the sense of conformable fractional derivative," *Alexandria Engineering Journal*, vol. 58, no. 4, pp. 1413–1420, 2019.

[27] M. A. Hammad, H. Alzaareer, H. Al-Zoubi, and H. Dutta, "Fractional Gauss hypergeometric differential equation," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 7, pp. 1113–1121, 2019.

[28] F. Martínez, I. Martínez, and S. Paredes, "A note on integrating factors of a conformable fractional differential equation," *International Journal of Applied Physics and Mathematics*, vol. 10, no. 4, pp. 143–152, 2020.

[29] F. Martínez, I. Martínez, M. K. A. Kaabar, and S. Paredes, "Some new results on conformable fractional power series," *Asia Pacific Journal of Mathematics*, vol. 7, no. 3, pp. 1–14, 2020.

[30] I. Abu Hammad and R. Khalil, “Fractional fourier series with applications,” *American Journal of Computational and Applied Mathematics*, vol. 4, no. 6, pp. 187–191, 2014.
[31] I. Abu Hammad and R. Khalil, “Some new results on conformable fractional differential equations,” *International Journal of Pure and Applied Mathematics*, vol. 94, no. 2, pp. 215–314, 2020.
[32] R. Khalil and H. Abu-Shaab, “Solution of some conformable fractional power series,” *Asia Pacific Journal of Mathematics*, vol. 103, no. 4, pp. 667–673, 2015.
[33] F. Martínez, I. Martínez, I. Martínez, M. K. A. Kaabar, and S. Paredes, “New results on complex conformable integral,” *AIMS Mathematics*, vol. 5, no. 6, pp. 7695–7710, 2020.
[34] Z. Korpinar, A. S. Alshomrani, M. Inc, A. S. Alshomrani, and D. Baleanu, “The deterministic and stochastic solutions of the Schrodinger equation with time conformable derivative in birefrigent fibers,” *AIMS Mathematics*, vol. 5, no. 3, pp. 2326–2345, 2020.
[35] Z. Korpinar, A. S. Alshomrani, and D. Baleanu, “On exact special solutions for the stochastic regularized long wave-Burgers equation,” *Advances in Difference Equations*, vol. 2020, no. 1, p. 433, 2020.
[36] M. Kaplan, “Applications of two reliable methods for solving a nonlinear conformable time-fractional equation,” *Optical and Quantum Electronics*, vol. 49, no. 9, p. 312, 2017.
[37] M. Kaplan and A. Akbulut, “Application of two different algorithms to the approximate long water wave equation with conformable fractional derivative,” *Arab Journal of Basic and Applied Sciences*, vol. 25, no. 2, pp. 77–84, 2018.
[38] M. M. A. Khater, Y.-M. Chu, R. A. M. Attia, M. Inc, and D. Lu, “On the analytical and numerical solutions in the quantum magnetoplasmas: the atangana conformable derivative $(1+3)$-ZK equation with power-law nonlinearity,” *Advances in Mathematical Physics*, vol. 2020, Article ID 5809289, 10 pages, 2020.
[39] A. Akbulut and M. Kaplan, “Auxiliary equation method for time-fractional differential equations with conformable derivative,” *Computers & Mathematics with Applications*, vol. 75, no. 3, pp. 876–882, 2018.
[40] D. Kumar and M. Kaplan, “New analytical solutions of $(2+1)$-dimensional conformable time fractional Zoomeron equation via two distinct techniques,” *Chinese Journal of Physics*, vol. 56, no. 5, pp. 2173–2185, 2018.
[41] D. Kumar, M. Kaplan, M. Haque, M. S. Osman, and D. Baleanu, “A variety of novel exact solutions for different models with the conformable derivative in shallow water,” *Frontiers in Physics*, vol. 8, no. 177, pp. 1–13, 2020.
[42] M. Spivak, *Calculus*, W.A. Benjamin, Inc., Menlo Park, CA, USA, 2nd edition, 1988.

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Research Article

Some Formulas for New Quadruple Hypergeometric Functions

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In this paper, we aim to introduce six new quadruple hypergeometric functions. Then, we investigate certain formulas and representations for these functions such as symbolic formulas, differential formulas, and integral representations.

1. Introduction

Hypergeometric functions of several variables play an important role in diverse areas of science and engineering. The developments in applied mathematics, mathematical physics, chemistry, combinatorics, statistics, numerical analysis, and other areas have led to increasing interest in the study of multiple hypergeometric functions. Many authors have studied a number of formulas involving hypergeometric functions (see, e.g., [1–6]).

In [7], Exton presented twenty-one complete hypergeometric functions in four variables denoted by symbols $K_1, K_2, \ldots, K_{21}$. In [8], Sharma and Parihar defined eighty-three complete quadruple hypergeometric functions, namely, $F_1^{(4)}, F_2^{(4)}, \ldots, F_{83}^{(4)}$. Bin-Saad and Younis [9] gave thirty new quadruple hypergeometric functions given by $X_1^{(4)}, X_2^{(4)}, \ldots, X_{30}^{(4)}$. In [10], the authors discovered the existence of twenty additional complete hypergeometric functions in four variables $X_{31}^{(4)}, X_{32}^{(4)}, \ldots, X_{50}^{(4)}$. Each quadruple hypergeometric function in [7–10] is of the form

$$X^{(4)}(.) = \sum_{m,n,p,q=0}^{\infty} \Omega(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \qquad (1)$$

where $\Omega(m,n,p,q)$ is a certain sequence of complex parameters, and there are twelve parameters in each series of $X^{(4)}(.)$ (eight $a's$ and four $c's$). The 1st, 2nd, 3rd, and 4th parameters in $X^{(4)}(.)$ are connected with integers m, n, p, and q, respectively. Each repeated parameter in the series $X^{(4)}(.)$ points out a term with double parameters in $\Omega(m,n,p,q)$. For example, $X^{(4)}(a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_5)$ means that $(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q$ includes the term. Similarly, $X^{(4)}(a_1, a_1, a_1, a_2, a_1, a_1, a_2, a_3)$ points out the term $(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q$, and $X^{(4)}(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4)$ shows the existence of the term $(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q$. Thus, it is possible to form various combinations of indices. There seems to be no way of independently establishing the number of distinct Gaussian hypergeometric series for any given integer $n \geq 2$ without explicitly stating all such series. Thus, in every situation with $n = 4$, one ought to begin by actually constructing the set just as in the case $n = 3$ (see [11]).

By using the conventions and notations above, we now introduce further quadruple hypergeometric functions as follows:

$$X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{86}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{n+q}(c_2)_m(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,y,z,u)==\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+p}(c_2)_{n+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\tag{2}$$

$$X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+q}(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+p+q}(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$X_{90}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c,c,c;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c)_{m+n+p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

for

$$\left(|x|<\frac{1}{4},|y|<1,|z|<1,|u|<1\right).\tag{3}$$

Here, $(a)_m$ is the Pochhammer symbol defined (for $a,m\in\mathbb{C}$), in terms of the familiar Gamma function Γ, by (see, e.g., [11], p. 2 and p. 5)

$$\begin{aligned}(a)_m&:=\frac{\Gamma(a+m)}{\Gamma(a)},\quad (a+m\in\mathbb{C}\setminus\mathbb{Z}_0^-)\\&=\begin{cases}1, & (m=0),\\ a(a+1)\ldots(a+m-1), & (m=n\in\mathbb{N}),\end{cases}\end{aligned}\tag{4}$$

where $\mathbb{C},\mathbb{Z}_0^-$, and $\mathbb{N}$ denote the sets of complex numbers, nonpositive integers, and positive integers, respectively.

We recall the Gauss hypergeometric function [12] which is defined by

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\quad(|x|<1).\tag{5}$$

Appell's double hypergeometric function F_2 is defined as follows [13]:

$$F_2(a,b,c;d,e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}.\tag{6}$$

In [14], Exton established twenty distinct triple hypergeometric functions, which are denoted by $X_1,X_2,\ldots,X_{20}$. We introduce the definitions of five of these functions in the following:

$$X_{15}(a_1,a_2,a_3;c_2,c_1;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_{n+p}(c_2)_m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{16}(a_1,a_2,a_3;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_{m+p}(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{17}(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\tag{7}$$

$$X_{18}(a_1,a_2,a_3,a_4;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{20}(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

The Lauricella functions of three variables F_M, F_N, F_p, F_S, and F_T are defined in [11, 15]:

$$\begin{aligned}
F_M(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\\
F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\\
F_P(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_2;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+p}(a_2)_n(b_1)_{m+n}(b_2)_p}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\\
F_S(a_1,a_2,a_2,b_1,b_2,b_3;c,c,c;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_{n+p}(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\\
F_T(a_1,a_2,a_2,b_1,b_2,b_1;c,c,c;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p}.
\end{aligned}\tag{8}$$

The structure of this paper is as follows. In Sections 2 to 5, we obtain several symbolic formulas, differentiation formulas, operator formulas, and integral representations for the hypergeometric functions of four variables $X_{85}^{(4)}, X_{86}^{(4)}, \ldots, X_{90}^{(4)}$.

2. Symbolic Formulas

First of all, we recall the following symbolic operators (see [16]):

$$D_\delta^m \delta^s = \frac{\Gamma(s+1)}{\Gamma(s-m+1)}\delta^{s-m},\tag{9}$$

$$D_\delta^{-m} \delta^s = \frac{\Gamma(s+1)}{\Gamma(s+m+1)}\delta^{s+m},\tag{10}$$

for

$$m\in\mathbb{N}\cup\{0\},\quad s\in\mathbb{C}-\{-1,-2,\ldots\},\tag{11}$$

where D_δ and D_δ^{-1} are the derivative and integral operator, respectively.

Now, we find the following formulas.

Theorem 1. *The following results hold true:*

$$\begin{aligned}
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{17}(a_1,a_2,a_3;c_1,c_2;\beta x,y,\alpha_1 z)\\
&\quad\times\left(\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}\right)=\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4c_1,c_2,c_3,c_1;\beta x,y,\alpha_1 z,u;),
\end{aligned}\tag{12}$$

$$\begin{aligned}
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)z\right]^{-a}X_{20}(a_1,a_2,a_3,a_4;c_1,c_2;x,\alpha_1 y,\alpha_2 u)\\
&\quad\times\left(\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c_3-1}\gamma^{a-1}\right)=\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c_3-1}\gamma^{a-1}X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,\alpha_1 y,z,\alpha_2 u).
\end{aligned}\tag{13}$$

Proof. To prove the result in equality (12) asserted in Theorem 1, let $\varnothing$ denote the left-hand side of equality (12). Then, employing the series representation of x_{17} and by using (9) and (10), we have

$$\begin{aligned}
\varnothing &= \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p(a)_q\beta^{-q}\gamma^{-q}}{(c_1)_m(c_2)_n(c_3)_p m!n!p!q!}x^m y^n z^p u^q\times D_{\alpha_1}^q D_{\alpha_2}^q D_\beta^{-q}D_\gamma^{-q}\left(\alpha_1^{a_3+p+q-1}\alpha_2^{a_4+q-1}\beta^{c+m+q-1}\gamma^{a-1}\right)\\
&=\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p+q}(a_4)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{(\beta x)^m}{m!}\frac{(y)^n}{n!}\frac{(\alpha_1 z)^p}{p!}\frac{u^q}{q!}
\end{aligned}$$

$$= \alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,y,z,u), \tag{14}$$

which completes the proof. Similarly, one can prove formulas (13) and (20). □

Theorem 2. *The following results hold true:*

$$\begin{aligned}
&\left[1-\left(D_\alpha^2\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha^2\right)x\right]^{-a}F_P(a_3,a_3,a_1,a_2,a_2,a_4;c_3,c_1,c_1;z,\alpha y,u)\\
&\quad\times\left(\alpha^{a_1-1}\beta^{c_2-1}\gamma^{a-1}\right)=\alpha^{a_1-1}\beta^{c_2-1}\gamma^{a-1}X_{86}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x,\alpha y,z,u),\\
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)y\right]_2^{-\alpha}F_1\left(\frac{a_1}{2},\frac{a_1+1}{2};c_2;4\alpha_1^2x\right)\\
&\quad\times F_2(a_3,a_2,a_4;c_3,c_1;\alpha_2 z,\beta u)\left(\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_1-1}\gamma^{a-1}\right)=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_1-1}\gamma^{a-1}\\
&X_{86}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;\alpha_1^2x,y,\alpha_2 z,\beta u\right).
\end{aligned}\tag{15}$$

Theorem 3. *The following results hold true:*

$$\begin{aligned}
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{16}(a_1,a_2,a_3;c_1,c_2;x,\beta y,\alpha_1 z)\\
&\quad\times\left(\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_2-1}\gamma^{a-1}\right)=\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_2-1}\gamma^{a-1}X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,\beta y,\alpha_1 z,u),\\
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)y\right]^{-a}F_N\left(a_4,\frac{a_1}{2},a_2,a_3,a_3,\frac{a_1+1}{2};c_2,c_1,c_1;\beta u,4\alpha_1^2x,\alpha_2 z\right)\\
&\quad\times\left(\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_2-1}\gamma^{a-1}\right)=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_2-1}\gamma^{a-1}X_{87}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;\alpha_1^2x,y,\alpha_2 z,\beta u\right).
\end{aligned}\tag{16}$$

Theorem 4. *The following results hold true:*

$$\begin{aligned}
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{15}(a_1,a_2,a_3;c_1,c_2;\beta x,y,\alpha_1 z)\\
&\quad\times\left(\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}\right)=\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;\beta x,y,\alpha_1 z,u),\\
&\left[1-\left(D_\alpha^2\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha^2\right)x\right]^{-a}F_M(a_4,a_2,a_2,a_3,a_3,a_1;c_1,c_2,c_2;\beta u,\alpha y,z)\\
&\quad\times\left(\alpha^{a_1-1}\beta^{c_1-1}\gamma^{a-1}\right)=\alpha^{a_1-1}\beta^{c_1-1}\gamma^{a-1}X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,\alpha y,z,\beta u).
\end{aligned}\tag{17}$$

Theorem 5. *The following results hold true:*

$$\begin{aligned}
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{16}(a_1,a_2,a_3;c_1,c_2;\beta x,y,\alpha_1\beta z)\\
&\quad\times\left(\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}\right)=\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_1-1}\gamma^{a-1}X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;\beta x,y,\alpha_1\beta z,u),\\
&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)y\right]^{-a}F_S\left(\frac{a_1}{2},a_3,a_3,\frac{a_1+1}{2},a_2,a_4;c_1,c_1,c_1;4\alpha_1 x,\alpha_2 z,u\right)\\
&\quad\times\left(\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_2-1}\gamma^{a-1}\right)=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_2-1}\gamma^{a-1}X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;\alpha_1 x,y,\alpha_2 z,u).
\end{aligned}\tag{18}$$

Theorem 6. *The following results hold true:*

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)z\right]^{-a}X_{18}\left(a_1,a_2,a_3,a_4;c;\beta x,\alpha_1\beta y,\alpha_2\beta u\right)$$
$$\times\left(\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma^{a-1}\right)=\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma^{a-1}X_{90}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c,c,c;\beta x,\alpha_1\beta y,z,\alpha_2\beta u\right), \quad (19)$$

$$\left[1-\left(D_{\alpha}^2\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^2\right)x\right]^{-a}F_T\left(a_4,a_2,a_2,a_3,a_1,a_3;c,c,c;\beta u,\alpha\beta y,\beta z\right)$$
$$\times\left(\alpha^{a_1-1}\beta^{c-1}\gamma^{a-1}\right)=\alpha^{a_1-1}\beta^{c-1}\gamma^{a-1}X_{90}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4c,c,c,c;x,\alpha\beta y,\beta z,\beta u;\right). \quad (20)$$

3. Differentiation Formulas

The results of this section can be derived from formula (9) by a direct evaluation.

Theorem 7. *The following derivative formulas hold true:*

$$D_{w_1}^{a_1-c}D_{w_2}^{a_2-c'}\left[w_1^{a_1-1}w_2^{a_2-1}X_{85}^{(4)}\left(c,c,c',a_3,c,c',a_3,a_4;c_1,c_2,c_3,c_1;w_1^2x,w_1w_2y,w_2z,u\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}X_{85}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;w_1^2x,w_1w_2y,w_2z,u\right),$$
$$D_{w_1}^{a_2-c}D_{w_2}^{a_3-c'}\left[w_1^{a_2-1}w_2^{a_3-1}X_{85}^{(4)}\left(a_1,a_1,c,c',a_1,c,c',a_4;c_1,c_2,c_3,c_1;x,w_1y,w_1w_2z,w_2u\right)\right]$$
$$=\frac{\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}X_{85}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,w_1y,w_1w_2z,w_2u\right), \quad (21)$$
$$D_{w_1}^{a_3-c}D_{w_2}^{a_4-c'}\left[w_1^{a_3-1}w_2^{a_4-1}X_{85}^{(4)}\left(a_1,a_1,a_2,c,a_1,a_2,c,c';c_1,c_2,c_3,c_1;x,y,w_1z,w_1w_2u\right)\right]$$
$$=\frac{\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}X_{85}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,y,w_1z,w_1w_2u\right).$$

Theorem 8. *The following differentiation formulas hold:*

$$D_x^{a_1-c}\left[x^{a_1-1}X_{86}^{(4)}\left(c,c,a_2,a_3,c,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x^2,xy,z,u\right)\right]$$
$$=\frac{\Gamma(a_1)}{\Gamma(c)}x^{c-1}X_{86}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x^2,xy,z,u\right),$$
$$D_x^{a_1-c}D_y^{a_2-c'}\left[x^{a_1-1}y^{a_2-1}X_{86}^{(4)}\left(c,c,c',a_3,c,c',a_3,a_4;c_2,c_1,c_3,c_1;x^2,xy,yz,u\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')}x^{c-1}y^{c'-1}X_{86}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x^2,xy,yz,u\right), \quad (22)$$
$$D_u^{a_4-c}\left[u^{a_4-1}X_{86}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,c;c_2,c_1,c_3,c_1;x,y,z,u\right)\right]$$
$$=\frac{\Gamma(a_4)}{\Gamma(c)}u^{c-1}X_{86}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x,y,z,u\right).$$

Theorem 9. *The following derivative formulas hold true:*

$$D_w^{a_2-c}D_z^{a_3-c'}\left[w^{a_2-1}z^{a_3-1}X_{87}^{(4)}(a_1,a_1,c,c',a_1,c,c',a_4;c_1,c_2,c_1,c_2;x,wy,wz,uz)\right]$$
$$=\frac{\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')}w^{c-1}z^{c'-1}X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,wy,wz,uz),$$
$$D_w^{a_1-c}D_y^{a_2-c'}D_z^{a_3-c''}\left[w^{a_1-1}y^{a_2-1}z^{a_3-1}X_{87}^{(4)}\left(c,c,c',c'',c,c',c'',a_4;c_1,c_2,c_1,c_2;w^2x,wy,yz,uz\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')\Gamma(c'')}w^{c-1}y^{c'-1}z^{c''-1}X_{87}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;w^2x,wy,yz,uz\right), \tag{23}$$
$$D_{w_1}^{a_1-c}D_{w_2}^{a_4-c'}\left[w_1^{a_1-1}w_2^{a_4-1}X_{87}^{(4)}\left(c,c,a_2,a_3,c,a_2,a_3c';c_1,c_2,c_1,c_2;w_1^2x,w_1y,z,w_2u\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_4)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}X_{87}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;w_1^2x,w_1y,z,w_2u\right).$$

Theorem 10. *The following derivative formulas hold true:*

$$D_y^{a_2-c}\left[y^{a_2-1}X_{88}^{(4)}(a_1,a_1,c,a_3,a_1,c,a_3,a_4;c_1,c_2,c_2,c_1;x,y,yz,u)\right]$$
$$=\frac{\Gamma(a_2)}{\Gamma(c)}y^{c-1}X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,y,yz,u),$$
$$D_z^{a_2-c}\left[z^{a_2-1}X_{88}^{(4)}(a_1,a_1,c,a_3,a_1,c,a_3,a_4;c_1,c_2,c_2,c_1;x,yz,z,u)\right]$$
$$=\frac{\Gamma(a_2)}{\Gamma(c)}z^{c-1}X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,yz,z,u),$$
$$D_{w_1}^{a_1-c}D_{w_2}^{a_2-c'}D_{w_3}^{a_3-c''}D_{w_4}^{a_4-c'''}\left[w_1^{a_1-1}w_2^{a_2-1}w_3^{a_3-1}w_4^{a_4-1}X_{88}^{(4)}\left(c,c,c',c'',c,c',c'',c''';c_1,c_2,c_2,c_1;w_1^2x,w_1w_2y,w_2w_3z,w_3w_4u\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')\Gamma\left(c'''\right)}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}w_4^{c'''-1}$$
$$\times X_{88}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;w_1^2x,w_1w_2y,w_2w_3z,w_3w_4u\right). \tag{24}$$

Theorem 11. *The following derivative formulas hold true:*

$$D_y^{a_1-c}\left[y^{a_1-1}X_{89}^{(4)}\left(c,c,a_2,a_3,c,a_2,a_3,a_4;c_1,c_2,c_1,c_1;xy^2,y,z,u\right)\right]$$
$$=\frac{\Gamma(a_1)}{\Gamma(c)}y^{c-1}X_{88}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;xy^2,y,z,u\right),$$
$$D_y^{a_1-c}D_w^{a_2-c'}\left[y^{a_1-1}w^{a_2-1}X_{89}^{(4)}\left(c,c,c',a_3,c,c',a_3,a_4;c_1,c_2,c_1,c_1;xy^2,wy,wz,u\right)\right]$$

$$= \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')} y^{c-1} w^{c'-1} X_{89}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_1; xy^2, wy, wz, u\right),$$

$$D_x^{a_1-c} D_{w_1}^{a_2-c''} D_z^{a_3-c'''} D_{w_2}^{a_4-c'''} \left[x^{a_1-1} w_1^{a_2-1} z^{a_3-1} w_2^{a_4-1} X_{89}^{(4)}\left(c, c, c', c'', c, c', c'', c''; c_1, c_2, c_1, c_1; x^2, w_1 xy, w_1 z, w_2 uz\right)\right]$$
$$= \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')\Gamma\left(c'''\right)} x^{c-1} w_1^{c'-1} z^{c''-1} w_2^{c'''-1} \times X_{89}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_1; x^2, w_1 xy, w_1 z, w_2 uz\right). \quad (25)$$

Theorem 12. *The following derivative formulas hold true:*

$$D_z^{a_2-c'} D_u^{a_3-c''} \left[z^{a_2-1} u^{a_3-1} X_{90}^{(4)}\left(a_1, a_1, c', c'', a_1, c', c'', a_4; c, c, c, c; x, yz, uz, u\right)\right]$$
$$= \frac{\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')} z^{c'-1} u^{c''-1} X_{90}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c, c, c, c; x, yz, uz, u\right),$$

$$D_y^{a_2-c'} D_w^{a_3-c''} D_u^{a_4-c'''} \left[y^{a_2-1} w^{a_3-1} u^{a_4-1} X_{90}^{(4)}\left(a_1, a_1, c', c'', a_1, c', c'', c'''; c, c, c, c; x, y, wyz, wu\right)\right]$$
$$= \frac{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c')\Gamma(c'')\Gamma\left(c'''\right)} y^{c'-1} w^{c''-1} u^{c'''-1} X_{90}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c, c, c, c; x, y, wyz, wu\right), \quad (26)$$

$$D_{w_1}^{a_1-c'} D_z^{a_3-c''} D_{w_2}^{a_4-c'''} \left[w_1^{a_1-1} z^{a_3-1} w_2^{a_4-1} X_{90}^{(4)}\left(c', c', a_2, c'', c', a_2, c'', c'''; c, c, c, c; w_1^2 x, w_1 y, z, w_2 uz\right)\right]$$
$$= \frac{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c')\Gamma(c'')\Gamma\left(c'''\right)} w_2^{c-1} z^{c''-1} w_2^{c'''-1} X_{90}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c, c, c, c; w_1^2 x, w_1 y, z, w_2 uz\right).$$

4. Integral Representations

In this section, we give integral representations of Laplace type for our new hypergeometric functions of four variables.

Theorem 13. *Each of the following integral representations holds true:*

$$X_{85}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_1; x, y, z, u\right)$$
$$= \frac{1}{\Gamma(a_1)\Gamma(a_3)} \int_0^\infty \int_0^\infty \times e^{-(s+t)} s^{a_1-1} t^{a_3-1} \Phi_3\left(a_4; c_1; tu, s^2 x\right) \Psi_2\left(a_4; c_2, c_3; sy, tz\right) \mathrm{d}s\mathrm{d}t \left(\mathrm{Re}(a_1) > 0, \mathrm{Re}(a_3) > 0\right), \quad (27)$$

$$X_{86}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c_2, c_1, c_3, c_1; x, y, z, u\right) = \frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\cdot \int_0^\infty \int_0^\infty \int_0^\infty \times e^{-(s+t+v)} s^{a_1-1} t^{a_2-1} v^{a_3-1} \Phi_3\left(a_4; c_1; vu, sty\right) {}_0F_1\left(-; c_2; s^2 x\right) {}_0F_1\left(-; c_3; tvz\right) \mathrm{d}s\mathrm{d}t\mathrm{d}v$$
$$\cdot \left(\mathrm{Re}(a_1) > 0, \mathrm{Re}(a_2) > 0, \mathrm{Re}(a_3) > 0\right), \quad (28)$$

$$X_{87}^{(4)}\left(a_1, a_1, a_2, a_3, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_2; x, y, z, u\right) = \frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\cdot \int_0^\infty \int_0^\infty \int_0^\infty \times e^{-(s+t+v)} s^{a_1-1} t^{a_2-1} v^{a_3-1} {}_0F_1\left(-; c_1; s^2 x + tvz\right) \Phi_3\left(a_4; c_2; vu, sty\right) \mathrm{d}s\mathrm{d}t\mathrm{d}v$$
$$\cdot \left(\mathrm{Re}(a_1) > 0, \mathrm{Re}(a_2) > 0, \mathrm{Re}(a_3) > 0\right), \quad (29)$$

$$X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,y,z,u)=\frac{1}{\Gamma(a_1)\Gamma(a_3)}\cdot\int_0^\infty\int_0^\infty \times e^{-(s+t)}s^{a_1-1}t^{a_3-1}\Phi_3\left(a_4;c_1;tu,s^2x\right){}_1F_1(a_2;c_2;sy+tz)\mathrm{d}s\mathrm{d}t\,(\mathrm{Re}(a_1)>0,\mathrm{Re}(a_3)>0), \tag{30}$$

$$X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;x,y,z,u)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)}\cdot\int_0^\infty\int_0^\infty\int_0^\infty \times e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_4-1}\Phi_3\left(a_3;c_1;tz+vu,s^2x\right){}_0F_1(-;c_2;sty)\mathrm{d}s\mathrm{d}t\mathrm{d}v\,(\mathrm{Re}(a_1)>0,\mathrm{Re}(a_2)>0,\mathrm{Re}(a_4)>0), \tag{31}$$

$$X_{90}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c,c,c;x,y,z,u)=\frac{1}{\Gamma(a_1)\Gamma(a_3)}\cdot\int_0^\infty\int_0^\infty \times e^{-(s+t)}s^{a_1-1}t^{a_3-1}\Phi_3^{(3)}\left(a_2,a_4;c;sy+tz,tu,s^2x\right)\mathrm{d}s\mathrm{d}t\,(\mathrm{Re}(a_1)>0,\mathrm{Re}(a_3)>0), \tag{32}$$

where ${}_0F_1, {}_1F_1, \Psi_2, \Phi_3$, and $\Phi_3^{(3)}$ are the confluent hypergeometric functions defined by (see [11])

$$\begin{aligned}
{}_0F_1(-;c;x) &= \sum_{m=0}^{\infty}\frac{1}{(c)_m}\frac{x^m}{m!},\\
{}_1F_1(a;c;x) &= \sum_{m=0}^{\infty}\frac{(a)_m}{(c)_m}\frac{x^m}{m!},\\
\Psi_2(a;b,c;x,y) &= \sum_{m=0}^{\infty}\frac{(a)_{m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\\
\Phi_3(a;c;x,y) &= \sum_{m,n=0}^{\infty}\frac{(a)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\\
\Phi_3^{(3)}(a,b;c;x,y,z) &= \sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.
\end{aligned} \tag{33}$$

Proof. It is noted that each of the integral representations (27) to (32) can be proved mainly by expressing the series definition of the involved special functions in each integrand, changing the order of the integral sign and the summation, and finally using the following well-known integral formula [12, 17]:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t,\quad (\Re(z)>0). \tag{34}$$

□

5. Operator Formulas

Here, we establish some operator identities for functions $X_{85}^{(4)}, X_{86}^{(4)}, \ldots, X_{90}^{(4)}$. We begin by recalling the following reciprocally inverse operators (see [3, 18]):

$$\begin{aligned}
H_{t_1,\ldots,t_i}(a,b) &:= \frac{\Gamma(b)\Gamma(a+\delta_1+\cdots+\delta_i)}{\Gamma(a)\Gamma(b+\delta_1+\cdots+\delta_i)}\\
&= \sum_{k_1,\ldots,k_i=0}^{\infty}\frac{(b-a)_{k_1+\cdots+k_i}(-\delta_1)_{k_1}\ldots(-\delta_i)_{k_i}}{(b)_{k_1+\cdots+k_i}k_1!\ldots k_i!},\\
\overline{H}_{t_1,\ldots,t_i}(a,b) &:= \frac{\Gamma(a)\Gamma(a+\delta_1+\cdots+\delta_i)}{\Gamma(b)\Gamma(a+\delta_1+\cdots+\delta_i)}\\
&= \sum_{k_1,\ldots,k_i=0}^{\infty}\frac{(b-a)_{k_1+\cdots+k_i}(-\delta_1)_{k_1}\ldots(-\delta_i)_{k_i}}{(1-a-\delta_1-\cdots-\delta_i)_{k_1+\cdots+k_i}k_1!\ldots k_i!},
\end{aligned} \tag{35}$$

where $\delta_j := t_j(\partial/\partial t_j)$, $j=1,\ldots,i; i\in\mathbb{N}:=\{1,2,3,\ldots\}$.

Theorem 14. *The following identities hold true:*

$$X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,y,z,u)=H_{y,z}(a_2,a)X_{85}^{(4)}(a_1,a_1,a,a_3,a_1,a,a_3,a_4;c_1,c_2,c_3,c_1;x,y,z,u), \tag{36}$$

$$X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_1;x,y,z,u)=H_z(c,c_3)X_{85}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c,c_1;x,y,z,u). \tag{37}$$

Theorem 15. *The following identities hold true:*

$$X_{86}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x,y,z,u)=\overline{H}_{z,u}(a,a_3)X_{86}^{(4)}(a_1,a_1,a_2,a,a_1,a_2,a,a_4;c_2,c_1,c_3,c_1;x,y,z,u), \tag{38}$$

$$X_{86}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_2,c_1,c_3,c_1;x,y,z,u)=\overline{H}_{x}(c_2,c)X_{86}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c_1,c_3,c_1;x,y,z,u). \tag{39}$$

Theorem 16. *The following identities hold true:*

$$X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,y,z,u)=H_{u}(a_4,a)X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a;c_1,c_2,c_1,c_2;x,y,z,u), \tag{40}$$

$$X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,y,z,u)=H_{x,z}(c,c_1)H_{y,u}(c',c_2)X_{87}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c',c,c';x,y,z,u). \tag{41}$$

Theorem 17. *The following identities hold true:*

$$X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,y,z,u)=\overline{H}_{y,z}(a,a_2)\overline{H}_{u}(a',a_4)X_{88}^{(4)}(a_1,a_1,a,a_3,a_1,a,a_3,a';c_1,c_2,c_2,c_1;x,y,z,u), \tag{42}$$

$$X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_2,c_1;x,y,z,u)=H_{x,u}(c,c_1)X_{88}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c_2,c_2,c;x,y,z,u). \tag{43}$$

Theorem 18. *The following identities hold true:*

$$X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;x,y,z,u)=H_{x,z,u}(c,c_1)X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c_2,c,c;x,y,z,u), \tag{44}$$

$$X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_1;x,y,z,u)=\overline{H}_{x,z,u}(c_1,c)X_{89}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c_2,c,c;x,y,z,u). \tag{45}$$

Theorem 19. *The following identities hold true:*

$$X_{90}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c,c,c;x,y,z,u)=\overline{H}_{u}(a,a_4)X_{90}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a;c,c,c,c;x,y,z,u), \tag{46}$$

$$X_{90}^{(4)}(a_1,a_1,a_2,a_3,a_1,a_2,a_3,a_4;c,c,c,c;x,y,z,u)=H_{z,u}(a_3,a)X_{90}^{(4)}(a_1,a_1,a_2,a,a_1,a_2,a,a_4;c,c,c,c;x,y,z,u). \tag{47}$$

Proof. Relations (37) to (47) can be proved by means of Mellin and Mellin–Barnes integral representation methods for hypergeometric functions (see [19]). The details of proofs are omitted. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and approved the final manuscript.

References

[1] M. G. Bin-Saad, K. S. Nisar, and J. A. Younis, *Fractional Differentiations and Integrations of Quadruple Hypergeometric Series*, Communications of the Korean Mathematical Society, Seoul, South Korea, 2020, in Press.

[2] Y. A. Brychkov and N. Saad, "On some formulas for the Appell function F_2 (a, b, b′;c, c′;w; z)," *Integral Transforms and Special Functions*, vol. 25, no. 2, pp. 111–123, 2014.

[3] A. Hasanov and J. A. Younis, "$H(\alpha,\beta)\overline{H}(\alpha,\beta)$Application of operators and to hypergeometric Gaussian functions in three variables," *Acta Universitatis Apulensis*, vol. 64, pp. 127–136, 2020.

[4] A. Hasanov and J. A. Younis, "Decomposition formulas for some Gaussian hypergeometric functions in three variables," *East-West Journal of Mathematics*, vol. 22, pp. 133–140, 2020.

[5] S. B. Opps, N. Saad, and H. M. Srivastava, "Recursion formulas for Appell's hypergeometric function with some applications to radiation field problems," *Applied Mathematics and Computation*, vol. 207, no. 2, pp. 545–558, 2009.

[6] V. Sahai and A. Verma, "Recursion formulas for the Srivastava-Daoust and related multivariable hypergeometric functions," *Asian-European Journal of Mathematics*, vol. 9, p. 4, 2016.

[7] H. Exton, *Multiple Hypergeometric Functions and Applications*, Halsted Press, New York, NY, USA, 1976.

[8] C. Sharma and C. L. Parihar, "Hypergeometric functions of four variables (I)," *The Journal of the Indian Academy of Mathematics*, vol. 11, pp. 121–133, 1989.

[9] M. G. Bin-Saad and J. A. Younis, "On connections between certain class of new quadruple and known triple hypergeomtric series," *Tamap Journal of Mathematics and Statistics*, vol. 2019, 2019.

[10] J. A. Younis and M. G. Bin-Saad, "Integral representations involving new hypergeometric functions of four variables," *Journal of Fractional Calculus and Applications*, vol. 10, pp. 77–91, 2019.

[11] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Lt1., Chichester, UK, 1984.

[12] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[13] P. Appell, "Sur les fonctions hypergéeométriques de deux variables," *Journal de Mathématiques Pures et Appliquées*, vol. 8, no. 3, pp. 173–216, 1882.

[14] H. Exton, "Hypergeometric functions of three variables," *Journal of Indian Academy of Mathematics*, vol. 4, pp. 113–119, 1982.

[15] G. Lauricella, "Sulle funzioni ipergeometriche a piu variabili," *Rendiconti del Circolo Matematico di Palermo*, vol. 7, no. S1, pp. 111–158, 1893.

[16] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.

[17] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, NY, USA, 1960.

[18] J. Choi and A. Hasanov, "Applications of the operator $H(\alpha,\beta)$ to the Humbert double hypergeometric functions," *Computers & Mathematics with Applications*, vol. 61, no. 3, pp. 663–671, 2011.

[19] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables., Halsted Press (Ellis Horwood Limited, Chichester)*, Wiley, New York, NY, USA, 1982.

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Research Article

Hadamard and Fejér–Hadamard Inequalities for Further Generalized Fractional Integrals Involving Mittag-Leffler Functions

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In this paper, generalized versions of Hadamard and Fejér–Hadamard type fractional integral inequalities are obtained. By using generalized fractional integrals containing Mittag-Leffler functions, some well-known results for convex and harmonically convex functions are generalized. The results of this paper are connected with various published fractional integral inequalities.

1. Introduction

First we give definitions of fractional integral operators which are useful in establishing the results of this paper. In the following, we give fractional integral operators defined by Andrić et al. in [1] via an extended generalized Mittag-Leffler function in their kernels.

Definition 1 (see [1]). Let $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\Re(\tau), \Re(\delta) > 0$, $\Re(c) > \Re(\rho) > 0$ with $\overline{p} \geq 0$, $\sigma, r > 0$ and $0 < k \leq r + \sigma$. Let $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ and $x \in [\varepsilon_1, \varepsilon_2]$. Then, the generalized fractional integral operators $\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\varepsilon_1^+}\varphi$ and $\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\varepsilon_2^-}\varphi$ are defined by

$$\left(\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\varepsilon_1^+}\varphi\right)(x;\overline{p}) = \int_{\varepsilon_1}^{x} (x-t)^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(x-t)^{\sigma};\overline{p}\right)\varphi(t)\mathrm{d}t,$$
$$\left(\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\varepsilon_2^-}\varphi\right)(x;\overline{p}) = \int_{x}^{\varepsilon_2} (t-x)^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(t-x)^{\sigma};\overline{p}\right)\varphi(t)\mathrm{d}t, \tag{1}$$

where

$$E^{\rho,r,k,c}_{\sigma,\tau,\delta}(t;\overline{p}) = \sum_{n=0}^{\infty} \frac{\beta_{\overline{p}}(\rho+nk, c-\rho)(c)_{nk}t^n}{\beta(\rho, c-\rho)\Gamma(\sigma n+\tau)(\delta)_{nr}}, \tag{2}$$

is the extended generalized Mittag-Leffler function and $\beta_{\overline{p}}$ is the extension of beta function which is defined as follows:

$$\beta_{\overline{p}}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-(\overline{p}/t(1-t))}\mathrm{d}t, \tag{3}$$

where $x, y, \overline{p}$ are positive real numbers.

Recently, Farid defined elegantly a unified integral operator in [2] (see, also [3]) as follows.

Definition 2. Let $\varphi, \theta\colon [\varepsilon_1, \varepsilon_2] \longrightarrow \mathbb{R}$, $0 < \varepsilon_1 < \varepsilon_2$ be the functions such that φ be positive and $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ and θ be a differentiable and strictly increasing function. Also, let χ/x be an increasing function on $[\varepsilon_1, \infty)$ and $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\Re(\tau), \Re(\delta) > 0$, $\Re(c) > \Re(\rho) > 0$ with $\overline{p} \geq 0$, $\sigma, r > 0$, and $0 < k \leq r + \sigma$.

Then, for $x \in [\varepsilon_1, \varepsilon_2]$, the integral operators $({}_{\theta}\Upsilon^{\chi,\rho,r,k,c}_{\sigma,\tau,\delta,\varepsilon_1^+}\varphi)$ and $({}_{\theta}\Upsilon^{\chi,\rho,r,k,c}_{\sigma,\tau,\delta,\varepsilon_1^-}\varphi)$ are defined by

$$\left({}_{\theta}\Upsilon^{\chi,\rho,r,k,c}_{\sigma,\tau,\delta,\varepsilon_1^+}\varphi\right)(x;\overline{p}) = \int_{\varepsilon_1}^{x} \frac{\chi(\theta(x)-\theta(t))}{\theta(x)-\theta(t)} E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(\theta(x)-\theta(t))^{\sigma};\overline{p}\right)\varphi(t)\mathrm{d}(\theta(t)),$$
$$\left({}_{\theta}\Upsilon^{\chi,\rho,r,k,c}_{\sigma,\tau,\delta,\varepsilon_2^-}\varphi\right)(x;\overline{p}) = \int_{x}^{\varepsilon_2} \frac{\chi(\theta(t)-\theta(x))}{\theta(t)-\theta(x)} E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(\theta(t)-\theta(x))^{\sigma};\overline{p}\right)\varphi(t)\mathrm{d}(\theta(t)). \tag{4}$$

The following definition of generalized fractional integral operators containing extended Mittag-Leffler function in the kernel can be extracted from Definition 2. It is generalization of Definition 1 by a monotonically increasing function.

Definition 3. Let $\varphi, \theta\colon [\varepsilon_1, \varepsilon_2] \longrightarrow \mathbb{R}$, $0 < \varepsilon_1 < \varepsilon_2$ be the functions such that φ be positive and $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ and θ be a differentiable and strictly increasing function. Also, let $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\mathfrak{R}(\tau), \mathfrak{R}(\delta) > 0$, $\mathfrak{R}(c) > \mathfrak{R}(\rho) > 0$ with $\overline{p} \geq 0$, $\sigma, r > 0$ and $0 < k \leq r + \sigma$. Then, for $x \in [\varepsilon_1, \varepsilon_2]$, fractional integral operators are defined by

$$\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega\varepsilon_1^+}\varphi\right)(x;\overline{p}) = \int_{\varepsilon_1}^{x} (\theta(x)-\theta(t))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(\theta(x)-\theta(t))^{\sigma};\overline{p}\right)\varphi(t)\mathrm{d}(\theta(t)), \tag{5}$$

$$\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega\varepsilon_2^-}\varphi\right)(x;\overline{p}) = \int_{x}^{\varepsilon_2} (\theta(t)-\theta(x))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(\theta(t)-\theta(x))^{\sigma};\overline{p}\right)\varphi(t)\mathrm{d}(\theta(t)). \tag{6}$$

The following remark provides connection of Definition 3 with existing fractional integral operators.

Remark 1

(i) If we set $\overline{p} = 0$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to fractional integral operators defined by Salim and Faraj in [4].

(ii) If we set $\delta = r = 1$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to the fractional integral operators ${}_{\theta}\Upsilon^{\rho,1,k,c}_{\sigma,\tau,1,\omega,\varepsilon_1^+}$ and ${}_{\theta}\Upsilon^{\rho,1,k,c}_{\sigma,\tau,1,\omega,\varepsilon_1^-}$ containing generalized Mittag-Leffler function $E^{\rho,1,k,c}_{\sigma,\tau,1}(t;\overline{p})$ defined by Rahman et al. in [5].

(iii) If we take $\overline{p} = 0, \delta = r = 1$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to fractional integral operators containing extended generalized Mittag-Leffler function introduced by Srivastava and Tomovski in [6].

(iv) If we set $\overline{p} = 0, \delta = r = k = 1$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to fractional integral operators defined by Prabhaker in [7].

(v) For $\overline{p} = \omega = 0$ and $\theta(x) = x$ in equations (5) and (6), these reduce to renowned Riemann–Liouville fractional integral operators [8].

The Riemann–Liouville fractional integrals for a function $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ of order $\tau \in \mathbb{R}$ $(\tau > 0)$ are defined by

$$I^{\tau}_{\varepsilon_1^+}\varphi(x) = \frac{1}{\Gamma(\tau)}\int_{\varepsilon_1}^{x} (x-t)^{\tau-1}\varphi(t)\mathrm{d}t, \quad x > \varepsilon_1,$$
$$I^{\tau}_{\varepsilon_2^-}\varphi(x) = \frac{1}{\Gamma(\tau)}\int_{x}^{\varepsilon_2} (t-x)^{\tau-1}\varphi(t)\mathrm{d}t, \quad x < \varepsilon_2. \tag{7}$$

After introducing generalized fractional integral operators, now we define notions of functions for which generalized fractional integral operators are utilized to get main results of this paper.

Definition 4 (see [9]). A function $\varphi\colon [\varepsilon_1, \varepsilon_2] \longrightarrow \mathbb{R}$ is said to be convex if

$$\varphi(tx_1 + (1-t)x_2) \leq t\varphi(x_1) + (1-t)\varphi(x_2), \tag{8}$$

holds for all $x_1, x_2 \in [\varepsilon_1, \varepsilon_2]$ and $t \in [0,1]$.

Definition 5 (see [10]). Let I be an interval such that $I \subseteq \mathbb{R}_+$. Then, a function $\varphi\colon I \longrightarrow \mathbb{R}$ is said to be harmonically convex, if

$$\varphi\left(\frac{ab}{ta + (1-t)b}\right) \leq t\varphi(b) + (1-t)\varphi(a), \tag{9}$$

holds for all $a, b \in I$ and $t \in [0,1]$.

Definition 6 (see [11]). Let $J \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. Then, a function $\varphi\colon J \longrightarrow \mathbb{R}$ is said to be p-convex, if

$$\varphi\left(\left[t\varepsilon_1^p + (1-t)\varepsilon_2^p\right]^{1/p}\right) \leq t\varphi(\varepsilon_1) + (1-t)\varphi(\varepsilon_2), \tag{10}$$

holds for $\varepsilon_1, \varepsilon_2 \in J$ and $t \in [0,1]$.

It is easy to see that for $p = 1$ and $p = -1$, the p-convexity reduces to convexity and harmonical convexity, respectively.

Definition 7 (see [11]). Let $p \in \mathbb{R}\backslash\{0\}$. Then, a function $\varphi\colon [\varepsilon_1, \varepsilon_2] \subset (0,\infty) \longrightarrow \mathbb{R}$ is said to be p-symmetric with respect to $[(\varepsilon_1^p + \varepsilon_2^p)/2]^{1/p}$ if

$$\varphi\left(t^{1/p}\right) = \varphi\left(\left[\varepsilon_1^p + \varepsilon_2^p - t\right]^{1/p}\right), \tag{11}$$

holds, for $t \in [\varepsilon_1, \varepsilon_2]$.

