# Mathematical Problems in Engineering 

# Theory, Methods, and Applications 

Special Issue<br>Stochastic Systems: Modeling, Analysis, Synthesis, Control, and Their Applications to Engineering

Guest Editors: Weihai Zhang, M. D. S. Aliyu, Yun-Gang Liu, and Xue-Jun Xie

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## Editorial

# Stochastic Systems: Modeling, Analysis, Synthesis, Control, and Their Applications to Engineering 

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Stochastic optimal control and filtering theory have been at the forefront of modern control theory and communication engineering. Filtering theory has played a significant role in space explorations, navigation, aerospace, radar, satellite and meteorological applications.

In recent years, stochastic control theory has been playing an important role in the study of biomathematics and mathematical finance, as well as adaptive and network controlled systems, which is a vital research field in modern control theory. This is primarily because of the fact that deterministic systems are to a large extent an abstraction, and all practical systems do have a certain degree of random and uncertain behavior in the form of noise, disturbance, and random failures. Moreover, due to this element of randomness, stochastic systems are more complicated than deterministic systems.

Therefore, a satisfactory resolution of many new, challenging, and complicated problems arising in the above areas, other engineering fields and scientific phenomenon, and involving modeling, identification, estimation, analysis and synthesis, require more advanced tools and rigorous investigations than hitherto available.

This special issue aims to introduce new developments in the theory of stochastic control systems with applications to engineering fields such as communication, networked control, system reliability, and mathematical finance. However, the main focus of the special issue is on stochastic modeling, analysis, and control, with particular emphasis on stability and stabilization, adaptive control, robust optimal control and filtering. Close to 50 papers were received, but only 30 could be accepted after a rigorous peer review to guarantee
the highest quality of the special issue. A quick summary of the final accepted papers and therefore the contents of the special issue is as follows.

There are four papers concerning nonlinear stochastic adaptive control listed as follows: "Nonsmooth adaptive control design for a large class of uncertain high-order stochastic nonlinear systems," by J. Zhang and Y. Liu; "Adaptive state-feedback stabilization for high-order stochastic nonlinear systems driven by noise of unknown covariance," by C.-R. Zhao et al.; "Adaptive output feedback control for a class of stochastic nonlinear systems with SiISS inverse dynamics," by N. Duan and H.-K. Liu; "High-order stochastic adaptive controller design with application to mechanical systems," by J. Tian et al.

The second set of three papers deal with the application of stochastic control theory to mathematical finance and are listed as follows: "Arbitrage-free conditions and hedging strategies for markets with penalty costs on short positions," by O. L. V. Costa and E. V. Queiroz Filho; "Multi-period mean-variance portfolio selection with uncertain time horizon when returns are serially correlated," by L. Zhang and Z. Li; "A fast Fourier transform technique for pricing European options with stochastic volatility and jump risk," by S.-m. Zhang and L.-h. Wang.

The third set of papers are concerned with stochastic robust optimal control and filtering, and there are five papers in this category listed as follows: "Robust $H_{\infty}$ filtering for general nonlinear stochastic state-delayed systems," by W. Zhang et al.; "Indefinite LQ control for discrete-time stochastic systems via semidefinite programming," by S. Zhou and W. Zhang; "Robust reliable $H_{\infty}$ control for nonlinear stochastic Markovian jump systems," by G. Chen and Y. Shen; "Weighted measurement fusion white noise deconvolution filter with correlated noise for multisensor stochastic systems," by X. Wang et al.; "Least-mean-square receding horizon estimation," by B. Kwon and S. Han.

The fourth category of three papers are devoted to stochastic stability and stabilization and are entitled "Robust stabilization for stochastic systems with time-delay and nonlinear uncertainties," by Z. Yan et al.; "New results on stability and stabilization of Markovian jump systems with partly known transition probabilities," by Y. Guo and F. Zhu; "Stochastic stability of damped Mathieu oscillator parametrically excited by a Gaussian noise," by C. Floris.

Then come the fifth set of four papers dealing entirely with networked control and communication. These are listed as follows: "Stability and stabilization of networked control systems with forward and backward random time delays," by Y.-G. Sun and Q.-Z. Gao; "Iterative learning control for remote control systems with communication delay and data dropout," by C. Liu et al; "Evaluation of network reliability for computer networks with multiple sources," by Y.-K. Lin and L. C.-L. Yeng; "Robust distributed Kalman filter for wireless sensor networks with uncertain communication channels," by Du. Y. Kim and M. Jeon.

In addition, there are two papers that are devoted to queuing systems and one paper to systems reliability. The first one is entitled " $A$ tandem $B M A P / G / 1 \rightarrow \bullet / M / N / 0$ queиe with group occupation of servers at the second station," by C. Kim et al., while the second one is entitled "Stochastic approximations and monotonicity of a single server feedback retrial queue," by M. Boualem et al. While the paper dealing with system reliability is entitled "Probabilistic approach to system reliability of mechanism with correlated failure models," by X. Huang and Y. Zhang.

The remaining set of papers deal with diverse topics and subjects, ranging from mobile robots to neural networks and from traffic control to navigation. We quickly recall their various titles here. One paper is concerned with the stabilization of nonholonomic mobile robot and is entitled "Stochastic stabilization of nonholonomic mobile robot with heading-angle-dependent disturbance," by Z. J. Wu and Y. H. Liu. The next paper deals with the synchronization of stochastic neural networks and is entitled "Master-slave synchronization
of stochastic neural networks with mixed time-varying delays," by Y. Ge et al. There is also one paper about BPP (Binomial-Poisson-Pascal) traffic entitled "Properties of recurrent equations for the full-availability group with BPP traffic," by M. Gła̧bowski et al.

In addition, one paper studies population dynamics and is entitled "Existence and uniqueness for stochastic age-dependent population with fractional Brownian motion," by Z. Qimin and L. Xining. Then another paper analyzes the relationship between extreme climate indices in China and is entitled "The use of geographically weighted regression for the relationship among extreme climate indices in China," by C. Wang et al. Further, there is one paper concerning INS/WSN-integrated navigation entitled "INS/WSN-integrated navigation utilizing LS-SVM and $H_{\infty}$ filtering," by Y. Xu et al., and then a paper dealing with wind-induced vibration control entitled "Wind-induced vibration control of Dalian International Trade Mansion by tuned liquid dampers," by H.-N. Li et al. The last paper discusses multiresolution analysis for random parameter fields and is entitled "Multiresolution analysis for stochastic finite element problems with wavelet-based Karhunen-Loève expansion," by C. Proppe.

## Acknowledgments

As the Lead Guest Editor of this special issue, I wish to express my profound gratitude to my three coeditors for accepting to undertake this project with me and the wonderful accomplishment that we have been able to achieve. I hope that the excellent work that has been assembled in this special issue will go a long way in stimulating further research in this important and active research area, as well as answering a lot of questions that were hitherto unanswered. We are also deeply appreciative of the efforts of the authors in submitting to the special issue regardless of the outcome of the review process. Finally, we also want to thank all the referees who have helped us in ensuring the highest quality of papers to be included in the special issue, and without whose help nothing would have been accomplished. The cooperation of the Editor-in-Chief of Mathematical Problems in Engineering and the staff of Hindawi Publishing Corporation is hereby also graciously acknowledged.

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Research Article

# Evaluation of Network Reliability for Computer Networks with Multiple Sources 

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#### Abstract

Evaluating the reliability of a network with multiple sources to multiple sinks is a critical issue from the perspective of quality management. Due to the unrealistic definition of paths of network models in previous literature, existing models are not appropriate for real-world computer networks such as the Taiwan Advanced Research and Education Network (TWAREN). This paper proposes a modified stochastic-flow network model to evaluate the network reliability of a practical computer network with multiple sources where data is transmitted through several light paths (LPs). Network reliability is defined as being the probability of delivering a specified amount of data from the sources to the sink. It is taken as a performance index to measure the service level of TWAREN. This paper studies the network reliability of the international portion of TWAREN from two sources (Taipei and Hsinchu) to one sink (New York) that goes through a submarine and land surface cable between Taiwan and the United States.


## 1. Introduction

The issue of the QoS [1] of networks has been studied in the past decades. QoS is an important element of understanding the efficiency of real-world computer networks. It refers to the ability to provide a predictable, consistent data transfer service and the ability to satisfy customers' application needs while maximizing the use of network resources, especially a network reliability analysis. One of the traditional issues in this area of network reliability research is known as the source-sink ( $s-t$ ) network reliability problem [2-16], which some articles refer to as two-terminal network reliability (TTNR) [14, 15]. In TTNR analyses, it is interesting to compute the network reliability in relation to the connecting paths between two specific network nodes, usually the source-sink ( $s-t$ ). Generally speaking, people are interested in obtaining the probability that the source connects the sink. Some researchers extend the study of TTNR to the $k$-terminal network reliability (KTNR) problem [17, 18], which contains at least one path from the source node to other $k$ nodes. Besides TTNR and

KTNR, there is an all-terminal network reliability (ATNR) (also called overall or uniform network reliability), which is calculated from the probability that each and every node in the network is connected to each other [19, 20]. In a binary-state flow network, the capacity of each arc has two levels 0 and a positive integer. System with various states is called a stochastic model [21-23]. For a more realistic system, the arc should have several possible states/capacities, and such a network is named a stochastic-flow network (or multistate network). The previous problems, TTNR, KTNR, and ATNR, are discussed for binary-state flow networks. However, this paper addresses the evaluation of the network reliability of a stochastic-flow network with multiple sources.

The Taiwan Advanced Research and Education Network (TWAREN) [24] is Taiwan's academic research network that mainly provides network communication services for Taiwan's research and academic society. It also offers a tunnel between Taiwan and the United States to connect the global research network through a land surface line and the Asia Pacific region's submarine cable. Since TWAREN's resources (i.e., bandwidth) are limited, it is a critical issue to find a technique to optimize its utility. Using efficient evaluation tools to understand TWAREN's performance to improve its infrastructure is one of the major tasks of Taiwan's National High Performance Computing Center (NCHC). To measure TWAREN's capability, network analysis is a useful tool. For a practical computer network, the transmission media (physical lines such as fiber optics or coaxial cables) may be modeled as arcs, while transmission facilities (switches or routers) may be modeled as nodes. In particular, the capacity of each arc should be stochastic due to the possibility of failure, partial failure, or maintenance. Thus, the computer network characterized by such arcs also has stochastic capacities and it is a typical stochastic-flow network [2-13, 25, 26]. Network reliability evaluation of a stochastic-flow network has been studied as a performance index for decades [2-13, 25, 26]. Most of these studies examined the network reliability from source node $s$ to sink node $t$ in terms of minimal paths (MPs), in which an MP is a path with proper subsets which are no longer paths $[2-4,6-9]$. This implies that an MP is a set that connects an $(s, t)$ pair, here not limited to one $(s, t)$ pair, without any surplus arcs from the perspective of the network topology.

Those previous studies assume that data can be sent through all possible MP from $s$ to $t$ according to the network topology, where each MP is composed of some physical lines (arcs). However, in a real computer system, data can only be sent through some unique light paths (LPs) between specific node pairs, where an LP is a virtual tunnel between two end-to-end nodes which combined by some segments (i.e., arcs or lines) and nodes; however, an MP is a path that connects a specific source and a specific sink, while an LP can be a link between any two nodes (not limited to source and sink pair). That is, data may be transmitted from source node $s$ to sink node $t$ via more than one LP. In particular, any segments that LP goes through cannot be divided during transmission. Therefore, the previous studies [2-4,69] based on MP to transmit data are not appropriate for TWAREN. In TWAREN, each LP is composed of a set of light path segments (LPSs) linking two nodes. In particular, each physical line can be divided into several LPS, and each LPS belongs to only one LP. Since TWAREN involves the light path, which cannot be divided through any part of its nodes or arcs during transmission, this kind of network model is different than the MP concept described in [2-4, 6-9]. Therefore, we implement a minimal light path (MLP) concept to find all LPs to evaluate TWAREN's network reliability. In this paper, the MLP is defined as a series of nodes and LPSs, from source node to sink node, which contains no cycle.

A revised stochastic-flow network with multiple sources is constructed to model the TWAREN in terms of LP. The difference of single source and multiple sources is that previous
one only dedicates on the network reliability between one source and one sink. But in real world, system may transfer the real time data to the sink that exceed the total capacity of LPS which are beside the single source node. That is, we need to transfer data from at least two source nodes, where the data from different source might influence each other, the theory that developed in traditional one source and one sink not applicable here. In generally, we have to transfer the real-time data from multiple sources to one sink to handle the practical worlds' data transmission. Therefore we consider multiple sources and implement the new technique in this paper to realize the operation of real system instead of single source. Besides, we need to deal with the assignment of multiple sources and the flow conflict on the intersectional arcs. The two-source case is firstly addressed for convenience. A general case with multiple sources can be extended by the proposed algorithm. Then we can evaluate the network reliability for the international part of TWAREN whose tunnel mainly connects to the global academic research network, especially the Internet2 Network [27]. Taiwan's largest network service provider (NSP), Chunghwa Telecom (CHT) [28], integrates those NSPs that the lines pass through to organize the whole portion of TWAREN's international infrastructure in two areas: on the land surface of both Taiwan and the United States and in the under-sea areas of the Asia Pacific, including the Japan-US submarine cable that disconnected when it was hit by the earthquake and tsunami in Japan on March 11th 2011. Nakagawa [29] has mentioned the influence of that earthquake regarding reliability, so we study this disaster's effect as well. In fact, when a line breaks, the NSP of these pass-through lines will offer serviceable lines as backups; therefore they offer some degree of the network reliability. However, in this study, we only concentrate on the portion that includes the regular lines to determine the factors that affect TWAREN's network reliability, as the NCHC's prime task, aside from improving TWAREN's overall performance, is to anticipate major factors which could fail the regular lines. The issue of the network reliability of the backup line [30-34] has not been considered yet.

This paper mainly emphasizes the network reliability that the network can send specified units of data from two source nodes (Taipei city and Hsinchu city) to a single sink node (New York) through TWAREN's light path. The remainder of this paper is organized as follows. The TWAREN network is introduced in Section 2. The research scope, problem formulation, the concept of the minimal light path and the evaluation technique, recursive sum of disjoint of products, (RSDP [9]) are all described in Section 3. Network reliability of TWAREN is evaluated in Section 4. A summary and conclusion are presented in Section 5.

## 2. TWAREN Network

### 2.1. Introduction to TWAREN

TWAREN has been funded by the National Science Council of Taiwan since 1998 and was built by the NCHC. Construction was completed at the end of 2003, and service and operation started in 2004. Today, more than 100 academic and research institutions connect with TWAREN in Taiwan and this number is increasing continuously. As well, since 2005, over 1,000 elementary schools and junior and senior high schools have been using TWAREN's internal backbone. TWAREN provides network infrastructure for general use but is also an integrated platform for network research. For instance, TWAREN was instrumental in developing applications and network technology such as IPv6, MPLS, VoIP, e-learning, multicast, multimedia, and performance measurement and has supported GRID computing
applications such as e-Learning Grid, Medical Grid, and EcoGrid. As promoting Taiwan as an international R\&D center is one of NCHC's objectives, a stable and reliable TWAREN is the foundation to achieve this goal.

Many countries fund national research and education network (NREN) infrastructure. TWAREN, Taiwan's NREN, connects to the international research community through global advanced networks, specifically the Internet2 Network [27] of the United States, the major NREN in the world. Therefore, network reliability analyses of TWAREN will help to continuously improve its infrastructure so it can continue to cooperate and connect globally.

### 2.2. TWAREN's Light Path

TWAREN is network that connects to the world-wide research network through light path international tunnel. TWAREN's physical topology is an optical infrastructure and its virtual topology is constructed by connecting light paths and routers. A light path is a tunnel between two sites connected by various cables and is an end-to-end, preallocated optical network resource, according to users' needs. It allows signals to be delivered sequentially without jitters and congestion. Each light path is generally a $155 \mathrm{Mbps} \sim 10 \mathrm{Gbps}$ dedicated channel that transports various applications.

Figure 1 is the light path international infrastructure that TWAREN leases from CHT, including major sites located at Taipei and Hsinchu in Taiwan, and Los Angeles, Chicago, and New York in the United States. This infrastructure contains the land surface and submarine cable between these cities. Each light path is denoted by $\mathrm{LP}_{i}$ where $i$ is the light path number, $i=1,2, \ldots, l$ with $l$ being the number of light path.

Most of these city sites connect to each other with 2.5 Gbps physical line connections, divided into four light path channels at 622 Mb bandwidths. The research scope of this paper is to study the network reliability of the transmission from two sources (Taipei city and Hsinchu city) to the sink node (New York) by means of the light path tunnel.

## 3. Problem Description and Model Formulation

### 3.1. Problem Description

This paper describes how the probability that a specified amount of data can be sent from Taipei and Hsinchu to New York via TWAREN is measured. This is referred to as network reliability. Also, Figure 1 is transformed into Figure 2 which is constructed by the light path segments and nodes.

### 3.2. Some Definitions

As Figure 2 shows, those cities or site devices defined as nodes are denoted by $n_{k}$, where $k=1,2, \ldots, p$ with $p$ being the number of nodes. For example, Taipei City is $n_{1}$ and TP- 1 is $n_{2}$. We denote each LPSs as $l_{i, j}$ where $l_{i, j} \in \mathrm{LP}_{i}$ means the $j$ th segment in $\mathrm{LP}_{i}\left(j=1,2, \ldots, r_{i}\right.$ with $r_{i}$ being the number of LPS in $\mathrm{LP}_{i}$ ). For example, in Figure 2, $\mathrm{LP}_{1}$ is a tunnel from Taipei $\left(n_{1}\right)$ to Chicago ( $n_{8}$ ), which is combined with three LPS $l_{1,1}, l_{1,2}$, and $l_{1,3}$, and goes through two nodes $n_{2}$ (TP-1) and $n_{6}$ (San Francisco). Its connection sequence is $n_{1} \leftrightarrow l_{1,1} \leftrightarrow n_{2} \leftrightarrow l_{1,2} \leftrightarrow$


Figure 1: TWAREN light path between Taiwan and US.
$n_{6} \leftrightarrow l_{1,3} \leftrightarrow n_{8}$. The capacity of each LP is 622 Mb , and each LP is combined by four 155 Mb channels. As each channel is regarded as one unit, there are 4 units for each LPs.

The physical line (PL) is the actual optical cable where the LP is located and used for data transmission. For example, $\operatorname{LPS} l_{1,3}, l_{4,4}, l_{11,1}$ is combined in one PL from San Francisco to Chicago, as shown as PL $P_{10}$ in Figure 3. The capacity of each PL is 2.5 G and is divided into four 622 Mb LP.

The capacity state of an LPS is the same as a PL either when connected or disconnected. Each LPS has two capacity states: 0 units ( 0 G ) and 4 units ( 622 Mb with four 155 MbLP ), respectively. That is, once the PL fails, all the LPSs that are located in this PL also fail. Those LPSs located in the same PL have the same disconnection probability (or, conversely, the same connection probability). For example, LPS $l_{1,3}, l_{4,4}, l_{11,1}$ located in one PL $P_{10}$ have the same disconnection probability.

### 3.3. Model Formulation

The stochastic-flow network evaluation technology developed in [3] is a method that is not suitable to be applied to TWAREN in Figure 2. There are some differences in this problem,


Figure 2: Revised network from Taipei and Hsinchu to New York using light path segments and nodes connection.
since each $\mathrm{LP}_{i}$ is combined with LPS $l_{i, j}$, which cannot be divided through any nodes. To create an easier expression, we re-sort all LPSs as $a_{1}, a_{2}, \ldots, a_{n}$, where $n$ is the total number of LPS, instead of $l_{i, j}$. Let $G=(A, N, M)$ be a stochastic flow network where $A=\left\{a_{i} \mid 1 \leq i \leq n\right\}$ is the set of LPS, $N$ is the set of nodes, and $M=\left(M^{1}, M^{2}, \ldots, M^{n}\right)$ with $M^{i}$ (an integer) being the maximum capacity of each LPS $a_{i}$. Such a $G$ is assumed to further satisfy the following assumptions.
(1) Each node is perfectly reliable.
(2) The capacity of each LPS is stochastic with a given probability distribution according to historical data.
(3) The capacities of different LPS are statistically independent.

Let Taipei be the first source node denoted by $s_{1}$, and let Hsinchu be the second source node denoted by $s_{2}$. Then let $S=\left\{s_{1}, s_{2}\right\}$. A minimal light path (MLP) is a series of LPSs from a source node to a sink node, which contains no cycle. In particular, any segment used by $\mathrm{LP}_{i}$ cannot be divided during transmission in $\mathrm{LP}_{i}$. That is, each LPS belongs to only one LP. Suppose $m l_{1}, m l_{2}, \ldots, m l_{r}$ are all MLPs from $s_{1}$ to $t$ and $m l_{r+1}, m l_{r+2}, \ldots, m l_{q}$ are all MLPs


Figure 3: Physical line connection.
from $s_{2}$ to $t$. Then, the stochastic flow network can be described by the capacity vector $X=$ $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and the flow vector $F=\left(f_{1}, f_{2}, \ldots, f_{q}\right)$ where $x_{i}$ denotes the current capacity of $a_{i}$, and $f_{j}$ denotes the current flow on $m l_{j}$. The following constraint shows that the flow through $a_{i}$ cannot exceed the maximum capacity of $a_{i}$ :

$$
\begin{equation*}
\sum_{j=1}^{q}\left\{f_{j} \mid a_{i} \in m l_{j}\right\} \leq M^{i} \tag{3.1}
\end{equation*}
$$

Let the total demand to New York be $p$. Then demand set $D_{p}=\left\{\left(d_{1}, d_{2}\right) \mid\left(d_{1}+d_{2}\right)=p\right\}$ where $d_{1}$ and $d_{2}$ are the demand from Taipei and Hsinchu to New York, respectively. To meet the demand pair $\left(d_{1}, d_{2}\right)$, the flow vector $F=\left(f_{1}, f_{2}, \ldots, f_{q}\right)$ has to satisfy

$$
\begin{align*}
& \sum_{j=1}^{r} f_{j}=d_{1} \\
& \sum_{j=r+1}^{q} f_{j}=d_{2} \tag{3.2}
\end{align*}
$$

For convenience, let $\mathbf{F}_{\left(d_{1}, d_{2}\right)}=\{F \mid F$ satisfy constraints (3.1) and (3.2) $\}$. For each $F \in \mathbf{F}_{\left(d_{1}, d_{2}\right)}$, the corresponding capacity vector $X_{F}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is generated via

$$
\begin{equation*}
x_{i}=\sum_{j=1}^{q}\left\{f_{j} \mid a_{i} \in m l_{j}\right\}, \quad x_{i}=1,2, \ldots, n \tag{3.3}
\end{equation*}
$$

Let $\Omega_{\left(d_{1}, d_{2}\right)}=\left\{X_{F} \mid F \in \mathbf{F}_{\left(d_{1}, d_{2}\right)}\right\}$ be such capacity vectors, and let $\Omega_{\left(d_{1}, d_{2}\right), \min }=\{X \mid X$ be $\leq$ with respect to in $\left.\Omega_{\left(d_{1}, d_{2}\right)}\right\}$ (where $Y \leq X$ if and only if $y_{i} \leq x_{i}$ for each $i=1,2, \ldots, n$ and $Y<X$, if and only if $Y \leq X$ and $y_{i}<x_{i}$ for at least one $\left.i\right)$. For convenience, each $X \in \Omega_{\left(d_{1}, d_{2}\right), \text { min }}$ is named a $\left(d_{1}, d_{2}\right)$-MLP in this paper. Suppose all MLPs have been precomputed. All $\left(d_{1}, d_{2}\right)$-MLP can be derived by the following steps.

Step 1. Do the following steps for each $\left(d_{1}, d_{2}\right) \in D_{p}$.
Step 2. Find all feasible solutions $F=\left(f_{1}, f_{2}, \ldots, f_{q}\right)$ of the constraints (3.1) and (3.2).
Step 3. Transform each $F$ into $X_{F}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ via (3.3) to get $\Omega_{\left(d_{1}, d_{2}\right)}$.
Step 4. Remove the nonminimal ones in $\Omega_{\left(d_{1}, d_{2}\right)}$ to obtain $\Omega_{\left(d_{1}, d_{2}\right), \text { min }}$, that is, $\left(d_{1}, d_{2}\right)$ MLP.
Step 5. Next $\left(d_{1}, d_{2}\right)$.
Step 6. End.

### 3.4. Network Reliability Evaluation

Network reliability $R_{D_{p}}$ is the probability that the system can transmit $p$ units of data to the sink, that is, $R_{D_{p}}=\sum_{\left(d_{1}, d_{2}\right) \in D_{p}} \operatorname{Pr}\left\{Y \in \Omega_{\left(d_{1}, d_{2}\right)} \mid Y \geq X\right.$ for a $\left(d_{1}, d_{2}\right)$-MLP X $\}$. If $\left\{X_{1}, X_{2}, \ldots, X_{h}\right\}$ is the set of minimal capacity vectors capable of satisfying any $\left(d_{1}, d_{2}\right) \in D_{p}$, then network reliability $R_{D_{p}}$ is

$$
\begin{equation*}
R_{D_{p}}=\operatorname{Pr}\left\{\bigcup_{v=1}^{h} Q_{v}\right\}, \tag{3.4}
\end{equation*}
$$

where $Q_{v}=\left\{X \mid X \geq X_{v}\right\}, v=1,2, \ldots, h$. Several methods such as the RSDP algorithm (Algorithm 1) [9], the inclusion-exclusion method (IEM) [10, 25], the disjoint-event method (DEM) [35], and state-space decomposition (SSD) [11, 12] may be applied to compute $R_{D_{p}}$. The IEM $[10,25]$ principle is a simple way to calculate network reliability, which basically is similar to the theorem in traditional probability theory that is recursively plus (inclusion) and minus (exclusion) the intersection portion, but easily results in memory overload as there are lots of input data. SSDs [12] are based upon the decomposition method, in which the state space is decomposed into three sets of states: acceptable $(A)$ sets, nonacceptable $(N)$ sets, and unspecified $(U)$ sets, which recursively decompose the $U$ sets into smaller $A, N$, and $U$ sets to get the whole system reliability in terms of the summation of the reliability of all A sets. Aven [12] proved that somehow SSD has much better performance than IEM [10, 25]. Zuo et al. [9] implemented a new technique RSDP; it calculates one record's reliability first and then continuously and, respectively, handles another single record that is minus the intersection portion with previous records that those reliability already been calculated, which quite different than the IEM that recursively plus and minus the intersection portions for all records. It has been proved by Zuo et al. [9] that RSDP has better efficiency than SSD [12] and easier than IEM [10, 25]. Therefore, recently most network reliability evaluation articles apply the RSDP to assess the related issue. It calculates the probability of a union

```
//Calculate the network reliability \(R_{D_{p}}\) for all \(\Omega_{\left(d_{1}, d_{2}\right), \text { min }}\)
function \(R_{D_{p}}=\operatorname{RSDP}\left(X_{1}, X_{2}, \ldots, X_{h}\right)\)
\(/ /\) Input \(h\) vectors ( \(X_{1}, X_{2}, \ldots, X_{h}\) ) and connection probability of each LPS
    for \(i=1: h\)
        if \(i==1\)
        \(R_{D_{p}}=\operatorname{Pr}\left(X \geq X_{i}\right) ;\)
    else
        Temp_R_1 \(=\operatorname{Pr}\left(X \geq X_{i}\right)\);
        If \(i==2\)
            Temp_R_2 \(=\operatorname{Pr}\left(X \geq \max \left(X_{1}, X_{i}\right)\right) ; / / \max \left(X_{1}, X_{i}\right)=\left(X_{1} \oplus X_{i}\right)\)
        else
            for \(j=1: i-1\)
                \(X_{j}=\max \left(X_{j}, X_{i}\right) ; / / \max \left(X_{j}, X_{i}\right)=\left(X_{j} \oplus X_{i}\right)\)
            end
            Temp_R_2 \(=\operatorname{RSDP}\left(X_{1}, X_{2}, \ldots, X_{i-1}\right)\);
        end
    end
    \(R_{D_{p}}=R_{D_{p}}+(\) Temp_R_1) - (Temp_R_2);
```

Algorithm 1: RSDP algorithm.
with $r$ vectors in terms of the probabilities unions with $(r-1)$ vectors or less by using a special maximum operator [9] " $\oplus$ ", which is defined as

$$
\begin{equation*}
X_{1,2}=X_{1} \oplus X_{2} \equiv\left(\max \left(x_{1 i}, x_{2 i}\right)\right), \quad \text { for } i=1,2, \ldots, n \tag{3.5}
\end{equation*}
$$

For example, if $X_{1}=(2,2,1,1,0)$ and $X_{2}=(3,0,1,0,1), X_{1,2}=X_{1} \oplus X_{2}=(\max (2,3), \max (2,0)$, $\max (1,1), \max (1,0), \max (0,1))=(3,2,1,1,1)$. The RSDP algorithm is presented as follows.

## 4. Case Study: TWAREN between Taiwan and the US through the Light Path

### 4.1. Level of Demand and MLP from Taipei and Hsinchu to New York

To calculate TWAREN's network reliability from Taipei and Hsinchu to New York, there must be a reasonable demand level. For each arc's capacity, each LP occupies a bandwidth 622 Mb , and each 622 Mb bandwidth has four 155 Mb channels. We regard each 155 Mb as one unit. Therefore, there are four units in each 622 Mb LP channel.

Let the total demand be $p=20$ units, that is, $5 \times 622 \mathrm{Mb}=3,110 \mathrm{Mb}$. For $\Omega_{\left(d_{1}, d_{2}\right), \mathrm{min}}$ the demand set $D_{20}=\{(20,0),(16,4),(12,8),(8,12),(4,16),(0,20)\}$, we try to evaluate $R_{D_{20}}=$ $\sum_{\left(d_{1}, d_{2}\right) \in D_{20}} \operatorname{Pr}\left\{Y \in \Omega_{\left(d_{1}, d_{2}\right)} \mid Y \geq X\right.$ for $a\left(d_{1}, d_{2}\right)$-MLP $\left.X\right\}$. In these cases, there are 10 MLPs from $n_{1}$ (Taipei) to $n_{9}$ (New York) as Table 1(a) and 10 MLPs from $n_{13}$ (Hsinchu) to $n_{9}$ (New York) as shown in Table 1(b).

Table 1: (a) All MLPs from Taipei $\left(n_{1}\right)$ to New York $\left(n_{9}\right)$. (b) All MLPs from Hsinchu ( $n_{13}$ ) to New York ( $n_{9}$ ).
(a)

| $\overline{\mathrm{MLP}}$ <br> no. | Light paths combination | Nodes \& LPS combination flow |
| :---: | :---: | :---: |
| $m l_{1}$ | $\begin{aligned} & \text { Taipei } \rightarrow \mathrm{LP}_{1} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{10} \rightarrow \text { New } \\ & \text { York } \end{aligned}$ | $\begin{aligned} & n_{1} \rightarrow l_{1,1} \rightarrow n_{2} \rightarrow l_{1,2} \rightarrow n_{6} \rightarrow l_{1,3} \rightarrow n_{8} \rightarrow \\ & l_{10,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{2}$ | $\text { Taipei } \rightarrow \mathrm{LP}_{1} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{13} \rightarrow \text { New }$ York | $\begin{aligned} & n_{1} \rightarrow l_{1,1} \rightarrow n_{2} \rightarrow l_{1,2} \rightarrow n_{6} \rightarrow l_{1,3} \rightarrow n_{8} \rightarrow \\ & l_{13,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{3}$ | Taipei $\rightarrow \mathrm{LP}_{4} \rightarrow$ Chicago $\rightarrow \mathrm{LP}_{10} \rightarrow$ New York | $\begin{aligned} & n_{1} \rightarrow l_{4,1} \rightarrow n_{2} \rightarrow l_{4,2} \rightarrow n_{5} \rightarrow l_{4,3} \rightarrow n_{6} \rightarrow \\ & l_{4,4} \rightarrow n_{8} \rightarrow l_{10,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{4}$ | $\text { Taipei } \rightarrow \mathrm{LP}_{4} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{13} \rightarrow \text { New }$ York | $\begin{aligned} & n_{1} \rightarrow l_{4,1} \rightarrow n_{2} \rightarrow l_{4,2} \rightarrow n_{5} \rightarrow l_{4,3} \rightarrow n_{6} \rightarrow \\ & l_{4,4} \rightarrow n_{8} \rightarrow l_{13,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{5}$ | Taipei $\rightarrow \mathrm{LP}_{3} \rightarrow$ New York | $\begin{aligned} & n_{1} \rightarrow l_{3,1} \rightarrow n_{2} \rightarrow l_{3,2} \rightarrow n_{3} \rightarrow l_{3,3} \rightarrow n_{7} \rightarrow \\ & l_{3,4} \rightarrow n_{9} \end{aligned}$ |
| $m l_{6}$ | $\begin{aligned} & \text { Taipei } \rightarrow \mathrm{LP}_{2} \rightarrow \text { Los } \\ & \text { Angeles } \rightarrow \mathrm{LP}_{12} \rightarrow \text { New York } \end{aligned}$ | $\begin{aligned} & n_{1} \rightarrow l_{2,1} \rightarrow n_{2} \rightarrow l_{2,2} \rightarrow n_{3} \rightarrow l_{2,3} \rightarrow n_{4} \rightarrow \\ & l_{2,4} \rightarrow n_{7} \rightarrow l_{12,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{7}$ | $\begin{aligned} & \text { Taipei } \rightarrow \mathrm{LP}_{2} \rightarrow \text { Los } \\ & \text { Angeles } \rightarrow \mathrm{LP}_{11} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{10} \rightarrow \text { New } \\ & \text { York } \end{aligned}$ | $\begin{aligned} & n_{1} \rightarrow l_{2,1} \rightarrow n_{2} \rightarrow l_{2,2} \rightarrow n_{3} \rightarrow l_{2,3} \rightarrow n_{4} \rightarrow \\ & l_{2,4} \rightarrow n_{7} \rightarrow l_{11,2} \rightarrow n_{6} \rightarrow l_{11,1} \rightarrow n_{8} \rightarrow \\ & l_{10,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{8}$ | Taipei $\rightarrow \mathrm{LP}_{2} \rightarrow$ Los <br> Angeles $\rightarrow \mathrm{LP}_{11} \rightarrow$ Chicago $\rightarrow \mathrm{LP}_{13} \rightarrow$ New York | $\begin{aligned} & n_{1} \rightarrow l_{2,1} \rightarrow n_{2} \rightarrow l_{2,2} \rightarrow n_{3} \rightarrow l_{2,3} \rightarrow n_{4} \rightarrow \\ & l_{2,4} \rightarrow n_{7} \rightarrow l_{11,2} \rightarrow n_{6} \rightarrow l_{11,1} \rightarrow n_{8} \rightarrow \\ & l_{13,1} \rightarrow n_{9} \end{aligned}$ |
| $\mathrm{ml}_{9}$ | Taipei $\rightarrow \mathrm{LP}_{1} \rightarrow$ Chicago $\rightarrow \mathrm{LP}_{11} \rightarrow$ Los Angeles $\rightarrow \mathrm{LP1}_{2} \rightarrow$ New York | $\begin{aligned} & n_{1} \rightarrow l_{1,1} \rightarrow n_{2} \rightarrow l_{1,2} \rightarrow n_{6} \rightarrow l_{1,3} \rightarrow n_{8} \rightarrow \\ & l_{11,1} \rightarrow n_{6} \rightarrow l_{11,2} \rightarrow n_{7} \rightarrow l_{12,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{10}$ | $\text { Taipei } \rightarrow \mathrm{LP}_{4} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{11} \rightarrow \text { Los }$ $\text { Angeles } \rightarrow \mathrm{LP}_{12} \rightarrow \text { New York }$ | $\begin{aligned} & n_{1} \rightarrow l_{4,1} \rightarrow n_{2} \rightarrow l_{4,2} \rightarrow n_{5} \rightarrow l_{4,3} \rightarrow n_{6} \rightarrow \\ & l_{4,4} \rightarrow n_{8} \rightarrow l_{11,1} \rightarrow n_{6} \rightarrow l_{11,2} \rightarrow n_{7} \rightarrow \\ & l_{12,1} \rightarrow n_{9} \end{aligned}$ |

(b)

| MLP <br> no. | Light paths combination | Nodes \& LPS combination flow |
| :---: | :---: | :---: |
| $m l_{11}$ | $\begin{aligned} & \text { Hsinchu } \rightarrow \mathrm{LP}_{5} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{10} \rightarrow \text { New } \\ & \text { York } \end{aligned}$ | $\begin{aligned} & n_{10} \rightarrow l_{5,1} \rightarrow n_{3} \rightarrow l_{5,2} \rightarrow n_{2} \rightarrow l_{5,3} \rightarrow n_{5} \rightarrow \\ & l_{5,4} \rightarrow n_{6} \rightarrow l_{5,5} \rightarrow n_{8} \rightarrow l_{10,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{12}$ | Hsinchu $\rightarrow \mathrm{LP}_{5} \rightarrow$ Chicago $\rightarrow \mathrm{LP}_{13} \rightarrow$ New York | $\begin{aligned} & n_{10} \rightarrow l_{5,1} \rightarrow n_{3} \rightarrow l_{5,2} \rightarrow n_{2} \rightarrow l_{5,3} \rightarrow n_{5} \rightarrow \\ & l_{5,4} \rightarrow n_{6} \rightarrow l_{5,5} \rightarrow n_{8} \rightarrow l_{13,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{13}$ | Hsinchu $\rightarrow \mathrm{LP}_{7} \rightarrow$ New York | $\begin{aligned} & n_{10} \rightarrow l_{7,1} \rightarrow n_{3} \rightarrow l_{7,2} \rightarrow n_{4} \rightarrow l_{7,3} \rightarrow n_{7} \rightarrow \\ & l_{7,4} \rightarrow n_{9} \end{aligned}$ |
| $m l_{14}$ | $\begin{aligned} & \text { Hsinchu } \rightarrow \mathrm{LP}_{6} \rightarrow \text { Los } \\ & \text { Angeles } \rightarrow \mathrm{LP}_{12} \rightarrow \text { New York } \end{aligned}$ | $n_{10} \rightarrow l_{6,1} \rightarrow n_{3} \rightarrow l_{6,2} \rightarrow n_{7} \rightarrow l_{12,1} \rightarrow n_{9}$ |
| $m l_{15}$ | $\begin{aligned} & \text { Hsinchu } \rightarrow \mathrm{LP}_{6} \rightarrow \text { Los } \\ & \text { Angeles } \rightarrow \mathrm{LP}_{11} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{10} \rightarrow \mathrm{New} \\ & \text { York } \end{aligned}$ | $\begin{aligned} & n_{10} \rightarrow l_{6,1} \rightarrow n_{3} \rightarrow l_{6,2} \rightarrow n_{7} \rightarrow l_{11,2} \rightarrow \\ & n_{6} \rightarrow l_{11,1} \rightarrow n_{8} \rightarrow l_{10,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{16}$ | $\begin{aligned} & \text { Hsinchu } \rightarrow \mathrm{LP}_{6} \rightarrow \text { Los } \\ & \text { Angeles } \rightarrow \mathrm{LP}_{11} \rightarrow \text { Chicago } \rightarrow \mathrm{LP}_{13} \rightarrow \text { New } \\ & \text { York } \end{aligned}$ | $\begin{aligned} & n_{10} \rightarrow l_{6,1} \rightarrow n_{3} \rightarrow l_{6,2} \rightarrow n_{7} \rightarrow l_{11,2} \rightarrow \\ & n_{6} \rightarrow l_{11,1} \rightarrow n_{8} \rightarrow l_{13,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{17}$ | $\begin{aligned} & \text { Hsinchu } \rightarrow \mathrm{LP}_{8} \rightarrow \text { Los } \\ & \text { Angeles } \rightarrow \mathrm{LP}_{12} \rightarrow \text { New York } \end{aligned}$ | $n_{10} \rightarrow l_{8,1} \rightarrow n_{3} \rightarrow l_{8,2} \rightarrow n_{7} \rightarrow l_{12,1} \rightarrow n_{9}$ |
| $m l_{18}$ | Hsinchu $\rightarrow \mathrm{LP}_{8} \rightarrow$ Los <br> Angeles $\rightarrow \mathrm{LP}_{11} \rightarrow$ Chicago $\rightarrow \mathrm{LP}_{10} \rightarrow$ New York | $\begin{aligned} & n_{10} \rightarrow l_{8,1} \rightarrow n_{3} \rightarrow l_{8,2} \rightarrow n_{7} \rightarrow l_{11,2} \rightarrow \\ & n_{6} \rightarrow l_{11,1} \rightarrow n_{8} \rightarrow l_{10,1} \rightarrow n_{9} \end{aligned}$ |
| $m l_{19}$ | Hsinchu $\rightarrow \mathrm{LP}_{8} \rightarrow$ Los <br> Angeles $\rightarrow \mathrm{LP}_{11} \rightarrow$ Chicago $\rightarrow \mathrm{LP}_{13} \rightarrow$ New York | $\begin{aligned} & n_{10} \rightarrow l_{8,1} \rightarrow n_{3} \rightarrow l_{8,2} \rightarrow n_{7} \rightarrow l_{11,2} \rightarrow \\ & n_{6} \rightarrow l_{11,1} \rightarrow n_{8} \rightarrow l_{13,1} \rightarrow n_{9} \end{aligned}$ |

(b) Continued.

| MLP <br> no. | Light paths combination | Nodes \& LPS combination flow |
| :--- | :--- | :--- |
|  | Hsinchu $\rightarrow \mathrm{LP}_{5} \rightarrow$ Chicago $\rightarrow \mathrm{LP}_{11} \rightarrow \mathrm{Los}$ | $n_{10} \rightarrow l_{5,1} \rightarrow n_{3} \rightarrow l_{5,2} \rightarrow n_{2} \rightarrow l_{5,3} \rightarrow n_{5} \rightarrow$ |
| $m l_{20}$ | Angeles $\rightarrow \mathrm{LP}_{12} \rightarrow$ New York | $l_{5,4} \rightarrow n_{6} \rightarrow l_{5,5} \rightarrow n_{8} \rightarrow l_{11,1} \rightarrow n_{6} \rightarrow$ |
|  | $l_{11,2} \rightarrow n_{7} \rightarrow l_{12,1} \rightarrow n_{9}$ |  |

Table 2: Connection probability of all physical lines and LPSs.

| $\begin{aligned} & \hline \text { PL } \\ & \text { no. } \end{aligned}$ | LPS in this PL | Disconnection starting time | Disconnection ending time | Disconnection duration | Connection probability ( $t=$ a week $=4032 \mathrm{mins}$ ) | Root caused |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $P_{1}$ | $\begin{aligned} & \hline l_{1,1} ; l_{2,1} ; \\ & l_{3,1} ; l_{4,1} \end{aligned}$ | $\begin{gathered} \hline \text { 2011/4/21 } \\ \text { 10:17:00 AM } \end{gathered}$ | $\begin{gathered} \hline \text { 2011/4/21 } \\ \text { 11:56:00 AM } \end{gathered}$ | 99 mins | $t-99 / t=0.98$ | Circuit broken |
| $P_{2}$ | $\begin{gathered} l_{2,2,2} ; l_{3,2} ; \\ l_{5,2} \end{gathered}$ | N/A | N/A | N/A | 1 | No |
| $P_{3}$ | $l_{2,3} ; l_{7,2}$ | N/A | N/A | N/A | 1 | No |
| $P_{4}$ | $l_{1,2}$ | $\begin{gathered} \text { 2010/9/28 } \\ \text { 10:08:00 AM } \end{gathered}$ | $\begin{gathered} \text { 2010/9/28 } \\ \text { 02:07:00 PM } \end{gathered}$ | 239 mins | $t-239 / t=0.94$ | Card broken |
| $P_{5}$ | $l_{4,2} ; l_{5,3}$ | $\begin{gathered} \text { 2011/4/21 } \\ \text { 22:01 PM } \end{gathered}$ | $\begin{gathered} \text { 2011/4/22 } \\ \text { 04:29:00 AM } \end{gathered}$ | 388 mins | $t-388 / t=0.90$ | Circuit broken |
| $P_{6}$ | $l_{4,3} ; l_{5,4}$ | $\begin{gathered} \text { 2009/5/29 } \\ 05: 16 \text { AM } \end{gathered}$ | $\begin{gathered} \text { 2009/5/29 } \\ \text { 08:27:00 AM } \end{gathered}$ | 191 mins | $t-191 / t=0.95$ | Circuit broken |
| $P_{7}$ | $\begin{gathered} l_{3,3 ;} ; l_{6,2} ; \\ l_{8,2} \end{gathered}$ | $\begin{gathered} \text { 2009/9/16 } \\ 04: 40: 00 \text { PM } \end{gathered}$ | $\begin{gathered} \text { 2009/9/16 } \\ \text { 05:01:00 PM } \end{gathered}$ | 211 mins | $t-211 / t=0.99$ | Circuit broken |
| $P_{8}$ | $l_{2,4} ; l_{7,3}$ | $\begin{gathered} \text { 2011/3/11 } \\ \text { 01:53:00 PM } \end{gathered}$ | $\begin{gathered} \text { 2011/3/12 } \\ \text { 01:37:00 AM } \end{gathered}$ | 704 mins | $t-704 / t=0.83$ | Japans' earthquake |
| $P_{9}$ | $l_{11,2}$ | $\begin{gathered} \text { 2011/2/17 } \\ \text { 01:19:00 AM } \end{gathered}$ | $\begin{gathered} \text { 2011/2/17 } \\ \text { 03:33:00 AM } \end{gathered}$ | 134 mins | $t-134 / t=0.97$ | Card disable |
| $P_{10}$ | $\begin{aligned} & l_{1,3} ; l_{4,4} ; \\ & l_{11,1} ; l_{5,5} \end{aligned}$ | $\begin{gathered} \text { 2010/5/25 } \\ \text { 04:28:00 AM } \end{gathered}$ | $\begin{gathered} \text { 2010/5/25 } \\ \text { 11:11:00 AM } \end{gathered}$ | 403 mins | $t-403 / t=0.90$ | Circuit broken |
| $P_{11}$ | $\begin{gathered} l_{3,4} ; l_{12,1} ; \\ l_{7,4} \end{gathered}$ | $\begin{gathered} \text { 2009/3/21 } \\ 08: 58: 00 \text { PM } \end{gathered}$ | $\begin{gathered} \text { 2009/3/22 } \\ \text { 03:23:00 AM } \end{gathered}$ | 385 mins | $t-385 / t=0.90$ | Card disable |
| $P_{12}$ | $l_{10,1} ; l_{13,1}$ | $\begin{gathered} \text { 2011/2/8 } \\ \text { 01:17:00 AM } \end{gathered}$ | $\begin{gathered} \text { 2011/2/8 } \\ \text { 11:07:00 AM } \end{gathered}$ | 590 mins | $t-590 / t=0.85$ | Card disable |
| $P_{13}$ | $\begin{aligned} & l_{5,1} ; l_{6,1} ; \\ & l_{7,1} ; l_{8,1} \\ & \hline \end{aligned}$ | N/A | N/A | N/A | 1 | No |

### 4.2. Probability of All LPSs Breaking

To compute the connection probability of each PL, we use the disconnection data from 2008 through 2011. The longest duration of every break for each physical line during the 168 hours of every week is used to determine the disconnection probability of each line. For example, as the physical line $P_{10}$ from San Francisco to Chicago broke for 403 minutes on 2010/5/25, its connection probability is $(168 \times 60-403) /(168 \times 60)=0.90$. Therefore, its disconnection probability is $(1-0.9)=0.1$. All the LPSs $l_{1,3}, l_{4,4}$ and $l_{11,1}$ located in this physical line $P_{10}$ have the same disconnection probability of 0.1 .

Table 2 shows all LPSs' connection probability after screening all physical lines' disconnection records and selecting the longest broken time for each. These breaks include disabled card devices, circuit failures, and breaks from March 11, 2011 Japanese earthquake

Table 3: $\operatorname{LPS} l_{i, j}$ redenoted by $a_{i}$ and its connection probability the same as the physical line $P_{i}$ it locates.

| Arc | $a_{1}$ | $a_{2}$ | $a_{3}$ | $a_{4}$ | $a_{5}$ | $a_{6}$ | $a_{7}$ | $a_{8}$ | $a_{9}$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| LPS | $l_{1,1}$ | $l_{1,2}$ | $l_{1,3}$ | $l_{2,1}$ | $l_{2,2}$ | $l_{2,3}$ | $l_{2,4}$ | $l_{3,1}$ | $l_{3,2}$ | $l_{3,3}$ | $l_{3,4}$ | $l_{4,1}$ | $l_{4,2}$ | $l_{4,3}$ | $l_{4,4}$ | $l_{10,1}$ | $l_{11,1}$ |
| Prob | 0.98 | 0.94 | 0.9 | 0.98 | 1 | 1 | 0.83 | 0.98 | 1 | 0.99 | 0.9 | 0.98 | 0.9 | 0.95 | 0.9 | 0.85 | 0.9 |
| Arc | $a_{18}$ | $a_{19}$ | $a_{20}$ | $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | $a_{25}$ | $a_{26}$ | $a_{27}$ | $a_{28}$ | $a_{29}$ | $a_{30}$ | $a_{31}$ | $a_{32}$ | $a_{33}$ |  |
| LPS | $l_{11,2}$ | $l_{12,1}$ | $l_{13,1}$ | $l_{5,1}$ | $l_{5,2}$ | $l_{5,3}$ | $l_{5,4}$ | $l_{5,5}$ | $l_{6,1}$ | $l_{6,2}$ | $l_{7,1}$ | $l_{7,2}$ | $l_{7,3}$ | $l_{7,4}$ | $l_{8,1}$ | $l_{8,2}$ |  |
| Prob | 0.97 | 0.9 | 0.85 | 1 | 1 | 0.9 | 0.95 | 0.9 | 1 | 0.99 | 1 | 1 | 0.83 | 0.9 | 1 | 0.99 |  |

and tsunami that caused the physical submarine line $P_{8}$ to break. This line uses a submarine cable connection between TP-3 and Los Angeles. Artificial devices, short circuits, and natural disasters simultaneously influence TWAREN's network reliability from Taipei and Hsinchu to New York. Since each failure of a node device has been included and recorded in the physical line's disconnection record, each node is supposed to be perfect with a reliability of 1 . For computational convenience, as described in Section 3.3, we converted LPS $l_{i, j}$ by using $a_{i}$ and the probability of $a_{i}$, as Table 3 shows.

### 4.3. Network Reliability Computation

When line breaks occur, the suppliers of these pass-through physical lines provide all serviceable lines as backup lines, therefore increasing the network reliability. In this study, we do not discuss the backup lines and concentrate only on the regular lines to determine those factors that affect their network reliability. Firstly, we focus on the demand set $D_{20}=$ $\{(20,0),(16,4),(12,8),(8,12),(4,16),(0,20)\}$, given all MLPs in Tables $1(\mathrm{a})$ and $1(\mathrm{~b})$ and by using the algorithm in Section 3.3 as follows to obtain $\Omega_{\left(d_{1}, d_{2}\right), \text { min }}$.

Step 1. Do the following steps for $(4,16) \in D_{20}$ (since there is no solution for $\Omega_{(0,20), \min }$ in this example, we only demonstrate $\Omega_{(4,16), \text { min }}$ here).

Step 2. Find all feasible solutions $F$ that satisfy constraints (4.1):

$$
\begin{gather*}
f_{1}+f_{2}+f_{9} \leq M^{1}=4, \\
f_{1}+f_{2}+f_{9} \leq M^{2}=4, \\
\vdots  \tag{4.1}\\
f_{17}+f_{18}+f_{19} \leq M^{32}=4, \\
f_{17}+f_{18}+f_{19} \leq M^{33}=4 \\
f_{1}+f_{2}+\cdots+f_{10}=d_{1}=4, \\
f_{11}+f_{12}+\cdots+f_{20}=d_{2}=16 .
\end{gather*}
$$

In this step, each $f_{i}$ has two values, say 0 and 4 , standing for the two capacity states of failure or success. From this, we obtain 4 flow vectors as shown in Table 4(a) (column 1).
Table 4: (a) Results of example for $\Omega_{(4,16), \min }$ in $D_{20}$. (b) Results of example for $\Omega_{(8,12), \min }$ in $D_{20}$. (c) Results of example for $\Omega_{(12,8), \text { min }}$ in $D_{20}$. (d) Results of example for $\Omega_{(16,4), \min }$ in $D_{20}$.
(a)

| $F$ | $X$ | (4,16)-MLP or <br> not? |
| :--- | :--- | :--- |
| $F_{1}=(0,0,0,0,4,0,0,0,0,0,0,4,4,0,4,0,4,0,0,0)$ | $X_{1}=(0,0,0,0,0,0,0,4,4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4)$ | $\mathrm{No}^{2}$ |
| $F_{2}=(0,0,0,0,4,0,0,0,0,0,0,4,4,4,0,0,0,4,0,0)$ | $X_{2}=(0,0,0,0,0,0,0,4,4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4)$ | $X_{1} \geqq X_{2}$ |
| $F_{3}=(0,0,0,0,4,0,0,0,0,0,4,0,4,0,0,4,4,0,0,0)$ | $X_{3}=(0,0,0,0,0,0,0,4,4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4)$ | $X_{2} \geqq X_{3}$ |
| $F_{4}=(0,0,0,0,4,0,0,0,0,0,4,0,4,4,0,0,0,0,4,0)$ | $X_{4}=(0,0,0,0,0,0,0,4,4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4)$ | No |

[^0](b) Continued

| $F$ | X | (8,12)-MLP or not? | Remark |
| :---: | :---: | :---: | :---: |
| $F_{23}=(4,0,0,0,4,0,0,0,0,0,0,4,4,0,0,0,4,0,0,0)$ | $X_{23}=(4,4,4,0,0,0,0,4,4,4,4,0,0,0,0,4,0,0,4,4,4,4,4,4,4,0,0,4,4,4,4,4,4)$ | Yes |  |
| $F_{24}=(4,0,0,0,4,0,0,0,0,0,0,4,4,4,0,0,0,0,0,0)$ | $X_{24}=(4,4,4,0,0,0,0,4,4,4,4,0,0,0,0,4,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0)$ | Yes |  |
| (c) |  |  |  |
| $F$ | X | $(12,8)$-MLP or not? | Remark |
| $F_{1}=(0,0,0,4,4,0,0,0,4,0,4,0,4,0,0,0,0,0,0,0)$ | $X_{1}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | No | $X_{1} \geqq X_{7}$ |
| $F_{2}=(0,0,0,4,4,0,4,0,0,0,0,0,4,0,0,0,4,0,0,0)$ | $X_{2}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | No | $X_{2} \geqq X_{4}$ |
| $F_{3}=(0,0,0,4,4,0,4,0,0,0,0,0,4,4,0,0,0,0,0,0)$ | $X_{3}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | No | $X_{3} \geqq X_{5}$ |
| $F_{4}=(0,0,0,4,4,4,0,0,0,0,0,0,4,0,0,0,0,4,0,0)$ | $X_{4}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | No | $X_{4} \geqq X_{8}$ |
| $F_{5}=(0,0,0,4,4,4,0,0,0,0,0,0,4,0,4,0,0,0,0,0)$ | $X_{5}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | No | $X_{5} \geqq X_{9}$ |
| $F_{6}=(0,0,0,4,4,4,0,0,0,0,4,0,4,0,0,0,0,0,0,0)$ | $X_{6}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | No | $X_{6} \geqq X_{12}$ |
| $F_{7}=(0,0,4,0,4,0,0,0,4,0,0,4,4,0,0,0,0,0,0,0)$ | $X_{7}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | No | $X_{7} \geqq X_{13}$ |
| $F_{8}=(0,0,4,0,4,0,0,4,0,0,0,0,4,0,0,0,4,0,0,0)$ | $X_{8}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | No | $X_{8} \geqq X_{10}$ |
| $F_{9}=(0,0,4,0,4,0,0,4,0,0,0,0,4,4,0,0,0,0,0,0)$ | $X_{9}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | No | $X_{9} \geqq X_{11}$ |
| $F_{10}=(0,0,4,0,4,4,0,0,0,0,0,0,4,0,0,0,0,0,4,0)$ | $X_{10}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | Yes |  |
| $F_{11}=(0,0,4,0,4,4,0,0,0,0,0,0,4,0,0,4,0,0,0,0)$ | $X_{11}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | Yes |  |
| $F_{12}=(0,0,4,0,4,4,0,0,0,0,0,4,4,0,0,0,0,0,0,0)$ | $X_{12}=(0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | Yes |  |
| $F_{13}=(0,4,0,0,4,0,0,0,0,4,4,0,4,0,0,0,0,0,0,0)$ | $X_{13}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | No | $X_{13} \geqq X_{19}$ |
| $F_{14}=(0,4,0,0,4,0,4,0,0,0,0,0,4,0,0,0,4,0,0,0)$ | $X_{14}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | No | $X_{14} \geqq X_{16}$ |
| $F_{15}=(0,4,0,0,4,0,4,0,0,0,0,0,4,4,0,0,0,0,0,0)$ | $X_{15}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | No | $X_{15} \geqq X_{17}$ |
| $F_{16}=(0,4,0,0,4,4,0,0,0,0,0,0,4,0,0,0,0,4,0,0)$ | $X_{16}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | No | $X_{16} \geqq X_{23}$ |
| $F_{17}=(0,4,0,0,4,4,0,0,0,0,0,0,4,0,4,0,0,0,0,0)$ | $X_{17}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | No | $X_{17} \geqq X_{24}$ |
| $F_{18}=(0,4,0,0,4,4,0,0,0,0,4,0,4,0,0,0,0,0,0,0)$ | $X_{18}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,0,0,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | No | $X_{18} \geqq X_{27}$ |
| $F_{19}=(0,4,4,0,4,0,0,0,0,0,0,0,4,0,0,0,0,0,0,4)$ | $X_{19}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | No | $X_{19} \geqq X_{22}$ |
| $F_{20}=(0,4,4,0,4,0,0,0,0,0,0,0,4,0,0,0,4,0,0,0)$ | $X_{20}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,0,0,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | No | $X_{20} \geqq X_{29}$ |
| $F_{21}=(0,4,4,0,4,0,0,0,0,0,0,0,4,4,0,0,0,0,0,0)$ | $X_{21}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,0,0,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | No | $X_{21} \geqq X_{30}$ |
| $F_{22}=(4,0,0,0,4,0,0,0,0,4,0,4,4,0,0,0,0,0,0,0)$ | $X_{22}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | No | $X_{22} \geqq X_{28}$ |
| $F_{23}=(4,0,0,0,4,0,0,4,0,0,0,0,4,0,0,0,4,0,0,0)$ | $X_{23}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | No | $X_{23} \geqq X_{25}$ |
| $F_{24}=(4,0,0,0,4,0,0,4,0,0,0,0,4,4,0,0,0,0,0,0)$ | $X_{24}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | No | $X_{24} \geqq X_{26}$ |
| $F_{25}=(4,0,0,0,4,4,0,0,0,0,0,0,4,0,0,0,0,0,4,0)$ | $X_{25}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | Yes |  |
| $F_{26}=(4,0,0,0,4,4,0,0,0,0,0,0,4,0,0,4,0,0,0,0)$ | $X_{26}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,4,4,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | Yes |  |
| $F_{27}=(4,0,0,0,4,4,0,0,0,0,0,4,4,0,0,0,0,0,0,0)$ | $X_{27}=(4,4,4,4,4,4,4,4,4,4,4,0,0,0,0,4,0,0,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | Yes |  |
| $F_{28}=(4,0,0,4,4,0,0,0,0,0,0,0,4,0,0,0,0,0,0,4)$ | $X_{28}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0,0,4,4,4,4,0,0)$ | Yes |  |
| $F_{29}=(4,0,0,4,4,0,0,0,0,0,0,0,4,0,0,0,4,0,0,0)$ | $X_{29}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,0,0,4,4,0,0,0,0,0,0,0,4,4,4,4,4,4)$ | Yes |  |
| $F_{30}=(4,0,0,4,4,0,0,0,0,0,0,0,4,4,0,0,0,0,0,0)$ | $X_{30}=(4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,0,0,4,4,0,0,0,0,0,4,4,4,4,4,4,0,0)$ | Yes |  |


| $F$ | $X$ | $(16,4)$-MLP or | Remark |
| :--- | :--- | :--- | :--- |
| $F_{1}=(0,1,1,0,1,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0)$ | $X_{1}=(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,1,1,0,0,0,0,0,0,0,1,1,1,1,0,0)$ | not? | $X_{1} \geqq X_{2}$ |
| $F_{2}=(1,0,0,1,1,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0)$ | $X_{2}=(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,1,1,0,0,0,0,0,0,0,1,1,1,1,0,0)$ | Yes |  |



Figure 4: (Demand, network reliability) for demand set being $D_{20}, D_{16}, D_{12}, D_{8}, D_{4}$.

Step 3. Transform each $F$ into LPS $X$ to get $\Omega_{4,16}$ by (4.2).
For $F_{1}=(0,0,0,0,4,0,0,0,0,0,0,4,4,0,4,0,4,0,0,0)$, the capacity vector $X_{1}$ is transformed by

$$
\begin{gather*}
x_{1}=f_{1}+f_{2}+f_{9}=0+0+0=0, \\
x_{2}=f_{1}+f_{2}+f_{9}=0+0+0=0, \\
\vdots  \tag{4.2}\\
x_{32}=f_{17}+f_{18}+f_{19}=4+0+0=4, \\
x_{33}=f_{17}+f_{18}+f_{19}=4+0+0=4 .
\end{gather*}
$$

Thus, $X_{1}=(0,0,0,0,0,0,0,4,4,4,4,0,0,0,0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4)$. Similarly, we obtain 4 capacity vectors as shown in Table 4(a) (column 2).

Step 4. The non-minimal ones in $\Omega_{(4,16)}$ are removed to obtain $\Omega_{(4,16), \min }$, that is, $(4,16)$-MLP as shown in Table 4(a) (column 3).

When repeating the previous steps, we can also obtain $\Omega_{(8,12), \text { min }}$ (resp., $\Omega_{(12,8), \text { min }}$ and $\Omega_{(16,4), \min }$ ) in Table 4(b) (resp. Tables 4(c) and 4(d)). In terms of RSDP [9], we calculate the network reliability $R_{D_{20}}=\sum_{\left(d_{1}, d_{2}\right) \in D_{20}} \operatorname{Pr}\left\{Y \in \Omega_{\left(d_{1}, d_{2}\right)} \mid Y \geq X\right.$ for a $\left(d_{1}, d_{2}\right)$-MLP X $\}$ $=0.4140$. Similarly, $R_{D_{16}}=0.8195, R_{D_{12}}=0.9707, R_{D_{8}}=0.9976$, and $R_{D_{4}}=0.9999$ can be evaluated, respectively. The network reliability can be observed to decrease as the total demand increases, as shown in Figure 4.

In regard to QoS, this is only a concern when there are insufficient networks resources. When there are enough resources and demand is low, for instance, as above with $D_{4}$, there are still plenty of resources to handle other transmission requests, so the network reliability is quite high. On the other hand, if demand is high, say above set $D_{20}$, the network reliability will be low, since there are not enough resources to handle other data transmissions. To maintain the network reliability, it is important to avoid full transmission loads or increase line capacity. Depending on the results of our analysis, we may decide to allocate more economic resources to TWAREN to maximize future network utilities.

## 5. Summary and Conclusion

Instead of the classical TTNR, KTNR, and ATNR analysis of a binary-state flow network, this paper evaluates the network reliability of a stochastic-flow network with multiple sources. It also designs an MLP-based network reliability evaluation technique for the international LP portion of TWAREN's academic and research network. This portion contains the domestic land surface line and the Asia Pacific submarine cables which connect to the global academic research network, including the Internet2 Network [27]. Since the LP cannot be divided through any of its nodes or LPSs during transmission, MLP is a new concept to evaluate the network reliability in an LP environment. MLP is used to discuss the flow assignment and to evaluate the network reliability. This research contributes by making real TWAREN data available to be analyzed in a stochastic-flow network model. By using the MLP analysis technique, we will know how to continuously adjust TWAREN's infrastructure to achieve higher network reliability. In this study, we concentrate on the portion of the network that includes regular lines and does not include backup cables yet. This allows us to determine those factors that influence the dedicated regular lines' network reliability. We also consider the effects of the earthquake that hit Japan on March 11, 2011. All factors are studied, including artificial, machine, and cable failures and natural disasters that simultaneously influence TWAREN's network reliability from the two source nodes, Taipei and Hsinchu, to the single sink node, New York. In addition, the MLP network reliability technique used in the multiple sources case will enable us to increase the efficiency of TWAREN and help us to learn how to improve its network infrastructure and performance in the near future. Subsequently, further study may be undertaken on the network reliability of TWAREN's multisource to multisink (terminal) issue.

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Research Article

# Weighted Measurement Fusion White Noise Deconvolution Filter with Correlated Noise for Multisensor Stochastic Systems 

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For the multisensor linear discrete time-invariant stochastic control systems with different measurement matrices and correlated noises, the centralized measurement fusion white noise estimators are presented by the linear minimum variance criterion under the condition that noise input matrix is full column rank. They have the expensive computing burden due to the high-dimension extended measurement matrix. To reduce the computing burden, the weighted measurement fusion white noise estimators are presented. It is proved that weighted measurement fusion white noise estimators have the same accuracy as the centralized measurement fusion white noise estimators, so it has global optimality. It can be applied to signal processing in oil seismic exploration. A simulation example for Bernoulli-Gaussian white noise deconvolution filter verifies the effectiveness.

## 1. Introduction

An important application background of white noise estimation is signal processing in oil seismic exploration. After the explosives buried underneath earth surface are exploded, analyzing the reflection coefficient formed by the reflections of every oil layer can be used to determine whether there is oil underground and the geometry shape of the oil field. The reflection coefficient can be described by Bernoulli-Gaussian white noise. So the white noise estimation problem can be used in the oil seismic exploration. This problem has been deeply researched by Mendel [1-3], but Mendel has not solved the system's measurement white noise estimation problem and also has not referred the problem of multisensor information fusion white noise deconvolution estimation.

In order to improve the estimation accuracy of single sensor to the white noise, in [4,5], when input noise and measurement noise are not correlated, the multisensor information
fusion white noise deconvolution filters are put forward, respectively, by Kalman filtering method and modern time series analysis method. In [6-8], multisensor information fusion white noise optimal filter is presented for the systems with correlated noise by Kalman filtering method. In [9], modern time series analysis method and Gevers-Wouters algorithm are used to present the information fusion white noise deconvolution filter for multisensor systems with correlated noises. Thus, the solution of Riccati equation is avoided and the selftuning filter with unknown model parameters can be designed. The shortcomings of methods presented in [3-9] are to require computing the high-dimension cross-covariance matrix and the fusion accuracy is global suboptimal.

Recently, the weighted measurement fusion (WMF) method has gained great attention. Its basic principle is to weigh local sensor measurements to obtain a lowdimensional measurement equation according to some fusion criterion and then use a single Kalman filter to obtain the final fused state estimation. It can, not only reduce the computing burden greatly, but also give a global optimal estimation. This can be explained as the accuracy of WMF and the centralized measurement fusion (CMF) filter [10] is the same. Therefore, it is globally optimal. Gan and Chris [10] put forward the WMF algorithm with the assumption that all the sensors have the same measurement matrix and the measurement noises of each sensor are uncorrelated. Using the Lagrange multiplier method, the WMF algorithm is presented when the measurement noises of each sensor are correlated [11]. Ran et al. $[12,13]$ put forward the WMF algorithm when the extended measurement matrix of all the sensors has full column rank or the measurement matrices of all the sensors have the maximal right factor. Self-tuning WMF Kalman filtering algorithm is presented in the work of Gao et al. [14] and Ran and Deng [15]. In [16, 17], full-rank decomposition and weighted least square theory is used under correlated noises and different sensor measurement matrices; the WMF algorithm is presented.

However, using WMF method to solve the white noise estimation value problem of multisensor system with correlated noise and with different measurement matrices in each sensor is always a difficult issue to be solved, since the present white noise estimation theory is not suitable for WMF method. In this paper, we use the WMF algorithm to solve the white noise fusion estimation problem of multisensor systems with correlated noises and different measurement matrices. Firstly, under the assumption that the noise input matrix is of full column-rank, we present the CMF white noise estimators by the extended measurements of all sensors. They have a large computing burden due to the high-dimension measurement matrix. Further, the WMF white noise deconvolution estimators are presented to reduce the computing burden, which have the global optimality.

The paper is structured as follows. The CMF white noise deconvolution estimators are presented in Section 2. The multisensor WMF white noise deconvolution estimators are designed in Section 3. A simulation example follows in Section 4. Some conclusions are given in the end.

## 2. Multisensor CMF White Noise Deconvolution Filter

Consider the discrete time-invariant linear stochastic control systems with $L$ sensors

$$
\begin{gather*}
x(t+1)=\Phi x(t)+B u(t)+\Gamma w(t),  \tag{2.1}\\
y_{i}(t)=H_{i} x(t)+v_{i}(t), \quad i=1, \ldots, L, \tag{2.2}
\end{gather*}
$$

where $x(t) \in R^{n}$ is the state, $y_{i}(t) \in R^{m_{i}}, i=1, \ldots, L$ are the measurements, $u(t) \in R^{p}$ is the known control input, $\Phi, B$, and $\Gamma$ are constant matrices with compatible dimensions. $H_{i} \in R^{m_{i} \times n}$ is the measurement matrix of the sensor $i . w(t)$ and $v_{i}(t)$ are correlated white noises with zero means, and

$$
E\left\{\left[\begin{array}{l}
w(t)  \tag{2.3}\\
v_{i}(t)
\end{array}\right]\left[\begin{array}{ll}
w^{T}(k) & v_{j}^{T}(k)
\end{array}\right]\right\}=\left[\begin{array}{cc}
Q_{w} & S_{j} \\
S_{i}^{T} & R_{i j}
\end{array}\right] \delta_{t k}
$$

where the symbol $E$ denotes the expectation, $\delta_{t t}=1, \delta_{t k}=0(t \neq k), R_{i i}=R_{i}$, the superscript $T$ denotes the transpose. Combining $L$ measurement equations of (2.2) yields the centralized measurement equation:

$$
\begin{equation*}
y^{(I)}(t)=H^{(I)}(t) x(t)+v^{(I)}(t) \tag{2.4}
\end{equation*}
$$

where

$$
\begin{equation*}
y^{(I)}(t)=\left[y_{1}^{T}(t), \ldots, y_{L}^{T}(t)\right]^{T}, \quad H^{(I)}=\left[H_{1}^{T}, \ldots, H_{L}^{T}\right]^{T}, \quad v^{(I)}(t)=\left[v_{1}^{T}(t), \ldots, v_{L}^{T}(t)\right]^{T} \tag{2.5}
\end{equation*}
$$

The fusion measurement white noise $v^{(I)}(t)$ has variance matrix $R^{(I)}=\left(R_{i j}\right)_{L L}$. The correlated function of $w(t)$ and $v^{(I)}(t)$ is $S=\left[S_{1}, \ldots, S_{L}\right]$.

To convert the systems (2.1) and (2.4) into the uncorrelated system, (2.1) is equivalent to

$$
\begin{equation*}
x(t+1)=\Phi x(t)+B u(t)+\Gamma w(t)+J\left[y^{(I)}(t)-H^{(I)} x(t)-v^{(I)}(t)\right] \tag{2.6}
\end{equation*}
$$

where $J$ is a pending matrix. (2.6) can be converted into

$$
\begin{equation*}
x(t+1)=\bar{\Phi} x(t)+\bar{u}(t)+\bar{w}(t), \tag{2.7}
\end{equation*}
$$

where $\bar{\Phi}=\Phi-J H^{(I)}, \bar{u}(t)=B u(t)+J y^{(I)}(t), \bar{w}(t)=\Gamma w(t)-J v^{(I)}(t) . J y^{(I)}(t)$ as output feedback becomes a part of the control item. Then primary system formulae (2.1) and (2.4) are equivalent to the system formed by formulae (2.4) and (2.7). To make $E\left[\bar{w}(t) v^{(I) \mathrm{T}}(t)\right]=0$, introduce $J=\Gamma S R^{(I)-1}$ which ensures that $\bar{w}(t)$ and $v^{(I)}(t)$ are not correlated. Then variance matrix of $\bar{w}(t)$ is yielded as $\bar{Q}_{w}=\Gamma\left(Q_{w}-S R^{(I)} S^{T}\right) \Gamma^{T}$.

Lemma 2.1 (see [18, 19]). Multisensor systems (2.1) and (2.4) with correlated noise have CMF global optimal input white noise deconvolution estimators $\widehat{w}(t \mid t+N)$ and the error variance matrices $P_{w}(t \mid t+N)$ as

$$
\begin{gather*}
\widehat{w}(t \mid t+N)=0, \quad(N<0) \\
\widehat{w}(t \mid t)=S\left[H^{(I)} P(t \mid t-1) H^{(I) T}+R^{(I)}\right]^{-1} \varepsilon^{(I)}(t) \\
P_{w}(t \mid t+N)=Q_{w}, \quad(N<0) \\
P_{w}(t \mid t)=Q_{w}-S\left[H^{(I)} P(t \mid t-1)^{(I) T}+R^{(I)}\right]^{-1} S^{T} \\
\widehat{w}(t \mid t+N)=\widehat{w}(t \mid t)+\sum_{i=1}^{N} M_{w}(t \mid t+i) \varepsilon^{(I)}(t+i), \quad(N>0) \\
P_{w}(t \mid t+N)=Q_{w}-\sum_{j=1}^{N} M(t \mid t+j) Q_{\varepsilon}^{(I)}(t+j) M^{T}(t \mid t+j), \quad(N>0) \\
M_{w}(t \mid t+1)=D_{w}(t, 1) H^{(I) T}\left[H^{(I)} P^{(I)}(t+1 \mid t) H^{(I) T}+R^{(I)}\right]^{-1}  \tag{2.8}\\
M_{w}(t \mid t+j)=D_{w}(t, 1)\left\{\prod_{i=1}^{j-1}\left[I_{n}-K_{f}^{(I)}(t+i) H^{(I)}\right]^{T} \bar{\Phi}^{T}\right\} H^{(I) T} \\
\times\left[H^{(I)} P^{(I)}(t+j \mid t+j-1) H^{(I) T}+R^{(I)}\right]^{-1} \\
D_{w}(t, 1)=-S K_{f}^{(I) T}(t) \bar{\Phi}^{T}+Q_{w} \Gamma^{T}-S J^{T} \\
\Psi_{p}^{(I)}(t)=\bar{\Phi}-\bar{K}_{p}^{(I)}(t) H^{(I)} \\
\bar{K}_{p}^{(I)}(t)=\bar{\Phi} K_{f}^{(I)}(t), K_{p}^{(I)}(t)=\bar{K}_{p}^{(I)}(t)+J \\
K_{f}^{(I)}(t)=P^{(I)}(t \mid t-1) H^{(I) T} Q_{\varepsilon}^{(I)-1}(t) \\
Q_{\varepsilon}^{(I)}(t)=H^{(I)} P^{(I)}(t \mid t-1) H^{(I) T}+R^{(I)} .
\end{gather*}
$$

$P^{(I)}(t+1 \mid t)$ satisfies Riccati equation:

$$
\begin{gather*}
P^{(I)}(t+1 \mid t)=\bar{\Phi}\left[P^{(I)}(t \mid t-1)-P^{(I)}(t \mid t-1) H^{(I) T}\left(H^{(I)} P^{(I)}(t \mid t-1) H^{(I) T}+R^{(I)}\right)^{-1}\right. \\
\left.\times H^{(I)} P^{(I)}(t \mid t-1)\right] \bar{\Phi}^{T}+\Gamma\left[Q_{w}-S R^{(I)-1} S^{T}\right] \Gamma^{T}  \tag{2.9}\\
\hat{x}^{(I)}(t+1 \mid t+1)=\widehat{x}^{(I)}(t+1 \mid t)+K_{f}^{(I)}(t+1) \varepsilon^{(I)}(t+1),  \tag{2.10}\\
\widehat{x}^{(I)}(t+1 \mid t)=\bar{\Phi} \widehat{x}^{(I)}(t \mid t)+B u(t)+J y^{(I)}(t),  \tag{2.11}\\
\varepsilon^{(I)}(t+1)=y^{(I)}(t+1)-H^{(I)} \widehat{x}^{(I)}(t+1 \mid t),  \tag{2.12}\\
K_{f}^{(I)}(t+1)=P^{(I)}(t+1 \mid t) H^{(I) T}\left[H^{(I)} P^{(I)}(t+1 \mid t) H^{(I) T}+R^{(I)}\right]^{-1}, \tag{2.13}
\end{gather*}
$$

$$
\begin{gather*}
P^{(I)}(t+1 \mid t+1)=\left[I_{n}-K^{(I)}(t+1) H^{(I)}\right] P^{(I)}(t+1 \mid t)  \tag{2.14}\\
\widehat{x}^{(I)}(t \mid t+N)=\widehat{x}^{(I)}(t \mid t+N-1)+K^{(I)}(t \mid t+N) \varepsilon^{(I)}(t+N), \quad N>0,  \tag{2.15}\\
P^{(I)}(t \mid t+N)=P^{(I)}(t \mid t)-\sum_{j=1}^{N} K^{(I)}(t \mid t+j) Q_{\varepsilon}^{(I)}(t+j) K^{(I) T}(t \mid t+j) . \tag{2.16}
\end{gather*}
$$

Smoothing gain has two computing methods:

$$
\begin{align*}
& K^{(I)}(t \mid t+N)=P^{(I)}(t \mid t-1)\left\{\prod_{j=0}^{N-1} \Psi_{p}^{(I) T}(t+j)\right\} H^{(I) T} Q_{\varepsilon}^{(I)-1}(t+N) \\
& K^{(I)}(t \mid t+N)=P^{(I)}(t \mid t) \bar{\Phi}^{T}\left\{\prod_{j=1}^{N-1} \Psi_{p}^{(I) T}(t+j)\right\} H^{(I) T} Q_{\varepsilon}^{(I)-1}(t+N) \tag{2.17}
\end{align*}
$$

Theorem 2.2. Supposing $\Gamma$ is full column-rank matrix, for systems (2.1) and (2.4), the optimal white noise deconvolution estimator of CMF input white noise $w(t)$ is given by

$$
\begin{gather*}
\widehat{w}^{(I)}(t \mid t+N)=0, \quad(N<0)  \tag{2.18}\\
\widehat{w}^{(I)}(t \mid t)=-A \widehat{x}^{(I)}(t \mid t)+C(t) \tag{2.19}
\end{gather*}
$$

And the variance matrices of estimator errors $\tilde{w}^{(I)}(t \mid t+N)=w(t)-\widehat{w}^{(I)}(t \mid t+N)$ are given as

$$
\begin{gather*}
P_{w}^{(I)}(t \mid t+N)=Q_{w} \quad(N<0)  \tag{2.20}\\
P_{w}^{(I)}(t \mid t)=A P^{(I)}(t \mid t) A^{T}+Q  \tag{2.21}\\
Q=Q_{w}-S R^{(I)-1} S^{T} \tag{2.22}
\end{gather*}
$$

defining

$$
\begin{gather*}
A=\Gamma^{+} J H^{(I)}=S R^{(I)-1} H^{(I)} \\
C(t)=\Gamma^{+}\left(J y^{(I)}(t)\right)=S R^{(I)-1} y^{(I)}(t), \tag{2.23}
\end{gather*}
$$

where $\Gamma^{+}$is the pseudo-inverse of $\Gamma$, that is,

$$
\begin{equation*}
\Gamma^{+}=\left(\Gamma^{T} \Gamma\right)^{-1} \Gamma^{T} \tag{2.24}
\end{equation*}
$$

Proof. Notice when $N<0, w(t) \perp L\left(y^{(I)}(t+N), y^{(I)}(t+N-1) \cdots\right)$, then (2.18) holds obviously. For (2.1), according to the projection theory [19], we have

$$
\begin{equation*}
\Gamma \widehat{w}^{(I)}(t \mid t)=\widehat{x}^{(I)}(t+1 \mid t)-\Phi \widehat{x}^{(I)}(t \mid t)-B u(t) \tag{2.25}
\end{equation*}
$$

From (2.11) and (2.24), we have

$$
\begin{equation*}
\widehat{w}^{(I)}(t \mid t)=\Gamma^{+}\left[\bar{\Phi} \widehat{x}^{(I)}(t \mid t)+B u(t)+J y^{(I)}(t)-\Phi \widehat{x}^{(I)}(t \mid t)-B u(t)\right] \tag{2.26}
\end{equation*}
$$

which can be simplified to

$$
\begin{equation*}
\widehat{w}^{(I)}(t \mid t)=\Gamma^{+}\left[-J H^{(I)} \widehat{x}^{(I)}(t \mid t)+J y^{(I)}(t)\right] \tag{2.27}
\end{equation*}
$$

Then, from the definition of (2.23), we easily obtain (2.19).
Subtracting (2.25) from (2.1) yields the error relation

$$
\begin{equation*}
\Gamma \tilde{w}^{(I)}(t \mid t)=\tilde{x}^{(I)}(t+1 \mid t)-\Phi \tilde{x}^{(I)}(t \mid t) \tag{2.28}
\end{equation*}
$$

Notice $\tilde{x}^{(I)}(t+N \mid t)=x(t)-\tilde{x}^{(I)}(t+N \mid t)$; there is a prediction error relation

$$
\begin{equation*}
\tilde{x}^{(I)}(t+1 \mid t)=\bar{\Phi} \tilde{x}^{(I)}(t \mid t)+\bar{w}(t) \tag{2.29}
\end{equation*}
$$

Substituting (2.29) into (2.28) yields

$$
\begin{equation*}
\Gamma \tilde{w}^{(I)}(t \mid t)=-J H^{(I)} \tilde{x}^{(I)}(t \mid t)+\bar{w}(t) \tag{2.30}
\end{equation*}
$$

And notice $\tilde{x}^{(I)}(t \mid t) \perp \bar{w}(t)$, then we have

$$
\begin{equation*}
\Gamma P_{w}^{(I)}(t \mid t) \Gamma^{T}=J H^{(I)} P^{(I)}(t \mid t) H^{(I) T} J^{T}+\bar{Q}_{w} \tag{2.31}
\end{equation*}
$$

So, (2.21) holds. From (2.18), we have (2.20). The proof is completed.
Theorem 2.3. For (2.1) and (2.4), when $\Gamma$ is a full column-rank matrix, one has optimal white noise deconvolution smoothers of CMF input white noise $w(t)$ as

$$
\begin{align*}
& \widehat{w}^{(I)}(t \mid t+N) \\
& \quad=\widehat{w}^{(I)}(t \mid t+N-1)+E^{(I)}(t) \Psi^{(I) T}(t+N, t+1) H^{(I) T} Q_{\varepsilon}^{(I)-1}(t+N) \varepsilon^{(I)}(t+N), \quad N>0 . \tag{2.32}
\end{align*}
$$

The error variance matrices are computed by

$$
\begin{align*}
P_{w}^{(I)}(t \mid t+N)= & P_{w}^{(I)}(t \mid t+N-1) \\
& -E^{(I)}(t) \Psi^{(I) T}(t+N, t+1) H^{(I) T} Q_{\varepsilon}^{(I)-1}(t+N) H^{(I)} \Psi^{(I)}(t+N, t+1) E^{(I) T}(t) . \tag{2.33}
\end{align*}
$$

Define

$$
\begin{gather*}
E^{(I)}(t)=-A P^{(I)}(t \mid t) \bar{\Phi}^{T}+Q \Gamma^{T},  \tag{2.34}\\
\Psi^{(I)}(t+N, t+N)=I_{n} \\
\Psi^{(I)}(t+N, i)=\Psi_{p}^{(I)}(t+N-1) \cdots \Psi_{p}^{(I)}(i) . \tag{2.35}
\end{gather*}
$$

Proof. For (2.1), the projection theory is used, and we have

$$
\begin{equation*}
\Gamma \widehat{w}^{(I)}(t \mid t+N)=\widehat{x}^{(I)}(t+1 \mid t+N)-\Phi \widehat{x}^{(I)}(t \mid t+N)-B u(t) . \tag{2.36}
\end{equation*}
$$

From (2.15), we have

$$
\begin{align*}
\Gamma \widehat{w}^{(I)}(t \mid t+N)= & \widehat{x}^{(I)}(t+1 \mid t+N-1)+K^{(I)}(t+1 \mid t+N) \varepsilon^{(I)}(t+N) \\
& -\Phi\left[\widehat{x}^{(I)}(t \mid t+N-1)+K^{(I)}(t \mid t+N) \varepsilon^{(I)}(t+N)\right]-B u(t) \tag{2.37}
\end{align*}
$$

From the recursive relation of (2.36), we have

$$
\begin{equation*}
\widehat{w}(t \mid t+N)=\widehat{w}^{(I)}(t \mid t+N-1)+\Gamma^{+}\left[K^{(I)}(t+1 \mid t+N)-\Phi K^{(I)}(t \mid t+N)\right] \varepsilon^{(I)}(t+N) \tag{2.38}
\end{equation*}
$$

then we obtain (2.32). In fact, from (2.17), we have

$$
\begin{align*}
\Gamma^{+} & {\left[K^{(I)}(t+1 \mid t+N)-\Phi K^{(I)}(t \mid t+N)\right] } \\
= & \Gamma^{+}\left[P^{(I)}(t+1 \mid t)\left\{\prod_{j=1}^{N-1} \Psi_{p}^{(I) T}(t+j)\right\} H^{(I)^{T}} Q_{\varepsilon}^{(I)-1}\right.  \tag{2.39}\\
& \left.\quad \times(t+N)-\Phi P^{(I)}(t \mid t)^{T} \times\left\{\prod_{j=1}^{N-1} \Psi_{p}^{(I) T}(t+j)\right\} H^{(I)^{T}} Q_{\varepsilon}^{(I)-1}(t+N)\right]
\end{align*}
$$

which is simplified to

$$
\begin{align*}
\Gamma^{+} & {\left[K^{(I)}(t+1 \mid t+N)-\Phi K^{(I)}(t \mid t+N)\right] } \\
& =\Gamma^{+}\left[P^{(I)}(t+1 \mid t)-\Phi P^{(I)}(t \mid t) \bar{\Phi}^{T}\right] \times\left\{\prod_{j=1}^{N-1} \Psi_{p}^{(I) T}(t+j)\right\} H^{(I) T} Q_{\varepsilon}^{(I)-1}(t+N) . \tag{2.40}
\end{align*}
$$

Define $E^{(I)}(t)$ as

$$
\begin{align*}
E^{(I)}(t) & =\Gamma^{+}\left[\bar{\Phi} P^{(I)}(t \mid t) \bar{\Phi}^{T}+\bar{Q}_{w}-\Phi P^{(I)}(t \mid t) \bar{\Phi}\right]  \tag{2.41}\\
& =-\Gamma^{+} J H^{(I)} P^{(I)}(t \mid t) \bar{\Phi}^{T}+\Gamma^{+} \bar{Q}_{w} .
\end{align*}
$$

Then we obtain (2.34). From the definition of (2.35), (2.32) is proved. Using $w(t)$ minus both sides of (2.32), and from $\tilde{w}^{(I)}(t \mid t+N) \perp \varepsilon^{(I)}(t+N)$, (2.33) is proved. This completes the proof.

Corollary 2.4. For multisensor systems with correlated noise (2.1) and (2.4), the nonrecursive white noise smoothers are given by

$$
\begin{equation*}
c \widehat{w}^{(I)}(t \mid t+N)=\widehat{w}^{(I)}(t \mid t)+E^{(I)}(t) \sum_{j=1}^{N} \Psi^{(I) T}(t+j, t+1) H^{(I) T} Q_{\varepsilon}^{(I)-1}(t+j) \varepsilon^{(I)}(t+j), \quad N>0 \tag{2.42}
\end{equation*}
$$

The error variances satisfy

$$
\begin{align*}
& P_{w}^{(I)}(t \mid t+N) \\
& \quad=P_{w}^{(I)}(t \mid t)-E^{(I)}(t)\left\{\sum_{j=1}^{N} \Psi^{(I) T}(t+j, t+1) H^{(I) T} Q_{\varepsilon}^{(I)-1}(t+j) H^{(I)} \Psi^{(I)}(t+j, t+1)\right\} E^{(I) T}(t) \tag{2.43}
\end{align*}
$$

Proof. (2.32) and (2.33) are iterated by $N$ times; (2.42) and (2.43) are obtained. This completes the proof.

## 3. Multisensor WMF White Noise Deconvolution Estimator

From [20], we know that any nonzero matrix $H^{(I)}$ has full-rank decomposition

$$
\begin{equation*}
H^{(I)}=F H^{(I I)} \tag{3.1}
\end{equation*}
$$

where $F$ is a full column-rank matrix with the rank $r$ and $H^{(I I)}$ is a full row-rank matrix with the rank $r$, then measurement model (2.4) can be represented as

$$
\begin{equation*}
y^{(I)}(t)=F H^{(I I)} x(t)+v^{(I)}(t) \tag{3.2}
\end{equation*}
$$

Given that $F$ is a full column-rank matrix, it follows that $F^{\mathrm{T}} R^{(I)} F$ is nonsingular. Then the weighted least squares (WLS) [21,22] method is used and the Gauss-Markov estimate of $H^{(I I)} x(t)$ is yielded as

$$
\begin{gather*}
y^{(I I)}(t)=\sum_{j=1}^{L} \Omega_{i} y_{i}(t)=\left(F^{T} R^{(I)-1} F\right)^{-1} F^{T} R^{(I)-1} y^{(I)}(t),  \tag{3.3}\\
{\left[\Omega_{1}, \Omega_{2}, \ldots, \Omega_{L}\right]=\left(F^{T} R^{(I)-1} F\right)^{-1} F^{T} R^{(I)-1}} \tag{3.4}
\end{gather*}
$$

then substituting (2.4) into (3.3) yields

$$
\begin{gather*}
y^{(I I)}(t)=H^{(I I)} x(t)+v^{(I I)}(t)  \tag{3.5}\\
v^{(I I)}(t)=\left(F^{T} R^{(I)-1} F\right)^{-1} F^{T} R^{(I)-1} v^{(I)}(t) \tag{3.6}
\end{gather*}
$$

The variance matrix $R^{(I I)}=E\left[v^{(I I)}(t) v^{(I I) T}(t)\right]$ of $v^{(I I)}(t)$ is given by $R^{(I I)}=\left(F^{T} R^{(I)-1} F\right)^{-1}$.
For systems (2.7) and (3.5) using standard Kalman filtering algorithm [19], we can obtain WMF Kalman estimators $\widehat{x}^{(I I)}(t \mid t)$, and its variance matrices $\mathrm{P}^{(I I)}(t \mid t)$, innovation $\varepsilon^{(\mathrm{II})}(t+j), j>0$. It is proved in [16] that the WMF Kalman filter $\hat{x}^{(I I)}(t \mid t)$ has the global optimality; that is, it is numerically identical to the CMF Kalman filter $\widehat{x}^{(I)}(t \mid t)$ if they have the same initial values.

The above WMF method can obviously reduce the computing burden since the dimension of the measurement vector for the CMF is $m \times 1, m=m_{1}+m_{2}+\cdots+m_{L}$, while that for the WMF is $r \times 1$, and $m$ is much larger than $r$ generally.

Theorem 3.1. For (2.7) and (3.5), when $\Gamma$ is a full column-rank matrix, one has WMF optimal nonrecursive smoothers of input white noise $w(t)$

$$
\begin{align*}
& \widehat{w}^{(I I)}(t \mid t+N) \\
& \quad=\widehat{w}^{(I I)}(t \mid t)+E^{(I I)}(t) \sum_{j=1}^{N} \Psi^{(I I) T}(t+j, t+1) H^{(I I) T} Q_{\varepsilon}^{(I I)-1}(t+j) \varepsilon^{(I I)}(t+j), \quad N>0 . \tag{3.7}
\end{align*}
$$

The error variance matrices satisfy

$$
\begin{align*}
P_{w}^{(I I)}(t \mid t+N)= & P_{w}^{(I I)}(t \mid t)-E^{(I I)}(t) \\
& \times\left\{\sum_{j=1}^{N} \Psi^{(I I) T}(t+j, t+1) H^{(I I) T} Q_{\varepsilon}^{(I I)-1}(t+j) H^{(I I)} \Psi^{(I I)}(t+j, t+1)\right\} \\
& \times E^{(I I) T}(t) \tag{3.8}
\end{align*}
$$

where

$$
\begin{gather*}
E^{(I I)}(t)=-A P^{(I I)}(t \mid t) \bar{\Phi}^{T}+Q \Gamma^{T}  \tag{3.9}\\
\Psi^{(I I)}(t+N, t+N)=I_{n} \\
\Psi^{(I I)}(t+N, i)=\Psi_{p}^{(I I)}(t+N-1) \cdots \Psi_{p}^{(I I)}(i) \tag{3.10}
\end{gather*}
$$

If $\widehat{x}^{(I)}(0 \mid 0)=\hat{x}^{(I I)}(0 \mid 0)$ is satisfied, then WMF is numerically equivalent to CMF, that is,

$$
\begin{array}{ll}
\widehat{w}^{(I)}(t \mid t+N)=\widehat{w}^{(I I)}(t \mid t+N) & \forall N, \forall t, \\
P_{w}^{(I)}(t \mid t+N)=P_{w}^{(I I)}(t \mid t+N) & \forall N, \forall t, \tag{3.11}
\end{array}
$$

where white noise filter is

$$
\begin{equation*}
\widehat{w}^{(I I)}(t \mid t)=-A \widehat{x}^{(I I)}(t \mid t)+C(t) . \tag{3.12}
\end{equation*}
$$

Proof. From [16], when $\hat{x}^{(I)}(0 \mid 0)=\hat{x}^{(I I)}(0 \mid 0)$, we have

$$
\begin{align*}
H^{(I) T} Q_{\varepsilon}^{(I)-1}(t) \varepsilon^{(I)}(t) & =H^{(I) T} Q_{\varepsilon}^{(I I)-1}(t) \varepsilon^{(I I)}(t), \\
\Psi_{p}^{(I)}(t) & =\Psi_{p}^{(I I)}(t),  \tag{3.13}\\
P^{(I)}(t \mid t) & =P^{(I I)}(t \mid t), \\
\widehat{x}^{(I)}(t \mid t) & =\widehat{x}^{(I I)}(t \mid t) .
\end{align*}
$$

So, we have $E^{(I)}(t)=E^{(I I)}(t)$. (2.19), (2.42), and (2.43) are compared with (3.7), (3.8), and (3.12), then (3.11) are obtained.

Corollary 3.2. WMF input white noise recursive smoothers are

$$
\begin{align*}
\widehat{w}^{(I I)}(t \mid t+N)= & \widehat{w}^{(I I)}(t+N-1)+E^{(I I)}(t) \Psi^{(I I) T}(t+N, t+1) H^{(I I) T} \\
& \times Q_{\varepsilon}^{(I I)-1}(t+N) \varepsilon^{(I I)}(t+N), \quad N>0 . \tag{3.14}
\end{align*}
$$

The error variance matrices satisfy

$$
\begin{align*}
P_{w}^{(I I)}(t \mid t+N)= & P_{w}^{(I I)}(t+N-1)-E^{(I I)}(t) \Psi^{(I I) T}(t+N, t+1) H^{(I I) T} \\
& \times Q_{\varepsilon}^{(I I)-1}(t+N) H^{(I I)} \Psi^{(I I)}(t+N, t+1) E^{(I I) T}(t), \tag{3.15}
\end{align*}
$$

then, one has the following relation:

$$
\begin{array}{ll}
\widehat{w}^{(I)}(t \mid t+N)=\widehat{w}^{(I I)}(t \mid t+N) & \forall N, \forall t, \\
P_{w}^{(I)}(t \mid t+N)=P_{w}^{(I I)}(t \mid t+N) & \forall N, \forall t . \tag{3.17}
\end{array}
$$

Proof. From Theorem 2.3 and (3.13), we have (3.14)-(3.17). This completes the proof.

## 4. Simulation Example

Consider the multisensor discrete linear stochastic ARMA signal system

$$
\begin{gather*}
A\left(q^{-1}\right) s(t)=C\left(q^{-1}\right) w(t), \\
y_{i}(t)=H_{0 i} s(t)+\xi_{i}(t), \quad i=1,2,3, \\
A\left(q^{-1}\right)=a_{0}+a_{1} q^{-1}+a_{2} q^{-2},  \tag{4.1}\\
C\left(q^{-1}\right)=c_{0}+c_{1} q^{-1},
\end{gather*}
$$



Figure 1: $w(t)$ and optimal fusion white noise filter $\widehat{w}(t \mid t), \widehat{w}^{(i)}(t \mid t), i=I, I I$.


- $\widehat{w}(t \mid t+1)$
- $\widehat{w}^{(I)}(t \mid t+1)$
- $\widehat{w}^{(I I)}(t \mid t+1)$

Figure 2: $w(t)$ and optimal fusion white noise smoother $\widehat{w}(t \mid t+1), \widehat{w}^{(i)}(t \mid t+1), i=I, I I$.
where $s(t) \in R$ is the signal, $y_{i}(t) \in R, i=1,2,3$ are the measurement signals. $\xi_{i}(t) \in R$, $i=1,2,3$ are Gaussian white noises with zero mean and variance matrix $Q_{\xi_{i}}$. And $w(t)=$ $b(t) g(t)$, where $b(t)$ is Bernoulli white noise satisfying $b(t)=1$ if $P(b(t)=1)=\lambda$, and $b(t)=0$ if $P(b(t)=0)=1-\lambda$, where $P$ denotes probability. $b(t)$ is independent of $g(t)$, then the variance matrix of $w(t)$ is $\sigma_{w}^{2}=\lambda \sigma_{g}^{2} \cdot q^{-1}$ is the back shift operator.

Our goal is to find the optimal white noise deconvolution estimators $\widehat{w}(t \mid t+N)$, $\widehat{w}^{(I)}(t \mid t+N)$, and $\widehat{w}^{(I I)}(t \mid t+N), N=0,1,2,3$, and the corresponding error variance matrices $P_{w}(t \mid t+N), P_{w}^{(I)}(t \mid t+N)$, and $P_{w}^{(I I)}(t \mid t+N)$, to test the estimation result values of these three algorithms equal and to compare the computing burden.
Table 1: Numeric equality of white noise deconvolution estimators of three algorithms.

| $t$ | 20 | 60 | 100 | 150 | 200 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $\widehat{w}(t \mid t)$ | 0.111998000967414 | -0.934613391503756 | -0.414208836483129 | -0.440806638705783 | -0.248866858088324 |
| $\widehat{w}^{(I)}(t \mid t)$ | 0.111998000967414 | -0.934613391503756 | -0.414208836483129 | -0.440806638705783 | -0.248866858088324 |
| $\widehat{w}^{(I I)}(t \mid t)$ | 0.111998000967414 | -0.934613391503756 | -0.414208836483129 | -0.440806638705783 | -0.248866858088324 |
| $\widehat{w}(t \mid t+1)$ | 0.118747413218407 | -0.971293527678591 | -0.453073939499306 | -0.38821382108702 | -0.219125568044873 |
| $\widehat{w}^{(I)}(t \mid t+1)$ | 0.118747413218407 | -0.971293527678591 | -0.453073939499306 | -0.38821382108702 | -0.219125568044873 |
| $\widehat{w}^{(I I)}(t \mid t+1)$ | 0.118747413218407 | -0.971293527678591 | -0.453073939499306 | -0.38821382108702 | -0.219125568044873 |
| $\widehat{w}(t \mid t+2)$ | 0.164558813520658 | -0.998738752610576 | -0.439294054718359 | -0.331015769821554 | -0.183694494396574 |
| $\widehat{w}^{(I)}(t \mid t+2)$ | 0.164558813520658 | -0.998738752610576 | -0.439294054718359 | -0.331015769821554 | -0.183694494396574 |
| $\widehat{w}^{(I I)}(t \mid t+2)$ | 0.164558813520658 | -0.998738752610576 | -0.439294054718359 | -0.331015769821554 | -0.183694494396574 |
| $\widehat{w}(t \mid t+3)$ | 0.186420977012657 | -1.01550762883661 | -0.448089267436535 | -0.314913935112578 | -0.163457575414262 |
| $\widehat{w}^{(I)}(t \mid t+3)$ | 0.186420977012657 | -1.01550762883661 | -0.448089267436535 | -0.314913935112578 | -0.163457575414262 |
| $\widehat{w}^{(I I)}(t \mid t+3)$ | 0.186420977012657 | -1.01550762883661 | -0.448089267436535 | -0.314913935112578 | -0.163457575414262 |



- $\widehat{w}(t \mid t+2)$
- $\widehat{w}^{(I)}(t \mid t+2)$
- $\widehat{w}^{(I I)}(t \mid t+2)$

Figure 3: $w(t)$ and optimal fusion white noise smoother $\widehat{\hat{w}}(t \mid t+2), \widehat{w}^{(i)}(t \mid t+2), i=I, I I$.


- $\widehat{w}(t \mid t+3)$
- $\widehat{w}^{(I)}(t \mid t+3)$
- $\widehat{w}^{(I I)}(t \mid t+3)$

Figure 4: $w(t)$ and optimal fusion white noise smoother $\widehat{w}(t \mid t+1), \widehat{w}^{(i)}(t \mid t+3), i=I, I I$.

The system (4.1) is converted to state space models

$$
\begin{gather*}
x(t+1)=\Phi x(t)+\Gamma w(t) \\
y_{i}(t)=H_{i} x(t)+v_{i}(t), \quad i=1,2,3  \tag{4.2}\\
s(t)=H x(t)+c_{0} w(t)
\end{gather*}
$$

where we define

$$
\begin{gather*}
c v_{i}(t)=H_{0 i} c_{0} w(t)+\xi_{i}(t), \\
\Phi=\left[\begin{array}{ll}
-a_{1} & 1 \\
-a_{2} & 0
\end{array}\right], \quad \Gamma=\left[\begin{array}{c}
c_{1}-a_{1} c_{0} \\
-a_{2} c_{0}
\end{array}\right], \quad H=[1,0], \quad H_{i}=H_{0 i} H . \tag{4.3}
\end{gather*}
$$

Table 2: Comparison of error variance for the local and fused estimators.

| $t=50$ | $P_{w}^{1}(t \mid t+N)$ | $P_{w}^{2}(t \mid t+N)$ | $P_{w}^{3}(t \mid t+N)$ | $P_{w}(t \mid t+N)$ | $P_{w}^{(I)}(t \mid t+N)$ | $P_{w}^{(I I)}(t \mid$ <br> $t+N)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $N=0$ | 0.28086 | 0.27750 | 0.32376 | 0.14117 | 0.14117 | 0.14117 |
| $N=1$ | 0.24502 | 0.24237 | 0.24057 | 0.13067 | 0.13067 | 0.13067 |
| $N=2$ | 0.23321 | 0.23071 | 0.21924 | 0.12607 | 0.12607 | 0.12607 |
| $N=3$ | 0.23214 | 0.22963 | 0.21913 | 0.12509 | 0.12509 | 0.12509 |

In the simulation, we set

$$
\begin{gather*}
c a_{0}=1, \quad a_{1}=0.2, \quad a_{2}=0.3, \quad c_{0}=1, \quad c_{1}=-0.6, \quad \lambda=0.25, \quad \sigma_{g}^{2}=4 \\
c Q_{\xi_{1}}=0.1, \quad Q_{\xi_{2}}=0.2, \quad Q_{\xi_{2}}=0.4, \quad H_{01}=0.7, \quad H_{02}=1, \quad H_{03}=1.2 \tag{4.4}
\end{gather*}
$$

So

$$
H^{(I)}=\left[\begin{array}{ccc}
0.7 & 1 & 1.2  \tag{4.5}\\
0 & 0 & 0
\end{array}\right]^{T}
$$

It is not full column rank, so Hermite standard model [20] can be used to compute the full-rank decomposition of $H^{(I)}, H^{(I)}=F H^{(I I)}$, where

$$
F=\left[\begin{array}{lll}
0.7 & 1 & 1.2
\end{array}\right]^{T}, \quad H^{(I I)}=\left[\begin{array}{ll}
1 & 0 \tag{4.6}
\end{array}\right] .
$$

The simulation results are presented by Figures 1,2,3 and 4, Tables 1 and 2. In Figures $1,2,3$ and 4 , the $y$-coordinates of endpoints on real lines represent real values $w(t), y$ coordinates of solid round points' centers represent $\widehat{w}(t \mid t+N), N=0,1,2,3, y$-coordinates of hollow round points' centers represent $\widehat{w}^{(I)}(t \mid t+N), N=0,1,2,3$. The $y$-coordinates of hollow squares' centers represent $\widehat{w}^{(I I)}(t \mid t+N), N=0,1,2,3$. It can be seen that the results are completely unanimous.

In Figure 1, it can be seen that at some specified time, the results of these three algorithms are also completely the same. Figure 2 represents the sameness of estimation error variances of the three fusion methods. The fusion accuracy is better than local estimation accuracy of every single sensor.

These results all represent that WMF is completely equivalent to CMF, and the fusion accuracy is higher than local estimation accuracy of each single sensor.

In the other aspect, it can be seen that when the filtering algorithm of CMF is used, the dimension of $y^{(I)}(t)$ is $3 \times 1$, the Kalman filtering needs to compute the inverse matrix of a $3 \times 3$ matrix at every time, while when the WMF Kalman filtering algorithm is used, the dimension of fusion measurement $y^{(I I)}(t)$ is $1 \times 1$, then the matrix inversion computation is converted into scalar division, so the WMF method can obviously reduce the computational burden.

## 5. Conclusions

White noise deconvolution problems have great applications background in oil seismic exploration. Under the condition that the white noise input matrix is full column-rank, the centralized measurement fusion and weighted measurement fusion white noise estimators are presented based on the projection theory, respectively. Their function equivalence has been proved. Furthermore, the proposed weighted measurement fusion white noise estimators can obviously reduce the computational burden.

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Research Article

# Robust Distributed Kalman Filter for Wireless Sensor Networks with Uncertain Communication Channels 

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#### Abstract

We address a state estimation problem over a large-scale sensor network with uncertain communication channel. Consensus protocol is usually used to adapt a large-scale sensor network. However, when certain parts of communication channels are broken down, the accuracy performance is seriously degraded. Specifically, outliers in the channel or temporal disconnection are avoided via proposed method for the practical implementation of the distributed estimation over large-scale sensor networks. We handle this practical challenge by using adaptive channel status estimator and robust L1-norm Kalman filter in design of the processor of the individual sensor node. Then, they are incorporated into the consensus algorithm in order to achieve the robust distributed state estimation. The robust property of the proposed algorithm enables the sensor network to selectively weight sensors of normal conditions so that the filter can be practically useful.


## 1. Introduction

The estimation problem for a multisensor environment has been investigated for two decades [1-5]. Mainly two schemes are discussed to design the system: centralized fusion and distributed fusion.

Centralized fusion is a fusion architecture composed of one fusion center linked with multiple sensors. This architecture does not require a particular fusion rule; instead observations from multiple sensors are stacked as one sensor measurement whose size is very large. It is relatively easy to implement; however, communication bottle neck problems, sensor scheduling, and lack of flexibility are known as disadvantages [1]. Furthermore, if
some communication channels between the central processor and sensor node are uncertain, it is difficult to manage in an adaptive manner.

On the other hand, decentralized fusion has been studied in the literature to tackle disadvantages of the centralized fusion [1]. Initial study of decentralized fusion is known as a decentralized Kalman filter in [2] which is mathematically equivalent to the centralized one. However, its usage is restricted only when sensor nodes are fully connected each other [2]. Namely, only one kind of network topology is allowed (e.g., all-to-all communication).

Based on this observation, the distributed fusion algorithm is suggested to consider issues in the practical implementation of the networked system [6]. For instance, the distributed fusion is required to resolve a network topology, measurement outliers, that is, sensor failure and limited communication bandwidth.

The network topology in multiagent systems has been actively investigated in control community and applied in the sensor network [3]. It was assumed that the network has fixed topology when sensor nodes are geometrically distributed and all communication channels operate in normal.

In parallel, regarding uncertain communication channel, single Kalman filter with intermittent observation gets explosive attentions in the network control system applications $[4,5]$. The main direction of conducted research was the stability analysis of the system under uncertain communication channels [4]. In this case, it is assumed that the communication channel uses TCP-like protocol that means that a packet is dropped based on its acknowledgement. Afterwards it has been extended for multisensor network systems [5]. The authors proposed the estimation algorithm using a sequential Kalman filtering using a set of recent observations collected from adjacent linked nodes. A tree topology is exploited for each node to understand the network topology of limited sensing range.

In another type of applications, however, we may not know the acknowledgement of the packet when the channel link between sensors is not reliable or changed due to the evolution of the network topology. For instance, mobile robots may change their topology based on the agents' location or there exit temporal disconnections in channels. In such cases, the packet arrival event should be estimated.

Previous research in data fusion and intermittent observation problem basically assumes that the noise statistic is known a priori, for example, Gaussian or bounded [4, 5, 7]. However, this assumption is often violated when outlier measurement happens.

The outlier can be originated from several practical challenges such as sensor failures, measurement outliers, or even intentional jamming. To solve this problem robust statistics has been investigated, for example, M-estimator [8]. L1-norm optimization was also considered as a solution because it is common that L1-norm optimization is robust against the outlier noise compared to the L2-norm optimization (e.g., Kalman filter). However, the use of L1norm is overlooked due to the computational complexity. With the help of advances in realtime convex optimization, L1-norm optimization is recently revisited and its application is rigorously investigated [9].

In this paper, we consider the estimation problem under multisensor environment with uncertain communication channels. Main tasks to solve this problem are (1) estimation of the channel status, (2) robust estimation to avoid outliers and uncertainty in communication channels, and (3) measurement fusion algorithm regarding to (1) and (2). The proposed work has a two-stage framework, that is, channel status estimation and L1-norm optimization-based outlier rejection. Note that to the author's knowledge there is no intensive research in the model-based state estimation problem (e.g., Kalman filtering) considering communication channel uncertainty over a large-scale sensor network.

The remainder of the paper is organized as follows. Section 2 describes the problem formulation and the Kalman-consensus filter is introduced as a basic framework. Proposed algorithm is subsequently explained in Section 3 including channel status estimation, sparse optimization via L1-norm optimization, and the modified robust Kalman-consensus filter as an overall algorithm. A test example is provided to demonstrate the efficacy of the proposed algorithm in Section 4. Then, conclusion is made in Section 5.

## 2. Problem Statement and Kalman-Consensus Filter

### 2.1. Problem Statement

Consider a time-invariant linear system with Jump Markov measurement model as

$$
\begin{gather*}
x_{t+1}=A x_{t}+w_{t} \\
y_{t}^{i}=C\left(r_{t}^{i}\right) x_{t}+v_{t}^{i}+z_{t}^{i}, \quad i=1, \ldots, M \tag{2.1}
\end{gather*}
$$

where $x_{t} \in \mathfrak{R}^{n}$ is the state (to be estimated) and $y_{t}^{i} \in \mathfrak{R}^{m}$ is the $i$ th measurement of a node in the network at time $t$, respectively. $A \in \mathfrak{R}^{n \times n}$ is the system matrix, and $C\left(\gamma_{t}^{i}\right) \in \mathfrak{R}^{m \times n}$ is the $i$ th measurement matrix of the node governed by a random latent variable $\gamma_{t}^{i}$ which describes the measurement mode $\gamma_{t}^{i}=1$ for packet received; $\gamma_{t}^{i}=0$, otherwise. $w_{t} \in \mathfrak{R}^{n}, v_{t}^{i} \in \mathfrak{R}^{m}$ and $z_{t}^{i} \in \mathfrak{R}^{m}$; are system noise, measurement noise, and sparse noise in the measurement, respectively. The sparse noise term $z_{t}^{i}$ models the outlier measurements whose magnitude is considerably large compared to the observation noise; thus it cannot be modeled as the standard Gaussian distribution. The process noise $w_{t}$ is independent identically distributed (i.i.d.) $N(0, Q)$ and the measurement noise $v_{t}^{i}$ is i.i.d. $N\left(0, R^{i}\right)$. Assume that the initial state $x_{0}$, the process noise, and the measurement noise are mutually uncorrelated each other.

Then, the main goal is to estimate the state $x_{0}$ given measurements from $M$ sensors up to time $t$, that is, $Y_{t}=\left\{y_{1: t}^{1}, \ldots, y_{1: t}^{M}\right\}$ where $y_{1: t}^{i}=\left\{y_{1}^{i}, \ldots, y_{t}^{i}\right\}$.

### 2.2. Kalman-Consensus Filter

In a large-scale sensor network, it is practically impossible that all the sensor nodes are fully connected each other. Therefore, there should be data fusion algorithm to adaptively aggregate sensor nodes into a globally reasonable estimate. In this study, we adopt a data fusion algorithm using a consensus protocol combined with Kalman filters of each node, called Kalman-consensus filter (KCF) [3]. Using a simple average consensus, individual decentralized Kalman filter called micro-Kalman filter communicates information with its neighbors and the state estimate. The flow of the information over the whole network is possible due to the graph Laplacian of the network topology. As illustrated in [3], even though the target state is partially observed with different groups of sensors and there is no fusion center, individual nodes agree with the converged estimate of the state.

Compared to other data fusion algorithms, KCF has advantages when error-cross covariance information is not available for pairs of sensor node. In addition, because the sensor network topology is incorporated in the data fusion algorithm, local information is propagated all over the network.

In KCF framework, the communication topology between sensor nodes is represented by the directed graph $G=(V, E)$, where $V=\{1,2, \ldots, M\}$ denotes the sensor node set. The edge set $E \subset V \times V$ describes the communication links between each pair of nodes. The neighbor of sensor node $i$ is defined as $L_{i}: J_{i}=L_{i} \cup\{i\}$. KCF is implemented based on the information from of Kalman filter for each node and the consensus protocol to approach the global estimate as follows.

Assume that the latent variable $\gamma_{t}^{i}$ is known. The estimation of the latent variable will be given in following section. Then, contribution terms of the information Kalman filter are calculated for each node as follows:

$$
\begin{gather*}
u_{t}^{i}=\left(C\left(\gamma_{t}^{i}\right)\right)^{T}\left(R^{i}\right)^{-1} y_{t}^{i} \\
U_{t}^{i}=\left(C\left(\gamma_{t}^{i}\right)\right)^{T}\left(R^{i}\right)^{-1} C\left(\gamma_{t}^{i}\right) . \tag{2.2}
\end{gather*}
$$

Based on the known communication topology G, each node broadcasts its message $m_{t}^{i}=$ $\left(u_{t}^{i}, U_{t}^{i}, \bar{x}_{t}^{i}\right)$, where $\bar{x}_{t}^{i}$ is the priori state estimate of the sensor node $i$ and collects message $m_{t}^{r}=\left(u_{t}^{r}, U_{t}^{r}, \bar{x}_{t}^{r}\right)$ from its neighbors. Then, all the contribution terms are aggregated as $g_{t}^{i}=$ $\sum_{r \in J_{i}} u_{t}^{r}, S_{t}^{i}=\sum_{r \in J_{i}} U_{t}^{r}$.

With the aggregated contribution terms, each node calculates Kalman-consensus estimate using update step and prediction step as follows.

Update:

$$
\begin{gather*}
M_{t}^{i}=\left(\left(P_{t}^{i}\right)^{-1}+S_{t}^{i}\right)^{-1}, \\
\widehat{x}_{t+1}^{i}=\bar{x}_{t}^{i}+M_{t}^{i}\left(g_{t}^{i}-S_{t}^{i} \bar{x}_{t}^{i}\right)+\varepsilon \frac{M_{t}^{i}}{1+\left\|M_{t}^{i}\right\|} \sum_{r \in J_{i}}\left(\bar{x}_{t}^{r}-\bar{x}_{t}^{i}\right) . \tag{2.3}
\end{gather*}
$$

Prediction:

$$
\begin{gather*}
P_{t+1}^{i} \longleftarrow A_{t} M_{t}^{i} A_{t}^{T}+Q, \\
\bar{x}_{t+1}^{i} \longleftarrow A_{t} \widehat{x}_{t+1}^{i}, \tag{2.4}
\end{gather*}
$$

where $\varepsilon$ is the discretization step size and $\|\cdot\|$ denotes the matrix norm.
KCF is easy to implement compared to other data fusion algorithms and scalable for large-scale networks [3]. However, it is assumed that the communication channel links are operating in normal which is often not the case in practical situations. When the topology has been changed, channel links are disconnected, or corrupted with outlier measurements, the performance of the consensus algorithm is not reliable anymore.

In this technical note, we propose the robust multisensor consensus estimator to avoid uncertainties in channel of the sensor network. We require three strategies to solve this problem: (1) channel status estimation, that is, mode estimation, (2) outlier rejection, and (3) data fusion. In the following section, we propose the robust Kalman-Consensus Filter based on the channel mode estimation, sparse optimization using L1-norm.

## 3. Main Results

### 3.1. Robust Kalman Filtering Using L1-Norm Optimization

Kalman filter is known as an optimal estimator in linear Gaussian system. However, when the linear Gaussian assumption is violated in practice, additional modification is necessary. Because the outlier measurement is not able to be modeled as Gaussian, we model the outlier measurement as $z_{t}^{i}$ which is sparse. To alleviate the additional sparse noise, we adopt the robust Kalman filtering via sparse optimization using L1-norm in [9].

The standard Kalman filtering without the sparse noise is given as follows.
Prediction:

$$
\begin{equation*}
\widehat{x}_{t \mid t-1}^{i}=A \widehat{x}_{t-1 \mid t-1}^{i} . \tag{3.1}
\end{equation*}
$$

Update:

$$
\begin{equation*}
\widehat{x}_{t \mid t}^{i}=\widehat{x}_{t \mid t-1}^{i}+\Phi_{t \mid t}^{i}\left(C^{i}\right)^{T}\left(C^{i} \Phi_{t \mid t}^{i}\left(C^{i}\right)^{T}+R^{i}\right)^{-1}\left(y_{t}^{i}-C^{i} \widehat{x}_{t \mid t-1}^{i}\right), \tag{3.2}
\end{equation*}
$$

where $\Phi_{t \mid t}^{i}$ is the state error covariance. Here, we assume that the channel mode is completely known, that is, $C\left(\gamma_{t}^{i}\right) \triangleq C_{t}^{i}$. In the least square problem, Kalman filter is to minimize the cost function defined as

$$
\begin{equation*}
\left(v_{t}^{i}\right)^{T}\left(R^{i}\right)^{-1} v_{t}^{i}+\left(x_{t}-\widehat{x}_{t \mid t-1}^{i}\right)^{T}\left(\Phi_{t \mid t}^{i}\right)^{-1}\left(x_{t}-\widehat{x}_{t \mid t-1}^{i}\right) \tag{3.3}
\end{equation*}
$$

subject to $y_{t}^{i}=C_{t}^{i} x_{t}+v_{t}^{i}$. The cost function of (3.1) is modified by adding regularization term considering the sparse noise $z_{t}^{i}$ as

$$
\begin{equation*}
\left(v_{t}^{i}\right)^{T}\left(R^{i}\right)^{-1} v_{t}^{i}+\left(x_{t}-\widehat{x}_{t \mid t-1}^{i}\right)^{T}\left(\Phi_{t \mid t}^{i}\right)^{-1}\left(x_{t}-\widehat{x}_{t \mid t-1}^{i}\right)+\lambda\left\|z_{t}^{i}\right\|_{1} \tag{3.4}
\end{equation*}
$$

subject to $y_{t}^{i}=C_{t}^{i} x_{t}+v_{t}^{i}+z_{t}^{i}$. The minimization problem is solved using convex optimization. $\lambda$ is the regularization parameter.

The cost function of (3.4) is rewritten using residual $e_{t}^{i}=y_{t}^{i}-C_{t}^{i} \widehat{x}_{t \mid t-1}^{i}$ as

$$
\begin{equation*}
J=\left(e_{t}^{i}-z_{t}^{i}\right)^{T} W\left(e_{t}^{i}-z_{t}^{i}\right)+\lambda\left\|z_{t}^{i}\right\|_{1} \tag{3.5}
\end{equation*}
$$

where $W=\left(I-C^{i} K_{t}^{i}\right)^{T}\left(R^{i}\right)^{-1}\left(I-C^{i} K_{t}^{i}\right)+\left(K_{t}^{i}\right)^{T}\left(\Phi_{t \mid t}^{i}\right)^{-1} K_{t}^{i} I$ is the identity matrix of appropriate dimension, and $K_{t}^{i}=\Phi_{t \mid t}^{i}\left(C_{t}^{i}\right)^{T}\left(C_{t}^{i} \Phi_{t \mid t}^{i}\left(C_{t}^{i}\right)^{T}+R^{i}\right)^{-1}$ is the Kalman gain. We solve the minimization of the cost function defined in (3.5) for the sparse noise as follows:

$$
\begin{equation*}
\widehat{z}_{t}^{i}=\underset{z_{t}^{i}}{\arg \min } J \tag{3.6}
\end{equation*}
$$

Then the Kalman filter estimate is represented by

$$
\begin{equation*}
\widehat{x}_{t \mid t}^{i}=\widehat{x}_{t \mid t-1}^{i}+K_{t}^{i}\left(e_{t}^{i}-\widehat{z}_{t}^{i}\right) . \tag{3.7}
\end{equation*}
$$

Unlike the implementation in [9], our goal is to estimate the sparse noises and reject them from the measurements for the channel mode estimation in the next stage.

### 3.2. Channel Mode Estimation

Motivated by the work [10, 11], we try to estimate the channel mode $\gamma_{t}$ using moving horizon strategy. We denote the channel mode for the sensor node $i$ as $\gamma_{t}^{i}$ which is the discrete random variable. Assume that the evolution of the channel mode has the first-order Markov chain, its transition probability $\pi_{1,1}$ is the probability that the packet will arrive between time steps, and conversely $\pi_{0,1}$ represents the probability that the channel is switched off between time steps. Then, Bayesian update of the channel mode probability, that is, $\operatorname{Pr}\left(\gamma_{s}^{i} \mid y_{s-1}^{i}\right)$ in the moving horizon $[t-\Delta, t]$ is provided as follows.

For each measurement mode, $l=1 \rightarrow \gamma_{t}^{i}=1$, and $l=0 \rightarrow r_{t}^{i}=0$.
Prediction:

$$
\begin{equation*}
\operatorname{Pr}\left(r_{s}^{i} \mid y_{s-1}^{i}\right)=\pi_{1,1} \operatorname{Pr}\left(\gamma_{s-1}^{i} \mid y_{s-1}^{i}\right)+\pi_{0,1}\left(1-\operatorname{Pr}\left(\gamma_{s-1}^{i} \mid y_{s-1}^{i}\right)\right) \tag{3.8}
\end{equation*}
$$

Update:

$$
\begin{equation*}
\operatorname{Pr}\left(r_{s}^{i} \mid y_{s}^{i}\right)=\frac{\Lambda_{s}^{i} \operatorname{Pr}\left(\gamma_{s}^{i} \mid y_{s-1}^{i}\right)}{1-\left(1-\Lambda_{s}^{i}\right) \operatorname{Pr}\left(\gamma_{s}^{i} \mid y_{s-1}^{i}\right)}, \quad s=t-\Delta, \ldots, t \tag{3.9}
\end{equation*}
$$

where the measurement likelihood of sensor node $i$ at time $s$ is defined as $\Lambda_{s}^{i} \triangleq y_{s}^{i}-C\left(\gamma_{s}^{i}\right) \hat{x}_{s \mid s-1}^{i}$ and $\hat{x}_{s \mid s-1}^{i}$ is the predicted state of local Kalman filter given in (3.1). Note that the recursion given in (3.8)-(3.9) is iterated in moving horizon $[t-\Delta, t]$ to obtain the channel mode estimate as $\widehat{\gamma}_{t}^{i}=1$, if $\operatorname{Pr}\left(\gamma_{t}^{i} \mid y_{t}^{i}\right)>$ Threshold $_{1}$, or $\widehat{\gamma}_{t}^{i}=0$, if $\operatorname{Pr}\left(\gamma_{t}^{i} \mid y_{t}^{i}\right)<$ Threshold $_{0}$. In the channel mode probability calculation, we assume that the channel modes is not switched to other mode again at least within $\alpha$ steps. It is similar to the mode observability assumption given in [10]. Compared to the given assumption in [10], our assumption is not strict because we are not trying to distinguish the sequence of the mode in the horizon but to obtain the stable estimate of the current mode.

### 3.3. Overall Algorithm

In previous subsections we have discussed about robust Kalman filtering via L1-norm optimization and the channel mode estimation based on the channel mode probability. In this subsection, we combine two methods and suggest a robust data fusion algorithm to construct the overall implementation of our algorithm.

The overall flow of the proposed algorithm is displayed in Figure 1. The dynamic state process is observed from multisensors. To efficiently reject the sparse measurement outliers,


Figure 1: Overall flow of the proposed algorithm.

L1-norm optimization is subsequently utilized. After trimming multiple measurements by rejection of outliers, we estimate the channel mode of each sensor node and finally fuse the set of estimate and measurement based on the consensus protocol which is explained in (2.2)-(2.4). According to the overall flow of the proposed algorithm described in Figure 1, we summarize the robust distributed fusion consensus filter in Algorithm 1.

Remark 3.1. Consider that error convergence of the algorithm L1-norm optimization and measurement mode estimation are main concerns. When we modified Kalman filtering update step with L1-norm optimization then, it is not Gaussian estimate anymore. So, it is not straightforward to readily analyze the error convergence in modified Kalman filtering. Therefore, it is remained as a future work for ours.

Remark 3.2. Considering the measurement mode observability, we follow the idea similar to [10] that there is a minimum dwell time of the measurement mode switching. Thus, we set the horizon window size $\Delta$ as the minimum dwell time of the measurement mode switching. In practice, this value is the design parameter for the network. However, the value of the delta is not that sensitive to the minimum dwell time of the channel, that is, tuning of the delta is not that sensitive. In addition, in cases where frequent switching happens, we regard it as outlier measurement and it will be handled via robust Kalman filtering step in L1-norm optimization.

Remark 3.3. In the experiment, we set the horizon size as 5 when the switching probability is $\pi_{1,0}=\pi_{0,1}=0.05$. In our experiment, if the mode is switched within $\tau$ step which is less than the predefined horizon size $\Delta$, then the performance of the channel mode estimator is not reliable. Thus, as already explained in Remark 3.1, frequent switching would be considered as permanent channel link break down. However, rather fast but not abnormally frequent switching can be handled via switching Kalman filters (e.g. interacting multiple model filter (IMM filter)).

## 4. Illustrative Example

In this section, we test the efficacy of the proposed algorithm with the state estimation problem using a large-scale sensor network.

Given the target dynamics of a circular movement [3]

$$
\begin{equation*}
x_{t+1}=A x_{t}+B w_{t} \tag{4.1}
\end{equation*}
$$

Given $\sum_{t \mid t-1}^{i}$, error covariance, $\widehat{x}_{t \mid t-1}^{i}$, previous estimate, consensus update parameter $\varepsilon$, and the window size $\Delta$.

1. Obtain measurement $y_{t}^{i}=C\left(\gamma_{t}^{i}\right) x_{t}+v_{t}^{i}+z_{t}^{i}, i=1, \ldots, N$.
2. For each measurement solve L1-norm optimization problem, reject outliers as given in (3.5) and then obtain the trimmed measurements: $\widehat{y}_{t}^{i}=y_{t}^{i}-\widehat{z}_{t}^{i}$.
3. Calculate the mode probability $\operatorname{Pr}\left(\gamma_{t}^{i} \mid \hat{y}_{t-\Delta: t}^{i}\right)$. Given $\operatorname{Pr}\left(\gamma_{t-\Delta}^{i} \mid \hat{y}_{t-\Delta}^{i}\right)$, For $s=t-\Delta: t$

Evaluate measurement likelihood for $\widehat{y}_{s}^{i}$.
Evaluate the Bayesian recursion (3.8)-(3.9).
End
Decide the channel mode $\widehat{\gamma}_{t}^{i}$ using threshold testing.
4. Compute contribution term of information state and matrix such that

$$
\begin{gathered}
u_{t}^{i}=\left(C_{t}^{i}\left(\hat{\gamma}_{t}^{i}\right)\right)^{T}\left(R^{i}\right)^{-1} \widehat{y}_{t}^{i} \\
U_{t}^{i}=\left(C_{t}^{i}\left(\widehat{\gamma}_{t}^{i}\right)\right)^{T}\left(R^{i}\right)^{-1} C_{t}^{i}\left(\hat{\gamma}_{t}^{i}\right) .
\end{gathered}
$$

5. Broadcast message $m_{t}^{i}=\left(u_{t}^{i}, U_{t}^{i}, \widehat{x}_{t \mid t-1}^{i}\right)$ to neighbors in $L_{i}$.
6. Collect messages $m_{t}^{r}=\left(u_{t}^{r}, U_{t}^{r}, \hat{x}_{t \mid t-1}^{r}\right)$ from neighbors.
7. Aggregate the information states and matrices of neighbors including node $i: J_{i}=L_{i} \cup\{i\}$ :

$$
g_{t}^{i}=\sum_{r \in J_{i}} u_{t}^{r}, \quad S_{t}^{i}=\sum_{r \in J_{i}} U_{t}^{r}
$$

8. Compute the Kalman-Consensus estimate:

$$
\begin{gathered}
\left(M_{t}^{i}\right)^{-1}=\left(\Phi_{t \mid t-1}^{i}\right)^{-1}+S_{t}^{i} \\
\widehat{x}_{t|t|}^{i}=\widehat{x}_{t \mid t-1}^{i}+M_{t}^{i}\left(g_{t}^{i}-S_{t}^{i} \hat{x}_{t \mid t-1}^{i}\right)+\varepsilon \frac{M_{t}^{i}}{1+\left\|M_{t}^{i}\right\| \|_{r \in J_{i}}\left(\widehat{x}_{t \mid t-1}^{r}-\widehat{x}_{t \mid t-1}^{i}\right) .} \\
\text { Prediction stage } \\
\Phi_{t+1 \mid t}^{i} \longleftarrow A M_{t}^{i} A^{T}+Q, \\
\widehat{x}_{t+1 \mid t}^{i} \longleftarrow A \widehat{x}_{t \mid t}^{i} .
\end{gathered}
$$

Algorithm 1: Robust distributed fusion algorithm for node $i$.
where $A_{0}=2\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right], B_{0}=5^{2} I_{2}, A=I_{2}+\varepsilon A_{0}+\left(\varepsilon^{2} / 2\right) A_{0}^{2}+\left(\varepsilon^{2} / 6\right) A_{0}^{3}$, and $B=\varepsilon B_{0}$. In addition, $I_{2}$ is a $2 \times 2$ identity matrix which is a discretized model with a step size $\varepsilon=0.015$, and the initial position and uncertainty are $x_{0}=(15,-10)^{T}$ and $P_{0}=10 I_{2}$, respectively. A moving target having a circular motion can then be observed via the large-scale sensor network of 100 sensor nodes as displayed in Figure 2. Here, the sensor nodes measure the target position with uncertain communication channel links between nodes as

$$
\begin{equation*}
y_{t}^{i}=C\left(r_{t}^{i}\right) x_{t}+v_{t}^{i}, \quad t=0,1, \ldots, i=1, \ldots, 100 \tag{4.2}
\end{equation*}
$$

where either

$$
C\left(r_{t}^{i}\right)=\left\{\begin{array}{ll}
{\left[\begin{array}{ll}
1 & 0
\end{array}\right],} & \text { if } r_{t}^{i}=1,  \tag{4.3}\\
{\left[\begin{array}{ll}
0 & 0
\end{array}\right],} & \text { if } r_{t}^{i}=0,
\end{array} \quad \text { or } \quad C\left(r_{t}^{i}\right)= \begin{cases}{\left[\begin{array}{ll}
0 & 1
\end{array}\right],} & \text { if } r_{t}^{i}=1 \\
{\left[\begin{array}{ll}
0 & 0
\end{array}\right],} & \text { if } r_{t}^{i}=0\end{cases}\right.
$$



Figure 2: A large-scale sensor network with 100 nodes.