Convex functions are equivalently studied by the Hadamard inequality.

Theorem 1. *Let φ: $[\varepsilon_1, \varepsilon_2] \longrightarrow \mathbb{R}$ be a convex function such that $\varepsilon_1 < \varepsilon_2$. Then, the following inequality holds:*

$$\varphi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \leq \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \varphi(x)\mathrm{d}x \leq \frac{\varphi(\varepsilon_1) + \varphi(\varepsilon_2)}{2}. \tag{12}$$

The Fejér–Hadamard inequality is a weighted version of the Hadamard inequality given by Fejér in [12].

Theorem 2. *Let φ: $[\varepsilon_1, \varepsilon_2] \longrightarrow \mathbb{R}$ be a convex function and g: $[\varepsilon_1, \varepsilon_2] \longrightarrow \mathbb{R}$ be non-negative, integrable, and symmetric about $(\varepsilon_1 + \varepsilon_2)/2$. Then, the following inequality holds:*

$$\begin{aligned} \varphi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \int_{\varepsilon_1}^{\varepsilon_2} g(x)\mathrm{d}x &\leq \int_{\varepsilon_1}^{\varepsilon_2} \varphi(x) g(x)\mathrm{d}x \\ &\leq \frac{\varphi(\varepsilon_1) + \varphi(\varepsilon_2)}{2} \int_{\varepsilon_1}^{\varepsilon_2} g(x)\mathrm{d}x. \end{aligned} \tag{13}$$

In recent decades, the Hadamard and the Fejér–Hadamard fractional integral inequalities have been studied extensively for different kinds of convex functions (see [1, 3, 13–21]). In this paper, we find Hadamard and Fejér–Hadamard inequalities for a generalized fractional integral operator involving an extended generalized Mittag-Leffler function.

In the upcoming section, we give two versions of the Hadamard inequality as well as two versions of the Fejér–Hadamard inequality. Their special cases are also discussed along with noticing connections with published results.

2. Main Results

First we give the following version of the Hadamard inequality.

Theorem 3. *Let φ, θ: $[\varepsilon_1, \varepsilon_2] \subset (0, \infty) \longrightarrow \mathbb{R}$, Range $(\theta) \subset [\varepsilon_1, \varepsilon_2]$ be the functions such that φ be positive and $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$, and θ be a differentiable and strictly increasing function. If φ is p-convex, $p \in \mathbb{R}\backslash\{0\}$, then the following inequalities for fractional integral operators (5) and (6) hold:*

(i) If $p > 0$, then

$$\begin{aligned} &\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+} 1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right) \\ &\leq \frac{1}{2}\Bigg(\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+} \varphi \circ \psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right) \\ &\quad + \left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-} \varphi \circ \psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1)); \overline{p}\right)\Bigg) \\ &\leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2}\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+} 1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right), \end{aligned} \tag{14}$$

where $\omega' = \omega/(\theta^p(\varepsilon_2) - \theta^p(\varepsilon_1))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_1^p, \varepsilon_2^p]$.

(ii) If $p < 0$, then

$$\begin{aligned} &\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left(\varepsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_-} 1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right) \\ &\leq \frac{1}{2}\Bigg(\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_+} \varphi \circ \psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1)); \overline{p}\right) \\ &\quad + \left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_-} \varphi \circ \psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right)\Bigg) \\ &\leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2}\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_-} 1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right), \end{aligned} \tag{15}$$

where $\omega' = \omega/(\theta^p(\varepsilon_1) - \theta^p(\varepsilon_2))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_2^p, \varepsilon_1^p]$.

Proof. (i) Since φ is p-convex over $[\varepsilon_1, \varepsilon_2]$, for all $x, y \in I$, we have

$$\varphi\left(\left(\frac{\theta^p(x)+\theta^p(y)}{2}\right)^{1/p}\right)\leq\frac{\varphi(\theta(x))+\varphi(\theta(y))}{2}. \tag{16}$$

Setting $\theta(x)=(t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2))^{1/p}$ and $\theta(y)=(t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1))^{1/p}$ in above inequality, we have

$$\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\leq\frac{\varphi\left(\left(t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2)\right)^{1/p}\right)+\varphi\left(\left(t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1)\right)^{1/p}\right)}{2}. \tag{17}$$

Multiplying both sides of (17) by $2t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}&2\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})dt\\&\leq\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})\varphi\left(\left(t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2)\right)^{1/p}\right)dt\\&\quad+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})\varphi\left(\left(t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1)\right)^{1/p}\right)dt.\end{aligned} \tag{18}$$

By choosing $\theta(x)=t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2)$ and $\theta(y)=t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1)$ in (18), we have

$$\begin{aligned}&2\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}(\theta^p(\varepsilon_2)-\theta(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega'(\theta^p(\varepsilon_2)-\theta(x))^\sigma;\overline{p}\right)1d(\theta(x))\\&\leq\int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}(\theta^p(\varepsilon_2)-\theta(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega'(\theta^p(\varepsilon_2)-\theta(x))^\sigma;\overline{p}\right)\varphi\left(\theta(x)^{1/p}\right)d(\theta(x))\\&\quad+\int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}(\theta(y)-\theta^p(\varepsilon_1))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega'(\theta(y)-\theta^p(\varepsilon_1))^\sigma;\overline{p}\right)\varphi\left(\theta(y)^{1/p}\right)d(\theta(y)),\end{aligned} \tag{19}$$

where $\omega'=\omega/(\theta^p(\varepsilon_2)-\theta^p(\varepsilon_1))^\sigma$.

This implies

$$\begin{aligned}&2\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left(\varepsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\&\leq\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\&\quad+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right).\end{aligned} \tag{20}$$

To prove the second inequality of (14), again from p-convexity of φ over $[\varepsilon_1,\varepsilon_2]$ and for $t\in[0,1]$, we have

$$\begin{aligned}&\varphi\left(\left(t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2)\right)^{1/p}\right)\\&\quad+\varphi\left(\left(t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1)\right)^{1/p}\right)\leq\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)).\end{aligned} \tag{21}$$

Multiplying both sides of (21) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})\varphi\left(\left(t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2)\right)^{1/p}\right)dt\\&\quad+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})\varphi\left(\left(t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1)\right)^{1/p}\right)dt\\&\leq(\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)))\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})dt.\end{aligned} \tag{22}$$

Setting $\theta(x)=t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2)$ and $\theta(y)=t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1)$ in (22), we have

$$\begin{aligned}&\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^{p}(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_2));\overline{p}\right)\\&\quad+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^{p}(\varepsilon_2))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_1));\overline{p}\right)\\&\leq(\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)))\\&\quad\cdot\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^{p}(\varepsilon_1))_+}1\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_2));\overline{p}\right).\end{aligned}\tag{23}$$

By combining (20) and (23), we get (14).

(ii) Proof is similar to the proof of (i). □

Remark 2

(i) By setting $\overline{p}=\omega=0$ and $\theta=I$, Theorem 9 of [11] is obtained.

(ii) By setting $\overline{p}=0$, $p=-1$, and $\theta=I$, Theorem 2.1 of [22] is obtained.

(iii) By setting $\theta=I$ and $p=-1$, Theorem 2.1 of [23] is obtained.

(iv) By setting $\omega=\overline{p}=0$, $p=-1$, and $\theta=I$, Theorem 4 of [18] is obtained.

(v) By setting $p=-1$, Theorem 2.1 of [24] is obtained.

(vi) By setting $p=-1$ and $\psi(x)=x$, Corollary 2.3 of [24] is obtained.

Corollary 1. *In (15), if we take $\omega=\overline{p}=0$, $p=-1$, and $\theta=I$, then we get the following Hadamard inequality for the RL fractional integrals:*

$$\begin{aligned}\varphi\left(\frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)&\leq\frac{\Gamma(\tau+1)}{2}\left(\frac{\varepsilon_1\varepsilon_2}{\varepsilon_2-\varepsilon_1}\right)^{\tau}\\&\quad\cdot\left(\left(I^{\tau}_{1\backslash\varepsilon_1^-}\varphi\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)+\left(I^{\tau}_{1\backslash\varepsilon_1^+}\varphi\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)\right)\\&\leq\frac{\varphi(\varepsilon_1)+\varphi(\varepsilon_2)}{2}.\end{aligned}\tag{24}$$

Now we obtain Fejér–Hadamard type fractional integral inequalities for p-convex function via generalized fractional integral operators; for this, first we prove the following lemma.

Lemma 1. *Let $\varphi,\theta\colon[\varepsilon_1,\varepsilon_2]\subset(0,\infty)\longrightarrow\mathbb{R}$, Range $(\theta)\subset[\varepsilon_1,\varepsilon_2]$ be the functions such that φ be positive and $\varphi\in L_1[\varepsilon_1,\varepsilon_2]$, $\varepsilon_1<\varepsilon_2$, and θ be a differentiable and strictly increasing function. If φ is p-convex, $p\in\mathbb{R}\backslash\{0\}$, and $\varphi(\theta^{1/p}(x))=\varphi((\theta^{p}(\varepsilon_1)+\theta^{p}(\varepsilon_2)-\theta(x))^{1/p})$, then for generalized fractional integral operators (5) and (6), we have*

(i) If $p>0$, then

$$\begin{aligned}&\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_2));\overline{p}\right)\\&=\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_2))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_1));\overline{p}\right)\\&=\frac{1}{2}\left(\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_2));\overline{p}\right)\right.\\&\quad\left.+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_2))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_1));\overline{p}\right)\right),\end{aligned}\tag{25}$$

with $\psi(t)=\theta^{1/p}(t)$, $t\in[\varepsilon_1^{p},\varepsilon_2^{p}]$.

(ii) If $p<0$, then

$$\begin{aligned}&\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_2))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_1));\overline{p}\right)\\&=\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_1))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_2));\overline{p}\right)\\&=\frac{1}{2}\left(\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_2))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_1));\overline{p}\right)\right.\\&\quad\left.+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^{p}(\varepsilon_1))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_2));\overline{p}\right)\right),\end{aligned}\tag{26}$$

with $\psi(t)=\theta^{1/p}(t)$, $t\in[\varepsilon_2^{p},\varepsilon_1^{p}]$.

Proof. (i) By definition of generalized fractional integral operators (5) and (6), we have

$$\begin{aligned}&\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^{p}(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^{p}(\varepsilon_2));\overline{p}\right)\\&=\int_{\theta^{-1}(\theta^{p}(\varepsilon_1))}^{\theta^{-1}(\theta^{p}(\varepsilon_2))}(\theta^{p}(\varepsilon_2)-\theta(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(\theta^{p}(\varepsilon_2)-\theta(x))^{\sigma};\overline{p}\right)\varphi\circ\psi(x)\mathrm{d}(\theta(x))\\&=\int_{\theta^{-1}(\theta^{p}(\varepsilon_1))}^{\theta^{-1}(\theta^{p}(\varepsilon_2))}(\theta^{p}(\varepsilon_2)-\theta(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(\theta^{p}(\varepsilon_2)-\theta(x))^{\sigma};\overline{p}\right)\varphi\left(\theta(x)^{1/p}\right)\mathrm{d}(\theta(x))\\&=\int_{\theta^{-1}(\theta^{p}(\varepsilon_1))}^{\theta^{-1}(\theta^{p}(\varepsilon_2))}(\theta^{p}(\varepsilon_2)-\theta(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(\theta^{p}(\varepsilon_2)-\theta(x))^{\sigma};\overline{p}\right)\varphi\left((\theta^{p}(\varepsilon_1)+\theta^{p}(\varepsilon_2)-\theta(x))^{1/p}\right)\mathrm{d}(\theta(x)).\end{aligned}\tag{27}$$

Setting $\theta(t)=\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x)$ in the above equation and using $\varphi(\theta^{1/p}(x))=\varphi((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x))^{1/p})$, we have

$$\begin{aligned}&\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\&=\int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}\left(\theta(t)-\theta^p(\varepsilon_1)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(\theta(t)-\theta^p(\varepsilon_1)\right)^\sigma;\overline{p}\right)\varphi\left(\theta(t)^{1/p}\right)\mathrm{d}(\theta(t))\\&=\int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}\left(\theta(t)-\theta^p(\varepsilon_1)\right)^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega\left(\theta(t)-\theta^p(\varepsilon_1)\right)^\sigma;\overline{p}\right)\varphi\circ\psi(t)\mathrm{d}(\theta(t)).\end{aligned}\tag{28}$$

This implies

$$\begin{aligned}&\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\&=\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right).\end{aligned}\tag{29}$$

By adding $\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)$ on both sides of (29), we have

$$\begin{aligned}&2\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\&=\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\&\quad+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right).\end{aligned}\tag{30}$$

From equations (29) and (30), the required result can be obtained.

(ii) Proof is on the same lines as the proof of (i). □

Theorem 4. *Let $\varphi,\theta,h\colon[\varepsilon_1,\varepsilon_2]\subset(0,\infty)\longrightarrow\mathbb{R}$, Range (θ), Range $(h)\subset[\varepsilon_1,\varepsilon_2]$ be the functions such that φ be positive and $\varphi\in L_1(\varepsilon_1,\varepsilon_2)$, $\varepsilon_1<\varepsilon_2$, and θ be a differentiable and strictly increasing function where h is a non-negative and integrable function. If φ is p-convex, $p\in\mathbb{R}\backslash\{0\}$, and $\varphi(\theta^{1/p}(x))=\varphi((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x))^{1/p})$, then the following inequalities for generalized fractional integral operators (5) and (6) hold:*

(i) If $p>0$, then

$$\begin{aligned}&\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\\&\cdot\left(\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\right)\\&\leq\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\\&\leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\\&\cdot\left(\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\right),\end{aligned}\tag{31}$$

where $\omega'=\omega/(\theta^p(\varepsilon_2)-\theta^p(\varepsilon_1))^\sigma$ and $\psi(t)=\theta^{1/p}(t)$ for all $t\in[\varepsilon_1^p,\varepsilon_2^p]$.

(ii) If $p<0$, then

$$\begin{aligned}&\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left(\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\right.\\&\left.\quad+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\right)\\&\leq\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\\&\quad+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\&\leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\left(\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\right),\end{aligned}\tag{32}$$

where $\omega' = \omega/(\theta^p(\varepsilon_1) - \theta^p(\varepsilon_2))^\sigma$ *and* $\psi(t) = \theta^{1/p}(t)$ *for all* $t \in [\varepsilon_2^p, \varepsilon_1^p]$.

Proof. (i) Multiplying both sides of (17) by $2t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma; \overline{p})h((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}
&2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma; \overline{p})h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt\\
&\leq \int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma; \overline{p})\varphi\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt\\
&\quad + \int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma; \overline{p})\varphi\left((t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}\right)h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt.
\end{aligned} \tag{33}$$

By choosing $\theta(x) = t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2)$, that is, $\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x) = t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1)$, in (33) and using $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, we have

$$\begin{aligned}
&2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\\
&\quad\cdot\int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}(\theta^p(\varepsilon_2) - \theta(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega'(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \overline{p}\right)h\circ\psi(x)d(\theta(x))\\
&\leq \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}(\theta^p(\varepsilon_2) - \theta(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega'(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \overline{p}\right)\varphi h\circ\psi(x)d(\theta(x))\\
&\quad + \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))}(\theta(x) - \theta^p(\varepsilon_1))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega'(\theta(x) - \theta^p(\varepsilon_1))^\sigma; \overline{p}\right)\varphi h\circ\psi(x)d(\theta(x)).
\end{aligned} \tag{34}$$

This implies

$$\begin{aligned}
&2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right)\\
&\leq \left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2)); \overline{p}\right)\\
&\quad + \left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1)); \overline{p}\right).
\end{aligned} \tag{35}$$

Using Lemma 1 (i) in above inequality, we have

$$\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left(\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\right.$$
$$\left.+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\right)$$
$$\leq\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)$$
$$+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right). \tag{36}$$

To prove the second inequality of (31), multiplying both sides of (21) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})h((t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2))^{1/p})$ and then integrating over $[0,1]$, we have

$$\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})\varphi\left((t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2))^{1/p}\right)h\left((t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt$$
$$+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})\varphi\left((t\theta^p(\varepsilon_2)+(1-t)\theta^p(\varepsilon_1))^{1/p}\right)h\left((t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt$$
$$\leq(\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)))\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^\sigma;\overline{p})h\left((t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt. \tag{37}$$

Setting $\theta(x)=t\theta^p(\varepsilon_1)+(1-t)\theta^p(\varepsilon_2)$ and using $\varphi(\theta^{1/p}(x))=\varphi((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x))^{1/p})$ in (37) and after simplification, we have

$$\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)$$
$$+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)$$
$$\leq(\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)))\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right). \tag{38}$$

Using Lemma 1 (i), inequality (38) becomes

$$\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)$$
$$+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)$$
$$\leq\frac{(\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)))}{2}$$
$$\cdot\left(\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\right.$$
$$\left.+\left({}_\theta\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\right). \tag{39}$$

By combining (36) and (39), we get (31).

(ii) Proof is similar to the proof of (i) by using (ii) of Lemma 1. □

Remark 3

(i) By setting $\overline{p}=0$ and $\theta=I$, Theorem 2.2 of [25] is obtained.
(ii) By setting $\overline{p}=\omega=0$ and $\theta=I$, Theorem 9 of [11] is obtained.
(iii) By setting $\overline{p}=0$, $h(x)=1$, $p=-1$, and $\theta=I$, Theorem 2.1 of [22] is obtained.
(iv) By setting $\theta=I$, $h(x)=1$, and $p=-1$, Theorem 2.1 of [23] is obtained.
(v) By setting $\omega=\overline{p}=0$, $h(x)=1$, $p=-1$, and $\theta=I$, Theorem 4 of [18] is obtained.
(vi) By setting $p=-1$, Theorem 2.5 of [24] is obtained.

Corollary 2. *If we put $p=-1$, $\overline{p}=0$, and $\theta=I$ in Theorem 4 (ii), we get the following Fejér–Hadamard inequalities for harmonically convex function via generalized fractional integral operators:*

$$\varphi\left(\frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)\left(\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',(1/\varepsilon_2)^+}h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',(1/\varepsilon_1)^-}h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)\right)$$
$$\leq\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',(1/\varepsilon_2)^+}\varphi h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',(1/\varepsilon_1)^-}\varphi h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right) \tag{40}$$
$$\leq\frac{\varphi(\varepsilon_1)+\varphi(\varepsilon_2)}{2}\left(\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',(1/\varepsilon_2)^+}h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',(1/\varepsilon_1)^-}h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)\right).$$

Now we give another version of the Hadamard inequality.

Theorem 5. *Let φ,θ: $[\varepsilon_1,\varepsilon_2]\subset(0,\infty)\longrightarrow\mathbb{R}$, Range $(\theta)\subset[\varepsilon_1,\varepsilon_2]$ be the functions such that φ be positive and $\varphi\in L_1[\varepsilon_1,\varepsilon_2]$, $\varepsilon_1<\varepsilon_2$, and θ be a differentiable and strictly increasing function. If φ is p-convex, $p\in\mathbb{R}\backslash\{0\}$, then for generalized fractional integral operators (5) and (6), we have*

(i) If $p>0$, then

$$\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)$$
$$\leq\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)$$
$$+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right) \tag{41}$$
$$\leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right),$$

where $\omega'=\omega/(\theta^p(\varepsilon_2)-\theta^p(\varepsilon_1))^\sigma$ and $\psi(t)=\theta^{1/p}(t)$ for all $t\in[\varepsilon_1^p,\varepsilon_2^p]$.

(ii) If $p<0$, then

$$\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)$$
$$\leq\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)$$
$$+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right) \tag{42}$$
$$\leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right),$$

where $\omega'=\omega/(\theta^p(\varepsilon_1)-\theta^p(\varepsilon_2))^\sigma$ and $\psi(t)=\theta^{1/p}(t)$ for all $t\in[\varepsilon_2^p,\varepsilon_1^p]$.

Proof. (i) Setting $\theta(x)=((t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2))^{1/p}$ and $\theta(y)=((t/2)\theta^p(\varepsilon_2)+((2-t)/2))\theta^p(\varepsilon_1))^{1/p}$ in (12), we have

$$\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \leq \frac{\varphi\left(\left((t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2)\right)^{1/p}\right)+\varphi\left(\left((t/2)\theta^p(\varepsilon_2)+((2-t)/2)\theta^p(\varepsilon_1)\right)^{1/p}\right)}{2}. \tag{43}$$

Multiplying both sides of (43) by $2t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}2\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\mathrm{d}t &\leq \int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2)\right)^{1/p}\right)\mathrm{d}t\\ &+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2)+((2-t)/2)\theta^p(\varepsilon_1)\right)^{1/p}\right)\mathrm{d}t.\end{aligned} \tag{44}$$

By choosing $\theta(x)=(t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2)$ and $\theta(y)=(t/2)\theta^p(\varepsilon_2)+((2-t)/2)\theta^p(\varepsilon_1)$ in (44) and by (5) and (6), we get first inequality of (41).

To prove the second inequality of (41), again from p-convexity of φ over $[\varepsilon_1,\varepsilon_2]$ and for $t\in[0,1]$, we have

$$\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)+\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_1)\right)^{1/p}\right)\leq\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)). \tag{45}$$

Multiplying both sides of (45) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)\mathrm{d}t\\ &+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_1)\right)^{1/p}\right)\mathrm{d}t\\ &\leq(\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)))\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\mathrm{d}t.\end{aligned} \tag{46}$$

Setting $\theta(x)=(t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2)$ and $\theta(y)=(t/2)\theta^p(\varepsilon_2)+((2-t)/2)\theta^p(\varepsilon_1)$ in (46) and using (5) and (6), we get second inequality of (41).

(ii) Proof is similar to the proof of (i). □

Remark 4

(i) By setting $\overline{p}=0$, $p=-1$, and $\theta=I$, Theorem 2.3 of [22] is obtained.

(ii) By setting $\theta=I$ and $p=-1$, Theorem 2.3 of [23] is obtained.

(iii) By setting $p=-1$, Theorem 2.7 of [24] is obtained.

Now we obtain another Fejér–Hadamard type fractional integral inequality for p-convex function via generalized fractional integral operators (5) and (6).

Theorem 6. *Let φ,θ,h: $[\varepsilon_1,\varepsilon_2]\subset(0,\infty)\longrightarrow\mathbb{R}$, Range (θ), Range $(h)\subset[\varepsilon_1,\varepsilon_2]$ be the functions such that φ be positive and $\varphi\in L_1(\varepsilon_1,\varepsilon_2)$, $\varepsilon_1<\varepsilon_2$, and θ be a differentiable and strictly increasing function where h is a non-negative and integrable function. If φ is p-convex, $p\in\mathbb{R}\backslash\{0\}$, and $\varphi(\theta^{1/p}(x))=\varphi((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x))^{1/p})$, then the following inequalities for generalized fractional integral operators (5) and (6) hold:*

(i) If $p>0$, then

$$\begin{aligned}
&\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\
&\leq\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\
&\quad+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\\
&\leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right),
\end{aligned}\tag{47}$$

where $\omega'=\omega/(\theta^p(\varepsilon_2)-\theta^p(\varepsilon_1))^{\sigma}$ *and* $\psi(t)=\theta^{1/p}(t)$ *for all* $t\in[\varepsilon_1^p,\varepsilon_2^p]$.

(ii) If $p<0$, *then*

$$\begin{aligned}
&\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\
&\leq\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\overline{p}\right)\\
&\quad+\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}\varphi h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right)\\
&\leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\left({}_{\theta}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,2^{\sigma}\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}h\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\overline{p}\right),
\end{aligned}\tag{48}$$

where $\omega'=\omega/(\theta^p(\varepsilon_1)-\theta^p(\varepsilon_2))^{\sigma}$ *and* $\psi(t)=\theta^{1/p}(t)$ *for all* $t\in[\varepsilon_2^p,\varepsilon_1^p]$.

Proof. (i) Multiplying (43) by $2t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})h(((t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2))^{1/p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}
&2\varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\\
&\quad\cdot\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)dt\\
&\leq\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)dt\\
&\quad+\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_1)\right)^{1/p}\right)h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)dt.
\end{aligned}\tag{49}$$

By choosing $\theta(x)=(t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2)$, that is, $\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x)=(t/2)\theta^p(\varepsilon_2)+((2-t)/2)\theta^p(\varepsilon_1)$, in (49) and using the condition $\varphi(\theta^{1/p}(x))=\varphi((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x))^{1/p})$, one can get first inequality of (47).

To prove the second inequality of (47), multiplying both sides of (45) by $t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};\overline{p})h(((t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2))^{1/p})$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)\mathrm{d}t\\
&+\int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^{\sigma};\overline{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_1)\right)^{1/p}\right)h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)\mathrm{d}t\\
&\leq(\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2)))\\
&\cdot\int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^{\sigma};\overline{p})h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1)+\left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)\mathrm{d}t.
\end{aligned}\tag{50}$$

Setting $\theta(x)=(t/2)\theta^p(\varepsilon_1)+((2-t)/2)\theta^p(\varepsilon_2)$ in (38) and using the condition $\varphi(\theta^{1/p}(x))=\varphi((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)-\theta(x))^{1/p})$, one can get second inequality of (47).

(ii) Proof is similar to the proof of (i). □

Remark 5

(i) By setting $p=-1$, $\overline{p}=0$, and $\theta=I$, Theorem 2.6 of [22] is obtained.

(ii) By setting $\theta=I$ and $p=-1$, Theorem 2.6 of [23] is obtained.

(iii) By setting $p=-1$, Theorem 2.10 of [24] is obtained.

Corollary 3. *When we set $p=-1$, $\omega=\overline{p}=0$, and $\theta=I$ in Theorem 6, then we get the following inequalities via RL fractional integrals.*

$$\begin{aligned}
&\varphi\left(\frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)\left(\left(I^{\tau}_{((\varepsilon_1+\varepsilon_2)/2\varepsilon_1\varepsilon_2)^+}h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left(I^{\tau}_{((\varepsilon_1+\varepsilon_2)/2\varepsilon_1\varepsilon_2)^-}h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)\right)\\
&\leq\left(I^{\tau}_{((\varepsilon_1+\varepsilon_2)/2\varepsilon_1\varepsilon_2)^+}\varphi h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left(I^{\tau}_{((\varepsilon_1+\varepsilon_2)/2\varepsilon_1\varepsilon_2)^-}\varphi h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)\\
&\leq\frac{\varphi(\varepsilon_1)+\varphi(\varepsilon_2)}{2}\left(\left(I^{\tau}_{((\varepsilon_1+\varepsilon_2)/2\varepsilon_1\varepsilon_2)^+}h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left(I^{\tau}_{((\varepsilon_1+\varepsilon_2)/2\varepsilon_1\varepsilon_2)^-}h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)\right).
\end{aligned}\tag{51}$$

3. Conclusion

We have established Hadamard and Fejér–Hadamard fractional integral inequalities for generalized fractional integrals of p-convex functions. The results of this paper hold simultaneously for convex and harmonically convex functions for different fractional integral operators containing Mittag-Leffler functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. Andrić, G. Farid, and J. Pečarić, "A further extension of Mittag-Leffler function," *Fractional Calculus and Applied Analysis*, vol. 21, no. 5, pp. 1377–1395, 2018.

[2] G. Farid, "A unified integral operator and further its consequences," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 1–7, 2020.

[3] Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and S. M. Kang, "Inequalities for a unified integral operator and associated results in fractional calculus," *IEEE Access*, vol. 7, pp. 126283–126292, 2019.

[4] T. O. Salim and A. W. Faraj, "A Generalization of Mittag-Leffler function and integral operator associated with integral calculus," *J. Frac. Calc. Appl*, vol. 3, no. 5, pp. 1–13, 2012.

[5] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen, and M. Arshad, "The extended generalized Mittag-Leffler function via fractional calculus," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 1, pp. 4244–4253, 2017.

[6] H. M. Srivastava and Ž. Tomovski, "Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel," *Applied Mathematics and Computation*, vol. 211, no. 1, pp. 198–210, 2009.

[7] T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.

[8] Z. Tomovski, R. Hiller, and H. M. Srivastava, "Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler function," *Integral Transforms and Special Functions*, vol. 21, no. 11, pp. 797–814, 2011.

[9] G. H. Toader, "Some generalization of convexity," *Proceedings of the Colloquium on Approximation and Optimization. Cluj-Napoca (Romania)*, University of Cluj-Napoca, Cluj-Napoca, pp. 329–338, 1984.

[10] I. Iscan, "Hermite-Hadamard type inequalities for harmonically convex functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 6, pp. 935–942, 2014.

[11] M. Kunt and İ. İşcan, "Hermite-Hadamard-Fejér type inequalities for p-convex functions via fractional integrals, Iran," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 42, no. 4, pp. 2079–2089, 2018.

[12] L. Fejér, "Überdie Fourierreihen II," *Math Naturwiss Anz Ungar Akad Wiss*, vol. 24, pp. 369–390, 1906.

[13] B. Ahmad, A. Alsaedi, M. Kirane, and B. T. Torebek, "Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals," *Journal of Computational and Applied Mathematics*, vol. 353, pp. 120–129, 2019.

[14] M. K. Bakula and J. Pečarić, "Note on some Hadamard-type inequalties," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 3, p. 74, 2004.

[15] H. Chen and U. N. Katugampola, "Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals," *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1274–1291, 2017.

[16] F. Chen and S. Wu, "Fejér and Hermite-Hadamard type Inequalities for harmonically convex functions," *Journal of Applied Mathematics*, vol. 2014, Article ID 386806, 2014.

[17] I. Iscan, "Ostrowski type inequalities for p-convex functions," *New Trends in Mathematical Science*, vol. 4, no. 3, p. 140, 2016.

[18] I. Iscan and S. Wu, "Hemite-Hadamard type inequalities for harmonically convex functions via fractional integrals," *Applied Mathematics and Computation*, vol. 238, pp. 237–244, 2014.

[19] S. Kang, G. Abbas, G. Farid, and W. Nazeer, "A generalized Fejér-Hadamard inequality for harmonically convex functions via generalized fractional integral operator and related results," *Mathematics*, vol. 6, no. 7, p. 122, 2018.

[20] M. Kunt, I. Iscan, N. Yazi, and U. Gozutok, "On new inequalities of Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals," *Springer Plus*, vol. 5, no. 1, pp. 1–19, 2016.

[21] Y. Rao, M. Yussouf, G. Farid, J. Pečarić, and I. Tlili, "Further generalizations of Hadamard and Fejér-Hadamard inequalities and error estimations," *Advances in Difference Equations*, vol. 2020, p. 421, 2020.

[22] G. Abbas and G. Farid, "Hadamard and Fejér-Hadamard type inequalities for harmonically convex functions via generalized fractional integrals," *The Journal of Analysis*, vol. 25, no. 1, pp. 107–119, 2017.

[23] G. Farid, A. U. Rehman, and S. Mehmood, "Hadamard and Fejér-Hadamard type integral inequalities for harmonically convex functions via an extended generalized Mittag-Leffler function," *Journal of Mathematics and Computer Science*, vol. 8, no. 5, pp. 630–643, 2018.

[24] X. Qiang, G. Farid, M. Yussouf, K. A. Khan, and A. U. Rehman, "New generalized fractional versions of Hadamard and Fejér inequalities for harmonically convex functions," *Journal of Inequalities and Applications*, vol. 2020, p. 191, 2020.

[25] G. Farid and G. Abbas, "Generalizaions of some Hermite-Hadamard-Fejér type inequalities for p-convex functions via generalized fractional integrals," *Journal of Fractional Calculus and Applications*, vol. 9, no. 2, pp. 56–76, 2018.

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Research Article

Inequalities for Riemann–Liouville Fractional Integrals of Strongly (s, m)-Convex Functions

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The results of this paper provide two Hadamard-type inequalities for strongly (s, m)-convex functions via Riemann–Liouville fractional integrals and error estimations of well-known fractional Hadamard inequalities. Their special cases are given and connected with the results of some published papers.

1. Introduction

The most prominent inequality for convex functions is the well-known Hadamard inequality stated in the following.

Theorem 1 (see [1]). *Let $f: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on the interval I, where $x, y \in I$ with $x < y$. Then, the following inequality holds:*

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\mathrm{d}u \leq \frac{f(x)+f(y)}{2}. \quad (1)$$

Convex functions are extended, generalized, and refined in different ways to define new types of convex functions. For instance, s-convex, m-convex, (s, m)-convex, strongly convex, and strongly (s, m)-convex functions are extensions of convex functions. The aim of this paper is to establish integral inequalities by using the class of strongly (s, m)-convex functions. We give definitions of (s, m)-convex and strongly (s, m)-convex functions as follows.

Definition 1 (see [2]). A function $f: [0, \infty) \longrightarrow \mathbb{R}$ is called (s, m)-convex in the second sense, if the following inequality holds:

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y), \quad (2)$$

for every $x, y \in [0, \infty)$, $t \in [0, 1]$ and $[s, m) \in (0, 1] \times [0, 1)$.

Definition 2 (see [3]). A function $f: [0, \infty) \longrightarrow \mathbb{R}$ is called strongly (s, m)-convex in the second sense with modulus $C \geq 0$, if the following inequality holds:

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y) - Cmt(1-t)(y-x)^2, \quad (3)$$

for every $x, y \in [0, \infty)$, $t \in [0, 1]$ and $[s, m) \in (0, 1] \times [0, 1)$.

By setting $(s, m) = (s, 1)$, $(s, m) = (1, m)$, and $(s, m) = (1, 1)$ in (2), we get s-convex [4], m-convex [5], and convex functions, respectively, while by setting $(s, m) = (s, 1)$, $(s, m) = (1, m)$, and $(s, m) = (1, 1)$ in (3), we get strongly s-convex [6], strongly m-convex [7], and strongly convex [6] functions, respectively.

Next we give definition of Riemann–Liouville fractional integrals $J^{\alpha}_{x^+} f$ and $J^{\alpha}_{y^-} f$ which are utilized to get the desired results of this paper.

Definition 3 (see [8]). Let $f \in L_1[x, y]$. Then, Riemann–Liouville fractional integral operators of order $\alpha > 0$ are given by

$$J_{x^+}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_x^u (u-t)^{\alpha-1} f(t)\mathrm{d}t, \quad u > x,$$
$$J_{y^-}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^y (t-u)^{\alpha-1} f(t)\mathrm{d}t, \quad u < y, \tag{4}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}\mathrm{d}t$ is the gamma function and $J_{x^+}^{0} f(u) = J_{y^-}^{0} f(u) = f(u)$.

The following special functions are also involved in the findings of this paper.

Definition 4. The beta function, also referred to as first type of Euler integral, is defined by

$$\beta(\alpha, s) = \int_0^1 t^{\alpha-1}(1-t)^{s-1}\mathrm{d}t, \tag{5}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(s) > 0$.

Close association of the beta function to the gamma function is an important factor of the beta function

$$\beta(\alpha, s) = \frac{\Gamma(\alpha)\Gamma(s)}{\Gamma(\alpha+s)}. \tag{6}$$

The beta function is symmetric, i.e., $\beta(\alpha, s) = \beta(s, \alpha)$. A generalization of the beta function, called the incomplete beta function, is defined by

$$\beta(b; \alpha, s) = \int_0^b t^{\alpha-1}(1-t)^{s-1}\mathrm{d}t, \tag{7}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(s) > 0$ with $0 < b < 1$. The incomplete beta function $\beta(b; \alpha, s)$ weakens to the ordinary $\beta(\alpha, s)$ (beta function) by setting $b = 1$.

In [8], the Hadamard inequality is studied for Riemann–Liouville fractional integrals which is stated in the following theorem.

Theorem 2. *Let $f\colon [x, y] \longrightarrow \mathbb{R}$ be a positive function with $0 \le x < y$ and $f \in L_1[x, y]$. If f is convex function on $[x, y]$, then the following inequality for fractional integrals holds:*

$$f\left(\frac{x+y}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^+}^{\alpha} f\right)(y) + \left(J_{y^-}^{\alpha} f\right)(x)\right] \le \frac{f(x)+f(y)}{2}, \tag{8}$$

with $\alpha > 0$.

The inequality stated in the aforementioned theorem motivates the researchers to work in this direction by establishing other kinds of inequalities for Riemann–Liouville fractional integrals. In the past decade, several classical inequalities have been extended via different kinds of fractional integral operators. The Hadamard inequality is one of the most studied inequalities for fractional integral operators. For some recent work, we refer the readers to [3, 8–17].

This paper is organized as follows. In Section 2, two versions of the Hadamard inequality for strongly (s, m)-convex functions via Riemann–Liouville fractional integrals are given. Their connection with the well-known results is established in the form of corollaries and remarks. In Section 3, the error estimations of Hadamard inequalities for Riemann–Liouville fractional integrals are obtained by using differentiable strongly (s, m)-convex functions.

2. Main Results

Theorem 3. *Let $f \in L_1[x, y]$ be a positive function with $0 \le x < y$. If f is strongly (s, m)-convex function on $[x, my]$ with modulus $C \ge 0$, $m \ne 0, 0 < s \le 1$, then the following fractional integral inequality holds:*

$$\begin{aligned}
&2^{s-1} f\left(\frac{x+my}{2}\right) + \frac{2^{s-3}Cm\alpha}{\alpha+2}\left((x-y)^2 + \frac{2(my-(x/m))^2}{\alpha(\alpha+1)} + \frac{2(x-y)(my-(x/m))}{(\alpha+1)}\right) \\
&\le \frac{\Gamma(\alpha+1)}{2(my-x)^{\alpha}}\left[J_{x^+}^{\alpha} f(my) + m^{\alpha+1} J_{y^-}^{\alpha} f\left(\frac{x}{m}\right)\right] \le \frac{\alpha(f(x)+mf(y))}{2(\alpha+s)} \\
&+ \frac{m\alpha\beta(\alpha, s+1)\left(f(y)+mf\left(x/m^2\right)\right)}{2} - \frac{Cm\alpha\left((y-x)^2 + m\left(y-\left(x/m^2\right)\right)^2\right)}{2(\alpha+1)(\alpha+2)},
\end{aligned} \tag{9}$$

with $\alpha > 0$.

Proof. Since f is strongly (s, m)-convex function, for $u, v \in [x, y]$, we have

$$f\left(\frac{u+mv}{2}\right) \le \frac{f(u)+mf(v)}{2^s} - \frac{Cm}{4}|u-v|^2. \tag{10}$$

By setting $u = xt + m(1-t)y$ and $v = yt + (1-t)(x/m)$, we have

$$f\left(\frac{x+my}{2}\right) \le \frac{1}{2^s} f(xt + m(1-t)y) + \frac{m}{2^s} f\left(yt + (1-t)\frac{x}{m}\right) - \frac{Cm}{4}\left|t(x-y) + (1-t)\left(my - \frac{x}{m}\right)\right|^2. \tag{11}$$

By multiplying inequality (11) with $t^{\alpha-1}$ on both sides and then integrating over the interval $[0, 1]$, we get

$$f\left(\frac{x+my}{2}\right)\int_0^1 t^{\alpha-1}\mathrm{d}t \le \frac{1}{2^s}\int_0^1 f(xt+m(1-t)y)t^{\alpha-1}\mathrm{d}t + \frac{m}{2^s}\int_0^1 f\left(yt+(1-t)\frac{x}{m}\right)t^{\alpha-1}\mathrm{d}t - \frac{Cm}{4}\int_0^1 \left|t(x-y)+(1-t)\left(my-\frac{x}{m}\right)\right|^2 t^{\alpha-1}\mathrm{d}t. \tag{12}$$

By change of variables, we will get

$$\frac{1}{\alpha}f\left(\frac{x+my}{2}\right) \le \frac{\Gamma(\alpha)}{2^s(my-x)^\alpha}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{x}(my-u)^{\alpha-1}f(u)\mathrm{d}u + \frac{m^{\alpha+1}}{\Gamma(\alpha)}\cdot\int_{(x/m)}^{y}\left(v-\frac{x}{m}\right)^{\alpha-1}f(v)\mathrm{d}v\right] - \frac{Cm}{4}\left(\frac{(x-y)^2}{\alpha+2} + \frac{2(my-(x/m))^2}{\alpha(\alpha+1)(\alpha+2)} + \frac{2(x-y)(my-(x/m))}{(\alpha+1)(\alpha+2)}\right). \tag{13}$$

Further, the above inequality takes the following form:

$$2^{s-1}f\left(\frac{x+my}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(my-x)^\alpha}\left[J_{x^+}^{\alpha}f(my) + m^{\alpha+1}J_{y^-}^{\alpha}f\left(\frac{x}{m}\right)\right] - \frac{2^{s-1}Cm\alpha}{4}\left(\frac{(x-y)^2}{\alpha+2} + \frac{2(my-(x/m))^2}{\alpha(\alpha+1)(\alpha+2)} + \frac{2(x-y)(my-(x/m))}{(\alpha+1)(\alpha+2)}\right). \tag{14}$$

From the definition of strongly (s,m)-convex function with modulus C, for $t\in[0,1]$, we have the following inequality:

$$\begin{aligned} &f(tx+m(1-t)y) + mf\left(yt+(1-t)\frac{x}{m}\right) \\ &\le t^s(f(x)+mf(y)) + m(1-t)^s\left(f(y)+mf\left(\frac{x}{m^2}\right)\right) \\ &- Cmt(1-t)\left((y-x)^2 + m\left(y-\frac{x}{m^2}\right)^2\right). \end{aligned} \tag{15}$$

By multiplying inequality (15) with $t^{\alpha-1}$ on both sides and then integrating over the interval $[0,1]$, we get

$$\begin{aligned} &\int_0^1 f(tx+m(1-t)y)t^{\alpha-1}\mathrm{d}t + m\int_0^1 f\left(yt+(1-t)\frac{x}{m}\right)t^{\alpha-1}\mathrm{d}t \\ &\le (f(x)+mf(y))\int_0^1 t^{s+\alpha-1}\mathrm{d}t + m\left(f(y)+mf\left(\frac{x}{m^2}\right)\right)\int_0^1 t^{\alpha-1}(1-t)^s\mathrm{d}t \\ &- Cm\left((y-x)^2 + m\left(y-\frac{x}{m^2}\right)^2\right)\int_0^1 t^{\alpha}(1-t)\mathrm{d}t. \end{aligned} \tag{16}$$

By change of variables, we will get

$$\frac{\Gamma(\alpha)}{(my-x)^{\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{x}(my-u)^{\alpha-1}f(u)du+\frac{m^{\alpha+1}}{\Gamma(\alpha)}\int_{(x/m)}^{y}\left(v-\frac{x}{m}\right)^{\alpha-1}f(v)dv\right]$$
$$\leq\frac{f(x)+mf(y)}{\alpha+s}+m\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\beta(s+1,\alpha)-\frac{Cm\left((y-x)^{2}+m\left(y-\left(x/m^{2}\right)\right)^{2}\right)}{(\alpha+1)(\alpha+2)}. \tag{17}$$

Further, the above inequality takes the following form:

$$\frac{\Gamma(\alpha+1)}{2(my-x)^{\alpha}}\left[J_{x^{+}}^{\alpha}f(my)+m^{\alpha+1}J_{y^{-}}^{\alpha}f\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\alpha(f(x)+mf(y))}{2(\alpha+s)}+\frac{m\alpha}{2}\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\beta(s+1,\alpha)$$
$$-\frac{Cm\alpha\left((y-x)^{2}+m\left(y-\left(x/m^{2}\right)\right)^{2}\right)}{2(\alpha+1)(\alpha+2)}. \tag{18}$$

From inequalities (14) and (18), one can get inequality (9). □

Remark 1

(i) For $s=1$ in (9), we have the result for strongly m-convex function [18].

(ii) For $m=1$ and $s=1$ in (9), we have the result for strongly convex function.

(iii) For $m=1$, $s=1$, and $C=0$, we get [[16], Theorem 2]

(iv) For $m=1$, $s=1$, $\alpha=1$, and $C=0$, we get the classical Hadamard inequality.

(v) For $m=1$ and $C=0$, we get [[17], Theorem 3].

Corollary 1. *For $m=1$, we have the result for Riemann–Liouville fractional integrals of strongly s-convex functions:*

$$2^{s-1}f\left(\frac{x+y}{2}\right)+\frac{2^{s-1}C\alpha(y-x)^{2}\left(\alpha^{2}-\alpha+2\right)}{4(\alpha+1)(\alpha+2)}$$
$$\leq\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J_{x^{+}}^{\alpha}f(y)+J_{y^{-}}^{\alpha}f(x)\right]$$
$$\leq\frac{f(x)+f(y)}{2}\left(\frac{\alpha}{\alpha+s}+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)-\frac{C\alpha(y-x)^{2}}{(\alpha+1)(\alpha+2)}. \tag{19}$$

Corollary 2. *For $\alpha=1$ and $m=1$, the following inequality holds for strongly s-convex function:*

$$2^{s-1}f\left(\frac{x+y}{2}\right)+\frac{2^{s-1}C(y-x)^{2}}{12}$$
$$\leq\frac{1}{y-x}\int_{x}^{y}f(u)du\leq\frac{f(x)+f(y)}{s+1}-\frac{C(y-x)^{2}}{6}. \tag{20}$$

In the next theorem, we give another version of the Hadamard inequality.

Theorem 4. *Under the assumptions of Theorem 3, the following fractional integral inequality holds:*

$$2^{s-1}f\left(\frac{x+my}{2}\right)$$
$$+\frac{Cm\alpha}{2^{4-s}}\left[\frac{(x-y)^{2}}{2(\alpha+2)}+\frac{(my-(x/m))^{2}\left(\alpha^{2}+5\alpha+8\right)}{2\alpha(\alpha+1)(\alpha+2)}+\frac{(x-y)(my-(x/m))(\alpha+3)}{(\alpha+1)(\alpha+2)}\right]$$
$$\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J_{((x+my)/2)^{+}}^{\alpha}f\right)(ym)+m^{\alpha+1}\left(J_{((x+ym)/2m)^{-}}^{\alpha}f\right)\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\alpha(f(x)+mf(y))}{2^{s+1}(\alpha+s)}+2^{\alpha-1}m\alpha\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\beta\left(\frac{1}{2};s+1,\alpha\right)$$
$$-\frac{Cm\alpha\left((y-x)^{2}+m\left(y-\left(x/m^{2}\right)\right)^{2}\right)(\alpha+3)}{8(\alpha+1)(\alpha+2)}, \tag{21}$$

with $\alpha > 0$.

Proof. Let $t \in [0,1]$. Using strong (s,m)-convexity of function f for $u = x(t/2) + m((2-t)/2)y$ and $v = ((2-t)/2)(x/m) + y(t/2)$ in inequality (10), we have

$$f\left(\frac{x+my}{2}\right) \le \frac{1}{2^s} f\left(x\frac{t}{2} + m\left(\frac{2-t}{2}\right)y\right) + \frac{m}{2^s} f\left(\left(\frac{2-t}{2}\right)\frac{x}{m} + y\frac{t}{2}\right) - \frac{Cm}{4}\left|\frac{t}{2}(x-y) + \frac{2-t}{2}\left(my - \frac{x}{m}\right)\right|^2. \tag{22}$$

By multiplying (22) with $t^{\alpha-1}$ on both sides and making integration over $[0,1]$, we get

$$f\left(\frac{x+my}{2}\right)\int_0^1 t^{\alpha-1}\mathrm{d}t \le \frac{1}{2^s}\int_0^1 f\left(x\frac{t}{2} + m\left(\frac{2-t}{2}\right)y\right)t^{\alpha-1}\mathrm{d}t + \frac{m}{2^s}\int_0^1 f\left(\left(\frac{2-t}{2}\right)\frac{x}{m} + y\frac{t}{2}\right)t^{\alpha-1}\mathrm{d}t - \frac{Cm}{4}\int_0^1\left|\frac{t}{2}(x-y) + \frac{2-t}{2}\left(my - \frac{x}{m}\right)\right|^2 t^{\alpha-1}\mathrm{d}t. \tag{23}$$

By using change of variables and computing the last integral, from (23), we get

$$\frac{2^s}{\alpha} f\left(\frac{x+my}{2}\right) \le \frac{2^\alpha\Gamma(\alpha)}{(my-x)^\alpha}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{((x+my)/2)}(my-u)^{\alpha-1}f(u)\mathrm{d}u + \frac{m^{\alpha+1}}{\Gamma(\alpha)}\int_{(x/m)}^{((ym+x)/2m)}\left(v - \frac{x}{m}\right)^{\alpha-1}f(v)\mathrm{d}v\right] - \frac{2^s Cm}{4}\left[\frac{(x-y)^2}{4(\alpha+2)} + \frac{(my-(x/m))^2(\alpha^2+5\alpha+8)}{4\alpha(\alpha+1)(\alpha+2)} + \frac{(x-y)(my-(x/m))(\alpha+3)}{2(\alpha+1)(\alpha+2)}\right]. \tag{24}$$

Further, it takes the following form:

$$2^{s-1} f\left(\frac{x+my}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^\alpha}\left[\left(J^\alpha_{((x+my)/2)^+}f\right)(ym) + m^{\alpha+1}\left(J^\alpha_{((ym+x)/2m)^-}f\right)\left(\frac{x}{m}\right)\right] - \frac{2^{s-1}Cm\alpha}{4}\left[\frac{(x-y)^2}{4(\alpha+2)} + \frac{(my-(x/m))^2(\alpha^2+5\alpha+8)}{4\alpha(\alpha+1)(\alpha+2)} + \frac{(x-y)(my-(x/m))(\alpha+3)}{2(\alpha+1)(\alpha+2)}\right]. \tag{25}$$

The first inequality of (21) can be seen in (25). Now we prove the second inequality of (21). Since f is strongly (s,m)-convex function and $t \in [0,1]$, we have the following inequality:

$$f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)+mf\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)\leq\left(\frac{t}{2}\right)^{s}(f(x)+mf(b))$$
$$+m\left(\frac{2-t}{2}\right)^{s}\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)-\frac{Cmt(2-t)}{4}\left[(y-x)^{2}+m\left(y-\frac{x}{m^{2}}\right)^{2}\right]. \tag{26}$$

By multiplying inequality (26) with $t^{\alpha-1}$ on both sides and making integration over $[0,1]$, we get

$$\int_0^1 f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)t^{\alpha-1}\mathrm{d}t+m\int_0^1 f\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)t^{\alpha-1}\mathrm{d}t$$
$$\leq\frac{1}{2^{s}}(f(x)+mf(y))\int_0^1 t^{s+\alpha-1}\mathrm{d}t+\frac{m}{2^{s}}\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\int_0^1(2-t)^{s}t^{\alpha-1}\mathrm{d}t \tag{27}$$
$$-\frac{Cm}{4}\left[(y-x)^{2}+m\left(y-\frac{x}{m^{2}}\right)\right]\int_0^1 t^{\alpha}(2-t)\mathrm{d}t.$$

By using change of variables and computing the last integral, from (27), we get

$$\frac{2^{\alpha}\Gamma(\alpha)}{(my-x)^{\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{((x+my)/2)}(my-u)^{\alpha-1}f(u)\mathrm{d}u+\frac{m^{\alpha+1}}{\Gamma(\alpha)}\int_{(x/m)}^{((my+x)/2m)}\left(v-\frac{x}{m}\right)^{\alpha-1}f(v)\mathrm{d}v\right]$$
$$\leq\frac{f(x)+mf(y)}{2^{s}(\alpha+s)}+2^{\alpha}m\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\beta\left(\frac{1}{2}\colon s+1,\alpha\right) \tag{28}$$
$$-\frac{Cm\left((y-x)^{2}+m\left(y-\left(x/m^{2}\right)\right)^{2}\right)(\alpha+3)}{4(\alpha+1)(\alpha+2)}.$$

Further, it takes the following form:

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}}f\right)(ym)+m^{\alpha+1}\left(J^{\alpha}_{((x+ym)/2m)^{-}}f\right)\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\alpha(f(x)+mf(y))}{2^{s+1}(\alpha+s)}+2^{\alpha-1}m\alpha\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\beta\left(\frac{1}{2}\colon s+1,\alpha\right) \tag{29}$$
$$-\frac{Cm\alpha\left((y-x)^{2}+m\left(y-\left(x/m^{2}\right)\right)^{2}\right)(\alpha+3)}{8(\alpha+1)(\alpha+2)}.$$

From inequalities (25) and (29), we have inequality (21). □

Remark 2

(i) For $s = 1$ in (21), we get the result for strongly m-convex function [18].

(ii) For $m = 1, s = 1$, and $C = 0$, we get [[16], Theorem 2]

(iii) For $m = 1$, $s = 1$, $\alpha = 1$, and $C = 0$, we get the classical Hadamard inequality.

Corollary 3. *For $m = 1$ and $s = 1$ in (21), we have the result for Riemann–Liouville fractional integrals of strongly convex function:*

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) + \frac{C(y-x)^2}{2(\alpha+1)(\alpha+2)} \\ & \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^+} f\right)(y) + \left(J^{\alpha}_{((x+y)/2)^-} f\right)(x)\right] \\ & \leq \frac{f(x)+f(y)}{2} - \frac{C\alpha(y-x)^2(\alpha+3)}{4(\alpha+1)(\alpha+2)}. \end{aligned} \tag{30}$$

Corollary 4. *For $m = 1$ in (21), we get the result for Riemann–Liouville fractional integrals of strongly s-convex function:*

$$\begin{aligned} & 2^{s-1} f\left(\frac{x+y}{2}\right) \\ & + \frac{2^s C(x-y)^2}{4(\alpha+1)(\alpha+2)} \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^+} f\right)(y) + \left(J^{\alpha}_{((x+y)/2)^-} f\right)(x)\right] \\ & \leq \alpha(f(x)+f(y))\left(\frac{1}{2^{s+1}(\alpha+s)} + 2^{\alpha-1}\beta\left(\frac{1}{2}; s+1, \alpha\right)\right) - \frac{C\alpha(y-x)^2(\alpha+3)}{4(\alpha+1)(\alpha+2)}. \end{aligned} \tag{31}$$

Corollary 5. *For $m = 1$ and $C = 0$ in (21), we get the result for Riemann–Liouville fractional integrals of s-convex function:*

$$\begin{aligned} 2^{s-1} f\left(\frac{x+y}{2}\right) & \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^+} f\right)(y) + \left(J^{\alpha}_{((x+y)/2)^-} f\right)(x)\right] \\ & \leq \alpha(f(x)+f(y))\left(\frac{1}{2^{s+1}(\alpha+s)} + 2^{\alpha-1}\beta\left(\frac{1}{2}; s+1, \alpha\right)\right). \end{aligned} \tag{32}$$

Corollary 6. *For $m = 1$ and $\alpha = 1$ in (1), we have the Hadamard inequality for strongly s-convex function:*

$$\begin{aligned} & 2^{s-1} f\left(\frac{x+y}{2}\right) + \frac{2^s C(x-y)^2}{24} \leq \frac{1}{y-x}\int_x^y f(u)\mathrm{d}u \\ & \leq \frac{f(x)+f(y)}{s+1} - \frac{C(y-x)^2}{6}. \end{aligned} \tag{33}$$

3. Error Estimations of Riemann–Liouville Fractional Integral Inequalities

The following two lemmas are very useful to obtain the results of this section.

Lemma 1 (see [8]). *Let $f\colon [x, y] \longrightarrow \mathbb{R}$ be a differentiable mapping on (x, y) with $x < y$. If $f' \in Ł[x, y]$, then the following fractional integral equality holds:*

$$\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha}f\right)(y)+\left(J_{y^{-}}^{\alpha}f\right)(x)\right]$$
$$=\frac{y-x}{2}\int_{0}^{1}\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(tx+(1-t)y)\mathrm{d}t. \tag{34}$$

Lemma 2 (see [10]). *Let $f\colon [x,y]\longrightarrow\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $f'\in[x,my], m\in(0,1]$, then the following equality for fractional integrals holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J_{((x+my)/2)^{+}}^{\alpha}f\right)(my)+m^{\alpha+1}\left(J_{((x+my)/2m)^{-}}^{\alpha}f\right)\left(\frac{x}{m}\right)\right]$$
$$-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)+mf\left(\frac{x+my}{2m}\right)\right]|$$
$$=\frac{mb-a}{4}\left[\int_{0}^{1}t^{\alpha}f'\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)\mathrm{d}t+\int_{0}^{1}t^{\alpha}f'\left(y\frac{t}{2}+\left(\frac{2-t}{2}\right)\frac{x}{m}\right)\mathrm{d}t\right]. \tag{35}$$

Theorem 5. *Let $f\colon [x,y]\longrightarrow\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $|f'|$ is a strongly (s,m)-convex function on $[x,my]$ with modulus $C\geq 0$, $m\neq 0$, and $0<s\leq 1$, then the following fractional integral inequality holds:*

$$\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha}f\right)(y)+\left(J_{y^{-}}^{\alpha}f\right)(x)\right]\right|$$
$$\leq\frac{y-x}{2}\left[\left|f'(x)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)\right.$$
$$+m\left|f'\left(\frac{y}{m}\right)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)$$
$$\left.-\frac{2Cm((y/m)-x)^{2}}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right], \tag{36}$$

with $\alpha>0$.

Proof. Since $|f'|$ is strongly (s,m)-convex function on $[x,y]$, for $t\in[0,1]$, we have

$$\left|f'(tx+(1-t)y)\right|\leq t^{s}\left|f'(x)\right|+m(1-t)^{s}\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^{2}. \tag{37}$$

By using Lemma 1 and (37), we have

$$\begin{aligned}
&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha}f\right)(y)+\left(J_{y^{-}}^{\alpha}f\right)(x)\right]\right|\\
&\leq\frac{y-x}{2}\int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(tx+m(1-t)\frac{y}{m}\right)\right|dt\\
&\leq\frac{y-x}{2}\int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left(t^{s}\left|f'(x)\right|+m(1-t)^{s}\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^{2}\right)dt\\
&\leq\frac{y-x}{2}\left[\int_{0}^{(1/2)}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(t^{s}\left|f'(x)\right|+m(1-t)^{s}\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^{2}\right)dt\right.\\
&\left.+\int_{(1/2)}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left(t^{s}\left|f'(x)\right|+m(1-t)^{s}\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^{2}\right)dt\right]\\
&\leq\frac{y-x}{2}\left[\left|f'(x)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)+m\left|f'\left(\frac{y}{m}\right)\right|\right.\\
&\left.\cdot\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)-\frac{2Cm((y/m)-x)^{2}}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right].
\end{aligned} \tag{38}$$

After simplifying the last inequality of (38), we get (36). □

Remark 3

(i) By setting $C=0$ in inequality (36), one can get result for (s,m)-convex function.

(ii) By setting $s=1$ in inequality (36), we get [[18], Theorem 8].

Corollary 7. *By taking $m=1$ in (36), we have the result for Riemann–Liouville fractional integrals of strongly s-convex function:*

$$\begin{aligned}
&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha}f\right)(y)+\left(J_{y^{-}}^{\alpha}f\right)(x)\right]\right|\\
&\leq\frac{y-x}{2}\left[\left|f'(x)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)\right.\\
&+\left|f'(y)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)\\
&\left.-\frac{2C(y-x)^{2}}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right].
\end{aligned} \tag{39}$$

Corollary 8. *By taking $m=1$ and $s=1$ in inequality (36), we have the result for Riemann–Liouville fractional integrals of strongly convex function:*

$$\begin{aligned}
&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha}f\right)(y)+\left(J_{y^{-}}^{\alpha}f\right)(x)\right]\right|\\
&\leq\frac{y-x}{2}\left[\frac{1-(1/2)^{\alpha}}{(\alpha+1)}\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{2C(y-x)^{2}}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right].
\end{aligned} \tag{40}$$

Corollary 9. *By taking $m = s = 1$ and $\alpha = 1$ in inequality (36), we get the following inequality:*

$$\left|\frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_x^y f(u)\mathrm{d}u\right| \leq \frac{y-x}{8}\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{C(y-x)^3}{32}. \tag{41}$$

Inequality (41) provides the refinement of [[19], Theorem 2.2].

Theorem 6. *Let $f\colon [x,y] \longrightarrow \mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $|f'|^q$ is strongly (s,m)-convex on $[x,my]$ with modulus $C \geq 0$, $(s,m) \in (0,1]^2$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^+}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^-}f\right)\left(\frac{x}{m}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)+mf\left(\frac{x+my}{2m}\right)\right]\right| \leq \frac{my-x}{4(\alpha+1)^{(1/p)}}\\
&\cdot\left[\left(\frac{|f'(x)|^q}{2^s(\alpha+s+1)}+2^{\alpha+1}m|f'(y)|^q\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{Cm(y-x)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right.\\
&\left.+\left(\frac{|f'(y)|^q}{2^s(\alpha+s+1)}+2^{\alpha+1}m\left|f'\left(\frac{x}{m^2}\right)\right|^q\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{Cm((x/m^2)-y)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right].
\end{aligned} \tag{42}$$

Proof. By applying Lemma 2 and strong (s,m)-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^+}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^-}f\right)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.\\
&\left.\left.+mf\left(\frac{x+my}{2m}\right)\right]\right| \leq \frac{my-x}{4}\left[\int_0^1\left|t^{\alpha}f'\left(\frac{t}{2}x+m\left(\frac{2-t}{2}\right)y\right)\right|\mathrm{d}t\right.\\
&\left.+\int_0^1\left|t^{\alpha}f'\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+\frac{t}{2}y\right)\right|\mathrm{d}t\right] \leq \frac{my-x}{4}\left[\left(\frac{|f'(x)|+|f'(y)|}{2^s}\right)\int_0^1 t^{\alpha+s}\mathrm{d}t\right.\\
&+\frac{m}{2^s}\left(|f'(y)|+\left|f'\left(\frac{x}{m^2}\right)\right|\right)\int_0^1 t^{\alpha}(2-t)^s\mathrm{d}t-\frac{Cm}{4}\left((y-x)^2+\left(\frac{x}{m^2}-y\right)^2\right)\\
&\left.\cdot\int_0^1 t^{\alpha+1}(2-t)\mathrm{d}t\right] = \frac{my-x}{4}\left[\left(\frac{|f'(x)|+|f'(y)|}{2^s(\alpha+s+1)}\right)+2^{\alpha+1}m\left(|f'(y)|+\left|f'\left(\frac{x}{m^2}\right)\right|\right)\right.\\
&\left.\cdot\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{Cm(\alpha+4)}{4(\alpha+2)(\alpha+3)}\left((y-x)^2+\left(\frac{x}{m^2}-y\right)^2\right)\right].
\end{aligned} \tag{43}$$

Now, for strong (s, m)-convexity of $|f'|^q$, $q > 1$, using power mean inequality, we get

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^+}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^-}f\right)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.\\
&\left.\left.+mf\left(\frac{x+my}{2m}\right)\right]\right|\leq\frac{my-x}{4}\left(\int_0^1 t^{\alpha}\mathrm{d}t\right)^{1-(1/q)}\left[\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}x+m\left(\frac{2-t}{2}\right)y\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right.\\
&\left.+\left(\int_0^1 t^{\alpha}\left|f'\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+\frac{t}{2}y\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right]\leq\frac{my-x}{4(\alpha+1)^{(1/p)}}\\
&\left(\frac{|f(x)|^q}{2^s}\int_0^1 t^{\alpha+s}\mathrm{d}t+\frac{m|f'(y)|^q}{2^s}\int_0^1 t^{\alpha}(2-t)^s\mathrm{d}t-\frac{Cm(y-x)^2}{4}\int_0^1 t^{\alpha+1}(2-t)\mathrm{d}t\right)^{(1/q)}\\
&+\left(\frac{|f'(y)|^q}{2^s}\int_0^1 t^{\alpha+s}\mathrm{d}t+\frac{m}{2^s}\left|f'\left(\frac{x}{m^2}\right)\right|^q\int_0^1 t^{\alpha}(2-t)^s\mathrm{d}t-\frac{Cm\left((x/m^2)-y\right)^2}{4}\int_0^1 t^{\alpha+1}(2-t)\mathrm{d}t\right)^{(1/q)}\\
&\leq\frac{my-x}{4(\alpha+1)^{(1/p)}}\left[\left(\frac{|f'(x)|^q}{2^s(\alpha+s+1)}+2^{\alpha+1}m|f'(y)|^q\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{Cm(y-x)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right.\\
&\left.+\left(\frac{|f'(y)|^q}{2^s(\alpha+s+1)}+2^{\alpha+1}m\left|f'\left(\frac{x}{m^2}\right)\right|^q\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{Cm\left((x/m^2)-y\right)^2(\beta+4)}{4(\beta+2)(\beta+3)}\right)^{(1/q)}\right].
\end{aligned}\tag{44}$$

Hence, we have inequality (42). □

Remark 4

(i) For $s = 1$ in inequality (42), we have the result for strongly m-convex function [18].

(ii) For $s = 1, m = 1$, and $C = 0$ in inequality (42), we get [[16], Theorem 5].

(iii) For $s = 1, m = 1, C = 0$, and $\alpha = 1$ in inequality (42), we get the inequality proved by Kirmaci in [20].

Corollary 10. *For $s = 1$ and $m = 1$ in inequality (42), we have the result for Riemann–Liouville fractional integrals of strongly convex function:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^+}f\right)(y)+\left(J^{\alpha}_{((x+y)/2)^-}f\right)(x)\right]-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4(\alpha+1)(2\alpha+4)^{(1/q)}}\\
&\cdot\left[\left(|f'(x)|^q(\alpha+1)+|f'(y)|^q(\alpha+3)-\frac{C(y-x)^2(\alpha+1)(\alpha+4)}{2(\alpha+3)}\right)^{(1/q)}\right.\\
&\left.+\left(|f'(y)|^q(\alpha+1)+|f'(x)|^q(\alpha+3)-\frac{C(y-x)^2(\alpha+1)(\alpha+4)}{2(\alpha+3)}\right)^{(1/q)}\right].
\end{aligned}\tag{45}$$

Corollary 11. *For $m = 1$ in inequality (42), we have the result for Riemann–Liouville fractional integrals of strongly s-convex function:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^{+}}f\right)(y)+\left(J^{\alpha}_{(x+y)/2^{-}}f\right)(x)\right]-f\left(\frac{x+y}{2}\right)\right|\\&\leq\frac{y-x}{4(\alpha+1)^{(1/p)}}\left[\left(\frac{|f'(x)|^{q}}{2^{s}(\alpha+s+1)}+2^{\alpha+1}|f'(y)|^{q}\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{C(y-x)^{2}(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right.\\&\left.+\left(\frac{|f''(y)|^{q}}{2^{s}(\alpha+s+1)}+2^{\alpha+1}|f'(x)|^{q}\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{C(y-x)^{2}(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right].\end{aligned}\tag{46}$$

Theorem 7. *Let $f\colon [x,y]\longrightarrow\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $|f'|^{q}$ is strongly (s,m)-convex function on $[x,my]$ with modulus $C\geq 0$, $(s,m)\in(0,1]^{2}$ for $q>1$, then the following fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^{-}}f\right)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.\\&\left.\left.+mf\left(\frac{x+my}{2m}\right)\right|\leq\frac{(my-x)\left(2^{s}(s+1)\right)^{(1/p)-1}}{4(\alpha p+1)^{(1/p)}}\right[\left(|f'(x)|^{q}+m\left(2^{s+1}-1\right)|f'(y)|^{q}\right.\\&\left.-\frac{2^{s}Cm(s+1)(y-x)^{2}}{6}\right)^{(1/q)}+\left(m\left(2^{s+1}-1\right)\left|f'\left(\frac{x}{m^{2}}\right)\right|^{q}+|f'(y)|^{q}\right.\\&\left.\left.-\frac{2^{s}Cm(s+1)\left((x/m^{2})-y\right)^{2}}{6}\right)^{(1/q)}\right]\leq\frac{(my-x)\left(2^{s}(s+1)\right)^{(1/p)-1}}{4(\alpha p+1)^{(1/p)}}\left[|f'(x)|+|f'(y)|\right.\\&\left.+m\left(2^{s+1}-1\right)\left(\left|f'\left(\frac{x}{m^{2}}\right)\right|+|f'(y)|\right)-\frac{2^{s}Cm(s+1)}{6}\left((y-x)^{2}+\left(\frac{x}{m^{2}}-y\right)^{2}\right)\right],\end{aligned}\tag{47}$$

where $\alpha>0$.

Proof. By applying Lemma 2 and then using Hölder inequality and strong (s,m)-convexity of $|f'|^{q}$, we get

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^{-}}f\right)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.\\&\left.\left.+mf\left(\frac{x+my}{2m}\right)\right]\right|\leq\frac{my-x}{4}\left(\int_{0}^{1}t^{p\alpha}\mathrm{d}t\right)^{1/p}\left[\left(\int_{0}^{1}\left|f'\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)\right|^{q}\mathrm{d}t\right)^{1/q}\right.\\&\left.+\left(\int_{0}^{1}\left|f'\left(y\frac{t}{2}+\left(\frac{2-t}{2}\right)\frac{x}{m}\right)\right|^{q}\mathrm{d}t\right)^{1/q}\right]\leq\frac{my-x}{4}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left[\left(|f'(x)|^{q}\int_{0}^{1}\left(\frac{t}{2}\right)^{s}\mathrm{d}t\right.\right.\\&+m|f'(y)|^{q}\int_{0}^{1}\left(\frac{2-t}{2}\right)^{s}\mathrm{d}t-\frac{Cm(y-x)^{2}}{4}\int_{0}^{1}t(2-t)\mathrm{d}t\right)^{1/q}+\left(|f'(y)|^{q}\int_{0}^{1}\left(\frac{t}{2}\right)^{s}\mathrm{d}t\right.\end{aligned}$$

$$+m\left|f'\left(\frac{x}{m^2}\right)\right|^q\int_0^1\left(\frac{2-t}{2}\right)^s dt-\frac{Cm((x/m^2)-y)^2}{4}\int_0^1 t(2-t)dt\Bigg)^{1/q}\Bigg]=\frac{(my-x)(s+1)^{(1/p)-1}}{2^{2-s((1/p)-1)}(\alpha p+1)^{(1/p)}}$$
$$\Bigg[\left(\left|f'(x)\right|^q+m\left|f'(y)\right|^q\left(2^{s+1}-1\right)-\frac{2^sCm(s+1)(y-x)^2}{6}\right)^{1/q}+\Bigg(m\left|f'\left(\frac{x}{m^2}\right)\right|^q\left(2^{s+1}-1\right)$$
$$+\left|f'(y)\right|^q-\frac{2^sCm(s+1)((x/m^2)-y)^2}{6}\Bigg)^{1/q}\Bigg]\leq\frac{(my-x)\left(2^s(s+1)\right)^{(1/p)-1}}{4(\alpha p+1)^{(1/p)}}\Big[\left|f'(x)\right|+\left|f'(y)\right|$$
$$+m\left(2^{s+1}-1\right)\left(\left|f'(y)\right|+\left|f'\left(\frac{x}{m^2}\right)\right|\right)-\frac{2^sCm(s+1)}{6}\left((y-x)^2+\left(\frac{x}{m^2}-y\right)^2\right)\Big]. \tag{48}$$

We have used $A^q + B^q \leq (A+B)^q$, for $A \geq 0, B \geq 0$. This completes the proof. □

Remark 5

(i) For $s = 1$ in inequality (47), we get [[18], Theorem 10].

(ii) For $s = 1$ and $C = 0$ in inequality (47), we get [[10], Theorem 2.7].

(iii) For $s = 1, m = 1$, and $C = 0$ in inequality (47), we get [[16], Theorem 6].

Corollary 12. *For $\alpha = 1$ and $m = 1$, we have the result for s-convex function:*

$$\left|\frac{1}{y-x}\int_x^y f(u)du-f\left(\frac{x+y}{2}\right)\right|\leq\frac{(y-x)\left(2^s(s+1)\right)^{(1/p)-1}}{4(p+1)^{(1/p)}}$$
$$\cdot\Bigg[\left(\left|f'(x)\right|^q+\left(2^{s+1}-1\right)\left|f'(y)\right|^q-\frac{2^sC(s+1)(y-x)^2}{6}\right)^{1/q}$$
$$+\left(\left(2^{s+1}-1\right)\left|f'(x)\right|^q+\left|f'(y)\right|^q-\frac{2^sC(s+1)(y-x)^2}{6}\right)^{1/q}\Bigg]$$
$$\leq\frac{(y-x)\left(2^s(s+1)\right)^{(1/p)-1}}{4(p+1)^{(1/p)}}\left[2^{s+1}\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{2^sC(s+1)}{3}(y-x)^2\right]. \tag{49}$$

Corollary 13. *For $\alpha = 1$ and $m = q = 1$, we have*

$$\left|\frac{1}{y-x}\int_x^y f(u)du-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4(s+1)}\left[2\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{C(s+1)}{3}(y-x)^2\right]. \tag{50}$$

Corollary 14. *For $\alpha = 1$ and $m = q = s = 1$, we have*

$$\left|\frac{1}{y-x}\int_x^y f(u)du-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4}\left[\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{C}{3}(y-x)^2\right]. \tag{51}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs*, Victoria University, Footscray, Australia, 2000.

[2] J. Park, "Some Hadamard's type inequalities for co-ordinated (s, m)-convex mappings in the second sense," *Far East Journal of Mathematical Sciences*, vol. 51, pp. 205–216, 2011.

[3] M. Bracamonte, J. Gimenez, and M. Vivas-Cortez, "Hermite-hadamard-fejer type inequalities for strongly (s, m)-Convex functions with modulus c, in second sense$(s, m)C$-convex functions with modulus in the second sense," *Applied*

Mathematics & Information Sciences, vol. 10, no. 6, pp. 2045–2053, 2016.
[4] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Mathematicae*, vol. 48, pp. 100–111, 1994.
[5] G. Toader, "Some generalizations of the convexity," in *Proceedings of the Colloquium on Approximation and Optimization*, pp. 329–338, University Cluj-Napoca, Trento, Italy, October 1985.
[6] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Mathematics Doklady*, vol. 7, pp. 72–75, 1966.
[7] T. Lara, N. Merentes, R. Quintero, and E. Rosales, "On strongly m-convex functions," *Mathematica Aeterna*, vol. 5, no. 3, pp. 521–535, 2015.
[8] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.
[9] S. I. Butt, A. O. Akdemir, J. Nasir, and F. Jarad, "Some Hermite-Jensen-Mercer like inequalities for convex functions through a certain generalized fractional integrals and related results," *Miskolc Mathematical Notes*, vol. 21, no. 2, pp. 689–715, 2020.
[10] G. Farid, A. U. Rehman, and B. Tariq, "On Hadamard-type inequalities for -convex functions via Riemann-Liouville fractional integrals," *Studia Universitatis Babes-Bolyai Mathematica*, vol. 62, no. 2, pp. 141–150, 2017.
[11] M. Gurbuz and M. E. Ozdemir, "On some inequalities for product of different kinds of convex functions," *Turkish Journal of Science*, vol. 5, no. 1, pp. 23–27, 2020.
[12] S. M. Kang, G. Abbas, G. Farid, and W. Nazeer, "A generalized Fejér–Hadamard inequality for harmonically convex functions via generalized fractional integral operator and related results," *Mathematics*, vol. 6, no. 7, p. 122, 2018.
[13] S. M. Kang, G. Farid, W. Nazeer, and S. Naqvi, "A version of the Hadamard inequality for Caputo fractional derivatives and related results," *Journal of Computational Analysis and Applications*, vol. 27, no. 6, pp. 962–972, 2019.
[14] S. Mehmood and G. Farid, "Fractional integrals inequalities for exponentially m-convex functions," *Open Journal of Mathematical Sciences*, vol. 4, pp. 78–85, 2020.
[15] X. Qiang, G. Farid, J. Pečarić, and S. B. Akbar, "Generalized fractional integral inequalities for exponentially (s, m)-convex functions," *Journal of Inequalities and Applications*, vol. 70, p. 2020, 2020.
[16] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2017.
[17] E. Set, M. Z. Sarikaya, M. E. Ozdemir, and H. Yildirim, "The Hermite-Hadamarad's inequality for some convex functions via fractional integrals and related results," *JAMSI*, vol. 10, no. 2, 2014.
[18] Y. C. Kwun, G. Farid, S. B. Akbar, and S. M. Kang, "Riemann-Liouville fractional version of Hadamard inequality for strongly m-convex functions," submitted.
[19] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.
[20] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.

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Research Article

(p, q)-Extended Struve Function: Fractional Integrations and Application to Fractional Kinetic Equations

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In this paper, the generalized fractional integral operators involving Appell's function $F_3(\cdot)$ in the kernel due to Marichev–Saigo–Maeda are applied to the (p, q)-extended Struve function. The results are stated in terms of Hadamard product of the Fox–Wright function ${}_r\psi_s(z)$ and the (p, q)-extended Gauss hypergeometric function. A few of the special cases (Saigo integral operators) of our key findings are also reported in the corollaries. In addition, the solutions of a generalized fractional kinetic equation employing the concept of Laplace transform are also obtained and examined as an implementation of the (p, q)-extended Struve function. Technique and findings can be implemented and applied to a number of similar fractional problems in applied mathematics and physics.

1. Introduction

The Struve functions are interesting special functions that also provide solutions to a variety of issues formulated in terms of discrete, integral, and differential equations of fractional order; thus, many authors have recently become interested in the domain of fractional calculus and its implementations. Therefore, an extremely large number of authors (for details, see [1–7]) have also researched, in detail, the features, implementations, and numerous extensions of different fractional calculus operators. The research monographs by Miller and Ross [8] can be referred to for comprehensive overview of fractional calculus operators (FCOs) together with their characteristics and potential applications. The (p, q)-variant (when $p = q$, p-variant) associated with a set of similar higher transcendental hypergeometric style special functions (see [9–13]) has recently been investigated by several authors. In specific, Maŝireviĉ et al. [14] introduced and analysed the (p, q)-extended Struve function $H_{\mu,p,q}(z)$ of the first kind of order δ with $\Re(\delta) > (-1/2)$ and $\min\{p, q\} \geq 0$ when $p = q = 0$ in this manner:

$$H_{\delta,p,q}(z) = \frac{2(z/2)^{\delta+1}}{\sqrt{\pi}\Gamma(\delta+(1/2))} \sum_{k=0}^{\infty} (-1)^k \mathfrak{B}\left(k+1, \delta+\frac{1}{2}; p, q\right) \frac{z^{2k}}{(2k+1)!}, \tag{1}$$

$$= \frac{z^{\delta+1}}{2^{\delta}\Gamma(\delta+(3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1, \delta+(1/2); p, q)}{(3/2)_k \mathfrak{B}(1, \delta+(1/2))k!} \left(\frac{-z^2}{4}\right)^k. \tag{2}$$

Choi et al. [15] introduced the (p,q)-extended beta function as

$$\mathfrak{B}(\varsigma,\vartheta\colon p,q)=\int_0^1 t^{\varsigma-1}(1-t)^{\vartheta-1}e^{-((p/t)+(q/1-t))}\mathrm{d}t,\quad (\min\{\Re(\varsigma),\Re(\vartheta)\}>0;\ \min\{\Re(p),\Re(q)\}\geq 0). \tag{3}$$

The more details and generalized form of the definitions (3) are considered in [16]. It is clear that the case $p=0=q$ automatically reduces the classical Struve function $H_\delta(z)$ of the first kind (see, e.g., [17] p. 328, equation (2)):

$$H_\delta(z)=\sum_{k=0}^{\infty}(-1)^k\frac{(z/2)^{2k+\delta+1}}{\Gamma(\delta+(3/2))\Gamma(\delta+k+(3/2))}. \tag{4}$$

The Struve function is widely studied in the reference to properties and applications in several papers (see details [18–22]).

FCO involving different special functions have established major significance and requirements in the simulation of related structures in diverse domain of engineering and science, such as quantum mechanics and turbulence, particle physics, nonlinear optimization system, and nonlinear control theory, controlled thermonuclear fusion, nonlinear natural processes, image processing, quantum mechanics, and astrophysics.

In the context of the success of Saigo operators [23, 24], in their study of different function spaces and their use in differential equations and integral equations, Saigo and Maeda [25] presented the corresponding generalized fractional differential and integral operators in any complex order with Appell's function $F_3(\cdot)$ in the kernel as follows. Let $\varsigma,\varsigma',\vartheta,\vartheta',\varpi\in\mathbb{C}$ and $x>0$, then the generalized fractional calculus operators are defined by the following equations:

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x)=\frac{x^{-\varsigma}}{\Gamma(\varpi)}\int_0^x(x-t)^{\varpi-1}t^{-\varsigma'}, \tag{5}$$

$$\times F_3\left(\varsigma,\varsigma',\vartheta,\vartheta';\varpi;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\mathrm{d}t,\quad (\Re(\varpi)>0)$$
$$=\left(\frac{d}{\mathrm{d}x}\right)^k\left(I_{0+}^{\varsigma,\varsigma',\vartheta+k,\vartheta',\varpi+k}f\right)(x), \tag{6}$$

$$(\Re(\varpi)\leq 0;k=[-\Re(\varpi)]+1);$$
$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x)=\frac{x^{-\varsigma'}}{\Gamma(\varpi)}\int_x^{\infty}(t-x)^{\varpi-1}t^{-\varsigma}, \tag{7}$$

$$\times F_3\left(\varsigma,\varsigma',\vartheta,\vartheta';\varpi;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\mathrm{d}t,\quad (\Re(\varpi)>0)$$
$$=\left(-\frac{d}{\mathrm{d}x}\right)^k\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta'+k,\varpi+k}f\right)(x), \tag{8}$$

$$(\Re(\varpi)\leq 0;k=[-\Re(\varpi)]+1),$$
$$\left(D_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x)=\left(I_{0+}^{-\varsigma',-\varsigma,-\vartheta',-\vartheta,-\varpi}f\right)(x), \tag{9}$$

$$=\left(\frac{d}{\mathrm{d}x}\right)^k\left(I_{0+}^{-\varsigma',-\varsigma,-\vartheta'+k,-\vartheta,-\varpi+k}f\right)(x),$$
$$(\Re(\varpi)>0;k=[\Re(\varpi)]+1);$$
$$\left(D_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x)=\left(I_{-}^{-\varsigma',-\varsigma,-\vartheta',-\vartheta,-\varpi}f\right)(x)$$
$$=\left(-\frac{d}{\mathrm{d}x}\right)^k\left(I_{-}^{-\varsigma',-\varsigma,-\vartheta',-\vartheta+k,-\varpi+k}f\right)(x),$$
$$(\Re(\varpi)>0;k=[\Re(\varpi)]+1). \tag{10}$$

The interested reader may refer to the monograph by Srivastava and Karlsson [26] for the concept of Appell function $F_3(\cdot)$.

The image formulas for a power function, under operators (5) and (7), are given by Saigo and Maeda [25] as follows:

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}x^{\tau-1}\right)(x)=x^{\tau-\varsigma-\varsigma'+\varpi-1}\times\Gamma\begin{bmatrix}\tau,\tau+\varpi-\varsigma-\varsigma'-\vartheta,\tau+\vartheta'-\varsigma'\\ \tau+\vartheta',\tau+\varpi-\varsigma-\varsigma',\tau+\varpi-\varsigma'-\vartheta\end{bmatrix}, \tag{11}$$

where $\Re(\tau)>\max\{0,\Re(\varsigma+\varsigma'+\vartheta-\varpi),\Re(\varsigma'-\vartheta')\}$ and $\Re(\varpi)>0$.

$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}x^{\tau-1}\right)(x)=x^{\tau+\varpi-\varsigma-\varsigma'-1}\times\frac{\Gamma(1-\tau-\varpi+\varsigma+\varsigma')\Gamma(1-\tau+\varsigma+\vartheta'-\varpi)\Gamma(1-\tau-\vartheta)}{\Gamma(1-\tau)\Gamma(1-\tau+\varsigma+\varsigma'+\vartheta'-\varpi)\Gamma(1-\tau+\varsigma-\vartheta)}, \tag{12}$$

where $\Re(\gamma)>0,\Re(\varpi)<1+\min\{\Re(\quad-\vartheta),\Re(\varsigma+\varsigma'-\varpi),\Re(\varsigma+\vartheta'-\varpi)\}$.

Here, we used the $\Gamma[:::]$ symbol, which represents a fraction of several of the Gamma functions.

We will need the definition of the Hadamard product (or convolution) of two analytical properties for our present investigation. It will help us decompose a newly generated function into two existing functions. In fact, if one of the two power series defines a whole function, then the Hadamard product series also defines a whole function. In reality, let

$$f(z)=\sum_{l=0}^{\infty}a_l z^l\left(|z|<\Re_f\right),\quad g(z)=\sum_{l=0}^{\infty}b_l z^l\left(|z|<\Re_g\right), \tag{13}$$

be two given power series whose radii of convergence are given by $\Re_f$ and $\Re_g$, respectively. Then, their Hadamard product is a power series defined by

$$(f * g)(z) = \sum_{l=0}^{\infty} a_l b_l z^l = (g * f)(z)(|x| < \mathfrak{R}), \tag{14}$$

whose radius of convergence $\mathfrak{R}$ is

$$\frac{1}{\mathfrak{R}} = \lim_{l\longrightarrow\infty} \sup\left(\left|a_l b_l\right|\right)^{(1/l)} \leq \left(\lim_{l\longrightarrow\infty} \sup\left(\left|a_l\right|\right)^{(1/l)}\right)\left(\lim_{l\longrightarrow\infty} \sup\left(\left|b_l\right|\right)^{(1/l)}\right) = \frac{1}{\mathfrak{R}_f \cdot \mathfrak{R}_g},$$
$$\mathfrak{R} \geq \mathfrak{R}_f \cdot \mathfrak{R}_g. \tag{15}$$

The results in Theorems 1 and 2 will be expressed in a Hadamard product of (p, q)-extended Gauss hypergeometric function (see [15], p. 354, equation (8)):

$${}_pF_q(c, b; a; z) = \sum_{l=0}^{\infty} \frac{\mathfrak{B}(b + l, a - b; p, q)}{\mathfrak{B}(b, a - b)} \frac{z^l}{l!} \, (|z| < 1, \mathfrak{R}(a) > \mathfrak{R}(b) > 0), \tag{16}$$

where $\mathfrak{B}(c, b)$ is the classical beta function [27] and Fox–Wright function $p\Psi q(z)(p, q \in \mathbb{N}_0)$ [28].

$$p\Psi q\left[\begin{matrix} (\varsigma_1, P_1), \ldots, (\varsigma_p, P_p); \\ (\vartheta_1, Q_1), \ldots, (\vartheta_q, Q_q); \end{matrix} \; z\right] = \sum_{k=0}^{\infty} \frac{\Gamma(\varsigma_1 + P_1 n), \ldots, \Gamma(\varsigma_p + P_p n)}{\Gamma(\vartheta_1 + Q_1 n), \ldots, \Gamma(\vartheta_p + Q_p n)} \frac{z^k}{k!},$$
$$\left(P_j \in \mathfrak{R}^+ (j = 1, \ldots, p), Q_j \in \mathfrak{R}^+ (j = 1, \ldots, q); 1 + \sum_{j=0}^{q} Q_j - \sum_{j=0}^{p} P_j \geq 0\right), \tag{17}$$

where the convergence condition holds true for

$$|z| < \nabla = \left(\prod_{j=1}^{p} P_j^{-P_j}\right) \cdot \left(\prod_{j=1}^{q} Q_j^{Q_j}\right). \tag{18}$$

In this paper, we aim to investigate compositions of the generalized fractional integration operators involving (p, q)-extended Struve function $H_{\delta,p,q}(z)$. Also, we consider (2) to achieve the solution of the generalized fractional kinetics equations (FKEs). Our approach here is based on Laplace transformation, and we plan to broaden our results by using the Sumudu transformation in a future career.

2. Fractional Integrations Approach

For this section, we assume that $\varsigma, \varsigma', \vartheta, \vartheta', \varpi, \tau, \delta, \omega \in \mathbb{C}$ such that $\mathfrak{R}(\varpi) > 0$, $\min\{\mathfrak{R}(p), \mathfrak{R}(q)\} > 0, \mathfrak{R}(\delta) > (-3/2)$. Furthermore, let the constants satisfy the condition $\varsigma_i, \vartheta_j \in \mathbb{C}$, and $P_i, Q_j \in \mathfrak{R}\,(P_i, Q_j \neq 0 i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$, such that condition (17) is also satisfied.

2.1. Left-Sided Generalized Fractional Integration of (p, q)-Extended Struve Function. In this segment, we establish image formulas for the (p, q)-extended Struve function involving left-sided operators of M-S-M fractional integral operators (5), in terms of the Hadamard product of the Fox–Wright function $r\psi s(z)$ and the (p, q)-extended Gauss hypergeometric function. These formulas are set out in the preceding theorems.

Theorem 1. *If $\mathfrak{R}(\varpi) > 0$, $\mathfrak{R}(\tau + \delta + 1) > \max\{0, \mathfrak{R}(\varsigma + \varsigma' + \vartheta - \varpi), \mathfrak{R}(\varsigma' - \vartheta')\}$, then the generalized fractional integration $I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}$ of the (p, q)-extended Struve function $H_{\delta,p,q}(z)$ is given by*

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right) = \sqrt{\pi}x^{\tau-\varsigma-\varsigma'+\varpi+\delta}\frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))} \times {}_pF_q\left[\begin{matrix}1,1; \\ \delta+(3/2);\end{matrix} -\frac{\omega^2x^2}{4}\right] * {}_3\Psi_4\left[\begin{matrix}(\tau+\delta+\varpi-\varsigma+\varsigma'-\vartheta+1,2), \\ ((3/2),1),(\tau+\delta+\vartheta'+1,2),\end{matrix}\right. \left.\begin{matrix}(\tau+\delta+\vartheta'-\varsigma'+1,2),(\tau+\delta+1,2); \\ (\tau+\delta+\varpi-\varsigma-\varsigma'+1,2),(\tau+\delta+\varpi-\varsigma'-\vartheta+1,2);\end{matrix} -\frac{\omega^2x^2}{4}\right], \tag{19}$$

where * indicates the Hadamard product in (14).

Proof. By applying (2) and (5), on the left side of (19), we have

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right)(x) = \frac{\omega^{\delta+1}}{2^{\delta}\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left(-\frac{\omega^2}{4}\right)^k \times\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau+\delta+1+2k-1}\right)\right)(x), \tag{20}$$

upon using the image formula (11):

$$\begin{aligned}&\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right)(x)\\ &= \frac{x^{\tau+\delta-\varsigma-\varsigma'+\varpi}\omega^{\delta+1}}{2^{\delta}\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left(\frac{-x^2\omega^2}{4}\right)^k\\ &\times\frac{\Gamma(\tau+\delta+\varpi-\varsigma+\varsigma'-\vartheta+2k+1)\Gamma(\tau+\delta+\vartheta'-\varsigma'+2k+1)\Gamma(\tau+\delta+2k+1)}{\Gamma(\tau+\delta+\vartheta'+2k+1)\Gamma(\tau+\delta+\varpi-\varsigma-\varsigma'+2k+1)\Gamma(\tau+\delta+\varpi-\varsigma'-\vartheta+2k+1)}.\end{aligned} \tag{21}$$

Presenting the last summation in (21) in terms of the Hadamard product (14) with the functions (16) and (17), we get the right side of (19).