In the observation model, individual sensor measures either $x$-position or $y$-position. For each sensor node we model the channel mode latent variable $\gamma_{t}^{i}$ that describes the channel condition. To simulate the true observation mode for each node, the mode switch in the communication channel link is modeled as $\operatorname{Pr}\left(\gamma_{t}^{i}=0\right) \equiv \operatorname{Pr}(u(0,1)<0.01)$, where $u(0,1)$ is a uniform random distribution. We also model the evolution of channel mode variable as the first-order Markov chain. In this case, the transition probability between modes is given a priori. The observation noise for each sensor is white Gaussian noise with $v_{t}^{i} \sim N\left(0,30^{2} \sqrt{i}\right)$. In addition, sparse noises are generated with the probability 0.05 , whose magnitude is 10 times larger than that of the measurement noise.

### 4.1. Comparison with KCF

In the experiment we compare our proposed algorithm with standard KCF. Figure 3 simply and clearly demonstrates that our algorithm is robust when there are practical challenges in the network.

To show more clearly the robustness against the outliers, we select one sensor node experiment. That is because in KCF framework, certain amount of uncertainty can be aggregated via consensus update. The comparison of estimated trajectory with the ground truth is given in Figure 4. Measurements are also displayed with outliers to show that the proposed estimation in sensor node considerably improved mean square error (MSE) as illustrated in Figure 5.

### 4.2. Comparison with Switching Kalman Filter

As mentioned in Remark 3.3, rather fast switching of the channel mode can be handled more accurately via the IMM filter that is known as switching Kalman filter [12]. From our experiments, the proposed method (i.e., observation mode estimation via moving horizon strategy) is more accurate when the actual switching of the channel occurs in more than $\Delta$ steps. It means that the moving horizon strategy guarantees us the stable estimate of the observation mode when the minimum dwell time assumption is held as described in Section 3.2. On the other hand, the IMM filter shows us slightly increased errors in this case


Figure 3: Comparison of estimated trajectories ((a) ground truth with KCF, (b) truth with robust KCF).


Figure 4: Comparison of estimated trajectories with measurements for single node (KF: micro KF; L1: proposed, obs: measurements).
because the IMM does not determine the exact mode as 1 or 0 . Instead, the mode probability is calculated and utilized for weighted averaging. However, the IMM filter is more robust in cases where mode switching frequently occurs. That is because there is no minimum dwell time assumption in the IMM filter.

In terms of computational complexity, the IMM filter is implemented using two parallel Kalman filters for each observation mode, that is, the complexity is approximately $O\left(2 n^{2}|E|+n^{2} N\right)$, where $n$ is the dimension of the state, $N$ is the number of nodes, and $|E|$ is the number of edge (e.g., links) in the network, as already mentioned in Section 2. In contrast,


Figure 5: Comparison of MSE.
the proposed algorithm requires a recursion for mode estimation within a horizon window; thus, the complexity is approximated as $O\left(n \Delta|E|+n^{2} N\right)$. Therefore, the proposed algorithm is less complex than the IMM algorithm. Note that the complexity of the Kalman filter is $O\left(n^{2}\right)$.

## 5. Conclusion

In this paper we propose a novel distributed data fusion algorithm that is robust against outlier measurements and channel uncertainty. Outliers are rejected from the L1-norm optimization algorithm and the channel uncertainty is reduced using the measurement mode estimation algorithm. For the implementation in a large-scale sensor network, we adopt the KCF framework and test the framework with an object state estimation problem. Results successfully demonstrate that the proposed framework is able to handle practical challenges.

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Research Article

# Multiresolution Analysis for Stochastic Finite Element Problems with Wavelet-Based Karhunen-Loève Expansion 

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#### Abstract

Multiresolution analysis for problems involving random parameter fields is considered. The random field is discretized by a Karhunen-Loève expansion. The eigenfunctions involved in this representation are computed by a wavelet expansion. The wavelet expansion allows to control the spatial resolution of the problem. Fine and coarse scales are defined, and the fine scales are taken into account by projection operators. The influence of the truncation level for the wavelet expansion on the computed reliability is documented.


## 1. Introduction

Reliability analysis for problems involving random parameter fields is concerned with the solution of stochastic elliptic boundary value problems. For the solution of stochastic boundary value problems, the random parameter field has to be discretized. This can be accomplished either by a Karhunen-Loève expansion or a projection on a polynomial basis [1, 2]. As a result of this discretization, the random parameter field can be approximated by an expression containing only a finite number of random variables. The reliability problem reduces then to the computation of failure probabilities with respect to a finite number of random variables.

After introduction of the discretization, the stochastic boundary value problem becomes equivalent to a deterministic one, for which the approximation on the physical domain, and on the stochastic domain can be treated differently. The deterministic boundary
value problem can be solved by standard approximation methods. On the physical domain, finite element (FE) approximations are prevalent. Galerkin projection methods or collocation schemes on the stochastic domain give rise to stochastic Galerkin methods [1] and stochastic collocation schemes [3-5]. Their theoretical foundation has been laid in [6-9], where local and global polynomial chaos expansions were investigated and where a priori error estimates have been proved for a fixed number of terms of the Karhunen-Loève expansion. Approximate solutions of the stochastic boundary value problem can be viewed as local stochastic response surfaces [10] that depend on three parameters: a discretization parameter for the physical domain, a discretization parameter for the stochastic domain, and the discretization level of the input random field.

Although the focus of many investigations concerning the approximate solution of stochastic boundary value problems lies on the computation of the first- and second-order moments of the solution, already, in [1], reliability computation techniques were described, that are based on series representations of the response distribution, on the reliability index, or on Monte Carlo simulation methods. In most of the papers that followed, for example, $[11,12]$ and references therein, either the reliability index, or simulation methods has been employed. Alternatively, once the random field is discretized, sampling-based response surfaces $[13,14]$ can be applied as well. Except for the stochastic Galerkin method, all methods provide nonintrusive algorithms, which allow combining the solution procedure with repetitive runs of an FE solver for deterministic problems.

In this paper, the focus is on the discretization of the random field itself. The discretization of the random field may enforce a rather fine FE mesh and thus increase the computational effort. The Karhunen-Loève expansion of the random field requires the solution of an eigenvalue problem for the determination of the expansion functions. Recently, a wavelet expansion of the eigenfunctions has been introduced [15]. The advantage of wavelet bases resides in the fact that they are of localized compact support, which lead to sparse representations of functions and integral operators. Moreover, the discrete wavelet transform provides an efficient means to solve the integral equation related to the determination of the eigenfunctions. Wavelet expansions have also been studied in the context of simulation of random fields $[16,17]$ and for the solution of stochastic dynamic systems by polynomial chaos expansion [18].

Here, the wavelet representation of the eigenfunctions is introduced into the stochastic FE approximation procedure in order to control the spatial resolution of the eigenfunctions. This leads to a multiresolution approximation scheme, where finer scales are taken into account by projection operators. The truncation level of the wavelet expansion has been identified as an additional parameter of the metamodel. As the multiresolution analysis is based on the eigenfunctions involved in the Karhunen-Loève expansion of the random field, it can be combined with stochastic Galerkin, stochastic collocation, or sampling-based methods. In this paper, only the latter are considered in the examples.

The paper is organized as follows: the next section gives an introduction to multiresolution analysis. Section 3 discusses random field discretization by Karhunen-Loève expansion with emphasis on wavelet approximations of the eigenfunctions. Section 4 provides basic information on the approximate solution of the stochastic boundary value problem. Section 5 explains the proposed multiresolution approximation, and Section 6 gives some details on reliability assessment. In Section 7, the proposed procedure is validated by examples and relative errors for the failure probability are given. Finally, conclusions are drawn in Section 8.

## 2. Multiresolution Analysis

A multiresolution analysis is a sequence of subspaces $V_{j} \subset L^{2}(\mathbb{R}),\{0\} \subset \cdots \subset V_{2} \subset V_{1} \subset V_{0} \subset$ $V_{-1} \subset V_{-2} \subset \cdots \subset L^{2}(\mathbb{R})$, with

$$
\begin{equation*}
\overline{\bigcup_{j \in \mathbb{Z}} V_{j}}=L^{2}(\mathbb{R}), \quad \bigcap_{j \in \mathbb{Z}} V_{j}=\{0\}, \quad f(\cdot) \in V_{j} \Longleftrightarrow f\left(2^{j} \cdot\right) \in V_{0} \tag{2.1}
\end{equation*}
$$

generated by a scaling function $\varphi \in L^{2}(\mathbb{R})$ via $\varphi_{j, k}(x)=2^{-j / 2} \varphi\left(2^{-j} x-k\right)$ and $V_{j}=$ $\overline{\operatorname{span}\left\{\varphi_{j, k} \mid k \in \mathbb{Z}\right\}}$. Especially, from $\varphi \in V_{0} \subset V_{-1}$, there is a sequence $\left\{h_{k}\right\}_{k \in \mathbb{Z}}$ of real numbers, such that

$$
\begin{equation*}
\varphi(x)=\sqrt{2} \sum_{k \in \mathbb{Z}} h_{k} \varphi(2 x-k) . \tag{2.2}
\end{equation*}
$$

For every $j \in \mathbb{Z}$, denote with $W_{j}$ the orthogonal complement of $V_{j}$ in $V_{j-1}: V_{j-1}=V_{j} \oplus W_{j}$. It follows that $V_{m}=\oplus_{j \geq m+1} W_{j}, L^{2}(\mathbb{R})=\oplus_{j \in \mathbb{Z}} W_{j}$ and that $f(\cdot) \in W_{j} \Leftrightarrow f\left(2^{j} \cdot\right) \in W_{0}$.

A wavelet is a function $\psi \in L^{2}(\mathbb{R})$ with

$$
\begin{equation*}
0<\int_{\mathbb{R}} \frac{|\widehat{\psi}(\omega)|^{2}}{|\omega|} \mathrm{d} \omega<\infty, \tag{2.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\widehat{\psi}(\omega)=\lim _{n \rightarrow \infty}(2 \pi)^{-1 / 2} \int_{-n}^{n} \psi(x) \exp (-i x \omega) \mathrm{d} x . \tag{2.4}
\end{equation*}
$$

It can be shown [19] that there exists a wavelet $\psi(x)$, such that $\psi_{j, k}(x)=2^{-j / 2} \psi\left(2^{-j} x-k\right)$ is an orthonormal basis for $W_{j}$. Thus, any function $f \in L^{2}(\mathbb{R})$ can be decomposed into

$$
\begin{equation*}
f=P_{m} f+\sum_{j=-\infty}^{m} \sum_{k \in \mathbb{Z}}\left\langle f, \psi_{j, k}\right\rangle \psi_{j, k}, \tag{2.5}
\end{equation*}
$$

where $P_{m}$ is the projection operator on $V_{m}$. As we are interested in functions with compact support, the second sum in (2.5) is finite. For $m=0$, we have after truncation at level $-v$

$$
\begin{equation*}
f(x)=\sum_{k \in \mathbb{Z}}\left\langle f, \varphi_{0, k}\right\rangle \varphi_{0, k}(x)+\sum_{j=-v}^{0} \sum_{k \in \mathbb{Z}}\left\langle f, \psi_{j, k}\right\rangle \psi_{j, k}+e(x), \tag{2.6}
\end{equation*}
$$

where the error $e(x)$ is an element of $W_{-(v+1)}$. Note that all sums are finite, if $f(x)$ has compact support.

The same representation can be obtained for functions $L^{2}\left(\mathbb{R}^{d}\right)$ with $d \in \mathbb{N}$. For example, for $d=2$, the functions

$$
\begin{equation*}
\phi_{j, k_{1}, k_{2}}(x, y)=2^{-j} \varphi\left(2^{-j} x-k_{1}\right) \varphi\left(2^{-j} y-k_{2}\right), \quad k_{1}, k_{2} \in \mathbb{Z}, \tag{2.7}
\end{equation*}
$$

constitute an orthonormal basis of $V_{j}$ and the functions

$$
\begin{array}{ll}
\psi_{j, k_{1}, k_{2}}^{h}(x, y)=2^{-j} \psi\left(2^{-j} x-k_{1}\right) \varphi\left(2^{-j} y-k_{2}\right), & k_{1}, k_{2} \in \mathbb{Z}, \\
\psi_{j, k_{1}, k_{2}}^{v}(x, y)=2^{-j} \varphi\left(2^{-j} x-k_{1}\right) \psi\left(2^{-j} y-k_{2}\right), & k_{1}, k_{2} \in \mathbb{Z},  \tag{2.8}\\
\psi_{j, k_{1}, k_{2}}^{d}(x, y)=2^{-j} \psi\left(2^{-j} x-k_{1}\right) \psi\left(2^{-j} y-k_{2}\right), & k_{1}, k_{2} \in \mathbb{Z},
\end{array}
$$

where superscript $h$ stands for horizontal, $v$ for vertical, and $d$ for diagonal translation of the unidimensional wavelet $\psi(x)$, constitute an orthonormal basis of $W_{j}$.

## 3. Karhunen-Loève Discretization of Random Fields

Let $D$ be a convex bounded open set in $\mathbb{R}^{n}$ and $(\Omega, \mathcal{F}, P)$ a complete probability space, where $\Omega$ is the set of outcomes, $\mathcal{F}$ the $\sigma$-field of events, and $P: \mathscr{F} \rightarrow[0: 1]$ a probability measure.

We consider a random field $\alpha: D \times \Omega \rightarrow \mathbb{R}$ that has a continuous and squareintegrable covariance function

$$
\begin{equation*}
C(x, y)=\int_{\Omega}(\alpha(x, \omega)-E[\alpha](x))(\alpha(y, \omega)-E[\alpha](y)) \mathrm{d} P(\omega), \tag{3.1}
\end{equation*}
$$

where the expectation operator $E[\alpha](x)=\int_{\Omega} \alpha(x, \omega) \mathrm{d} P(\omega)$ denotes the mean value of the random field. It is assumed that $\alpha(x, \omega)$ is bounded and coercive, that is, there exist positive constants $a_{\text {min }}, a_{\text {max }}$, such that

$$
\begin{equation*}
P\left(\omega \in \Omega: a_{\min }<\alpha(x, \omega)<a_{\max } \forall x \in D\right)=1 . \tag{3.2}
\end{equation*}
$$

Due to the properties of the covariance function, the operator $T: L^{2}(D) \rightarrow L^{2}(D)$,

$$
\begin{equation*}
T w=\int_{D} C(x, y) w(x) \mathrm{d} x \tag{3.3}
\end{equation*}
$$

is compact and self-adjoint and thus admits a spectrum of decreasing nonnegative eigenvalues $\left\{\lambda_{i}\right\}_{i=1}^{\infty}$,

$$
\begin{equation*}
\int_{D} C(x, y) w_{i}(x) \mathrm{d} x=\lambda_{i} w_{i}(y) \tag{3.4}
\end{equation*}
$$

The corresponding eigenfunctions $\left\{w_{i}(x)\right\}_{i=1}^{\infty}$ are orthonormal in $L^{2}(D)$. The random variables given by

$$
\begin{equation*}
\xi_{i}(\omega)=\frac{1}{\sqrt{\lambda_{i}}} \int_{D}(\alpha(x, \omega)-E[\alpha](x)) w_{i}(x) \mathrm{d} x \tag{3.5}
\end{equation*}
$$

are uncorrelated (but in general not independent), have zero mean and unit variance, and allow to represent the random field by the Karhunen-Loève expansion

$$
\begin{equation*}
\alpha(x, \omega)=E[\alpha](x)+\sum_{i=1}^{\infty} \sqrt{\lambda_{i}} \xi_{i}(\omega) w_{i}(x) \tag{3.6}
\end{equation*}
$$

that converges in $L^{2}(D \times \Omega)$ [20]. Conditions for stronger convergence properties are given in [8]. The Karhunen-Loève expansion is usually truncated by retaining only the first $M$ terms. In order to keep the computational effort small, a fast decay of the spectrum of (3.3) is important. It is shown in [21] that fast eigenvalue decay corresponds to smoothness of the covariance function.

The solution of the Fredholm integral equation (3.4) can be computed by means of the wavelet basis introduced before (cf., e.g., [15]). For the $k$ th eigenfunction $w_{k}(x)$, an approximation with a finite number of basis functions $\Psi_{i}(x), i=1,2, \ldots, n$, of $V_{0}$ and $W_{j}$, $0 \leq j \leq-v$, is given by

$$
\begin{equation*}
\tilde{w}_{k}(x)=\sum_{i=1}^{n} d_{i}^{(k)} \Psi_{i}(x) \tag{3.7}
\end{equation*}
$$

A Galerkin technique can be applied in order to compute the eigenvalue $\lambda_{k}$ and the coefficients of the normalized $\left(\int_{D} \tilde{w}_{k}^{2}(x) d x=1\right)$ approximate eigenfunctions. To this end, the representation (3.7) is inserted into the Fredholm integral equation (3.4) and the equation is multiplied with a basis function $\Psi_{j}(y)$. After integration over the domain $D$, one obtains the algebraic eigenvalue problem

$$
\begin{equation*}
\sum_{i=1}^{n}\left(\iint_{D} C(x, y) \Psi_{i}(x) \Psi_{j}(y) \mathrm{d} x \mathrm{~d} y d_{i}^{(k)}\right)=\lambda_{k} d_{j}^{(k)}, \quad j=1,2, \ldots n \tag{3.8}
\end{equation*}
$$

where the coefficients on the left hand side can be obtained from discrete wavelet transforms of the covariance function $C(x, y)$.

More details and convergence studies for this approximation of the eigenfunctions may be found in [15]. Here, the decomposition is introduced in order to obtain a coarse approximation (by taking scales until a certain level $-\mu>-v$ into account) and a fine approximation (scales from $-\mu-1$ to $-v$ ).

## 4. Stochastic Linear Elliptic Boundary Value Problems

Consider the following model problem with stochastic operator and deterministic input function on $D \times \Omega$ : find $u: D \times \Omega \rightarrow \mathbb{R}$, such that $P$ almost surely:

$$
\begin{equation*}
-\nabla \cdot \alpha(x, \omega) \nabla u(x, \omega)=g(x) \quad \text { on } D, \quad u(x, \omega)=0 \quad \text { on } \partial D . \tag{4.1}
\end{equation*}
$$

It is assumed that the deterministic input function $g(x)$ is square-integrable.

We are interested in the probability that a functional $F(u)$ of the solution $u(x, \omega)$ exceeds a threshold $F_{0}$, that is, we want to evaluate the integral

$$
\begin{equation*}
P_{F}=\int_{\Omega} x_{\left(F_{0}, \infty\right)}(F(u(x, \omega))) \mathrm{d} P(\omega), \tag{4.2}
\end{equation*}
$$

where $X_{I}(\cdot)$, the indicator function, assumes the value 1 in the interval $I$ and vanishes elsewhere.

The variational formulation of the stochastic boundary value problem necessitates the introduction of the Sobolev space $H_{0}^{1}(D)$ of functions having generalized derivatives in $L^{2}(D)$ and vanishing on the boundary $\partial D$ with norm $\|u\|_{H_{0}^{1}(D)}=\left(\int_{D}|\nabla u|^{2} \mathrm{~d} x\right)^{1 / 2}$, the space $L_{P}^{2}(\Omega)$ of square integrable random variables and the tensor product space $H_{0}^{1}(D) \otimes L_{p}^{2}(\Omega)$ of $H_{0}^{1}(D)$ valued random fields with finite second-order moments, equipped with the inner product

$$
\begin{equation*}
(u, v)_{H_{0}^{1}(D) \otimes L^{2}(\Omega)}=\int_{\Omega} \int_{D} \nabla u(x, \omega) \cdot \nabla v(x, \omega) \mathrm{d} x \mathrm{~d} P(\omega) \tag{4.3}
\end{equation*}
$$

The variational formulation of the stochastic linear elliptic boundary value problem (4.1) then reads find $u \in H_{0}^{1}(D) \otimes L_{P}^{2}(\Omega)$, such that for all $v \in H_{0}^{1}(D) \otimes L_{P}^{2}(\Omega)$ :

$$
\begin{equation*}
\int_{\Omega} \int_{D} \alpha(x, \omega) \nabla u \cdot \nabla v \mathrm{~d} x \mathrm{~d} P(\omega)=\int_{\Omega} \int_{D} g(x) v(x, \omega) \mathrm{d} x \mathrm{~d} P(\omega) \tag{4.4}
\end{equation*}
$$

The assumptions on the random field $\alpha(x, \omega)$ guarantee the continuity and coercivity of the bilinear form in (4.4). Existence and uniqueness of a solution to (4.4) follow from the LaxMilgram lemma.

For a prescribed, uniformly bounded random field $\alpha(x, \omega)$, the random variables $\xi_{i}(\omega)$ in (3.6) would be dependent non-Gaussian random variables whose joint distribution function is very difficult to identify. If, on the other hand, independent but bounded distributions are prescribed for $\xi_{i}(\omega), i=1,2, \ldots, M$, the random field $\alpha(x, \omega)$ is not necessarily bounded for $M \rightarrow \infty$. Thus, one is left with Gaussian distributions for $\xi_{i}(\omega)$ and $\alpha(x, \omega)$, with transformations of Gaussian random fields [22] or with some situations, where nonnegative distributions for $\xi_{i}(\omega)$ lead to meaningful (e.g., Erlang-) distributions for $\alpha(x, \omega)$.

Some authors [11, 23] consider models that contain a finite number of random variables as parameters of a boundary value problem. For this kind of problems, Babuška et al. [8] investigate convergence properties of several approximation schemes. The examples studied in this paper belong to this class of problems. It is henceforth assumed that the random field $\alpha(x, \omega)$ is given by a truncated sum

$$
\begin{equation*}
\alpha_{M}(x, \omega)=E[\alpha](x)+\sum_{i=1}^{M} \sqrt{\lambda_{i}} \xi_{i}(\omega) w_{i}(x) \tag{4.5}
\end{equation*}
$$

where $\xi_{i}(\omega)$ are continuous and independent random variables with zero mean and unit variance, where $\Gamma_{i}=\xi_{i}(\Omega)$ are bounded intervals in $\mathbb{R}$ and the parameters $\lambda_{i}$ and functions
$w_{i}(x)$ are the first $M$ eigenvalues and corresponding eigenfunctions of the operator (3.3). In the examples, $\xi_{i}(\omega)$ are Gaussian random variables truncated at $\pm 3 \sigma, \sigma=1$.

The representation of $\alpha_{M}(x, \omega)$ by a finite number of random variables allows to consider a deterministic auxiliary problem:

$$
\begin{equation*}
-\nabla \cdot\left(E[\alpha](x)+\sum_{i=1}^{M} \sqrt{\lambda_{i}} y_{i} w_{i}(x)\right) \nabla u(x, y)=g(x), \quad \text { for }(x, y) \in \mathrm{D} \times \Gamma \tag{4.6}
\end{equation*}
$$

where $\Gamma=\prod_{i=1}^{M} \Gamma_{i} \subset \mathbb{R}^{M}$. From the expression for $u(x, y)$, one obtains $u(x, \omega)$ by replacing the vector $y$ with the vector of random variables $\xi_{i}(\omega), i=1,2, \ldots, M$.

This equation is discretized on finite dimensional approximation spaces for the physical domain. For $H_{0}^{1}(D)$, a family of standard FE approximation spaces $X_{h} \subset H_{0}^{1}(D)$ of continuous piecewise linear functions in a regular triangulation $\tau_{h}$ of $D$ with mesh parameter $h$ is considered.

Denote with $N_{i}(x), i=1,2, \ldots, N$, a basis of $X_{h} \subset H_{0}^{1}(D)$. The solution $u(x, y)$ is approximated by

$$
\begin{equation*}
u(x, y)=\sum_{i=1}^{N} u_{i}(y) N_{i}(x) \tag{4.7}
\end{equation*}
$$

For a fixed value $y$, the unknown coefficients $u_{i}(y), i=1,2, \ldots, N$, can be computed from the solution of the FE problem

$$
\begin{equation*}
\left(\mathbf{K}^{(0)}+\sum_{s=1}^{M} \mathbf{K}^{(s)} y_{s}\right) \mathbf{u}(y)=\mathbf{g} \tag{4.8}
\end{equation*}
$$

where the matrices $\mathbf{K}^{(0)}, \mathbf{K}^{(s)}, s=1,2, \ldots, M$ and the vector $\mathbf{g}$ are given by

$$
\begin{gather*}
K_{i j}^{(0)}=\int_{D} E[\alpha](x) \nabla N_{i}(x) \cdot \nabla N_{j}(x) \mathrm{d} x,  \tag{4.9}\\
K_{i j}^{(s)}=\sqrt{\lambda_{s}} \int_{D} w_{s}(x) \nabla N_{i}(x) \cdot \nabla N_{j}(x) \mathrm{d} x  \tag{4.10}\\
g_{i}=\int_{D} g(x) \mathrm{N}_{i}(x) \mathrm{d} x, \quad i, j=1,2, \ldots, N, \tag{4.11}
\end{gather*}
$$

and $\mathbf{u}(y)$ is the vector containing the nodal displacements $u_{i}(y), i=1,2, \ldots, N$, for the fixed value $y$. The matrices $\mathbf{K}^{(s)}, s=1,2, \ldots, M$, can be interpreted, for example, as FE stiffness matrices for a spatial variation of elastic properties.

## 5. Multiresolution Approximation

By means of multiresolution analysis, each of the matrices $\mathbf{K}^{(s)}$ in (4.10) can be decomposed as $\mathbf{K}^{(s)}=\mathbf{K}_{c}^{(s)}+\mathbf{K}_{f}^{(s)}$, where $\mathbf{K}_{c}^{(s)}$ contains the low-frequency content, expressed by the
multiresolution scheme until the level $-\mu$, while $\mathbf{K}_{f}^{(s)}$ takes the high frequency content of the levels $-\mu-1$ until $-v$ into account.

Given a fixed value for $y$, the decomposition of $\mathbf{K}^{(s)}$ leads to a decomposition of the global stiffness matrix:

$$
\begin{equation*}
\left(\mathbf{K}_{c}+\mathbf{K}_{f}\right) \mathbf{u}=\mathbf{g}, \tag{5.1}
\end{equation*}
$$

where $\mathbf{K}_{c}=\mathbf{K}^{0}+\sum_{s=1}^{M} \mathbf{K}_{c}^{(s)}$ and $\mathbf{K}_{f}=\sum_{s=1}^{M} \mathbf{K}_{f}^{(s)}$. Also the solution $\mathbf{u}$ is decomposed into $\mathbf{u}_{c}$ and $\mathbf{u}_{f}$, where $\mathbf{u}_{c}$ is represented on a coarse mesh and $\mathbf{u}_{f}$ on a fine mesh. With the help of an interpolation matrix $\mathbf{P}$, the solution $\mathbf{u}$ is written as $\mathbf{u}=\mathbf{P} \mathbf{u}_{c}+\mathbf{u}_{f}$. Inserting these decompositions into equation (5.1) leads to

$$
\begin{equation*}
\mathbf{K}_{c} \mathbf{P} \mathbf{u}_{c}+\mathbf{K}_{c} \mathbf{u}_{f}+\mathbf{K}_{f} \mathbf{P} \mathbf{u}_{c}+\mathbf{K}_{f} \mathbf{u}_{f}=\mathbf{g} . \tag{5.2}
\end{equation*}
$$

This equation is split into

$$
\begin{gather*}
\mathbf{K}_{c} \mathbf{P} \mathbf{u}_{c}=\mathbf{g}  \tag{5.3}\\
\mathbf{K}_{c} \mathbf{u}_{f}+\mathbf{K}_{f} \mathbf{P} \mathbf{u}_{c}+\mathbf{K}_{f} \mathbf{u}_{f}=\mathbf{0} \tag{5.4}
\end{gather*}
$$

Multiplication of the first equation with $\mathbf{P}^{T}$ projects this equation onto the coarse mesh:

$$
\begin{equation*}
\mathbf{P}^{T} \mathbf{K}_{c} \mathbf{P} \mathbf{u}_{c}=\mathbf{P}^{T} \mathbf{g} \tag{5.5}
\end{equation*}
$$

For $\mathbf{u}_{f}$, a coarse mesh approximation is computed in the following manner:
(i) the term $\mathbf{K}_{f} \mathbf{u}_{f}$ in (5.4) is neglected;
(ii) $\mathbf{u}_{f}$ is represented on the coarse mesh by $\tilde{\mathbf{u}}_{f}$, thus $\mathbf{u}_{f}=\mathbf{P} \tilde{\mathbf{u}}_{f}$.

After multiplication of (5.4) with $\mathbf{P}^{T}$, one then obtains

$$
\begin{equation*}
\mathbf{P}^{T} \mathbf{K}_{c} \mathbf{P} \tilde{\mathbf{u}}_{f}=-\mathbf{P}^{T} \mathbf{K}_{f} \mathbf{P} \mathbf{u}_{c} \tag{5.6}
\end{equation*}
$$

for the coarse mesh approximation $\tilde{\mathbf{u}}_{f}$ of $\mathbf{u}_{f}$. Thus, the linear system of equations in (5.5) and (5.6) has to be resolved only on the reduced set of degrees of freedom given by the coarse mesh, which saves a considerable amount of CPU time. The interpolation matrix $\mathbf{P}$ can be generated from FE interpolation functions.

## 6. Reliability Assessment

Once the algebraic problems are solved and the correction to the coarse scale solution is computed, an approximation for $u(x, \omega)$ has been obtained. The approximation quality depends on the following parameters:
(i) partition of the physical domain and the stochastic domain,
(ii) truncation of the Karhunen-Loève expansion $(M)$,
(iii) wavelet scales $(\mu, v)$.

The wavelet scales $\mu, \nu$ appear as an additional parameter that influences the approximation quality.

For solving reliability problems, (4.8) yields a functional relationship between the input random variables and $u(x, y)$. It is then possible to compute the most probable point of failure (MPP), that is, the point $\xi \in \Gamma$ with $F(u(x, \xi))=F_{0}$ with lowest Euclidean norm. The norm of the MPP may serve as a control variable for the approximation quality and for the adaptation of the parameters mentioned above.

The MPP may also be useful for the evaluation of the integral in (4.2) by means of variance reduced Monte Carlo simulation (importance sampling). To this end, a sampling density $\tilde{p}(y)$ is introduced by shifting the original probability density function $p(y)$ of the random variables $\xi_{i}(\omega), i=1, \ldots, M$, to the previously obtained MPP, and (4.2) is approximated by

$$
\begin{equation*}
P_{F} \approx \sum_{j=1}^{N_{s}} X_{\left(F_{0}, \infty\right)}\left(F\left(u\left(x, y^{j}\right)\right)\right) \frac{p\left(y^{j}\right)}{\tilde{p}\left(y^{j}\right)} \tilde{p}\left(y^{j}\right) \tag{6.1}
\end{equation*}
$$

where the sampling points $y^{j}, j=1,2, \ldots, N_{s}$, are generated according to $\tilde{p}(y)$.

## 7. Examples

### 7.1. Example 1: Clamped Square Plate

The first example deals with a standard problem for stochastic FE techniques, a clamped thin square plate under uniform in-plane tension $q$ (cf. [1]). The problem is depicted in Figure 1. The plate has unit length $l$. The product of Young's modulus and the thickness of the plate are assumed to be an isotropic normal random field with covariance function

$$
\begin{equation*}
C\left(x_{1}, y_{1} ; x_{2}, y_{2}\right)=\sigma^{2} \exp \left(-\frac{\left|x_{1}-x_{2}\right|}{l_{x}}-\frac{\left|y_{1}-y_{2}\right|}{l_{y}}\right) \tag{7.1}
\end{equation*}
$$

with standard deviation $\sigma=0.2$ and unit mean value. Poisson's ratio is set to 0.3 .
A first investigation is concerned with the efficiency of the multiresolution analysis. To this end, relative errors for the maximum longitudinal displacement have been computed by comparing the coarse mesh-coarse level $(\mu=-1)$ solution and the coarse solution corrected by the fine level solution $(v=-4)$ to the fine mesh-fine level reference solution (i.e., $\mu=0, v=-4$ ). Dividing the relative error of the coarse scale solution by the relative


Figure 1: Thin square plate under uniform in-plane tension.


Figure 2: Efficiency of the multiresolution scheme with respect to the number of Karhunen-Loève expansion terms.
error of the corrected solution yields an improvement factor that indicates the efficiency of the multiresolution correction. For a coarse mesh of $2 \times 2$ quadrilateral elements only and a fine mesh of $4 \times 4$ quadrilateral elements, Figures 2 and 3 display the improvement factors for a varying number of Karhunen-Loève expansion terms and for a variation of the correlation length $l_{c}=l_{x}=l_{y}$, respectively. Here and in the following, the fine mesh is generated from the coarse mesh by halving the edge length of the elements. For Figure 2, the correlation length was $l_{c}=0.5$, and, for Figure 3, 12 Karhunen-Loève expansion terms have been retained. The figures reveal that the multiresolution scheme is more efficient if fluctuations become more important, that is for a large number of Karhunen-Loève expansion terms and low correlation lengths.

Next, the influence of the multiresolution correction on the prediction of failure has been investigated. A threshold value of 1.5 is assumed for the maximum longitudinal displacement, and failure occurs if the maximum longitudinal displacement exceeds this threshold value.

The random field has been discretized by a Karhunen-Loève expansion with $M=4$ terms for a correlation length $l_{c}=1$. The eigenfunctions were computed either on a coarse ( $8 \times 8$ elements) or a fine ( $16 \times 16$ elements) mesh. A reference solution MPP has been computed on the fine mesh including up to seven scales for the wavelet expansion of the eigenfunctions, and a reference result for the failure probability has been obtained on the fine mesh from importance sampling with 30000 samples. These results allow to evaluate the


Figure 3: Efficiency of the multiresolution scheme with respect to the correlation length.


Figure 4: Relative error of the norm for the MPP.
error due to the multiresolution scheme. To this end, relative errors for the Euclidean norm of the MPP and the failure probability with respect to the reference results have been computed from the multiresolution scheme with one coarse level $(\mu=-1)$ and up to seven fine levels. Figure 4 presents the development of the relative errors for the norm of the design point.

Figure 5 displays the relative error of the failure probability. From the figure, an exponential decrease of the error with respect to the number of fine scales involved can be deduced. The coarse mesh solution needed less than half of the CPU time of the fine scale solution.

### 7.2. Example 2: Soil-Structure Interaction

The second example, previously introduced in [24], considers a soil structure interaction problem. In contrast to the first example, it deals with an anisotropic autocorrelation function of the random field and an anisotropic FE mesh. The settlement of a foundation (width $2 B=10 \mathrm{~m}$ ), represented by a uniform pressure of 0.2 MPa , on an elastic soil layer of thickness


Figure 5: Relative error of the failure probability.


Figure 6: Soil structure interaction problem.
$t=30 \mathrm{~m}$ lying on a rigid substratum, is investigated. A plane strain deformation of the soil is assumed with linear elastic material properties (cf. Figure 6). The Young's modulus is assumed to be a random field with mean value $50 \mathrm{MPa}, 20 \%$ coefficient of variation, and an exponential correlation function as in (7.1) but with correlation length $l_{x}=250 \mathrm{~m}$, $l_{y}=100 \mathrm{~m}$. Poisson's ratio is set to 0.3. Six truncated Gaussian random variables $(M=6)$ for the representation of the random field have been considered.

Starting from the center of the foundation, the right half of the soil layer is meshed by quadrilateral finite elements until a length of $L=60 \mathrm{~m}$, as indicated in Figure 6. The coarse mesh consists of 24 elements, while the fine mesh is divided into 96 elements. The reliability problem deals with the displacement of point A in Figure 6, situated at the center of the foundation and on the surface of the soil, where the vertical displacements of the corresponding deterministic problem (i.e., considering only the mean value of Young's modulus) attain their maximum. Failure is defined as exceedance of a limit value of 0.1 m for the vertical displacement at point A. Figures 7 and 8 summarize the behavior of the relative errors for the Euclidean norm of the MPP and for the failure probability, respectively. As in the previous example, $\mu$ has been set to -1 and $v$ has been varied. It can be seen that, while the error in the norm is relatively small, rather large errors for the probability of failure may occur. As in the previous example, the relative error for the failure probability decreases


Figure 7: Relative error of the norm for the MPP.


Figure 8: Relative error of the failure probability.
exponentially with the wavelet level $-v$; however, this decrease starts only after taking a certain number of fine scales (4 and more) into account.

## 8. Conclusions

In this paper, wavelet-based Karhunen-Loève expansion has been extended to multiresolution analysis of stochastic FE problems. The proposed procedure allows to include fine scale corrections of the FE solution by means of coarse mesh computations. Improvement due to these corrections has been highlighted by examples, and it could be seen that the relative error of the failure probability decreases exponentially with the wavelet level.

Although the procedure has been combined in this paper to importance samplingbased computation of failure probabilities, it applies to stochastic Galerkin and stochastic collocation methods as well, because it is solely based on the decomposition of the KarhunenLoève eigenfunctions. However, it relies on linear problems.

Typical applications for multiresolution analysis in the context of stochastic FE methods arise in multiscale analysis of material properties, especially if there is scale coupling
and length scales that correspond to the length scales of the representative volume element have to be taken into account. For this kind of application, the limitation of wavelets to rather simple geometries, as those in the examples, will not be a drawback, since simple geometries are frequently used on the micro- and mesoscale and more complex geometries could be embedded into simple ones for the solution of the Fredholm integral equation.

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Research Article

# Robust Reliable $H_{\infty}$ Control for Nonlinear Stochastic Markovian Jump Systems 

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#### Abstract

The robust reliable $H_{\infty}$ control problem for a class of nonlinear stochastic Markovian jump systems (NSMJSs) is investigated. The system under consideration includes Itô-type stochastic disturbance, Markovian jumps, as well as sector-bounded nonlinearities and norm-bounded stochastic nonlinearities. Our aim is to design a controller such that, for possible actuator failures, the closed-loop stochastic Markovian jump system is exponential mean-square stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$. Based on the Lyapunov stability theory and Itô differential rule, together with LMIs techniques, a sufficient condition for stochastic systems is first established in Lemma 3. Then, using the lemma, the sufficient conditions of the solvability of the robust reliable $H_{\infty}$ controller for linear SMJSs and NSMJSs are given. Finally, a numerical example is exploited to show the usefulness of the derived results.


## 1. Introduction

In the past few decades, Markovian jump systems (MJSs) have been considerably studied since this kind of hybrid systems consists of a number of subsystems and a switch signal, which includes applications in safety-critical and high-integrity systems (e.g., aircraft, chemical plants, nuclear power station, robotic manipulator systems, large-scale flexible structures for space stations such as antenna, and solar arrays) typically systems, which may experience abrupt changes in their structure, see, for example, [1] and the references therein. And now, some results of stability and stabilization for Itô type stochastic Markovian jump systems are also available in many papers, see, for example, [2-4] and the references therein.

The analysis and synthesis problems of Markovian jump systems (MJSs) or stochastic Markovian jump systems (SMJSs) have attracted plenty of attention from many researchers. Many important and remarkable achievements reasonable have obtained. If the control
systems possess integrity against actuator and sensor failures, we called reliable control systems or fault-tolerant control systems [5]. Recently, the robust reliable control and filtering problems for time-delay systems or Markovian jump systems (MJSs) have attracted considerable attention, and several approaches have been developed, see, for example, [611] and the references therein. Via linear matrix inequalities (LMIs), the authors designed the robust reliable $H_{\infty}$ controller for uncertain nonlinear systems [6]. In [7], for admissible uncertainties as well as actuator failures occurring among a prespecified subset of actuators, Zhang et al. studied the reliable dissipative control of Markovian jump impulsive systems. The reliable $H_{\infty}$ control problem for discrete-time piecewise linear systems with infinite distributed delays have been investigated in [8]. Recently, the study of stochastic $H_{\infty}$ filtering for the systems governed by stochastic Itô-type equations has attracted a great deal of attention, and Zhang and Chen [9] firstly solved the nonlinear stochastic delayfree $H_{\infty}$ filtering problem by means of a stochastic bounded real lemma derived in [10]. The reliable $H_{\infty}$ filtering problems for discrete time-delay systems with randomly occurred nonlinearities [11] and discrete time-delay Markovian jump systems with partly unknown transition probabilities [12] also has been studied, respectively. The reliable control problem for a class of Markovian jump systems with interval time-varying delays and stochastic failure is studied in [13]. In recent years, the research begins to focusing on robust reliable control problems for stochastic systems or stochastic switched nonlinear systems, see, for example, $[14-16]$ and the references therein.

However, all the aforementioned results are mainly focusing on the reliable control and filtering problems of discrete-time-delay systems and Markovian jump systems. Up to now, to the best of the authors' knowledge, the robust reliable $H_{\infty}$ control problem for nonlinear stochastic Markovian jump systems (NSMJSs) has not been fully investigated, which is an open problem and gives the motivation of our present investigation. In this paper, our aim is to design a robust reliable $H_{\infty}$ controller for NSMJSs, such that the NSMJSs are globally mean exponential stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$.

### 1.1. Notations

Throughout this paper, for symmetric matrices $X$ and $Y$, the notation $X \geq Y$ (resp., $X>Y$ ) means that the Matrix $X-Y$ is positive semidefinite (respectively, positive definite). $I$ is an identity matrix with appropriate dimensions; the subscript " $T$ " represents the Transposition. $E(\cdot)$ denotes the expectation operator with respect to some probability measure $P . \ell_{2}[0, \infty)$ is the space of square integrable vector functions over $[0, \infty)$; let $(\Omega, \mathcal{F}, P)$ be a complete probability space which is relative to an increasing family $\left(\mathcal{F}_{t}\right)_{t>0}$ of $\sigma$ algebras $\left(\mathcal{F}_{t}\right)_{t>0} \subset \mathcal{F}$, where $\Omega$ is the samples space, $\mathcal{F}$ is $\sigma$ algebra of subsets of the sample space, and $P$ is the probability measure on $\mathcal{F} .\|\cdot\|_{E_{2}}=\|E(\cdot)\|_{2}$, while $\|\cdot\|_{2}$ stands for the usual $\mathcal{L}_{2}[0, \infty)$ norm, $R^{n}$ and $R^{n \times m}$ denote the $n$ dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. In this paper, we provide all spaces $\mathbb{K}^{k}, k \geq 1$ with the usual inner product $\langle\cdot, \cdot\rangle$ and its corresponding 2-norm $\|\cdot\|$. Let $L^{2}\left(\Omega, \mathbb{K}^{k}\right)$ denote the space of square-integrable $\mathbb{K}^{k}$-valued functions on the probability space $(\Omega, \mathcal{F}, P)$. For any $0<T<\infty$, we write $[0, T]$ for the closure of the open interval $(0, T)$ in $R$ and denote by $L_{2}^{n}\left([0, T] ; L^{2}\left(\Omega, \mathbb{K}^{k}\right)\right)$ the space of the nonanticipative stochastic processes $y(\cdot)=(y(\cdot))_{t \in[0, T]}$ with respect to $\left(\mathcal{F}_{t}\right)_{t \in[0, T]}$ satisfying $\|y(\cdot)\|_{L_{2}^{n}}^{2}=E\left(\int_{0}^{T}\|y(t)\|^{2} \mathrm{~d} t\right)=\int_{0}^{T} E\left(\|y(t)\|^{2}\right) \mathrm{d} t<\infty . V(x(t), t, r(t)=i)=V(x(t), t, i)$, $A(r(t)=i)=A_{i} B(r(t)=i)=B_{i}, A_{0}(r(t)=i)=A_{0 i}, B_{0}(r(t)=i)=B_{0 i}, C(r(t)=i)=$ $C_{i}, D(r(t)=i)=D_{i}$.

## 2. Problem Formulation and Failure Model

In this paper, we mainly consider the following nonlinear stochastic Markovian jump systems (NSMJSs) with actuator failures:

$$
\begin{align*}
\mathrm{d} x(t)= & {\left[A(r(t)) x(t)+B(r(t)) u^{f}(t, r(t))+E(r(t)) v(t)+f(r(t), x(t))\right] \mathrm{d} t } \\
& +\left[C(r(t)) x(t)+D(r(t)) u^{f}(t, r(t))+H(r(t)) v(t)+g(r(t), x(t))\right] \mathrm{d} w(t),  \tag{2.1}\\
z(t)= & J(r(t)) x(t), \\
x\left(t_{0}\right)= & x_{0},
\end{align*}
$$

where $x(t) \in R^{n}$ is the system state, $u^{f}(t) \in R^{l}$ is the control input of actuator fault, $v(t) \in R^{q}$ is the exogenous disturbance input of the systems which belong to $\Omega_{2}[0, \infty)$, $z(t) \in R^{r}$ is the system control output, $w(t)$ is a zero mean real scalar Weiner processes on a probability space $(\Omega, \mathcal{F}, P)$ relative to an increase family $\left(\mathcal{F}_{t}\right)_{t>0}$ of $\sigma$ algebras $\left(\mathcal{F}_{t}\right)_{t>0} \subset \mathcal{F}$. $A_{i}, B_{i}, E_{i}, C_{i}, D_{i}, F_{i}, H_{i}, J_{i}$ are the known real constant matrices with appropriate dimensions. Morever, we assume that

$$
\begin{equation*}
E(\mathrm{~d} w(t))=0, \quad E\left((\mathrm{~d} w(t))^{2}\right)=\mathrm{d} t \tag{2.2}
\end{equation*}
$$

Let $r(t), t \geq 0$, be a right-continuous Markovian chain on the probability space taking values in a finite state space $S=1,2, \ldots, N$ with generator $\Gamma=\left(\lambda_{i j}\right)_{N \times N}$ given by

$$
P\{r(t+\Delta)=j \mid r(t)=i\}= \begin{cases}\lambda_{i j} \Delta+o(\Delta) & \text { if } i \neq j  \tag{2.3}\\ 1+\lambda_{i i} \Delta+o(\Delta) & \text { if } i=j\end{cases}
$$

where $\Delta>0$. Here $\lambda_{i j} \geq 0$ is the transition rate from manner $i$ to manner $j$, if $i \neq j$ while $\lambda_{i i}=-\sum_{j \neq i} \lambda_{i j}$. We assume that the Markovian chain $r(\cdot)$ is independent of the Wienner process $w(\cdot)$. It is well known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jump in any finite subinterval of $R_{+}(:=[0,+\infty))$.
$f(\cdot, \cdot): S \times R^{n} \rightarrow R^{n}$ is a unknown nonlinear function which describes the system nonlinearity satisfying the following sector-bounded conditions:

$$
\begin{equation*}
\left(f_{i}(x(t))-T_{1 i} x\right)^{T}\left(f_{i}(x(t))-T_{2 i} x\right) \leq 0, \quad i \in S \tag{2.4}
\end{equation*}
$$

$g(\cdot, \cdot): S \times R^{n} \rightarrow R^{n}$ also is a unknown nonlinear function which describes the stochastic nonlinearity satisfying the following:

$$
\begin{equation*}
g_{i}^{T}(x(t)) g_{i}(x(t)) \leq x^{T} G_{i}^{T} G_{i} x, \quad i \in S, \tag{2.5}
\end{equation*}
$$

where $T_{1 i}, T_{2 i}, G_{i}$ are known real constant matrices with approximate dimensions.

Remark 2.1. The nonlinearities $f_{i}(x(t))$ are bounded by sectors, which belong to [ $L_{1 i}, L_{2 i}$ ], and are very general that include the usual Lipschitz conditions as a special case which is considerable investigated and includes several other classes well studied nonlinear systems [17-19]. The nonlinearities $g_{i}(x(t))$ satisfy the norm-bounded conditions.

When the actuator experiences failure, we use $u^{f}(t, r(t))$ to describe the control signal form actuators. Consider the following actuator failure model with failure parameter $F_{i}$ :

$$
\begin{equation*}
u_{i}^{f}(t)=F_{i} u_{i}(t), \tag{2.6}
\end{equation*}
$$

where $F_{i}$ is the actuator fault matrix with

$$
\begin{equation*}
F_{i}=\operatorname{diag}\left(f_{i 1}, f_{i 2}, \ldots, f_{i m}\right), \quad 0 \leq \underline{f}_{i j} \leq f_{i j} \leq \bar{f}_{i j}, \bar{f}_{i j} \geq 1, j=1,2, \ldots, m \tag{2.7}
\end{equation*}
$$

In which the variables $f_{i j}$ quantify the failures of the actuators. $f_{i j}=0$ means that $j$ th actuator completely fails, and $f_{i j}=1$ means that the $j$ th actuator is normal.

Define the following:

$$
\begin{align*}
& F_{0 i}=\operatorname{diag}\left(f_{0 i 1}, f_{0 i 2}, \ldots, f_{0 i m}\right)=\frac{\bar{F}_{i}+\underline{F}_{i}}{2}, \quad f_{0 i j}=\frac{\underline{f}_{i j}+\bar{f}_{i j}}{2}  \tag{2.8}\\
& \tilde{F}_{0 i}=\operatorname{diag}\left(\tilde{f}_{0 i 1}, \tilde{f}_{0 i 2}, \ldots, \tilde{f}_{0 i m}\right)=\frac{\bar{F}_{i}-\underline{F}_{i}}{2}, \quad f_{0 i j}=\frac{\bar{f}_{i j}-\underline{f}_{i j}}{2} \tag{2.9}
\end{align*}
$$

and hence, the matrix $F_{i}$ can be rewritten as

$$
\begin{equation*}
F_{i}=F_{0 i}+\Delta_{i}=F_{0 i}+\operatorname{diag}\left(\varphi_{i 1}, \varphi_{i 2}, \ldots, \varphi_{i m}\right), \quad\left|\varphi_{i j}\right| \leq \tilde{f}_{i j}, \quad j=1,2, \ldots, m \tag{2.10}
\end{equation*}
$$

In this paper, our aim is to design the controller $u_{i}(t)=K_{i} x(t), i \in S$, such that the closed-loop systems satisfy the following conditions:
(i) without the exogenous disturbance input (i.e., $v(t)=0$ ), the closed-loop control systems (2.1) are globally exponentially stable with convergence rate $\alpha>0$;
(ii) with zero initial condition (i.e., $x\left(t_{0}\right)=0$ ) and nonzero exogenous disturbance input (i.e., $v(t) \neq 0$ ), the following inequality holds:

$$
\begin{equation*}
\|z\|_{E_{2}}<\gamma\|v\|_{2}\left(\text { i.e., } \int_{0}^{T} z^{T}(t) z(t) \mathrm{d} t \leq r^{2} \int_{0}^{T} v^{T}(t) v(t) \mathrm{d} t\right) . \tag{2.11}
\end{equation*}
$$

If the above two conditions hold, we also called the systems that are exponential meansquare stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$.

## 3. Main Results

Lemma 3.1 (Schur complement lemma [20]). For a given matrix $S=\left(\begin{array}{c}S_{1} \\ * \\ *\end{array} S_{2}\right)$ with $S_{1}^{T}=$ $S_{1}, S_{2}^{T}=S_{2}$, the following conditions are equivalent:
(1) $S<0$,
(2) $S_{2}<0, S_{1}-S_{3} S_{2}^{-1} S_{3}^{T}<0$,
(3) $S_{1}<0, S_{2}-S_{3} S_{1}^{-1} S_{3}^{T}<0$.

Lemma 3.2 (see [21]). Let $x \in \mathbb{R}^{n}$ and $y \in \mathbb{R}^{n}$. Then, for any positive scalar $\varepsilon$, we have

$$
\begin{equation*}
x^{T} y+y^{T} x \leq \varepsilon x^{T} x+\varepsilon^{-1} y^{T} y \tag{3.1}
\end{equation*}
$$

### 3.1. Robust Reliable $H_{\infty}$ for LSMJSs

To obtain our main results, we first consider the following linear stochastic Markovian jump systems (LSMJSs) without control input:

$$
\begin{gather*}
\mathrm{d} x(t)=\left[A_{i} x(t)+E_{i} v(t)\right] \mathrm{d} t+\left[C_{i} x(t)+H_{i} v(t)\right] \mathrm{d} w(t), \\
z(t)=J_{i} x(t),  \tag{3.2}\\
x\left(t_{0}\right)=x_{0} .
\end{gather*}
$$

Lemma 3.3. Suppose that $P(t, r(t))>0$ is continuously differentiable, then the systems (3.2) are exponential mean-square stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$ if and only if the following matrix functional inequalities hold:

$$
\Xi_{i}(t)=\left(\begin{array}{ccc}
M_{i}(t)+J_{i}^{T} J_{i} & P_{i} E_{i} & C_{i}^{T}  \tag{3.3}\\
* & -r^{2} I & H_{i}^{T} \\
* & * & -P_{i}^{-1}(t)
\end{array}\right)<0, \quad i \in S
$$

where $M_{i}(t)=A_{i}^{T} P_{i}(t)+P_{i}(t) A_{i}+\dot{P}(t)+\sum_{j \in S} \lambda_{i j} P_{j}(t)$.
Proof. At first, let $v(t)=0$, and defining the following Lyapunov function:

$$
\begin{equation*}
V(x(t), t, i)=V(x(t), t, r(t)=i)=x^{T}(t) P(t, r(t)=i) x(t)=x^{T}(t) P_{i}(t) x(t) . \tag{3.4}
\end{equation*}
$$

By Itô formula, we get the following:

$$
\begin{equation*}
\varrho V(x(t), t, i)=x^{T}(t)\left(M_{i}(t)+C_{i}^{T} P_{i}(t) C_{i}\right) x(t) \tag{3.5}
\end{equation*}
$$

the matrix function inequalities (3.3) imply that $£ V(x(t), t, i)<0$, and let $a_{i}=\lambda_{\max }\left(-\Xi_{i}(t)\right)$, $a=\max _{i \in S}\left(a_{i}\right)$, where $\lambda_{\max }(\cdot)$ means the maximum eigenvalue of matrix $(\cdot)$, and we have

$$
\begin{equation*}
\rho V(x(t), t, i) \leq-a x^{T}(t) x(t) \tag{3.6}
\end{equation*}
$$

Hence

$$
\begin{align*}
d\left[e^{\alpha t} V(x(t), t, i)\right] & =\alpha e^{\alpha t} V(x(t), t, i)+e^{\alpha t} d V(x(t), t, i) \\
& \leq(b \alpha-a) e^{\alpha t}\|x(t)\|^{2}+e^{\alpha t} 2 x^{T}(t) P_{i}(t) C_{i} x(t) d w(t) \tag{3.7}
\end{align*}
$$

where $b_{i}=\sup _{t \geq t_{0}}\left\{\lambda_{\max }\left(P_{i}(t)\right)\right\}$, and $b=\max _{i \in S}\left(b_{i}\right)$. Integrating the both sides of above inequality from $t_{0}$ to $T$ and taking expectation, we obtain that

$$
\begin{equation*}
E e^{\alpha T}\left[V(x(T), T, i)-V\left(x_{0}, t_{0}, i\right)\right] \leq(b \alpha-a) E \int_{t_{0}}^{T} e^{\alpha s}\|x(s)\|^{2} \mathrm{~d} s \tag{3.8}
\end{equation*}
$$

Set $\alpha=a / b$, and the following inequality is obtained:

$$
\begin{equation*}
e^{\alpha T} \min _{i \in S} \lambda_{\min }\left(P_{i}(T)\right) E\|x(T)\|^{2} \leq E\left[e^{\alpha T} V(x(T), T, i)\right] \leq E V\left(x_{0}, t_{0}, i\right) \tag{3.9}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
E\|x(T)\|^{2} \leq E V\left(x_{0}, t_{0}, i\right) \frac{1}{\min _{i \in S} \Lambda_{\min }\left(P_{i}(T)\right)} e^{-\alpha T} \tag{3.10}
\end{equation*}
$$

That is to say that the stochastic systems are globally exponentially stable with convergence rate $\alpha>0$.

Then, considering the stochastic $H_{\infty}$ performance level for the resulting systems (3.2) with nonzero exogenous disturbance input $(v(t) \neq 0)$, for any $t>0$, we define that

$$
\begin{equation*}
J(t)=E\left\{\int_{0}^{t}\left[z^{T}(s) z(s)-r^{2} v^{T}(s) v(s)\right] \mathrm{d} s\right\} \tag{3.11}
\end{equation*}
$$

By general Itô formula, we get he following:

$$
\begin{align*}
J(t) & =E\left\{\int_{t_{0}}^{t}\left[z^{T}(s) z(s)-\gamma^{2} v^{T}(s) v(s)+\rho V(x(s), s, i)\right] \mathrm{d} s\right\}-E(V(x(t), t, i)) \\
& \leq E\left\{\int_{0}^{t}\left[z^{T}(s) z(s)-r^{2} v^{T}(s) v(s)+\rho V(x(s), s, i)\right] \mathrm{d} s\right\} \leq E\left\{\int_{0}^{t} \eta^{T}(s) \Omega_{i}(s) \eta(s) \mathrm{d} s\right\} \tag{3.12}
\end{align*}
$$

where $\eta^{T}(t)=\left(x^{T}(t) v^{T}(t)\right), \Omega_{i}(t)=\left(\begin{array}{cc}M_{i}(t)+J_{i}^{T} J_{i} P_{i}(t) E_{i} \\ E_{i}^{T} P_{i}(t) & -r^{2} I\end{array}\right)+\binom{C_{i}^{T}}{H_{i}^{T}} P_{i}(t)\binom{C_{i}^{T}}{H_{i}^{T}}^{T}$ From (3.3) we know that $\Omega(t)<0$, which implies that

$$
\begin{equation*}
J(t)<0 \tag{3.13}
\end{equation*}
$$

Therefore, the inequality $\|z\|_{E_{2}}<\gamma\|v\|_{2}$ holds. The proof is completed.

In the following time, we consider the following linear stochastic Markovian jump systems (LSMJSs) under the state feedback controller:

$$
\begin{gather*}
\mathrm{d} x(t)=\left[\left(A_{i}+B_{i} F_{i} K_{i}\right) x(t)+E_{i} v(t)\right] \mathrm{d} t+\left[\left(C_{i}+D_{i} F_{i} K_{i}\right) x(t)+H_{i} v(t)\right] \mathrm{d} w(t), \\
z(t)=J_{i} x(t),  \tag{3.14}\\
x\left(t_{0}\right)=x_{0} .
\end{gather*}
$$

Theorem 3.4. If there exist the positive matrices $X_{i}>0$, and the constant matrices $Y_{i}$ with approximate dimensions, such that the following LMIs hold

$$
\Theta_{i}=\left(\begin{array}{cccc}
\Theta_{i 1} & E_{i} & \Theta_{i 2} & \Theta_{i 3}  \tag{3.15}\\
* & -r^{2} I & H_{i}^{T} & 0 \\
* & * & -X_{i} & 0 \\
* & * & * & \Theta_{i 4}
\end{array}\right)<0, \quad i \in S
$$

where $\Theta_{i 1}=X_{i} A_{i}^{T}+A_{i} X_{i}+B_{i} F_{i} Y_{i}+Y_{i}^{T} F_{i}^{T} B_{i}^{T}+\lambda_{i i} X_{i}, \Theta_{i 2}=X_{i} C_{i}^{T}+Y_{i}^{T} F_{i}^{T} D_{i}^{T}$,

$$
\begin{gather*}
\Theta_{i 3}=\left(\sqrt{\lambda_{i 1}} X_{i} \cdots \sqrt{\lambda_{i, i-1}} X_{i} \sqrt{\lambda_{i, i+1}} X_{i} \cdots \sqrt{\lambda_{i N}} X_{i} \quad X_{i} J_{i}^{T}\right),  \tag{3.16}\\
\Theta_{i 4}=\operatorname{diag}\left(-X_{1}, \ldots,-X_{i-1},-X_{i+1}, \ldots,-X_{N},-I\right)
\end{gather*}
$$

then the LSMJSs (3.14) are exponential mean-square stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$. In this case, the desired controllers are given as follows:

$$
\begin{equation*}
K_{i}=Y_{i} X_{i}^{-1} \tag{3.17}
\end{equation*}
$$

Proof. Defining the following Lyapunov function:

$$
\begin{equation*}
V(x(t), t, i)=V(x(t), t, r(t)=i)=x^{T}(t) P_{i} x(t) \tag{3.18}
\end{equation*}
$$

By Lemma 3.3, and similar to the proof of Lemma 3.3, we can get the following:

$$
\begin{equation*}
\bumpeq V(x(t), t, i) \leq \eta^{T}(t) \Xi_{i} \eta(t) \tag{3.19}
\end{equation*}
$$

where $\Xi_{i}=\left(\begin{array}{ccc}M_{i} & P_{i} E_{i} & C_{i}^{T}+K_{i}^{T} F_{i}^{T} D_{i}^{T} \\ * & -\gamma^{2} I & H_{i}^{T} \\ * & * & -P_{i}^{-1}\end{array}\right) M_{i}=\left(A_{i}+B_{i} F_{i} K_{i}\right)^{T} P_{i}+P_{i}\left(A_{i}+B_{i} F_{i} K_{i}\right)+\sum_{j \in S} \lambda_{i j} P_{j}$.

Using Schur complement lemma together with contragredient transformation, we know that LMIs (3.15) imply that $\Xi_{i}<0$. So we have

$$
\begin{align*}
J(t) & =E\left\{\int_{0}^{t}\left[z^{T}(s) z(s)-r^{2} v^{T}(s) v(s)\right] \mathrm{d} s\right\} \\
& =E\left\{\int_{t_{0}}^{t}\left[z^{T}(s) z(s)-r^{2} v^{T}(s) v(s)+\rho V(x(s), s, i)\right] \mathrm{d} s\right\}-E(V(x(t), t, i))  \tag{3.20}\\
& \leq E\left\{\int_{0}^{t}\left[z^{T}(s) z(s)-r^{2} v^{T}(s) v(s)+\varrho V(x(s), s, i)\right] \mathrm{d} s\right\}<0
\end{align*}
$$

Therefore, the inequality $\|z\|_{E_{2}}<\gamma\|v\|_{2}$ holds. The proof is completed.
Theorem 3.5. If there exist the positive matrices $X_{i}>0$, the positive diagonal matrices $R_{i}>0$, and the constant matrices $Y_{i}$ with approximate dimensions, such that the following LMIs hold:

$$
\widetilde{\Theta}_{i}=\left(\begin{array}{cccccc}
\widetilde{\Theta}_{i 1} & E_{i} & \tilde{\Theta}_{i 2} & \Theta_{i 3} & B_{i} R_{i} & Y_{i}^{T}  \tag{3.21}\\
* & -\gamma^{2} I & H_{i}^{T} & 0 & 0 & 0 \\
* & * & -X_{i} & 0 & D_{i} R_{i} & 0 \\
* & * & * & \Theta_{i 4} & 0 & 0 \\
* & * & * & * & -R_{i} & 0 \\
* & * & * & * & * & -R_{i} \widetilde{F}_{i 0}^{-2}
\end{array}\right)<0, \quad i \in S
$$

where $\tilde{\Theta}_{i 1}=X_{i} A_{i}^{T}+A_{i} X_{i}+B_{i} F_{i 0} Y_{i}+Y_{i}^{T} F_{i 0}^{T} B_{i}^{T}+\lambda_{i i} X_{i}, \widetilde{\Theta}_{i 2}=X_{i} C_{i}^{T}+Y_{i}^{T} F_{i 0}^{T} D_{i}^{T}$, Then the LSMJSs (3.14) are exponential mean-square stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$. In this case, the desired controllers are given as follows:

$$
\begin{equation*}
K_{i}=Y_{i} X_{i}^{-1} \tag{3.22}
\end{equation*}
$$

Proof. Noticing (2.10), we can see that $\Theta_{i}$ in (3.15) can be rewritten as

$$
\Theta_{i}=\Theta_{i 0}+\left[\begin{array}{llll}
B_{i}^{T} & 0 & D_{i}^{T} & 0
\end{array}\right]^{T} \Delta_{i}\left[\begin{array}{llll}
Y_{i} & 0 & 0 & 0
\end{array}\right]+\left[\begin{array}{llll}
Y_{i} & 0 & 0 & 0
\end{array}\right]^{T} \Delta_{i}\left[\begin{array}{llll}
B_{i}^{T} & 0 & D_{i}^{T} & 0 \tag{3.23}
\end{array}\right]
$$

where $\Theta_{i 0}=\left(\begin{array}{cccc}\tilde{\Theta}_{i 1} & E_{i} & \tilde{\Theta}_{i 2} & \Theta_{i 3} \\ * & -\gamma^{2} & H_{i}^{T} & 0 \\ * & * & -X_{i} & 0 \\ * & * & * & \Theta_{i 4}\end{array}\right)$.
By Lemma 3.2, we have

$$
\Theta_{i} \leq \Theta_{i 0}+\left[\begin{array}{llll}
B_{i}^{T} & 0 & D_{i}^{T} & 0
\end{array}\right]^{T} R_{i}\left[\begin{array}{llll}
B_{i}^{T} & 0 & D_{i}^{T} & 0
\end{array}\right]+\left[\begin{array}{llll}
Y_{i} & 0 & 0 & 0
\end{array}\right]^{T} R_{i}^{-1} F_{0 i}^{2}\left[\begin{array}{llll}
Y_{i} & 0 & 0 & 0 \tag{3.24}
\end{array}\right]
$$

by Schur complement, we know that $\widetilde{\Theta}_{i}<0$ imply that $\Theta_{i}<0$. Therefore, we can know from Theorem 3.4 that the LSMJSs (3.14) are stabilizable with convergence rate $\alpha$ and disturbance attenuation $\gamma$. This completes the proof.

### 3.2. Robust Reliable $H_{\infty}$ for NSMJSs

In this section, we consider the following nonlinear stochastic Markovian jump systems (NSMJSs) under the state feedback controller:

$$
\begin{align*}
\mathrm{d} x(t)= & {\left[\left(A_{i}+B_{i} F_{i} K_{i}\right) x(t)+E_{i} v(t)+f_{i}(x(t))\right] \mathrm{d} t } \\
& +\left[\left(C_{i}+D_{i} F_{i} K_{i}\right) x(t)+H_{i} v(t)+g_{i}(x(t))\right] \mathrm{d} w(t), \\
z(t)= & H_{i} x(t),  \tag{3.25}\\
x\left(t_{0}\right)= & x_{0} .
\end{align*}
$$

Theorem 3.6. If there exist the positive matrices $X_{i}>0$, and the constant matrices $Y_{i}$ with approximate dimensions, for the positive constant $\varepsilon_{i}$ and the given scalar $\lambda_{i}$, such that the following LMIs hold:

$$
\bar{\Theta}_{i}=\left(\begin{array}{cccccc}
\Theta_{i 1} & E_{i} & I-\lambda_{i} X_{i} \widehat{T}_{i 2} & \Theta_{i 2}^{T} & \Theta_{i 2}^{T} & \bar{\Theta}_{i 3}  \tag{3.26}\\
* & -\gamma^{2} I & 0 & H_{i}^{T} & H_{i}^{T} & 0 \\
* & * & -\lambda_{i} I & 0 & 0 & 0 \\
* & * & * & -X_{i} & 0 & 0 \\
* & * & * & * & -\varepsilon_{i} I & 0 \\
* & * & * & * & * & \bar{\Theta}_{i 4}
\end{array}\right)<0, \quad i \in S,
$$

where $\bar{\Theta}_{i 3}=\left(\varepsilon_{i} G_{i} \lambda_{i} X_{i} \widehat{T}_{i 1} \Theta_{i 3}\right), \bar{\Theta}_{i 4}=\operatorname{diag}\left(-\varepsilon_{i} I,-\lambda_{i} \widehat{T}_{i 1}, \Theta_{i 4}\right), \widehat{T}_{i 1}=\left(T_{i 1}^{T} T_{i 2}+T_{i 2}^{T} T_{i 1}\right) / 2, \widehat{T}_{i 2}=$ $-\left(T_{i 1}^{T}+T_{i 2}^{T}\right) / 2$, then the NSMJSs (3.25) are exponential mean-square stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$. In this case, the desired controllers are given as follows:

$$
\begin{equation*}
K_{i}=Y_{i} X_{i}^{-1} \tag{3.27}
\end{equation*}
$$

Proof. Defining the following Lyapunov function:

$$
\begin{equation*}
V(x(t), t, i)=V(x(t), t, r(t)=i)=x^{T}(t) P_{i} x(t) \tag{3.28}
\end{equation*}
$$

by Itô formula, we get the following:

$$
\begin{align*}
£ V(x(t), t, i)= & 2 x^{T}(t) P_{i}\left[\left(A_{i}+B_{i} F_{i} K_{i}\right) x(t)+E_{i} v(t)+f_{i}(x(t))\right]+\sum_{j \in S} \lambda_{i j} x^{T}(t) P_{j} x(t) \\
& +\left[\left(C_{i}+D B_{i} F_{i} K_{i}\right) x(t)+H_{i} v(t)+g_{i}(x(t))\right]^{T} \\
& \times P_{i}\left[\left(C_{i}+D B_{i} F_{i} K_{i}\right) x(t)+H_{i} v(t)+g_{i}(x(t))\right] \\
\leq & \sigma^{T}(t) \Sigma_{i} \sigma(t)+x^{T}(t) G_{i}^{T} P_{i} G_{i} x(t)+2\left[\left(C_{i}+D B_{i} F_{i} K_{i}\right) x(t)+H_{i} v(t)\right]^{T} P_{i} g_{i}(x(t)), \tag{3.29}
\end{align*}
$$

where $\sigma^{T}(t)=\left[x^{T}(t), v^{T}(t), f_{i}^{T}(x(t))\right], \Sigma_{i}=\left(\begin{array}{cc}M_{i} & P_{i} E_{i} P_{i} \\ E_{i}^{T} P_{i}^{T} & 0 \\ P_{i}^{T} & 0\end{array}\right)+\left[\begin{array}{c}C_{i}^{T}+K_{i}^{T} F_{T}^{T} D_{i}^{T} \\ H_{T}^{T} \\ 0\end{array}\right] P_{i}\left[\begin{array}{c}C_{i}^{T}+K_{i}^{T} F_{i}^{T} D_{i}^{T} \\ H_{i}^{T} \\ 0\end{array}\right]^{T}$.
By Lemma 3.2, it follows that

$$
\begin{align*}
& 2\left[\left(C_{i}+D B_{i} F_{i} K_{i}\right) x(t)+H_{i} v(t)\right]^{T} P_{i} g_{i}(x(t)) \\
& \leq \sigma^{T}(t)\left[\begin{array}{c}
C_{i}^{T}+K_{i}^{T} F_{i}^{T} D_{i}^{T} \\
H_{i}^{T} \\
0
\end{array}\right] \varepsilon_{i}^{-1} I\left[\begin{array}{c}
C_{i}^{T}+K_{i}^{T} F_{i}^{T} D_{i}^{T} \\
H_{i}^{T} \\
0
\end{array}\right]^{T} \sigma(t)+x^{T}(t)\left(\varepsilon_{i} P_{i} G_{i}\right)^{T} \varepsilon_{i}^{-1} I\left(\varepsilon_{i} P_{i} G_{i}\right) x(t), \tag{3.30}
\end{align*}
$$

from (2.4) $\left(f_{i}(x(t))-T_{1 i} x\right)^{T}\left(f_{i}(x(t))-T_{2 i} x\right) \leq 0, i \in S$ which are equivalent to

$$
\left[\begin{array}{c}
x(t)  \tag{3.31}\\
f(x(t))
\end{array}\right]^{T}\left[\begin{array}{cc}
\widehat{T}_{i 1} & \widehat{T}_{i 2} \\
T_{i 2}^{T} & I
\end{array}\right]\left[\begin{array}{c}
x(t) \\
f(x(t))
\end{array}\right] \leq 0, \quad i \in S .
$$

Considering the stochastic $H_{\infty}$ performance level for the resulting systems (3.25) with nonzero exogenous disturbance input $(v(t) \neq 0)$, for any $t>0$, we define that

$$
\begin{equation*}
J(t)=E\left\{\int_{0}^{t}\left[z^{T}(s) z(s)-\gamma^{2} v^{T}(s) v(s)\right] \mathrm{d} s\right\} . \tag{3.32}
\end{equation*}
$$

By general Itô formula, for a given positive scalar $\lambda$, we get the following:
$J(t)$

$$
\begin{align*}
& =E\left\{\int_{t_{0}}^{t}\left[z^{T}(s) z(s)-\gamma^{2} v^{T}(s) v(s)+\varrho V(x(s), s, i)\right] \mathrm{d} s\right\}-E(V(x(t), t, i)) \\
& \leq E\left\{\int_{0}^{t}\left[z^{T}(s) z(s)-\gamma^{2} v^{T}(s) v(s)+\varrho V(x(s), s, i)-\lambda_{i}\left(f_{i}(x(t))-T_{1 i} x(t)\right)^{T}\left(f_{i}(x(t))-T_{2 i} x(t)\right)\right] \mathrm{d} s\right\} \\
& \leq E\left\{\int_{0}^{t} \sigma^{T}(s) \bar{\Omega}_{i} \sigma(s) \mathrm{d} s\right\}, \tag{3.33}
\end{align*}
$$

where

$$
\begin{align*}
\bar{\Omega}_{i}= & \Sigma_{i}+\left(\begin{array}{ccc}
\left(\varepsilon_{i} P_{i} G_{i}\right)^{T} \varepsilon_{i}^{-1} I\left(\varepsilon_{i} P_{i} G_{i}\right)+J_{i}^{T} J_{i} & 0 & 0 \\
0 & -\gamma^{2} I & 0 \\
0 & 0 & 0
\end{array}\right) \\
& +\left[\begin{array}{c}
C_{i}^{T}+K_{i}^{T} F_{i}^{T} D_{i}^{T} \\
H_{i}^{T} \\
0
\end{array}\right] \varepsilon_{i}^{-1} I\left[\begin{array}{ccc}
C_{i}^{T}+K_{i}^{T} F_{i}^{T} D_{i}^{T} \\
H_{i}^{T} \\
0
\end{array}\right]+\left(\begin{array}{ccc}
-\lambda \widehat{T}_{i 1} & 0 & -\lambda \widehat{T}_{i 2} \\
0 & 0 & 0 \\
-\lambda \widehat{T}_{i 2}^{T} & 0 & -\lambda I
\end{array}\right) . \tag{3.34}
\end{align*}
$$

By Schur complement lemma, we see that $\bar{\Omega}_{i}<0$ is equivalent to the following matrix inequalities:

$$
\left(\begin{array}{ccccccc}
M_{i}-\lambda_{i} \widehat{T}_{i 1} & E_{i} & P_{i}-\lambda_{i} \widehat{T}_{i 2} & X_{i}^{-1} \Theta_{i 2}^{T} & X_{i}^{-1} \Theta_{i 2}^{T} & \varepsilon_{i} P_{i} G_{i} & J_{i}^{T}  \tag{3.35}\\
* & -r^{2} I & 0 & H_{i}^{T} & H_{i}^{T} & 0 & 0 \\
* & * & -\lambda_{i} I & 0 & 0 & 0 & 0 \\
* & * & * & -P_{i}^{-1} & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_{i} I & 0 & 0 \\
* & * & * & * & * & -\varepsilon_{i} I & 0 \\
* & * & * & * & * & * & -I
\end{array}\right)<0, \quad i \in S,
$$

which is implied in LIMs (3.26). Hence $J(t)<0$.
Therefore, the inequality $\|z\|_{E_{2}}<\gamma\|v\|_{2}$ holds. The proof is completed.
Similar to the proof of Theorem 3.5, we can get the following theorem without proof immediately.


Figure 1: The Markovian chain $r(t)$.

Theorem 3.7. If there exist the positive matrices $X_{i}>0$, and the constant matrices $Y_{i}$ with approximate dimensions, for the positive constant $\varepsilon_{i}$ and the given scalar $\Lambda_{i}$, such that the following LMIs hold

$$
\widehat{\Theta}_{i}\left(\begin{array}{cccccccc}
\widetilde{\Theta}_{i 1} & E_{i} & I-\lambda_{i} X_{i} \widehat{T}_{i 2} & \widetilde{\Theta}_{i 2}^{T} & \widetilde{\Theta}_{i 2}^{T} & B_{i} R_{i} & Y_{i}^{T} & \bar{\Theta}_{i 3}  \tag{3.36}\\
* & -\gamma^{2} I & 0 & H_{i}^{T} & H_{i}^{T} & 0 & 0 & 0 \\
* & * & -\lambda_{i} I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -X_{i} & 0 & D_{i} R_{i} & 0 & 0 \\
* & * & * & * & -\varepsilon_{i} I & 0 & 0 & 0 \\
* & * & * & * & * & -R_{i} & 0 & 0 \\
* & * & * & * & * & * & -R_{i} \tilde{F}_{i 0}^{-2} & 0 \\
* & * & * & * & * & * & * & \bar{\Theta}_{i 4}
\end{array}\right)<0, \quad i \in S .
$$

then the NSMJSs (3.27) are exponential mean-square stable with convergence rate a and disturbance attenuation $\gamma$. In this case, the desired controllers are given as follows:

$$
\begin{equation*}
K_{i}=Y_{i} X_{i}^{-1} . \tag{3.37}
\end{equation*}
$$

## 4. Numerical Example with Simulation

In this section, we will give an example to show the usefulness of the derived results and the effectiveness of the proposed methods (Figure 1).


Figure 2: The state curve of uncontrolled LSMJSs (3.14).

Consider linear SMJSs (3.14) with $S=\{1,2\}$, and the system parameters are given as follows:

$$
\begin{gather*}
A_{1}=\left(\begin{array}{ccc}
0.3 & 0.3 & 0.5 \\
-0.2 & 0 & -0.3 \\
0.1 & 0 & 0.3
\end{array}\right), \quad A_{2}=\left(\begin{array}{ccc}
0.5 & 0.2 & 0.2 \\
-0.2 & 0 & -0.4 \\
0.2 & 0 & 0.2
\end{array}\right), \\
C_{1}=\left(\begin{array}{ccc}
0.5 & 0.2 & 0.1 \\
0 & 0.2 & -0.1 \\
0.3 & -0.1 & -0.3
\end{array}\right), \quad C_{2}=\left(\begin{array}{ccc}
0.2 & 0.1 & 0.3 \\
0.1 & -0.3 & 0.5 \\
0 & 0.1 & -0.5
\end{array}\right),  \tag{4.1}\\
B_{1}=\operatorname{diag}(0.5,0.4,0.5), \quad B_{2}=\operatorname{diag}(0.5,0.4,0.5), \\
E_{1}=E_{2}=(0.3,0.1,0.5)^{T}, \quad H_{2}=H_{1}=(0.2,0.1,0.3)^{T}, \\
D_{2}=D_{1}=\operatorname{diag}(0.2,0.3,0.4), \quad J_{1}=(0.3,0.2,0.6), \\
J_{2}=(0.1,-0.1,0.4), \quad \gamma=0.9 .
\end{gather*}
$$

The actuator failure parameters are as follows:

$$
\begin{equation*}
0.2 \leq f_{i 1} \leq 0.4, \quad 0.1 \leq f_{i 2} \leq 0.7, \quad 0.1 \leq f_{i 3} \leq 0.9, \quad i \in S=\{1,2\} . \tag{4.2}
\end{equation*}
$$

From (2.8) and (2.9), we have

$$
\begin{equation*}
F_{10}=F_{20}=\operatorname{diag}(0.3,0.4,0.5), \quad \tilde{F}_{10}=\tilde{F}_{20}=\operatorname{diag}(0.1,0.3,0.4) \tag{4.3}
\end{equation*}
$$



Figure 3: The state curve of closed-loop LSMJSs (3.14).


Figure 4: The curve of $|z(t)|^{2}-\gamma^{2}|v(t)|$ for controlled LSMJSs (3.14).

From Figure 2, we can see that the uncontrolled LSMJSs are not stable, according to Theorem 3.5. By using the LMI toolbox, the controller parameters can be calculated as follows:

$$
K_{1}=\left(\begin{array}{ccc}
-56.2264 & -6.3843 & -67.8069  \tag{4.4}\\
-1.1129 & -8.9588 & -3.6802 \\
-0.9754 & 0.1795 & -3.2600
\end{array}\right), \quad K_{2}=\left(\begin{array}{ccc}
-41.7846 & 6.0578 & -200.8802 \\
-1.1365 & -7.5245 & -11.0209 \\
0.1171 & -0.4055 & -0.7561
\end{array}\right)
$$

Figures 3 and 4 give the simulation results of the response for the closed-loop LSMJSs, which confirm that the closed-loop LSMJSs are exponential mean-square stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$.

## 5. Conclusions

In this paper, we have studied the robust reliable $H_{\infty}$ control problems for a class of NSMJSs. The system under study contains Itô-type stochastic disturbance, Markovian jumps, sectorbounded nonlinearities, and norm-bounded stochastic nonlinearities. Based on the Lyapunov stability theory and Itô differential rule, sufficient condition which ensures exponential meansquare stable with convergence rate $\alpha$ and disturbance attenuation $\gamma$ for SMJSs has been established in Lemma 3.3. By the lemma, together with the LMIs techniques, the sufficient conditions for the designation of the robust reliable $H_{\infty}$ controller of linear SMJSs and NSMJSs have been obtained in terms of LMIs. Finally, a numerical example has been given to show the usefulness of the derived results and the effectiveness of the proposed methods. It is possible to extend our main results to the NSMJSs with time delay by using delay-dependent techniques, which is one of the future research topics.

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Research Article

# Probabilistic Approach to System Reliability of Mechanism with Correlated Failure Models 

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#### Abstract

In this paper, based on the kinematic accuracy theory and matrix-based system reliability analysis method, a practical method for system reliability analysis of the kinematic performance of planar linkages with correlated failure modes is proposed. The Taylor series expansion is utilized to derive a general expression of the kinematic performance errors caused by random variables. A proper limit state function (performance function) for reliability analysis of the kinematic performance of planar linkages is established. Through the reliability theory and the linear programming method the upper and lower bounds of the system reliability of planar linkages are provided. In the course of system reliability analysis, the correlation of different failure modes is considered. Finally, the practicality, efficiency, and accuracy of the proposed method are shown by a numerical example.


## 1. Introduction

Mechanisms are the skeletons of modern mechanical products and devices. The kinematic accuracy of mechanisms greatly influences the performance and reliability of the mechanical products and devices. Traditionally, in mechanism synthesis, a designer often tries to choose proper mechanism configurations and component dimensions to make the designed mechanism meet prespecified requirements. However, in the physical realization of any constituent member, primary errors always occur due to technological features of production. Once a theoretical solution is translated into physical reality, a theoretically feasible mechanism might be unable to meet practical requirements because of the effects of uncertain factors (e.g. manufacturing tolerances, elastic deformations and joint clearances). Since these uncertain factors are inevitable, it is necessary to build a proper mode to quantify the effects of the uncertain factors on the accuracy of mechanisms and optimally allocate the working ranges of the mechanisms [1-3].

In recent years, with the continual increase of the demands of consumers on the kinematic and dynamic performance of mechanical products, the theory of mechanical reliability
is more and more widely applied in mechanism analysis and synthesis. Mechanism reliability can simply be defined as the capability that a mechanism performs its prespecified movement accurately, timely, and coordinately throughout its lifetime. Based on the analysis of the sources of original errors, Sergeyev [4] clarified the main failure modes of mechanisms and presented an analytical method for reliability analysis of mechanisms preliminarily. Subsequently, there have been various attempts to derive the reliability of the kinematic and dynamic accuracy of mechanisms, such as the linear regression method [5], the mean value first-order second-moment method [6], the advanced first-order second-moment method [7], the hybrid dimension reduction method [8], and the Monte-Carlo simulation method [9].

As shown in practical engineering, most performance deficiencies of mechanical products are found in the stage of systematic analysis, therefore it's important to build a proper system reliability analysis model to evaluate the performance quality of mechanical equipments and products. Recently, Zhang et al. [9] studied the method for system reliability analysis of mechanisms without considering the interactions of failure modes. However, to the best of the authors' knowledge, system reliability analysis of mechanisms with correlated failure modes has not been reported yet. Combining the mechanism theory and system reliability analysis method, this paper proposes a general method for system reliability analysis of planar mechanisms with correlated failure modes.

## 2. Reliability Analysis for Kinematic Performance of Planar Linkages

The kinematic performance function of planar linkages can be expressed as [10]

$$
\begin{equation*}
\mathbf{Q}=\mathbf{Q}(\mathbf{V}, \mathbf{L}, \mathbf{U}) \tag{2.1}
\end{equation*}
$$

where $\mathbf{Q}_{s \times 1}$ is the performance parameter vector. For example, for a function generator, $\mathbf{Q}_{s \times 1}$ may be referred to the positions of the output link, and for a path generator, it may be the coordinates of a point on the output link. $\mathbf{V}_{m \times 1}$ is the input (independent) variable vector, $\mathbf{L}_{p \times 1}$ is the effective dimension variable vector, and $\mathbf{U}_{n \times 1}$ is the output (dependent) variable vector which can be obtained by solving the loop closure equations of planar linkages

$$
\begin{equation*}
\mathbf{F}(\mathbf{L}, \mathbf{U}, \mathbf{V})=0 \tag{2.2}
\end{equation*}
$$

The performance errors of the mechanism under consideration can be obtained as

$$
\begin{equation*}
\Delta \mathbf{Q}=\frac{\partial \mathbf{Q}}{\partial \mathbf{L}^{T}} \Delta \mathbf{L}+\frac{\partial \mathbf{Q}}{\partial \mathbf{U}^{T}} \Delta \mathbf{U}+\frac{\partial \mathbf{Q}}{\partial \mathbf{V}^{T}} \Delta \mathbf{V} \tag{2.3}
\end{equation*}
$$

where $\partial \mathbf{Q} / \partial \mathbf{L}^{T}, \partial \mathbf{Q} / \partial \mathbf{U}^{T}$ and $\partial \mathbf{Q} / \partial \mathbf{V}^{T}$ are Jacobian matrices, whose values are got at the mean values of the random variables. $\Delta \mathbf{L}_{p \times 1}, \Delta \mathbf{U}_{n \times 1}$, and $\Delta \mathbf{V}_{m \times 1}$ are the tolerance vectors of random design variables. $\Delta \mathbf{V}_{m \times 1}$ and $\Delta \mathbf{L}_{p \times 1}$ are determined by several objective factors such as the machining accuracy, the assembly accuracy and the operation precision. From (2.3), $\Delta \mathbf{U}_{n \times 1}$ can be obtained:

$$
\begin{equation*}
\Delta \mathbf{U}=-\left[\frac{\partial \mathbf{F}}{\partial \mathbf{U}^{T}}\right]^{-1}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{V}^{T}} \Delta \mathbf{V}+\frac{\partial \mathbf{F}}{\partial \mathbf{L}^{T}} \Delta \mathbf{L}\right) \tag{2.4}
\end{equation*}
$$

A limit state is defined as a condition in which a mechanism becomes unsuitable for its intended motion (i.e., a violation of the serviceability limit state). The corresponding limit state functions (performance functions) when the mechanism meets the requirements of the upper and lower limits are

$$
\begin{align*}
& \mathbf{g}_{U}(\mathbf{X})=\boldsymbol{\varepsilon}-\Delta \mathbf{Q}=\boldsymbol{\varepsilon}-\mathbf{J} \Delta \mathbf{X},  \tag{2.5}\\
& \mathbf{g}_{L}(\mathbf{X})=\Delta \mathbf{Q}+\boldsymbol{\varepsilon}=\mathbf{J} \Delta \mathbf{X}+\boldsymbol{\varepsilon},
\end{align*}
$$

where

$$
\begin{equation*}
\mathbf{J}=\left[\frac{\partial \mathbf{Q}}{\partial \mathbf{V}^{T}}-\frac{\partial \mathbf{Q}}{\partial \mathbf{U}^{T}}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}^{T}}\right)^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{V}^{T}}, \frac{\partial \mathbf{Q}}{\partial \mathbf{L}^{T}}-\frac{\partial \mathbf{Q}}{\partial \mathbf{U}^{T}}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}^{T}}\right)^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{L}^{T}}\right] . \tag{2.6}
\end{equation*}
$$

$\mathbf{g}_{U}$ and $\mathbf{g}_{L}$ are called the upper and lower limit state functions, $\mathbf{X}=\left[V_{1}, \ldots, V_{m}, L_{1}, \ldots, L_{p}\right]^{T}$ is the basic variables, $\varepsilon$ is the allowable errors, and $\Delta \mathbf{X}=\left[\Delta V_{1}, \ldots, \Delta V_{m}, \Delta L_{1}, \ldots, \Delta L_{p}\right]^{T}$ are used to represent the random error vector of basic variables.

The kinematic reliability of a mechanism is the probability that the mechanism realizes its required motion within a specified tolerance limit. The lower limit reliability of the $k$ th dependent variable, $Q_{k}$, is defined as:

$$
\begin{equation*}
R_{L}^{(k)}=\int_{g_{L}^{(k)}(\mathbf{X})>0} f(\mathbf{X}) \mathrm{d} \mathbf{X}, \tag{2.7}
\end{equation*}
$$

where $f(\mathbf{X})$ is the joint probability density function of multidimensional basic random variables, $\mathbf{X}$ and $g_{L}^{(k)}(\mathbf{X})=\Delta Q_{k}+\varepsilon_{k}=\mathbf{J}_{k} \Delta \mathbf{X}+\varepsilon_{k}$ is the limit state function of the $k$ th dependent variable, $Q_{k}$. Note that $\mathbf{J}_{k}$ is the $k$ th row of matrix $\mathbf{J}$.

The mean value, $\mu_{L}^{(k)}$, and variance, $\left(\sigma_{L}^{(k)}\right)^{2}$, of the limit state function, $g_{L}^{(k)}(\mathbf{X})$, can be expressed as

$$
\begin{align*}
\mu_{L}^{(k)} & =E\left[g_{L}^{(k)}(\mathbf{X})\right]=\mathbf{J}_{k} E(\Delta \mathbf{X})+\varepsilon_{k} \\
\left(\sigma_{L}^{(k)}\right)^{2} & =\operatorname{Var}\left[g_{L}^{(k)}(\mathbf{X})\right]=\mathbf{J}_{k}^{[2]} \operatorname{Cs}(\operatorname{Cov}(\Delta \mathbf{X})) \tag{2.8}
\end{align*}
$$

where $E(\Delta \mathbf{X})$ and $\operatorname{Cov}(\Delta \mathbf{X})$ are the mean value vector and covariance matrix of primary errors, respectively, $\mathbf{J}_{k}$ is the $k$ th row of matrix $\mathbf{J},(\cdot)^{[2]}=(\cdot) \otimes(\cdot)$ is the second-order Kronecker power of $(\cdot)$, and $\otimes$ represents Kronecker product [11]

$$
\mathbf{A}_{p \times q} \otimes \mathbf{B}_{s \times t}=\left[\begin{array}{cccc}
a_{11} \mathbf{B} & a_{12} \mathbf{B} & \ldots & a_{1 q} \mathbf{B}  \tag{2.9}\\
a_{21} \mathbf{B} & a_{22} \mathbf{B} & \ldots & a_{2 q} \mathbf{B} \\
\vdots & \vdots & \ddots & \vdots \\
a_{p 1} \mathbf{B} & a_{p 2} \mathbf{B} & \ldots & a_{p q} \mathbf{B}
\end{array}\right]_{p s \times q t}
$$

$\operatorname{cs}(\cdot)$ is the column string of $(\cdot)$, the column sequence

$$
\begin{equation*}
\operatorname{cs}\left(\mathbf{A}_{p \times q}\right)=\sum_{j=1}^{q}\left(\mathbf{e}_{q \times 1}^{j} \otimes \mathbf{I}_{p \times p}\right) \mathbf{A}_{p \times q} \mathbf{e}_{q \times 1}^{j}, \tag{2.10}
\end{equation*}
$$

where $\mathbf{I}_{p \times p}$ is identity matrix with $p \times p$ dimensions, and $\mathbf{e}_{q \times 1}^{j}$ is the $j$ th elementary vector with $q \times 1$ dimensions, all zeros except 1 in the $j$ th position.

In the mechanism literature, the distribution of the random variables is always assumed independent normal [4-8]. The distance from the "minimum" tangent plane to the failure surface may be used to approximate the actual failure surface, and the reliability index of the $i$ th output variable is defined as:

$$
\begin{equation*}
\beta_{L}^{(k)}=\frac{\mu_{L}^{(k)}}{\sigma_{L}^{(k)}} \tag{2.11}
\end{equation*}
$$

which can be used to reflect the position (the distance from the original point) and dispersion degree of the safety margin. When the primary errors are normally and independently distributed, the unary estimator of the kinematic performance reliability of planar linkages is represented as follows:

$$
\begin{equation*}
R_{L}^{(k)}=\Phi\left(\beta_{L}^{(k)}\right) \tag{2.12}
\end{equation*}
$$

where $\Phi(\cdot)$ is the standard normal distribution function.
The correlation coefficient between performance functions $g_{L}^{(k)}$ and $g_{L}^{(t)}$ is

$$
\begin{equation*}
\operatorname{Cov}\left(g_{L}^{(k)}, g_{L}^{(t)}\right)=E\left[\left(g_{L}^{(k)}-\bar{g}_{L}^{(k)}\right)\left(g_{L}^{(t)}-\bar{g}_{L}^{(t)}\right)\right]=\sum_{i=1}^{m+l} \sum_{j=1}^{m+l} \frac{\partial g_{L}^{(k)}}{\partial X_{i}} \frac{\partial g_{L}^{(t)}}{\partial X_{j}} \operatorname{Cov}\left(X_{i}, X_{j}\right) \tag{2.13}
\end{equation*}
$$

The correlation coefficient between performance functions $g_{L}^{(k)}$ and $g_{L}^{(t)}$ is

$$
\begin{equation*}
\rho_{L}^{(k, t)}=\frac{\operatorname{Cov}\left(g_{L}^{(k)}, g_{L}^{(t)}\right)}{\sqrt{\operatorname{Var}\left(g_{L}^{(k)}\right) \operatorname{Var}\left(g_{L}^{(t)}\right)}} \tag{2.14}
\end{equation*}
$$

Then the joint reliability of $g_{L}^{(k)}$ and $g_{L}^{(t)}$ can be estimated by the joint normal distribution function:

$$
\begin{align*}
R_{L}^{(k, t)} & =1-\iint_{-\infty}^{0} f_{k t}\left(g_{L}^{(k)}, g_{L}^{(t)}\right) d g_{L}^{(k)} d g_{L}^{(t)}=1-\int_{-\infty}^{-\beta_{L}^{t}} \int_{-\infty}^{-\beta_{L}^{k}} \phi_{L}^{(k, t)}(u, v) \mathrm{d} u \mathrm{~d} v \\
& =1-\int_{-\infty}^{-\beta_{L}^{t}} \Phi\left[\frac{-\beta_{L}^{k}-\rho_{L}^{(k, t)} v}{\sqrt{1-\left(\rho_{L}^{(k, t)}\right)^{2}}}\right] \phi(v) \mathrm{d} v \tag{2.15}
\end{align*}
$$

where

$$
\begin{align*}
f_{k t}\left(g_{L}^{(k)}, g_{L}^{(t)}\right)= & \frac{1}{2 \pi \sigma_{L}^{(k)} \sigma_{L}^{(t)} \sqrt{\left[1-\left(\rho_{L}^{(k, t)}\right)^{2}\right]}} \\
& \times \exp \left\{-\frac{1}{2\left[1-\left(\rho_{L}^{(k, t)}\right)^{2}\right]}\left[\frac{\left(g_{L}^{(k)}-\mu_{L}^{(k)}\right)^{2}}{\operatorname{Var}\left(g_{L}^{(k)}\right)}-2 \rho_{L}^{(k, t)} \frac{\left(g_{L}^{(k)}-\mu_{L}^{(k)}\right)\left(g_{L}^{(t)}-\mu_{L}^{(t)}\right)}{\sigma_{L}^{(k)} \sigma_{L}^{(t)}}\right.\right. \\
& \left.\left.+\frac{\left(g_{L}^{(t)}-\mu_{L}^{(t)}\right)^{2}}{\operatorname{Var}\left(g_{L}^{(t)}\right)}\right]\right\} \tag{2.16}
\end{align*}
$$

is the joint PDF of $g_{L}^{(k)}$ and $g_{L}^{(t)}$.
In the same way, the reliability corresponding to each failure modes and the joint reliability between each two failure modes while the mechanism satisfies the upper limits could be obtained. Then the reliability corresponding to each failure model and the joint reliability between two failure models while the mechanism meets the upper and lower limits can be derived as

$$
\begin{gather*}
R^{k}=R_{L}^{(k)}+R_{U}^{(k)}-1, \\
R^{(k, t)}=R_{L}^{(k, t)}+R_{U}^{(k, t)}-1 . \tag{2.17}
\end{gather*}
$$

## 3. System Reliability of Linkage Performance

For convenience system reliability analysis of structures with multi-failure modes is often performed by consuming that the failure modes are independent between each other. In most cases, however, the failure modes of a mechanism (e.g., the position and pose of a rigid-body guidance mechanism) are correlated. Consequently, it is of great meaning to propose an accurate and efficient system reliability analysis method to evaluate the working state of the mechanism. Ditlevsen [12] presented the well-known "narrow bounds theory" for computing system reliability. The correlation between each of the two failure modes is considered in Ditlevsen's method, making it more physically reasonable. And then the bounds method in which the system reliability is estimated by computing the bound values developed continuously and received wide acceptance [13-15]. In this section, a practical method for system reliability analysis of mechanisms is proposed by using the linear programming.

Linear programming solves the problem of minimizing or maximizing a linear function, whose variables are subject to linear equality and inequality constraints. And the linear programming for solving the possible bounds on the system reliability of linkages can be presented as follows:

$$
\begin{array}{cc}
\min (\max ) & \mathbf{c}^{T} \mathbf{p} \\
\text { s.t. } & \mathbf{a}_{1} \mathbf{p}=\mathbf{b}_{1}  \tag{3.1}\\
& \mathbf{a}_{2} \mathbf{p} \geq \mathbf{b}_{2}
\end{array}
$$

where $\mathbf{p}$ is the design variable vector of the linear programming, $\mathbf{c}$ is a vector of coefficients, $\mathbf{c}^{T} \mathbf{p}$ is the linear objective function, and $\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{b}_{1}$, and $\mathbf{b}_{2}$ are the coefficient matrices and vectors that represent the equality and inequality constraints, respectively.

In the proposed system reliability analysis method, the kinematic failure space of a mechanism can be divided into $2^{n}$ mutually exclusive and collectively exhaustive (MECE) events according to the number of failure modes, $n$. Typically, for a system with three failure modes, the performance sample space can be depicted as Figure 1 by defining kinematic safety of the mechanism as event $S$ and defining the performance $Q_{i}$ meets the requirement of performance quality as event $E_{i}$. The space $S$ is divided into 8 MECE events, $\left\{e_{1}=E_{1} E_{2} E_{3}, e_{2}=\bar{E}_{1} E_{2} E_{3}, e_{3}=E_{1} \bar{E}_{2} E_{3}, e_{4}=E_{1} E_{2} \bar{E}_{3}, e_{5}=\bar{E}_{1} \bar{E}_{2} E_{3}, e_{6}=\bar{E}_{1} E_{2} \bar{E}_{3}, e_{7}=\right.$ $E_{1} \bar{E}_{2} \bar{E}_{3}$, and $\left.e_{8}=\bar{E}_{1} \bar{E}_{2} \bar{E}_{3}\right\}$. Let $p_{i}=P\left(e_{i}\right), i=1,2, \ldots, 8$ denotes the probability of the $i$ th basic MECE event. These probabilities serve as the design variables in the linear programming problem to be formulated. According to the basic definition of probability,

$$
\begin{gather*}
\sum_{i=1}^{8} p_{i}=1, \\
p_{i} \geq 0, \quad i=(1,2, \ldots, 8) . \tag{3.3}
\end{gather*}
$$

The constraint (3.2) is analogous to the equality constraints in linear programming (3.1) with $\mathbf{a}_{1}$ being a row vector of 1's and $\mathbf{b}_{1}=1$, whereas (3.3) is analogous to the inequality constraints with $\mathbf{a}_{2}$ being an $8 \times 8$ identity matrix and $\mathbf{b}_{2}$ a $8 \times 1$ vector of 0 's.

As can be seen from Figure 1,

$$
\begin{gather*}
P\left(E_{i}\right)=P_{i}=\sum_{r: e_{r} \subseteq E_{i}} p_{r},  \tag{3.4}\\
P\left(E_{i} E_{j}\right)=P_{i j}=\sum_{r: e_{r} \subseteq E_{i} E_{j}} p_{r}, \tag{3.5}
\end{gather*}
$$

scilicet,

$$
\begin{gather*}
P\left(E_{1}\right)=P_{1}=p_{1}+p_{3}+p_{4}+p_{7}, \\
P\left(E_{2}\right)=P_{2}=p_{1}+p_{2}+p_{4}+p_{6}, \\
P\left(E_{3}\right)=P_{3}=p_{1}+p_{2}+p_{3}+p_{5},  \tag{3.6}\\
P\left(E_{1} E_{2}\right)=P_{12}=p_{1}+p_{4}, \\
P\left(E_{1} E_{3}\right)=P_{13}=p_{1}+p_{3}, \\
P\left(E_{2} E_{3}\right)=P_{23}=p_{1}+p_{2} .
\end{gather*}
$$



Figure 1: Sample space for the linkage.

Equations (3.4) and (3.5) provide linear equality constraints on the design variable $\mathbf{p}$ with $\mathbf{a}_{1}$ a matrix having elements of 0 or 1 and $\mathbf{b}_{1}$ a vector listing the known reliability. With the increase of the known or computed reliability, such as the uni-, bi-, and sometimes trimode reliability, the upper and lower bounds of the system reliability obtained by the linear programming become increasingly accuracy. However, there is always a tradeoff between complexity and accuracy, and with the increase of the constraints, the convergence of the linear programming becomes more and more difficult.

By now, the coefficient matrices and vectors of the constraint functions of linear programming (3.1) to obtain the upper and lower bounds of the system reliability of the mechanism are completely established, which are

$$
\begin{gather*}
\mathbf{a}_{1}=\left[\begin{array}{llllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],  \tag{3.7}\\
\mathbf{b}_{1}=\left[\begin{array}{lllllll}
1 & P_{1} & P_{2} & P_{3} & P_{12} & P_{13} & P_{23}
\end{array}\right]^{T}, \\
\mathbf{a}_{2}=\mathbf{I}_{8} \times 8,
\end{gather*}
$$

where $\mathbf{I}_{8 \times 8}$ is identity matrix with $8 \times 8$ dimensions.

Table 1: Coefficients of the object functions $\mathrm{c}^{T} \mathrm{p}$.

|  | $c_{1}$ | $c_{2}$ | $c_{3}$ | $c_{4}$ | $c_{5}$ | $c_{6}$ | $c_{7}$ | $c_{8}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{E}_{1} \cup \mathrm{E}_{2} \cup \mathrm{E}_{3}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| $\mathrm{E}_{1} \cap \mathrm{E}_{2} \cap \mathrm{E}_{3}$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathrm{E}_{1} \cap \mathrm{E}_{2} \cup \mathrm{E}_{3}$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $\left(\mathrm{E}_{1} \cup \mathrm{E}_{2}\right) \cap\left(\mathrm{E}_{2} \cup \bar{E}_{3}\right)$ | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |



Figure 2: Double-rocker four-bar linkage with driving crank.

According to the relationship between failure modes (series or parallel), there are four different kinds of systems. As shown in Table 1, the coefficient $c_{i}(i=1,2, \ldots, 8)$ of the object functions of linear programming (3.1) for each kind of system can be determined, respectively. So far, the linear programming to derive the lower and upper bounds of the system reliability of a mechanism with random parameters is completely established.

## 4. Numerical Examples

Consider the vector loop as shown in Figure 2, the nominal geometry characteristics of the double-rocker four-bar linkage with driving crank are shown as: $r_{1}=2.36 \mathrm{~cm}, r_{2}=1.33 \mathrm{~cm}$, $r_{3}=5.08 \mathrm{~cm}, r_{4}=3.94 \mathrm{~cm}, r_{5}=1.00 \mathrm{~cm}, r_{6}=0.45 \mathrm{~cm}, r_{7}=1.50 \mathrm{~cm}, r_{8}=1.00 \mathrm{~cm}, r_{9}=6.00 \mathrm{~cm}$, and $\alpha_{9}=30^{\circ}$. Among them, $r_{1}, r_{2}, r_{3}$, and $r_{4}$ are random variables, which are normally and independently distributed, and the variation coefficient of the random variables are supposed to be $c=0.001$. All other variables are deterministic parameters. According to the working condition, the maximum allowable values of the kinematic performance errors vector are
$\varepsilon=[0.015 \mathrm{rad}, 0.8 \mathrm{~mm}, 0.6 \mathrm{~mm}]^{T}$. The double-rocker four-bar linkage can work normally, only if all the kinematic performance quality requirements are satisfied (i.e., the linkage system is series). It is required to solve the system reliability of the double-rocker four-bar linkage in its working range ( $\alpha_{6}=150^{\circ} \sim 270^{\circ}$ ).

As shown in Figure 1, in the double-rocker four-bar linkage with driving crank, the effective dimension variable vector is $\mathbf{L}=\left[r_{1}, r_{2}, r_{3}, r_{4}, r_{5}, r_{6}, r_{7}, r_{8}, r_{9}, \alpha_{9}\right]^{T}$, the input (independent) and output (dependent) variable vectors are $V=\left[\alpha_{6}\right]$ and $\mathbf{U}=$ $\left[\alpha_{2}, \alpha_{3}, \alpha_{4}, \alpha_{7}\right]^{T}$, respectively, the performance parameter vector is $\mathbf{Q}=\left[\alpha_{3}, M_{x}, M_{y}\right]^{T}$, then the closure equations of the planar linkage are

$$
\mathbf{F}=\left[\begin{array}{c}
r_{6} \cos \alpha_{6}+r_{7} \cos \alpha_{7}-r_{8} \cos \alpha_{2}-r_{5}  \tag{4.1}\\
r_{6} \sin \alpha_{6}+r_{7} \sin \alpha_{7}-r_{8} \sin \alpha_{2} \\
r_{2} \cos \alpha_{2}+r_{3} \cos \alpha_{3}-r_{4} \cos \alpha_{4}-r_{1} \\
r_{2} \sin \alpha_{2}+r_{3} \sin \alpha_{3}-r_{4} \sin \alpha_{4}
\end{array}\right] .
$$

The kinematic performance functions of the linkage are

$$
\mathbf{Q}=\left[\begin{array}{c}
\alpha_{3}+\alpha_{9}  \tag{4.2}\\
r_{2} \cos \alpha_{2}+r_{9} \cos \left(\alpha_{3}+\alpha_{9}\right) \\
r_{2} \sin \alpha_{2}+r_{9} \sin \left(\alpha_{3}+\alpha_{9}\right)
\end{array}\right]
$$

Suppose that $\mathbf{X}=\left[r_{1}, r_{2}, r_{3}, r_{4}\right]^{T}$ is the random variable vector, then the Jacobian matrices are derived as:

$$
\begin{gather*}
\frac{\partial \mathbf{F}}{\partial \mathbf{X}^{T}}=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & \cos \alpha_{2} & \cos \alpha_{3} & -\cos \alpha_{4} \\
0 & \sin \alpha_{2} & \sin \alpha_{3} & -\sin \alpha_{4}
\end{array}\right] \\
\frac{\partial \mathbf{F}}{\partial \mathbf{U}^{T}}=\left[\begin{array}{cccc}
r_{8} \sin \alpha_{2} & 0 & 0 & -r_{7} \sin \alpha_{7} \\
-r_{8} \cos \alpha_{2} & 0 & 0 & r_{7} \cos \alpha_{7} \\
-r_{2} \sin \alpha_{2} & -r_{3} \sin \alpha_{3} & r_{4} \sin \alpha_{4} & 0 \\
r_{2} \cos \alpha_{2} & r_{3} \cos \alpha_{3} & -r_{4} \cos \alpha_{4} & 0
\end{array}\right],  \tag{4.3}\\
\left.\frac{\partial \mathbf{Q}}{\frac{\partial \mathbf{Q}}{\partial \mathbf{X}^{T}}=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \cos \alpha_{2} & 0 & 0 \\
0 & \sin \alpha_{2} & 0 & 0
\end{array}\right],} \begin{array}{cccc}
0 & 1 & 0 & 0 \\
-r_{2} \sin \alpha_{2} & -r_{9} \sin \left(\alpha_{3}+\alpha_{9}\right) & 0 & 0 \\
r_{2} \cos \alpha_{2} & -r_{9} \cos \left(\alpha_{3}+\alpha_{9}\right) & 0 & 0
\end{array}\right]
\end{gather*}
$$



Figure 3: System reliability of the double-rocker four-bar linkage with driving crank.

By substituting (4.3) into (2.3), the kinematic performance error vector, $\Delta \mathbf{Q}$, of the linkage can be obtained. Then (2.12) can be used to obtain the reliability corresponding to each failure mode. The covariance matrix of the limit sate functions of the planar linkage can be derived from (3.5), and then the joint reliability between each two failure modes can also be derived from (2.15).

The upper and lower bounds of the system reliability of the double-rocker four-bar linkage can be obtained by solving the linear program (3.1), and the results are, respectively, shown as the pan dash line and triangle dash dot line in Figure 3. Besides the system reliability of the manipulator using Monte-Carlo simulation with $10^{6}$ samples is shown as the point solid line. What needs to be specially notified is that, in order to demonstrate the proposed mechanism system reliability analysis method, the truncation errors caused by the first-order Taylor expansion are omitted in the Monte-Carlo simulation. Comparing with the results of numerical simulation, the kinematic performance system reliability of the doublerocker four-bar linkage obtained by the proposed method is of high accuracy.

## 5. Conclusions

Using the mechanism accuracy theory and (system) reliability analysis method, this paper proposes a general method for system reliability analysis of planar linkages with correlated failure modes. The proposed method is applicable to any system defined as a logical expression of kinematic failure modes of planar linkages. This includes series and parallel systems, as well as general systems. Utilization of the first-order Taylor expansion technique in error estimation of kinematic performance of mechanisms must result in a certain degree of truncation errors. And these errors will increase with the increase of sensitivity of performance functions to design parameters. The accuracy of system reliability analysis can be improved by increasing of the order of Taylor expansion. However, in this process, the
complexity of calculation will greatly increase. Further studies are needed to provide a more precise and robust method for reliability analysis of kinematic accuracy of mechanisms.

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Research Article

# Multi-Period Mean-Variance Portfolio Selection with Uncertain Time Horizon When Returns Are Serially Correlated 

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We study a multi-period mean-variance portfolio selection problem with an uncertain time horizon and serial correlations. Firstly, we embed the nonseparable multi-period optimization problem into a separable quadratic optimization problem with uncertain exit time by employing the embedding technique of Li and Ng (2000). Then we convert the later into an optimization problem with deterministic exit time. Finally, using the dynamic programming approach, we explicitly derive the optimal strategy and the efficient frontier for the dynamic mean-variance optimization problem. A numerical example with $\operatorname{AR}(1)$ return process is also presented, which shows that both the uncertainty of exit time and the serial correlations of returns have significant impacts on the optimal strategy and the efficient frontier.

## 1. Introduction

The portfolio selection problem, which is one of great importance from both theoretical and practical perspectives, aims to find the best allocation of wealth among different assets in financial market. The mean-variance analysis pioneered by Markowitz [1] is one of the most widely used frameworks dealing with portfolio selection problems. In the past few decades, the mean-variance model stimulates a great deal of extensions and applied researches under single-period setting. Due to the nonseparability of multi-period mean-variance models, only up to 2000, Li and Ng [2] develop the embedding technique and solve a multi-period meanvariance portfolio selection problem analytically. In their work, the returns of risky assets are
assumed to be independent and identically distributed, and this assumption is also adopted by lots of the later literature, such as Guo and Hu [3].

A large number of empirical analyses of the assets price dynamics show that there exist salient serial correlations in the returns of financial assets, and the correlation structure is very complicated. The ARMA model is developed to study the feature of the financial assets returning with serial correlations in the field of econometrics, and it is widely used in the empirical research of financial market. Hakansson $[4,5]$ had already taken the impact of serial correlations into account on his portfolio selection problems and had investigated the myopic optimal portfolio strategies when there existed serial correlations of yields and not. However, due to the complexity of multi-period portfolio selection problem with serial correlations of returns, there are little relevant literature and results focused on the impact of serial correlations on the optimal portfolio selection strategy. Balvers and Mitchell [6] first derived an analytical solution for a dynamic portfolio selection problem with autocorrelation assets returns, where the utility function was a negative exponential function, and the assets returns were subject to the normal ARMA $(1,1)$ process. Dokuchaev [7] analyzed a discretetime portfolio selection model with serial correlations and found the correlation structure which ensured the optimal strategy being myopic for both the power and the log utility functions. Çelikyurt and Özekici [8] studied such models with the assumption that the market evolution followed a Markov chain and the states were observable, whose objective functions depended on the mean and the variance of the terminal wealth. Çanakoğlu and Özekici [9] considered the utility maximization problem with imperfect information modulated by a hidden Markov chain, and obtained the explicit characterization of the optimal strategy and the value function. Wei and Ye [10] extended the work of [8] to take the risk control over bankruptcy into consideration. Xu and Li [11] investigated a multiperiod mean-variance portfolio selection problem with one risky asset whose returns were serially correlated. By using the embedding technique of Li and Ng [2] and the dynamic programming approach, they obtained the explicit optimal strategy and proposed a measure of the risky asset value. To our knowledge, up to now, quite a few papers consider serial correlations of returns under dynamic portfolio selection framework.

On the other hand, the literature mentioned above makes an important hypothesis, implicitly or explicitly, that an investor knows her/his final exit time exactly at the moment of entering the market and making investment decisions, that is, the investment horizon is deterministic, either finite or infinite. In practice, however, the investor's exit time may be impacted by many exogenous and endogenous factors. An investor may exit from the market when she/he faces an unexpected need of huge consumption, sudden death, job loss, early retirement, investment target achieved, and so forth. Thus, it is more practical to weaken the restrictive assumption that the investment horizon is deterministic. If the exit time is uncertain, it is a random variable. As far as we know, study on the uncertain exit time can be dated back to Yarri [12], who studied an optimal consumption problem with an uncertain investment horizon. Hakansson [13] extended the work of [12] to a multi-period setting with a risky asset and an uncertain time horizon. Merton [14] addressed a dynamic optimal investment and consumption problem, and the uncertain retiring time was defined as the first jump of an independent Poisson process. Karatzas and Wang [15] considered an optimal investment problem in complete markets with the assumption that the exit time was a stopping time of the asset price filtration. Martellini and Uros̆ević [16] extended the original model of [1] to a static mean-variance model in which the exit time was dependent on asset returns. Guo and Hu [3] analyzed a multi-period mean-variance investment problem with uncertainty time of exiting. Huang et al. [17] dealt with the portfolio selection problem with
uncertain time horizon by adopting the worst-case CVaR methodology. Blanchet-Scalliet et al. [18] extended the optimal investment problem of [14] to allow the investor's time horizon to be stochastic and correlate to the returns of risky assets. Yi et al. [19] considered a multiperiod asset-liability management problem with an uncertain investment horizon under the meanvariance framework.

To the best of our knowledge, there is no work that considers the multiperiod meanvariance portfolio selection with an uncertain investment horizon and serially correlate returns at the same time. In the present paper, we try to tackle such problem. We assume that the distribution of the exit time is known, and the serial correlations of risky asset returns are settled the same as Hakansson [5] and Xu and Li [11]. We first embed our nonseparable problem, in the sense of dynamic programming, into a separable one by employing the embedding technique of Li and Ng [2]; then transform the separable problem with uncertain exit time into one with deterministic time horizon; finally solve the problem with deterministic time horizon by using the dynamic programming approach.

The rest of the paper is organized as follows. Section 2 formulates our problem and embeds it into a separable auxiliary problem. In Section 3, we solve the tractable auxiliary problem. In Section 4, we derive the optimal strategy and the efficient frontier of the original problem. Section 5 extends the results to the case of multiple risky assets. Section 6 gives a numerical simulation to show the impacts of exit time and serial correlations on the meanvariance efficient frontier. Finally, we conclude the paper in Section 7.

## 2. Modeling

We consider a financial market consisting of a risky asset and a riskless asset. The return rates of the riskless asset and risky asset at period $t+1$ (the time interval from time $t$ to time $t+1$ ) are denoted by $r_{t}^{0}$ and $r_{t}$, respectively. It is assumed that $r_{t}^{0}$ is a constant and $r_{t}$ is a $(t+1)$-measurable random variable. The risky asset will not degenerate into the riskless asset at any period, and its return rates $\left\{r_{t}, t=0,1, \ldots\right\}$ are correlated, that is, the value of $r_{t}$ is dependent on the values of $r_{s}, s<t$, which are the realized returns of risky asset at the past periods. Thus, at time $t$, the expectation of a random variable, denoted by $E_{t}$, is a conditional expectation based on all of the history information up to time $t$.

We assume that an investor, who joins the market at time 0 with the initial wealth $x_{0}$, may invest her/his wealth among the risky asset and the riskless asset within a time horizon of $T$ periods. At the beginning of each period $t(t=1, \ldots, T)$, the investor may adjust the amount invested in the risky and riskless assets by transaction. However, she/he may be forced to leave the financial market at time $\tau$ before $T$ by some uncontrollable reasons. The uncertain exit time $\tau$ is supposed to be an exogenously random variable with the discrete probability distribution $\tilde{p}_{t}=\operatorname{Pr}\{\tau=t\}, t=1,2, \ldots$. Therefore, the actual exit time of the investor is $T \wedge \tau:=\min \{T, \tau\}$, and its probability distribution is

$$
p_{t}:=\operatorname{Pr}\{T \wedge \tau=t\}= \begin{cases}\tilde{p}_{t}, & t=1, \ldots, T-1,  \tag{2.1}\\ 1-\sum_{t=1}^{T-1} \tilde{p}_{t}, & t=T .\end{cases}
$$

Let $u_{t}$ be the amount invested in the risky asset at the beginning of period $t+1$. The investment series over $T$ periods, $u:=\left\{u_{0}, u_{1}, \ldots, u_{T-1}\right\}$, is called an investment strategy.

Define the excess return of risky asset at period $t+1(t=0,1, \ldots, T-1)$ as $R_{t}=r_{t}-r_{t}^{0}$, which is assumed to be nondegenerated before time $t+1$, that is, the risky asset will not degenerate into the riskless asset at period $t+1$. Let $x_{t}$ be the wealth of the investor at time $t(t=0,1, \ldots, T)$. If the investment strategy $u$ is used in a self-financing way, the wealth dynamics can be described mathematically as

$$
\begin{equation*}
x_{t+1}=r_{t}^{0} x_{t}+R_{t} u_{t}, \quad t=0,1, \ldots, T-1 \tag{2.2}
\end{equation*}
$$

The multi-period mean-variance portfolio selection problem with uncertain exit time and serially correlate returns now can be formulated as

$$
P(\omega) \begin{cases}\max _{u} & E_{0}\left(x_{T \wedge \tau}\right)-\omega \operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)  \tag{2.3}\\ \text { s.t. } & x_{t+1}=r_{t}^{0} x_{t}+R_{t} u_{t}, \quad t=0,1, \ldots, T-1,\end{cases}
$$

where $\omega$ is a given positive constant, representing the degree of the investor's risk aversion, and $\operatorname{Var}_{0}$ is the variance conditional on the information available at time 0 . There are some other assumptions with respect to model $P(\omega)$, which are summarized as follows: (a) short selling is permitted at any periods for the risky asset; (b) transaction costs and fees are negligible; (c) the investor can borrow and lend the riskless asset at any periods without limitation.

Recall that the mean-variance model $P(\omega)$ is difficult to solve due to its nonseparable structure in the sense of dynamic programming, which is one of the most powerful and universal methodologies for optimization problems with separable nature. Fortunately, Li and Ng [2] propose an embedding technique, and this technique is also applicable to solve the current problem with uncertain exit time and serially correlate returns. Instead of solving problem $P(\omega)$ directly, we first consider the following auxiliary problem:

$$
A(\lambda, \omega) \begin{cases}\max _{u} & E_{0}\left(\lambda x_{T \wedge \tau}-\omega x_{T \wedge \tau}^{2}\right)  \tag{2.4}\\ \text { s.t. } & x_{t+1}=r_{t}^{0} x_{t}+R_{t} u_{t}, \quad t=0,1, \ldots, T-1\end{cases}
$$

for a given constant $\lambda>0$.
Let $\Psi_{A}(\lambda, \omega)$ and $\Psi_{P}(\omega)$ be the optimal solution sets of problem $A(\lambda, \omega)$ and $P(\omega)$, respectively, namely,

$$
\begin{align*}
\Psi_{A}(\lambda, \omega) & =\{u \mid u \text { is an optimal solution of } A(\lambda, \omega)\}, \\
\Psi_{P}(\omega) & =\{u \mid u \text { is an optimal solution of } P(\omega)\} \tag{2.5}
\end{align*}
$$

The following two theorems can be proven by a similar method to that described in Li and Ng [2], and so their proofs are omitted.

Theorem 2.1. For any optimal solution $u^{*}$ of $\Psi_{P}(\omega), u^{*}$ is the optimal solution of $\Psi_{A}\left(\lambda^{*}, \omega\right)$ with $\lambda^{*}=1+\left.2 \omega E_{0}\left(x_{T \wedge \tau}\right)\right|_{u^{*}}$.

Theorem 2.2. If $u^{*} \in \Psi_{A}\left(\lambda^{*}, \omega\right)$, a necessary condition for $u^{*} \in \Psi_{P}(\omega)$ is $\lambda^{*}=1+\left.2 \omega E_{0}\left(x_{T \wedge \tau}\right)\right|_{u^{*}}$.

## 3. Analytical Solution of Auxiliary Problem $A(\lambda, \omega)$

In this section, we translate the auxiliary problem $A(\lambda, \omega)$ into a portfolio selection problem with certain exit time and then solve it by using the dynamic programming approach.

Since

$$
\begin{align*}
E_{0}\left[\lambda x_{T \wedge \tau}-\omega x_{T \wedge \tau}^{2}\right] & =\sum_{t=1}^{T} E_{0}\left[\lambda x_{T \wedge \tau}-\omega x_{T \wedge \tau}^{2} \mid T \wedge \tau=t\right] P(T \wedge \tau=t) \\
& =E_{0}\left[\sum_{t=1}^{T}\left(\lambda x_{t}-\omega x_{t}^{2}\right) p_{t}\right], \tag{3.1}
\end{align*}
$$

problem $A(\lambda, \omega)$ can be written equivalently as

$$
A(\lambda, \omega) \begin{cases}\max _{u} & E_{0}\left[\sum_{t=1}^{T}\left(\lambda x_{t}-\omega x_{t}^{2}\right) p_{t}\right]  \tag{3.2}\\ \text { s.t. } & x_{t+1}=r_{t}^{0} x_{t}+R_{t} u_{t}, \quad t=0,1, \ldots, T-1 .\end{cases}
$$

Define the value function

$$
\begin{align*}
f_{t}^{*}\left(x_{t}\right) & =\max _{u_{t}} f_{t}\left(x_{t}\right) \\
& =\max _{u_{t}} E_{t}\left[\sum_{s=t}^{T}\left(\lambda x_{s}-\omega x_{s}^{2}\right) p_{s}\right] \tag{3.3}
\end{align*}
$$

as the optimal expected utility using the optimal strategy conditional on the information available at time $t(t=0,1, \ldots, T-1)$, and the boundary condition is

$$
\begin{equation*}
f_{T}^{*}\left(x_{T}\right)=\left(\lambda x_{T}-\omega x_{T}^{2}\right) p_{T} . \tag{3.4}
\end{equation*}
$$

According to the dynamic programming principle, we have the Bellman equation

$$
\begin{align*}
f_{t}^{*}\left(x_{t}\right) & =\max _{u_{t}} f_{t}\left(x_{t}\right) \\
& =\max _{u_{t}} E_{t}\left[\left(\lambda x_{t}-\omega x_{t}^{2}\right) p_{t}+f_{t+1}^{*}\left(x_{t+1}\right)\right], \tag{3.5}
\end{align*}
$$

for $t=0,1, \ldots, T-1$.

First, we give the following notations:

$$
\begin{align*}
& \theta_{t}=\frac{E_{t}^{2}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}, \quad t=0,1, \ldots, T-1,  \tag{3.6}\\
& \Xi_{t}=E_{t}\left[\frac{\lambda^{2}}{4 \omega} \sum_{s=t}^{T-1} \theta_{s}\right], \quad t=0,1, \ldots, T-1,  \tag{3.7}\\
& \omega_{t}=p_{t}+\left(r_{t}^{0}\right)^{2}\left[E_{t}\left(\omega_{t+1}\right)-\frac{E_{t}^{2}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}\right], \quad t=0,1, \ldots, T-1, \quad \omega_{T}=p_{T},  \tag{3.8}\\
& \lambda_{t}=p_{t}+r_{t}^{0}\left[E_{t}\left(\lambda_{t+1}\right)-\frac{E_{t}\left(\omega_{t+1} R_{t}\right) E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}\right], \quad t=0,1, \ldots, T-1, \lambda_{T}=p_{T} . \tag{3.9}
\end{align*}
$$

For notational simplicity, we define $\sum_{j=s}^{t}(\cdot)_{j}=0$ and $\prod_{j=s}^{t}(\cdot)_{j}=1$ if $s>t$.
Note that $R_{t}$ and $R_{t+1}$ are not independent of each other for $t=0,1, \ldots, T-1$, so both $\omega_{t+1}$ and $\lambda_{t+1}$ are dependent on the risky asset return at period $t+1, R_{t}$. Then, for $t=0,1, \ldots, T-$ 1,

$$
\begin{equation*}
E_{t}\left(\omega_{t+1} R_{t}\right) \neq E_{t}\left(\omega_{t+1}\right) E_{t}\left(R_{t}\right), \quad E_{t}\left(\lambda_{t+1} R_{t}\right) \neq E_{t}\left(\lambda_{t+1}\right) E_{t}\left(R_{t}\right) \tag{3.10}
\end{equation*}
$$

The following lemma comes from Xu and Li [11]. For the completeness, we provide its proof here.

Lemma 3.1. Let $x$ be a nondegenerated random variable, and let $\xi$ be a positive random variable under the information at time $t$, then $E_{t}\left(x^{2} \xi\right) E_{t}(\xi)>\left(E_{t}(x \xi)\right)^{2}$.
Proof. Since $\xi$ is a positive random variable, we can define a new probability measure $Q$ as

$$
\begin{equation*}
\mathrm{d} Q \triangleq \frac{\xi}{E_{t}(\xi)} \mathrm{d} P \tag{3.11}
\end{equation*}
$$

where $P$ is the original measure. Since $x$ is a nondegenerated random variable, we have, under measure $Q$,

$$
\begin{equation*}
\operatorname{Var}_{t}^{Q}(x)=E_{t}^{Q}\left(x^{2}\right)-\left(E_{t}^{Q}(x)\right)^{2}>0 \tag{3.12}
\end{equation*}
$$

Transforming the above inequality to under measure $P$, we obtain

$$
\begin{equation*}
E_{t}\left(x^{2} \frac{\xi}{E_{t}(\xi)}\right)-\left(E_{t}\left(x \frac{\xi}{E_{t}(\xi)}\right)\right)^{2}>0 \tag{3.13}
\end{equation*}
$$

Multiplying both sides by $\left(E_{t}(\xi)\right)^{2}$ in the above inequality produces

$$
\begin{equation*}
E_{t}\left(x^{2} \xi\right) E_{t}(\xi)>\left(E_{t}(x \xi)\right)^{2} \tag{3.14}
\end{equation*}
$$

This completes the proof.

Theorem 3.2. For $t=0,1, \ldots, T-1, \omega_{t}>0, \theta_{t} \geq 0$, and $\Xi_{t} \geq 0$.
Proof. We use induction. For $t=T-1$, since the return of the risky asset at period $T, R_{T-1}$, is a nondegenerated random variable, then

$$
\begin{equation*}
\operatorname{Var}_{T-1}\left(R_{T-1}\right)=E_{T-1}\left(R_{T-1}^{2}\right)-\left(E_{T-1}\left(R_{T-1}\right)\right)^{2}>0 \tag{3.15}
\end{equation*}
$$

$E_{T-1}\left(R_{T-1}^{2}\right)>0$. So

$$
\begin{align*}
& 0 \leq \theta_{T-1}=\frac{E_{T-1}^{2}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}=p_{T} \frac{E_{T-1}^{2}\left(R_{T-1}\right)}{E_{T-1}\left(R_{T-1}^{2}\right)}<p_{T}, \\
& \Xi_{T-1}=E_{T-1}\left(\frac{\lambda^{2}}{4 \omega} \theta_{T-1}\right) \geq 0 . \tag{3.16}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\omega_{T-1}=p_{T-1}+\left(r_{T-1}^{0}\right)^{2}\left(p_{T}-\frac{E_{T-1}^{2}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}\right)>0 \tag{3.17}
\end{equation*}
$$

Suppose that $\omega_{s}>0, \theta_{s} \geq 0$, and $\Xi_{s} \geq 0$ hold true for $s=t+1, \ldots, T-2, T-1$, then for period $t$,

$$
\begin{equation*}
\theta_{t}=\frac{E_{t}^{2}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} \geq 0 \tag{3.18}
\end{equation*}
$$

By Lemma 3.1, we can easily see that

$$
\begin{equation*}
E_{t}\left(\omega_{t+1}\right)>\frac{E_{t}^{2}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} \tag{3.19}
\end{equation*}
$$

Hence, we obtain

$$
\begin{align*}
& \Xi_{t}=E_{t}\left(\sum_{s=t}^{T-1} \frac{\lambda^{2}}{4 \omega} \theta_{s}\right)=\frac{\lambda^{2}}{4 \omega} \theta_{t}+E_{t}\left(\Xi_{t+1}\right) \geq 0 \\
& \omega_{t}=p_{t}+\left(r_{t}^{0}\right)^{2}\left[E_{t}\left(\omega_{t+1}\right)-\frac{E_{t}^{2}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}\right]>0 \tag{3.20}
\end{align*}
$$

By induction, it shows that for $t=0,1, \ldots, T-1, \omega_{t}>0, \theta_{t} \geq 0$, and $\Xi_{t} \geq 0$.
The analytical optimal strategy and the value function of problem $A(\lambda, \omega)$ can be derived by using dynamic programming approach, which are summarized in the following theorem.

Theorem 3.3. The optimal strategy and the value functions of problem $A(\lambda, \omega)$ are, respectively, given by

$$
\begin{gather*}
u_{t}^{*}=\frac{\lambda}{2 \omega} \frac{E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}-\frac{E_{t}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} r_{t}^{0} x_{t}, \quad t=0,1, \ldots, T-1,  \tag{3.21}\\
f_{t}^{*}\left(x_{t}\right)=-\omega \omega_{t} x_{t}^{2}+\lambda \lambda_{t} x_{t}+\Xi_{t}, \quad t=0,1, \ldots, T-1 \tag{3.22}
\end{gather*}
$$

where $\Xi_{t}, \omega_{t}$, and $\lambda_{t}$ are given as defined in (3.7)-(3.9).
Proof. We will show that the above recursive formulas hold true by induction starting with the boundary condition $f_{T}\left(x_{T}\right)=\left(\lambda x_{T}-\omega x_{T}^{2}\right) p_{T}$. Note that for $t=T-1$,

$$
\begin{align*}
f_{T-1}^{*}\left(x_{T-1}\right)= & \max _{u_{T-1}} f_{T-1}\left(x_{T-1}\right) \\
= & \max _{u_{T-1}} E_{T-1}\left[\left(\lambda x_{T-1}-\omega x_{T-1}^{2}\right) p_{T-1}+f_{T}^{*}\left(x_{T}\right)\right] \\
= & \max _{u_{T-1}} E_{T-1}\left[\left(\lambda x_{T-1}-\omega x_{T-1}^{2}\right) p_{T-1}+\left(\lambda x_{T}-\omega x_{T}^{2}\right) p_{T}\right]  \tag{3.23}\\
= & \max _{u_{T-1}}\left(\lambda x_{T-1}-\omega x_{T-1}^{2}\right) p_{T-1}+\lambda\left[r_{T-1}^{0} p_{T} x_{T-1}+E_{T-1}\left(p_{T} R_{T-1}\right) u_{T-1}\right] \\
& -\omega\left[\left(r_{T-1}^{0}\right)^{2} p_{T} x_{T-1}^{2}+2 r_{T-1}^{0} E_{T-1}\left(p_{T} R_{T-1}\right) u_{T-1} x_{T-1}+E_{T-1}\left(p_{T} R_{T-1}^{2}\right) u_{T-1}^{2}\right] .
\end{align*}
$$

Since $E_{T-1}\left(R_{T-1}^{2}\right)>0$ by assumption, the function $f_{T-1}\left(x_{T-1}\right)$ is a concave function of $u_{T-1}$. The first-order condition gives

$$
\begin{equation*}
\lambda E_{T-1}\left(p_{T} R_{T-1}\right)-2 \omega\left[r_{T-1}^{0} E_{T-1}\left(p_{T} R_{T-1}\right) x_{T-1}+E_{T-1}\left(p_{T} R_{T-1}^{2}\right) u_{T-1}\right]=0 \tag{3.24}
\end{equation*}
$$

which yields the optimal solution $u_{T-1}^{*}$ as

$$
\begin{equation*}
u_{T-1}^{*}=\frac{\lambda}{2 \omega} \frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}-\frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)} r_{T-1}^{0} x_{T-1} \tag{3.25}
\end{equation*}
$$

Substituting $u_{T-1}^{*}$ back into $f_{T-1}\left(x_{T-1}\right)$, it follows that

$$
\begin{align*}
& f_{T-1}^{*}\left(x_{T-1}\right)=\left(\lambda x_{T-1}-\omega x_{T-1}^{2}\right) p_{T-1} \\
&+\lambda\left[r_{T-1}^{0} p_{T} x_{T-1}+E_{T-1}\left(p_{T} R_{T-1}\right)\left(\frac{\lambda}{2 \omega} \frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}-\frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)} r_{T-1}^{0} x_{T-1}\right)\right] \\
&-\omega\left[\left(r_{T-1}^{0}\right)^{2} p_{T} x_{T-1}^{2}+2 r_{T-1}^{0} E_{T-1}\left(p_{T} R_{T-1}\right)\right. \\
& \times\left(\frac{\lambda}{2 \omega} \frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}-\frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)} r_{T-1}^{0} x_{T-1}\right) x_{T-1} \\
&\left.+E_{T-1}\left(p_{T} R_{T-1}^{2}\right)\left(\frac{\lambda}{2 \omega} \frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}-\frac{E_{T-1}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)} r_{T-1}^{0} x_{T-1}\right)^{2}\right] \\
&=-\omega\left[p_{T-1}+\left(r_{T-1}^{0}\right)^{2}\left(p_{T}-\frac{E_{T-1}^{2}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}\right)\right] x_{T-1}^{2} \\
&+\lambda\left[p_{T-1}+r_{T-1}^{0}\left(p_{T}-\frac{E_{T-1}^{2}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)}\right)\right] x_{T-1}^{2}+\frac{\lambda^{2}}{4 \omega} \frac{E_{T-1}^{2}\left(p_{T} R_{T-1}\right)}{E_{T-1}\left(p_{T} R_{T-1}^{2}\right)} \\
&=- \\
&=-\omega \omega_{T-1} x_{T-1}^{2}+\lambda \lambda_{T-1} x_{T-1}+\frac{\lambda^{2}}{4 \omega} \theta_{T-1}  \tag{3.26}\\
&=- x_{T-1}^{2}+\lambda \lambda_{T-1} x_{T-1}+\Xi_{T-1} .
\end{align*}
$$

Hence, the conclusion holds true for $t=T-1$.
Now we assume that the conclusion holds true for time $t+1$, in other words,

$$
\begin{gather*}
f_{t+1}^{*}\left(x_{t+1}\right)=-\omega \omega_{t+1} x_{t+1}^{2}+\lambda \lambda_{t+1} x_{t+1}+\Xi_{t+1}, \\
u_{t+1}^{*}=\frac{\lambda}{2 \omega} \frac{E_{t+1}\left(\lambda_{t+2} R_{t+1}\right)}{E_{t+1}\left(\omega_{t+2} R_{t+1}^{2}\right)}-\frac{E_{t+1}\left(\omega_{t+2} R_{t+1}\right)}{E_{t+1}\left(\omega_{t+2} R_{t+1}^{2}\right)} r_{t+1}^{0} x_{t+1}, \tag{3.27}
\end{gather*}
$$

then the optimization problem at time $t$ for given state $x_{t}$ is

$$
\begin{aligned}
f_{t}^{*}\left(x_{t}\right) & =\max _{u_{t}} f_{t}\left(x_{t}\right) \\
& =\max _{u_{t}} E_{t}\left[\left(\lambda x_{t}-\omega x_{t}^{2}\right) p_{t}+f_{t+1}^{*}\left(x_{t+1}\right)\right] \\
& =\max _{u_{t}} E_{t}\left[\left(\lambda x_{t}-\omega x_{t}^{2}\right) p_{t}-\omega \omega_{t+1} x_{t+1}^{2}+\lambda \lambda_{t+1} x_{t+1}+\Xi_{t+1}\right]
\end{aligned}
$$

$$
\begin{align*}
=\max _{u_{t}}\{ & \left(\lambda x_{t}-\omega x_{t}^{2}\right) p_{t}-\omega\left[\left(r_{t}^{0}\right)^{2} x_{t}^{2} E_{t}\left(\omega_{t+1}\right)+2 r_{t}^{0} E_{t}\left(\omega_{t+1} R_{t}\right) u_{t} x_{t}+E_{t}\left(\omega_{t+1} R_{t}^{2}\right) u_{t}^{2}\right] \\
& \left.+\lambda\left(r_{t}^{0} x_{t} E_{t}\left(\lambda_{t+1}\right)+E_{t}\left(\lambda_{t+1} R_{t}\right) u_{t}\right)+E_{t}\left(\Xi_{t+1}\right)\right\} \tag{3.28}
\end{align*}
$$

Noting that $E_{t}\left(\omega_{t+1} R_{t}^{2}\right)>0$ by Theorem 3.2, the function $f_{t}\left(x_{t}\right)$ is also a concave function of $u_{t}$. The first-order condition yields

$$
\begin{equation*}
u_{t}^{*}=\frac{\lambda}{2 \omega} \frac{E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}-\frac{E_{t}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} r_{t}^{0} x_{t} \tag{3.29}
\end{equation*}
$$

Therefore, for the above given $u_{t}^{*}$,

$$
\begin{align*}
f_{t}^{*}\left(x_{t}\right)= & \left(\lambda x_{t}-\omega x_{t}^{2}\right) p_{t} \\
& +\lambda\left[r_{t}^{0} x_{t} E_{t}\left(\lambda_{t+1}\right)+E_{t}\left(\lambda_{t+1} R_{t}\right)\left(\frac{\lambda}{2 \omega} \frac{E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}-\frac{E_{t}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} r_{t}^{0} x_{t}\right)\right] \\
- & \omega\left[\left(r_{t}^{0}\right)^{2} x_{t}^{2} E_{t}\left(\omega_{t+1}\right)+2 r_{t}^{0} E_{t}\left(\omega_{t+1} R_{t}\right)\left(\frac{\lambda}{2 \omega} \frac{E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}-\frac{E_{t}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} r_{t}^{0} x_{t}\right) x_{t}\right. \\
& \left.\quad+E_{t}\left(\omega_{t+1} R_{t}^{2}\right)\left(\frac{\lambda}{2 \omega} \frac{E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}-\frac{E_{t}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} r_{t}^{0} x_{t}\right)^{2}\right]+E_{t}\left(\Xi_{t+1}\right) \\
=- & \omega\left[p_{t}+\left(r_{t}^{0}\right)^{2}\left(E_{t}\left(\omega_{t+1}\right)-\frac{E_{t}^{2}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}\right)\right] x_{t}^{2} \\
& +\lambda\left[p_{t}+r_{t}^{0}\left(E_{t}\left(\lambda_{t+1}\right)-\frac{E_{t}\left(\omega_{t+1} R_{t}\right) E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}\right)\right] x_{t}+\frac{\lambda^{2}}{4 \omega} \theta_{t}+E_{t}\left(\Xi_{t+1}\right) \\
= & -\omega \omega_{t} x_{t}^{2}+\lambda \lambda_{t} x_{t}+\Xi_{t} . \tag{3.30}
\end{align*}
$$

Hence, the conclusion is true for $t$. By induction, the theorem is true.

## 4. Optimal Strategy and the Efficient Frontier of the Original Problem $P(\omega)$

If we insert the optimal strategy given in Theorem 3.3 into the dynamic process of wealth, $x_{T}$ and $x_{T}^{2}$ can be expressed as

$$
\begin{aligned}
x_{T} & =r_{T-1}^{0} x_{T-1}+R_{T-1} u_{T-1}^{*} \\
& =\left[1-\frac{R_{T-1} E_{T-1}\left(\omega_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}\right] r_{T-1}^{0} x_{T-1}+\frac{\lambda}{2 \omega} \frac{R_{T-1} E_{T-1}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}
\end{aligned}
$$

$$
\begin{align*}
x_{T}^{2}= & {\left[1-\frac{2 R_{T-1} E_{T-1}\left(\omega_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}+\frac{R_{T-1}^{2} E_{T-1}^{2}\left(\omega_{T} R_{T-1}\right)}{E_{T-1}^{2}\left(\omega_{T} R_{T-1}^{2}\right)}\right]\left(r_{T-1}^{0}\right)^{2} x_{T-1}^{2} } \\
& +\frac{\lambda}{\omega}\left[\frac{R_{T-1} E_{T-1}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}-\frac{R_{T-1}^{2} E_{T-1}\left(\omega_{T} R_{T-1}\right) E_{T-1}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}^{2}\left(\omega_{T} R_{T-1}^{2}\right)}\right] r_{T-1}^{0} x_{T-1} \\
& +\frac{\lambda^{2}}{4 \omega^{2}} \frac{R_{T-1}^{2} E_{T-1}^{2}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}^{2}\left(\omega_{T} R_{T-1}^{2}\right)}, \tag{4.1}
\end{align*}
$$

$$
\begin{align*}
\lambda_{T} x_{T}= & {\left[\lambda_{T}-\frac{\lambda_{T} R_{T-1} E_{T-1}\left(\omega_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}\right] r_{T-1}^{0} x_{T-1}+\frac{\lambda}{2 \omega} \frac{\lambda_{T} R_{T-1} E_{T-1}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}, } \\
\omega_{T} x_{T}^{2}= & {\left[\omega_{T}-\frac{2 \omega_{T} R_{T-1} E_{T-1}\left(\omega_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}+\frac{\omega_{T} R_{T-1}^{2} E_{T-1}^{2}\left(\omega_{T} R_{T-1}\right)}{E_{T-1}^{2}\left(\omega_{T} R_{T-1}^{2}\right)}\right]\left(r_{T-1}^{0}\right)^{2} x_{T-1}^{2} } \\
& +\frac{\lambda}{\omega}\left[\frac{\omega_{T} R_{T-1} E_{T-1}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)}-\frac{\omega_{T} R_{T-1}^{2} E_{T-1}\left(\omega_{T} R_{T-1}\right) E_{T-1}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}^{2}\left(\omega_{T} R_{T-1}^{2}\right)}\right] r_{T-1}^{0} x_{T-1}  \tag{4.2}\\
& +\frac{\lambda^{2}}{4 \omega^{2}} \frac{\omega_{T} R_{T-1}^{2} E_{T-1}^{2}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}^{2}\left(\omega_{T} R_{T-1}^{2}\right)} .
\end{align*}
$$

Taking expectations on both sides of (4.2) based on the information available at time $T-1$, we conclude that

$$
\begin{align*}
E_{T-1}\left(\lambda_{T} x_{T}\right) & =\lambda_{T-1} x_{T-1}-p_{T-1} x_{T-1}+\frac{\lambda}{2 \omega} \theta_{T-1}  \tag{4.3}\\
E_{T-1}\left(\omega_{T} x_{T}^{2}\right) & =\omega_{T-1} x_{T-1}^{2}-p_{T-1} x_{T-1}^{2}+\frac{\lambda^{2}}{4 \omega^{2}} \theta_{T-1} \tag{4.4}
\end{align*}
$$

where

$$
\begin{equation*}
\theta_{T-1}=\frac{E_{T-1}^{2}\left(\lambda_{T} R_{T-1}\right)}{E_{T-1}\left(\omega_{T} R_{T-1}^{2}\right)} \tag{4.5}
\end{equation*}
$$

The above equations are recursive equations, and by taking expectations on both sides of (4.3) and (4.4) at time $T-2, \ldots, 1,0$ repeatedly, we obtain

$$
\begin{gather*}
E_{0}\left(x_{T \wedge \tau}\right)=\sum_{t=1}^{T} p_{t} E_{0}\left(x_{t}\right)=\lambda_{0} x_{0}+\frac{\lambda}{2 \omega} \sum_{t=1}^{T} E_{0}\left(\theta_{t-1}\right)=\lambda_{0} x_{0}+\frac{\lambda}{2 \omega} \Theta,  \tag{4.6}\\
E_{0}\left(x_{T \wedge \tau}^{2}\right)=\sum_{t=1}^{T} p_{t} E_{0}\left(x_{t}^{2}\right)=\omega_{0} x_{0}^{2}+\frac{\lambda^{2}}{4 \omega^{2}} \sum_{t=1}^{T} E_{0}\left(\theta_{t-1}\right)=\omega_{0} x_{0}^{2}+\frac{\lambda^{2}}{4 \omega^{2}} \Theta, \tag{4.7}
\end{gather*}
$$

where

$$
\begin{equation*}
\theta_{t}=\frac{E_{t}^{2}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}, \quad \Theta=\sum_{t=1}^{T} E_{0}\left(\theta_{t-1}\right) . \tag{4.8}
\end{equation*}
$$

With the results (4.6) and (4.7), the variance of the terminal wealth $x_{T \wedge \tau}$ under the optimal strategy (3.21) can be written as

$$
\begin{align*}
\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right) & =E_{0}\left(x_{T \wedge \tau}^{2}\right)-\left[E_{0}\left(x_{T \wedge \tau}\right)\right]^{2} \\
& =E_{0}\left(\sum_{t=1}^{T} p_{t} x_{t}^{2}\right)-\left[E_{0}\left(\sum_{t=1}^{T} p_{t} x_{t}\right)\right]^{2} \\
& =\omega_{0} x_{0}^{2}+\frac{\lambda^{2}}{4 \omega^{2}} \Theta-\left[\lambda_{0} x_{0}+\frac{\lambda}{2 \omega} \Theta\right]^{2}  \tag{4.9}\\
& =\frac{\lambda^{2}}{4 \omega^{2}}\left(\Theta-\Theta^{2}\right)-\frac{\lambda}{\omega} \lambda_{0} \Theta x_{0}+\left(\omega_{0}-\lambda_{0}^{2}\right) x_{0}^{2} .
\end{align*}
$$

Lemma 4.1. $0<\Theta<1, \omega_{0}-\lambda_{0}^{2} /(1-\Theta)>0$.
Proof. First of all, we claim that $\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)>0$, since it measures the risk of investor at the time of exiting market, and the risky asset cannot degenerate into the riskless asset. Especially, when $x_{0}=0, \operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)$ can be reduced to

$$
\begin{equation*}
\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)=\frac{\lambda^{2}}{4 \omega^{2}}\left(\Theta-\Theta^{2}\right)>0 \tag{4.10}
\end{equation*}
$$

and it is easy to show that $0<\Theta<1$.
The expression of $\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)$ can be further converted into

$$
\begin{equation*}
\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)=\left(\Theta-\Theta^{2}\right)\left[\frac{\lambda}{2 \omega}-\frac{\lambda_{0}}{1-\Theta} x_{0}\right]^{2}+\left(\omega_{0}-\frac{\lambda_{0}^{2}}{1-\Theta}\right) x_{0}^{2}>0 \tag{4.11}
\end{equation*}
$$

Since we know that $0<\Theta<1$, the above inequality implies $\omega_{0}-\lambda_{0}^{2} /(1-\Theta)>0$, and we finish the proof of Lemma 4.1.

According to Theorem 2.2, a necessary condition for the optimal solution of auxiliary problem $A\left(\lambda^{*}, \omega\right)$ to attain the optimality of problem $P(\omega)$ at the same time is

$$
\begin{equation*}
\lambda^{*}=1+\left.2 \omega E_{0}\left(x_{T \wedge \tau}\right)\right|_{u *}=1+2 \omega\left(\lambda_{0} x_{0}+\frac{\lambda^{*}}{2 \omega} \Theta\right) . \tag{4.12}
\end{equation*}
$$

We can easily obtain

$$
\begin{equation*}
\lambda^{*}=\frac{1+2 \omega \lambda_{0} x_{0}}{1-\Theta} . \tag{4.13}
\end{equation*}
$$

Finally, substituting (4.13) back into (3.21) yields the analytically optimal strategy of the original problem $P(\omega)$, which is summarized in the following theorem.

Theorem 4.2. For the mean-variance problem $P(\omega)$, the optimal strategy is given by

$$
\begin{equation*}
u_{t}^{*}=\frac{1+2 \omega \lambda_{0} x_{0}}{2 \omega(1-\Theta)} \frac{E_{t}\left(\lambda_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)}-\frac{E_{t}\left(\omega_{t+1} R_{t}\right)}{E_{t}\left(\omega_{t+1} R_{t}^{2}\right)} r_{t}^{0} x_{t}, \quad t=0,1, \ldots, T-1 \tag{4.14}
\end{equation*}
$$

where $\Theta, \lambda_{0}, \omega_{t+1}$, and $\lambda_{t+1}$ are given as defined.
Referring to (4.6),

$$
\begin{equation*}
\frac{\lambda^{*}}{2 \omega}=\frac{E_{0}\left(x_{T \wedge \tau}\right)-\lambda_{0} x_{0}}{\Theta} \tag{4.15}
\end{equation*}
$$

Substituting (4.15) back into (4.11), the relationship between $\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)$ and $E_{0}\left(x_{T \wedge \tau}\right)$ can be shown as follows:

$$
\begin{equation*}
\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)=\frac{(1-\Theta)}{\Theta}\left[E_{0}\left(x_{T \wedge \tau}\right)-\frac{\lambda_{0} x_{0}}{1-\Theta}\right]^{2}+\left(\omega_{0}-\frac{\lambda_{0}^{2}}{1-\Theta}\right) x_{0}^{2} \tag{4.16}
\end{equation*}
$$

Therefore, the efficient frontier of the original problem $P(\omega)$ is given by (4.16) for

$$
\begin{equation*}
E_{0}\left(x_{T \wedge \tau}\right) \in\left[\frac{\lambda_{0} x_{0}}{1-\Theta},+\infty\right) \tag{4.17}
\end{equation*}
$$

From the efficient frontier (4.16) of the optimal dynamic mean-variance portfolio selection problem with an uncertain exit time, when returns are serially correlated, we can obtain the trade-off between the return and the risk when investor exits from market. Since all of the parameters $\Theta, \lambda_{0}$, and $\omega_{0}$ are functions of $p_{t}$ and $R_{t}$ for $t=0,1, \ldots, T-1$, both the exiting time and the correlations of the risky asset returns have impacts on the optimal strategy and the efficient frontier, and this is quite different from the cases with deterministic terminal time, and the risky asset returns at different periods are independent.

Remark 4.3. In Xu and Li [11], a multi-period portfolio selection problem with serial correlation and a certain exit time is studied. If $p_{1}=p_{2}=\cdots=p_{T-1}=0, p_{T}=1$, and $r_{t}^{0}=r, \quad t=0,1, \ldots, T-1$ in our model, our result is exactly the same as the one of Xu and Li [11]. So we generalize the model and results of Xu and Li [11] to the case with an uncertain investment horizon.

## 5. Extension to the Situation with Multiple Risky Assets

The results in the previous sections can be extended to the general situation with multiple risky assets. Suppose that there are $n$ risky assets and one riskless asset with period- $t+1$ returns $r_{t}^{i}(i=1,2, \ldots, n)$ and $r_{t}^{0}$, respectively. Define $e_{t}^{i}=r_{t}^{i}-r_{t}^{0}, e_{t}=\left(e_{t}^{1}, e_{t}^{2}, \ldots, e_{t}^{n}\right)^{\prime}$ and
$U_{t}=\left(u_{t}^{1}, u_{t}^{2}, \ldots, u_{t}^{n}\right)^{\prime}$ for $i=1,2, \ldots, n$ and $t=0,1, \ldots, T-1$, where $u_{t}^{i}$ is the amount invested in the $i$ th risky asset at time $t$. In this case, the wealth dynamics is described by

$$
\begin{equation*}
x_{t+1}=r_{t}^{0} x_{t}+e_{t}^{\prime} U_{t}, \quad t=0,1, \ldots, T-1 \tag{5.1}
\end{equation*}
$$

Accordingly, the multi-period mean-variance portfolio selection problem with an uncertain exit time and serial correlations can be formulated as

$$
\widehat{P}(\omega) \begin{cases}\max _{U} & E_{0}\left(x_{T \wedge \tau}\right)-\omega \operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)  \tag{5.2}\\ \text { s.t. } & x_{t+1}=r_{t}^{0} x_{t}+e_{t}^{\prime} U_{t}, \quad t=0,1, \ldots, T-1\end{cases}
$$

where $\omega \geq 0$ is a pregiven parameter, representing the degree of the investor's risk aversion.
With the same method as in the previous section, we can show the following theorem.
Theorem 5.1. For problem $\widehat{P}(\omega)$, the optimal investment strategy is given by

$$
\begin{equation*}
U_{t}^{*}=\frac{1+2 \omega \tilde{\lambda}_{0} x_{0}}{2 \omega(1-\tilde{\Theta})} E_{t}^{-1}\left(\tilde{\omega}_{t+1} e_{t} e_{t}^{\prime}\right) E_{t}\left(\tilde{\lambda}_{t+1} e_{t}\right)-E_{t}^{-1}\left(\tilde{\omega}_{t+1} e_{t} e_{t}^{\prime}\right) E_{t}\left(\tilde{\omega}_{t+1} e_{t}\right) r_{t}^{0} x_{t} \tag{5.3}
\end{equation*}
$$

for $t=0,1, \ldots, T-1$, and the efficient frontier is given by

$$
\begin{equation*}
\operatorname{Var}_{0}\left(x_{T \wedge \tau}\right)=\frac{(1-\tilde{\Theta})}{\tilde{\Theta}}\left[E_{0}\left(x_{T \wedge \tau}\right)-\frac{\tilde{\lambda}_{0} x_{0}}{1-\widetilde{\Theta}}\right]^{2}+\left(\tilde{\omega}_{0}-\frac{\tilde{\lambda}_{0}^{2}}{1-\widetilde{\Theta}}\right) x_{0}^{2} \tag{5.4}
\end{equation*}
$$

where

$$
\begin{align*}
& \tilde{\theta}_{t}=E_{t}\left(\tilde{\lambda}_{t+1} e_{t}^{\prime}\right) E_{t}^{-1}\left(\tilde{\omega}_{t+1} e_{t} e_{t}^{\prime}\right) E_{t}\left(\tilde{\lambda}_{t+1} e_{t}\right), \quad \tilde{\Theta}=\sum_{t=1}^{T} E_{0}\left(\tilde{\theta}_{t-1}\right), \\
& \tilde{\omega}_{t}=p_{t}+\left(r_{t}^{0}\right)^{2}\left[E_{t}\left(\tilde{\omega}_{t+1}\right)-E_{t}\left(\tilde{\omega}_{t+1} e_{t}^{\prime}\right) E_{t}^{-1}\left(\tilde{\omega}_{t+1} e_{t} e_{t}^{\prime}\right) E_{t}\left(\tilde{\omega}_{t+1} e_{t}\right)\right], \quad \tilde{\omega}_{T}=p_{T},  \tag{5.5}\\
& \tilde{\lambda}_{t}=p_{t}+r_{t}^{0}\left[E_{t}\left(\tilde{\lambda}_{t+1}\right)-E_{t}\left(\tilde{\lambda}_{t+1} e_{t}^{\prime}\right) E_{t}^{-1}\left(\tilde{\omega}_{t+1} e_{t} e_{t}^{\prime}\right) E_{t}\left(\tilde{\omega}_{t+1} e_{t}\right)\right], \quad \tilde{\lambda}_{T}=p_{T},
\end{align*}
$$

for $t=0,1, \ldots, T-1$.
Remark 5.2. When the returns rates of the $n$ risky assets are statistically independent, our results reduce to the results of Guo and Hu [3]. That is, we extend the model and results of Guo and Hu [3] to the case with serially correlate returns.

## 6. Numerical Example

In the previous sections, we derive the optimal strategies and the mean-variance efficient frontiers of two optimal portfolio selection problems with serial correlations and uncertain


Figure 1: Efficient frontiers with different probability distributions of exit time.
exit time. In this section, we provide a numerical example to demonstrate the impacts of the uncertainty of exit time and the serial correlations of returns on the efficient frontier. In the example, we only consider one risky asset for simplicity and assume that its return rate is subject to $\operatorname{AR}(1)$ progress

$$
\begin{equation*}
r_{t}=\mu+\rho\left(r_{t-1}-\mu\right)+\sqrt{1-\rho^{2}} \sigma y_{t} \tag{6.1}
\end{equation*}
$$

where $\mu$ is the unconditional expectation of $r_{t}, \rho \in(-1,1)$ is the first-order autocorrelation coefficient, $\sigma$ is the unconditional standard deviation of $r_{t}, y_{t}$ is a random variable with standard normal distribution, and $y_{t}$ is independent of $y_{s}(s<t)$.

To examine the impact of the uncertainty of exit time on the efficient frontier clearly, we compare the efficient frontiers under three different probability distributions of uncertain exit time $t=T \wedge \tau: P_{1}=\left(p_{0}, p_{1}, p_{2}, p_{3}\right)=(0,0.09,0.2,0.71), P_{2}=\left(p_{0}, p_{1}, p_{2}, p_{3}\right)=(0,0,0.5,0.5)$, and $P_{3}=\left(p_{0}, p_{1}, p_{2}, p_{3}\right)=(0,0.2,0.1,0.7)$. The remaining parameters are set as $\mu=0.03, b=0.5$, $\sigma=0.02, x_{0}=1$, and $T=3$. Figure 1 implies when the investor exits from market later, the investor enjoys more expected wealth returns at the same level of risk than the one terminates the investment earlier.

Furthermore, to test the impact of the correlation coefficient on the efficient frontier, we compare the efficient frontiers under three different settings of correlation coefficient $\rho$ : $\rho_{1}=0.1, \rho_{2}=0.4$, and $\rho_{3}=0.7$. The other parameters are given as $\mu=0.02, \sigma=0.02$, $x_{0}=1$, and $T=3$. It is obvious from Figure 2 that the investor who takes the risky asset with larger correlation coefficient will suffer less risk than the one with less correlate risky asset to achieve the same expected wealth return.


Figure 2: Efficient frontiers with different values of correlation coefficient.

## 7. Conclusion

In this paper, we consider an optimal portfolio selection problem under multi-period setting and mean-variance framework for an investor, who does not know with certainty when she/he will exit the market in which the capital returns are serially correlated. The problem is much more complicated than the case with deterministic exit time and/or with serially noncorrelate assets returns. By applying the dynamic programming approach and the embedding technique of Li and Ng [2], both the optimal strategy and the efficient frontier of the problem are derived explicitly. Our results include, as special cases, the ones of Li and Ng [2], Guo and Hu [3], and Xu and Li [11]. In addition, a numerical example with AR (1) return process is presented. It shows that both the serial correlations of assets returns and the uncertainty of exit time have significant impacts on the optimal strategy and the efficient frontier.

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Research Article

# New Results on Stability and Stabilization of Markovian Jump Systems with Partly Known Transition Probabilities 

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This paper investigates the problem of stability and stabilization of Markovian jump linear systems with partial information on transition probability. A new stability criterion is obtained for these systems. Comparing with the existing results, the advantage of this paper is that the proposed criterion has fewer variables, however, does not increase any conservatism, which has been proved theoretically. Furthermore, a sufficient condition for the state feedback controller design is derived in terms of linear matrix inequalities. Finally, numerical examples are given to illustrate the effectiveness of the proposed method.

## 1. Introduction

Over the past few decades, Markov jump systems (MJSs) have drawn much attention of researchers throughout the world. This is due to their important roles in many practical systems. That is, MJSs are quite appropriate to model the plants whose structures are subject to random abrupt changes, which may result from random component failures, abrupt environment changes, disturbance, changes in the interconnections of subsystems, and so forth [1].

Since the transition probabilities in the jumping process determine the behavior of the MJSs, the main investigation on MJSs is to assume that the information on transition probabilities is completely known (see, e.g., [2-5]). However, in most cases, the transition probabilities of MJSs are not exactly known. Whether in theory or in practice, it is necessary to further consider more general jump systems with partial information on transition probabilities. Recently, [6-9] considered the general MJSs with partly unknown transition probabilities. But in these papers, when the terms containing unknown transition probabilities were separated from others, the fixed connection weighting matrices were introduced,
which may lead to the conservatism. As noticed, it, currently [10], have achieved an excellent work of reducing the conservatism. The basic idea is to introduce free-connection weighting matrices to substitute the fixed connection weighting matrices. However, this means that the method of [10] has to increase the number of decision variables. As shown in [11], more decision variables imply the augmentation of the numerical burden. Therefore, developing some new methods without introducing any additional variable meanwhile without increasing conservatism will be a valuable work, which motivates the present study.

In this paper, we are concerned with the problem of the stability and stabilization of MJSs with partly unknown transition probabilities. By fully unitizing the relationship among the transition rates of various subsystems, we obtain a new stability criterion. The proposed criterion avoids introducing any connection weighting matrix; however, do not increase any conservatism comparing to that of [10], which has been proved theoretically. More importantly, because the proposed stability criterion need not introduce any slack matrix, the relationships among Lyapunov matrices are highlighted. Therefore, it helps us to understand the effect of the unknown transition probabilities on the stability. Then, based on the proposed stability criterion, the condition for the controller design is derived in terms of LMIs. Finally, numerical examples are given to illustrate the effectiveness of the proposed method.

## Notation

In this paper, $\mathbb{R}^{n}$ and $\mathbb{R}^{n \times m}$ denote the $n$-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. $\mathbb{Z}^{+}$represents the set of positive integers. The notation $P>0(P \geq 0)$ means that $P$ is a real symmetric and positive definite (semipositivedefinite) matrix. For notation $(\Omega, \mathcal{F}, D), \Omega$ represents the sample space, $\mathcal{F}$ is the $\sigma$-algebra of subsets of the sample space, and $D$ is the probability measure on $\mathcal{F} . E\{\cdot\}$ stands for the mathematical expectation. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Problem Formulation

Consider the following stochastic system with Markovian jump parameters:

$$
\begin{equation*}
\dot{x}(t)=A\left(r_{t}\right) x(t)+B\left(r_{t}\right) u(t) \tag{2.1}
\end{equation*}
$$

where $x(t) \in \mathbb{R}^{n}$ is the state vector, $x_{0}$ denotes initial condition, and $\left\{r_{t}\right\}, t \geq 0$ is a rightcontinuous Markov process on the probability space taking values in a finite state space $\mathbb{S}=$ $\{1,2, \ldots, N\}$ with generator $\Lambda=\left\{\pi_{i j}\right\}, i, j \in \mathbb{S}$, given by

$$
\operatorname{Pr}\left\{r_{t+\Delta}=j \mid r_{t}=i\right\}= \begin{cases}\pi_{i j} \Delta+o(\Delta), & i \neq j  \tag{2.2}\\ 1+\pi_{i j} \Delta+o(\Delta), & i=j\end{cases}
$$

where $\Delta>0, \lim _{\Delta \rightarrow 0} O(\Delta) / \Delta=0$, and $\pi_{i j} \geq 0$, for $i \neq j$, is the transition rate from mode $i$ at time $t$ to mode $j$ at time $t+\Delta$, and $\pi_{i i}=-\sum_{j=1, i \neq j}^{N} \pi_{i j} . A\left(r_{t}\right)$ are known matrix functions of the Markov process.

Since the transition probability depends on the transition rates for the continuous-time MJSs, the transition rates of the jumping process are considered to be partly accessible in this
paper. For instance, the transition rate matrix $\Lambda$ for system (2.1) with $N$ operation modes may be expressed as

$$
\left[\begin{array}{ccccc}
\pi_{11} & ? & \pi_{13} & \cdots & ?  \tag{2.3}\\
? & ? & ? & \cdots & \pi_{2 N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
? & \pi_{N 2} & \pi_{N 3} & \cdots & \pi_{N N}
\end{array}\right]
$$

where "?" represents the unknown transition rate.
For notational clarity, for all $i \in \mathbb{S}$, we denote $\mathbb{S}=\mathbb{S}_{k}^{i} \cup \mathbb{S}_{u k}^{i}$ with $\mathbb{S}_{k}^{i} \triangleq\left\{j: \pi_{i j}\right.$ is known for $j \in \mathbb{S}\}$ and $\mathbb{S}_{u k}^{i} \triangleq\left\{j: \pi_{i j}\right.$ is unknown for $\left.j \in \mathbb{S}\right\}$.

Moreover, if $\mathbb{S}_{k}^{i} \neq \emptyset$, it is further described as

$$
\begin{equation*}
\mathbb{S}_{k}^{i}=\left\{k_{1}^{i}, k_{2}^{i}, \ldots, k_{m}^{i}\right\}, \tag{2.4}
\end{equation*}
$$

where $m$ is a nonnegative integer with $1 \leq m \leq N$ and $k_{j}^{i} \in \mathbb{Z}^{+}, 1 \leq k_{j}^{i} \leq N, j=1,2, \ldots, m$ represent the $j$ th known element of the set $\mathbb{S}_{k}^{i}$ in the $i$ th row of the transition rate matrix $\Lambda$.

For the underlying systems, the following definitions will be adopted in the rest of this paper. More details refer to [2].

Definition 2.1. The system (2.1) with $u(t)=0$ is said to be stochastically stable if the following inequality holds

$$
\begin{equation*}
E\left\{\int_{0}^{\infty}\|x(t)\|^{2} d t \mid x_{0}, r_{0}\right\}<\infty, \tag{2.5}
\end{equation*}
$$

for every initial condition $x_{0} \in \mathbb{R}^{n}$ and $r_{0} \in \mathbb{S}$.
To this end, we introduce the following result on the stability analysis of systems (2.1).
Lemma 2.2 (see [2]). The system (2.1) with $u(t)=0$ is stochastically stable if and only if there exists a set of symmetric and positive-definite matrices $P_{i}, i \in \mathbb{S}$, satisfying

$$
\begin{equation*}
A_{i}^{T} P_{i}+P_{i} A_{i}+\sum_{j=1}^{N} \pi_{i j} P_{j}<0 \tag{2.6}
\end{equation*}
$$

Remark 2.3. Since the unknown transition rates may have infinitely admissible values, it is impossible to be directly used the inequalities of Lemma 2.2 to test the stability of the system.

## 3. Stochastic Stability Analysis

In this section, a stochastic stability criterion for MJSs is given without any additional weighting matrix.

Theorem 3.1. The system (2.1) with a partly unknown transition rate matrix (2.3) and $u(t)=0$ is stochastically stable if there exist matrices $P_{i}>0$, such that the following LMIs are feasible for $i=1,2, \ldots, N$. If $i \notin \mathbb{S}_{k^{\prime}}^{i}$

$$
\begin{align*}
& P_{i} A_{i}+A_{i}^{T} P_{i}+P_{k}^{i}-\pi_{k}^{i} P_{i}<0  \tag{3.1}\\
& P_{i}-P_{j} \geq 0, \quad \forall j \in \mathbb{S}_{u k}^{i}, j \neq i \tag{3.2}
\end{align*}
$$

$$
\text { If } i \in \mathbb{S}_{k^{\prime}}^{i}
$$

$$
\begin{equation*}
P_{i} A_{i}+A_{i}^{T} P_{i}+D_{k}^{i}-\pi_{k}^{i} P_{j}<0, \quad \forall j \in \mathbb{S}_{u k}^{i} \tag{3.3}
\end{equation*}
$$

where $D_{k}^{i}=\sum_{j \in \mathbb{S}_{k}^{i}} \pi_{i j} P_{j}$ and $\pi_{k}^{i}=\sum_{j \in \mathbb{S}_{k}^{i}} \pi_{i j}$.
Proof. Based on Lemma 2.2, we know that the system (2.1) with $u(t)=0$ is stochastically stable if (2.6) holds. Now we prove that (3.1)-(3.3) guarantee that (2.6) holds by the following two cases.

Case $I\left(i \notin \mathbb{S}_{k}^{i}\right)$. In this case, (2.6) can be rewritten as

$$
\begin{equation*}
\Phi_{i} \triangleq P_{i} A_{i}+A_{i}^{T} P_{i}+p_{k}^{i}+\pi_{i \mathrm{i}} P_{i}+\sum_{j \in \mathbb{S}_{u k k^{i}}, j \neq i} \pi_{i j} P_{j}<0 . \tag{3.4}
\end{equation*}
$$

Note that in this case $\sum_{j \in \mathbb{S}_{u k^{\prime}}^{i} j \neq i} \pi_{i j}=-\pi_{i i}-\pi_{k}^{i}$ and $\pi_{i j} \geq 0, j \in \mathbb{S}_{u k^{\prime}}^{i} j \neq i$; then from (3.2), we have

$$
\begin{align*}
\Phi_{i} & \leq P_{i} A_{i}+A_{i}^{T} P_{i}+p_{k}^{i}+\pi_{i i} P_{i}+\sum_{j \in \mathbb{S}_{u k}^{i}, j \neq i} \pi_{i j} P_{i} \\
& =P_{i} A_{i}+A_{i}^{T} P_{i}+p_{k}^{i}+\pi_{i i} P_{i}+\left(-\pi_{i i}-\pi_{k}^{i}\right) P_{i}  \tag{3.5}\\
& =P_{i} A_{i}+A_{i}^{T} P_{i}+p_{k}^{i}-\pi_{k}^{i} P_{i} .
\end{align*}
$$

Therefore, if $i \notin \mathbb{S}_{k}^{i}$, inequalities (3.1) and (3.2) imply that (2.6) holds.
Case II $\left(i \in \mathbb{S}_{k}^{i}\right)$. Note that in this case $-\pi_{k}^{i}=\sum_{j \in \mathbb{S}_{u k}^{i}} \pi_{i j}$ and $\pi_{i j} \geq 0, j \in \mathbb{S}_{u k}^{i}$. So, if $-\pi_{k}^{i}=0$, then we must have $\mathbb{S}_{u k}^{i}=\emptyset$. Therefore, (2.6) becomes

$$
\begin{equation*}
P_{i} A_{i}+A_{i}^{T} P_{i}+p_{k}^{i}<0 \tag{3.6}
\end{equation*}
$$

which is equivalent to (3.3) noticing $-\pi_{k}^{i}=0$.

Else, if $-\pi_{k}^{i} \neq 0$, then we must have $-\pi_{k}^{i}>0$. Then, (2.6) can be rewritten as

$$
\begin{equation*}
\sum_{j \in S_{u k}^{i}} \pi_{i j}\left\{\frac{\left(P_{i} A_{i}+A_{i}^{T} P_{i}+p_{k}^{i}\right)}{-\pi_{k}^{i}}+P_{j}\right\}<0 \tag{3.7}
\end{equation*}
$$

Obviously, (3.3) implies that (3.7) holds. Then, if $i \notin \mathbb{S}_{k^{\prime}}^{i}$ (3.3) implies that (2.6) holds.
Therefore, if LMIs (3.1)-(3.3) hold, we conclude that system (2.1) is stochastically stable according to Lemma 2.2. The proof is completed.

Theorem 3.1 proposed in this paper does not introduce any free variable. It involves $N n(n+1) / 2$ variables, while Theorem 3.3 in [10] involves $N n(n+1)$ variables. Namely, the number of variables in this paper is only half of [10]. Generally, reducing the number of decision variables easily results in increasing conservatism of stability criteria. However, Theorem 3.1 in this paper does not increase conservatism while with less variables. To show this, we rewrite it as follows.

Theorem 3.2 (see [10]). The system (2.1) with partly unknown transition rate matrix (2.3) and $u(t)=0$ is stochastically stable if there exist matrices $P_{i}>0, W_{i}=W_{i}^{T}$, such that the following LMIs are feasible for $i=1,2, \ldots, N$,

$$
\begin{gather*}
P_{i} A_{i}+A_{i}^{T} P_{i}+\sum_{j \in \mathbb{S}_{k}^{i}} \pi_{i j}\left(P_{j}-W_{i}\right)<0  \tag{3.8}\\
P_{j}-W_{i} \leq 0, \quad j \in \mathbb{S}_{u k}^{i}, j \neq i  \tag{3.9}\\
P_{j}-W_{i} \geq 0, \quad j \in \mathbb{S}_{u k^{\prime}}^{i} j=i \tag{3.10}
\end{gather*}
$$

Now we have the following conclusion.
Theorem 3.3. Suppose for system (2.1) with partly unknown transition rate matrix (2.3) there exist $P_{i}>0$ and $W_{i}=W_{i}^{T}, i=1,2, \ldots, N$, such that (3.8)-(3.10) hold; then the matrices $P_{i}>0$, $i=1,2, \ldots, N$, satisfy (3.1)-(3.3).

Proof. If $i \notin \mathbb{S}_{k^{\prime}}^{i}$ (3.9)-(3.10) imply that (3.2) and the following inequality hold:

$$
\begin{equation*}
P_{i} \geqslant W_{i} \tag{3.11}
\end{equation*}
$$

From (3.11) and (3.8), we can obtain (3.1).
If $i \in \mathbb{S}_{k^{\prime}}^{i}$ (3.9) and (3.10) guarantee that

$$
\begin{equation*}
P_{j}-W_{i} \leq 0, \quad j \in \mathbb{S}_{u k}^{i} \tag{3.12}
\end{equation*}
$$

In addition, under this circumstance, we have

$$
\begin{equation*}
\sum_{j \in S_{k}^{i}} \pi_{i j} \leq 0 \tag{3.13}
\end{equation*}
$$

Then, From (3.12), (3.13), and (3.8), we obtain that (3.3) holds. The proof is completed.
Remark 3.4. The stability condition in [10] and that in Theorem 3.1 are derived via different techniques. Now Theorem 3.3 proves that the former can be simplified to the latter without increasing any conservatism. More importantly, because Theorem 3.1 of this paper does not involve any slack matrix, the relationships among Lyapunov matrices are highlighted. Therefore, it is clearer how the unknown transition probabilities affect on the stability.

## 4. State-Feedback Stabilization

In this section, the stabilization problem of system (2.1) with control input $u(t)$ is considered. The mode-dependent controller with the following form is designed:

$$
\begin{equation*}
u(t)=K\left(r_{t}\right) x(t) \tag{4.1}
\end{equation*}
$$

where $K\left(r_{t}\right)$ for all $r_{t} \in \mathbb{S}$ are the controller gains to be determined. In the following, for given $r_{t}=i \in \mathbb{S}, K\left(r_{t}\right)=K_{i}$.

Using (4.1), the system (2.1) is represented as

$$
\begin{equation*}
\dot{x}(t)=\left[A\left(r_{t}\right)+B\left(r_{t}\right) K\left(r_{t}\right)\right] x(t) \tag{4.2}
\end{equation*}
$$

The following theorem is proposed to design the mode-dependent stabilizing controller with the form (4.1) for system (2.1).

Theorem 4.1. The closed-loop system (4.2) with a partly unknown transition rate matrix (2.3) is stochastically stable if, there exist matrices $Q_{i}>0$ and $Y_{i}, i=1,2, \ldots, N$ such that the following LMIs are feasible for $i=1,2, \ldots, N$.

If $i \notin \mathbb{S}_{k^{\prime}}^{i}$

$$
\begin{gather*}
{\left[\begin{array}{cc}
\Xi_{i}-\pi_{k}^{i} Q_{i} & \Pi_{1 i} \\
* & -\Psi_{1 i}
\end{array}\right]<0,}  \tag{4.3}\\
{\left[\begin{array}{cc}
-Q_{i} & Q_{i} \\
* & -Q_{j}
\end{array}\right] \leq 0, \quad \forall j \in \mathbb{S}_{u k}^{i} \& j \neq i .} \tag{4.4}
\end{gather*}
$$

$$
\begin{align*}
& \text { If } i \in \mathbb{S}_{k^{\prime}}^{i} \\
& \qquad\left[\begin{array}{ccc}
\Xi_{i}+\pi_{i i} Q_{i} & \Pi_{2 i} & \sqrt{\left(-\pi_{k}^{i}\right)} Q_{i} \\
* & -\Psi_{2 i} & 0 \\
* & * & -Q_{j}
\end{array}\right]<0, \quad \forall j \in \mathbb{S}_{u k^{\prime}}^{i} \tag{4.5}
\end{align*}
$$

where

$$
\begin{gather*}
\Xi_{i}=A_{i} Q_{i}+Q_{i} A_{i}^{T}+B_{i} Y_{i}+Y_{i}^{T} B_{i}^{T}, \\
\Pi_{1 i}=\left[\sqrt{\pi_{i k}{ }_{1}^{i}} Q_{i} \sqrt{\pi_{i k_{2}^{i}}} Q_{i} \cdots \sqrt{\pi_{i k_{m}^{i}}} Q_{i}\right], \\
\Psi_{1 i}=\operatorname{diag}\left\{Q_{k_{1}^{i}}, Q_{k_{2}^{i}}, \ldots, Q_{k_{m}^{i}}\right\},  \tag{4.6}\\
\Pi_{2 i}=\left[\sqrt{\pi_{i k_{1}^{i}}} Q_{i} \cdots \sqrt{\pi_{i k_{l-1}^{i}}} Q_{i} \sqrt{\pi_{i k_{l+1}^{i}}} Q_{i} \cdots \sqrt{\pi_{i k_{m}^{i}}} Q_{i}\right], \\
\Psi_{2 i}=\operatorname{diag}\left\{Q_{k_{1}^{i}}, \ldots, Q_{k_{l-1}^{i}}, Q_{k_{l+1}^{i}}, \ldots, Q_{k_{m}^{i}}\right\},
\end{gather*}
$$

with $k_{1}^{i}, k_{2}^{i}, \ldots, k_{m}^{i}$ described in (2.4) and $k_{l}^{i}=i$.
Moreover, if (4.3)-(4.5) are true, the stabilization controller gains from (4.1) are given by

$$
\begin{equation*}
K_{i}=Y_{i} Q_{i}^{-1} \tag{4.7}
\end{equation*}
$$

Proof. It is clear that the system (4.2) is stable if the following conditions are satisfied.
If $i \notin \mathbb{S}_{k^{\prime}}^{i}$

$$
\begin{gather*}
P_{i}\left(A_{i}+B_{i} K_{i}\right)+\left(A_{i}+B_{i} K_{i}\right)^{T} P_{i}+p_{\mathrm{k}}^{i}-\pi_{k}^{i} P_{i}<0,  \tag{4.8}\\
P_{j}-P_{i} \leq 0, \quad \forall j \in \mathbb{S}_{u k^{\prime}}^{i} j \neq i . \tag{4.9}
\end{gather*}
$$

If $i \in \mathbb{S}_{k}^{i}$,

$$
\begin{equation*}
P_{i}\left(A_{i}+B_{i} K_{i}\right)+\left(A_{i}+B_{i} K_{i}\right)^{T} P_{i}+p_{k}^{i}-\pi_{k}^{i} P_{j}<0, \quad \forall j \in \mathbb{S}_{u k}^{i} \tag{4.10}
\end{equation*}
$$

Pre- and postmultiply the left sides of (4.8)-(4.10) by $P_{i}^{-1}$, respectively, and introduce the following new variables:

$$
\begin{equation*}
Q_{i}=P_{i}^{-1}, \quad Y_{i}=K_{i} P_{i}^{-1} \tag{4.11}
\end{equation*}
$$

Then, inequalities (4.8)-(4.10) are equivalent to the following matrix inequalities, respectively.

If $i \notin \mathbb{S}_{k^{\prime}}^{i}$

$$
\begin{gather*}
\Xi_{i}-\pi_{k}^{i} Q_{i}+\Pi_{1 i} \Psi_{1 i}^{-1} \Pi_{1 i}^{T}<0  \tag{4.12}\\
Q_{i} P_{j} Q_{i}-Q_{i} \leq 0, \quad \forall j \in \mathbb{S}_{u k^{\prime}}^{i} j \neq i \tag{4.13}
\end{gather*}
$$

If $i \in \mathbb{S}_{k^{\prime}}^{i}$

$$
\begin{equation*}
\Xi_{i}+\pi_{i i} Q_{i}+\Pi_{i 2} \Psi_{2 i}^{-1} \Pi_{i 2}^{T}+\left(-\pi_{k}^{i}\right) Q_{i} P_{j} Q_{i}<0, \quad \forall j \in \mathbb{S}_{u k}^{i} \tag{4.14}
\end{equation*}
$$

By applying Schur complement, inequalities (4.12)-(4.14) are equivalent to LMIs (4.3)-(4.5), respectively.

Therefore, if LMIs (4.3)-(4.5) hold, the closed-loop system (4.2) is stochastically stable according to Theorem 3.1. Then, system (2.1) can be stabilized with the state feedback controller (4.1), and the desired controller gains are given by (4.7). The proof is completed.

Remark 4.2. The number of variables involved in Theorem 4.1 in this paper is also $N n(n+1) / 2$ less than that in the corresponding result of [10]. Furthermore, it can be seen that no conservativeness is introduced when deriving Theorem 4.1 from Theorem 3.1. Therefore, the stabilization method presented in Theorem 4.1 is not more conservative than that of [10], too.

## 5. Numerical Example

In this section, an example is provided to illustrate the effectiveness of our results.
Consider the following MJSs, which borrowed from [10] with small modifications,

$$
\begin{array}{cc}
A_{1}=\left[\begin{array}{cc}
32 & -7.30 \\
1.48 & 0.81
\end{array}\right], & A_{2}=\left[\begin{array}{cc}
0.89 & -3.11 \\
1.48 & 0.21
\end{array}\right] \\
A_{3}=\left[\begin{array}{cc}
-0.11 & -0.85 \\
2.31 & -0.10
\end{array}\right], & A_{4}=\left[\begin{array}{cc}
-0.17 & -1.48 \\
1.59 & -0.27
\end{array}\right], \\
B_{1}=\left[\begin{array}{c}
0.57 \\
1.23
\end{array}\right], & B_{2}=\left[\begin{array}{c}
0.78 \\
-0.49
\end{array}\right]  \tag{5.1}\\
B_{3}=\left[\begin{array}{c}
1.34 \\
0.39
\end{array}\right], & B_{4}=\left[\begin{array}{c}
-0.38 \\
1.07
\end{array}\right]
\end{array}
$$

The partly transition rate matrix $\Lambda$ is considered as

$$
\Lambda=\left[\begin{array}{cccc}
-1.3-a & 0.2 & ? & ?  \tag{5.2}\\
? & ? & 0.3 & 0.3 \\
0.6 & ? & -1.5 & ? \\
0.4 & ? & ? & ?
\end{array}\right]
$$



Figure 1: State response of the open-loop system with 1000 random samplings.
where the parameter $a$ in matrix $\Lambda$ can take different values for extensive comparison purpose.

We consider the stabilization of this system corresponding to different $a$ by using different approaches. Considering the precision of comparison, we let $a$ increase starting from 0 with a very small constant increment 0.01 . Using the LMI toolbox in MATLAB, both the LMIs in Theorem 5 of [10] and the ones in Theorem 4.1 of this paper are feasible for all $a=0,0.01, \ldots, 1.64$, and are infeasible when $a$ increases to 1.65 . It can be seen that for this example the stabilization method in our paper is not conservative than that in [10].

Now by some simulation results, we further show the effectiveness of the stabilization method of this paper. For example, when $a=1.64$, in our method, the controller gains are obtained as

$$
\begin{gather*}
K_{1}=\left[\begin{array}{ll}
-15143 & 5252
\end{array}\right], \quad K_{2}=\left[\begin{array}{ll}
-873700 & 392930
\end{array}\right], \\
K_{3}=\left[\begin{array}{ll}
-1086.6 & -290.8
\end{array}\right], \quad K_{4}=\left[\begin{array}{ll}
440620 & -300480
\end{array}\right] . \tag{5.3}
\end{gather*}
$$

Figure 1 is the state response cures in 1000 random sampling with initial condition $x_{0}=[1-1]^{T}$ when $u(t)=0$. In each random sampling, the transition rate matrix is randomly generated but satisfies the partly transition rate matrix $\Lambda$ in (5.2). Figure 1 shows that the open-loop system is unstable.

Applying the controllers in (5.3), the trajectory simulation of state response for the closed-loop system with 1000 random sampling is shown in Figure 2 under the given initial condition $x_{0}=[1-1]^{T}$. In this case, the transition rate matrix is also randomly generated but satisfies the partly transition rate matrix $\Lambda$ in (5.2). Figure 2 shows that the stabilizing controller effectively keeps the running reliability of the system.

## 6. Conclusions

This paper has considered the problem of stability and stabilization of a class continuous-time MJSs with unknown transition rates. A new stability criterion has been proposed. The merit


Figure 2: State response of the closed-loop system with 1000 random samplings.
of the proposed criterion is that it has less decision variables without increasing conservatism comparing those in the literature to date. Then, the mode-dependent state feedback controller designing method has been proposed, which possess the same merit as the stability criterion. Numerical examples have been given to illustrate the effectiveness of the results in this paper.

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Research Article

# Stochastic Stability of Damped Mathieu Oscillator Parametrically Excited by a Gaussian Noise 

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#### Abstract

This paper analyzes the stochastic stability of a damped Mathieu oscillator subjected to a parametric excitation of the form of a stationary Gaussian process, which may be both white and coloured. By applying deterministic and stochastic averaging, two Itô's differential equations are retrieved. Reference is made to stochastic stability in moments. The differential equations ruling the response statistical moment evolution are written by means of Itô's differential rule. A necessary and sufficient condition of stability in the moments of order $r$ is that the matrix $\mathbf{A}_{r}$ of the coefficients of the ODE system ruling them has negative real eigenvalues and complex eigenvalues with negative real parts. Because of the linearity of the system the stability of the first two moments is the strongest condition of stability. In the case of the first moments (averages), critical values of the parameters are expressed analytically, while for the second moments the search for the critical values is made numerically. Some graphs are presented for representative cases.


## 1. Introduction

The so-called Mathieu equation has attracted the attention of the scholars in the past, and many papers and books are available on this subject such as $[1,2]$. The interest in this equation stems from two facts: first, its dynamics is very rich; second, it describes the vibrations of important mechanical systems such as a prismatic bar stretched or compressed by a sinusoidal axial force, a pendulum whose support is subjected to a sinusoidal motion, and an elliptic membrane $[3,4]$. In general, attention is focused on the stability of the motion.

When the excitation acting on the Mathieu oscillator is a stochastic process, its response is a stochastic process too. Thus, the analyst is concerned with the problem of the statistical characterization of the response, and in the case of a system prone to instability the problem of the stochastic stability must be solved. Nevertheless, studies on the Mathieu systems with stochastic excitation are not numerous. Probably, the first authors who
addressed the problem were Stratonovich and Romanovski [5]. Analyses of the Mathieutype equations with external random excitation were presented by Dimentberg [6] and by Cai and Lin [7].

A Mathieu system with random parametrical excitation was considered by Rong et al. [8] and by Xie [9] (see the references cited herein too). The first authors use the method of multiple scale to determine the equations of the amplitude and the phase of the response. The check for almost-sure stability is made by computing the largest Lyapunov exponent. On the contrary, in ([9, Chapter 7]) the method of stochastic averaging is used, and the stability in moments is analyzed. It recalled that there are different definitions of stochastic stability, for which the reader is referred to $[10,11]$. According to the selected stability definition, different stability bounds are found. Thus, it is not surprising that the results of [8, 9] are different. However, the conclusions that are reached are opposite: in [8] it is claimed that the stochastic parametric excitation is always detrimental with respect to the stability, while in [9] it is affirmed that there are cases in which the stochastic excitation stabilizes the system.

In this paper, the Mathieu damped oscillator parametrically excited by a Gaussian noise is considered again. A suitable coordinate transformation is made, and deterministic and stochastic averaging are applied in such a way that the second-order motion equation is replaced by two first-order equations, are interpreted in Itô's sense. As the Fokker-PlanckKolmogorov equation for the response probability density function (PDF) is not analytically solvable, the stochastic stability in moments is considered as in [9, 12]: the differential equations governing the response moment evolution are written by means of the rules of Itô's stochastic differential calculus [13-16]. A necessary and sufficient condition of stability in the moments of order $r$ is that the matrix $\mathbf{A}_{r}$ of the coefficients of the ODE system governing them has negative real eigenvalues and complex eigenvalues with negative real parts. Because of the linearity of the system the stability of the first two moments is the strongest condition of stability so that only the first two moments are considered ([11, Chapter 7]). Analytical stability bounds are established for the first moments, while the stability of the second moments is searched numerically.

## 2. Statement of the Problem

Consider the Mathieu-type stochastic differential equation

$$
\begin{equation*}
\ddot{X}(t)+2 \varepsilon \zeta_{0} \omega_{0} \dot{X}(t)+\omega_{0}^{2}[1+\varepsilon \beta \sin \Omega t+\sqrt{\varepsilon} F(t)] X(t)=0, \tag{2.1}
\end{equation*}
$$

where $\zeta_{0}$ is the ratio of critical damping, $\omega_{0}$ is the undamped pulsation of the oscillator, $\varepsilon$ is a small parameter, $\beta$ and $\Omega$ are the amplitude and the pulsation of the sinusoidal term, respectively, and $F(t)$ is a stochastic stationary Gaussian process with zero mean. Because of the parametric excitation $F(t)$ the response $X(t)$ is a stochastic process too.

The pulsations $\Omega$ and $\omega_{0}$ can be linked by writing

$$
\begin{equation*}
\omega_{0}^{2}=\frac{\Omega^{2}}{4}+\varepsilon \delta \tag{2.2}
\end{equation*}
$$

where $\delta$ is a detuning parameter. By inserting (2.2) in (2.1), this becomes

$$
\begin{equation*}
\ddot{X}(t)+\frac{\Omega^{2}}{4} X(t)=-\varepsilon\left[2 \zeta_{0} \omega_{0} \dot{X}(t)+\left(\delta+\omega_{0}^{2} \beta \sin \Omega t\right) X(t)\right]-\sqrt{\varepsilon} \omega_{0}^{2} X(t) F(t) \tag{2.3}
\end{equation*}
$$

The problem of the statistical characterization of (2.3) will be tackled by using the stochastic averaging method $[6,11,17-20]$. However, in the case of a harmonic term like that in (2.3), the classic coordinate transformation $X(t)=A(t) \cos \phi(t)$, where $A(t)$ and $\phi(t)$ are stochastic processes, does not work well as the joint probability density function (PDF) of the response is not separable ([11, chapter 7]). A suitable coordinate transformation was proposed in $[6,17,18]$, which reads as

$$
\begin{equation*}
X(t)=X_{1}(t) \cos \left(\frac{\Omega t}{2}\right)+X_{2}(t) \sin \left(\frac{\Omega t}{2}\right) \tag{2.4}
\end{equation*}
$$

According to the principles of deterministic averaging for weakly nonlinear systems [21] it is assumed that

$$
\begin{gather*}
\dot{X}(t)=-\frac{1}{2} \Omega X_{1}(t) \sin \left(\frac{\Omega t}{2}\right)+\frac{1}{2} \Omega X_{2}(t) \cos \left(\frac{\Omega t}{2}\right) \\
\ddot{X}(t)=-\frac{1}{2} \Omega \dot{X}_{1} \sin \left(\frac{\Omega t}{2}\right)-\frac{1}{4} \Omega^{2} X_{1} \cos \left(\frac{\Omega t}{2}\right)+\frac{1}{2} \Omega \dot{X}_{2} \cos \left(\frac{\Omega t}{2}\right)-\frac{1}{4} \Omega^{2} X_{2} \sin \left(\frac{\Omega t}{2}\right) . \tag{2.5}
\end{gather*}
$$

The expressions in (2.5) are not exact but only approximate. The exact expression of the first derivative is $-(1 / 2) \Omega X_{1} \sin (\Omega t / 2)+\dot{X}_{1} \cos (\Omega t / 2)+(1 / 2) \Omega X_{2} \cos (\Omega t / 2)+\dot{X}_{2} \sin (\Omega t / 2)$. According to the method of deterministic averaging [21] the second and fourth terms are neglected so that the derivative retains the same form as it would have if $X_{1}$ and $X_{2}$ were constant.

By inserting (2.4), (2.5) into (2.3), after some algebra we obtain a pair of first-order stochastic differential equations:

$$
\begin{align*}
\dot{X}_{1}(t)=-\varepsilon & \left\{2 \zeta_{0} \omega_{0}\left[X_{1} \sin ^{2}\left(\frac{\Omega t}{2}\right)-\frac{1}{2} X_{2} \sin \Omega t\right]-\frac{\delta}{\Omega} X_{1} \sin \Omega t-\frac{2 \delta}{\Omega} X_{2} \sin ^{2}\left(\frac{\Omega t}{2}\right)\right. \\
& \left.-\frac{\omega_{0} \beta}{2} X_{1} \sin ^{2} \Omega t-\omega_{0} \beta X_{2} \sin ^{2}\left(\frac{\Omega t}{2}\right) \sin \Omega t\right\}  \tag{2.6}\\
& +\sqrt{\varepsilon} \omega_{0}\left[\frac{1}{2} X_{1} \sin \Omega t+X_{2} \sin ^{2}\left(\frac{\Omega t}{2}\right)\right] F(t), \\
\dot{X}_{2}(t)=\varepsilon & \left\{2 \zeta_{0} \omega_{0}\left[\frac{1}{2} X_{1} \sin \Omega t-X_{2} \cos ^{2}\left(\frac{\Omega t}{2}\right)\right]-\frac{\delta}{\Omega} X_{2} \sin \Omega t-\frac{2 \delta}{\Omega} X_{1} \cos ^{2}\left(\frac{\Omega t}{2}\right)\right. \\
& \left.-\frac{\omega_{0} \beta}{2} X_{2} \sin ^{2} \Omega t-\omega_{0} \beta X_{1} \cos ^{2}\left(\frac{\Omega t}{2}\right) \sin \Omega t\right\}  \tag{2.7}\\
& -\sqrt{\varepsilon} \omega_{0}\left[X_{1} \cos ^{2}\left(\frac{\Omega t}{2}\right)+\frac{X_{2}}{2} \sin \Omega t\right] F(t) .
\end{align*}
$$

The method of stochastic averaging is applied to (2.6), (2.7). In the field of stochastic dynamics this method of analysis was proposed first by Stratonovich [17, 18] (see also $[11,19,20])$ as an extension to stochastic systems of the deterministic method by Bogoliubov and Mitropolsky [21] (see Mettler [22], too). Then, it was rigorously demonstrated by Khasminskii [23]. It involves two phases that can be performed in either order: in the former, the deterministic terms that do not contain the forcing functions are averaged. In the second phase, the terms containing the forcing functions are reduced to Gaussian stationary white noises.

The deterministic averaging of the terms in brace brackets in (2.6), (2.7) requires the evaluation of integrals like $\Theta^{-1} \cdot \int_{0}^{\Theta}(\bullet) \mathrm{d} t[6,17,18]$, where $\Theta=2 \pi / \Omega$. It is obtained that

$$
\begin{align*}
& \dot{X}_{1}(t)=-\varepsilon\left(\zeta_{0} \omega_{0} X_{1}-\frac{\delta}{\Omega} X_{2}-\frac{\omega_{0} \beta}{4} X_{1}\right)+\sqrt{\varepsilon} \omega_{0}\left[\frac{1}{2} X_{1} \sin \Omega t+X_{2} \sin ^{2}\left(\frac{\Omega t}{2}\right)\right] F(t) \\
& \dot{X}_{2}(t)=-\varepsilon\left(\zeta_{0} \omega_{0} X_{2}+\frac{\delta}{\Omega} X_{1}+\frac{\omega_{0} \beta}{4} X_{2}\right)-\sqrt{\varepsilon} \omega_{0}\left[X_{1} \cos ^{2}\left(\frac{\Omega t}{2}\right)+\frac{1}{2} X_{2} \sin \Omega t\right] F(t) \tag{2.8}
\end{align*}
$$

Now, the forcing terms must be worked out. In the classic stochastic averaging method it is required that the process $F(t)$ is broadbanded: in order to remove this restriction, a different way will be followed. Stratonovich [17, Volume 1, Chapter 7] suggested that, given a stochastic process $F(t)$, which may be even narrowbanded, the product $\sin \Omega t F(t)$ is replaceable by a stationary Gaussian white noise $W_{1}(t)$ having the autocorrelation function $K_{0} \delta(\tau)$. The constant $K_{0}$ is given by the following integral: $K_{0}=1 / 2 \int_{-\infty}^{+\infty} R_{F F}(\tau) \cos \Omega \tau \mathrm{d} \tau=$ $\pi S_{F F}(\Omega)$, where $R_{F F}(\tau)$ and $S_{F F}(\Omega)$ are the auto-correlation function and the power spectral density (PSD) of $F(t)$, respectively. However, the forcing terms in (2.8) contain the contributions $F(t) \sin ^{2}(\Omega t / 2), F(t) \cos ^{2}(\Omega t / 2)$ too. These contributions are replaced by a stationary Gaussian white noise $W_{2}(t)$, whose intensity is computed by adapting to the present case the derivation of Stratonovich [17, Volume 1, Section 7.2]. We have

$$
\begin{equation*}
E\left[W_{2}(t) W_{2}(t+\tau)\right]=\delta(\tau) \int_{-\infty}^{+\infty} R_{F F}(\tau) \sin ^{2}\left(\frac{\Omega t}{2}\right) \sin ^{2}\left[\frac{\Omega}{2}(t+\tau)\right] \mathrm{d} \tau \tag{2.9}
\end{equation*}
$$

By expanding the product of the trigonometric functions in the integral (2.9), we obtain: $\sin ^{2}(\Omega t / 2) \sin ^{2}[\Omega / 2(t+\tau)]=1 / 4\{1-\cos [\Omega(t+\tau)]-\cos \Omega t+0.5 \cos \Omega \tau+0.5 \cos [\Omega(2 t+\tau)]\}$. In the last expression, the second and the third addenda, when averaged, give rise to zero, while the fifth term is a faster oscillatory one that according to Stratonovich can be neglected. Thus, the integral (2.9) is equal to

$$
\begin{equation*}
\int_{-\infty}^{+\infty} R_{F F}(\tau)\left(\frac{1}{4}+\frac{1}{8} \cos \Omega \tau\right) \mathrm{d} \tau=\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega) \tag{2.10}
\end{equation*}
$$

The expansion of $\int_{-\infty}^{+\infty} R_{F F}(\tau) \cos ^{2}(\Omega t / 2) \cos ^{2}[\Omega / 2(t+\tau)] \mathrm{d} \tau$ leads to the same result. In much the same way, it results that $E\left[W_{1}(t) W_{2}(t+\tau)\right]=0$, that is, they are uncorrelated.

Performing these operations, (2.8) simplify into

$$
\begin{align*}
\dot{X}_{1}(t)= & -\varepsilon \omega_{0}\left(\zeta_{0}-\frac{\beta}{4}\right) X_{1}+\frac{\varepsilon \delta}{\Omega} X_{2}+X_{1} \omega_{0} \sqrt{\varepsilon \frac{\pi}{4} S_{F F}(\Omega)} W_{1}(t) \\
& +X_{2} \omega_{0} \sqrt{\varepsilon\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right]} W_{2}(t)  \tag{2.11}\\
\dot{X}_{2}(t)= & -\varepsilon \omega_{0}\left(\zeta_{0}+\frac{\beta}{4}\right) X_{2}-\frac{\varepsilon \delta}{\Omega} X_{1}+X_{2} \omega_{0} \sqrt{\varepsilon \frac{\pi}{4} S_{F F}(\Omega)} W_{1}(t) \\
& +X_{1} \omega_{0} \sqrt{\varepsilon\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right]} W_{2}(t)
\end{align*}
$$

For the sake of simplicity, the white noises in the previous equations have unit intensities. More-over, the forcing terms in the second equation have the sign plus as the white noise processes $\pm \sqrt{K} W(t)$ have the same probabilistic characteristics ([24, chapter 6, section 6.3.2]).

In order to transform (2.11) into two Itô-type stochastic differential equations, the socalled Wong-Zakai-Stratonovich corrective terms must be added to the drift terms [25, 26]. They are computed according to the following formula:

$$
\begin{equation*}
m_{c i}=\frac{1}{2} \sum_{1}^{n} k \sum_{1}^{m}{ }_{j} g_{k j} \frac{\partial g_{i j}}{\partial z_{k}} \tag{2.12}
\end{equation*}
$$

where in the present case $n=m=2$ and $g_{k j}, g_{i j}$ are elements of the diffusion matrix

$$
\mathbf{G}=\left[\begin{array}{ll}
X_{1} \omega_{0} \sqrt{\varepsilon \frac{\pi}{4} S_{F F}(\Omega)} & X_{2} \omega_{0} \sqrt{\varepsilon\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right]}  \tag{2.13}\\
X_{2} \omega_{0} \sqrt{\varepsilon \frac{\pi}{4} S_{F F}(\Omega)} & X_{1} \omega_{0} \sqrt{\varepsilon\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right]}
\end{array}\right]
$$

Hence, the two Itô stochastic differential equations that govern the problem are

$$
\begin{align*}
\mathrm{d} X_{1}= & {\left[-\varepsilon \omega_{0}\left(\zeta_{0}-\frac{\beta}{4}\right)+\varepsilon \frac{\pi}{4} \omega_{0}^{2} \bar{S}\right] X_{1} \mathrm{~d} t+\varepsilon \frac{\delta}{\Omega} X_{2} \mathrm{~d} t+X_{1} \omega_{0} \sqrt{\varepsilon \frac{\pi}{4} S_{F F}(\Omega)} \mathrm{d} B_{1}(t) } \\
& +X_{2} \omega_{0} \sqrt{\varepsilon\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right]} \mathrm{d} B_{2}(t) \\
\mathrm{d} X_{2}= & {\left[-\varepsilon \omega_{0}\left(\zeta_{0}+\frac{\beta}{4}\right)+\varepsilon \frac{\pi}{4} \omega_{0}^{2} \bar{S}\right] X_{2} \mathrm{~d} t-\varepsilon \frac{\delta}{\Omega} X_{1} \mathrm{~d} t }  \tag{2.14}\\
& +X_{2} \omega_{0} \sqrt{\varepsilon \frac{\pi}{4} S_{F F}(\Omega)} \mathrm{d} B_{1}(t)+X_{1} \omega_{0} \sqrt{\varepsilon\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right]} \mathrm{d} B_{2}(t)
\end{align*}
$$

where $\bar{S}=S_{F F}(0)+S_{F F}(\Omega)$ and $B_{1}(t)$ and $B_{2}(t)$ are two standard Wiener processes (Brownian motion), for which the formal relationship $\mathrm{d} B_{i} / \mathrm{d} t=W_{i}(t)(i=1,2)$ holds.

From a theoretical point of view the probabilistic characterization of the random vector $X=\left\{X_{1}, X_{2}\right\}^{t}$ should be obtained by solving the so-called Fokker-Planck-Kolmogorov (FPK) equation in the joint PDF $p_{\mathrm{X}}$ of the state variables [11, 17, 18, 24]

$$
\begin{equation*}
\frac{\partial p_{\mathbf{X}}}{\partial t}=-\frac{\partial}{\partial x_{i}}\left(m_{i} p_{\mathbf{X}}\right)+\frac{1}{2} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left(b_{i j} p_{\mathbf{X}}\right) \tag{2.15}
\end{equation*}
$$

where the summation rule with respect to a repeated index has been adopted, and $m_{1}=$ $\left[-\varepsilon \omega_{0}\left(\zeta_{0}-\beta / 4\right)+\varepsilon(\pi / 4) \omega_{0}^{2} \bar{S}\right] X_{1}+(\varepsilon \delta / \Omega) X_{2}, m_{2}=\left[-\varepsilon \omega_{0}\left(\zeta_{0}+\beta / 4\right)+\varepsilon(\pi / 4) \omega_{0}^{2} \bar{S}\right] X_{2}-$ $(\varepsilon \delta / \Omega) X_{1}, b_{i j}=\left[G^{t}\right]_{i j}$, with the matrix $G$ being defined in (2.13), while the apex " $t$ " denotes transpose. Unfortunately, as in many other cases the FPK equation (2.15) does not have an analytical solution because the excitation is a parametric one. Thus, another method has to be chosen in order to characterize the probabilistic characteristics of the response. The moment equation approach is used herein.

## 3. Moment Equation Approach

In (2.14) the excitation is multiplicative only, and there is not an external excitation. Thus, zero solution $X_{1}=X_{2}=0$ for all $t$ satisfies them, even if the multiplicative excitation is a stochastic process. We are concerned with the stability of zero solution. There exist different definitions of stochastic stability, for which the reader is referred to Chap. 6 of [11]. Here, the stability of zero solution is analyzed in the moments. In fact, the stability of the first two moments is the strongest for linear autonomous systems under multiplicative Gaussian excitation [11, 27]. Thus, the stability of the first two moments only will be considered here. As the response of (2.14) is not Gaussian, in order to characterize it statistically, from a theoretical point of view the knowledge of the infinite hierarchy of the moments would be necessary. Nevertheless, the system of (2.14) is linearly parametric: it has been shown that the equations for the statistical moments of such a type of systems are a close set [28], and they can be solved in succession. Thus, the convergence of the moments of first and second order is a necessary condition for being stable the moments of higher orders.

In the field of dynamic stability analyses, a system is stable when it comes back to the initial configuration after being subjected to a small perturbation. Zero statistical moments $E\left[X_{1}^{r_{1}} X_{2}^{r_{2}}\right\rfloor\left(r_{1}+r_{2}=r\right)$ correspond to a zero solution. When the zero solution is perturbed, the response moments are no longer zero: if the stochastic Mathieu oscillator is stable, once the perturbation is removed, they decay to zero, otherwise they grow without limits. The first step is to write the ordinary differential equations (ODEs) ruling the time evolution of the response moments. Since (2.14) is an Itô system, use is made of Itô's differential rule [13-15], which reads as

$$
\begin{equation*}
\mathrm{d} \psi=\frac{\partial \psi}{\partial t}+\frac{\partial \psi}{\partial x_{i}} \mathrm{~d} x_{i}+\frac{1}{2} \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}} \mathrm{~d} x_{i} \mathrm{~d} x_{j} \tag{3.1}
\end{equation*}
$$

In (3.1) $\Psi$ is a nonanticipating function of the state variables $x_{i}$ and the summation rule is adopted. It is recalled that (3.1) retains the terms of order $\mathrm{d} t$ only, as $\mathrm{d} B$ is of order $\sqrt{\mathrm{d} t}$. In order to write the moment equations, the appropriate nonanticipating function to be chosen
is $\psi=X_{1}^{i} X_{2}^{j}$, where $i, j$ are positive integers or zero with the constraint $i+j=r$, with $r$ being the order of the moments to be computed.

By choosing $\psi=X_{1}^{i} X_{2}^{j}$ with $i+j=1$, the ODEs for the first order moments are

$$
\begin{align*}
& \dot{E}\left[X_{1}\right]=-\varepsilon\left[\omega_{0}\left(\zeta_{0}-\frac{\beta}{4}\right)-\omega_{0}^{2} \frac{\pi}{4} \bar{S}\right] E\left[X_{1}\right]+\frac{\varepsilon \delta}{\Omega} E\left[X_{2}\right]  \tag{3.2}\\
& \dot{E}\left[X_{2}\right]=-\varepsilon\left[\omega_{0}\left(\zeta_{0}+\frac{\beta}{4}\right)-\omega_{0}^{2} \frac{\pi}{4} \bar{S}\right] E\left[X_{2}\right]-\frac{\varepsilon \delta}{\Omega} E\left[X_{1}\right]
\end{align*}
$$

where the dots mean derivatives with respect to time and $\bar{S}=S_{F F}(0)+S_{F F}(\Omega)$. Analogously, the ODEs ruling the second moments are found to be

$$
\begin{align*}
\dot{E}\left[X_{1}^{2}\right]= & {\left[-2 \varepsilon \omega_{0}\left(\zeta_{0}-\frac{\beta}{4}\right)+\varepsilon \omega_{0}^{2} \frac{\pi}{2} S_{F F}(0)+\varepsilon \omega_{0}^{2} \frac{3}{2} \pi S_{F F}(\Omega)\right] E\left[X_{1}^{2}\right] } \\
& +2 \frac{\varepsilon \delta}{\Omega} E\left[X_{1} X_{2}\right]+\varepsilon \omega_{0}^{2}\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right] E\left[X_{2}^{2}\right] \\
= & a_{11} E\left[X_{1}^{2}\right]+a_{12} E\left[X_{1} X_{2}\right]+a_{22} E\left[X_{2}^{2}\right], \\
\dot{E}\left[X_{1} X_{2}\right]= & -\frac{\varepsilon \delta}{\Omega} E\left[X_{1}^{2}\right]+\left\{-2 \varepsilon \omega_{0} \zeta_{0}+\varepsilon \omega_{0}^{2} \frac{\pi}{2}\left[S_{F F}(0)+S_{F F}(\Omega)\right]\right\} E\left[X_{1} X_{2}\right]+\frac{\varepsilon \delta}{\Omega} E\left[X_{2}^{2}\right]  \tag{3.3}\\
= & a_{21} E\left[X_{1}^{2}\right]+a_{22} E\left[X_{1} X_{2}\right]+a_{23} E\left[X_{2}^{2}\right], \\
\dot{E}\left[X_{2}^{2}\right]= & \varepsilon \omega_{0}^{2}\left[\frac{\pi}{2} S_{F F}(0)+\frac{\pi}{4} S_{F F}(\Omega)\right] E\left[X_{1}^{2}\right]-2 \frac{\varepsilon \delta}{\Omega} E\left[X_{1} X_{2}\right] \\
& +\left\{-2 \varepsilon \omega_{0}\left(\zeta_{0}+\frac{\beta}{4}\right)+\varepsilon \omega_{0}^{2}\left[\frac{\pi}{2} S_{F F}(0)+\frac{3}{4} \pi S_{F F}(\Omega)\right]\right\} E\left[X_{2}^{2}\right] \\
= & a_{11} E\left[X_{1}^{2}\right]+a_{12} E\left[X_{1} X_{2}\right]+a_{22} E\left[X_{2}^{2}\right] .
\end{align*}
$$

By inspecting (3.3), it is noted that the forcing terms are absent, which due to the fact that in (2.14) the excitation is purely parametric. Thus, (3.3) can be written in compact matrix form as

$$
\begin{equation*}
\dot{\mathbf{m}}_{r}(t)=\mathbf{A}_{r} \mathbf{m}_{r} \tag{3.4}
\end{equation*}
$$

where $\mathbf{m}_{r}(t)$ is a vector collecting all the moments of order $r$ of the system states and $\mathbf{A}_{r}$ is a matrix of the coefficients $a_{i j}$ as shown in (3.3). The solution to (3.4) is

$$
\begin{equation*}
\mathbf{m}_{r}(t)=\mathbf{m}_{0} \exp \left(\mathbf{A}_{r} \mathbf{t}\right) \tag{3.5}
\end{equation*}
$$

where $\mathbf{m}_{0}$ is a vector whose entries are the initial conditions for the moments. These constitute the perturbation to zero solution.

It is well known (e.g., see [4]) that, as $t$ grows, the response moments decay to zero whenever the matrix $\mathbf{A}_{r}$ has negative real eigenvalues and complex eigenvalues with negative real parts. Otherwise, the moments increase without limits. Thus, the condition of stability in moments is that the eigenvalues of the matrix $\mathbf{A}_{r}$ have negative real parts. In
this way, the stochastic problem is led to the classic deterministic problem of studying the eigenvalues of a matrix. Since the matrix $\mathbf{A}_{r}$ depends on the system parameters (2.1), there exist critical values of these quantities for which the real part of almost an eigenvalue is zero. Increasing them further, a real part becomes positive, and the moments grow to infinity.

In order to study the eigenvalues of $\mathbf{A}_{r}$, its characteristic equation is formed:

$$
\begin{equation*}
\operatorname{det}\left(\lambda \mathbf{I}_{r}-\mathbf{A}_{r}\right)=0 \tag{3.6}
\end{equation*}
$$

where $\mathbf{I}_{r}$ is a unit matrix having the same order as $\mathbf{A}_{r}$. Equation (3.6) must specialized for the order $r$ of the moments. As previously advanced, due to the linearity of the system and to the Gaussianity of the input, the moment stability analysis is limited to the first two moments.

Coming back to the first-order moments, from (3.2) the characteristic equation looks like

$$
\operatorname{det}\left[\begin{array}{cc}
\lambda+\varepsilon\left[\omega_{0}\left(\zeta_{0}-\frac{\beta}{4}\right)-\omega_{0}^{2} \frac{\pi}{4} \bar{S}\right] & -\frac{\varepsilon \delta}{\Omega}  \tag{3.7}\\
\frac{\varepsilon \delta}{\Omega} & \lambda+\varepsilon\left[\omega_{0}\left(\zeta_{0}+\frac{\beta}{4}\right)-\omega_{0}^{2} \frac{\pi}{4} \bar{S}\right]
\end{array}\right]=0
$$

where from (2.2) $\delta=\varepsilon^{-1}\left(\omega_{0}^{2}-\Omega^{2} / 4\right)$. The roots of (3.7) are

$$
\begin{equation*}
\lambda_{1,2}=\varepsilon\left(-\zeta_{0} \omega_{0}+\frac{1}{4} \pi \omega_{0}^{2} \bar{S} \pm \frac{1}{4} \sqrt{\omega_{0}^{2} \beta^{2}-16 \frac{\delta^{2}}{\Omega^{2}}}\right) \tag{3.8}
\end{equation*}
$$

In examining the eigenvalues given by (3.8), it is necessary to distinguish between the case of real eigenvalues and that of complex conjugate ones. The eigenvalues are real numbers when

$$
\begin{equation*}
\beta>4 \frac{\delta}{\omega_{0} \Omega} \tag{3.9}
\end{equation*}
$$

where for the sake of simplicity $\beta$ is assumed to be positive. Otherwise, they are complex conjugate numbers. Clearly, in the latter case the stability condition for the first moments is

$$
\begin{equation*}
\zeta_{0}>\frac{\pi}{4} \omega_{0} \bar{S} \tag{3.10}
\end{equation*}
$$

Equation (3.10) requires that the oscillator is damped and the amount of damping depends on both the oscillator frequency and the intensity of the exciting noise. This requirement may be rather restrictive when $\omega_{0}$ is not small.

Now, let us consider the case in which (3.9) is satisfied. From (3.8) it is seen that there is passage to instability when the eigenvalue with the sign plus before the square root becomes zero, that is,

$$
\begin{equation*}
-\zeta_{0} \omega_{0}+\frac{1}{4} \pi \omega_{0}^{2} \bar{S}+\frac{1}{4} \omega_{0} \sqrt{\omega_{0}^{2} \beta^{2}-16 \frac{\delta^{2}}{\Omega^{2}}}=0 \tag{3.11}
\end{equation*}
$$

Solving (3.11) with respect to $\bar{S}$, we obtain the critical value of this quantity:

$$
\begin{equation*}
\bar{S}_{\mathrm{cr}}=\frac{4 \zeta_{0}}{\pi \omega_{0}}-\frac{1}{\pi \omega_{0}} \sqrt{\beta^{2}-16 \frac{\delta^{2}}{\omega_{0}^{2} \Omega^{2}}} \tag{3.12}
\end{equation*}
$$

where $\beta$ must be larger than the value in the right-hand side of (3.9). By comparing (3.12) with (3.10) it is found that the critical value given by the former is always smaller than that given by the latter. Keeping into account that $\bar{S}=S_{F F}(0)+S_{F F}(\Omega)$, the critical intensity of the excitation depends on the form of power spectral intensity of this: in Section 4 the cases of white and coloured noises will be considered.

As regards the second moment stability, since the moment equations are three, (3.3), the characteristic equation associated with the matrix $\mathbf{A}_{2}$ is of the third order. The roots of such a type of equation have analytical expressions. However, they are rather cumbersome as many parameters enter them. Thus, it has been preferred to proceed numerically. By using a computer algebra software a parameter is varied till one out of the three eigenvalues becomes zero: the corresponding value of the parameter is the critical one. Routh-Hurwitz criteria $[4,29]$ are not used. This is why in another study devoted to the stability of elastic columns with memory-dependent damping [12] it has been found that these criteria may overevaluate the critical values by a $30 \%$.

## 4. Stability Analyses

The present section is devoted to the applications of the theory previously explained. Three cases are considered for the parametric excitation $F(t)$ : (1) stationary Gaussian white noise; (2) coloured Gaussian process obtained by passing a stationary Gaussian white noise through a first-order linear filter; (3) coloured Gaussian process obtained by passing a stationary Gaussian white noise through a second-order linear filter. This choice makes the comparisons easier as the three stochastic processes have a common parameter that is the intensity $\sigma^{2}$ of the white noise.

The power spectral density (PSD) of a stationary stochastic process $X(t)$ is defined as

$$
\begin{equation*}
S_{X X}=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} R_{X X}(\tau) \cdot \exp (-i w \tau) \mathrm{d} \tau \tag{4.1}
\end{equation*}
$$

where $R_{X X}(\tau)$ is the autocorrelation function of the process and $i=\sqrt{-1}$. Since the autocorrelation function of a stationary Gaussian white noise $W(t)$ can be expressed as $R_{W W}(\tau)=2 \pi \sigma^{2} \delta(\tau)\left(\delta(\tau)\right.$ is Dirac's delta), its PSD is $\sigma^{2}$, constant on the whole real axis. Thus, for a white noise the first criterion of stability for the first-order moments (3.10) gives

$$
\begin{equation*}
\sigma_{\mathrm{cr}}^{2}=\frac{2 \zeta_{0}}{\pi \omega_{0}} \tag{4.2}
\end{equation*}
$$

The critical value of $\sigma^{2}$ increases linearly with the system ratio of critical damping $\zeta_{0}$ and is inversely proportional to system pulsation $\omega_{0}$, but it does not depend on $\Omega$.


Figure 1: First moment stability: plot $\sigma_{\mathrm{cr}}^{2}$ of as a function of $\Omega$ from (4.5), first-order filter, $\alpha=10, \omega_{0}=2 \pi$, $\zeta_{0}=0.05$.

Now, the white noise is passed through a first-order linear filter (the Langevin equation) to have the excitation $F(t)$, that is,

$$
\begin{equation*}
\dot{F}(t)+\alpha F(t)=\sqrt{2 \pi} \sigma W(t) \tag{4.3}
\end{equation*}
$$

where $W(t)$ has unit strength. The PSD of $F(t)$ is

$$
\begin{equation*}
S_{F F}(\omega)=\frac{2 \sigma^{2} \alpha}{\alpha^{2}+\omega^{2}} \tag{4.4}
\end{equation*}
$$

The critical value $\sigma_{\mathrm{cr}}^{2}$ is found by solving (3.10) with respect to $\bar{S}$ and taking the expression of this quantity into account, that is,

$$
\begin{equation*}
\bar{S}_{\mathrm{cr}}=\frac{2 \sigma_{\mathrm{cr}}^{2}}{\alpha}+\frac{2 \sigma_{\mathrm{cr}}^{2} \alpha}{\alpha^{2}+\Omega^{2}}=\frac{4 \zeta_{0}}{\pi \omega_{0}} \tag{4.5}
\end{equation*}
$$

The result of (4.5) is plotted in Figure 1 as a function of $\Omega$ for $\alpha=10, \omega_{0}=2 \pi$, and $\zeta_{0}=0.05$. It is found that $\sigma_{\mathrm{cr}}^{2}$ is an increasing function of $\Omega$.

The third type of excitation is obtained by passing the white noise through a secondorder linear filter, that is,

$$
\begin{equation*}
\ddot{F}(t)+2 \zeta_{f} \omega_{f} \dot{F}(t)+\omega_{f}^{2} F(t)=\sqrt{2 \pi} \sigma W(t) \tag{4.6}
\end{equation*}
$$

In this case the PSD of $F(t)$ is

$$
\begin{equation*}
S_{F F}(\omega)=\frac{2 \sigma^{2}}{\left(\omega_{f}^{2}-\omega^{2}\right)^{2}+4 \zeta_{f}^{2} \omega_{f}^{2} \omega^{2}} \tag{4.7}
\end{equation*}
$$



Figure 2: First moment stability: plot of $\sigma_{\mathrm{cr}}^{2}$ as a function of $\Omega$ from (4.8), second-order filter: $\omega_{0}=2 \pi$, $\zeta_{0}=0.05$, and $\zeta_{f}=0.10$; top plot $\omega_{f}=\omega_{0}$, bottom plot $\omega_{f}=2 \omega_{0}$.


Figure 3: First moment stability: plot of $\bar{S}_{\text {cr }}$ as function of $\Omega$ according to (3.12) for $\omega_{0}=2 \pi, \zeta_{0}=0.05$, and $\beta=\kappa \delta /\left(\omega_{0} \Omega\right)$ with $\kappa=4.01$ (red line), 4.10 (black), 4.5 (blue), and 5.0 (violet).
$\sigma_{\text {cr }}^{2}$ for the first condition of stability is computed from

$$
\begin{equation*}
\bar{S}_{\mathrm{cr}}=\frac{2 \sigma_{\mathrm{cr}}^{2}}{\omega_{f}^{4}}+\frac{2 \sigma_{\mathrm{cr}}^{2}}{\left(\omega_{f}^{2}-\Omega^{2}\right)^{2}+4 \zeta_{f}^{2} \omega_{f}^{2} \Omega^{2}}=\frac{4 \zeta_{0}}{\pi \omega_{0}} . \tag{4.8}
\end{equation*}
$$

The results of (4.8) are plotted in Figure 2 for $\omega_{0}=2 \pi, \zeta_{0}=0.05$, and $\zeta_{f}=0.05$; the top plot refers to $\omega_{f}=\omega_{0}$ and the bottom plot to $\omega_{f}=2 \omega_{0}$. Both plots show resonance with a marked minimum when $\Omega$ equates $\omega_{f}$.

As regards the second condition of stability, $\bar{S}_{\text {cr }}$ from (3.12) is plotted in Figure 3 for $\omega_{0}=2 \pi, \zeta_{0}=0.05$. Since this condition is valid only when $\beta$ is larger than the right-hand side of (3.9), the plots are drawn for $\beta=\kappa \delta /\left(\omega_{0} \Omega\right)$ with $\mathcal{\kappa}=4.01,4.10,4.5$, and 5.0. Equation (3.12) gives a result with physical meaning only when $\bar{S}_{\mathrm{cr}}$ is positive. From the plots-the


Figure 4: (a) Comparison among the PSDs of the white noise (green line) and those given by (4.4) (black line) and by (4.7) (red and blue lines) for $\sigma^{2}=0.01, \alpha=10, \zeta_{f}=0.10$, and $\omega_{f}=2 \pi$ or $4 \pi$. (b) Comparison between the PSDs deriving from (4.4) and (4.7): from (4.4) red line, from (4.7) blue and green lines. (c) Comparison between the PSDs deriving from (4.7): blue line $\omega_{f}=2 \pi$, black line $\omega_{f}=4 \pi$.
bottom is a detail-it is seen that the interval of validity is small, and it narrows as $\kappa$ increases. All the curves have a peak for $\Omega=2 \omega_{0}=4 \pi$.

Before presenting the results of the stability analyses for the second-order moments, it is necessary to compare the three PSD functions used in the analyses: they are depicted in Figure 4.The parameter $\sigma^{2}$ is common to the three PDFs, and it constitutes the strength of the white noise. The plots clearly show as the white noise is by far the most severe excitation that may excite a dynamic system, if the colored excitations are obtained by passing it through a linear filter. As a consequence of this only observation, it cannot produce stabilizing effects on a dynamic system. The Wiener process yielded by the first-order filter of (4.3) (the Langevin equation) is the second in order of severity. As regards the processes generated by the secondorder filter of (4.6), they have less strength as the abscissae of the peaks move farther from


Figure 5: Second moment stability, white noise excitation: plot of $\sigma_{\mathrm{cr}}^{2}$ as a function of $\Omega$ for $\varepsilon=1, \omega_{0}=2 \pi$, $\zeta_{0}=0.05 ; \beta=0.1$ green line, and $\beta=0.15$ black line.


Figure 6: Second moment stability, white noise excitation: plot of $\beta_{\text {cr }}$ as a function of $\Omega$ for $\varepsilon=1, \omega_{0}=2 \pi$, $\zeta_{0}=0.05 ; \sigma^{2}=0.0005$ green line, and $\sigma^{2}=0.001$ black line.
the origin. These remarks allow to explain the results of the analyses keeping in mind that the excitation affects the moments equations through $S_{F F}(0)$ and $S_{F F}(\Omega)$. The following analyses are of two types: (1) search of the critical value of $\sigma^{2}$ keeping $\beta$, the amplitude of the sinusoidal term in (2.1), constant; (2) search of the critical value of $\beta$ keeping $\sigma^{2}$ constant. For the sake of simplicity in all analyses the parameter $\varepsilon$ is worth one. It is recalled that there is passage to instability when an eigenvalue becomes zero, which happens when $\sigma^{2}$ equates $\sigma_{\mathrm{cr}}^{2}$ or $\beta$ equates $\beta_{\text {cr }}$.

For white noise excitation the plots of $\sigma_{\mathrm{cr}}^{2}$ and $\beta_{\mathrm{cr}}$ as a function of $\Omega$ are in Figures 5 and 6 , respectively, where $\omega_{0}=2 \pi$ and $\zeta_{0}=0.05$. Both $\sigma_{\text {cr }}^{2}$ and $\beta_{\text {cr }}$ are constant with respect to $\Omega$, being the stable regions below the straight lines. This result is not surprising: in fact, the excitation enters the moment equations through $S_{F F}(0)$ and $S_{F F}(\Omega)$, which are equal and do


Figure 7: White noise excitation: $E\left[X_{1}^{2}\right]=z_{1}$ with respect to time, $\omega_{0}=2 \pi, \zeta_{0}=0.05, \beta=0.1, \varepsilon=1$; (a) $\sigma^{2}=0.00123<\sigma_{\mathrm{cr}}^{2} ;(\mathrm{b}) \sigma^{2}=0.00130>\sigma_{\mathrm{cr}}^{2}$.


Figure 8: Second moment stability, excitation given by the first-order filter (4.3): plot of $\sigma_{\mathrm{cr}}^{2}$ as a function of $\Omega$ for $\beta=0.1, \alpha=10, \varepsilon=1, \omega_{0}=2 \pi, \zeta_{0}=0.05$.
not vary with $\Omega$ in the case of white noise, whose PSD is constant on the whole frequency axis. $\sigma_{\mathrm{cr}}^{2}$ increases as $\beta$ decreases (Figure 5), and $\beta_{\mathrm{cr}}$ has a similar trend (Figure 6).

Figure 7 shows $E\left[X_{1}^{2}\right]$ as a function of time for $\beta=0.1, \sigma^{2}=0.0012$ (top plot), and $\sigma^{2}=0.00130$ (bottom plot). The moment equations (3.3) are numerically integrated by means of a fourth-order Runge-Kutta method. The initial perturbation is given by means of the initial conditions $E\left[X_{1}^{2}(0)\right]=E\left[X_{2}^{2}(0)\right]=0.1$. In the former case $\sigma^{2}$ is lesser than $\sigma_{c r}^{2}$, and $E\left[X_{1}^{2}\right\rfloor$ starts from the prescribed value 0.1 , then it decays to zero and does not show any oscillation. In the latter case $\sigma^{2}$ is larger than $\sigma_{\mathrm{cr}}^{2}$, and $E\left\lfloor X_{1}^{2}\right\rfloor$ grows without limit. The absence of oscillations is due to the fact that in applying the stochastic averaging the oscillatory terms are cancelled.

As regards the excitation given by the first-order filter of (4.3), the plots of $\sigma_{\mathrm{cr}}^{2}$ and $\beta_{\text {cr }}$ as a function of $\Omega$ are shown in Figures 8 and 9, respectively, where $\omega_{0}=2 \pi, \zeta_{0}=0.05$, and $\alpha=10$. Both quantities increase as $\Omega$ increases. In fact, the PSD (4.4) is a monotonically


Figure 9: Second moment stability, excitation given by the first-order filter (4.3): plot of $\beta_{\text {cr }}$ as a function of $\Omega$ for $\sigma^{2}=0.005, \alpha=10, \varepsilon=1, \omega_{0}=2 \pi, \zeta_{0}=0.05$.


Figure 10: Excitation from the second order filter (4.6): plots of $\sigma_{\text {cr }}^{2}$ as a function of $\Omega: \varepsilon=1, \omega_{0}=2 \pi$, $\zeta_{0}=0.05, \beta=0.10, \zeta_{f}=0.10$, black line $\omega_{f}=2 \pi$, blue line $\omega_{f}=4 \pi$.
decreasing func-tion of $\Omega$, which causes the excitation to diminish with $\Omega$. The curve in the plot of Figure 8 is obtained by keeping $\beta$ equal to 0.1 . The increase of $\sigma_{\mathrm{cr}}^{2}$ is marked and reaches a $70 \%$ as $\Omega$ passes from $0.5 \pi$ to $6.5 \pi$. The curve of $\beta_{\text {cr }}$ refers to $\sigma^{2}=0.0005$ : in the same interval the increase is less marked and amounts to about $28 \%$. The analyses are made in a discrete series of values of $\Omega$ obtaining a set of points in the plane $\left(\sigma^{2}, \Omega\right)$ or $(\beta, \Omega)$. Then, the curves are traced by means of the routine Spline of MAPLE.

The plots for the excitation given by the second-order filter (4.6) are shown in Figures 10 and 11 and are obtained for this data set: $\varepsilon=1, \omega_{0}=2 \pi, \zeta_{0}=0.05, \beta=0.10$ in Figure 10 for $\sigma_{\mathrm{cr}}^{2}, \sigma^{2}=0.01$ in Figure 11 for $\beta_{\mathrm{cr}}, \zeta_{f}=0.10$, black line $\omega_{f}=2 \pi$, blue line $\omega_{f}=4 \pi$. In both plots the curve deriving from $\omega_{f}=4 \pi$ is the higher since the excitation strength is smaller, as already explained. In both plots there is a marked valley for $\Omega=\omega_{f}$ : since $S_{F F}(\Omega)$ assumes


Figure 11: Excitation from the second-order filter (4.6): plots of $\beta_{\text {cr }}$ as a function of $\Omega: \varepsilon=1, \omega_{0}=2 \pi$, $\zeta_{0}=0.05, \sigma^{2}=0.01, \zeta_{f}=0.10$, black line $\omega_{f}=2 \pi$, blue line $\omega_{f}=4 \pi$.
its largest value (see Figure 4), there is a kind of stochastic resonance [30], even if the system remains stable.

## 5. Summary and Conclusions

In this paper the issue of stability of the stochastic Mathieu oscillator is addressed. In order to solve the problem, a suitable coordinate transformation and stochastic averaging are applied to the original dynamic system. In this way, two first-order stochastic differential equations are obtained. Then, they are transformed into two Itô-type stochastic equations. For such a type of stochastic differential equations Itô's differential rule is applicable and allows one to derive the ODEs ruling the time evolution of the response statistical moments.

In the dynamic system considered here the excitation is merely linearly parametric. Hence, the moments ODEs are linear and homogeneous, and the ODEs for the moments of order $r$ are uncoupled from the ODEs for moments of different order. Because of this feature the ODEs for the moments of order $r$ admit zero solution, which means that the system is at rest, and the parametric excitation does not disturb this state. The system is stable in moments when these return to zero after the application of an external perturbation; otherwise, it is unstable and the moments diverge. The necessary and sufficient condition for the stability of the moments of order $r$ is that all the eigenvalues of the matrix $\mathbf{A}_{r}$ of the coefficients of the ODEs written for those moments have negative real part. The passage to the instability requires that an eigenvalue becomes zero. In this way, the stochastic problem is reduced to the deterministic problem of the study of the eigenvalues of a matrix.

The procedure developed in this paper is applied to three types of Gaussian stationary stochastic excitations: (1) white noise; (2) white noise filtered through a first-order filter; (3) white noise filtered through a second-order filter. In order to make it possible to compare the results, in cases (2) and (3) the source white noise is the same as that of case (1), and it is by far the more severe excitation (Figure 4).

As regards the stability of the first-order moments, it is substantially led by the system damping. Several parameters affect the stability of the second-order moments: since the char-
acteristic equation of the matrix $\mathbf{A}_{2}$ is of the third order, so that its roots have complicated expressions, it has been chosen to proceed numerically. The analyses were aimed at finding the critical values of the white noise strength $\sigma^{2}$ and of the amplitude $\beta$ of the sinusoidal term in the motion equation (2.1). Differently from the deterministic Mathieu oscillator the pulsation $\Omega$ of this term has little or no effects on the bounds of stability, which is due to the presence of damping. Vice versa, in the case of excitation deriving from a second-order filter there is a stochastic resonance when $\Omega$ equates the peak frequency $\omega_{f}$ of the PSD of the excitation. In any case, the critical values $\sigma_{\text {cr }}^{2}$ and $\beta_{\text {cr }}$ are by far smaller for the white noise excitation since as observed it is the strongest excitation. Thus, the statement of [9,31] that a white noise excitation may cause a stabilizing effect is not found true. The results of these papers agree with those of [8]: a direct comparison of the results is not possible as the system studied in that reference is a little different and a quite different method of analysis is used there.

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Research Article

# Robust Stabilization for Stochastic Systems with Time-Delay and Nonlinear Uncertainties 

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#### Abstract

This paper deals with the problem of robust stabilization of stochastic systems with time-delay and nonlinear uncertainties via memory state feedback. Based on Lyapunov krasoviskii functional, some sufficient conditions on local (global) stabilization are given in terms of matrix inequalities. In particular, these stabilizable conditions for a class of nonlinear stochastic time-delay systems are derived in the form of linear matrix inequalities, which have the advantage of easy computation. Moreover, the corresponding results are further extended to the stochastic multiple time-delays systems. Finally, an example is presented to show the superiority of memory state feedback controller to memoryless state feedback controller.


## 1. Introduction

Stochastic differential delay equations are one of the most useful stochastic models in applications, for example, aircraft, chemical or process control system, and distributed networks. It is known that time-delay is, in many cases, a source of poor system performance or instability. Hence, the stability and stabilization of stochastic time delay systems have been recently attracting the attention of a number of researchers, see [1-18] and the references therein. In [2], the authors studied the stability of linear stochastic systems with uncertain time delay by generalized Riccati equation approach, while [3] extended the results of [2] to nonlinear case via linear matrix inequalities. Reference [4] investigated the problems of stabilization for a class of linear stochastic systems with norm-bounded uncertainties and state delay, and it developed two criteria for the stability analysis: delay-dependent and delay-independent. The memoryless nonfragile state feedback control law for nonlinear stochastic time-delay systems was designed in [5], in which new sufficient conditions for the existence of such controllers were presented based on the linear matrix inequalities approach.

Reference [6] was concerned with the stability analysis and proposed improved delaydependent stability criteria for uncertain stochastic systems with interval time-varying delay. The study of exponential stability of stochastic delay-differential equations was discussed in [7-9]. The output feedback stabilization of stochastic nonlinear time-delay systems was investigated in [10], and some stabilization criterions for nonlinear stochastic time-delay systems with state and control-dependent noise were given in $[11,12]$ by means of matrix inequalities.

We usually design memoryless state feedback controller for the stabilization of systems because of its advantage of easy implementation. However, its performance, for time-delay systems, cannot be better than a memory state feedback controller which utilizes the available information of the size of delay. Reference [19] has given a general form of a memory state feedback (delayed feedback) controller:

$$
\begin{equation*}
u(t)=G x(t)+\int_{t-\tau}^{t} G_{1}(s) x(s) d s \tag{1.1}
\end{equation*}
$$

But the task of storing all the previous states $x(\cdot)$ and computing the values of time-varying gain matrices $G_{1}(\cdot)$ makes the practical realization of infinite-dimensional controller (1.1) very difficult. For these reasons, the controller

$$
\begin{equation*}
u(t)=G x(t)+G_{2} x(t-\tau) \tag{1.2}
\end{equation*}
$$

could be considered as a compromise between the performance improvement and the implementation simplicity. Reference [20] gave the sufficient conditions for the stabilization of deterministic state-delayed systems. References [21] and [22] designed a memory state feedback controller for neutral time-delay systems and singular timedelay systems, respectively. Reference [23] studied the stabilization problem for a class of discrete-time Markovian jump linear systems with time-delays both in the system state and in the mode signal via time-delayed controller and obtained a sufficient condition. What [13] actually studied is the stabilization problem of linear stochastic time-delay systems using generalized Riccati equation method. Up to now, to the best of the authors' knowledge, the issue on memory state feedback stabilization of stochastic systems with time-delay and nonlinear uncertainties has not been fully investigated in previous literatures.

In this paper, we consider the problem on robust stabilization for stochastic systems with time-delay and nonlinear uncertainties via memory state feedback. This problem contains three inevitable aspects of practical application: timedelay, nonlinear uncertainties and more effective controller, which is more complex than the stabilization of pure stochastic systems via memoryless control. These complexities result in some difficulties of memory stabilizing controller design. By the Itô formula, mathematical expectation properties, and matrix transformation, some sufficient conditions are obtained on locally and globally asymptotic stabilization in probability by means of matrix inequalities. Especially for a class of nonlinear stochastic time-delay systems, a sufficient condition for the existence of memory state feedback stabilizing controller is obtained in terms of LMIs, which has the advantage of easy computation. Meanwhile, a memoryless state feedback controller is also given as a special case of memory state feedback controller. Moreover, the robust stabilization problem for stochastic multiple time-delays systems is further studied and a general sufficient condition is derived.

The paper is organized as follows. Some preliminaries and problem formulations are presented in Section 2. In Section 3, main results are given. Section 4 presents one example to illustrate the effectiveness of our developed results. Section 5 concludes this paper.

Notation 1. $A^{\prime}$ : the transpose of matrix $A ; A \geq 0(A>0): A$ is positive semidefinite (positive definite) symmetric matrix; $I$ : identity matrix; $\|\cdot\|$ : Euclidean norm; $L_{\mp}^{2}\left([0, \infty), \mathbf{R}^{l}\right)$ : space of nonanticipative stochastic process $y(t) \in \mathbf{R}^{l}$ with respect to an increasing $\sigma$-algebra $\mathcal{F}_{t}(t \geq 0)$ satisfying $E \int_{0}^{\infty}\|y(t)\|^{2} d t<\infty$. $I_{n \times n}: n \times n$ identity matrix.

## 2. Preliminaries and Problem Statement

Consider the following continuous nonlinear stochastic time-delay systems:

$$
\begin{align*}
d x(t)= & \left(A x(t)+B x(t-\tau)+B_{1} u(t)+H_{0}(x(t), x(t-\tau), u(t))\right) d t \\
& +\left(C x(t)+D x(t-\tau)+D_{1} u(t)+H_{1}(x(t), x(t-\tau), u(t))\right) d w(t),  \tag{2.1}\\
x(t)= & \varnothing(t) \in L^{2}\left(\omega, \mathcal{F}_{0}, C\left([-\tau, 0], R^{n}\right)\right), \quad t \in[-\tau, 0],
\end{align*}
$$

where $x(t) \in \mathbf{R}^{n}$ and $u(t) \in \mathbf{R}^{m}$ are system state and control input, respectively; $w(t)$ is 1-dimensional standard Wiener process defined on the probability space $\left(\Omega, \mathcal{F}, \mathcal{F}_{t}, P\right)$ with $F_{t}=\sigma\{w(s): 0 \leq s \leq t\} ; H_{i}(0, \cdot, \cdot)=0, i=0,1 ; A, B, B_{1}, C, D$, and $D_{1}$ are constant matrices; $\tau>$ 0 is a certain timedelay. Under very mild conditions on $H_{i}(0, \cdot, \cdot), i=0,1,(2.1)$ exists a unique global solution [1]. It should be pointed out that any general nonlinear stochastic system which is sufficiently differentiable can take the form of (2.1) via Taylor's series expansion at the origin.

Next, we give the following definitions essential for the paper.
Definition 2.1 (see [1]). System (2.1) with $u(t)=0$ is said to be stable in probability, if for any $\epsilon>0$,

$$
\begin{equation*}
\lim _{x \rightarrow 0} P\left(\sup _{t \geq 0}\|x(t)\|>\epsilon\right)=0 \tag{2.2}
\end{equation*}
$$

Additionally, if we also have

$$
\begin{equation*}
\lim _{x_{0} \rightarrow 0} P\left(\lim _{t \rightarrow \infty} x(t)=0\right)=1 \tag{2.3}
\end{equation*}
$$

then system (2.1) with $u(t)=0$ is said to be locally asymptotically stable in probability.
If (2.2) holds and

$$
\begin{equation*}
P\left(\lim _{t \rightarrow \infty} x(t)=0\right)=1 \tag{2.4}
\end{equation*}
$$

for all $x_{0} \in \mathbf{R}^{n}$, then system (2.1) with $u(t)=0$ is said to be globally asymptotically stable in probability.

Definition 2.2. If there exists a constant memory state feedback control law

$$
\begin{equation*}
u(t)=K_{1} x(t)+K_{2} x(t-\tau) \tag{2.5}
\end{equation*}
$$

such that the equilibrium point of the closed-loop system

$$
\begin{align*}
d x(t)= & \left(\left(A+B_{1} K_{1}\right) x(t)+\left(B+B_{1} K_{2}\right) x(t-\tau)+H_{0}\left(x(t), x(t-\tau), K_{1} x(t)+K_{2} x(t-\tau)\right)\right) d t \\
& +\left(\left(C+D_{1} K_{1}\right) x(t)+\left(D+D_{1} K_{2}\right) x(t-\tau)\right. \\
& \left.+H_{1}\left(x(t), x(t-\tau), K_{1} x(t)+K_{2} x(t-\tau)\right)\right) d w(t), \\
x(t)= & \varnothing(t), \quad[-\tau, 0] \tag{2.6}
\end{align*}
$$

is asymptotically stable in probability [1] for all $\tau>0$, then stochastic time-delay differential system (2.1) is called locally robustly stabilizable. If (2.6) is robustly stable [2], that is, the equilibrium point of (2.6) is asymptotically stable in the large [1] for all $\tau>0$, (2.1) is globally robustly stabilizable.

Remark 2.3. Definition 2.2 gives locally (globally) robustly stabilizable of stochastic timedelay systems via memory state feedback control law $u(t)=K_{1} x(t)+K_{2} x(t-\tau)$, which is more general than that via memoryless state feedback control law [12]. This is because Definition 2.2 reduces to the corresponding definition under memoryless state feedback control law when $K_{2}=0$.

The aim of this paper is to find a constant memory state feedback control law (2.5), such that the equilibrium point of (2.6) is asymptotically stable in probability for all $\tau>0$.

## 3. Main Results

In this section, we will give some sufficient conditions of the stabilization of system (2.1). Without loss of generality, we can give the following assumption for nonlinear function $H_{i}$.

Assumption 3.1. There exists an $\epsilon>0$, such that

$$
\begin{equation*}
\sup \left\|H_{i}\left(x, y, K_{1} x+K_{2} y\right)\right\| \leq \epsilon(\|x\|+\|y\|), \quad i=0,1 \tag{3.1}
\end{equation*}
$$

holds for all $x, y \in \mathbf{U}$ (a neighborhood of the origin).
The following general theorem is presented, which yields several applicable corollaries.

Theorem 3.2. If (3.1) holds and $K_{1}, K_{2} \in \mathbf{R}^{m \times n}, P>0, Q>0$ are the solutions of the following matrix inequality

$$
\begin{equation*}
Z+Z_{1}<0 \tag{3.2}
\end{equation*}
$$

then system (2.1) can be locally robustly stabilized by (2.5). If $\mathbf{U}$ is replaced by $\mathbf{R}^{n}$, then system (2.1) can be globally robustly stabilized by the same controller.

In (3.2), $Z$ and $Z_{1}$ are defined by

$$
\begin{gather*}
Z=\left[\begin{array}{cc}
\Sigma_{1} & \Sigma_{2} \\
* & \left(D+D_{1} K_{2}\right)^{\prime} P\left(D+D_{1} K_{2}\right)-Q
\end{array}\right] \\
Z_{1}=\left[\begin{array}{cc}
\Sigma_{3} & 0 \\
0 & \Sigma_{4}
\end{array}\right] \tag{3.3}
\end{gather*}
$$

where

$$
\begin{gather*}
\Sigma_{1}=P\left(A+B_{1} K_{1}\right)+\left(A+B_{1} K_{1}\right)^{\prime} P+Q+\left(C+D_{1} K_{1}\right)^{\prime} P\left(C+D_{1} K_{1}\right), \\
\Sigma_{2}=P\left(B+B_{1} K_{2}\right)+\left(C+D_{1} K_{1}\right)^{\prime} P\left(D+D_{1} K_{2}\right),  \tag{3.4}\\
\Sigma_{3}=\epsilon\left(3+3\|C\|+3\left\|D_{1}\right\| \cdot\left\|K_{1}\right\|+3 \epsilon+\|D\|+\epsilon\left\|D_{1}\right\| \cdot\left\|K_{2}\right\|\right)\|P\| I, \\
\Sigma_{4}=\epsilon\left(3+\|D\|+\left\|D_{1}\right\| \cdot\left\|K_{2}\right\|+\|C\|+\left\|D_{1}\right\| \cdot\left\|K_{1}\right\|+2 \epsilon\right)\|P\| I .
\end{gather*}
$$

Proof. Choose the following Lyapunov-Krasoviskii functional:

$$
\begin{equation*}
V(t, x)=x^{\prime}(t) P x(t)+\int_{0}^{\tau} x^{\prime}(t-s) Q x(t-s) d s \tag{3.5}
\end{equation*}
$$

where $P>0$ and $Q>0$ are the solutions of (3.2). Let $\Omega$ be the infinitesimal generator of the closed-loop system (2.6), then, by the Itô's formula, we have

$$
\begin{align*}
\varrho V(t, x(t))= & x^{\prime}(t)\left[P\left(A+B_{1} K_{1}\right)+\left(A+B_{1} K_{1}\right)^{\prime} P+Q+\left(C+D_{1} K_{1}\right)^{\prime} P\left(C+D_{1} K_{1}\right)\right] x(t) \\
& +2 x^{\prime}(t)\left[P\left(B+B_{1} K_{2}\right)+\left(C+D_{1} K_{1}\right)^{\prime} \cdot P\left(D+D_{1} K_{2}\right)\right] x(t-\tau)+x^{\prime}(t-\tau) \\
& \cdot\left[\left(D+D_{1} K_{2}\right)^{\prime} P\left(D+D_{1} K_{2}\right)-Q\right] x(t-\tau) \\
& +2 H_{0}^{\prime} P x(t)+2 H_{1}^{\prime} P\left(D+D_{1} K_{2}\right) x(t-\tau)  \tag{3.6}\\
& +2 H_{1}^{\prime} P\left(C+D_{1} K_{1}\right) x(t)+H_{1}^{\prime} P H_{1} \\
= & {\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right]^{\prime} Z\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right]+2 H_{0}^{\prime} P x(t)+H_{1}^{\prime} P H_{1} } \\
& +2 H_{1}^{\prime} P\left(D+D_{1} K_{2}\right) x(t-\tau)+2 H_{1}^{\prime} P\left(C+D_{1} K_{1}\right) x(t) .
\end{align*}
$$

In addition, by (3.1), we obtain

$$
\begin{align*}
2 H_{0}^{\prime} P x & (t)+2 H_{1}^{\prime} P\left(D+D_{1} K_{2}\right) x(t-\tau)+2 H_{1}^{\prime} P\left(C+D_{1} K_{1}\right) x(t)+H_{1}^{\prime} P H_{1} \\
\leq & 2 \epsilon\|P\|\left(1+\|C\|+\left\|D_{1}\right\| \cdot\left\|K_{1}\right\|+\epsilon\right)\|x(t)\|^{2} \\
& +2 \epsilon\|P\|\left(1+\|C\|+\left\|D_{1}\right\| \cdot\left\|K_{1}\right\|+\|D\|+\left\|D_{1}\right\| \cdot\left\|K_{2}\right\|+\epsilon\right)\|x(t)\| \cdot\|x(t-\tau)\|  \tag{3.7}\\
& +\left(2 \epsilon+\epsilon^{2}\right)\|P\| \cdot\|x(t-\tau)\|^{2}
\end{align*}
$$

By inequality $|a b| \leq(1 / 2)\left(a^{2}+b^{2}\right)$, then (3.7) becomes

$$
\begin{align*}
& 2 H_{0}^{\prime} P x(t)+2 H_{1}^{\prime} P\left(D+D_{1} K_{2}\right) x(t-\tau)+2 H_{1}^{\prime} P\left(C+D_{1} K_{1}\right) x(t)+H_{1}^{\prime} P H_{1} \\
& \leq \epsilon\left(3+3\|C\|+3\left\|D_{1}\right\| \cdot\left\|K_{1}\right\|+3 \epsilon+\|D\|+\left\|D_{1}\right\| \cdot\left\|K_{2}\right\|\right)\|P\| I\|x(t)\|^{2} \\
&+\epsilon\left(3+\|D\|+\left\|D_{1}\right\| \cdot\left\|K_{2}\right\|+\left\|K_{2}\right\|\|C\|+\left\|D_{1}\right\| \cdot\left\|K_{1}\right\|+2 \epsilon\right)\|P\| I\|x(t-\tau)\|^{2}  \tag{3.8}\\
& \quad= {\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right]^{\prime} Z_{1}\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right] }
\end{align*}
$$

Substituting (3.8) into (3.6), it follows

$$
\mathscr{L} V(t, x(t)) \leq\left[\begin{array}{c}
x(t)  \tag{3.9}\\
x(t-\tau)
\end{array}\right]^{\prime}\left(Z+Z_{1}\right)\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right]
$$

According to (3.2), that is $\rho V(t, x(t))<0$ in the domain $\{t>0\} \times \mathbf{U}$ for $x \neq 0$, so the local stabilization of Theorem 3.2 is obtained by Corollary of [1]. By the same discussion, the global stabilization conditions can be also obtained by Theorem 4.4 of [1].

From Theorem 3.2, we can derive some useful results, which can be expressed in terms of LMIs.

Corollary 3.3. If $H_{i} \equiv 0, i=0,1$, and the matrix inequality $Z<0$ has solutions $P>0, Q>0$ and $K_{1}, K_{2} \in \mathbf{R}^{m \times n}$, then the linear stochastic time-delay system

$$
\begin{equation*}
d x(t)=\left(A x(t)+B x(t-\tau)+B_{1} u(t)\right) d t+\left(C x(t)+D x(t-\tau)+D_{1} u(t)\right) d w(t) \tag{3.10}
\end{equation*}
$$

is globally robustly stabilizable. If $D=0, D_{1}=0$, and the following $L M I$ :

$$
\left[\begin{array}{cccc}
\Sigma_{5} & \widehat{P} & \widehat{P} C^{\prime} & B_{1} X  \tag{3.11}\\
\widehat{P} & -\widehat{Q} & 0 & 0 \\
C \widehat{P} & 0 & -\widehat{P} & 0 \\
X B_{1}^{\prime} & 0 & 0 & -\widehat{Q}
\end{array}\right]<0
$$

has solutions $\widehat{P}>0, Y \in \mathbf{R}^{m \times n}, X \in \mathbf{R}^{n \times m}$, and $\widehat{Q}>0$, then

$$
\begin{equation*}
d x(t)=\left(A x(t)+B x(t-\tau)+B_{1} u(t)\right) d t+C x(t) d w(t) \tag{3.12}
\end{equation*}
$$

is globally robustly stabilizable, where $\Sigma_{5}=A \widehat{P}+\widehat{P} A^{\prime}+B_{1} Y+Y^{\prime} B_{1}^{\prime}+B \widehat{Q} B^{\prime}+B X B_{1}^{\prime}+B_{1} X B^{\prime}$, and a stabilizing feedback control law $u(t)=K_{1} x(t)+K_{2} x(t-\tau)=Y \widehat{P}^{-1} x(t)+X \widehat{Q}^{-1} x(t-\tau)$.

Proof. If $H_{i}(\cdot, \cdot, \cdot)=0, i=0,1$, we can take $\epsilon=0$ in (3.1), then $\mathcal{\varrho} V(t, x(t))<0$ for $(t, x) \in t>$ $0 \times \mathbf{R}^{n}$, except possibly at $x=0$.

Thus, the first part of Corollary 3.3 is proved.
Furthermore, if $D=0, D_{1}=0$, (3.2) degenerates into

$$
Z=\left[\begin{array}{cc}
\Sigma_{6} & P\left(B+B_{1} K_{2}\right)  \tag{3.13}\\
\left(B+B_{1} K_{2}\right)^{\prime} P & -Q
\end{array}\right]<0,
$$

where $\Sigma_{6}=P\left(A+B_{1} K_{1}\right)+\left(A+B_{1} K_{1}\right)^{\prime} P+Q+C^{\prime} P C$.
According to Schur's complement, (3.13) is equivalent to

$$
\begin{equation*}
\Sigma_{6}+P\left(B+B_{1} K_{2}\right) Q^{-1}\left(B+B_{1} K_{2}\right)^{\prime} P<0 . \tag{3.14}
\end{equation*}
$$

Then, pre- and post-(3.14) by $P^{-1}$, we have

$$
\begin{equation*}
P^{-1} \Sigma_{6} P^{-1}+\left(B+B_{1} K_{2}\right) Q^{-1}\left(B+B_{1} K_{2}\right)^{\prime}<0 \tag{3.15}
\end{equation*}
$$

Setting $\widehat{P}=P^{-1}, Y=K_{1} P^{-1}, \widehat{Q}=Q^{-1}$, and $X=K_{2} Q^{-1}$. Again, by Schur's complement, (3.15) is equivalent to (3.11). Thus, the second part of Corollary 3.3 is also proved.

Remark 3.4. Reference [13] considered the analogous problem to Corollary 3.3 by delay feedback, where the main result is expressed by means of generalized algebraic Riccati equations (GAREs) GAREs. However, Corollary 3.3 gives a sufficient condition in terms of LMIs which are easy to be solved.

Corollary 3.5. If the following LMI:

$$
\left[\begin{array}{cccccc}
\Gamma_{1}^{*} & \sqrt{2} \widehat{P} C^{\prime} & \widehat{P} & B^{\prime} & 0 & 0  \tag{3.16}\\
\sqrt{2} C \widehat{P} & -\widehat{P} & 0 & 0 & 0 & 0 \\
\widehat{P} & 0 & -I & 0 & 0 & 0 \\
B & 0 & 0 & -I & \sqrt{2} D^{\prime} & K_{2}^{\prime} B_{1}^{\prime} \\
0 & 0 & 0 & \sqrt{2} D & -\widehat{P} & 0 \\
0 & 0 & 0 & B_{1} K_{2} & 0 & -\widehat{P}
\end{array}\right]<0,
$$

has solution $\widehat{P}>0, Y$, and $K_{2}$, then the stochastic linear time-delay controlled system

$$
\begin{equation*}
d x(t)=\left(A x(t)+B x(t-\tau)+B_{1} u(t)\right) d t+\left(C x(t)+D x(t-\tau)+D_{1} u(t)\right) d w(t) \tag{3.17}
\end{equation*}
$$

is globally robustly stabilizable, where $\Gamma_{1}^{*}=A^{\prime} \widehat{P}+\widehat{P} A+B_{1} Y+Y^{\prime} B_{1}^{\prime}+\widehat{P}$. Moreover, the stabilizing feedback control law

$$
\begin{equation*}
u(t)=Y \widehat{P}^{-1} x(t)+K_{2} x(t-\tau) . \tag{3.18}
\end{equation*}
$$

Proof. Applying the well-known inequality

$$
\begin{equation*}
X^{\prime} Y+Y^{\prime} X \leq \gamma X^{\prime} X+\gamma^{-1} Y^{\prime} Y, \quad \forall \gamma>0 \tag{3.19}
\end{equation*}
$$

and supposing $\gamma=1$ for simplicity, we have

$$
\begin{align*}
2 x^{\prime}(t) & P B_{1} K_{2} x(t-\tau)+2 x^{\prime}(t)\left(C+D_{1} K_{1}\right)^{\prime} P \cdot\left(D+D_{1} K_{2}\right) x(t-\tau) \\
\leq & x^{\prime}(t)\left[P+\left(C+D_{1} K_{1}\right)^{\prime} P\left(C+D_{1} K_{1}\right)\right] x(t)  \tag{3.20}\\
& +x^{\prime}(t-\tau)\left[K_{2}^{\prime} B_{1}^{\prime} P B_{1} K_{2}+\left(D+D_{1} K_{2}\right)^{\prime} P \cdot\left(D+D_{1} K_{2}\right)\right] x(t-\tau)
\end{align*}
$$

Let $\Gamma_{1}=\Sigma_{1}+P+\left(C+D_{1} K_{1}\right)^{\prime} P\left(C+D_{1} K_{1}\right), \Gamma_{2}=2\left(D+D_{1} K_{2}\right)^{\prime} P\left(D+D_{1} K_{2}\right)+K_{2}^{\prime} B_{1}^{\prime} P B_{1} K_{2}-Q$. Then,

$$
Z \leq\left[\begin{array}{cc}
\Gamma_{1} & P B  \tag{3.21}\\
* & \Gamma_{2}
\end{array}\right]=\Gamma .
$$

Obviously, if $\Gamma<0$, then $Z<0$. Applying the Theorem 3.2, the closed-loop system of (3.17) is robustly stable [2].