Now, we discuss the special cases of (19) as follows.

For $\varsigma=\varsigma+\vartheta,\varsigma'=\vartheta'=0,\vartheta=-\beta,\varpi=\varsigma$, we obtain the following relationship:

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x) = \left(I_{0+}^{\varsigma,\vartheta,\beta}f\right)(x), \tag{22}$$

where the operator $I_{0+}^{\varsigma,\vartheta,\beta}(\cdot)$ express the Saigo fractional integral operator [23], which is defined by

$$\left(I_{0+}^{\varsigma,\vartheta,\beta}f\right)(x) = \frac{x^{-\varsigma-\vartheta}}{\Gamma(\varsigma)}\int_0^x (x-t)_2^{\varsigma-1}F_1\left(\varsigma+\vartheta,-\beta;\varsigma;1-\frac{t}{x}\right)f(t)\mathrm{d}t, \quad \Re(\varsigma)>0. \tag{23}$$

□

Corollary 1. *Let* $\Re(\varsigma)>0, \Re(\tau+\delta+1)>\max[0,\Re(\vartheta-\beta)]$, *then there holds the following formula:*

$$\left(I_{0+}^{\varsigma,\vartheta,\beta}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right) = \sqrt{\pi}x^{\tau+\delta-\vartheta}\frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))_p}F_q\left[\begin{matrix}1,1; \\ \delta+(3/2);\end{matrix} -\frac{\omega^2x^2}{4}\right]$$
$$* {}_2\Psi_3\left[\begin{matrix}(\tau+\delta-\vartheta+\beta+1,2),(\tau+\delta+1,2); \\ \left(\frac{3}{2},1\right),(\tau+\delta-\vartheta+1,2),(\tau+\delta+\varsigma+\beta+1,2);\end{matrix} -\frac{\omega^2x^2}{4}\right]. \tag{24}$$

2.2. Right-Sided Generalized Fractional Integration of the (p,q)*-Extended Struve Function.* In this portion, we establish image formulas for the (p,q)-extended Struve function containing right-sided operators of M-S-M fractional integral operators (7), in terms of the Hadamard product of the Fox–Wright function ${}_r\psi_s(z)$ and the (p,q)-extended Gauss hypergeometric function. These formulas are set out in the preceding theorems.

Theorem 2. *If* $\Re(\tau-\delta)<2+\min\{\Re(-\vartheta),\Re(\varsigma+\varsigma'-\varpi),\Re(\varsigma-\vartheta'-\varpi)\}$, $\Re(\varpi)>0$, *then the generalized fractional integration* $I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}$ *of the* (p,q)*-extended Struve function* $H_{\delta,p,q}(z)$ *is given by*

$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau-1}H_{\delta,p,q}\left(\frac{\omega}{t}\right)\right)\right) = \sqrt{\pi}x^{\tau-\varsigma-\varsigma'+\varpi-\delta-2}\frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))}$$
$$\times {}_pF_q\left[\begin{matrix}1,1; \\ \delta+(3/2);\end{matrix} -\frac{\omega^2}{4x^2}\right] * {}_3\Psi_4\left[\begin{matrix}(2-\tau+\delta-\varpi+\varsigma+\varsigma',2), \\ ((3/2),1),(2-\tau+\delta,2),\end{matrix}\right. \tag{25}$$
$$\left.\begin{matrix}(2-\tau+\delta+\varsigma+\vartheta'-\varpi,2),(2-\tau+\delta-\vartheta,2); \\ (2-\tau+\delta-\varpi+\varsigma+\varsigma'+\vartheta',2),(2-\tau+\delta+\varsigma-\vartheta,2);\end{matrix} -\frac{\omega^2}{4x^2}\right].$$

Proof. By applying (2) and (7) on the left-hand side of (25), we get

$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau-1}H_{\delta,p,q}\left(\frac{\omega}{t}\right)\right)\right)(x) = \frac{\omega^{\delta+1}}{2^{\delta}\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left(-\frac{\omega^2}{4}\right)^k$$
$$\times\left(I_{0-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau-\delta-2k-2}\right)\right)(x), \tag{26}$$

and upon using the image formula (12) yields

$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right)(x)$$
$$= \frac{x^{\tau+\delta-\varsigma-\varsigma'+\varpi}\omega^{\delta+1}}{2^{\delta}\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left[\frac{-x^2\omega^2}{4}\right]^k \tag{27}$$
$$\times\frac{\Gamma(2-\tau+\delta-\varpi+\varsigma+\varsigma'+2k)\Gamma(2-\tau+\delta+\varsigma+\vartheta'-\varpi+2k)\Gamma(2-\tau+\delta-\vartheta+2k)}{\Gamma(2-\tau+\delta+2k)\Gamma(2-\tau+\delta-\varpi+\varsigma+\varsigma'+\vartheta'+2k)\Gamma(2-\tau+\delta+\varsigma-\vartheta+2k)}.$$

Interpreting the right-hand side of (27) in terms of the Hadamard product (14) with the functions (16) and (17), we get the right side of (25).

When we let $\varsigma = \varsigma + \vartheta, \varsigma' = \vartheta' = 0, \vartheta = -\beta, \varpi = \varsigma$, then we obtain the relationship

$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi} f\right)(x) = \left(I_{-}^{\varsigma,\vartheta,\beta} f\right)(x), \tag{28}$$

where the Saigo fractional integral operator [23] is represented as

$$\left(I_{-}^{\varsigma,\vartheta,\beta} f\right)(x) = \frac{1}{\Gamma(\varsigma)} \int_x^\infty (t-x)^{\varsigma-1} t^{-\varsigma-\vartheta}{}_2F_1\left(\varsigma+\vartheta, -\beta; \varsigma; 1-\frac{x}{t}\right) f(t)\mathrm{d}t. \tag{29}$$

□

Corollary 2. *If* $\Re(\varsigma) > 0, \Re(\tau - \delta) < 2 + \min[\Re(\vartheta), \Re(\beta)]$, *then we have*

$$\left(I_{-}^{\varsigma,\vartheta,\beta}\left(t^{\tau-1} H_{\delta,p,q}\left(\frac{\omega}{t}\right)\right)\right) = \sqrt{\pi} x^{\tau-\vartheta-\delta-2} \frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))_p} F_q\left[\begin{matrix} 1, 1; \\ \delta+(3/2); \end{matrix} -\frac{\omega^2}{4x^2}\right] * {}_2\Psi_3\left[\begin{matrix} (2-\tau+\delta+\vartheta, 2), (2-\tau+\delta+\beta, 2); \\ \left(\frac{3}{2}, 1\right), (2-\tau+\delta, 2), (2-\tau+\delta+\varsigma+\vartheta+\beta, 2); \end{matrix} -\frac{\omega^2}{4x^2}\right]. \tag{30}$$

In the next part, we derived the generalized fractional kinetic equations (FKEs) and take into account the Laplace transformation technique to produce outcomes.

3. Generalized Fractional Kinetic Equations Involving (p, q)-Extended Struve Function

The generalized FKEs involving the (p, q)-extended Struve function with the Laplace transform (LT) is derived in this section. FKEs were extensively reviewed in a variety of articles [29–35].

Let $\mathfrak{N}(t)$ be an arbitrary reaction that depends on time, d is a destruction rate, and p is a production rate of $\mathfrak{N}$, then the mathematical representation of these three ratios is described by Haubold and Mathai [36] as a fractional differential equation:

$$\frac{\mathrm{d}\mathfrak{N}}{\mathrm{d}t} = -d(\mathfrak{N}_t) + p(\mathfrak{N}_t), \tag{31}$$

where $\mathfrak{N}_t(t^*) = \mathfrak{N}(t - t^*)$ for $t^* > 0$. Also, [36] have researched that equation (31) would become the following differential equation if spatial fluctuation or inhomogeneities in quantity $\mathfrak{N}(t)$ are ignored:

$$\frac{\mathrm{d}\mathfrak{N}_i}{\mathrm{d}t} = -c_i \mathfrak{N}_i(t), \tag{32}$$

with $\mathfrak{N}_i(t = 0) = \mathfrak{N}_0$. Solution of equation (32) is given by

$$\mathfrak{N}_i(t) = \mathfrak{N}_0 e^{-c_i t}. \tag{33}$$

Alternatively, if we eliminate the index i and integrate (32), we get

$$\mathfrak{N}(t) - \mathfrak{N}_0 = c_0 \mathfrak{D}_t^{-1} \mathfrak{N}(t), \tag{34}$$

where ${}_0\mathfrak{D}_t^{-1}$ is the standard integral operator. The fractional generalization of equation (34) was defined by Haubold and Mathai [36] as

$$\mathfrak{N}(t) - \mathfrak{N}_0 = c^v {}_0\mathfrak{D}_t^{-v} \mathfrak{N}(t), \tag{35}$$

where ${}_0\mathfrak{D}_t^{-v}$ is given by

$${}_0\mathfrak{D}_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-x)^{v-1} f(x)\mathrm{d}x, \quad \Re(v) > 0. \tag{36}$$

Definition 1. The Mittag–Leffler function is generalized by Wiman [28] in the following form:

$$E_{\varsigma,\vartheta}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\varsigma l + \vartheta)}, \ (z, \varsigma, \vartheta \in \mathbb{C}; \Re(\varsigma) > 0, \Re(\vartheta) > 0). \tag{37}$$

The results of this section, solutions of generalized FKESs, will be expressed based on the generalized Mittag–Leffler function which is defined in (37).

Theorem 3. *If* $d > 0, v > 0$, *with* $\min\{p, q\} \geq 0$ *and* $\Re(\delta) > -(1/2)$, *the solution of fractional kinetic equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0 H_{\delta,p,q}(t) = -d^v {}_0\mathfrak{D}_t^{-v} \mathfrak{N}(t) \tag{38}$$

becomes

$$\mathfrak{N}(t) = \mathfrak{N}_0 \frac{t^{\delta+1}}{2^{\delta}\Gamma(\delta+(3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)\Gamma(\delta+2k+2)}{(3/2)_k \mathfrak{B}(1,\delta+(1/2))k!} \left(\frac{-t^2}{4}\right)^k \times E_{\nu,\delta+2k+2}(-d^{\nu}t^{\nu}). \quad (39)$$

Proof. The LT of the Riemann–Liouville (RL) fractional integral operator is given by Srivastava and Saxena [37] as

$$L\{{}_0\mathfrak{D}_t^{-\nu} f(t); s\} = s^{-\nu}F(s). \quad (40)$$

Now, applying the LT to both sides of (38) and using (2) and (40), we have

$$L\{\mathfrak{N}(t); s\} = \mathfrak{N}_0 L\{H_{\delta,p,q}(t); s\} - d^{\nu} L\{{}_0\mathfrak{D}_t^{-\nu}\mathfrak{N}(t); s\}, \quad (41)$$

which gives

$$\mathfrak{N}(s) = \mathfrak{N}_0 \int_0^{\infty} e^{-st} \frac{t^{\delta+1}}{2^{\delta}\Gamma(\delta+(3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)}{(3/2)_k \mathfrak{B}(1,\delta+(1/2))k!} \left(\frac{-t^2}{4}\right) dt - d^{\nu}s^{-\nu}\mathfrak{N}(s), \quad (42)$$

which implies that

$$\mathfrak{N}(s) + d^{\nu}s^{-\nu}\mathfrak{N}(s) = \frac{\mathfrak{N}_0}{2^{\delta}\Gamma(\delta+(3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)}{(3/2)_k \mathfrak{B}(1,\delta+(1/2))k!} \frac{(-1)^k}{4^k} \times \int_0^{\infty} e^{-st} t^{2k+\delta+1} dt. \quad (43)$$

After some simple calculation, we get

$$\begin{aligned} \mathfrak{N}(s)(1+d^{\nu}s^{-\nu}) &= \frac{\mathfrak{N}_0}{2^{\delta}\Gamma(\delta+(3/2))} \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)}{(3/2)_k \mathfrak{B}(1,\delta+(1/2))k!} \frac{\Gamma(\delta+2k+2)}{s^{\delta+2k+2}}, \\ \mathfrak{N}(s) &= \frac{\mathfrak{N}_0}{2^{\delta}\Gamma(\delta+(3/2))} \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)\Gamma(\delta+2k+2)}{(3/2)_k \mathfrak{B}(1,\delta+(1/2))k!} \\ &\quad \times s^{-(\delta+2k+2)} \sum_{l=0}^{\infty} (1)_l \frac{\left[-(s/d)^{-\nu}\right]^l}{l!}. \end{aligned} \quad (44)$$

Taking inverse LT on both sides of (44) and using $L^{-1}(s^{-\nu}) = (t^{\nu-1}/\Gamma(\nu))$ for $\mathfrak{R}(\nu) > 0$, we get

$$\mathfrak{N}(t) = \frac{\mathfrak{N}_0}{2^{\delta}\Gamma(\delta+(3/2))} \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)\Gamma(\delta+2k+2)}{(3/2)_k \mathfrak{B}(1,\delta+(1/2))k!} \times \sum_{l=0}^{\infty} (-1)^l d^{\nu l} \frac{t^{\delta+2k+\nu l+1}}{\Gamma(\delta+\nu l+2k+2)}. \quad (45)$$

□

Interpreting the right-hand side of (45) in the view of (37), we obtain the needful result (39).

Theorem 4. *If $d>0, v>0$, with $\min\{p,q\}\geq 0$ and $\Re(\delta)>-(1/2)$, then the solution of*

$$\mathfrak{N}(t)-\mathfrak{N}_0 H_{\delta,p,q}(d^v t^v)=-d^v{}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \quad (46)$$

is given by

$$\mathfrak{N}(t)=\mathfrak{N}_0\frac{(d^v t^v)^{\delta+1}}{2^\delta\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)\Gamma(2vk+\delta v+v+1)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left(\frac{-(d^v t^v)^2}{4}\right)^k \times E_{v,\delta v+2vk+v+1}(-d^v t^v). \quad (47)$$

Proof. Taking the LT on both sides of (46), using the definition of (p,q)-extended Struve functions (2) and (40), and after doing simple calculation and taking inverse LT term written in the view of (37), we obtain the needful result (47). □

Theorem 5. *If $d>0, v>0$, with $\min\{p,q\}\geq 0, a\neq d$ and $\Re(\delta)>-(1/2)$, the solution of fractional kinetic equation*

$$\mathfrak{N}(t)-\mathfrak{N}_0 H_{\delta,p,q}(d^v t^v)=-a^v{}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \quad (48)$$

becomes

$$\mathfrak{N}(t)=\mathfrak{N}_0\frac{(d^v t^v)^{\delta+1}}{2^\delta\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(1/2);p,q)\Gamma(2vk+\delta v+v+1)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left(\frac{-(d^v t^v)^2}{4}\right)^k \times E_{v,\delta v+2vk+v+1}(-a^v t^v). \quad (49)$$

Proof. In similar way of proof of Theorem 4, we can get solution (49). Therefore, we omitted the proof.

Now by setting $p=0$, $q=0$, on equation (3), then results of Theorems 3–5 are adjusted on Corollaries 3–5. □

Corollary 3. *If $d>0, v>0$, and $\Re(\delta)>-(1/2)$, the solution of fractional kinetic equation*

$$\mathfrak{N}(t)-\mathfrak{N}_0 H_{\delta,0,0}(t)=-d^v{}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \quad (50)$$

becomes

$$\mathfrak{N}(t)=\mathfrak{N}_0\frac{t^{\delta+1}}{\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\Gamma(\delta+2k+2)}{\Gamma(\delta+k+(3/2))}\left(\frac{-t^2}{4}\right)^k \times E_{v,\delta+2k+2}(-d^v t^v). \quad (51)$$

Corollary 4. *If $d>0, v>0$, with $\min\{p,q\}\geq 0$ and $\Re(\delta)>-(1/2)$, then the solution of*

$$\mathfrak{N}(t)-\mathfrak{N}_0 H_{\delta,p,q}(d^v t^v)=-d^v{}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \quad (52)$$

is given by

$$\mathfrak{N}(t)=\mathfrak{N}_0\frac{(d^v t^v)^{\delta+1}}{\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\Gamma(2vk+\delta v+v+1)}{\Gamma(\delta+k+(3/2))}\left(\frac{-(d^v t^v)^2}{4}\right)^k \times E_{v,\delta v+2vk+v+1}(-d^v t^v). \quad (53)$$

Corollary 5. *If $d>0, v>0$, with $\min\{p,q\}\geq 0, a\neq d$ and $\Re(\delta)>-(1/2)$, the solution of fractional kinetic equation*

$$\mathfrak{N}(t)-\mathfrak{N}_0 H_{\delta,p,q}(d^v t^v)=-a^v{}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \quad (54)$$

becomes

$$\mathfrak{N}(t)=\mathfrak{N}_0\frac{(d^v t^v)^{\delta+1}}{2^\delta\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\Gamma(2vk+\delta v+v+1)}{\Gamma(\delta+k+(3/2))}\left(\frac{-(d^v t^v)^2}{4}\right)^k \times E_{v,\delta v+2vk+v+1}(-a^v t^v). \quad (55)$$

4. Conclusion

In this article, the authors have established the generalized fractional integrations of the (p,q)-extended Struve function. The achieved results are expressed in terms of Hadamard product of the Fox–Wright function ${}_r\psi_s(z)$ and the (p,q)-extended Gauss hypergeometric function. The solutions of fractional kinetic equations are obtained with the support of Laplace transforms to show the possible application of the (p,q)-extended Struve function. As the solution of the equations is common and can derive several new and existing FKE solutions involving different types of special functions, the results obtained in this study are significant.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] M. Al-Qurashi, S. Rashid, Y. Karaca, Z. Hammouch, D. Baleanu, and Y. M. Chu, “Achieving more precise bounds basedon double and triple integral as proposed by generalized pro-portional fractional operators in the hilfer sense,” *Fractals*, 2020.

[2] K. K. Ali, M. S. Osman, H. M. Baskonus, N. S. Elazabb, and E. İlhan, “Analytical and numerical study of the HIV-1 infection of $CD4^{+4}$ T-cells conformable fractional mathematical model that causes acquired immunodeficiency syndrome with the effect of antiviral drug therapy,” *Mathematical Methods in the Applied Sciences*, pp. 1–17, 2020.

[3] V. V. Au, H. Jafari, Z. Hammouch, and N. H. Tuan, “On a final value problem for a nonlinear fractional pseudo-parabolic equation,” *Electronic Research Archive*, vol. 29, no. 1, pp. 1709–1734, 2021.

[4] W. Gao, P. Veeresha, D. G. Prakasha, and H. M. Baskonus, “New numerical simulation for fractional Benney-Lin equation arising in falling film problems using two novel techniques,” *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 210–243, 2020.

[5] W. Gao, P. Veeresha, D. G. Prakasha, B. Senel, and H. M. Baskonus, “Iterative method applied to the fractional nonlinear systems arising in thermoelasticity with Mittag-Leffler kernel,” *Fractals*, vol. 28, no. 8, Article ID 2040040, 2020.

[6] W. Gao, G. Yel, H. Mehmet Baskonus, and C. Cattani, “Complex solitons in the conformable (2+1)-dimensional Ablowitz-Kaup-Newell-segur equation,” *AIMS Mathematics*, vol. 5, no. 1, pp. 507–521, 2020.

[7] Y.-M. Li, S. Rashid, Z. Hammouch, D. Baleanu, and Y. M. Chu, “New newton’s type estimates pertaining to local fractional integral via generalized P-convexity with applications,” *Fractals*, 2020.

[8] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, Hoboken, NJ, USA, 1993.

[9] M. Aslam Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, “Extension of Euler’s beta function,” *Journal of Computational and Applied Mathematics*, vol. 78, no. 1, pp. 19–32, 1997.

[10] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, “Extended hypergeometric and confluent hypergeometric functions,” *Applied Mathematics and Computation*, vol. 159, no. 2, pp. 589–602, 2004.

[11] M. Aslam Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press, Boca Raton, FL, USA, 2002.

[12] J. Choi, R. K. Parmar, and T. K. Pogány, “Mathieu-type series built by (p, q)-extended gaussian hypergeometric function,” *Bulletin of the Korean Mathematical Society*, vol. 54, no. 3, pp. 789–797, 2017.

[13] M.-J. Luo, R. K. Parmar, and R. K. Raina, “On extended Hurwitz-Lerch zeta function,” *Journal of Mathematical Analysis and Applications*, vol. 448, no. 2, pp. 1281–1304, 2017.

[14] D. J. Maŝirevîc, R. K. Parmar, and T. K. Pogány, “(p,q)-extended Bessel and modified Bessel functions of the first kind,” *Results in Mathematics*, vol. 72, no. 1-2, pp. 617–632, 2017.

[15] J. Choi, A. K. Rathie, and R. K. Parmar, “Extension of extended beta, hypergeometric and confluent hypergeometric functions,” *Honam Mathematical Journal*, vol. 36, no. 2, pp. 339–367, 2014.

[16] H. M. Srivastava, R. K. Parmar, and P. Chopra, “A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions,” *Axioms*, vol. 1, no. 3, pp. 238–258, 2012.

[17] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, MA, USA, 2nd edition, 1944.

[18] K. N. Bhowmick, “Some relations between a generalized Struve’s function and hypergeometric functions,” *Vijnana Parishad Anusandhan Patrika*, vol. 5, pp. 93–99, 1962.

[19] J. Choi and K. S. Nisar, “Certain families of integral formulas involving Struve functions,” *Boletim da Sociedade Paranaense de Matematica*, vol. 37, no. 3, pp. 27–35, 2019.

[20] K. S. Nisar, D. L. Suthar, S. D. Purohit, and M. Aldhaifallah, “Some unified integral associated with the generalized Struve function,” *Proceedings of the Jangjeon Mathematical Society*, vol. 20, no. 2, pp. 261–267, 2017.

[21] R. K. Parmar and J. Choi, “Fractional calculus of the (p, q)-extended struve function,” *Far East Journal of Mathematical Sciences (FJMS)*, vol. 103, no. 2, pp. 541–559, 2018.

[22] D. L. Suthar, D. Baleanu, S. D. Purohit, and F. Uçar, “Certain k-fractional calculus operators and image formulas of k-Struve function,” *AIMS Mathematics*, vol. 5, no. 3, pp. 1706–1719, 2020.

[23] M. Saigo, “A remark on integral operators involving the Gauss hypergeometric functions,” *Mathematical reports of College of General Education, Kyushu University*, vol. 11, no. 2, pp. 135–143, 1978.

[24] M. Saigo, “A certain boundary value problem for the Euler-Darboux equation,” *Japanese Journal of Mathematics*, vol. 24, no. 4, pp. 377–385, 1979.

[25] M. Saigo and N. Maeda, “More generalization of fractional calculus,” *Transform Methods & Special Functions*, pp. 386–400, Bulgarian Academy of Science, Sofia, Bulgaria, 1998.

[26] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussion Hypergeometric Series*, John Wiley & Sons, New York, NY, USA, 1985.

[27] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, England, 2010.

[28] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.

[29] H. Habenom, D. L. Suthar, and M. Gebeyehu, “Application of Laplace transform on fractional kinetic equation pertaining to the generalized Galué type Struve function,” *Advances in Mathematical Physics*, vol. 2019, Article ID 5074039, 8 pages, 2019.

[30] K. S. Nisar, S. D. Purohit, and S. R. Mondal, “Generalized fractional kinetic equations involving generalized Struve function of the first kind,” *Journal of King Saud University-Science*, vol. 28, no. 2, pp. 167–171, 2016.

[31] R. K. Saxena, A. M. Mathai, and H. J. Haubold, “On fractional kinetic equations,” *Astrophysics and Space Science*, vol. 282, no. 1, pp. 281–287, 2002.

[32] R. K. Saxena and S. L. Kalla, "On the solutions of certain fractional kinetic equations," *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 504–511, 2008.

[33] D. L. Suthar, H. Habenom, and K. S. Nisar, "Solutions of fractional kinetic equation and the generalized galué type struve function," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 7, pp. 1167–1184, 2019.

[34] D. L. Suthar, D. Kumar, and H. Habenom, "Solutions of fractional kinetic equation associated with the generalized multiindex bessel function via laplace transform," *Differential Equations and Dynamical Systems*, 2019.

[35] D. L. Suthar, S. D. Purohit, and S. Araci, "Solution of fractional kinetic equations associated with the p,q-mathieu-type series," *Discrete Dynamics in Nature and Society*, vol. 2020, Article ID 8645161, 7 pages, 2020.

[36] H. J. Haubol and A. M. Mathai, "The fractional kinetic equation and thermonuclear functions," *Astrophysics and Space Science*, vol. 327, pp. 53–63, 2000.

[37] H. M. Srivastava and R. K. Saxena, "Operators of fractional integration and their applications," *Applied Mathematics and Computation*, vol. 118, no. 1, pp. 1–52, 2001.

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Research Article

A Nonlinear Implicit Fractional Equation with Caputo Derivative

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In this paper, we study a nonlinear implicit differential equation with initial conditions. The considered problem involves the fractional Caputo derivatives under some conditions on the order. We prove an existence and uniqueness analytic result by application of Banach principle. Then, another result that deals with the existence of at least one solution is delivered and some sufficient conditions for this result are established by means of the fixed point theorem of Schaefer. At the end, we discuss two examples to illustrate the applicability of the main results.

1. Introduction

The theory of differential equations of fractional order and fractional calculus is very important since they can be used in analyzing and modeling real word phenomena. Recently, several researchers are interested in the important progress of differential equations of fractional order. For more information on these works and their applications, one can consult the references [1–9]. In particular, research on the existence of unique solutions for fractional differential equations is of big importance since it helps physicians to better understand the behaviour of real phenomena. See, for more details, the references [10–14].

The motivation for this work arises from both the development of the theory of fractional calculus itself and its wide applications to various fields of science, such as physics, chemistry, biology, electromagnetism of complex media, robotics, and economics.

Much attention has been paid to the existence and uniqueness of solutions of fractional dynamical systems [15–18] due to the fact that existence is the fundamental problem and a necessary condition for considering some other properties for fractional dynamical systems, such as controllability and stability. Chai [19] provided sufficient conditions for the existence of solutions to a class of anti-periodic boundary value problems for fractional differential equations, while Sheng and Jiang [20] considered a class of initial value problems for fractional differential systems. There are several operators studied in the field of fractional calculus, for example, see [21–26], but the difference in this work is that the operator considered is in the sense of Caputo derivative.

Motivated by the works of Benchohra et al. [27], we will establish in this paper existence and uniqueness results of the solutions of the fractional dynamical system with Caputo fractional derivative

$$\begin{cases} D^{\alpha}x(t) - AD^{\beta}x(t) = f\left(t, x(t), D^{\beta}x(t), D^{\alpha}x(t)\right), & t \in I = [0,1], \\ x(0) = x_0, \\ x'(0) = x_0', \\ x''(0) = x_0'', \\ x'''(0) = x_0'', \end{cases} \tag{1}$$

where D^{α} is in the sense of Caputo, $f\colon I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given function, $x_0, x_0', x_0'' \in \mathbb{R}^n$, A is an $n \times n$ matrix, and $1 < \beta < 2$, $3 < \alpha < 4$, with $\beta + 2 < \alpha$.

Rest of the paper is organised as follows: in Section 2, we recall some results and definitions which we use for the proof of our main results. In Section 3, we give and prove the main theorems of this paper, and we discuss some illustrative examples.

2. Preliminaries

In this section, we introduce some definitions, lemmas, and preliminaries facts which are used throughout this paper. See [7] for more information. Let $|.|$ be a suitable norm in $\mathbb{R}^n$ and $\|.\|$ be the matrix norm. We denote by $C(I, \mathbb{R}^n)$ the Banach space of continuous functions from I to $\mathbb{R}^n$ with the norm $\|x\|_{\infty} = \sup\{|x|,\ x \in I\}$. We denote by $L^1(I, \mathbb{R}^n)$ the space of Lebesgue-integrable function $x\colon I \longrightarrow \mathbb{R}^n$ with the norm

$$\|x\|_{L^1} = \int_0^1 |x(t)|\mathrm{d}t. \tag{2}$$

Let

$$X = \left\{ \begin{array}{l} x \in C(J, \mathbb{R}^n), \\ x'' \in C(I; \mathbb{R}^n) \end{array} \right\}, \tag{3}$$

with the norm

$$\|x\|_X = \|x\|_{\infty} + \left\|x''\right\|_{\infty}. \tag{4}$$

Definition 1. The Riemann–Liouville integral of order $\alpha > 0$ for a continuous function $\varphi \in L^1((0,1], \mathbb{R})$ is given by

$$I^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)\mathrm{d}\tau, \quad \forall t \in (0,1], \tag{5}$$

with $\Gamma(\alpha)\colon = \int_0^{\infty} e^{-u}u^{\alpha-1}\mathrm{d}u$.

Definition 2. If $\varphi \in C^n([0,1], \mathbb{R})$ and $n-1 < \alpha \leq n$, then the Caputo fractional derivative is given by

$$D^{\alpha}\varphi(t) = I^{n-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\varphi(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}\varphi^{(n)}(s)\mathrm{d}s. \tag{6}$$

Lemma 1. *Let $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$, then the general solution of $D^{\alpha}u(t) = 0$ is given by*

$$u(t) = \sum_{i=0}^{n-1} c_i t^i, \tag{7}$$

such that $c_i \in \mathbb{R}$, $\quad i = 0, 1, 2, \ldots, n-1$.

Lemma 2. *Taking $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$, we have*

$$I^{\alpha}D^{\alpha}u(t) = u(t) + \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!}t^k, \tag{8}$$

with $t > 0$, $n-1 < \alpha < n$.

Lemma 3. *Let $1 < \beta < 2$ and $3 < \alpha < 4$. Then, it holds*

$$I^{\alpha}D^{\beta}u(t) = I^{\alpha-\beta}u(t) - \frac{u(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{u'(0)t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}. \tag{9}$$

Proof. For this proof, we use the same method in [28]. We have

$$I^{\alpha}D^{\beta}u(t)=\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}\int_0^t(t-\tau)^{\alpha-1}\int_0^t(\tau-s)^{1-\beta}u''(s)\mathrm{d}s\mathrm{d}\tau,$$
$$=\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}\int_0^t u''(s)\mathrm{d}s\int_s^t(t-\tau)^{\alpha-1}(\tau-s)^{1-\beta}\mathrm{d}\tau. \tag{10}$$

With the change of variable $\tau=s+(t-s)\eta$, we have

$$\int_s^t(t-\tau)^{\alpha-1}(\tau-s)^{1-\beta}\mathrm{d}\tau=\frac{\Gamma(\alpha)\Gamma(2-\beta)}{\Gamma(\alpha-\beta+2)}(t-s)^{\alpha-\beta+1}. \tag{11}$$

Now, we get

$$I^{\alpha}D^{\beta}u(t)=\frac{1}{\Gamma(\alpha-\beta+2)}\int_0^t(t-s)^{\alpha-\beta+1}u''(s)\mathrm{d}s=I^{\alpha-\beta}u(t)-\frac{u(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\frac{u'(0)t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}. \tag{12}$$

□

Definition 3. Let X be a Banach space. Then, a map $T\colon X\longrightarrow X$ *is* called a contraction mapping on X if there exists $q\in[0,1)$ such that

$$\|T(x)-T(y)\|\leq q\|x-y\|, \tag{13}$$

for all $x,y\in X$.

Theorem 1 (Banach's fixed point theorem, see [29]). *Let Ω be a nonempty closed subset of a Banach space X. Then, any contraction mapping T of Ω into itself has a unique fixed point.*

Theorem 2 (Schaefer's fixed point theorem, see [29]). *Let X be a Banach space, and let $N\colon X\longrightarrow X$ be a completely continuous operator. If the set $E=\{y\in X\colon y=\lambda Ny \text{ for some } \lambda\in(0,1)\}$ is bounded, then N has fixed points.*

3. Main Results

We begin this section by some results that help us for solving the problem considered in (1).

Lemma 4. *For any $x\in X$ and $1<\beta<2$, we have*

$$\left\|D^{\beta}x\right\|_{\infty}\leq\frac{1}{\Gamma(3-\beta)}\|x''\|_{\infty}\leq\frac{1}{\Gamma(3-\beta)}\|x''\|_{X}. \tag{14}$$

Proof. By the definition of the operator D^{β}, we have

$$\left|D^{\beta}x(t)\right|_{\infty}=\frac{1}{\Gamma(2-\beta)}\left|\int_0^t(t-s)^{1-\beta}x''(s)\mathrm{d}s\right|\leq\|x''\|_{\infty}\frac{1}{\Gamma(2-\beta)}\int_0^1(1-s)^{1-\beta}\mathrm{d}s\leq\frac{1}{\Gamma(3-\beta)}\|x''\|_{\infty}. \tag{15}$$

□

Lemma 5. *Let* $1<\beta<2$, $3<\alpha<4$, *and* $G\in C(I,\mathbb{R}^n)$. *Then, we can state that the problem,*

$$\begin{cases} D^{\alpha}x(t)-AD^{\beta}x(t)=G(t), \quad t\in I=[0,1],\\ x(0)=x_0,\\ x'(0)=x_0',\\ x''(0)=x_0'',\\ x'''(0)=x_0'', \end{cases} \tag{16}$$

has for solution the following function

$$\begin{aligned} x(t)={}&x_0+x_0't+\frac{1}{2}x_0''t^2+\frac{1}{6}x_0''t^3-\frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x_0-\frac{At^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}x_0'\\ &+\frac{A}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}G(s)\mathrm{d}s. \end{aligned} \tag{17}$$

Proof. By applying I^{α} to both sides of equation (16), we have

$$I^{\alpha}D^{\alpha}x(t)-AI^{\alpha}D^{\beta}x(t)=I^{\alpha}G(t), \tag{18}$$

and using the property established in Lemmas 2 and 3, we find that

$$\begin{aligned} x(t)={}&x(0)+x'(0)t+\frac{1}{2}x''(0)t^2+\frac{1}{6}x'''(0)t^3-\frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x(0)-\frac{At^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}x'(0)\\ &+\frac{A}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}G(s)\mathrm{d}s. \end{aligned} \tag{19}$$

Some of the initial conditions allow us to have the result.

Conversely, assume that $x(t)$ satisfy the equation (16), then we see easily the initial conditions.

We use the fact $D^{\alpha}I^{\alpha}G(t)=G(t)$ and $D^{\alpha}C=0$, where C is a constant; we get

$$D^{\alpha}x(t)-AD^{\beta}x(t)=G(t), \quad t\in I=[0,1]. \tag{20}$$

Let us now transform the above problem to a fixed point one. Consider the nonlinear operator $T\colon X\longrightarrow X$ defined by

$$\begin{aligned} Tx(t)={}&x_0+x_0't+\frac{1}{2}x_0''t^2+\frac{1}{6}x_0''t^3-\frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x_0-\frac{At^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}x_0'\\ &+\frac{A}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(t,x(t),D^{\beta}x(t),D^{\alpha}x(t)\big)\mathrm{d}s. \end{aligned} \tag{21}$$

To prove the main results, we need to work with the following hypotheses:

(H1) The function f defined on $I\times\mathbb{R}^{3n}$ is continuous.

(H2) There exist nonnegative constants c_1, c_2, and $c_3<1$ such that, for any $t\in I$, $x_1,x_2,x_3,x_1^*,x_2^*,x_3^*\in\mathbb{R}^n$,

$$\left|f(t,x_1,x_2,x_3)-f(t,x_1^*,x_2^*,x_3^*)\right|\le c_1\left|x_1-x_1^*\right|+c_2\left|x_2-x_2^*\right|+c_3\left|x_3-x_3^*\right|. \tag{22}$$

Also, we consider the quantities

$$D_1 = \frac{\|A\|}{\Gamma(\alpha-\beta+1)} + \frac{c_1\Gamma(3-\beta)+c_3\|A\|+c_2}{(1-c_3)\Gamma(\alpha+1)\Gamma(3-\beta)},$$
$$D_2 = \frac{\|A\|}{\Gamma(\alpha-\beta-1)} + \frac{c_1\Gamma(3-\beta)+c_3\|A\|+c_2}{(1-c_3)\Gamma(\alpha-1)\Gamma(3-\beta)}. \tag{23}$$

The first main result deals with the existence of a unique solution for (1). It is based on the application of Banach fixed point theorem for contraction mappings. □

Theorem 3. *If the conditions (H1) and (H2) are satisfied and $D<1$ $(D:=D_1+D_2)$, then problem (1) has a unique solution on I.*

Proof. It is sufficient for us to prove that H is a contraction mapping.

Let $(x,y)\in X^2$. Then, we can write

$$|Tx(t)-Ty(t)| \le \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x(s)-y(s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (s-t)^{\alpha-1}|g(s)-h(s)|\mathrm{d}s, \tag{24}$$

where $g,h\in C(I,\mathbb{R}^n)$ defined by $g(t)=f(t,x(t),D^\beta x(t),g(t)+AD^\beta x(t))$ and $h(t)=f(t,y(t),D^\beta y(t),h(t)+AD^\beta y(t))$.

From (H2) for each $t\in I$, we have

$$|g(t)-h(t)| \le c_1|x(t)-y(t)| + c_2\left|D^\beta(x(t)-y(t))\right| + c_3|g(t)-h(t)| + c_3\|A\||x(t)-y(t)|, \tag{25}$$

and using Lemma 4, we have

$$|g(t)-h(t)| \le \frac{c_1\Gamma(3-\beta)+c_3\|A\|+c_2}{(1-c_3)\Gamma(3-\beta)}\|x-y\|_X. \tag{26}$$

Therefore, we have for each $t\in I$,

$$\|Tx-Ty\|_\infty \le \left[\frac{\|A\|}{\Gamma(\alpha-\beta+1)} + \frac{c_1\Gamma(3-\beta)+c_3\|A\|+c_2}{(1-c_3)\Gamma(\alpha+1)\Gamma(3-\beta)}\right]\|x-y\|_X \le D_1\|x-y\|_X. \tag{27}$$

On the other hand, we have

$$(Tx)''(t) = x_0'' + x_0''t - \frac{At^{\alpha-\beta-2}}{\Gamma(\alpha-\beta-1)}x_0 - \frac{At^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}x_0' + \frac{A}{\Gamma(\alpha-\beta-2)}\int_0^t (t-s)^{\alpha-\beta-3}x(s)\mathrm{d}s$$
$$+ \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3} f\left(t,x(t),D^\beta x(t),D^\alpha x(t)\right)\mathrm{d}s, \tag{28}$$

which is clear in $C(I,\mathbb{R}^n)$.

Then, with the same arguments as before, we have

$$\left\|(Tx)'' - (Ty)''\right\|_\infty \le \left[\frac{\|A\|}{\Gamma(\alpha-\beta-1)} + \frac{c_1\Gamma(3-\beta) + c_3\|A\| + c_2}{(1-c_3)\Gamma(\alpha-1)\Gamma(3-\beta)}\right]\|x-y\|_X \le D_2\|x-y\|_X. \tag{29}$$

Thus, we have

$$\|Tx - Ty\|_X \le (D_1 + D_2)\|x-y\|_X. \tag{30}$$

Since $D<1$, then the operator T is contraction. Hence, by Banach's contraction principle, T has a unique fixed point which is the unique solution of problem (1).

The following main result deals with the existence of at least one solution of the studied problem. □

Theorem 4. *Under the hypotheses (H1) and (H2), problem (1) has at least one solution $u(t)$, $t\in I$.*

Proof. Let us prove the result by considering the following steps:

Continuous of T: if the proof is trivial, then it is omitted (we just apply the fact that f is continuous.

Uniform boundness of T: let us take $r>0$ and consider the (bounded) ball B_r: $=\{x\in X; \|x\|_X\le r\}$. For $y\in B_r$, we can write

$$|Ty(t)| \le |x_0| + \frac{1}{2}|x_0''| + \frac{1}{6}|x_0''| + \frac{\|A\||x_0|}{\Gamma(\alpha-\beta+1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha-\beta+2)} + \frac{\|A\|}{\Gamma(\alpha-\beta-1)}\|y\|_\infty + \frac{1}{\Gamma(\alpha+1)}\|g\|_\infty. \tag{31}$$

With a simple calculus, we get

$$\|g\|_\infty \le \frac{c_1\Gamma(3-\beta) + c_3\|A\| + c_2}{(1-c_3)\Gamma(3-\beta)}\|y\|_X + m^*, \tag{32}$$

where $m^* = \sup_{t\in I}|f(t,0,0,0)|$.

Then, we have

$$\|Ty\|_\infty \le |x_0| + \frac{1}{2}|x_0''| + \frac{1}{6}|x_0''| + \frac{\|A\||x_0|}{\Gamma(\alpha-\beta+1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha-\beta+2)} + \frac{m^*}{\Gamma(\alpha+1)} + D_1 r < +\infty, \tag{33}$$

and also we have

$$\left\|(Ty)''\right\|_\infty \le |x_0''| + |x_0'' + \frac{\|A\||x_0|}{\Gamma(\alpha-\beta-1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha-\beta)} + \frac{m^*}{\Gamma(\alpha-1)} + D_2 r < +\infty. \tag{34}$$

The above two inequalities show that $\|Ty\|_X < +\infty$.

Consequently, T is uniformly bounded.

Equicontinuity of T: we prove that, for any bounded set B_r for instance, we obtain that $T(B_r)$ is an equicontinuous set of X.

Take $t_1, t_2 \in [0,1]$, $t_1<t_2$ and consider the above (bounded) ball B_r of X. So, by considering $y\in B_r$, we can state that

$$\begin{aligned}|Ty(t_2) - Ty(t_1)| \le{}& |x_0'||t_2 - t_1| + \frac{1}{2}|x_0''||t_2^2 - t_1^2| + \frac{1}{6}|x_0''||t_2^3 - t_1^3| + \frac{\|A\||x_0'|}{\Gamma(\alpha-\beta+1)}\left|t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\right| \\ &+ \frac{\|A\||x_0''|}{\Gamma(\alpha-\beta+2)}\left|t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1}\right| + \frac{\|A\|r}{\Gamma(\alpha-\beta+1)}\left|t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\right| + \frac{M}{\Gamma(\alpha+1)}\left|t_2^{\alpha} - t_1^{\alpha}\right|,\end{aligned} \tag{35}$$

where $M = (c_1\Gamma(3-\beta) + c_3\|A\| + c_2 / (1-c_3)\Gamma(3-\beta))r + m^*$.

As $t_2 \longrightarrow t_1$, the right-hand side of the above inequality tends to zero, and we have also

$$\begin{aligned}\left|(Ty)''(t_2) - (Ty)''(t_1)\right| &\leq \left|x_0''\right|\left|t_2 - t_1\right| + \frac{\|A\|\left|x_0'\right|}{\Gamma(\alpha-\beta-1)}\left|t_2^{\alpha-\beta-2} - t_1^{\alpha-\beta-2}\right| + \frac{\|A\|\left|x_0''\right|}{\Gamma(\alpha-\beta)}\left|t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\right| \\ &+ \frac{\|A\|r}{\Gamma(\alpha-\beta-1)}\left|t_2^{\alpha-\beta-2} - t_1^{\alpha-\beta-2}\right| + \frac{M}{\Gamma(\alpha-1)}\left|t_2^{\alpha-2} - t_1^{\alpha-2}\right|.\end{aligned} \quad (36)$$

As $t_2 \longrightarrow t_1$, the right-hand side of the above inequality tends to zero. From a consequence of the Ascoli-Arzela's theorem, we conclude that T is completely continuous.

Boundness of A_γ: the set $A_\gamma := \{x \in X : x = \gamma Tx, \gamma \in]0,1[\}$ is bounded.

Let $y \in A_\gamma$. Then, we have $y = \gamma Ty$ for some $0 < \gamma < 1$. Hence, we can write

$$\|y\|_\infty \leq \gamma\left(\left|x_0\right| + \frac{1}{2}\left|x_0''\right| + \frac{1}{6}\left|x_0''\right| + \frac{\|A\|\left|x_0\right|}{\Gamma(\alpha-\beta+1)} + \frac{\|A\|\left|x_0'\right|}{\Gamma(\alpha-\beta+2)} + \frac{m^*}{\Gamma(\alpha+1)} + D_1 r\right), \quad (37)$$

$$\left\|(y)''\right\|_\infty \leq \gamma\left(\left|x_0''\right| + \left|x_0''\right| + \frac{\|A\|\left|x_0\right|}{\Gamma(\alpha-\beta-1)} + \frac{\|A\|\left|x_0'\right|}{\Gamma(\alpha-\beta)} + \frac{m^*}{\Gamma(\alpha-1)} + D_2 r\right). \quad (38)$$

From (37) and (38), we state that $\|y\|_X < \infty$. The set is thus bounded.