Then, pre- and post-multiplying $\Gamma<0$ by $\operatorname{diag}\left\{P^{-1}, I\right\}$, and by Schur's complement, we have $\Gamma<0$ is equivalent to

$$
\left[\begin{array}{cccccc}
\Gamma_{1}^{*} & \sqrt{2} P^{-1} C^{\prime} & P^{-1} & B^{\prime} & 0 & 0  \tag{3.22}\\
\sqrt{2} C P^{-1} & -P^{-1} & 0 & 0 & 0 & 0 \\
P^{-1} & 0 & -Q^{-1} & 0 & 0 & 0 \\
B & 0 & 0 & -Q & \sqrt{2} D^{\prime} & K_{2}^{\prime} B_{1}^{\prime} \\
0 & 0 & 0 & \sqrt{2} D & -P^{-1} & 0 \\
0 & 0 & 0 & B_{1} K_{2} & 0 & -P^{-1}
\end{array}\right]<0,
$$

where $\Gamma_{1}^{*}=A P^{-1}+P^{-1} A+B_{1} K_{1} P^{-1}+P^{-1} K_{1}^{\prime} B_{1}^{\prime}+P^{-1}$. Set $\widehat{P}=P^{-1}, Y=K_{1} P^{-1}=K_{1} \widehat{P}, Q=I$, (3.22) is equivalent to (3.16). This ends the proof of Corollary 3.5.

Below, for $D=0, D_{1}=0$, we give another sufficient condition for the local (global) stabilization of system (2.1) in the terms of LMIs.

Theorem 3.6. For $D=0, D_{1}=0$ in (2.1), suppose (3.1) holds for all $x, y \in \mathbf{U}\left(x, y \in \mathbf{R}^{n}\right)$. If the LMIs:

$$
\left[\begin{array}{ccccc}
\Pi_{1} & \widehat{P} & \widehat{P} & \sqrt{2} \widehat{P} C^{\prime} & B+B_{1} K_{2} \\
\hat{P} & -\widehat{Q} & 0 & 0 & 0 \\
\widehat{P} & 0 & -\frac{\alpha}{6 \epsilon^{2}} I & 0 & 0 \\
\sqrt{2} C \widehat{P} & 0 & 0 & -\widehat{P} & 0  \tag{3.26}\\
B^{\prime}+K_{2}^{\prime} B_{1}^{\prime} & 0 & 0 & 0 & -\epsilon^{2} I
\end{array}\right]<0,
$$

have solutions $\widehat{P}>0, \alpha, \widehat{Q}>0, K_{2}$, and $Y \in \mathbf{R}^{m \times n}$, then system (2.1) can be locally (globally) robustly stabilized by

$$
\begin{equation*}
u(t)=Y \widehat{P}^{-1} x(t)+K_{2} x(t-\tau) \tag{3.27}
\end{equation*}
$$

where $\Pi_{1}=A \widehat{P}+\widehat{P} A^{\prime}+B_{1} Y+Y^{\prime} B^{\prime}{ }_{1}+\widehat{P}$.
Proof. Applying the well-known inequality (3.19) again and supposing $\gamma=1$ for simplicity, we have (if $0<P \leq I / \alpha$ for some $\alpha>0$ )

$$
\begin{align*}
& 2 H_{0}^{\prime} P x(t)+2 H_{1}^{\prime} P C x(t)+H_{1}^{\prime} P H_{1} \\
& \quad \leq \frac{6 \epsilon^{2}}{\alpha}\left(\|x(t)\|^{2}+\|x(t-\tau)\|^{2}\right)+x^{\prime}(t)\left(P+C^{\prime} P C\right) x(t) \tag{3.28}
\end{align*}
$$

which holds because

$$
\begin{align*}
2 H_{0}^{\prime} P x= & H_{0}^{\prime} P^{1 / 2} \cdot P^{1 / 2} x+x^{\prime} P^{1 / 2} \cdot P^{1 / 2} H_{0}^{\prime} \\
\leq & H_{0}^{\prime} P H_{0}+x^{\prime} P x \\
\leq & \frac{2 \epsilon^{2}}{\alpha}\left(\|x(t)\|^{2}+\|x(t-\tau)\|^{2}\right)+x^{\prime} P x, \\
H_{1}^{\prime} P H_{1} \leq & \frac{2 \epsilon^{2}}{\alpha}\left(\|x(t)\|^{2}+\|x(t-\tau)\|^{2}\right),  \tag{3.29}\\
2 H_{1}^{\prime} P C x= & H_{1}^{\prime} P^{1 / 2} \cdot P^{1 / 2} C x+x^{\prime} C^{\prime} P^{1 / 2} \cdot P^{1 / 2} H_{1}^{\prime} \\
\leq & \frac{2 \epsilon^{2}}{\alpha}\left(\|x(t)\|^{2}+\|x(t-\tau)\|^{2}\right) \\
& +x^{\prime}(t) C^{\prime} P C x(t) .
\end{align*}
$$

Substituting (3.28) into (3.6), it follows that

$$
\mathscr{\rho} V(t, x(t)) \leq\left[\begin{array}{c}
x(t)  \tag{3.30}\\
x(t-\tau)
\end{array}\right]^{\prime} \hat{Z}\left[\begin{array}{c}
x(t) \\
x(t-\tau)
\end{array}\right],
$$

where

$$
\hat{Z}=\left[\begin{array}{cc}
\Pi_{1}^{*}+\frac{6}{\alpha} \epsilon^{2} I & P\left(B+B_{1} K_{2}\right)  \tag{3.31}\\
* & \frac{6}{\alpha} \epsilon^{2} I-Q
\end{array}\right] .
$$

Considering (3.24), (3.25), and (3.26), it follows that

$$
\widehat{Z} \leq\left[\begin{array}{cc}
\Pi_{1}^{*}+\frac{6}{\alpha} \epsilon^{2} I & P\left(B+B_{1} K_{2}\right)  \tag{3.32}\\
* & -\epsilon^{2} I
\end{array}\right]
$$

Let

$$
Z_{1}=\left[\begin{array}{cc}
\Pi_{1}^{*}+\frac{6}{\alpha} \epsilon^{2} I & P\left(B+B_{1} K_{2}\right)  \tag{3.33}\\
* & -\epsilon^{2} I
\end{array}\right]
$$

where $\Pi_{1}^{*}=P\left(A+B_{1} K_{1}\right)+\left(A+B_{1} K_{1}\right)^{\prime} P+Q+P+2 C^{\prime} P C$.
Obviously, if $Z_{1}<0$, then $\widehat{Z}<0$. So if (3.1) holds for all $x \in \mathbf{U}\left(x \in \mathbf{R}^{n}\right)$, and $\widehat{Z}<0$, then system (2.1) can be locally (globally) robustly stabilized by $u(t)=K_{1} x(t)+K_{2} x(t-\tau)$.

Note that $Z_{1}<0$ is equivalent to that

$$
\begin{equation*}
P\left(A+B_{1} K_{1}\right)+\left(A+B_{1} K_{1}\right)^{\prime} P+Q+P+2 C^{\prime} P C+\frac{6}{\alpha} \epsilon^{2} I+P\left(B+B_{1} K_{2}\right) \epsilon^{-2} I\left(K_{2}^{\prime} B_{1}^{\prime}+B^{\prime}\right) P<0 \tag{3.34}
\end{equation*}
$$

Then pre- and postmultiply (3.34) by $P^{-1}$, we have

$$
\begin{align*}
& \left(A+B_{1} K_{1}\right) P^{-1}+P^{-1}\left(A+B_{1} K_{1}\right)^{\prime}+P^{-1} Q P^{-1} \\
& \quad+P^{-1}+2 P^{-1} C^{\prime} P C P^{-1}+P^{-1} \frac{6}{\alpha} \epsilon^{2} I P^{-1}  \tag{3.35}\\
& \quad+\left(B+B_{1} K_{2}\right) \epsilon^{-2} I\left(K_{2}^{\prime} B_{1}^{\prime}+B^{\prime}\right)<0 .
\end{align*}
$$

Setting $\widehat{P}=P^{-1}, Y=K_{1} P^{-1}=K_{1} \widehat{P}$, and $\widehat{Q}=Q^{-1}$ by the Schur's complement, (3.35) is equivalent to (3.23). Thus, the theorem is proved.

In the special case when $K_{2}=0$, our results reduce the corresponding results in memoryless state feedback case. The following theorem gives a sufficient condition for the existence of memoryless state feedback controller of system (2.1) with $D=0, D_{1}=0$.

Theorem 3.7. For $D=0, D_{1}=0$ in (2.1), suppose there exists an $\epsilon>0$,

$$
\begin{equation*}
\sup \left\|H_{i}\left(x, y, K_{1} x\right)\right\| \leq \epsilon(\|x\|+\|y\|), \quad i=0,1 \tag{3.36}
\end{equation*}
$$

holds for all $x, y \in \mathbf{U}\left(x, y \in \mathbf{R}^{n}\right)$, if the LMIs

$$
\left[\begin{array}{ccccc}
\Pi_{1} & \widehat{P} & \widehat{P} & \sqrt{2} \widehat{P} C^{\prime} & B  \tag{3.37}\\
\widehat{P} & -\widehat{Q} & 0 & 0 & 0 \\
\widehat{P} & 0 & -\frac{\alpha}{6 \epsilon^{2}} I & 0 & 0 \\
\sqrt{2} C \widehat{P} & 0 & 0 & -\widehat{P} & 0 \\
B^{\prime} & 0 & 0 & 0 & -\epsilon^{2} I
\end{array}\right]<0
$$

and (3.24), (3.25), and (3.26) have solutions $\widehat{P}>0, \widehat{Q}>0, \alpha$, and $Y \in \mathbf{R}^{m \times n}$, then system (2.1) can be locally(globally) robustly stabilized by

$$
\begin{equation*}
u(t)=Y \widehat{P}^{-1} x(t) \tag{3.38}
\end{equation*}
$$

Proof. It is derived by the same procedure as the proof of Theorem 3.6.
By the above discussion about stochastic systems with single delay (2.1), we further study robust stabilization for the following stochastic systems with multiple delays

$$
\begin{align*}
d x(t)= & {\left[A x(t)+\sum_{j=1}^{q} B_{j} x\left(t-\tau_{j}\right)+\sum_{j=1}^{q} B_{1 j} u_{j}(t)+H_{0}\left(x(t), x\left(t-\tau_{j}\right), u_{j}(t)\right)\right] d t } \\
& +\left[C x(t)+\sum_{j=1}^{q} D_{j} x\left(t-\tau_{j}\right)+\sum_{j=1}^{q} D_{1 j} u_{j}(t)+H_{1}\left(x(t), x\left(t-\tau_{j}\right), u_{j}(t)\right)\right] d w(t)  \tag{3.39}\\
x(t)= & \varnothing(t) \in L^{2}\left(\omega, \mathscr{F}_{0}, C\left([-h, 0], R^{n}\right)\right), \quad t \in[-h, 0]
\end{align*}
$$

where $\tau_{j}>0, j=1, \ldots, q$, denote the state delay; $h=\max \left\{\tau_{j}, j \in[1, q]\right\}$.
For system (3.39), the following memory state feedback control law is adopted:

$$
\begin{equation*}
u_{j}(t)=K_{j 1} x(t)+K_{j 2} x\left(t-\tau_{j}\right), \quad j=1, \ldots, q . \tag{3.40}
\end{equation*}
$$

Applying control law (3.40) to system (3.39), the resulting closed-loop system is given by

$$
\begin{align*}
d x(t)= & {\left[\bar{A} x(t)+\sum_{j=1}^{q} \bar{B} x\left(t-\tau_{j}\right)+H_{0}\left(x(t), x\left(t-\tau_{j}\right), K_{j 1} x(t)+K_{j 2} x\left(t-\tau_{j}\right)\right)\right] d t } \\
& +\left[\bar{C} x(t)+\sum_{j=1}^{q} \bar{D} x\left(t-\tau_{j}\right)+H_{1}\left(x(t), x\left(t-\tau_{j}\right), K_{j 1} x(t)+K_{j 2} x\left(t-\tau_{j}\right)\right)\right] d w(t), \\
x(t)= & \varnothing(t) \in L^{2}\left(\omega, \mathcal{F}_{0}, C\left([-h, 0], R^{n}\right)\right), \quad t \in[-h, 0], \tag{3.41}
\end{align*}
$$

where $\bar{A}=A+\sum_{j=1}^{q} B_{1 j} K_{j 1}, \bar{B}=B_{j}+B_{1 j} K_{j 2}, \overline{\mathrm{C}}=C+\sum_{j=1}^{q} D_{1 j} K_{j 1}, \bar{D}=D_{j}+D_{1 j} K_{j 2}$.
By the same analysis as Theorem 3.2, we obtain the following theorem which gives a general sufficient condition for the robust stabilization of stochastic multiple time-delays system (3.39).

Theorem 3.8. If (3.1) holds, and $K_{j 1}, K_{j 2}, P>0$, and $Q>0$ are the solutions of the following matrix inequality

$$
\begin{equation*}
Z^{0}+Z^{1}<0, \tag{3.42}
\end{equation*}
$$

then system (3.39) can be locally robustly stabilized by $u_{j}(t)=K_{j 1} x(t)+K_{j 2} x\left(t-\tau_{j}\right)$. Especially if $\mathbf{U}$ is replaced by $\mathbf{R}^{n}$, then system (3.39) can be globally robustly stabilized by the same controller.

In (3.42), $Z^{0}$ and $Z^{1}$ are defined by

$$
\begin{align*}
& Z^{0}=\left[\begin{array}{cc}
Z_{11}^{0} & Z_{12}^{0} \\
* & Z_{22}^{0}
\end{array}\right],  \tag{3.43}\\
& Z^{1}=\left[\begin{array}{cc}
Z_{11}^{1} & 0 \\
0 & Z_{22}^{1}
\end{array}\right],
\end{align*}
$$

where

$$
\begin{aligned}
Z_{11}^{0}= & P\left(A+\sum_{j=1}^{q} B_{1 j} K_{j 1}\right)+\left(A+\sum_{j=1}^{q} B_{1 j} K_{j 1}\right)^{\prime} P+Q \\
& +\left(C+\sum_{j=1}^{q} D_{1 j} K_{j 1}\right)^{\prime} P\left(C+\sum_{j=1}^{q} D_{1 j} K_{j 1}\right), \\
Z_{12}^{0}= & P \sum_{j=1}^{q}\left(B_{j}+B_{1 j} K_{j 2}\right)+\left(C+\sum_{j=1}^{q} D_{1 j} K_{j 1}\right)^{\prime} P \sum_{j=1}^{q}\left(D_{j}+D_{1 j} K_{j 2}\right),
\end{aligned}
$$

$$
\begin{align*}
& Z_{22}^{0}= {\left[\sum_{j=1}^{q}\left(D_{j}+D_{1 j} K_{j 2}\right)\right]^{\prime}\left[\sum_{j=1}^{q}\left(D_{j}+D_{1 j} K_{j 2}\right)\right]-Q, } \\
& Z_{11}^{1}=\epsilon\left(3+3\|C\|+3\left\|\sum_{j=1}^{q} D_{1 j}\right\| \cdot\left\|\sum_{j=1}^{q} K_{j 1}\right\|+3 \epsilon\right. \\
&\left.+\|D\|+\epsilon\left\|\sum_{j=1}^{q} D_{1 j}\right\| \cdot\left\|\sum_{j=1}^{q} K_{j 2}\right\|\right)\|P\| I, \\
& Z_{22}^{1}=\epsilon\left(3+\|D\|+\left\|\sum_{j=1}^{q} D_{1 j}\right\| \cdot\left\|\sum_{j=1}^{q} K_{j 2}\right\|+\|C\|\right. \\
&\left.\quad+\left\|\sum_{j=1}^{q} D_{1 j}\right\| \cdot\left\|\sum_{j=1}^{q} K_{j 1}\right\|+2 \epsilon\right)\|P\| I . \tag{3.44}
\end{align*}
$$

Remark 3.9. From Theorem 3.7, some useful results can be easily derived for stochastic multiple timedelays systems (3.39), which are similar to the results obtained for stochastic single time-delay systems (2.1).

## 4. Numerical Example

Now, we present one example to illustrate the effectiveness of our developed result (Theorem 3.6) in testing the stabilization of nonlinear stochastic time-delay system (2.1). In (2.1), take $D=0, D_{1}=0$, and

$$
\begin{gather*}
A=\left[\begin{array}{ll}
-5.00 & 2.23 \\
-1.56 & 2.15
\end{array}\right], \quad B=\left[\begin{array}{cc}
-0.24 & 0.89 \\
1.22 & -0.76
\end{array}\right], \\
B_{1}=\left[\begin{array}{c}
-2.25 \\
4.48
\end{array}\right], \quad C=\left[\begin{array}{cc}
-0.05 & -0.15 \\
0.15 & -0.10
\end{array}\right], \\
H_{0}=\left[\begin{array}{c}
\sin \left(u(t) x_{2}(t-\tau)\right) x_{1}(t) \\
\cos \left(u(t) x_{1}(t-\tau)\right) x_{2}(t)
\end{array}\right],  \tag{4.1}\\
H_{1}=\left[\begin{array}{c}
\exp \left(-\left(u(t)+x_{1}(t-\tau)+x_{2}(t-\tau)\right)^{2}\right) x_{2}(t) \\
\exp \left(-\left(u^{2}(t) x_{1}^{2}(t-\tau)\right)\right) x_{1}(t)
\end{array}\right], \\
\phi(0)=\left[\begin{array}{ll}
10 & 8
\end{array}\right]^{\prime}, \quad \tau=0.5 .
\end{gather*}
$$

Obviously, (2.1) holds for all $x \in \mathbf{R}^{n}$ with $\epsilon=1$.

Case 1 (Memory State Feedback Stabilization). Substituting all the above data into (3.23) and then solving the LMIs (3.23), (3.24), (3.25), and (3.26) by LMIs Toolbox, we can obtain solutions

$$
\begin{gather*}
\widehat{P}=\left[\begin{array}{cc}
0.4625 & -0.0626 \\
-0.0626 & 0.3383
\end{array}\right]>0 \\
\alpha=0.9015 \\
Y=\left[\begin{array}{ll}
-0.1377 & -0.6674
\end{array}\right]  \tag{4.2}\\
K_{2}=\left[\begin{array}{ll}
-0.2391 & 0.2149
\end{array}\right] \\
\widehat{Q}=\left[\begin{array}{ll}
0.1141 & 0.0052 \\
0.0052 & 0.0974
\end{array}\right]>0
\end{gather*}
$$

So by Theorem 3.6, system (2.1) can be globally robustly stabilized by

$$
\begin{align*}
u(t)= & Y \widehat{P}^{-1} x(t)+K_{2} x(t-\tau) \\
= & -0.5790 x_{1}(t)-2.0795 x_{2}(t)  \tag{4.3}\\
& -0.2391 x_{1}(t-\tau)+0.2149 x_{2}(t-\tau)
\end{align*}
$$

The state trajectories of close-loop system (2.6) and control curve in memory state feedback case are illustrated as Figure 1, from which, we see that the closed-loop system (2.6) takes only one second to have been stable.

Case 2 (Memory-Less State Feedback Stabilization). Solving LMIs (3.37), (3.24), (3.25), and (3.26), we obtain

$$
\begin{gather*}
\widehat{P}=\left[\begin{array}{cc}
0.5609 & -0.1964 \\
-0.1964 & 0.4117
\end{array}\right]>0 \\
Y=\left[\begin{array}{ll}
0.0602 & -0.5168
\end{array}\right] \\
\widehat{Q}=\left[\begin{array}{cc}
0.0923 & 0.0017 \\
0.0017 & 0.0790
\end{array}\right]>0  \tag{4.4}\\
\alpha=0.9243
\end{gather*}
$$

So by Theorem 3.7, system (2.1) can be globally robustly stabilized by

$$
\begin{equation*}
u(t)=Y \widehat{P}^{-1} x(t)=-0.4449 x_{1}(t)-1.5770 x_{2}(t) \tag{4.5}
\end{equation*}
$$



Figure 1: State trajectories and control input in memory state feedback case.

The state trajectories of close-loop system (2.6) and control curve in memoryless state feedback case are illustrated as Figure 2, from which, it can be seen that the closed-loop system (2.6) takes 1.5 seconds to have been stable.

From the two simulation results, the time needed to stabilize system using memory state feedback controller is less than that using memory-less state feedback controller, which shows the advantage of memory state feedback control.

## 5. Conclusions

This paper has discussed memory state feedback stabilization of stochastic systems with time-delay and nonlinear uncertainties. Some sufficient conditions have been given for the existence of a memory state feedback stabilizing control law in terms of linear matrix inequalities, which have the advantage of easy computation. The corresponding results to stochastic single time-delay systems have been further extended to the stochastic multiple time-delays systems. The results obtained in this paper can be reduced to the corresponding results in memoryless state feedback case and may also be extended to other stochastic system model.


Figure 2: State trajectories and control input in memoryless state feedback case.

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Research Article

# Stochastic Approximations and Monotonicity of a Single Server Feedback Retrial Queue 

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This paper focuses on stochastic comparison of the Markov chains to derive some qualitative approximations for an $M / G / 1$ retrial queue with a Bernoulli feedback. The main objective is to use stochastic ordering techniques to establish various monotonicity results with respect to arrival rates, service time distributions, and retrial parameters.

## 1. Introduction

Retrial queueing systems are described by the feature that the arriving customers (or calls) who find the server busy join the orbit to try again for their requests in a random order and at random time intervals. Retrial queues are widely and successfully used as mathematical models of several computer systems and telecommunication networks. For excellent and recent bibliographies on retrial queues, the reader is referred to [1-3].

Most of the queueing systems with repeated attempts assume that each customer in the retrial group seeks service independently of each other after a random time exponentially distributed with rate $\theta$ so that the probability of repeated attempt during the interval $(t, t+\Delta t)$ given that there were $n$ customers in orbit at time $t$ is $n \theta \Delta t+\circ(\Delta t)$. This discipline for access to the server from the retrial group is called classical retrial policy $[4,5]$.

Several papers on retrial queues have analyzed the systems without customer feedback. A more practical retrial queue with the Bernoulli feedback of the customers occurs in many real world situations: for instance, in communication networks where data transmissions need to be guaranteed to be error free within some specified probability, feedback schemes are used to request retransmission of packets that are lost or received in a corrupted form.

Because of complexity of retrial queueing models, analytic results are generally difficult to obtain. In contrast, there are a great number of numerical and approximation methods which are of practical importance. One important approach is monotonicity which allow to establish some stochastic bounds helpful in understanding complicated models by more simpler models for which an evaluation can be made using the stochastic comparison method based on the general theory of stochastic orderings [6].

Stochastic orders represent an important tool for many problems in probability and statistics. They lead to powerful approximation methods and bounds in situations where realistic stochastic models are too complex for rigorous treatment. They are also helpful in situations where fundamental model distributions are only known partially. Further details and applications about these stochastic orders may be found in [6-8].

There exists a flourishing literature on stochastic comparison methods and monotonicity of queues. Oukid and Aissani [9] obtain lower bound and new upper bound for the mean busy period of GI/GI/1 queue with breakdowns and FIFO discipline. Boualem et al. [10] investigate some monotonicity properties of an $M / G / 1$ queue with constant retrial policy in which the server operates under a general exhaustive service and multiple vacation policy relative to strong stochastic ordering and convex ordering. These results imply in particular simple insensitive bounds for the stationary queue length distribution. More recently, Taleb and Aissani [11] investigate some monotonicity properties of an unreliable $M / G / 1$ retrial queue relative to strong stochastic ordering and increasing convex ordering.

In this work, we use the tools of the qualitative analysis to investigate various monotonicity properties for an $M / G / 1$ retrial queue with classical retrial policy and Bernoulli feedback relative to strong stochastic ordering, increasing convex ordering and the Laplace ordering. Instead of studying a performance measure in a quantitative fashion, this approach attempts to reveal the relationship between the performance measures and the parameters of the system.

The rest of the paper is organized as follows. In Section 2, we describe the mathematical model in detail and derive the generating function of the stationary distribution. In Section 3, we present some useful lemmas that will be used in what follows. Section 4 focusses on stochastic monotonicity of the transition operator of the embedded Markov chain and gives comparability conditions of two transition operators. Stochastic bounds for the stationary number of customers in the system are discussed in Section 5. In Section 6, we obtain approximations for the conditional distribution of the stationary queue given that the server is idle.

## 2. Description and Analysis of the Queueing System

We consider a single server retrial queue with the Bernoulli feedback at which customers arrive from outside the system according to a Poisson process with rate $\lambda$. An arriving customer receives immediate service if the server is idle, otherwise he leaves the service area temporarily to join the retrial group. Any orbiting customer produces a Poisson stream of repeated calls with intensity $\theta$ until the time at which he finds the server idle and starts his service. The service times follow a general probability law with distribution function $B(x)$ having finite mean $\beta_{1}$ and Laplace-Stieltjes transform $\widetilde{\beta}(s)$. After the customer is completely served, he will decide either to join the retrial group again for another service with probability $p(0<p<1)$ or to leave the system forever with probability $\bar{p}(=1-p)$.

We finally assume that the input flow of primary arrivals, intervals between repeated attempts and service times, are mutually independent.

The state of the system at time $t$ can be described by the Markov process $R(t)=$ $\{C(t), N(t), \zeta(t)\}_{(t \geq 0)}$, where $C(t)$ is the indicator function of the server state: $C(t)$ is equal to 0 or 1 depending on whether the server is free or busy at time $t$ and $N(t)$ is the number of customers occupying the orbit. If $C(t)=1$, then $\zeta(t)$ corresponds to the elapsed time of the customer being served at time $t$.

Note that the stationary distribution of the system state (the stationary joint distribution of the server state and the number of customers in the orbit) was found in [12], using the supplementary variables method. In this section, we are interested in the embedded Markov chain. To this end, we describe the structure of the latter, determine its ergodicity condition, and obtain its stationary distribution.

### 2.1. Embedded Markov Chain

Let $\tau_{n}$ be the time of the $n$th departure and $D_{n}$ the number of customers in the orbit just after the time $\tau_{n}$, then $C\left(\tau_{n}^{+}\right)=0$ and $N\left(\tau_{n}^{+}\right)=D_{n}, \forall n \geq 1$. We have the following fundamental recursive equation:

$$
\begin{equation*}
D_{n+1}=D_{n}+v_{n+1}-\delta_{D_{n+1}}+\eta, \tag{2.1}
\end{equation*}
$$

where (i) $v_{n+1}$ is the number of primary customers arriving at the system during the service time which ends at $\tau_{n+1}$. It does not depend on events which have occurred before the beginning of the $(n+1)$ st service. Its distribution is given by:

$$
\begin{equation*}
k_{j}=P\left(v_{n+1}=j\right)=\int_{0}^{\infty} \frac{(\lambda x)^{j}}{j!} e^{-\lambda x} d B(x), \quad j \geq 0 \tag{2.2}
\end{equation*}
$$

with generating function $K(z)=\sum_{j \geq 0} k_{j} z^{j}=\tilde{\beta}(\lambda(1-z))$,
(ii) the Bernoulli random variable $\delta_{D_{n+1}}$ is equal to 1 or 0 depending on whether the customer who leaves the service area at time $\tau_{n+1}$ proceeds from the orbit or otherwise. Its conditional distribution is given by

$$
\begin{equation*}
P\left(\delta_{D_{n+1}}=1 / D_{n}=l\right)=\frac{l \theta}{\lambda+l \theta^{\prime}}, \quad P\left(\delta_{D_{n+1}}=0 / D_{n}=l\right)=\frac{\lambda}{\lambda+l \theta^{\prime}} \tag{2.3}
\end{equation*}
$$

(iii) the random variable $\eta$ is 0 or 1 depending on whether the served customer leaves the system or goes to orbit. We have also that $P[\eta=1]=p$ and $P[\eta=0]=\bar{p}(=1-p)$.

The sequence $\left\{D_{n}, n \geq 1\right\}$ forms an embedded Markov chain with transition probability matrix $\mathbf{P}=\left(p_{i j}\right)_{i, j \geq 0}$, where $p_{i j}=P\left(D_{n+1}=j / D_{n}=i\right)$, defined by

$$
\begin{equation*}
p_{i j}=\frac{\lambda \bar{p}}{\lambda+i \theta} k_{j-i}+\frac{i \theta \bar{p}}{\lambda+i \theta} k_{j-i+1}+\frac{\lambda p}{\lambda+i \theta} k_{j-i-1}+\frac{i \theta p}{\lambda+i \theta} k_{j-i} . \tag{2.4}
\end{equation*}
$$

Note that $p_{i j} \neq 0$ only for $i=0,1, \ldots, j+1$.
Theorem 2.1. The embedded Markov chain $\left\{D_{n}, n \geq 1\right\}$ is ergodic if and only if $\rho=\lambda \beta_{1}+p<1$.

Proof. It is not difficult to see that $\left\{D_{n}, n \geq 1\right\}$ is irreducible and aperiodic. To find a sufficient condition, we use Foster's criterion which consists to show the existence of a nonnegative function $f(s), s \in S$, and $\epsilon>0$ such that the mean drift $x_{s}=E\left[f\left(D_{n+1}\right)-f\left(D_{n}\right) / D_{n}=s\right]$ is finite for all $s \in S$ and $x_{s} \leq-\epsilon$ for all $s \in S$ except perhaps a finite number. In our case, we consider the function $f(s)=s$ for all $s \in \mathcal{S}$. Then, the mean drift is given by

$$
\begin{equation*}
x_{s}=E\left[f\left(D_{n+1}\right)-f\left(D_{n}\right) / D_{n}=s\right]=E\left[v_{n+1}-\delta_{D_{n+1}}+\eta / D_{n}=s\right]=\lambda \beta_{1}-\frac{s \theta}{\lambda+s \theta}+p \tag{2.5}
\end{equation*}
$$

Let $x=\lim _{s \rightarrow \infty} x_{s}$. Then $x=\lambda \beta_{1}-1+p<0$. Therefore, the sufficient condition is $\lambda \beta_{1}+P<1$.
To prove that the previous condition is also a necessary condition for ergodicity of our embedded Markov chain, we apply Kaplan's condition: $x_{i}<\infty$, for all $i \geq 0$, and there is an $i_{0}$ such that $x_{i} \geq 0$, for $i \geq i_{0}$. In our case, this condition is verified because $p_{i j}=0$ for $j<i-1$ and $i>0$ (see (2.4)).

### 2.2. Generating Function of the Stationary Distribution

Now, under the condition $\rho<1$, we find the stationary distribution $\pi_{m}=\lim _{n \rightarrow \infty} P\left(D_{n}=m\right)$. Using (2.4), one can obtain Kolmogorov equations for the distribution $\pi_{m}$

$$
\begin{equation*}
\pi_{m}=\sum_{n=0}^{m} \frac{\lambda \bar{p}}{\lambda+n \theta} \pi_{n} k_{m-n}+\sum_{n=1}^{m+1} \frac{n \theta \bar{p}}{\lambda+n \theta} \pi_{n} k_{m-n+1}+\sum_{n=0}^{m-1} \frac{\lambda p}{\lambda+n \theta} \pi_{n} k_{m-n-1}+\sum_{n=0}^{m} \frac{n \theta p}{\lambda+n \theta} \pi_{n} k_{m-n} \tag{2.6}
\end{equation*}
$$

Because of presence of convolutions, these equations can be transformed with the help of the generating functions $\pi(z)=\sum_{m \geq 0} \pi_{m} z^{m}$ and $L(z)=\sum_{m \geq 0}\left(\pi_{m} / \lambda+m \theta\right) z^{m}$ to

$$
\begin{align*}
\pi(z) & =\widetilde{\beta}(\lambda-\lambda z)\left[\lambda \bar{p} L(z)+\theta z p L^{\prime}(z)+\theta \bar{p} L^{\prime}(z)+\lambda p z L(z)\right]  \tag{2.7}\\
& =\widetilde{\beta}(\lambda-\lambda z)(\bar{p}+p z)\left[\lambda L(z)+\theta L^{\prime}(z)\right]
\end{align*}
$$

Since

$$
\begin{equation*}
\pi(z)=\frac{\lambda \pi_{m}}{\lambda+m \theta} z^{m}+\frac{m \theta \pi_{m}}{\lambda+m \theta} z^{m}=\lambda L(z)+\theta z L^{\prime}(z) \tag{2.8}
\end{equation*}
$$

from (2.7) and (2.8), we have

$$
\begin{equation*}
\theta L^{\prime}(z)[(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)-z]=\lambda L(z)[1-(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)] \tag{2.9}
\end{equation*}
$$

We consider now the function $f(z)=(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)-z$.

It is easy to show that

$$
\begin{gather*}
f(1)=\tilde{\beta}(0)-1=1-1=0, \\
f^{\prime}(z)=-\lambda(\bar{p}+p z) \tilde{\beta}^{\prime}(\lambda-\lambda z)+p \widetilde{\beta}(\lambda-\lambda z)-1, \\
f^{\prime}(1)=-\lambda \tilde{\beta}^{\prime}(0)+p \tilde{\beta}(0)-1=\rho-1<0,  \tag{2.10}\\
f^{\prime \prime}(z)=\lambda^{2}(\bar{p}+p z) \tilde{\beta}^{\prime \prime}(\lambda-\lambda z) \geq 0 .
\end{gather*}
$$

Therefore the function $f(z)$ is decreasing on the interval $[0,1], z=1$ is the only zero there and for $z \in[0,1)$ the function is positive, that is, (as $\rho<1$ ) for $z \in[0,1$ ) we have: $z<$ $(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z) \leq 1$.

Besides,

$$
\begin{equation*}
\frac{1-(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)}{(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)-z}=\frac{\rho}{1-\rho}<\infty, \tag{2.11}
\end{equation*}
$$

that is, the function $1-(\bar{p}+p z) \widetilde{\beta}(\lambda-\lambda z) /(\bar{p}+p z) \widetilde{\beta}(\lambda-\lambda z)-z$ can be defined at the point $z=1$ as $\rho / 1-\rho$.

This means that for $z \in[0,1]$ we can rewrite (2.9) as follows:

$$
\begin{equation*}
L^{\prime}(z)=\frac{\lambda}{\theta} \frac{1-(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)}{(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)-z} L(z) . \tag{2.12}
\end{equation*}
$$

The solution of the differential equation (2.12) is given by

$$
\begin{equation*}
L(z)=L(1) \exp \left(-\frac{\lambda}{\theta} \int_{z}^{1} \frac{1-(\bar{p}+p u) \tilde{\beta}(\lambda-\lambda u)}{(\bar{p}+p u) \tilde{\beta}(\lambda-\lambda u)-u} d u\right) \tag{2.13}
\end{equation*}
$$

From (2.8), we have

$$
\begin{equation*}
\pi(z)=\frac{\lambda(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)(1-z)}{(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)-z} L(1) \exp \left(-\frac{\lambda}{\theta} \int_{z}^{1} \frac{1-(\bar{p}+p u) \tilde{\beta}(\lambda-\lambda u)}{(\bar{p}+p u) \tilde{\beta}(\lambda-\lambda u)-u} d u\right) \tag{2.14}
\end{equation*}
$$

We obtain from the normalization condition $\pi(1)=1$ that $L(1)=(1-\rho) / \lambda$.
Finally we get the following formula for the generating function of the steady state queue size distribution at departure epochs (which is known in the literature as the stochastic decomposition property):

$$
\begin{equation*}
\pi(z)=\left[\frac{(1-\rho) \tilde{\beta}(\lambda-\lambda z)(1-z)}{(\bar{p}+p z) \tilde{\beta}(\lambda-\lambda z)-z}\right]\left[(\bar{p}+p z) \exp \left(-\frac{\lambda}{\theta} \int_{z}^{1} \frac{1-(\bar{p}+p u) \tilde{\beta}(\lambda-\lambda u)}{(\bar{p}+p u) \tilde{\beta}(\lambda-\lambda u)-u} d u\right)\right] . \tag{2.15}
\end{equation*}
$$

It is easy to see that the right hand part of expression (2.15) can be decomposed into two factors. The first factor is the generating function for the number of customers in $M / G / 1$
queueing system with Bernoulli feedback (see [13]); the remaining one is the generating function for the number of customers in the retrial queue with feedback given that the server is idle [12]. One can see that formula (2.15) is cumbersome (it includes integrals of Laplace transform, solutions of functional equations). It is why we use, in the rest of the paper, the general theory of stochastic orderings to investigate the monotonicity properties of the system relative to the strong stochastic ordering, convex ordering, and Laplace ordering.

## 3. Preliminaries

### 3.1. Stochastic Orders and Ageing Notions

First, let us recall some stochastic orders and ageing notions which are most pertinent to the main results to be developed in the subsequent section.

Definition 3.1. For two random variables $X$ and $Y$ with densities $f$ and $g$ and cumulative distribution functions $F$ and $G$, respectively, let $\bar{F}=1-F$ and $\bar{G}=1-G$ be the survival functions. As the ratios in the statements below are well defined, $X$ is said to be smaller than $Y$ in:
(a) stochastic ordering (denoted by $X \leq_{\mathrm{st}} Y$ ) if and only if $\bar{F}(x) \leq \bar{G}(x), \forall x \geq 0$,
(b) increasing convex ordering (denoted by $X \leq_{\text {icx }} Y$ ) if and only if $\int_{x}^{+\infty} \bar{F}(u) d(u) \leq$ $\int_{x}^{+\infty} \bar{G}(u) d(u), \forall x \geq 0$,
(c) Laplace ordering (denoted by $X \leq_{L} Y$ ) if and only if $\int_{0}^{+\infty} \exp (-s x) d F(x) \geq$ $\int_{0}^{+\infty} \exp (-s x) d G(x), \forall s \geq 0$.

If the random variables of interest are of discrete type and $\omega=\left(\omega_{n}\right)_{n \geq 0}, \beta=\left(\beta_{n}\right)_{n \geq 0}$ are the corresponding distributions, then the definitions above can be given in the following forms:
(a) $\omega \leq_{s t} \beta$ if and only if $\bar{\omega}_{m}=\sum_{n \geq m} \omega_{n} \leq \bar{\beta}_{m}=\sum_{n \geq m} \beta_{n}$, for all $m$,
(b) $\omega \leq_{\text {icx }} \beta$ if and only if $\overline{\bar{\omega}}_{m}=\sum_{n \geq m} \sum_{k \geq n} \omega_{k} \leq \overline{\bar{\beta}}_{m}=\sum_{n \geq m} \sum_{k \geq n} \beta_{k}$, for all $m$,
(c) $\omega \leq_{L} \beta$ if and only if $\sum_{n \geq 0} \omega_{n} z^{n} \geq \sum_{n \geq 0} \beta_{n} z^{n}$, for all $z \in[0,1]$.

For a comprehensive discussion on these stochastic orders see [6-8].
Definition 3.2. Let $X$ be a positive random variable with distribution function $F$ :
(a) $F$ is HNBUE (harmonically new better than used in expectation) if and only if $F \leq_{\text {icx }} F^{*}$,
(b) $F$ is HNWUE (harmonically new worse than used in expectation) if and only if $F \geq_{\text {icx }} F^{*}$,
(c) $F$ is of class $£$ if and only if $F \geq_{S} F^{*}$,
where $F^{*}$ is the exponential distribution function with the same mean as $F$.

### 3.2. Some Useful Lemmas

Consider two $M / G / 1$ retrial queues with classical retrial policy and Bernoulli feedback with parameters $\lambda^{(i)}$ and $B^{(i)}, i=1,2$. Let $k_{j}^{(i)}=\int_{0}^{+\infty}\left(\left(\lambda^{(i)} x\right)^{j} / j!\right) e^{-\lambda^{(i)} x} d B^{(i)}(x)$ be the distribution of the number of primary calls which arrive during the service time of a call in the $i$ th system.

The following lemma turns out to be a useful tool for showing the monotonicity properties of the embedded Markov chain.

Lemma 3.3. If $\lambda^{(1)} \leq \lambda^{(2)}$ and $B^{(1)} \leq_{s} B^{(2)}$, then $\left\{k_{n}^{(1)}\right\} \leq_{s}\left\{k_{n}^{(2)}\right\}$, where $\leq_{s}$ is either $\leq_{\text {st }}$ or $\leq_{\text {icx }}$.
Proof. To prove that $\left\{k_{n}^{(1)}\right\} \leq_{s}\left\{k_{n}^{(2)}\right\}$, we have to establish the usual numerical inequalities:

$$
\begin{align*}
& \bar{k}_{n}^{(1)}=k_{m}^{(1)} \leq \bar{k}_{n}^{(2)},\left(\text { for } \leq_{s}=\leq_{\text {st }}-\text { ordering }\right) \\
& \overline{\bar{k}}_{n}^{(1)}=\bar{k}_{m}^{(1)} \leq \overline{\bar{k}}_{n}^{(2)},\left(\text { for } \leq_{s}=\leq_{\text {icx }}-\text { ordering }\right) . \tag{3.1}
\end{align*}
$$

The rest of the proof is known in the more general setting of a random summation.
The next lemma is key to proving the main result in Section 6.
Lemma 3.4. If $\lambda^{(1)} \leq \lambda^{(2)}$ and $B^{(1)} \leq_{L} B^{(2)}$, then $\left\{k_{n}^{(1)}\right\} \leq_{L}\left\{k_{n}^{(2)}\right\}$.
Proof. We have, $K^{(i)}(z)=\sum_{n \geq 0} k_{n}^{(i)} z^{n}=\tilde{\beta}^{(i)}\left(\lambda^{(i)}(1-z)\right), i=1,2$, where $K^{(1)}(z), K^{(2)}(z)$ are the corresponding distributions of the number of new arrivals during a service time.

Let $\lambda^{(1)} \leq \lambda^{(2)}, B^{(1)} \leq_{L} B^{(2)}$. To prove that $\left\{k_{n}^{(1)}\right\} \leq_{L}\left\{k_{n}^{(2)}\right\}$, we have to establish that

$$
\begin{equation*}
\tilde{\beta}^{(1)}\left(\lambda^{(1)}(1-z)\right) \geq \tilde{\beta}^{(2)}\left(\lambda^{(2)}(1-z)\right) . \tag{3.2}
\end{equation*}
$$

The inequality $B^{(1)} \leq_{L} B^{(2)}$ means that $\tilde{\beta}^{(1)}(s) \geq \widetilde{\beta}^{(2)}(s)$ for all $s \geq 0$.
In particular, for $s=\lambda^{(1)}(1-z)$ we have

$$
\begin{equation*}
\tilde{\beta}^{(1)}\left(\lambda^{(1)}(1-z)\right) \geq \tilde{\beta}^{(2)}\left(\lambda^{(1)}(1-z)\right) . \tag{3.3}
\end{equation*}
$$

Since any Laplace transform is a decreasing function, $\lambda^{(1)} \leq \lambda^{(2)}$ implies that

$$
\begin{equation*}
\tilde{\beta}^{(2)}\left(\lambda^{(1)}(1-z)\right) \geq \tilde{\beta}^{(2)}\left(\lambda^{(2)}(1-z)\right) \tag{3.4}
\end{equation*}
$$

By transitivity, (3.3) and (3.4) give (3.2).

## 4. Stochastic Monotonicity of Transition Operator

Let $\mathbf{Q}$ be the transition operator of an embedded Markov chain, which associates to every distribution $\omega=\left\{p_{i}\right\}_{i \geq 0}$, a distribution $\mathbf{Q}_{\omega}=\left\{q_{j}\right\}_{j \geq 0}$ such that $q_{j}=\sum_{i} p_{i} p_{i j}$.

Corollary 4.1 (see [6]). The operator $\mathbf{Q}$ is monotone with respect to $\leq_{\text {st }}$ if and only if $\bar{p}_{n m}-\bar{p}_{n-1 m} \geq 0$, and is monotone with respect to $\leq_{i c x}$ if and only if $\overline{\bar{p}}_{n-1 m}+\overline{\bar{p}}_{n+1 m}-2 \overline{\bar{p}}_{n m} \geq 0, \forall n, m$. Here, $\bar{p}_{n m}=$ $\sum_{l=m}^{\infty} p_{n l}$ and $\overline{\bar{p}}_{n m}=\sum_{l=m}^{\infty} \bar{p}_{n l}$.

Theorem 4.2. The transition operator $\mathbf{Q}$ of the embedded chain $\left\{D_{n}, n \geq 1\right\}$ is monotone with respect to the orders $\leq_{\text {st }}$ and $\leq_{\text {icx }}$.

Proof. We have

$$
\begin{align*}
& \bar{p}_{n m}=p_{n k}=\bar{k}_{m-n}+\frac{\lambda p}{\lambda+n \theta} k_{m-n-1}-\frac{n \theta \bar{p}}{\lambda+n \theta} k_{m-n}  \tag{4.1}\\
& \overline{\bar{p}}_{n m}=\bar{r}_{n k}=\overline{\bar{k}}_{m-n}+\frac{\lambda p}{\lambda+n \theta} \bar{k}_{m-n-1}-\frac{n \theta \bar{p}}{\lambda+n \theta} \bar{k}_{m-n}
\end{align*}
$$

Thus

$$
\begin{align*}
\bar{p}_{n m}-\bar{p}_{n-1 m}= & \frac{\lambda^{2} \bar{p}+(n-1) \lambda \theta+n(n-1) \theta^{2} p}{(\lambda+n \theta)(\lambda+(n-1) \theta)} k_{m-n} \\
& +\frac{(n-1) \theta \bar{p}}{\lambda+(n-1) \theta} k_{m-n+1}+\frac{\lambda p}{\lambda+n \theta} k_{m-n-1} \geq 0 \\
\overline{\bar{p}}_{n-1 m}+\overline{\bar{p}}_{n+1 m}-2 \overline{\bar{p}}_{n m}= & \frac{\lambda p}{\lambda+(n+1) \theta} k_{m-n-2}+\frac{(n-1) \theta \bar{p}}{\lambda+(n-1) \theta} k_{m-n}  \tag{4.2}\\
& +\frac{\lambda^{2} \bar{p}+(n-p) \lambda \theta+n(n+1) \theta^{2} p}{(\lambda+n \theta)(\lambda+(n+1) \theta)} k_{m-n-1} \\
& +\frac{2 \theta^{2}}{(\lambda+n \theta)(\lambda+(n-1) \theta)(\lambda+(n+1) \theta)} \bar{k}_{m-n} \geq 0
\end{align*}
$$

Based on Corollary 4.1 we obtain the stated result.
In Theorem 4.3, we give comparability conditions of two transition operators. Consider two $M / G / 1$ retrial queues with classical retrial policy and feedback with parameters $\lambda^{(1)}, \theta^{(1)}, p^{(1)}, B^{(1)}$, and $\lambda^{(2)}, \theta^{(2)}, p^{(2)}, B^{(2)}$, respectively. Let $\mathbf{Q}^{1}, \mathbf{Q}^{2}$ be the transition operators of the corresponding embedded Markov chains.

Theorem 4.3. If $\lambda^{(1)} \leq \lambda^{(2)}, \theta^{(1)} \geq \theta^{(2)}, p^{(1)} \leq p^{(2)}, B^{(1)} \leq_{s} B^{(2)}$, where $\leq_{s}$ is either $\leq_{s t}$ or $\leq_{\mathrm{icx}}$, then $\mathbf{Q}^{1} \leq_{s} \mathbf{Q}^{2}$, that is, for any distribution $\omega$, one has $\mathbf{Q}^{1} \omega \leq_{s} \mathbf{Q}^{2} \omega$.

Proof. From Stoyan [6], we wish to establish that

$$
\begin{align*}
& \bar{p}_{n m}^{(1)} \leq \bar{p}_{n m^{\prime}}^{(2)} \quad \forall n, m, \quad\left(\text { for } \leq_{s}=\leq_{\text {st }}-\text { ordering }\right)  \tag{4.3}\\
& \overline{\bar{p}}_{n m}^{(1)} \leq \overline{\bar{p}}_{n m^{\prime}}^{(2)} \quad \forall n, m, \quad\left(\text { for } \leq_{s}=\leq_{\mathrm{icx}}-\text { ordering }\right) \tag{4.4}
\end{align*}
$$

To prove inequality (4.3), we have (for $i=1,2$ )

$$
\begin{equation*}
\bar{p}_{n m}^{(i)}=\bar{k}_{m-n}^{(i)}+\frac{\lambda^{(i)} p^{(i)}}{\lambda^{(i)}+n \theta^{(i)}} k_{m-n-1}^{(i)}-\frac{n \theta^{(i)} \bar{p}^{(i)}}{\lambda^{(i)}+n \theta^{(i)}} k_{m-n}^{(i)} . \tag{4.5}
\end{equation*}
$$

By hypothesis, we have that

$$
\lambda^{(1)} \leq \lambda^{(2)}, \quad \theta^{(1)} \geq \theta^{(2)} \Longrightarrow\left\{\begin{array}{l}
\frac{\lambda^{(1)}}{\theta^{(1)}} \leq \frac{\lambda^{(2)}}{\theta^{(2)}}, \text { or }  \tag{4.6}\\
\frac{\theta^{(1)}}{\lambda^{(1)}} \geq \frac{\theta^{(2)}}{\lambda^{(2)}}
\end{array}\right.
$$

Since the function $G(x)=x /(x+n)$ is increasing, we have

$$
\begin{equation*}
G\left(\frac{\lambda^{(1)}}{\theta^{(1)}}\right)=\frac{\lambda^{(1)}}{\lambda^{(1)}+n \theta^{(1)}} \leq G\left(\frac{\lambda^{(2)}}{\theta^{(2)}}\right)=\frac{\lambda^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} \tag{4.7}
\end{equation*}
$$

Moreover, $p^{(1)} \leq p^{(2)}$. Then

$$
\begin{equation*}
\frac{\lambda^{(1)}}{\lambda^{(1)}+n \theta^{(1)}} p^{(1)} \leq \frac{\lambda^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} p^{(2)} \tag{4.8}
\end{equation*}
$$

Similarly, the function $H(x)=x /(1+x)$ is increasing, we have

$$
\begin{equation*}
H\left(\frac{n \theta^{(1)}}{\lambda^{(1)}}\right)=\frac{n \theta^{(1)}}{\lambda^{(1)}+n \theta^{(1)}} \geq H\left(\frac{n \theta^{(2)}}{\lambda^{(2)}}\right)=\frac{n \theta^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} . \tag{4.9}
\end{equation*}
$$

Besides, $p^{(1)} \leq p^{(2)}$ implies that $\bar{p}^{(1)} \geq \bar{p}^{(2)}$. Hence

$$
\begin{equation*}
-\frac{n \theta^{(1)}}{\lambda^{(1)}+n \theta^{(1)}} \bar{p}^{(1)} \leq-\frac{n \theta^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} \bar{p}^{(2)} \tag{4.10}
\end{equation*}
$$

Using inequalities (4.8)-(4.10) and Lemma 3.3 (for $\leq_{s}=\leq_{s t}$-ordering) we get

$$
\begin{align*}
\bar{p}_{n m}^{(1)}= & \bar{k}_{m-n}^{(1)}+\frac{\lambda^{(1)} p^{(1)}}{\lambda^{(1)}+n \theta^{(1)}} k_{m-n-1}^{(1)}-\frac{n \theta^{(1)} \bar{p}^{(1)}}{\lambda^{(1)}+n \theta^{(1)}} k_{m-n}^{(1)} \\
\leq & \bar{k}_{m-n}^{(1)}+\frac{\lambda^{(2)} p^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} k_{m-n-1}^{(1)}-\frac{n \theta^{(2)} \bar{p}^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} k_{m-n}^{(1)}  \tag{4.11}\\
= & \frac{\lambda^{(2)} \bar{p}^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} \bar{k}_{m-n}^{(1)}+\frac{n \theta^{(2)} \bar{p}^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} \bar{k}_{m-n+1}^{(1)} \\
& +\frac{\lambda^{(2)} p^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} \bar{k}_{m-n-1}^{(1)}+\frac{n \theta^{(2)} p^{(2)}}{\lambda^{(2)}+n \theta^{(2)}} \bar{k}_{m-n}^{(1)} \leq \bar{p}_{n m}^{(2)} .
\end{align*}
$$

Following the technique above and using Lemma 3.3 (for $\leq_{s}=\leq_{i c x}$-ordering), we establish inequality (4.4).

## 5. Stochastic Bounds for the Stationary Distribution

Consider two $M / G / 1$ retrial queues with classical retrial policy and feedback with parameters $\lambda^{(1)}, \theta^{(1)}, p^{(1)}, B^{(1)}$ and $\lambda^{(2)}, \theta^{(2)}, p^{(2)}, B^{(2)}$, respectively, and let $\pi_{n}^{(1)}, \pi_{n}^{(2)}$ be the corresponding stationary distributions of the number of customers in the system.

Theorem 5.1. If $\lambda^{(1)} \leq \lambda^{(2)}, \theta^{(1)} \geq \theta^{(2)}, p^{(1)} \leq p^{(2)}, B^{(1)} \leq_{s} B^{(2)}$, then $\left\{\pi_{n}^{(1)}\right\} \leq_{s}\left\{\pi_{n}^{(2)}\right\}$, where $\leq_{s}$ is one of the symbols $\leq_{\text {st }}$ or $\leq_{\text {icx }}$.

Proof. Using Theorems 4.2 and 4.3 which state, respectively, that $\mathbf{Q}^{i}$ are monotone with respect to the order $\leq_{s}$ and $\mathbf{Q}^{1} \leq_{s} \mathbf{Q}^{2}$, we have by induction $\mathbf{Q}^{1, n} \omega \leq_{s} \mathbf{Q}^{2, n} \omega$ for any distribution $\omega$, where $\mathbf{Q}^{i, n}=\mathbf{Q}^{i}\left(\mathbf{Q}^{i, n-1} \omega\right)$. Taking the limit, we obtain the stated result.

Based on Theorem 5.1 we can establish insensitive stochastic bounds for the generating function of the stationary distribution of the embedded Markov chain defined in (2.15).

Theorem 5.2. For any M/G/1 retrial queue with classical retrial policy and Bernoulli feedback the distribution $\pi_{n}$ is greater relative to the increasing convex ordering than the distribution with the generating function

$$
\begin{equation*}
\pi^{*}(z)=\left[\frac{(1-\rho) e^{\lambda \beta_{1}(z-1)}(1-z)}{(\bar{p}+p z) e^{\lambda \beta_{1}(z-1)}-z}\right]\left[(\bar{p}+p z) \exp \left(\frac{\lambda}{\theta} \int_{1}^{z} \frac{1-(\bar{p}+p u) e^{\lambda \beta_{1}(u-1)}}{(\bar{p}+p u) e^{\lambda \beta_{1}(u-1)}-u} d u\right)\right] . \tag{5.1}
\end{equation*}
$$

Proof. Consider an auxiliary $M / D / 1$ retrial queue with classical retrial policy and feedback having the same arrival rate $\lambda$, retrial rate $\theta$, mean service time $\beta_{1}$, and probability $p$, as those of the $M / G / 1$ retrial queue with classical retrial policy and Bernoulli feedback. The service times follow a deterministic low with distribution function:

$$
B^{\star}(x)= \begin{cases}0, & \text { if } x \leq \beta_{1},  \tag{5.2}\\ 1, & \text { if } x>\beta_{1} .\end{cases}
$$

From Stoyan [6], it is known that $B^{\star}(x) \leq_{\mathrm{icx}} B(x)$. Therefore, the required result follows from Theorem 5.1.

Theorem 5.3. If in the $M / G / 1$ retrial queue with classical retrial policy and feedback the service time distribution $B(x)$ is HNBUE (or HNWUE), then $\left\{\pi_{n}\right\} \leq_{i c x}\left\{\pi_{n}^{*}\right\}$ (or $\left\{\pi_{n}^{*}\right\} \leq \leq_{\text {icx }}\left\{\pi_{n}\right\}$ ), where $\left\{\pi_{n}^{*}\right\}$ is the stationary distribution of the number of customers in the $M / M / 1$ retrial queue with classical retrial policy and Bernoulli feedback with the same parameters as those of the $M / G / 1$ retrial queue with classical retrial policy and Bernoulli feedback.

Proof. Consider an auxiliary $M / M / 1$ retrial queue with classical retrial policy and Bernoulli feedback with the same arrival rate $\lambda$, probability $p$, retrial rate $\theta$, and mean service time $\beta_{1}$ as in the $M / G / 1$ retrial queue with classical retrial policy and Bernoulli feedback, but with exponentially distributed service time $B^{*}(x)=1-\exp \left(-\left(x / \beta_{1}\right)\right)$. If $B(x)$ is HNBUE, then $B(x) \leq_{\mathrm{icx}} B^{*}(x)$ (if $B(x)$ is HNWUE, then $B^{*}(x) \leq_{\mathrm{icx}} B(x)$ ). Therefore, by using Theorem 5.1, we deduce the statement of this theorem.

## 6. Stochastic Approximations for the Conditional Distribution

We consider the conditional distribution $\varphi_{n}$ of the stationary queue given that the server is idle. This distribution has also appeared in the stochastic decomposition law for the stationary queue length, see equation (2.15). As we saw its generating function $\phi(z)=$ $\sum_{n \geq 0} \varphi_{n} z^{n}$ was given by

$$
\begin{equation*}
\phi(z)=(\bar{p}+p z) \exp \left(-\frac{\lambda}{\theta} \int_{z}^{1} \frac{1-(\bar{p}+p u) \tilde{\beta}(\lambda-\lambda u)}{(\bar{p}+p u) \widetilde{\beta}(\lambda-\lambda u)-u} d u\right) . \tag{6.1}
\end{equation*}
$$

Theorem 6.1. Suppose we have two $M / G / 1$ retrial queues with classical retrial policy and Bernoulli feedback with parameters $\lambda^{(1)}, \theta^{(1)}, p^{(1)}, B^{(1)}$ and $\lambda^{(2)}, \theta^{(2)}, p^{(2)}, B^{(2)}$, respectively. If $\lambda^{(1)} \leq$ $\lambda^{(2)}, \theta^{(1)} \geq \theta^{(2)}, p^{(1)} \leq p^{(2)}, B^{(1)}(x) \leq_{L} B^{(2)}(x)$, then $\varphi_{n}^{(1)} \leq_{L} \varphi_{n}^{(2)}$.

Proof. By Lemma 3.4, we have $\widetilde{\beta}^{(1)}\left(\lambda^{(1)}(1-z)\right) \geq \widetilde{\beta}^{(2)}\left(\lambda^{(2)}(1-z)\right)$.
Moreover, one has $p^{(1)} \leq p^{(2)} \Rightarrow \bar{p}^{(1)}+p^{(1)} z \geq \bar{p}^{(2)}+p^{(2)} z$, for all $z \in[0,1]$.
This implies that

$$
\begin{equation*}
\frac{1-\left(\bar{p}^{(1)}+p^{(1)} u\right) \tilde{\beta}^{(1)}\left(\lambda^{(1)}-\lambda^{(1)} u\right)}{\left(\bar{p}^{(1)}+p^{(1)} u\right) \tilde{\beta}^{(1)}\left(\lambda^{(1)}-\lambda^{(1)} u\right)-u} \leq \frac{1-\left(\bar{p}^{(2)}+p^{(2)} u\right) \tilde{\beta}^{(2)}\left(\lambda^{(2)}-\lambda^{(2)} u\right)}{\left(\bar{p}^{(2)}+p^{(2)} u\right) \tilde{\beta}^{(2)}\left(\lambda^{(2)}-\lambda^{(2)} u\right)-u} . \tag{6.2}
\end{equation*}
$$

Besides, $\lambda^{(1)} \leq \lambda^{(2)}$ and $\theta^{(1)} \geq \theta^{(2)} \Rightarrow\left(\lambda^{(1)} / \theta^{(1)}\right) \leq\left(\lambda^{(2)} / \theta^{(2)}\right)$ and thus

$$
\begin{equation*}
\frac{\lambda^{(1)}}{\theta^{(1)}} \int_{z}^{1} \frac{1-\left(\bar{p}^{(1)}+p^{(1)} u\right) \tilde{\beta}^{(1)}\left(\lambda^{(1)}-\lambda^{(1)} u\right)}{\left(\bar{p}^{(1)}+p^{(1)} u\right) \tilde{\beta}^{(1)}\left(\lambda^{(1)}-\lambda^{(1)} u\right)-u} d u \leq \frac{\lambda^{(2)}}{\theta^{(2)}} \int_{z}^{1} \frac{1-\left(\bar{p}^{(2)}+p^{(2)} u\right) \tilde{\beta}^{(2)}\left(\lambda^{(2)}-\lambda^{(2)} u\right)}{\left(\bar{p}^{(2)}+p^{(2)} u\right) \tilde{\beta}^{(2)}\left(\lambda^{(2)}-\lambda^{(2)} u\right)-u} d u \tag{6.3}
\end{equation*}
$$

Therefore

$$
\begin{align*}
& \exp \left(-\frac{\lambda^{(1)}}{\theta^{(1)}} \int_{z}^{1} \frac{1-\left(\bar{p}^{(1)}+p^{(1)} u\right) \tilde{\beta}^{(1)}\left(\lambda^{(1)}-\lambda^{(1)} u\right)}{\left(\bar{p}^{(1)}+p^{(1)} u\right) \tilde{\beta}^{(1)}\left(\lambda^{(1)}-\lambda^{(1)} u\right)-u} d u\right)  \tag{6.4}\\
& \quad \geq \exp \left(-\frac{\lambda^{(2)}}{\theta^{(2)}} \int_{z}^{1} \frac{1-\left(\bar{p}^{(2)}+p^{(2)} u\right) \tilde{\beta}^{(2)}\left(\lambda^{(2)}-\lambda^{(2)} u\right)}{\left(\bar{p}^{(2)}+p^{(2)} u\right) \tilde{\beta}^{(2)}\left(\lambda^{(2)}-\lambda^{(2)} u\right)-u} d u\right)
\end{align*}
$$

By combining this latter inequality with the inequality: $\bar{p}^{(1)}+p^{(1)} z \geq \bar{p}^{(2)}+p^{(2)} z$, we get $\phi^{(1)}(z) \geq \phi^{(2)}(z)$ for all $z \in[0,1]$, which means the stochastic inequality $\left\{\varphi_{n}^{(1)}\right\} \leq_{L}\left\{\varphi_{n}^{(2)}\right\}$.

Theorem 6.2. For any $M / G / 1$ retrial queue with classical retrial policy and Bernoulli feedback the distribution $\varphi_{n}$ is less relative to the Laplace ordering than the distribution with the generating function

$$
\begin{equation*}
(\bar{p}+p z) \exp \left(-\frac{\lambda}{\theta} \int_{z}^{1} \frac{1-(\bar{p}+p u) e^{\lambda \beta_{1}(u-1)}}{(\bar{p}+p u) e^{\lambda \beta_{1}(u-1)}-u} d u\right) \tag{6.5}
\end{equation*}
$$

and if $B(x)$ is of class $\mathcal{L}$ then the distribution $\varphi_{n}$ is greater relative to the ordering $\leq_{L}$ than the corresponding distribution in the $M / M / 1$ queue with classical retrial policy and Bernoulli feedback.

Proof. Consider an auxiliary $M / D / 1$ and $M / M / 1$ retrial queues with classical retrial policy and Bernoulli feedback with the same arrival rates $\lambda$, probability $p$, retrial rates $\theta$, and mean service times $\beta_{1}$.

Since $B(x)$ is always less, relative to the ordering $\leq_{L}$, than a deterministic distribution with the same mean value, based on Theorem 6.1 we obtain the stated result.

If $B(x)$ is of class $\mathcal{L}$ then $B(x)$ is greater relative to the ordering $\leq_{L}$ than the exponential distribution with the same mean, based on Theorem 6.1 we can guarantee the second inequality.

## 7. Conclusion and Further Research

In this paper, we prove the monotonicity of the transition operator of the embedded Markov chain relative to strong stochastic ordering and increasing convex ordering. We obtain comparability conditions for the distribution of the number of customers in the system. Inequalities are derived for conditional distribution of the stationary queue given that the server is idle. The obtained results allow us to place in a prominent position the insensitive bounds for both the stationary distribution and the conditional distribution of the stationary queue of the considered model.

Monotonicity results are of importance in robustness analysis: if there is insecurity on the input of the model, then our order results provide information on what kind of deviation from the nominal model to expect. Moreover, in gradient estimation one has to control the growth of the cycle length as function of a change of the model. More precisely, the results established in this paper allow to bound the measure-valued derivative of the stationary distribution where the derivative can be translated into unbiased (higher order) derivative estimators with respect to some parameter (e.g., arrival rate $(\lambda)$ or retrial rate $(\theta)$ parameter). Such bounds can be used to derive information on the speed of convergence of the gradient estimator. Finally, under some conditions (order holds in the strong sense), those results imply a fast convergence of the gradient estimator of the stationary distribution [14-16].

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Research Article

# Existence and Uniqueness for Stochastic Age-Dependent Population with Fractional Brownian Motion 

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#### Abstract

A model for a class of age-dependent population dynamic system of fractional version with Hurst parameter $h \in(1 / 2,1]$ is established. We prove the existence and uniqueness of a mild solution under some regularity and boundedness conditions on the coefficients. The proofs of our results combine techniques of fractional Brownian motion calculus. Ideas of the finite-dimensional approximation by the Galerkin method are used.


## 1. Introduction

Stochastic differential equations have been found in many applications in areas such as economics, biology, finance, ecology, and other sciences [1-3]. In recent years, existence, uniqueness, stability, invariant measures, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by many authors. For example, it is well known that these topics have been developed mainly by using two different methods, that is, the semigroup approach [4,5] (e.g., Taniguchi et al. [4] using semigroup methods discussed existence, uniqueness, pth moment, and almost sure Lyapunov exponents of mild solutions to a class of stochastic partial functional differential equations with finite delays) and the variational one (e.g., Krylov and Rozovskii [6] and Pardoux [7]). On the other hand, although stochastic partial functional differential equations also seem very important as stochastic models of biological, chemical, physical, and economical systems, the corresponding properties of these systems have not been studied in great detail (cf. [8, 9]). As a matter of fact, there exists extensive literature on the related topics for deterministic age-dependent population dynamic system. There has been much recent interest in application of deterministic age-structures mathematical models with
diffusion. For example, Cushing [10] investigated hierarchical age-dependent populations with intraspecific competition or predation.

There has been much recent interest in application of stochastic population dynamics. For example, Qimin and Chongzhao gave a numerical scheme and showed the convergence of the numerical approximation solution to the true solution to stochastic age-structured population system with diffusion [11]. In papers [12, 13], Qi-Min et al. discussed the existence and uniqueness for stochastic age-dependent population equation, when diffusion coefficient $k=0$ and $k \neq 0$, respectively. Numerical analysis for stochastic age-dependent population equation has been studied by Zhang and Han [14]. In papers [11-14], the random disturbances are described by stochastic integrals with respect to Wiener processes.

However, the Wiener process is not suitable to replace a noise process if long-rang dependence is modeled. It is then desirable to replace the Wiener process by fractional Brownian motion. But this process is not a semimartingale, so that it is not possible to apply the Itô calculus. A stochastic analysis with respect to fractional Brownian motion is faced with difficulties.

Next, the stochastic continuous time age-dependent model is derived. In [12], the nonlinear age-dependent population dynamic with diffusion can be written in the following form:

$$
\begin{gather*}
\frac{\partial P(r, t, x)}{\partial t}+\frac{\partial P(r, t, x)}{\partial r}-k_{1}(r, t) \Delta P(r, t, x) \\
=-\mu_{1}(r, t, x) P(r, t, x)+f_{1}(r, t, x)+g_{1}(r, t, x) \frac{d w(t)}{d t}, \quad \text { in } Q_{A}=(0, A) \times Q \\
P(0, t, x)=\int_{0}^{A} \beta_{1}(r, t, x) P(r, t, x) d r, \quad \text { in }(0, T) \times \Gamma  \tag{1.1}\\
P(r, 0, x)=P_{0}(r, x), \quad \text { in }(0, A) \times \Gamma \\
P(r, t, x)=0, \quad \text { on } \Sigma_{A}=(0, A) \times(0, T) \times \partial \Gamma \\
y(t, x)=\int_{0}^{A} P(r, t, x) d r, \quad \text { in } Q
\end{gather*}
$$

where $t \in(0, T), r \in(0, A), x \in \Gamma \subset R^{N}(1 \leq N \leq 3), Q=(0, T) \times \Gamma, P(r, t, x)$ denotes the population density of age $r$ at time $t$ in spatial position, $x, \beta_{1}(r, t, x)$ denotes the fertility rate of females of age $r$ at time $t$, in spatial position $x, \mu_{1}(r, t, x)$ denotes the mortality rate of age $r$ at time $t$, in spatial position $x, \Delta$ denotes the Laplace operator with respect to the space variable, and $k_{1}(r, t)>0$ is the diffusion coefficient. $f_{1}(r, t, x)+g_{1}(r, t, x)(d w(t) / d t)$ denotes effects of external environment for population system, such as emigration and earthquake have. The effects of external environment the deterministic and random parts which depend on $r, t$, and $x . w(t)$ is a standard Wiener process.

In this paper, suppose that $f_{1}(r, t, x)$ is stochastically perturbed, with

$$
\begin{equation*}
f_{1}(r, t, x) \longrightarrow f_{1}(r, t, x)+g_{1}(r, t, x) d B^{h}(t) \tag{1.2}
\end{equation*}
$$

where $B^{h}(t)$ is fractional Brownian motions with the Hurst constant $h$. Then this environmentally perturbed system may be described by the Itô equation

$$
\begin{gather*}
\frac{\partial P(r, t, x)}{\partial t}+\frac{\partial P(r, t, x)}{\partial r}-k_{1}(r, t) \Delta P(r, t, x)  \tag{1.3}\\
=-\mu_{1}(r, t, x) P(r, t, x)+f_{1}(r, t, x)+g_{1}(r, t, x) d B^{h}(t), \quad \text { in } Q_{A}=(0, A) \times Q, \\
P(0, t, x)=\int_{0}^{A} \beta_{1}(r, t, x) P(r, t, x) d r, \quad \text { in }(0, T) \times \Gamma  \tag{1.4}\\
P(r, 0, x)=P_{0}(r, x), \quad \text { in }(0, A) \times \Gamma  \tag{1.5}\\
P(r, t, x)=0, \quad \text { on } \Sigma_{A}=(0, A) \times(0, T) \times \partial \Gamma  \tag{1.6}\\
y(t, x)=\int_{0}^{A} P(r, t, x) d r, \quad \text { in } Q \tag{1.7}
\end{gather*}
$$

new stochastic differential equations (1.3)-(1.7) for an age-dependent population are derived. It is an extension of (1.1).

Our work differs from these references [11-14]. In papers [11-14], the random disturbances are described by stochastic integrals with respect to Wiener processes. In this paper, we study a stochastic age-dependent population dynamic system with an additive noise in the form of a stochastic integral with respect to a Hilbert space-valued fractional Borwnian motion. It is well known that a fractional Brownian motion $B^{h}$ is a semimartingale if and only if $h=1 / 2$, that is, in the case of a classical Brownian motion. For $h=1 / 2$, Qimin and Chongzhao discussed the existence and uniqueness for stochastic age-dependent population equation [12]. In this paper, we shall discuss the existence and uniqueness for a stochastic age-dependent population equation with fractional Brownian motions with $h \in[1 / 2,1]$. The discussion uses ideas of the finite-dimensional approximation by the Galerkin method.

In Section 2, we begin with some preliminary results which are essential for our analysis and introduce the definition of a solution with respect to stochastic age-dependent populations. In Section 3, we shall prove existence and uniqueness of solution for stochastic age-dependent population equation (1.3).

## 2. Preliminaries

Consider stochastic age-structured population system with diffusion (1.3). A is the maximal age of the population species, so

$$
\begin{equation*}
P(r, t, x)=0, \quad \forall r \geq A . \tag{2.1}
\end{equation*}
$$

By (1.7), integrating on $[0, A]$ to (1.3) and (1.5) with respect to $r$, we obtain the following system

$$
\begin{gather*}
\frac{\partial y}{\partial t}-k(t) \Delta y+\mu(t, x) y-\beta(t, x) y \\
=f(t, x)+g(t, x) \frac{d B^{h}(t)}{d t}, \quad \text { in } Q=(0, T) \times \Gamma,  \tag{2.2}\\
y(0, x)=y_{0}(x), \quad \text { in } \Gamma, \\
y(t, x)=0, \quad \text { on } \Sigma=(0, T) \times \partial \Gamma,
\end{gather*}
$$

where

$$
\begin{equation*}
\beta(t, x) \equiv\left(\int_{0}^{A} \beta_{1}(r, t, x) P(r, t, x) d r\right)\left(\int_{0}^{A} P(r, t, x) d r\right)^{-1} \tag{2.3}
\end{equation*}
$$

where $\int_{0}^{A} P(r, t, x) d r=y(t, x)$ is the total population, and the birth process is described by the nonlocal boundary conditions $\int_{0}^{A} \beta_{1}(r, t, x) P(r, t, x) d r$ clearly, $\beta(t, x)$ denotes the fertility rate of total population at time $t$ and in spatial position $x$.

$$
\begin{equation*}
\mu(t, x) \equiv\left(\int_{0}^{A} \mu_{1}(r, t, x) P(r, t, x) d r\right)\left(\int_{0}^{A} P(r, t, x) d r\right)^{-1} \tag{2.4}
\end{equation*}
$$

where $\mu(t, x)$ denotes the mortality rate at time $t$ and in spatial position $x$

$$
\begin{align*}
& f(t, x) \equiv \int_{0}^{A} f_{1}(r, t, x) d r  \tag{2.5}\\
& g(t, x) \equiv \int_{0}^{A} g_{1}(r, t, x) d r
\end{align*}
$$

Let

$$
\begin{equation*}
V=H^{1}(\Gamma) \equiv\left\{\varphi \mid \varphi \in L^{2}(\Gamma), \frac{\partial \varphi}{\partial x_{i}} \in L^{2}(\Gamma), \text { where } \frac{\partial \varphi}{\partial x_{i}} \text { are generalized partial derivatives }\right\} \tag{2.6}
\end{equation*}
$$

Then $V^{\prime}=H^{-1}(\Gamma)$ the dual space of $V$. We denote by $|\cdot|$ and $\|\cdot\|$ the norms in $V$ and $V^{\prime}$ respectively, by $\langle\cdot, \cdot\rangle$ the duality product between $V, V^{\prime}$, and by $(\cdot, \cdot)$ the scalar product in $H$.

We consider stochastic age-structured population system with diffusion of the form

$$
\begin{gather*}
d_{t} y(t)-k \Delta y(t) d t+\mu(t, x) y(t) d t-\beta(t, x) y(t) d t \\
=f(t, x) d t+g(t, x) d B^{h}(t), \quad \text { in } Q=(0, T) \times \Gamma, \\
y(0, x)=y_{0}(x), \quad \text { in } \Gamma,  \tag{2.7}\\
y(t, x)=0, \quad \text { on } \Sigma=(0, T) \times \partial \Gamma,
\end{gather*}
$$

where $d_{t} y(t)$ is the differential of $y(t, x)$ relative to $t$, that is, $\left(d_{t} y(t)=\partial y(t) / \partial t\right) d t, y(t):=$ $y(t, x) . T>0, A>0$.

The integral version of (2.7) is given by the equation

$$
\begin{equation*}
y(t)-y(0)-\int_{0}^{t} k \Delta y(s) d s-\int_{0}^{t}(\beta(s, x)-\mu(s, x)) y(s) d s=\int_{0}^{t} f(s, x) d s+\int_{0}^{t} g(s, x) d B^{h}(s) \tag{2.8}
\end{equation*}
$$

here $y(t, x)=0$, on $\Sigma=(0, T) \times \partial \Gamma$.
Let $B_{j}^{h}(t)_{t \geq 0}(j=1,2, \ldots)$ be independent centered Gaussian processes with $B_{j}^{h}(0)=0$ on a given probability space $(\Omega, \mathcal{F}, P)$, where we assume that

$$
\begin{gather*}
E\left(B_{j}^{h}(t)-B_{j}^{h}(s)\right)^{2}=|t-s|^{2 h} \mu_{j} \quad(j=1,2, \ldots), \\
\mu_{j}>0, \quad \sum_{j=1}^{\infty} \mu_{j}<\infty \tag{2.9}
\end{gather*}
$$

and $h \in[1 / 2,1]$.
The processes $B_{j}^{h}(t)_{t \geq 0}$ are independent fractional Brownian motions with the Hurst constant $h$ and $E\left(B_{j}^{h}(1)\right)^{2}=\mu_{j}(j=1,2, \ldots)$.

It follows from Kleptsyna et al. (cf. [15]) that

$$
\begin{equation*}
B_{j}^{h}(t)=\left(\int_{-\infty}^{0}\left(|t-r|^{h-1 / 2}-|r|^{h-1 / 2}\right) d W_{j}(r)+\int_{0}^{t}|t-r|^{h-1 / 2} d W_{j}(r)\right) \tag{2.10}
\end{equation*}
$$

where $\left(W_{j}(t)\right)_{t \geq 0}(j=1,2, \ldots)$ are real independent Wiener processes with $E W_{j}^{2}(t)=\mu_{j} t$.
Let $K$ be a separable Hilbert space with the scalar product $(\cdot, \cdot)_{K}$, and $\left(e_{j}\right)_{j=1,2, \ldots .}$ denotes a complete orthogonal system in $K$, Then

$$
\begin{equation*}
\sum_{j=1}^{\infty} E\left\|B_{j}^{h}(t) e_{j}\right\|_{K}^{2}=t^{2 h} \sum_{j=1}^{\infty} \mu_{j}<\infty \tag{2.11}
\end{equation*}
$$

and $B^{h}(t)=\sum_{j=1}^{\infty} B_{j}^{h}(t) e_{j}$ is called a $K$-valued fractional Brownian motion where the sum is defined mean square.

Definition 2.1. A $H$-valued continuous stochastic process $(y(t))_{t \in[0, T]}$ with $y(t) \in V$ (P-a.s) is a solution of (2.7) if it holds for $v \in V$ and all $t \in[0, T]$ that

$$
\begin{align*}
(y(t), v)_{H}= & (y(0), v)_{H}+\int_{0}^{t}\langle k \Delta y(s), v\rangle d s+\int_{0}^{t}(\beta(s, x) y(s)-\mu(s, x) y(s), v)_{H} d s \\
& +\int_{0}^{t}(f(s, x), v)_{H} d s+\int_{0}^{t}\left(g(s, x) d B^{h}(s), v\right)_{H}, \quad P-a . s . \tag{2.12}
\end{align*}
$$

The objective in this paper is that we hopefully find a unique process $y(t)$ such that (2.7) holds For this objective, we assume that the following conditions are satisfied:
(1) $\mu(t, x), \beta(t, x)$ and $k(r, t)$ are nonnegative measurable, and

$$
\begin{gather*}
0 \leq k_{0} \leq k(t)<\infty \quad \text { in }(0, A) \times(0, T) \\
0 \leq \mu_{0} \leq \mu(t, x)<\infty  \tag{2.13}\\
0 \leq \beta(t, x) \leq \beta_{0}<\infty \quad \text { in }(0, A) \times \Gamma \\
\text { in }(0, A) \times \Gamma .
\end{gather*}
$$

(2) Let $f(t, x)$ and $g(t, x)$ be measurable functions which are defined on $Q$ with

$$
\begin{equation*}
|f(t, x)| \bigvee|g(t, x)| \leq K \tag{2.14}
\end{equation*}
$$

where $K$ is a positive constant.

## 3. Existence and Uniqueness of Solutions

Consider also the $K$-valued fractional Brownian motion $B^{h, n}(t)=\sum_{i=1}^{n} B_{i}^{h}(t) e_{i}$. Obviously, the following lemma holds.

If the process $(y(t))_{t \in[0, T]}$ is a solution of (2.7), then the process $Z(t)=y(t)-$ $\int_{0}^{t} g(s) d B^{h}(s)$ solves

$$
\begin{align*}
& \quad \begin{aligned}
d_{t} Z(t) & -k \Delta Z(t) d t+\mu(t, x)\left(Z(t)+\int_{0}^{t} g(s) B^{h}(s)\right) d s-\beta(t, x)\left(Z(t)+\int_{0}^{t} g(s) d B^{h}(s)\right) d t \\
& =f(t, x) d t+k \Delta \int_{0}^{t} g(s) d B^{h}(s) d t, \quad \text { in } Q=(0, T) \times \Gamma, \\
Z(0, x) & =Z_{0}(x), \quad \text { in } \Gamma, \\
Z(t, x) & =0, \quad \text { on } \quad \Sigma=(0, T) \times \partial \Gamma,
\end{aligned}
\end{align*}
$$

where $Z(t):=Z(t, x)$. If $Z(t)$ is a solution of (3.1), then exists a process $y(t)_{t \in[0, T]}$ so that $Z(t)$ can be written as $Z(t)=y(t)-\int_{0}^{t} g(s, x) d B^{h}(s)$, and consequently $y(t)$ solves (2.7).

As a result, we shall consider (3.1) instead of (2.7). It is noted that, for fixed $\omega \in \Omega$, (3.1) is a deterministic problem.

Lemma 3.1. Problem (3.1) has, for fixed $\omega \in \Omega$, a unique solution $Z(t)$, and there exists a nonnegative random variable $\eta$ with finite expectation such that

$$
\begin{equation*}
\sup _{0 \leq s \leq T}|Z(s)|^{2}+k_{0} \int_{0}^{T}\|Z(s)\|^{2} d s \leq \eta \tag{3.2}
\end{equation*}
$$

where for fixed $\omega \in \Omega, Z(t)$ is continuous with respect to $t$ in $H$.

Proof. The Galerkin approximations are defined by $Z_{n}(t)=\sum_{i=1}^{n} Z_{n, i}(t) v_{i}$, where $Z_{n, i}(t)$ solves the stochastic equations

$$
\begin{align*}
Z_{n, i}(t)= & \left(y(0), v_{i}\right)_{H}+\int_{0}^{t}\left\langle k \Delta\left(\sum_{k=1}^{n} Z_{n, k}(s) v_{k}\right), v_{i}\right\rangle d s \\
& +\int_{0}^{t}\left\langle(\beta(s, x)-\mu(s, x)) \sum_{k=1}^{n} Z_{n, k}(s) v_{k}, v_{i}\right\rangle d s  \tag{3.3}\\
& +\int_{0}^{t}\left(f(s, x), v_{i}\right)_{H} d s+\int_{0}^{t}\left\langle(\beta(s, x)-\mu(s, x))_{0}^{s} g(u, x) d B^{h, n}(u), v_{i}\right\rangle d s \\
& +\int_{0}^{t}\left\langle k \Delta\left(\int_{0}^{s} g(u, x) d B^{h, n}(u)\right), v_{i}\right\rangle d s . \quad(i=1,2, \ldots, n) .
\end{align*}
$$

It follows from the assumption (2) that (3.3) can be solved for every $\omega$ by the method of successive approximation, and the iterates are measurable with respect to $\omega$. Consequently, $\left(Z_{n, i}(t)\right)_{t \in[0, T]}(i=1,2, \ldots, n)$ are stochastic processes since $y_{0}$ is a random $H$-valued variable and $\left(B^{h, n}(t)\right)_{t \in[0, T]}$ is a stochastic process. It follows from (3.3) that

$$
\begin{align*}
Z_{n}(t)= & \sum_{i=1}^{n}\left(y(0), v_{i}\right)_{H} v_{i}+\int_{0}^{t} \sum_{i=1}^{n}\left\langle k \Delta Z_{n}(s), v_{i}\right\rangle v_{i} d s \\
& +\int_{0}^{t} \sum_{i=1}^{n}\left((\beta(s, x)-\mu(s, x)) \sum_{i=1}^{n} Z_{n}(s), v_{i}\right) v_{i} d s  \tag{3.4}\\
& +\int_{0}^{t}\left(f(s, x), v_{i}\right)_{H} v_{i} d s+\int_{0}^{t}\left\langle(\beta(s, x)-\mu(s, x)) \int_{0}^{s} g(u, x) d B^{h, n}(u), v_{i}\right\rangle v_{i} d s \\
& +\int_{0}^{t}\left\langle k \Delta\left(\int_{0}^{s} g(u, x) d B^{h, n}(u)\right), v_{i}\right\rangle v_{i} d s .
\end{align*}
$$

Using the chain rule, we get the following

$$
\begin{align*}
\left|Z_{n}(t)\right|^{2}= & \sum_{j=1}^{n}\left(y(0), v_{j}\right)_{H}^{2}+2 \int_{0}^{t} k\left\langle\Delta Z_{n}(s), Z_{n}(s)\right\rangle d s \\
& +2 \int_{0}^{t}\left\langle(\beta(s, x)-\mu(s, x)) Z_{n}(s), Z_{n}(s)\right\rangle d s+2 \int_{0}^{t}\left(f(s, x), Z_{n}(s)\right) d s  \tag{3.5}\\
& +2 \int_{0}^{t}\left((\beta(s, x)-\mu(s, x)) \int_{0}^{s} g(u, x) d B^{h, n}(u) d s, Z_{n}(s)\right) d s \\
& +2 \int_{0}^{t} k\left\langle\Delta\left(\int_{0}^{s} g(u, x) d B^{h, n}(u)\right), Z_{n}(s)\right\rangle d s .
\end{align*}
$$

If we set $B(t) \equiv 0$ in of Qimin and Chongzhao [11], under assumptions (1)-(2), then this result implies that

$$
\begin{equation*}
\sup _{0 \leq s \leq T}\left|Z_{n}(s)\right|^{2}+k_{0} \int_{0}^{t}\left\|Z_{n}(s)\right\|^{2} d s \leq \eta \tag{3.6}
\end{equation*}
$$

The following result is an analogous to that of Theorem 4 in [1]. In the Galerkin approximation, we have

$$
\begin{equation*}
E\left|Z_{n}(t)-Z(t)\right|^{2}+E \int_{0}^{t}\left\|Z_{n}(s)-Z(s)\right\|^{2} d s \longrightarrow 0 \tag{3.7}
\end{equation*}
$$

for all $t \in[0, T]$ and for $n \rightarrow \infty . Z(t)$ is a $H$-valued continuous process with $Z(t) \in V$ for all $t \in[0, T] P$-a.s., and $Z(t)$ is a $P$-a.s. unique solution.

Now let $\left(B^{h}(t)\right)_{t \in[0, T]}$ be a $H$-valued fractional Brownian motion with $\sum_{j=1}^{\infty} \lambda_{j} \mu_{j}<\infty$ and $\sum_{j=1}^{\infty} \lambda_{j} \mu_{j}^{1 / 2}<\infty$. We consider the finite-dimensional approximation

$$
\begin{equation*}
\sum_{j=1}^{n} \int_{0}^{t} g(s, x) d B_{j}^{h}(s) v_{j} d s \tag{3.8}
\end{equation*}
$$

in mean square of the stochastic integral $\int_{0}^{t} g(u, x) d B^{h}(u)$. Obviously this is a stochastic integral with respect to the $V$-valued Brownain motion $B^{h, n}(u)=\sum_{j=1}^{n} B_{j}^{h}(u) v_{j}$. Consequently, the corresponding Galerkin equations for (2.7) are given by

$$
\begin{align*}
& \left.\begin{array}{l}
d_{t} y^{m}(t)-k \Delta y^{m}(t) d t+\mu(t, x) y^{m}(t) d t-\beta(t, x) y^{m}(t) d t \\
\quad=
\end{array}\right) \\
& \begin{aligned}
& y^{m}(0)=\sum_{j=1}^{m}\left(y_{0}, v_{j}\right) v_{j}, \quad \text { in } \Gamma, \\
& y^{m}(t, x)=0, \quad \text { on } \Sigma=(0, T) \times \partial \Gamma, \\
& d_{t} y^{n}(t)-k \Delta y^{n} d t+\mu(t, x) y^{n}(t) d t-\beta(t, x) y^{n}(t) d t \\
& \quad=f(t, x) d t+g(t, x) d B^{h, n}(t), \quad \text { in }(0, T) \times \Gamma, \\
& y^{n}(0)= \sum_{j=1}^{n}\left(y_{0}, v_{j}\right) v_{j}, \quad \text { in } \Gamma, \\
& y^{n}(t, x)=0, \quad \text { on } \Sigma=(0, T) \times \partial \Gamma .
\end{aligned}
\end{align*}
$$

Lemma 3.1 shows that these problems have solutions.
Theorem 3.2. If $\sum_{j=1}^{\infty} \lambda_{j} \mu_{j}<\infty$ and $\sum_{j=1}^{\infty} \lambda_{j} \mu_{j}^{1 / 2}<\infty$, then there exists a $P$-a.s unique solution $(y(t))_{t \in[0, T]}$ of (2.7) with

$$
\begin{equation*}
E|y(t)|^{2}+k_{0} E \int_{0}^{t}\|Z(s)\|^{2} d s \leq M_{t, h}, \quad \forall t \in[0, T] \tag{3.11}
\end{equation*}
$$

where $M_{t, h}$ is a positive constant.

Proof. We choose $n>m$ with $n=m+p$ and define

$$
\begin{equation*}
Z^{m, p}(t)=y^{m+p}(t)-y^{m}(t)-\int_{0}^{t} g(u, x) d B^{h, m}(u)+\int_{0}^{t} g(u, x) d B^{h, m+p}(u) \tag{3.12}
\end{equation*}
$$

Then

$$
\begin{align*}
\left|Z^{m, p}(t)\right|^{2} \leq & \left|y^{m+p}(0)-y^{m}(0)\right|^{2}+2 \int_{0}^{t} k\left\langle\Delta Z^{m, p}(s), Z^{m, p}(s)\right\rangle d s \\
& +2 \int_{0}^{t}\left|\left((\beta(s, x)-\mu(s, x))\left(y^{m+p}(s)-y^{m}(s)\right), Z^{m, p}(s)\right)\right| d s \\
& +2 \int_{0}^{t}\left|\left(f(s, x), Z^{m, p}\right)(s)\right| d s  \tag{3.13}\\
& +2 \int_{0}^{t}\left|\left(k \Delta \int_{0}^{s} g(u, x) d\left(B^{h, m+p}(u)-B^{h, m}(u)\right), Z^{m, p}(s)\right)\right| d s
\end{align*}
$$

However, by Lemma 2.2 [14] and assumptions (1)-(2), we have

$$
\begin{align*}
& \int_{0}^{t}\left|\left(k \Delta \int_{0}^{s} g(u, x) d\left(B^{h, m+p}(u)-B^{h, n}(u)\right), Z^{m, p}\right)\right| d s \\
& \quad \leq k_{0} \sum_{j=m+1}^{n} \lambda_{j} E \int_{0}^{t}\left|\int_{0}^{s} g(u, x) d B_{j}^{h}(u)\left(v_{j}, Z^{m, p}(s)\right)\right| d s \\
& \quad \leq \frac{1}{2} k_{0} \sum_{j=m+1}^{n} \lambda_{j} \int_{0}^{t} E\left\|\int_{0}^{s} g(u, x) v_{j} d B_{j}^{h}(u)\right\|^{2} d s+\frac{1}{2} \int_{0}^{t} E\left|Z^{m, p}(s)\right|^{2} d s  \tag{3.14}\\
& \quad \leq \frac{1}{2} k_{0} T^{2 h} K^{2} T \sum_{j=m+1}^{n} \lambda_{j} \mu_{j}+\frac{1}{2} \int_{0}^{t} E\left|Z^{m, p}(s)\right|^{2} d s .
\end{align*}
$$

Further,

$$
\begin{align*}
& E\left((\beta(s, x)-\mu(s, x))\left(y^{m+p}(s)-y^{m}(s)\right), Z^{m, p}(s)\right) d s \\
& \quad \leq E\left(\left|\beta_{0}-\mu_{0}\right|\left|Z^{m, p}(s)\right|^{2}+\left|\beta_{0}-\mu_{0}\right|\left|\int_{0}^{s} g(u, x) d\left(B^{h, m+p}(u)-B^{h, n}(u)\right)\right|\left|Z^{m, p}(s)\right|\right) \\
& \quad \leq 2\left|\beta_{0}-\mu_{0}\right| E\left|Z^{m, p}(s)\right|^{2}+\left|\beta_{0}-\mu_{0}\right| K^{2} T \sum_{j=m+1}^{m+p} \mu_{j} \tag{3.15}
\end{align*}
$$

Consequently, in view of (3.13),

$$
\begin{align*}
E\left|Z^{n}(s)\right|^{2}+2 k_{0} E \int_{0}^{t}\left\|Z^{n}(s)\right\|^{2} d s \leq & \left(2\left|\beta_{0}-\mu_{0}\right|+K^{2}+1\right) \int_{0}^{t} E\left|Z^{n}(s)\right|^{2} d s \\
& +k_{0} C_{h} K^{2} T \sum_{j=m+1}^{m+p} \lambda_{j} \mu_{j}+2\left|\beta_{0}-\mu_{0}\right| T K^{2} \sum_{j=m+1}^{m+p} \mu_{j} \tag{3.16}
\end{align*}
$$

Then, the Gronwall's lemma implies that

$$
\begin{equation*}
E\left|Z^{m, p}(s)\right|^{2} \longrightarrow 0, \quad E \int_{0}^{t}\left\|Z^{m, p}(s)\right\|^{2} \longrightarrow 0 \tag{3.17}
\end{equation*}
$$

for $m, p \rightarrow \infty$ for all $t \in[0, T]$. In particular, there exists a process $(Z(t))_{t \in[0, T]}$ with $E \mid Z^{m}(t)-$ $\left.Z(t)\right|^{2} \rightarrow 0$ for $m \rightarrow \infty$, and consequently, there exists a process $y(t)$ with $E\left|y^{m}(t)-y(t)\right|^{2} \rightarrow 0$ for $m \rightarrow \infty$. We must now show that $(y(t))_{t \in[0, T]}$ is solution of (2.7). We have

$$
\begin{align*}
& E\left|y^{n}(t)-y^{m}(t)+\int_{0}^{s} g(u, a) d\left(\bar{B}^{h, m+p}(u)-\bar{B}^{h, n}(u)\right)\right|^{2}+2 k_{0} E \int_{0}^{t}\left\|y^{n}(s)-y^{m}(s)\right\|^{2} d s \\
& \leq \\
& \quad 2 E \int_{0}^{t} k\left\langle\Delta\left(y^{n}(s)-y^{m}(s)\right), \int_{0}^{s} g(u, x) d\left(B^{h, n}(u)-B^{h, m}(u)\right)\right\rangle d s \\
& \quad+2 E \int_{0}^{t}\left((\beta(s, x)-\mu(s, x)) y^{n}-y^{m}(s), y^{n}(s)-y^{m}(s)+\int_{0}^{s} g(u, x) d\left(B^{h, n}(u)-B^{h, m}(u)\right) d s\right.  \tag{3.18}\\
& \quad+2 E \int_{0}^{t}\left(f(s, x), y^{n}(s)-y^{m}(s)+\int_{0}^{s} g(u, x) d\left(B^{h, m+p}(u)-B^{h, n}(u)\right)\right) d s .
\end{align*}
$$

Let $\varepsilon>0$ be chosen arbitrary. Then there exists $p_{0}>0$ so that $\sum_{j=p+1}^{m+p} \lambda_{j} \mu_{j}^{1 / 2}<\varepsilon$ for all $p>p_{0}$. Let $y^{n, r}(t)=\sum_{j=1}^{r} y_{j}^{n, r}(t) v_{j}$ and $y^{m, r}(t)=\sum_{j=1}^{r} y_{j}^{m, r}(t) v_{j}$ be the $r$ th Galerkin approximation of $y^{n}(t)$ and $y^{m}(t)$, respectively. For $r=m+p$, we have

$$
\begin{align*}
& \left|E \int_{0}^{t} k\left\langle\Delta\left(y^{n, r}(s)-y^{m, r}(s)\right), \int_{0}^{s} g(u, x) d\left(B^{h, n}(u)-B^{h, m}(u)\right)\right\rangle d s\right| \\
& \quad \leq k_{0}\left|E \sum_{i=p+1}^{m+p} \int_{0}^{t} \lambda_{i}\left(y_{i}^{n, r}(s)-y_{i}^{m, r}(s)\right) \int_{0}^{s} g(u, x) d B_{i}^{h}(u) d s\right| \\
&  \tag{3.19}\\
& \quad \leq k_{0} E \sum_{i=p+1}^{m+p}\left(\int_{0}^{t}\left|y_{i}^{n, r}(s)-y_{i}^{m, r}(s)\right|^{2} d s\right)^{1 / 2} \lambda_{i}\left(E \int_{0}^{t}\left|\int_{0}^{s} g(u, x) d B_{i}^{h}(u)\right|^{2} d s\right)^{1 / 2} \\
& \quad \leq \text { const. } k_{0} E \sum_{i=p+1}^{m+p} \lambda_{i} \mu_{i}^{1 / 2} \\
& \quad<\text { const. } \times \varepsilon .
\end{align*}
$$

Consequently, the first term on the right-hand side of (3.10) is also less than const. $\times \varepsilon$. It is clear that the second term and third term on the right-hang side of (3.18) tends to zero. Then (3.18) gives

$$
\begin{equation*}
E \int_{0}^{t}\left\|y^{m+p}(s)-y^{p}(s)\right\|^{2} \longrightarrow 0 \tag{3.20}
\end{equation*}
$$

for $m, p \rightarrow \infty$, there is $\left(y^{m}(t)\right)$ is also a Cauchy sequence in $L_{V}^{2}(\Omega \times[0, T])$ for all $t \in[0, T]$. Let $\bar{y}$ be the limit a of this sequence. Then it follows from the properties of a Gelfand triple that

$$
\begin{equation*}
E \int_{0}^{t}\left|y_{n}(s)-\bar{y}(s)\right|^{2} \leq M E \int_{0}^{t}\left\|y_{n}(s)-\bar{y}(s)\right\|^{2} \longrightarrow 0 \tag{3.21}
\end{equation*}
$$

for $n \rightarrow \infty$, where $M$ is a positive constant. Consequently, $\bar{y}(s)=y(s)$ (a.s) and it follows from (3.9) that

$$
\begin{equation*}
d_{t} y(t)-k \Delta y(t) d s-(\beta(s, x)-\mu(s, x)) y(t) d s=f(s, x) d s+g(s, x) d B^{h}(t) \tag{3.22}
\end{equation*}
$$

hence, we have proved Theorem 3.2.

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Research Article

# Iterative Learning Control for Remote Control Systems with Communication Delay and Data Dropout 

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Iterative learning control (ILC) is applied to remote control systems in which communication channels from the plant to the controller are subject to random data dropout and communication delay. Through analysis, it is shown that ILC can achieve asymptotical convergence along the iteration axis, as far as the probabilities of the data dropout and communication delay are known a priori. Owing to the essence of feedforward-based control ILC can perform trajectory-tracking tasks while both the data-dropout and the one-step delay phenomena are taken into consideration. Theoretical analysis and simulations validate the effectiveness of the ILC algorithm for networkbased control tasks.

## 1. Introduction

Iterative leaning control (ILC) is a control method that achieves perfect trajectory tracking when the system operates repeatedly. ILC has made significant progresses over the past two decades [1-3] and covered a wide scope of research issues such as continuous-time nonlinear system control [4], discrete-time nonlinear system [5], the initial reset problem [6, 7], stochastic process control [8], state delays [9], and data dropout [10].

On the other hand, the research on networked control systems has attracted much attention [11, 12] over the past few years. In network control, two frequently encountered issues are data dropout and communication delays, which are causes of poor performance of remote control systems. A central research area in remote control systems is to evaluate
and compensate data dropout and time-delay factors [13-16]. Since data dropout and delay are random and time varying by nature, the existing control methods for deterministic data dropout and communication delays cannot be directly applied. Significant research efforts have been made on the control problems for networked systems with random data dropout and communication delays that are modeled in various ways in terms of the probability and characteristics of sources and destinations, for instance [10, 17].

It is in general still an open research area in ILC when remote control systems problems are concerned, except for certain pioneer works that address linear systems associated with either random data dropout [10, 18] or random communication delays [17, 19-21]. This paper investigates the implementation of ILC in a remote control systems setting, specifically focusing on compensation when both random data dropout and delays occur at the communication channels between the plant output and the controller.

Since ILC is in principle a feedforward technique, it is possible to send the controller signal before the task is executed. This would not be possible for feedback-based control systems. Hence, the data dropout can be circumvented to certain extent by using network protocols that assure the delivery of data packets. Likewise, the large delay due to large data package can also be avoided when the package is used for repeated task executions, namely, in future executions. ILC task is carried out in a finite-time interval, hence the time-domain stability is not a concern. Thus, unlike most network control works that focus on the stability issue, ILC can be applied to address trajectory-tracking tasks and the learning convergence is achieved in the iteration domain.

On the other hand, the use of data in the feedforward fashion would require the temporal analysis and management of data packages as well as resending the missing data package, which may not be available in certain remote control systems tasks. In this work, we adopt an ILC scheme that uses pastcontrol signals, as well as the error signals that are perturbed by the data dropout and communication delay. The ILC law adopts classical Dtype algorithm and a revised learning gain that takes into consideration the probabilities of both data-dropout and communication-delay factors. As a result, the output tracking errors can be made to converge along the iteration axis. The ILC scheme can be applied to linear discrete-time plants with trajectory-tracking tasks.

The paper is organized as below. Section 2 formulates the remote control systems problem. Sections 3 and 4 prove the convergence property of ILC for linear discrete-time plants. Section 5 presents a numerical examples.

Throughout the paper, the following notations are used. Let $\mathcal{\varepsilon}[\cdot]$ be the expected value of a random variable, $D[\cdot]$ the probability of an event, $\|\cdot\|_{2}$ the Euclidean norm of a vector, and $\|\cdot\|$ the maximal singular value of a matrix. Let $\mathbf{z}(t)$ is a discrete time signal with $t \in$ $\{0,1, \ldots, T\}$. For any $a>1$ and any $\lambda>1$, define

$$
\begin{equation*}
\|\mathbf{z}\|_{(\lambda, a)} \triangleq \sup _{t \in[0, T]} a^{-\lambda t}\|\mathbf{z}(t)\|_{2} \tag{1.1}
\end{equation*}
$$

where $[0, T]=\{0,1, \ldots, T\}$.

## 2. Problem Formulation

Consider a deterministic discrete-time linear time-invariant dynamics system:

$$
\begin{gather*}
\mathbf{x}_{i}(t+1)=A \mathbf{x}_{i}(t)+B \mathbf{u}_{i}(t) \\
\mathbf{y}_{i}(t)=C \mathbf{x}_{i}(t), \tag{2.1}
\end{gather*}
$$



Figure 1: The schematic diagram of the remote control system.
where " $i$ " and " " 1 " denote the iteration index and discrete time, respectively. $\mathbf{x}_{i}(t) \in \mathbb{R}^{n}, \mathbf{u}_{i}(t) \in$ $\mathbb{R}^{p}$, and $\mathbf{y}_{i}(t) \in \mathbb{R}^{r}$ for all $t \in[0, T]$ are system states, inputs, and outputs, respectively, at the $i$ th iteration. $A, B$, and $C$ are constant matrices with appropriate dimensions.

The schematic diagram of the remote control systems under consideration is shown in Figure 1.

It should be noted that the open-loop system from the ILC input to the plant output is deterministic. The randomness occurs during the data transmission from the plant output to the ILC input. There are two approaches in analyzing the closed-loop system. The first approach is to treat the entire closed-loop system as a random or stochastic process. In such circumstances, the topology of the overall system keeps changing and the control process is either a Markovian jump process or a switching process. Another approach, which is adopted in this work, is to retain the essentially deterministic structure of the original openloop system, meanwhile model the random data dropout and communication delay into two random factors with known probability distributions. As a consequence, the signals used in ILC, $\tilde{\mathbf{y}}_{i}(t)$ are the modulated plant output with the two random factors.

When the control process is deterministic, an effective ILC law for the linear system (2.1) is

$$
\begin{equation*}
\mathbf{u}_{i+1}(t)=\mathbf{u}_{i}(t)+L \mathbf{e}_{i}(t+1), \tag{2.2}
\end{equation*}
$$

where $\mathbf{u}_{i+1}(t)$ and $\mathbf{u}_{i}(t)$ are control inputs at the $(i+1)$ th and $i$ th iterations, namely, the present trial and the previous trial, respectively. $\mathbf{e}_{i}(t+1)=\mathbf{y}_{d}(t+1)-\mathbf{y}_{i}(t+1)$ is the output tracking error at the time $(t+1)$ th time instance of the $i$ th iteration. $L$ is a learning gain matrix.

Remark 2.1. Note that in the ILC law (2.2), the control signal of the present iteration, $\mathbf{u}_{i+1}(t)$, consists of both the pastcontrol input, $\mathbf{u}_{i}(t)$ and the past error with one-step temporal advance, $\mathbf{e}_{i}(t+1)$. The current-cycle feedback errors, such as $\mathbf{e}_{i+1}(t)$, are not used. Since ILC does not require the current-cycle feedback nor the temporal stability, it is an effective control method for remote control systems problems with random data dropout and communication delay.

To facilitate the ILC design and convergence analysis, data dropout and one-step communication delay are formulated. First formulate the data-dropout problem. Denote $\gamma(t)$ a stochastic variable with Bernoulli distribution taking binary values 0 and 1 , where $\gamma(t)=0$ denotes an occurrence of data dropout and $\gamma(t)=1$ denotes a normal data communication. The probabilities of $\gamma(t)$ are

$$
\begin{gather*}
p[\gamma(t)=1]=\varepsilon[\gamma(t)]=\bar{\gamma},  \tag{2.3}\\
p[\gamma(t)=0]=1-\varepsilon[\gamma(t)]=1-\bar{\gamma},
\end{gather*}
$$

where $\bar{\gamma}>0$ is a known constant. Here, we assume that $\gamma(t)$ is a stationery stochastic process, thus the data dropout rate is independent of the time $t$. In subsequent derivations, we treat $\gamma$ as time invariant.

When the data dropout occurs in multiple communication channels, we can similarly define $\mathcal{\varepsilon}\left[\gamma_{j}\right]=\bar{\gamma}_{j}>0$ for the $j$ th communication channel. Thus, denote

$$
\Gamma=\operatorname{diag}\left(\gamma_{j}\right)=\left[\begin{array}{cccc}
r_{1} & 0 & \cdots & 0  \tag{2.4}\\
0 & \gamma_{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_{r}
\end{array}\right]
$$

the corresponding mathematical expectation is

$$
\begin{equation*}
\mathcal{\varepsilon}[\Gamma]=\bar{\Gamma}, \tag{2.5}
\end{equation*}
$$

where $\bar{\Gamma}>0$ is known a priori.
Due to the data dropout, the plant output received by the controller at the $(i+1)$ th iteration is

$$
\begin{equation*}
\Gamma_{i+1} \mathbf{y}_{i+1}(t) \tag{2.6}
\end{equation*}
$$

Generally speaking, the occurrences of data dropouts at two iterations are uncorrelated, thus independent. On the other hand, ILC law at the current iteration, the $(i+1)$ th iteration, uses only signals of the previous iteration, namely, $i$ th iteration, as shown in (2.2). Thus $\mathbf{y}_{i+1}(t)$ with the control input $\mathbf{u}_{i+1}(t)$ contains data dropouts upto the $i$ th iteration. Therefore, $\Gamma_{i+1}$ and $\mathbf{y}_{i+1}(t)$ are independent, that is,

$$
\begin{align*}
\varepsilon\left[\Gamma_{i+1} \mathbf{y}_{i+1}(t)\right] & =\varepsilon\left[\Gamma_{i+1}\right] \varepsilon\left[\mathbf{y}_{i+1}(t)\right] \\
& =\bar{\Gamma}_{i+1} \varepsilon\left[\mathbf{y}_{i+1}(t)\right] . \tag{2.7}
\end{align*}
$$

Without the loss of generality, we assume $\mathcal{\varepsilon}\left[\Gamma_{i}\right]=\bar{\Gamma}$, namely, the data dropout rate is invariant at different iterations.

Next formulate the one-step communication delay problem. Denote $w(t)$ is a random delay factor with Bernoulli distribution, which takes binary values 0 and 1 that indicate, respectively, the presence and absence of an one-step communication delay. Here we assume that $w(t)$ is a stationery stochastic process, thus the occurrence of the communication delay is independent of the time $t$. In subsequent derivations we treat $w$ as time invariant. With multiple communication channels, we define matrix $W=\operatorname{diag}\left(w_{j}\right)$, where $w_{j}$ denotes the occurrence of communication delay at the $j$ th communication channel. Denote $\mathcal{\varepsilon}[w]=\bar{w}$ and $\mathcal{\varepsilon}[W]=\bar{W}$. The plant output received by ILC with possible communication delay is formulated by

$$
\begin{equation*}
\mathbf{y}_{i}^{o}(t)=W_{i} \mathbf{y}_{i}(t)+\left[1-W_{i}\right] \mathbf{y}_{i}(t-1) \tag{2.8}
\end{equation*}
$$

where $W_{i}$ is the communication delay at the $i$ th iteration. Without the loss of generality, we assume $\varepsilon\left[W_{i}\right]=\bar{W}$, namely, the probability of the communication delay is invariant at different iterations. Analogous to data dropout, assume that communication delay at any two iterations are independent, then $W_{i+1}$ and $W_{i}$ are independent, so are $W_{i+1}$ and $\mathbf{y}_{i+1}(t)$ because $\mathbf{y}_{i+1}(t)$ contains communication delays upto the $i$ th iteration through the ILC law (2.2).

It is worthwhile noting that stochastic variables $\gamma$ and $w$ are not completely independent. A delayed or nondelayed communication occurs only when $\gamma=1$, that is, no data dropout. Hence, we should have the condition probability for data transmission without delay

$$
\begin{equation*}
\operatorname{Prob}[\gamma=1, w=1]=p[\gamma=1] p[w=1 \mid \gamma=1]=\overline{\gamma w} \tag{2.9}
\end{equation*}
$$

and the condition probability for data transmission with one-step delay

$$
\begin{align*}
D[\gamma=1, w=0] & =p[\gamma=1] p[w=0 \mid \gamma=1] \\
& =p[\gamma=1](1-p[w=1 \mid \gamma=1])=\bar{\gamma}(1-\bar{w}) . \tag{2.10}
\end{align*}
$$

As a consequence, we have

$$
\begin{equation*}
\varepsilon[\gamma w]=\overline{\gamma w} . \tag{2.11}
\end{equation*}
$$

The relationship between data drop out and communication delay, (2.11), can be extended to multiple channels at the $i$ th iteration

$$
\begin{equation*}
\varepsilon\left[\Gamma_{i} W_{i}\right]=\overline{\Gamma W} \tag{2.12}
\end{equation*}
$$

At the $i$ th iteration, the output signals perturbed by data dropout and one-step communication delay can be expressed as

$$
\begin{equation*}
\tilde{\mathbf{y}}_{i}(t)=\Gamma_{i} \mathbf{y}_{i}^{o}(t)=\Gamma_{i}\left[W_{i} \mathbf{y}_{i}(t)+\left(I-W_{i}\right) \mathbf{y}_{i}(t-1)\right] \tag{2.13}
\end{equation*}
$$

where $I$ is a unity matrix of appropriate dimensions. The mathematical expectation of $\tilde{\mathbf{y}}_{i}(t)$ can be derived using the independence property between $\Gamma_{i}, W_{i}$, and $\mathbf{y}_{i}$, as well as the relationship (2.12)

$$
\begin{align*}
\varepsilon\left[\tilde{\mathbf{y}}_{i}(t)\right] & =\varepsilon\left\{\Gamma_{i}\left[W \mathbf{y}_{i}(t)+(I-W) \mathbf{y}_{i}(t-1)\right]\right\} \\
& =\bar{\Gamma}\left\{\bar{W} \varepsilon\left[\mathbf{y}_{i}(t)\right]+(I-\bar{W}) \varepsilon\left[\mathbf{y}_{i}(t-1)\right]\right\} . \tag{2.14}
\end{align*}
$$

The objective of control design is to seek an appropriate ILC law that can take into consideration data dropout and communication delay concurrently. The following ILC law is adopted

$$
\begin{equation*}
\mathbf{u}_{i+1}(t)=\mathbf{u}_{i}(t)+L \tilde{e}_{i}(t+1) \tag{2.15}
\end{equation*}
$$

where

$$
\begin{align*}
\tilde{\mathbf{e}}_{i}(t+1) & =\Gamma_{i}\left[\mathbf{y}_{d}(t+1)-\tilde{\mathbf{y}}_{i}(t+1)\right] \\
& =\Gamma_{i}\left[\mathbf{y}_{d}(t+1)-W_{i} \mathbf{y}_{i}(t+1)-\left(I-W_{i}\right) \mathbf{y}_{i}(t)\right]  \tag{2.16}\\
& =\Gamma_{i}\left[W_{i} \mathbf{e}_{i}(t+1)+\left(I-W_{i}\right) \mathbf{e}_{i}(t)+\left(I-W_{i}\right) \boldsymbol{\delta}(t)\right]
\end{align*}
$$

where $\boldsymbol{\delta}(t) \triangleq \mathbf{y}_{d}(t+1)-\mathbf{y}_{d}(t)$.

## 3. Convergence Analysis for Left-Invertible Systems $\boldsymbol{r} \geq \boldsymbol{p}$

In this section, we derive the convergence property of the ILC (2.15) in the presence of data dropout and communication delays.

In ILC, the learning convergence can be derived in terms of either the output tracking error, $\mathbf{e}_{i}(t)$, or the input tracking error, $\Delta \mathbf{u}_{i}(t)$. In this section, we prove the learning convergence property of $\Delta \mathbf{u}_{i}(t)$.

Assumption 3.1. For a given output reference trajectory $\mathbf{y}_{d}(t)$, which is realizable, there exists a unique desired control input $\mathbf{u}_{d}(t) \in \mathbb{R}^{p}$ such that

$$
\begin{gather*}
\mathbf{x}_{d}(t+1)=A \mathbf{x}_{d}(t)+B \mathbf{u}_{d}(t),  \tag{3.1}\\
\mathbf{y}_{d}(t)=C \mathbf{x}_{d}(t)
\end{gather*}
$$

where $\mathbf{u}_{d}(t)$ is uniformly bounded for all $t \in[0, T]$. It is assumed that for all $i \in \mathbb{Z}_{+}, \mathbf{x}_{i}(0)$ is a random variable with $\mathcal{E}\left[\mathbf{x}_{i}(0)\right]=\mathbf{x}_{0}=\mathbf{x}_{d}(0)$.

Define the input and state errors

$$
\begin{align*}
\Delta \mathbf{u}_{i+1} & \triangleq \mathbf{u}_{d}(t)-\mathbf{u}_{i+1}(t)  \tag{3.2}\\
\Delta \mathbf{x}_{i+1} & \triangleq \mathbf{x}_{d}(t)-\mathbf{x}_{i+1}(t)
\end{align*}
$$

then from (2.1) and (3.1), we have

$$
\begin{gather*}
\Delta \mathbf{x}_{i}(t+1)=A \Delta \mathbf{x}_{i}(t)+B \Delta \mathbf{u}_{i}(t) \\
\mathbf{e}_{i}(t)=C \Delta \mathbf{x}_{i}(t) \tag{3.3}
\end{gather*}
$$

From (2.15), using the relationship (2.12), we have

$$
\begin{align*}
\varepsilon\left[\tilde{\mathbf{e}}_{i+1}(t+1)\right] & =\varepsilon\left[\Gamma_{i}\left[\mathbf{y}_{d}(t+1)-W_{i} \mathbf{y}_{i}(t+1)-\left(I-W_{i}\right) \mathbf{y}_{i}(t)\right]\right] \\
& =\varepsilon\left[\Gamma_{i}\left[W_{i} \mathbf{e}_{i}(t+1)+\left(I-W_{i}\right) \mathbf{e}_{i}(t)+\left(I-W_{i}\right)\left(\mathbf{y}_{d}(t+1)-\mathbf{y}_{d}(t)\right)\right]\right]  \tag{3.4}\\
& =\bar{\Gamma}\left\{\bar{W} \varepsilon\left[\mathbf{e}_{i}(t+1)\right]+(I-\bar{W}) \varepsilon\left[\mathbf{e}_{i}(t)\right]+(I-\bar{W}) \varepsilon[\delta(t)]\right\} .
\end{align*}
$$

Theorem 3.2. Suppose that the update law (2.15) is applied to the networked control system and satisfied the Assumption 3.1. If there exist $\rho$ satisfying

$$
\begin{equation*}
\left\|I_{p}-L \overline{\Gamma W} C B\right\| \leq \rho<1, \tag{3.5}
\end{equation*}
$$

then the input error along the iteration axis, $\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i}(t)\right]$, converges to a bound that is proportional to the factor $\boldsymbol{\delta}(t)$.

Proof. First, subtracting $\mathbf{u}_{d}(t)$ from both sides of the ILC law (2.15) yields

$$
\begin{equation*}
\Delta \mathbf{u}_{i+1}(t)=\Delta \mathbf{u}_{i}(t)-L \widetilde{\mathbf{e}}_{i}(t+1) \tag{3.6}
\end{equation*}
$$

Applying the ensemble operator $\varepsilon[\cdot]$ to both sides of (3.6) and substituting the relationship (3.4) with $\mathbf{e}_{i}(t)=C \Delta \mathbf{x}_{i}(t)$, we obtain

$$
\begin{align*}
\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i+1}(t)\right]= & \mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i}(t)\right]-L \overline{\Gamma W} C \mathcal{E}\left[\Delta \mathbf{x}_{i}(t+1)\right] \\
& -L \bar{\Gamma}(I-\bar{W}) \subset \mathcal{E}\left[\Delta \mathbf{x}_{i}(t)\right]-L \bar{\Gamma}(I-\bar{W}) \boldsymbol{\delta}(t) \tag{3.7}
\end{align*}
$$

Substituting the state error dynamics (3.3) into (3.7) leads to the following relationship:

$$
\begin{align*}
\varepsilon\left[\Delta \mathbf{u}_{i+1}(t)\right]= & \left(I_{p}-L \overline{\Gamma W} C B\right) \varepsilon\left[\Delta \mathbf{u}_{i}(t)\right] \\
& -L[\overline{\Gamma W} C A+\bar{\Gamma}(I-\bar{W}) C] \varepsilon\left[\Delta \mathbf{x}_{i}(t)\right]-L \bar{\Gamma}(I-\bar{W}) \boldsymbol{\delta}(t) \tag{3.8}
\end{align*}
$$

Define $\rho \triangleq\left\|I_{p}-L \overline{\Gamma W} C B\right\|$.
Now let us handle the second term on the right hand side of (3.8), which is related to $\Delta \mathbf{x}_{i}(t)$. Applying the ensemble operation to the following relationship:

$$
\begin{equation*}
\Delta \mathbf{x}_{i}(t)=A^{t}\left[\mathbf{x}_{d}(0)-\mathbf{x}_{i}(0)\right]+\sum_{k=0}^{t-1} A^{t-1-k} B \Delta \mathbf{u}_{i}(k) \tag{3.9}
\end{equation*}
$$

Substituting the relation (3.9) into (3.8), taking the norm $\|\cdot\|_{2}$ on both sides, the following relationship is derived:

$$
\begin{align*}
\left\|\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i+1}(t)\right]\right\|_{2} \leq & \rho\left\|\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i}(t)\right]\right\|_{2}+\|L \overline{\Gamma W} C A+L \bar{\Gamma}(I-\bar{W}) C\| a^{t}\left\|\mathbf{x}_{d}(0)-\mathcal{E}\left[\mathbf{x}_{i}(0)\right]\right\|_{2} \\
& +\|L \overline{\Gamma W} C A+L \bar{\Gamma}(I-\bar{W}) C\|\|B\| \sum_{k=0}^{t-1} a^{t-1-k}\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}(k)\right]\right\|_{2}  \tag{3.10}\\
& +\|L \bar{\Gamma}(I-\bar{W})\|\|\boldsymbol{\delta}(t)\|_{2}
\end{align*}
$$

where $a \geq\|A\|$ and in this work we choose $a>1$ if $\|A\| \leq 1$.

In order to handle the exponential term with $a^{t}$ in (3.11), we introduce the $\lambda$ norm. From Assumption 3.1, multiplying both sides of (3.10) by $a^{-\lambda t}$ and taking the supermum over $[0, T]$ yield

$$
\begin{align*}
\sup _{t \in[0, T]} a^{-\lambda t}\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i+1}(t)\right]\right\|_{2} \leq & \rho \sup _{t \in[0, T]} a^{-\lambda t}\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}(t)\right]\right\|_{2} \\
& +\beta_{1} \sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=0}^{t-1} a^{t-1-k}\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}(k)\right]\right\|_{2}  \tag{3.11}\\
& +\beta_{2} \sup _{t \in[0, T]} a^{-\lambda t}\|\boldsymbol{\delta}(t)\|_{2}
\end{align*}
$$

where $\beta_{1}=\|L \overline{\Gamma W} C A+L \bar{\Gamma}(I-\bar{W}) C\|\|B\|, \beta_{2}=\|L \bar{\Gamma}(I-\bar{W})\|$, and $\eta_{i}=\left\|\mathbf{x}_{d}(0)-\varepsilon\left[\mathbf{x}_{i}(0)\right]\right\|$ that is independent of $t$. Since $\|\boldsymbol{\delta}(t)\|$ is bounded, so is

$$
\begin{equation*}
\mu=\sup _{t \in[0, T]} a^{-\lambda t}\|\boldsymbol{\delta}(t)\|_{2} \tag{3.12}
\end{equation*}
$$

Substituting the properties of Lemma A. 1 into (3.11) yields

$$
\begin{equation*}
\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i+1}\right]\right\|_{(\lambda, a)} \leq\left(\rho+\beta_{1} \frac{1-a^{-(\lambda-1) T}}{a^{\lambda}-a}\right)\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}\right]\right\|_{(\lambda, a)}+\beta_{2} \mu \tag{3.13}
\end{equation*}
$$

Since $0 \leq \rho<1$, it is possible to choose $\lambda$ sufficiently large such that

$$
\begin{equation*}
\rho_{1}=\rho+\beta_{1} \frac{1-a^{-(\lambda-1) T}}{a^{\curlywedge}-a}<1 \tag{3.14}
\end{equation*}
$$

Therefore we can rewrite (3.13) as

$$
\begin{equation*}
\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i+1}\right]\right\|_{(\lambda, a)} \leq \rho_{1}\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}\right]\right\|_{(\lambda, a)}+\beta_{2} \mu \tag{3.15}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}\right]\right\|_{(\lambda, a)} \leq \frac{\beta_{2} \mu}{1-\rho_{1}} \tag{3.16}
\end{equation*}
$$

Note that $\mu$ is proportional to $\boldsymbol{\delta}(t)$, namely, the maximum difference between $\mathbf{y}_{d}(t+$ $1)-\mathbf{y}_{d}(t)$ in $t \in[0, T]$, which is bounded and small when the reference trajectory is smooth or the sampling interval is sufficiently small. When the probability associated with the data communication delay, $\bar{W}$, is known a priori, we can further revise the reference trajectory to an augmented one, such that the resulting $\boldsymbol{\delta}(t)=0$.

Corollary 3.3. Revising the original reference $\mathbf{y}_{d}(t)$ into an augmented one $\mathbf{y}_{d}^{a}(t)=\bar{W} \mathbf{y}_{d}(t)+(I-$ $\bar{W}) \mathbf{y}_{d}(t-1)$, then $\boldsymbol{\delta}(t)=0$ and the ILC (2.15) ensures a zero-tracking error.

Proof. Note that $\mathcal{\varepsilon}\left[\mathbf{y}_{i}^{o}(t)\right]=\bar{W} \varepsilon\left[\mathbf{y}_{i}(t)\right]+(I-\bar{W}) \mathcal{E}\left[\mathbf{y}_{i}(t-1)\right]$. Suppose that $\mathbf{y}_{i}(t)=\mathbf{y}_{d}(t)$, then the delay-perturbed output should be $\bar{W} \varepsilon\left[y_{d}(t)\right]+(I-\bar{W}) \varepsilon\left[y_{d}(t-1)\right]$. In other words, the augmented reference trajectory for $\mathbf{y}_{i}^{o}(t+1)$ should be $\mathbf{y}_{d}^{a}(t)=\bar{W} \mathbf{y}_{d}(t)+(I-\bar{W}) \mathbf{y}_{d}(t-1)$. As a result, $\mathbf{y}_{i}^{o}(t)=\mathbf{y}_{d}^{a}(t)$ implies $\mathbf{y}_{i}(t)=\mathbf{y}_{d}(t)$. Now replacing $\mathbf{y}_{d}(t+1)$ in (2.16) with $\mathbf{y}_{d}^{a}(t+1)$, we can derive

$$
\begin{align*}
\widetilde{\mathbf{e}}_{j}(t+1) & =\Gamma_{j}\left[\mathbf{y}_{d}^{a}(t+1)-\mathbf{y}_{j}^{o}(t+1)\right] \\
& =\Gamma_{j}\left[W_{j} \mathbf{y}_{d}(t+1)-W_{j} \mathbf{y}_{j}(t+1)+\left(I-W_{j}\right) \mathbf{y}_{d}(t-1)-\left(I-W_{j}\right) \mathbf{y}_{j}(t)\right]  \tag{3.17}\\
& =\Gamma_{j} W_{j} \mathbf{e}_{j}(t+1)+\Gamma_{j}\left(I-W_{j}\right) \mathbf{e}_{j}(t) .
\end{align*}
$$

Comparing the above expression with (2.16), we conclude that $\boldsymbol{\delta}(t)=0$, subsequently $\mu=0$, which implies a zero-tracking error according to (3.16).

## 4. Convergence Analysis for Right-Invertible Systems $\boldsymbol{r} \leq \boldsymbol{p}$

In this section, we prove the learning convergence property of $\mathbf{e}_{i}(t)$.
Assumption 4.1. $\left(I_{r}-C B L \overline{\Gamma W}\right)^{-1}$ always exists.
Theorem 4.2. Suppose that the update law (2.15) is applied to the networked control system and satisfied the Assumption 4.1. If

$$
\begin{equation*}
\rho^{\prime} \triangleq\left\|I_{r}-C B L \overline{\Gamma \bar{W}}\right\|<1 \tag{4.1}
\end{equation*}
$$

then the tracking error along the iteration axis, $\mathfrak{\varepsilon}\left[\mathbf{e}_{i}(t)\right]$, converges to a bound that is proportional to the factor $\boldsymbol{\delta}(t)$.

Proof. First note the relationship:

$$
\begin{align*}
\mathbf{e}_{i+1}(t+1) & =\mathbf{y}_{d}(t+1)-\mathbf{y}_{i+1}(t+1) \\
& =\mathbf{e}_{i}(t+1)+\mathbf{y}_{i}(t+1)-\mathbf{y}_{i+1}(t+1)  \tag{4.2}\\
& =\mathbf{e}_{i}(t+1)+C A\left[\mathbf{x}_{i}(t)-\mathbf{x}_{i+1}(t)\right]+C B\left[\mathbf{u}_{i}(t)-\mathbf{u}_{i+1}(t)\right] \\
\mathbf{x}_{i}(t) & =A^{t} \mathbf{x}_{i}(0)+\sum_{k=0}^{t-1} A^{t-1-k} B \mathbf{u}_{i}(k) \tag{4.3}
\end{align*}
$$

Substituting ILC law (2.15), (2.16), and (4.3) into (4.2) yields

$$
\begin{aligned}
\mathbf{e}_{i+1}(t+1)= & \mathbf{e}_{i}(t+1)+C A^{t+1}\left[\mathbf{x}_{i}(0)-\mathbf{x}_{i+1}(0)\right] \\
& -C \sum_{k=0}^{t-1} A^{t-k} B L \widetilde{\mathbf{e}}_{i}(k+1)-C B L \widetilde{\mathbf{e}}_{i}(t+1)
\end{aligned}
$$

$$
\begin{align*}
= & \mathbf{e}_{i}(t+1)+C A^{t+1}\left[\mathbf{x}_{i}(0)-\mathbf{x}_{i+1}(0)\right] \\
& -C \sum_{k=0}^{t-1} A^{t-k} B L \Gamma_{i}\left[W_{i} \mathbf{e}_{i}(k+1)+\left(I-W_{i}\right) \mathbf{e}_{i}(k)+\left(I-W_{i}\right) \boldsymbol{\delta}(k)\right] \\
& -C B L \Gamma_{i}\left[W_{i} \mathbf{e}_{i}(t+1)+\left(I-W_{i}\right) \mathbf{e}_{i}(t)+\left(I-W_{i}\right) \boldsymbol{\delta}(t)\right] \\
= & \left(I_{r}-C B L \Gamma_{i} W_{i}\right) \mathbf{e}_{i}(t+1)+C A^{t+1}\left[\mathbf{x}_{i}(0)-\mathbf{x}_{i+1}(0)\right] \\
& -C \sum_{k=0}^{t} A^{t-k} B L \Gamma_{i} W_{i} \mathbf{e}_{i}(k+1)-C \sum_{k=0}^{t-1} A^{t-k-1} B L \Gamma_{i}\left(I-W_{i}\right) \mathbf{e}_{i}(k) \\
& -C \sum_{k=0}^{t} A^{t-k} B L \Gamma_{i}\left(I-W_{i}\right) \boldsymbol{\delta}(k) . \tag{4.4}
\end{align*}
$$

Assumption 4.3. Assume $\mathcal{\varepsilon}\left[\mathbf{x}_{i+1}(0)\right]-\mathcal{\varepsilon}\left[\mathbf{x}_{i}(0)\right]=0$.
Applying the ensemble operator $\varepsilon[\cdot]$ to both sides of (4.4) and substituting the relationship (4.2), we obtain

$$
\begin{align*}
\varepsilon\left[\mathbf{e}_{i+1}(t+1)\right]= & \left(I_{r}-C B L \overline{\Gamma W}\right) \varepsilon\left[\mathbf{e}_{i}(t+1)\right]-C \sum_{k=0}^{t-1} A^{t-k-1} B L \overline{\Gamma W} \varepsilon\left[\mathbf{e}_{i}(k+1)\right] \\
& -C \sum_{k=0}^{t} A^{t-k} B L \bar{\Gamma}(I-\bar{W}) \varepsilon\left[\mathbf{e}_{i}(k)\right]-C \sum_{k=0}^{t} A^{t-k} B L \bar{\Gamma}(I-\bar{W}) \varepsilon[\boldsymbol{\delta}(t)] . \tag{4.5}
\end{align*}
$$

Taking the norm $\|\cdot\|_{2}$ on both sides of (4.5), the following relationship is derived

$$
\begin{align*}
\left\|\mathcal{E}\left[\mathbf{e}_{i+1}(t+1)\right]\right\|_{2} \leq & \left\|I_{r}-C B L \overline{\Gamma \bar{W}}\right\|\left\|\mathcal{\varepsilon}\left[\mathbf{e}_{i}(t+1)\right]\right\|_{2}+\|C\| \sum_{k=0}^{t-1} a^{t-k-1}\|B L\| \overline{\Gamma \bar{W}}\left\|\mathcal{\varepsilon}\left[\mathbf{e}_{i}(k+1)\right]\right\|_{2} \\
& +\|C\| \sum_{k=0}^{t} a^{t-k}\|B L\| \bar{\Gamma}(I-\bar{W})\left\|\mathcal{\varepsilon}\left[\mathbf{e}_{i}(k)\right]\right\|_{2} \\
& +\|C\| \sum_{k=0}^{t} a^{t-k}\|B L\| \bar{\Gamma}(I-\bar{W})\|\mathcal{\varepsilon}[\boldsymbol{\delta}(t)]\|_{2} \tag{4.6}
\end{align*}
$$

where $a \geq\|A\|$ and in this work we choose $a>1$ if $\|A\| \leq 1$.

In order to handle the exponential term with $a^{t}$ in (4.5), we introduce the $\lambda$ norm. Multiplying both sides of (4.5) by $a^{-\lambda t}$ and taking the supermum over $[0, T]$ yield

$$
\begin{align*}
\sup _{t \in[0, T]} a^{-\lambda t}\left\|\mathcal{E}\left[\mathbf{e}_{i+1}(t)\right]\right\|_{2} \leq & \sup _{t_{1} \in[0, T]}\|I-C B L \overline{\Gamma W}\| \sup _{t \in[0, T]} a^{-\lambda t}\left\|\mathcal{E}\left[\mathbf{e}_{i}(t)\right]\right\|_{2} \\
& +\|C\| \sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=1}^{t} a^{t-k}\|B L\| \overline{\Gamma W}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2} \\
& +\|C\| \sup _{t \in[0, T]} a^{-\lambda} \sum_{k=0}^{t-1} a^{t-k-1}\|B L\| \bar{\Gamma}(I-\bar{W})\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2}  \tag{4.7}\\
& +\|C\| \sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=0}^{t} a^{t-k}\|B L\| \bar{\Gamma}(I-\bar{W})\|\mathcal{E}[\boldsymbol{\delta}(t)]\|_{2} .
\end{align*}
$$

Substituting the properties of Lemma A. 2 into (4.7) yields

$$
\begin{equation*}
\left\|\varepsilon\left[\mathbf{e}_{i+1}\right]\right\|_{(\lambda, a)} \leq\left(\rho^{\prime}+\beta_{4} \frac{1-a^{-(\lambda-1) T}}{a^{(\lambda-1)}-1}\right)\left\|\varepsilon\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)}+\beta_{5} \mu, \tag{4.8}
\end{equation*}
$$

where $\beta_{3} \triangleq\|C\|\|B L\| \bar{\Gamma}$ and $\beta_{4} \triangleq\|C\|\|B L\| \bar{\Gamma}(I-\bar{W})\left(\left(1-a^{-(\lambda-1) T}\right) /\left(a^{(\lambda-1)}-1\right)\right)$.
Since $0 \leq \rho^{\prime}<1$, it is possible to choose $\lambda$ sufficiently large such that

$$
\begin{equation*}
\rho_{2}=\rho^{\prime}+\beta_{3} \frac{1-a^{-(\lambda-1) T}}{a^{(\lambda-1)}-1}<1 . \tag{4.9}
\end{equation*}
$$

Therefore, we can rewrite (4.8) as

$$
\begin{equation*}
\left\|\mathcal{E}\left[\mathbf{e}_{i+1}\right]\right\|_{(\lambda, a)} \leq \rho_{2}\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)}+\beta_{4} \mu, \tag{4.10}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(1, a)} \leq \frac{\beta_{4} \mu}{1-\rho_{2}} . \tag{4.11}
\end{equation*}
$$

## 5. Numerical Examples

Consider the following linear discrete-time system:

$$
\begin{gather*}
\mathbf{x}_{i}(t+1)=\left[\begin{array}{ccc}
0.50 & -0.25 & 1.00 \\
0.15 & 0.30 & -0.50 \\
-0.75 & 0.25 & -0.25
\end{array}\right] \mathbf{x}_{i}(t)+\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] u_{i}(t),  \tag{5.1}\\
y_{i}(t)=\left[\begin{array}{lll}
0 & 0 & 1.0
\end{array}\right] \mathbf{x}_{i}(t),
\end{gather*}
$$

with the initial condition $\mathbf{x}_{i}(0)=0$. The desired trajectory is $y_{d}(t)=\sin (2 \pi t / 50)$. The tracking period is $\{0,1, \ldots, 49\}$. The control profile of the first iteration is $u_{0}(t)=0$ for $t=0,1, \ldots, 49$. Two sets of probabilities for the data dropout rate and communication delay are considered, which are $\bar{\gamma}=0.9, \bar{w}=0.9, \bar{\gamma}=0.6$, and $\bar{w}=0.6$, respectively. The learning gain is $L=0.5$, which yields $\|I-L \overline{\gamma w} C B\|=0.595<1$, and $\|I-L \overline{\gamma w} C B\|=0.82<1$ with respect to the two sets of probabilities. The tracking performance of two ILC algorithms is given in Figure 2, where Max Error denotes the maximum absolute error of each iteration.

## 6. Conclusion

In this work, we address a class of networked control system problems with random data dropout and communication delay. D-type ILC is applied to handle this remote control systems problem with repeated tracking tasks. Through analysis, we illustrate the desired convergence property of the ILC. Although we focus on one-step communication delay in this work, the results could be extended to multiple delays, which is one of our ongoing research topics. In our future work, we will also explore the extension to more generic nonlinear dynamic processes.

## Appendix

Lemma A.1. For all $a>1$, for all $\lambda>1$, for all $i \in \mathbb{Z}_{+}$, the inequality:

$$
\begin{equation*}
\sup _{t \in[0, T]} a^{-\lambda t} \sum_{\tau=0}^{t-1} a^{t-1-\tau}\left\|\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i}(\tau)\right]\right\|_{2} \leq \frac{1-a^{-(\lambda-1) T}}{a^{\lambda}-a}\left\|\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i}\right]\right\|_{(\lambda, a)} \tag{A.1}
\end{equation*}
$$

holds.
Proof. Consequently

$$
\begin{align*}
\sup _{t \in[0, T]} a^{-\lambda t} \sum_{\tau=0}^{t-1} a^{t-1-\tau}\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}(\tau)\right]\right\|_{2} & =a^{-1} \sup _{t \in[0, T]} a^{-t(\lambda-1)} \sum_{\tau=0}^{t-1} a^{-\lambda \tau}\left\|\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i}(\tau)\right]\right\| a^{(\lambda-1) \tau} \\
& \leq a^{-1}\left\|\mathcal{\varepsilon}\left[\Delta \mathbf{u}_{i}\right]\right\|_{(\lambda, a)} \sup _{t \in[0, T]} a^{-t(\lambda-1)} \frac{a^{(\lambda-1) t}-1}{a^{\lambda-1}-1}  \tag{A.2}\\
& \leq\left(\frac{1-a^{-(\lambda-1) T}}{a^{\lambda}-a}\right)\left\|\mathcal{E}\left[\Delta \mathbf{u}_{i}\right]\right\|_{(\lambda, a)}
\end{align*}
$$

Lemma A.2. For all $a>1$, for all $\lambda>1$, for all $i \in \mathbb{Z}_{+}$, the inequalities

$$
\begin{align*}
& \sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=0}^{t-1} a^{t-1-k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2} \leq \frac{1-a^{-(\lambda-1) T}}{a^{(\lambda-1)}-1}\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)}, \\
& \sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=1}^{t} a^{t-1-k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2} \leq \frac{1-a^{-(\lambda-1) T}}{a^{(\lambda-1)}-1}\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)} \tag{A.3}
\end{align*}
$$

hold.


Figure 2: The tracking error profiles for the discrete-time linear system with data dropout and one-step communication delay. (a) is learning results with the data dropout rate $\bar{\gamma}=0.9$ and communication delay rate $\bar{w}=0.9$. (b) is learning results with the data dropout rate $\bar{\gamma}=0.6$ and communication delay rate $\bar{w}=0.6$.

Proof. Consequently

$$
\begin{align*}
\sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=0}^{t-1} a^{t-1-k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2} & \leq \sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=0}^{t} a^{t-k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2} \\
& =\sup _{t \in[0, T]} a^{-t(\lambda-1)} \sum_{k=0}^{t} a^{-\lambda k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\| a^{(\lambda-1) k} \\
& \leq\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)} \sup _{t \in[0, T]} a^{-t(\lambda-1)} \frac{a^{(\lambda-1) t}-1}{a^{\lambda-1}-1} \\
& \leq\left(\frac{1-a^{-(\lambda-1) T}}{a^{\lambda-1}-1}\right)\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)} \\
\sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=1}^{t} a^{t-1-k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2} & \leq \sup _{t \in[0, T]} a^{-\lambda t} \sum_{k=0}^{t} a^{t-k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\|_{2}  \tag{A.4}\\
& =\sup _{t \in[0, T]} a^{-t(\lambda-1)} \sum_{k=0}^{t} a^{-\lambda k}\left\|\mathcal{E}\left[\mathbf{e}_{i}(k)\right]\right\| a^{(\lambda-1) k} \\
& \leq\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)} \sup _{t \in[0, T]} a^{-t(\lambda-1)} \frac{a^{(\lambda-1) t}-1}{a^{\lambda-1}-1} \\
& \leq\left(\frac{1-a^{-(\lambda-1) T}}{a^{\lambda-1}-1}\right)\left\|\mathcal{E}\left[\mathbf{e}_{i}\right]\right\|_{(\lambda, a)} .
\end{align*}
$$

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Research Article

# INS/WSN-Integrated Navigation Utilizing LS-SVM and $H_{\infty}$ Filtering 

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In order to achieve continuous navigation capability in areas such as tunnels, urban canyons, and indoors a new approach using least squares support vector machine (LS-SVM) and $H_{\infty}$ filter (HF) for integration of INS/WSN is proposed. In the integrated system, HF estimates the errors of position and velocity while the signals in WSNs are available. Meanwhile, the compensation model is trained by LS-SVM with corresponding HF states. Once outages of the signals in WSNs, the model is used to correct INS solution as HF does. Moreover, due to device reasons, there are slight fluctuations in sampling period in practice. For overcoming this problem of integrated navigation, the theoretical analysis and implementation of HF for an integrated navigation system with stochastic uncertainty are also given. Simulation shows the performance of HF is more robust compared with INS-only solution and Kalman filter (KF) solution, and the prediction of LS-SVM has the smallest error compared with INS-only and back propagation (BP), the improvement is particularly obvious.

## 1. Introduction

The demand for location-based services (LBSs) has been driving the need for the accurate positioning techniques in the past and is expected to remain the same in the future [1, 2]. Wireless sensor network (WSN) has boomed in the last decades, it shows great potential to develop positioning system in the environments such as tunnels, urban canyons, and indoors, where the Global Positioning Systems (GPS) cannot provide a solution with consistent and long-term stable accuracy due to satellite signal blockage [3-7]. So, the physical location becomes one of key applications in WSNs recently. Most of the current wireless localization in WSN employs the measurement of one or several physical parameters of the radio signal
transmitted between the reference nodes (RNs) and blind nodes (BNs) [8]. For example, Patwari et al. employed the measurements of time of arrival (TOA) and received signal strength (RSS) to estimate relative location in WSNs in 2003 [9]. In 2002, Al-Jazzar and Caffery Jr. estimated node location for nonline of sight (NLOS) environments by TOA [10], then Al-Jazzar et al. used a joint TOA/AOA (so-called angle of arrival) constrained minimization method for locating wireless devices in nonline-of-sight environment in 2009 [11]. Alsindi et al. employed TOA for ranging in indoor multipath environments in 2009 [12]. The mainstream method is to use electromagnetic waves for indoor localization, but due to the high propagation speed, the accuracy is of the order of several meters. On the other hand, some researchers employ ultrasonic waves to achieve high accuracy with narrow bandwidth and narrow directional characteristics, for example, a fully distributed localization system based on ultrasound is proposed by Minami et al., and the accuracy of localization is about 20 cm with 24 devices [13]. Although WSN is capable of indoor wireless localization with the characteristics of low power, low cost, and low complexity, it requires high density of RNs for high accuracy due to its short-distance communication. Therefore, it has to employ a large number of RNs to keep localization accuracy if localization area is large.

Differing from WSN-based wireless localization requiring RNs, INS is a self-contained system incorporating three orthogonal accelerometers and two orthogonal gyroscopes [1]. It is capable of providing positioning information independently. However, for the INS accuracy deteriorates with time due to possible inherent sensor errors (white noise, correlated random noise, bias instability, and angle random walk) that exhibit considerable long-term growth [14-17], it is just a short-term compensation to GPS outages, and the INS cannot maintain long-term high accuracy when GPS signals are unavailable. Thus, INS is poor in long-term self-contained navigation.

Aiming at continuous navigation capability, many intelligent integration navigation approaches have been employed. For instance, Kim et al. presented an integrated GPS/INS/Vision system for helicopter navigation [18], Berefelt et al. used GPS/INS navigation in urban environment [19]. In this mode, training the compensation model by Artificial Intelligence (AI) techniques is widely used to improve the performance of integrated navigation, especially the neural network (NN), as in, for example, the use of the NN for denoising inertial outputs based on microelectromechanical system (MEMS) in [20], and the NN for the compensation model in [21]. However, to achieve good performance it has to select sufficient data samples of good quality for the NN [22], and it is poor in high dimension input spaces. Current algorithms for good quality samples of the compensation model are mainly based on integration filter, as the core of an integrated system, the integration filter should be carefully designed. The KF is one of the most common examples for filtering. With the stochastic state space model of the system and measurement outputs, it is able to achieve the optimal estimation of states in multi-input, multioutput (MIMO) systems [23]. However, due to the noises of system and measurement should be corrupted by white noise and the state estimation is approached with the minimization of the covariance of the estimation error, the KF is not suitable for nonlinear systems. Through the first-order linearization of the nonlinear system, extended KF (EKF) is able to achieve nonlinear estimation. However, for the state distribution is assumed as a Gaussian random variable (GRV), it may generate large errors in the true posterior mean and covariance of the transformed GRV, which can lead to suboptimal performance and sometimes divergence of the filter [24]. Moreover, the system with GRV is often unavailable in practice.

In this paper, we present INS/WSN integration using LS-SVM and HF for longdistance continuous navigation in areas such as tunnels, urban canyons, and indoors. Aiming at the robust performance of filtering, the HF is employed to estimate errors of position and velocity while signals in WSNs are available. Meanwhile, compensation model is trained by LS-SVM, which is used to correct the INS errors during signals in WSN outages. Simulation is employed to evaluate the performance of the proposed method. The results of filtering are compared with the INS-only solution and KF solution, moreover, the results of prediction are compared with the INS-only method and BP method. The remainder of the paper is organized as follows: HF for integration and LS-SVM model are described in Section 2 and Section 3, respectively. Section 4 gives the hybrid method for INS/WSN integration. Simulations and the analyses of experiment results based on semiphysical can be obtained in detail in Section 5. Finally, the conclusions are given.

For convenience, this paper adopts the following notations.
$A^{\prime}$ : transpose of a matrix or vector $A$.
$A>0(A \geq 0): A$ is positive definite (positive semidefinite) symmetric matrix.
$S_{n}$ : the set of all real symmetric matrices.
$R^{n}$ : $n$-dimensional Euclidean space.
$I_{n \times n}: n \times n$ identity matrix.

## 2. $H_{\infty}$ Fusion Filter for Integration

### 2.1. Stochastic Uncertain System

In order to achieve robust performance, HF is widely analyzed and used in the nonlinear systems [25-27]. The HF is to design an estimator to estimate the unknown state combination with measurement output [28]. In contrast with the KF and EKF, one of the main advantages of HF is that it is not necessary to know exactly the statistical properties of the noise but only on the assumption of the noise with bounded energy [29], which makes this technique useful in certain practical applications. For the above-mentioned reasons, HF technique has been extensively developed in the last decade, and many HF-based methods have been proposed, especially in the field of stochastic systems. For instance, Xu and Chen proposed an $H_{\infty}$ filtering for uncertain impulsive stochastic systems under sampled measurements in [30]. Zhang and Chen studied the exact observability of stochastic systems in [29], and, then, they solved the problem of filtering for nonlinear stochastic uncertain system [28].

As WSN-based wireless localization is a relative localization, HF uses relative errors of position and velocity of BN as the state vector. Ideally, the relative position errors of BN measured by INS at k state are able to be illustrated in (2.1):

$$
\begin{align*}
& e_{x, k+1}=e_{x, k}+T \cdot e_{v x, k}+\omega_{x, k}  \tag{2.1}\\
& e_{y, k+1}=e_{y, k}+T \cdot e_{v y, k}+\omega_{y, k}
\end{align*}
$$

where $\left(e_{x}, e_{y}\right)$ is the relative position error of BN at k moment, $\left(e_{v x}, e_{v y}\right)$ is the velocity error of BN at k moment, and $T$ is ideal sample time. Due to the limitation of timing device, there
will be a stochastic error for ideal sample time in practice. It leads the system in (2.1) to be a stochastic uncertain system, which is expressed as follow:

$$
\begin{align*}
& e_{x, k+1}=e_{x, k}+\left((T+\delta t) \cdot \beta_{k}\right) \cdot e_{v x, k}+\omega_{x, k} \\
& e_{y, k+1}=e_{y, k}+\left((T+\delta t) \cdot \beta_{k}\right) \cdot e_{v y, k}+\omega_{y, k} \tag{2.2}
\end{align*}
$$

Here $\delta t$ is the stochastic uncertainty of system and $\beta_{k}$ is a standard random scalar sequences with zero mean. Thus, the model of the system in (2.2) can be written in matrix form:

$$
\left[\begin{array}{c}
e_{x, k+1}  \tag{2.3}\\
e_{y, k+1} \\
e_{v x, k+1} \\
e_{v y, k+1}
\end{array}\right]=\left\{\left[\begin{array}{cccc}
1 & 0 & T & 0 \\
0 & 1 & 0 & T \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]+\left[\begin{array}{cccc}
0 & 0 & \delta t & 0 \\
0 & 0 & 0 & \delta t \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \beta_{k}\right\}\left[\begin{array}{c}
e_{x, k} \\
e_{y, k} \\
e_{v x, k} \\
e_{v y, k}
\end{array}\right]+\left[\begin{array}{c}
\omega_{x, k} \\
\omega_{y, k} \\
\omega_{v x, k} \\
\omega_{v y, k}
\end{array}\right]
$$

Here, we denote (2.3) as (2.4).

$$
\begin{equation*}
X_{k+1}=\left(A+E \cdot \beta_{k}\right) X_{k}+B \omega_{k} \tag{2.4}
\end{equation*}
$$

where $\omega_{k} \in R^{m}$ is stochastic process noise which belongs to $l_{2}[0, \infty)$.
The observation vectors of the HF are formed by differencing the INS and WSN positions ( $r_{\text {WSN }}, r_{\text {INS }}$ ) and the velocities $\left(v_{\text {WSN }}, v_{\text {INS }}\right)$. Thus, the observation equation is illustrated as (2.5):

$$
\left[\begin{array}{c}
\Delta r_{x, k}  \tag{2.5}\\
\Delta r_{y, k} \\
\Delta v_{x, k} \\
\Delta v_{y, k}
\end{array}\right]=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]\left[\begin{array}{c}
e_{x, k} \\
e_{y, k} \\
e_{v x, k} \\
e_{v y, k}
\end{array}\right]+\left[\begin{array}{c}
v_{x, k} \\
v_{y, k} \\
v_{v x, k} \\
v_{v y, k}
\end{array}\right]
$$

where $\Delta r_{x, k}=r_{x, k(\mathrm{INS})}-r_{x, k(\mathrm{WSN})}, \Delta r_{y, k}=r_{y, k(\mathrm{INS})}-r_{y, k(\mathrm{WSN})}, \Delta v_{x, k}=v_{x, k(\mathrm{INS})}-v_{x, k(\mathrm{WSN})}$, and $\Delta v_{y, k}=v_{y, k(\mathrm{INS})}-v_{y, k(\mathrm{WSN})}$.

Here, we denote (2.5) as (2.6).

$$
\begin{equation*}
Y_{k}=C X_{k}+D v_{k} \tag{2.6}
\end{equation*}
$$

where $v_{k} \in R^{m}$ is measurement noise which belongs to $l_{2}[0, \infty)$. For the convenience, we assume that

$$
\begin{equation*}
E \sum_{k=0}^{\infty} \omega_{k}^{\prime} \omega_{k}<\infty, \quad E \sum_{k=0}^{\infty} v_{k}^{\prime} v_{k}<\infty \tag{2.7}
\end{equation*}
$$

So, the stochastic uncertain system can be simply expressed as:

$$
\begin{gather*}
X_{k+1}=\left(A+E \cdot \beta_{k}\right) X_{k}+B_{1} \varsigma_{k} \\
Y_{k}=C X_{k}+D_{1} \varsigma_{k}  \tag{2.8}\\
Z_{k}=L X_{k} .
\end{gather*}
$$

Here, $Z_{k}$ is the state combination to be estimated, $B_{1}=\left[\begin{array}{ll}B & 0\end{array}\right], D_{1}=\left[\begin{array}{ll}0 & D\end{array}\right]$, and $\varsigma_{k}=$ $\left[\begin{array}{ll}\omega_{k}^{\prime} & v^{\prime}{ }_{k}\end{array}\right]^{\prime}$.

## 2.2. $H_{\infty}$ Filter Formulation

In this section, we investigate the design of a linear estimator for $Z_{k}$ of the following form:

$$
\begin{gather*}
\widehat{X}_{k+1}=A_{f} \widehat{X}_{k}+B_{f} Y_{k} \\
\widehat{Z}_{k}=L \widehat{X}_{k}, \quad k=0,1,2, \ldots, \tag{2.9}
\end{gather*}
$$

where $\widehat{X}_{k}$ and $\widehat{Z}_{k}$ are the estimates of $X_{k}$ and $Z_{k}$, respectively, and $\left\{A_{f}, B_{f}, L\right\}$ are the constant matrices. Here, we define state error vector and measurement error vector, respectively, as follows:

$$
\begin{equation*}
e_{k}=X_{k}-\widehat{X}_{k}, \tilde{Z}_{k}=Z_{k}-\widehat{Z}_{k} \tag{2.10}
\end{equation*}
$$

Let $\Xi=E \beta_{k}$ and $A_{f}=A-B_{f} C+\Xi$, then we can obtain the following equation with (2.8), (2.9), and (2.10):

$$
\begin{gather*}
e_{k+1}=\left(\tilde{A}+E \beta_{k}\right) e_{k}+\tilde{B}_{S_{k}}  \tag{2.11}\\
\tilde{Z}_{k}=L e_{k}
\end{gather*}
$$

where $\tilde{A}=A-B_{f} C$ and $\widetilde{B}=B_{1}-B_{f} D_{1}$.
For a given scalar $\gamma>0$, the performance index is illustrated as (2.12):

$$
\begin{equation*}
J=E \sum_{k=0}^{\infty}\left(\tilde{Z}_{k}^{\prime} \tilde{Z}_{k}-r^{2} s^{\prime}{ }_{k} \varsigma_{k}\right) . \tag{2.12}
\end{equation*}
$$

In this paper, we look for an $H_{\infty}$ filter satisfies that for all nonzero $\omega_{k}$ and $v_{k}$ with the initial state $X_{k}=0, J<0$, and the system (2.11) is asymptotically stable.

### 2.3. Asymptotic Stability

For future convenience, we give the following lemmas which are very useful for the proof of our main theorem.

Lemma 2.1 (see [31]).

$$
\begin{gather*}
X_{k+1}=\left(A+E \beta_{k}\right) X_{k}+B \varsigma_{k}  \tag{2.13}\\
Z_{k}=L X_{k}
\end{gather*}
$$

for any $\gamma>0$, the system (2.13) is asymptotically stable and $J$ in (2.12) is negative for all nonzero $\varsigma_{k} \in l_{2}[0, \infty)$ if there exists $P=P^{\prime}>0$ that satisfies the inequality

$$
\begin{equation*}
-P+A^{\prime} P A+A^{\prime} P B \Theta^{-1} B^{\prime} P A+L^{\prime} L+E^{\prime} P E<0 \tag{2.14}
\end{equation*}
$$

and also satisfies $\Theta>0$, where $\Theta=r^{2} I-B^{\prime} P B$.
Lemma 2.2 (Schur's complement). For real matrices $N, M=M^{\prime}, R=R^{\prime}<0$, the following two conditions are equivalent:
(1) $M-N R^{-1} N^{\prime}<0$,
(2) $\left[\begin{array}{ll}M & N \\ N^{\prime} & R\end{array}\right]<0$.

Consider the system of (2.11). We arrive at the following result.
Theorem 2.3. The condition for system of (2.11) to be asymptotically stable and $\gamma$ of (2.12) to be existed is that there exists $P=P^{\prime}>0$ and $Q$ satisfies the inequality (2.15):

$$
\left[\begin{array}{cccccc}
-P & 0 & A^{\prime} P-C^{\prime} Q^{\prime} & A^{\prime} P-C^{\prime} Q^{\prime} & L^{\prime} & E^{\prime} P  \tag{2.15}\\
0 & -r^{2} I & B_{1}^{\prime} P-D_{1}^{\prime} Q^{\prime} & 0 & 0 & 0 \\
P A-Q C & P B_{1}-Q D_{1} & -P & 0 & 0 & 0 \\
P A-Q C & 0 & 0 & -P & 0 & 0 \\
L & 0 & 0 & 0 & -I & 0 \\
P E & 0 & 0 & 0 & 0 & -P
\end{array}\right]<0
$$

Proof. Consider the system of (2.11) and apply Lemma 2.1, given $\gamma>0$, a necessary and sufficient condition for $J$ in (2.12) to be negative for all nonzero $\varsigma_{k} \in l_{2}[0, \infty)$ which is that there exists $P=P^{\prime}>0$ to

$$
\begin{equation*}
-P+\widetilde{A}^{\prime} P \tilde{A}+\widetilde{A}^{\prime} P \widetilde{B} \tilde{\Theta}^{-1} \widetilde{B}^{\prime} P \tilde{A}+L^{\prime} L+E^{\prime} P E<0 \tag{2.16}
\end{equation*}
$$

where $\tilde{\Theta}=r^{2} I-\widetilde{B}^{\prime} P \widetilde{B}, P>0$, and $\tilde{\Theta}>0$.
Defining $\Delta=-P+\tilde{A}^{\prime} P \tilde{A}+L^{\prime} L+E^{\prime} P E$, so

$$
\begin{equation*}
\Delta+\tilde{A}^{\prime} P \tilde{B} \widetilde{\Theta}^{-1} \widetilde{B}^{\prime} P \tilde{A}<0 \tag{2.17}
\end{equation*}
$$

Applying Schur's complement, inequality (2.17) and the following inequality are equivalent:

$$
\left[\begin{array}{cc}
\Delta & \tilde{A}^{\prime} P \tilde{B}  \tag{2.18}\\
\widetilde{B}^{\prime} P \tilde{A} & -\left(\gamma^{2} I-\tilde{B}^{\prime} P \tilde{B}\right)
\end{array}\right]<0 .
$$

And inequality (2.18) can be rewritten as inequality (2.19):

$$
\left[\begin{array}{cc}
\Delta & 0  \tag{2.19}\\
0 & -\gamma^{2} I
\end{array}\right]-\left[\begin{array}{c}
\tilde{A}^{\prime} \\
\tilde{B}^{\prime}
\end{array}\right] P(-P)^{-1} P\left[\begin{array}{cc}
\tilde{A} & \tilde{B}
\end{array}\right]<0
$$

Using Schur's complement again, inequality (2.19) is able to be written as (2.20):

$$
\left[\begin{array}{ccc}
\Delta & 0 & \tilde{A}^{\prime} P  \tag{2.20}\\
0 & -\gamma^{2} I & \widetilde{B}^{\prime} P \\
P \tilde{A} & P \widetilde{B} & -P
\end{array}\right]<0
$$

Then, by Lemma 2.1 and $\Delta=-P+\widetilde{A}^{\prime} P \tilde{A}+L^{\prime} L+E^{\prime} P E$, we can obtain inequality (2.21) readily:

$$
\left[\begin{array}{cccccc}
-P & 0 & \tilde{A}^{\prime} P & \tilde{A}^{\prime} P & L^{\prime} & E^{\prime} P  \tag{2.21}\\
0 & -r^{2} I & \widetilde{B}^{\prime} P & 0 & 0 & 0 \\
P \widetilde{A} & P \widetilde{B} & -P & 0 & 0 & 0 \\
P \widetilde{A} & 0 & 0 & -P & 0 & 0 \\
L & 0 & 0 & 0 & -I & 0 \\
P E & 0 & 0 & 0 & 0 & -P
\end{array}\right]<0 .
$$

Now, we substitute (2.11) into inequality (2.21) and define that $Q=P B_{f}$, then we can obtain inequality (2.15) readily.

Moreover, the matrix inequality is able to be written as following if we set $\gamma^{2}=\bar{\gamma}$ :

$$
\begin{equation*}
\psi(P, Q, \bar{\gamma})<0 \tag{2.22}
\end{equation*}
$$

Thus, the solving of the filter is transformed to the following optimisation problem:

$$
\begin{equation*}
\min _{p_{1}>0, Q} \bar{\gamma}, \tag{2.23}
\end{equation*}
$$

subject to LMIs(2.22).

So, the filter is asymptotically stable and there exists the minimum performance index $\sqrt{\bar{\gamma}}$, the parameters of the HF can obtain by (2.23):

$$
\begin{equation*}
B_{f}=P^{-1} Q, \quad A_{f}=A-P^{-1} Q C+\Xi \tag{2.24}
\end{equation*}
$$

For the mean value of $\Xi$ is zero, the $A_{f}$ is also able to denote as follows:

$$
\begin{equation*}
A_{f}=A-P^{-1} Q C \tag{2.25}
\end{equation*}
$$

## 3. LS-SVM Model and Training Algorithm

LS-SVM is powerful to estimate for nonlinear. It is also able to extract the optimal solution with small training data. The LS-SVM algorithm is employed here to improve the accuracy of the INS-only solution during WSN outages.

### 3.1. LS-SVM Regression Algorithm

Equation (3.1) shows the optimal linear regression function which is built in feature space. Where $b$ is the bias term and $\omega$ is weight vector. Given a training set $\left\{x_{k}, y_{k}\right\}_{k=1}^{n}$, the LS-SVM algorithm maps a higher dimensional feature space $\psi(x)=\left\{\phi\left(x_{1}\right), \phi\left(x_{2}\right), \ldots, \phi\left(x_{n}\right)\right\}$ with nonlinear function $\varphi(x)$ :

$$
\begin{equation*}
f(x)=\{\omega, \phi(x)\}+b=\sum_{i=1}^{N} \omega_{i} \varphi_{i}(x)+b \tag{3.1}
\end{equation*}
$$

The optimisation problem is

$$
\begin{equation*}
\min _{\omega, b, e} J(\omega, e)=\frac{1}{2} \omega^{T} \omega+\eta \frac{1}{2} \sum_{k=1}^{N} e_{k}^{2} \tag{3.2}
\end{equation*}
$$

due to the equality constraints

$$
\begin{equation*}
y_{k}=\omega^{T} \varphi\left(x_{k}\right)+b+e_{k}, \quad k=1, \ldots, N \tag{3.3}
\end{equation*}
$$

To solve the optimisation problem abovementioned, the Lagrangian function is introduced:

$$
\begin{equation*}
L\left(\omega, b, e, \alpha_{i}\right)=J(\omega, e)-\sum_{i=1}^{N} \alpha_{i} \omega^{T} \phi\left(x_{i}\right)+b+e_{i}-y_{i} \tag{3.4}
\end{equation*}
$$

where $\alpha_{i}$ are the Lagrange multipliers, according to Karush Kuhn Tucker (KKT) optimization conditions which are illustrated in (3.5):

$$
\begin{equation*}
\frac{\partial L}{\partial \omega}=0, \quad \frac{\partial L}{\partial b}=0, \quad \frac{\partial L}{\partial e_{i}}=0, \quad \frac{\partial L}{\partial \alpha_{i}}=0 \tag{3.5}
\end{equation*}
$$

$$
\begin{align*}
& \text { So, } \\
& \omega=\sum_{i=1}^{N} \alpha_{i} \phi\left(x_{i}\right), \quad \sum_{i=1}^{N} \alpha_{i}=0, \quad \alpha_{i}=\eta e_{i}, \omega^{T} \phi\left(x_{i}\right)+b+e_{i}-y_{i}=0, i=1,2, \ldots, N . \tag{3.6}
\end{align*}
$$

The solution of (3.6), $\alpha_{i}$ and $b$, can be computed from the input of the sample sets when the LS-SVM is trained. Applying the Mercer condition one obtains [22]:

$$
\begin{equation*}
K\left(x_{k}, x_{l}\right)=\varphi\left(x_{k}\right)^{T} \varphi\left(x_{l}\right), \quad k, l=1,2, \ldots, N \tag{3.7}
\end{equation*}
$$

Thus, the LS-SVM model for nonlinear estimation is illustrated as (3.8).

$$
\begin{equation*}
y(x)=\sum_{i=1}^{N} \alpha_{i} K\left(x, x_{i}\right)+b \tag{3.8}
\end{equation*}
$$

The RBF kernel is used as the kernel function of the LS-SVM in this paper:

$$
\begin{equation*}
K\left(x, x_{i}\right)=\exp \left(-\frac{\left\|x-x_{i}\right\|^{2}}{2 \sigma^{2}}\right) \tag{3.9}
\end{equation*}
$$

As mentioned above, regularisation parameter $(\eta)$ and kernel width $(\sigma)$ need to be selected. In order for an optimal combination determined before the LS-SVM is trained, we use a simplified cross-validation method developed by Xu et al. [22], which defines a training set, consisting of the validation subsets and the verification subsets. Validation subset is used to train LS-SVM with some empirical combinations of tuning parameters. The primary parameters are those combinations which make the output of the LS-SVM approach the given accuracy. On the other hand, verification set is used to further train LS-SVM. As a result, the final selection of tuning parameters is made, and the system model is also obtained.

### 3.2. The Input/Output Design of LS-SVM

Due to the position and velocity changes with time, there is an HF states variation. It has been found that there is a correlation between states measured by INS and the HF states. Although modeling this correlation is difficult, it is able to build correlation with designed LS-SVM after adequate training. When the signals in WSNs are unavailable, with the input of INS's own estimation error of position and velocity, the LS-SVM is able to output the correction value for position and velocity, respectively, which is used to compensate the INS solution (as the integration HF does when the signals in WSN are available). As mentioned above, the


Figure 1: Configuration of the LS-SVM/HF hybrid system.
estimation errors of position and velocity measured by INS are selected as the input of the LS-SVM, which is illustrated as (3.10):

$$
\begin{equation*}
\mathrm{LS}^{-S V M} \mathrm{in}=\left\{\delta r_{x}, \delta r_{y}, \delta v_{x}, \delta v_{y}\right\} \tag{3.10}
\end{equation*}
$$

And the output of the LS-SVM can be simplified as:

$$
\begin{equation*}
\mathrm{LS}^{-S V M}{ }_{\text {out }}=\left\{\Delta r_{x}, \Delta r_{y}, \Delta v_{x}, \Delta v_{y}\right\} \tag{3.11}
\end{equation*}
$$

The structure of LS-SVM is consistently implemented for the training and prediction stages.

## 4. LS-SVM and HF Hybrid Method for Integration System

In this section, the LS-SVM/HF architecture is designed. The integration navigation consists of two stages. One is the LS-SVM/HF hybrid system. The other is the LS-SVM-based prediction during WSN outages.

### 4.1. The LS-SVM/HF Hybrid System for INS/WSN

The LS-SVM is in the training mode when the signals in WSNs are available. Figure 1 displays the configuration of the integration system for training of LS-SVM. INS estimates the errors of position and velocity in two directions which are continuously input to the LS-SVM for training. Meanwhile, the output of HF is employed for the target vectors of the training. The differences of position ( $r_{\text {WSN }}, r_{\text {INS }}$ ) and velocity ( $v_{\text {WSN }}, v_{\text {INS }}$ ) between INS and WSN are used for the observation vectors of the HF.

### 4.2. The Configuration of the LS-SVM-Based Prediction during WSN Outages.

The integrated system becomes a stand-alone INS without WSN signal. The LS-SVM is in the prediction mode now, and the output of the LS-SVM is used for error compensation.


Figure 2: Configuration of the LS-SVM-based prediction during WSN outages.


Figure 3: Definition of simulation scenarios.

The errors of position and velocity estimated by INS are continuously input to the LS-SVM as it was done during the training stage. The configuration of the LS-SVM-based prediction during WSN outage is illustrated in Figure 2.

## 5. Simulation and Performance

### 5.1. Assumptions

In order to assess the performance of the proposed method, the simulation is implemented. A $700 \mathrm{~m} \times 450 \mathrm{~m}$ area is defined as simulation scenario. In simulation, we assume that a BN moves from start point $(650,0)$ to end point $(130,400)$ along the red-dotted line in Figure 3. Based on the real-time data measured by INS, two areas (denote as green) is set as training area, where the signals in WSN are available. The scale of one training area is about $150 \mathrm{~m} \times$ 150 m (denote as no. 1 training area), and the other one is about $100 \mathrm{~m} \times 100 \mathrm{~m}$ (denote as no. 2 training area). The range between RNs is 5 m , and the communication range is 11 m . The sampling period ( $T$ ) in (2.3) is set to 1s. Here, we assume that the WSNs employ ultrasonic waves for localization, which is similar to [13], and the accuracy of localization is about 20 cm .

Table 1: LS-SVM training results with the validation set.

| No. | $\sigma$ | $\eta$ |  | Errors with validation set <br> Mean error <br> $(\mathrm{m})$ |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $\delta\left(\delta r_{x}\right)(\mathrm{m})$ | $\delta\left(\delta r_{y}\right)(\mathrm{m} / \mathrm{s})$ | $\delta\left(\delta v_{y}\right)(\mathrm{m} / \mathrm{s})$ | Mean error <br> $(\mathrm{m} / \mathrm{s})$ |  |  |
| 1 | 1 | 1 | 0.5991 | 0.2046 | 0.3558 | 0.1338 | 0.0864 | 0.1101 |
| 2 | 1 | 100 | 0.2354 | 0.1126 | 0.1711 | 0.0913 | 0.0968 | 0.0941 |
| 3 | 1 | 1000 | 0.2331 | 0.1067 | 0.1699 | 0.0932 | 0.0959 | 0.0946 |
| 4 | 30 | 100 | 0.2862 | 0.1067 | 0.1967 | 0.0924 | 0.0914 | 0.0919 |
| 5 | 30 | 1000 | 0.2542 | 0.1071 | 0.1807 | 0.0921 | 0.0926 | 0.0924 |
| 6 | 50 | 100 | 0.2871 | 0.1071 | 0.1972 | 0.0922 | 0.0912 | 0.0917 |
| 7 | 50 | 1000 | 0.2610 | 0.1073 | 0.1839 | 0.0918 | 0.0924 | 0.0921 |
| 8 | 100 | 100 | 0.2933 | 0.1068 | 0.2009 | 0.0923 | 0.0903 | 0.0913 |
| 9 | 100 | 1000 | 0.2641 | 0.1085 | 0.1852 | 0.0915 | 0.0923 | 0.0919 |
| 10 | 1000 | 1000 | 0.2903 | 0.1063 | 0.1452 | 0.0922 | 0.0900 | 0.0911 |

### 5.2. LS-SVM Training

In this paper, we define an independent training set with 100 points, the first 50 points are the validation subset, and other points are the verification subset. LS-SVM training results with the validation set are listed in Table 1. From Table 1, the final selection of tuning parameters is $\sigma=1000$ and $\eta=1000$. Finally, using both the validation and verification sets as well as the selected tuning parameters, the LS-SVM was trained again to obtain the compensation model.

### 5.3. Performance Analysis

According to Theorem 2.3, we readily obtain the following parameters for HF which is used in training area.

To the HF used in the first training area,

$$
\begin{gather*}
E=\left[\begin{array}{cccc}
0 & 0 & -0.0012 & 0 \\
0 & 0 & 0 & -0.0012 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \quad r=4.0055, \\
A_{f}=\mathrm{e}^{-11}\left[\begin{array}{cccc}
-0.0044 & -0.0694 & 0.0658 & 0.0082 \\
0.0066 & -0.0146 & -0.0296 & 0.0379 \\
-0.0588 & 0.1083 & -0.0353 & -0.0142 \\
0.0553 & -0.0232 & -0.0009 & -0.0316
\end{array}\right], \quad B_{f}=\left[\begin{array}{cccc}
1 & 0 & 0.2 & 0 \\
0 & 1 & 0 & 0.2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] . \tag{5.1}
\end{gather*}
$$

Table 2: The mean errors of position and velocity for the INS-only, KF, and HF methods in the training area.

| Mode | $\delta\left(\delta r_{x}\right)(\mathrm{m})$ |  | $\delta\left(\delta r_{y}\right)(\mathrm{m})$ |  | $\delta\left(\delta v_{x}\right)$ |  | $(\mathrm{m} / \mathrm{s})$ | $\delta\left(\delta v_{y}\right)(\mathrm{m} / \mathrm{s})$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | No. 1 area | No. 2 area | No. 1 area | No. 2 area | No. 1 area | No. 2 area | No. 1 area | No. 2 area |  |
| INS-only | 7.8315 | 40.6179 | 4.7923 | 36.0891 | 0.6584 | 2.0649 | 1.5777 | 2.0649 |  |
| KF | 2.9170 | 14.9663 | 1.7916 | 13.2591 | 0.2715 | 0.3155 | 0.5984 | 0.8856 |  |
| HF | 0.3881 | 0.4958 | 0.5338 | 0.8464 | 0.1064 | 0.1041 | 0.1008 | 0.0932 |  |

To the HF used in the second training area,

$$
\begin{gather*}
E=e^{-3}\left[\begin{array}{cccc}
0 & 0 & 0.8409 & 0 \\
0 & 0 & 0 & 0.8409 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \quad r=4.0055,  \tag{5.2}\\
A_{f}=e^{-11}\left[\begin{array}{cccc}
0.0857 & -0.0230 & -0.0524 & -0.0106 \\
-0.0087 & -0.0476 & 0.0416 & 0.0153 \\
-0.0626 & 0.1602 & -0.0407 & -0.0571 \\
-0.0153 & -0.0912 & 0.0568 & 0.0495
\end{array}\right], \quad B_{f}=\left[\begin{array}{cccc}
1 & 0 & 0.21 & 0 \\
0 & 1 & 0 & 0.21 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] .
\end{gather*}
$$

Figures 4 and 5 display the position errors in $x$-direction and $y$-direction in the first training area. The HF result is compared with the INS-only solution and the KF solution. In those figures one can see that both the KF and HF can reduce the position errors in $x$ direction and $y$ direction, respectively, and that the HF solution has the smallest error. Simulation result shows that the proposed HF method is very effective as it decreases the mean errors of position by about $85 \%$ in $x$ direction and by about $80 \%$ in $y$ direction compared with KF.

To further clearly demonstrate how the proposed HF improves the accuracy of the solution, the velocity errors in $x$-direction and $y$-direction for the INS-only, KF, and HF methods are shown in Figures 6 and 7. Note that the errors for the HF are smaller than the ones for the KF and INS-only methods both in $x$ direction and $y$ direction, confirming that the proposed algorithm can improve system performance. Simulation result shows that the proposed HF method decreases the velocity errors by about $70 \%$ in $x$ direction and in $y$ direction errors by about $75 \%$ compared with KF.

The mean errors of position and velocity in $x$ direction and $y$ direction in the second training area are illustrated in Table 2. We can see that the improvement is also particularly obvious.

In order to assess the performance of the hybrid method, two WSN outages are simulated. The LS-SVM result is compared with the INS-only solution and the BP solution during these outages. The position errors in $x$ direction and $y$ direction after the first training area derived from the INS-only (in green), BP (in blue), and the LS-SVM (in red) methods are shown in Figures 8 and 9, respectively. The BP method has the same input/output as the LS-SVM. In Figures 8 and 9, one can see that both the BP and LS-SVM are able to reduce the position errors, and that the HF solution has the smallest error, confirming that the proposed algorithm can improve system performance. From these outage results it can be seen that


Figure 4: The position errors in $x$ direction for the INS-only, KF, and HF methods in the first training area.


Figure 5: The position errors in $y$ direction for the INS-only, KF, and HF methods in the first training area.
the proposed method decreases by about $80 \%$ position errors in $x$ direction and $70 \%$ position errors in $y$ direction compared with $B P$.

Figures 10 and 11 display the velocity errors in $x$-direction and $y$-direction in the first $50-$ second WSN outages area. The LS-SVM results are compared with the INS-only solution and the BP solution. In those figures one can see that both the LS-SVM and BP can reduce


Figure 6: The velocity errors in $x$ direction for the INS-only, KF, and HF methods in the first training area.


Figure 7: The velocity errors in $y$ direction for the INS-only, KF, and HF methods in the first training area.
the position errors, and that the LS-SVM solution has the smallest error. However, there are some fluctuations in BP's error. The mean errors of position and velocity in $x$ direction and $y$ direction in the second WSN outages area are illustrated in Table 3. Simulation result shows that the proposed LS-SVM method is very effective as it decreases the mean errors of velocity by about $40 \%$ in $x$ direction and by about $70 \%$ in $y$ direction compared with BP.


Figure 8: The position errors in $x$ direction for the INS-only, HF + BP, and HF + LS-SVM methods in the first WSN outages.


Figure 9: The position errors in $y$ direction for the INS-only, HF + BP, and HF + LS-SVM methods in the first WSN outages.

As mentioned above, the improvement is particularly obvious in the prediction period. The prediction of the LS-SVM is able to maintain a higher accuracy and reduce the influence of accuracy deterioration caused by the INS.


Figure 10: The velocity errors in $x$ direction for the INS-only, HF + BP, and HF + LS-SVM methods in the first WSN outages.


Figure 11: The velocity errors in $y$ direction for the INS-only, HF + BP, and HF + LS-SVM methods in the first WSN outages.

## 6. Conclusions

This work proposes an integrated INS/WSN system using LS-SVM and HF. The input and output of an LS-SVM are selected on the basis of correlations between the estimation errors measured by INS and the HF states. When the signals in WSN are available, the HF

Table 3: The mean errors of position and velocity for the INS-only, KF, and HF methods in the prediction area.

| Mode | $\delta\left(\delta r_{x}\right)(\mathrm{m})$ |  | $\delta\left(\delta r_{y}\right)(\mathrm{m})$ |  | $\delta\left(\delta v_{x}\right)(\mathrm{m} / \mathrm{s})$ |  | $\delta\left(\delta v_{y}\right)(\mathrm{m} / \mathrm{s})$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | No. 1 area | No. 2 area | No. 1 area | No. 2 area | No. 1 area | No. 2 area | No. 1 area | No. 2 area |
| INS-only | 12.1524 | 72.4795 | 8.7879 | 70.0089 | 0.8897 | 3.0793 | 2.0265 | 6.0012 |
| BP | 5.2628 | 36.4784 | 4.9441 | 36.5174 | 0.6502 | 1.1008 | 0.8096 | 1.6045 |
| LS-SVM | 0.5916 | 5.2938 | 0.4676 | 4.6548 | 0.0786 | 0.0092 | 0.0754 | 0.0567 |

is employed to provide optimal estimation of position and velocity errors, which is used to update the INS solution. Meanwhile, mapping model between the estimation errors of INS and the HF states is trained by the LS-SVM. Based on the real-time data measured by INS, WSNs enabled and outages areas are simulated. The results show an improved overall performance in comparison with the results of the INS-only and KF solutions, and the prediction of the LS-SVM has a higher accuracy than the prediction of the BP.

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Research Article

# Arbitrage-Free Conditions and Hedging Strategies for Markets with Penalty Costs on Short Positions 

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#### Abstract

We consider a discrete-time financial model in a general sample space with penalty costs on short positions. We consider a friction market closely related to the standard one except that withdrawals from the portfolio value proportional to short positions are made. We provide necessary and sufficient conditions for the nonexistence of arbitrages in this situation and for a self-financing strategy to replicate a contingent claim. For the finite-sample space case, this result leads to an explicit and constructive procedure for obtaining perfect hedging strategies.


## 1. Introduction

In recent years, applications of stochastic analysis and control have entered in the field of financial engineering in an effective and rapid way, due mainly to the powerful tools that can be brought from these disciplines into almost all aspects of fields like, for instance, in the study of arbitrage, hedging, pricing, and portfolio optimization. One of the classical problems in portfolio optimization is the mean-variance portfolio selection problem, which was transformed with the seminal work of Markowitz in [1] (see also [2]). Since then, the amount of research on this subject has increased in order to provide the development of sophisticated analytical and numerical methods for financial engineering models with more realistic assumptions see, for instance, [3-8], among others. More recently, the multiperiod meanvariance problem was tackled by [9] and later extended in several directions see, for instance, [10-25]. The pioneering work of Black and Scholes [26] and Merton [27] provided a major change in the area of pricing of derivative securities, showing that the analysis should be based on nonarbitrage considerations rather than on preference-related concepts such as
expected values. From the works of Harrison and Kreps [28] and Harrison and Pliska [29], it became apparent that semimartingale theory provided a natural framework for the analysis of financial markets and pricing.

Another fundamental result on the study of arbitrage, hedging, and pricing of financial markets is the Dalang-Morton-Willinger theorem, also known as the fundamental theorem of asset pricing. It states that in a frictionless security market, the existence of an equivalent martingale measure for the discounted price process is equivalent to the absence of arbitrage (see, e.g., [30]). Recently, there has been a number of papers dealing with contingent claim valuation and extending versions of the aforementioned theorem in several directions (see, e.g., [31-38]). The subject of pricing derivatives with transaction costs and portfolio selection with transaction costs is of practical importance and has been in evidence over the last years. Two types of transaction costs are considered; fixed costs, which are paid whenever there is a change of position, and proportional costs, which are charged according to the volume traded. Different approaches to the problem of pricing derivatives with transaction costs and the portfolio choice problem under transaction costs can be found in the literature see, for instance, $[32,37,39-44]$. The results in [45] provided a version of the fundamental theorem of asset pricing within a short sales constraints framework and possible infinite number of transactions within a finite period of time, using the free-lunch notion, a stronger notion of the no arbitrage condition. The case of closed cone constraints on the amount invested in the risky assets, which includes restrictions on short sales, has been studied in [46] for the case in which the price process is positive and under a nondegeneracy hypothesis on the price process. In [47], these results were generalized, and the fundamental theorem of asset pricing was stated under polyhedral convex cone constraints and using the classical notion of no arbitrage instead of free lunch. The general short sales constraints in [45, 47] were considered by separating the price process into two sort of securities; those which cannot be held in negative amounts and those that can only be held in negative amounts. The no arbitrage condition in this case is shown in [47] to be equivalent to the existence of a positive interest rate process and an equivalent probability measure $\mathbf{Q}$ under which the discounted price processes of securities that cannot be sold short are supermartingales, and the discounted price processes of securities that can only be sold short are submartingales.

In this paper, we consider a model closely related to the standard one (see [30,48, 49]), except that a withdrawal directly proportional to the amount on short positions is made from the portfolio. As far as the authors are aware of, this model has not been studied before (see also Remark 3.3). Theorem 3.1 provides necessary and sufficient conditions for the nonexistence of arbitrages directly in terms of the price process and penalty costs at time $t$ and can be seen as a natural extension of the standard fundamental theorem of asset pricing (see, e.g., [30, 48, 49]). When the penalty costs go to zero, our result reduces to that presented in [30]. From this result we derive a sufficient condition for nonarbitrage and for a self-financing strategy to consistently replicate a contingent claim (e.g., any other superreplicating selffinancing strategy will have an initial value greater than that of the replicanting strategy). For the finite sample space case, this result yields an explicit and constructive procedure for obtaining perfect hedging strategies.

This paper is organized in the following way. Section 2 presents some notation, definitions, and the financial model. Section 3 contains the main results of the paper. In Section 4, we present an explicit and constructive procedure for obtaining perfect hedging strategies for the case in which the sample space is finite, as well as some numerical examples. Section 5 concludes the paper. The proof of some auxiliary results and the main results are presented in the appendix.

## 2. Notation, Definitions and Problem Formulation

Let the real $d$-dimensional vector space be denoted by $\mathbb{R}^{d}$ and for $x \in \mathbb{R}^{d}$ we will write $x_{i}$ for the $i$ th component of the vector. The superscript will be omitted for the case $d=1$. For $x, y$ in $\mathbb{R}^{d}$, we set

$$
\begin{equation*}
x \cdot y=\sum_{i=1}^{d} x_{i} y_{i} \tag{2.1}
\end{equation*}
$$

We write $x \geq 0$ to denote that all components of $x$ are positive, that is, $x_{i} \geq 0$ for $i=1, \ldots, d$. For $x \in \mathbb{R}^{d}$, we set the following vectors in $\mathbb{R}^{d}: x^{+} \geq 0$ such that its $i$ th component $x_{i}^{+}$is equal to $x_{i}$ if $x_{i} \geq 0$, zero otherwise, $x^{-}=(-x)^{+} \geq 0$ (therefore, $x=x^{+}-x^{-}$and $x_{i}^{+} x_{i}^{-}=0$ for each $i=1, \ldots, d)$. The vector formed by 1 in all components will be represented by $e$, and the vector with 1 at the $i$ th component and 0 elsewhere by $b^{i}$. For a real number $a$, we define $a^{\oplus}=1 / a$ if $a \neq 0$, zero otherwise.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a filtration $\left\{\mathcal{F}_{t}\right\}, t=$ $0,1, \ldots, T$. For $\mathcal{G}$ a sub- $\sigma$-algebra of $\mathcal{F}$, we denote by $\mathcal{L}_{0}^{d}(\Omega, \mathcal{G}, \mathbf{P})$ (or simply $\mathcal{L}_{0}^{d}(\mathcal{G})$ ) the space of $\mathcal{G}$-measurable random variables with values in $\mathbb{R}^{d}$, which is a complete topological vector space if equipped with the topology of convergence measure. As any sequence converging in probability contains a subsequence converging almost surely (a.s.), we can assume without loss of generality that any convergent sequence in $\mathscr{L}_{0}^{d}(\mathcal{G})$ will converge a.s. For any probability measure $\mathbf{Q}, E_{\mathbf{Q}}(\cdot)$ denotes the expectation with respect to $\mathbf{Q}$, and we write $\mathbf{Q} \sim \mathbf{P}(\mathbf{Q} \ll \mathbf{P})$ whenever the probability measure $\mathbf{Q}$ is equivalent to (absolutely continuous with respect to) $\mathbf{P}$. For any $A \in \mathscr{F}, 1_{A}$ denotes the indicator function of the set $A$. Let $\mathscr{L}_{0}^{d+}(\mathcal{G})$ be the set of random vectors $X \in \mathcal{L}_{0}^{d}(\mathcal{G})$ such that $\mathbf{P}(X \geq 0)=1$. The space of integrable random vectors in $\mathscr{L}_{0}^{d}(\mathcal{G})$ will be denoted by $\mathscr{L}_{1}^{d}(\mathcal{G})$ and the space of essentially bounded random vectors in $\mathcal{L}_{0}^{d}(\mathcal{G})$ by $£_{\infty}^{d}(\mathcal{G})$.

Consider given stochastic processes $S=\{S(t) ; t=0, \ldots, T\}$ and $D=\{D(t) ; t=$ $1, \ldots, T\}$ taking values in $\mathbb{R}^{d}$ with $S(t) \in \mathcal{P}_{0}^{d}\left(\mathscr{F}_{t}\right)$ and $S_{1}(t)=1$ for each $t=0, \ldots, T$, and $D(t) \in \mathcal{L}_{0}^{d+}\left(\mathcal{F}_{t}\right)$ for each $t=1, \ldots, T$. We define for $t=1, \ldots, T, \Delta S(t):=S(t)-S(t-1)$. A trading strategy $H=(H(1), \ldots, H(T))$ is defined such that each $H(t)$ is a $d \times 2$-dimensional random matrix with columns $H^{+}(t) \in \mathcal{L}_{0}^{d+}\left(\mathcal{F}_{t-1}\right)$ and $H^{-}(t) \in \mathcal{L}_{0}^{d+}\left(\mathcal{F}_{t-1}\right)$. It describes an investor's portfolio as carried forward from time $t=0$ to time $t=T$. In the model of a security market, $S$ describes the evolution of the prices of $d$ securities, $H^{+}(t)$ represents the number of units of each security hold in a long position from time $t-1$ to time $t, H^{-}(t)$ represents the number of units of each security hold in a short selling position from time $t-1$ to time $t$, and $D$ the evolution of the penalty costs and possible spread costs between borrowing and lending rates due to short selling positions on each security $i$.

Associated to a trading strategy $H$, we have the value process $V^{H}:=\left(V^{H}(0), \ldots\right.$, $V^{H}(T)$ ) describing the total value of the portfolio at each time $t$. For notational simplicity, we will omit the superscript $H$ whenever no confusion arises. The portfolio value can be written, at time $t=0$, as

$$
\begin{equation*}
V(0)=\left(H^{+}(1)-H^{-}(1)\right) \cdot S(0) \tag{2.2}
\end{equation*}
$$

and at times $t=1, \ldots, T$, as

$$
\begin{equation*}
V(t)=\left(H^{+}(t)-H^{-}(t)\right) \cdot S(t)-H^{-}(t) \cdot D(t) \tag{2.3}
\end{equation*}
$$

The quantity $V(t)$ represents the value of the portfolio at time $t$ just before any change of ownership positions take place at that time. The penalty costs due to short selling positions are represented by the costs:

$$
\begin{equation*}
H^{-}(t) \cdot D(t) \tag{2.4}
\end{equation*}
$$

The value of the portfolio at time $t+1$ just after the change of ownership positions is $\left(H^{+}(t+\right.$ $\left.1)-H^{-}(t+1)\right) \cdot S(t)$. We consider in this paper self-financing trading strategies, so that no money is added or withdrawn from the portfolio between times $t=0$ to time $t=T$. Any change in the portfolio's value is due to a gain or loss in the investments, and penalty costs due to the short selling positions. Thus, we must have

$$
\begin{equation*}
V(t)=\left(H^{+}(t+1)-H^{-}(t+1)\right) \cdot S(t) \tag{2.5}
\end{equation*}
$$

From (2.2)-(2.5), we have for $t=1, \ldots, T$ that

$$
\begin{equation*}
V(t)=V(t-1)+\left(H^{+}(t)-H^{-}(t)\right) \cdot \Delta S(t)-H^{-}(t) \cdot D(t) \tag{2.6}
\end{equation*}
$$

We notice that the penalties can be seen as withdrawals from the portfolio value proportional to the short selling positions.

We conclude this section with the definition of an arbitrage opportunity. We say that there is an arbitrage opportunity if for some self-financing trading strategy $H$, we have a.s. that
(i) $V(0) \leq 0$,
(ii) $V(T) \geq 0$, and
(iii) $E(V(T))>0$.

## 3. Main Results

In this section, we present the main results of the paper. We start with Theorem 3.1, which provides necessary and sufficient conditions for the nonexistence of arbitrages and can be seen as a natural extension of the standard fundamental theorem of asset pricing (see, e.g., [30, 48, 49]). As pointed out in Remark 3.2, when the penalty costs go to zero, our result reduces to that presented in [30]. In Remark 3.3, we point out the differences between our result and previous results presented in the literature. As usual in this kind of problems, the hardest part of the proof is to show that a certain set is closed (see Proposition A. 1 in the appendix). In the sequence, we derive a sufficient condition for nonarbitrage and for a selffinancing strategy to consistently replicate a contingent claim. In Section 4, we consider the finite sample space case so that the results in this section yield an explicit and constructive procedure for obtaining perfect hedging strategies.

The following theorem provides necessary and sufficient conditions for the nonexistence of arbitrages. The proof can be found in the appendix. In what follows, we recall that $\mathcal{L}_{\infty}^{+}\left(\mathcal{F}_{t}\right)$ represents the space of essentially bounded $\mathcal{F}_{t}$-measurable random variables $Z$ such that $\mathbf{P}(Z \geq 0)=1$.

Theorem 3.1. The following statements are equivalent:
(i) there are no arbitrage opportunities,
(ii) for any self-financing strategy $H$, one has a.s. that

$$
\begin{gather*}
\left.\quad \begin{array}{l}
\left.H^{+}(t)-H^{-}(t)\right) \cdot S(t-1) \leq 0 \\
\left(H^{+}(t)-H^{-}(t)\right) \cdot \Delta S(t)-H^{-}(t) \cdot D(t) \geq 0
\end{array}\right\} \Longrightarrow\left(H^{+}(t)-H^{-}(t)\right) \cdot \Delta S(t)-H^{-}(t) \cdot D(t)=0, ~ \tag{3.1}
\end{gather*}
$$

(iii) there exists a stochastic process $\{r(t)\}$ with $r(t) \in \mathcal{L}_{\infty}^{+}\left(\mathcal{F}_{t}\right)$ for each $t=0,1, \ldots, T-1$ and a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that $0<d \mathbf{Q} / d \mathbf{P} \in \perp_{\infty}^{+}(\mathcal{F}), S(t), D(t)$ are integrable with respect to $\mathbf{Q}$ and for each $t=0, \ldots, T-1$,

$$
\begin{equation*}
E_{\mathbf{Q}}\left(\Delta S(t+1) \mid \mathcal{F}_{t}\right) \leq r(t) S(t) \leq E_{\mathbf{Q}}\left(\Delta S(t+1)+D(t+1) \mid \mathcal{F}_{t}\right) \text { a.s. } \tag{3.2}
\end{equation*}
$$

Remark 3.2. For the case in which $D(t)=0$, our results reduce to the well-known fundamental theorem of asset pricing with finite-discrete time and infinite state space, see [30] (recall that $S_{1}(t)=1, \Delta S_{1}(t)=0$, and if $D_{1}(t)=0$, then (3.2) implies that $\left.r(t)=0\right)$.

Remark 3.3. In [45, 47], the authors consider a financial market with two sort of securities, those that cannot be held in negative amounts and represented by $\bar{S}(t)$, and those that can only be held in negative amounts and represented by $\widetilde{S}(t)$. To write our problem in the above set-up, we would need to define the fictitious price processes $\bar{S}(t), \widetilde{S}(t)$ as: $\bar{S}(0)=\widetilde{S}(0)=S(0)$ and for $t=1, \ldots, T, \bar{S}(t)=S(t), \widetilde{S}(t)=S(t)+D(1)+\cdots+D(t)$. Notice that there is no discounting to be applied since we are considering that $S_{1}(t)=1$. By doing this, we would have that for $t=1, \ldots, T, \Delta \bar{S}(t)=\Delta S(t), \Delta \widetilde{S}(t)=\Delta S(t)+D(t)$ and this would yield that $V(t)=V(t-1)+\left(H^{+}(t)-H^{-}(t)\right) \cdot \Delta S(t)-H^{-}(t) \cdot D(t)$, which is similar to (2.6). Note, however, that the models are different since we cannot guarantee that a self-financing strategy $H$ for the above model will be self-financing for our model, and vice versa. Indeed, for the above model, the self-financing condition would read as $\left(H^{+}(t+1)-H^{-}(t+1)\right) \cdot S(t)-H^{-}(t+1) \cdot \sum_{k=1}^{t} D(k)=$ $\left(H^{+}(t)-H^{-}(t)\right) \cdot S(t)-H^{-}(t) \cdot \sum_{k=1}^{t} D(k)$, while for our model it would be $\left(H^{+}(t+1)-H^{-}(t+1)\right)$. $S(t)=\left(H^{+}(t)-H^{-}(t)\right) \cdot S(t)-H^{-}(t) \cdot D(t)$. This also occurs with the nonarbitrage conditions of the two models. Indeed, the nonarbitrage condition presented in [45,47] states that $\bar{S}(t)$ is a supermartingale and $\widetilde{S}(t)$ is a submartingale, which would involve the sum of the terms $(k)$. On the other hand, for our model, the nonarbitrage condition (3.2) involves only the state price $S(t), S(t+1)$ and penalty costs $D(t+1)$ emphasizing the difference between the two models.

In what follows, we define

$$
\begin{equation*}
\mathcal{K}=\{(\{r(t)\}, \mathbf{Q}) ; r(t) \text { and } \mathbf{Q} \text { satisfying condition (iii) of Theorem 3.1\}. } \tag{3.3}
\end{equation*}
$$

We recall next that a contingent claim (random variable) $X \in \mathscr{L}_{0}\left(\mathscr{F}_{T}\right)$ is marketable if for some self-financing strategy $H$ we have that a.s. $X=V^{H}(T)$ and, in this case, $H$ is said to replicate $X$. We say that $H$ superreplicates $X$ if a.s. we have that $V^{H}(T) \geq X$. We have the following corollary (see the proof in the appendix).

Corollary 3.4. Suppose that $H$ superreplicates $X$ and there is no arbitrage. Then, for any $(\{r(t)\}$, $\mathbf{Q}) \in \nless K$ one has a.s. that

$$
\begin{equation*}
E_{\mathbf{Q}}\left(\left.\frac{X}{(1+r(T)) \cdots(1+r(t))} \right\rvert\, \mathscr{F}_{t}\right) \leq V(t) \tag{3.4}
\end{equation*}
$$

Writing $\Gamma(t)=\left\{\omega ; V(t)(\omega)-E_{\mathbf{Q}}\left(X /(1+r(T)) \cdots(1+r(t)) \mid \mathcal{F}_{t}\right)(\omega)>0\right\}$ one has that if $\mathbf{P}(\Gamma(T))>$ 0 then $\mathbf{P}(\Gamma(t))>0$ for every $t=T-1, \ldots, 0$.

We will be interested now in deriving a condition such that the pricing of a marketable contigent claim $X \in \Omega_{0}\left(\mathscr{F}_{T}\right)$ is obtained from a self-financing strategy $H$ that replicates $X$ with $H^{+}(t) \cdot H^{-}(t)=0$ for $t=1, \ldots, T$, so that logical pricing can be obtained in this way. Let us define $\mathbb{J}:=\left\{a=\{a(t)\}_{t=0}^{T-1}\right.$; for $t=0, \ldots, T-1, a(t) \in \mathcal{L}_{0}^{d+}\left(\mathscr{F}_{t}\right), a_{i}(t)=0$ or 1$\}$. For a self-financing strategy $H$ satisfying $H^{+}(t) \cdot H^{-}(t)=0$, we set $a^{H}=\left\{a^{H}(t)\right\} \in \mathbb{J}$ as $a_{i}^{H}(t)=$ $1_{\left\{H_{i}^{-}(t+1)>0\right\}}$.

Definition 3.5. For $a=\{a(t)\} \in \mathbb{J}$, set

$$
\begin{gather*}
\Theta^{a}=\left\{\mathbf{Q} \sim \mathbf{P}, \frac{d \mathbf{Q}}{d \mathbf{P}} \in \mathscr{L}_{\infty}^{+}(\mathscr{F}) ; S(t), D(t) \text { are integrable with respectto } \mathbf{Q},\right. \\
\text { for } t=0, \ldots, T, E_{\mathbf{Q}}\left(\Delta S_{i}(t+1) \mid \mathscr{F}_{t}\right)=0 \text { a.s. on }\left\{a_{i}(t)=0\right\}, \text { and }  \tag{3.5}\\
\left.E_{\mathbf{Q}}\left(\Delta S_{i}(t+1)+D_{i}(t+1) \mid \mathcal{F}_{t}\right)=0 \text { a.s. on }\left\{a_{i}(t)=1\right\}\right\} .
\end{gather*}
$$

We have the following proposition showing that any $a=\{a(t)\} \in \mathbb{J}$ will lead to an element in $\mathcal{K}$ (see the proof in the appendix).

Proposition 3.6. If for some $a=\{a(t)\} \in \mathbb{J}$ one has $\Theta^{a} \neq \emptyset$, then there are no arbitrages.
Finally, we have the following result, presenting a sufficient condition for a self-financing strategy to consistently replicate a contingent claim (i.e., any other superreplicating self-financing strategy will have an initial value greater than that of the replicating strategy). The proof can be found in the appendix.

Proposition 3.7. If $H$ is a self-financing strategy that replicates $X$ with $H^{+}(t) \cdot H^{-}(t)=0$ for $t=1$, $\ldots, T$ and $\Theta^{a^{H}} \neq \emptyset$, then for any superreplicating strategy $\widehat{H}$ for $X$ one has a.s. for $t=0,1, \ldots, T$ that $V^{\widehat{H}}(t) \geq V^{H}(t)$ and if $\mathbf{P}\left(V^{\widehat{H}}(T)>X\right)>0$, then $\left(V^{\widehat{H}}(t)>V^{H}(t)\right)>0$.

## 4. A Numerical Procedure

In this section, we consider the finite-state space case and present an algorithm for obtaining the hedging strategy for a marketable claim $X$ satisfying the conditions of Proposition 3.7. We assume here that $\Omega=\left\{\omega_{1}, \ldots, \omega_{\kappa}\right\}$ and that $D(t)$ is $\mathcal{F}_{t-1}$-measurable. We consider the single period case only, and suppress the time dependence whenever it is possible. The multiperiod case follows in a similar way, by using the information structure described in [50] or [51], and
by applying backwards in time, the procedure described here for the single period case and each node of the information structure. Define the following matrix $A$ : $A=\left(A_{1} A_{2}\right)$, where

$$
\mathrm{A}_{1}=\left(\begin{array}{cccc}
1 & S_{2}(1)\left(\omega_{1}\right) & \ldots & S_{d}(1)\left(\omega_{1}\right)  \tag{4.1}\\
\vdots & \vdots & \ddots & \vdots \\
1 & S_{2}(1)\left(\omega_{\kappa}\right) & \ldots & S_{d}(1)\left(\omega_{\kappa}\right)
\end{array}\right), \quad A_{2}=-\left(A_{1}+D\right)
$$

with $D=e \mathbf{d}$ (recall that $e$ is the vector formed by 1 in all components), $\mathbf{d}=\left(g_{1} \ldots g_{d}\right)$. Let the vector $x \in \mathbb{R}^{\kappa}$ be such that $x_{j}=X\left(\omega_{j}\right), j=1, \ldots, \kappa$. We have that $X$ is marketable if and only if there exists $H^{+}, H^{-}$that satisfy the system:

$$
\begin{gather*}
A\binom{H^{+}}{H^{-}}=A_{1}\left(H^{+}-H^{-}\right)-D H^{-}=x  \tag{4.2}\\
H^{+} \geq 0, \quad H^{-} \geq 0 \tag{4.3}
\end{gather*}
$$

For the case in which $\kappa=d$ and $A_{1}$ has an inverse, we have the following explicit and constructive procedure for obtaining a trading strategy $H$ that replicates $X$ with $H^{+} \cdot H^{-}=0$. Since $D=e d$ and $A_{1}^{-1} e=b^{1}$ (recall that $b^{1}$ is the vector formed by 1 at the 1 st component, and 0 elsewhere), we have premultiplying (4.2) by $A_{1}^{-1}$ that $H^{+}, H^{-}$satisfy (4.2) if and only if satisfy

$$
\begin{equation*}
\left(H^{+}-H^{-}\right)-A_{1}^{-1} e \mathbf{d} H^{-}=\left(H^{+}-H^{-}\right)-b^{1} \mathbf{d} H^{-}=A_{1}^{-1} x . \tag{4.4}
\end{equation*}
$$

Set $y=A_{1}^{-1} x$. Let us obtain $H$ that satisfies (4.2), (4.3) with $H^{+} \cdot H^{-}=0$. Define for $i=2$, $\ldots, d: H_{i}^{+}=y_{i}, H_{i}^{-}=0$ if $y_{i} \geq 0$, otherwise, $H_{i}^{-}=-y_{i}, H_{i}^{+}=0$. For $i=1$, calculate

$$
\begin{equation*}
z=y_{1}+\sum_{i=2}^{d} g_{i} H_{i}^{-} \tag{4.5}
\end{equation*}
$$

In order to have (4.4) satisfied, we must have $\left(H_{1}^{+}-H_{1}^{-}\right)-g_{1} H_{1}^{-}=z$. If $z \geq 0$, then set $H_{1}^{+}=z$, $H_{1}^{-}=0$, otherwise, set $H_{1}^{+}=0$ and $H_{1}^{-}=-z /\left(1+g_{1}\right)$. Thus, we have obtained in this way a trading strategy $H$ that replicates $X$ with $H^{+} \cdot H^{-}=0$.

Finally, notice that for the discrete sample space, the set $\mathbb{J}$ is finite, and thus if we assume that for every $a \in \mathbb{J}, \Theta^{a} \neq \emptyset$, then the conditions of Proposition 3.7 will be satisfied. With the above procedure, we have a seller price and a buyer price for each contingent claim. The seller price, denoted by $V_{s}(0)$, is obtained by applying to $X$ backwards in time the algorithm presented above. The buyer price, denoted by $V_{b}(0)$, is obtained by applying the backward algorithm to $-X$, and taking $V_{b}(0)=-V(0)$. We illustrate this procedure next for the binomial case.

Example 4.1. Let us consider the binomial model, which consists of a single risky security satisfying

$$
\begin{equation*}
S_{2}(t)=\frac{1}{B(t)} u^{N(t)} d^{t-N(t)} S_{2}(0), \tag{4.6}
\end{equation*}
$$

$t=1, \ldots, T$, where $0<d<1<u$ and $N=\{N(t) ; t=1, \ldots, T\}$ is a binomial process with parameter $p, 0<p<1$, and the bank account is given by $B(t)=\left(1+r_{f}\right)^{t}, t=0,1, \ldots, T$. The penalty costs are assumed to be of the form:

$$
\begin{equation*}
D_{1}(t)=\alpha_{1} \frac{1}{1+r_{f}}, \quad D_{2}(t)=\alpha_{2} \frac{S_{2}(t-1)}{1+r_{f}} \tag{4.7}
\end{equation*}
$$

It is easy to see that in this case $\mathbb{J}=\{(0,0),(0,1),(1,0),(1,1)\}$, and we have the following possibilities for $\Theta^{a}=\left\{\pi_{1}, \pi_{2}\right\}$, where $\pi_{1}$ is associated to the probability measure $\mathbf{Q}$ when the stocks goes up, $\pi_{2}$ when the stocks goes down:
(i) $a=(0,0)$; in this case,

$$
\begin{equation*}
\pi_{1}=\frac{1+r_{f}-d}{u-d}, \quad \pi_{2}=\frac{u-\left(1+r_{f}\right)}{u-d} \tag{4.8}
\end{equation*}
$$

(ii) $a=(1,0)$; in this case,

$$
\begin{equation*}
\pi_{1}=\frac{1+r_{f}-\alpha_{1}-d}{u-d}, \quad \pi_{2}=\frac{u-\left(1+r_{f}-\alpha_{1}\right)}{u-d} \tag{4.9}
\end{equation*}
$$

(iii) $a=(0,1)$; in this case,

$$
\begin{equation*}
\pi_{1}=\frac{1+r_{f}+\alpha_{0}-d}{u-d}, \quad \pi_{2}=\frac{u-\left(1+r_{f}+\alpha_{0}\right)}{u-d} ; \tag{4.10}
\end{equation*}
$$

(iv) $a=(1,1)$; in this case,

$$
\begin{equation*}
\pi_{1}=\frac{1+r_{f}+\alpha_{0}-\alpha_{1}-d}{u-d}, \quad \pi_{2}=\frac{u-\left(1+r_{f}+\alpha_{0}-\alpha_{1}\right)}{u-d} \tag{4.11}
\end{equation*}
$$

From above, it is clear that the condition which guarantees that $\Theta^{a} \neq \emptyset$, and thus that the conditions of Proposition 3.7, will be satisfied, is given by $u>1+r_{f}+\alpha_{1}$ and $d<1+r_{f}-\alpha_{2}$.

Let us consider the following numerical example. Suppose that $S_{2}(0)=5, u=4 / 3$, $d=8 / 9, \alpha_{1}=\alpha_{2}=1 / 30, r_{f}=1 / 9$. For this case, we have $1+r_{f}+\alpha_{1}=103 / 90<u=4 / 3$, and $1+r_{f}-\alpha_{2}=97 / 90>d=8 / 9$, and the conditions of Proposition 3.7 will be verified. Let us consider the following option: $X=\max \{S(2)-5,0\}$. By applying the backward procedure described above, we obtain that the seller price for $X$ is $V_{S}(0)=1.3272$, with the following hedging strategy: $H_{1}^{+}(0)=0, H_{1}^{-}(0)=2.796, H_{2}^{+}(0)=0.8246, H_{2}^{-}(0)=0$, and for the case in which the risky security goes up, $H_{1}^{+}(1)=0, H_{2}^{-}(1)=3.932, H_{2}^{+}(1)=1.0, H_{2}^{-}(1)=0, V(1)=$ 2.2977, while for the case in which it goes down, $H_{1}^{+}(0)=0, H_{1}^{-}(0)=1.4563, H_{2}^{+}(0)=0.4687$, $H_{2}^{-}(0)=0$, and $V(1)=0.4652$.

By repeating the procedure now for $-X$, we obtain that the buyer price for $X$ is $V_{b}(0)=$ 0.9355 , with the following hedging strategy: $H_{1}^{+}(0)=2.6926, H_{1}^{-}(0)=0, H_{2}^{+}(0)=0, H_{2}^{-}(0)=$ 0.7256 , and for the case in which the risky security goes up, $H_{1}^{+}(1)=4.23, H_{1}^{-}(1)=0, H_{2}^{+}(1)=$ $0, H_{2}^{-}(1)=1.0, V(1)=1.9667$, while for the case in which it goes down, $H_{1}^{+}(0)=1.5562$,
$H_{1}^{-}(0)=0, H_{2}^{+}(0)=0, H_{2}^{-}(0)=0.4688$, and $V(1)=0.3542$. As expected, $V_{b}(0)=0.9355<$ $V_{s}(0)=1.3272$.

## 5. Conclusions

In this paper, we study a discrete time with infinite sample space financial model with penalty costs on short selling positions. Unlike previous works, we consider only one price structure for both short and long positions, with the penalties being withdrawals from the portfolio proportional to the short selling position. Our main result, Theorem 3.1, provides necessary and sufficient conditions for the nonexistence of arbitrages and can be seen as an extension of the standard fundamental theorem of asset pricing. When the penalty costs go to zero our result reduces to that presented in [30]. We also present a sufficient condition for a selffinancing strategy to consistently replicate a contingent claim. For the finite-sample space case, this result leads to an explicit and constructive procedure for obtaining perfect hedging strategies. Some examples are presented to illustrate the possible applications of the model.

## Appendices

We present in this appendix the proof of the main results in Section 3. First, we need some auxiliary results, presented next. In what follows, we recall that $\mathcal{L}_{0}^{d}(\mathcal{G})$ represents the space of $\mathcal{G}$-measurable random vectors with values in $\mathbb{R}^{d}, \mathscr{L}_{0}^{d+}(\mathcal{G}) \subset \mathscr{L}_{0}^{+}(\mathcal{G})$, the space of $\mathcal{G}$-measurable random vectors $Z$ such that $\mathbf{P}(Z \geq 0)$ and, for simplicity, $\Omega_{0}(\mathcal{G})=\Omega_{0}^{1}(\mathcal{G}), \Omega_{0}^{+}(\mathcal{G})=\Omega_{0}^{1+}(\mathcal{G})$. The definition for $\complement_{0}^{d}(\mathcal{F}), \complement_{0}^{d+}(\mathcal{F})$ and $\mathscr{L}_{0}(\mathcal{F}), \mathfrak{L}_{0}^{+}(\mathcal{F})$ is similar.

## A. Some Auxiliary Results

Let $Y \in \mathscr{L}_{0}^{d}(\mathcal{G}), \in \mathscr{L}_{0}^{d}(\mathcal{F})$, and $D \in \mathscr{L}_{0}^{d+}(\mathcal{F})$. We set

$$
\begin{align*}
K_{Y, X} & =\left\{(\alpha \cdot Y, \alpha \cdot X) ; \alpha \in \mathfrak{L}_{0}^{d+}(\mathcal{G})\right\}  \tag{A.1}\\
N_{Y, X, D} & =\left\{(\beta \cdot Y, \beta \cdot(X+D)) ; \beta \in \mathscr{L}_{0}^{d+}(\mathcal{G})\right\},  \tag{A.2}\\
J_{Y, X, D} & =K_{Y, X}-N_{Y, X, D},  \tag{A.3}\\
A_{Y, X, D} & =J_{Y, X, D}-\left\{\{0\} \times \mathscr{L}_{0}^{+}(\mathcal{F})\right\} . \tag{A.4}
\end{align*}
$$

The following propositions will be crucial for the developing of our results and are based on the arguments presented in [47, 48, 52].

Proposition A.1. The following statements are equivalent:
(i) $A_{Y, X, D} \cap\left\{\{0\} \times \mathscr{L}_{0}^{+}(\mathscr{F})\right\}=(0,0)$,
(ii) $A_{Y, X, D} \cap\left\{\{0\} \times \mathscr{\perp}_{0}^{+}(\mathscr{F})\right\}=(0,0)$ and $A_{Y, X, D}$ is closed.

Proof. We have to show that (i) implies that $A_{Y, X, D}$ is closed. For this, we consider sequences $\left\{\alpha^{n}\right\},\left\{\beta^{n}\right\}$ in $\mathscr{L}_{0}^{d+}(\mathcal{G}),\left\{\rho^{n}\right\}$ in $\mathscr{L}_{0}^{+}(\mathscr{F})$, and $x \in \boldsymbol{L}_{0}(\mathcal{G}), \psi \in \mathscr{L}_{0}(\mathcal{F})$ such that a.s.,

$$
\begin{gather*}
\lim _{n \rightarrow \infty}\left(\alpha^{n}-\beta^{n}\right) \cdot Y=X \\
\lim _{n \rightarrow \infty}\left\{\left(\alpha^{n}-\beta^{n}\right) \cdot X-\beta^{n} \cdot D-\rho^{n}\right\}=\psi . \tag{A.5}
\end{gather*}
$$

If we can find $\tilde{\alpha}$ and $\tilde{\beta}$ in $\mathscr{L}_{0}^{d+}(\mathcal{G})$ such that $\mathbf{P}$-a.s.

$$
\begin{gather*}
(\tilde{\alpha}-\tilde{\beta}) \cdot Y=X,  \tag{A.6}\\
(\tilde{\alpha}-\tilde{\beta}) \cdot X-\tilde{\beta} \cdot D \geq \psi, \tag{A.7}
\end{gather*}
$$

then the result is proved since in this case, setting $\rho=(\tilde{\alpha}-\tilde{\beta}) \cdot X-\widetilde{\beta} \cdot D-\psi$, we have from (A.7) that $\rho \in \mathscr{\Omega}_{0}^{+}(\mathcal{F})$ and $\psi=(\widetilde{\alpha}-\widetilde{\beta}) \cdot X-\widetilde{\beta} \cdot D-\rho$, thus $(X, \psi) \in A_{Y, X, D}$. We set $\Omega_{0} \in \mathcal{F}$ such that the limits (A.5) hold and $\left\{\alpha^{n}\right\},\left\{\beta^{n}\right\},\left\{\rho^{n}\right\}, D$ are always nonnegative. It is easy to see that $\left(\Omega_{0}\right)=1$.

We define next $\bar{\alpha}^{n}=\left(\alpha^{n}-\beta^{n}\right)^{+}$and $\bar{\beta}^{n}=\left(\alpha^{n}-\beta^{n}\right)^{-}$so that $\bar{\alpha}^{n}-\bar{\beta}^{n}=\alpha^{n}-\beta^{n}, \bar{\alpha}_{i}^{n}{ }_{\beta}^{n}=0$, $i=1, \ldots, d,\left\|\bar{\alpha}^{n}-\bar{\beta}^{n}\right\|^{2}=\left\|\bar{\alpha}^{n}\right\|^{2}+\left\|\bar{\beta}^{n}\right\|^{2}$, and on $\Omega_{0}$ that

$$
\begin{gather*}
0 \leq \bar{\alpha}^{n} \leq \alpha^{n}, \quad 0 \leq \bar{\beta}^{n} \leq \beta^{n}, \\
\left(\bar{\alpha}^{n}-\bar{\beta}^{n}\right) \cdot X-\bar{\beta}^{n} \cdot D \geq\left(\alpha^{n}-\beta^{n}\right) \cdot X-\beta^{n} \cdot D . \tag{A.8}
\end{gather*}
$$

Set $\varsigma=\liminf _{n \rightarrow \infty}\left\|\alpha^{n}-\beta^{n}\right\|=\liminf _{n \rightarrow \infty}\left(\left\|\bar{\alpha}^{n}\right\|^{2}+\left\|\bar{\beta}^{n}\right\|^{2}\right)^{1 / 2}$ and $\Omega_{1}=\left\{\omega \in \Omega_{0} ; \varsigma(\omega)<\right.$ $\infty\}$. From Lemma 2 of [52], we can find subsequences $\left\{\tilde{\alpha}^{k}\right\},\left\{\tilde{\beta}^{k}\right\}$ of, respectively, $\left\{\alpha^{n}\right\},\left\{\beta^{n}\right\}$ such that on $\Omega_{1}, \lim _{k \rightarrow \infty} \tilde{\alpha}^{k}=\tilde{\alpha}$ and $\lim _{k \rightarrow \infty} \widetilde{\beta}^{k}=\tilde{\beta}$ for some $\tilde{\alpha}, \tilde{\beta}$ in $\mathscr{\Omega}_{0}^{d+}(\mathcal{G})$. Set $\left\{\tilde{\rho}^{k}\right\}$ the corresponding subsequence of $\left\{\rho^{n}\right\}$. It follows that on $\Omega_{1}, \lim _{k \rightarrow \infty}\left(\tilde{\alpha}^{k}-\widetilde{\beta}^{k}\right) \cdot Y=(\tilde{\alpha}-\widetilde{\beta}) \cdot Y=x$ from (A.5), and (20),

$$
\begin{align*}
(\tilde{\alpha}-\tilde{\beta}) \cdot X-\tilde{\beta} \cdot D & =\lim _{k \rightarrow \infty}\left\{\left(\tilde{\alpha}^{k}-\tilde{\beta}^{k}\right) \cdot X-\tilde{\beta}^{k} \cdot D\right\} \\
& \geq \liminf _{k \rightarrow \infty}\left\{\left(\tilde{\alpha}^{k}-\tilde{\beta}^{k}\right) \cdot X-\tilde{\beta}^{k} \cdot D-\tilde{\rho}^{k}\right\} \geq \psi . \tag{A.9}
\end{align*}
$$

If $\mathbf{P}\left(\Omega_{1}\right)=1$, then from (A.6) and (A.7) the result is proved. Otherwise, we define $\Omega_{2}=\{\omega \in$ $\left.\Omega_{0} ; \varsigma(\omega)=\infty\right\}$. As in [47, 48], we form partitions of $\Omega$, and argue on each separate partition as an autonomous space, considering the appropriate restrictions of the random vectors and traces of the $\sigma$-algebras.

On $\Omega_{2}$, we define $g^{n}=\left\|\bar{\alpha}^{n}-\bar{\beta}^{n}\right\|^{\oplus} \bar{\alpha}^{n}, f^{n}=\left\|\bar{\alpha}^{n}-\bar{\beta}^{n}\right\|^{\oplus} \bar{\beta}^{n}$, and $v^{n}=\left\|\bar{\alpha}^{n}-\bar{\beta}^{n}\right\|^{\oplus} \rho^{n}$. From (A.5), (A.8), it follows that on $\Omega_{2}$,

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left(g^{n}-f^{n}\right) \cdot Y=0 \\
& \liminf _{n \rightarrow \infty}\left\{\left(g^{n}-f^{n}\right) \cdot X-f^{n} \cdot D-v^{n}\right\} \geq \lim _{n \rightarrow \infty}\left\|\alpha^{n}-\beta^{n}\right\|^{\oplus}\left\{\left(\alpha^{n}-\beta^{n}\right) \cdot X-\beta^{n} \cdot D-\rho^{n}\right\}=0 \tag{A.10}
\end{align*}
$$

Since on $\Omega_{2},\left\|g^{n}\right\| \leq 1$ and $\left\|f^{n}\right\| \leq 1$, we have again from Lemma 2 of [52] that we can find convergent subsequences $\left\{\tilde{g}^{k}\right\},\left\{\tilde{f}^{k}\right\}$ of, respectively, $\left\{g^{n}\right\},\left\{f^{n}\right\}$, with limits, respectively, $\tilde{g} \geq 0$ and $\tilde{f} \geq 0$. We denote by $\left\{\tilde{\alpha}^{k}\right\},\left\{\widetilde{\beta}^{k}\right\},\left\{\tilde{\rho}^{k}\right\}$ the corresponding subsequences of $\left\{\bar{\alpha}^{n}\right\},\left\{\bar{\beta}^{n}\right\}$, $\left\{\rho^{n}\right\}$, obtained from the association with $\left\{g^{n}\right\}$ and $\left\{f^{n}\right\}$. Since for each $i=1, \ldots, d, g_{i}^{n} f_{i}^{n}=0$, it follows that for each $i=1, \ldots, d$,

$$
\begin{equation*}
\tilde{g}_{i} \tilde{f}_{i}=0 \tag{A.11}
\end{equation*}
$$

From (A.10) we have, that on $\Omega_{2}$,

$$
\begin{align*}
(\tilde{g}-\tilde{f}) \cdot Y & =\lim _{k \rightarrow \infty}\left(\tilde{g}^{k}-\tilde{f}^{k}\right) \cdot Y=0  \tag{A.12}\\
(\tilde{g}-\tilde{f}) \cdot X-\tilde{f} \cdot D & =\lim _{k \rightarrow \infty}\left\{\left(\tilde{g}^{k}-\tilde{f}^{k}\right) \cdot X-\tilde{f}^{k} \cdot D\right\} \geq 0 \tag{A.13}
\end{align*}
$$

and from (i), it follows that (A.12) and (A.13) imply that

$$
\begin{equation*}
(\tilde{g}-\tilde{f}) \cdot X-\tilde{f} \cdot D=0 \tag{A.14}
\end{equation*}
$$

We also have on $\Omega_{2}$ that

$$
\begin{equation*}
1=\|\tilde{g}-\tilde{f}\|^{2}=\|\tilde{g}\|^{2}+\|\tilde{f}\|^{2} \tag{A.15}
\end{equation*}
$$

We proceed now by applying induction on $d$. Suppose first that $d=1$. We can find a partition of $\Omega_{2}$ into 2 disjoint sets, defined by $\Omega_{2}^{\tilde{g}}=\left\{\omega \in \Omega_{2} ; \widetilde{g}(\omega)>0\right\}$, and $\Omega_{2}^{\tilde{f}}=\left\{\omega \in \Omega_{2} ; \tilde{f}(\omega)>0\right\}$. From (A.11) and (A.15) we have that indeed $\Omega_{2}^{\tilde{g}}$ and $\Omega_{2}^{\tilde{f}}$ form a disjoint partition of $\Omega_{2}$. From (A.12) and (A.14) we have that on $\Omega_{2}^{\tilde{g}}$ (recalling that in this case $\tilde{f}=0$ ), $Y=0$ and $X=0$, and that $\lim _{k \rightarrow \infty} \widetilde{\beta}^{k}=0\left(\right.$ since $\tilde{\alpha}^{k} \tilde{\beta}^{k}=0$ and $\left.\tilde{g}>0\right)$. This implies that on $\Omega_{2}^{\tilde{g}}, \lim _{k \rightarrow \infty}\left(\tilde{\alpha}^{k}-\tilde{\beta}^{k}\right) Y=0$ and as in (A.9), $0=\lim _{k \rightarrow \infty}\left(\left(\widetilde{\alpha}^{k}-\widetilde{\beta}^{k}\right) X-\widetilde{\beta}^{k} D\right) \geq \liminf _{k \rightarrow \infty}\left(\left(\widetilde{\alpha}^{k}-\widetilde{\beta}^{k}\right) X-\widetilde{\beta}^{k} D-\tilde{\rho}^{k}\right) \geq \psi$, and (A.6), (A.7) hold with $\tilde{\alpha}=0, \tilde{\beta}=0$. Similarly, on $\Omega_{2}^{\tilde{f}}, Y=0, X+D=0$, and $\lim _{k \rightarrow \infty} \tilde{\alpha}^{k}=0$, so that again (A.6), (A.7) hold with $\tilde{\alpha}=0, \tilde{\beta}=0$. This completes the proof for $d=1$.

Suppose now that the equivalence between (i) and (ii) holds for $d-1$, and that (i) holds for. Define the $2 d$ disjoint sets

$$
\begin{align*}
& \Omega_{2 i}^{\tilde{g}}=\left\{\omega \in \Omega_{2} ; \tilde{g}_{j}(\omega)=0, \tilde{f}_{j}(\omega)=0, j \leq i-1, \tilde{g}_{i}(\omega)>0\right\} \\
& \Omega_{2 i}^{\tilde{f}}=\left\{\omega \in \Omega_{2} ; \tilde{g}_{j}(\omega)=0, \tilde{f}_{j}(\omega)=0, j \leq i-1, \widetilde{g}_{i}(\omega)=0, \tilde{f}_{i}(\omega)>0\right\} . \tag{A.16}
\end{align*}
$$

From (A.11) and (A.15), we have that indeed $\Omega_{2 i}^{\tilde{g}}$ and $\Omega_{2 i}^{\tilde{f}}, i=1, \ldots, d$, form a disjoint partition of $\Omega_{2}$. For $i$ fixed, we will consider first a disjoint partition of $\Omega_{2 i}^{\tilde{g}}$. Consider all subsets, indexed by $s$, formed as $v_{s}^{i} \subseteq\{i+1, \ldots, d\}, u_{s}^{i} \subseteq\{i+1, \ldots, d\}$, with $v_{s}^{i} \cap u_{s}^{i}=\emptyset$. Write $w_{s}^{i}=\{i+1, \ldots, d\}-$ $\left(v_{s}^{i} \cup u_{s}^{i}\right)$, and consider a disjoint partition of $\Omega_{2 i}^{\tilde{g}}$ formed by the sets:

$$
\begin{equation*}
\Omega_{2 i v_{s}^{i} u_{s}^{i}}^{\tilde{\tilde{g}}}=\left\{\omega \in \Omega_{2 i}^{\tilde{g}} ; \tilde{g}_{j}(\omega)>0, j \in v_{s}^{i}, \quad \tilde{f}_{r}(\omega)>0, r \in u_{s}^{i}, \tilde{g}_{m}(\omega)=0, \tilde{f}_{m}(\omega)=0, m \in w_{s}^{i}\right\} \tag{A.17}
\end{equation*}
$$

We fix now $v_{s}^{i}, u_{s}^{i}$ and for notational simplicity, we write $v=v_{s}^{i}, u=u_{s}^{i}, w=w_{s}^{i}, \Omega^{\prime}=\Omega_{2 i v_{s}^{i} u_{s}^{i}}^{\tilde{g}}$, and $\mathcal{F}^{\prime}, \mathcal{G}^{\prime}$, respectively, the corresponding trace of the $\sigma$-algebras $\mathcal{F}, \mathcal{G}$ on $\Omega^{\prime}$. Let us consider that $\mathbf{P}\left(\Omega^{\prime}\right)>0$ (otherwise, it could be discarded) and write $\mathbf{P}^{\prime}(\cdot)=\mathbf{P}(\cdot) / \mathbf{P}\left(\Omega^{\prime}\right)$. On the set $\Omega^{\prime}$, we have from (A.12) and (A.14) that

$$
\begin{align*}
& Y_{i}=-\frac{1}{\widetilde{g}_{i}}\left\{\sum_{j \in v} \widetilde{g}_{j} Y_{j}-\sum_{j \in u} \tilde{f}_{j} Y_{j}\right\},  \tag{A.18}\\
& X_{i}=-\frac{1}{\widetilde{g}_{i}}\left\{\sum_{j \in v} \widetilde{g}_{j} X_{j}-\sum_{j \in u} \tilde{f}_{j}\left(X_{j}+D_{j}\right)\right\} . \tag{A.19}
\end{align*}
$$

Set the $d$-1-dimensional random vectors $Y^{\prime}, X^{\prime}, D^{\prime}$ as follows: for $j=1, \ldots, i-1, Y_{j}^{\prime}=Y_{j}$, $X_{j}^{\prime}=X_{j}$ and $D_{j}^{\prime}=D_{j}$, for $j=i+1, \ldots, d, Y_{j-1}^{\prime}=Y_{j}$, and

$$
\begin{align*}
& X_{j-1}^{\prime}= \begin{cases}X_{j}+D_{j}, & j \in u, \\
X_{j}, & j \notin u, j \geq i+1,\end{cases}  \tag{A.20}\\
& D_{j-1}^{\prime}= \begin{cases}0, & j \in u \cup v, \\
D_{j}, & j \notin u \cup v, j \geq i+1 .\end{cases}
\end{align*}
$$

For $\ell=1,2, \ldots$, define

$$
\begin{equation*}
\tau^{\ell}=\inf \left\{k ; \tilde{g}_{j}^{k}>\tilde{g}_{j}\left(1-\frac{1}{\ell}\right), \forall j \in v \cup\{i\}, \tilde{f}_{r}^{k}>\tilde{g}_{r}\left(1-\frac{1}{\ell}\right), \forall r \in u\right\} \tag{A.21}
\end{equation*}
$$

On $\Omega^{\prime}$, we have that $\tau^{\ell}<\infty, \tilde{f}_{j}^{\tau^{\ell}}=0$ for all $j \in v \cup\{i\}$ and $\widetilde{g}_{r}^{\tau^{\ell}}=0$ for all $r \in u$, and consequently $\widetilde{\beta}_{j}^{\tau^{e}}=0$ for all $j \in v \cup\{i\}$ and $\widetilde{\alpha}_{r}^{\tau^{l}}=0$ for all $r \in u$. Define $\widehat{\alpha}^{\ell}=\widetilde{\alpha}^{\tau^{e}}, \hat{\beta}^{\ell}=\widetilde{\beta}^{\tau^{e}}$, and $\hat{\rho}^{\ell}=\tilde{\rho}^{\tau^{\ell}}$. From (A.19), we obtain that

$$
\begin{equation*}
\left(\widehat{\alpha}^{\ell}-\widehat{\beta}^{\ell}\right) \cdot X-\widehat{\beta}^{\ell} \cdot D=\left(\delta^{\ell}-\varepsilon^{\ell}\right) \cdot X^{\prime}-\varepsilon^{\ell} \cdot D^{\prime}, \tag{A.22}
\end{equation*}
$$

where for $j=1, \ldots, i-1, \delta_{j}^{\ell}=\widehat{\alpha}_{j}^{l}, \varepsilon_{j}^{\ell}=\widehat{\beta}_{j}^{\ell}$, for $j \in w, \delta_{j-1}^{\ell}=\widehat{\alpha}_{j}^{\ell}, \varepsilon_{j-1}^{\ell}=\hat{\beta}_{j}^{\ell}$, for $j \in v$, $\delta_{j-1}^{\ell}=\left(\widehat{\alpha}_{j}^{\ell}-\left(\widetilde{g}_{j} / \widetilde{g}_{i}\right) \widehat{\alpha}_{i}^{\ell}\right)^{+}, \varepsilon_{j-1}^{\ell}=\left(\widehat{\alpha}_{j}^{\ell}-\left(\widetilde{g}_{j} / \widetilde{g}_{i}\right) \widehat{\alpha}_{i}^{\ell}\right)^{-}$, and for $j \in u, \delta_{j-1}^{\ell}=\left(-\widehat{\beta}_{j}^{\ell}+\left(\tilde{f}_{j} / \widetilde{g}_{i}\right) \widehat{\alpha}_{i}^{\ell}\right)^{+}$, $\varepsilon_{j-1}^{\ell}=\left(-\widehat{\beta}_{j}^{\ell}+\left(\tilde{f}_{j} / \widetilde{g}_{i}\right) \widehat{\alpha}_{i}^{\ell}\right)^{-}$. We notice that $\delta^{\ell}$ and $\varepsilon^{\ell}$ belong to $\mathcal{L}_{0}^{d-1+}\left(\Omega^{\prime}, \mathcal{G}^{\prime}, \mathbf{P}^{\prime}\right)$. Similarly, from (A.18), we have that

$$
\begin{equation*}
\left(\hat{\alpha}^{l}-\widehat{\beta}^{l}\right) \cdot Y=\left(\delta^{\ell}-\varepsilon^{\ell}\right) \cdot Y^{\prime} . \tag{A.23}
\end{equation*}
$$

Next, we show that $J_{Y^{\prime}, X^{\prime}, D^{\prime}} \subset J_{Y, X, D}$. For this, we establish a mapping $(\alpha, \beta)=F(\delta, \varepsilon)$ such that $\alpha$ and $\beta$ belong to $\mathscr{L}_{0}^{d+}\left(\Omega^{\prime}, \mathcal{G}^{\prime}, \mathbf{P}^{\prime}\right)$ whenever $\delta$ and $\varepsilon$ belong to $\mathscr{L}_{0}^{d-1+}\left(\Omega^{\prime}, \mathcal{G}^{\prime}, \mathbf{P}^{\prime}\right)$, and that

$$
\begin{align*}
(\delta-\varepsilon) \cdot Y^{\prime} & =(\alpha-\beta) \cdot Y  \tag{A.24}\\
(\delta-\varepsilon) \cdot X^{\prime}-\varepsilon \cdot D^{\prime} & =(\alpha-\beta) \cdot X-\beta \cdot D . \tag{A.25}
\end{align*}
$$

Indeed, setting $\beta_{i}=0$

$$
\begin{equation*}
\alpha_{i}=\max \left\{0,\left\{\left(\varepsilon_{j-1}-\delta_{j-1}\right) \frac{\widetilde{g}_{i}}{\widetilde{g}_{j}} \text { for } j \in v\right\},\left\{\left(\delta_{r-1}-\varepsilon_{r-1}\right) \frac{\widetilde{g}_{i}}{\tilde{f}_{r}} \text { for } r \in u\right\}\right\}, \tag{A.26}
\end{equation*}
$$

and for $j=1, \ldots, i-1, \alpha_{j}=\delta_{j}, \beta_{j}=\varepsilon_{j}$, for $j \in w, \alpha_{j}=\delta_{j-1}, \beta_{j}=\varepsilon_{j-1}$, for $j \in v, \alpha_{j}=$ $\left(\delta_{j-1}-\varepsilon_{j-1}\right)+\left(\widetilde{g}_{j} / \widetilde{g}_{i}\right) \alpha_{i}, \beta_{j}=0$, and for $j \in u, \alpha_{j}=0, \beta_{j}=-\left(\delta_{j-1}-\varepsilon_{j-1}\right)+\left(\tilde{f}_{j} / \widetilde{g}_{i}\right) \alpha_{i}$, we get from (A.26) that $\alpha, \beta$ belong to $\mathcal{L}_{0}^{d+}\left(\Omega^{\prime}, \mathcal{G}^{\prime}, \mathbf{P}^{\prime}\right)$ whenever $\delta$ and $\varepsilon$ belong to $\mathcal{L}_{0}^{d-1+}\left(\Omega^{\prime}, \mathcal{G}^{\prime}, \mathbf{P}^{\prime}\right)$, and from (A.18) and (A.19), we get that (A.24) and (A.25) are satisfied, yielding the desired inclusion. From this and (i), we can conclude that $A_{Y^{\prime}, X^{\prime}, D^{\prime}} \cap\left\{\{0\} \times \mathfrak{\Omega}_{0}^{+}\left(\Omega^{\prime}, \mathscr{F}^{\prime}, \mathbf{P}^{\prime}\right)\right\}=(0,0)$ and by the induction hypothesis, we get that $A_{Y^{\prime}, X^{\prime}, D^{\prime}}$ is closed. Therefore, for some $\tilde{\delta}$ and $\tilde{\varepsilon}$ belonging to $\mathscr{L}_{0}^{d-1+}\left(\Omega^{\prime}, \mathcal{G}^{\prime}, \mathbf{P}^{\prime}\right)$, and some $\tilde{p} \in \mathscr{\perp}_{0}^{+}\left(\Omega^{\prime}, \mathcal{F}^{\prime}, \mathbf{P}^{\prime}\right)$, we have from (A.22) and (A.23), and taking $(\widetilde{\alpha}, \widetilde{\beta})=F(\tilde{\delta}, \tilde{\varepsilon})$, that

$$
\begin{align*}
X & =\lim _{\ell \rightarrow \infty}\left(\widehat{\alpha}^{\ell}-\widehat{\beta}^{\ell}\right) \cdot Y=\lim _{\ell \rightarrow \infty}\left(\delta^{\ell}-\varepsilon^{\ell}\right) \cdot Y^{\prime}=(\delta-\varepsilon) \cdot Y^{\prime} \\
& =\left(\tilde{\delta}^{\ell}-\tilde{\varepsilon}^{\ell}\right) \cdot Y^{\prime}=(\tilde{\alpha}-\tilde{\beta}) \cdot Y, \\
\psi & =\lim _{\ell \rightarrow \infty}\left(\left(\widehat{\alpha}^{\ell}-\widehat{\beta}^{\ell}\right) \cdot X-\widehat{\beta}^{\ell} \cdot D-\widehat{\rho}^{\ell}\right) \\
& =\lim _{\ell \rightarrow \infty}\left(\left(\delta^{\ell}-\varepsilon^{\ell}\right) \cdot X^{\prime}-\varepsilon^{\ell} \cdot D^{\prime}-\widehat{\rho}^{\ell}\right) \\
& =(\widetilde{\delta}-\tilde{\varepsilon}) \cdot X^{\prime}-\tilde{\varepsilon}^{\ell} \cdot D^{\prime}-\tilde{p}=(\tilde{\alpha}-\tilde{\beta}) \cdot X-\tilde{\beta} \cdot D^{\prime}-\tilde{p}, \tag{A.27}
\end{align*}
$$

showing that (A.6) and (A.7) are satisfied on $\Omega^{\prime}$.

The proof for the sets $\Omega_{2 i v_{s}^{i} u_{s}^{i}}^{\tilde{f}}$ goes along the same lines, bearing in mind that (A.18) and (A.19) are replaced, respectively, by $Y_{i}=1 / \tilde{f}_{i}\left\{\sum_{j \in v} \tilde{g}_{j} Y_{j}-\sum_{j \in u} \tilde{f}_{j} Y_{j}\right\}$ and $X_{i}+D_{i}=$ $\left(1 / \tilde{f}_{i}\right)\left\{\sum_{j \in v} \tilde{g}_{j} X_{j}-\sum_{j \in u} \tilde{f}_{j}\left(X_{j}+D_{j}\right)\right\}$, and that the mapping $F(\delta, \varepsilon)$ is defined such that instead of (A.26), we take $\alpha_{i}=0$ and $\beta_{i}=\max \left\{0,\left\{\left(\varepsilon_{j-1}-\delta_{j-1}\right)\left(\tilde{f}_{i} / \tilde{g}_{j}\right)\right.\right.$ for $\left.j \in v\right\},\left\{\left(\delta_{r-1}-\varepsilon_{r-1}\right)\left(\tilde{f}_{i} / \tilde{f}_{r}\right)\right.$ for $r \in u\}\}$. This completes the proof of the proposition.

In the next proposition let us consider that $Y \in \mathscr{L}_{0}^{d}(\mathcal{G})$ is such that $Y_{1}=1$ and $X \in$ $\mathscr{L}_{0}^{d}(\mathscr{F})$ is such that $X_{1}=0$. Again we suppose that $\in \mathscr{L}_{0}^{d+}(\mathscr{F})$.

Proposition A.2. The following statements are equivalent:
(i) for any $\alpha, \beta$ in $\perp_{0}^{d+}(\mathcal{G})$, one has a.s. that

$$
\left.\begin{array}{l}
(\alpha-\beta) \cdot Y \leq 0  \tag{A.28}\\
(\alpha-\beta) \cdot X-\beta \cdot D \geq 0
\end{array}\right\} \Longrightarrow(\alpha-\beta) \cdot X-\beta \cdot D=0
$$

(ii) there exists $r \in \mathcal{L}_{\infty}^{+}(\mathcal{G})$ and a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that $0<R=d \mathbf{Q} / d \mathbf{P} \in$ $\perp_{\infty}^{+}(\mathcal{F}), Y, X, D$ are integrable with respect to $\mathbf{Q}$, and

$$
\begin{equation*}
E_{\mathbf{Q}}(X \mid \mathcal{G}) \leq r Y \leq E_{\mathbf{Q}}(X+D \mid \mathcal{G}) \quad \text { a.s. } \tag{A.29}
\end{equation*}
$$

Proof. First, we note that (i) is equivalent to $A_{Y, X, D} \cap\left\{\{0\} \times \mathscr{\perp}_{0}^{+}(\mathcal{F})\right\}=(0,0)$. Indeed, if (i) holds, then clearly $A_{Y, X, D} \cap\left\{\{0\} \times \mathscr{L}_{0}^{+}(\mathscr{F})\right\}=(0,0)$. Conversely, suppose that for some $\alpha, \beta$ in $\mathscr{L}_{0}^{d+}(\mathcal{G})$, $(\alpha-\beta) \cdot Y \leq 0$ and $(\alpha-\beta) \cdot X-\beta \cdot D \geq 0$ a.s. Then, by taking $\widetilde{\alpha}_{1}=\alpha_{1}-(\alpha-\beta) \cdot Y \geq 0, \tilde{\alpha}_{i}=\alpha_{i}$, and recalling that $Y_{1}=1$ and $X_{1}=0$ we get that $(\tilde{\alpha}-\beta) \cdot Y=0$ and $(\tilde{\alpha}-\beta) \cdot X-\beta \cdot D=(\alpha-\beta) \cdot X-\beta \cdot D \geq 0$ a.s., which implies that $(\alpha-\beta) \cdot X-\beta \cdot D=0$ a.s.

Let us show first that (ii) implies (i). Consider $\alpha, \beta$ in $\mathcal{L}_{0}^{d+}(\mathcal{G})$ such that $(\alpha-\beta) \cdot Y=0$ and $(\alpha-\beta) \cdot X-\beta D \geq 0$ a.s. From (A.29), we get that a.s.,

$$
\begin{align*}
0 & \leq E_{\mathbf{Q}}((\alpha-\beta) \cdot X-\beta \cdot D \mid \mathcal{G})=\alpha \cdot E_{\mathbf{Q}}(X \mid \mathcal{G})-\beta \cdot E_{\mathbf{Q}}((X+D) \mid \mathcal{G}) \\
& \leq r(\alpha-\beta) \cdot Y=0 \tag{A.30}
\end{align*}
$$

and thus $E_{\mathbf{Q}}((\alpha-\beta) \cdot X-\beta \cdot D)=0$. From the fact that $\mathbf{Q} \sim \mathbf{P}$, we get that $(\alpha-\beta) \cdot X-\beta \cdot D \geq 0$ a.s., which implies that $(\alpha-\beta) \cdot X-\beta \cdot D=0$ a.s. and again, from $\mathbf{Q} \sim \mathbf{P}$, that $(\alpha-\beta) \cdot X-\beta \cdot D=0$ a.s., showing that (ii) implies (i).

Next, we show that (i) implies (ii). In what follows, we recall that for a complete probability space $\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right)$, we denote by $\mathcal{L}_{1}^{d}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right)$ the space of integrable $\mathcal{G}$-measurable random variables with values in $\mathbb{R}^{d}$, and $Z \in \mathcal{L}_{1}^{d+}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right)$ if $Z \in \mathcal{L}_{1}^{d}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right)$ and $\mathbf{P}^{\prime}(Z \geq 0)=1$. We remind that if $\widetilde{\mathbf{P}} \ll \mathbf{P}$ then $\widetilde{\mathbf{P}} \sim \mathbf{P}$ if and only if $d \widetilde{\mathbf{P}} / d \mathbf{P}>0$ a.s., and that for any random variable $\eta$ there exists an equivalent probability measure $\widetilde{\mathbf{P}}$ with bounded density such that $\eta$ is integrable under $\widetilde{\mathbf{P}}$ (see [52]). Consider a change of probability measure $d \mathbf{P}^{\prime}=p d \mathbf{P}$ with $0<p \in \mathcal{L}_{\infty}(\mathcal{F})$ such that $Y \in \mathcal{L}_{1}^{d}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right), X \in \mathcal{L}_{1}^{d}\left(\Omega, \mathcal{F}, \mathbf{P}^{\prime}\right)$, and $D \in \mathcal{L}_{1}^{d+}\left(\Omega, \mathscr{F}, \mathbf{P}^{\prime}\right)$. From (i) (which is invariant under equivalent change of probability) and Proposition A.1, we get that $V:=A_{Y, X, D} \cap\left(\mathscr{L}_{1}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right) \times \mathfrak{L}_{1}\left(\Omega, \mathcal{F}, \mathbf{P}^{\prime}\right)\right)$ is a closed convex set of $\mathscr{L}_{1}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right) \times \mathfrak{L}_{1}\left(\Omega, \mathcal{F}, \mathbf{P}^{\prime}\right)$,
and that $V \cap\{0\} \times \mathscr{L}_{1}^{+}\left(\Omega, \mathcal{F}, \mathbf{P}^{\prime}\right)=(0,0)$. Therefore, for any $A \in \mathcal{F}$, with $(A)>0,\left(0,1_{A}\right) \in$ $\{0\} \times \Omega_{1}^{+}\left(\Omega, \mathcal{F}, \mathbf{P}^{\prime}\right)$, and thus does not belong to the set $V$. By the Hahn-Banach Theorem, $\left(0,1_{A}\right)$ can be strongly separated from $V$ by a nonzero linear continuous functional, so that there exists $r_{A} \in \Omega_{\infty}(\mathcal{G}), Z_{A} \in \complement_{\infty}(\mathcal{F}),\left(r_{A}, Z_{A}\right) \neq(0,0)$, such that

$$
\begin{equation*}
\sup _{(\varphi, \theta) \in V} E^{\prime}\left(r_{A} \varphi+Z_{A} \theta\right)<E^{\prime}\left(Z_{A} 1_{A}\right) \tag{A.31}
\end{equation*}
$$

where $E^{\prime}$ denotes the expectation with respect to $\mathbf{P}^{\prime}$. We note that $\varphi \in \mathcal{L}_{1}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right)$ and $\theta \in$ $\mathcal{L}_{1}\left(\Omega, \mathcal{F}, \mathbf{P}^{\prime}\right)$ are such that $\varphi=(\alpha-\beta) \cdot \gamma$ and $\theta=(\alpha-\beta) \cdot X-\beta \cdot D-\rho$, with, $\beta \in \mathcal{L}_{0}^{d+}(\mathcal{G})$ and $\rho \in \mathcal{L}_{0}^{+}(\mathcal{G})$. By taking $\varphi=0$ and $\theta=-n 1_{B}$ for any $B \in \mathcal{F}$ and positive integer $n$ (just take $\alpha=\beta=0$ and $\left.\rho=n 1_{B}\right)$, we have from (A.31) that $-n E^{\prime}\left(Z_{A} 1_{B}\right)<E^{\prime}\left(Z_{A} 1_{A}\right)$ which implies that $Z_{A} \geq 0$ a.s. Normalizing, we assume that $Z_{A} \leq 1$. Similarly by taking $\varphi=n 1_{B}$ for any $B \in \mathcal{G}$ and $\theta=0$ (this is possible since $Y_{1}=1$ and $X_{1}=0$, just take $\alpha_{1}=n 1_{B}, \alpha_{i}=0$ for $i=2, \ldots, d$, and $\beta=0, \rho=0$ ), we get from the same reasons as before that $-r_{A} \geq 0$. Considering now $\varphi=\alpha \cdot Y$ and $\theta=\alpha \cdot X$ we have from (A.31) and the same arguments as before that for every $\alpha \in \mathcal{L}_{\infty}^{d+}(\mathcal{G}), E^{\prime}\left(\alpha \cdot\left(r_{A} Y+Z_{A} X\right)\right) \leq 0$, which implies that a.s.,

$$
\begin{equation*}
E^{\prime}\left(Z_{A} X \mid G\right) \leq-r_{A} Y \tag{A.32}
\end{equation*}
$$

Similarly, considering now $\varphi=-\beta \cdot Y$ and $\theta=-\beta \cdot(X+D)$, we have from (A.31) and the same arguments as before that for every $\beta \in \mathcal{L}_{\infty}^{d+}(\mathcal{G}), E^{\prime}\left(\beta \cdot\left(r_{A} Y+Z_{A}(X+D)\right)\right) \geq 0$, which implies that a.s.

$$
\begin{equation*}
E^{\prime}\left(Z_{A}(X+D) \mid G\right) \geq-r_{A} Y \tag{A.33}
\end{equation*}
$$

Consider the family of measures:

$$
\begin{equation*}
Q=\left\{Q_{A} ; d Q_{A}=Z_{A} d \mathbf{P}^{\prime}, \forall A \in \mathscr{F} \text { such that } \mathbf{P}^{\prime}(A)>0\right\} \tag{A.34}
\end{equation*}
$$

Clearly, $Q$ is dominated by $\mathbf{P}^{\prime}$ (i.e., $Q_{A} \ll \mathbf{P}^{\prime}$ for every $A \in \mathscr{F}$ such that $\left.\mathbf{P}^{\prime}(A)>0\right)$. From the Halmos-Savage Theorem, $Q$ contains a countable equivalent family $\tilde{Q}=\left\{Q_{A_{k}} ; k \in \mathbb{I}\right\}$, where $\mathbb{I}$ is a countable set. Define $\Gamma=\left\{\omega ; Z_{A_{k}}(\omega)=0\right.$ for every $\left.k \in \mathbb{I}\right\}$. Since $Q$ and $\tilde{Q}$ are equivalent and $Q_{A_{k}}(\Gamma)=0$ for every $k \in \mathbb{I}$, it follows that $Q_{A}(\Gamma)=0$ for every $A \in \mathcal{F}$ such that $\mathbf{P}^{\prime}(A)>0$. We show next that $\mathbf{P}^{\prime}(\Gamma)=0$. Suppose by contradiction that $\mathbf{P}^{\prime}(\Gamma)>0$, so that $Q_{\Gamma}(\Gamma)=E^{\prime}\left(Z_{\Gamma} 1_{\Gamma}\right)=0$. From (A.31), $\sup _{(\varphi, \theta) \in V} E^{\prime}\left(r_{\Gamma} \varphi+Z_{\Gamma} \theta\right)<0$ which is clearly an absurd (just take $=0, \theta=0$ ). This shows that $\mathbf{P}^{\prime}(\Gamma)=0$. Define

$$
\begin{align*}
& Z=C \sum_{k \in \mathbb{I}} \frac{Z_{A_{k}}}{2^{k}} \in \perp_{\infty}^{+}\left(\Omega, \mathcal{F}, \mathbf{P}^{\prime}\right) \\
& r=C \sum_{k \in \mathbb{I}} \frac{\left(-r_{A_{k}}\right)}{2^{k}} \in \mathscr{L}_{\infty}^{+}\left(\Omega, \mathcal{G}, \mathbf{P}^{\prime}\right) \tag{A.35}
\end{align*}
$$

where $C=\left(\sum_{k \in \mathbb{I}} E^{\prime}\left(Z_{A_{k}}\right) / 2^{k}\right)^{-1}>0$. Note that $\mathbf{P}^{\prime}(Z>0)=1$ since $\mathbf{P}^{\prime}(\Gamma)=0$, and thus $C$ is well defined. Define the probability measure $\mathbf{Q}$ as $=Z d \mathbf{P}^{\prime}$. It is easy to see that $\mathbf{Q} \sim \mathbf{P}^{\prime}$
(since $\mathbf{P}^{\prime}(Z>0)=1$ ). Finally, from (A.32), (A.33), (A.35), and the bounded convergence theorem, we obtain (A.29) with $R=p Z$, completing the proof of the proposition.

## B. Proof of the Main Results

We present next the proof of Theorem 3.1.
Proof. Let us show that (i) implies (ii). The proof of this fact follows from the contrapositive. Suppose we can find $\widehat{H}=\left(\widehat{H}^{+} \widehat{H}^{-}\right)$with $\widehat{H}^{+}, \widehat{H}^{-} \in \mathscr{L}_{0}^{d+}\left(\mathcal{F}_{t}\right)$ such that a.s., $\left(\widehat{H}^{+}-\widehat{H}^{-}\right) \cdot S(t)=0$, $\mathcal{R}=\left(\widehat{H}^{+}-\widehat{H}^{-}\right) \cdot \Delta S(t+1)-\widehat{H}^{-} \cdot D(t+1) \geq 0$ and $\mathbf{P}(\Gamma)>0$, where $\Gamma=\{\omega ; \mathcal{R}(\omega)>0\}$. Then, we can define an arbitrage opportunity $H$ for the multiperiod market as follows: $H(s)=0$ for $s \leq t, H(t+1)(\omega)=\widehat{H}$ for $\omega \in \Gamma, H(t+1)(\omega)=0$ for $\omega \notin \Gamma, H_{1}^{+}(s)(\omega)=\mathcal{R}(\omega)$ for $\omega \in \Gamma$ and $s \geq t+2$, and $H_{i}^{+}(s)(\omega)=0, H_{i}^{-}(s)(\omega)=0$ in all the other situations. Then, clearly $V(s)=0$ for $s=0, \ldots, t-1,(t)=0$, and from (2.5), (2.6), $V(s)(\omega)=\mathcal{R}(\omega)$ for $s=t+1, \ldots, T, \omega \in \Gamma$, and equal to zero for $\omega \notin \Gamma$, showing the result.

Let us show now that (ii) implies (iii). We will show by backward induction on time $t=T, \ldots, 1$ that we can find random variables $0<\widetilde{R}(t) \in \mathcal{L}_{\infty}^{+}\left(\mathcal{F}_{t}\right), 0 \leq \widetilde{r}(t-1) \in \mathcal{L}_{\infty}^{+}\left(\mathcal{F}_{t-1}\right)$, and random vectors $\tilde{X}(t) \in \mathcal{L}_{0}^{d}\left(\mathcal{F}_{t}\right), \tilde{D}(t) \in \mathcal{L}_{0}^{d+}\left(\mathcal{F}_{t}\right)$ that satisfy $E(\|\tilde{X}(t)\| \tilde{R}(t))<\infty$, $E(\|\widetilde{D}(t)\| \widetilde{R}(t))<\infty, E(\|S(t-1)\| \widetilde{R}(t))<\infty$, and a.s., $E\left(\widetilde{R}(t) \mid \mathcal{F}_{t-1}\right)=1$,

$$
\begin{equation*}
E\left(\tilde{X}(t) \widetilde{R}(t) \mid \mathcal{F}_{t-1}\right) \leq \tilde{r}(t-1) S(t-1) \leq E\left(\widetilde{R}(t)(\tilde{X}(t)+\widetilde{D}(t)) \mid \mathcal{F}_{t-1}\right) \tag{B.1}
\end{equation*}
$$

For $t=T$, the result follows from Proposition A. 2 with $\mathcal{F}=\mathcal{F}_{T}, \mathcal{G}=\mathcal{F}_{T-1}, Y=S(T-1)$, $X=\widetilde{X}(T)=\Delta S(T), D=\widetilde{D}(T)=D(T)$, and $\widetilde{R}(T)=R, \widetilde{r}(T)=r$. Suppose the result holds for $t+1$. Define $\widetilde{X}(t)=\Delta S(t) E\left(\widetilde{R}(T) \cdots \widetilde{R}(t+1) \mid \mathcal{F}_{t}\right)$ and $\widetilde{D}(t)=D(t) E\left(\widetilde{R}(T) \cdots \widetilde{R}(t+1) \mid \mathcal{F}_{t}\right)$. Again, the result follows from Proposition A. 2 with $\mathcal{F}=\mathcal{F}_{t}, \mathcal{G}=\mathcal{F}_{t-1}, Y=S(t-1), X=\tilde{X}(t)$, $D=\widetilde{D}(t)$, and $\tilde{R}(t)=R, \tilde{r}(t-1)=r$, completing the induction argument. Set $\tilde{R}(0)=1$, and define for $t=1, \ldots, T,(t-1)=\widetilde{R}(0) \cdots \widetilde{R}(t-1) \widetilde{r}(t-1), d \mathbf{Q}=\widetilde{R}(0) \cdots \widetilde{R}(T) d \mathbf{P}$. It follows that a.s.,

$$
\begin{align*}
E_{\mathbf{Q}}\left(\Delta S(t) \mid \mathcal{F}_{t-1}\right) & =\tilde{R}(0) \cdots \tilde{R}(t-1) E\left(\Delta S(t) \tilde{R}(t) \cdots \tilde{R}(T) \mid \mathcal{F}_{t-1}\right) \\
& =\widetilde{R}(0) \cdots \tilde{R}(t-1) E\left(\Delta S(t) \widetilde{R}(t) E\left(\widetilde{R}(t+1) \cdots \tilde{R}(T) \mid \mathcal{F}_{t}\right) \mid \mathcal{F}_{t-1}\right) \\
& =\widetilde{R}(0) \cdots \tilde{R}(t-1) E\left(\widetilde{R}(t) \tilde{X}(t) \mid \mathcal{F}_{t-1}\right) \\
& \leq \widetilde{R}(0) \cdots \tilde{R}(t-1) \widetilde{r}(t-1) S(t-1)=r(t-1) S(t-1) \tag{B.2}
\end{align*}
$$

and similarly, $E_{\mathbf{Q}}\left((\Delta S(t)+D(t)) \mid \mathscr{F}_{t-1}\right) \geq r(t-1) S(t-1)$, showing (iii).
Finally, let us show that (iii) implies (i). Indeed, from (2.5), (2.6), and (3.2), we have a.s. that

$$
\begin{aligned}
E_{\mathbf{Q}}\left[V(t+1) \mid \mathscr{F}_{t}\right]= & V(t)+H^{+}(t+1) \cdot E_{\mathbf{Q}}\left[\Delta S(t+1) \mid \mathscr{F}_{t}\right] \\
& -H^{-}(t+1) \cdot E_{\mathbf{Q}}\left[\Delta S(t+1)+D(t+1) \mid \mathscr{F}_{t}\right]
\end{aligned}
$$

$$
\begin{align*}
& \leq(1+r(t))\left(H^{+}(t+1)-H^{-}(t+1)\right) \cdot S(t) \\
& =(1+r(t)) V(t) \tag{B.3}
\end{align*}
$$

Suppose by contradiction that an arbitrage opportunity $H$ exists and (iii) holds. Let us denote by $V=(V(0), \ldots, V(T))$ the value process associated to the trading strategy $H$. Then, $V(0)=$ $0, V(T) \geq 0$, and $(V(T))>0$. We have now by backward induction in time that $V(t) \geq 0$ a.s. for all $t=T, \ldots, 0$. Indeed, from the definition of an arbitrage, the result is clearly true for $t=T$. Suppose $V(t) \geq 0$. From (B.3), it follows that $V(t) \geq 0$ (since $r(t) \geq 0)$, showing the desired result. We show now by forward induction in time that, in fact, $V(t)=0$ a.s. for all $t=0, \ldots, T$. Indeed, for $t=0$, the result follows from the definition of an arbitrage. Suppose $V(t)=0$. From (B.3) and recalling that $V(t+1) \geq 0$ we get that $E_{\mathrm{Q}}\left[V(t+1) \mid \mathcal{F}_{t}\right]=0$. Taking the expected value, we obtain that $E_{\mathbf{Q}}[V(t+1)]=0$, which shows that $V(t+1)=0$ a.s., completing the induction argument. In particular, we have that $V(T)=0$, in contradiction with the fact that $E(V(T))>0$, showing the desired result.

Next, we present the proof of Corollary 3.4.
Proof. By backward induction on $t$, the result is true for $t=T$ by assumption. Suppose it holds for $t+1$, that is, a.s.

$$
\begin{equation*}
E_{\mathbf{Q}}\left(\left.\frac{X}{(1+r(T)) \cdots(1+r(t+1))} \right\rvert\, \mathscr{F}_{t+1}\right) \leq V(t+1) \tag{B.4}
\end{equation*}
$$

Then, from (B.3) and (B.4), we have a.s. that

$$
\begin{align*}
& E_{\mathbf{Q}}\left(\left.\frac{X}{(1+r(T)) \cdots(1+r(t+1))} \right\rvert\, \mathcal{F}_{t}\right) \frac{1}{1+r(t)} \\
& \quad=E_{\mathbf{Q}}\left(\left.E_{\mathbf{Q}}\left(\left.\frac{X}{(1+r(T)) \cdots(1+r(t+1))} \right\rvert\, \mathcal{F}_{t+1}\right) \right\rvert\, \mathcal{F}_{t}\right) \frac{1}{1+r(t)} \\
& \quad \leq E_{\mathbf{Q}}\left(V(t+1) \mid \mathcal{F}_{t}\right) \frac{1}{1+r(t)} \leq V(t) \tag{B.5}
\end{align*}
$$

showing (3.4). We apply backward induction on $t$ to show that $\mathbf{P}(\Gamma(t))>0$. For $t=T$, the result is true by assumption. Suppose it holds for $t+1$. Thus we get that $\mathbf{Q}\left(\Gamma(t+1) \mid \mathcal{F}_{t}\right)>0$ a.s. Therefore, we have from (B.3) that a.s.,

$$
\begin{align*}
& E_{\mathbf{Q}}\left(\left.(1+r(t)) V(t)-\frac{X}{(1+r(T)) \cdots(1+r(t+1))} \right\rvert\, \mathcal{F}_{t}\right) \\
& \quad \geq E_{\mathbf{Q}}\left(\left.\left\{V(t+1)-E_{\mathbf{Q}}\left(\left.\frac{X}{(1+r(T)) \cdots(1+r(t+1))} \right\rvert\, \mathcal{F}_{t+1}\right)\right\} 1_{\Gamma(t+1)} \right\rvert\, \mathcal{F}_{t}\right)>0 \tag{B.6}
\end{align*}
$$

which implies, after dividing by $\left(1+r(t)\right.$ that $E_{\mathbf{Q}}\left(V(t)-X /(1+r(T)) \cdots(1+r(t)) \mid \mathcal{F}_{t}\right)>0$ a.s., yielding the desired result.