Consequently, thanks to Schaefer fixed point theorem, we deduce that T has at least one fixed point. Thus, problem (1) has a solution. □

Example 1. Let us consider the following example:

$$\begin{cases} D^\alpha x(t) - AD^\beta x(t) = f\left(t, x(t), D^\beta x(t), D^\alpha x(t)\right), \quad t \in I = [0,1], \\ x(0) = \left(\frac{1}{2}, 0\right), \\ x'(0) = \left(0, \frac{1}{2}\right), \\ x''(0) = \left(\frac{1}{2}, 0\right), \\ x'''(0) = \left(0, \frac{1}{2}\right), \end{cases} \quad (39)$$

where

$$f\colon [0,1]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$
$$(t,u,v,w)\mapsto\left(\frac{1}{10e^{t+5}(1+\|u\|+\|v\|+\|w\|)},\frac{1}{10e^{t+7}(1+\|u\|+\|v\|+\|w\|)}\right), \tag{40}$$

with $\|u\| = \max\{x_1, x_2\}$, $u = (x_1, x_2)$. We take $A = \begin{pmatrix} (1/20) & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha = (15/4)$, and $\beta = (3/2)$.

We can see clearly that the function f is continuous.

For any $u, v, w, \overline{u}, \overline{v}, \overline{w} \in \mathbb{R}^2$ and $t \in [0,1]$,

$$\left|\frac{1}{10e^{t+5}(1+\|\overline{u}\|+\|\overline{v}\|+\|\overline{w}\|)}-\frac{1}{10e^{t+5}(1+\|\overline{u}\|+\|\overline{v}\|+\|\overline{w}\|)}\right|\le\frac{1}{10e^5}(|u-\overline{u}|+|v-\overline{v}|+|w-\overline{w}|),$$
$$\left|\frac{1}{10e^{t+7}(1+\|\overline{u}\|+\|\overline{v}\|+\|\overline{w}\|)}-\frac{1}{10e^{t+7}(1+\|\overline{u}\|+\|\overline{v}\|+\|\overline{w}\|)}\right|\le\frac{1}{10e^7}(|u-\overline{u}|+|v-\overline{v}|+|w-\overline{w}|), \tag{41}$$

which give

$$|f(t,u,v,w)-f(t,\overline{u},\overline{v},\overline{w})|\le c_1|u-\overline{u}|+c_2|v-\overline{v}|+c_3|w-\overline{w}|, \tag{42}$$

where $c_1 = c_2 = c_3 = (1/10e^5)$.

Hence, the hypotheses (H1) and (H2) are satisfied.

With a simple computation, we get $D_1 = 0,0807$ and $D_2 = 0,24$, which imply $D < 1$.

Thus, all the assumptions from (H1)–(H3) are satisfied. From Theorem 3, we conclude that equation (1) has a unique solution.

4. Conclusion

In this work, we consider a nonlinear implicit fractional differential equation and we use the Caputo derivative operator. We prove two theorems and an example to illustrate our results. In the first theorem, we prove the existence and uniqueness of the solution and the second theorem deals with the existence of at least one solution. The methods used are the Banach's fixed point theorem and Schaefer's fixed point theorem. Here, two Caputo derivative operators of different fractional orders were used in the considered equation and it would be relevant to generalize this idea by considering several Caputo operators of different fractional orders.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no known conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] S. Abbas and M. Benchohra, "On the generalized Ulam–Hyers–Rassias stability for Darboux problem for partial fractional implicit differential equations," *Applied Mathematics E-Notes*, vol. 14, pp. 20–28, 2014.

[2] S. Abbas, M. Benchohra, J. Graef, and J. Henderson, "Implicit fractional differential and integral equations: existence and stability," in *De Gruyter Series in Nonlinear Analysis and Applications*Vol. 26, De Gruyter, Berlin, Germany, 2018.

[3] S. Abbas, M. Benchohra, and G. M. N'Guérékata, "Topics in fractional differential equations," in *Developments in Mathematics*Vol. 27, Springer, New York, NY, USA, 2012.

[4] S. Abbas, M. Benchohra, and G. M. N'Guérékata, "Advanced fractional differential and integral equations," in *Mathematics Research Developments*Nova Science Publishers, Inc., New York, NY, USA, 2015.

[5] M. A. Dokuyucu and H. Dutta, "A fractional order model for Ebola virus with the new Caputo fractional derivative without singular kernel," *Chaos, Solitons & Fractals*, vol. 134, Article ID 109717, 2020.

[6] M. A. Dokuyucu, “Caputo and atangana-baleanu-caputo fractional derivative applied to Garden equation,” *Turish Journal of Sciences*, vol. 5, no. 1, pp. 1–7, 2020.

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, “Theory and applications of fractional differential equations,” in *North-Holland Mathematics Studies*Elsevier Science B.V., Amsterdam, Netherlands, 2006.

[8] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, NY, USA, 1993.

[9] N. Sene and A. Ndiaye, “Fractional model for a class of diffusion-reaction equation represented by the fractional-order derivative,” *Fractal and Fractional*, vol. 4, no. 2, 15 pages, Article ID 8815377, 2020.

[10] R. P. Agarwal, D. O’Regan, and S. Staněk, “Positive solutions for mixed problems of singular fractional differential equations,” *Mathematische Nachrichten*, vol. 285, no. 1, 2012.

[11] Z. Bai and W. Sun, “Existence and multiplicity of positive solutions for singular fractional boundary value problems,” *Computers & Mathematics with Applications*, vol. 63, no. 9, 2012.

[12] Z. Bekkouche, Z. Dahmani, and G. Zhang, “Solutions and stabilities for a 2D-non homogeneous Lane-Emden fractional system,” *International Journal of Open Problems in Computer Science and Mathematics*, vol. 11, no. 2, 2018.

[13] Y. Gouari, Z. Dahmani, and A. Ndiaye, “A generalized sequential problem of Lane Emden type via fractional calculus,” *Moroccan Journal of Pure and Applied Analysis*, vol. 6, no. 2, Article ID 168183, 2020.

[14] R. W. Ibrahim, “Stability of a fractional differential equation,” *International Journal of Mathematical, Computational, Physical and Quantum Engineering*, vol. 7, no. 3, 2013.

[15] M. Feckan, Y. Zhou, and J. Wang, “On the concept and existence of solution of an impulsive fractional differential equations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, p. 1627, 1996.

[16] T. L. Guo and W. Jiang, “Impulsive fractional functional differential equations,” *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3414–3424, 2012.

[17] J. Wang, X. Li, and W. Wei, “On the natural solution of an impulsive fractional differential equation of order $q \in (1;2)$,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, pp. 1381–1391, 2012.

[18] Y. Zhou, F. Jiao, and J. Li, “Existence and uniqueness for fractional neutral differential equations with infinite delay,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3249–3256, 2009.

[19] G. Chai, “Existence results for anti-periodic boundary value problems of fractional differential equations,” *Advances in Difference Equations*, vol. 2013, no. 35, pp. 1–15, 2013.

[20] J. Sheng and W. Jiang, “Existence and uniqueness of the solution of fractional damped dynamical systems,” *Advances in Difference Equations*, vol. 2017, no. 16, pp. 1–14, 2017.

[21] A. Keten, M. Yavuz, and D. Baleanu, “Nonlocal cauchy problem via a fractional operator involving power kernel in Banach spaces,” *Fractal and Fractional*, vol. 3, no. 2, p. 27, 2019.

[22] M. A. Khan, Z. Hammouch, and D. Baleanu, “Modeling the dynamics of hepatitis E via the Caputo-Fabrizio derivative,” *Mathematical Modelling of Natural Phenomena*, vol. 14, no. 3, p. 19, 2019.

[23] P. Veeresha, H. M. Baskonus, D. G. Prakasha, W. Gao, and G. Yel, “Regarding new numerical solution of fractional schistosomiasis disease arising in biological phenomena,” *Chaos Solitons Fractals*, vol. 133, Article ID 109661, 2020.

[24] X.-J. Yang, M. Abdel-Aty, and C. Cattani, “A new general fractional-order derivataive with Rabotnov fractional-exponential kernel applied to model the anomalous heat transfer,” *Thermal Science*, vol. 23, no. 3, pp. 1677–1681, 2019.

[25] M. Yavuz, “Characterizations of two different fractional operators without singular kernel,” *Mathematical Modelling of Natural Phenomena*, vol. 14, no. 3, p. 13, 2019.

[26] A. Yokus and S. Gulbahar, “Numerical solutions with linearization techniques of the fractional Harry Dym equation,” *Applied Mathematics and Nonlinear Sciences*, vol. 4, no. 1, pp. 35–41, 2019.

[27] S. Abbas, M. Benchohra, S. Bouriah, and J. Henderson, “Existence and uniqueness results for nonlinear implicit fractional systems,” *Journal of Nonlinear Evolution Equations and Applications*, no. 5, pp. 81–94, 2019.

[28] X. L. Dind, “Controllability and optimality of linear time-invariant neutral control systems with different fractional orders,” *Acta Mathematica Scientia*, vol. 35B, no. 5, pp. 1003–1013, 2015.

[29] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, NY, USA, 2003.

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Research Article

Existence and Stability for a Nonlinear Coupled p-Laplacian System of Fractional Differential Equations

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In this paper, we study the nonlinear coupled system of equations with fractional integral boundary conditions involving the Caputo fractional derivative of orders θ_1 and θ_2 and Riemann–Liouville derivative of orders ϱ_1 and ϱ_2 with the p-Laplacian operator, where $n-1<\theta_1,\theta_2,\varrho_1,\varrho_2\leq n$, and $n\geq 3$. With the help of two Green's functions ($G^{\varrho_1}(w,\mathfrak{T}), G^{\varrho_2}(w,\mathfrak{T})$), the considered coupled system is changed to an integral system. Since topological degree theory is more applicable in nonlinear dynamical problems, the existence and uniqueness of the suggested coupled system are treated using this technique, and we find appropriate conditions for positive solutions to the proposed problem. Moreover, necessary conditions are highlighted for the Hyer–Ulam stability of the solution for the specified fractional differential problems. To confirm the theoretical analysis, we provide an example at the end.

1. Introduction

The theoretical development of fractional calculus and its applications is more important to model nonlinear complex problems with the arbitrary fractional order. The subject of fractional differential equations (FDEs) has become an important area in real life because of their ability to model a lot of physical phenomena associated with rapid and concise changes with their significance in science and engineering through the past three decades, such as chemistry, physics, biology, engineering, visco-elasticity, electrotechnical, signal processing, electrochemistry, and controllability (see the details, [1–9], and the reference therein). In the near time, the nonlinear fractional partial differential equations are the most applied research area in which most authors and scientists are focused for their investigation. In this case, the Caputo derivative plays a great role to analyze the specific application of nonlinear PDEs. in [10], the authors have studied the cancer treatment model based on Caputo–Fabrizio fractional derivative. After integrating the model into the Caputo–Fabrizio fractional derivative, they have analyzed the existence of the solution as well. The Caputo–Fabrizio fractional derivative is implemented in [11] for the modeling and characterizing of the alcoholism. By applying the fixed-point theorem, they have studied the existence and uniqueness of the alcoholism model. The spread of the SIQR model is investigated by [12] using the Caputo derivative. They have justified the stability and uniqueness of the nonvirus equilibrium and virus equilibrium point.

For this problem, different authors proposed different numerical solution techniques. The analysis with the nonlinear time-fractional HIV/AIDS transmission model is considered in [13], in which the numerical solution is found using the fractional variational iteration method with convergence analysis. The nonlinear garden equation is studied in [14] based on the Atangana–Baleau Caputo derivative. He has highlighted the fixed-point theorem for proving the existence and uniqueness of the garden equation.

One of the main difficulties for the solution of the nonlinear fractional PDEs is to analyze the existence theory of solutions. Sufficient conditions for the existence and uniqueness of solutions (EUS) have been obtained by using different nonlinear analysis techniques and fixed-point theorems (for more details, read [15–18]). Also, the boundary value problems with various boundary conditions for many ordinary differential equations are studied [19–23]. However, the theory of boundary value problems for nonlinear FDEs is still not discussed more, and many problems of this theory require to be explored. On the contrary, the investigation of coupled systems of the differential equations is also significant because systems of this kind appear in various applied nature problems (refer [24–28]).

The topological degree theory is a useful tool in nonlinear analysis with numerous applications to operatorial equations, optimization theory, fractal theory, and other topics. We will see the following consideration of topological degree theory with boundary conditions based on the Caputo fractional derivative by different authors. Isaia [29] applied the topological degree theory to establish sufficient conditions for the existence of a solution for the following nonlinear integral equations:

$$\pi(w) = \mathscr{Z}(w, \pi(w)) + \int_a^b \mathscr{Q}(w, \mathfrak{J}, \pi(\mathfrak{J}))d\mathfrak{J}, \quad w \in [a, b], \tag{1}$$

where $\mathscr{Z}: [a, b] \times IR \longrightarrow IR$ and $\mathscr{Q}: [a, b] \times [a, b] \times IR \longrightarrow IR$ are continuous functions. In their study [30], Wang et al. used the topological degree method to obtain some existence conditions of the solution for the following nonlocal Cauchy problem:

$$\begin{cases} {}^cD^{\varrho}\pi(w) = \mathscr{Z}(w, \pi(w)), & 0 \leq w \leq W, \\ \pi(0) + h(\pi) = \pi_0, \end{cases} \tag{2}$$

where ${}^cD^{\varrho}$ denotes the Caputo fractional derivative with order $\varrho \in (0, 1)$ and $\mathscr{Z}: C([0, w], \mathbb{R}) \longrightarrow \mathbb{R}$ and $\pi_0 \in \mathbb{R}$ are continuous. The nonlocal term $h: C([0, w], \mathbb{R}) \longrightarrow \mathbb{R}$ is a given function. Proceeding on the same fashion, Shah and Khan [31] proved the EUS for a coupled system under the fractional derivatives by using the technique of degree theory given as follows:

$$\begin{cases} {}^cD^{\varrho_1}\pi_1(w) = \mathscr{Z}_1(w, \pi_1(w), \pi_2(w)), & w \in [0, 1], \\ {}^cD^{\varrho_2}\pi_2(w) = \mathscr{Z}_2(w, \pi_1(w), \pi_2(w)), & w \in [0, 1], \\ \alpha_1\pi_1(0) - \beta_1\pi_1(\theta) - \delta_1\pi_1(1) = \phi_1(\pi_1), \\ \alpha_2\pi_2(0) - \beta_2\pi_2(\vartheta) - \delta_2\pi_2(1) = \phi_2(\pi_2), \end{cases} \tag{3}$$

where $\varrho_1, \varrho_2, \theta, \vartheta \in (0, 1)$, $\mathscr{Z}_1, \mathscr{Z}_2 \in [0, 1] \times R^2 \longrightarrow R$, and $\phi_1, \phi_2: C([0, 1], \mathbb{R}) \longrightarrow \mathbb{R}$ are continuous. Khan et al. [32] used the above-mentioned method to study the following coupled system in the sense of Caputo derivatives with p-Laplacian:

$$\begin{cases} D_{0_+}^{\theta_1}\left(\phi_p\left(D_{0_+}^{\varrho_1}\pi_1(w)\right)\right) + \mathscr{Z}_1(w, \pi_2(w)) = 0, \\ D_{0+}^{\theta_2}\left(\phi_p\left(D_{0_+}^{\varrho_2}\pi_2(w)\right)\right) + \mathscr{Z}_2(w, \pi_1(w)) = 0, \\ D_{0_+}^{\varrho_1}\pi_1(0) = \left(\phi_p\left(D_{0_+}^{\varrho_1}\pi_1(w)\right)\right)'|_{w=0} = D_{0_+}^{\delta_1}\pi_1(w)|_{w=\eta_1} = 0, \\ \pi_1(1) = \dfrac{\Gamma(2-\delta_1)}{\eta_1^{1-\delta_1}} J^{\varrho_1-\delta_1}\phi_q\left(J_{0_+}^{\theta_1}\mathscr{Z}_1(w, \pi_2(w))\right)|_{w=\eta_1}, \\ D_{0_+}^{\varrho_2}\pi_2(0) = \left(\phi_p\left(D_{0_+}^{\varrho_2}\pi_1(w)\right)\right)'|_{w=0} = D_{0_+}^{\delta_2}\pi_2(w)|_{w=\eta_2} = 0, \\ \pi_2(1) = \dfrac{\Gamma(2-\delta_2)}{\eta_2^{1-\delta_2}} J^{\varrho_2-\delta_2}\phi_q\left(J_{0_+}^{\theta_2}\mathscr{Z}_1(w, \pi_1(w))\right)|_{w=\eta_2}, \end{cases} \tag{4}$$

where $\varrho_i, \theta_i \in (1, 2]$ and $\delta_i, \eta_i \in (0, 1]$, for $i = 1$ and 2. The study of positive solutions to boundary value problems for fractional-order differential equations using the topological degree theory technique is rarely available in the literature, so this research field needs further elaboration. Most papers that dealt the topological degree theory with fractional orders belong to $(0, 1)$ or $(1, 2]$. For the uniqueness and existence analysis of nonlinear fractional differential equations, the case only Caputo fractional derivative is used frequently.

Thus, our motivation to this study is developing a sufficient condition for the coupled nonlinear fractional derivative that is based on both Caputo and Riemann–Liouville derivatives. The fractional order in our study is expanded to

$(n-1, n]$, and we have used a technique of topological degree theory for the analysis of existence and uniqueness of our coupled system defined below. Besides, we have investigated Hyers–Ulam stability to the nonlinear coupled system of fractional-ordered ordinary differential equations with boundary conditions designed by the following:

$$\begin{cases} {}^cD^{\theta_1}\left(\phi_p\left({}^RD^{\varrho_1}\pi_1(w)\right)\right) = \mathscr{X}_1(w,\pi_2(w)), \quad w \in [0,1], \\ {}^cD^{\theta_2}\left(\phi_p\left({}^RD^{\varrho_2}\pi_2(w)\right)\right) = \mathscr{X}_2(w,\pi_1(w)), \quad w \in [0,1], \\ \left(\phi_p\left({}^RD^{\varrho_1}\pi_1(w)\right)\right)^{(i)}|_{w=0} = 0, \quad i = 0,1,2,\ldots,n-1, \\ I^{k-\varrho_1}\pi_1(w)|_{w=0} = 0, \quad k = 1,2,4,\ldots,n, \\ D^{\lambda}\pi_1(w)|_{w=1} = \dfrac{1}{\Gamma(\lambda)}\displaystyle\int_0^W (W-\mathfrak{T})^{\lambda-1}\varphi(\pi_1(\mathfrak{T}))d\mathfrak{T}, \\ \left(\phi_p\left({}^RD^{\varrho_2}\pi_2(w)\right)\right)^{(j)}|_{w=0} = 0, \quad j = 0,1,2,\ldots,n-1, \\ I^{h-\varrho_2}\pi_2(w)|_{w=0} = 0, \quad h = 1,2,4,\ldots,n, \\ D^{\sigma}\pi_2(w)|_{w=1} = \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^W (W-\mathfrak{T})^{\sigma-1}\varpi(\pi_2(\mathfrak{T}))d\mathfrak{T}, \end{cases} \tag{5}$$

where $\varrho_1, \varrho_2, \theta_1, \theta_2 \in (n-1, n]$, $n \geq 3$, $1 < \lambda$, $\sigma \leq 2$, ${}^cD^{\theta_1}$ and ${}^cD^{\theta_2}$ denote the Caputo fractional derivatives, ${}^RD^{\varrho_1}$ and ${}^RD^{\varrho_2}$ are the Riemann–Liouville fractional derivatives, and $\mathscr{X}_1, \mathscr{X}_2$: $[0,1] \times IR \longrightarrow IR$ are nonlinear functions, and the boundary functions $\varphi, \varpi \in L[0,1]$. ϕ_p represents the p-Laplacian operator such that $\phi_p(\nu) = \nu|\nu|^{p-2}$, and $\phi_q = \phi_p^{-1}$ denotes the inverse of p-Laplacian, where $(1/p) + (1/q) = 1$. Since it is difficult to find the exact solution of the nonlinear differential equations, stability and uniqueness have played a great role to get the approximate solution for the given nonlinear problems. Therefore, scientists and researchers have given attention to study the various forms of stability to the nonlinear problems in the sense of Ulam and their multiple types in the last few decades. We observe that the concept of Hyers–Ulam stability is fundamental in realistic problems, such as numerical analysis, biology, and economics (see [33–38]).

The remaining part of this manuscript is structured as follows. In Section 2, we have introduced some basic definitions and lemmas that we need to prove our main results. By using the topological degree theory, the results of existence and uniqueness for the solutions are obtained in Section 3. In Section 4, we investigate the stability of Hyers–Ulam to our proposed coupled system. The theoretical results are demonstrated by providing an example in Section 5, and finally, we have drawn the conclusion in Section 6.

2. Preliminaries

In this section, we introduce some basic notions, definitions, and important lemmas which are used in this article.Let $\Pi = C([0,1], IR)$ be a Banach space for all continuous functions π: $[0,1] \longrightarrow IR$ with the norm $\|\pi\| = \sup\{|\pi(w)|: 0 \leq w \leq 1\}$. Further, $\Omega = \Pi \times \Pi$ is also a Banach space under the norms $\|(\pi_1, \pi_2)\| = \|\pi_1\| + \|\pi_2\|$ and $|(\pi_1, \pi_2)| = \max\{\|\pi_1\|, \|\pi_2\|\}$. The family of each bounded set of $P(\Omega)$ symbolized by $\mathfrak{B}$.

Definition 1. For $\mathscr{X}(w)$: $(0, +\infty) \longrightarrow \mathbb{R}$, the Caputo fractional derivative of noninteger order $\varrho > 0$ is known by

$${}^cD^{\varrho}\mathscr{X}(w) = \frac{1}{\Gamma(n-\varrho)}\int_0^w (w-\mathfrak{T})^{n-\varrho-1}\mathscr{X}^{(n)}(\mathfrak{T})d\mathfrak{T}, \tag{6}$$

where $n-1 < \varrho < n$, the integral in the right side is pointwise defined on $(0, \infty)$, and $\mathscr{X}(w)$ is a continuous function.

Definition 2. For $\mathscr{X}(w)$: $(0, +\infty) \longrightarrow \mathbb{R}$, the Riemann–Liouville fractional derivative of noninteger order $\varrho > 0$ is known by

$${}^RD^{\varrho}\mathscr{X}(w) = \frac{1}{\Gamma(n-\varrho)}\left(\frac{d}{dw}\right)^n\int_0^w (w-\mathfrak{T})^{n-\varrho-1}\mathscr{X}(\mathfrak{T})d\mathfrak{T}, \tag{7}$$

where $n-1 < \varrho < n$, the integral in the right side is pointwise defined on $(0, \infty)$, and $\mathscr{X}(w)$ is a continuous function.

Definition 3. For $\mathscr{X}(w)$: $(0, +\infty) \longrightarrow \mathbb{R}$, the Riemann–Liouville fractional integral of order $\varrho > 0$ is defined by

$$I^{\varrho}\mathscr{X}(w) = \frac{1}{\Gamma(\varrho)}\int_0^w (w-\mathfrak{T})^{\varrho-1}\mathscr{X}(\mathfrak{T})d\mathfrak{T}, \tag{8}$$

where the integral on the right side is pointwise defined on $(0, +\infty)$ and $\Gamma(\varrho)$ indicates the Gamma function defined as

$$\Gamma(\varrho) = \int_0^{\infty} e^{-\mathfrak{T}}\mathfrak{T}^{\varrho-1}d\mathfrak{T}. \tag{9}$$

Lemma 1 (see [39]). *Let $\varrho > 0$ and $\mathscr{X} \in C(0,1) \cap L^1(0,1)$. Then, the general solution of the fractional differential equation $D^{\varrho}\mathscr{X}(w) = \pi(w)$ is given by*

$$\mathscr{X}(w) = \pi(w) + c_0 + c_1 w + c_2 w^2 + \cdots + c_{n-1}w^{n-1}, \tag{10}$$

for $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$.

Lemma 2 (see [2, 8]). *Let $\theta \in (n-1, n]$, $\mathscr{X} \in C^{n-1}$, and ${}^cD^{\theta}$ is the fractional derivative for Caputo, then*

$$I^{\theta c}D^{\theta}\mathscr{X}(w) = \mathscr{X}(w) + a_1 + a_2 w + a_3 w^2 + a_4 w^3 + \cdots + a_n w^{n-1}, \tag{11}$$

for $a_i \in \mathbb{R}$ and $i = 1, 2, 3, 4, \ldots, n$.

Lemma 3 (see [2, 8]). *Let $\varrho \in (n-1, n]$, $\mathscr{X} \in C^{n-1}$, and ${}^RD^{\varrho}$ is the fractional derivative for Riemman–Liouville, then*

$$I^{\varrho R}D^{\varrho}\mathscr{Z}(w) = \mathscr{Z}(w) + b_1 w^{\varrho-1} + b_2 w^{\varrho-2} + b_3 w^{\varrho-3} + a_4 w^{\varrho-4} + \cdots + b_n w^{\varrho-n}, \tag{12}$$

for $b_i \in \mathbb{R}$ *and* $i = 1, 2, 3, 4, \ldots, n$.

Lemma 4 (see [22]). *For* $\varrho, \varepsilon > 0$, *the following relations are satisfying:*

$$D^{\varrho}\xi^{\varepsilon} = \frac{\Gamma(\varepsilon+1)}{\Gamma(1+\varepsilon-\varrho)}\xi^{\varepsilon-\varrho}, \qquad I^{\varrho}\xi^{\varepsilon} = \frac{\Gamma(\varepsilon+1)}{\Gamma(1+\varepsilon+\varrho)}\xi^{\varepsilon+\varrho}. \tag{13}$$

Definition 4. The Kuratowski measure of noncompactness is the map $\vartheta\colon \mathfrak{B} \longrightarrow (0, \infty)$ known as $\vartheta(\mathscr{B}) = \inf\{d > 0\colon \mathscr{B}$ which admits a finite cover by sets of diameter $\leq d\}$, where $\mathscr{B} \in \mathfrak{B}$.

Proposition 1 (see [40]). *The Kuratowski measure of* ϑ *satisfies the following properties:*

(a) *The Kuratowski measure* $\vartheta(\mathscr{B}) = 0$; *for a relative compact* $\mathscr{B}$

(b) ϑ *is a seminorm, i.e.,* $\vartheta(\mu\mathscr{B}) = |\mu|\vartheta(\mathscr{B}), \mu \in IR$, *and* $\vartheta(\mathscr{B}_1 + \mathscr{B}_2) \leq \vartheta(\mathscr{B}_1) + \vartheta(\mathscr{B}_2)$

(c) $\mathscr{B}_1 \subset \mathscr{B}_2$ *implies* $\vartheta(\mathscr{B}_1) \leq \vartheta(\mathscr{B}_2)$; $\vartheta(\mathscr{B}_1 \bigcup \mathscr{B}_2) = \sup\{\vartheta(\mathscr{B}_1), \vartheta(\mathscr{B}_2)\}$

(d) $\vartheta(conv\,\mathscr{B}) = \vartheta(\mathscr{B})$

(e) $\vartheta(\overline{\mathscr{B}}) = \vartheta(\mathscr{B})$

Definition 5. Suppose that the function $\mathscr{F}\colon \Psi \longrightarrow \Pi$ is a continuous and bounded map, where $\Psi \subset \Pi$. $\mathscr{F}$ is called ϑ–Lipschitz with $\eta \geq 0$, and if $\vartheta(\mathscr{F}(\mathscr{B})) \leq \eta\vartheta(\mathscr{B}), \forall \mathscr{B} \subset \Psi$ is bounded.

Moreover, if $\eta < 0$, then $\mathscr{F}$ will be a strict ϑ–contraction.

Definition 6. The function $\mathscr{F}$ is called ϑ–condensing, and if $\vartheta(\mathscr{F}(\mathscr{B})) < \vartheta(\mathscr{B}), \forall \mathscr{B} \subset \Psi$ is bounded with $\vartheta(\mathscr{B}) > 0$.

In other words,$\vartheta(\mathscr{F}(\mathscr{B})) \geq \vartheta(\mathscr{B})$ implies $\vartheta(\mathscr{B}) = 0$.

We indicate that the class of each ϑ–condensing mappings $\mathscr{F}\colon \Psi \longrightarrow \Pi$ by $C_{\vartheta}(\Psi)$ and the class of each strict ϑ–contractions $\mathscr{F}\colon \Psi \longrightarrow \Pi$ by $\zeta C_{\vartheta}(\Psi)$.

We remark that $C_{\vartheta}(\Psi)^{\zeta C_{\vartheta}(\Psi)}$, and every $\mathscr{F} \in C_{\vartheta}(\Psi)$ is ϑ–Lipschitz with $\eta = 1$. As well, we recall that $\mathscr{F}\colon \Psi \longrightarrow \Pi$ is Lipschitz if $\exists\, \eta > 0$ such that $\|\mathscr{F}(\pi) - \mathscr{F}(\overline{\pi})\| \leq \eta\|\pi - \overline{\pi}\|$, and $\forall \pi, \overline{\pi} \in \Psi$. Also, $\mathscr{F}$ is a strict contraction under the condition $\eta < 1$.

Proposition 2 (see [31]). *Let* $\mathscr{F}, \mathscr{G}\colon \Psi \longrightarrow \Pi$ *be* ϑ*–Lipschitz operators with constants* η_1 and η_2, *respectively, then* $\mathscr{F} + \mathscr{G}\colon \Psi \longrightarrow \Pi$ *is* ϑ*–Lipschitz with constants* $\eta_1 + \eta_2$.

Proposition 3 (see [41]). *The operator* $\mathscr{F}\colon \Psi \longrightarrow \Pi$ *is compact if and only if* $\mathscr{F}$ *is* ϑ*–Lipschitz with* $\eta = 0$.

Proposition 4 (see [31]). *The operator* $\mathscr{F}\colon \Psi \longrightarrow \Pi$ *is Lipschitz with constant* η *if and only if* $\mathscr{F}$ *is* ϑ*–Lipschitz with constant* η.

Lemma 5 (see [39]). *Let* ϕ_p *be a nonlinear p-Laplacian operator.*

(1) If $1 < p \leq 2$, $j_1 j_2 > 0$, and $|j_1|, |j_2| \geq \rho > 0$, then

$$\left|\phi_p(j_1) - \phi_p(j_2)\right| \leq (p-1)\rho^{p-2}\left|j_1 - j_2\right|. \tag{14}$$

(2) If $p > 2$ and $|j_1|, |j_2| \leq \rho^*$, then

$$\left|\phi_p(j_1) - \phi_p(j_2)\right| \leq (p-1)\rho^{*p-2}\left|j_1 - j_2\right|. \tag{15}$$

Theorem 1 (see [29]). *Let* $\mathscr{F}\colon \Omega \longrightarrow \Omega$ *be a* ϑ*–contraction, and* $\Xi = \{\omega \in \Omega\colon \exists, 0 \leq \rho \leq 1$ *such that* $\omega = \rho\mathscr{F}\omega$. *If* $\Xi \subset \Omega$ *is a bounded set, there exists* $r > 0$ *such that* $\Xi \subset \mathscr{B}_r(0)$, *then the degree* $\deg(I - \rho\mathscr{F}, \mathscr{B}_r(0), 0) = 1, \forall \rho \in [0, 1]$.

Thus, $\mathscr{F}$ has at least one fixed point, and the set of the fixed points of $\mathscr{F}$ lies in $\mathscr{B}_r(0)$.

The above theorem that we mentioned plays a substantial role in obtaining our main results.

3. Main Results

In the current section, we establish some appropriate conditions for proposed coupled system (5).

Theorem 2. *Let* $\mathscr{Z}\colon [0,1] \longrightarrow IR$ *be a* ϱ_1 *times' integrable function. Then, for* $\varrho_1 \in (3, n]$ *and positive integer* $n \geq 4$, *the solution of the boundary value problem is as follows:*

$$\begin{cases} {}^{c}D^{\theta_1}\left(\phi_p\left({}^{R}D^{\varrho_1}\pi_1(w)\right)\right) = \mathscr{Z}_1(w, \pi_2(w)), & w \in [0,1], \\ \left(\phi_p\left({}^{R}D^{\varrho_1}\pi_1(w)\right)\right)^{(i)}|_{w=0} = 0, & i = 0, 1, 2, \ldots, n-1, \\ I^{k-\varrho_1}\pi_1(w)|_{w=0} = 0, & k = 1, 2, 4, \ldots, n, \\ D^{\lambda}\pi_1(w)|_{w=1} = \dfrac{1}{\Gamma(\lambda)}\displaystyle\int_0^W (W - \mathfrak{I})^{\lambda-1}\varphi(\pi_1(\mathfrak{I}))d\mathfrak{I}, \end{cases} \tag{16}$$

is given by

$$\pi_1(w) = \frac{\Gamma(\varrho_1 - \lambda - 2)w^{\varrho_1-3}}{\Gamma(\varrho_1 - 2)\Gamma(\lambda)}\int_0^W (W - \mathfrak{I})^{\lambda-1}\varphi(\pi_1(\mathfrak{I}))d\mathfrak{I} + \int_0^1 G^{\varrho_1}(w, \mathfrak{I})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{I}} (\mathfrak{I} - \tau)^{\theta_1-1}\mathscr{Z}_1(\tau, \pi_2(\tau))d\tau\right)d\mathfrak{I}, \tag{17}$$

where $G^{\varrho_1}(w, \mathfrak{I})$ *is the Green's function provided by*

$$G^{\varrho_1}(w,\mathfrak{T}) = \begin{cases} \dfrac{(w-\mathfrak{T})^{\varrho_1-1}}{\Gamma(\varrho_1)} - \dfrac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}(1-\mathfrak{T})^{\varrho_1-\lambda-1}}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda)}, & 0\le \mathfrak{T}\le w\le 1,\\ \dfrac{-\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}(1-\mathfrak{T})^{\varrho_1-\lambda-1}}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda)}, & 0\le w\le \mathfrak{T}\le 1. \end{cases} \tag{18}$$

Proof. Applying the integral operator I^{θ_1} and using Lemma 2 on (16), we get

$$\phi_p\left[{}^RD^{\varrho_1}\pi_1(w)\right] = I^{\theta_1}\mathscr{Z}_1(w,\pi_2(w)) + a_1 + a_2 w + a_3 w^2 + a_4 w^3 + \cdots + a_n w^{n-1}. \tag{19}$$

Using the condition $(\phi_p[{}^RD^{\varrho_1}\pi_1(w)])^{(i)}|_{w=0} = 0$, for $i = 0,1,2,\ldots,n-1$, in (19), we obtain $a_1 = a_2 = a_3 = \cdots = a_n = 0$, and then, we get

$$\phi_p\left[{}^RD^{\varrho_1}\pi_1(w)\right] = I^{\theta_1}\mathscr{Z}_1(w,\pi_2(w)). \tag{20}$$

From (20), we have

$${}^RD^{\varrho_1}\pi_1(w) = \phi_q\left(I^{\theta_1}(\mathscr{Z}_1(w,\pi_2(w))\right). \tag{21}$$

Applying the operator I^{ϱ_1} and using Lemma 3 in (21), we get

$$\pi_1(w) = I^{\varrho_1}\left(\phi_q\left(I^{\theta_1}(\mathscr{Z}_1(w,\pi_2(w)))\right)\right) + b_1 w^{\varrho_1-1} + b_2 w^{\varrho_1-2} + b_3 w^{\varrho_1-3} + b_4 w^{\varrho-4} + \cdots + b_n w^{\varrho_1-n}. \tag{22}$$

Using the condition $I^{k-\varrho_1}\pi(w)|_{w=0} = 0, k = 1,2,4,\ldots,n$, we get $b_1 = b_2 = b_4 = \ldots = b_n = 0$, and then, we obtain

$$\pi_1(w) = I^{\varrho_1}\left(\phi_q\left(I^{\theta_1}(\mathscr{Z}_1(w,\pi_2(w)))\right)\right) + b_3 w^{\varrho_1-3}. \tag{23}$$

Using the condition $D^{\lambda}\pi_1(w)|_{w=1} = {}_0I_W^{\lambda}\varphi(\pi_1) = (1/\Gamma(\lambda))\int_0^W (W-\mathfrak{T})^{\lambda-1}\varphi(\pi_1(\mathfrak{T}))d\mathfrak{T}$ and Lemma 4 in (22), we get

$$b_3 = \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)}\left[{}_0I_W^{\lambda}\varphi(\pi_1) - I^{\varrho_1-\lambda}\left(\phi_q\left(I^{\theta_1}(\mathscr{Z}_1(1,\pi_2(1)))\right)\right)\right]. \tag{24}$$

Putting the value of b_3 in (23), we get

$$\begin{aligned}\pi_1(w) &= \frac{1}{\Gamma(\varrho_1)}\int_0^w (w-\mathfrak{T})^{\varrho_1-1}\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\tau)^{\theta_1-1}\mathscr{Z}_1((\tau,\pi_2(\tau))d\tau\right)d\mathfrak{T}\\ &+ \frac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{T})^{\lambda-1}\varphi(\pi_1(\mathfrak{T}))d\mathfrak{T}\\ &- \frac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda)}\int_0^1 (1-\mathfrak{T})^{\varrho_1-\lambda-1}\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau,\pi_2(\tau))d\tau\right)d\mathfrak{T},\end{aligned} \tag{25}$$

which can be written after rearranging as follows:

$$\begin{aligned}\pi_1(w) &= \frac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{T})^{\lambda-1}\varphi(\pi_1(\mathfrak{T}))d\mathfrak{T}\\ &+ \int_0^1 G^{\varrho_1}(w,\mathfrak{T})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\tau)^{\theta_1-1}\mathscr{Z}_1((\tau,\pi_2(\tau))d\tau\right)d\mathfrak{T},\end{aligned} \tag{26}$$

where $G^{\varrho_1}(w,\mathfrak{T})$ is the Green's function defined in (18). □

In view of Theorem 2, the identical coupled system of Hammerstein-kind integral equations to fractional differential equation coupled system (5) is given as follows:

$$\begin{cases}\pi_1(w) = \dfrac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\displaystyle\int_0^W (W-\mathfrak{T})^{\lambda-1}\varphi(\pi_1(\mathfrak{T}))d\mathfrak{T}\\ \quad + \displaystyle\int_0^1 G^{\varrho_1}(w,\mathfrak{T})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau,\pi_2(\tau))d\tau\right)d\mathfrak{T},\\ \pi_2(w) = \dfrac{\Gamma(\varrho_2-\sigma-2)w^{\varrho_2-3}}{\Gamma(\varrho_2-2)\Gamma(\sigma)}\displaystyle\int_0^W (W-\mathfrak{T})^{\sigma-1}\varpi(\pi_2(\mathfrak{T}))d\mathfrak{T}\\ \quad + \displaystyle\int_0^1 G^{\varrho_2}(w,\mathfrak{T})\phi_q\left(\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\tau)^{\theta_2-1}\mathscr{Z}_2(\tau,\pi_1(\tau))d\tau\right)d\mathfrak{T},\end{cases} \tag{27}$$

where $G^{\varrho_2}(w,\mathfrak{T})$ is the Green's function provided by

$$G^{\varrho_2}(w,\mathfrak{J}) = \begin{cases} \dfrac{(w-\mathfrak{J})^{\varrho_2-1}}{\Gamma(\varrho_2)} - \dfrac{\Gamma(\varrho_2-\sigma-2)w^{\varrho_2-3}(1-\mathfrak{J})^{\varrho_2-\sigma-1}}{\Gamma(\varrho_2-2)\Gamma(\varrho_2-\sigma)}, & 0\le\mathfrak{J}\le w\le 1, \\ \dfrac{-\Gamma(\varrho_2-\sigma-2)w^{\varrho_2-3}(1-\mathfrak{J})^{\varrho_2-\sigma-1}}{\Gamma(\varrho_2-2)\Gamma(\varrho_2-\sigma)}, & 0\le w\le\mathfrak{J}\le 1. \end{cases} \tag{28}$$

From $G^{\varrho_1}(w,\mathfrak{J})$ and $G^{\varrho_2}(w,\mathfrak{J})$ obviously,

$$\max_{w\in[0,1]}\left|G^{\varrho_1}(w,\mathfrak{J})\right| = \frac{\Gamma(\varrho_1-\lambda-2)(1-\mathfrak{J})^{\varrho_1-\lambda-1}}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda)},$$
$$\max_{w\in[0,1]}\left|G^{\varrho_2}(w,\mathfrak{J})\right| = \frac{\Gamma(\varrho_2-\sigma-2)(1-\mathfrak{J})^{\varrho_2-\sigma-1}}{\Gamma(\varrho_2-2)\Gamma(\varrho_2-\sigma)}, \quad \mathfrak{J}\in[0,1]. \tag{29}$$

We define the operators $\mathcal{F}_1\colon \Pi_1 \longrightarrow \Pi_1$ and $\mathcal{F}_2\colon \Pi_2 \longrightarrow \Pi_2$ as

$$\mathcal{F}_1(\pi_1)(w) = \frac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{J})^{\lambda-1}\varphi(\pi_1(\mathfrak{J}))\mathrm{d}\mathfrak{J},$$
$$\mathcal{F}_2(\pi_2)(w) = \frac{\Gamma(\varrho_2-\sigma-2)w^{\varrho_2-3}}{\Gamma(\varrho_2-2)\Gamma(\sigma)}\int_0^W (W-\mathfrak{J})^{\sigma-1}\varpi(\pi_2(\mathfrak{J}))\mathrm{d}\mathfrak{J},$$
$$\mathcal{G}_1,\mathcal{G}_2\colon \Omega\longrightarrow\Omega, \tag{30}$$

as

$$\mathcal{G}_1(\pi_2)(w) = \int_0^1 G^{\varrho_1}(w,\mathfrak{J})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_1-1}\mathcal{Z}_1(\tau,\pi_2(\tau))\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J},$$
$$\mathcal{G}_2(\pi_1)(w) = \int_0^1 G^{\varrho_2}(w,\mathfrak{J})\phi_q\left(\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_2-1}\mathcal{Z}_2(\tau,\pi_1(\tau))\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J}. \tag{31}$$

Therefore, we have $\mathcal{F}(\pi_1,\pi_2) = (\mathcal{F}_1,\mathcal{F}_2)(\pi_1,\pi_2)$, $\mathcal{G}(\pi_1,\pi_2) = (\mathcal{G}_1,\mathcal{G}_2)(\pi_1,\pi_2)$, and $\mathcal{T}(\pi_1,\pi_2) = \mathcal{F}(\pi_1,\pi_2)+\mathcal{G}(\pi_1,\pi_2)$. Thus, the equivalent operator equation for the toppled system of Hammerstein-kind integral equations (27) is provided by

$$(\pi_1,\pi_2) = \mathcal{T}(\pi_1,\pi_2) = \mathcal{F}(\pi_1,\pi_2)+\mathcal{G}(\pi_1,\pi_2). \tag{32}$$

Consequently, the solutions of system (27) are the fixed points of operator equation (32).

Now, we need to list the following assumptions to complete our results.