In what follows, we present the proof of Propositions 3.6 and 3.7.

Proof. We take $r(t)=0$ and $\mathbf{Q} \in \Theta^{a}$ and show that $(\{r(t)\}, \mathbf{Q}) \in \mathcal{K}$. Indeed, recalling that $D(t+1) \geq 0$ we have a.s. that on $\left\{a_{i}(t)=0\right\}$ :

$$
\begin{equation*}
E_{\mathbf{Q}}\left(\Delta S_{i}(t+1) \mid \mathscr{F}_{t}\right)=0 \leq E_{\mathbf{Q}}\left(D_{i}(t+1) \mid \mathscr{F}_{t}\right)=E_{\mathbf{Q}}\left(\Delta S_{i}(\mathrm{t}+1)+D_{i}(t+1) \mid \mathscr{F}_{t}\right) \tag{B.7}
\end{equation*}
$$

and similarly on $\left\{a_{i}(t)=1\right\}$,

$$
\begin{equation*}
E_{\mathbf{Q}}\left(\Delta S_{i}(t+1)+D_{i}(t+1) \mid \mathscr{F}_{t}\right)=0 \geq E_{\mathbf{Q}}\left(\Delta S_{i}(t+1) \mid \mathscr{F}_{t}\right) . \tag{B.8}
\end{equation*}
$$

Proof. Consider $r(t)=0$ and $\mathbf{Q} \in \Theta^{a^{H}}$. As shown in Proposition $3.6,(\{r(t)\}, \mathbf{Q}) \in \mathcal{K}$. From the hypothesis that $H_{i}^{+}(t) H_{i}^{-}(t)=0$ and (2.6), we have a.s. that

$$
\begin{align*}
E_{\mathbf{Q}}\left(V^{H}(t) \mid \mathscr{F}_{t-1}\right)= & V^{H}(t-1)+\sum_{\left\{i \in\left\{a^{H}(t-1)=0\right\}\right\}} H_{i}^{+}(t) E_{\mathbf{Q}}\left(\Delta S_{i}(t) \mid \mathscr{F}_{t-1}\right) \\
& -\sum_{\left\{i \in\left\{a^{H}(t-1)=1\right\}\right\}} H_{i}^{-}(t) E_{\mathbf{Q}}\left(\Delta S_{i}(t)+D_{i}(t) \mid \mathcal{F}_{t-1}\right) \\
= & V^{H}(t-1) \tag{B.9}
\end{align*}
$$

and thus a.s., $V^{H}(t)=E_{\mathbf{Q}}\left(X \mid \mathcal{F}_{t}\right)$ for all $t=0, \ldots, T-1$. From Corollary 3.4, we have the result.

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Research Article

# Least-Mean-Square Receding Horizon Estimation 

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We propose a least-mean-square (LMS) receding horizon (RH) estimator for state estimation. The proposed LMS RH estimator is obtained from the conditional expectation of the estimated state given a finite number of inputs and outputs over the recent finite horizon. Any a priori state information is not required, and existing artificial constraints for easy derivation are not imposed. For a general stochastic discrete-time state space model with both system and measurement noise, the LMS RH estimator is explicitly represented in a closed form. For numerical reliability, the iterative form is presented with forward and backward computations. It is shown through a numerical example that the proposed LMS RH estimator has better robust performance than conventional Kalman estimators when uncertainties exist.

## 1. Introduction

Several criteria have been often employed for the design of optimal estimators. In particular, the mean-square-error criterion is the most popular and has many applications since it offers a simple closed-form solution as well as important geometric and physical interpretations. It is well known that the optimal estimators based on the mean-square-error criterion is obtained from the conditional expectation of the estimated variable given the known measurements.

For state estimation, many trials have been conducted to obtain a receding horizon (RH) estimator based on the mean-square-error criterion. At the current time $k$, the least-mean-square (LMS) RH estimator is to estimate the state $x_{k-h}$ at the time $k-h$ from the inputs


$$
\begin{equation*}
\widehat{x}_{k-h \mid k}=\mathrm{E}\left[x_{k-h} \mid u_{k-N} \cdots u_{k-1}, y_{k-N} \cdots y_{k-1}\right] \tag{1.1}
\end{equation*}
$$



Receding horizon estimator
Figure 1: The structure of the receding horizon estimator ( $D$ is a unit delay component).
where $h$ and $N$, that is, the lag size and the memory size are design parameters to be determined, respectively. Available inputs and outputs are regarded as given conditions and the corresponding conditional expectation of the state at time $k-h$ is obtained as its optimal estimate. Practically, inputs and outputs are known variables and hence can be considered as conditions. The structure of the LMS RH estimator (1.1) is depicted in Figure 1. As mentioned before, the LMS RH estimator (1.1) minimizes the mean-square-error criterion, $\mathrm{E}\left[\left(\widehat{x}_{k-h \mid k}-x_{k-h}\right)^{T}\left(\widehat{x}_{k-h \mid k}-x_{k-h}\right) \mid u_{k-N} \cdots u_{k-1}, y_{k-N} \cdots y_{k-1}\right]$. For $h \geq 2, h=1$, and $h \leq 0$, the estimators (1.1) are often called the smoothers, the filters, and the predictors, respectively. "Receding horizon" is traced from the fact that the finite-time horizon, where inputs and outputs necessary for estimating unknown states are available, recedes with time. In designing the controls, receding horizon schemes have already been popular [1, 2]. If $N$ approaches $\infty$, the estimator (1.1) reduces to the well-known stationary Kalman estimator with infinite-memory. So, the LMS RH estimator (1.1) can be also called a finite memory estimator for the finite and fixed memory size $N$. It has been illustrated through numerical simulation and analysis that the LMS RH estimators with finite memory have better robust performance than conventional Kalman estimators with infinite memory [3, 4]. Also in input/output models arising in signal processing area, it is acknowledged that the finitememory or finite-impulse-response (FIR) filters have been preferable for practical reasons [5].

In spite of the good performance and the usefulness of the LMS RH estimators, no one has proposed a general result on the conditional expectation (1.1). Since it was difficult to obtain the conditional expectation (1.1) for a general state space model, some assumptions have been made to simplify the problem and then obtain a solution easily. At first, system or measurement noise were set to zero, which offers a closed-form solution easily [6-10]. In [11, 12], a priori information on the initial state on the horizon was assumed to be known for obtaining a solution easily. Instead of directly obtaining a closed-form solution, the duality of a control and the complicated scattering theory from a physical phenomenon were used to show the feasibility of implementation for a general state space model [13, 14]. For easy derivation, the Kalman filter was also employed with infinite covariance [4]. However, this approach is so heuristic that the optimality is not guaranteed in the sense of the mean-square-error criterion. Besides, the system matrix is required to be nonsingular. As in other conventional estimators, there were also some trials that unbiased and linear constraints are imposed to obtain the optimal estimator [3,15-17]. However, external control inputs are not considered [3] and the system matrix is required to be nonsingular [15]. Furthermore, it is not
guaranteed that even though such constraints are removed, the optimality is still preserved. Though computing the conditional expectation (1.1) looks like a very simple problem, there is no result on a closed-form solution corresponding to (1.1) for a general state space model without any artificial assumptions and requirements.

In this paper, existing artificial assumptions for obtaining a solution easily are not made and any conditions are not required. Unlike the existing results, the system matrix is not required to be nonsingular. Both system and measurement noise are considered together with external control inputs. The LMS RH estimator will be directly obtained from the conditional expectation (1.1), which automatically guarantees its optimality. It turns out that the proposed LMS RH estimator has the deadbeat property and the linear structure with inputs and outputs over the recent finite horizon.

The rest of this paper is organized as follows. In Section 2, the LMS RH estimator is obtained from the conditional expectation and its iterative computation is introduced in Section 3. A numerical simulation is carried out in Section 4 to illustrate the performance of the proposed LMS RH estimator. Finally, conclusions are presented in Section 5.

## 2. LMS RH Estimator

Consider a linear discrete-time state space model with an external control input:

$$
\begin{gather*}
x_{i+1}=A x_{i}+B u_{i}+\mathrm{G} w_{i} \\
y_{i}=C x_{i}+v_{i}, \tag{2.1}
\end{gather*}
$$

where $x_{i} \in \Re^{n}, u_{i} \in \mathfrak{R}^{l}$, and $y_{i} \in \mathfrak{R}^{q}$ are the state, the input, and the output, respectively, and the system noise $w_{i} \in \mathfrak{R}^{p}$ and the measurement noise $v_{i} \in \mathfrak{R}^{q}$ are assumed to be zero-mean white Gaussian with

$$
\mathbf{E}\left[\left[\begin{array}{c}
w_{i}  \tag{2.2}\\
v_{i}
\end{array}\right]\left[\begin{array}{ll}
w_{i}^{T} & v_{i}^{T}
\end{array}\right]\right]=\left[\begin{array}{cc}
Q & S \\
S^{T} & R
\end{array}\right], \quad Q \geq 0, R>0
$$

where nonzero $S$ often happens when I/O models are converted to state space models. It is also assumed that $(A, C)$ of the system is observable. Through this paper, the current time is denoted by $k$. For mathematical tractability, the state space model (2.1) is equivalently changed to

$$
\begin{gather*}
x_{i+1}=A_{s} x_{i}+B u_{i}+G w_{s, i}+G S R^{-1} y_{i}  \tag{2.3}\\
y_{i}=C x_{i}+v_{i}
\end{gather*}
$$

where $A_{s} \triangleq A-G S R^{-1} C$ and $w_{s, i} \triangleq w_{i}-S R^{-1} v_{i}$. It can be easily shown that $w_{s, i}$ and $v_{i}$ in (2.3) are not correlated. In other words, we have

$$
\mathbf{E}\left[\left[\begin{array}{c}
w_{s, i}  \tag{2.4}\\
v_{i}
\end{array}\right]\left[\begin{array}{ll}
w_{s, i}^{T} & v_{i}^{T}
\end{array}\right]\right]=\left[\begin{array}{cc}
Q-S R^{-1} S^{T} & O \\
O & R
\end{array}\right],
$$

where $Q-S R^{-1} S^{T} \geq 0$. It is noted that off-diagonal blocks are filled with zeros.

For simple representations, several variables are defined as

$$
\begin{align*}
& \tilde{C}_{N} \triangleq\left[\begin{array}{c}
C \\
C A_{s} \\
C A_{s}^{2} \\
\vdots \\
C A_{s}^{N-1}
\end{array}\right], \quad Y_{k-1} \triangleq\left[\begin{array}{c}
y_{k-N} \\
y_{k-N+1} \\
y_{k-N+2} \\
\vdots \\
y_{k-1}
\end{array}\right],  \tag{2.5}\\
& \widetilde{B}_{N} \triangleq\left[\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
C B & 0 & \cdots & 0 & 0 \\
C A_{s} B & C B & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
C A_{s}^{N-2} B & C A_{s}^{N-3} B & \cdots & C B & 0
\end{array}\right],  \tag{2.6}\\
& Q_{N} \triangleq \overbrace{Q-S R^{-1} S^{T} \oplus \cdots \oplus Q-S R^{-1} S^{T}}^{N},  \tag{2.7}\\
& R_{N} \triangleq \overbrace{R \oplus R \oplus \cdots \oplus R}^{N}, \tag{2.8}
\end{align*}
$$

where $\oplus$ denotes the direct sum of the matrices. Additionally, $W_{k-1}, V_{k-1}$, and $U_{k-1}$ are defined by replacing $y$ in (2.5) with $w_{s}, v$, and $u$, respectively. $\widetilde{G}_{N}$ and $\tilde{S}_{N}$ are also defined by replacing $B$ in $\widetilde{B}_{N}$ with $G$ and $G S R^{-1}$, respectively.

First, we consider the case of $0 \leq h \leq N-1$ in (1.1). A prediction problem for $h \leq-1$ will be discussed later on.

By using the defined variables, the state $x_{k-h}$ to be estimated is represented in terms of the initial state $x_{k-N}$ on the horizon, inputs, and system noise on the recent finite horizon [ $k-N, k-1$ ] as

$$
\begin{equation*}
x_{k-h}=A_{s}^{N-h} x_{k-N}+L_{g, N} W_{k-1}+L_{b, N} U_{k-1}+L_{s, N} Y_{k-1} \tag{2.9}
\end{equation*}
$$

where $L_{g, N}$ is given by

$$
L_{g, N} \triangleq\left[\begin{array}{lllll}
A_{s}^{N-h-1} G & A_{s}^{N-h-2} G & \cdots & G & \overbrace{O \cdots O}^{h} \tag{2.10}
\end{array}\right]
$$

and $L_{b, N}$ and $L_{s, N}$ are obtained by replacing $G$ in $L_{g, N}$ with $B$ and $G S R^{-1}$, respectively. From (2.9), the conditional expectation $\mathrm{E}\left[x_{k-h} \mid U_{k-1}, Y_{k-1}\right]$ in (1.1) can be represented as

$$
\begin{align*}
\mathrm{E}\left[x_{k-h} \mid U_{k-1}, \Upsilon_{k-1}\right]= & A_{s}^{N-h} \mathbf{E}\left[x_{k-N} \mid U_{k-1}, Y_{k-1}\right]  \tag{2.11}\\
& +L_{g, N} \mathbf{E}\left[W_{k-1} \mid U_{k-1}, Y_{k-1}\right]+L_{b, N} U_{k-1}+L_{s, N} Y_{k-1}
\end{align*}
$$

Note that if $\mathbf{E}\left[x_{k-N} \mid U_{k-1}, Y_{k-1}\right]$ and $\mathbf{E}\left[W_{k-1} \mid U_{k-1}, Y_{k-1}\right]$ are known, $\mathrm{E}\left[x_{k-h} \mid U_{k-1}, Y_{k-1}\right]$ can be obtained. In order to compute $\mathrm{E}\left[x_{k-N} \mid U_{k-1}, Y_{k-1}\right]$ and $\mathrm{E}\left[W_{k-1} \mid U_{k-1}, Y_{k-1}\right]$, we first try
to find the relation among $x_{k-N}, W_{k-1}, U_{k-1}$, and $Y_{k-1}$. For this purpose, the system (2.3) is represented in a batch form on the recent finite horizon $[k-N, k-1]$ as follows:

$$
\begin{equation*}
Y_{k-1}=\widetilde{C}_{N} x_{k-N}+\tilde{B}_{N} U_{k-1}+\tilde{G}_{N} W_{k-1}+\widetilde{S}_{N} Y_{k-1}+V_{k-1} \tag{2.12}
\end{equation*}
$$

from which we obtain

$$
\left[\begin{array}{c}
W_{k-1}  \tag{2.13}\\
V_{k-1}
\end{array}\right]=\left[\begin{array}{cc}
I & 0 \\
-\widetilde{G}_{N} & -\tilde{C}_{N}
\end{array}\right]\left[\begin{array}{l}
W_{k-1} \\
x_{k-N}
\end{array}\right]+\left[\begin{array}{c}
0 \\
Y_{k-1}-\widetilde{S}_{N} Y_{k-1}-\widetilde{B}_{N} U_{k-1}
\end{array}\right]
$$

We can see from (2.13) that $W_{k-1}$ and $V_{k-1}$ are linearly, more correctly affinely, dependent on $W_{k-1}$ and $x_{k-N}$. The joint probability density function of $W_{k-1}$ and $x_{k-N}$, that is, $f_{w x}\left(W_{k-1}, x_{k-N}\right)$, can be expressed as $k f_{w v}\left(W_{k-1}, V_{k-1} \mid U_{k-1}, Y_{k-1}\right)=k f_{w v}\left(W_{k-1}, Y_{k-1}-\right.$ $\left.\widetilde{C}_{N} x_{k-N}-\widetilde{B}_{N} U_{k-1}-\widetilde{G}_{N} W_{k-1}-\widetilde{S}_{N} Y_{k-1}\right)$ for an appropriate scaling factor $k$. How to choose $k$ will be discussed later on. By using the probability density function of $W_{k-1}$ and $V_{k-1}$ given as

$$
\begin{align*}
f_{w v}\left(W_{k-1}, V_{k-1}\right)= & \frac{1}{\sqrt{(2 \pi)^{(p+q) N} \operatorname{det}\left(Q_{N}\right) \operatorname{det}\left(R_{N}\right)}}  \tag{2.14}\\
& \times \exp \left(-\frac{1}{2}\left[\begin{array}{c}
W_{k-1} \\
V_{k-1}
\end{array}\right]^{T}\left[\begin{array}{cc}
Q_{N} & 0 \\
0 & R_{N}
\end{array}\right]^{-1}\left[\begin{array}{c}
W_{k-1} \\
V_{k-1}
\end{array}\right]\right),
\end{align*}
$$

$\mathrm{E}\left[x_{k-N} \mid U_{k-1}, Y_{k-1}\right]$ can be computed from

$$
\begin{align*}
\mathrm{E}[ & \left.x_{k-N} \mid U_{k-1}, Y_{k-1}\right] \\
& =\int x_{k-N} f_{w x}\left(W_{k-1}, x_{k-N} \mid U_{k-1}, Y_{k-1}\right) d W_{k-1} d x_{k-N} \\
& =\int x_{k-N} k f_{w v}\left(W_{k-1}, Y_{k-1}-\widetilde{C}_{N} x_{k-N}-\widetilde{B}_{N} U_{k-1}-\tilde{G}_{N} W_{k-1}-\widetilde{S}_{N} Y_{k-1}\right) d W_{k-1} d x_{k-N}, \tag{2.15}
\end{align*}
$$

where a constant $k$ is chosen to satisfy the following normalization condition:

$$
\begin{align*}
& \int f_{w x}\left(W_{k-1}, x_{k-N} \mid U_{k-1}, Y_{k-1}\right) d W_{k-1} d x_{k-N} \\
& \quad=k \int f_{w v}\left(W_{k-1}, Y_{k-1}-\widetilde{C}_{N} x_{k-N}-\widetilde{B}_{N} U_{k-1}-\widetilde{G}_{N} W_{k-1}-\widetilde{S}_{N} Y_{k-1}\right) d W_{k-1} d x_{k-N}=1 \tag{2.16}
\end{align*}
$$

In the same way, $\mathrm{E}\left[W_{k-1} \mid U_{k-1}, Y_{k-1}\right]$ can be obtained. It is noted that $f_{w v}\left(W_{k-1}, Y_{k-1}-\right.$ $\tilde{C}_{N} x_{k-N}-\widetilde{B}_{N} U_{k-1}-\widetilde{G}_{N} W_{k-1}-\widetilde{S}_{N} Y_{k-1}$ ) is an exponential function with the following exponent:

$$
\left[\begin{array}{c}
Y_{k-1}^{T}\left(I-\tilde{S}_{N}\right)^{T}  \tag{2.17}\\
U_{k-1}^{T} \\
x_{k-N}^{T} \\
W_{k-1}^{T}
\end{array}\right]\left[\begin{array}{cccc}
R_{N}^{-1} & -R_{N}^{-1} \tilde{B}_{N} & -R_{N}^{-1} \tilde{G}_{N} & -R_{N}^{-1} \tilde{C}_{N} \\
* & \tilde{B}_{N}^{T} R_{N}^{-1} \tilde{B}_{N} & \widetilde{B}_{N}^{T} R_{N}^{-1} \widetilde{G}_{N} & \widetilde{B}_{N}^{T} R_{N}^{-1} \widetilde{C}_{N} \\
* & * & W_{1,1} & W_{1,2} \\
* & * & * & W_{2,2}
\end{array}\right]\left[\begin{array}{c}
Y_{k-1}^{T}\left(I-\tilde{S}_{N}\right)^{T} \\
U_{k-1}^{T} \\
x_{k-N}^{T} \\
W_{k-1}^{T}
\end{array}\right]
$$

where $*$ denotes symmetric parts and $W_{1,1}, W_{1,2}$, and $W_{2,2}$ are given by

$$
\begin{align*}
W_{1,1} & =\tilde{C}_{N}^{T} R_{N}^{-1} \tilde{C}_{N} \\
W_{1,2} & =\tilde{C}_{N}^{T} R_{N}^{-1} \tilde{G}_{N}  \tag{2.18}\\
W_{2,2} & =\tilde{G}_{N}^{T} R_{N}^{-1}+Q_{N}^{-1} .
\end{align*}
$$

By completing the square of the terms of the exponent and recalling the integration of Gaussian functions from $-\infty$ to $\infty$, we can compute $\mathrm{E}\left[x_{k-N} \mid U_{k-1}, Y_{k-1}\right]$ and $\mathrm{E}\left[W_{k-1} \mid\right.$ $\left.U_{k-1}, Y_{k-1}\right]$. To begin with, we introduce the following completion of squares:

$$
\left[\begin{array}{l}
a  \tag{2.19}\\
b
\end{array}\right]^{T}\left[\begin{array}{cc}
\alpha & \beta \\
\beta^{T} & \gamma
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]=\left(b+\gamma^{-1} \beta^{T} a\right)^{T} \gamma\left(b+\gamma^{-1} \beta^{T} a\right)+a^{T}\left(\alpha-\beta \gamma^{-1} \beta^{T}\right) a
$$

to obtain

$$
\frac{\int b \exp \left(\left[\begin{array}{l}
a  \tag{2.20}\\
b
\end{array}\right]^{T}\left[\begin{array}{cc}
\alpha & \beta \\
\beta^{T} & \gamma
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]\right) d b}{\int \exp \left(\left[\begin{array}{l}
a \\
b
\end{array}\right]^{T}\left[\begin{array}{c}
\alpha \\
\beta^{T} \beta
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]\right) d b}=-\gamma^{-1} \beta^{T} a,
$$

for some vectors $a$ and $b$, and some matrices $\alpha, \beta$, and $\gamma$ of appropriate dimensions. The relation (2.20) can be easily obtained in a similar way to the mean of normal distribution. From the following correspondences:

$$
\begin{gather*}
a \longleftarrow\left[\begin{array}{cc}
Y_{k-1}^{T}\left(I-\tilde{S}_{N}\right)^{T} \\
U_{k-1}^{T}
\end{array}\right], \quad b \longleftarrow\left[\begin{array}{c}
x_{k-N}^{T} \\
W_{k-1}^{T}
\end{array}\right], \quad \alpha \longleftarrow\left[\begin{array}{cc}
R_{N}^{-1} & -R_{N}^{-1} \tilde{B}_{N} \\
* & \widetilde{B}_{N}^{T} R_{N}^{-1} \tilde{B}_{N}
\end{array}\right],  \tag{2.21}\\
\beta \longleftarrow\left[\begin{array}{cc}
-R_{N}^{-1} \tilde{G}_{N} & -R_{N}^{-1} \tilde{C}_{N} \\
\tilde{B}_{N}^{T} R_{N}^{-1} \widetilde{G}_{N} & \tilde{B}_{N}^{T} R_{N}^{-1} \widetilde{C}_{N}
\end{array}\right], \quad \gamma \longleftarrow\left[\begin{array}{cc}
W_{1,1} & W_{1,2} \\
* & W_{2,2}
\end{array}\right],
\end{gather*}
$$

we have

$$
\left[\begin{array}{l}
\mathrm{E}\left[x_{k-N} \mid U_{k-1}, Y_{k-1}\right]  \tag{2.22}\\
\mathrm{E}\left[W_{k-1} \mid U_{k-1}, Y_{k-1}\right]
\end{array}\right]=\left[\begin{array}{ll}
W_{1,1} & W_{1,2} \\
W_{1,2}^{T} & W_{2,2}
\end{array}\right]^{-1}\left[\begin{array}{c}
\tilde{C}_{N} \\
\widetilde{\mathrm{G}}_{N}
\end{array}\right] \times R_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\tilde{B}_{N} U_{k-1}\right)
$$

where $W_{1,1}, W_{1,2}$, and $W_{2,2}$ are given by (2.18). It is noted that if $(A, C)$ of the system (2.1) is observable, $W_{1,1}$ is positive definite and $W_{2,2}-W_{1,2}^{T} W_{1,1}^{-1} W_{1,2}$ is also positive definite, which implies that the block matrix including $W_{1,1}, W_{1,2}$, and $W_{2,2}$ is guaranteed to be positive definite and hence nonsingular. Substituting (2.22) into (2.11), we can obtain $\mathbf{E}\left[x_{k-h} \mid\right.$ $\left.U_{k-1}, Y_{k-1}\right]$. Through a long and tedious algebraic calculation, we have the covariance matrix of the mean-square-error $\mathrm{E}\left[\left(\widehat{x}_{k-h \mid k}-x_{k-h}\right)\left(\widehat{x}_{k-h \mid k}-x_{k-h}\right)^{\mathrm{T}}\right]$ as follows:

$$
P=\Xi_{N}\left[\begin{array}{ll}
W_{1,1} & W_{1,2}  \tag{2.23}\\
W_{1,2}^{T} & W_{2,2}
\end{array}\right]^{-1} \Xi_{N^{\prime}}^{T}
$$

where $\Xi_{N}$ is defined by

$$
\Xi_{j} \triangleq\left[\begin{array}{llllll}
A_{s}^{N-h} & A_{s}^{N-h-1} G & A_{s}^{N-h-2} G & \cdots & G & \overbrace{O O}^{j-N+h} \tag{2.24}
\end{array}\right]
$$

for $N-h \leq j \leq N$.
What we have done until now can be summarized in the following theorem.
Theorem 2.1. Suppose that $(A, C)$ of the system (2.1) is observable. Then, the LMS RH estimator (1.1) is given by

$$
\begin{align*}
\widehat{x}_{k-h \mid k}= & \Xi_{N}\left[\begin{array}{ll}
W_{1,1} & W_{1,2} \\
W_{1,2}^{T} & W_{2,2}
\end{array}\right]^{-1}\left[\begin{array}{l}
\tilde{C}_{N} \\
\widetilde{G}_{N}
\end{array}\right] R_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right)  \tag{2.25}\\
& +L_{b, N} U_{k-1}+L_{s, N} Y_{k-1}
\end{align*}
$$

where $\widehat{x}_{k-h \mid k}$ denotes $\mathbf{E}\left[x_{k-h} \mid U_{k-1}, Y_{k-1}\right]$ and $W_{1,1}, W_{1,2}$, and $W_{2,2}$ are defined in (2.18).
The corresponding covariance matrix is given as (2.23).
It is noted that the proposed LMS RH estimator (2.25) is designed without requirements of the removal of some noise and the nonsingular system matrix $A$. In previous work, those requirements are adopted to solve the problems more easily. If constraints or assumptions of existing results are applied to the proposed result, the latter reduces to the former. For example, $Q$ and $\widetilde{B}_{N}$ can be set to zero for comparison with the results on systems without system noise and inputs, respectively.

As seen in (2.25), the LMS RH estimator is linear with respect to inputs and outputs on the recent finite horizon $[k-N, k-1]$. So, "finite memory" may be called "finite impulse response" that is often used in linear signal processing systems. In addition, the deadbeat property is guaranteed in the following theorem.

Theorem 2.2. If no noise is applied, the LMS RH estimator (2.25) is a deadbeat estimator, that is, $x_{k-h \mid k}=x_{k-h}$ at the fixed-lag $(h \geq 1)$ or the current $(h=0)$ time.

Proof. The LMS RH estimator (2.25) can be rearranged in terms of $Y_{k-1}$ and $U_{k-1}$, that is, $H Y_{k-1}+L U_{k-1}$ for some gain matrices $H$ and $L$. Substituting $Y_{k-1}$ in (2.12) into $H Y_{k-1}+L U_{k-1}$, using (2.9), and removing all noise, we have

$$
\begin{align*}
H \Upsilon_{k-1}+L U_{k-1}= & \left(H \tilde{C}_{N}-A_{s}^{N-h}\right) x_{k-N}+x_{k-h}+\left(H \widetilde{B}_{N}-L_{b, N}+L\right) U_{k-1}  \tag{2.26}\\
& +\left(H \widetilde{S}_{N}-L_{s, N}\right) \Upsilon_{k-1}
\end{align*}
$$

It can be easily seen that $H \tilde{C}_{N}=A_{s}^{N-h}$ and terms associated with inputs $U_{k-1}$ and outputs $Y_{k-1}$ become zero. It follows then that we have $H Y_{k-1}+L U_{k-1}=x_{k-h}$. This completes the proof.

According to the deadbeat property, the LMS RH estimator tracks down the real state exactly if no noise is applied.

The prediction problem for $h \leq-1$ can be easily solved from the previous results for $h=0$. Since $x_{k-h}$ is represented as

$$
\begin{equation*}
x_{k-h}=A^{-h} x_{k}+\sum_{i=0}^{-h-1} A^{i} B u_{k+i}+\sum_{i=0}^{-h-1} A^{i} G w_{k+i} \tag{2.27}
\end{equation*}
$$

we have

$$
\begin{equation*}
\mathrm{E}\left[x_{k-h} \mid U_{k-1}, Y_{k-1}\right]=A^{-h} \mathbf{E}\left[x_{k} \mid U_{k-1}, Y_{k-1}\right]+\sum_{i=0}^{-h-1} A^{i} B u_{k+i} \tag{2.28}
\end{equation*}
$$

which means that $\mathrm{E}\left[x_{k-h} \mid U_{k-1}, Y_{k-1}\right]$ can be obtained from $\mathrm{E}\left[x_{k} \mid U_{k-1}, Y_{k-1}\right]$. Setting $h$ in (2.25) to zero, we can compute $\mathrm{E}\left[x_{k} \mid U_{k-1}, Y_{k-1}\right]$ easily.

## 3. Iterative Computation

The LMS FM estimator (2.25) is of the compact and simple form. However, this form requires the inverse of big matrices, which may lead to long computation time and large numerical errors. To overcome these weak points, in this section, we provide an effective iterative form of (2.25).

First, we represent (2.25) in another form for getting recursive equations. Recalling the following fact:

$$
\begin{gather*}
\left(\tilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{C}_{N}\right)^{-1}=\left(W_{1,1}-W_{1,2} W_{2,2}^{-1} W_{1,2}^{T}\right)^{-1}  \tag{3.1}\\
\tilde{C}_{N}^{T} \Pi_{N}^{-1}=\tilde{C}_{N}^{T} R_{N}^{-1}-W_{1,2} W_{2,2}^{-1} \widetilde{G}_{N}^{T} R_{N}^{-1}
\end{gather*}
$$

we obtain

$$
\begin{align*}
& {\left[\begin{array}{c}
\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \\
0
\end{array}\right]+\left[\begin{array}{c}
\tilde{C}_{N}^{T} \\
\widetilde{G}_{N}^{T}
\end{array}\right] R_{N}^{-1}} \\
& \quad=\left[\begin{array}{cc}
2 \widetilde{C}_{N}^{T} R_{N}^{-1}-W_{1,2} W_{2,2}^{-1} \widetilde{G}_{N}^{T} R_{N}^{-1} \\
\tilde{G}_{N}^{T} R_{N}^{-1}
\end{array}\right]=\left[\begin{array}{cc}
2 I & -W_{1,2} W_{2,2}^{-1} \\
0 & I
\end{array}\right]\left[\begin{array}{l}
\widetilde{C}_{N}^{T} \\
\widetilde{G}_{N}^{T}
\end{array}\right] R_{N}^{-1} \\
& {\left[\begin{array}{cc}
W_{1,1} & +\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N} \\
W_{1,2}^{T} & W_{2,2} \\
W_{2,2}
\end{array}\right]^{-1}\left[\begin{array}{cc}
2 I & -W_{1,2} W_{2,2}^{-1} \\
0 & I
\end{array}\right]}  \tag{3.2}\\
& \quad=\left\{\left[\begin{array}{cc}
\frac{1}{2} I & \frac{1}{2} W_{1,2} W_{2,2}^{-1} \\
0 & I
\end{array}\right]\left[\begin{array}{cc}
W_{1,1}+\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N} & W_{1,2} \\
W_{1,2}^{T} & W_{2,2}
\end{array}\right]\right\}^{-1}=\left[\begin{array}{ll}
W_{1,1} & W_{1,2} \\
W_{1,2}^{T} & W_{2,2}
\end{array}\right]^{-1} .
\end{align*}
$$

Note that nonsingularity of $\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}$ is guaranteed for the observability of $(A, C)$. From (3.2), it can be easily seen that the LMS RH (2.25) can be represented as

$$
\begin{align*}
\widehat{x}_{k-h \mid k}= & \Xi_{N}\left[\begin{array}{cc}
W_{1,1}+\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N} & W_{1,2} \\
W_{1,2}^{T} & W_{2,2}
\end{array}\right]^{-1} \\
& \times\left(\left[\begin{array}{c}
\widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right) \\
0
\end{array}\right]+\left[\begin{array}{c}
\widetilde{C}_{N}^{T} \\
\widetilde{G}_{N}^{T}
\end{array}\right] R_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right)\right) \\
& +L_{s, N} Y_{k-1}+L_{b, N} U_{k-1} . \tag{3.3}
\end{align*}
$$

Now, we consider two recursive equations for the batch form (3.3). One is for obtaining $\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}$ and $\widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right)$. The other is for (3.3) given $\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}$ and $\widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right)$. The first one is computed in a backward time and the second one in a forward time. Next subsections deal with each recursive equation.

### 3.1. Recursive Equation for Backward Computation

Here, how to obtain $\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}$ and $\widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right)$ in (3.3) will be discussed. These values will be computed in a backward time. $\left(\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}\right)^{-1} \widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\right.$ $\left.\widetilde{B}_{N} U_{k-1}\right)$ and $\left(\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}\right)^{-1}$ will be denoted by $\beta_{0 \mid k}$ and $P_{0}$, respectively, for consistency with the next section. The following theorem provides the main result.

Theorem 3.1. $P_{0}=\left(\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}\right)^{-1}$ and $\beta_{0 \mid k}=P_{0} \widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right)$ can be computed recursively as follows:

$$
\begin{align*}
P_{0} & =\left(C^{T} R^{-1} C+\widehat{P}_{N}\right)^{-1},  \tag{3.4}\\
\beta_{0 \mid k} & =P_{0}\left(C^{T} R^{-1} y_{k-N}+\alpha_{N \mid k}\right), \tag{3.5}
\end{align*}
$$

where $\widehat{P}_{N}$ and $\alpha_{N \mid k}$ are obtained from

$$
\begin{align*}
\widehat{P}_{j+1}= & A_{s}^{T} C^{T} R^{-1} C A_{s}+A_{s}^{T} \widehat{P}_{j} A_{s}-A_{s}^{T}\left(C^{T} R^{-1} C+\widehat{P}_{j}\right) G \\
& \times\left\{Q_{s}^{-1}+G^{T}\left(C^{T} R^{-1} C+\widehat{P}_{j}\right) G\right\}^{-1} G^{T}\left(C^{T} R^{-1} C+\widehat{P}_{j}\right) A_{s}  \tag{3.6}\\
\alpha_{j+1 \mid k}= & \left\{A_{s}^{T}-A_{s}^{T}\left(C^{T} R^{-1} C+\widehat{P}_{j}\right) G\left\{Q_{s}^{-1}+G^{T}\left(C^{T} R^{-1} C+\widehat{P}_{j}\right) G\right\}^{-1} G^{T}\right\}  \tag{3.7}\\
& \times\left\{\alpha_{j \mid k}+C^{T} R^{-1} y_{k-j}-\left(\widehat{P}_{j}+C^{T} R^{-1} C\right)\left(B u_{k-j-1}+G S R^{-1} y_{k-j-1}\right)\right\},
\end{align*}
$$

for $1 \leq j \leq N-1, Q_{s}=Q-S R^{-1} S^{T}$, and $\widehat{P}_{1}$ and $\alpha_{1 \mid k}$ are zero matrices with appropriate dimensions.
Proof. Before going into a main proof, we introduce some variables $\widehat{C}_{j}, \widehat{N}_{j}, \widehat{\Pi}_{j}$, and $\widehat{P}_{j}$ as

$$
\begin{align*}
& \widehat{C}_{j} \triangleq\left[\begin{array}{c}
C A_{s} \\
\vdots \\
C A_{s}^{j-3} \\
C A_{s}^{j-2} \\
C A_{s}^{j-1}
\end{array}\right]=\left[\begin{array}{c}
C \\
\widehat{C}_{j-1}
\end{array}\right] A,  \tag{3.8}\\
& \widehat{N}_{j} \triangleq\left[\begin{array}{cccc}
C G & 0 & \cdots & 0 \\
C A_{s} G & C G & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
C A_{s}^{j-2} G & C A_{s}^{j-3} G & \cdots & C G
\end{array}\right]=\left[\begin{array}{c|c}
C G & 0 \\
\hline \widehat{C}_{j-1} G & \widehat{N}_{j-1}
\end{array}\right],  \tag{3.9}\\
& \widehat{\Pi}_{j} \triangleq \widehat{N}_{j} Q_{j-1} \widehat{N}_{j}^{T}+R_{j-1} \in \Re^{q(j-1) \times q(j-1),}  \tag{3.10}\\
& \widehat{P}_{j} \triangleq \widehat{C}_{j}^{T} \widehat{\Pi}_{j}^{-1} \widehat{C}_{j}, \tag{3.11}
\end{align*}
$$

for $2 \leq j \leq N$. In terms of $\widehat{\Pi}_{j}$ in (3.10) and $\widehat{P}_{j}$ in (3.11), $\beta_{0 \mid k}$ and $P_{0}$ can be represented as

$$
\begin{gather*}
\beta_{0 \mid k}=P_{0}\left(C^{T} R^{-1} y_{k-N}+\widehat{C}_{N}^{T} \hat{\Pi}_{N}^{-1}\left(\left(I^{o}-\tilde{S}_{N}^{o}\right) \hat{Y}_{N}-\widetilde{B}_{N}^{o} \widehat{U}_{N}\right)\right)=P_{0}\left(C^{T} R^{-1} y_{k-N}+\alpha_{N \mid k}\right), \\
P_{0}=\left(C^{T} R^{-1} C+\widehat{P}_{N}\right)^{-1}, \tag{3.12}
\end{gather*}
$$

where $I^{o}, \tilde{S}_{j}^{o}$, and $\widetilde{B}_{j}^{o}$ are obtained by removing the first row blocks of $I, \widetilde{S}_{j}$, and $\widetilde{B}_{j}$, respectively, and $\widehat{Y}_{j}, \widehat{U}_{j}$, and $\alpha_{j \mid k}$ are given by

$$
\begin{align*}
& \widehat{Y}_{j} \triangleq\left[\begin{array}{lll}
y_{k-j}^{T} & \cdots & y_{k-1}^{T}
\end{array}\right]^{T}, \quad \widehat{U}_{j} \triangleq\left[\begin{array}{lll}
u_{k-j}^{T} \cdots & u_{k-1}^{T}
\end{array}\right]^{T},  \tag{3.13}\\
& \alpha_{j \mid k} \triangleq \widehat{C}_{j}^{T} \widehat{\Pi}_{j}^{-1}\left(\left(I^{o}-\widetilde{S}_{j}^{o}\right) \widehat{Y}_{j}-\widetilde{B}_{j}^{o} \widehat{U}_{j}\right), \tag{3.14}
\end{align*}
$$

for $2 \leq j \leq N$. In order to obtain $P_{0}$ and $\beta_{0 \mid k}$, we have only to know $\widehat{P}_{N}$ and $\alpha_{N \mid k}$. Now, in order to get $\widehat{P}_{N}$ and $\alpha_{N \mid k}$, we try to find recursive equations for $\widehat{P}_{j}$ and $\alpha_{j \mid k}$ on $1 \leq j \leq N$. By using recursions in (3.8) and (3.9), we have the following equality:

$$
\widehat{\Pi}_{j+1}^{-1}=\widehat{\Delta}_{j}^{-1}-\widehat{\Delta}_{j}^{-1}\left[\begin{array}{c}
C  \tag{3.15}\\
\widehat{C}_{j}
\end{array}\right] G\left\{Q_{s}^{-1}+G^{T}\left[\begin{array}{c}
C \\
\widehat{C}_{j}
\end{array}\right]^{T} \widehat{\Delta}_{j}^{-1}\left[\begin{array}{c}
C \\
\widehat{C}_{j}
\end{array}\right] G\right\}^{-1} \times G^{T}\left[\begin{array}{c}
C \\
\widehat{C}_{j}
\end{array}\right]^{T} \widehat{\Delta}_{j}^{-1}
$$

where $\widehat{\Delta}_{j}$ is given by

$$
\widehat{\Delta}_{j} \triangleq\left[\begin{array}{cc}
R & 0  \tag{3.16}\\
0 & \widehat{\Pi}_{j}
\end{array}\right]
$$

Pre- and postmultiplying (3.15) by

$$
\widehat{C}_{j+1}^{T}=A_{s}^{T}\left[\begin{array}{c}
C  \tag{3.17}\\
\widehat{C}_{j}
\end{array}\right]^{T}, \quad \widehat{C}_{j+1}=\left[\begin{array}{c}
C \\
\widehat{C}_{j}
\end{array}\right] A_{s}
$$

respectively, we have (3.6). Pre- and postmultiplying (3.15) by

$$
\begin{gather*}
\widehat{C}_{j+1}^{T}=A_{s}^{T}\left[\begin{array}{c}
C \\
\widehat{C}_{j}
\end{array}\right]^{T}  \tag{3.18}\\
\left(I^{o}-\widetilde{S}_{j+1}^{o}\right) \widehat{Y}_{j+1}-\widetilde{B}_{j+1}^{o} \widehat{U}_{j+1}=\left[\begin{array}{c}
y_{k-j} \\
\left(I^{o}-\widetilde{S}_{j}^{o}\right) \widehat{Y}_{j}-\widetilde{B}_{j}^{o} \widehat{U}_{j}
\end{array}\right]-\left[\begin{array}{c}
C \\
\widehat{C}_{j}
\end{array}\right]\left(B u_{k-j-1}+G S R^{-1} y_{k-j-1}\right), \tag{3.19}
\end{gather*}
$$

respectively, we have (3.7). Note that (3.6) and (3.7) hold for $i \geq 2$. From (3.11) and (3.14), $\widehat{P}_{2}$ and $\alpha_{2 \mid k}$ can be written as

$$
\begin{align*}
\widehat{P}_{2} & =\widehat{C}_{2}^{T} \widehat{\Pi}_{2}^{-1} \widehat{C}_{2}=A_{s}^{T} C^{T}\left(C G Q_{s} G^{T} C^{T}+R\right)^{-1} C A_{s}^{T}  \tag{3.20}\\
\alpha_{2 \mid k} & =\widehat{C}_{2}^{T} \widehat{\Pi}_{2}^{-1}\left(\left(I^{o}-\widetilde{S}_{2}^{o}\right) \widehat{Y}_{2}-\widetilde{B}_{2}^{o} \widehat{U}_{2}\right) \\
& =A_{s}^{T} C^{T}\left(C G Q_{s} G^{T} C^{T}+R\right)^{-1}\left(y_{k-1}-C G S R^{-1} y_{k-2}-C B u_{k-2}\right) . \tag{3.21}
\end{align*}
$$

If $\widehat{P}_{1}$ and $\alpha_{1 \mid k}$ are set to zero matrices with appropriate dimensions, $\widehat{P}_{2}$ in (3.20) and $\alpha_{2 \mid k}$ in (3.21) can be calculated from (3.6) and (3.7). Thus, we can say that (3.6) and (3.7) hold for $i \geq 1$ and are initiated with $\widehat{P}_{1}=0$ and $\alpha_{1 \mid k}=0$. After obtaining $\widehat{P}_{N}$ from (3.6), we can calculate $P_{0}$ from (3.4). $\beta_{0 \mid k}$ in (3.5) can be obtained from $\alpha_{N \mid k}$ that comes from (3.7).

This completes the proof.

### 3.2. Recursive Equation for Forward Computation

Here, the recursive equation for (3.3) on the horizon is derived under the assumption that $P_{0}=\left(\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}\right)^{-1}$ and $\beta_{0 \mid k}=P_{0} \widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right)$ are given. $P_{0}$ and $\beta_{0 \mid k}$ in Section 3.1 are computed in a backward time while variables introduced in this section are computed in a forward time. Before proceeding to a main result, we introduce some variables and the necessary lemma.
$L_{G, i}, M_{i}$, and $N_{i}$ are defined as

$$
\begin{align*}
& L_{G, i} \triangleq\left[\begin{array}{llllll}
A_{s}^{i} & A_{s}^{i-1} G & A_{s}^{i-2} G & \cdots & A_{s} G & G
\end{array}\right],  \tag{3.22}\\
& M_{i} \triangleq\left[\begin{array}{cc}
\widetilde{C}_{i}^{T} R_{i}^{-1} \tilde{C}_{i}+P_{0}^{-1} & \widetilde{C}_{i}^{T} R_{i}^{-1} \tilde{G}_{i}^{o} \\
\widetilde{G}_{i}^{o T} R_{i}^{-1} \tilde{C}_{i} & \widetilde{G}_{i}^{o T} R_{i}^{-1} \tilde{G}_{i}^{o}+Q_{i-1}^{-1}
\end{array}\right],  \tag{3.23}\\
& N_{i} \triangleq\left[\begin{array}{cc}
M_{i} & 0 \\
0 & Q_{s}^{-1}
\end{array}\right]=\left[\begin{array}{cc}
\widetilde{C}_{i}^{T} R_{i}^{-1} \widetilde{C}_{i}+P_{0}^{-1} & \tilde{C}_{i}^{T} R_{i}^{-1} \tilde{G}_{i} \\
\widetilde{G}_{i}^{T} R_{i}^{-1} \widetilde{C}_{i} & \widetilde{G}_{i}^{T} R_{i}^{-1} \widetilde{G}_{i}+Q_{i}^{-1}
\end{array}\right], \tag{3.24}
\end{align*}
$$

where $2 \leq i \leq N$ and $\tilde{G}_{i}^{o}$ is the matrix obtained by removing the last zero column block from $\tilde{G}_{i}$, that is, $\tilde{G}_{i}=\left[\tilde{G}_{i}^{o} \mid 0\right]$. In particular, $L_{G, 1}, M_{1}$, and $N_{1}$ are defined as $\left[A_{s} G\right], C^{T} R^{-1} C+P_{0}^{-1}$ and $M_{1} \oplus Q_{s}^{-1}$. The following lemma shows how variables $L_{G, i}, N_{i}$, and $M_{i}$ are related to one another.

Lemma 3.2. The following relation is satisfied:

$$
\begin{equation*}
N_{i}+L_{G, i}^{T} C^{T} R^{-1} C L_{G, i}=M_{i+1} \tag{3.25}
\end{equation*}
$$

where $L_{G, i}, N_{i}$, and $M_{i}$ are defined in (3.22)-(3.24), respectively.
Proof. If $\Gamma_{i}$ is defined as

$$
\Gamma_{i} \triangleq\left[\begin{array}{lllll}
A_{s}^{i-1} G & A_{s}^{i-2} G & \cdots & A_{s} G & G \tag{3.26}
\end{array}\right]
$$

$L_{G, i}$ can be represented as

$$
L_{G, i}=\left[\begin{array}{ll}
A_{s}^{i} & \Gamma_{i} \tag{3.27}
\end{array}\right],
$$

from which we have

$$
L_{G, i}^{T} C^{T} R^{-1} C L_{G, i}=\left[\begin{array}{cc}
A_{s}^{T i} C^{T} R^{-1} C A_{s}^{i} & A_{s}^{T i} C^{T} R^{-1} C \Gamma_{i}  \tag{3.28}\\
\Gamma_{i}^{T} C^{T} R^{-1} C A_{s}^{i} & \Gamma_{i}^{T} C^{T} R^{-1} C \Gamma_{i}
\end{array}\right] .
$$

The four block elements in $M_{i+1}$ can be expressed recursively as

$$
\begin{align*}
& \tilde{C}_{i+1}^{T} R_{i+1}^{-1} \tilde{C}_{i+1}=\tilde{C}_{i}^{T} R_{i}^{-1} \tilde{C}_{i}+A_{s}^{T i} C^{T} R^{-1} C A_{s}^{i} \\
& \widetilde{C}_{i+1}^{T} R_{i+1}^{-1} \tilde{G}_{i+1}^{o}=\left[\begin{array}{cc}
\tilde{C}_{i}^{T} R_{i}^{-1} \tilde{G}_{i}^{o} & 0
\end{array}\right]+A_{s}^{T i} C^{T} R^{-1} C \Gamma_{i},  \tag{3.29}\\
& \tilde{G}_{i+1}^{o T} R_{i+1}^{-1} \tilde{G}_{i+1}^{o}=\left[\begin{array}{ccc}
\tilde{G}_{i}^{o T} R_{i}^{-1} \tilde{G}_{i}^{o} & 0 \\
\hline 0 & 0
\end{array}\right]+\Gamma_{i}^{T} C^{T} R^{-1} C \Gamma_{i} .
\end{align*}
$$

Using (3.28) and (3.29), we have $N_{i}+L_{G, i}^{T} C^{T} R^{-1} C L_{G, i}=M_{i+1}$. This completes the proof.
Lemma 3.2 is useful for breaking up big matrices of (3.3) into small matrices. We now exploit the recursive equations for two following quantities:

$$
\beta_{j \mid k}=L_{G, j} N_{j}^{-1}\left(\left[\begin{array}{c}
\tilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{Y}_{m, N-1}  \tag{3.30}\\
0
\end{array}\right]+\left[\begin{array}{c}
\tilde{C}_{j}^{T} \\
\tilde{G}_{j}^{T}
\end{array}\right] R_{j}^{-1} \tilde{Y}_{m, j-1}\right)+L_{B, j} \tilde{U}_{j-1}+L_{S, j} \tilde{Y}_{j-1}
$$

for $1 \leq j \leq N$

$$
\gamma_{j \mid k}=\Xi_{j} N_{j}^{-1}\left(\left[\begin{array}{c}
\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{Y}_{m, N-1}  \tag{3.31}\\
0
\end{array}\right]+\left[\begin{array}{c}
\tilde{C}_{j}^{T} \\
\tilde{G}_{j}^{T}
\end{array}\right] R_{j}^{-1} \tilde{Y}_{m, j-1}\right)+L_{b, j} \tilde{U}_{j-1}+L_{s, j} \tilde{Y}_{j-1}
$$

for $N-h \leq j \leq N$, where $L_{b, j}$ and $L_{s, j}$ are defined in a form (2.24), and $\tilde{Y}_{j-1}, \tilde{U}_{j-1}, L_{B, j}, L_{S, j}$, and $\tilde{Y}_{m, j-1}$ are given by

$$
\left.\left.\begin{array}{c}
\tilde{Y}_{j-1} \triangleq\left[y_{k-N}^{T} \cdots y_{k-N+j-1}^{T}\right]^{T}, \quad \tilde{U}_{j-1} \triangleq\left[\begin{array}{lll}
u_{k-N}^{T} \cdots & \cdots u_{k-N+j-1}^{T}
\end{array}\right]^{T} \\
L_{B, j} \triangleq\left[\begin{array}{lll}
A_{s}^{j-1} B & A_{s}^{j-2} B & \cdots
\end{array}\right]  \tag{3.32}\\
L_{S, j} \triangleq[
\end{array} A_{s}^{j-1} G S R^{-1} \quad A_{s}^{j-2} G S R^{-1} \cdots \cdots \quad G S R^{-1}\right],\right] ~ \tilde{Y}_{m, j-1} \triangleq\left(I-\widetilde{S}_{j}\right) \tilde{Y}_{j-1}-\widetilde{B}_{j} \tilde{U}_{j-1} .
$$

Note that $\beta_{0 \mid k}=\widetilde{C}_{N}^{T} \Pi_{N}^{-1}\left(\left(I-\widetilde{S}_{N}\right) Y_{k-1}-\widetilde{B}_{N} U_{k-1}\right), \gamma_{N-h \mid k}=\beta_{N-h \mid k}$, and $\widehat{x}_{k-h \mid k}=\gamma_{N \mid k}$. However, $\beta_{j \mid k}$ is recursively computed from $j=0$ to $j=N$ and $\gamma_{j \mid k}$ from $j=N-h$ to $j=N$ with the initial value $\gamma_{N-h \mid k}=\beta_{N-h \mid k}$. Even though $\beta_{j \mid k}$ does not look useful on $[k-h, k]$, it is still used to compute $\gamma_{j \mid k}$ on that horizon. Now, we try to find out recursive equations for $\beta_{j \mid k}$ and $\gamma_{j \mid k}$ in what follows.

### 3.2.1. Recursive Equation for $\beta_{j \mid k}$ on $0 \leq j \leq N$

Using Lemma 3.2, we will obtain a recursive equation for $\beta_{j \mid k}$, which is introduced in the following theorem.

Theorem 3.3. On $0 \leq j \leq N, \beta_{j \mid k}$ in (3.30) can be computed as follows:

$$
\begin{equation*}
\beta_{j+1 \mid k}=A_{s} \beta_{j \mid k}+B u_{k-N+j}+G S R^{-1} y_{k-N+j}+A_{s} P_{j} C^{T}\left(R+C P_{j} C^{T}\right)^{-1}\left(y_{k-N+j}-C \beta_{j \mid k}\right) \tag{3.33}
\end{equation*}
$$

where $P_{j}$ is given by

$$
\begin{equation*}
P_{j+1}=A_{s} P_{j} A_{s}^{T}+G Q_{s} G^{T}-A_{s} P_{j} C^{T}\left(R+C P_{j} C^{T}\right)^{-1} C P_{j} A_{s}^{T} \tag{3.34}
\end{equation*}
$$

and initial conditions are set to $P_{0}=\left(\tilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{C}_{N}\right)^{-1}$ and $\beta_{0 \mid k}=P_{0} \tilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{Y}_{m, N-1}$ computed in Section 3.1.

Proof. First, we will obtain a closed-form of $P_{j}$ in (3.34) and then, using the closed-form of $P_{j}$, we show that $\beta_{j \mid k}$ in (3.30) can be computed recursively from (3.33).

Using the defined variables (3.22) and (3.24), we assume that $P_{j}$ in (3.34) is of the form:

$$
\begin{equation*}
P_{j}=L_{G, j} N_{j}^{-1} L_{\mathrm{G}, j}^{T} \tag{3.35}
\end{equation*}
$$

By an induction method, we will prove (3.35). For the first step, we check (3.35) for $j=1$. Given the initial value $P_{0}$, we transform $P_{1}$ to the form (3.35) by (3.34) as follows:

$$
P_{1}=\left[\begin{array}{ll}
A_{s} & G
\end{array}\right]\left[\begin{array}{cc}
P_{0}^{-1}+C^{T} R^{-1} C & 0  \tag{3.36}\\
0 & Q_{s}^{-1}
\end{array}\right]^{-1}\left[\begin{array}{ll}
A_{s} & G
\end{array}\right]^{T}
$$

Equation (3.38) can be written in terms of $L_{G, i}$ and $N_{i}$ as $P_{1}=L_{G, 1} N_{1}^{-1} L_{G, 1}^{T}$. Now, we check $P_{j+1}$ under the assumption that $P_{j}=L_{G, j} N_{j}^{-1} L_{G, j}^{T}$. From (3.34), we have

$$
\begin{equation*}
P_{j+1}=A_{s} L_{G, j} M_{j+1}^{-1} L_{\mathrm{G}, j}^{T} A_{s}^{T}+G Q_{s} G^{T}=L_{\mathrm{G}, j+1} N_{j+1}^{-1} L_{\mathrm{G}, j+1}^{T} \tag{3.37}
\end{equation*}
$$

Thus, we can see that $P_{j}$ in (3.34) can be represented as the form (3.35) in terms of $\mathrm{L}_{\mathrm{G}, j}$ and $N_{j}$. Using this result, we are in a position to show that $\beta_{j \mid k}$ in (3.30) can be computed recursively from (3.33). We can rewrite $\beta_{j \mid k}$ in (3.30) as

$$
\begin{equation*}
\beta_{j \mid k}=L_{G, j} N_{j}^{-1} T_{j}+L_{B, j} \tilde{U}_{j-1}+L_{S, j} \tilde{Y}_{j-1} \tag{3.38}
\end{equation*}
$$

where $T_{j}$ is given by

$$
T_{j} \triangleq\left[\begin{array}{c}
P_{0}^{-1}  \tag{3.39}\\
0
\end{array}\right] \beta_{0 \mid k}+\left[\begin{array}{c}
\tilde{C}_{j}^{T} \\
\tilde{G}_{j}^{T}
\end{array}\right] R_{j}^{-1} \tilde{Y}_{m, j-1}
$$

As in a derivation of $P_{j}$ in a batch form, we show by an induction method that $\beta_{j \mid k}$ in (3.38) is equivalent to the one in (3.33). First, we show that $\beta_{1 \mid k}$ can be obtained from $P_{0}=\left(\tilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{C}_{N}\right)^{-1}$ and $\beta_{0 \mid k}=P_{0} \widetilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{Y}_{m, N-1}$ :

$$
\begin{equation*}
\beta_{1 \mid k}=A_{s}\left(P_{0}^{-1}+C^{T} R^{-1} C\right)^{-1}\left(P_{0}^{-1} \beta_{0 \mid k}+C^{T} R^{-1} y_{k-N}\right)+B u_{k-N}+G S R^{-1} y_{k-N} \tag{3.40}
\end{equation*}
$$

where, in terms of $L_{G, i}, N_{i}$, and $T_{i}, \beta_{1 \mid k}$ can be written as $\beta_{1 \mid k}=L_{G, 1} N_{1}^{-1} T_{1}+L_{B, 1} \tilde{U}_{0}+L_{S, 1} \tilde{Y}_{0}$. Next, we show that $\beta_{j+1 \mid k}$ in the form (3.38) can be obtained from $\beta_{j \mid k}$ of the batch form, that is, $\beta_{j \mid k}=L_{G, j} N_{j}^{-1} T_{j}+L_{B, j} \tilde{U}_{j-1}+L_{S, j} \tilde{Y}_{j-1}$. First, note that we have the following relations:

$$
\begin{align*}
& \left\{A_{s}-A_{s} P_{j} C^{T}\left(R+C P_{j} C^{T}\right)^{-1} C\right\} L_{G, j} N_{j}^{-1} T_{j}=A_{s} L_{G, j} M_{j+1}^{-1} T_{j}  \tag{3.41}\\
& A_{s} P_{j} C^{T}\left(R+C P_{j} C^{T}\right)^{-1} y_{k-N+j}=A_{s} L_{G, j} M_{j+1}^{-1} L_{G, j}^{T} C^{T} R^{-1} y_{k-N+j}
\end{align*}
$$

Substituting (3.41) into (3.33) yields

$$
\begin{align*}
\beta_{j+1 \mid k}= & A_{s} L_{G, j} M_{j+1}^{-1} T_{j}+A_{s} L_{G, j} M_{j+1}^{-1} L_{G, j}^{T} C^{T} R^{-1} y_{k-N+j} \\
& -A_{s} L_{G, j} M_{j+1}^{-1} L_{G, j}^{T} C^{T} R^{-1} C\left(L_{B, j} \tilde{U}_{j-1}+L_{S, j} \tilde{Y}_{j-1}\right)+L_{B, j+1} \tilde{U}_{j}+L_{S, j+1} \tilde{Y}_{j}, \\
= & {\left.\left[\begin{array}{ll}
A_{s} L_{G, j} & G
\end{array}\right]\left[\begin{array}{cc}
M_{j+1}^{-1} & 0 \\
0 & Q_{s}
\end{array}\right]\left\{\left[\begin{array}{c}
P_{0}^{-1} \\
0
\end{array}\right] \beta_{0 \mid k}+\left[\begin{array}{c}
\tilde{C}_{j+1}^{T} \\
{\left[\widetilde{G}_{j+1}^{o} \mid 0\right.}
\end{array}\right]^{T}\right] R_{j+1}^{-1} \tilde{Y}_{m, j}\right\} }  \tag{3.42}\\
& +L_{B, j+1} \tilde{U}_{j}+L_{S, j+1} \tilde{Y}_{j}=L_{G, j+1} N_{j+1}^{-1} T_{j+1}+L_{B, j+1} \tilde{U}_{j}+L_{S, j+1} \tilde{Y}_{j,}
\end{align*}
$$

where $G$ and $Q_{s}$ in the first and second matrix blocks on the right-hand side of the first equality have no effect on the equation. This completes the proof.

It is observed that the recursive equations with (3.33) and (3.34) are the same as the Kalman filter with initial conditions $P_{0}=\left(\widetilde{C}_{N}^{T} \Pi_{N}^{-1} \widetilde{C}_{N}\right)^{-1}$ and $\beta_{0 \mid k}=P_{0} \widetilde{C}_{N}^{T} \Pi_{N}^{-1} \tilde{Y}_{m, N-1} \cdot \beta_{j \mid k}$ on $1 \leq j \leq N$ provides initial values and inputs for a recursive equation of $\gamma_{j \mid k}$ (3.31), which will be investigated in what follows.

### 3.2.2. Recursive Equation for $\gamma_{j \mid k}$ on $N-h \leq j \leq N$

Here, we discuss the recursive equation for $\gamma_{j \mid k}$ on $N-h \leq j \leq N$. As mentioned before, the recursive equation for $\gamma_{j \mid k}$ starts from $j=N-h$ with the initial value $\gamma_{N-h \mid k}=\beta_{N-h \mid k}$. On $N-h \leq j \leq N, \gamma_{j \mid k}$ can be recursively computed with the help of $\beta_{j \mid k}$. However, $\gamma_{N \mid k}=\widehat{x}_{k-h \mid k}$ is what we want to find out finally. The recursive equation for $\gamma_{j \mid k}$ is given in the following theorem.

Theorem 3.4. On $N-h \leq j \leq N-1, \gamma_{j \mid k}$ in (3.31) can be computed as follows:

$$
\begin{equation*}
\gamma_{j+1 \mid k}=\gamma_{j \mid k}+\not 火_{j} C^{T}\left(C P_{j} C^{T}+R\right)^{-1}\left(y_{k-N+j}-C \beta_{j \mid k}\right) \tag{3.43}
\end{equation*}
$$



Figure 2: An iterative form of the LMS RH estimator.
where $\gamma_{N-h \mid k}=\beta_{N-h \mid k}, \beta_{j \mid k}$ and $P_{j}$ are obtained from (3.33) and (3.34), respectively, and $\mathcal{K}_{j}$ is given by

$$
\begin{equation*}
\mathscr{K}_{j}=\mathscr{K}_{j-1}\left(I-C^{T}\left(C P_{j-1} C^{T}+R\right)^{-1} C P_{j-1}\right) A^{T}, \tag{3.44}
\end{equation*}
$$

with the initial condition $\boldsymbol{K}_{N-h}=P_{N-h}$.
Proof. Using Lemma 3.2, $\gamma_{j+1 \mid k}$ in (3.31) for $j \geq N-h$ can be represented as

$$
\begin{equation*}
\gamma_{j+1 \mid k}=\gamma_{j \mid k}+\Xi_{j} N_{j}^{-1} L_{G, j}^{T} C^{T}\left(C P_{j} C^{T}+R\right)^{-1}\left(y_{k-N+j}-C \beta_{j \mid k}\right) . \tag{3.45}
\end{equation*}
$$

If we denote $\Xi_{j} N_{j}^{-1} L_{G, j}^{T}$ by $\mathscr{K}_{j}$, we have only to prove (3.44). Since $\Xi_{N-h}=L_{G, N-h}, \mathcal{K}_{N-h}$ is equal to $P_{N-h}$. Using Lemma 3.2, we can represent $\mathcal{K}_{j}$ in a recursive form as follows:

$$
\begin{equation*}
\mathscr{K}_{j}=\mathscr{K}_{j-1}\left(I-C^{T}\left(C P_{j} C^{T}+R\right)^{-1} C P_{j-1}\right) A^{T} . \tag{3.46}
\end{equation*}
$$

This completes the proof.
It is noted that the recursive equation (3.43) in Theorem 3.4 is a fixed-point smoother of the state $x_{k-h}$, which runs with a recursive equation (3.33) in Theorem 3.3. Variables for recursive equations in Sections 3.1 and 3.2 are visualized in Figure 2. Starting from $\widehat{P}_{1}=0$ and $\alpha_{1 \mid k}=0$, we compute $\widehat{P}_{N}$ and $\alpha_{N \mid k}$ recursively in a backward time. From $\widehat{P}_{N}$ and $\alpha_{N \mid k}$, we compute $P_{0}$ and $\beta_{0 \mid k}$, from which we drive the forward recursive equation for $\beta_{j \mid k}$ to get $\gamma_{N \mid k}=\widehat{x}_{k-h \mid k}$.


Figure 3: Estimation errors of LMS RH and Kalman estimators when temporary uncertainties exist.

## 4. Simulation

In this section, a numerical example is given to demonstrate the performance of the proposed LMS RH estimator. Suppose that we have a state space model represented as

$$
\begin{gather*}
x_{i+1}=\left[\begin{array}{cc}
1.5400+2 \delta_{i} & -0.7379 \\
0.7379 & \delta_{i}
\end{array}\right] x_{i}+\left[\begin{array}{l}
0.4921 \\
0.7594
\end{array}\right] u_{i}+\left[\begin{array}{l}
1 \\
1
\end{array}\right] w_{i}  \tag{4.1}\\
y_{i}=\left[\begin{array}{ll}
1+\delta_{i} & \left.1+\delta_{i}\right]
\end{array}\right] x_{i}+v_{i}
\end{gather*}
$$

where $\delta_{i}$ is an uncertain model parameter given as

$$
\delta_{i}= \begin{cases}0.1, & 200 \leq i \leq 220  \tag{4.2}\\ 0, & \text { otherwise }\end{cases}
$$

The system noise covariance $Q$ and the measurement noise covariance $R$ are set to $0.01^{2}$ and $0.027^{2}$, respectively. The memory size and the fixed-lag size are taken as $N=10$ and $h=3$, respectively. A sinusoidal input is applied as an input.

We carry out a simulation for the system (4.1) with temporary modeling uncertainties (4.2). In Figure 3, we compare the estimation errors of the LMS RH estimator with the fixedlag Kalman estimator [18]. When uncertainties do not exist, the fixed-lag Kalman estimator has the smaller estimation error than the proposed LMS RH estimator. It can, however, be seen that the estimation error of the LMS RH estimator is considerably smaller than that of the fixed-lag Kalman estimator when modeling uncertainties exist. Actually, one of poles of the fixed-lag Kalman estimator is so close to a unit that even small uncertainties have a good chance of divergence. Additionally, we can see that the estimation error of the LMS RH estimator converges much more rapidly than that of the fixed-lag Kalman estimator after temporary modeling uncertainty disappears. The slow response is also related to the pole
near a unit. To be summarized, we can say that the proposed LMS RH estimator is more robust than other estimators with infinite memory when applied to systems with modeling uncertainties.

## 5. Conclusions

In this paper, we proposed a receding horizon ( RH ) estimator based on the mean-squareerror criterion for a discrete-time state space model, called a least-mean-square (LMS) RH estimator. An unknown state was estimated by making use of the finite number of inputs and outputs over the recent finite horizon without any arbitrary assumptions and any $a$ priori state information. The proposed LMS RH estimator was obtained from the conditional expectations of the initial state and the system noise on the corresponding horizon. It was shown that the LMS RH estimator has a deadbeat property and has good robust performance through a numerical example.

To the best of authors' knowledge, the proposed LMS RH estimator would be the most general version among existing RH or finite-memory estimators in the mean-square-error sense. Furthermore, the LMS RH estimator could be extended to other stochastic systems with imperfect communications, uncertainties, and so on [19, 20].

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Research Article

# High-Order Stochastic Adaptive Controller Design with Application to Mechanical System 

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The main purpose of this paper is to apply stochastic adaptive controller design to mechanical system. Firstly, by a series of coordinate transformations, the mechanical system can be transformed to a class of special high-order stochastic nonlinear system, based on which, a more general mathematical model is considered, and the smooth state-feedback controller is designed. At last, the simulation for the mechanical system is given to show the effectiveness of the design scheme.

## 1. Introduction

In recent years, the study for deterministic high-order nonlinear systems has achieved remarkable development, see, for example, [1-3] and references herein. Inspired by these interesting and important results, it is natural to generalize their results to the following stochastic high-order nonlinear systems which are neither necessarily feedback linearizable nor affine in the control input:

$$
\begin{gather*}
d z=f_{0}\left(z, x_{1}\right) d t+g_{0}^{T}\left(z, x_{1}\right) d \omega, \\
d x_{i}=\left(d_{i}\left(\bar{x}_{i}, t\right) x_{i+1}^{p_{i}}+f_{i}\left(z, \bar{x}_{i}\right)\right) d t+g_{i}^{T}\left(z, \bar{x}_{i}\right) d \omega, \quad i=1, \ldots, n-1,  \tag{1.1}\\
d x_{n}=\left(d_{n}\left(\bar{x}_{n}, t\right) u^{p_{n}}+f_{n}\left(z, \bar{x}_{n}\right)\right) d t+g_{n}^{T}\left(z, \bar{x}_{n}\right) d \omega,
\end{gather*}
$$

where $\left(z^{T}, x_{1}, \ldots, x_{n}\right)^{T} \in \mathbb{R}^{m+n}$, and $u \in \mathbb{R}$ are the measurable state and the input of system, respectively, $\bar{x}_{i}=\left(x_{1}, \ldots, x_{i}\right)^{T}, i=1, \ldots, n, z=\left(z_{1}, \ldots, z_{m}\right)^{T} \in \mathbb{R}^{m}$ is referred to as the state of


Figure 1: A mechanical system.
the stochastic inverse dynamics, $\omega$ is an $r$-dimensional standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ being a $\sigma$-algebra, and $P$ being a probability measure, $p_{i} \geq 1, i=1, \ldots, n$ are odd integers, and the functions $f_{i}(\cdot)$ and $g_{i}(\cdot)$, $i=0,1, \ldots, n$ are assumed to be smooth, vanishing at the origin $\left(z^{T}, \bar{x}_{n}^{T}\right)=\left(0_{1 \times m}, 0_{1 \times n}\right)$.

For (1.1) with $d_{i}(\cdot)=1$, Xie and Tian in [4] considered the state-feedback stabilization problem for the first time. After considering the stabilization of high-order stochastic nonlinear systems, [5] further addressed the problem of state-feedback inverse optimal stabilization in probability, that is, the designed stabilizing backstepping controller is also optimal with respect to meaningful cost functionals. When $d_{i}(\cdot) \neq 1$, [6] designed an adaptive state-feedback controller for a class of stochastic nonlinear uncertain systems with $0<\lambda_{i} \leq$ $d_{i}(\cdot) \leq \mu_{i} \leq \mu$, and [7] designed a smooth adaptive state-feedback controller for high-order stochastic systems with $\lambda_{i}\left(\bar{x}_{i}\right) \leq d_{i}(\cdot) \leq \bar{\mu}_{i}\left(\bar{x}_{i}, \theta\right)$ by using the parameter separation lemma and some flexible algebraic techniques. Recently, more excellent results [8-28] were achieved by Xie and his group.

However, all these theoretical results mentioned above are demonstrated only by some numerical simulation examples. Since many practical application systems in aerospace industry, industrial process control, and so forth, can be described by (or transformed to) stochastic high-order nonlinear systems, so it is very necessary to apply the control schemes to these systems. Based on this reason, we consider a practical example of mechanical movement in this paper. By a series of coordinate transformations, the mechanical system can be transformed to a high-order stochastic nonlinear system, based on which, we consider a more general mathematical model and design a smooth state-feedback control law. At last, the simulation for the mechanical system is given to show the effectiveness of the design scheme.

This paper is organized as follows. Section 2 gives a practical example. Section 3 provides preliminary knowledge and presents problem statement. Controller design and stability analysis are given in Section 4. The simulation for the practical example is provided to demonstrate the control scheme in Section 5. Section 6 gives some concluding remarks.

## 2. A Practical Example

Let us consider the following mechanical system which consists of two masses $m_{1}$ and $m_{2}$ on a horizontal smooth surface as shown in Figure 1. The mass $m_{1}$ is interconnected to the wall by a linear spring and to the mass $m_{2}$ by a nonlinear spring which has cubic force-deformation relation. Let $x$ be the displacement of mass $m_{1}$ and $y$ the displacement of mass $m_{2}$ such that at $x=0$ and $y=0$, that is, the springs are unstretched. A control force $u$ acts on $m_{1}$.

Where the units of $m_{1}, x$, and $u$ are " kg ", " m ", and " N ", respectively, and $y_{1}=x-y$. The equations of motion for the system are described by

$$
\begin{gather*}
\ddot{y}=\frac{k_{1}}{m_{2}}(x-y)^{3}  \tag{2.1}\\
\ddot{x}=-\frac{k}{m_{1}} x-\frac{k_{1}}{m_{1}}(x-y)^{3}+\frac{u}{m_{1}},
\end{gather*}
$$

where $k$ and $k_{1}$ are the spring coefficients, and their units are " $\mathrm{N} / \mathrm{m}$ " and " $\mathrm{N} / \mathrm{m}$ ", respectively.

Introducing the smooth change of coordinates

$$
\begin{gather*}
x_{1}=y, \quad x_{2}=\dot{x}_{1}=\dot{y} \\
x_{3}=(x-y) \sqrt[3]{\frac{k_{1}}{m_{1}}}, \quad x_{4}=\dot{x}_{3} \tag{2.2}
\end{gather*}
$$

one gets

$$
\begin{align*}
y=x_{1}, & \dot{y}
\end{align*}=x_{2}, ~\left(x_{3}=\frac{x_{4}}{\sqrt[3]{k_{1} / m_{1}}}+x_{2} .\right.
$$

The linear spring constant $k$ has a specific nominal value $k_{0}=1.5$ which is considered uncertain, and $k \in[0.75,2.25]$. Let $\Delta(t)=k(t)-k_{0}$. For all $t \geq 0, \Delta(t)$ is the Gaussian white noise process with $E \Delta(t)=0$ and $E \Delta^{2}(t)=\sigma^{2}$. We can choose the value of parameter $\sigma$ such that $k(t)$ obeys the bound $0.75 \leq k \leq 2.25$ with a sufficiently high probability. This model of spring rate variations leads to an uncertain stochastic system. By (2.2), one chooses the smooth state-feedback control

$$
\begin{equation*}
u=m_{1} \frac{v}{\sqrt[3]{k_{1} / m_{1}}}+\frac{m_{1}+m_{2}}{m_{2}} m_{1} x_{3}^{3} \tag{2.4}
\end{equation*}
$$

which together with the property of $\Delta(t)$ leads to

$$
\begin{gather*}
d x_{1}=x_{2} d t \\
d x_{2}=\frac{m_{1}}{m_{2}} x_{3}^{3} d t \\
d x_{3}=x_{4} d t  \tag{2.5}\\
d x_{4}=v d t+k_{0} f(x) d t+\sigma f(x) d \omega \\
y=x_{1}
\end{gather*}
$$

where $f(x)=-x_{3} / m_{1}-\sqrt[3]{\left(k_{1} / m_{1}\right)}\left(x_{1} / m_{1}\right)$, and $\omega$ is standard Wiener process.

This stochastic high-order nonlinear systems can be generalized to a more general system which will be given in the following section.

## 3. Preliminary Knowledge and Problem Statement

### 3.1. Preliminary Knowledge

In this section, we will introduce the concept of input-to-state practical stability (ISpS) in probability.

Consider the following stochastic nonlinear system

$$
\begin{equation*}
d x=f(x, u) d t+g^{T}(x, u) d \omega, \quad x(0)=x_{0} \in \mathbb{R}^{n} \tag{3.1}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m}$ are the state and the input of system, respectively. The Borel measurable functions $f: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n}$ and $g: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n \times r}$ are locally Lipschitz in $x$, and $\omega \in \mathbb{R}^{r}$ is an $r$ dimensional independent standard Wiener process defined on the complete probability space $(\Omega, \not \subset, P)$.

The following definitions and lemmas will be used throughout the paper.
Definition 3.1 (see [29]). For any given $V(x) \in \mathcal{C}^{2}$, associated with stochastic system (3.1), the differential operator $\perp$ is defined as follows:

$$
\begin{equation*}
\mathscr{L} V(x)=\frac{\partial V(x)}{\partial x} f(x, u)+\frac{1}{2} \operatorname{Tr}\left\{g(x, u) \frac{\partial^{2} V(x)}{\partial x^{2}} g^{T}(x, u)\right\} \tag{3.2}
\end{equation*}
$$

Definition 3.2 (see [30]). The stochastic system (3.1) is input-to-state practically stable (ISpS) in probability if for any $\varepsilon>0$, there exist a class $\nless \mathcal{L}$-function $\beta(\cdot)$, a class $\mathcal{K}_{\infty}$-function $\gamma(\cdot)$, and a constant $d_{0}$ such that

$$
\begin{equation*}
P\left\{|x(t)|<\beta\left(\left|x_{0}\right|, t\right)+\gamma\left(\left|u_{t}\right|\right)+d_{0}\right\} \geq 1-\varepsilon, \quad x_{0} \in \mathbb{R}^{n} \backslash\{0\} . \tag{3.3}
\end{equation*}
$$

Lemma 3.3 (see [30]). For system (3.1), if there exist a $\mathcal{C}^{2}$ function $V(x)$, class $\mathcal{K}_{\infty}$ functions $\alpha_{1}$, $\alpha_{2}, X$, a class $\nless$ function $\alpha$, and a constant $\bar{d}$ such that

$$
\begin{gather*}
\alpha_{1}(|x|) \leq V(x) \leq \alpha_{2}(|x|),  \tag{3.4}\\
\rho V(x) \leq-\alpha(|x|)+x(|u|)+\bar{d} \tag{3.5}
\end{gather*}
$$

then
(1) There exists an almost surely unique solution on $[0, \infty)$;
(2) The system (3.1) is ISpS in probability.

Lemma 3.4 (see [6]). Let $x$ and $y$ be real variables. Then, for any positive integers $m, n$ and any nonnegative smooth function $b(\cdot)$, the following inequality holds:

$$
\begin{equation*}
\left|x^{m} y^{n}\right| \leq \frac{m}{m+n} b(\cdot)|x|^{m+n}+\frac{n}{m+n} b(\cdot)^{-m / n}|y|^{m+n} \tag{3.6}
\end{equation*}
$$

Lemma 3.5 (see [2]). For real variables $x \geq 0, y>0$, and real number $m \geq 1$, the following inequality holds:

$$
\begin{equation*}
x \leq y+\left(\frac{x}{m}\right)^{m}\left(\frac{m-1}{y}\right)^{m-1} \tag{3.7}
\end{equation*}
$$

### 3.2. Problem Statement

From (2.5), we introduce a more general class of stochastic nonlinear systems as follows:

$$
\begin{gather*}
d x_{i}=d_{i}(x) x_{i+1}^{p_{i}} d t+f_{i}\left(\bar{x}_{i+1}\right) d t+g_{i}\left(\bar{x}_{i}\right)^{T} d \omega, \quad i=1, \ldots, n-1, \\
d x_{n}=d_{n}(x) u^{p_{n}} d t+f_{n}\left(\bar{x}_{n}\right) d t+g_{n}\left(\bar{x}_{n}\right)^{T} d \omega  \tag{3.8}\\
y=x_{1}
\end{gather*}
$$

where $x=\left(x_{1}, \ldots, x_{n}\right)^{T} \in \mathbb{R}^{n}, u, y \in \mathbb{R}$ are the state, the input, and the measurable output of system, respectively, $\bar{x}_{i}=\left(x_{1}, \ldots, x_{i}\right)^{T}, p_{i}, i=1, \ldots, n$, are positive odd integers, $f_{i}(\cdot)$ : $\mathbb{R}^{i+1} \rightarrow \mathbb{R}$ and $g_{i}(\cdot): \mathbb{R}^{i} \rightarrow \mathbb{R} \times \mathbb{R}^{r}$ are smooth functions with $f_{i}(0)=0$ and $g_{i}(0)=0, d_{i}(x)$ is unknown control coefficient with known sign, and $\omega$ is an $r$-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$.

The following assumptions are made on system (3.8).
A1: for each $d_{i}(x)$, there exist unknown constant $\theta^{\prime}>0$ and known nonnegative smooth functions $b_{i}\left(\bar{x}_{i}\right)$ and $\bar{b}_{i}\left(\bar{x}_{i+1}\right)$ such that

$$
\begin{equation*}
0 \leq b_{i}\left(\bar{x}_{i}\right) \leq d_{i}(x) \leq \theta^{\prime} \bar{b}_{i}\left(\bar{x}_{i+1}\right) \tag{3.9}
\end{equation*}
$$

A2: for functions $f_{i}(\cdot), g_{i}(\cdot), i=1,2, \ldots, n$, there exist known nonnegative smooth functions $\varphi_{i j}\left(\bar{x}_{i}\right)$ and $\bar{\psi}_{i}\left(\bar{x}_{i}\right)$ such that

$$
\begin{align*}
\left|f_{i}\left(\bar{x}_{i+1}\right)\right| & \leq \sum_{j=0}^{p_{i}-1}\left|x_{i+1}\right|^{j} \varphi_{i j}\left(\bar{x}_{i}\right),  \tag{3.10}\\
\left|g_{i}\left(\bar{x}_{i}\right)\right| & \leq\left(\left|x_{1}\right|^{\left(p_{i}+1\right) / 2}+\cdots+\left|x_{i}\right|^{\left(p_{i}+1\right) / 2}\right) \bar{\psi}_{i}\left(\bar{x}_{i}\right) .
\end{align*}
$$

A3: the reference signal $y_{r}$ and its derivative $\dot{y}_{r}$ are bounded.
The objective of this paper is to design an adaptive controller such that the closedloop system is ISpS in probability and the tracking error $\xi_{1}=y-y_{r}$ can be regulated to a neighborhood of the origin with radius as small as possible.

## 4. Controller Design and Stability Analysis

With the aid of Lemmas 3.3-3.5, we are ready to present the main results of this paper. In this section, we show that under A1-A3, it is possible to construct a globally stabilizing, state-feedback smooth controller for system (3.8). Introduce the odd positive integer $p=$ $\max _{i=1, \ldots, n}\left\{p_{i}\right\}$, and the following coordinate change

$$
\begin{gather*}
\xi_{1}=x_{1}-y_{r} \\
\xi_{i}=x_{i}-x_{i}^{*}\left(\bar{x}_{i-1}, y_{r}, \widehat{\theta}\right), \quad i=2, \ldots, n \tag{4.1}
\end{gather*}
$$

where $x_{i}^{*}\left(\bar{x}_{i-1}, y_{r}, \widehat{\theta}\right), i=2, \ldots, n$, are virtual smooth controllers to be designed later, $\theta:=$ $\max \left\{\theta^{\prime}, \theta^{\prime}(p+3) /\left(p-p_{i}+3\right)\right\}$, and $\widehat{\theta}$ denotes the estimate of $\theta$. Then, according to Itô differentiation rule, one has

$$
\begin{align*}
d \xi_{1}= & d_{1} x_{2}^{p_{1}} d t+f_{1} d t+g_{1}^{T} d \omega-\dot{y}_{r} d t \\
d \xi_{i}= & d_{i} x_{i+1}^{p_{i}} d t+f_{i} d t-\sum_{k=1}^{i-1} \frac{\partial x_{i}^{*}}{\partial x_{k}}\left(x_{k+1}^{p_{k}}+f_{k}\right) d t-\frac{1}{2} \sum_{j, k=1}^{i-1} \frac{\partial^{2} x_{i}^{*}}{\partial x_{j} \partial x_{k}} g_{j}^{T} g_{k} d t \\
& -\frac{\partial x_{i}^{*}}{\partial \widehat{\theta}} \hat{\theta} d t-\frac{\partial x_{i}^{*}}{\partial y_{r}} \dot{y}_{r} d t+\left(g_{i}^{T}-\sum_{k=1}^{i-1} \frac{\partial x_{i}^{*}}{\partial x_{k}} g_{k}^{T}\right) d \omega, \quad i=2, \ldots, n-1,  \tag{4.2}\\
d \xi_{n}= & d_{n} u^{p_{n}} d t+f_{n} d t-\sum_{k=1}^{n-1} \frac{\partial x_{n}^{*}}{\partial x_{k}}\left(x_{k+1}^{p_{k}}+f_{k}\right) d t-\frac{1}{2} \sum_{j, k=1}^{n-1} \frac{\partial^{2} x_{n}^{*}}{\partial x_{j} \partial x_{k}} g_{j}^{T} g_{k} d t \\
& -\frac{\partial x_{n}^{*}}{\partial \widehat{\theta}} \dot{\hat{\theta}} d t-\frac{\partial x_{n}^{*}}{\partial y_{r}} \dot{y}_{r} d t+\left(g_{n}^{T}-\sum_{k=1}^{n-1} \frac{\partial x_{n}^{*}}{\partial x_{k}} g_{k}^{T}\right) d \omega .
\end{align*}
$$

Let $G_{i}^{T}=g_{i}^{T}-\sum_{k=1}^{i-1}\left(\partial x_{i}^{*} / \partial x_{k}\right) g_{k}^{T}, i=2, \ldots, n$. Next, we design the controller step by step by backstepping.

Step 1. Consider the 1st Lyapunov candidate function

$$
\begin{equation*}
V_{1}\left(\xi_{1}, \tilde{\theta}\right)=\frac{1}{p-p_{1}+4} \xi_{1}^{p-p_{1}+4}+\frac{1}{2} \tilde{\theta}^{2} \tag{4.3}
\end{equation*}
$$

where $\tilde{\theta}=\theta-\widehat{\theta}$ is the parameter estimation error. In view of (3.2), (4.1), and (4.2), one has

$$
\begin{align*}
\varrho V_{1}\left(\xi_{1}, \tilde{\theta}\right) & =\xi_{1}^{p-p_{1}+3}\left(d_{1}(x) x_{2}^{p_{1}}+f_{1}\left(\bar{x}_{2}\right)-\dot{y}_{r}\right)+\frac{1}{2} \operatorname{Tr}\left\{g_{1}\left(x_{1}\right)\left(p-p_{1}+3\right) \xi_{1}^{p-p_{1}+2} g_{1}^{T}\left(x_{1}\right)\right\}-\tilde{\theta} \dot{\theta} \\
& \leq d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{p_{1}}+\left|\xi_{1}\right|^{p-p_{1}+3}\left|f_{1}\left(\bar{x}_{2}\right)-\dot{y}_{r}\right|+\frac{1}{2}\left(p-p_{1}+3\right) \xi_{1}^{p-p_{1}+2}\left|g_{1}\left(x_{1}\right)\right|^{2}-\tilde{\theta} \dot{\hat{\theta}} \tag{4.4}
\end{align*}
$$

By Lemma 3.4 and A2, there exist nonnegative smooth functions $\bar{\varphi}_{1}\left(x_{1}\right)$ and $\psi_{1}\left(x_{1}\right)$ such that

$$
\begin{align*}
\left|f_{1}\left(\bar{x}_{2}\right)\right| & \leq \sum_{j=0}^{p_{1}-1}\left|x_{2}\right|^{j} \varphi_{1 j}\left(x_{1}\right)=\sum_{j=0}^{p_{1}-1}\left|x_{2}\right|^{j}\left(\varphi_{1 j}^{1 /\left(p_{1}-j\right)}\left(x_{1}\right)\right)^{p_{1}-j} \\
& \leq \sum_{j=0}^{p_{1}-1}\left(\frac{j}{p_{1}}\left(\frac{1}{2 j} b_{1}\left(x_{1}\right)\right)\left|x_{2}\right|^{p_{1}}+\frac{p_{1}-j}{p_{1}}\left(\frac{2 j}{b_{1}\left(x_{1}\right)}\right)^{j /\left(p_{1}-j\right)} \varphi_{1 j}^{p_{1} /\left(p_{1}-j\right)}\left(x_{1}\right)\right) \\
& \leq \frac{b_{1}\left(x_{1}\right)}{2}\left|x_{2}\right|^{p_{1}}+\bar{\varphi}_{1}\left(x_{1}\right), \\
\left|g_{1}\left(x_{1}\right)\right| & \leq\left|\xi_{1}\right|^{\left(p_{1}+1\right) / 2} \psi_{1}\left(x_{1}\right), \tag{4.5}
\end{align*}
$$

which together with the boundedness of $\dot{y}_{r}$ imply that

$$
\begin{equation*}
\left|f_{1}-\dot{y}_{r}\right| \leq \frac{b_{1}\left(x_{1}\right)}{2}\left|x_{2}\right|^{p_{1}}+\varphi_{1}^{\prime}\left(x_{1}, y_{r}\right) \tag{4.6}
\end{equation*}
$$

where $\varphi_{1}^{\prime}\left(x_{1}, y_{r}\right)$ is a nonnegative smooth function, $\psi_{1}\left(x_{1}\right)=\bar{\psi}_{1}\left(x_{1}\right)$. Then, for any real number $\delta_{1}>0$, choosing $a=\left|\xi_{1}^{p-p_{1}+3}\right| \varphi_{1}^{\prime}\left(x_{1}, y_{r}\right), b=\delta_{1}, m=(p+3) /\left(p-p_{1}+3\right)$, by Lemma 3.5, there is a smooth function $\phi_{11}\left(x_{1}, y_{r}\right)$ such that

$$
\begin{align*}
& \left|\xi_{1}\right|^{p-p_{1}+3}\left|f_{1}-\dot{y}_{r}\right| \\
& \quad \leq\left|\xi_{1}\right|^{p-p_{1}+3}\left(\frac{b_{1}\left(x_{1}\right)}{2}\left|x_{2}\right|^{p_{1}}+\varphi_{1}^{\prime}\left(x_{1}, y_{r}\right)\right) \leq\left|\xi_{1}\right|^{p-p_{1}+3} \frac{b_{1}\left(x_{1}\right)}{2}\left|x_{2}\right|^{p_{1}} \\
& \quad+\delta_{1}+\left(\frac{\left(p-p_{1}+3\right)\left|\xi_{1}\right|^{p-p_{1}+3} \varphi_{1}^{\prime}\left(x_{1}, y_{r}\right)}{p+3}\right)^{(p+3) /\left(p-p_{1}+3\right)} \times\left(\frac{p_{1}}{\delta_{1}\left(p-p_{1}+3\right)}\right)^{p_{1} /\left(p-p_{1}+3\right)} \\
& \quad=\left|\xi_{1}\right|^{p-p_{1}+3} \frac{b_{1}\left(x_{1}\right)}{2}\left|x_{2}\right|^{p_{1}}+\xi_{1}^{p+3} \phi_{11}\left(x_{1}, y_{r}\right)+\delta_{1} \tag{4.7}
\end{align*}
$$

where $\phi_{11}\left(x_{1}, y_{r}\right)=\left(\left(p-p_{1}+3\right) \varphi_{1}^{\prime}\left(x_{1}, y_{r}\right) /(p+3)\right)^{(p+3) /\left(p-p_{1}+3\right)}\left(p_{1} / \delta_{1}\left(p-p_{1}+3\right)\right)^{p_{1} /\left(p-p_{1}+3\right)}$. Substituting (4.5) and (4.7) into (4.4), and adding and subtracting $\left(b_{1}\left(x_{1}\right) / 2\right) \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}$ on the right-hand side of (4.4), we have

$$
\begin{align*}
\varrho V_{1} \leq & d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{p_{1}}+\frac{b_{1}\left(x_{1}\right)}{2}\left|\xi_{1}^{p-p_{1}+3} x_{2}^{p_{1}}\right|+\xi_{1}^{p+3} \phi_{11}\left(x_{1}, y_{r}\right)+\delta_{1} \\
& +\frac{p-p_{1}+3}{2} \xi_{1}^{p-p_{1}+2} \xi_{1}^{p_{1}+1} \xi_{1}^{2}\left(x_{1}\right)-\tilde{\theta} \dot{\theta} \\
= & d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{p_{1}}+\frac{b_{1}\left(x_{1}\right)}{2}\left|\xi_{1}\right|^{p-p_{1}+3}\left|x_{2}\right|^{p_{1}}+\frac{b_{1}\left(x_{1}\right)}{2} \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}  \tag{4.8}\\
& -\frac{b_{1}\left(x_{1}\right)}{2} \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}+\xi_{1}^{p+3} \phi_{11}\left(x_{1}, y_{r}\right)+\xi_{1}^{p+3} \phi_{12}\left(x_{1}\right)+\delta_{1}-\tilde{\theta} \dot{\theta}
\end{align*}
$$

where $\phi_{12}\left(x_{1}\right)=\left(\left(p-p_{1}+3\right) / 2\right) \xi_{1}^{p-p_{1}} \psi_{1}^{2}\left(x_{1}\right)$. Suppose the virtual smooth controller $x_{2}^{*}=$ $-\xi_{1} \beta_{1}\left(x_{1}, y_{r}, \widehat{\theta}\right)$ with $\beta_{1}\left(x_{1}, y_{r}, \widehat{\theta}\right)>0$, which together with A1 lead to

$$
\begin{gather*}
0 \leq-b_{1}\left(x_{1}\right) \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}} \leq-d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}, \\
-\frac{b_{1}\left(x_{1}\right)}{2} \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}+\frac{b_{1}\left(x_{1}\right)}{2}\left|\xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}\right| \leq-d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}},  \tag{4.9}\\
-\frac{b_{1}\left(x_{1}\right)}{2} \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}} \leq-d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}-\frac{b_{1}\left(x_{1}\right)}{2}\left|\xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}\right| .
\end{gather*}
$$

Substituting (4.9) into (4.8), one can obtain

$$
\begin{align*}
\mathscr{L} V_{1} \leq & d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{p_{1}}+\frac{b_{1}\left(x_{1}\right)}{2}\left|\xi_{1}^{p-p_{1}+3} x_{2}^{p_{1}}\right|+\frac{b_{1}\left(x_{1}\right)}{2} \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}} \\
& -d_{1}(x) \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}-\frac{b_{1}\left(x_{1}\right)}{2}\left|\xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}}\right|+\delta_{1}-\tilde{\theta} \dot{\theta} \\
& -\bar{c}_{1} \theta \xi_{1}^{p+3}+\bar{c}_{1} \theta \xi_{1}^{p+3}+\xi_{1}^{p+3} \phi_{11}\left(x_{1}, y_{r}\right)+\xi_{1}^{p+3} \phi_{12}\left(x_{1}\right)  \tag{4.10}\\
\leq & -\bar{c}_{1} \theta \xi_{1}^{p+3}+\left(\theta \bar{b}_{1}\left(\bar{x}_{2}\right)+\frac{b_{1}\left(x_{1}\right)}{2}\right)\left|\xi_{1}\right|^{p-p_{1}+3}\left|x_{2}^{p_{1}}-x_{2}^{* p_{1}}\right|+\frac{b_{1}\left(x_{1}\right)}{2} \xi_{1}^{p-p_{1}+3} x_{2}^{* p_{1}} \\
& +\bar{c}_{1} \hat{\theta} \xi_{1}^{p+3}+\xi_{1}^{p+3} \phi_{11}\left(x_{1}, y_{r}\right)+\xi_{1}^{p+3} \phi_{12}\left(x_{1}\right)+\delta_{1}+\tilde{\theta}\left(\tau_{1}-\dot{\hat{\theta}}\right)
\end{align*}
$$

where $\tau_{1}=\bar{c}_{1} \xi_{1}^{p+3}$ is a nonnegative smooth function. Choose $x_{2}^{*}$ as follows:

$$
\begin{align*}
& x_{2}^{*}\left(x_{1}, y_{r}, \hat{\theta}\right)=-\xi_{1} \beta_{1}\left(x_{1}, y_{r}, \hat{\theta}\right) \\
& \beta_{1}\left(x_{1}, y_{r}, \widehat{\theta}\right)=\left(\frac{2}{b_{1}\left(x_{1}\right)}\left(c_{1}+\phi_{11}\left(x_{1}, y_{r}\right)+\phi_{12}\left(x_{1}\right)+\bar{c}_{1} \sqrt{1+\hat{\theta}^{2}}\right)\right)^{1 / p_{1}} \tag{4.11}
\end{align*}
$$

where $\beta_{1}\left(x_{1}, y_{r}, \widehat{\theta}\right) \geq 0$ is a smooth function. Then,

$$
\begin{equation*}
\varrho V_{1} \leq-c_{1} \xi_{1}^{p+3}-\bar{c}_{1} \theta \xi_{1}^{p+3}+\left(\theta \bar{b}_{1}\left(\bar{x}_{2}\right)+\frac{b_{1}\left(x_{1}\right)}{2}\right)\left|\xi_{1}\right|^{p-p_{1}+3}\left|x_{2}^{p_{1}}-x_{2}^{* p_{1}}\right|+\delta_{1}+\tilde{\theta}\left(\tau_{1}-\dot{\hat{\theta}}\right) \tag{4.12}
\end{equation*}
$$

Step i. $2 \leq i \leq n$ : Assume that at Step $i-1$, there exists a smooth state-feedback virtual control

$$
\begin{equation*}
x_{i}^{*}\left(\bar{x}_{i-1}, y_{r}, \widehat{\theta}\right)=-\beta_{i-1}\left(\bar{x}_{i-1}, y_{r}, \widehat{\theta}\right) \xi_{i-1} \tag{4.13}
\end{equation*}
$$

such that

$$
\begin{align*}
\mathscr{L} V_{i-1} \leq & -\sum_{j=1}^{i-1}\left(c_{j}-\sum_{k=j+1}^{i-1} c_{k j}\right) \xi_{j}^{p+3}-\sum_{j=1}^{i-1}\left(\bar{c}_{j}-\sum_{k=j+1}^{i-1} \bar{c}_{k j}\right) \theta \xi_{j}^{p+3} \\
& +\left(\theta \bar{b}_{i-1}+\frac{b_{i-1}}{2}\right)\left|\xi_{i-1}\right|^{p-p_{i-1}+3}\left|x_{i}^{p_{i-1}}-x_{i}^{* p_{i-1}}\right|+\sum_{j=1}^{i-1} \delta_{j}+\left(\tilde{\theta}+\sum_{k=2}^{i-1} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\right)\left(\tau_{i-1}-\dot{\hat{\theta}}\right), \tag{4.14}
\end{align*}
$$

where $\beta_{i-1}>0$ is a smooth function, and $V_{i-1}=(1 / 4) \sum_{k=1}^{i-1} \xi_{k}^{p-p_{k}+4}+(1 / 2) \widetilde{\theta}^{2}$. We will prove that (4.14) still holds for Step $i$.

Define the $i$ th Lyapunov candidate function

$$
\begin{equation*}
V_{i}=V_{i-1}+\frac{1}{4} \xi_{i}^{p-p_{i}+4} \tag{4.15}
\end{equation*}
$$

From (4.2) and (4.14), it follows that

$$
\begin{align*}
\varrho V_{i} \leq & -\sum_{j=1}^{i-1}\left(c_{j}-\sum_{k=j+1}^{i-1} c_{k j}\right) \xi_{j}^{p+3}-\sum_{j=1}^{i-1}\left(\bar{c}_{j}-\sum_{k=j+1}^{i-1} \bar{c}_{k j}\right) \theta \xi_{j}^{p+3} \\
& +\left(\theta \bar{b}_{i-1}+\frac{b_{i-1}}{2}\right)\left|\xi_{i-1}\right|^{p-p_{i-1}+3}\left|x_{i}^{p_{i-1}}-x_{i}^{* p_{i-1}}\right|+\sum_{j=1}^{i-1} \delta_{j} \\
& +\left(\tilde{\theta}+\sum_{k=2}^{i-1} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\right)\left(\tau_{i-1}-\dot{\theta}\right)+\xi_{i}^{p-p_{i}+3} d_{i}(x) x_{i+1}^{p_{i}}+\xi_{i}^{p-p_{i}+3}  \tag{4.16}\\
& \times\left(f_{i}-\sum_{k=1}^{i-1} \frac{\partial x_{i}^{*}}{\partial x_{k}}\left(d_{k}(x) x_{k}^{p_{k}}+f_{k}\right)-\frac{1}{2} \sum_{j, k=1}^{i-1} \frac{\partial^{2} x_{i}^{*}}{\partial x_{j} \partial x_{k}} g_{j}^{T} g_{k}-\frac{\partial x_{i}^{*}}{\partial y_{r}} \dot{y}_{r}-\frac{\partial x_{i}^{*}}{\partial \widehat{\theta}} \dot{\theta}\right) \\
& +\frac{1}{2} \operatorname{Tr}\left\{G_{i}\left(p-p_{i}+3\right) \xi_{i}^{p-p_{i}+2} G_{i}^{T}\right\} .
\end{align*}
$$

By A2 and Lemma 3.4, there is a smooth nonnegative function $\bar{\varphi}_{i}\left(\bar{x}_{i}\right)$ such that

$$
\begin{equation*}
\left|f_{i}\left(\bar{x}_{i+1}\right)\right| \leq \sum_{j=0}^{p_{i}-1}\left|x_{i+1}\right|^{j} \varphi_{i j}\left(\bar{x}_{i}\right) \leq \frac{b_{i}\left(\bar{x}_{i}\right) \mid x_{i+1}^{p_{i}}}{2}+\bar{\varphi}_{i}\left(\bar{x}_{i}\right) \tag{4.17}
\end{equation*}
$$

then,

$$
\begin{equation*}
\left|f_{i}-\sum_{k=1}^{i-1} \frac{\partial x_{i}^{*}}{\partial x_{k}} f_{k}-\frac{1}{2} \sum_{j, k=1}^{i-1} \frac{\partial^{2} x_{i}^{*}}{\partial x_{j} \partial x_{k}} g_{j}^{T} g_{k}-\frac{\partial x_{i}^{*}}{\partial y_{r}} \dot{y}_{r}\right| \leq \frac{b_{i}\left(\bar{x}_{i}\right) \mid x_{i+1}^{p_{i}}}{2}+\varphi_{i}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right) \tag{4.18}
\end{equation*}
$$

where $\varphi_{i}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right)$ is a smooth function. By A2, (4.1) and (4.13), there exists a nonnegative smooth function $\psi_{i}^{\prime}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right)$ such that

$$
\begin{equation*}
\left|G_{i}\left(\bar{x}_{i}\right)\right| \leq\left(\left|\xi_{j}\right|^{\left(p_{i}+1\right) / 2}+\cdots+\left|\xi_{i}\right|^{\left(p_{i}+1\right) / 2}\right) \psi_{i}^{\prime}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) . \tag{4.19}
\end{equation*}
$$

By (4.13), we have

$$
\begin{align*}
& \left(\theta \bar{b}_{i-1}\left(\bar{x}_{i}\right)+\frac{b_{i-1}\left(x_{i-1}\right)}{2}\right)\left|\xi_{i-1}\right|^{p-p_{i-1}+3}\left|x_{i}^{p_{i-1}}-x_{i}^{* p_{i-1}}\right| \\
& \quad=\left(\theta \bar{b}_{i-1}\left(\bar{x}_{i}\right)+\frac{b_{i-1}\left(x_{i-1}\right)}{2}\right) \sum_{k=1}^{p_{i-1}} C_{p_{i-1}}^{k}\left|\xi_{i}\right|^{k}\left|\xi_{1}\right|^{p-k+3} \beta_{1}^{p_{i-1}-k} \\
& \quad \leq \sum_{k=1}^{i-1} c_{i k 1} \xi_{k}^{p+3}+\sum_{k=1}^{i-1} \bar{c}_{i k} \theta \xi_{k}^{p+3}+\varphi_{i 1}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) \xi_{i}^{p+3}+\theta \varphi_{i 2} \xi_{i}^{p+3}, \tag{4.20}
\end{align*}
$$

where $\varphi_{i 1}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right)$ and $\varphi_{i 2}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right)$ are two smooth functions. From A1, (4.1), and (4.13), it follows that

$$
\begin{align*}
& \left|-\xi_{i}^{p-p_{i}+3} \sum_{k=1}^{i-1} \frac{\partial x_{i}^{*}}{\partial x_{k}} d_{k}(x) x_{k}^{p_{k}}\right| \\
& \quad \leq \theta^{\prime}\left|\xi_{i}\right|^{p-p_{i}+3} \sum_{k=1}^{i-1} b_{k-1}\left(\bar{x}_{k}\right)\left|\frac{\partial x_{i}^{*}}{\partial x_{k}}\right| \xi_{k}+\left.x_{k}^{*}\right|^{p_{k}} \\
& \quad \leq \theta^{\prime \prime(p+3) /\left(p-p_{i}+3\right)} \xi_{i}^{p+3} \varphi_{i 3}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right)+\delta_{i 1} \\
& \quad \leq \theta \xi_{i}^{p+3} \varphi_{i 3}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right)+\delta_{i 1},  \tag{4.21}\\
& \left|\xi_{i}\right|^{p-p_{i}+3}\left|f_{i}-\sum_{k=1}^{i-1} \frac{\partial x_{i}^{*}}{\partial x_{k}} f_{k}-\frac{1}{2} \sum_{j, k=1}^{i-1} \frac{\partial^{2} x_{i}^{*}}{\partial x_{j} \partial x_{k}} g_{j}^{T} g_{k}-\frac{\partial x_{i}^{*}}{\partial y_{r}} \dot{y}_{r}\right| \\
& \quad \leq\left|\xi_{i}\right|^{p-p_{i}+3}\left(\frac{b_{i}\left(\bar{x}_{i}\right)\left|x_{i+1}\right|^{p_{i}}}{2}+\varphi_{i}^{\prime}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right)\right)  \tag{4.22}\\
& \quad \leq \frac{b_{i}\left(\bar{x}_{i}\right)}{2}\left|\xi_{i}\right|^{p-p_{i}+3}\left|x_{i+1}\right|^{p_{i}}+\varphi_{i 4}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) \xi_{i}^{p+3}+\delta_{i 2},
\end{align*}
$$

where $\varphi_{i 3}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right)$ and $\varphi_{i 4}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right)$ are two smooth functions. From (4.19), one can obtain

$$
\frac{1}{2} \operatorname{Tr}\left\{G_{i}\left(p-p_{i}+3\right) \mathfrak{\xi}_{i}^{p-p_{i}+2} G_{i}^{T}\right\}
$$

$$
\begin{align*}
& \leq \frac{p-p_{i}+3}{2} \xi_{2}^{p-p_{i}+2}\left(\left|\xi_{1}\right|^{\left(p_{i}+1\right) / 2}+\cdots+\left|\xi_{i}\right|^{\left(p_{i}+1\right) / 2}\right)^{2} \psi_{i}^{\prime 2}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) \\
& \leq \sum_{k=1}^{i-1} c_{i k 2} \xi_{k}^{p+3}+\varphi_{i 5}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) \xi_{i}^{p+3}, \tag{4.23}
\end{align*}
$$

where $\varphi_{i 5}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right)$ is a smooth nonnegative function. Substituting (4.20)-(4.23) into (4.16), one gets

$$
\begin{align*}
\mathscr{L} V_{i} \leq & -\sum_{j=1}^{i-1}\left(c_{j}-\sum_{k=j+1}^{i-1} c_{k j}\right) \xi_{j}^{p+3}-\sum_{j=1}^{i}\left(\bar{c}_{j}-\sum_{k=j+1}^{i-1} \bar{c}_{k j}\right) \theta \xi_{j}^{p+3}+\sum_{j=1}^{i-1} c_{i j}^{\prime} \xi_{j}^{p+3} \\
& +\sum_{j=1}^{i-1} \delta_{j}+\theta \sum_{j=1}^{i-1} \bar{c}_{i j} \xi_{j}^{p+3}+h_{i 1}^{\prime} \xi_{i}^{p+3}+\theta h_{i 2} \xi_{i}^{p+3}-\bar{c}_{i} \theta \xi_{i}^{p+3}+\bar{c}_{i} \theta \xi_{i}^{p+3} \\
& +\xi_{i}^{p-p_{i}+3} d_{i}(x) x_{i+1}^{p_{i}}+\frac{b_{i}\left(\bar{x}_{i}\right)}{2}\left|\xi_{i}\right|^{p-p_{i}+3}\left|x_{i+1}\right|^{p_{i}}+\frac{b_{i}\left(\bar{x}_{i}\right)}{2} \xi_{i}^{p-p_{i}+3} x_{3}^{* p_{i}} \\
& -\frac{b_{i}\left(\bar{x}_{i}\right)}{2} \xi_{i}^{p-p_{i}+3} x_{3}^{* p_{i}}+\left(\tilde{\theta}+\sum_{k=2}^{i-1} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\right)\left(\tau_{i-1}-\dot{\theta}\right)-\xi_{i}^{p-p_{i}+3} \frac{\partial x_{i}^{*}}{\partial \widehat{\theta}} \hat{\theta}, \tag{4.24}
\end{align*}
$$

where

$$
\begin{gather*}
c_{i j}^{\prime}=c_{i j 1}+c_{i j 2}, \quad j=1, \ldots, i-1,  \tag{4.25}\\
h_{i 1}^{\prime}=\varphi_{i 1}+\varphi_{i 4}+\varphi_{i 5}, \quad h_{i 2}=\varphi_{i 2}+\varphi_{i 3} .
\end{gather*}
$$

Suppose the virtual smooth controller $x_{i+1}^{*}=-\xi_{i} \beta_{i}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right)$ with $\beta_{i}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right)>0$, which together with A2 render

$$
\begin{equation*}
-\frac{b_{i}\left(\bar{x}_{i}\right)}{2} \xi_{i}^{p-p_{i}+3} x_{i+1}^{* p_{i}} \leq-d_{i}(x) \xi_{i}^{p-p_{i}+3} x_{i+1}^{* p_{i}}-\frac{b_{i}\left(\bar{x}_{i}\right)}{2}\left|\xi_{i}^{p-p_{i}+3} x_{i+1}^{* p_{i}}\right| . \tag{4.26}
\end{equation*}
$$

Substituting (4.26) into (4.24) leads to

$$
\begin{aligned}
\varrho V_{i} \leq & -\sum_{j=1}^{i-1}\left(c_{j}-\sum_{k j j+1}^{i-1} c_{k j}\right) \xi_{j}^{p+3}-\sum_{j=1}^{i}\left(\bar{c}_{j}-\sum_{k=j+1}^{i} \bar{c}_{k j}\right) \theta \xi_{j}^{p+3}+\sum_{j=1}^{i-1} c_{i j}^{\prime} \xi_{j}^{p+3} \\
& +h_{i 1}^{\prime} \xi_{i}^{\xi^{p+3}}+(\hat{\theta}+\tilde{\theta}) h_{i 2} \xi_{i}^{p+3}+\bar{c}_{i}(\hat{\theta}+\tilde{\theta}) \xi_{i}^{p+3}+\xi_{i}^{p-p_{i}+3} d_{i}(x) x_{i+1}^{p_{i}} \\
& +\frac{b_{i}\left(\bar{x}_{i}\right)}{2}\left|\xi_{i}\right|^{p-p_{i}+3}\left|x_{i+1}\right|^{p_{i}}+\frac{b_{i}\left(\bar{x}_{i}\right)}{2} \xi_{i}^{p-p_{i}+3} x_{3}^{* p_{i}}-d_{i}(x) \xi_{i}^{p-p_{i}+3} x_{i+1}^{* p_{2}}
\end{aligned}
$$

$$
\begin{align*}
& -\frac{b_{i}\left(\bar{x}_{i}\right)}{2}\left|\xi_{i}^{p-p_{i}+3} x_{i+1}^{* p_{i}}\right|+\left(\tilde{\theta}+\sum_{k=2}^{i-1} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \hat{\theta}}\right)\left(\tau_{i-1}-\dot{\hat{\theta}}\right)-\xi_{i}^{p-p_{i}+3} \frac{\partial x_{i}^{*}}{\partial \widehat{\theta}} \hat{\theta} \\
& +\sum_{k=2}^{i-1} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\left(h_{i 2}+\bar{c}_{i}\right) \xi_{i}^{p+3}-\sum_{k=2}^{i-1} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\left(h_{i 2}+\bar{c}_{i}\right) \xi_{i}^{p+3} \\
& +\xi_{i}^{p-p_{i}+3} \frac{\partial x_{i}^{*}}{\partial \widehat{\theta}} \tau_{i}-\xi_{i}^{p-p_{i}+3} \frac{\partial x_{i}^{*}}{\partial \widehat{\theta}} \tau_{i} \tag{4.27}
\end{align*}
$$

where $\tau_{i}=\tau_{i-1}+\left(h_{i 2}+\bar{c}_{i}\right) \xi_{i}^{p+3}$. For (4.27), we have

$$
\begin{align*}
& \left|-\sum_{k=2}^{i-1} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\left(h_{i 2}+\bar{c}_{i}\right) \xi_{i}^{\xi^{p+3}}\right| \leq \varphi_{i 6}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right) \xi_{i}^{p+3},  \tag{4.28}\\
& \left|-\xi_{i}^{p-p_{i}+3} \frac{\partial x_{i}^{*}}{\partial \widehat{\theta}} \tau_{i}\right| \leq \sum_{k=1}^{i-1} c_{i k} \xi_{k}^{p+3}+\varphi_{i 7}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right) \xi_{i}^{p+3},
\end{align*}
$$

where $c_{i k 3}$ is a design parameter, $\varphi_{i 6}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right)$ and $\varphi_{i 7}\left(\bar{x}_{i}, y_{r}, \widehat{\theta}\right)$ are the smooth functions. Let $c_{i j}=c_{i j}^{\prime}+c_{i j 3}, h_{i 1}=h_{i 1}^{\prime}+\varphi_{i 6}+\varphi_{i 7}$. (4.27) becomes

$$
\begin{align*}
\mathscr{L} V_{i} \leq & -\sum_{j=1}^{i-1}\left(c_{j}-\sum_{k=j+1}^{i-1} c_{k j}\right) \xi_{j}^{p+3}-\sum_{j=1}^{i}\left(\bar{c}_{j}-\sum_{k=j+1}^{i} \bar{c}_{k j}\right) \theta \xi_{j}^{p+3}+\sum_{j=1}^{i-1} c_{i j} \xi_{j}^{p+3} \\
& +h_{i 1}^{\prime} \xi_{i}^{p+3}+\hat{\theta} h_{i 2} \xi_{i}^{p+3}+\left(\theta \bar{\theta}_{i}\left(\bar{x}_{i+1}\right)+\frac{b_{i}\left(x_{i}\right)}{2}\right)\left|\xi_{i}\right|^{p-p_{i}+3}\left|x_{i+1}^{p_{i}}-x_{i+1}^{* p_{i}}\right| \\
& +\frac{b_{i}\left(\bar{x}_{i}\right)}{2} \xi_{i}^{p-p_{i}+3} x_{i+1}^{* p_{i}}+\left(\tilde{\theta}+\sum_{k=2}^{i} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\right)\left(\tau_{i}-\dot{\hat{\theta}}\right)+\sum_{j=1}^{i} \delta_{j} \\
\leq & -\sum_{j=1}^{i}\left(c_{j}-\sum_{k=j+1}^{i} c_{k j}\right) \xi_{j}^{p+3}-\sum_{j=1}^{i}\left(\bar{c}_{j}-\sum_{k=j+1}^{i} \bar{c}_{k j}\right) \theta \xi_{j}^{p+3} \\
& +\left(\theta \bar{b}_{i}\left(\bar{x}_{i+1}\right)+\frac{b_{i}\left(x_{i}\right)}{2}\right)\left|\xi \xi_{i}\right|^{p-p_{i}+3}\left|x_{i+1}^{p_{i}}-x_{i+1}^{* p_{i}}\right|+\sum_{j=1}^{i} \delta_{j}+\left(\tilde{\theta}+\sum_{k=2}^{i} \xi_{k}^{p-p_{k}+3} \frac{\partial x_{k}^{*}}{\partial \widehat{\theta}}\right)\left(\tau_{i}-\dot{\hat{\theta}}\right), \tag{4.29}
\end{align*}
$$

by choosing

$$
\begin{align*}
x_{i+1}^{*}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) & =-\xi_{i} \beta_{i}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right), \\
\beta_{i}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) & =\left(\frac{2}{b_{i}\left(\bar{x}_{i}\right)}\left(c_{i}+h_{i 1}+\left(h_{i 2}+\bar{c}_{i}\right) \sqrt{1+\hat{\theta}^{2}}\right)\right)^{1 / p_{i}} \tag{4.30}
\end{align*}
$$

where $\beta_{i}\left(\bar{x}_{i}, y_{r}, \hat{\theta}\right) \geq 0$ is a smooth function.


Figure 2: Gives the response of the closed-loop system, from which, the effectiveness of the controller is demonstrated.

Finally, when $i=n, x_{n+1}=x_{n+1}^{*}=u$ is the actual control. By choosing the actual control law and the adaptive law:

$$
\begin{equation*}
u\left(\bar{x}_{n}, y_{r}, \widehat{\theta}\right)=-\beta_{n}\left(\bar{x}_{n}, y_{r}, \hat{\theta}\right) \xi_{n,} \quad \dot{\hat{\theta}}=\tau_{n}=\sum_{k=1}^{n} H_{k 2} \xi_{k}^{p+3} \tag{4.31}
\end{equation*}
$$

where $\beta_{n} \geq 0$ and $H_{12}, \ldots, H_{n 2}$ are smooth functions, one gets

$$
\begin{equation*}
\perp V_{n} \leq-\sum_{j=1}^{n}\left(c_{j}-\sum_{k=j+1}^{n} c_{k j}\right) \xi_{j}^{p+3}-\sum_{j=1}^{n}\left(\bar{c}_{j}-\sum_{k=j+1}^{n} \bar{c}_{k j}\right) \theta \xi_{j}^{p+3}+\sum_{j=1}^{n} \delta_{j} \tag{4.32}
\end{equation*}
$$

Theorem 4.1. If A1-A3 hold for the high-order stochastic nonlinear system (3.8), under the smooth adaptive state-feedback controller (4.32), the closed-loop system is ISpS in probability, and the tracking error $\xi_{1}=y-y_{r}$ can be regulated to a neighborhood of the origin in probability with radius as small as possible (Figure 2).

Proof. For $V_{n}=\sum_{i=1}^{n}(1 / 4) \xi_{i}^{p-p_{i}+4}+(1 / 2) \tilde{\theta}^{2}$, it is obvious that $V_{n}$ satisfies (3.4). Choosing all the design parameters $c_{j}$ and $\bar{c}_{j}$ to satisfy

$$
\begin{equation*}
c_{j}>\sum_{k=j+1}^{n} c_{k j}, \quad \bar{c}_{j}>\sum_{k=j+1}^{n} \bar{c}_{k j}, \quad j=1, \ldots, n \tag{4.33}
\end{equation*}
$$

such that (3.5) holds, and then using Lemma 3.3, one can prove Theorem 4.1.

## 5. Simulation

Now, we apply the control scheme to the mechanical system (2.5). Let $\xi_{1}=x_{1}-y_{r}$ be the tracking error, where $y_{r}=\sin t$ is a bounded smooth reference signal. For (2.5), $d_{i}(\cdot)=1$, and $p=\max \{1,3\}=3$.

Choose $V_{1}\left(\xi_{1}\right)=\left(1 /\left(p-p_{1}+4\right)\right) \xi_{1}^{p-p_{1}+4}=\xi_{1}^{6} / 6$. Then,

$$
\begin{equation*}
\rho V_{1}\left(\xi_{1}\right)=\xi_{1}^{5}\left(x_{2}-\dot{y}_{r}\right) \tag{5.1}
\end{equation*}
$$

The smooth virtual controller can be chosen as $x_{2}^{*}\left(x_{1}, y_{r}\right)=-c_{1} \xi_{1}+\dot{y}_{r}$, which renders

$$
\begin{equation*}
\mathcal{L} V_{1}\left(\xi_{1}\right)=-c_{1} \xi_{1}^{6}+\xi_{1}^{5}\left(x_{2}-x_{2}^{*}\right) \tag{5.2}
\end{equation*}
$$

Next, defining $V_{2}\left(\xi_{1}, \xi_{2}\right)=V_{1}+\left(1 /\left(p-p_{2}+4\right)\right) \xi_{2}^{p-p_{2}+4}=\xi_{1}^{6} / 6+\xi_{2}^{4} / 4$, a direct calculation gives

$$
\begin{equation*}
\rho V_{2}=-c_{1} \xi_{1}^{6}+\xi_{1}^{5} \xi_{2}+\xi_{2}^{3}\left(x_{3}^{3}-\frac{\partial x_{2}^{*}}{\partial x_{1}} x_{2}-\frac{\partial x_{2}^{*}}{\partial y_{r}} \dot{y}_{r}\right)=-c_{1} \xi_{1}^{6}+\xi_{1}^{5} \xi_{2}+\xi_{2}^{3}\left(x_{3}^{3}-h_{2}\right) \tag{5.3}
\end{equation*}
$$

where $\xi_{2}=x_{2}-x_{2}^{*}$. By Lemma 3.5, choosing $m=3 / 2$, one can obtain that for any constant $\delta_{2}>0$,

$$
\begin{equation*}
\left|\xi_{2}^{4} h_{2}\right| \leq \delta_{2}+\left(\frac{2 \xi_{2}^{4} h_{2}}{3}\right)^{3 / 2}\left(\frac{1}{2 \delta_{2}}\right)^{1 / 2} \leq \delta_{2}+\xi_{2}^{6} \varphi_{2}\left(\bar{x}_{2}\right) \tag{5.4}
\end{equation*}
$$

Then, by (5.4) and (5.5), it is easy to see that

$$
\begin{equation*}
\varrho V\left(\xi_{1}, \xi_{2}\right)=-\left(c_{1}-c_{21}\right) \xi_{1}^{6}-c_{2} \xi_{2}^{6}+\xi_{2}^{3}\left(x_{3}^{3}-x_{3}^{* 3}\right)+\delta_{2} \tag{5.5}
\end{equation*}
$$

by choosing $x_{3}^{*}\left(x_{1}, x_{2}, y_{r}\right)=-\xi_{2}\left(c_{2}+d_{2}+\varphi_{2}\right)^{1 / 3}$.
Defining $\xi_{3}=x_{3}-x_{3}^{*}$ and the Lyapunov function $V_{3}\left(\xi_{1}, \xi_{2}, \xi_{3}\right)=V_{2}\left(\xi_{1}, \xi_{2}\right)+(1 / 6) \xi_{3}^{6}$, one gets

$$
\begin{align*}
\mathscr{L} V_{3} & \leq-\left(c_{1}-c_{21}\right) \xi_{1}^{6}-c_{2} \xi_{2}^{6}+\xi_{2}^{3}\left(x_{3}^{3}-x_{3}^{* 3}\right)+\delta_{2}+\xi_{3}^{5}\left(x_{4}-\frac{\partial x_{3}^{*}}{\partial x_{1}} x_{2}-\frac{\partial x_{3}^{*}}{\partial x_{2}} x_{3}^{3}-\frac{\partial x_{3}^{*}}{\partial y_{r}} \dot{y}_{r}\right) \\
& \leq-\left(c_{1}-c_{21}\right) \xi_{1}^{6}-c_{2} \xi_{2}^{6}+\delta_{31}+\xi_{3}^{6} \varphi_{31}+\delta_{32}+\xi_{3}^{6} \varphi_{32}+\xi_{3}^{5}\left(x_{4}-x_{4}^{*}\right)+\xi_{3}^{5} x_{4}^{*}  \tag{5.6}\\
& =-\left(c_{1}-c_{21}\right) \xi_{1}^{6}-c_{2} \xi_{2}^{6}-c_{3} \xi_{3}^{6}+\xi_{3}^{5}\left(x_{4}-x_{4}^{*}\right)+\delta_{2}+\delta_{3}
\end{align*}
$$

by choosing $x_{4}^{*}\left(x_{1}, x_{2}, x_{3}, y_{r}\right)=-\xi_{3}\left(c_{3}+\varphi_{31}+\varphi_{32}\right)$. At last, choosing $\xi_{4}=x_{4}-x_{4}^{*}$, $V_{4}\left(\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}\right)=V_{3}\left(\xi_{1}, \xi_{2}, \xi_{3}\right)+(1 / 6) \xi_{4}^{6}$, a direct calculation gives

$$
\begin{align*}
V_{4} \leq & -\left(c_{1}-c_{21}\right) \xi_{1}^{6}-c_{2} \xi_{2}^{6}-c_{3} \xi_{3}^{6}+\xi_{3}^{5}\left(x_{4}-x_{4}^{*}\right)+\delta_{2}+\delta_{3} \\
& +\xi_{4}^{5}\left(v+k_{0} f-\frac{\partial x_{4}^{*}}{\partial x_{1}} x_{2}-\frac{\partial x_{4}^{*}}{\partial x_{2}} x_{3}^{3}-\frac{\partial x_{4}^{*}}{\partial x_{3}} x_{4}-\frac{\partial x_{4}^{*}}{\partial y_{r}} \dot{y}_{r}\right)+5 \xi_{4}^{4} \sigma^{2} f^{2}  \tag{5.7}\\
\leq & -\left(c_{1}-c_{21}\right) \xi_{1}^{6}-c_{2} \xi_{2}^{6}-c_{3} \xi_{3}^{6}-c_{4} \xi_{4}^{6}+\delta_{2}+\delta_{3}+\delta_{41}+\xi^{6} \varphi_{41}+\delta_{42}+\xi^{6} \varphi_{42}+\xi_{4}^{5} v \\
= & -\left(c_{1}-c_{21}\right) \xi_{1}^{6}-c_{2} \xi_{2}^{6}-c_{3} \xi_{3}^{6}-c_{4} \xi_{4}^{6}+\delta_{2}+\delta_{3}+\delta_{4},
\end{align*}
$$

by choosing

$$
\begin{equation*}
v=-\xi_{4}\left(c_{4}+\varphi_{41}+\varphi_{42}\right) . \tag{5.8}
\end{equation*}
$$

Choose the design parameters $\sigma=0.125, \delta_{2}=0.01, \delta_{3}=0.01$, and $\delta_{4}=0.01$. Moreover, to satisfy (5.3), we choose $c_{1}=1>c_{21}=5 / 6, c_{2}=1.5, c_{3}=0.5$ and $c_{4}=0.5$. Choose the initial values $x_{1}(0)=0.45, x_{2}(0)=0.5, x_{3}(0)=0.5$, and $x_{4}(0)=0.5$.

## 6. Concluding Remarks

In this paper, a mechanical system is firstly introduced. Then, by a series of coordinate transformations, the mechanical system can be transformed to a class of high-order stochastic nonlinear system, based on which, a more general mathematical model is considered and the smooth state-feedback controller is designed which guarantees that the tracking error $\xi_{1}=y-y_{r}$ can be regulated to a neighborhood of the origin in probability with radius as small as possible. At last, the simulation is given to show the effectiveness of the design scheme.

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Research Article

# Adaptive Output Feedback Control for a Class of Stochastic Nonlinear Systems with SiISS Inverse Dynamics 

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#### Abstract

The adaptive stabilization scheme based on tuning function for stochastic nonlinear systems with stochastic integral input-to-state stability (SiISS) inverse dynamics is investigated. By combining the stochastic LaSalle theorem and small-gain type conditions on SiISS, an adaptive output feedback controller is constructively designed. It is shown that all the closed-loop signals are bounded almost surely and the stochastic closed-loop system is globally stable in probability.


## 1. Introduction

Global stabilization control design of stochastic nonlinear systems is one of the most important topics in nonlinear control theory, which has received and is increasingly receiving a great deal of attention; see, for example, [1-37] and the references therein. For a class of stochastic nonlinear systems with stochastic inverse dynamics, much progress has been made in the design of the global stabilization controller [12, 13, 15, 16, 24, 25, 31]. However, all these controllers are only robust against stochastic inverse dynamics with stringent stability margin. To weaken the stringent condition on stochastic inverse dynamics, a natural idea is to benefit from input-to-state stability (ISS) in [38] and integral input-to-state stability (iISS) in [39] which are now recognized as the central unifying concepts in feedback design and stability analysis of deterministic nonlinear systems. Tsinias in [21], Tang and Basar in [19] first proposed the concept of stochastic input-to-state stability (SISS) independently. Further in-depth study on SISS and its applications are presented in [9-11, 18, 22]. Motivated by these aforementioned important results, [34] showed that SISS condition can be weakened to stochastic integral input-to-state stability (SiISS) and developed a unifying output feedback framework for global regulation.

Nonlinear small-gain theorem plays an important role in the controller design and stability analysis of deterministic nonlinear systems in [40,41]. While for stochastic nonlinear systems, there are fewer results on the small-gain theorem. [25] firstly established a gain-function-based stochastic nonlinear small-gain theorem for ISpS in probability. In some succeeding research work, [9, 11, 35] presented some Lyapunov-based small-gain type conditions on SISS and SiISS, respectively. [4] further discusses the relationship of smallgain type conditions on SiISS and studies the problem of GAS in probability via output feedback.

In this paper, inspired by [4], a more general class of stochastic nonlinear systems with uncertain parameters and stochastic integral input-to-state stability (SiISS) inverse dynamics is investigated. By combining the stochastic LaSalle theorem and small-gain type conditions on SiISS, an adaptive output feedback controller is proposed to guarantee that all the closedloop signals are bounded almost surely and the stochastic closed-loop system is globally stable in probability.

The paper is organized as follows. Section 2 begins with the mathematical preliminaries. Section 3 presents statement of the problem. The design of adaptive output feedback controller is given in Section 4. The corresponding analysis is given in Section 5. Section 6 concludes the paper.

## 2. Mathematical Preliminaries

The following notations are used throughout the paper. $\mathbb{R}_{+}$stands for the set of all nonnegative real numbers, $\mathbb{R}^{n}$ is the $n$-dimensional Euclidean space, and $\mathbb{R}^{n \times m}$ is the space of real $n \times m$-matrices. For $x=\left(x_{1}, \ldots, x_{n}\right)$, one defines $\bar{x}_{i}=\left(x_{1}, \ldots, x_{i}\right), i=1, \ldots, n-1$. $C^{i}$ denotes the family of all the functions with continuous $i$ th partial derivatives. $L^{1}\left(\mathbb{R}_{+} ; \mathbb{R}_{+}\right)$is the family of all functions $l: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$such that $\int_{0}^{\infty} l(t) d t<\infty$. For a given vector or matrix $X, X^{T}$ denotes its transpose, $\operatorname{Tr}\{X\}$ denotes its trace when $X$ is square. $|X|$ denotes the Euclidean norm of a vector $X$, and $\|X\|=\left(\operatorname{Tr}\left\{X X^{T}\right\}\right)^{1 / 2}$ for a matrix $X$. $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, which are continuous, strictly increasing, and vanishing at zero; $\mathcal{K}_{\infty}$ is the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{K} \perp$ denotes the set of all functions $\beta(s, t)$ : $\mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, which are of $\mathcal{K}$ for each fixed $t$ and decrease to zero as $t \rightarrow \infty$ for each fixed $s$.

Consider the stochastic nonlinear delay-free system

$$
\begin{equation*}
d x=f(x, t) d t+g(x, t) d w, \quad \forall x(0)=x_{0} \in \mathbb{R}^{n} \tag{2.1}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}, w$ is an $m$-dimensional standard Wiener process defined in a complete probability space $\left\{\Omega, \mathcal{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right\}$ with $\Omega$ being a sample space, $\mathcal{F}$ being a $\sigma$-field, $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ being a filtration, and $P$ being the probability measure. Borel measurable functions $f: \mathbb{R}^{n} \times$ $\mathbb{R}_{+} \rightarrow \mathbb{R}^{n}$ and $g: \mathbb{R}^{n} \times \mathbb{R}_{+} \rightarrow \mathbb{R}^{n \times m}$ are piecewise continuous in $t$ and locally bounded and locally Lipschitz continuous in $x$ uniformly in $t \in \mathbb{R}_{+}$. Let $£ V(x)$ denote infinitesimal generator of function $V \in C^{2}$ along stochastic system (2.1) with the definition of

$$
\begin{equation*}
\varrho V(x)=\frac{\partial V(x)}{\partial x} f(x, t)+\frac{1}{2} \operatorname{Tr}\left\{g^{T}(x, t) \frac{\partial^{2} V(x)}{\partial x^{2}} g(x, t)\right\} \tag{2.2}
\end{equation*}
$$

Definition 2.1 (see [34]). Stochastic process $\{\xi(t)\}_{t \geq t_{0}}$ is said to be bounded almost surely if $\sup _{t \geq t_{0}}|\xi(t)|<\infty$ a.s.

Lemma 2.2 (Stochastic LaSalle Theorem [14]). For system (2.1), if there exist functions $V \in C^{2}$, $\alpha_{1}, \alpha_{2} \in \mathcal{K}_{\infty}, l \in L^{1}\left(\mathbb{R}_{+} ; \mathbb{R}_{+}\right)$, and a continuous nonnegative function $W: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}^{n}, t \geq 0$,

$$
\begin{equation*}
\alpha_{1}(|x|) \leq V(x) \leq \alpha_{2}(|x|), \quad \varrho V(x) \leq-W(x)+l(t) \tag{2.3}
\end{equation*}
$$

then for each $x_{0} \in \mathbb{R}^{n}$,
(i) system (2.1) has a unique strong solution on $[0, \infty)$, and solution $x(t)$ is bounded almost surely;
(ii) when $f(0, t) \equiv 0, g(0, t) \equiv 0, l(t) \equiv 0$, the equilibrium $x=0$ is globally stable in probability.

In the following, we cite two small-gain type conditions on SiISS in [35].
Consider the following stochastic nonlinear system

$$
\begin{equation*}
d x=f(x, v, t) d t+g(x, v, t) d w, \quad \forall x(0)=x_{0} \in \mathbb{R}^{n} \tag{2.4}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}$ is the state, $v \in \mathbb{R}^{m}$ is the input, and $w$ is an $r$-dimensional standard Wiener process defined as in (2.1). Borel measurable functions $f: \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}_{+} \rightarrow \mathbb{R}^{n}$ and $g: \mathbb{R}^{n} \times$ $\mathbb{R}^{m} \times \mathbb{R}_{+} \rightarrow \mathbb{R}^{n \times r}$ are locally bounded and locally Lipschitz continuous with respect to $(x, v)$ uniformly in $t \in \mathbb{R}_{+}$.

Definition 2.3 (see [34]). System (2.4) is said to be stochastic integral input-to-state stable (SiISS) using Lyapunov function if there exist functions $V \in C^{2}\left(\mathbb{R}^{n} ; \mathbb{R}\right), \alpha_{1}, \alpha_{2}, \gamma \in \mathcal{K}_{\infty}$, and a merely positive definite continuous function $\alpha$ such that

$$
\begin{equation*}
\alpha_{1}(|x|) \leq V(x) \leq \alpha_{2}(|x|), \quad \rho V(x) \leq-\alpha(|x|)+\gamma(|v|) . \tag{2.5}
\end{equation*}
$$

The function $V$ satisfying (2.5) is said to be a SiISS-Lyapunov function, and $(\alpha, \gamma)$ in (2.5) is called the SiISS supply rate of system (2.4).

Lemma 2.4 (see [35]). For system (2.4) satisfying (2.5), if there exists a positive definite continuous function $\tilde{\alpha}$ such that $\lim \sup _{s \rightarrow 0_{+}} \tilde{\alpha}(s) / \alpha(s)<\infty, \limsup _{s \rightarrow \infty} \tilde{\alpha}(s) / \alpha(s)<\infty$, then there exists a function $\tilde{\gamma} \in \mathcal{K}_{\infty}$ such that $(\tilde{\alpha}, \tilde{\gamma})$ is a new SiISS supply rate of system (2.4). Moreover, if $\limsup \sin _{s \rightarrow 0+} \gamma(s) / s^{m}<\infty$, then $\lim \sup _{s \rightarrow 0+} \tilde{\gamma}(s) / s^{m}<\infty$, where $m$ is any positive integer.

The following lemma shows that the condition at infinity can be removed if more prior information on stochastic system is known.

Assumption $H$. For functions $g, V, \alpha$ given in (2.4), (2.5) with $\lim _{\inf }^{s \rightarrow \infty}, \alpha(s)=\infty$, there exist known smooth positive definite functions $\phi_{1}, \phi_{2}$ such that $\|g(x, v, t)\| \leq \phi_{1}(|x|),|\partial V(x) / \partial x| \leq$ $\phi_{2}(|x|)$ and $\lim \sup _{s \rightarrow 0+} \phi_{1}^{2}(s) \phi_{2}^{2}(s) / \alpha(s)<\infty$.

Lemma 2.5 (see [35]). For system (2.4) satisfying (2.5) and Assumption H, if there exists a positive definite function $\tilde{\alpha}$ such that

$$
\begin{equation*}
\limsup _{s \rightarrow 0+} \frac{\tilde{\alpha}(s)}{\alpha(s)}<\infty, \quad \int_{0}^{\infty} e^{-\int_{0}^{s}\left(1 / \zeta\left(\alpha_{1}^{-1}(\tau)\right)\right) d \tau}\left[\xi\left(\alpha_{1}^{-1}(s)\right)\right]^{\prime} d s<\infty, \tag{2.6}
\end{equation*}
$$

where $\xi(\cdot) \geq 0, \zeta(\cdot)>0$ are smooth increasing functions with $\xi(s) \alpha(s) \geq 2 \widetilde{\alpha}(s), \zeta(s) \alpha(s) \geq$ $\phi_{1}^{2}(s) \phi_{2}^{2}(s)$ for any $s \geq 0$, then there exists a function $\tilde{\gamma} \in \aleph_{\infty}$ such that $(\tilde{\alpha}, \tilde{\gamma})$ is a new SiISS supply rate of system (2.4). Moreover, if $\lim \sup _{s \rightarrow 0+} \gamma(s) / s^{m}<\infty$, then $\lim \sup _{s \rightarrow 0+} \tilde{\gamma}(s) / s^{m}<\infty$, where $m$ is any positive integer.

Lemma 2.6. Let $x, y$ be real variables, then for any positive integers $m, n$ and continuous function $a(\cdot) \geq 0, a(\cdot) x^{m} y^{n} \leq b|x|^{m+n}+(n /(m+n))((m+n) / m)^{-m / n} a^{(m+n) / n}(\cdot) b^{m / n}|y|^{m+n}$, where $b>0$ is any real number.

## 3. Problem Statement

In this paper, we consider a class of stochastic nonlinear systems described by

$$
\begin{gather*}
d \eta=\varphi_{0}\left(\eta, x_{1}\right) d t+\psi_{0}\left(\eta, x_{1}\right) d w, \\
d x_{1}=x_{2} d t+\varphi_{1}(\eta, x) d t+\psi_{1}(\eta, x) d w, \\
\vdots  \tag{3.1}\\
d x_{n-1}=x_{n} d t+\varphi_{n-1}(\eta, x) d t+\psi_{n-1}(\eta, x) d w, \\
d x_{n}=u d t+\varphi_{n}(\eta, x) d t+\psi_{n}(\eta, x) d w, \\
y=x_{1},
\end{gather*}
$$

where $\left(x_{2}, \ldots, x_{n}\right) \in \mathbb{R}^{n-1}, u, y \in \mathbb{R}$ represent the unmeasurable state, the control input, and the measurable output, respectively. $\eta \in \mathbb{R}^{n_{0}}$ is referred to as the stochastic inverse dynamics. The initial value $\left(\eta^{T}(0), x_{1}(0), \ldots, x_{n}(0)\right)$ can be chosen arbitrarily. $w$ is an $m$-dimensional standard Wiener process defined as in (2.1). Uncertain functions $\varphi_{i}: \mathbb{R}^{n_{0}} \times \mathbb{R}^{n} \rightarrow \mathbb{R}, \psi_{i}$ : $\mathbb{R}^{n_{0}} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}, 1 \leq i \leq n$, are smooth functions. It is assumed that $\varphi_{0}$ and $\psi_{0}$ are locally Lipschitz continuous functions.

The research purpose of this paper is to design an adaptive output feedback controller for system (3.1) by using stochastic LaSalle theorem and small-gain type conditions on SiISS, in such a way that, for all initial conditions, the solutions of the closed-loop system are bounded almost surely and the closed-loop systems are globally stable in probability. To achieve the control purpose, we need the following assumptions.

Assumption 3.1. For each $1 \leq i \leq n$, there exist the unknown constant $\theta_{i}>0$, the known nonnegative smooth functions $\varphi_{i 0}, \varphi_{i 1}, \psi_{i 0}$ and $\psi_{i 1}$ with $\varphi_{i j}(0)=0, \varphi_{i j}(0)=0, j=0,1$, such that $\left|\varphi_{i}(\eta, x)\right| \leq \theta_{i}\left(\varphi_{i 0}(|\eta|)+\varphi_{i 1}\left(x_{1}\right)\right),\left|\psi_{i}(\eta, x)\right| \leq \theta_{i}\left(\psi_{i 0}(|\eta|)+\psi_{i 1}\left(x_{1}\right)\right)$.

For Assumption 3.1, there exist smooth functions $\bar{\varphi}_{i 1}$ and $\bar{\psi}_{i 1}$ satisfying

$$
\begin{equation*}
\varphi_{i 1}\left(x_{1}\right)=x_{1} \bar{\varphi}_{i 1}\left(x_{1}\right), \quad \psi_{i 1}\left(x_{1}\right)=x_{1} \bar{\psi}_{i 1}\left(x_{1}\right), \tag{3.2}
\end{equation*}
$$

which will be frequently used in the subsequent sections.
Assumption 3.2. For the $\eta$-subsystem, there exists an SiISS-Lyapunov function $V_{0}(\eta)$. Namely, $V_{0}$ satisfies $\underline{\alpha}(|\eta|) \leq V_{0}(\eta) \leq \bar{\alpha}(|\eta|), ~ \varrho V_{0}(\eta) \leq-\alpha(|\eta|)+\gamma\left(\left|x_{1}\right|\right)$, where $\underline{\alpha}, \bar{\alpha}, \gamma$ are class $\not_{\infty}$ functions, and $\alpha$ is merely a continuous positive definite function.

## 4. Design of an Adaptive Output Feedback Controller

### 4.1. Reduced-Order Observer Design

Introduce the following reduced-order observer:

$$
\begin{gather*}
\dot{\hat{x}}_{i}=\widehat{x}_{i+1}+k_{i+1} y-k_{i}\left(\widehat{x}_{1}+k_{1} y\right), \quad 1 \leq i \leq n-2, \\
\dot{\hat{x}}_{n-1}=u-k_{n-1}\left(\widehat{x}_{1}+k_{1} y\right), \tag{4.1}
\end{gather*}
$$

where $k=\left(k_{1}, \ldots, k_{n-1}\right)^{T}$ is chosen such that $A_{0}=\left[\begin{array}{cc}I_{n-2} \\ -k & \ldots . .0\end{array}\right]$ is asymptotically stable. Define the error variable

$$
\begin{equation*}
e_{i}=\frac{1}{\theta^{*}}\left(x_{i+1}-\widehat{x}_{i}-k_{i} x_{1}\right), \quad 1 \leq i \leq n-1, \theta^{*}=\max \left\{1, \theta_{1}, \ldots, \theta_{n}\right\} \tag{4.2}
\end{equation*}
$$

By (3.1), (4.1), and (4.2), one gets

$$
\begin{equation*}
d e_{i}=\left(e_{i+1}-k_{i} e_{1}\right) d t+\frac{1}{\theta^{*}}\left(\varphi_{i+1}(\eta, x)-k_{i} \varphi_{1}(\eta, x)\right) d t+\frac{1}{\theta^{*}}\left(\psi_{i+1}(\eta, x)-k_{i} \psi_{1}(\eta, x)\right) d w \tag{4.3}
\end{equation*}
$$

whose compact form is

$$
\begin{equation*}
d e=\left(A_{0} e+\Phi(\eta, x)\right) d t+\Psi(\eta, x) d w \tag{4.4}
\end{equation*}
$$

where

$$
\begin{gather*}
e=\left(e_{1}, \ldots, e_{n-1}\right)^{T}, \\
\Phi(\eta, x)=\frac{1}{\theta^{*}}\left(\varphi_{2}(\eta, x)-k_{1} \varphi_{1}(\eta, x), \varphi_{3}(\eta, x)-k_{2} \varphi_{1}(\eta, x), \ldots, \varphi_{n}(\eta, x)-k_{n-1} \varphi_{1}(\eta, x)\right)^{T}, \\
\Psi(\eta, x)=\frac{1}{\theta^{*}}\left(\psi_{2}(\eta, x)-k_{1} \psi_{1}(\eta, x), \psi_{3}(\eta, x)-k_{2} \psi_{1}(\eta, x), \ldots, \psi_{n}(\eta, x)-k_{n-1} \psi_{1}(\eta, x)\right)^{T} . \tag{4.5}
\end{gather*}
$$

### 4.2. The Design of Adaptive Backstepping Controller

From (3.1), (4.1), (4.2), and (4.4), the interconnected system is represented as

$$
\begin{gather*}
d \eta=\varphi_{0}(\eta, y) d t+\psi_{0}(\eta, y) d w, \\
d e=\left(A_{0} e+\Phi(\eta, x)\right) d t+\Psi(\eta, x) d w, \\
d y=\left(\widehat{x}_{1}+k_{1} y+\theta^{*} e_{1}+\varphi_{1}(\eta, x)\right) d t+\psi_{1}(\eta, x) d w, \\
d \widehat{x}_{1}=\left(\hat{x}_{2}+k_{2} y-k_{1}\left(\hat{x}_{1}+k_{1} y\right)\right) d t,  \tag{4.6}\\
d \widehat{x}_{2}=\left(\hat{x}_{3}+k_{3} y-k_{2}\left(\hat{x}_{1}+k_{1} y\right)\right) d t, \\
\vdots \\
d \widehat{x}_{n-1}=\left(u-k_{n-1}\left(\widehat{x}_{1}+k_{1} y\right)\right) d t .
\end{gather*}
$$

Next, we will develop an adaptive backstepping controller by using the backstepping method. Firstly, a coordinate transformation is introduced

$$
\begin{equation*}
z_{1}=y, \quad z_{i+1}=\widehat{x}_{i}-\alpha_{i}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{i-1}, \widehat{\theta}\right), \quad i=1, \ldots, n-1 \tag{4.7}
\end{equation*}
$$

Step 1. By (4.6) and (4.7), one has

$$
\begin{equation*}
d z_{1}=\left(\alpha_{1}+z_{2}+k_{1} y+\theta^{*} e_{1}+\varphi_{1}(\eta, x)\right) d t+\psi_{1}(\eta, x) d w \tag{4.8}
\end{equation*}
$$

Since $A_{0}$ is asymptotically stable, there exists a positive definite matrix $P$ such that $P A_{0}+$ $A_{0}^{T} P=-I_{n-1}$. Choose

$$
\begin{equation*}
V_{1}\left(e, z_{1}, \hat{\theta}\right)=\frac{\delta}{2}\left(e^{T} P e\right)^{2}+\frac{1}{2 \Gamma} \tilde{\theta}^{2}+\frac{1}{4} z_{1}^{4}, \quad \delta>0, \Gamma>0 \tag{4.9}
\end{equation*}
$$

where $\widetilde{\theta}=\widehat{\theta}-\theta, \widehat{\theta}$ is the estimate of $\theta=\max \left\{\theta^{*}, \theta^{*(4 / 3)}, \theta^{* 4}\right\}$. In view of (2.2), (4.4), (4.8), and (4.9), then

$$
\begin{aligned}
\rho V_{1}= & 2 \delta e^{T} P e e^{T} P\left(A_{0} e+\Phi(\eta, x)\right)+\frac{1}{2} \operatorname{Tr}\left\{\Psi^{T}(\eta, x)\left(4 \delta\left(e^{T} P\right)^{T} e^{T} P+2 \delta e^{T} P e P\right) \Psi(\eta, x)\right\} \\
& +\frac{1}{\Gamma} \tilde{\theta} \dot{\theta}+z_{1}^{3}\left(\alpha_{1}+z_{2}+k_{1} y+\theta^{*} e_{1}+\varphi_{1}(\eta, x)\right)+\frac{3}{2} z_{1}^{2} \operatorname{Tr}\left\{\psi_{1}^{T}(\eta, x) \psi_{1}(\eta, x)\right\}
\end{aligned}
$$

$$
\begin{align*}
= & \delta e^{T} P e e^{T}\left(P A_{0}+A_{0}^{T} P\right) e+2 \delta e^{T} P e e^{T} P \Phi(\eta, x) \\
& +\operatorname{Tr}\left\{\Psi^{T}(\eta, x)\left(2 \delta\left(e^{T} P\right)^{T} e^{T} P+\delta e^{T} P e P\right) \Psi(\eta, x)\right\} \\
& +\frac{1}{\Gamma} \tilde{\theta} \dot{\theta}+z_{1}^{3}\left(\alpha_{1}+z_{2}+k_{1} y+\theta^{*} e_{1}+\varphi_{1}(\eta, x)\right)+\frac{3}{2} z_{1}^{2} \operatorname{Tr}\left\{\psi_{1}^{T}(\eta, x) \psi_{1}(\eta, x)\right\} \\
\leq & -\delta \lambda_{\min }(P)|e|^{4}+2 \delta\|P\|^{2}|e|^{3}|\Phi(\eta, x)|+3 \delta\|P\|^{2}|e|^{2}\|\Psi(\eta, x)\|^{2}+\frac{1}{\Gamma} \tilde{\theta} \dot{\theta} \\
& +z_{1}^{3}\left(\alpha_{1}+z_{2}+k_{1} y+\theta^{*} e_{1}+\varphi_{1}(\eta, x)\right)+\frac{3}{2} z_{1}^{2} \operatorname{Tr}\left\{\psi_{1}^{T}(\eta, x) \psi_{1}(\eta, x)\right\} \tag{4.10}
\end{align*}
$$

Applying Assumption 3.1, (3.2), (4.3), (4.7), and Lemma 2.6, it follows that

$$
\begin{gather*}
2 \delta\|P\|^{2}|e|^{3} \Phi|(\eta, x)| \leq a_{01}|e|^{4}+\bar{a}_{01}|\Phi(\eta, x)|^{4}, \\
3 \delta\|P\|^{2}|e|^{2}\|\Psi(\eta, x)\|^{2} \leq a_{02}|e|^{4}+\bar{a}_{02}\|\Psi(\eta, x)\|^{4}, \\
z_{1}^{3} z_{2} \leq a_{10} z_{1}^{4}+\bar{a}_{10} z_{2}^{4}, \quad \theta^{*} z_{1}^{3} e_{1} \leq a_{11} e_{1}^{4}+\theta^{*(4 / 3)} \gamma_{11}\left(z_{1}\right) z_{1}^{4} \leq a_{11} e_{1}^{4}+\theta \gamma_{11}\left(z_{1}\right) z_{1}^{4},  \tag{4.11}\\
z_{1}^{3} \varphi_{1}(\eta, x) \leq\left|z_{1}\right|^{3} \theta^{*}\left(\varphi_{10}(|\eta|)+\varphi_{11}\left(z_{1}\right)\right) \leq a_{12} \varphi_{10}^{4}(|\eta|)+\theta \gamma_{12}\left(z_{1}\right) z_{1}^{4}, \\
\frac{3}{2} z_{1}^{2} \operatorname{Tr}\left\{\psi_{1}^{T}(\eta, x) \psi_{1}(\eta, x)\right\} \leq 3 z_{1}^{2} \theta^{* 2}\left(\psi_{10}^{2}(|\eta|)+\psi_{11}^{2}\left(z_{1}\right)\right) \leq a_{13} \psi_{10}^{4}(|\eta|)+\theta \gamma_{13}\left(z_{1}\right) z_{1}^{4},
\end{gather*}
$$

where $\bar{a}_{01}, \bar{a}_{02}, \gamma_{11}, \gamma_{12}$, and $\gamma_{13}$ are smooth functions, $a_{01}, a_{02}, a_{10}, \bar{a}_{10}, a_{11}, a_{12}, a_{13}>0$ are constants.

Using $\left(a_{1}+\cdots+a_{n}\right)^{2} \leq(x+a)^{n}=n \sum_{i=1}^{n} a_{i}^{2},(a+b)^{4} \leq 8 a^{4}+8 b^{4}, y=x_{1}$, Assumption 3.1, (3.2), and (4.5), one gets

$$
\begin{gather*}
|\Phi(\eta, x)|^{4} \leq 64(n-1)\left(\left(k_{1}^{4}+\cdots+k_{n-1}^{4}\right)\left(\varphi_{10}^{4}+y^{4} \bar{\varphi}_{11}^{4}\right)+\varphi_{20}^{4}+\cdots+\varphi_{n 0}^{4}+y^{4}\left(\bar{\varphi}_{21}^{4}+\cdots+\bar{\varphi}_{n 1}^{4}\right)\right) \\
\|\Psi(\eta, x)\|^{4} \leq 64(n-1)\left(\left(k_{1}^{4}+\cdots+k_{n-1}^{4}\right)\left(\psi_{10}^{4}+y^{4} \bar{\psi}_{11}^{4}\right)+\psi_{20}^{4}+\cdots+\psi_{n 0}^{4}+y^{4}\left(\bar{\psi}_{21}^{4}+\cdots+\bar{\psi}_{n 1}^{4}\right)\right) \tag{4.12}
\end{gather*}
$$

Substituting (4.11)-(4.12) into (4.10) and using $z_{1}=y$ lead to

$$
\begin{align*}
\mathscr{L} V_{1} \leq & -a_{00}|e|^{4}+a_{11} e_{1}^{4}+\Delta_{1}(|\eta|)+\bar{a}_{10} z_{2}^{4}+z_{1}^{3}\left(\alpha_{1}+k_{1} z_{1}+\Delta_{00}\left(z_{1}\right) z_{1}+a_{10} z_{1}\right) \\
& +\theta\left(\gamma_{11}\left(z_{1}\right)+\gamma_{12}\left(z_{1}\right)+\gamma_{13}\left(z_{1}\right)\right) z_{1}^{4}+\frac{1}{\Gamma} \tilde{\theta} \dot{\theta} \tag{4.13}
\end{align*}
$$

where

$$
\begin{align*}
a_{00}= & \delta \lambda_{\min }(P)-a_{01}-a_{02} \\
\Delta_{1}(|\eta|)= & 64(n-1) \bar{a}_{01}\left(\left(k_{1}^{4}+\cdots+k_{n-1}^{4}\right) \varphi_{10}^{4}(|\eta|)+\varphi_{20}^{4}(|\eta|)+\cdots+\varphi_{n 0}^{4}(|\eta|)\right) \\
& +64(n-1) \bar{a}_{02}\left(\left(k_{1}^{4}+\cdots+k_{n-1}^{4}\right) \psi_{10}^{4}(|\eta|)+\psi_{20}^{4}(|\eta|)+\cdots+\psi_{n 0}^{4}(|\eta|)\right) \\
& +a_{12} \varphi_{10}^{4}(|\eta|)+a_{13} \psi_{10}^{4}(|\eta|),  \tag{4.14}\\
\Delta_{00}\left(z_{1}\right)= & 64(n-1) \bar{a}_{01}\left(\left(k_{1}^{4}+\cdots+k_{n-1}^{4}\right) \bar{\varphi}_{11}^{4}\left(z_{1}\right)+\bar{\varphi}_{21}^{4}\left(z_{1}\right)+\cdots+\bar{\varphi}_{n 1}^{4}\left(z_{1}\right)\right) \\
& +64(n-1) \bar{a}_{02}\left(\left(k_{1}^{4}+\cdots+k_{n-1}^{4}\right) \bar{\psi}_{11}^{4}\left(z_{1}\right)+\bar{\psi}_{21}^{4}\left(z_{1}\right)+\cdots+\bar{\psi}_{n 1}^{4}\left(z_{1}\right)\right) .
\end{align*}
$$

Choosing the virtual control $\alpha_{1}(\cdot)$ and the tuning function $\tau_{1}(\cdot)$

$$
\begin{gather*}
\alpha_{1}(y, \hat{\theta})=-z_{1}\left(c_{1}+\nu_{1}\left(z_{1}\right)+k_{1}+\Delta_{00}\left(z_{1}\right)+a_{10}+\widehat{\theta}\left(\gamma_{11}\left(z_{1}\right)+\gamma_{12}\left(z_{1}\right)+\gamma_{13}\left(z_{1}\right)\right)\right),  \tag{4.15}\\
\tau_{1}(y)=\Gamma\left(\gamma_{11}\left(z_{1}\right)+\gamma_{12}\left(z_{1}\right)+\gamma_{13}\left(z_{1}\right)\right) z_{1}^{4},
\end{gather*}
$$

one gets

$$
\begin{equation*}
\varrho V_{1} \leq-c_{1} z_{1}^{4}-v_{1}\left(z_{1}\right) z_{1}^{4}-a_{00}|e|^{4}+a_{11} e_{1}^{4}+\Delta_{1}(|\eta|)+\bar{a}_{10} z_{2}^{4}+\frac{1}{\Gamma} \tilde{\theta}\left(\dot{\theta}-\tau_{1}\right) \tag{4.16}
\end{equation*}
$$

where $c_{1}>0$ is design parameter and $v_{1}\left(z_{1}\right)>0$ is a smooth function to be chosen later.
Step $i(i=2, \ldots, n)$. For notation coherence, denote $u=\alpha_{n}, z_{n+1}=0$. At this step, we can obtain a property similar to (4.16), which is presented by the following lemma.

Lemma 4.1. For the $i$ th Lyapunov function $V_{i}\left(e, \bar{z}_{i}, \widehat{\theta}\right)=(\delta / 2)\left(e^{T} P e\right)^{2}+(1 / 2 \Gamma) \tilde{\theta}^{2}+\sum_{j=1}^{i}\left(z_{j}^{4} / 4\right)$, there are the virtual control law $\alpha_{i}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{i-1}, \widehat{\theta}\right)$ and the tuning function $\tau_{i}$ with the form

$$
\begin{align*}
& \begin{array}{l}
\alpha_{i}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{i-1}, \widehat{\theta}\right)=-\Omega_{i}-z_{i}\left(c_{i}+\bar{a}_{i-1,0}+a_{i 0}+\hat{\theta}\left(\gamma_{i 1}+\gamma_{i 2}+\gamma_{i 3}+\gamma_{i 4}\right)\right. \\
\\
\left.\quad+\Gamma\left(\gamma_{i 1}+\gamma_{i 2}+\gamma_{i 3}+\gamma_{i 4}\right) \sum_{j=1}^{i-1} \frac{\partial \alpha_{j}}{\partial \widehat{\theta}} z_{j+1}^{3}\right)+\frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \tau_{i}, \\
\tau_{i}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{i-1}, \widehat{\theta}\right)=\tau_{i-1}+\Gamma\left(\gamma_{i 1}+\gamma_{i 2}+\gamma_{i 3}+\gamma_{i 4}\right) z_{i}^{4},
\end{array}
\end{align*}
$$

such that

$$
\begin{align*}
\mathscr{L} V_{i} \leq & -\sum_{j=1}^{i} c_{j} z_{j}^{4}+\sum_{j=2}^{i} b_{j} z_{1}^{4}-v_{1}\left(z_{1}\right) z_{1}^{4}-a_{00}|e|^{4}+\sum_{j=1}^{i} a_{j 1} e_{1}^{4}+\Delta_{i}(|\eta|)+\bar{a}_{i 0} z_{i+1}^{4} \\
& +\left(\frac{1}{\Gamma} \tilde{\theta}-\sum_{j=1}^{i-1} \frac{\partial \alpha_{j}}{\partial \widehat{\theta}} z_{j+1}^{3}\right)\left(\dot{\hat{\theta}}-\tau_{i}\right), \tag{4.18}
\end{align*}
$$

where $c_{i}>0$ is the designed parameters, $a_{i 0}, \bar{a}_{i-1,0}, a_{i 1}, b_{i}$ are some positive constants, $\gamma_{i 1}, \gamma_{i 2}, \gamma_{i 3}, \gamma_{i 4}$ are smooth nonnegative functions, whose choices are given in the proof.

Proof. See the appendix.
Therefore, at the end of the recursive procedure, the controller can be constructed as

$$
\begin{equation*}
u=\alpha_{n}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{n-1}, \widehat{\theta}\right), \quad \dot{\hat{\theta}}=\tau_{n}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{n-1}, \widehat{\theta}\right) \tag{4.19}
\end{equation*}
$$

Choosing parameters $\delta, a_{01}, a_{02}, a_{11}, \ldots, a_{n 1}, c_{2}, \ldots, c_{n}$ to satisfy

$$
\begin{equation*}
a_{0}=\delta \lambda_{\min }(P)-a_{01}-a_{02}-\sum_{i=1}^{n} a_{i 1}>0, \quad c_{2}, \ldots, c_{n}>0, \tag{4.20}
\end{equation*}
$$

by (4.16) and (4.18), one has

$$
\begin{equation*}
\mathcal{L} V_{n} \leq-\sum_{i=1}^{n} c_{i} z_{i}^{4}+\sum_{j=2}^{n} b_{j} z_{1}^{4}-v_{1}\left(z_{1}\right) z_{1}^{4}-a_{0}|e|^{4}+\Delta_{n}(|\eta|) \tag{4.21}
\end{equation*}
$$

where

$$
\begin{align*}
V_{n}(e, z, \hat{\theta})= & \frac{\delta}{2}\left(e^{T} P e\right)^{2}+\frac{1}{2 \Gamma} \tilde{\theta}^{2}+\frac{1}{4} \sum_{i=1}^{n} z_{i}^{4} \\
\Delta_{n}(|\eta|)= & 64(n-1) \bar{a}_{01}\left(\left(k_{1}^{4}+\ldots+k_{n-1}^{4}\right) \varphi_{10}^{4}(|\eta|)+\varphi_{20}^{4}(|\eta|)+\ldots+\varphi_{n 0}^{4}(|\eta|)\right) \\
& +64(n-1) \bar{a}_{02}\left(\left(k_{1}^{4}+\ldots+k_{n-1}^{4}\right) \psi_{10}^{4}(|\eta|)+\psi_{20}^{4}(|\eta|)+\ldots+\psi_{n 0}^{4}(|\eta|)\right) \\
& +\sum_{i=1}^{n}\left(a_{i 2} \varphi_{10}^{4}(|\eta|)+a_{i 3} \psi_{10}^{4}(|\eta|)\right)+\sum_{j=2}^{n} a_{j 4} \psi_{10}^{4}(|\eta|) . \tag{4.22}
\end{align*}
$$

## 5. Stability Analysis

We state the main theorems in this paper. This section is divided into two parts.

### 5.1. Case I

Theorem 5.1. Assume that Assumptions 3.1 and 3.2 hold with the following properties:

$$
\begin{equation*}
\limsup _{s \rightarrow 0+} \frac{\Delta_{n}(|\eta|)}{\alpha(s)}<\infty, \quad \limsup _{s \rightarrow \infty} \frac{\Delta_{n}(|\eta|)}{\alpha(s)}<\infty \tag{5.1}
\end{equation*}
$$

If $\lim \sup _{s \rightarrow 0+} \gamma(s) / s^{4}<\infty$ in Assumption 3.2, by appropriately choosing the positive smooth function $\mathcal{v}_{1}(\cdot)$ in (4.15) and the parameters $\delta, a_{01}, a_{02}, a_{11}, \ldots, a_{n 1}, c_{2}, \ldots, c_{n}$ to satisfy (4.20), then
(i) the closed-loop system consisting of (3.1), (4.1), (4.2), (4.7), (4.15), (4.17), and (4.19) has a unique and almost surely bounded strong solution on $[0, \infty)$;
(ii) for each initial value $(\eta(0), x(0), \widehat{x}(0), \widehat{\theta}(0))$, the equilibrium $(\eta, x)=(0,0)$ is globally stable in probability, where $\widehat{x}=\left(\widehat{x}_{1}, \ldots, \widehat{x}_{n-1}\right)$.

Proof. For any constant $\epsilon>0$, by (5.1), one has

$$
\begin{equation*}
\limsup _{s \rightarrow 0+} \frac{(1+\epsilon) \Delta_{n}(|\eta|)}{\alpha(s)}<\infty, \quad \limsup _{s \rightarrow \infty} \frac{(1+\epsilon) \Delta_{n}(|\eta|)}{\alpha(s)}<\infty \tag{5.2}
\end{equation*}
$$

For the $\eta$-subsystem, by Lemma 2.4, there exists a positive and radially unbounded Lyapunov function $\tilde{V}_{0}(\eta) \in C^{2}$ and $\tilde{\gamma}=\rho \gamma$ such that

$$
\begin{align*}
£ \tilde{V}_{0}(\eta) \leq & -(1+\epsilon) \Delta_{n}(|\eta|)+\tilde{\gamma}\left(\left|z_{1}\right|\right)  \tag{5.3}\\
& \limsup _{s \rightarrow 0+} \frac{\tilde{\gamma}(s)}{s^{4}}<\infty \tag{5.4}
\end{align*}
$$

where $\rho$ is a positive constant satisfying $(1+\epsilon) \Delta_{n}(|\eta|) \leq \rho \alpha(s)$ for all $s \geq 0$. Choosing the following Lyapunov function for the entire closed-loop system

$$
\begin{equation*}
V(\eta, e, z, \hat{\theta})=\tilde{V}_{0}(\eta)+V_{n}(e, z, \hat{\theta}) \tag{5.5}
\end{equation*}
$$

and combining (4.21) and (5.3), one obtains

$$
\begin{equation*}
\rho V(\eta, e, z, \hat{\theta}) \leq-\sum_{i=1}^{n} c_{i} z_{i}^{4}-\left(v_{1}\left(z_{1}\right)-\sum_{j=2}^{n} b_{j}\right) z_{1}^{4}-a_{0}|e|^{4}-\epsilon \Delta_{n}(|\eta|)+\widetilde{\gamma}\left(\left|z_{1}\right|\right) \tag{5.6}
\end{equation*}
$$

By (5.4), there always exists a smooth function $\mathcal{v}_{1}(\cdot)$ to satisfy the following two inequalities:

$$
\begin{equation*}
z_{1}^{4}\left(v_{1}\left(z_{1}\right)-\sum_{j=2}^{n} b_{j}\right) \geq \tilde{\gamma}\left(\left|z_{1}\right|\right), \quad v_{1}\left(z_{1}\right) \geq \sum_{j=2}^{n} b_{j} \tag{5.7}
\end{equation*}
$$

Substituting (5.7) into (5.6) leads to

$$
\begin{align*}
\mathscr{\varrho}(\eta, e, z, \widehat{\theta}) & \leq-\sum_{i=1}^{n} c_{i} z_{i}^{4}-a_{0}|e|^{4}-\epsilon \Delta_{n}(|\eta|)  \tag{5.8}\\
& \triangleq-W(\eta, e, z)
\end{align*}
$$

Noting that $\varphi_{i 0}(\cdot)$ and $\psi_{i 0}(\cdot)$ are nonnegative smooth functions and using (4.22), it follows that $\Delta_{n}(\cdot)$ and $W(\cdot)$ are continuous nonnegative functions. By (5.5), (5.8), and Lemma 2.2, one concludes that all the solutions of the closed-loop system are bounded almost surely, the equilibrium $(\eta, e, z)=(0,0,0)$ is globally stable in probability. By (3.1), (4.1), (4.2), (4.7), (4.15), (4.17), (4.19), and the almost sure boundedness of all the signals, it is not difficult to recursively prove that the equilibrium $(\eta, x)=(0,0)$ is globally stable in probability.

### 5.2. Case II

If more information about $\alpha$ in Assumption 3.2 is known, that is, $\lim _{\inf }^{s \rightarrow \infty}, ~ \alpha(s)=\infty$, further results under the weaker conditions is given as follows.

Assumption 5.2. For functions $\psi_{0}$ and $V_{0}$ given by (4.22) and Assumption 3.2, there exist known smooth nonnegative functions $\tilde{\psi}_{1}$ and $\tilde{\psi}_{2}$ with $\tilde{\psi}_{1}(0)=\tilde{\psi}_{2}(0)=0$, such that $\left\|\psi_{0}\left(\eta, x_{1}\right)\right\| \leq \tilde{\psi}_{1}(|\eta|)$ and $\left|\partial V_{0}(\eta) / \partial \eta\right| \leq \tilde{\psi}_{2}(|\eta|)$.

Lemma 5.3. For $\Delta_{n}, \underline{\alpha}, \alpha, \tilde{\psi}_{1}$ and $\tilde{\psi}_{2}$ given by (4.22), Assumptions 3.2 and 5.2, if

$$
\begin{equation*}
\limsup _{s \rightarrow 0+} \frac{\Delta_{n}(s)+\tilde{\psi}_{1}^{2}(s) \tilde{\psi}_{2}^{2}(s)}{\alpha(s)}<\infty, \quad \int_{0}^{\infty} e^{-\int_{0}^{s}\left(1 / \zeta_{1}\left(\underline{\alpha}^{-1}(\tau)\right)\right) d \tau}\left[\xi_{1}\left(\underline{\alpha}^{-1}(s)\right)\right]^{\prime} d s<\infty, \tag{5.9}
\end{equation*}
$$

where $\xi_{1}(\cdot) \geq 0$ and $\zeta_{1}(\cdot)>0$ are smooth increasing functions satisfying

$$
\begin{equation*}
\xi_{1}(s) \alpha(s) \geq 2(1+\epsilon) \Delta_{n}(s), \quad \zeta_{1}(s) \alpha(s) \geq \tilde{\psi}_{1}^{2}(s) \tilde{\psi}_{2}^{2}(s), \quad \forall s \geq 0 \tag{5.10}
\end{equation*}
$$

$\epsilon$ is any positive constant. Then there exists a function $\tilde{\gamma} \in \mathcal{K}_{\infty}$ such that $\left((1+\epsilon) \Delta_{n}, \tilde{\gamma}\right)$ is a new SiISS supply rate of the $\eta$-subsystem in (3.1). Moreover, if $r$ in Assumptions 3.2 satisfies $\lim \sup _{s \rightarrow 0^{+}} \gamma(s) / s^{4}<\infty$, then $\lim \sup _{s \rightarrow 0+} \tilde{\gamma}(s) / s^{4}<\infty$.

Since the condition (5.9) is weaker than (5.1), by using Lemma 5.3, we give further results under the weaker condition (5.9).

Theorem 5.4. Suppose that Assumptions 3.1-5.2 and the conditions of Lemma 5.3 hold. If $\limsup \operatorname{sut}_{s \rightarrow+} \gamma(s) / s^{4}<\infty$ in Assumption 3.2, by appropriately choosing the positive smooth function $v_{1}(\cdot)$ in (4.15) and the parameters $\delta, a_{01}, a_{02}, a_{11}, \ldots, a_{n 1}, c_{2}, \ldots, c_{n}$ to satisfy (4.20), then
(1) the closed-loop system consisting of (3.1), (4.1), (4.2), (4.7), (4.15), (4.17), and (4.19) has a unique and almost surely bounded strong solution on $[0, \infty)$;
(2) for each initial value $(\eta(0), x(0), \widehat{x}(0), \widehat{\theta}(0))$, the equilibrium $(\eta, x)=(0,0)$ is globally stable in probability.

## 6. Conclusions

This paper further considers a more general class of stochastic nonlinear systems with uncertain parameters and SiISS inverse dynamics. By combining the stochastic LaSalle theorem and small-gain type conditions on SiISS, an adaptive output feedback controller is designed to guarantee that all the closed-loop signals are bounded almost surely and the stochastic closed-loop system is globally stable in probability.

There are two remaining problems to be investigated: (1) an essential problem is to find a practical example with explicit physical meaning for system (3.1). A preliminary attempt on high-order stochastic nonlinear system can be found in [28]. (2) How to design an output feedback controller by using this method in this paper for system (3.1) in which the drift and diffusion vector fields depend on the unmeasurable states besides the measurable output?

## Appendix

Proof of Lemma 4.1. We prove Lemma 4.1 by induction. Assume that at Step $i-1$, there are virtual control laws

$$
\begin{align*}
\alpha_{i-1}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{i-2}, \widehat{\theta}\right)= & -\Omega_{i-1}-z_{i-1} \\
& \times\left(c_{i-1}+\bar{a}_{i-2,0}+a_{i-1,0}+\bar{a}_{i-1,1}+\widehat{\theta}\left(\gamma_{i-1,1}+\gamma_{i-1,2}+\gamma_{i-1,3}+\gamma_{i-1,4}\right)\right. \\
& \left.+\Gamma\left(\gamma_{i-1,1}+\gamma_{i-1,2}+\gamma_{i-1,3}+\gamma_{i-1,4}\right)+\sum_{j=1}^{i-2} \frac{\partial \alpha_{j}}{\partial \widehat{\theta}} z_{j+1}^{3}\right)+\frac{\partial \alpha_{i-2}}{\partial \widehat{\theta}} \tau_{i-1}, \\
\tau_{i-1}\left(y, \widehat{x}_{1}, \ldots, \widehat{x}_{i-2}, \hat{\theta}\right)= & \tau_{i-2}+\Gamma\left(\gamma_{i-1,1}+\gamma_{i-1,2}+\gamma_{i-1,3}+\gamma_{i-1,4}\right) z_{i-1}^{4} \tag{A.1}
\end{align*}
$$

such that $V_{i-1}\left(e, \bar{z}_{i-1}, \widehat{\theta}\right)=(\delta / 2)\left(e^{T} P e\right)^{2}+(1 / 2 \Gamma) \widetilde{\theta}^{2}+\sum_{j=1}^{i-1}\left(z_{j}^{4} / 4\right)$ satisfies

$$
\begin{align*}
\mathscr{L} V_{i-1} \leq & -\sum_{j=1}^{i-1} c_{j} z_{j}^{4}+\sum_{j=2}^{i-1} b_{j} z_{1}^{4}-v_{1}\left(z_{1}\right) z_{1}^{4}-a_{00}|e|^{4}+\sum_{j=1}^{i-1} a_{j 1} e_{1}^{4}+\Delta_{i-1}(|\eta|) \\
& +\bar{a}_{i-1,0} z_{i}^{4}+\left(\frac{1}{\Gamma} \tilde{\theta}-\sum_{j=1}^{i-2} \frac{\partial \alpha_{j}}{\partial \widehat{\theta}} z_{j+1}^{3}\right)\left(\dot{\hat{\theta}}-\tau_{i-1}\right) \tag{A.2}
\end{align*}
$$

where $c_{i-1}>0$ are the designed parameters, $a_{i-1,0}, \bar{a}_{i-1,0}, a_{i-1,1}, b_{i-1}$ are some positive constants, and $\gamma_{i-1,1}, \gamma_{i-1,2}, \gamma_{i-1,3}, \gamma_{i-1,4}$ are smooth nonnegative functions.In the sequel, we will prove that Lemma 4.1 still holds for Step $i$. Choosing

$$
\begin{equation*}
V_{i}\left(e, \bar{z}_{i}, \widehat{\theta}\right)=V_{i-1}\left(e, \bar{z}_{i-1}, \widehat{\theta}\right)+\frac{1}{4} z_{i}^{4} \tag{A.3}
\end{equation*}
$$

with the use of (4.6), (4.7), (4.15), and (A.1), the ItÔ differential of $z_{i}$ is given as follows:

$$
\begin{align*}
d z_{i}= & \left(\alpha_{i}+z_{i+1}+\Omega_{i}-\frac{\partial \alpha_{i-1}}{\partial y} \theta^{*} e_{1}-\frac{\partial \alpha_{i-1}}{\partial y} \varphi_{1}(\eta, x)-\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}\right) d t \\
& -\frac{1}{2} \frac{\partial^{2} \alpha_{i-1}}{\partial y^{2}} \psi_{1}(\eta, x) \psi_{1}^{T}(\eta, x) d t-\frac{\partial \alpha_{i-1}}{\partial y} \psi_{1}(\eta, x) d w \tag{A.4}
\end{align*}
$$

where $\Omega_{i}=k_{i} y-k_{i-1}\left(\widehat{x}_{1}+k_{1} y\right)-\sum_{j=1}^{i-2}\left(\partial \alpha_{i-1} / \partial \widehat{x}_{j}\right)\left(\widehat{x}_{j+1}+k_{j+1} y-k_{j}\left(\widehat{x}_{1}+k_{1} y\right)\right)-\left(\partial \alpha_{i-1} / \partial y\right)\left(\widehat{x}_{1}+\right.$ $k_{1} y$ ). Using (2.2) and (A.2)-(A.4), we arrive at

$$
\begin{align*}
\varrho V_{i} \leq & -\sum_{j=1}^{i-1} c_{j} z_{j}^{4}+\sum_{j=2}^{i-1} b_{j} z_{1}^{4}-v_{1}\left(z_{1}\right) z_{1}^{4}-a_{00}|e|^{4}+\sum_{j=1}^{i-1} a_{j 1} e_{1}^{4}+\Delta_{i-1}(|\eta|)+\bar{a}_{i-1,0} z_{i}^{4} \\
& +\left(\frac{1}{\Gamma} \tilde{\theta}-\sum_{j=1}^{i-2} \frac{\partial \alpha_{j}}{\partial \widehat{\theta}} z_{j+1}^{3}\right)\left(\dot{\hat{\theta}}-\tau_{i-1}\right) \\
& +z_{i}^{3}\left(\alpha_{i}+\Omega_{i}-\frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \dot{\theta}+z_{i+1}-\frac{\partial \alpha_{i-1}}{\partial y} \theta^{*} e_{1}-\frac{\partial \alpha_{i-1}}{\partial y} \varphi_{1}(\eta, x)-\frac{1}{2} \frac{\partial^{2} \alpha_{i-1}}{\partial y^{2}} \psi_{1}(\eta, x) \psi_{1}^{T}(\eta, x)\right) \\
& +\frac{3}{2} z_{i}^{2} \operatorname{Tr}\left\{\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{2} \psi_{1}^{T}(\eta, x) \psi_{1}(\eta, x)\right\} . \tag{A.5}
\end{align*}
$$

Now, we estimate the last five terms, respectively, in the right-hand side of (A.5). According to Assumption 3.1, (3.2), (4.7), and Lemma 2.6, there exist positive real numbers $a_{i 0}, \bar{a}_{i 0}, a_{i 1}$, $a_{i 2}, a_{i 3}, a_{i 4}, b_{i 2}, b_{i 3}, b_{i 4}$, smooth nonnegative functions $\gamma_{i 1}, \gamma_{i 2}, \gamma_{i 3}, \gamma_{i 4}$ such that

$$
\begin{align*}
& z_{i}^{3} z_{i+1} \leq a_{i 0} z_{i}^{4}+\bar{a}_{i 0} z_{i+1}^{4}, \quad-z_{i}^{3} \frac{\partial \alpha_{i-1}}{\partial y} \theta^{*} e_{1} \leq a_{i 1} e_{1}^{4}+\theta \gamma_{i 1}\left(\bar{z}_{i}\right) z_{i}^{4}, \\
& -z_{i}^{3} \frac{\partial \alpha_{i-1}}{\partial y} \varphi_{1}(\eta, x) \leq\left|z_{i}\right|^{3}\left(1+\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{2}\right)^{1 / 2}\left(\varphi_{10}(|\eta|)+\varphi_{11}\left(z_{1}\right)\right) \\
& \leq a_{i 2} \varphi_{10}^{4}(|\eta|)+b_{i 2} z_{1}^{4}+\theta \gamma_{i 2}\left(\bar{z}_{i}\right) z_{i}^{4}, \\
& -\frac{1}{2} \frac{\partial^{2} \alpha_{i-1}}{\partial y^{2}} z_{i}^{3} \psi_{1}(\eta, x) \psi_{1}^{T}(\eta, x) \leq\left|z_{i}\right|^{3}\left|\frac{\partial^{2} \alpha_{i-1}}{\partial y^{2}}\right|\left(\psi_{10}^{2}(|\eta|)+\psi_{11}^{2}\left(z_{1}\right)\right)  \tag{A.6}\\
& \leq a_{i 3} \psi_{10}^{4}(|\eta|)+b_{i 3} z_{1}^{4}+\theta \gamma_{i 3}\left(\bar{z}_{i}\right) z_{i}^{4}, \\
& \begin{aligned}
\frac{3}{2} z_{i}^{2} \operatorname{Tr}\left\{\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{2} \psi_{1}^{T}(\eta, x) \psi_{1}(\eta, x)\right\} & \leq 3 z_{i}^{2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{2}\left(\psi_{10}^{2}(|\eta|)+\psi_{11}^{2}\left(z_{1}\right)\right) \\
& \leq a_{i 4} \psi_{10}^{4}(|\eta|)+b_{i 4} z_{1}^{4}+\theta \gamma_{i 4}\left(\bar{z}_{i}\right) z_{i}^{4} .
\end{aligned}
\end{align*}
$$

Choosing $\alpha_{i}$ and $\tau_{i}$ as (4.17) and substituting (A.6) into (A.5), (4.18) holds, where $c_{i}>0$ is a design parameter,

$$
\begin{equation*}
\Delta_{i}(|\eta|)=\Delta_{i-1}(|\eta|)+a_{i 2} \varphi_{10}^{4}(|\eta|)+a_{i 3} \psi_{10}^{4}(|\eta|)+a_{i 4} \psi_{10}^{4}(|\eta|), \quad b_{i}=b_{i 2}+b_{i 3}+b_{i 4} \tag{A.7}
\end{equation*}
$$

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Research Article

# A Fast Fourier Transform Technique for Pricing European Options with Stochastic Volatility and Jump Risk 

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#### Abstract

We consider European options pricing with double jumps and stochastic volatility. We derived closed-form solutions for European call options in a double exponential jump-diffusion model with stochastic volatility(SVDEJD). We developed fast and accurate numerical solutions by using fast Fourier transform (FFT) technique. We compared the density of our model with those of other models, including the Black-Scholes model and the double exponential jump-diffusion model. At last, we analyzed several effects on option prices under the proposed model. Simulations show that the SVDEJD model is suitable for modelling the long-time real-market changes and stock returns are negatively correlated with volatility. The model and the proposed option pricing method are useful for empirical analysis of asset returns and managing the corporate credit risks.


## 1. Introduction

The classical Black-Scholes (BS) model [1] has long been known to result in systematically biased option valuation. By adding jumps to the archetypal price process with Gaussian innovations Merton [2] is able to partly explain the observed deviations from the benchmark model which are characterized by fat tail and excess kurtosis in the returns distribution. For an overview of "stylized facts" on asset returns see Cont [3]. Statistical properties of implied volatilities are summarized in Cont et al. [4]. In the sequel also other authors develop more realistic models, for example, the pure jump models of Eberlein and Keller [5], Madan et al. [6], and Duffie et al. [7], stochastic volatility models of Steven [8], and stochastic volatility model with normal jumps of Bates [9] and Keppo et al. [10]. The double exponential jump-diffusion (DEJD) model, recently proposed by Kou [11], generates a highly skewed and leptokurtic distribution and is capable of matching key features of stock and index
returns. Moreover, the DEJD model leads to tractable pricing formulas for exotic and path dependent options [12]. Accordingly, the DEJD model has gained wide acceptance. However, the DEJD model cannot capture the volatility clustering effects, which can be captured by stochastic volatility models [13]. Jump-diffusion models and the stochastic volatility model complement each other: the stochastic volatility model can incorporate dependent structures better, while the DEJD model has better analytical tractability, especially for path-dependent options. Since allowing interest rates to be stochastic does not improve pricing performance any further [14], the model that combines stochastic volatility and double exponential jumpdiffusion (SVDEJD) may be more reasonable.

In the BS setting, the probability measure has a well-known analytic form [15], but, under stochastic volatility, it can only be obtained numerically. Monte Carlo simulation and the finite difference method are usually used to value the options. But, the two techniques require substantially more computing time and thus are difficult to be applied in real option pricing. Recently, being fast, accurate, and easy to implement, Fourier transforms have been widely used in valuing financial derivatives, for example, Carr and Madan [16] propose Fourier transforms with respect to log-strike price; Duffie et al. [7] offer a comprehensive survey that the Fourier methods are applicable to a wide range of stochastic processes; Carr and Wu [17] apply the transforms to time-changed Lévy processes and the class of generalized affine models. Hurd and Zhou [18] express the spread option payoff in terms of the gamma function and FFT technique. For an overview of option pricing using Fourier transforms, see Schmelzle [19].

The current paper extends the study of option pricing under the DEJD model in three ways. First, we propose a model which combines the double jumps and stochastic volatility. Second, using the martingale method, Fourier transform formula, and FeynmanKac theorem, we obtain a closed-form solution for European call options pricing under the proposed model. Third, we obtain fast and accurate numerical solutions for European call options pricing by FFT technique.

The rest of the paper is organized as follows. Section 2 develops the underlying pricing model. Section 3 derives a closed-form solution for European call options pricing under the proposed model. Section 4 provides approximation solutions for European call options pricing by FFT technique. Section 5 numerically compares the density of the solutions to the alternative models and analyzes several effects on potion prices. Section 6 concludes. Applied program codes in Matlab package are presented in the appendix.

## 2. The Model

We consider an arbitrage-free, frictionless financial market where only riskless asset $B$ and risky asset $S$ are traded continuously up to a fixed horizon date $T$. Let $\left\{\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{0 \leq t \leq T}, P\right\}$ be a complete probability space with a filtration satisfying the usual conditions, that is, the filtration is continuous on the right and $\mathscr{F}_{0}$ contains all $P$-null sets. Suppose $W(t), W_{v}(t)$ are both standard Brownian motion, which is $\mathcal{F}_{t}$ adapted, and $W(t)$ has correlation $\rho$ with $W_{v}(t)$.

Let $S(t)$ represent the price for a stock or a stock portfolio. Generally, instantaneous variance of asset returns in financial markets shows randomness; thus, the continuous part of the price process, defined as $S^{c}(t)$, is

$$
\begin{equation*}
d S^{c}(t)=r S^{c}(t) d t+\sigma \sqrt{V(t)} S^{c}(t) d W(t) \tag{2.1}
\end{equation*}
$$

where $r$ is risk-free rate and $\sigma$ is nonnegative constant, and suppose $S^{c}(0)=s$, which can be set equal to 1 without any loss of generality. The size of the diffusion component is determined by $V(t)$, which represents, absent of any jump occurring, the level of (stochastic) return variance attributable to diffusion variations. For tractability, let $V(t)$ follow a squareroot process:

$$
\begin{equation*}
d V(t)=\left(\theta_{v}-\alpha_{v} V(t)\right) d t+\sigma_{v} \sqrt{V(t)} d W_{v}(t) \tag{2.2}
\end{equation*}
$$

where nonnegative constants $\theta_{v}, \theta_{v} / \alpha_{v}$, and $\sigma_{v}$, respectively, reflect the speed of adjustment, the long-run mean, and the variation coefficient of $V(t)$, and suppose $V(0)=V_{0}$.

It has been suggested from extensive empirical studies that markets tend to have both overreaction and underreaction to various good or bad news. One may interpret the jump part of the model as the market response to outside news. Good or bad news arrives according to a Poisson process, and the asset price changes in response according to the jump size distribution. According to Kou [11], the jumps in the log-price are modeled as a sequence of i.i.d. nonnegative random variables that occur at times determined by an independent Poisson process $N(t)$ with constant intensity $\lambda>0$ such that $Y=\ln U$ has an asymmetric double exponential distribution with the density

$$
\begin{equation*}
f_{Y}(y)=p \eta_{1} e^{-\eta_{1} y} \mathbf{1}_{y \geq 0}+q \eta_{2} e^{\eta_{2} y} \mathbf{1}_{\mathrm{y}<0}, \quad \eta_{1}>1, \quad \eta_{2}>0, \tag{2.3}
\end{equation*}
$$

where 1 denotes the indicator function, so $\mathbf{1}_{y} \geq 0$ equals 1 if $y \geq 0$, but 0 otherwise. $p, q \geq$ $0, p+q=1$ are up-move jump and down-move jump, respectively. Except for $W(t)$ which has correlation with $W_{v}(t)$, all sources of randomness, $W(t), W_{v}(t), N(t), Y_{j}$, and $N(t)$, are assumed to be independent.

Because of jumps and stochastic volatility, the risk-neutral probability measure is not unique. Following Naik and Lee [20] and Kou [11], by using the rational expectations argument with a HARA-type utility function for the representative agent, one can choose a particular risk-neutral measure $P^{*}$ so that the equilibrium price of an option is given by the expectation under this risk-neutral measure of the discounted option payoff. Throughout this paper, we assume that there exists a martingale probability measure $P^{*}$ being equivalent to $P$. Let $X(t)$ be the sum of all the jumps which occur up to and including time $t, J(t)=$ $\exp [X(t)]-E[\exp (X(t))]$, we have

$$
\begin{equation*}
J(t)=\exp \left[X(t)-\lambda t\left(\frac{p \eta_{1}}{\eta_{1}-1}+\frac{q \eta_{2}}{\eta_{2}+1}-1\right)\right] \tag{2.4}
\end{equation*}
$$

Obviously, $J(t)$ is a $P$-martingale. Finally, the price process $S(t)$ is defined as

$$
\begin{equation*}
S(t)=S^{c}(t) J(t) \tag{2.5}
\end{equation*}
$$

Remark 2.1. The model contains most existing models as special cases. For example, we obtain (1) the BS model by setting $\lambda=0$ and $\theta_{v}=\alpha_{v}=\sigma_{v}=0$; (2) the SV model by setting $\lambda=0$;
(3) the DEJD model by setting $\theta_{v}=\alpha_{v}=\sigma_{v}=0$.

Let $d W(t)=\rho d W_{v}(t)+\sqrt{1-\rho^{2}} d Z(t)$, where $Z(t)$ is standard Brownian motion that is $\mathcal{F}_{t}$ adapted, and independent of $W_{v}(t), N(t)$, and random variables $U_{j}$. From Itŏ's formula, we have

$$
\begin{align*}
\ln S(t)= & \ln J(t)+\ln S^{c}(t) \\
= & X(t)-\lambda t\left(\frac{p \eta_{1}}{\eta_{1}-1}+\frac{q \eta_{2}}{\eta_{2}+1}-1\right)+r t \\
& +\left[\rho \sigma \int_{0}^{t} \sqrt{V(t)} d W_{v}(t)-\frac{1}{2} \sigma^{2} \rho^{2} \int_{0}^{t} V(t) d t\right]  \tag{2.6}\\
& +\left[\sqrt{1-\rho^{2}} \sigma \int_{0}^{t} \sqrt{V(t)} d Z(t)-\frac{1}{2} \sigma^{2}\left(1-\rho^{2}\right) \int_{0}^{t} V(t) d t\right] \\
= & X(t)-\lambda t\left(\frac{p \eta_{1}}{\eta_{1}-1}+\frac{q \eta_{2}}{\eta_{2}+1}-1\right)+r t+\xi_{t}+s_{t} .
\end{align*}
$$

## 3. A Closed-Form Solution of European Option Pricing

In this section, we derive closed-form solution of a European call option pricing under the SVDEJD model. For a European put option, we can obtain easily corresponding result by the put-call parity [1]. For this purpose, we need the following results.

Lemma 3.1. Supposing the variance process $V(t)$ follows (2.2) and $s_{1}, s_{2}$ are any complex, one has

$$
\begin{equation*}
E\left\{\exp \left[-s_{1} \int_{0}^{T} V(t) d t-s_{2} V(t)\right]\right\}=\exp \left[A(T)-B(T) V_{0}\right] \tag{3.1}
\end{equation*}
$$

where

$$
\begin{gather*}
A(T)=\frac{2 \theta_{v}}{\sigma_{v}^{2}} \ln \left[\frac{2 \gamma e^{(1 / 2)\left(\alpha_{v}-\gamma\right) T}}{2 \gamma e^{-\gamma T}+\left(\alpha_{v}+\gamma+\sigma_{v}^{2} s_{2}\right)\left(1-e^{-\gamma T}\right)}\right], \\
B(T)=\frac{\left(1-e^{-\gamma T}\right)\left(2 s_{1}-\alpha_{v} s_{2}\right)+\gamma s_{2}\left(1+e^{-\gamma T}\right)}{2 \gamma e^{-\gamma T}+\left(\alpha_{v}+\gamma+\sigma_{v}^{2} s_{2}\right)\left(1-e^{-\gamma T}\right)},  \tag{3.2}\\
\gamma=\sqrt{\alpha_{v}^{2}+2 \sigma_{v}^{2} s_{1}} .
\end{gather*}
$$

Proof. Let $F(V, 0, T)=E\left\{\exp \left[-s_{1} \int_{0}^{T} V(t) d t-s_{2} V(t)\right]\right\}$. Because of the affine structure of the variance process (2.2), we obtain that $F(V, 0, T)$ has a solution of the following form:

$$
\begin{equation*}
F(V, 0, T)=\exp \left[A(T)-B(T) V_{0}\right] \tag{3.3}
\end{equation*}
$$

From the Feynman-Kac formula, $F(V, 0, T)$ is the solution of the following backward Parabolic partial differential equation with the Cauchy problem:

$$
\begin{gather*}
\frac{\partial F}{\partial t}+\left(\theta_{v}-\alpha_{v} V\right) \frac{\partial F}{\partial V}+\frac{1}{2} \sigma_{v}^{2} V \frac{\partial^{2} F}{\partial V^{2}}-s_{1} V F=0  \tag{3.4}\\
F(V, 0,0)=\exp \left(-s_{2} V_{0}\right)
\end{gather*}
$$

Putting (3.3) in (3.4), we have

$$
\begin{gather*}
A_{t}(T)-\theta_{v} B(T)=0, \quad A(0)=0 \\
-B_{t}(T)+\frac{1}{2} \sigma_{v}^{2} B^{2}(T)+\alpha_{v} B(T)-s_{1}=0, \quad B(0)=s_{2} \tag{3.5}
\end{gather*}
$$

Solving (3.5), we can obtain the result of Lemma 3.1.
Lemma 3.2. Supposing the asset price $S(T)$ follows (2.6) and $z$ is any complex, one has

$$
\begin{align*}
E\{\exp [-r T+z \ln S(T)]\}=\exp \{ & (z-1) r T+\lambda T\left(\frac{p \eta_{1}}{\eta_{1}-z}+\frac{q \eta_{2}}{\eta_{2}+z}-1\right) \\
& -z \lambda T\left(\frac{p \eta_{1}}{\eta_{1}-1}+\frac{q \eta_{2}}{\eta_{2}+1}-1\right)-z \frac{\rho \sigma}{\sigma_{v}}\left(V(T)+\theta_{v} T\right) \\
& +\frac{2 \theta_{v}}{\sigma_{v}^{2}} \ln \left[\frac{2 \gamma e^{(1 / 2)\left(\alpha_{v}-\gamma\right) T}}{2 \gamma e^{-\gamma T}+\left(\alpha_{v}+\gamma+\sigma_{v}^{2} s_{2}\right)\left(1-e^{-\gamma T}\right)}\right] \\
& \left.-\frac{\left(1-e^{-\gamma T}\right)\left(2 s_{1}-\alpha_{v} s_{2}\right)+\gamma s_{2}\left(1+e^{-\gamma T}\right)}{2 \gamma e^{-\gamma T}+\left(\alpha_{v}+\gamma+\sigma_{v}^{2} s_{2}\right)\left(1-e^{-\gamma T}\right)} V_{0}\right\}, \tag{3.6}
\end{align*}
$$

where

$$
\begin{equation*}
s_{1}=-(z-1) z \frac{1}{2} \sigma^{2}\left(1-\rho^{2}\right)-z\left(\frac{\rho \sigma}{\sigma_{v}} \alpha_{v}-\frac{1}{2} \sigma^{2} \rho^{2}\right), \quad s_{2}=-z \frac{\rho \sigma}{\sigma_{v}} \tag{3.7}
\end{equation*}
$$

Proof. Let $\phi(z)=E\{\exp [-r T+z \ln S(T)]\}$. Because $N(t)$ is independent of $W(t), W_{v}(t)$, and $Z(t)$, we have

$$
\begin{equation*}
\phi(T)=e^{(z-1) r T} E\left[e^{z \ln J(T)}\right] E\left[e^{z\left(\xi_{T}+\zeta_{T}\right)}\right]=e^{(z-1) r T} C(T) D(T) \tag{3.8}
\end{equation*}
$$

From (2.2) and (2.3), we have

$$
\begin{equation*}
C(T)=\exp \left[\lambda T\left(\frac{p \eta_{1}}{\eta_{1}-z}+\frac{q \eta_{2}}{\eta_{2}+z}-1\right)-z \lambda T\left(\frac{p \eta_{1}}{\eta_{1}-1}+\frac{\mathrm{q} \eta_{2}}{\eta_{2}+1}-1\right)\right] \tag{3.9}
\end{equation*}
$$

Because $W_{v}(t)$ is a standard Brownian motion, we have

$$
\begin{equation*}
E\left(\varsigma_{T}\right)=-\frac{1}{2} \sigma^{2}\left(1-\rho^{2}\right) \int_{0}^{T} V(T) d t, \operatorname{Var}\left(\varsigma_{T}\right)=\sigma^{2}\left(1-\rho^{2}\right) \int_{0}^{T} V(T) d t \tag{3.10}
\end{equation*}
$$

Then,

$$
\begin{align*}
D(T)= & E\left\{\exp \left[(z-1) z \frac{1}{2} \sigma^{2}\left(1-\rho^{2}\right) \int_{0}^{T} V(t) d t+z \xi_{T}\right]\right\} \\
= & \exp \left\{-z \frac{\rho \sigma}{\sigma_{v}}\left[V(T)+\theta_{v} T\right]\right\}  \tag{3.11}\\
& \times E\left\{\exp \left[(z-1) z \frac{1}{2} \sigma^{2}\left(1-\rho^{2}\right)+z\left(\frac{\rho \sigma}{\sigma_{v}} \alpha_{v}-\frac{1}{2} \sigma^{2} \rho^{2}\right)\right] \int_{0}^{T} V(t) d t-z \frac{\rho \sigma}{\sigma_{v}} V_{0}\right\}
\end{align*}
$$

Let $s_{1}=-(z-1) z(1 / 2) \sigma^{2}\left(1-\rho^{2}\right)-z\left(\left(\rho \sigma / \sigma_{v}\right) \alpha_{v}-(1 / 2) \sigma^{2} \rho^{2}\right)$, and $s_{2}=-z\left(\rho \sigma / \sigma_{v}\right)$. From Lemma 3.1, we have

$$
\begin{align*}
D(T)=\exp \{ & -z \frac{\rho \sigma}{\sigma_{v}}\left(\mathrm{~V}(T)+\theta_{v} T\right)+\frac{2 \theta_{v}}{\sigma_{v}^{2}} \ln \left[\frac{2 \gamma e^{(1 / 2)\left(\alpha_{v}-\gamma\right) T}}{2 \gamma e^{-\gamma T}+\left(\alpha_{v}+\gamma+\sigma_{v}^{2} s_{2}\right)\left(1-e^{-\gamma T}\right)}\right]  \tag{3.12}\\
& \left.-\frac{\left(1-e^{-\gamma T}\right)\left(2 s_{1}-\alpha_{v} s_{2}\right)+\gamma s_{2}\left(1+e^{-\gamma T}\right)}{2 \gamma e^{-\gamma T}+\left(\alpha_{v}+\gamma+\sigma_{v}^{2} s_{2}\right)\left(1-e^{-\gamma T}\right)} V_{0}\right\}
\end{align*}
$$

From (3.8), (3.9), (3.10) and (3.12), we can obtain the required Lemma 3.2.
Lemma 3.3. Suppose $\varphi(u)=E[\exp (i u \ln S(T)]$ is the characteristic function of $\ln S(T)$; then

$$
\begin{align*}
\varphi(u)= & {\left[\frac{2 \delta}{2 \delta+\left(\alpha_{v}-\delta-i u \rho \sigma \sigma_{v}\right)\left(1-e^{-\delta T}\right)}\right]^{2 \theta_{v} / \sigma_{v}^{2}} } \\
& \times \exp \left\{i u \ln S(t)+\frac{\theta_{v}\left(\alpha_{v}-\delta\right) T}{\sigma_{v}^{2}}-\frac{i u \theta_{v} \sigma \rho T}{\sigma_{v}}\right.  \tag{3.13}\\
& +\lambda T\left[\frac{p \eta_{1}}{\eta_{1}-i u}+\frac{q \eta_{2}}{\eta_{2}+i u}-1-i u\left(\frac{p \eta_{1}}{\eta_{1}-1}+\frac{q \eta_{2}}{\eta_{2}+1}-1\right)\right] \\
& \left.+i u r T+\epsilon V_{0}\right\}
\end{align*}
$$

where

$$
\begin{align*}
& \delta=\sqrt{\left(\alpha_{v}-i u \rho \sigma \sigma_{v}\right)^{2}+i u(1-i u) \sigma^{2} \sigma_{v}^{2}} \\
& \epsilon=\frac{i u(i u-1) \sigma^{2}\left(1-e^{-\delta T}\right)}{2 \delta+\left(\alpha_{v}-\delta-i u \rho \sigma \sigma_{v}\right)\left(1-e^{-\delta T}\right)} \tag{3.14}
\end{align*}
$$

Proof. Let $\phi(z)=E\{\exp [-r T+z \ln S(T)]\}$. Because

$$
\begin{equation*}
\varphi(u)=E[\exp (i u \ln S(T))]=\frac{E\{\exp [-r T+i u \ln S(T)]\}}{E[\exp (-r T)]}=\frac{\phi(i u)}{\phi(0)} \tag{3.15}
\end{equation*}
$$

from Lemma 3.2, we can obtain the required Lemma 3.3.
Theorem 3.4. Let $k$ denote the log of the strike price $K, x_{T}=\ln (S(T))$, and $C_{T}(k)$ the desired value of a T-maturity call option with strike $\exp (k)$. Assume that, under $P^{*}$, the underlying nondividendpaying stock price $S(t)$ and its components are given by (2.1)-(2.5), $\varphi(u)$ is the characteristic function of $x_{T}, q(x)$ is the density of $x_{T}$; then the initial call value $C_{T}(k)$ is written as

$$
\begin{align*}
C_{T}(k)= & \frac{1}{2}\left(S(t)-e^{-r T} K\right) \\
& +\frac{1}{\pi} \int_{0}^{\infty} S(t) \Re\left[\frac{e^{i u k} \varphi_{T}(u-i)}{i u}\right]-e^{-r T} K \Re\left[\frac{e^{i u k} \varphi_{T}(u)}{i u}\right] d u, \tag{3.16}
\end{align*}
$$

where $\mathfrak{R}[\cdot]$ represents real part.
Proof. From the risk-neutral theory, we have

$$
\begin{align*}
C_{T}(k) & =E\left[e^{-r T}(S(T)-K)^{+}\right] \\
& =e^{-r T} \int_{0}^{+\infty}(S(T)-K)^{+} q(S(T)) d S(T)  \tag{3.17}\\
& =e^{-r T} \int_{k}^{+\infty} e^{x_{T}} q(x) d x-e^{-r T} K \int_{k}^{+\infty} q(x) d x \\
& =S \Pi_{1}-e^{-r T} \Pi_{2}
\end{align*}
$$

Introducing a change of measure from $P^{*}$ to $Q^{*}$ by a Radon-Nikodym derivative, we get

$$
\begin{equation*}
\frac{d Q^{*}}{d P^{*}}=\frac{e^{x_{T}}}{E\left[e^{x_{T}}\right]} \tag{3.18}
\end{equation*}
$$

With this new measure $Q^{*}$, the Fourier transform of $\Pi_{1}$ is defined as

$$
\begin{equation*}
E^{Q^{*}}\left[e^{i u x_{T}}\right]=\frac{\varphi(u-i)}{\varphi(-i)} \tag{3.19}
\end{equation*}
$$

Because of the no-arbitrage condition, we can obtain

$$
\begin{equation*}
\Pi_{1}=\frac{1}{2}+\frac{1}{\pi} \int_{0}^{\infty} \Re\left[\frac{e^{-i u k} \varphi_{T}(u-i)}{i u \varphi_{T}(-i)}\right] d u \tag{3.20}
\end{equation*}
$$

From the Fourier transform formula, the probability density for our model is given by

$$
\begin{equation*}
q(x)=\frac{1}{\pi} \int_{0}^{+\infty} e^{-i u k} \varphi(u) d u \tag{3.21}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\Pi_{2}=\int_{k}^{+\infty}\left(\frac{1}{\pi} \int_{0}^{+\infty} e^{-i u k} \varphi(u) d u\right) d x \tag{3.22}
\end{equation*}
$$

Changing the order of integration, we have

$$
\begin{equation*}
\Pi_{2}=\frac{1}{2}+\frac{1}{\pi} \int_{0}^{\infty} \mathfrak{R}\left[\frac{e^{-i u k} \varphi_{\mathrm{T}}(u)}{i u}\right] d u \tag{3.23}
\end{equation*}
$$

From (3.17), (3.20), and (3.23), we can obtain the required Theorem 3.4.
Remark 3.5. In (3.16), $C_{T}(k)$ tends to $S_{0}$ not zero as $k$ goes to $-\infty$. Hence, $C_{T}(k)$ is not $L^{1}$ (absolutely integrable) and a Fourier transform does not exist.

## 4. Fast Fourier Transform for European Option Pricing

Since the integrand in (3.16) is singular at the required evaluation point $u=0$, the FFT cannot be applied directly to evaluate the integrals we mentioned above. Therefore, instead of solving for the risk-neutral exercise probabilities of finishing in-the-Money (ITM), Carr and Madan [16] introduce a new technique with the key idea to calculate the Fourier transform of a modified call option price with respect to the logarithmic strike price. With this specification and a FFT routine, a whole range of option prices can be obtained within a single Fourier inversion. In this section, we develop the numerical solutions of the prices by using the idea of Carr and Madan [16].

### 4.1. Fourier Transform of ITM and at-the-Money (ATM) Option Prices

By introducing an exponential damping factor $e^{\alpha k}$ with $\alpha>0$, it is possible to make the integrand in (3.16) be square integrable. We modified the pricing function (3.16) by

$$
\begin{equation*}
C_{T}(k)=\frac{\exp (-\alpha k)}{\pi} \int_{0}^{\infty} e^{-i v k} \psi_{T}(v) d v, \tag{4.1}
\end{equation*}
$$

where $\psi_{T}(v)=e^{-r T} \varphi_{T}(v-(\alpha+1) i) /\left(\alpha^{2}+\alpha-v^{2}+i(2 \alpha+1) v\right)$.
This method is viable when $\alpha$ is chosen in a way that the damped option price is well behaved. Damping the option price with $e^{\alpha k}$ makes it integrable for the negative axis $k<0$. On the other hand, for $k>0$ the option prices increase by the exponential $e^{\alpha k}$, which
influences the integrability for the positive axis. A sufficient condition of $c_{T}(k)$ to be integrable for both sides (square integrability) is given by $\psi(0)$ being finite, that is,

$$
\begin{equation*}
\psi(0)=\frac{e^{-r T} \varphi_{T}(-(\alpha+1) i)}{\alpha^{2}+\alpha}<\infty . \tag{4.2}
\end{equation*}
$$

Thus we need $\varphi_{T}(-(\alpha+1) i)<\infty$, which is equivalent to

$$
\begin{equation*}
E^{Q}\left[S(T)^{1+\alpha}<\infty\right] \tag{4.3}
\end{equation*}
$$

Therefore, $c_{T}(k)$ is well behaved when the moments of order $1+\alpha$ of the underlying asset exist and are finite. If not all moments of $S(T)$ exist, this will impose an upper bound on $\alpha$. We find that one quarter of this upper bound serves as a good choice for $\alpha$.

Using the trapezoid rule for the integral on the right-hand side of (4.1) and setting $v_{j}=\eta(j-1)$, an approximation for $C_{T}(k)$ is

$$
\begin{equation*}
C_{T}(k) \approx \frac{\exp (-\alpha k)}{\pi} \sum_{j=1}^{N} e^{-i v_{j} k} \psi_{T}\left(v_{j}\right) \eta . \tag{4.4}
\end{equation*}
$$

The FFT returns $N$ values of $k$, and we employ a regular spacing of size $h$ so that our values for $k$ are

$$
\begin{equation*}
k_{u}=-b+h(u-1) \quad \text { for } u=1, \ldots, N . \tag{4.5}
\end{equation*}
$$

This gives us log-strike levels ranging from $-b$ to $b$, where

$$
\begin{equation*}
b=\frac{1}{2} N h . \tag{4.6}
\end{equation*}
$$

In order to apply FFT we define

$$
\begin{equation*}
\eta h=\frac{2 \pi}{N} . \tag{4.7}
\end{equation*}
$$

To obtain an accurate integration with larger values of $\eta$, we incorporate Simpson's rule weightings into our summation. From (4.1)-(4.7)and Simpson's rule weightings, we obtain ATM and ITM call value as

$$
\begin{equation*}
C\left(k_{u}\right)=\frac{\exp \left(-\alpha k_{u}\right)}{\pi} \sum_{j=1}^{N} e^{-i(2 \pi / N)(j-1)(i-1)} e^{i b v_{j}} \psi\left(v_{j}\right) \frac{\eta}{3}\left[3+(-1)^{j}-\omega_{j-1}\right] \tag{4.8}
\end{equation*}
$$

where $\omega_{n}$ is the Kronecker delta function that is unity for $n=0$ and zero otherwise. The summation in (4.8) is an exact application of the FFT.

### 4.2. Fourier Transform of out-of-the-Money (OTM) Option Prices

In the previous section call values are calculated by an exponential function to obtain square integrable function whose Fourier transform is an analytic function of the characteristic function of the log-price. But, for very short maturities, the call value approaches its non analytic intrinsic value causing the integrand in the Fourier inversion to be high oscillate, and therefore difficult to integrate numerically. We introduce an alternative approach that works with time values only, which is quite similar to the previous approach. But in this case the call price is obtained via the Fourier transform of a modified time value, where the modification involves a hyperbolic sine function instead of an exponential function.

Let $z_{T}(k)$ denote the time value of an OTM option, that is, for $k<x_{T}$ we have the put price for $z_{T}(k)$ and for $k<x_{T}$ we have the call price. Scaling $S_{0}=1$ for simplicity, $z_{T}(k)$ is defined by

$$
\begin{equation*}
z_{T}(k)=e^{-r T} \int_{-\infty}^{\infty}\left[\left(e^{k}-e^{x_{T}}\right) \mathbf{1}_{x_{T}<k, k<0}+\left(e^{x_{T}}-e^{k}\right) \mathbf{1}_{x_{T}>k, k>0}\right] q(x) d x, \tag{4.9}
\end{equation*}
$$

where $q(x)$ is the risk-neutral density of the log-price $x_{T}$. Let $\zeta_{T}(u)$ be the Fourier transform of $z_{T}(k)$ :

$$
\begin{equation*}
\zeta_{T}(u)=\int_{-\infty}^{\infty} e^{i u k} z_{T}(k) d k . \tag{4.10}
\end{equation*}
$$

By considering a damping function $\sinh (\alpha k)$, the time value of an option follows a Fourier inversion:

$$
\begin{equation*}
z_{T}(k)=\frac{1}{\sinh (\alpha k)} \frac{1}{\pi} \int_{0}^{\infty} e^{-i u k} \Upsilon_{T}(u) d u \tag{4.11}
\end{equation*}
$$

where $\Upsilon_{T}(u)=\left(\zeta_{T}(u-i \alpha)-\zeta_{T}(u+i \alpha)\right) / 2$.
The use of the FFT for calculating OTM option prices is similar to (4.8). The only differences are that they replace the multiplication by $\exp \left(-\alpha k_{u}\right)$ with a division by $\sinh (\alpha k)$ and the function call to $\psi(v)$ is replaced by a function call to $\Upsilon_{T}(u)$.

## 5. Simulation Studies

In this section, to compare across the BS, DEJD, and SVDEJD models, we analyze the probability densities of these models. Then, we analyze mainly the impact of $\rho$ and volatility of volatility $\sigma_{v}$ on option pricing under the SVDEJD model. For our FFT methods, we used $N=4096$ points in our quadrature, implying a log-strike spacing of $h=\pi / 300=0.01047$, which is adequate for practice. For the choice of the dampening coefficient in the transform of the modified call price, we used a value of $\alpha=2.55$. For the modified time value, we used $\alpha=1.55$. Other parameter values used in the computation are listed in Table 1. (We have used analytic moments to set plausible parameter values for the model. For a formal econometric estimator, one could use these moments to develop a generalized method of moments estimator within the framework of Hall and Inoue [21].)

Table 1: Default parameters for simulation of option prices.

| Parameter | Value |
| :--- | :---: |
| Probability of upward | $p=0.6$ |
| Volatility of asset price | $\sigma=0.16$ |
| Mean of the exponential distribution of upward | $\eta_{1}=40$ |
| Mean of the exponential distribution of downward | $\eta_{2}=40$ |
| Intensity of the Poisson process | $\lambda=10$ |
| Interest rate | $r=0.05$ |
| Initial asset price | $S_{0}=100$ |
| Initial variance | $V_{0}=1$ |
| Rate of reversion | $a_{v}=0.3$ |
| Long-run variance | $\theta_{v}=0.6$ |
| Volatility of volatility | $\sigma_{v}=0.25$ |
| Correlation between returns and volatility | $\rho=-0.8$ |

### 5.1. Probability Densities under Alternative Models

We compare the probability densities of the SVDEJD model, the BS model, and the DEJD model to verify the rationality of our model. Suppose $\varphi(u)$ is the characteristic function of $x_{T}$ and $q(x)$ the probability density of our model. From FFT algorithm, $q(x)$ can be approximated by

$$
\begin{equation*}
q(x) \approx \frac{1}{\pi} \sum_{j=1}^{N} e^{-i(2 \pi / N)(j-1)(k-1)} \varphi(u) \quad k=1, \ldots, N . \tag{5.1}
\end{equation*}
$$

The density $q$ has the mean and variance given by

$$
\begin{gather*}
E_{q}(Q)=\frac{\varphi^{\prime}(0)}{i}  \tag{5.2}\\
\operatorname{Var}_{q}(Q)=-\varphi^{\prime \prime}(0)+\left(\varphi^{\prime}(0)\right)^{2}
\end{gather*}
$$

Figure 1 shows the figures of the probability density $q(x)$, compared with the normal density with the same mean and variance given by (5.2). The first figure compares the overall shapes of the densities of the SVDEJD model and the BS model, the second one details the shapes around the peak areas, and the last one shows the right tail. From Figure 1, we can see that the leptokurtic and skewness feature of the density of our model is quite evident. Moreover, additional numerical plots suggest that the feature of skewness becomes more significant if $|\rho|$ increases, which is impossible for the DEJD model.

We also compare the short-term and long-term densities of the SVDEJD model, the BS model, and the DEJD one. Figure 2 shows their densities under $T=3$ months and $T=2$ years. From Figure 2, we can see that the SVDEJD model and the DEJD model generate virtually identical densities for short-term options, with a slight departure occurring between the two densities in the upper tail. This means that differential pricing performance between the SVDEJD model and the DEJD model is unlikely to occur when they are applied to price


Figure 1: Comparison of the probability densities of the SVDEJD model and the BS model. Except for the maturity time $T=2$ years, the parameters used here are shown in Table 1.
short-term OTM puts and that only when they are applied to deep ITM puts (and deep OTM calls) can differences be observed between these models. Yet, compared to the BS model density, the densities of the two models are distinctly different: they all have leptokurtic and skewness feature. Therefore, the two models can potentially correct the BS model's tendency to underprice deep OTM puts and overprice deep OTM calls. The long-term density curves in Figure 2 still show significantly different pricing structures between the BS and its two alternatives. But, more importantly, the densities of the SVDEJD model and the DEJD model also exhibit different shapes now. The SVDEJD density has higher peak and assigns more weight to both the entire lower tail and the far upper tail, but less weight to those payoffs than the DEJD.

Our simulation studies have demonstrated that the SVDEJD model has better performance than the DEJD one on pricing long-term options, while both the DEJD model and the SVDEJD model have better performance than the BS model.


Figure 2: Comparison of the short-term and long-term probability densities of the SVDEJD model, the BS model, and the DEJD model. Except for the maturity time $T=3$ months and $T=2$ years, the parameters used here are shown in Table 1.

Table 2: The effects of volatility of volatility, exercise price $K$, maturity time $T$, and correlation $\rho$ on option values.

| Strike price | $\rho=-0.8$ |  | $\rho=0$ |  | $\rho=0.8$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\sigma_{v}=0.15$ | $\sigma_{v}=0.25$ | $\sigma_{v}=0.15$ | $\sigma_{v}=0.25$ | $\sigma_{v}=0.15$ | $\sigma_{v}=0.25$ |
| $T=3$ months |  |  |  |  |  |  |
| 90 | 11.5004 | 11.5122 | 11.4827 | 11.4829 | 11.4647 | 11.4527 |
| 95 | 7.5901 | 7.6000 | 7.5745 | 7.5741 | 7.5585 | 7.5472 |
| 100 | 4.4325 | 4.4323 | 4.4315 | 4.4307 | 4.4303 | 4.4287 |
| 105 | 2.5442 | 2.5347 | 2.5573 | 2.5567 | 2.5567 | 2.5780 |
| 110 | 1.1258 | 1.1109 | 1.1476 | 1.1476 | 1.1690 | 1.1830 |
| 115 | 0.5169 | 0.5043 | 0.5361 | 0.5364 | 0.5532 | 0.5681 |
| $T=2$ years |  |  |  |  |  |  |
| 90 | 22.5229 | 22.5682 | 22.4438 | 22.4398 | 22.3592 | 22.2959 |
| 95 | 19.4525 | 19.4825 | 19.3935 | 19.3872 | 19.3293 | 19.2776 |
| 100 | 16.6650 | 16.6750 | 16.6337 | 16.6255 | 16.5979 | 16.5633 |
| 105 | 14.3313 | 14.3225 | 14.3271 | 14.3179 | 14.3189 | 14.3018 |
| 110 | 11.9850 | 11.9507 | 12.0183 | 12.0083 | 12.0476 | 12.0549 |
| 115 | 10.1354 | 10.0809 | 10.1994 | 10.1896 | 10.2594 | 10.2871 |

### 5.2. Effects of the Main Parameter on Option Values

In Table 2, we use the SVDEJD model to examine the effects of volatility of volatility $\sigma_{v}$, the correlation coefficient $\rho$, exercise price $K$, and maturity time $T$ on option values. We analyze the prices of three-month call options and two-year call options. To examine the effect of the negative correlation coefficient, we have calculated the model with $\rho=-0.8, \rho=0$, and $\rho=0.8$. The prices for three-month call options associated with volatility of volatility $\sigma_{v}=0.15$
and $\sigma_{v}=0.25$ are relatively close. With $\rho=-0.8$, the largest price difference is 0.0149 ; with $\rho=0$, the largest price difference is only 0.0008 ; with $\rho=0.8$, the largest price difference is 0.0213 . However, the difference is significantly larger when longer-time horizons such as two-year call options are valued. With $\rho=-0.8$, the largest price difference is an increase on 0.0149 to 0.0545 and the effect of volatility of volatility is an increase for ITM calls and a decrease for OTM calls. With $\rho=0$, the largest price difference is increase of 0.0008 to 0.01 and the effect of volatility of volatility is a small decreas that is negligible for option values. With $\rho=0.8$, the largest price difference is an increase of 0.0213 to 0.0633 and the effect of volatility of volatility is a decrease for ITM and ATM calls and an increase for OTM calls. The correlation parameter $\rho$ has several effects depending on the relation between the strike price and the current stock price. A negative $\rho$ tends to produce higher values for ITM calls and lower values for OTM money calls.

We have also compared the model with the BS model, which can be interpreted as a first-order approximation with no jumps, and $\rho=0$. A common practice is to set the implied volatility in the BS model so that the model matches the price for the option with a strike price closest to the current stock price. For some comparisons not reported here, we have set the implied volatility in the BS model so that it matches the price generated by the stochastic volatility model for an ATM option. The BS implied volatility is very close to the expected volatility under the risk-neutral distribution when short-term options are valued. When longer-term options are used, there is a significant difference between the BS implied volatility and the expected volatility. As an approximation, the BS model tends to undervalue ITM calls and overvalue OTM calls.

## 6. Conclusion

The SVDEJD model incorporates several important features of stock returns. We derive a closed-form solution for European call options in the model by using the martingale method, Fourier inversion transform formula, and Feynman-Kac theorem. Using FFT, we obtain fast and accurate numerical solution to European option under the model. The comparison of densities of the alternative models shows that the SVDEJD model has better pricing performance on long-time options. An analysis of the model reveals that volatility of volatility $\sigma_{v}$ and the correlation coefficient $\rho$ have significant impact on option values, especially long-time option, stock returns are negatively correlated with volatility, and these negative correlations are important for option valuation.

## Appendix

## A.1. Matlab Codes for ITM and ATM Options Pricing by FFT

function CV =inSVDexpJ(ata1, ata2, lamta, sigma, thetav, alphav,
rho, sigmav, r, p, s0, v0, strike, T)
$\mathrm{x} 0=\log (\mathrm{s} 0)$
alpha $=2.55$
$\mathrm{N}=4096$
$c=600$

```
eta =c/N
b}=pi/et
u}=[0:N-1\mp@subsup{]}{}{*}\mathrm{ eta
lamda = 2*b/N
position = (log(strike ) +b)/lamda +1
v = u - (alpha+1)*i
k=p*ata1/ (ata1-1)+(1-p)*ata2/ (ata2+1)-1
l=p*ata1./ (ata1-i*v)+(1-p)*ata2./ (ata2+i*v)
m=sqrt((alphav-i* v*rho*sigma*sigmav).. 2+i**.* * (1-i**)* (sigma * sigmav) }\mp@subsup{}{}{\wedge}2
n=2*m+(alphav-m-i*v*rho*sigma*sigmav).*(1-exp(-m*T))
A=(2*m./n).^(2*thetav/sigmav }\mp@subsup{}{}{\wedge}2
B=i*v**0+(thetav*(alphav-m)*T)/sigmav^2-(i* ***ro**igma*thetav*T)/sigmav...
+lamta*T*(l-i* v*k-1)+i* ****
C=(i*v.* (i*v-1)*sigma^2.* (1-exp(-m*T)))./n
charFunc=A.* exp(C*v0+B)
ModifiedCharFunc = charFunc*exp(-r*T)./(alpha^2+alpha - u.^2 + i* (2*alpha
+1)*u)
SimpsonW = 1/3* (3 + (-1).^[1:N] - [1, zeros(1,N-1)])
FftFunc = exp(i*b*u).*ModifiedCharFunc*eta.*SimpsonW
payoff = real(fft(FftFunc))
CallValueM = (exp(-log(strike)*alpha) )'* payoff / pi
format short
CV= CallValueM(round(position)).
```


## A.2. Matlab Codes for OTM Options Pricing by FFT

function $C V=$ outSVDexpJ(ata1,ata2,lamta,sigma,thetav,alphav, rho,sigmav,r,p,s0,v0,strike,T)
$\mathrm{x} 0=\log (\mathrm{s} 0)$
alpha $=1.55$
$\mathrm{N}=4096$
$c=600$
eta $=c / N$
b=pi/eta
$\mathrm{u}=[0: \mathrm{N}-1]^{*}$ eta
lamda $=2 *$ b/N
position $=(\log ($ strike $)+b) /$ lamda +1
$\mathrm{w} 1=\mathrm{u}-\mathrm{i}^{*}$ alpha

```
w2 = u+i**alpha
v1 = u-i*alpha -i
v2 = u+i*alpha -i
k=p*ata1/(ata1-1)+(1-p)*ata2/ (ata2+1)-1
11=\mp@subsup{p}{}{*}\mathrm{ ata1./ (ata1-i**1)+(1-p)*ata2./ (ata2+i*v1)}
m1=sqrt((alphav-i*v1*rho*sigma*sigmav).^2+i** v1.*(1-i*v1)*(sigma*sigmav)}\mp@subsup{)}{}{\wedge}2
n1=2*m1+(alphav-m1-i**1*rho*sigma*sigmav).*(1-exp(-m1*T))
A1=(2*m1./n1).^(2*thetav/sigmav^2)
```



```
+lamta*T*(11-i*v1*k-1)+i** v1*r*T
C1=(i** 1.* (i**v1-1)*sigma^2.*(1-exp(-m1*T)))./n1
charFunc1=A1.* exp(C1*v0+B1)
ModifiedCharFunc1 = exp(--**T)*(1./(1+i**w1)\ldots
-exp(r*T)./(i**1) - charFunc1./(w1.^2 - i* w1))
12=\mp@subsup{p}{}{*}\mathrm{ ata1./ (ata1-i*v2)+(1-p)*ata2./ (ata2+i**2)}
m2=sqrt((alphav-i**2* rho*sigma*sigmav).^2+i** 2.* (1-i*v2)*(sigma*sigmav)^2)
n2=2*m2+(alphav-m2-i**2**rho*sigma*sigmav).* (1-exp(-m2*T))
A2=(2*m2./n2).^(2*thetav/sigmav }\mp@subsup{}{}{\wedge}2
B2=i** 2 **x0+(thetav*(alphav-m2)*T)/sigmav 2-(i*v2* rho*sigma* thetav*T)/sigmav... 
+lamta*T**(12-i*v2**-1)+i*v2***T
C2=(i*v2.* (i** v2-1)**igma^2.* (1-exp(-m2*T)))./n2
charFunc2=A2.*}\operatorname{*exp}(\textrm{C}2**v0+B2
ModifiedCharFunc2 = exp(-r*T)*}(1./(1+\mp@subsup{i}{}{*}w2)-\operatorname{exp}(\mp@subsup{r}{}{*}T)./ (i**2)..
- charFunc2./(w2.^2 - i* w2))
ModifiedCharFuncCombo = (ModifiedCharFunc1 - ModifiedCharFunc2)/2
SimpsonW = 1/3*(3 + (-1).^[1:N] - [1, zeros(1,N-1)])
FftFunc = exp(i*b*u).*ModifiedCharFuncCombo*eta.*SimpsonW
payoff = real(fft(FftFunc))
CallValueM = payoff/pi/sinh(alpha*log(strike))
format short
CV= CallValueM(round(position)).
```


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Research Article

# Nonsmooth Adaptive Control Design for a Large Class of Uncertain High-Order Stochastic Nonlinear Systems 

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This paper investigates the problem of the global stabilization via partial-state feedback and adaptive technique for a class of high-order stochastic nonlinear systems with more uncertainties/unknowns and stochastic zero dynamics. First of all, two stochastic stability concepts are slightly extended to allow the systems with more than one solution. To solve the problem, a lot of substantial technical difficulties should be overcome since the presence of severe uncertainties/unknowns, unmeasurable zero dynamics, and stochastic noise. By introducing the suitable adaptive updated law for an unknown design parameter and appropriate control Lyapunov function, and by using the method of adding a power integrator, an adaptive continuous (nonsmooth) partial-state feedback controller without overparameterization is successfully designed, which guarantees that the closed-loop states are bounded and the original system states eventually converge to zero, both with probability one. A simulation example is provided to illustrate the effectiveness of the proposed approach.

## 1. Introduction

In the past decades, stability and stabilization for stochastic nonlinear systems have been vigorously developed [1-13]. As the early investigation in the area, in [1-3], some quite fundamental notations have been proposed to characterize different types of stochastic stability and, meanwhile for which, sufficient conditions have been separately provided. As the recent investigation, works [4] and $[3,5]$ considered stabilization problems by using Sontag's formula and backstepping method, respectively, and stimulated a series of subsequent works [6-13].

The control designs for classes of high-order nonlinear systems have received intense investigation recently and developed the so-called method of adding a power integrator which
is based on the idea of the stable domain [14] and can be viewed as the latest achievement of the traditional backstepping method [15]. By applying such skillful method, smooth statefeedback control design can be achieved when some severe conditions are imposed on systems (see, e.g., $[16,17]$ ), while, without those conditions, only nonsmooth state-feedback control can be possibly designed (see, e.g., [18-23]). As a natural extension, the outputfeedback case was considered in [24], for less available information, which is a more interesting and difficult subject of intensive study. Another extension is the control design for high-order stochastic nonlinear systems, which attract plenty of attention because of the presence of stochastic disturbance and cannot be solved by simply extending the methods for deterministic systems (see, e.g., [25-31]). To the authors' knowledge, this issue has not been richly investigated and on which many significant problems remain unsolved.

This paper considers the global stabilization for the high-order stochastic nonlinear systems described by (3.1) below, relaxes the assumptions imposed on the systems in [2528], and obtains much more general results than the previous ones. Since the presence of system uncertainties, some nontrivial obstacles will be encountered during control design, which force many skillful adaptive techniques to be employed in this paper. Furthermore, for the stabilization problem, finding a suitable and available control Lyapunov function is necessary and important. In this paper, a novel control Lyapunov function is first successfully constructed, which is available for the stabilization of system (3.1) and different from those introduced in [25-28] which are unusable here. Then, by using the method of adding a power integrator, an adaptive continuous partial-state feedback controller is successfully achieved to guarantee that for any initial condition the original system states are bounded and can be regulated to the origin almost surely.

The contributions of the paper are highlighted as follows.
(i) The systems under investigation are more general than those studied in closely related works [25-28]. Different from [26], the zero dynamics of the systems are unmeasurable and disturbed by stochastic noise. Moreover, the restrictions on the system nonlinear terms are weaker than those in [25-28], and in particular, the assumption in [27] that the low bounds of unknown control coefficients are known has been removed.
(ii) The paper considerably generalizes the results in $[17,22]$, and more importantly, no overparameterization problem is present in the adaptive control scheme. In fact, the paper presents the stochastic counterpart of the result in [22] under quite weak assumptions. Particularly, the paper develops the adaptive control scheme without overparameterization (one parameter estimate is enough). Furthermore, it is easy to see that the scheme developed can be used to eliminate the overparameterization problem in $[17,21,22]$ (reduce the number of parameter estimates from $n+1$ to 1 ).
(iii) The formulation of zero dynamics is typical and suggestive. In fact, to make the formulation of zero dynamics more representational, we adopt partial assumptions on zero dynamics in $[8,9]$. It is worth pointing out that the formulation of the gain functions of stochastic disturbance is somewhat general than those in [8, 9].

The remainder of this paper is organized as follows. Section 2 presents some necessary notations, definition and preliminary results. Section 3 describes the systems to be studied, formulates the control problem, and presents some useful propositions. Section 4 gives the main contributions of this paper and presents the design scheme to the controller. Section 5 gives a simulation example to demonstrate the effectiveness of the theoretical results. The paper ends with an Appendices A and B.

## 2. Notations and Preliminary Results

Throughout the paper, the following notations are adopted. $\mathbf{R}^{n}$ denotes the real $n$-dimensional space. $\mathbf{R}_{\text {odd }}^{\geq 1}$ denotes the set $\left\{q_{1} / q_{2} \mid q_{1}\right.$ and $q_{2}$ are odd positive integers, and $\left.q_{1} \geq q_{2}\right\} . \mathbf{R}^{+}$ denotes the set of all positive real numbers. For a given vector or matrix $X, X^{T}$ denotes its transpose, $\operatorname{Tr}\{X\}$ denotes its trace when $X$ is square, and $\|X\|$ denotes the Euclidean norm when $X$ is a vector. $C^{k}$ denotes the set of all functions with continuous partial derivatives up to the $k$ th order. $\mathcal{K}$ denotes the set of all functions from $\mathbf{R}^{+}$to $\mathbf{R}^{+}$, which are continuous, strictly increasing, and vanishing at zero, and $\boldsymbol{K}_{\infty}$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded.

Consider the general stochastic nonlinear system

$$
\begin{equation*}
\mathrm{d} x(t)=f(t, x) \mathrm{d} t+g(t, x) \mathrm{d} w \tag{2.1}
\end{equation*}
$$

where $x \in \mathbf{R}^{n}$ is the system state vector with the initial condition $x(0)=x_{0}$; drift term $f: \mathbf{R}^{+} \times \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ and diffusion term $g: \mathbf{R}^{+} \times \mathbf{R}^{n} \rightarrow \mathbf{R}^{n} \times \mathbf{R}^{m}$ are piecewise continuous and continuous with respect to the first and second arguments, respectively, and satisfy $f(t, 0) \equiv 0$ and $g(t, 0) \equiv 0 ; w(t) \in \mathbf{R}^{m}$ is an independent standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ a $\sigma$-algebra on $\Omega$, and $P$ a probability measure.

Since both $f(\cdot)$ and $g(\cdot)$ are only continuous, not locally Lipschitz, system (2.1) may not have the solution in the classical sense as in [7, 9]. However, the system always has weak solutions which are essentially different from the classical (or strong) solution since the former may not be unique and may be defined on a different probability space $\left(\Omega^{\prime}, \mathscr{F}^{\prime}, P^{\prime}\right)$. The following definition gives the rigorous characterization of the weak solution of system (2.1), and for more details of weak solution, we refer the reader to [32,33].

Definition 2.1. For system (2.1), if a continuous stochastic process $x(t)$ defined on a probability space $\left(\Omega_{x}, \mathcal{F}_{x}, P_{x}\right)$ with a filtration $\left\{\mathcal{F}_{x, t}\right\}_{t \geq 0}$ and an $m$-dimensional Brownian motion $w(t)$ adapted to $\left\{\mathcal{F}_{x, t}\right\}_{t \geq 0}$, such that for all $t \in\left[0, \tau_{x,+\infty}\right)$, the integrals below are well-defined and $x(t)$ satisfies

$$
\begin{equation*}
x(t)=x_{0}+\int_{0}^{t} f(s, x(s)) \mathrm{d} s+\int_{0}^{t} g(s, x(s)) \mathrm{d} w(s) \tag{2.2}
\end{equation*}
$$

then $x(t)$ is called a weak solution of system (2.1), where $\tau_{x,+\infty}$ denotes either $+\infty$ or the finite explosion time of solution $x(t)$ (i.e., $\tau_{x,+\infty}=\lim _{r \rightarrow+\infty} \inf \{s \geq 0:\|x(s)\| \geq r\}$ ).

To characterize the stability of the origin solution of system (2.1), as well as the common statistic property of all possible weak solutions of the system, we slightly extend the classical stochastic stability concepts of globally stable in probability and globally asymptotically stable in probability given in [7]. This extension is inspired by the deterministic analog in [34] and allows the above two stability concepts applicable to the systems with more than one weak solution.

Definition 2.2. The origin solution of system (2.1) is globally stable in probability if, for all $\epsilon>0$, for any weak solution $x(t)$ which is defined on its corresponding probability space $\left(\Omega_{x}, \mathcal{F}_{x}, P_{x}\right)$, there exists a class $\nless$ function $\gamma_{x}(\cdot)$ such that

$$
\begin{equation*}
P_{x}\left\{\|x(t)\| \leq \gamma_{x}\left(\left\|x_{0}\right\|\right)\right\} \geq 1-\epsilon, \quad \forall t \geq 0, \forall x_{0} \in \mathbf{R}^{n} \backslash\{0\} \tag{2.3}
\end{equation*}
$$

and globally asymptotically stable in probability if it is globally stable in probability and for any weak solution $x(t)$,

$$
\begin{equation*}
P_{x}\left\{\lim _{t \rightarrow+\infty}\|x(t)\|=0\right\}=1, \quad \forall x_{0} \in \mathbf{R}^{n} \tag{2.4}
\end{equation*}
$$

More importantly, we have the following theorem, which can be regarded as the version of Theorem 2.1 of [7] in the setting of more than one weak solution, provides the sufficient conditions for the above two extended stability concepts, and consequently will play a key role in the later development. By comparison, one can see that Theorem 2.3 preserves the main conclusion of Theorem 2.1 of [7] except for the uniqueness of strong solution. By some minor/trivial modifications to the proofs of Theorem 3.19 in [35] (or that of Lemma 2 in [36]) and Theorem 2.4 in [37], it is not difficult to prove Theorem 2.3.

Theorem 2.3. For system (2.1), suppose that there exists a $\mathcal{C}^{2}$ function $V(\cdot)$ which is positive definite and radially unbounded, such that

$$
\begin{equation*}
\mathcal{L} V(x):=\frac{\partial V}{\partial x} f(s, x)+\frac{1}{2} \operatorname{Tr}\left\{g^{T}(s, x) \frac{\partial^{2} V}{\partial x^{2}} g(s, x)\right\} \leq-W(x), \quad \forall s \geq 0, \forall x \in \mathbf{R}^{n} \tag{2.5}
\end{equation*}
$$

where $W(\cdot)$ is continuous and nonnegative. Then the origin solution of (2.1) is globally stable in probability. Furthermore, if $W(\cdot)$ is positive definite, then for any weak solution $x(t)$ defined on probability space $\left(\Omega_{x}, \mathcal{F}_{x}, P_{x}\right)$, there holds $P_{x}\left\{\lim _{t \rightarrow+\infty}\|x(t)\|=0\right\}=1$.

Proof. From Theorem 2.3 in [33, page 159], it follows that system (2.1) has at least one weak solution. We use $x(t)$ to denote anyone of the weak solutions, which is defined on its corresponding probability space $\left(\Omega_{x}, \mathcal{F}_{x}, P_{x}\right)$ and on $\left[0, \tau_{x,+\infty}\right)$ where $\tau_{x,+\infty}$ denotes either $+\infty$ or the finite explosion time of the weak solution $x(t)$.

First, quite similar to the proof of Theorem 3.19 in [35, page 95-96] or that of Lemma 2 in [36], we can prove that $P_{x}\left\{\tau_{x,+\infty}=+\infty\right\}=1$ (namely, all weak solutions of system (2.1) are defined on $[0,+\infty)$ ) and that the origin solution of system (2.1) is globally stable in probability.

Second, very similar to the proof of Theorem 2.4 in [37, page 114-115], we can show that if $W(\cdot)$ is positive definite, then for any weak solution $x(t)$, it holds $P_{x}\left\{\lim _{t \rightarrow+\infty}\|x(t)\|=\right.$ $0\}=1$.

We next provide three lemmas which will play an important role in the later development. In fact, Lemma 2.4 can be directly deduced from the well-known Young's Inequality, and the proofs of Lemmas 2.5 and 2.6 can be found in [19, 20].

Lemma 2.4. For any $c>0, d>0, \varepsilon>0$, there holds

$$
\begin{equation*}
|x|^{c}|y|^{d} \leq \frac{c}{c+d} \varepsilon|x|^{c+d}+\frac{d}{c+d} \varepsilon^{-c / d}|y|^{c+d}, \quad \forall x \in \mathbf{R}, \quad \forall y \in \mathbf{R} . \tag{2.6}
\end{equation*}
$$

Lemma 2.5. For any continuous function $g: \mathbf{R}^{m} \times \mathbf{R}^{n} \rightarrow \mathbf{R}$, there are smooth functions $a: \mathbf{R}^{m} \rightarrow$ $\mathbf{R}^{+}, b: \mathbf{R}^{n} \rightarrow \mathbf{R}^{+}, c: \mathbf{R}^{m} \rightarrow[1,+\infty)$, and $d: \mathbf{R}^{n} \rightarrow[1,+\infty)$ such that

$$
\begin{equation*}
|g(x, y)| \leq a(x)+b(y), \quad|g(x, y)| \leq c(x) d(y), \quad \forall x \in \mathbf{R}^{m}, \forall y \in \mathbf{R}^{n} \tag{2.7}
\end{equation*}
$$

Lemma 2.6. For any $p \geq 1$, and any $x \in \mathbf{R}, y \in \mathbf{R}$, there hold

$$
\begin{align*}
|x+y|^{p} & \leq 2^{p-1}\left|x^{p}+y^{p}\right| \\
(|x|+|y|)^{1 / p} & \leq|x|^{1 / p}+|y|^{1 / p} \leq 2^{(p-1) / p}(|x|+|y|)^{1 / p} \tag{2.8}
\end{align*}
$$

and, in particular, if $p \in \mathbf{R}_{\text {odd }}^{\geq 1},|x-y|^{p} \leq 2^{p-1}\left|x^{p}-y^{p}\right|$.

## 3. System Model and Control Objective

In this paper, we consider the global adaptive stabilization for a class of uncertain high-order stochastic nonlinear systems in the following form:

$$
\begin{gather*}
\mathrm{d} \eta=f_{0}(x, \eta) \mathrm{d} t+g_{0}(x, \eta) \mathrm{d} w, \\
\mathrm{~d} x_{1}=d_{1}(x, \eta) x_{2}^{p_{1}} \mathrm{~d} t+f_{1}(x, \eta) \mathrm{d} t+g_{1}^{T}(x, \eta) \mathrm{d} w, \\
\vdots  \tag{3.1}\\
\mathrm{~d} x_{n-1}=d_{n-1}(x, \eta) x_{n}^{p_{n-1}} \mathrm{~d} t+f_{n-1}(x, \eta) \mathrm{d} t+g_{n-1}^{T}(x, \eta) \mathrm{d} w, \\
\mathrm{~d} x_{n}=d_{n}(x, \eta) u^{p_{n}} \mathrm{~d} t+f_{n}(x, \eta) \mathrm{d} t+g_{n}^{T}(x, \eta) \mathrm{d} w,
\end{gather*}
$$

where $\eta \in \mathbf{R}^{m_{1}}$ is the unmeasurable system state vector, called zero dynamics; $x=$ $\left[x_{1}, \ldots, x_{n}\right]^{T} \in \mathbf{R}^{n}$ and $u \in \mathbf{R}$ are the measurable system state vector and the control input, respectively; the system initial condition is $\eta(0)=\eta_{0}, x(0)=x_{0} ; p_{i} \in \mathbf{R}_{\text {odd }}^{\geq 1}, i=1, \ldots, n$ are said the system high orders; $f_{0}: \mathbf{R}^{n} \times \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{m_{1}}, f_{i}: \mathbf{R}^{n} \times \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}, i=1, \ldots, n$ and $g_{0}: \mathbf{R}^{n} \times \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{m_{1} \times m}, g_{i}: \mathbf{R}^{n} \times \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{m}, i=1, \ldots, n$ are unknown continuous functions, called the system drift and diffusion terms, respectively; $d_{i}: \mathbf{R}^{n} \times \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}, i=1, \ldots, n$ are uncertain and continuous, called the control coefficients; $w \in \mathbf{R}^{m}$ is an independent standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ a $\sigma$-algebra on $\Omega$, and $P$ a probability measure. Besides, for the simplicity of expression in later use, let $x_{n+1}=u$ and $x_{[k]}=\left[x_{1}, \ldots, x_{k}\right]^{T}$.

Differential equations (3.1) describe a large class of uncertain high-order stochastic nonlinear systems, for which some tedious technical difficulties will be encountered in control design mainly due to the presence of the stochastic zero dynamics and the uncertainties/ unknowns in the control coefficients, the system drift, and diffusion terms. In the recent works [25-28], with measurable inverse dynamics or deterministic zero dynamics and by
imposing somewhat severe restrictions on $p_{i}{ }^{\prime} \mathrm{s}, d_{i}{ }^{\prime} \mathrm{s}, f_{i}{ }^{\prime} \mathrm{s}$, and $g_{i}{ }^{\prime}$ s in system (3.1), smooth stabilizing controllers have been designed. The purpose of this paper is to relax these restrictions and solve the stabilization problem of the more general system (3.1) under the following three assumptions.

Assumption 3.1. There exists a $\mathcal{C}^{2}$ function $\tilde{V}_{0}: \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{+}$such that

$$
\begin{gather*}
\kappa_{1}(\|\eta\|) \leq \tilde{V}_{0}(\eta) \leq \kappa_{2}(\|\eta\|) \\
\rho \tilde{V}_{0}(\eta)=\frac{\partial \tilde{V}_{0}}{\partial \eta} f_{0}+\frac{1}{2} \operatorname{Tr}\left\{g_{0}^{T} \frac{\partial^{2} \tilde{V}_{0}}{\partial \eta^{2}} g_{0}\right\} \leq-v_{1}(\eta)\|\eta\|^{4}+\bar{b} \bar{\alpha}\left(x_{1}\right) x_{1}^{4}  \tag{3.2}\\
\left\|g_{0}^{T} \frac{\partial \tilde{V}_{0}}{\partial \eta^{T}}\right\|^{2} \leq v_{2}(\eta)\|\eta\|^{4}+\bar{b} \bar{\alpha}\left(x_{1}\right) x_{1}^{4}
\end{gather*}
$$

where $\mathcal{K}_{i}, i=1,2$ are $\mathcal{K}_{\infty}$ functions; $\boldsymbol{\nu}_{1}: \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{+} \backslash\{0\}, \nu_{2}: \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{+}$, and $\bar{\alpha}: \mathbf{R} \rightarrow \mathbf{R}^{+}$are continuous functions; and $\bar{b}>0$ is an unknown constant.

Assumption 3.2. For each $i=1, \ldots, n, f_{i}$ and $g_{i}$ satisfy

$$
\begin{equation*}
\left|f_{i}(x, \eta)\right| \leq b_{f_{i}} \sum_{j=1}^{l_{i}}\left|x_{i+1}\right|^{q_{i j}} \bar{f}_{i j}\left(x_{[i]}, \eta\right), \quad\left\|g_{i}(x, \eta)\right\| \leq b_{g_{i}} \bar{g}_{i}\left(x_{[i]}, \eta\right) \tag{3.3}
\end{equation*}
$$

where $\bar{f}_{i j}: \mathbf{R}^{i} \times \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{+}$and $\bar{g}_{i}: \mathbf{R}^{i} \times \mathbf{R}^{m_{1}} \rightarrow \mathbf{R}^{+}$are known $\mathcal{C}^{1}$ functions with $\bar{f}_{i j}(0,0)=0$ and $\bar{g}_{i}(0,0)=0 ; b_{f_{i}}>0$ and $b_{g_{i}}>0$ are unknown constants; $l_{i}$ is some positive integer; $q_{i j}{ }^{\prime} \mathrm{s}$ satisfy $0 \leq q_{i 1}<\cdots<q_{i l_{i}}<p_{i}$.

Assumption 3.3. For each $d_{i}, i=1, \ldots, n$, its sign is known, and there are unknown constants $a>0$ and $\bar{a}>0$, known smooth functions $\lambda_{i}: \mathbf{R}^{i} \rightarrow \mathbf{R}^{+} \backslash\{0\}$, and $\mu_{i}: \mathbf{R}^{i+1} \rightarrow \mathbf{R}^{+}$such that

$$
\begin{equation*}
0<a \lambda_{i}\left(x_{[i]}\right) \leq\left|d_{i}(x, \eta)\right| \leq \bar{a} \mu_{i}\left(x_{[i+1]}\right) \tag{3.4}
\end{equation*}
$$

where $x_{[n+1]}=x$ when $i=n$.
Above three assumptions are common and similar to the ones usually imposed on the high-order nonlinear systems (see, e.g., [17, 20]). Based on Assumption 3.2 and Lemma 2.5, we obtain the following proposition which dominates the growth properties of $f_{i}$ 's and $g_{i}$ 's and will play a key role in overcoming the obstacle caused by system uncertainties/unknowns. The proof is omitted here since it is quite similar to that of Proposition 2 in [22].

Proposition 3.4. For each $i=1, \ldots, n$, there exist smooth functions $\delta: \mathbf{R}^{m_{1}} \rightarrow[1,+\infty), \varphi_{i}: \mathbf{R}^{i} \rightarrow$ $\mathbf{R}^{+}$, and $\phi_{i}: \mathbf{R}^{i} \rightarrow \mathbf{R}^{+}$, such that

$$
\begin{gather*}
\left|f_{i}(x, \eta)\right| \leq \frac{\left|d_{i}(x, \eta)\right|}{2}\left|x_{i+1}\right|^{p_{i}}+\bar{\Theta}\left(\delta(\eta)\|\eta\|+\varphi_{i}\left(x_{[i]}\right) \sum_{k=1}^{i}\left|x_{k}\right|\right) \\
\left\|g_{i}(x, \eta)\right\| \leq \bar{\Theta}\left(\delta(\eta)\|\eta\|+\phi_{i}\left(x_{[i]}\right) \sum_{k=1}^{i}\left|x_{k}\right|\right) \tag{3.5}
\end{gather*}
$$

where $\bar{\Theta} \geq \max \{1, \bar{a}\}$ is obviously an unknown constant.
Remark 3.5. It is worth pointing out that in the recent related work [30], to ensure continuously differential output feedback control design, somewhat stronger assumptions have been imposed on the system drift and diffusion terms. For example, different from Proposition 1, Assumption 1 in [30] requires that the powers of $\left|x_{k}\right|, k=1, \ldots, i$ are larger than one in the upper bound estimations of $f_{i}(\cdot)$ and $g_{i}(\cdot)$ (the case of $f_{n}(\cdot)$ and $g_{n}(\cdot)$ is more evident).

Furthermore, as done in $[8,9]$, to ensure the stabilizability of system (3.1), it is necessary to make the following restriction on $\mathcal{\kappa}_{1}, \mathcal{\nu}_{1}$, and $\mathcal{v}_{2}$ in Assumption 3.1, and $\delta$ in Proposition 3.4.

Assumption 3.6. For some $l \in(0,1)$, there exist $\zeta(\cdot)$ and $\xi(\cdot)$ which are continuous, positive, and monotone increasing functions satisfying $\zeta(\|\eta\|) \geq\left(l \nu_{1}(\eta)+\nu_{2}(\eta)\right) / 2(1-l) \nu_{1}(\eta)$ and $\xi(\|\eta\|) \geq \delta^{4}(\eta) / \nu_{1}(\eta)$, such that

$$
\begin{equation*}
\int_{0}^{+\infty} e^{-\int_{0}^{r}\left(1 / \zeta\left(\kappa_{1}^{-1}(s)\right)\right) \mathrm{d} s} \mathrm{~d} \xi\left(\kappa_{1}^{-1}(r)\right)<+\infty, \tag{3.6}
\end{equation*}
$$

where $\kappa_{1}^{-1}(\cdot)$ denotes the inverse function of $\kappa_{1}(\cdot)$.
To understand well the academic meaning of the control problem to be studied, and in particular the generality and different nature of system (3.1) compared with the exiting works, we make the following four remarks corresponding to above four assumptions, respectively.

Remark 3.7. Assumption 3.1 indicates that the unmeasurable zero dynamics possesses the Stochastic ISS (Input-State Stability) type property, like in [8, 9], and the restriction on $g_{0}$ is somewhat weaker than that in $[8,9]$ since the additional term $b_{2} \bar{\alpha}_{2}\left(x_{1}\right) x_{1}^{4}$ in the estimation of $\left\|g_{0}^{T}\left(\partial \tilde{V}_{0} / \partial \eta^{T}\right)\right\|^{2}$.

Remark 3.8. Assumption 3.2 demonstrates that the power of $x_{i+1}$ in $f_{i}(x, \eta)$ must be strictly less than the corresponding system high order. This is necessary to realize the stabilization of the system by using the domination approach of [18]. Moreover, thanks to no further restrictions on $\bar{f}_{i j}$ 's or $\bar{g}_{i}$ 's, Assumption 3.2 is more possibly met than those in [25-28].

Remark 3.9. Assumption 3.3 shows that the control coefficients $d_{i}$ 's never vanish and otherwise system (3.1) would be uncontrollable somewhere. Besides, from this assumption, one can easily see that the signs of $d_{i}$ 's remain unchanged. Furthermore, the unknown constant
" $a$ " makes system (3.1) more general than those studied in [25-28] where the lower bounds of uncertain control coefficients $d_{i}$ 's are required to be precisely known.

Remark 3.10. In fact, Assumption 3.6 is similar to the corresponding one in $[8,9]$. From above formulation of the system, it can be seen that the unwanted effects of $\eta$, that is, " $v_{2}(\eta)\|\eta\|^{4 "}$ in Assumption 3.1 and " $\bar{\Theta} \delta(\eta)\|\eta\|$ " in Proposition 3.4, can only be dominated by the term " $-\mathcal{v}_{1}(\eta)\|\eta\|^{4 "}$ in Assumption 3.1, and therefore some requirements should be imposed on these three terms. For the sake of stabilization, we make Assumption 3.6, which clearly includes a special case where $\nu_{1}=\nu_{2}=\delta^{4}$ since at this moment $\zeta$ and $\xi$ can be constants and (3.6) obviously holds.

As the recent development on high-order control systems, works [17,21,22] proposed a novel adaptive control technique, which is powerful to successfully overcome the technical difficulties in stabilizing system (3.1) caused by the weaker conditions on unknown control coefficients. Inspired by these works, the paper extends the stabilization results in [17, 22] from deterministic systems to stochastic ones, under quite weaker assumptions than those in [25-28]. More importantly, instead of simple generalization, motivated by the novel adaptive technique for deterministic nonlinear systems [23], we develop the adaptive control scheme without overparameterization that occurred in $[17,21,22]$. (In fact, the number of parameter estimates is reduced from $n+1$ to 1 .)

For details, in this paper, the main objective is to design a controller in the following form:

$$
\begin{equation*}
\dot{\hat{\delta}}=\psi(x, \widehat{\delta}), \quad u=\varphi(x, \widehat{\delta}) \tag{3.7}
\end{equation*}
$$

where $\widehat{\delta}(t) \in \mathbf{R}$, and $\psi$ is a smooth function, while $\varphi$ is a continuous function, such that all closed-loop states are bounded almost surely, and furthermore, the original system is globally asymptotically stable in probability.

Finally, for the sake of the later control design, we obtain the following proposition by the technique of changing supply functions [20,38]. The proof of Proposition 3.11 is mainly inspired by $[9,38]$ and placed in Appendix A.

Proposition 3.11. Define $V_{0}(\eta)=\int_{0}^{\tilde{V}_{0}(\eta)} q(s) \mathrm{d} s$, and $q: \mathbf{R}^{+} \rightarrow \mathbf{R}^{+}$. Then, under Assumptions 3.1 and 3.6 , one can construct a suitable $q(s)$ which is $\mathcal{C}^{1}$, monotone increasing, such that
(i) $V_{0}(\eta)$ is $\mathcal{C}^{2}$, positive definite, and radially unbounded;
(ii) there exist a smooth function $\bar{\alpha}_{0}: \mathbf{R} \rightarrow[1,+\infty)$ and an unknown constant $b>0$ such that

$$
\begin{equation*}
\varrho V_{0}=\frac{\partial V_{0}(\eta)}{\partial \eta} f_{0}+\frac{1}{2} \operatorname{Tr}\left\{g_{0}^{T} \frac{\partial^{2} V_{0}}{\partial \eta^{2}} g_{0}\right\} \leq-(n+1) \delta^{4}(\eta)\|\eta\|^{4}+b \bar{\alpha}_{0}\left(x_{1}\right) x_{1}^{4} \tag{3.8}
\end{equation*}
$$

## 4. Partial-State Feedback Adaptive Stabilizing Control

Since the signs of $d_{i}$ 's are known and remain unchanged, without loss of generality, suppose $d_{i}>0, i=1, \ldots, n$. The following theorem summarizes the main result of this paper.

Theorem 4.1. Consider system (3.1) and suppose Assumptions 3.1-3.3 and 3.6 hold. Then there exists an adaptive continuous partial-state feedback controller in the form (3.7), such that
(i) the origin solution of the closed-loop system is globally stable in probability;
(ii) the states of the original system converge to the origin, and the other states of the closed-loop system converge to some finite value, both with probability one.

About the main theorem, we have the following remark.
Remark 4.2. From Claim (i) and the former part of Claim (ii), we easily know that the original system is globally asymptotically stable in probability.

Proof. To complete the proof, we will first construct an adaptive continuous controller in the form (3.7) for system (3.1). Then by applying Theorem 2.3, it will be shown that the theorem holds for the closed-loop system.

First, let us define $\Theta=\bar{\Theta}^{4} \max \left\{b / a, 1 / a^{4}, a^{2}\right\}$, where $a$ and $b$ are the same as in Assumption 3.3 and Proposition 3.11, respectively. The estimate of $\Theta$ is denoted by $\widehat{\Theta}(t)$, for which the following updating law will be designed:

$$
\begin{equation*}
\dot{\hat{\Theta}}=\tau(x, \widehat{\Theta}), \quad \hat{\Theta}(0)=1 \tag{4.1}
\end{equation*}
$$

where $\tau(x, \widehat{\Theta})$ is a to-be-determined nonnegative smooth function which ensures that $\widehat{\Theta}(t) \geq$ 1 , for all $t \geq 0$.

We would like to give some inequalities on above defined $\Theta$ for the sake of use in the later control design. Noting $\bar{\Theta} \geq 1$ (see Proposition 3.4) and $\max \left\{1 / a^{4}, a^{2}\right\} \geq 1$, for all $a>0$, it is clear that $\Theta \geq 1$. Moreover, since $p_{i} \geq 1, i=1, \ldots, n-1$, there hold $-1<\left(4-4 p_{1} \cdots p_{i}\right) /\left(4 p_{1} \cdots p_{i}-1\right) \leq 0<4 /\left(3 p_{1} \cdots p_{i}-1\right) \leq 2, i=1, \ldots, n-1$, and hence $\Theta \geq \bar{\Theta}^{4} a^{\left(4-4 p_{1} \cdots p_{i}\right) /\left(4 p_{1} \cdots p_{i}-1\right)}$ and $\Theta \geq \bar{\Theta}^{4} a^{4 /\left(3 p_{1} \cdots p_{i}-1\right)}$.

Remark 4.3. As will be seen, mainly because that the definition of new unknown parameter $\Theta$ is essentially different form that in [17, 21, 22], the overparameterization problem that occurred in the works is successfully overcome.

Next, we introduce the following new variables:

$$
\begin{equation*}
z_{1}=x_{1}, \quad z_{i}=x_{i}^{p_{1} \cdots p_{i-1}}-\alpha_{i-1}^{p_{1} \cdots p_{i-1}}\left(x_{[i-1]}, \widehat{\Theta}\right), \quad i=2, \ldots, n, \tag{4.2}
\end{equation*}
$$

and the actual control law $u=\alpha_{n}(x, \widehat{\Theta})$, where $\alpha_{i}: \mathbf{R}^{i} \times \mathbf{R} \rightarrow \mathbf{R}, i=1, \ldots, n$ are continuous functions satisfying $\alpha_{i}(0, \widehat{\Theta})=0$, for all $\widehat{\Theta} \in \mathbf{R}$. In the following, a recursive design procedure is provided to construct the virtual and actual controllers $\alpha_{i}$ 's. For completing the control design, we also introduce a sequence of functions $\left\{W_{i}, i=1, \ldots, n\right\}$ as follows:

$$
\begin{equation*}
W_{1}=\frac{1}{4} z_{1}^{4}, \quad W_{i}=\int_{\alpha_{i-1}}^{x_{i}}\left(s^{p_{1} \cdots p_{i-1}}-\alpha_{i-1}^{p_{1} \cdots p_{i-1}}\right)^{4-1 /\left(p_{1} \cdots p_{i-1}\right)} \mathrm{d} s, \quad i=2,3, \ldots, n . \tag{4.3}
\end{equation*}
$$

Similar to the corresponding proof in [18], it is easy to verify that, for each $i=1, \ldots, n, W_{i}$ is $\mathcal{C}^{2}$ in all its arguments, $W_{i}=0$ when $z_{i}=0, W_{i}>0$ when $z_{i} \neq 0$, and $W_{i} \rightarrow+\infty$ as $\left|z_{i}\right| \rightarrow+\infty$.