(H_1) For $\hbar,\pi_1,\ell,\pi_2\in IR$, the nonlocal functions φ and ϖ satisfy $\|\varphi(\hbar)-\varphi(\pi_1)\|\le K_\varphi\|\hbar-\pi_1\|$ and $\|\varpi(\ell)-\varpi(\pi_2)\|\le K_\varpi\|\ell-\pi_2\|$ such that $K_\varphi,K_\varpi\in[0,1)$

(H_2) With the positive constants given $C_\varphi, C_\varpi, N_\varphi, N_\varpi$, and $q_1\in[0,1)$, the nonlocal functions φ and ϖ for $\pi_1,\pi_2\in IR$ satisfy the following growth conditions $|\varphi(\pi_1)|\le C_\varphi|\pi_1|^{q_1}+N_\varphi$ and $|\varpi(\pi_2)|\le C_\varpi|\pi_2|^{q_1}+N_\varpi$

(H_3) With the presence of constants $g,h,N_{\mathcal{Z}_1},N_{\mathcal{Z}_2}$, and $q_2\in[0,1)$, the nonlinear functions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ for $\pi_1,\pi_2\in IR$ satisfy the following growth conditions:

$$\left|\mathcal{Z}_1(w,\pi_2)\right|\le\phi_p\left(g|\pi_2|^{q_2}+N_{\mathcal{Z}_1}\right),$$
$$\left|\mathcal{Z}_2(w,\pi_1)\right|\le\phi_p\left(h|\pi_1|^{q_2}+N_{\mathcal{Z}_2}\right). \tag{33}$$

(H_4) For $\hbar,\pi_1,\ell,\pi_2\in IR$, there exists positive constants $L_{\mathcal{Z}_1}$ and $L_{\mathcal{Z}_2}$ such that

$$\left|\mathcal{Z}_1(w,\ell)-\mathcal{Z}_1(w,\pi_2)\right|\le L_{\mathcal{Z}_1}\left|\ell-\pi_2\right|,$$
$$\left|\mathcal{Z}_2(w,\hbar)-\mathcal{Z}_2(w,\pi_1)\right|\le L_{\mathcal{Z}_2}\left|\hbar-\pi_1\right|. \tag{34}$$

Theorem 3. *Assume that (H_1) and (H_2) hold true. Then, the operator $\mathcal{F}$ is Lipschitz and satisfies the following growth condition:*

$$\left\|\mathcal{F}(\pi_1,\pi_2)\right\|\le C_{\mathcal{F}}\left\|(\pi_1,\pi_2)\right\|^{q_1}+N_{\mathcal{F}}, \quad \forall(\pi_1,\pi_2)\in\Omega. \tag{35}$$

Proof. By assumption (H_1), we get

$$\begin{aligned}\left|\mathcal{F}_1(\pi_1)(w)-\mathcal{F}_1(\overline{\pi_1})(w)\right| &= \left|\frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{I})^{\lambda-1}\left[\varphi(\pi_1)-\varphi(\overline{\pi_1})\right]\mathrm{d}\mathfrak{I}\right| \\ &\leq \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{I})^{\lambda-1}\left|\varphi(\pi_1)-\varphi(\overline{\pi_1})\right|\mathrm{d}\mathfrak{I},\end{aligned} \tag{36}$$

which yields

$$\left\|\mathcal{F}_1(\pi_1)-\mathcal{F}_1(\overline{\pi_1})\right\| \leq \overline{K}_\varphi\left\|\pi_1-\overline{\pi_1}\right\|, \tag{37}$$

where

$$\overline{K}_\varphi = \frac{K_\varphi\Gamma(\varrho_1-\lambda-2)W^\lambda}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)} \in [0,1). \tag{38}$$

To get the growth condition, consider

$$\begin{aligned}\left|\mathcal{F}_1(\pi_1)(w)\right| &= \left|\frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{I})^{\lambda-1}\varphi(\pi_1(\mathfrak{I}))\mathrm{d}\mathfrak{I}\right| \\ &\leq \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{I})^{\lambda-1}\left|\varphi(\pi_1(\mathfrak{I}))\right|\mathrm{d}\mathfrak{I},\end{aligned} \tag{39}$$

which means that

$$\left\|\mathcal{F}_1\pi_1\right\|_c \leq \frac{\Gamma(\varrho_1-\lambda-2)W^\lambda}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)}\left[C_\varphi\|\pi_1\|_c^{q_1}+N_\varphi\right]. \tag{40}$$

In a similar manner, we have

$$\begin{aligned}\left|\mathcal{F}_2(\pi_2)(w)\right| &= \left|\frac{\Gamma(\varrho_2-\sigma-2)}{\Gamma(\varrho_2-2)\Gamma(\sigma)}\int_0^W (W-\mathfrak{I})^{\sigma-1}\varpi(\pi_2(\mathfrak{I}))\mathrm{d}\mathfrak{I}\right| \\ &\leq \frac{\Gamma(\varrho_2-\sigma-2)}{\Gamma(\varrho_2-2)\Gamma(\sigma)}\int_0^W (W-\mathfrak{I})^{\sigma-1}\left|\varpi(\pi_2(\mathfrak{I}))\right|\mathrm{d}\mathfrak{I},\end{aligned} \tag{41}$$

which implies that

$$\left\|\mathcal{F}_2\pi_2\right\|_c \leq \frac{\Gamma(\varrho_2-\sigma-2)W^\sigma}{\Gamma(\varrho_2-2)\Gamma(\sigma+1)}\left[C_\varpi\|\pi_2\|_c^{q_1}+N_\varpi\right]. \tag{42}$$

Now,

$$\begin{aligned}\left\|\mathcal{F}(\pi_1,\pi_2)\right\|_c &\leq \frac{\Gamma(\varrho_1-\lambda-2)W^\lambda}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)}\left[C_\varphi\|\pi_1\|_c^{q_1}+N_\varphi\right] \\ &\quad+\frac{\Gamma(\varrho_2-\sigma-2)W^\sigma}{\Gamma(\varrho_2-2)\Gamma(\sigma+1)}\left[C_\varpi\|\pi_2\|_c^{q_1}+N_\varpi\right] \\ &\leq \left(\frac{\Gamma(\varrho_1-\lambda-2)W^\lambda}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)}C_\varphi\|\pi_1\|_c^{q_1}\right. \\ &\quad\left.+\frac{\Gamma(\varrho_2-\sigma-2)W^\sigma}{\Gamma(\varrho_2-2)\Gamma(\sigma+1)}C_\varpi\|\pi_2\|_c^{q_1}\right) \\ &\quad+\left(\frac{\Gamma(\varrho_1-\lambda-2)W^\lambda N_\varphi}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)}+\frac{\Gamma(\varrho_2-\sigma-2)W^\sigma N_\varpi}{\Gamma(\varrho_2-2)\Gamma(\sigma+1)}\right).\end{aligned} \tag{43}$$

Thus,

$$\begin{aligned}\left\|\mathcal{F}(\pi_1,\pi_2)\right\|_c &\leq C_{\mathcal{F}}\left[\|\pi_1\|^{q_1}+\|\pi_2\|^{q_1}\right]+N_{\mathcal{F}} \\ &= C_{\mathcal{F}}\left\|(\pi_1,\pi_2)\right\|^{q_1}+N_{\mathcal{F}},\end{aligned} \tag{44}$$

where

$$\begin{aligned}C_{\mathcal{F}} &= \max\left\{\frac{\Gamma(\varrho_1-\lambda-2)W^\lambda}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)}C_\varphi, \frac{\Gamma(\varrho_2-\sigma-2)W^\sigma}{\Gamma(\varrho_2-2)\Gamma(\sigma+1)}C_\varpi\right\}, \\ N_{\mathcal{F}} &= \frac{\Gamma(\varrho_1-\lambda-2)W^\lambda N_\varphi}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)}+\frac{\Gamma(\varrho_2-\sigma-2)W^\sigma N_\varpi}{\Gamma(\varrho_2-2)\Gamma(\sigma+1)}.\end{aligned} \tag{45}$$

□

Theorem 4. *Suppose that (H_3) is satisfied. Then, the operator $\mathcal{G}$ is continuous and satisfies the following growth condition:*

$$\|G(\pi_1,\pi_2)\| \leq \Lambda\|(\pi_1,\pi_2)\|^{q_2} + \Theta, \quad \text{for all } (\pi_1,\pi_2) \in \Omega, \tag{46}$$

where $\Lambda = \gamma(g+h)$ and $\Theta = \gamma(N_{\mathcal{Z}_1} + N_{\mathcal{Z}_2})$ such that

$$\begin{aligned}\gamma = \max\Bigg\{&\left(\frac{1}{\Gamma(\varrho_1+1)} + \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda+1)}\right)\left[\frac{1}{\Gamma(\theta_1+1)}\right]^{q-1},\\ &\cdot\left(\frac{1}{\Gamma(\varrho_2+1)} + \frac{\Gamma(\varrho_2-\sigma-2)}{\Gamma(\varrho_2-2)\Gamma(\varrho_2-\sigma+1)}\right)\left[\frac{1}{\Gamma(\theta_2+1)}\right]^{q-1}\Bigg\}.\end{aligned} \tag{47}$$

Proof. Let $\mathfrak{B}_r = \{(\pi_1,\pi_2) \in \Omega: \|(\pi_1,\pi_2)\| \leq r\}$ be a bounded set with a sequence (π_{1_n},π_{2_n}) converging to (π_1,π_2) in $\mathfrak{B}_r$. In order to show that $\mathcal{G}$ is continuous, we have to prove that $\|\mathcal{G}(\pi_{1_n},\pi_{2_n}) - \mathcal{G}(\pi_1,\pi_2)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Let us choose the following:

$$\begin{aligned}&\left|\left(\mathcal{G}_1(\pi_{2_n}) - \mathcal{G}_1(\pi_2)\right)(w)\right|\\ &= \Bigg|\int_0^1 G^{\varrho_1}(w,\mathfrak{F})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_1-1}\mathcal{Z}_1(\tau,\pi_{2_n}(\tau))\mathrm{d}\tau\right)\mathrm{d}\mathfrak{F}\\ &\quad - \int_0^1 G^{\varrho_1}(w,\mathfrak{F})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_1-1}\mathcal{Z}_1(\tau,\pi_2(\tau)\mathrm{d}\tau)\mathrm{d}\mathfrak{F}\right)\Bigg|\\ &\leq \int_0^1\|G^{\varrho_1}(w,\mathfrak{F})\|\phi_q\left[\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_1-1}\mathcal{Z}_1\left(\tau,\pi_{2_n}(\tau)\mathrm{d}\tau\right)\right]\\ &\quad - \phi_q\left[\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_1-1}\mathcal{Z}_1(\tau,\pi_2(\tau)\mathrm{d}\tau)\right]|\mathrm{d}\mathfrak{F}\\ &\leq (q-1)\rho_1^{q-2}\int_0^1\|G^{\varrho_1}(w,\mathfrak{F})\|\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_1-1}\mathcal{Z}_1\left(\tau,\pi_{2_n}(\tau)\mathrm{d}\tau\right)\\ &\quad - \frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_1-1}\mathcal{Z}_1(\tau,\pi_2(\tau)\mathrm{d}\tau)|\mathrm{d}\mathfrak{F}\\ &\leq (q-1)\rho_1^{q-2}\int_0^1\left|G^{\varrho_1}(w,\mathfrak{F})\right|\left[\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_1-1}\left|\mathcal{Z}_1\left(\tau,\pi_{2_n}(\tau)\right) - \mathcal{Z}_1(\tau,\pi_2(\tau))\right|\mathrm{d}\tau\right]\mathrm{d}\mathfrak{F}.\end{aligned} \tag{48}$$

The continuity of $\mathcal{Z}_1$ implies that $|\mathcal{Z}_1(\tau,\pi_{2_n}(\tau) - \mathcal{Z}_1(\tau,\pi_2(\tau))| \longrightarrow 0$ as $n \longrightarrow \infty$, and then,

$$\left\|\mathcal{G}_1\left(\pi_{2_n}\right)(w) - \mathcal{G}_1(\pi_2)(w)\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{49}$$

and also, we can in the same way prove that

$$\begin{aligned}&\left\|\mathcal{G}_2\left(\pi_{1_n}\right)(w) - \mathcal{G}_2(\pi_1)(w)\right\|\\ &\leq (q-1)\rho_2^{q-2}\int_0^1\left|G^{\varrho_2}(w,\mathfrak{F})\right|\left[\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{F}}(\mathfrak{F}-\tau)^{\theta_2-1}|\mathcal{Z}_2\left(\tau,\pi_{1_n}(\tau)\right)\right.\\ &\quad \left. - \mathcal{Z}_2(\tau,\pi_1(\tau))|\mathrm{d}\tau\right]\mathrm{d}\mathfrak{F}.\end{aligned} \tag{50}$$

The continuity of $\mathscr{Z}_2$ implies that $|\mathscr{Z}_2(\tau, \pi_{1_n}(\tau) - \mathscr{Z}_2(\tau, \pi_1(\tau))| \longrightarrow 0$ as $n \longrightarrow \infty$, and then,

$$\left\|\mathscr{G}_2\left(\pi_{1_n}\right)(w) - \mathscr{G}_2(\pi_1)(w)\right\| \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty. \tag{51}$$

Thus, from (49) and (51), we have

$$\begin{aligned}
&\left\|\mathscr{G}\left(\pi_{1_n}, \pi_{2_n}\right)(w) - \mathscr{G}(\pi_1, \pi_2)(w)\right\| \\
&\leq (q-1)\rho_1^{q-2}\int_0^1 |G^{\varrho_1}(w, \mathfrak{J})|\left[\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_1-1}|\mathscr{Z}_1\left(\tau, \pi_{2_n}(\tau)\right)\right. \\
&\quad \left. - \mathscr{Z}_1(\tau, \pi_2(\tau))|\mathrm{d}\tau\right]\mathrm{d}\mathfrak{J} \\
&\quad + (q-1)\rho_2^{q-2}\int_0^1 |G^{\varrho_2}(w, \mathfrak{J})|\left[\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_2-1}|\mathscr{Z}_2\left(\tau, \pi_{1_n}(\tau)\right)\right. \\
&\quad \left. - \mathscr{Z}_2(\tau, \pi_1(\tau))|\mathrm{d}\tau\right]\mathrm{d}\mathfrak{J}.
\end{aligned} \tag{52}$$

From the continuity of $\mathscr{Z}_1$ and $\mathscr{Z}_2$ and (52), we have

$$\left\|\mathscr{G}\left(\pi_{1_n}, \pi_{2_n}\right)(w) - \mathscr{G}(\pi_1, \pi_2)(w)\right\| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{53}$$

To calculate (46) for $\mathscr{G}$, using assumption (H_3) and (29), we obtain

$$\begin{aligned}
|\mathscr{G}_1(\pi_2)(w)| &= \left|\int_0^1 \mathscr{G}^{\varrho_1}(w, \mathfrak{J})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau, \pi_2(\tau)\mathrm{d}\tau)\mathrm{d}\mathfrak{J}\right)\right| \\
&\leq \int_0^1 |G^{\varrho_1}(w, \mathfrak{J})|\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_1-1}\left|\mathscr{Z}_1(\tau, \pi_2(\tau))\right|\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J} \\
&\leq \int_0^1 |G^{\varrho_1}(w, \mathfrak{J})|\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_1-1}\phi_p\left(g|\pi_2|^{q_2} + N_{\mathscr{Z}_1}\right)\right)\mathrm{d}\mathfrak{J} \\
&\leq \left(\frac{1}{\Gamma(\varrho_1+1)} + \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda+1)}\right)\left[\frac{1}{\Gamma(\theta_1+1)}\right]^{q-1}\left(g|\pi_2|^{q_2} + N_{\mathscr{Z}_1}\right).
\end{aligned} \tag{54}$$

From assumption (H_3) and (29), we get

$$\begin{aligned}
|\mathscr{G}_2(\pi_1)(w)| &= \left|\int_0^1 \mathscr{G}^{\varrho_2}(w, \mathfrak{J})\phi_q\left(\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_2-1}\mathscr{Z}_2(\tau, \pi_1(\tau)d\tau)\right)\mathrm{d}\mathfrak{J}\right| \\
&\leq \int_0^1 |G^{\varrho_2}(w, \mathfrak{J})|\phi_q\left(\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_2-1}\left|\mathscr{Z}_2(\tau, \pi_1(\tau))\right|\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J} \\
&\leq \int_0^1 |G^{\varrho_2}(w, \mathfrak{J})|\phi_q\left(\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_2-1}\phi_p\left(h|\pi_1|^{q_2} + N_{\mathscr{Z}_2}\right)\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J}.
\end{aligned} \tag{55}$$

Then,

$$\begin{aligned}
|\mathscr{G}_2(\pi_1)(w)| &\leq \int_0^1 |G^{\varrho_2}(w, \mathfrak{J})|\phi_q\left(\frac{1}{\Gamma(\theta_2)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\tau)^{\theta_2-1}\phi_p\left(h|\pi_1|^{q_2} + N_{\mathscr{Z}_2}\right)\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J} \\
&\leq \left(\frac{1}{\Gamma(\varrho_2+1)} + \frac{\Gamma(\varrho_2-\sigma-2)}{\Gamma(\varrho_2-2)\Gamma(\varrho_2-\sigma+1)}\right)\left[\frac{1}{\Gamma(\theta_2+1)}\right]^{q-1}\left(h|\pi_1|^{q_2} + N_{\mathscr{Z}_2}\right).
\end{aligned} \tag{56}$$

By the help of (54) and (55), we have obtained

$$\begin{aligned}\|\mathscr{G}(\pi_1,\pi_2)\| &= \|\mathscr{G}_1(\pi_2)\| + \|\mathscr{G}_2(\pi_1)\| \\ &\le \gamma\left(g\|\pi_2\|^{q_2} + N_{\mathscr{Z}_1}\right) + \gamma\left(h\|\pi_1\|^{q_2} + N_{\mathscr{Z}_2}\right) \\ &\le \gamma(g+h)\left(\|\pi_2\|^{q_2} + \|\pi_1\|^{q_2}\right) + \gamma\left(N_{\mathscr{Z}_1} + N_{\mathscr{Z}_2}\right) \\ &= \Lambda\|(\pi_1,\pi_2)\|^{q_2} + \Theta.\end{aligned} \tag{57}$$

□

Theorem 5. *The operator $\mathscr{G}: \Omega \longrightarrow \Omega$ is ϑ–Lipschitz with constant zero and is compact.*

Proof. Take a bounded set E and a sequence (π_{1_n}, π_{2_n}) such that $E \subset \mathfrak{B}_r \subseteq \Omega$. Then, using (46), we have

$$\left\|\mathscr{G}\left(\pi_{1_n}, \pi_{2_n}\right)\right\| \le \Lambda\|(\pi_1,\pi_2)\| + \Theta, \quad \text{for all } (\pi_1,\pi_2) \in \Omega, \tag{58}$$

which means that $\mathscr{G}$ is bounded. Now, for all $(\pi_{1_n}, \pi_{2_n}) \in E$, we have, for $0 \le w_1 < w_2 \le 1$,

$$\begin{aligned}&\left|\mathscr{G}_1\pi_{2_n}(w_1) - \mathscr{G}_1\pi_{2_n}(w_2)\right| \\ &= \Big|\int_0^1 G^{\varrho_1}(w_1,\mathfrak{I})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{I}}(\mathfrak{I}-\tau)^{\theta_1-1}\mathscr{Z}_1((\tau,\pi_2(\tau))\mathrm{d}\tau)\mathrm{d}\mathfrak{I}\right) \\ &\quad - \int_0^1 G^{\varrho_1}(w_2,\mathfrak{I})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{I}}(\mathfrak{I}-\tau)^{\theta_1-1}\mathscr{Z}_1((\tau,\pi_2(\tau))\mathrm{d}\tau)\mathrm{d}\mathfrak{I}\right)\Big| \\ &\le \int_0^1\left|G^{\varrho_1}(w_1,\mathfrak{I}) - G^{\varrho_1}(w_2,\mathfrak{I})\right|\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{I}}(\mathfrak{I}-\tau)^{\theta_1-1}\phi_p\left(g\|\pi_1\|^{q_2} + N_{\mathscr{Z}_1}\right)\mathrm{d}\tau\right)\mathrm{d}\mathfrak{I}.\end{aligned} \tag{59}$$

Hence, it follows that

$$\begin{aligned}&\left|\mathscr{G}_1\pi_{2_n}(w_1) - \mathscr{G}_1\pi_{2_n}(w_2)\right| \\ &\le \left[\frac{(w_1^{\varrho_1} - w_2^{\varrho_1})}{\Gamma(\varrho_1+1)} + \frac{\left(w_1^{\varrho_1-3} - w_2^{\varrho_1-3}\right)}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda+1)}\right]\left[\frac{1}{\Gamma(\theta_1+1)}\right]^{q-1} \\ &\quad \times\left(g\|\pi_1\|^{q_2} + N_{\mathscr{Z}_1}\right).\end{aligned} \tag{60}$$

Similarly, we have

$$\begin{aligned}&\left|\mathscr{G}_2\pi_{1_n}(w_1) - \mathscr{G}_2\pi_{1_n}(w_2)\right| \\ &\le \left[\frac{(w_1^{\varrho_2} - w_2^{\varrho_2})}{\Gamma(\varrho_2+1)} + \frac{\left(w_1^{\varrho_2-3} - w_2^{\varrho_2-3}\right)}{\Gamma(\varrho_2-2)\Gamma(\varrho_2-\sigma+1)}\right]\left[\frac{1}{\Gamma(\theta_2+1)}\right]^{q-1} \\ &\quad \times\left(h\|\pi_2\|^{q_2} + N_{\mathscr{Z}_2}\right).\end{aligned} \tag{61}$$

Both the right sides of (55) and (61) tend to be zero as $w_1 \longrightarrow w_2$. Therefore, the operators $\mathscr{G}_1$ and $\mathscr{G}_2$ are equicontinuous, and hence, $\mathscr{G} = (\mathscr{G}_1, \mathscr{G}_2)$ is equicontinuous on E. Thus, $\mathscr{G}(E)$ is compact by the theorem of Arzela–Ascoli. Moreover, through Proposition 3, $\mathscr{G}$ is ϑ–Lipschitz with constant zero. □

Theorem 6. *Suppose that (H_1)–(H_3) are satisfied with $\Lambda + C_{\mathscr{F}} \le 1$. Then, the toppled system (5) has at least one solution $(\pi_1,\pi_2) \in \Omega$. Furthermore, the set of solutions of (5) is bounded in Ω.*

Proof. With the help of Theorem 3, $\mathscr{F}$ is ϑ–Lipschitz with constant $0 \le C_{\mathscr{F}} < 1$, and $\mathscr{G}$ is ϑ–Lipschitz with constant zero by Theorem 5. Thus, $\mathscr{T}$ is strictly ϑ–condensing with constant η by Proposition 2. Now, let us set that we have to show that $\mathfrak{B}$ is bounded in Ω. In fact,

$$\begin{aligned}\|(\pi_1,\pi_2)\| &= \|\kappa\mathscr{T}(\pi_1,\pi_2)\| \le \|\mathscr{T}(\pi_1,\pi_2)\| \\ &\le \left(\|\mathscr{F}(\pi_1,\pi_2)\| + \|\mathscr{G}(\pi_1,\pi_2)\|\right) \\ &\le C_{\mathscr{F}}\|(\pi_1,\pi_2)\|^{q_1} + N_{\mathscr{F}} + \Lambda\|(\pi_1,\pi_2)\|^{q_2} + \Theta \\ &= (C_{\mathscr{F}} + \Lambda)\|(\pi_1,\pi_2)\|^{q_3} + N_{\mathscr{F}} + \Theta, \\ &\quad \text{where } q_3 = \max\{q_1,q_2\}.\end{aligned} \tag{62}$$

Obviously, $\|\pi_1,\pi_2\|$ is bounded. If not correct, take $\|\pi_1,\pi_2\| = \mathscr{S}$ such that $\mathscr{S} \longrightarrow \infty$ and $q_3 \in (0,1)$. Consequently,

$$\begin{aligned}&1 \le (C_{\mathscr{F}} + \Lambda)\frac{\|(\pi_1,\pi_2)\|^{q_3}}{\|(\pi_1,\pi_2)\|} + \frac{N_{\mathscr{F}} + \Theta}{\|(\pi_1,\pi_2)\|}, \\ &1 \le \frac{(C_{\mathscr{F}} + \Lambda)\mathscr{S}^{q_3}}{\mathscr{S}} + \frac{N_{\mathscr{F}} + \Theta}{\mathscr{S}}, \\ &1 \le \frac{(C_{\mathscr{F}} + \Lambda)}{\mathscr{S}^{1-q_3}} + \frac{N_{\mathscr{F}} + \Theta}{\mathscr{S}} \longrightarrow 0 \quad \text{as } \mathscr{S} \longrightarrow \infty,\end{aligned} \tag{63}$$

which is a contradiction. So, $\mathfrak{B}$ is bounded. Thus, by Theorem 7, we conclude that $\mathscr{T}$ has at least one fixed point and that is a solution of system (5), and the set of solutions is bounded in Ω. □

Theorem 7. *Suppose that* (H_1)–(H_4) *hold and* $\chi < 1$, *where*

$$\begin{aligned}\chi = {} & \frac{K_{\varphi}\Gamma(\varrho_1 - \lambda - 2)W^{\lambda}}{\Gamma(\varrho_1 - 2)\Gamma(\lambda + 1)} + \frac{K_{\varpi}\Gamma(\varrho_2 - \sigma - 2)W^{\sigma}}{\Gamma(\varrho_2 - 2)\Gamma(\sigma + 1)} \\ & + (q-1)\rho_1^{q-2}L_{\mathcal{Z}_1}\left(\frac{1}{\Gamma(\varrho_1 + 1)} + \frac{\Gamma(\varrho_1 - \lambda - 2)}{\Gamma(\varrho_1 - 2)\Gamma(\varrho_1 - \lambda + 1)}\right)\left(\frac{1}{\Gamma(\theta_1 + 1)}\right) \\ & + (q-1)\rho_2^{q-2}L_{\mathcal{Z}_2}\left(\frac{1}{\Gamma(\varrho_2 + 1)} + \frac{\Gamma(\varrho_2 - \sigma - 2)}{\Gamma(\varrho_2 - 2)\Gamma(\varrho_2 - \sigma + 1)}\right)\left(\frac{1}{\Gamma(\theta_2 + 1)}\right).\end{aligned} \tag{64}$$

Then, toppled system (5) has a unique solution.

Proof. Let (π_1, π_2) and $(\overline{\pi_1}, \overline{\pi_2}) \in \Omega$ are two solutions, then

$$\begin{aligned}\left|\mathcal{T}(\pi_1, \pi_2) - \mathcal{T}(\overline{\pi_1}, \overline{\pi_2})\right| & = \left|[\mathcal{F}(\pi_1, \pi_2) + \mathcal{G}(\pi_1, \pi_2)] - [\mathcal{F}(\overline{\pi_1}, \overline{\pi_2}) + \mathcal{G}(\overline{\pi_1}, \overline{\pi_2})]\right| \\ & \leq \left|\mathcal{F}(\pi_1, \pi_2) - \mathcal{F}(\overline{\pi_1}, \overline{\pi_2})\right| + \left|\mathcal{G}(\pi_1, \pi_2) - \mathcal{G}(\overline{\pi_1}, \overline{\pi_2})\right|,\end{aligned} \tag{65}$$

and after simplification, we obtain

$$\begin{aligned}& \left\|\mathcal{T}(\pi_1, \pi_2) - \mathcal{T}(\overline{\pi_1}, \overline{\pi_2})\right\| \\ & \leq \Bigg(\frac{K_{\varphi}\Gamma(\varrho_1 - \lambda - 2)W^{\lambda}}{\Gamma(\varrho_1 - 2)\Gamma(\lambda + 1)} + \frac{K_{\varpi}\Gamma(\varrho_2 - \sigma - 2)W^{\sigma}}{\Gamma(\varrho_2 - 2)\Gamma(\sigma + 1)} \\ & \quad + (q-1)\rho_1^{q-2}L_{\mathcal{Z}_1}\left(\frac{1}{\Gamma(\varrho_1 + 1)} + \frac{\Gamma(\varrho_1 - \lambda - 2)}{\Gamma(\varrho_1 - 2)\Gamma(\varrho_1 - \lambda + 1)}\right)\left(\frac{1}{\Gamma(\theta_1 + 1)}\right) \\ & \quad + (q-1)\rho_2^{q-2}L_{\mathcal{Z}_2}\left(\frac{1}{\Gamma(\varrho_2 + 1)} + \frac{\Gamma(\varrho_2 - \sigma - 2)}{\Gamma(\varrho_2 - 2)\Gamma(\varrho_2 - \sigma + 1)}\right)\left(\frac{1}{\Gamma(\theta_2 + 1)}\right)\Bigg) \\ & \quad \times \left\|(\pi_1, \pi_2) - (\overline{\pi_1}, \overline{\pi_2})\right\|,\end{aligned} \tag{66}$$

which implies that

$$\left\|\mathcal{T}(\pi_1, \pi_2) - \mathcal{T}(\overline{\pi_1}, \overline{\pi_2})\right\| \leq \chi\left\|(\pi_1, \pi_2) - (\overline{\pi_1}, \overline{\pi_2})\right\|. \tag{67}$$

Thus, the operator $\mathcal{T}$ is a contraction as $\chi < 1$, and by the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point, and then, considered toppled system (5) has a unique solution. □

4. Hyers–Ulam Stability

In this section, we investigate the stability of Hyers–Ulam for the suggested toppled system.

Definition 7. We say that the toppled system of Hammerstein-kind integral equations (27) is Hyers–Ulam stable if there exists positive constants $a, b, c,$ and d such that, for each $\xi_1, \xi_2 > 0$ and any solution (π_1^*, π_2^*) of the system

$$\begin{cases} \left|\pi_1^*(w) - \dfrac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\displaystyle\int_0^W (W-\mathfrak{J})^{\lambda-1}\varphi(\pi_1^*(\mathfrak{J}))d\mathfrak{J}\right. \\ \left. + \displaystyle\int_0^1 G^{\varrho_1}(w,\mathfrak{J})\phi_q\left(\dfrac{1}{\Gamma(\theta_1)}\displaystyle\int_0^{\mathfrak{J}} (\mathfrak{J}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau,\pi_2^*(\tau))d\tau\right)d\mathfrak{J}\right| \le \xi_1, \\ \left|\pi_2^*(w) - \dfrac{\Gamma(\varrho_2-\sigma-2)w^{\varrho_2-3}}{\Gamma(\varrho_2-2)\Gamma(\sigma)}\displaystyle\int_0^W (W-\mathfrak{J})^{\sigma-1}\varpi(\pi_2^*(\mathfrak{J}))d\mathfrak{J}\right. \\ \left. + \displaystyle\int_0^1 G^{\varrho_2}(w,\mathfrak{J})\phi_q\left(\dfrac{1}{\Gamma(\theta_2)}\displaystyle\int_0^{\mathfrak{J}} (\mathfrak{J}-\tau)^{\theta_2-1}\mathscr{Z}_2(\tau,\pi_1^*(\tau))d\tau\right)d\mathfrak{J}\right| \le \xi_2, \end{cases} \tag{68}$$

there exists (π_1, π_2) which is the unique solution of (27) satisfying that

$$\begin{aligned} |\pi_1(w) - \pi_1^*(w)| &\le a\xi_1 + b\xi_2, \\ |\pi_2(w) - \pi_2^*(w)| &\le c\xi_1 + d\xi_2. \end{aligned} \tag{69}$$

Theorem 8. *The toppled system (5) is Hyers–Ulam stable under hypotheses* (H_1)–(H_4).

Proof. With the help of Definition 7 and Theorem 7, suppose that (π_1, π_2) to be the correct solution and the pair (π_1^*, π_2^*) be the other solution of system (27). Then, we have, from the first equation of (27),

$$\begin{aligned} |\pi_1(w) - \pi_1^*(w)| = {} & |\frac{\Gamma(\varrho_1-\lambda-2)w^{\varrho_1-3}}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{J})^{\lambda-1}\varphi(\pi_1(\mathfrak{J})\mathrm{d}\mathfrak{J} \\ & + \int_0^1 G^{\varrho_1}(w,\mathfrak{J})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}} (\mathfrak{J}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau,\pi_2(\tau))\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J} \\ & - \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{J})^{\lambda-1}\varphi(\pi_1^*(\mathfrak{J})\mathrm{d}\mathfrak{J} \\ & - \int_0^1 G^{\varrho_1}(w,\mathfrak{J})\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}} (\mathfrak{J}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau,\pi_2^*(\tau))\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J}| \\ & \le \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{J})^{\lambda-1}(\varphi(\pi_1)-\varphi(\pi_1^*))\mathrm{d}\mathfrak{J} \\ & + \int_0^1 \|G^{\varrho_1}(w,\mathfrak{J})\|\phi_q\left[\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}} (\mathfrak{J}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau,\pi_2(\tau))\mathrm{d}\tau\right] \\ & - \phi_q\left[\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}} (\mathfrak{J}-\tau)^{\theta_1-1}\mathscr{Z}_1(\tau,\pi_2^*(\tau))\mathrm{d}\tau\right]|\mathrm{d}\mathfrak{J} \\ & \le \frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\lambda)}\int_0^W (W-\mathfrak{J})^{\lambda-1}|(\varphi(\pi_1)-\varphi(\pi_1^*))|\mathrm{d}\mathfrak{J} \\ & + (q-1)\rho_1^{q-2}\int_0^1 |G^{\varrho_1}(w,\mathfrak{J})|\left(\frac{1}{\Gamma(\theta_1)}\int_0^{\mathfrak{J}} (\mathfrak{J}-\tau)^{\theta_1-1}\right. \\ & \left. \times |\mathscr{Z}_1(\tau,\pi_2(\tau)) - \mathscr{Z}_1(\tau,\pi_2^*(\tau))|\mathrm{d}\tau\right)\mathrm{d}\mathfrak{J}. \end{aligned} \tag{70}$$

Then,

$$\begin{aligned}\left|\pi_1(w)-\pi_1^*(w)\right| &\le \frac{\Gamma(\varrho_1-\lambda-2)W^{\lambda}K_{\varphi}}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)}\left\|\pi_1(w)-\pi_1^*(w)\right\| \\ &+(q-1)\rho_1^{q-2}L_{\mathcal{Z}_1}\left(\frac{1}{\Gamma(\varrho_1+1)}+\frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda+1)}\right)\left(\frac{1}{\Gamma(\theta_1+1)}\right) \\ &\times\left\|\pi_2(w)-\pi_2^*(w)\right\|\le a\xi_1+b\xi_2,\end{aligned} \tag{71}$$

where

$$\begin{aligned}a&=\frac{\Gamma(\varrho_1-\lambda-2)W^{\lambda}K_{\varphi}}{\Gamma(\varrho_1-2)\Gamma(\lambda+1)},\\ b&=(q-1)\rho_1^{q-2}L_{\mathcal{Z}_1}\left(\frac{1}{\Gamma(\varrho_1+1)}+\frac{\Gamma(\varrho_1-\lambda-2)}{\Gamma(\varrho_1-2)\Gamma(\varrho_1-\lambda+1)}\right)\left(\frac{1}{\Gamma(\theta_1+1)}\right).\end{aligned} \tag{72}$$

Similarly, we get

$$\left|\pi_2(w)-\pi_2^*(w)\right|\le c\xi_1+d\xi_2, \tag{73}$$

where

$$\begin{aligned}c&=\frac{\Gamma(\varrho_2-\sigma-2)W^{\sigma}K_{\varpi}}{\Gamma(\varrho_2-2)\Gamma(\sigma+1)},\\ d&=(q-1)\rho_2^{q-2}L_{\mathcal{Z}_2}\left(\frac{1}{\Gamma(\varrho_2+1)}+\frac{\Gamma(\varrho_2-\sigma-2)}{\Gamma(\varrho_2-2)\Gamma(\varrho_2-\sigma+1)}\right)\left(\frac{1}{\Gamma(\theta_2+1)}\right).\end{aligned} \tag{74}$$

Hence, by (71) and (73), integral equations' toppled system (27) is Hyers–Ulam stable. Thus, proposed toppled system (5) is Hyers–Ulam stable. □

5. Illustrative Example

In this section, we introduce an application of our results, which were proved in Sections 3 and 4.

Example 1. Consider the following toppled fractional system with the p-Laplacian operator and integral boundary conditions for $n=5$:

$$\begin{cases} {}^{c}D^{14/3}\left(\phi_4\left({}^{R}D^{13/3}\pi_1(w)\right)=\dfrac{-21w}{12}+\dfrac{1}{10}\cos(\pi_2(w)), & w\in[0,1],\\ {}^{c}D^{14/3}\left(\phi_4\left({}^{R}D^{13/3}\pi_2(w)\right)=\dfrac{32}{15}+\dfrac{1}{10}\sin(\pi_1(w)), & w\in[0,1],\\ \left(\phi_4\left({}^{R}D^{13/3}\pi_1(w)\right)\right)^{(i)}|_{w=0}=0, & i=0,1,2,3,4,\\ I^{k-(13/3)}\pi_1(w)|_{w=0}=0, & k=1,2,4,5,\\ D^{3/2}\pi_1(w)|_{w=1}=\dfrac{1}{\Gamma(3/2)}\displaystyle\int_0^1\dfrac{(1-\mathfrak{I})^{(1/2)}\cos(\pi_1)}{6}\mathrm{d}\mathfrak{I}, & \\ \left(\phi_4\left({}^{R}D^{13/3}\pi_2(w)\right)^{(j)}\right|_{w=0}=0, & j=0,1,2,3,4,\\ I^{h-(13/3)}\pi_2(w)|_{w=0}=0, & h=1,2,4,5,\\ D^{(3/2)}\pi_2(w)|_{w=1}=\dfrac{1}{\Gamma(3/2)}\Bigg)\displaystyle\int_0^1\dfrac{(1-\mathfrak{I})^{(1/2)}\cos(\pi_2)}{6}\mathrm{d}\mathfrak{I}, & \end{cases} \tag{75}$$

where $\theta_1 = \theta_2 = (14/3)$, $p = 4$, $\varrho_1 = \varrho_2 = (13/3)$, and $\lambda = \sigma = (3/2)$. Then, we obtain $K_\varphi = K_\varpi = (1/6)$ and $L_{\mathscr{X}_1} = L_{\mathscr{X}_2} = (1/10)$. Via simple calculation and taking $\rho_1 = \rho_2 = (1/2)$, we get $\chi = 0.1767 < 1$. Hence, by Theorem 7, toppled system (75) has a unique solution. With comparable fashion, it is easy to verify the fulfillment of the conditions of Theorem 6. Likewise, the conditions of Theorem 8 can be easily confirmed, and consequently, the solution of system (75) is Hyers–Ulam stable.

6. Conclusion

In this study, we analyzed the stability and uniqueness solution of Caputo and Riemann–Liouville fractional derivatives with fractional orders $n - 1 < \theta_1, \theta_2, \varrho_1, \varrho_2 \leq n$, and $n \geq 3$. By using the topological degree theory, we have proved sufficient conditions for the EUS of the coupled system of fractional differential equations with integral boundary conditions involving the p-Laplacian operator. Also, we have found appropriate conditions for Hyers–Ulam stability of the solution for the considered system. At the end, we have provided an example that supported our results as we have done in Section 5 to confirm the theoretical analysis.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] R. P. Agarwal, D. O'Regan, and S. Staněk, "Positive solutions for mixed problems of singular fractional differential equations," *Mathematische Nachrichten*, vol. 285, no. 1, pp. 27–41, 2012.

[2] W. Al-Sadi, H. Zhenyou, and A. Alkhazzan, "Existence and stability of a positive solution for nonlinear hybrid fractional differential equations with singularity," *Journal of Taibah University for Science*, vol. 13, no. 1, pp. 951–960, 2019.

[3] A. Hallaci, H. Boulares, and M. Kurulay, "On the study of nonlinear fractional differential equations on unbounded interval," *General Letters in Mathematics*, vol. 5, no. 3, pp. 111–117, 2018.

[4] H. Jafari, H. Jassim, S. P. Moshokoa, V. M. Ariyan, and F. Tchier, "Reduced differential transform method for partial differential equations within local fractional derivative operators," *Advances in Mechanical Engineering*, vol. 8, no. 4, Article ID 1687814016633013, 2016.

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and applications of fractional differential equations," *North-Holland. Mathematics Studies*, vol. 204, 2006.

[6] C. Kou, H. Zhou, and Y. Yan, "Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 17, pp. 5975–5986, 2011.

[7] C. P. Li and F. R. Zhang, "A survey on the stability of fractional differential equations," *The European Physical Journal Special Topics*, vol. 193, no. 1, pp. 27–47, 2011.

[8] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, vol. 198, , 1999.

[9] B. Zhou, L. Zhang, E. Addai, and N. Zhang, "Multiple positive solutions for nonlinear high-order riemann–liouville fractional differential equations boundary value problems with p-laplacian operator," *Boundary Value Problems*, vol. 2020, no. 1, p. 26, 2020.

[10] M. A. Dokuyucu, E. Celik, H. Bulut, and H. M. Baskonus, "Cancer treatment model with the Caputo-Fabrizio fractional derivative," *The European Physical Journal Plus*, vol. 133, no. 3, pp. 1–6, 2018.

[11] M. A. Dokuyucu, "A fractional order alcoholism model via Caputo-Fabrizio derivative," *AIMS Mathematics*, vol. 5, no. 2, pp. 781–797, 2020.

[12] K. İlknur, E. Akçetin, and P. Yaprakdal, "Numerical approximation for the spread of SIQR model with Caputo fractional order derivative," *Turkish Journal of Science*, vol. 5, no. 2, pp. 124–139, 2020.

[13] M. Goyal, H. M. Baskonus, and A. Prakash, "Regarding new positive, bounded and convergent numerical solution of nonlinear time fractional HIV/AIDS transmission model," *Chaos, Solitons and Fractals*, vol. 139, Article ID 110096, 2020.

[14] M. A. Dokuyucu, "Caputo and atangana-baleanu-caputo fractional derivative applied to garden equation"" *Turkish Journal of Science*, vol. 5, no. 1, pp. 1–7, 2020.

[15] B. Li and H. Gou, "Existence of solutions for impulsive fractional evolution equations with periodic boundary condition," *Advances in Difference Equations*, vol. 2017, no. 1, pp. 1–22, 2017.

[16] M. M. Matar, I. A. Amra, and J. Alzabut, "Existence of solutions for tripled system of fractional differential equations involving cyclic permutation boundary conditions," *Boundary Value Problems*, vol. 2020, no. 1, pp. 1–13, 2020.

[17] J. A. Nanware and D. B. Dhaigude, "Existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions," *Journal of Nonlinear Sciences and Applications*, vol. 7, no. 4, pp. 246–254, 2014.

[18] H. Yang, R. P. Agarwal, and H. K. Nashine, "Coupled fixed point theorems with applications to fractional evolution equations," *Advances in Difference Equations*, vol. 2017, no. 1, p. 239, 2017.

[19] Z. Ali, A. Zada, and K. Shah, "On ulam's stability for a coupled systems of nonlinear implicit fractional differential equations," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 5, pp. 2681–2699, 2019.

[20] K. Shah, "Nonlocal boundary value problems for nonlinear toppled system of fractional differential equations," *Hacettepe Journal of Mathematics and Statistics*, vol. 49, no. 1, pp. 316–337, 2020.

[21] K. Shah, A. Ali, and R. A. Khan, "Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems," *Boundary Value Problems*, vol. 2016, no. 1, pp. 1–12, 2016.

[22] Y. Tian, Y. Wei, and S. Sun, "Multiplicity for fractional differential equations with p-laplacian," *Boundary Value Problems*, vol. 2018, no. 1, p. 127, 2018.

[23] J. R. L. Webb, "Positive solutions of some higher order nonlocal boundary value problems," *Electronic Journal of*

Qualitative Theory of Differential Equations, vol. 29, pp. 1–15, 2010.

[24] H. Khan, W. Chen, A. Khan, T. S. Khan, and Q. M. Al-Madlal, "Hyers–ulam stability and existence criteria for coupled fractional differential equations involving p-laplacian operator," *Advnces in Difference Equations*, vol. 1, p. 455, 2018.

[25] A. Khan, K. Shah, Y. Liand, and T. S. Khan, "Ulam type stability for a coupled system of boundary value problems of nonlinear fractional differential equations," *Journal of Function Spaces*, vol. 2017, Article ID 3046013, 8 pages, 2017.

[26] R. A. Khan and K. Shah, "Multiple positive solutions to a coupled systems of nonlinear fractional differential equations," *SpringerPlus*, vol. 5, no. 1, pp. 1–20, 2016.

[27] K. Shah and R. A. Khan, "Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions," *Differential Equations and Applications*, vol. 7, no. 2, pp. 245–262, 2015.

[28] K. Shah, H. Khalil, and R. A. Khan, "Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations," *Chaos, Solitons and Fractals*, vol. 77, pp. 240–246, 2015.

[29] F. Isaia, "On a nonlinear integral equation without compactness," *Acta Mathematica Universitatis Comenianae*, vol. 75, no. 2, pp. 233–240, 2006.

[30] J. Wang, Y. Zhou, and W. Wei, "Study in fractional differential equations by means of topological degree methods," *Numerical Functional Analysis and Optimization*, vol. 33, no. 2, pp. 216–238, 2012.

[31] K. Shah and R. A. Khan, "Existence and uniqueness results to a coupled system of fractional order boundary value problems by topological degree theory," *Numerical Functional Analysis and Optimization*, vol. 37, no. 7, pp. 887–899, 2016.

[32] H. Khan, Y. Li, H. Sun, and A. Khan, "Existence of solution and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 10, pp. 5219–5229, 2017.

[33] N. Ahmad, Z. Ali, K. Khan, A. Zada, and R. Ghaus, "Analysis of implicit type nonlinear dynamical problem of impulsive fractional differential equations," *Complexity*, vol. 2018, Article ID 6423974, 15 pages, 2018.

[34] A. Ali, F. Rabiei, and K. Shah, "On Ulam's type stability for a class of impulsive fractional differential equations with nonlinear integral boundary conditions," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 9, pp. 4760–4775, 2017.

[35] M. Benchohra and S. Bouriah, "Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order," *Moroccan Journal of Pure and Applied Analysis*, vol. 1, no. 1, pp. 22–37, 2015.

[36] H. Khan, C. Tunc, W. Chen, and A. Khan, "Existence theorems and hyers–ulam stability for a class of hybrid fractional differential equations with p-laplacian operator," *Journal of Applied Analysis and Computation*, vol. 8, no. 4, pp. 1211–1226, 2018.

[37] J. Wang, M. Fečkan, and Y. Tian, "Stability analysis for a general class of non-instantaneous impulsive differential equations"" *Mediterranean Journal of Mathematics*, vol. 14, no. 2, p. 46, 2017.

[38] J. Wang, K. Shah, and A. Ali, "Existence and hyers-ulam stability of fractional nonlinear impulsive switched coupled evolution equations," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 6, pp. 2392–2402, 2018.

[39] H. Khan, Y. Li, W. Chen, D. Baleanu, and A. Khan, "Existence theorems and hyers-ulam stability for a coupled system of fractional differential equations with p-laplacian operator," *Boundary Value Problems*, vol. 2017, no. 1, pp. 1–16, 2017.

[40] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, NY, USA, 1985.

[41] M. U. Rehman and R. A. Khan, "Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations"" *Applied Mathematics Letter*, vol. 23, no. 9, p. 239, 2010.

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Research Article

Quantum Inequalities of Hermite–Hadamard Type for r-Convex Functions

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In this present study, we first establish Hermite–Hadamard type inequalities for r-convex functions via q^{κ_2}-definite integrals. Then, we prove some quantum inequalities of Hermite–Hadamard type for product of two r-convex functions. Finally, by using these established inequalities and the results given by (Brahim et al. 2015), we prove several quantum Hermite–Hadamard type inequalities for coordinated r-convex functions and for the product of two coordinated r-convex functions.

1. Introduction

Quantum calculus research is an unlimited analysis of calculus and is known as q-calculus. We get the initial mathematical formulas in q-calculus as q reaches 1^-. The commencement of the analysis of q-calculus was initiated by Euler (1707–1783). The aforementioned results lead to an intensive investigation on q-calculus in the twentieth century. The concept of q-calculus is used in many areas in mathematics and physics such as theory, orthogonal polynomials, integration, basic hypergeometric functions, mechanical theory, and quantum and relativity theory. For more information about q-calculus, one can refer to [1–10].

Mathematically, convexity is very simple and natural which plays a very important role in various fields of pure and applied science, such as in the field of practicality, engineering science, and management science. In the recent past, the classical concept of convexity has been extended and generalized in different directions. Another factor that makes the theory of the most popular convex works is its relationship to the concept of inequality. Many inequalities can be achieved using the definition of convex functions. One of the widely studied inequalities involving convex works is the Hermite–Hadamard inequality, which is the first basic result of convex design with natural geometric descriptions and multiple uses and has attracted great interest in elementary mathematics. Many mathematicians have devoted their efforts to generalization, refinement, modelling, and multiplication of various fields of work such as the use of convex mappings (see, e.g., [11], p.137, and [12]).

The classical Hermite–Hadamard inequality states that if $\mathcal{F}\colon I \longrightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \le \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\mathrm{d}\varkappa \le \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \quad (1)$$

The inequality holds in the reversed direction if $\mathcal{F}$ is concave. We see that the Hermite–Hadamard inequality can be regarded as a refinement of the concept of integration and is easily followed by Jensen's inequality. The Hermite–Hadamard inequality of convex works has received renewed attention in recent years and has been studied in significant and practical variations.

In [13], Pachpatte proved the following inequalities for products of convex functions.

Theorem 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be real-valued, nonnegative, and convex functions on $[\kappa_1, \kappa_2]$. Then, we have*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)d\varkappa \leq \frac{1}{3}\mathcal{A}(\kappa_1,\kappa_2)+\frac{1}{6}\mathcal{B}(\kappa_1,\kappa_2), \quad (2)$$

$$2F\left(\frac{\kappa_1+\kappa_2}{2}\right)\mathcal{G}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)d\varkappa + \frac{1}{6}\mathcal{A}(\kappa_1,\kappa_2)+\frac{1}{3}\mathcal{B}(\kappa_1,\kappa_2), \quad (3)$$

where $\mathcal{A}(\kappa_1,\kappa_2)=F(\kappa_1)\mathcal{G}(\kappa_1)+F(\kappa_2)\mathcal{G}(\kappa_2)$ *and* $\mathcal{B}(\kappa_1,\kappa_2)=F(\kappa_1)\mathcal{G}(\kappa_2)+F(\kappa_2)\mathcal{G}(\kappa_1)$.

A positive function is called r-convex on $[\kappa_1,\kappa_2]$, if for all $\varkappa, y\in[\kappa_1,\kappa_2]$ and $\xi\in[0,1]$,

$$F(\xi\varkappa+(1-\xi)y) \leq \begin{cases} \left(\xi(F(\varkappa))^r+(1-\xi)(F(y))^r\right)^{1/r}, & \text{if } r\neq 0,\\ F(\varkappa)^{\xi}F(y)^{(1-\xi)}, & \text{if } r=0. \end{cases} \quad (4)$$

It is obvious if $r=1$, then the inequality classical convex functions. It should be noted that if F is r-convex function, then F is convex function. We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions [14].

In [15], the definition of r-convex functions on coordinates is given, such that

Definition 1. A function $F\colon \Delta\colon = [\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]\longrightarrow\mathbb{R}_+$ will be called r-convex on Δ for all $\xi,\lambda\in[0,1]$ and $(\varkappa,y),(u,v)\in\Delta$, if the following inequality holds:

$$F(\xi\varkappa+(1-\xi)y,\lambda u+(1-\lambda)v) \leq \begin{cases} \left[\xi\lambda F^r(\varkappa,u)+\xi(1-\lambda)F^r(\varkappa,v)+(1-\xi)\lambda F^r(y,u)+(1-\xi)(1-\lambda)F^r(y,v)\right]^{1/r}, & \text{if } r\neq 0,\\ F^{\xi\lambda}(\varkappa,u)F^{\xi(1-\lambda)}(\varkappa,v)F^{(1-\xi)\lambda}(y,u)F^{(1-\xi)(1-\lambda)}(y,v), & \text{if } r=0. \end{cases} \quad (5)$$

It is simply to see that if we choose $r=0$, we have co-ordinated log-convex functions and if we choose $r=1$, we have coordinated convex functions. In [15], Ekinci et al. also prove several Hermite–Hadamard type inequalities for co-ordinated r-convex functions. In literature, many studies have been done on r-convex functions. For some of them, one can see [16–23].

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we present some required definitions and related inequalities about q-calculus. For more information about q-calculus, one can refer to [1–10, 24, 25]. Also, here and further, we use the following notation (see [5]):

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\cdots+q^{n-1},\quad q\in(0,1). \quad (6)$$

In [4], Jackson gave the q-Jackson integral from 0 to κ_2 for $0<q<1$ as follows:

$$\int_0^{\kappa_2} F(\varkappa)d_q\varkappa=(1-q)\kappa_2\sum_{n=0}^{\infty}q^nF(\kappa_2q^n), \quad (7)$$

provided the sum converges absolutely.

Jackson in [4] gave the q-Jackson integral in a generic interval $[\kappa_1,\kappa_2]$ as

$$\int_{\kappa_1}^{\kappa_2} F(\varkappa)d_q\varkappa=\int_0^{\kappa_2} F(\varkappa)d_q\varkappa-\int_0^{\kappa_1} F(\varkappa)d_q\varkappa. \quad (8)$$

Definition 2 (see [9]). For a continuous function $F\colon[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}$, then q-derivative of F at $\varkappa\in[\kappa_1,\kappa_2]$ for $0<q<1$ is characterized by the expression

$${}_{\kappa_1}D_qF(\varkappa)=\frac{F(\varkappa)-F(q\varkappa+(1-q)\kappa_1)}{(1-q)(\varkappa-\kappa_1)},\quad \varkappa\neq\kappa_1. \quad (9)$$

Since $F\colon[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}$ is a continuous function, thus we have ${}_{\kappa_1}D_qF(\kappa_1)=\lim_{\varkappa\longrightarrow\kappa_1}{}_{\kappa_1}D_qF(\varkappa)$. The function F is said to be q-differentiable on $[\kappa_1,\kappa_2]$ if ${}_{\kappa_1}D_qF(\xi)$ exists for all $\varkappa\in[\kappa_1,\kappa_2]$. If $\kappa_1=0$ in (9), then ${}_0D_qF(\varkappa)=D_qF(\varkappa)$, where $D_qF(\varkappa)$ is familiar q-derivative of F at $\varkappa\in[\kappa_1,\kappa_2]$ defined by the expression (see [5])

$$D_qF(\varkappa)=\frac{F(\varkappa)-F(q\varkappa)}{(1-q)\varkappa},\quad \varkappa\neq 0. \quad (10)$$

Definition 3 (see [9]). Let $F\colon[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}$ be a continuous function. Then, the q_{κ_1}-definite integral on $[\kappa_1,\kappa_2]$ and $0<q<1$ are defined as

$$\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)_{\kappa_1} d_q \varkappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n \digamma(q^n\kappa_2 + (1-q^n)\kappa_1) = (\kappa_2-\kappa_1)\int_0^1 \digamma((1-\xi)\kappa_1 + \xi\kappa_2) d_q\xi. \tag{11}$$

In [26], Alp et al. proved the following q_{κ_1}-Hermite–Hadamard inequality for convex functions in the setting of quantum calculus.

Theorem 2. *If $\digamma\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ is a convex differentiable function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then, q-Hermite–Hadamard inequalities are as follows:*

$$\digamma\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)_{\kappa_1} d_q\varkappa \le \frac{q\digamma(\kappa_1)+\digamma(\kappa_2)}{1+q}. \tag{12}$$

On the other hand, Bermudo et al. gave the following new definition and related Hermite–Hadamard type inequalities.

Definition 4 (see [27]). Let $\digamma\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ be a continuous function. Then, the q^{κ_2}-definite integral on $[\kappa_1,\kappa_2]$ for $0<q<1$ is defined as

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)^{\kappa_2} d_q\varkappa &= (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n \digamma(q^n\kappa_1 + (1-q^n)\kappa_2)\\ &= (\kappa_2-\kappa_1)\int_0^1 \digamma(\xi\kappa_1 + (1-\xi)\kappa_2) d_q\xi.\end{aligned} \tag{13}$$

Theorem 3 (see [27]). *If $\digamma\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ is a convex differentiable function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then, q-Hermite–Hadamard inequalities are as follows:*

$$\digamma\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)^{\kappa_2} d_q\varkappa \le \frac{\digamma(\kappa_1)+q\digamma(\kappa_2)}{1+q}. \tag{14}$$

From Theorem 2 and Theorem 3, one can get the following inequalities.

Corollary 1 (see [27]). *For any convex function $\digamma\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ and $0<q<1$, we have*

$$\digamma\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) + \digamma\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) \le \frac{1}{\kappa_2-\kappa_1}\left\{\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)_{\kappa_1} d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)^{\kappa_2} d_q\varkappa\right\} \le \digamma(\kappa_1)+\digamma(\kappa_2), \tag{15}$$

$$\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)_{\kappa_1} d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)^{\kappa_2} d_q\varkappa\right\} \le \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2}. \tag{16}$$

Brahim et al. prove the following lemma and theorem for r-convex functions.

Lemma 1 (see [28]). *For $p\ge 1$ and $0<q<1$, the following inequality is valid:*

$$\int_0^1 (1-\xi)^p d_q\xi \le \frac{q}{[p+1]_q}. \tag{17}$$

Theorem 4 (see [28]). *Let $\digamma\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex on $[\kappa_1,\kappa_2]$. Then, the following inequality holds for $0<r_1\le 1$ and $0<q<1$:*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)\kappa_1 d_q\varkappa \le \frac{1}{[1/r_1+1]_q}\left([q\digamma(\kappa_1)]^{r_1} + [\digamma(\kappa_2)]^{r_1}\right)^{1/r_1}. \tag{18}$$

Theorem 5 (see [28]). *Let $F, \mathcal{G}\colon [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1, \kappa_2]$. Then, the following inequality holds for $0 < r_1, r_2 \leq 2$ and $0 < q < 1$:*

$$\frac{2}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)\kappa_1 d_q\varkappa \leq \frac{1}{[2/r_1 + 1]_q} \cdot \left(\left[q^{1/2}F(\kappa_1)\right]^{r_1} + \left[F(\kappa_2)\right]^{r_1}\right)^{2/r_1} + \frac{1}{[2/r_2 + 1]_q}\left(\left[q^{1/2}\mathcal{G}(\kappa_1)\right]^{r_2} + \left[\mathcal{G}(\kappa_2)\right]^{r_2}\right)^{2/r_2}. \quad (19)$$

Theorem 6 (see [28]). *Let $F, \mathcal{G}\colon [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, the following inequality holds if $r_1 > 1$ and $1/r_1 + 1/r_2 = 1$:*

$$\frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)_{\kappa_1} d_q\varkappa \leq \left(\frac{\left[qF(\kappa_1)\right]^{r_1} + \left[F(\kappa_2)\right]^{r_1}}{[2]_q}\right)^{1/r_1} \cdot \left(\frac{\left[q\mathcal{G}(\kappa_1)\right]^{r_2} + \left[\mathcal{G}(\kappa_1)\right]^{r_2}}{[2]_q}\right)^{1/r_2}. \quad (20)$$

In [29], Latif defined the $q_{\kappa_1\kappa_3}$-integral and related properties for two variable functions as follows.

Definition 5. Suppose that $F\colon \Delta \longrightarrow \mathbb{R}$ is continuous function and $0 < q_1, q_2 < 1$. Then, the definite $q_{\kappa_1\kappa_3}$-integral on Δ is defined by

$$\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{y} F(\xi, s)_{\kappa_3} d_{q_2}s\, _{\kappa_1}d_{q_1}\xi = (1 - q_1)(1 - q_2)(\varkappa - \kappa_1)(y - \kappa_3) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n\varkappa + (1 - q_1^n)\kappa_1, q_2^m y + (1 - q_2^m)\kappa_3\right), \quad (21)$$

for $(\varkappa, y) \in \Delta$.

In [29], Latif et al. also proved a q-Hermite–Hadamard inequality for coordinated convex functions.

By Definitions 4 and 5, Budak et al. defined the following $q_{\kappa_1}^{\kappa_4}$, $q_{\kappa_3}^{\kappa_2}$ and $q^{\kappa_2\kappa_4}$ integrals.

Definition 6 (see [30]). Suppose that $F\colon \Delta \longrightarrow \mathbb{R}$ is a continuous function and $0 < q_1, q_2 < 1$. Then, the following $q_{\kappa_1}^{\kappa_4}$, $q_{\kappa_3}^{\kappa_2}$, and $q^{\kappa_2\kappa_4}$ integrals on Δ are defined by

$$\begin{aligned}
\int_{\kappa_1}^{\varkappa}\int_{y}^{\kappa_4} F(\xi, s)\, ^{\kappa_4}d_{q_2}s\, _{\kappa_1}d_{q_1}\xi &= (1 - q_1)(1 - q_2)(\varkappa - \kappa_1)(\kappa_4 - y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_2^m\varkappa + (1 - q_2^m)\kappa_1, q_1^n y + (1 - q_1^n)\kappa_4\right),\\
\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{y} F(\xi, s)\, _{\kappa_3}d_{q_2}s\, ^{\kappa_2}d_{q_1}\xi &= (1 - q_1)(1 - q_2)(\kappa_2 - \varkappa)(y - \kappa_3) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_2^m\varkappa + (1 - q_2^m)\kappa_2, q_1^n y + (1 - q_1^n)\kappa_3\right),
\end{aligned} \quad (22)$$

$$\int_{\varkappa}^{\kappa_2}\int_{y}^{\kappa_4} F(\xi, s)\, ^{\kappa_4}d_{q_2}s\, ^{\kappa_2}d_{q_1}\xi = (1 - q_1)(1 - q_2)(\kappa_2 - \varkappa)(\kappa_4 - y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_2^m\varkappa + (1 - q_2^m)\kappa_2, q_1^n y + (1 - q_1^n)\kappa_4\right), \quad (23)$$

respectively, for $(\varkappa, y) \in \Delta$.

Budak et al. also proved some quantum Hermite–Hadamard type inequalities for coordinated convex functions. For other similar quantum inequalities, please see [31,32].

In this paper, we first prove the new variant of results of Brahim et al. for q^{κ_2}-integrals. We also obtain quantum versions of the inequalities in [15].

3. Quantum Hermite–Hadamard Type Inequalities for r-Convex Functions

In this section, we obtain some quantum inequalities of Hermite–Hadamard type for r-convex functions and for product of two r-convex functions.

Theorem 7. *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ be a r_1-convex function on $[\kappa_1, \kappa_2]$. Then, the following inequality holds for $0 < r_1 \leq 1$:*

$$\frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\kappa_2 d_q\varkappa \leq \frac{1}{[1/r_1 + 1]_q}\left(\left[F(\kappa_1)\right]^{r_1} + \left[qF(\kappa_2)\right]^{r_1}\right)^{1/r_1}, \quad (24)$$

where $0 < q < 1$.

Proof. According to definition r_1-convex, for all $\xi \in [0, 1]$, we have

$$F(\xi\kappa_1 + (1 - \xi)\kappa_2) \leq \left(\xi\left[F(\kappa_1)\right]^{r_1} + (1 - \xi)\left[F(\kappa_2)\right]^{r_1}\right)^{1/r_1}. \quad (25)$$

By integrating the inequality on $[0, 1]$, we obtain

$$\int_0^1 F(\xi\kappa_1+(1-\xi)\kappa_2)d_q\xi \le \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}d_q\xi. \tag{26}$$

From Definition 4, we get

$$\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2}d_q\varkappa \le \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}d_q\xi. \tag{27}$$

Using Minkowski's inequality for right side of inequality (26),

$$\begin{aligned}&\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}d_q\xi\\ &\le \left(\left(\int_0^1 \xi^{1/r_1}d_q\xi\right)^{r_1}[F(\kappa_1)]^{r_1}+\left(\int_0^1 (1-\xi)^{1/r_1}d_q\xi\right)^{r_1}[F(\kappa_2)]^{r_1}\right)^{1/r_1}.\end{aligned} \tag{28}$$

By Lemma 1, we have

$$\left(\int_0^1 \xi^{1/r_1}d_q\xi\right)^{r_1} = \left(\frac{1}{[1/r_1+1]_q}\right)^{r_1}, \tag{29}$$

$$\left(\int_0^1 (1-\xi)^{1/r_1}d_q\xi\right)^{r_1} \le \left(\frac{q}{[1/r_1+1]_q}\right)^{r_1}. \tag{30}$$

Thus, by substituting (29) and (30) in (28), we obtain

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2}d_q\varkappa &\le \left(\left(\frac{1}{[1/r_1+1]_q}\right)^{r_1}[F(\kappa_1)]^{r_1}+\left(\frac{q}{[1/r_1+1]_q}\right)^{r_1}[F(\kappa_2)]^{r_1}\right)^{1/r_1}\\ &\le \frac{1}{[1/r_1+1]_q}\left([F(\kappa_1)]^{r_1}+q^{r_1}[F(\kappa_2)]^{r_1}\right)^{1/r_1}.\end{aligned} \tag{31}$$

The proof is completed. □

Remark 1. If we take the limit $q \longrightarrow 1^-$ in Theorem 7, then Theorem 7 reduces to Theorem 2.1 in [33].

Remark 2. If we choose $r_1 = 1$ in Theorem 7, then inequality (24) reduces to the second inequality in (14).

Theorem 8. *Let $F, \mathcal{G}$: $[\kappa_1,\kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1,\kappa_2]$. Then, the following inequality holds for $0 < r_1, r_2 \le 2$:*

$$\frac{2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)^{\kappa_2}d_q\varkappa \le \frac{1}{[2/r_1+1]_q}\left([F(\kappa_1)]^{r_1}+[q^{1/2}F(\kappa_2)]^{r_1}\right)^{2/r_1}+\frac{1}{[2/r_2+1]_q}\left([\mathcal{G}(\kappa_1)]^{r_2}+[q^{1/2}\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2}, \tag{32}$$

where $0 < q < 1$.

Proof. By the assumptions that F is an r_1-convex function and $\mathcal{G}$ is an r_2-convex function, we can write

$$F(\xi\kappa_1+(1-\xi)\kappa_2) \le \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}, \tag{33}$$

$$\mathcal{G}(\xi\kappa_1+(1-\xi)\kappa_2) \le \left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2}, \tag{34}$$

for all $\xi \in [0,1]$ and $r_1, r_2 > 0$.

Then,

$$\begin{aligned}&F(\xi\kappa_1+(1-\xi)\kappa_2)\mathcal{G}(\xi\kappa_1+(1-\xi)\kappa_2)\\ &\le \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\\ &\quad\cdot\left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2}.\end{aligned} \tag{35}$$

Integrating both sides with respect to ξ on $[0,1]$ and from Definition 4, we obtain

$$\begin{aligned}&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)^{\kappa_2} d_q\varkappa\\&\leq \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2} d_q\xi.\end{aligned}\tag{36}$$

Using Cauchy's inequality for right side of inequality (36), we obtain

$$\begin{aligned}&\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2} d_q\xi\\&\leq\frac{1}{2}\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{2/r_1} d_q\xi+\frac{1}{2}\int_0^1 \left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2} d_q\xi.\end{aligned}\tag{37}$$

By using Minkowski's inequality, we have

$$\begin{aligned}&\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{2/r_1} d_q\xi\\&\leq\left(\left(\int_0^1 \xi^{2/r_1}d_q\xi\right)^{r_1/2}[F(\kappa_1)]^{r_1}+\left(\int_0^1 (1-\xi)^{2/r_1}d_q\xi\right)^{r_1/2}[F(\kappa_2)]^{r_1}\right)^{2/r_1}\\&=\left(\left(\frac{1}{[2/r_1+1]_q}\right)^{r_1/2}[F(\kappa_1)]^{r_1}++\left(\frac{q}{[2/r_1+1]_q}\right)^{r_1/2}[F(\kappa_2)]^{r_1}\right)^{2/r_1}.\end{aligned}\tag{38}$$

Similarly, we have

$$\begin{aligned}&\int_0^1 \left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2} d_q\xi\\&\leq\left(\left(\frac{1}{[2/r_2+1]_q}\right)^{r_2/2}[\mathcal{G}(\kappa_1)]^{r_2}\right.\\&\left.+\left(\frac{q}{[2/r_2+1]_q}\right)^{r_2/2}[\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2}.\end{aligned}\tag{39}$$

Thus, from the inequalities (36)–(39), we obtain the desired result. □

Remark 3. If we take the limit $q\longrightarrow 1^-$ in Theorem 8, then Theorem 8 reduces to Theorem 2.3 in [33].

Corollary 2. *If we choose $r_1=r_2=2$ in Theorem 8, then we have the inequality*

$$\begin{aligned}&\frac{2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)\kappa_2 d_q\varkappa\\&\leq\frac{[F(\kappa_1)]^2+q[F(\kappa_2)]^2}{[2]_q}+\frac{[\mathcal{G}(\kappa_1)]^2+q[\mathcal{G}(\kappa_2)]^2}{[2]_q}.\end{aligned}\tag{40}$$

Particularly, if $F(\varkappa)=\mathcal{G}(\varkappa)$ for all $\varkappa\in[\kappa_1,\kappa_2]$, then we get

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}[F(\varkappa)]^2\kappa_2 d_q\varkappa\leq\frac{[F(\kappa_1)]^2+q[F(\kappa_2)]^2}{[2]_q}.\tag{41}$$

Theorem 9. *Let $F,\mathcal{G}\colon[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1,\kappa_2]$. Then, we get the following inequality:*

$$\begin{aligned}&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)^{\kappa_2} d_q\varkappa\\&\leq\left(\frac{[F(\kappa_1)]^{r_1}+[qF(\kappa_2)]^{r_1}}{[2]_q}\right)^{1/r_1}\left(\frac{[\mathcal{G}(\kappa_1)]^{r_2}+[q\mathcal{G}(\kappa_2)]^{r_2}}{[2]_q}\right)^{1/r_2},\end{aligned}\tag{42}$$

where $0<q<1$ and $1/r_1+1/r_2=1$ with $r_1>1$.

Proof. From (36), we have

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathscr{G}(\varkappa)^{\kappa_2} d_q\varkappa \leq \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\left(\xi[\mathscr{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathscr{G}(\kappa_2)]^{r_2}\right)^{1/r_2} d_q\xi. \quad (43)$$

Using Hölder inequality for quantum integrals, we have

$$\begin{aligned}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathscr{G}(\varkappa)\kappa_2 d_q\varkappa &\leq \left(\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right) d_q\xi\right)^{1/r_1}\left(\int_0^1 \left(\xi[\mathscr{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathscr{G}(\kappa_2)]^{r_2}\right) d_q\xi\right)^{1/r_2} \\ &= \left(\frac{[F(\kappa_1)]^{r_1}+[qF(\kappa_2)]^{r_1}}{[2]_q}\right)^{1/r_1}\left(\frac{[\mathscr{G}(\kappa_1)]^{r_2}+[q\mathscr{G}(\kappa_2)]^{r_2}}{[2]_q}\right)^{1/r_2}.\end{aligned} \quad (44)$$

This completes the proof. □

Remark 4. If we take the limit $q\longrightarrow 1^-$ in Theorem 9, then Theorem 9 reduces to Theorem 2.6 in [33].

Corollary 3. *If we choose $r_1=r_2=2$ in Theorem 9, then we have the inequality*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathscr{G}(\varkappa)\kappa_2 d_q\varkappa \leq \sqrt{\frac{[F(\kappa_1)]^2+[qF(\kappa_2)]^2}{[2]_q}}\sqrt{\frac{[\mathscr{G}(\kappa_1)]^2+[q\mathscr{G}(\kappa_2)]^2}{[2]_q}}. \quad (45)$$

Particularly, if $F(\varkappa)=\mathscr{G}(\varkappa)$ for all $\varkappa\in[\kappa_1,\kappa_2]$, then we get

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} [F(\varkappa)]^{2\kappa_2} d_q\varkappa \leq \frac{[F(\kappa_1)]^2+[qF(\kappa_2)]^2}{[2]_q}. \quad (46)$$

4. Quantum Hermite–Hadamard Type Inequalities for Coordinated r-Convex Functions

In this section, we present several Hermite–Hadamard type inequalities for coordinated r-convex functions via $q^{\kappa_2\kappa_4}$, $q_{\kappa_1}^{\kappa_4}$, $q_{\kappa_3}^{\kappa_2}$, and $q_{\kappa_1\kappa_3}$ integrals. We also prove some quantum inequalities of Hermite–Hadamard type for the product of two coordinated r-convex functions.where $0<r_1\leq 1$ and $0<q_1,q_2<1$.

Theorem 10. *Suppose that $F:\Delta\longrightarrow\mathbb{R}_+$ is a positive coordinated r_1-convex function on Δ. Then, one has the inequality*

$$\begin{aligned}&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)^{\kappa_2} d_{q_1}\varkappa\kappa_4 d_{q_2}y \\ &\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)} \\ &\int_{\kappa_1}^{\kappa_2}\left([F(\varkappa,\kappa_3)]^{r_1}+[q_2F(\varkappa,\kappa_4)]^{r_1}\right)^{1/r_1\kappa_2} d_{q_1}\varkappa \\ &+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)} \\ &\int_{\kappa_3}^{\kappa_4}\left([F(\kappa_1,y)]^{r_1}+[q_1F(\kappa_2,y)]^{r_1}\right)^{1/r_1\kappa_4} d_{q_2}\varkappa,\end{aligned} \quad (47)$$

Proof. Since $F:\Delta\longrightarrow\mathbb{R}_+$ is a coordinated r_1-convex function, then the partial mappings,

$$F_\varkappa:[\kappa_3,\kappa_4]\longrightarrow\mathbb{R}_+, F_\varkappa(v)=F(\varkappa,v), \quad (48)$$

$$F_y:[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}_+, F_y(u)=F(u,y), \quad (49)$$

are r_1-convex. By inequality (24), we can write

$$\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F_{\varkappa}(y)\,{}_{\kappa_4}d_{q_2}y \leq \frac{1}{[1/r_1+1]_{q_2}}\left(\left[F_{\varkappa}(\kappa_3)\right]^{r_1}+\left[q_2F_{\varkappa}(\kappa_4)\right]^{r_1}\right)^{1/r_1}, \quad (50)$$

i.e.,

$$\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}_{\kappa_4}d_{q_2}y \leq \frac{1}{[1/r_1+1]_{q_2}}\left(\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}. \quad (51)$$

Dividing both sides of the inequality $(\kappa_2-\kappa_1)$ and q^{κ_2}-integrating with respect to $\varkappa$ on $[\kappa_1,\kappa_2]$, we get

$$\begin{aligned}&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}^{\kappa_2}d_{q_1}\varkappa\,{}_{\kappa_4}d_{q_2}y\\ &\leq \frac{1}{[1/r_1+1]_{q_2}}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}\,{}^{\kappa_2}d_{q_1}\varkappa\right].\end{aligned} \quad (52)$$

By a similar argument for the mapping

$$F_y\colon [\kappa_1,\kappa_2]\longrightarrow \mathbb{R}_+, F_y(u)=F(u,y), \quad (53)$$

we have

$$\begin{aligned}&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}^{\kappa_2}d_{q_1}\varkappa\,{}^{\kappa_4}d_{q_2}y\\ &\leq \frac{1}{[1/r_1+1]_{q_1}}\left[\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1F(\kappa_2,y)\right]^{r_1}\right)^{1/r_1}\,{}^{\kappa_4}d_{q_2}y\right].\end{aligned} \quad (54)$$

By adding inequalities (52) and (54), we can obtain inequality (47). □

Remark 5. If we take the limit $q_1\longrightarrow 1^-$ and $q_2\longrightarrow 1^-$ in Theorem 10, then Theorem 10 reduces to Theorem 5 in [15].

Remark 6. If we choose $r_1=1$ in Theorem 10, then inequality (47) reduces to the third inequality of Theorem 3.6 in [30].

Theorem 11. *Suppose that $F\colon \Delta\longrightarrow \mathbb{R}_+$ is a positive coordinated r_1-convex function on Δ. Then, one has the inequality*

$$\begin{aligned}&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}_{\kappa_1}d_{q_1}\varkappa\,{}_{\kappa_3}d_{q_2}y\\ &\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\\ &\cdot\left(\left[q_2F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_1}d_{q_1}\varkappa\\ &+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\\ &\cdot\left(\left[q_1F(\kappa_1,y)\right]^{r_1}+\left[F(\kappa_2,y)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_3}d_{q_2}y,\end{aligned} \quad (55)$$

where $0<r_1\leq 1$ and $0<q_1,q_2<1$.

Proof. The proof is similar to the proof of Theorem 10 by using Theorem 4. □

Theorem 12. *Suppose that $F\colon \Delta\longrightarrow \mathbb{R}_+$ is a positive coordinated convex function on Δ. Then, one has the inequality*

$$\begin{aligned}&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}_{\kappa_1}d_{q_1}\varkappa\,{}_{\kappa_4}d_{q_2}y\\ &\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\\ &\cdot\left(\left[q_2F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_1}d_{q_1}\varkappa\\ &+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\\ &\cdot\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1F(\kappa_2,y)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_4}d_{q_2}y,\end{aligned} \quad (56)$$

where $0<r_1\leq 1$ and $0<q_1,q_2<1$.

Proof. The proof is similar to the proof of Theorem 10 by using Theorems 4 and 7. □

Theorem 13. *Suppose that $F: \Delta \longrightarrow \mathbb{R}_+$ is a positive coordinated r_1-convex function on Δ. Then, one has the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\chi,y)^{\kappa_2} d_{q_1}\chi^{\kappa_3} d_{q_2} y \\
&\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2} \\
&\left([F(\chi,\kappa_3)]^{r_1}+[q_2 F(\chi,\kappa_4)]^{r_1}\right)^{1/r_1 \kappa_2} d_{q_1}\chi \\
&+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4} \\
&\left([q_1 F(\kappa_1,y)]^{r_1}+[F(\kappa_2,y)]^{r_1}\right)_{\kappa_3}^{1/r_1} d_{q_2},
\end{aligned} \tag{57}$$

where $0<r_1\leq 1$ *and* $0<q_1,q_2<1$.

Proof. The proof is similar to the proof of Theorem 10 by using Theorems 4 and 7. □

Theorem 14. *Suppose that $F,\mathcal{G}: \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2-convex function, respectively, on Δ. Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4} d_{q_2}y^{\kappa_2} d_{q_1}\varkappa \\
&\leq \frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1} \\
&\int_{\kappa_1}^{\kappa_2}\left([F(\varkappa,\kappa_3)]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1} d_{q_1}\varkappa \\
&+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1} \\
&\int_{\kappa_1}^{\kappa_2}\left([\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2 \kappa_2} d_{q_1}\varkappa \\
&+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3} \\
&\int_{\kappa_3}^{\kappa_4}\left([F(\kappa_1,y)]^{r_1}+\left[q_1^{1/2}F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1 \kappa_4} d_{q_2}y \\
&+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3} \\
&\int_{\kappa_3}^{\kappa_4}\left([\mathcal{G}(\kappa_1,y)]^{r_2}+\left[q_1^{1/2}\mathcal{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2 \kappa_4} d_{q_2}y,
\end{aligned} \tag{58}$$

where $0<r_1$, $r_2\leq 2$, *and* $0<q_1,q_2<1$.

Proof. Since $F: \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex on Δ, then the partial mappings,

$$\begin{aligned}
&F_\varkappa: [\kappa_3,\kappa_4] \longrightarrow \mathbb{R}_+, F_\varkappa(v) = F(\varkappa,v), \\
&F_y: [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}_+, F_y(u) = F(u,y),
\end{aligned} \tag{59}$$

are r_1-convex on Δ. On the other hand, if $\mathcal{G}$ is a coordinated r_2-convex function, then the partial mappings,

$$\begin{aligned}
&\mathcal{G}_\varkappa: [\kappa_3,\kappa_4] \longrightarrow \mathbb{R}_+, \mathcal{G}_\varkappa(v) = \mathcal{G}(\varkappa,v), \\
&\mathcal{G}_y: [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}_+, \mathcal{G}_y(u) = \mathcal{G}(u,y),
\end{aligned} \tag{60}$$

are r_2-convex on Δ. From inequality (32), we get

$$\begin{aligned}
&\frac{2}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F_\varkappa(y)\mathcal{G}_\varkappa(y)^{\kappa_4} d_{q_2}y \leq \frac{1}{[2/r_1+1]_{q_2}} \\
&\left([F_\varkappa(\kappa_3)]^{r_1}+\left[q_2^{1/2}F_\varkappa(\kappa_4)\right]^{r_1}\right)^{2/r_1} \\
&+\frac{1}{[2/r_2+1]_{q_2}}\left([\mathcal{G}_\varkappa(\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}_\varkappa(\kappa_4)\right]^{r_2}\right)^{2/r_2},
\end{aligned} \tag{61}$$

i.e.,

$$\begin{aligned}
&\frac{2}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)\kappa_4 d_{q_2}y \\
&\leq \frac{1}{[2/r_1+1]_{q_2}}\left([F(\varkappa,\kappa_3)]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1} \\
&+\frac{1}{[2/r_2+1]_{q_2}}\left([\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2}.
\end{aligned} \tag{62}$$

Dividing both sides of the inequality $(\kappa_2-\kappa_1)$ and q^{κ_2}-integrating with respect to $\varkappa$ on $[\kappa_1,\kappa_2]$, we have

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4} d_{q_2}y^{\kappa_2} d_{q_1}\varkappa \\
&\leq \frac{1}{2}\frac{1}{[2/r_1+1]_{q_2}}\left[\frac{1}{\kappa_2-\kappa_1}\right. \\
&\left.\int_{\kappa_1}^{\kappa_2}\left([F(\varkappa,\kappa_3)]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1 \kappa_2} d_{q_1}\varkappa\right] \\
&+\frac{1}{2}\frac{1}{[2/r_2+1]_{q_2}}\left[\frac{1}{\kappa_2-\kappa_1}\right. \\
&\left.\int_{\kappa_1}^{\kappa_2}\left([\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2 \kappa_2} d_{q_1}\varkappa\right].
\end{aligned} \tag{63}$$

By a similar argument, we obtain

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{2}\frac{1}{[2/r_1+1]_{q_1}}\left[\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1^{1/2}F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1\,\kappa_4}d_{q_2}y\right]$$
$$+\frac{1}{2}\frac{1}{[2/r_2+1]_{q_1}}\left[\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[\mathcal{G}(\kappa_1,y)\right]^{r_2}+\left[q_1^{1/2}\mathcal{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2\,\kappa_4}d_{q_2}y\right]. \tag{64}$$

By adding inequalities (63) and (64), we obtain the required result. □

Remark 7. If we take the limit $q_1\longrightarrow 1^-$ and $q_2\longrightarrow 1^-$ in Theorem 14, then Theorem 14 reduces to Theorem 6 in [15].

Corollary 4. *If we choose $r_1=r_2=2$ in Theorem 14, then we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{4(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[F(\varkappa,\kappa_3)]^2+q_2[F(\varkappa,\kappa_4)]^2}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{4(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[\mathcal{G}(\varkappa,\kappa_3)]^2+q_2[\mathcal{G}(\varkappa,\kappa_4)]^2}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{4(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[F(\kappa_1,y)]^2+q_1[F(\kappa_2,y)]^2}{[2]_{q_1}}\right)^{\kappa_4}d_{q_2}y$$
$$+\frac{1}{4(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[\mathcal{G}(\kappa_1,y)]^2+q_1[\mathcal{G}(\kappa_2,y)]^2}{[2]_{q_1}}\right)^{\kappa_4}d_{q_2}y. \tag{65}$$

Particularly, if $F(\varkappa,y)=\mathcal{G}(\varkappa,y)$ for all $(\varkappa,y)\in\Delta$, then we get

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}[F(\varkappa,y)]^{2\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[F(\varkappa,\kappa_3)]^2+q_2[F(\varkappa,\kappa_4)]^2}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[F(\kappa_1,y)]^2+q_1[F(\kappa_2,y)]^2}{[2]_{q_1}}\right)^{\kappa_4}d_{q_2}y. \tag{66}$$

Theorem 15. *Suppose that $F,\mathcal{G}\colon\Delta\longrightarrow\mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2-convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)_{\kappa_3}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1\,\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[\mathcal{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2\,\kappa_2}d_{q_1}\varkappa \tag{67}$$
$$+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}F(\kappa_1,y)\right]^{r_1}+\left[F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1}{}_{\kappa_3}d_{q_2}y$$
$$+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}\mathcal{G}(\kappa_1,y)\right]^{r_2}+\left[\mathcal{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2}{}_{\kappa_3}d_{q_2}y,$$

where $0 < r_1,\ r_2 \leq 2$, *and* $0 < q_1, q_2 < 1$.

Proof. The proof is similar to the proof of Theorem 14 by using Theorems 5 and 8. □

Theorem 16. *Suppose that* $F, \mathscr{G}\colon \Delta \longrightarrow \mathbb{R}_+$ *is a coordinated* r_1*-convex function and coordinated* r_2*-convex function, respectively, on* Δ. *Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa, y)\mathscr{G}(\varkappa, y)\,{}_{\kappa_3}d_{q_2}y\,{}_{k_1}d_{q_1}\varkappa\\
&\leq \frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}\mathscr{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[\mathscr{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}F(\kappa_1,y)\right]^{r_1}+\left[F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1}{}_{\kappa_3}d_{q_2}y\\
&+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}\mathscr{G}(\kappa_1,y)\right]^{r_2}+\left[\mathscr{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2}{}_{\kappa_3}d_{q_2}y,
\end{aligned}\tag{68}$$

where $0 > r_1,\ r_2 \leq 2$, *and* $0 < q_1, q_2 < 1$.

Proof. The proof is similar to the proof of Theorem 14 by using Theorems 5 and 8. □

Theorem 17. *Suppose that* $F, \mathscr{G}\colon \Delta \longrightarrow \mathbb{R}_+$ *is a coordinated* r_1*-convex function and coordinated* r_2*-convex function, respectively, on* Δ. *Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa, y)\mathscr{G}(\varkappa, y)\,{}^{\kappa_4}d_{q_2}y\,{}_{k_1}d_{q_1}\varkappa\\
&\leq \frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\\
&\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\\
&\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}\mathscr{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[\mathscr{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\\
&\int_{\kappa_3}^{\kappa_4}\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1^{1/2}F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1}{}^{\kappa_4}d_{q_2}y\\
&+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\\
&\int_{\kappa_3}^{\kappa_4}\left(\left[\mathscr{G}(\kappa_1,y)\right]^{r_2}+\left[q_1^{1/2}\mathscr{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2}{}^{\kappa_4}d_{q_2}y,
\end{aligned}\tag{69}$$

where $0 < r_1,\ r_2 \leq 2$, *and* $0 < q_1, q_2 < 1$.

Proof. The proof is similar to the proof of Theorem 14 by using Theorem 5. □

Theorem 18. *Suppose that* $F, \mathscr{G}\colon \Delta \longrightarrow \mathbb{R}_+$ *is a coordinated* r_1*-convex function and coordinated* r_2 *-convex function, respectively, on* Δ. *Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa, y)\mathscr{G}(\varkappa, y)\,{}^{\kappa_4}d_{q_2}\,{}^{\kappa_2}d_{q_1}\varkappa\\
&\leq \frac{1}{2}\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\frac{\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2F(\varkappa,\kappa_4)\right]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}^{\kappa_2}d_{q_1}\varkappa\right)\\
&\times\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\frac{\left[\mathscr{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[q_2\mathscr{G}(\varkappa,\kappa_4)\right]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}^{\kappa_2}d_{q_1}\varkappa\right)\\
&+\frac{1}{2}\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\frac{\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1F(\kappa_2,y)\right]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}^{\kappa_4}d_{q_2}\right)\\
&\times\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\frac{\left[\mathscr{G}(\kappa_1,y)\right]^{r_2}+\left[q_1\mathscr{G}(\kappa_2,y)\right]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}^{\kappa_4}d_{q_2}\right).
\end{aligned}\tag{70}$$

where $0 < q_1, q_2 < 1$, *and* $1/r_1 + 1/r_2 = 1$ *with* $r_1 > 1$.

Proof. By applying inequality (42) for the partial mapping $F_\varkappa$ and $\mathscr{G}_\varkappa$, we can write

$$\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y \leq \left(\frac{[F(\varkappa,\kappa_3)]^{r_1}+[q_2F(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1} \cdot \left(\frac{[\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[q_2\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}. \tag{71}$$

By using q^{κ_2}-integral, we obtain

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa \leq \left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[F(\varkappa,\kappa_3)]^{r_1}+[q_2F(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}^{\kappa_2}d_{q_1}\varkappa\right) \times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[q_2\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa\right). \tag{72}$$

Similarly, by applying inequality (42) for the partial mapping F_y and $\mathcal{G}_y$, we can write

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa \leq \left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[F(\kappa_1,y)]^{r_1}+[q_1F(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}^{\kappa_4}d_{q_2}y\right) \times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[\mathcal{G}(\kappa_1,y)]^{r_2}+[q_1\mathcal{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}^{\kappa_4}d_{q_2}y\right). \tag{73}$$

By adding inequalities (72) and (73), we obtain the desired result (70). □

Remark 8. If we take the limit $q_1 \longrightarrow 1^-$ and $q_2 \longrightarrow 1^-$ in Theorem 18, then Theorem 18 reduces to Theorem 7 in [15].

Corollary 5. *If we choose $r_1 = r_2 = 2$ in Theorem 18, then we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa \leq\frac{1}{2}\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\sqrt{\frac{[F(\varkappa,\kappa_3)]^2+[q_2F(\varkappa,\kappa_4)]^2}{[2]_{q_2}}}{}^{\kappa_2}d_{q_1}\varkappa\right) \times\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\sqrt{\frac{[\mathcal{G}(\varkappa,\kappa_3)]^2+[q_2\mathcal{G}(\varkappa,\kappa_4)]^2}{[2]_{q_2}}}{}^{\kappa_2}d_{q_1}\varkappa\right) +\frac{1}{2}\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\sqrt{\frac{[F(\kappa_1,y)]^2+[q_1F(\kappa_2,y)]^2}{[2]_{q_1}}}{}^{\kappa_4}d_{q_2}y\right) \times\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\sqrt{\frac{[\mathcal{G}(\kappa_1,y)]^2+[q_1\mathcal{G}(\kappa_2,y)]^2}{[2]_{q_1}}}{}^{\kappa_4}d_{q_2}y\right). \tag{74}$$

Theorem 19. *Suppose that $F,\mathcal{G}\colon \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2-convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)_{k_3}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa \leq\frac{1}{2}\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[F(\varkappa,\kappa_3)]^{r_1}+[q_2F(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}^{\kappa_2}d_{q_1}\varkappa\right) \times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[q\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}^{\kappa_2}d_{q_1}\varkappa\right) +\frac{1}{2}\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1F(\kappa_1,y)]^{r_1}+[F(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}_{k_3}d_{q_2}y\right) \times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1\mathcal{G}(\kappa_1,y)]^{r_2}+[\mathcal{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}_{k_3}d_{q_2}y\right). \tag{75}$$

where $0 < q_1, q_2 < 1$, and $1/r_1 + 1/r_2 = 1$ with $r_1 > 1$.

Proof. The proof is similar to the proof of Theorem 18 by using Theorems 6 and 9. □

Theorem 20. *Suppose that $F,\mathcal{G}\colon \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2-convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y_{\kappa_1}d_{q_1}\varkappa \leq\frac{1}{2}\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2F(\varkappa,\kappa_3)]^{r_1}+[F(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}_{\kappa_1}d_{q_1}\varkappa\right) \times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}_{\kappa_1}d_{q_1}\varkappa\right) +\frac{1}{2}\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[F(\kappa_1,y)]^{r_1}+[q_1F(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}^{\kappa_4}d_{q_2}y\right) \times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[\mathcal{G}(\kappa_1,y)]^{r_2}+[q_1\mathcal{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}^{\kappa_4}d_{q_2}y\right), \tag{76}$$

where $0 < q_1, q_2 < 1$, and $1/r_1 + 1/r_2 = 1$ with $r_1 > 1$.

Proof. The proof is similar to the proof of Theorem 18 by using Theorems 6 and 9. □

Theorem 21. *Suppose that $F,\mathcal{G}\colon \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2 -convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathscr{G}(\varkappa,y)\,{}_{\kappa_3}d_{q_2}y\,{}_{\kappa_1}d_{q_1}\varkappa$$
$$\leq\frac{1}{2}\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2F(\varkappa,\kappa_3)]^{r_1}+[F(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}_{\kappa_1}d_{q_1}\varkappa\right)$$
$$\times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2\mathscr{G}(\varkappa,\kappa_3)]^{r_2}+[\mathscr{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}_{\kappa_1}d_{q_1}\varkappa\right)$$
$$+\frac{1}{2}\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1F(\kappa_1,y)]^{r_1}+[F(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}_{\kappa_3}d_{q_2}y\right)$$
$$\times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1\mathscr{G}(\kappa_1,y)]^{r_2}+[\mathscr{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}_{\kappa_3}d_{q_2}y\right). \tag{77}$$

where $0<q_1,q_2<1$, *and* $1/r_1+1/r_2=1$ *with* $r_1>1$.

Proof. The proof is similar to the proof of Theorem 18 by using Theorem 6. □

5. Conclusions

In this study, we present several quantum Hermite–Hadamard type inequalities for r-convex functions and coordinated r-convex functions. We also give some quantum inequalities for the product of two r-convex functions and for the product of two coordinated r-convex functions. In the future work, we can establish the similar quantum inequalities by using generalized r-convex functions.

Data Availability

No datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] H. Budak, S. Erden, and M. A. Ali, "Simpson and newton type inequalities for convex functions via newly defined quantum integrals," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 1, pp. 378–390, 2021.

[2] T. Ernst, *A Comprehensive Treatment q-Calculus*, Birkhäuser, Basel, Switzerland, 2012.

[3] H. Gauchman, "Integral inequalities in q-calculus," *Computers & Mathematics with Applications*, vol. 47, no. 2-3, pp. 281–300, 2004.

[4] F. H. Jackson, "On a q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[5] V. Kac and P. Cheung, *Quantum Calculus*, Universitext. Springer-Verlag, Berlin, Germany, 2002.

[6] M. A. Noor, K. I. Noor, and M. U. Awan, "Some quantum estimates for Hermite-Hadamard inequalities," *Applied Mathematics and Computation*, vol. 251, pp. 675–679, 2015.

[7] M. A. Noor, K. I. Noor, and M. U. Awan, "Some quantum integral inequalities via preinvex functions," *Applied Mathematics and Computation*, vol. 269, pp. 242–251, 2015.

[8] W. Sudsutad, S. K. Ntouyas, and J. Tariboon, "Quantum integral inequalities for convex functions," *Journal of Mathematical Inequalities*, vol. 9, no. 3, pp. 781–793, 2015.

[9] J. Tariboon and S. K. Ntouyas, "Quantum calculus on finite intervals and applications to impulsive difference equations," *Advancements in Difference Equiations*, vol. 282, p. 19, 2013.

[10] H. Zhuang, W. Liu, and J. Park, "Some quantum estimates of Hermite-Hadmard inequalities for quasi-convex functions," *Mathematics*, vol. 7, no. 2, p. 18, 2019.

[11] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, MA, USA, 1992.

[12] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs*, Victoria University, Footscray, Australia, 2000.

[13] B. G. Pachpatte, "On some inequalities for convex functions," *Research Group in Mathematical Inequalities and Applications*, vol. 6, no. E, 2003.

[14] C. E. M. Pearce, J. Pečarić, and V. Šimić, "Stolarsky means and hadamard's inequality," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 1, pp. 99–109, 1998.

[15] A. Ekinci, A. O. Akdemir, and M. E. Ozdemir, "On Hadamard-type Inequalities for co–ordinated r-convex functions," *AIP Conference Proceedings*, vol. 1833, no. 1, 2017.

[16] M. Avriel, "R-convex functions," *Mathematical Programming*, vol. 2, no. 1, pp. 309–323, 1972.

[17] M. Avriel, "Solution of certain nonlinear programs involvingr-convex functions," *Journal of Optimization Theory and Applications*, vol. 11, no. 2, pp. 159–174, 1973.

[18] B. Bayraktar, "Some integral inequalities for functions whose absolute values of the third derivative is concave and r-convex," *Turkish Journal of Inequalities*, vol. 4, no. 2, pp. 59–78, 2020.

[19] D.-D. Gao, B.-Y. Xi, Y. Wu, and B.-N. Guo, "On integral inequalities of Hermite-Hadamard type for coordinated r-mean convex functions," *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 873–885, 2019.

[20] M. Karagozlu and M. A. Ardıc, "New integral inequalities for r-convex function," in *Proceedings of the 1st International Conference on Mathematical and Related Sciences (2018)*, Antalya, Turkey, September 2018.

[21] G. S. Yang and D. Y. Hwang, "Refinementes of Hadamard inequality for r-convex functions," *Indian Journal of Pure and Applied Mathematics*, vol. 32, no. 10, pp. 1571–15791, 2001.

[22] G. Zabandan, A. Bodaghi, and A. Kilicman, "The Hermite-Hadamard inequality for r-convex functions," *Journal of Inequalities and Applications*, vol. 215, p. 8, 2012.

[23] Y. X. Zhao, S. Y. Wang, and L. Coladas Uria, "Characterizations of r-convex functions," *Journal of Optimization Theory and Applications*, vol. 145, no. 1, pp. 186–195, 2010.

[24] M. W. Alomari, "On q-Bernoulli inequality," *Turkish Journal of Science*, vol. 3, no. 1, pp. 320–39, 2018.

[25] T. Ernst, *The History of q-calculus and a New Method*, Department of Mathematics, Uppsala University, Uppsala, Sweden, 2000.

[26] N. Alp, M. Z. Sarıkaya, M. Kunt, and İ. İşcan, "q -Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions," *Journal of King Saud University - Science*, vol. 30, no. 2, pp. 193–203, 2018.

[27] S. Bermudo, P. Kórus, and J. E. Nápoles Valdés, "On q-Hermite-Hadamard inequalities for general convex functions," *Acta Mathematica Hungarica*, vol. 162, no. 1, pp. 364–374, 2020.

[28] K. Brahim, R. Latifa, and S. Taf, "Hermite-Hadamard type inequalities for r-convex functions in q-calculus," *Le Matematiche*, vol. 70, no. 2, pp. 295–303, 2015.

[29] M. A. Latif, S. S. Dragomir, and E. Momoniat, "Some q -analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane," *Journal of King Saud University-Science*, vol. 29, no. 3, pp. 263–273, 2017.

[30] H. Budak, M. A. Ali, and M. Tarhanaci, "Some new quantum hermite-hadamard-like inequalities for coordinated convex functions," *Journal of Optimization Theory and Applications*, vol. 186, no. 3, pp. 899–910, 2020.

[31] M. A. Ali, H. Budak, Z. Zhang, and H. Yildirim, "Some new Simpson's type inequalities for coordinated convex functions in quantum calculus," *Mathematical Methods in the Applied Sciences*, vol. 7, 2020.

[32] H. Budak, M. A. Ali, and T. Tunc, "Quantum Ostrowski type integral inequalities for functions of two variables," *Mathematical Methods in the Applied Sciences*, vol. 13, 2020.

[33] N. P. N. Ngoc, N. V. Vinh, and P. T. T. Hien, "Integral inequalities of hadamard type for r-convex functions," *International Mathematical Forum*, vol. 33–36, no. 4, pp. 1723–1728, 2009.

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Research Article

Nonlocal Fractional Hybrid Boundary Value Problems Involving Mixed Fractional Derivatives and Integrals via a Generalization of Darbo's Theorem

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In this work, a new existence result is established for a nonlocal hybrid boundary value problem which contains one left Caputo and one right Riemann–Liouville fractional derivatives and integrals. The main result is proved by applying a new generalization of Darbo's theorem associated with measures of noncompactness. Finally, an example to justify the theoretical result is also presented.

1. Introduction

In the past years, fractional differential equations have attracted a lot of attention from many research studies as they have played a key role in many basic sciences such as chemistry, control theory, biology, and other arenas [1–3]. In addition, boundary conditions of differential models are the strongest tools to extend applications of those equations [4–6]. In fact, fractional differential equations can be extended by creating different types of boundary conditions. Newly, many authors have studied various types of boundary conditions to obtain new results of differential models.

The following hybrid differential equation was studied by Dhage and Lakshmikantham [7]:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{x(t)}{h(t,x(t))}\right] = \omega(t,x(t)), \quad a.\mathrm{e}\, \mathrm{t} \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where h and ω are continuous functions from $J \times \mathbb{R}$ into $\mathbb{R}\smallsetminus\{0\}$ and $\mathbb{R}$, respectively. Based on the above work, the Caputo hybrid boundary value problem of the form

$$\begin{cases} {}^{C}D_{0^+}^{p}\left[\dfrac{x(t)}{h(t,x(t))}\right] = \omega(t,x(t)), \quad a.\mathrm{e}\, \mathrm{t} \in I := [0,L], \\ a_1\dfrac{x(0)}{h(0,x(0))} + a_2\dfrac{x(L)}{h(L,x(L))} = d, \end{cases} \tag{2}$$

was studied by Hilal and Kajouni [8] in which $0 < p < 1$, h and ω are continuous functions from $J \times \mathbb{R}$ into $\mathbb{R}\smallsetminus\{0\}$ and $\mathbb{R}$, respectively, and a_1, a_2, and d are real constants with $a_1 + a_2 \neq 0$. For some recent results on hybrid fractional differential equations, see [9–12].

In [13], the authors proved the following integro-differential equation:

$$
\begin{aligned}
&{}^{c}D_{1_-}^{\alpha_1}\,{}^{\mathrm{RL}}D_{0_+}^{\alpha_2}u(t)+\theta I_{-1}^{p}I_{0_+}^{q}f_1(t,u(t))=f(t,u(t)),\quad t\in[0,1],\\
&u(0)=u(\xi)=0,\\
&u(1)=\delta u(\mu),\quad 0<\xi<\mu<1,
\end{aligned}\tag{3}
$$

where ${}^{c}D_{1_-}^{\alpha_1}$ and ${}^{\mathrm{RL}}D_{0_+}^{\alpha_2}$ indicate right Caputo and left Riemann–Liouville fractional derivatives of orders $\alpha_1\in(1,2]$ and $\alpha_2\in(0,1]$, respectively, $f_1,f_2\colon[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions, and the symbols I_{-1}^{p} and $I_{0^+}^{q}$ denote both right and left Riemann–Liouville fractional integrals of orders $p,q>0$, respectively. Ahmad et al. [13] applied Banach and Krasnosel'skiĭ fixed point theorems as well Leray–Schauder nonlinear alternative to obtain main results. We point out that fractional differential equations containing mixed fractional derivatives appear in the study of variational principles [14].

For some recent results for boundary value problems involving left or/and right fractional derivatives, we refer to the papers [15–31] and references therein.

In the present paper, we combine mixed fractional derivatives and hybrid fractional differential equations. More precisely, we investigate the existence of solutions for the following hybrid boundary value problem which contains both left Caputo and right Riemann–Liouville fractional derivatives and integrals and nonlocal hybrid conditions of the form:

$$
\begin{aligned}
&{}^{c}D_{1_-}^{\alpha_1}\,{}^{\mathrm{RL}}D_{1_+}^{\alpha_2}\frac{u(t)}{g(t,u(t))}+\theta I_{-1}^{p}I_{0^+}^{q}f_1(t,u(t))=f_2(t,y(t)),\quad t\in J\coloneqq[0,1],\\
&\frac{u(0)}{g(0,u(0))}=\frac{u(\xi)}{g(\xi,u(\xi))}=0,\\
&\frac{u(1)}{g(1,u(1))}=\delta\frac{u(\mu)}{g(\mu,u(\mu))},\quad 0<\xi<\mu<1,
\end{aligned}\tag{4}
$$

where ${}^{c}D_{1_-}^{\alpha_1}$ and ${}^{\mathrm{RL}}D_{0_+}^{\alpha_2}$ are right Caputo and left Riemann–Liouville fractional derivatives of orders $\alpha_1\in(1,2]$ and $\alpha_2\in(0,1]$, respectively, and the symbols I_{-1}^{p} and $I_{0^+}^{q}$ denote both right and left Riemann–Liouville fractional integrals of orders $p,q>0$, respectively, $f_1,f_2\in C(J\times\mathbb{R},\mathbb{R})$, $g\in C(J\times\mathbb{R},\mathbb{R}\smallsetminus\{0\})$, and $\delta,\theta\in\mathbb{R}$. An existence result is obtained via a new extension of Darbo's theorem associated with measures of noncompactness.

The structure of the paper has been organized as follows. Section 2 presents some basic definitions and lemmas which will be applied in the future. In Section 3, we prove an existence result for problem (4). Finally, we present an example to illustrate the obtained result.

2. Preliminaries

Now, some basic notations are recalled from [2].

Definition 1. For an integrable function $\phi\colon(0,\infty)\longrightarrow\mathbb{R}$, we define the left and right Riemann–Liouville fractional integrals of order $\beta>0$, respectively, by

$$
\begin{aligned}
I_{0^+}^{\beta}\phi(t)&=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\phi(s)\mathrm{d}s,\\
I_{1-}^{\beta}\phi(t)&\int_t^1\frac{(s-t)^{\beta-1}}{\Gamma(\beta)}\phi(s)\mathrm{d}s.
\end{aligned}\tag{5}
$$

Definition 2. For the function $\phi\colon(0,\infty)\longrightarrow\mathbb{R}$ in which $\phi\in C^n(0,\infty)$, we define the left Riemann–Liouville fractional derivative and the right Caputo fractional derivative of order $\beta\in(n-1,n]$, respectively, by

$$
\begin{aligned}
{}^{\mathrm{RL}}D_{0^+}^{\beta}\phi(t)&=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t\frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)}\phi(s)\mathrm{d}s,\\
{}^{c}D_{1-}^{\beta}\phi(t)&=(-1)^n\int_t^1\frac{(s-t)^{n-\beta-1}}{\Gamma(n-\beta)}\phi^{(n)}(s)\mathrm{d}s.
\end{aligned}\tag{6}
$$

Lemma 1. *If $p>0$ and $q>0$, then the following relations hold almost everywhere on $[a,b]$:*

$$
\begin{aligned}
I_{1-}^{p}I_{-1}^{q}f(x)&=I_{1-}^{p+q}f(x),\\
I_{0^+}^{p}I_{0^+}^{q}f(x)&=I_{0^+}^{p+q}f(x).
\end{aligned}\tag{7}
$$

As the technique of measure of noncompactness will be applied to obtain our main result, we recall some basic facts about the notion of measure of noncompactness.

Assume that Z is the real Banach space with the norm $\|\cdot\|$ and zero element θ. For a nonempty subset X of Z, the closure and the closed convex hull of X will be denoted by $\overline{X}$ and $\mathrm{Conv}(X)$, respectively. Also, M_Z and N_Z denote the family of all nonempty and bounded subsets of Z and its subfamily consisting of all relatively compact sets, respectively.

Definition 3 (see [32]). We say that a mapping $h\colon M_Z\longrightarrow[0,\infty)$ is a measure of noncompactness, if the following conditions hold true:

(1) The family Ker$h = \{X \in M_Z : h(X) = 0\}$ is nonempty and Ker $h \subseteq N_Z$
(2) $X_1 \subseteq Y_1 \Rightarrow h(X_1) \leq h(Y_1)$
(3) $h(\overline{X}) = h(X)$
(4) $h(\text{Conv}(X)) = h(X)$
(5) $h(\alpha X + (1-\alpha)Y) \leq \alpha h(X) + (1-\alpha)h(Y)$ for $\alpha \in [0,1]$
(6) For the sequence $\{X_n\}$ of closed sets from M_Z in which $X_{n+1} \subseteq X_n$ for $n = 1, 2, \ldots$ and $\lim_{n\longrightarrow\infty} h(X_n) = 0$, we have $\cap_{n=1}^{\infty} X_n \neq \emptyset$

In [33], some generalizations of Darbo's theorem have been proved by Samadi and Ghaemi. Also, in [34], Darbo's theorem was extended, and the following result was presented which is basis for our main result.

Theorem 1. *Let T be a continuous self-mapping operator on the set D, where D denotes a nonempty, bounded, closed, and convex subset of a Banach space Z. Assume that, for all nonempty subset X of D, we have*

$$\theta_1((h(X)) + \theta_2(h(T(X))) \leq \theta_2(h(X))), \tag{8}$$

where h is an arbitrary measure of noncompactness defined in Z and $(\theta_1, \theta_2) \in U$. Then, T has a fixed point in D.

In Theorem 1, let U indicate the set of all pairs (θ_1, θ_2) where the following conditions hold true:

(U_1) $\theta_1(t_n) \nrightarrow 0$ for each strictly increasing sequence $\{t_n\}$
(U_2) θ_2 is strictly increasing function
(U_3) If $\{\alpha_n\}$ be a sequence of positive numbers, then $\lim_{n\longrightarrow\infty}\alpha_n = 0 \Leftrightarrow \lim_{n\longrightarrow\infty}\theta_2(\alpha_n) = -\infty$
(U_4) Let $\{l_n\}$ be a decreasing sequence in which $l_n \longrightarrow 0$ and $\theta_1(l_n) < \theta_2(l_n) - \theta_2(l_{n+1})$, then we have $\sum_{n=1}^{\infty} l_n < \infty$

Next, the definition of a measure of noncompactness in the space $C([0,1])$ is recalled which will be applied later. Fix $Y \in M_{C[0,1]}$, and for $\varepsilon > 0$ and $y \in Y$, we define

$$\begin{aligned}
&\varphi(y,\varepsilon) = \sup\{|y(t) - y(s)|: t, s \in [0,1], \quad |t-s| \leq \varepsilon\},\\
&\varphi(Y,\varepsilon) = \sup\{\varphi(y,\varepsilon): y \in Y\},\\
&\varphi_0(Y) = \lim_{\varepsilon\longrightarrow 0} \varphi(Y,\varepsilon).
\end{aligned} \tag{9}$$

Banas and Goebel [32] proved that $\varphi_0(Y)$ is a measure of noncompactness in the space $C([0,1])$.

Lemma 2 (see [32]). *The measure of noncompactness φ_0 on $C(I)$ satisfies the following condition:*

$$\varphi_0(XY) \leq \|X\|\varphi_0(Y) + \|Y\|\varphi_0(X), \tag{10}$$

for all $X, Y \subseteq C(I)$.

3. Main Existence Result

In this section, an existence result of problem (4) is investigated. In view of [13], Lemma 2, we present the following lemma which is an essential tool in our consideration.

Lemma 3. *Let $H_1, F_1 \in C[0,1] \cap L(0,1)$, $g \in C([0,1], \mathbb{R}\setminus\{0\})$, and $\Lambda \neq 0$. Then, the solution of the problem*

$$\begin{cases}
{}^{c}D_{1_-}^{\alpha_1}\, {}^{\text{RL}}D_{0^+}^{\beta} \dfrac{u(t)}{g(t,u(t))} + \lambda I_{-1}^{p} I_{0_+}^{q} H_1(t) = F_1(t), \quad t \in [0,1],\\[2ex]
\dfrac{u(0)}{g(0,u(0))} = \dfrac{u(\xi)}{g(\xi,u(\xi))} = 0,\\[2ex]
\dfrac{u(1)}{g(1,u(1))} = \delta \dfrac{u(\mu)}{g(\mu,u(\mu))}, \quad 0 < \xi < \mu < 1,
\end{cases} \tag{11}$$

has the form:

$$u(t) = g(t,u(t)) \left[\begin{array}{c} \displaystyle\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left(I_{1_-}^{\alpha} F_1(s) - \lambda I_{1_-}^{\alpha_1+p} I_{0_+}^{q} H_1(s)\right) ds + a_1(t) \left\{ \begin{array}{c} \displaystyle\delta \int_0^{\mu} \frac{(\mu-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left(I_{1_-}^{\alpha} F_1(s) - \lambda I_{1_-}^{\alpha_1+p} I_{0_+}^{q} H(s)\right) ds \\[2ex] \displaystyle - \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left(I_{1_-}^{\alpha_1} F_1(s) - \lambda I_{1_-}^{\alpha_1+p} I_{0_+}^{q} H_1(s)\right) ds \end{array} \right\} \\[4ex] \displaystyle + a_2(t) \int_0^{\xi} \frac{(\xi-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left(I_{1_-}^{\alpha_1} F_1(s) - \lambda I_{1_-}^{\alpha_1+p} I_{0_+}^{q} H_1(s)\right) ds \end{array} \right], \tag{12}$$

where

$$a_1(t) = \frac{1}{\Lambda}\left[\xi^{\alpha_2+1}t^{\alpha_2} - \xi^{\alpha_2}t^{\alpha_2+1}\right],$$
$$a_2(t) = \frac{1}{\Lambda}\left[t^{\alpha_2}\left(1 - \delta\mu^{\alpha_2+1}\right) - t^{\alpha_2+1}\left(1 - \delta\mu^{\alpha_2}\right)\right], \tag{13}$$
$$\Lambda = \xi^{\alpha_2+1}\left(1 - \delta\mu^{\alpha_2}\right) - \xi^{\alpha_2}\left(1 - \delta\mu^{\alpha_2+1}\right).$$

Now, the hypotheses which will be applied to prove the main result of this section are presented.

(H_1) g: $[0,1]\times\mathbb{R} \longrightarrow \mathbb{R}\setminus\{0\}$ is a continuous function, and there exists a positive real number $d > 0$ provided that

$$\left|g(t, x_1) - g(t, x_2)\right| \le e^{-d}\left|x_1 - x_2\right|, \tag{14}$$

where $t \in I$ and $x_1, x_2 \in \mathbb{R}$. Moreover, assume that $\overline{g} = \sup\{|g(t,0)|; t \in [0,1]\}$.

(H_2) f_1, f_2: $[0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions provided that

$$\begin{gathered} \left|f_1(t,u)\right| \le M_1, \\ \left|f_1(t,u) - f_1(t,v)\right| \le k_1|u - v|, \\ \left|f_2(t,u)\right| \le M_2, \\ \left|f_2(t,u) - f_2(t,v)\right| \le k_2|u - v|, \end{gathered} \tag{15}$$

where $M_1, M_2, k_1, k_2 \ge 0$ and $u, v \in \mathbb{R}$.

(H_3) The inequality

$$\left[e^{-d}r_0 + \overline{g}\right]\left\{\frac{M_1}{\Gamma(\alpha_1 + 1)} + \frac{|\theta|M_2}{\Gamma(\alpha_1 + p + 1)\Gamma(q + 1)}\right\}\Delta \le r_0, \tag{16}$$

has a positive solution r_0. Also, assume that

$$\left\{\frac{M_1}{\Gamma(\alpha_1 + 1)} + \frac{|\theta|M_2}{\Gamma(\alpha_1 + p + 1)\Gamma(q + 1)}\right\}\Delta < 1, \tag{17}$$

where

$$\Delta = \frac{1}{\Gamma(\alpha_2 + 1)}\left[1 + \overline{a_1}\left(|\delta|\mu^{\alpha_2} + 1\right) + \overline{a_2}\xi^{\alpha_2}\right],$$
$$\overline{a_1} = \max_{t\in[0,1]}\left|a_1(t)\right|, \tag{18}$$
$$\overline{a_2} = \max_{t\in[0,1]}\left|a_2(t)\right|.$$

Theorem 2. *Suppose that the hypotheses* $(H_1) - (H_3)$ *are true. Then, the hybrid boundary value problem (4) has at least one solution on* $[0,1]$.

Proof. Due to Lemma 3, assume that the operator T has been defined on $C(I)$, $I \coloneqq [0,1]$ as follows:

$$T_1(u)(t) = \left(F_1u(t) + \overline{F_1}u(t)\right)\left(G_1u(t)\right), \tag{19}$$

where

$$G_1u(t) = g(t, u(t)),$$
$$F_1u(t) = \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1} f_2(s,u(s)) - \theta I_{1_-}^{\alpha_1+p} I_{0_+}^{q} f_1(s,u(s))\right)ds,$$
$$\overline{F_1}u(t) = a_1(t)\left\{\begin{aligned} &\delta\int_0^\mu \frac{(\mu-s)^{\alpha_1-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1} f_2(s,u(s)) - \theta I_{1_-}^{\alpha_1+p} I_{0_+}^{q} f_2(s,u(s))\right)ds \\ &-\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1} f_2(s,u(s)) - \theta I_{1_-}^{\alpha_1+p} I_{0_+}^{q} f_1(s,u(s))\right)ds \end{aligned}\right\} \tag{20}$$
$$+ a_2(t)\int_0^\xi \frac{(\xi-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1} f_2(s,u(s)) - \theta I_{1_-}^{\alpha_1+p} I_{0_+}^{q} f_1(s,u(s))\right)ds.$$

First, we show that $T_1u \in C(I)$ in which $u \in C(I)$. In view of assumption (H$_1$), we conclude that $G_1u \in C(I)$, $u \in C(I)$. Consequently, by proving $F_1u, \overline{F_2}u \in C(I)$, the claim is obtained. Let l_n be a sequence in $[0,1]$ such that $l_n \longrightarrow l$. Then, due to our assumptions, we get

$$\left|F_1u(l_n)-F_1u(l)\right| \le \left|\begin{array}{l} \displaystyle\int_0^l \frac{\left[(l_n-s)^{\alpha_2-1}-(l-s)^{\alpha_2-1}\right]}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1}f_2(s,u(s))-\theta I_{1_-}^{\alpha_1+p}I_{0_+}^{q}f_1(s,u(s))\right)\mathrm{d}s \\ \displaystyle+\int_l^{l_n}\frac{(l_n-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1}f_2(s,u(s))-\theta I_{1_-}^{\alpha_1+p}I_{0_+}^{q}f_1(s,u(s))\right)\mathrm{d}s \end{array}\right|$$
$$\le \frac{M_1}{\Gamma(\alpha_2)}\left|\int_0^{l_n}\left[(l_n-s)^{\alpha_2-1}-(l-s)^{\alpha_2-1}\right]I_{1-}^{\alpha_1}(1)\mathrm{d}s\right|$$
$$+M_1\left|\frac{1}{\Gamma(\alpha_2)}\int_l^{l_n}(l_n-s)^{\alpha_2-1}I_{1-}^{\alpha_1}(1)\mathrm{d}s\right|+\frac{|\theta|M_2}{\Gamma(\alpha_2)}\left|\int_0^l\left[(l_n-s)^{\alpha_2-1}-(l-s)^{\alpha_2-1}\right]I_{1_-}^{\alpha_1+p}I_{0_+}^{q}(1)\mathrm{d}s\right|$$
$$+\frac{|\theta|M_2}{\Gamma(\alpha_2)}\left|\int_l^{l_n}(l_n-s)^{\alpha_2-1}I_{1_-}^{\alpha_1+p}I_{0_+}^{q}(1)\mathrm{d}s\right|\le\left\{\frac{M_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}+\frac{|\theta|M_2}{\Gamma(q+1)\Gamma(\alpha_2+1)\gamma(\alpha_1+p+1)}\right\}$$
$$\times\left[2(l_n-l)^{\alpha_2}+\left|l_n^{\alpha_2}-l^{\alpha_2}\right|\right]\longrightarrow 0. \tag{21}$$

Hence, $F_1u\in C(I)$. To obtain that $\overline{F_1}u\in C(I)$, by the definitions of a_1 and a_2, we have $|a_1(l_n)-a_1(l)|\longrightarrow 0$ and $|a_2(l_n)-a_2(l)|\longrightarrow 0$. Hence, we have $|\overline{F_1}u(l_n)-\overline{F_1}u(l)|\longrightarrow 0$. Consequently, $T_1u\in C(I)$ for all $x\in C(I)$.

Now, we prove that the ball $D_{r_0}=\{u\in C(I)\colon \|u\|\le r_0\}$ is mapped into itself by the operator T. Let us fix $u\in C(I)$. Hence, due to existence assumptions, for $t\in I$, we have

$$\left|(T_1u)(t)\right|\le\left[e^{-d}|u(t)|+\overline{g}\right]$$
$$\cdot\left[\begin{array}{l}\displaystyle\left|\int_0^t\frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1}f_2(s,u(s))-\theta I_{1_-}^{\alpha_1+p}I_{0_+}^{q}f_1(s,u(s))\right)\mathrm{d}s\right| \\ +\left|\begin{array}{l} a_1(t)\left\{\begin{array}{l}\displaystyle\delta\int_0^\mu\frac{(\mu-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1}f_2(s,u(s))-\theta I_{1_-}^{\alpha_1+p}I_{0_+}^{q}f_2(s,u(s))\right)\mathrm{d}s \\ \displaystyle-\int_0^1\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1}f_2(s,u(s))-\theta I_{1_-}^{\alpha_1+p}I_{0_+}^{q}f_1(s,u(s))\right)\mathrm{d}s\end{array}\right\} \\ \displaystyle+\left|a_2(t)\int_0^\xi\frac{(\xi-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(I_{1_-}^{\alpha_1}f_2(s,u(s))-\theta I_{1_-}^{\alpha_1+p}I_{0_+}^{q}f_1(s,u(s))\right)\mathrm{d}s\right|\end{array}\right|\end{array}\right] \tag{22}$$
$$\le\left[e^{-d}r_0+\overline{g}\right]\left\{\frac{M_1}{\Gamma(\alpha_1+1)}+\frac{|\theta|M_2}{\Gamma(\alpha_1+p+1)\Gamma(q+1)}\right\}\Delta.$$

Consequently, according to assumption (H_3) we conclude that T maps the ball D_{r_0} into itself.

Now, the continuity property of the operator T is considered on the ball D_{r_0}. To do this, fix $\varepsilon>0$ and take $u,v\in D_{r_0}$ such that $\|u-v\|\le\varepsilon$. Then, for $t\in I$, we have

$$\begin{aligned}\left|(T_1u)(t)-(T_1v)(t)\right| &= \left|F_1u(t)G_1u(t)-F_1v(t)G_1v(t)+\overline{F_1}u(t)G_1u(t)-\overline{F_1}v(t)G_1v(t)\right|\\ &\leq \left|F_1u(t)G_1u(t)-F_1v(t)G_1u(t)\right|+\left|F_1v(t)G_1u(t)-F_1v(t)G_1v(t)\right|\\ &\quad+\left|\overline{F_1}u(t)G_1u(t)-\overline{F_1}v(t)G_1u(t)\right|+\left|\overline{F_1}\,v(t)G_1u(t)-\overline{F_1}v(t)G_1v(t)\right|\\ &\leq \left|g(t,u(t))-g(t,v(t))\right|\left\|F_1v(t)\right\|+\left|g(t,u(t))\right|\left|F_1u(t)-F_1v(t)\right|\\ &\quad+\left|g(t,u(t))-g(t,v(t))\right|\left\|\overline{F_1}v(t)\right\|+\left|g(t,u(t))\right|\left|\overline{F_1}u(t)-\overline{F_1}v(t)\right|.\end{aligned} \tag{23}$$

Then, we have

$$\begin{aligned}\left|(T_1u)(t)-(T_1v)(t)\right| &\leq e^{-d}\varepsilon\left(F_1v(t)+\overline{F_1}v(t)\right)+|g(t,u(t))|\varepsilon\left[\int_0^t\frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(k_1I_{1_-}^{\alpha_1}(1)+k_2|\theta|I_{1_-}^{\alpha_1+p}I_{0_+}^{q}(1)\right)\mathrm{d}s\right]\\ &\quad+|g(t,u(t))|\varepsilon\left[\begin{array}{l}|a_1(t)|\left\{\begin{array}{l}\delta\displaystyle\int_0^\mu\frac{(\mu-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(k_1I_{1_-}^{\alpha_1}(1)+k_2|\theta|I_{1_-}^{\alpha_1+p}I_{0_+}^{q}(1)\right)\mathrm{d}s\\ +\displaystyle\int_0^1\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(k_1I_{1_-}^{\alpha_1}(1)+k_2|\theta|I_{1_-}^{\alpha+p}I_{0_+}^{q}(1)\right)\mathrm{d}s\end{array}\right\}\\ +|a_2(t)|\displaystyle\int_0^\xi\frac{(\xi-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left(k_1I_{1_-}^{\alpha_1}(1)+k_2|\theta|I_{1_-}^{\alpha_1+p}I_{0_+}^{q}(1)\right)\mathrm{d}s\end{array}\right]\\ &\leq \varepsilon\left[\frac{k_1}{\Gamma(\alpha_1+1)}+\frac{|\theta|k_2}{\Gamma(\alpha_1+p+1)\Gamma(q+1)}\right]\left\{e^{-d}\Delta+\left(e^{-d}r_0+\overline{g}\right)(\Delta+1)\right\}.\end{aligned} \tag{24}$$

Consequently, the continuity property of T is obtained on the ball D_{r_0}.

To finish the proof, condition (8) of Theorem 1 is proved. Consider X as a nonempty subset of the ball D_{r_0} and assume that $u\in X$, $\varepsilon>0$ be arbitrarily constant. Choose $l_1,l_2\in[0,1]$ such that $l_1<l_2$ and $|l_2-l_1|<\varepsilon$. Taking into account our assumptions, we get

$$\begin{aligned}\left(\left|(G_1u)(l_1)-(G_1u)(l_2)\right|\right) &= \left|g(l_1,u(l_1))-g(l_2),u(l_2)\right|\\ &\leq \left|g(l_1,u(l_1))-g(l_1),u(l_2)\right|\\ &\quad+\left|g(l_1,u(l_2))-g(l_2),u(l_2)\right|\\ &\leq e^{-d}\varphi(X,\varepsilon)+\varphi(g,\varepsilon),\end{aligned} \tag{25}$$

where

$$\varphi(g,\varepsilon)=\sup\left\{\left|g(l_1,u)-g(l_2,u)\right|;l_1,l_2\in I,\left|l_1-l_2\right|<\varepsilon,u\in[-r_0,r_0]\right\}. \tag{26}$$

Consequently,

$$\varphi(G_1X,\varepsilon)\leq e^{-d}\varphi(X,\varepsilon)+\varphi(g,\varepsilon). \tag{27}$$

As g is uniformly continuous on $I\times[-r_0,r_0]$, we have $\varphi(g,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Thus, from (27), we conclude that

$$\varphi_0(G_1X)\leq e^{-d}\varphi_0(X). \tag{28}$$

Next, we estimate $\varphi_0(F_1X)$ and $\varphi_0(\overline{F_1}(X))$. In view of (21), since F_1x is uniformly continuous on $[0,1]$, then for fixed $\varepsilon>0$, there exists $\delta>0$ such that, for $l_1,l_2\in I$ with $|l_2-1_1|<\delta\leq\varepsilon$, we have

$$\varphi_0(F_1X)\leq\varepsilon. \tag{29}$$

Besides, since a_1 and a_2 are uniformly continuous on $[0,1]$, for $l_1,l_2\in[0,1]$ with $|l_2-1_1|<\delta\leq\varepsilon$, we have $|a_2(l_2)-a_2(l_1)|<\varepsilon$ and also $|a_1(l_2)-a_1(l_1)|<\varepsilon$. Consequently, we conclude that $\varphi_0(\overline{F_1}(X))=0$. Now, we estimate $\varphi_0(T_1X)$ for $X\subseteq D_{r_0}$. By applying (28) and (29) and Lemma 2 and using the fact that $\varphi_0(\overline{F_1}(X))=0$, we get

$$\begin{aligned}\varphi_0(T_1X) &= \varphi_0(G_1)X(F_1)(X) + \overline{F_1}((X))\\ &\le \left(\|F_1X\| + \|\overline{F_1}(X)\|\right)\varphi_0(G_1X)\\ &\quad + \|G_1X\|(\varphi_0)(F_1X) + \varphi_0(\overline{F_1})((X))\\ &\le e^{-d}\varphi_0(X)\left\{\frac{M_1}{\Gamma(\alpha_1+1)} + \frac{|\theta|M_2}{\Gamma(\alpha_1+p+1)\Gamma(q+1)}\right\}\Delta.\end{aligned} \tag{30}$$

Consequently, we derive that

$$d + \ln(\varphi_0(T_1X)) \le \ln(\varphi_0(X)). \tag{31}$$

Thus, we conclude the contractive condition in Theorem 1 with $\theta_1(t) = d$ and $\theta_2(t) = \ln(t)$. Thus, by Theorem 1, at least one solution is obtained for the operator T in D_{r_0} which is a solution of problem (4) and the proof is completed. □

Now, the following example is investigated to show the applicability of the obtained result.

Example 1. Consider the following hybrid boundary value problem:

$$\begin{aligned}&D_{1^-}^{(3/2)}D_{0^+}^{(1/2)}\frac{u(t)}{\left(e^{-d}/1+t+|u(t)|\right)} + 2I_{1^-}^{4/3}I_{0^+}^{5/4}\frac{e^{-t}}{100}\cos u(t) = \frac{e^{-t}}{100}\sin u(t),\\ &\frac{u(0)}{\left(e^{-d}/1+|u(0)|\right)} = \frac{u(2/3)}{\left(e^{-d}/t+1+|u(2/3)|\right)} = 0, \quad \frac{u(1)}{\left(e^{-d}/2+|u(1)|\right)} = \frac{1}{2}\frac{u(3/4)}{\left(e^{-d}/1+(3/4)+|u(3/4)|\right)} = 0.\end{aligned} \tag{32}$$

By putting

$$\begin{aligned}g(t,u(t)) &= \frac{e^{-d}}{1+t+|u(t)|},\\ f_1(t,u(t)) &= \frac{e^{-t}}{100}\cos u(t),\\ f_2(t,u(t)) &= \frac{e^{-t}}{100}\sin u(t),\end{aligned} \tag{33}$$

in problem (4), we conclude the above hybrid boundary value problem as a special case of problem (4). Now, the conditions of Theorem 2 are checked. For all $l \in [0,1]$ and $u_1, u_2 \in \mathbb{R}$, we have

$$\left|g(l,u_1) - g(l,u_2)\right| \le e^{-d}\left|u_1 - u_2\right|. \tag{34}$$

Moreover, we have $\overline{g} = \sup\{|g(l,0); l \in [0,1]\} = e^{-d}$. Besides, the functions f_1 and f_2 are continuous, and for all $l \in [0,1]$ and $u, v \in \mathbb{R}$, we have

$$\begin{aligned}\left|f_2(t,u)\right| &\le \frac{1}{100},\\ \left|f_2(t,u) - f_2(t,v)\right| &\le \frac{1}{100}|u-v|,\\ \left|f_1(t,u)\right| &\le \frac{1}{100},\\ \left|f_1(t,u) - f_1(t,v)\right| &\le \frac{1}{100}|u-v|.\end{aligned} \tag{35}$$

In this example $\alpha_1 = 3/2$, $\alpha_2 = 1/2$, $\theta = 2$, $p = 4/3$, $q = 5/4$, $\mu = 3/43/4$, $\delta = 1/2$, and $\xi = 2/3$. Hence, $|\Lambda| \approx 0.242$, $\overline{a_1} = 1.121$ and $\overline{a_2} = 1.168$. Consequently, the existent inequality in condition (H_3) has the form:

$$\left[e^{-d}r_0 + \frac{1}{100}\right]\left\{\frac{(1/100)}{\Gamma(5/2)} + \frac{(2/100)}{\Gamma(23/6)\Gamma(9/4)}\right\}\Delta \le r_0. \tag{36}$$

Obviously, the above inequality has the positive solution r_0, for example $r_0 = 1$. Moreover, according to the obtained values we have

$$\left\{\frac{M_1}{\Gamma(\alpha_1+1)} + \frac{|\theta|M_2}{\Gamma(\alpha_1+p+1)\Gamma(q+1)}\right\}\Delta < 1. \tag{37}$$

Thus, we conclude all conditions of Theorem 2, and hence at least one solution of the mapping T_1 is obtained on $[0,1]$ which is a solution of problem (32).

4. Conclusion

We have studied a nonlocal hybrid boundary value problem which contains both left Caputo and right Riemann–Liouville fractional derivatives and integrals and nonlocal hybrid conditions. An existence result is proved by applying a new generalization of Darbo's fixed point theorem associated with measures of noncompactness. The result obtained in this paper is new and significantly contributes to the existing literature on the topic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, USA, 1999.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies*, Vol. 204, Elsevier, Amesterdam, The Netherlands, 2006.

[3] B. Ahmad, A. Alsaedi, S. K. Ntouyas, and J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, Switzerland, 2017.

[4] A. Ergun, "Integral representation for solution of discontinuous diffusion operator with jump conditions," *Cumhuriyet Science Journal*, vol. 39, no. 1, pp. 842–863, 2018.

[5] A. Ergun and R. Amirov, "Dırect and inverse problems for diffusion operator with discontinuity points," *TWMS Journal of Pure and Applied Mathematics*, vol. 9, no. 1, pp. 9–21, 2019.

[6] R. Amirov, A. Ergun, and S. Durak, "Half-inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse," *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 915–924, 2021.

[7] B. C. Dhage and V. Lakshmikantham, "Basic results on hybrid differential equations," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 3, pp. 414–424, 2010.

[8] K. Hilal and A. Kajouni, "Boundary value problems for hybrid differential equations with fractional order," *Advances in Differential Equations*, vol. 2015, p. 183, 2015.

[9] E. T. Karimov, B. López, and K. Sadarangani, "About the existence of solutions for a hybrid nonlinear generalized fractional pantograph equation," *Fractional Differential Calculus*, vol. 6, no. 1, pp. 95–110, 2016.

[10] N. Mahmudov and M. Matar, "Existence of mild solutions for hybrid differential equations with arbitrary fractional order," *TWMS Journal of Pure and Applied Mathematics*, vol. 8, pp. 160–169, 2017.

[11] S. Sitho, S. K. Ntouyas, and J. Tariboon, "Existence results for hybrid fractional integro-differential equations, Bound," *Value Problems*, vol. 2015, p. 113, 2015.

[12] S. Sun, Y. Zhao, Z. Han, and Y. Li, "The existence of solutions for boundary value problem of fractional hybrid differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 12, pp. 4961–4967, 2012.

[13] B. Ahmad, A. Broom, A. Alsaedi, and S. K. Ntouyas, "Nonlinear integro-differential equations involving mixed right and left fractional derivatives and integrals with nonlocal boundary data," *Mathematics*, vol. 8, no. 3, pp. 1–13, 2020.

[14] O. P. Agrawal, "Formulation of Euler-Lagrange equations for fractional variational problems," *Journal of Mathematical Analysis and Applications*, vol. 272, no. 1, pp. 368–379, 2002.

[15] L. Zhang, B. Ahmad, and G. Wang, "The existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative," *Applied Mathematics Letters*, vol. 31, pp. 1–6, 2014.

[16] R. Khaldi and A. Guezane-Lakoud, "Higher order fractional boundary value problems for mixed type derivatives," *Journal of Nonlinear Functions and Analysis*, vol. 30, p. 9, 2017.

[17] A. G. Lakoud, R. Khaldi, and A. Kilicman, "Existence of solutions for a mixed fractional boundary value problem," *Advances in Difference Equations*, vol. 2017, p. 164, 2017.

[18] A. G. Lakoud, R. Khaldi, and D. F. M. Torres, "On a fractional oscillator equation with natural boundary conditions," *Progress in Fractional Differentiation and Applications*, vol. 3, no. 3, pp. 191–197, 2017.

[19] S. K. Ntouyas and H. H. Al-Sulami, "A study of coupled systems of mixed order fractional differential equations and inclusions with coupled integral fractional boundary conditions," *Advances in Difference Equations*, vol. 2020, p. 73, 2020.

[20] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, "Existence theory for nonlocal boundary value problems involving mixed fractional derivatives," *Nonlinear Analysis: Modelling and Control*, vol. 24, pp. 937–957, 2019.

[21] M. Bouaouid, M. Hannabou, and K. Hilal, "Nonlocal conformable-fractional differential equations with a measure of noncompactness in Banach spaces," *Journal of Mathematics*, vol. 2020, Article ID 5615080, , 2020.

[22] M. A. Ragusa and A. Scapellato, "Mixed Morrey spaces and their applications to partial differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 151, pp. 51–65, 2017.

[23] M. A. Ragusa and A. Tachikawa, "Regularity for minimizers for functionals of double phase with variable exponents," *Advances in Nonlinear Analysis*, vol. 9, pp. 710–728, 2020.

[24] E. Set, A. O. Akdemir, and I. Mumcu, "Chebyshev type inequalities for conformable fractional integrals," *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 1227–1236, 2019.

[25] M. A. Dokuyucu, E. Celik, H. Bulut, and H. M. Baskonus, "Cancer treatment model with the Caputo-Fabrizio fractional derivative," *European Physical Journal-Plus*, vol. 133, p. 92, 2018.

[26] I. Koca, E. Akcetin, and P. Yaprakdal, "Numerical approximation for the spread of SIQR model with Caputo fractional order derivative," *Turkish Journal of Science*, vol. 5, no. 2, pp. 124–139, 2020.

[27] M. A. Dokuyucu and H. Dutta, "A fractional order model for Ebola Virus with the new Caputo fractional derivative without singular kernel," *Chaos, Solitons & Fractals*, vol. 134, Article ID 109717, 2020.

[28] M. A. Dokuyucu, "Caputo and Atangana-Baleanu-Caputo fractional derivative applied to Garden equation," *Turkish Journal of Science*, vol. 5, no. 1, pp. 1–7, 2020.

[29] P. Agarwal and A. A. El-Sayed, "Vieta–Lucas polynomials for solving a fractional-order mathematical physics model," *Advances in Differential Equations*, vol. 2020, p. 626, 2020.

[30] N. A. Shah, P. Agarwal, J. D. Chung, E. R. El-Zahar, and Y. S. Hamed, "Analysis of optical solitons for nonlinear Schrödinger equation with detuning term by iterative transform method," *Symmetry*, vol. 12, no. 11, p. 1850, 2020.

[31] P. Agarwal, M. Akbar, R. Nawaz, and M. Jleli, "Solutions of system of Volterra intefro-differential equations using optimal homotopy asymptotic method," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 3, pp. 1–11, 2020.

[32] J. Banas and K. Goebel, "Measure of noncompactness in Banach spaces," in *Lecture Notes in Pure and Applied Mathematics*Vol. 60, Dekker, New York, NY, USA, 1980.

[33] A. Samadi and M. B. Ghaemi, "An extension of Darbo fixed point theorem and its applications to coupled fixed point and integral equations," *Filomat*, vol. 28, no. 4, pp. 879–886, 2014.

[34] A. Samadi, "Applications of measure of noncompactness to coupled fixed points and systems of integral equations," *Miskolc Mathematical Notes*, vol. 19, no. 1, pp. 537–553, 2018.