Step 1. Choose $V_{1}=V_{0}+W_{1}+(a / 2) \tilde{\Theta}^{2}$ to be the candidate Lyapunov function for this step, where $\tilde{\Theta}=\Theta-\widehat{\Theta}$ denotes the parameter estimation error. Then, along the trajectories of system (3.1), we have

$$
\begin{equation*}
\rho V_{1}=\rho V_{0}+z_{1}^{3}\left(d_{1} \alpha_{1}^{p_{1}}+d_{1} z_{2}+f_{1}\right)+\frac{3}{2} z_{1}^{2} g_{1}^{T} g_{1}-a \tilde{\Theta} \dot{\hat{\Theta}} \tag{4.4}
\end{equation*}
$$

By Proposition 3.4 and Lemma 2.4, we have following estimations:

$$
\begin{align*}
z_{1}^{3} f_{1} & \leq \frac{d_{1}}{2}\left|z_{1}\right|^{3}\left|x_{2}\right|^{p_{1}}+z_{1}^{3} \bar{\Theta}\left(\delta(\eta)\|\eta\|+\left|x_{1}\right| \varphi_{1}\left(x_{1}\right)\right) \\
& \leq \frac{d_{1}}{2}\left|z_{1}\right|^{3}\left|x_{2}\right|^{p_{1}}+\frac{1}{4} \delta^{4}(\eta)\|\eta\|^{4}+\frac{3}{4} \bar{\Theta}^{4 / 3} z_{1}^{4}+\bar{\Theta} \varphi_{1}\left(x_{1}\right) z_{1}^{4}  \tag{4.5}\\
\frac{3}{2} z_{1}^{2} g_{1}^{T} g_{1} & \leq \frac{3}{2} z_{1}^{2} \bar{\Theta}^{2}\left(\delta(\eta)\|\eta\|+\left|x_{1}\right| \phi_{1}\left(x_{1}\right)\right)^{2} \\
& \leq \frac{1}{2} \delta^{4}(\eta)\|\eta\|^{4}+\frac{9}{2} \bar{\Theta}^{4} z_{1}^{4}+3 \bar{\Theta}^{2} \phi_{1}^{2}\left(x_{1}\right) z_{1}^{4}
\end{align*}
$$

from which, (4.4), Proposition 3.11, and the facts $\bar{\Theta} \geq 1, a \Theta \geq \max \left\{b, \bar{\Theta}^{4}, 1 / a^{3}\right\} \geq 1$, it follows that

$$
\begin{align*}
£ V_{1} \leq & -n \delta^{4}(\eta)\|\eta\|^{4}+d_{1} z_{1}^{3} z_{2}+\frac{d_{1}}{2}\left|z_{1}\right|^{3}\left|x_{2}\right|^{p_{1}}+d_{1} z_{1}^{3} \alpha_{1}^{p_{1}}+a \Theta \rho_{1}\left(x_{1}\right) z_{1}^{4}-a \tilde{\Theta} \dot{\Theta} \\
\leq & -n \delta^{4}(\eta)\|\eta\|^{4}-\frac{n}{a^{3}} z_{1}^{4}+d_{1} z_{1}^{3} z_{2}+\frac{d_{1}}{2}\left|z_{1}\right|^{3}\left|x_{2}\right|^{p_{1}}+d_{1} z_{1}^{3} \alpha_{1}^{p_{1}} \\
& +a \Theta\left(n-1+\frac{5}{4}+\rho_{1}\left(x_{1}\right)\right) z_{1}^{4}-a \tilde{\Theta} \dot{\Theta}  \tag{4.6}\\
\leq & -n \delta^{4}(\eta)\|\eta\|^{4}-\frac{n}{a^{3}} z_{1}^{4}+d_{1} z_{1}^{3} z_{2}+\frac{d_{1}}{2}\left|z_{1}\right|^{3}\left(\left|x_{2}\right|^{p_{1}}+\operatorname{sign}\left(z_{1}\right) \alpha_{1}^{p_{1}}\right) \\
& +a z_{1}^{3}\left(\frac{d_{1}}{2 a} \alpha_{1}^{p_{1}}+\hat{\Theta}\left(n-1+\frac{5}{4}+\rho_{1}\left(x_{1}\right)\right) z_{1}\right)+a \tilde{\Theta}\left(\tau_{1}\left(x_{1}, \widehat{\Theta}\right)-\dot{\Theta}\right)
\end{align*}
$$

where $\rho_{1}\left(x_{1}\right)=6+\varphi_{1}\left(x_{1}\right)+\bar{\alpha}_{0}\left(x_{1}\right)+3 \phi_{1}^{2}\left(x_{1}\right)$ and $\tau_{1}=\left(n-1+(5 / 4)+\rho_{1}\left(x_{1}\right)\right) z_{1}^{4}$. It will be seen from the later design steps that a series of nonnegative smooth functions $\tau_{k}\left(x_{[k]}, \widehat{\Theta}\right)$, $k=2, \ldots, n$, are introduced so as to finally obtain the updating law of $\widehat{\Theta}$, that is, $\dot{\hat{\Theta}}=\tau=\tau_{n}$.

Mainly based on (4.6), the virtual continuous controller $\alpha_{1}$ is chosen such that

$$
\begin{equation*}
\alpha_{1}^{p_{1}}=-2 \widehat{\Theta} \lambda_{1}\left(x_{1}\right)^{-1}\left(n-1+\frac{5}{4}+\rho_{1}\left(x_{1}\right)\right) z_{1}=:-h_{1}\left(x_{1}, \widehat{\Theta}\right) z_{1} \tag{4.7}
\end{equation*}
$$

and such choice makes (4.6) become

$$
\begin{equation*}
\varrho V_{1} \leq-n \delta^{4}(\eta)\|\eta\|^{4}-\frac{n}{a^{3}} z_{1}^{4}+a \tilde{\Theta}\left(\tau_{1}-\dot{\hat{\Theta}}\right)+\frac{3}{2} \bar{\Theta} \mu_{1}\left(x_{[2]}\right)\left|z_{1}^{3} z_{2}\right| \tag{4.8}
\end{equation*}
$$

Remark 4.4. It is necessary to mention that in the first design step, functions $\rho_{1}$ and $h_{1}$ have been provided with explicit expressions in order to deduce the completely explicit virtual controller $\alpha_{1}$. However, in the later design steps, sometimes for the sake of briefness, we will not explicitly write out the functions which are easily defined.

Inductive Steps. Suppose that the first $k-1(k=2, \ldots, n)$ design steps have been completed. In other words, we have found appropriate functions $\alpha_{i}, \tau_{i}, i=1, \ldots, k-1$ satisfying $\alpha_{i}^{p_{1} \cdots p_{i}}=$ $-h_{i}\left(x_{[i]}, \widehat{\Theta}\right) z_{i}$ and $\tau_{i}=\sum_{j=1}^{i}\left(n-j+(5 / 4)+\rho_{j}\left(x_{[j]}, \widehat{\Theta}\right)\right) z_{j}^{4}$ for known nonnegative smooth functions $h_{i}, \rho_{j}, j=1, \ldots, i$, such that

$$
\begin{align*}
\mathscr{L} V_{k-1} \leq & -(n-k+2) \delta^{4}(\eta)\|\eta\|^{4}-\frac{n-k+2}{a^{3}} \sum_{i=1}^{k-1} z_{i}^{4}+\left(a \tilde{\Theta}-\sum_{i=1}^{k-1} \frac{\partial W_{i}}{\partial \hat{\Theta}}\right)\left(\tau_{k-1}-\dot{\Theta}\right)  \tag{4.9}\\
& +\frac{3}{2} \bar{\Theta} \mu_{k-1}\left(x_{[k]}\right)\left|z_{k-1}\right|^{\left(4 p_{1} \cdots p_{k-2}-1\right) / p_{1} \cdots p_{k-2}}\left|x_{k}^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}}\right|,
\end{align*}
$$

for the candidate Lyapunov function $V_{k-1}\left(x_{[k-1]}, \widehat{\Theta}\right)$.
Let $V_{k}=V_{k-1}+W_{k}$ be the candidate Lyapunov function for step $k$. Then, along the trajectories of system (3.1), we have

$$
\begin{align*}
\varrho W_{k}= & \frac{\partial W_{k}}{\partial \widehat{\Theta}} \dot{\Theta}+d_{k} z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left(x_{k+1}^{p_{k}}-\alpha_{k}^{p_{k}}\right)+d_{k} z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \alpha_{k}^{p_{k}} \\
& +z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} f_{k}+\sum_{i=1}^{k-1} \frac{\partial W_{k}}{\partial x_{i}}\left(d_{i} x_{i+1}^{p_{i}}+f_{i}\right)+\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} g_{i}^{T} \frac{\partial^{2} W_{k}}{\partial x_{i} \partial x_{j}} g_{j} . \tag{4.10}
\end{align*}
$$

Just as in the first step, in order to design $\alpha_{k}$, one should appropriately estimate the last four terms on the right-hand side of above equality and the last term on the right-hand side of (4.9), as formulated in the following proposition whose proof is placed in Appendix B.

Proposition 4.5. There exists nonnegative smooth function $\rho_{k}: \mathbf{R}^{k} \times \mathbf{R} \rightarrow \mathbf{R}^{+}$, such that

$$
\begin{align*}
& z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} f_{k}+\sum_{i=1}^{k-1} \frac{\partial W_{k}}{\partial x_{i}}\left(d_{i} x_{i+1}^{p_{i}}+f_{i}\right)+\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} g_{i}^{T} \frac{\partial^{2} W_{k}}{\partial x_{i} \partial x_{j}} g_{j} \\
& +\frac{3}{2} \bar{\Theta} \mu_{k-1}\left(x_{[k]}\right)\left|z_{k-1}\right|^{\left(4 p_{1} \cdots p_{k-2}-1\right) / p_{1} \cdots p_{k-2}}\left|x_{k}^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}}\right| \\
& \quad \leq \frac{d_{k}}{2}\left|z_{k}\right|^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left|x_{k+1}\right|^{p_{k}}+\delta^{4}(\eta)\|\eta\|^{4}+\frac{3}{4 a^{3}} \sum_{i=1}^{k-1} z_{i}^{4}+a \Theta z_{k}^{4} \rho_{k}\left(x_{[k]}, \hat{\Theta}\right) . \tag{4.11}
\end{align*}
$$

Then, by (4.9), (4.10), and Proposition 4.5, we have

$$
\begin{align*}
\mathscr{L} V_{k} \leq & -(n-k+1) \delta^{4}(\eta)\|\eta\|^{4}-\frac{n-k+5 / 4}{a^{3}} \sum_{i=1}^{k-1} z_{i}^{4}+\left(a \tilde{\Theta}-\sum_{i=1}^{k-1} \frac{\partial W_{i}}{\partial \widehat{\Theta}}\right)\left(\tau_{k-1}-\dot{\Theta}\right) \\
& +\frac{\partial W_{k}}{\partial \widehat{\Theta}} \dot{\Theta}+d_{k} z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left(x_{k+1}^{p_{k}}-\alpha_{k}^{p_{k}}\right)+\frac{d_{k}}{2} z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left|x_{k+1}\right|^{p_{k}} \\
& +d_{k} z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \alpha_{k}^{p_{k}}+a \Theta z_{k}^{4} \rho_{k}\left(x_{[k]}, \widehat{\Theta}\right) \\
\leq & -(n-k+1) \delta^{4}(\eta)\|\eta\|^{4}-\frac{n-k+5 / 4}{a^{3}} \sum_{i=1}^{k} z_{i}^{4}+\left(a \tilde{\Theta}-\sum_{i=1}^{k} \frac{\partial W_{i}}{\partial \widehat{\Theta}}\right)\left(\tau_{k}-\dot{\Theta}\right) \\
& +a z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left(\frac{d_{k}}{2 a} \alpha_{k}^{p_{k}}+\widehat{\Theta} z_{k}^{1 / p_{1} \cdots p_{k-1}}\left(n-k+\frac{5}{4}+\rho_{k}\left(x_{[k]}, \widehat{\Theta}\right)\right)\right) \\
& +d_{k} z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left(x_{k+1}^{p_{k}}-\alpha_{k}^{p_{k}}\right)+\left.\frac{d_{k}}{2}\left|z_{k}\right|\right|^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left(\left|x_{k+1}\right|^{p_{k}}+\operatorname{sign}\left(z_{k}\right) \alpha_{k}^{p_{k}}\right) \\
& +\frac{\partial W_{k}}{\partial \widehat{\Theta}} \tau_{k}+\sum_{i=1}^{k-1} \frac{\partial W_{i}}{\partial \widehat{\Theta}}\left(\tau_{k}-\tau_{k-1}\right), \tag{4.12}
\end{align*}
$$

where $\tau_{k}=\tau_{k-1}+\left(n-k+(5 / 4)+\rho_{k}\right) z_{k}^{4}$.
Observing that a nonnegative smooth function $\gamma_{k}: \mathbf{R}^{k} \times \mathbf{R} \rightarrow \mathbf{R}$ can be easily constructed such that

$$
\begin{equation*}
\frac{\partial W_{k}}{\partial \widehat{\Theta}} \tau_{k}+\sum_{i=1}^{k-1} \frac{\partial W_{i}}{\partial \widehat{\Theta}}\left(\tau_{k}-\tau_{k-1}\right) \leq \frac{1}{4 a^{3}} \sum_{i=1}^{k} z_{i}^{4}+a z_{k}^{4} \gamma_{k}\left(x_{[k]}, \widehat{\Theta}\right) \tag{4.13}
\end{equation*}
$$

if we design the continuous virtual controller $\alpha_{k}$ such that

$$
\begin{align*}
\alpha_{k}^{p_{1} \cdots p_{k}} & =-\lambda_{k}^{-p_{1} \cdots p_{k-1}}\left(2 \widehat{\Theta}\left(n-k+\frac{5}{4}+\rho_{k}\left(x_{[k]}, \widehat{\Theta}\right)\right)+\gamma_{k}\left(x_{[k]}, \widehat{\Theta}\right)\right)^{p_{1} \cdots p_{k-1}} z_{k}  \tag{4.14}\\
& =:-h_{k}\left(x_{[k]}, \widehat{\Theta}\right) z_{k}
\end{align*}
$$

(obviously, $h_{k}$ is a strictly positive smooth function), then (4.12) becomes

$$
\begin{align*}
\varrho V_{k} \leq & -(n-k+1) \delta^{4}(\eta)\|\eta\|^{4}-\frac{n-k+1}{a^{3}} \sum_{i=1}^{k} z_{i}^{4}+\left(a \tilde{\Theta}-\sum_{i=1}^{k} \frac{\partial W_{i}}{\partial \widehat{\Theta}}\right)\left(\tau_{k}-\dot{\Theta}\right)  \tag{4.15}\\
& +\frac{3}{2} \bar{\Theta} \mu_{k}\left(x_{[k+1]}\right)\left|z_{k}\right|^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left|x_{k+1}^{p_{k}}-\alpha_{k}^{p_{k}}\right| .
\end{align*}
$$

Noting the arguments in the last design step, we choose the adaptive actual continuous controller $u: \mathbf{R}^{n} \times \mathbf{R} \rightarrow \mathbf{R}$ as follows:

$$
\begin{gather*}
u=\alpha_{n}(x, \widehat{\Theta}) \\
\dot{\Theta}=\tau(x, \widehat{\Theta})=\tau_{n}(x, \widehat{\Theta}), \tag{4.16}
\end{gather*}
$$

from which, (4.15) with $k=n$ and the aforementioned $x_{n+1}=u, x_{[n+1]}=x$, it follows that

$$
\begin{align*}
\mathscr{L} V_{n} & \leq-\delta^{4}(\eta)\|\eta\|^{4}-\frac{1}{a^{3}} \sum_{i=1}^{n} z_{i}^{4} \\
& =-\delta^{4}(\eta)\|\eta\|^{4}-\frac{1}{a^{3}} x_{1}^{4}-\frac{1}{a^{3}} \sum_{i=2}^{n}\left(x_{i}^{p_{1} \cdots p_{i-1}}-\alpha_{i-1}^{p_{1} \cdots p_{i-1}}\right)^{4}  \tag{4.17}\\
& =-W(\eta, x),
\end{align*}
$$

where $W(\eta, x)$ is a smooth function.
With the adaptive controller (4.16) in loop, we know that $[0, \ldots, 0, \Theta]^{T} \in \mathbf{R}^{n+m_{1}+1}$ is the origin solution of the closed-loop system. Thus, from Theorem 2.3 and (4.17), it follows that the origin solution is globally stable in probability; furthermore, since $W(\eta, x)$ is positive definite which can be deduced from the expressions of $W(\eta, x)$ and $\alpha_{i}^{p_{1} \cdots p_{i}}(i=1, \ldots, n-1)$, it follows that $P\left\{\lim _{t \rightarrow+\infty}(\|\eta(t)\|+\|x(t)\|)=0\right\}=1$, and in terms of the similar proof of Theorem 3.1 in [6], one can see that the state $\hat{\Theta}$ converges to some finite value with probability one.

We would like to point out that the adaptive control scheme given above can be used to remove the overparameterization in the recent works [17, 21, 22], where the number of parameter estimates is not less than $n+1$. For this aim, it suffices to introduce another new unknown parameter like $\Theta$ defined before, and the design steps are quite similar to those developed earlier and do not need further discussion.

## 5. A Simulation Example

Consider the following three-dimensional uncertain high-order stochastic nonlinear system:

$$
\begin{gather*}
\mathrm{d} \eta=-\left(1+\eta^{4}\right) \eta \mathrm{d} t+\theta x_{1} \sin x_{2} \mathrm{~d} t+x_{1} \mathrm{~d} w \\
\mathrm{~d} x_{1}=\theta\left(2-0.2 \sin x_{2}\right) x_{2}^{3} \mathrm{~d} t+\theta x_{1} \cos (3 \eta) \mathrm{d} t+2 \eta x_{1} \mathrm{~d} w  \tag{5.1}\\
\mathrm{~d} x_{2}=2 \theta u \mathrm{~d} t+2 \theta \eta x_{1} \mathrm{~d} t+\theta \eta^{2} \mathrm{~d} w
\end{gather*}
$$

where $\theta>0$ is an unknown constant.
It is easy to verify that system (5.1) satisfies Assumptions 3.1 and 3.6 with $\tilde{V}_{0}(\eta)=$ $\kappa_{1}(\eta)=\kappa_{2}(\eta)=\eta^{4}, \nu_{1}(\eta)=v_{2}(\eta)=1+\eta^{4}$, and $\delta^{4}(\eta)=\left(1+\eta^{2}\right)^{2}$. Assumption 3.2 holds with $\left|\theta x_{1} \cos (3 \eta)\right| \leq \theta\left|x_{1}\right|,\left|2 \eta x_{1}\right| \leq \delta(\eta)|\eta|+\left|x_{1}\right| \sqrt{1+x_{1}^{2}},\left|2 \theta \eta x_{1}\right| \leq \theta\left(\delta(\eta)|\eta|+\left|x_{1}\right| \sqrt{1+x_{1}^{2}}\right)$,


Figure 1: The trajectories of $\eta, x_{1}, x_{2}$.


Figure 2: The trajectory of $\widehat{\Theta}$.
and $\left|\theta \eta^{2}\right| \leq \theta \delta(\eta)|\eta|$. Assumption 3.3 holds with $a \lambda_{1}\left(x_{1}\right)=a \lambda_{2}\left(x_{[2]}\right)=\theta$, and $\bar{a} \mu_{1}\left(x_{1}\right)=$ $\bar{a} \mu_{2}\left(x_{[2]}\right)=2.2 \theta$. Therefore, in terms of the design steps developed in Section 4 , an adaptive partial-state feedback stabilizing controller can be explicitly given.

Let $\theta=1.2$ and the initial states be $\eta(0)=2, x_{1}(0)=1$, and $x_{2}(0)=-2.5$. Using MATLAB, Figures 1 and 2 are obtained to exhibit the trajectories of the closed-loop system states. (To show the transient behavior more clearly, logarithmic X-coordinates have been adopted.) From these figures, one can see that $\eta, x_{1}$, and $x_{2}$ are regulated to zero while $\widehat{\Theta}$ converges to a finite value, all with probability one.

## 6. Concluding Remarks

In this paper, the partial-state feedback stabilization problem has been investigated for a class of high-order stochastic nonlinear systems under weaker assumptions than the existing
works. By introducing the novel adaptive updated law and appropriate control Lyapunov function, and using the method of adding a power integrator, we have designed an adaptive continuous partial-state feedback controller without overparameterization and given a simulation example to illustrate the effectiveness of the control design method. It has been shown that, with the designed controller in loop, all the original system states are regulated to zero and the other closed-loop states are bounded almost surely for any initial condition. Along this direction, there are a lot of other interesting research problems, such as output-feedback control for the systems studied in the paper, which are now under our further investigation.

## Appendices

## A. The Proof of Proposition 3.11

It is easy to verify that the first assertion of Proposition 3.11 holds when $q(s)$ is chosen to be positive, $\mathcal{C}^{1}$, and monotone increasing. Thus, in the rest of the proof, we will find such $q(s)$ to guarantee the correctness of the second assertion.

First, as defined in Proposition 3.11, $V_{0}(\eta)=\int_{0}^{\tilde{V}_{0}(\eta)} q(s) \mathrm{d} s$, where $q(s)$ is $\mathcal{C}^{1}$ and, for simplicity, $\dot{q}(s):=\mathrm{d} q(s) / \mathrm{d} s$. Thus by Assumption 3.1, we have

$$
\begin{align*}
\mathscr{L} V_{0}(\eta) & =\frac{\partial V_{0}(\eta)}{\partial \eta} f_{0}+\frac{1}{2} \operatorname{Tr}\left\{g_{0}^{T} \frac{\partial^{2} V_{0}(\eta)}{\partial \eta^{2}} g_{0}\right\} \\
& =q\left(\tilde{V}_{0}(\eta)\right) \frac{\partial \widetilde{V}}{\partial \eta} f_{0}+\frac{1}{2} \operatorname{Tr}\left\{g_{0}^{T}\left(\dot{q}\left(\tilde{V}_{0}(\eta)\right) \frac{\partial \tilde{V}_{0}}{\partial \eta^{T}} \frac{\partial \tilde{V}_{0}}{\partial \eta}+q\left(\tilde{V}_{0}(\eta)\right) \frac{\partial^{2} \tilde{V}_{0}}{\partial \eta^{2}}\right) g_{0}\right\} \\
& =q\left(\tilde{V}_{0}(\eta)\right)\left(\frac{\partial \tilde{V}_{0}}{\partial \eta} f_{0}+\frac{1}{2} \operatorname{Tr}\left\{g_{0}^{T} \frac{\partial \tilde{V}_{0}}{\partial \eta^{2}} g_{0}\right\}\right)+\frac{1}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right)\left\|g_{0}^{T} \frac{\partial \tilde{V}_{0}}{\partial \eta^{T}}\right\|^{2} \\
& \leq q\left(\tilde{V}_{0}(\eta)\right)\left(-v_{1}(\eta)\|\eta\|^{4}+\bar{b} \bar{\alpha}\left(x_{1}\right) x_{1}^{4}\right)+\frac{1}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right)\left(v_{2}(\eta)\|\eta\|^{4}+\bar{b} \bar{\alpha}\left(x_{1}\right) x_{1}^{4}\right) . \tag{A.1}
\end{align*}
$$

The following proceeds in two different cases in which $l$ is the same as in Assumption 3.6.
(i) Case of $l v_{1}(\eta)\|\eta\|^{4} \geq \bar{b} \bar{\alpha}\left(x_{1}\right) x_{1}^{4}$

For this case, from (A.1), we have

$$
\begin{equation*}
\rho V_{0}(\eta) \leq-(1-l) q\left(\tilde{V}_{0}(\eta)\right) v_{1}(\eta)\|\eta\|^{4}+\frac{l}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right) v_{1}(\eta)\|\eta\|^{4}+\frac{1}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right) v_{2}(\eta)\|\eta\|^{4} \tag{A.2}
\end{equation*}
$$

Let $l_{1}(s)=1 / \zeta\left(\kappa_{1}^{-1}(s)\right), l_{2}(s)=\xi\left(\kappa_{1}^{-1}(s)\right) / \zeta\left(\kappa_{1}^{-1}(s)\right)$ for the same $\xi$ and $\zeta$ as in Assumption 3.6, and as done in [9], denote

$$
\begin{equation*}
q(s)=\frac{n+1}{1-l} e^{e_{0}^{s} l_{1}(\tau) \mathrm{d} \tau}\left(\frac{1-l}{n+1} q(0)-\int_{0}^{s} l_{2}(r) e^{-\int_{0}^{r} l_{1}(\tau) \mathrm{d} \tau} \mathrm{~d} r\right) \tag{A.3}
\end{equation*}
$$

by $q(0)=((n+1) /(1-l))\left(\xi(0)+\int_{0}^{+\infty} e^{-\int_{0}^{r}\left(1 / \zeta\left(\kappa_{1}^{-1}(r)\right)\right) \mathrm{d} \tau} \mathrm{d} \xi\left(\kappa_{1}^{-1}(r)\right)\right) \geq 0$. Then, it is easy to see that

$$
\begin{align*}
\dot{q}(s) & =l_{1}(s) q(s)-\frac{n+1}{1-l} l_{2}(s) \\
& =\frac{n+1}{1-l} l_{1}(s) e^{\int_{0}^{s} l_{1}(\tau) \mathrm{d} \tau}\left(\frac{1-l}{n+1} q(0)-\int_{0}^{s} l_{2}(r) e^{-\int_{0}^{r} l_{1}(\tau) \mathrm{d} \tau} \mathrm{~d} r-\frac{l_{2}(s)}{l_{1}(s)} e^{-\int_{0}^{s} l_{1}(\tau) \mathrm{d} \tau}\right) . \tag{A.4}
\end{align*}
$$

Moreover, noting the above definitions of $l_{1}, l_{2}$ and using integration by parts, we have for all $s \geq 0$

$$
\begin{align*}
& \int_{0}^{s} l_{2}(r) e^{-\int_{0}^{r} l_{1}(\tau) \mathrm{d} \tau} \mathrm{~d} r+\frac{l_{2}(s)}{l_{1}(s)} e^{-\int_{0}^{s} l_{1}(\tau) \mathrm{d} \tau} \\
& \quad=-\left.\xi\left(k_{1}^{-1}(r)\right) e^{-\int_{0}^{r} l_{1}(\tau) \mathrm{d} \tau}\right|_{0} ^{\mathrm{s}}+\int_{0}^{s} e^{-\int_{0}^{r} l_{1}(\tau) \mathrm{d} \tau} \mathrm{~d} \xi\left(\kappa_{1}^{-1}(r)\right)+\xi\left(k_{1}^{-1}(s)\right) e^{-\int_{0}^{s} l_{1}(\tau) \mathrm{d} \tau}  \tag{A.5}\\
& \quad=\xi(0)+\int_{0}^{s} e^{-\int_{0}^{r}\left(1 / \zeta\left(\kappa_{1}^{-1}(r)\right)\right) \mathrm{d} \tau} \mathrm{~d} \xi\left(\kappa_{1}^{-1}(r)\right) \leq \frac{1-l}{n+1} q(0),
\end{align*}
$$

which together with (A.4) concludes that $\dot{q}(s) \geq 0$, for all $s \in \mathbf{R}^{+}$, and therefore, $q(s)$ is positive, $\mathcal{C}^{1}$, and monotone increasing on $\mathbf{R}^{+}$.

Furthermore, from (A.4) and the definitions of $l_{1}, l_{2}, \xi$, and $\zeta$ we yield

$$
\begin{equation*}
(1-l) q\left(\tilde{V}_{0}(\eta)\right) v_{1}(\eta)-\frac{l}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right) v_{1}(\eta)-\frac{1}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right) v_{2}(\eta) \geq(n+1) \delta^{4}(\eta) \tag{A.6}
\end{equation*}
$$

which together with (A.2) results in

$$
\begin{equation*}
\varrho V_{0}(\eta) \leq-(n+1) \delta^{4}(\eta)\|\eta\|^{4} \tag{A.7}
\end{equation*}
$$

This shows that the second assertion of Proposition 3.11 holds for this case.
(ii) Case of $l \nu_{1}(\eta)\|\eta\|^{4}<\bar{b} \bar{\alpha}\left(x_{1}\right) x_{1}^{4}$

In this case, it is not hard to find a $\mathcal{K}_{\infty}$ function $\kappa_{\eta}(\cdot)$ and an unknown constant $\bar{b}_{1}>0$ satisfying $\|\eta\| \leq \bar{b}_{1} \kappa_{\eta}\left(\left|x_{1}\right|\right)$. Then from (A.1), we get

$$
\begin{align*}
\mathscr{L} V_{0}(\eta) \leq & -(1-l) q\left(\tilde{V}_{0}(\eta)\right) v_{1}(\eta)\|\eta\|^{4}+\frac{1}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right) v_{2}(\eta)\|\eta\|^{4}+\bar{b} q\left(\tilde{V}_{0}(\eta)\right) \bar{\alpha}\left(x_{1}\right) x_{1}^{4} \\
& +\frac{1}{2} \bar{b} \dot{q}\left(\tilde{V}_{0}(\eta)\right) \bar{\alpha}\left(x_{1}\right) x_{1}^{4} \tag{A.8}
\end{align*}
$$

Choosing the same $q(s)$ as in the first case and in view of (A.6), we have

$$
\begin{equation*}
(1-l) q\left(\tilde{V}_{0}(\eta)\right) v_{1}(\eta)-\frac{1}{2} \dot{q}\left(\tilde{V}_{0}(\eta)\right) v_{2}(\eta) \geq(n+1) \delta^{4}(\eta) \tag{A.9}
\end{equation*}
$$

From this, (A.4) and (A.8), it follows that

$$
\begin{align*}
\varrho V_{0}(\eta) & \leq-(n+1) \delta^{4}(\eta)\|\eta\|^{4}+\bar{b} q\left(\tilde{V}_{0}(\eta)\right) \bar{\alpha}\left(x_{1}\right) x_{1}^{4}+\frac{1}{2} \bar{b} \dot{q}\left(\tilde{V}_{0}(\eta)\right) \bar{\alpha}\left(x_{2}\right) x_{1}^{4}  \tag{A.10}\\
& \leq-(n+1) \delta^{4}(\eta)\|\eta\|^{4}+\bar{b}\left(1+\frac{1}{2} l_{1}(0)\right) q\left(\kappa_{2}\left(\bar{b}_{1} \kappa_{\eta}\left(\left|x_{1}\right|\right)\right)\right) \bar{\alpha}\left(x_{1}\right) x_{1}^{4}
\end{align*}
$$

which shows that second assertion of Proposition 3.11 holds for this case by letting $b=\bar{b}(1+$ $\left.q\left(\kappa_{2}\left(\bar{b}_{1}^{2}\right)\right)\right)\left(1+(1 / 2) l_{1}(0)\right)$ and $\bar{\alpha}_{0}\left(x_{1}\right)=\bar{\alpha}\left(x_{1}\right)\left(1+q\left(\kappa_{2}\left(\kappa_{\eta}^{2}\left(\left|x_{1}\right|\right)\right)\right)\right)$. (Since $0 \leq \bar{b}_{1} \kappa_{\eta}\left(\left|x_{1}\right|\right) \leq$ $(1 / 2)\left(\bar{b}_{1}^{2}+\kappa_{\eta}^{2}\left(\left|x_{1}\right|\right)\right) \leq \bar{b}_{1}^{2}$ when $\bar{b}_{1} \geq \kappa_{\eta}^{2}\left(\left|x_{1}\right|\right)$, and otherwise $0 \leq \bar{b}_{1} \kappa_{\eta}\left(\left|x_{1}\right|\right) \leq \kappa_{\eta}^{2}\left(\left|x_{1}\right|\right)$, from the fact that $q(\cdot)$ and $\kappa_{2}(\cdot)$ are positive and monotone increasing functions on $\mathbf{R}^{+}$, it follows that $q\left(\kappa_{2}\left(\bar{b}_{1} \kappa_{\eta}\left(\left|x_{1}\right|\right)\right)\right) \leq q\left(\kappa_{2}\left(\bar{b}_{1}^{2}\right)\right)+q\left(\kappa_{2}\left(\kappa_{\eta}^{2}\left(\left|x_{1}\right|\right)\right)\right) \leq\left(1+q\left(\kappa_{2}\left(\bar{b}_{1}^{2}\right)\right)\right)\left(1+q\left(\kappa_{2}\left(\kappa_{\eta}^{2}\left(\left|x_{1}\right|\right)\right)\right)\right)$.

## B. The Proof of Proposition 4.5

We first prove the following proposition.
Proposition B.1. For $k=2, \ldots, n$, there exist smooth nonnegative functions $\sigma_{k}\left(x_{[k]}, \widehat{\Theta}\right)$, $C_{k}\left(x_{[k]}, \widehat{\mathrm{O}}\right)$, and $D_{k}\left(x_{[k+1]}, \widehat{\mathrm{\Theta}}\right)$, such that

$$
\begin{gather*}
\sum_{r=1}^{k}\left|x_{r}\right| \leq \sigma_{k}\left(x_{[k]}, \widehat{\Theta}\right) \sum_{r=1}^{k}\left|z_{r}\right|^{1 /\left(p_{1} \cdots p_{k-1}\right)}, \\
\left\|\frac{\partial \alpha_{k}^{p_{1} \cdots p_{k}}}{\partial x_{i}} g_{i}\right\| \leq \bar{\Theta} C_{k}\left(x_{[k]}, \widehat{\Theta}\right)\left(\delta(\eta)\|\eta\|+\sum_{r=1}^{k}\left|z_{r}\right|\right),  \tag{B.1}\\
\left|\frac{\partial^{2} \alpha_{k}^{p_{1} \cdots p_{k}}}{\partial x_{i} \partial x_{j}} g_{i}^{T} g_{j}\right| \leq \bar{\Theta}^{2} C_{k}\left(x_{[k]}, \widehat{\Theta}\right)\left(\delta^{2}(\eta)\|\eta\|^{2}+\sum_{r=1}^{k}\left|z_{r}\right|\right), \\
\left|\frac{\partial \alpha_{k}^{p_{1} \cdots p_{k}}}{\partial x_{i}}\left(d_{i} x_{i+1}^{p_{i}}+f_{i}\right)\right| \leq \bar{\Theta} D_{k}\left(x_{[k+1]}, \widehat{\Theta}\right)\left(\delta(\eta)\|\eta\|+\sum_{r=1}^{i+1}\left|z_{r}\right|\right),
\end{gather*}
$$

where $i=1, \ldots, k, j=1, \ldots, k$, and $\bar{\Theta}$ is the same as in Proposition 3.4.
Proof. The first claim obviously holds when $k=2$, because of the following inequality:

$$
\begin{equation*}
\left|x_{1}\right|+\left|x_{2}\right| \leq\left|z_{1}\right|+\left|z_{2}+z_{1} h_{1}\left(x_{1}, \widehat{\Theta}_{1}\right)\right|^{1 / p_{1}} \leq \sigma_{2}\left(x_{[2]}, \widehat{\Theta}_{1}\right)\left(\left|z_{1}\right|^{1 / p_{1}}+\left|z_{2}\right|^{1 / p_{1}}\right) \tag{B.2}
\end{equation*}
$$

and can be easily proven in the same way of Lemma 3.4 in [19].
Based on Lemma 2.4 and Proposition 3.4, the proof for the last three claims is straightforward (though somewhat tedious) and quite similar to the proof of Lemma 3.5 in [19] and is omitted here.

Next, in view of Proposition B.1, we complete the Proof of Proposition 4.5 by estimating each term of the left-hand side of (4.11).

From Propositions 3.4 and B.1, we have

$$
\begin{align*}
z_{k}^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} f_{k} \leq & \frac{d_{k}}{2}\left|z_{k}\right|^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left|x_{k+1}\right|^{p_{k}}+\bar{\Theta}\left|z_{k}\right|^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \delta(\eta)\|\eta\| \\
& +\bar{\Theta} \phi_{k}\left(x_{[k]}\right) \sigma_{k}\left(x_{[k]}, \widehat{\Theta}\right)\left|z_{k}\right|^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \sum_{j=1}^{k}\left|z_{i}\right|^{1 / p_{1} \cdots p_{k-1}}  \tag{B.3}\\
\leq & \frac{d_{k}}{2}\left|z_{k}\right|^{\left(4 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}}\left|x_{k+1}\right|^{p_{k}}+\frac{1}{3} \delta^{4}(\eta)\|\eta\|^{4}+\frac{1}{6 a^{3}} \sum_{i=1}^{k-1} z_{i}^{4} \\
& +a \Theta z_{k}^{4} \rho_{k, 1}\left(x_{[k]}, \widehat{\Theta}\right)
\end{align*}
$$

where and whereafter $\rho_{k, i}\left(x_{[k]}, \widehat{\Theta}\right), i=1, \ldots, 4$ are nonnegative smooth functions and can be easily obtained by Lemma 2.4, and for the notional convenience, their explicit expressions are omitted.

From Lemma 2.6, Propositions 3.4 and B.1, and the expression of $W_{k}$ given by (4.3), we have

$$
\begin{aligned}
& \sum_{i=1}^{k-1} \frac{\partial W_{k}}{\partial x_{i}}\left(d_{i} x_{i+1}^{p_{i}}+f_{i}\right) \\
& \quad \leq 4 \sum_{i=1}^{k-1}\left|\int_{\alpha_{k-1}}^{x_{k}}\left(s^{p_{1} \cdots p_{k-1}}-\alpha_{k-1}^{p_{1} \cdots p_{k-1}}\right)^{\left(3 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \mathrm{~d} s\right| \cdot\left|\frac{\partial \alpha_{k-1}^{p_{1} \cdots p_{k-1}}}{\partial x_{i}}\left(d_{i} x_{i+1}^{p_{i}}+f_{i}\right)\right| \\
& \quad \leq \sum_{i=1}^{k-1} 8\left|z_{k}\right|^{3} \bar{\Theta} D_{k-1}\left(x_{[k]}, \widehat{\Theta}\right)\left(\delta(\eta)\|\eta\|+\sum_{j=1}^{i+1}\left|z_{j}\right|\right) \\
& \leq \frac{1}{3} \delta^{4}(\eta)\|\eta\|^{4}+\frac{1}{6 a^{3}} \sum_{i=1}^{k-1} z_{i}^{4}+a \Theta z_{k}^{4} \rho_{k, 2}\left(x_{[k]}, \widehat{\Theta}\right), \\
& \begin{array}{l}
\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} g_{i}^{T} \frac{\partial^{2} W_{k}}{\partial x_{i} \partial x_{j}} g_{j} \\
\leq
\end{array} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1}\left|\int_{\alpha_{k-1}}^{x_{k}}\left(s^{p_{1} \cdots p_{k-1}}-\alpha_{k-1}^{p_{1} \cdots p_{k-1}}\right)^{\left(3 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \mathrm{~d} s\right| \cdot\left|\frac{\partial^{2} \alpha_{k-1}^{p_{1} \cdots p_{k-1}}}{\partial x_{i} \partial x_{j}} g_{i}^{T} g_{j}\right| \\
& \quad+6 \sum_{i=1}^{k-1} \sum_{j=1}^{k-1}\left|\int_{\alpha_{k-1}}^{x_{k}}\left(s^{p_{1} \cdots p_{k-1}}-\alpha_{1}^{p_{1} \cdots p_{k-1}}\right)^{\left(2 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \mathrm{~d} s\right| \cdot\left\|\frac{\partial \alpha_{k-1}^{p_{1} \cdots p_{k-1}}}{\partial x_{i}} g_{i}\right\| \\
& \quad\left\|\frac{\partial \alpha_{k-1}^{p_{1} \cdots p_{k-1}}}{\partial x_{j}} g_{j}\right\|+4\left|z_{k}\right|^{\left(3 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} \sum_{i=1}^{k-1}\left\|\frac{\partial \alpha_{k-1}^{p_{1} \cdots p_{k-1}}}{\partial x_{i}} g_{i}\right\| \cdot\left\|g_{k}\right\|
\end{aligned}
$$

$$
\begin{align*}
\leq & 4(k-1)^{2}\left|z_{k}\right|^{3} \bar{\Theta}^{2} C_{k-1}\left(x_{[k-1]}, \widehat{\Theta}\right)\left(\delta^{2}(\eta)\|\eta\|^{2}+\sum_{r=1}^{k-1}\left|z_{r}\right|\right) \\
& +12(k-1)^{2} z_{k}^{2} \bar{\Theta}^{2} C_{k-1}^{2}\left(x_{[k-1]}, \widehat{\Theta}\right)\left(\delta(\eta)\|\eta\|+\sum_{r=1}^{k-1}\left|z_{r}\right|\right)^{2} \\
& +4(k-1) \bar{\Theta}\left|z_{k}\right|^{\left(3 p_{1} \cdots p_{k-1}-1\right) / p_{1} \cdots p_{k-1}} C_{k-1}\left(x_{[k-1]}, \widehat{\Theta}\right)\left(\delta(\eta)\|\eta\|+\sum_{r=1}^{k-1}\left|z_{r}\right|\right)\left\|g_{k}\right\| \\
& +2 p_{1} \cdots p_{k-1}\left|z_{k}\right|^{\left(3 p_{1} \cdots p_{k-1}-1\right) /\left.p_{1} \cdots p_{k-1}\left|x_{k}\right|\right|_{1 \cdots p_{k-1}-1} ^{p_{1}}\left\|g_{k}\right\|^{2}} \\
\leq & \frac{1}{3} \delta^{4}(\eta)\|\eta\|^{4}+\frac{1}{6 a^{3}} \sum_{i=1}^{k-1} z_{i}^{4}+a \Theta z_{k}^{4} \rho_{k, 3}\left(x_{[k]}, \widehat{\Theta}\right) . \tag{B.4}
\end{align*}
$$

For the last term, by Lemma 2.6, we get

$$
\begin{align*}
& \frac{3}{2} \bar{\Theta} \mu_{k-1}\left(x_{[k]}\right)\left|z_{k-1}\right|^{\left(4 p_{1} \cdots p_{k-2}-1\right) / p_{1} \cdots p_{k-2}} \cdot\left|x_{k}^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}}\right| \\
& \quad \leq 3 \bar{\Theta} \mu_{k-1}\left(x_{[k]}\right)\left|z_{k-1}\right|^{\left(4 p_{1} \cdots p_{k-2}-1\right) /\left(p_{1} \cdots p_{k-2}\right)} \cdot\left|z_{k}\right|^{1 /\left(p_{1} \cdots p_{k-2}\right)}  \tag{B.5}\\
& \quad \leq \frac{1}{4 a^{3}} z_{k-1}^{4}+a \Theta z_{k}^{4} \rho_{k, 4}\left(x_{[k]}, \widehat{\Theta}\right)
\end{align*}
$$

So far, by choosing $\rho_{k}=\sum_{i=1}^{4} \rho_{k, i}$, the proof of Proposition 4.5 is finished.

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Research Article

# Stability and Stabilization of Networked Control System with Forward and Backward Random Time Delays 

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#### Abstract

This paper deals with the problem of stabilization for a class of networked control systems (NCSs) with random time delay via the state feedback control. Both sensor-to-controller and controller-toactuator delays are modeled as Markov processes, and the resulting closed-loop system is modeled as a Markovian jump linear system (MJLS). Based on Lyapunov stability theorem combined with Razumikhin-based technique, a new delay-dependent stochastic stability criterion in terms of bilinear matrix inequalities (BMIs) for the system is derived. A state feedback controller that makes the closed-loop system stochastically stable is designed, which can be solved by the proposed algorithm. Simulations are included to demonstrate the theoretical result.


## 1. Introduction

Feedback control systems in which the control loops are closed through a real-time network are called networked control systems (NCSs) [1]. Recently, much attention has been paid to the study of stability analysis and controller design of NCSs [2,3] due to their low cost, reduced weight and power requirements, simple installation and maintenance, and high reliability. Consequently, NCSs have been applied to various areas such as mobile sensor networks [4], remote surgery [5], haptics collaboration over the Internet [6-8], and automated highway systems and unmanned aerial vehicles [9, 10]. However, the sampling data and controller signals are transmitted through a network, so network-induced delays in NCSs are always inevitable $[11,12]$.

One of the main issues in NCSs is network-induced delays, which are usually the major causes for the deterioration of system performance and potential system instability
[13]. For different scheduling protocols, the network-induced delay may be constant, or time-varying, but in most cases, it is random [14]. Hence, systems with random time delay attract considerable attention [15-18]. Based on stochastic control theory and a separation property, the effect of random delay is treated as an LQG problem in [15]. However, the network-induced random delay has to be less than one sampling interval. The results in [15] have recently been extended to the case with longer delays in [16]. It is noted that the given controller depends only on sensor-to-controller delay. In [17], a control problem for Bernoulli binary random delay is considered, and a linear matrix inequalities (LMIs) problem for the analysis of stochastic exponential mean square stability is established. The model-based NCSs with random transmission delay is studied in [18]. Sufficient conditions for almost sure stability and stochastic exponential mean square stability are presented.

On the other hand, the study of stochastic systems has attracted a great deal of attention [19-38]. Some of these results are applied to networked control systems with random time delays [39-43]. In [39, 40], the network-induced random delays are modeled as Markov chains such that the closed-loop systems are jump linear systems with one mode. It is noticed that in [39], the state feedback gain is mode independent, and in [40], the state feedback gain only depends on the delay from sensor to controller. Recently, stabilization of networked control systems with the sensor-to-controller and controller-to-actuator delays are considered in [41]. In [42, 43], a class of Markovian jump linear systems with time delays both in the system state and in the mode signal is considered. Based on Lyapunov method, a timedelayed, mode-dependent, and state feedback controller such that the closed-loop system is stochastically stable is designed. It is noticed that the time delay in the mode signal is constant in $[42,43]$, and the time delay in the mode signal is random. It is worth pointing out that in all of the aforementioned papers, the plant is in the discrete-time domain. To the best of the authors' knowledge, the stability and stabilization problems for NCSs with the plant being in the continuous-time domain have not been fully investigated to date. Especially for the case where both sensor-to-controller and controller-to-actuator network-induced delays are random and longer than one sampling interval, very few results related to NCSs have been available in the literature so far, which motivates the present study.

The aim of this paper is to consider a class of networked control systems with sensors and actuators connected to a controller via two communication networks in the continuous-time domain. Two Markov processes are introduced to describe sensor-tocontroller transmission delay and the controller-to-actuator transmission delay. Based on Lyapunov stability theorem, a method for designing a mode-dependent state feedback controller that stabilizes this class of networked control systems is proposed. The existence of such a controller is given in terms of BMIs, which can be solved by the proposed algorithm.

This paper is organized as follows. In Section 2, the problem is stated and some useful definitions and lemmas are given, and then the main results of this paper are given in Section 3. Simulation results are presented in Section 4. Finally, the conclusions are provided in Section 5.

Notation. $\mathbb{R}^{n}$ denotes the $n$-dimensional Euclidean space, and $I$ is identity matrix. $A^{T}$ stands for the transpose of the corresponding matrix $A$. The notation $A \geq 0(A>0)$ means that the matrix $A$ is a positive semidefinite (positive definite) matrix. For an arbitrary matrix $Y$ and two symmetric matrices $X$ and $Z,\left[\begin{array}{lll}X & Y \\ * & Z\end{array}\right]$ denotes a symmetric matrix, where $*$ denotes a block matrix entry implied by symmetry, and $\|\cdot\|$ refers to the Euclidean norm for vectors and induced 2-norm for matrices. $\mathbb{E}(\cdot)$ stands for the mathematical expectation operator, and $\mathbb{P}(\cdot)$ for probability operator.


Figure 1: Illustration of NCSs over communication network.

## 2. Problem Formulation

Consider linear systems described by the differential equation

$$
\begin{equation*}
\dot{x}=A x(t)+B u(t), \tag{2.1}
\end{equation*}
$$

where $x(t) \in \mathbb{R}^{n}$ is the state vector, and $u(t) \in \mathbb{R}^{m}$ is the control input. Matrices $A$ and $B$ are known matrices of appropriate dimensions.

The plant is interconnected by a controller over a communication network, see Figure 1. The sensor and controller are periodically sampled with the sampling interval $T$. We describe the sensor-to-controller transmission delay as $\tau_{\mathrm{sc}}\left(r_{t}\right)$ and the controller-to-actuator transmission delay as $\tau_{\mathrm{ca}}\left(\eta_{t}\right)$. The mode switching of $\tau_{\mathrm{sc}}\left(r_{t}\right)$ is governed by the continuoustime discrete-state Markov process $r_{t}$ taking the values in the finite set $\varsigma_{r}:=\left\{1, \ldots, N_{r}\right\}$ with generator $\Lambda=\left(\lambda_{i j}\right), i, j \in \zeta_{r}$ given by

$$
\mathbb{P}\left[r_{t+h}=j \mid r_{t}=i\right]= \begin{cases}\lambda_{i j} h+o(h), & i \neq j,  \tag{2.2}\\ 1+\lambda_{i i} h+o(h), & i=j,\end{cases}
$$

where $\lambda_{i j}$ is the transition rate from mode $i$ to $j$ with $\lambda_{i j} \geq 0$ when $i \neq j$ and $\lambda_{i i}=-\sum_{j=1, j \neq i}^{N_{r}} \lambda_{i j}$, and $o(h)$ is such that $\lim _{h \rightarrow 0} o(h) / h=0$. The mode switching of $\tau_{\mathrm{ca}}\left(\eta_{t}\right)$ is governed by the continuous-time discrete-state Markov process $\eta_{t}$ taking the values in the finite set $\varsigma_{\eta}:=\left\{1, \ldots, N_{\eta}\right\}$ with generator $\Pi=\left(\pi_{k l}\right), k, l \in \varsigma_{\eta}$ given by

$$
\mathbb{P}\left[\eta_{t+h}=l \mid \eta_{t}=k\right]= \begin{cases}\pi_{k l} h+o(h), & k \neq l,  \tag{2.3}\\ 1+\pi_{k k} h+o(h), & k=l,\end{cases}
$$

with $\pi_{k l} \geq 0$ and $\pi_{k k}=-\sum_{l=1, l \neq k}^{N_{n}} \pi_{k l}$.
Throughout the paper, the following assumption is needed for the considered networked control systems.


Figure 2: Illustration of the time delay.

Assumption 2.1. The switching difference of consecutive delays is less than one sampling interval, that is,

$$
\begin{gather*}
\mathbb{P}\left(\left|\tau_{\mathrm{sc}}\left(r_{t_{k+1}}\right)-\tau_{\mathrm{sc}}\left(r_{t_{k}}\right)\right| \geq T\right)=0, \\
\mathbb{P}\left(\left|\tau_{\mathrm{ca}}\left(\eta_{t_{k+1}}\right)-\tau_{\mathrm{ca}}\left(\eta_{t_{k}}\right)\right| \geq T\right)=0, \tag{2.4}
\end{gather*}
$$

where $t_{k}=k T$ is the $k$ th sampling instant.
Remark 2.2. Although Assumption 2.1 restricts that the switching difference of consecutive delays is less than one sampling interval $T$, this does not imply that the network delay $\tau_{\mathrm{sc}}\left(r_{t_{k}}\right)$ and $\tau_{\mathrm{ca}}\left(\eta_{t_{k}}\right)$ are less than $T$.

According to Figure 1, for $t_{k} \leq t<t_{k+1}$, the control law has the form:

$$
\begin{equation*}
u(t)=K\left(r_{t}, \eta_{t}\right) x\left(t_{k}-\tau_{\mathrm{sc}}\left(r_{t}\right)-\tau_{\mathrm{ca}}\left(\eta_{t}\right)\right) \tag{2.5}
\end{equation*}
$$

Define the time delay $\tau\left(r_{t}, \eta_{t}\right)$ as follows:

$$
\begin{equation*}
\tau\left(r_{t}, \eta_{t}\right)=t-t_{k}+\tau_{\mathrm{sc}}\left(r_{t}\right)+\tau_{\mathrm{ca}}\left(\eta_{t}\right) \tag{2.6}
\end{equation*}
$$

which can be illustrated by Figure 2.
Then, we have

$$
\begin{equation*}
u(t)=K\left(r_{t}, \eta_{t}\right) x\left(t-\tau\left(r_{t}, \eta_{t}\right)\right) \tag{2.7}
\end{equation*}
$$

The associated upper bounds of $\tau\left(r_{t}, \eta_{t}\right)$ are defined as

$$
\begin{equation*}
\bar{\tau}=T+\max _{i \in \varsigma_{r}} \tau_{\mathrm{sc}}(i)+\max _{k \in \varsigma_{\eta}} \tau_{\mathrm{ca}}(k) \tag{2.8}
\end{equation*}
$$

Applying controller (2.7) to the open-loop system (2.1) results in the closed-loop networked control system

$$
\begin{gather*}
\dot{x}(t)=A x(t)+B K\left(r_{t}, \eta_{t}\right) x\left(t-\tau\left(r_{t}, \eta_{t}\right)\right), \\
x(\theta)=\phi(\theta), \quad \theta \in[-\bar{\tau}, 0], \tag{2.9}
\end{gather*}
$$

where $\phi(\theta), \theta \in[-\bar{\tau}, 0]$ is the initial function.
We have the following stochastic stability concept for system (2.9).
Definition 2.3. The system (2.9) is said to be stochastically stable if there exists a constant $\mathbb{T}\left(r_{0}, \eta_{0}, \phi(\cdot)\right)$ such that

$$
\begin{equation*}
\mathbb{E}\left[\int_{0}^{\infty}\|x(s)\|^{2} d s \mid\left(r_{0}, \eta_{0}, \phi(\cdot)\right)\right] \leq \mathbb{T}\left(r_{0}, \eta_{0}, \phi(\cdot)\right) \tag{2.10}
\end{equation*}
$$

for any initial condition $x\left(r_{0}, \eta_{0}, \phi(\cdot)\right)$.
The following lemmas will be essential for the proofs in Section 3.
Lemma 2.4 (see [44]). Given any real matrices $\Sigma_{1}, \Sigma_{2}, \Sigma_{3}$ of appropriate dimensions and a scalar $\epsilon>0$ such that $\Sigma_{3}=\Sigma_{3}^{T}>0$, Then the following inequality holds:

$$
\begin{equation*}
\sum_{1}^{T} \sum_{2}+\sum_{2}^{T} \sum_{1} \leq \epsilon \sum_{1}^{T} \sum_{3} \sum_{1}+\epsilon^{-1} \sum_{2}^{T} \sum_{3}^{-1} \sum_{2} . \tag{2.11}
\end{equation*}
$$

For the delay functional differential equation,

$$
\begin{equation*}
\dot{x}(t)=f\left(t, x_{t}\right), \tag{2.12}
\end{equation*}
$$

where

$$
\begin{equation*}
f:[0,+\infty) \times C\left([-\tau, 0], \mathbb{R}^{n}\right) \longrightarrow \mathbb{R}^{n} \tag{2.13}
\end{equation*}
$$

is completely continuous, $f(t, 0)=0$, and $x_{t}(\theta)$ is defined as

$$
\begin{equation*}
x_{t}(\theta)=x(t+\theta), \quad \theta \in[-\tau, 0] . \tag{2.14}
\end{equation*}
$$

Then we have the following Razumikhin lemma.
Lemma 2.5 (see [45]). Suppose that $u, v, w: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$are continuous, strictly monotonous increasing functions, then $u(s), v(s)$, and $w(s)$ are positive for $s>0$, and $u(0)=v(0)=0$. If there is a continuous function $V:[-\tau,+\infty) \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{+}$such that

$$
\begin{equation*}
u(\|x\|) \leq V(t, x) \leq v(\|x\|), \quad t \in[-\tau,+\infty), x \in \mathbb{R} \tag{2.15}
\end{equation*}
$$

and there is a continuous nondecreasing function $p(s)>s$ for $s>0$, and for any $t_{0} \in \mathbb{R}^{+}$,

$$
\begin{equation*}
\dot{V}(t, x) \leq-w(\|x\|) \tag{2.16}
\end{equation*}
$$

if

$$
\begin{equation*}
V(x(t+\theta, t+\theta))<p(V(t, x)), \quad \theta \in[-\tau, 0], t \geq t_{0} \tag{2.17}
\end{equation*}
$$

then the zero solution of (2.12) is uniformly asymptotically stable.

## 3. Main Results

The following theorem provides sufficient conditions for existence of a mode-dependent state feedback controller for the system (2.9).

Theorem 3.1. Consider the closed-loop system (2.9) satisfying Assumption 2.1. If there exist symmetric matrix $Q(i, k)>0$, matrix $Y(i, k)$, and positive scalar $\epsilon_{1}, \epsilon_{2}$ such that the following matrix inequalities hold for all $i \in \varsigma_{r}$ and $k \in \varsigma_{\eta}$,

$$
\begin{align*}
& {\left[\begin{array}{ccc}
J(i, k) & \varphi_{1}(i, k) & \varphi_{2}(i, k) \\
* & -\psi_{1} & 0 \\
* & * & -\psi_{2}
\end{array}\right]<0,}  \tag{3.1}\\
& {\left[\begin{array}{cc}
-\varepsilon_{1} Q(i, k) & A Q(i, k) \\
* & -Q(i, k)
\end{array}\right]<0,}  \tag{3.2}\\
& {\left[\begin{array}{cc}
-\varepsilon_{2} Q(i, k) & B Y(i, k) \\
* & -Q(i, k)
\end{array}\right]<0,} \tag{3.3}
\end{align*}
$$

where

$$
\begin{gather*}
J(i, k)=Q(i, k) A^{T}+A Q(i, k)+Y^{T}(i, k) B^{T}+B Y(i, k)+\bar{\tau}\left(\epsilon_{1}+3 \epsilon_{2}\right) Q(i, k)+\lambda_{i i} Q(i, k)+\pi_{k k} Q(i, k), \\
\varphi_{1}(i, k)=\left[\sqrt{\lambda_{i, 1}} Q(i, k), \ldots, \sqrt{\lambda_{i, i-1}} Q(i, k), \sqrt{\lambda_{i, i+1}} Q(i, k), \ldots, \sqrt{\lambda_{i, N_{r}}} Q(i, k)\right], \\
\varphi_{2}(i, k)=\left[\sqrt{\pi_{k, 1}} Q(i, k), \ldots, \sqrt{\pi_{k, k-1}} Q(i, k), \sqrt{\pi_{k, k+1}} Q(i, k), \ldots, \sqrt{\pi_{k, N_{\eta}}} Q(i, k)\right], \\
\psi_{1}=\operatorname{diag}\left[Q(1, k), \ldots, Q(i-1, k), Q(i+1, k), \ldots, Q\left(N_{r}, k\right)\right], \\
\psi_{2}=\operatorname{diag}\left[Q(i, 1), \ldots, Q(i, k-1), Q(i, k+1), \ldots, Q\left(i, N_{\eta}\right)\right], \tag{3.4}
\end{gather*}
$$

with $P(i, k)=Q^{-1}(i, k)$, then the system is stochastically stable with the state feedback gain:

$$
\begin{equation*}
K(i, k)=Y(i, k) Q^{-1}(i, k) \tag{3.5}
\end{equation*}
$$

Proof. Consider the following Lyapunov candidate:

$$
\begin{equation*}
V\left(x(t), r_{t}, \eta_{t}\right)=x^{T}(t) P\left(r_{t}, \eta_{t}\right) x(t) \tag{3.6}
\end{equation*}
$$

where $P\left(r_{t}, \eta_{t}\right)$ is the positive symmetric matrix. From (3.6), it follows that

$$
\begin{equation*}
\beta_{1}\|x(t)\|^{2} \leq V\left(x(t), r_{t}, \eta_{t}\right) \leq \beta_{2}\|x(t)\|^{2}, \tag{3.7}
\end{equation*}
$$

where

$$
\begin{align*}
& \beta_{1}=\min _{r_{t} \in S_{r}, \eta_{t} \in S_{\eta}} \lambda_{\min }\left(P\left(r_{t}, \eta_{t}\right)\right), \\
& \beta_{2}=\max _{r_{t} \in S_{r}, \eta_{t} \in S_{\eta}} \lambda_{\max }\left(P\left(r_{t}, \eta_{t}\right)\right) . \tag{3.8}
\end{align*}
$$

Note that

$$
\begin{align*}
x\left(t-\tau\left(r_{r}, \eta_{t}\right)\right) & =x(t)-\int_{-\tau\left(r_{r}, \eta_{t}\right)}^{0} \dot{x}(t+\theta) d \theta  \tag{3.9}\\
& =x(t)-\int_{-\tau\left(r_{r}, \eta_{t}\right)}^{0}\left[A x(t+\theta),+B K\left(r_{t}, \eta_{t}\right) x\left(t-\tau\left(r_{r}, \eta_{t}\right)+\theta\right)\right] d \theta
\end{align*}
$$

Thus, the closed-loop system (2.9) can be rewritten as

$$
\begin{equation*}
\dot{x}(t)=\left[A+B K\left(r_{t}, \eta_{t}\right)\right] x(t)-B K\left(r_{t}, \eta_{t}\right) \int_{-\tau\left(r_{r}, \eta_{t}\right)}^{0}\left[A x(t+\theta)+B K\left(r_{t}, \eta_{t}\right) x\left(t-\tau\left(r_{r}, \eta_{t}\right)+\theta\right)\right] d \theta \tag{3.10}
\end{equation*}
$$

Let $\mathcal{L}(\cdot)$ be the weak infinitesimal generator of $\left\{x(t), r_{t}, \eta_{t}, t \geq 0\right\}$, then for $r_{t}=i \in \varsigma_{r}, \eta_{t}=k \in$ $\varsigma_{\eta}$, we have

$$
\begin{align*}
& \mathscr{L} V(x(t), i, k) \\
& =\dot{x}^{T}(t) P(i, k) x(t)+x^{T}(t) P(i, k) \dot{x}(t)+\sum_{j=1}^{N_{r}} \lambda_{i j} x^{T}(t) P(j, k) x(t)+\sum_{l=1}^{N_{n}} \pi_{k l} x^{T}(t) P(i, l) x(t) \\
& +x^{T}(t)\left[A^{T} P(i, k)+P(i, k) A+K^{T}(i, k) B^{T} P(i, k)+P(i, k) B K(i, k)+\sum_{j=1}^{N_{r}} \lambda_{i j} P(j, k)+\sum_{l=1}^{N_{n}} \pi_{k l} P(i, l)\right] x(t) \\
& -2 \int_{-\tau(i, k)}^{0}\left\{x^{T}(t) P(i, k) B K(i, k) \times[A x(t+\theta)+B K(i, k) x(t-\tau(i, k)+\theta)]\right\} d \theta . \tag{3.11}
\end{align*}
$$

According to Lemma 2.4, we have

$$
\begin{align*}
& -2 \int_{-\tau(i, k)}^{0}\left\{x^{T}(t) P(i, k) B K(i, k) \times[A x(t+\theta)+B K(i, k) x(t-\tau(i, k)+\theta)]\right\} d \theta \\
& \leq \tau(i, k)\left[\epsilon_{1}^{-1} x^{T}(t) P(i, k) B K(i, k) A P^{-1}(i, k) \times A^{T} K^{T}(i, k) B^{T} P(i, k) x(t)+\epsilon_{2}^{-1} x^{T}(t)\right. \\
& \\
& \left.\quad \times P(i, k) B K(i, k) B K(i, k) P^{-1}(i, k) K^{T}(i, k) \times B^{T} K^{T}(i, k) B^{T} P(i, k) x(t)\right]  \tag{3.12}\\
& \quad+\epsilon_{1} \int_{-\tau(i, k)}^{0} x^{T}(t+\theta) P(i, k) x(t+\theta) d \theta+\epsilon_{2} \int_{-\tau(i, k)}^{0} x^{T}(t-\tau(i, k)+\theta) P(i, k) x(t-\tau(i, k)+\theta) d \theta .
\end{align*}
$$

From (3.2), (3.3), and Lemma 2.5, we can obtain

$$
\begin{gather*}
A P^{-1}(i, k) A^{T}<\epsilon_{1} P^{-1}(i, k),  \tag{3.13}\\
B K(i, k) P^{-1}(i, k) K^{T}(i, k) B^{T}<\epsilon_{2} P^{-1}(i, k),
\end{gather*}
$$

which yields

$$
\begin{align*}
& -2 \int_{-\tau(i, k)}^{0}\left\{x^{T}(t) P(i, k) B K(i, k) \times[A x(t+\theta)+B K(i, k) x(t-\tau(i, k)+\theta)]\right\} d \theta \\
& \leq 2 \tau(i, k) \epsilon_{2} x^{T}(t) P(i, k) x(t)+\epsilon_{1} \int_{-\tau(i, k)}^{0} x^{T}(t+\theta) P(i, k) x(t+\theta) d \theta  \tag{3.14}\\
& \quad+\epsilon_{2} \int_{-\tau(i, k)}^{0} x^{T}(t-\tau(i, k)+\theta) P(i, k) \times x(t-\tau(i, k)+\theta) d \theta .
\end{align*}
$$

Following Lemma 2.5 , for $-2 \bar{\tau} \leq \theta \leq 0$, we assume that for any $\delta>1$, the following inequality holds:

$$
\begin{equation*}
V\left(x(t+\theta), r_{t+\theta}, \eta_{t+\theta}\right)<\delta V\left(x(t), r_{t}, \eta_{t}\right) \tag{3.15}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\mathfrak{L V}(x(t), i, k) \leq x^{T}(t) \not{\ell}(\tau(i, k), \delta) x(t), \tag{3.16}
\end{equation*}
$$

where $\mathscr{H}(\tau(i, k), \delta)$ is given by

$$
\begin{align*}
& \mathscr{H}(\tau(i, k), \delta) \\
&= A^{T} P(i, k)+P(i, k) A+K^{T}(i, k) B^{T} P(i, k)+P(i, k) B K(i, k)+\sum_{j=1}^{N_{r}} \lambda_{i j} P(j, k)  \tag{3.17}\\
&+\sum_{l=1}^{N_{n}} \pi_{k l} P(i, l)+2 \tau(i, k) \epsilon_{2} P(i, k)+\tau(i, k) \epsilon_{1} \delta P(i, k)+\tau(i, k) \epsilon_{2} \delta P(i, k),
\end{align*}
$$

for some positive scalars $\epsilon_{1}$ and $\epsilon_{2}$. before and after multiplying $\mathscr{H}(\tau(i, k), \delta)$ by $Q(i, k)=$ $P^{-1}(i, k)$ and its transpose, it gives

$$
\begin{align*}
& \widetilde{\mathscr{H}}(\tau(i, k), \delta) \\
&= Q(i, k) A^{T}+A Q(i, k)+Q(i, k) K^{T}(i, k) B^{T}+B K(i, k) Q(i, k)+Q(i, k) \sum_{j=1}^{N_{r}} \lambda_{i j} P(j, k) Q(i, k) \\
&+Q(i, k) \sum_{l=1}^{N_{n}} \pi_{k l} P(i, l) Q(i, k)+2 \tau(i, k) \epsilon_{2} Q(i, k)+\tau(i, k) \epsilon_{1} \delta Q(i, k)+\tau(i, k) \epsilon_{2} \delta Q(i, k) . \tag{3.18}
\end{align*}
$$

Since

$$
\begin{equation*}
0 \leq \tau(i, k) \leq \bar{\tau}, \tag{3.19}
\end{equation*}
$$

we have from (3.16) that

$$
\begin{equation*}
\bumpeq V(x(t), i, k) \leq x^{T}(t) \mathscr{H}(\bar{\tau}, \delta) x(t) . \tag{3.20}
\end{equation*}
$$

From (3.1) and Lemma 2.5, it follows that

$$
\begin{equation*}
\widetilde{\mathscr{X}}(\bar{\tau}, \delta=1)<0, \tag{3.21}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
\mathscr{H}(\bar{\tau}, \delta=1)<0 . \tag{3.22}
\end{equation*}
$$

Using the continuity properties of the eigenvalues of $\mathscr{H}$ with respect to $\delta$, then there exists a $\delta>1$ sufficiently small such that (3.21) still holds. Thus, for such a $\delta$, we have

$$
\begin{equation*}
\mathscr{H}(\bar{\tau}, \delta)<0, \tag{3.23}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\mathcal{L V}\left(x(t), r_{t}, \eta_{t}\right) \leq-\beta\|x(t)\|^{2} \tag{3.24}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta=\min _{i \in \varsigma_{r}, k \in \varsigma_{n}}\left[\lambda_{\min }(-\mathscr{H}(\bar{\tau}, \delta))\right]>0 . \tag{3.25}
\end{equation*}
$$

Applying Dynkin's formula, we have

$$
\begin{align*}
\mathbb{E}[V(x(t), i, k)]-\mathbb{E}\left[V\left(x_{0}, r_{0}, \eta_{0}\right)\right] & =\mathbb{E}\left\{\int_{0}^{t}\left[\perp V\left(x(s), r_{s}, \eta_{s}\right) d s \mid x_{0}, r_{0}, \eta_{0}\right]\right\}  \tag{3.26}\\
& \leq-\beta \mathbb{E}\left\{\int_{0}^{t}\left[\|x(s)\|^{2} d s \mid x_{0}, r_{0}, \eta_{0}\right]\right\}
\end{align*}
$$

Note that

$$
\begin{equation*}
\mathbb{E}[V(x(t), i, k)] \geq 0 \tag{3.27}
\end{equation*}
$$

Then we can obtain

$$
\begin{align*}
\beta \mathbb{E}\left\{\int_{0}^{t}\left[\|x(s)\|^{2} d s \mid x_{0}, r_{0}, \eta_{0}\right]\right\} & \leq \mathbb{E}[V(x(t), i, k)]+\beta \mathbb{E}\left\{\int_{0}^{t}\left[\|x(s)\|^{2} d s \mid x_{0}, r_{0}, \eta_{0}\right]\right\}  \tag{3.28}\\
& \leq \mathbb{E}\left[V\left(x_{0}, r_{0}, \eta_{0}\right)\right]
\end{align*}
$$

This completes the proof.
Remark 3.2. In case of constant transmission delay, that is, $\tau_{\mathrm{sc}}\left(r_{t}\right)=\tau_{\mathrm{sc}}, \tau_{\mathrm{ca}}\left(\eta_{t}\right)=\tau_{\mathrm{ca}}, \lambda_{i j}=0$, and $\pi_{k l}=0$, Theorem 3.1 can be directly applied to systems with constant delay.

It should be noted that the terms $\epsilon_{1} Q(i, k)$ and $\epsilon_{2} Q(i, k)$ in (3.1)-(3.3) are bilinear. Therefore, we propose the following algorithm to solve these bilinear matrix inequality problems.

Step 1. Set $Q_{0}(i, k)>0$, and $Y_{0}(i, k)$ such that the following LMI holds:

$$
\left[\begin{array}{ccc}
\widetilde{J}(i, k) & \varphi_{1}(i, k) & \varphi_{2}(i, k)  \tag{3.29}\\
* & -\psi_{1} & 0 \\
* & * & -\psi_{2}
\end{array}\right]<0
$$

where

$$
\begin{equation*}
\tilde{J}(i, k)=Q(i, k) A^{T}+A Q(i, k)+Y^{T}(i, k) B^{T}+B Y(i, k)+\lambda_{i i} Q(i, k)+\pi_{k k} Q(i, k) \tag{3.30}
\end{equation*}
$$

Step 2. For $Q(i, k)>0$ given in the previous step, find $\epsilon_{1 s,} \epsilon_{2 s}$, and $Y_{s}(i, k)$ by solving the following convex optimization problem:

$$
\begin{equation*}
\max _{Y(i, k), \epsilon_{1}, \epsilon_{2}} \bar{\tau}\left(Y(i, k), \epsilon_{1}, \epsilon_{2}\right) \tag{3.31}
\end{equation*}
$$

s.t. (3.1)-(3.3) hold for $Q(i, k)>0$ fixed.

Step 3. For $Y(i, k), \epsilon_{2}$, and $\epsilon_{1}$ given in the previous step, find $Q_{s}(i, k)>0$ by solving the following quasiconvex optimization problem

$$
\begin{align*}
\max _{Q(i, k)>0} & \bar{\tau}(Q(i, k)),  \tag{3.32}\\
\text { s.t. } & (3.1)-(3.3) \text { hold for } Y(i, k), \epsilon_{2}, \text { and } \epsilon_{1} \text { fixed. }
\end{align*}
$$

Step 4. Return to step 2 until the convergence of $\bar{\tau}$ is attained with a desired precision.
Remark 3.3. For a given $Q(i, k)$, the considered optimization problem consists of minimizing an eigenvalue problem which is a convex one. On the other hand, for given $Y(i, k), \epsilon_{1}$ and $\epsilon_{2}$, the considered optimization problem consists of minimizing a generalized eigenvalue problem which is a quasiconvex optimization problem. Therefore, the proposed algorithm gives a suboptimal solution.

## 4. Simulations

In this section, simulations of the position control for robotic manipulator ViSHaRD3 [46] are included to illustrate the effectiveness of the proposed method. Combining computed torque feedback approach [47] with friction compensation, the system is decoupled into three systems. The first and second joints of the ViSHaRD3 are

$$
\frac{d}{d t}\left[\begin{array}{l}
q  \tag{4.1}\\
\dot{q}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
1 & -50
\end{array}\right]\left[\begin{array}{l}
q \\
\dot{q}
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] u(t)
$$

and the third is

$$
\frac{d}{d t}\left[\begin{array}{l}
q  \tag{4.2}\\
\dot{q}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
1 & -40
\end{array}\right]\left[\begin{array}{l}
q \\
\dot{q}
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] u(t)
$$

For simplicity, we only discuss the third joint of ViSHaRD3. Suppose that the sampling interval is $T=0.01 \mathrm{~s}$, and the Markov process $r_{t}$ that governs the mode switching of the SC delay takes values in $\varsigma_{r}=\{1,2\}$ and has the generator

$$
\Lambda=\left[\begin{array}{cc}
-3 & 3  \tag{4.3}\\
2 & -2
\end{array}\right]
$$

and the Markov process $\eta_{t}$ that governs the mode switching of the CA delay takes values in $\eta_{r}=\{1,2\}$ and has the generator

$$
\Pi=\left[\begin{array}{cc}
-1 & 1  \tag{4.4}\\
2 & -2
\end{array}\right]
$$

Associated with modes 1 and 2, let the system have time delay $\tau_{\mathrm{sc}}(1)=0.03 \mathrm{~s}$, $\tau_{\mathrm{ca}}(1)=0.02 \mathrm{~s}$ and $\tau_{\mathrm{sc}}(2)=0.025 \mathrm{~s}, \tau_{\mathrm{ca}}(2)=0.015 \mathrm{~s}$, respectively. From (2.8), we have $\bar{\tau}=0.06 \mathrm{~s}$,


Figure 3: State response of closed-loop system.
and the initial condition is $\phi(\theta)=[-1,0]^{T}, \theta \in[-0.06,0]$. By the proposed algorithm and Theorem 3.1, we can obtain the controllers as follows:

$$
\begin{align*}
& K(1,1)=\left[\begin{array}{ll}
-645.0596 & -15.9109
\end{array}\right], \\
& K(1,2)=\left[\begin{array}{ll}
-623.3689 & -15.4999
\end{array}\right],  \tag{4.5}\\
& K(2,1)=\left[\begin{array}{ll}
-575.1361 & -14.2296
\end{array}\right], \\
& K(2,2)=\left[\begin{array}{ll}
-616.8428 & -15.3049
\end{array}\right] .
\end{align*}
$$

The simulations of the state response and the control input for the closed-loop system are depicted in Figures 3 and 4, respectively, which shows that the system is stochastically stable.

## 5. Conclusions

In this paper, a technique of designing a mode-dependent state feedback controller for networked control systems with random time delays has been proposed. The main contribution of this paper is that both the sensor-to-controller and controller-to-actuator delays have been taken into account. Two Markov processes have been used to model these two time delays. Based on Lyapunov stability theorem combined with Razumikhin-based technique, some new delay-dependent stability criteria in terms of BMIs for the system are derived. A state feedback controller that makes the closed-loop system stochastically stable is


Figure 4: Control input of closed-loop system.
designed, which can be solved by the proposed algorithm. Simulations results are presented to illustrate the validity of the design methodology.

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## Research Article

# Indefinite LQ Control for Discrete-Time Stochastic Systems via Semidefinite Programming 

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#### Abstract

This paper is concerned with a discrete-time indefinite stochastic LQ problem in an infinite-time horizon. A generalized stochastic algebraic Riccati equation (GSARE) that involves the MoorePenrose inverse of a matrix and a positive semidefinite constraint is introduced. We mainly use a semidefinite-programming- (SDP-) based approach to study corresponding problems. Several relations among SDP complementary duality, the GSARE, and the optimality of LQ problem are established.


## 1. Introduction

Stochastic linear quadratic (LQ) control problem was pioneered by Wonham [1] and has become one of the most popular research field of modern control theory; see, for example, [2-12] and the references therein. In the most early literature about stochastic LQ issue, it is always assumed that the control weighting matrix $R$ is positive definite and the state weighting matrix $Q$ is positive semidefinite as the deterministic LQ problem does. However, a surprising fact was found that, different from deterministic LQ problem, for a stochastic LQ modeled by a stochastic Itô-type differential system, the original LQ optimization may still be well posed even if the cost weighting matrices $Q$ and $R$ are indefinite [5]. Follow-up research was carried out, and a lot of important results were obtained. In [6-9], continuoustime indefinite stochastic LQ control problem was studied. For the discrete-time case, there have been some works. For example, the system with only control-dependent noises was studied in [10]. The finite time and infinite horizon indefinite stochastic LQ control problem with state- and control-dependent noises were, respectively, studied in [11, 12].

In this paper, we study discrete-time indefinite stochastic LQ control problem over an infinite time horizon. The system involves multiplicative noises in both the state and the
control. We mainly use the SDP approach introduced in $[9,13]$ to discuss the corresponding problem. We first introduce a generalized stochastic algebraic Riccati equation (GSARE) that involves the Moore-Penrose inverse of a matrix. The potential relations among LQ problem, SDP, and GSARE are studied. What we have obtained extends the results of [9] from continuous-time case to discrete-time case.

The remainder of this paper is organized as follows. In Section 2, we formulate the discrete-time indefinite stochastic LQ problem and present some preliminaries including generalized stochastic algebraic Riccati equation, SDP, and some lemmas. Section 3 contains the main results. Some relations among the optimality of the LQ problem, the complementary optimal solutions of the SDP and its dual problem, and the solvability of the GSARE are established. Some comments are given in Section 4.

Notations. $\mathcal{R}^{n}: n$-dimensional Euclidean space. $\mathcal{R}^{n \times m}$ : the set of all $n \times m$ matrices. $\mathcal{S}^{n}$ : the set of all $n \times n$ symmetric matrices. $A^{\prime}$ : the transpose of matrix $A$. $A \geq 0(A>0)$ : $A$ is positive semidefinite (positive definite). $I$ : the identity matrix. $\mathcal{R}$ : the set of all real numbers. $N:=$ $\{0,1,2, \ldots\}$ and $N_{t}:=\{0,1,2, \ldots, t\}$. $\operatorname{Tr}(M)$ : the trace of a square matrix $M$. $\mathcal{A}^{\text {adj }}$ : the adjoint mapping of a mapping $\mathcal{A}$.

## 2. Preliminaries

### 2.1. Problem Statement

Consider the following discrete-time stochastic system:

$$
\begin{gather*}
x(t+1)=A x(t)+B u(t)+[C x(t)+D u(t)] w(t),  \tag{2.1}\\
x(0)=x_{0}, \quad t=0,1,2, \ldots,
\end{gather*}
$$

where $x(t) \in \mathcal{R}^{n}, u(t) \in \mathcal{R}^{m}$ are, respectively, the system state and control input. $x_{0} \in \mathcal{R}^{n}$ is the initial state and $w(t) \in \mathcal{R}$ is the noise. $A, C \in \mathcal{R}^{n \times n}$ and $B, D \in \mathcal{R}^{n \times m}$ are constant matrices. $\{w(t), t \in N\}$ is a sequence of real random variables defined on a complete probability space $\left(\Omega, \mathcal{F}, \mathcal{F}_{t}, P\right)$ with $\mathcal{F}_{t}=\sigma\left\{w(s): s \in N_{t}\right\}$, which is a wide sense stationary, second-order process with $E[w(t)]=0$ and $E[w(s) w(t)]=\delta_{s t}$, where $\delta_{s t}$ is the Kronecker function. $u(t)$ belongs to $\mathscr{L}_{4}^{2}\left(\mathcal{R}^{m}\right)$, the space of all $R^{m}$-valued, $\mathcal{F}_{t}$-adapted measurable processes satisfying $E\left(\sum_{t=0}^{\infty}\|u(t)\|^{2}\right)<\infty$. We assume that the initial state $x_{0}$ is independent of the noise $w(t), t \in$ $N$.

We first give the following definitions.
Definition 2.1. System (2.1) is called mean square stabilizable if there exists a feedback control $u(t)=K x(t)$ such that for any initial state $x_{0}$, the closed-loop system

$$
\begin{align*}
x(t+1)= & (A+B K) x(t)+(C+D K) x(t) w(t) \\
& x(0)=x_{0}, \quad t=0,1,2, \ldots \tag{2.2}
\end{align*}
$$

is asymptotically mean square stable, that is, the corresponding state of (2.2) satisfies $\lim _{t \rightarrow \infty} E\|x(t)\|^{2}=0$, where $K \in \mathcal{R}^{m \times n}$ is a constant matrix.

For system (2.1), we define the admissible control set

$$
U_{a d}=\left\{\begin{array}{l}
u(t) \in \mathscr{L}_{\mp}^{2}\left(\mathcal{R}^{m}\right)  \tag{2.3}\\
u(t) \text { is mean square stabilizing control. }
\end{array}\right.
$$

The cost functional associated with system (2.1) is

$$
\begin{equation*}
J\left(x_{0}, u\right)=\sum_{t=0}^{\infty} E\left[x^{\prime}(t) Q x(t)+u^{\prime}(t) R u(t)\right] \tag{2.4}
\end{equation*}
$$

where $Q$ and $R$ are symmetric matrices with appropriate dimensions and may be indefinite. The LQ optimal control problem is to minimize the cost functional $J\left(x_{0}, u\right)$ over $u \in U_{a d}$. We define the optimal value function as

$$
\begin{equation*}
V\left(x_{0}\right)=\inf _{u \in U_{a d}} J\left(x_{0}, u\right) . \tag{2.5}
\end{equation*}
$$

Since the weighting matrices $Q$ and $R$ may be indefinite, the LQ problem is called an indefinite stochastic LQ control problem.

Definition 2.2. The LQ problem is called well posed if

$$
\begin{equation*}
-\infty<V\left(x_{0}\right)<\infty, \quad \forall x_{0} \in \mathcal{R}^{n} . \tag{2.6}
\end{equation*}
$$

If there exists an admissible control $u^{*}$ such that $V\left(x_{0}\right)=J\left(x_{0}, u^{*}\right)$, the LQ problem is called attainable and $V\left(x_{0}\right)$ is the optimal cost value. $u^{*}(t), t \in N$, is called an optimal control, and $x^{*}(t), t \in N$, corresponding to $u^{*}(t)$ is called the optimal trajectory.

Stochastic algebraic Riccati equation (SARE) is a primary tool in solving stochastic LQ control problems. In [12], the following discrete SARE:

$$
\begin{gather*}
-P+A^{\prime} P A+C^{\prime} P C+Q-\left(A^{\prime} P B+C^{\prime} P D\right)\left(R+B^{\prime} P B+D^{\prime} P D\right)^{-1}\left(B^{\prime} P A+D^{\prime} P C\right)=0,  \tag{2.7}\\
R+B^{\prime} P B+D^{\prime} P D>0,
\end{gather*}
$$

was studied. The constraint that $R+B^{\prime} P B+D^{\prime} P D>0$ is demanded in (2.7). In fact, the corresponding LQ problem may have optimal control even if the condition is not satisfied. In this paper, we introduce the following generalized stochastic algebraic Riccati equation (GSARE),

$$
\begin{align*}
\mathcal{R}(P) \equiv-P+A^{\prime} P A+C^{\prime} P C+Q- & \left(A^{\prime} P B+C^{\prime} P D\right)\left(R+B^{\prime} P B+D^{\prime} P D\right)^{+}\left(B^{\prime} P A+D^{\prime} P C\right)=0  \tag{2.8}\\
& R+B^{\prime} P B+D^{\prime} P D \geq 0
\end{align*}
$$

which weakens the positive definiteness constraint of $R+B^{\prime} P B+D^{\prime} P D$ to positive semidefiniteness constraint and replaces the inverse by Moore-Penrose inverse. Hence, (2.8) is an extension of (2.7).

### 2.2. Semidefinite Programming

In this subsection, we will introduce SDP and its dual. SDP is a special conic optimization problem and is defined as follows.

Definition 2.3 (see [14]). Suppose that $U$ is a finite-dimensional vector space with an inner product $\langle\cdot, \cdot\rangle_{\text {ש }}$ and $S$ is a space of block diagonal symmetric matrices with an inner product $\langle\cdot, \cdot\rangle_{\mathcal{S}} \cdot \mathcal{A}: \mathcal{U} \rightarrow \mathcal{S}$ is a linear mapping, and $A_{0} \in \mathcal{S}$. The following optimization problem:

$$
\begin{align*}
\min & \langle c, x\rangle_{V} \\
\text { s.t. } & A(x)=\mathscr{A}(x)+A_{0} \geq 0 \tag{2.9}
\end{align*}
$$

is called a semidefinite programming (SDP). From convex duality, the dual problem associated with the SDP is defined as

$$
\begin{align*}
\max & -\left\langle A_{0}, Z\right\rangle_{S^{\prime}} \\
\text { s.t. } & \mathscr{A}^{\text {adj }}=c, Z \geq 0 \tag{2.10}
\end{align*}
$$

In the context of duality, we refer to the SDP (2.9) as the primal problem associated with (2.10).

Consider the following SDP problem:
$(P) \max \operatorname{Tr}(P)$,

$$
\text { s.t. } \quad A(P)=\left[\begin{array}{cc}
-P+A^{\prime} P A+C^{\prime} P C+Q & A^{\prime} P B+C^{\prime} P D  \tag{2.11}\\
B^{\prime} P A+D^{\prime} P C & R+B^{\prime} P B+D^{\prime} P D
\end{array}\right] \geq 0 .
$$

By the definition of SDP, we can get the dual problem of (2.11).
Proposition 2.4. The dual problem of (2.11) can be formulated as
(D) min $\operatorname{Tr}(Q S+R T)$,

$$
\text { s.t. }\left\{\begin{array}{l}
-S+A S A^{\prime}+C S C^{\prime}+B U A^{\prime}+D U C^{\prime}  \tag{2.12}\\
\quad+A U^{\prime} B^{\prime}+C U^{\prime} D^{\prime}+B T B^{\prime}+D T D^{\prime}+I=0 \\
Z=\left[\begin{array}{ll}
S & U^{\prime} \\
U & T
\end{array}\right] \geq 0
\end{array}\right.
$$

Proof. The objective of the primal problem can be rewritten as maximizing $\langle I, P\rangle_{S^{n}}$. The dual variable $Z=\left[\begin{array}{cc}S_{S}^{\prime} \\ U & U^{\prime}\end{array}\right] \geq 0$, where $(S, T, U) \in S^{n} \times S^{m} \times \mathcal{R}^{m \times n}$. The LMI constraint in the primal problem can be represented as

$$
A(P)=\mathscr{A}(P)+A_{0}=\left[\begin{array}{cc}
-P+A^{\prime} P A+C^{\prime} P C & A^{\prime} P B+C^{\prime} P D  \tag{2.13}\\
B^{\prime} P A+D^{\prime} P C & B^{\prime} P B+D^{\prime} P D
\end{array}\right]+\left[\begin{array}{ll}
Q & 0 \\
0 & R
\end{array}\right]
$$

According to the definition of adjoint mapping, we have $\langle\mathcal{A}(P), Z\rangle_{\mathcal{S}^{n+m}}=\left\langle P, \mathcal{A}^{\text {adj }}(Z)\right\rangle_{\mathcal{S}^{n}}$, that is, $\operatorname{Tr}[\mathcal{A}(P) Z]=\operatorname{Tr}\left[P A^{\text {adj }}(Z)\right]$. It follows $\mathcal{A}^{\text {adj }}(Z)=-S+A S A^{\prime}+C S C^{\prime}+B U A^{\prime}+D U C^{\prime}+$ $A U^{\prime} B^{\prime}+C U^{\prime} D^{\prime}+B T B^{\prime}+D T D^{\prime}$. By Definition 2.3, the objective of the dual problem is to minimize $\left\langle A_{0}, Z\right\rangle_{S^{n+m}}=\operatorname{Tr}\left(A_{0} Z\right)=\operatorname{Tr}(Q S+R T)$. On the other hand, we will state that the constraints of the dual problem (2.10) are equivalent to the constraints of (2.12). Obviously, $\mathscr{A}^{\text {adj }}(Z)=-I$ is equivalent to the equality constraint of (2.12). This ends the proof.

The primal problem (2.9) is said to satisfy the Slater condition if there exists a primal feasible solution $x^{0}$ such that $A\left(x^{0}\right)>0$, that is, the primal problem (2.9) is strictly feasible. The dual problem (2.10) is said to satisfy the Slater condition if there is a dual feasible solution $Z^{0}$ satisfying $Z^{0}>0$, that is, the dual problem (2.10) is strictly feasible.

Let $p^{*}$ and $d^{*}$ denote the optimal values of $\operatorname{SDP}$ (2.9) and the dual SDP (2.10), respectively. Let $\mathbf{X}_{\text {opt }}$ and $\mathbf{Z}_{\text {opt }}$ denote the primal and dual optimal sets. Then, we have the following proposition (see [13, Theorem 3.1]).

Proposition 2.5. $p^{*}=d^{*}$ if either of the following conditions holds.
(1) The primal problem (2.9) satisfies Slater condition.
(2) The dual problem (2.10) satisfies Slater condition.

If both conditions hold, the optimal sets $\mathbf{X}_{\text {opt }}$ and $\mathbf{Z}_{\text {opt }}$ are nonempty. In this case, a feasible point $x$ is optimal if and only if there is a feasible point $Z$ satisfying the complementary slackness condition:

$$
\begin{equation*}
A(x) Z=0 \tag{2.14}
\end{equation*}
$$

### 2.3. Some Definitions and Lemmas

The following definitions and lemmas will be used frequently in this paper.
Definition 2.6. For any matrix $M$, there exists a unique matrix $M^{+}$, called the Moore-Penrose inverse of $M$, satisfying

$$
\begin{equation*}
M M^{+} M=M, \quad M^{+} M M^{+}=M^{+}, \quad\left(M M^{+}\right)^{\prime}=M M^{+}, \quad\left(M^{+} M\right)^{\prime}=M^{+} M \tag{2.15}
\end{equation*}
$$

Lemma 2.7 (extended Schur's lemma). Let matrices $M=M^{\prime}, N$, and $R=R^{\prime}$ be given with appropriate dimensions. Then, the following conditions are equivalent:
(1) $M-N R^{+} N^{\prime} \geq 0, R \geq 0$, and $N\left(I-R R^{+}\right)=0$,
(2) $\left[\begin{array}{ll}M & N \\ N^{\prime} & R\end{array}\right] \geq 0$,
(3) $\left[\begin{array}{ll}R & N^{\prime} \\ N & M\end{array}\right] \geq 0$.

Lemma 2.8 (see [7]). For a symmetric matrix $S$, we have
(1) $S^{+}=\left(S^{+}\right)^{\prime}$,
(2) $S \geq 0$ if and only if $S^{+} \geq 0$,
(3) $S S^{+}=S^{+} S$.

Lemma 2.9 (see [12]). In system (2.1), suppose $T \in N$ is given, and $P(t) \in \mathcal{S}^{n}, t=0,1, \ldots, T+1$, is an arbitrary family of matrices, then, for any $x(0) \in \boldsymbol{R}^{n}$, we have

$$
\sum_{t=0}^{T} E\left[\begin{array}{l}
x(t)  \tag{2.16}\\
u(t)
\end{array}\right]^{\prime} Q[P(t)]\left[\begin{array}{l}
x(t) \\
u(t)
\end{array}\right]=E\left[x^{\prime}(T+1) P(T+1) x(T+1)-x^{\prime}(0) P(0) x(0)\right]
$$

where

$$
Q[P(t)]=\left[\begin{array}{cc}
-P(t)+A^{\prime} P(t+1) A+C^{\prime} P(t+1) C & A^{\prime} P(t+1) B+C^{\prime} P(t+1) D  \tag{2.17}\\
B^{\prime} P(t+1) A+D^{\prime} P(t+1) C & B^{\prime} P(t+1) B+D^{\prime} P(t+1) D
\end{array}\right] .
$$

Lemma 2.10. System (2.1) is mean square stabilizable if and only if one of the following conditions holds.
(1) There are a matrix $K$ and a symmetric matrix $P>0$ such that

$$
\begin{equation*}
-P+(A+B K) P(A+B K)^{\prime}+(C+D K) P(C+D K)^{\prime}<0 \tag{2.18}
\end{equation*}
$$

Moreover, the stabilizing feedback control is given by $u(t)=K x(t)$.
(2) For any matrix $Y>0$, there is a matrix $K$ such that the following matrix equation:

$$
\begin{equation*}
-P+(A+B K) P(A+B K)^{\prime}+(C+D K) P(C+D K)^{\prime}+Y=0 \tag{2.19}
\end{equation*}
$$

has a unique positive definite solution $P>0$. Moreover, the stabilizing feedback control is given by $u(t)=K x(t)$.
(3) The dual problem $(D)$ satisfies the Slater condition.

Proof. (1) and (2) can be derived from Proposition 2.2 in [15]. (3) is a discrete edition of Theorem 6 in [7]. The proof is similar to Theorem 6 in [7] and is omitted.

To this end, we need the following assumptions throughout the paper.
Assumption 2.11. System (2.1) is mean square stabilizable.
Assumption 2.12. The feasible set of $(P)$ is nonempty.

## 3. Main Results

In this section, we will establish the relationship among the optimality of the LQ problem, the SDP, and the GSARE.

The following theorem reveals the relation between the SDP complementary optimal solutions and the GSARE.

Theorem 3.1. If a feasible solution of $(P), P^{*}$, satisfies $\mathcal{R}\left(P^{*}\right)=0$, and the feedback control

$$
\begin{equation*}
u(t)=K^{*} x(t)=-\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right) x(t), \quad t \in N \tag{3.1}
\end{equation*}
$$

is stabilizing, then there exist complementary optimal solutions of $(P)$ and $(D)$. In particular, $P^{*}$ is optimal to $(P)$, and there is a complementary dual optimal solution $Z^{*}$ of $(D)$, such that $S^{*}>0$.

Proof. By the stability assumption of the control $u(t)=K^{*} x(t)$ and Lemma 2.10, the equation

$$
\begin{equation*}
-Y+\left(A+B K^{*}\right) Y\left(A+B K^{*}\right)^{\prime}+\left(C+D K^{*}\right) Y\left(C+D K^{*}\right)^{\prime}+I=0 \tag{3.2}
\end{equation*}
$$

has a positive solution $Y^{*}>0$. Let

$$
\begin{equation*}
S^{*}=Y^{*}, \quad U^{*}=K^{*} S^{*}=K^{*} Y^{*}, \quad T^{*}=K^{*} U^{* \prime}=K^{*} Y^{*} K^{* \prime} \tag{3.3}
\end{equation*}
$$

that is,

$$
Z^{*}=\left[\begin{array}{cc}
S^{*} & U^{* \prime}  \tag{3.4}\\
U^{*} & T^{*}
\end{array}\right]=\left[\begin{array}{cc}
Y^{*} & Y^{*} K^{* \prime} \\
K^{*} Y^{*} & K^{*} Y^{*} K^{* \prime}
\end{array}\right]=\left[\begin{array}{cc}
I & 0 \\
K^{*} & I
\end{array}\right]\left[\begin{array}{cc}
Y^{*} & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{cc}
I & K^{* \prime} \\
0 & I
\end{array}\right] \geq 0
$$

By (3.2) and (3.3), we have

$$
\begin{equation*}
-S^{*}+A S^{*} A^{\prime}+C S^{*} C^{\prime}+B U^{*} A^{\prime}+D U^{*} C^{\prime}+A U^{* \prime} B^{\prime}+C U^{* \prime} D^{\prime}+B T^{*} B^{\prime}+D T^{*} D^{\prime}+I=0 \tag{3.5}
\end{equation*}
$$

which shows $Z^{*}$ is a feasible solution of $(D) . A\left(P^{*}\right) \geq 0$ because $P^{*}$ is a feasible solution of $(P)$. By Lemmas 2.7 and 2.8,

$$
\begin{align*}
A^{\prime} P^{*} B+C^{\prime} P^{*} D & =\left(A^{\prime} P^{*} B+C^{\prime} P^{*} D\right)\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}  \tag{3.6}\\
& =\left(A^{\prime} P^{*} B+C^{\prime} P^{*} D\right)\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)
\end{align*}
$$

In addition, we have

$$
\begin{equation*}
-P^{*}+A^{\prime} P^{*} A+C^{\prime} P^{*} C+Q=K^{* \prime}\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) K^{*} \tag{3.7}
\end{equation*}
$$

by $\mathcal{R}\left(P^{*}\right)=0$ and $K^{*}=-\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right)$. Therefore, we have

$$
\begin{align*}
A\left(P^{*}\right) Z^{*} & =\left[\begin{array}{cc}
-P^{*}+A^{\prime} P^{*} A+C^{\prime} P^{*} C+Q & A^{\prime} P^{*} B+C^{\prime} P^{*} D \\
B^{\prime} P^{*} A+D^{\prime} P^{*} C & R+B^{\prime} P^{*} B+D^{\prime} P^{*} D
\end{array}\right]\left[\begin{array}{cc}
S^{*} & U^{* \prime} \\
U^{*} & T^{*}
\end{array}\right] \\
& =\left[\begin{array}{cc}
I & -K^{* \prime} \\
0 & I
\end{array}\right]\left[\begin{array}{cc}
R\left(P^{*}\right) & 0 \\
0 & R+B^{\prime} P^{*} B+D^{\prime} P^{*} D
\end{array}\right]\left[\begin{array}{cc}
I & 0 \\
-K^{*} & I
\end{array}\right]\left[\begin{array}{cc}
S^{*} & U^{* \prime} \\
U^{*} & T^{*}
\end{array}\right]  \tag{3.8}\\
& =\left[\begin{array}{cc}
I & -K^{* \prime} \\
0 & I
\end{array}\right]\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] .
\end{align*}
$$

Obviously, $P^{*}$ and $Z^{*}$ are complementary optimal solutions to $(P)$ and $(D) . P^{*}$ is optimal to $(P)$, and $Z^{*}$ is optimal to $(D) . S^{*}>0$ is trivial because $S^{*}=Y^{*}>0$.

In above, the assumption that the control in (3.1) is stabilizing is not automatically satisfied. The following theorem reveals that we can obtain a stabilizing feedback control by the dual SDP.

Theorem 3.2. Suppose that $Z=\left[\begin{array}{cc}S & U^{\prime} \\ U & T\end{array}\right]$ is a feasible solution of $(D)$ with $S>0$, then the feedback control $u(t)=U S^{-1} x(t)$ is stabilizing.

Proof. First, we have $Z \geq 0$ because $Z$ is feasible to $(D)$. By Lemma 2.7, the inequality $T$ $U S^{-1} U^{\prime} \geq 0$ holds. By simple calculations, we have

$$
\begin{align*}
& A S A^{\prime}+B U A^{\prime}+A U^{\prime} B^{\prime}+B U S^{-1} U^{\prime} B^{\prime}=\left(A+B U S^{-1}\right) S\left(A+B U S^{-1}\right)^{\prime} \\
& C S C^{\prime}+D U C^{\prime}+C U^{\prime} D^{\prime}+D U S^{-1} U^{\prime} D^{\prime}=\left(C+D U S^{-1}\right) S\left(C+D U S^{-1}\right)^{\prime} \tag{3.9}
\end{align*}
$$

Hence,

$$
\begin{align*}
0= & -S+A S A^{\prime}+C S C^{\prime}+B U A^{\prime}+D U C^{\prime}+A U^{\prime} B^{\prime}+C U^{\prime} D^{\prime}+B T B^{\prime}+D T D^{\prime}+I \\
\geq & -S+A S A^{\prime}+C S C^{\prime}+B U A^{\prime}+D U C^{\prime}+A U^{\prime} B^{\prime}+C U^{\prime} D^{\prime} \\
& +B U S^{-1} U^{\prime} B^{\prime}+D U S^{-1} U^{\prime} D^{\prime}+I  \tag{3.10}\\
> & -S+\left(A+B U S^{-1}\right) S\left(A+B U S^{-1}\right)^{\prime}+\left(C+D U S^{-1}\right) S\left(C+D U S^{-1}\right)^{\prime}
\end{align*}
$$

Above inequality shows (2.18) has a positive definite solution $S>0$ with $K=U S^{-1}$. According to Lemma 2.10, $u(t)=K x(t)=U S^{-1} x(t)$ is stabilizing.

The following theorem shows the relationship between the optimality of the LQ problem and the solution of GSARE.

Theorem 3.3. If LQ problem (2.1)-(2.5) is attainable with respect to any $x_{0} \in \mathcal{R}^{n}$, then ( $P$ ) must have an optimal solution $P^{*}$ such that $\mathcal{R}\left(P^{*}\right)=0$.

Proof. Since the LQ problem is attainable, then the optimal value must be of the quadratic form [16]:

$$
\begin{equation*}
\inf _{u \in U_{a d}} J\left(x_{0}, u\right)=x_{0}^{\prime} M x_{0}, \quad \forall x_{0} \in \mathcal{R}^{n} . \tag{3.11}
\end{equation*}
$$

Let $\left(x^{*}(\cdot), u^{*}(\cdot)\right)$ be an optimal pair for the initial state $x_{0}$. Let $T \rightarrow \infty$ and $P(t)=P$ in (2.16), where $P$ is an any feasible solution of $(P)$, then we have

$$
x_{0}^{\prime} P x_{0}+\sum_{t=0}^{\infty} E\left[\begin{array}{l}
x(t)  \tag{3.12}\\
u(t)
\end{array}\right]^{\prime} Q(P)\left[\begin{array}{l}
x(t) \\
u(t)
\end{array}\right]=0
$$

Then, a completion square means

$$
\begin{align*}
J\left(x_{0}, u^{*}\right)= & \sum_{t=0}^{\infty} E\left[x^{* \prime}(t) Q x^{*}(t)+u^{* \prime}(t) R u^{*}(t)\right] \\
= & x_{0}^{\prime} P x_{0}+\sum_{t=0}^{\infty} E\left\{\left[u^{*}(t)-K x^{*}(t)\right]^{\prime}\left(R+B^{\prime} P B+D^{\prime} P D\right)\left[u^{*}(t)-K x^{*}(t)\right]\right.  \tag{3.13}\\
& \left.\quad+x^{* \prime}(t) \mathcal{R}(P) x^{*}(t)\right\},
\end{align*}
$$

where $K=-\left(R+B^{\prime} P B+D^{\prime} P D\right)^{+}\left(B^{\prime} P A+D^{\prime} P C\right)$. Since $P$ is feasible to $(P)$, we have $R+B^{\prime} P B+$ $D^{\prime} P D \geq 0$ and $R(P) \geq 0$ by Lemma 2.7. Then, the inequality

$$
\begin{equation*}
x_{0}^{\prime} M x_{0} \equiv J\left(x_{0}, u^{*}\right) \geq x_{0}^{\prime} P x_{0} \tag{3.14}
\end{equation*}
$$

holds for any feasible solution $P$ to $(P)$. This shows that $M$ must be optimal to $(P)$. Moreover, taking $P=M$ in (3.13) and considering $J\left(x_{0}, u^{*}\right)=x_{0}^{\prime} M x_{0}$, we know that $E x^{* \prime}(t) \mathcal{R}(M) x^{*}(t)=$ 0 for $t \in N$. Setting $t=0$ and noticing that $x_{0}$ is arbitrary, it follows that $\mathcal{R}(M)=0$.

Below, we will show $M$ is a feasible solution of $(P)$. We consider the following SDP and its dual under a perturbation $\varepsilon>0$ :

$$
\begin{align*}
\left(P_{\varepsilon}\right) \max & \operatorname{Tr}(P), \\
\text { s.t. } & {\left[\begin{array}{cc}
-P+A^{\prime} P A+C^{\prime} P C+Q+\varepsilon I & A^{\prime} P B+C^{\prime} P D \\
B^{\prime} P A+D^{\prime} P C & R+\varepsilon I+B^{\prime} P B+D^{\prime} P D
\end{array}\right] \geq 0, }  \tag{3.15}\\
\left(D_{\varepsilon}\right) \min \quad & \operatorname{Tr}[(Q+\varepsilon I) S+(R+\varepsilon I) T], \\
\text { s.t. } & \left\{\begin{aligned}
-S+A S A^{\prime}+C S C^{\prime}+B U A^{\prime}+D U C^{\prime} \\
+A U^{\prime} B^{\prime}+C U^{\prime} D^{\prime}+B T B^{\prime}+D T D^{\prime}+I=0, \\
Z=\left[\begin{array}{ll}
S & U^{\prime} \\
U & T
\end{array}\right] \geq 0 .
\end{aligned}\right. \tag{3.16}
\end{align*}
$$

Obviously, $\left(P_{\varepsilon}\right)$ satisfies the Slater condition because we assume that the feasible set of $(P)$ is nonempty and $\left(D_{\varepsilon}\right)$ also satisfies the Slater condition by the mean square stabilizability assumption and Lemma 2.10. Hence, the complementary optimal solutions exist by Proposition 2.5. Take any dual feasible solution $Z^{0}=\left[\begin{array}{lll}S_{0}^{0} & u^{0^{0}} \\ U^{0} & T^{0}\end{array}\right]$. By the weak duality in conic optimization problems, we have

$$
\begin{equation*}
\operatorname{Tr}(P) \leq \operatorname{Tr}\left[(Q+\varepsilon I) S^{0}+(R+\varepsilon I) T^{0}\right] \tag{3.17}
\end{equation*}
$$

Let $P^{0}$ be a feasible solution of $(P)$, then $P^{0}$ is feasible to $\left(P_{\varepsilon}\right)$ for all $\varepsilon \geq 0$. Similar to Theorem 10 in [7], we conclude that, for any $\varepsilon>0$, there exists the unique optimal solution of $\left(P_{\varepsilon}\right)$, denoted by $P_{\varepsilon}^{*}$, and $P_{\varepsilon}^{*} \geq P^{0}$.

Together with (3.17), we know that $P_{\varepsilon}^{*}$ are contained in a compact set with $0 \leq \varepsilon \leq$ $\varepsilon_{0}\left(\varepsilon_{0}>0\right.$ is a constant). Then, take a convergent subsequence satisfying $\lim _{i \rightarrow \infty} P_{\varepsilon_{i}}^{*}=P_{0}^{*}$ with $\varepsilon_{i} \rightarrow 0$ as $i \rightarrow \infty$. Obviously, $P_{0}^{*}$ is feasible to $(P)$ because the feasible region of $\left(P_{\varepsilon}\right)$ monotonically shrinks as $\varepsilon \downarrow 0$. Define the perturbed cost functional

$$
\begin{equation*}
J_{\varepsilon}\left(x_{0}, u\right)=\sum_{t=0}^{\infty} E\left[x^{\prime}(t) Q_{\varepsilon} x(t)+u^{\prime}(t) R_{\varepsilon} u(t)\right] \tag{3.18}
\end{equation*}
$$

where $R_{\varepsilon}=R+\varepsilon I, Q_{\varepsilon}=Q+\varepsilon I$. By (3.13), we have

$$
\begin{align*}
J_{\varepsilon}\left(x_{0}, u\right)=x_{0}^{\prime} P_{\varepsilon}^{*} x_{0}+\sum_{t=0}^{\infty} E\{ & {\left[u(t)-K_{\varepsilon} x(t)\right]^{\prime}\left(R_{\varepsilon}+D^{\prime} P_{\varepsilon}^{*} D+B^{\prime} P_{\varepsilon}^{*} B\right)\left[u(t)-K_{\varepsilon} x(t)\right] }  \tag{3.19}\\
+ & \left.x^{\prime}(t) \mathcal{R}_{\varepsilon}\left(P_{\varepsilon}^{*}\right) x(t)\right\}
\end{align*}
$$

for any $u \in U_{a d}$, where $K_{\varepsilon}=-\left(R_{\varepsilon}+B^{\prime} P_{\varepsilon}^{*} B+D^{\prime} P_{\varepsilon}^{*} D\right)^{+}\left(B^{\prime} P_{\varepsilon}^{*} A+D^{\prime} P_{\varepsilon}^{*} C\right)$ and $\mathcal{R}_{\varepsilon}\left(P_{\varepsilon}^{*}\right)$ is the form of $\mathcal{R}\left(P_{\varepsilon}^{*}\right)$ with $Q$ and $R$ replaced by $Q_{\varepsilon}$ and $R_{\varepsilon}$. Then, by Theorems 10 and 12 in [7],

$$
\begin{equation*}
\inf _{u \in U_{a d}} J_{\varepsilon}\left(x_{0}, u\right)=x_{0}^{\prime} P_{\varepsilon}^{*} x_{0} \tag{3.20}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
x_{0}^{\prime} P_{\varepsilon_{i}}^{*} x_{0}=\inf _{u \in U_{a d}} J_{\varepsilon_{i}}\left(x_{0}, u\right) \geq \inf _{u \in U_{a d}} J\left(x_{0}, u\right)=x_{0}^{\prime} M x_{0} \tag{3.21}
\end{equation*}
$$

Taking limit, we have $x_{0}^{\prime} P_{0}^{*} x_{0} \geq x_{0}^{\prime} M x_{0}$. On the other hand, $x_{0}^{\prime} M x_{0} \geq x_{0}^{\prime} P_{0}^{*} x_{0}$ because $P_{0}^{*}$ is feasible to $(P)$ and (3.14). So $M=P_{0}^{*}$. The feasibility of $M$ is proved. The proof is completed.

The following theorem studies the converse of Theorem 3.3.
Theorem 3.4. If a feasible solution of $(P), P^{*}$, satisfies $\mathcal{R}\left(P^{*}\right)=0$ and the feedback control $u^{*}(t)=$ $-\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right) x(t)$ is stabilizing, then it must be optimal for LQ problem (2.1)-(2.5).

Proof. For any $u \in U_{a d}$, we have

$$
\begin{equation*}
J\left(x_{0}, u\right)=x_{0}^{\prime} P^{*} x_{0}+\sum_{t=0}^{\infty} E\left[u(t)-K^{*} x(t)\right]^{\prime}\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)\left[u(t)-K^{*} x(t)\right] \tag{3.22}
\end{equation*}
$$

by (3.13) and $\mathcal{R}\left(P^{*}\right)=0$, where $K^{*}=-\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right)$. Because $u^{*}(t)=K^{*} x(t)$ is stabilizing, $u^{*}(t)$ must be optimal.

The following theorem shows we can get the optimal feedback control by SDP dual optimal solution.

Theorem 3.5. Assume that $(P)$ and $(D)$ have complementary optimal solutions $P^{*}$ and $Z^{*}$ with $S^{*}>$ 0 . Then, $\mathcal{R}\left(P^{*}\right)=0$ and LQ problem (2.1)-(2.5) has an attainable optimal feedback control given by $u^{*}(t)=U^{*}\left(S^{*}\right)^{-1} x^{*}(t)$.

Proof. From the proof of Theorem 3.1, we have

$$
A\left(P^{*}\right)=\left[\begin{array}{cc}
I & -K^{* \prime}  \tag{3.23}\\
0 & I
\end{array}\right]\left[\begin{array}{cc}
R\left(P^{*}\right) & 0 \\
0 & R+B^{\prime} P^{*} B+D^{\prime} P^{*} D
\end{array}\right]\left[\begin{array}{cc}
I & 0 \\
-K^{*} & I
\end{array}\right],
$$

where $K^{*}=-\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right)$. By complementary slackness condition $A\left(P^{*}\right) Z^{*}=0$ and the invertibility of $\left[\begin{array}{cc}I & -K^{* \prime} \\ 0 & I\end{array}\right]$, we have

$$
\begin{align*}
& {\left[\begin{array}{cc}
R\left(P^{*}\right) & 0 \\
0 & R+B^{\prime} P^{*} B+D^{\prime} P^{*} D
\end{array}\right]\left[\begin{array}{cc}
I & 0 \\
-K^{*} & I
\end{array}\right]\left[\begin{array}{cc}
S^{*} & U^{* \prime} \\
U^{*} & T^{*}
\end{array}\right]} \\
& \quad=\left[\begin{array}{cc}
\mathcal{R}\left(P^{*}\right) S^{*} & \mathcal{R}\left(P^{*}\right) U^{* \prime} \\
-\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)\left(K^{*} S^{*}-U^{*}\right) & -\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)\left(K^{*} U^{* \prime}-T^{*}\right)
\end{array}\right]  \tag{3.24}\\
& \quad=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] .
\end{align*}
$$

So $\mathcal{R}\left(P^{*}\right) S^{*}=0, \mathcal{R}\left(P^{*}\right) U^{* \prime}=0$. On the other hand, $T^{*}-U^{*}\left(S^{*}\right)^{+} U^{* \prime} \geq 0, S^{*} \geq 0$ and $U^{*}=$ $U^{*} S^{*}\left(S^{*}\right)^{+}$from $Z^{*} \geq 0$ and Lemma 2.7. From the equality constraint in (2.12) and the above results, we have

$$
\begin{align*}
& 0= \mathcal{R}\left(P^{*}\right)\left[-S^{*}+A S^{*} A^{\prime}+C S^{*} C^{\prime}+B U^{*} A^{\prime}+D U^{*} C^{\prime}+A U^{* \prime} B^{\prime}+C U^{* \prime} D^{\prime}\right. \\
&\left.+B T^{*} B^{\prime}+D T^{*} D^{\prime}+I\right] \mathcal{R}\left(P^{*}\right) \\
& \geq \mathcal{R}\left(P^{*}\right)\left[A S^{*} A^{\prime}+C S^{*} C^{\prime}+B U^{*} A^{\prime}+D U^{*} C^{\prime}+A U^{* \prime} B^{\prime}+C U^{* \prime} D^{\prime}\right. \\
&\left.+B U^{*}\left(S^{*}\right)^{+} U^{* \prime} B^{\prime}+D U^{*}\left(S^{*}\right)^{+} U^{* \prime} D^{\prime}+I\right] \mathcal{R}\left(P^{*}\right)  \tag{3.25}\\
&= {\left[\mathcal{R}\left(P^{*}\right)\right]^{2}+\mathcal{R}\left(P^{*}\right)\left[\left(C S^{*}+D U^{*}\right)\left(S^{*}\right)^{+}\left(C S^{*}+D U^{*}\right)^{\prime}\right.} \\
& \geq\left.+\left(A S^{*}+B U^{*}\right)\left(S^{*}\right)^{+}\left(A S^{*}+B U^{*}\right)^{\prime}\right] \mathcal{R}\left(P^{*}\right) \\
& \geq
\end{align*}
$$

The last inequality holds because $\left(S^{*}\right)^{+} \geq 0$ from Lemma 2.8. It follows that $\mathcal{R}\left(P^{*}\right)=0$.
For any $u \in U_{a d}$, by (3.13), we get

$$
\begin{gather*}
J\left(x_{0}, u\right)=x_{0}^{\prime} P x_{0}+\sum_{t=0}^{\infty} E\left\{[u(t)-K x(t)]^{\prime}\left(R+B^{\prime} P B+D^{\prime} P D\right)[u(t)-K x(t)]\right.  \tag{3.26}\\
\left.+x^{\prime}(t) R(P) x(t)\right\},
\end{gather*}
$$

where $P$ is any feasible solution of $(P)$ and $K=-\left(R+B^{\prime} P B+D^{\prime} P D\right)^{+}\left(B^{\prime} P A+D^{\prime} P C\right) . \mathcal{R}(P) \geq 0$ because of the feasibility of $P$. Then,

$$
\begin{equation*}
J\left(x_{0}, u\right) \geq x_{0}^{\prime} P x_{0} \tag{3.27}
\end{equation*}
$$

On the other hand, $u^{*}(t)=U^{*}\left(S^{*}\right)^{-1} x^{*}(t)$ is stabilizing by Theorem 3.2. Let $u(t)=u^{*}(t)$ and $P=P^{*}$ in (3.26), then it follows that

$$
\begin{equation*}
J\left(x_{0}, u^{*}\right)=x_{0}^{\prime} P^{*} x_{0}+\sum_{t=0}^{\infty} E\left[u^{*}(t)-K^{*} x^{*}(t)\right]^{\prime}\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)\left[u^{*}(t)-K^{*} x^{*}(t)\right] \tag{3.28}
\end{equation*}
$$

where $K^{*}=-\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right)$. Below we prove $J\left(x_{0}, u^{*}\right)=x_{0}^{\prime} P^{*} x_{0}$. Applying complementary slackness condition $A\left(P^{*}\right) Z^{*}=0$ and above proof, we have

$$
\left.\begin{array}{l}
{\left[\begin{array}{cc}
\mathcal{R}\left(P^{*}\right) & 0 \\
0 & R+B^{\prime} P^{*} B+D^{\prime} P^{*} D
\end{array}\right]\left[\begin{array}{cc}
I & 0 \\
-K^{*} & I
\end{array}\right]\left[\begin{array}{cc}
S^{*} & U^{* \prime} \\
U^{*} & T^{*}
\end{array}\right]} \\
\quad=\left[\begin{array}{c}
\mathcal{R}\left(P^{*}\right) S^{*} \\
\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) U^{*}+\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right) S^{*} \\
\mathcal{R}\left(P^{*}\right) U^{* \prime}
\end{array}\right. \\
\quad\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) T+\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right) U^{* \prime} \tag{3.29}
\end{array}\right] .
$$

Hence, $\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) U^{*}=-\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right) S^{*}$. Then,

$$
\begin{align*}
{\left[u^{*}(t)-\right.} & \left.K^{*} x^{*}(t)\right]^{\prime}\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)\left[u^{*}(t)-K^{*} x^{*}(t)\right] \\
= & u^{* \prime}(t)\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) u^{*}(t)+2 u^{* \prime}(t)\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right) x^{*}(t) \\
& +x^{* \prime}(t)\left(A^{\prime} P^{*} B+C^{\prime} P^{*} D\right)\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)^{+}\left(B^{\prime} P^{*} A+D^{\prime} P^{*} C\right) x^{*}(t) \\
= & u^{* \prime}(t)\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) u^{*}(t)-2 u^{* \prime}(t)\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) U^{*}\left(S^{*}\right)^{-1} x^{*}(t) \\
& +x^{* \prime}(t)\left(S^{*}\right)^{-1} U^{* \prime}\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right) U^{*}\left(S^{*}\right)^{-1} x^{*}(t) \\
= & {\left[u^{*}(t)-U^{*}\left(S^{*}\right)^{-1} x^{*}(t)\right]^{\prime}\left(R+B^{\prime} P^{*} B+D^{\prime} P^{*} D\right)\left[u^{*}(t)-U^{*}\left(S^{*}\right)^{-1} x^{*}(t)\right] } \\
= & 0 . \tag{3.30}
\end{align*}
$$

It follows from (3.27) and (3.28) that

$$
\begin{equation*}
J\left(x_{0}, u^{*}\right)=x_{0}^{\prime} P^{*} x_{0} \leq J\left(x_{0}, u\right), \quad \forall u \in U_{a d} . \tag{3.31}
\end{equation*}
$$

The optimality of $u^{*}(t)$ is proved.

## 4. Conclusion

In this paper, we use the SDP approach to study discrete-time indefinite stochastic LQ control problem. Some relations are given and are summarized as follows. The condition that LQ problem is attainable can induce that $(P)$ has an optimal solution $P^{*}$ satisfying GSARE (Theorem 3.3). Theorems 3.4 and 3.5 give two suffcient conditions for LQ problem attainability by GSARE and complementary optimal solutions of $(P)$ and $(D)$. Moreover, by dual SDP, we can get stabilized feedback control (Theorem 3.2). What we have obtained can be viewed as a discrete-time version of [9]. Of course, there are many open problems to be solved. For instance, the indefinite LQ problems for Markovian jumps or time-variant system merit further study.

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Research Article

# A Tandem BMAP/G/1 $\rightarrow \bullet / M / N / 0$ Queue with Group Occupation of Servers at the Second Station 

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#### Abstract

We consider a two-stage tandem queue with single-server first station and multiserver second station. Customers arrive to Station 1 according to a batch Markovian arrival process (BMAP). A batch may consist of heterogeneous customers. The type of a customer is determined upon completion of a service at Station 1. The customer's type is classified based on the number of servers required to process the request of the customer at Station 2. If the required number of servers is not available, the customer may leave the system forever or block Station 1 by waiting for the required number of servers. We determine the stationary distribution of the system states at embedded epochs and derive the Laplace-Stieltjes transform of the sojourn time distribution. Some key performance measures are calculated, and illustrative numerical results are presented.


## 1. Introduction

Queueing networks are widely used in capacity planning and performance evaluation of computer and communication systems, service centers, and manufacturing systems among several others. Some examples of their application to real systems can be found in [1]. Tandem queues can be used for modeling real-life two-node networks as well as for the validation of general decomposition algorithms in networks (see, e.g., $[2,3]$ ). Thus, tandem queueing systems have found much interest in the literature. An extensive survey of early papers on tandem queues can be seen in [4]. Most of these papers are devoted to exponential queueing models. Over the last two decades or so, the efforts of many investigators in tandem queues were in weakening the distribution assumptions on the service times as well as on the arrivals. In particular, the arrival process should be able to capture any correlation and burstiness that are commonly seen in the traffic of modern communication networks [3]. Such an arrival
process was introduced in [5] and ever since this process is referred to as a batch Markovian arrival process (BMAP). In this paper, we deal with a tandem queue under the assumption that the customers arrive according to a BMAP.

Tandem queues with the BMAP input were considered in [6-10]. The papers $[6,7,10$ ] are devoted to the MAP/PH/1 $\rightarrow \bullet / \mathrm{G} / 1$ system with blocking. In [8], the tandem queues $\mathrm{BMAP} / \mathrm{G} / 1 / N \rightarrow \bullet / \mathrm{PH} / 1 / \mathrm{M}-1$ with losses are studied. The tandem-queue of the BMAP/G/1 $\rightarrow \bullet / \mathrm{PH} / 1 / M-1$ type with losses and feedback has been studied in [9].

In the present paper, we consider a $\mathrm{BMAP} / G / 1 \rightarrow \bullet / M / N / 0$ tandem queueing system where the (possibly heterogeneous) customers arrive in batches of random sizes to Station 1. Here the customers receive service individually, and upon completion of a service the customer's type is determined. This type identification is necessary to determine the nature of service, if any, offered at Station 2. The customer's type is classified based on the number of servers (resources) required to process the request of the customer. The simultaneous initiation or occupation of several servers to a customer's request is typical for the so-called nonelastic traffic in communication networks. If the required number of servers is not available at that instant of the request, the customer either leaves the system forever, or awaits until the requirement is met through the release of the sufficient number of servers. In the latter case, Station 1 will be blocked.

Possible applications of the tandem queue under study lie in the modeling of the distributed server application or web server application, see, for example, [11]. Station 1 is interpreted as an authentication or an access step while Station 2 represents the computing step or data base server if the processing of a job is produced by several parallel threads. This tandem queue can model also multiaddress transmission of information. Station 1 is interpreted as a transmission channel while Station 2 regulates the transmission rate by providing necessary transmission windows (timers that are switched on at the moment of a message transmission and switched off when the receipt of this message is acknowledged or time-out expires). The performance evaluation of wireless IP networks providing heterogeneous multimedia services with different QoS demands, (see [12, Chapter 8]), is the other possible application of the model under study.

In this paper, we derive the stability condition of the model under study, and briefly touch calculation of the stationary distribution of the system states at the service completion epochs at Station 1 and calculation of the system performance measures. Furthermore, we derive the Laplace-Stieltjes transform of the virtual and the actual sojourn time distributions at both stations and in the whole system. The procedures for calculation of the moments of the virtual sojourn time distribution and the mean actual sojourn time are discussed. Some numerical results illustrating the behavior of the system characteristics are presented. The problem of optimal design is numerically investigated.

To the best of our knowledge, the results of our paper are novel even for the case of homogeneous customers. The most important and valuable, from the mathematical point of view, result concerns the sojourn time distribution. Previously, the sojourn time distribution in tandem queues with MAP input was considered only in $[13,14]$. There, the service time distribution at both the single-server stations is of phase type which allows the authors to model the sojourn time as the time until absorption in suitably defined quasi-birth-and-death processes and continuous-time Markov chains. Because we assume general service time distribution at Station 1, we need to analyze a more complicated stochastic process.

The rest of the paper is organized as follows. In Section 2, the mathematical model is described. In Section 3, the results concerning the stationary distribution of the embedded Markov chain in Station 1 service completion epochs are presented. In Section 4, we focus
on the analysis of the virtual and actual sojourn time distributions and their moments. In Section 5, the numerical results are presented. The paper is concluded with Section 6. Appendices contain auxiliary results, proofs, and formulas useful for computations.

## 2. The Mathematical Model

We consider a tandem queue consisting of two stations, say, Station 1 and Station 2. We assume that there is no buffer between the two stations. Station 1 is represented by the BMAP/G/1 queue. That is, the arrivals to Station 1 are described by a BMAP. The BMAP is defined by the underlying process $\mathcal{v}_{t}, t \geq 0$, which is an irreducible continuous time Markov chain with state space $\{0, \ldots, W\}$, and with the matrix generating function $D(z)=$ $\sum_{k=0}^{\infty} D_{k} z^{k},|z| \leq 1$. Arrivals occur only at epochs of the jumps in the underlying process $\nu_{t}, t \geq 0$. The intensities of the transitions of the process $\nu_{t}$ accompanied by a batch of size $k$ are defined by the matrices $D_{k}, k \geq 0$. The matrix $D(1)$ is the infinitesimal generator of the process $\nu_{t}$. The stationary distribution vector $\boldsymbol{\theta}$ of this process satisfies the equations $\boldsymbol{\theta} D(1)=\mathbf{0}, \boldsymbol{\theta} \mathbf{e}=1$, where $\mathbf{e}$ is a column vector consisting of $1^{\prime} \mathrm{s}$, and $\mathbf{0}$ is a row vector of 0 . s .

The average intensity $\lambda$ (fundamental rate) of the BMAP is given by $\lambda=\left.\boldsymbol{\theta} D^{\prime}(z)\right|_{z=1} \mathbf{e}$. We assume that $\lambda<\infty$. The average intensity $\lambda_{b}$ of group arrivals is defined by $\lambda_{b}=\boldsymbol{\theta}\left(-D_{0}\right) \mathbf{e}$. The coefficient of variation, $c_{\mathrm{var}}$, of intervals between successive group arrivals is defined by $c_{\text {var }}^{2}=2 \lambda_{b} \boldsymbol{\theta}\left(-D_{0}\right)^{-1} \mathbf{e}-1$. The coefficient of correlation $c_{\text {cor }}$ of the successive intervals between group arrivals is given by $c_{\text {cor }}=\left(\lambda_{b} \boldsymbol{\theta}\left(-D_{0}\right)^{-1}\left(D(1)-D_{0}\right)\left(-D_{0}\right)^{-1} \mathbf{e}-1\right) / c_{\text {var }}^{2}$. For more information about the BMAP and related research see, for example, [5, 15].

All arriving customers enter into Station 1. The successive service times of customers at Station 1 are independent random variables with general distribution $B(t)$, Laplace-Stieltjes transform $\beta(s)=\int_{0}^{\infty} e^{-s t} d B(t)$, and finite first moment $b_{1}=\int_{0}^{\infty} t d B(t)$.

After receiving a service at Station 1, the customer proceeds to Station 2. At this station, there are $N$ identical servers. Each of these servers offers services that are exponentially distributed with parameter $\mu$. Customers are heterogeneous with respect to the number of servers that are required to process a customer at Station 2 . With probability $q_{m}, q_{m} \geq 0, m=$ $\overline{0, N}, \sum_{m=0}^{N} q_{m}=1$, the customer will require exactly $m$ servers to provide a service at Station 2 and will be called type $m$ customer. Here and in the sequel, notation such as $m=\overline{0, N}$, means that $m$ assumes values from the set $\{0,1, \ldots, N\}$. Note that customers who are all in the same batch (at the time of arriving) may belong to different types after receiving service at Station 1. Type 0 customer leaves the system for good after the service at Station 1 . We assume that $q_{0} \neq 1$. Otherwise, the queue under consideration will be reduced to the BMAP/G/1 queue which has been studied extensively.

If the customer is of type $m, m=\overline{1, N}$, and the required number of servers is available, the customer's service will begin immediately. Each of these $m$ servers processes the customer's request independently of the others, and furthermore any server who becomes free after completing his/her share of the processing will be available to process waiting or future customers' requests.

If the required number of servers is not available, with probability $\gamma, 0 \leq \gamma \leq 1$, the customer will choose to leave the system for good and with probability $1-\gamma$ will decide to wait until the required number of servers is available. In the latter case, the customer will block Station 1 since we assume that there is no buffer between the two stations. Such an assumption of blocking and loss will allow us to unify these two classes of models which are studied separately in the literature.

In the following, we are interested in the steady state analysis of the model under study. For further use in the sequel, we introduce the following notation:
(i) $I$ is an identity matrix of appropriate dimension;
(ii) $\otimes$ and $\oplus$ are symbols of the Kronecker product and sum of matrices;
(iii) $\tilde{D}_{k}=I_{N+1} \otimes D_{k}, k \geq 0, \tilde{D}(z)=\sum_{k=0}^{\infty} \tilde{D}_{k} z^{k},|z| \leq 1$;
(iv) $P(j, t), j \geq 0$, is a matrix function defined by the expansion $\sum_{j=0}^{\infty} P(j, t) z^{j}=e^{D(z) t}$;
(v) $F(t)=\left(F_{r, r^{\prime}}(t)\right)_{r, r^{\prime}=\overline{0, N}}$, where $F_{r, r^{\prime}}(t)=0$ for $r \leq r^{\prime}$ and for $r>r^{\prime}, F_{r, r^{\prime}}(t)$ is the generalized Erlang distribution function with the Laplace-Stieltjes transform $f_{r, r^{\prime}}(s)=\prod_{l=r^{\prime}+1}^{r} l \mu(l \mu+s)^{-1} ;$
(vi) $Q_{m}, m=\overline{1,4}$, are square matrices: $Q_{2}=\operatorname{diag}\left\{\sum_{m=N-r+1}^{N} q_{m}, r=\overline{0, N}\right\}$,

$$
Q_{1}=\left(\begin{array}{cccc}
q_{0} & q_{1} & \ldots & q_{N}  \tag{2.1}\\
0 & q_{0} & \ldots & q_{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & q_{0}
\end{array}\right), \quad Q_{3}=\left(\begin{array}{cccc}
0 & \ldots & 0 & q_{N} \\
0 & \ldots & 0 & q_{N-1} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & q_{0}
\end{array}\right), \quad Q_{4}=\left(\begin{array}{ccccc}
q_{0} & q_{1} & \ldots & q_{N-1} & q_{N} \\
q_{0} & q_{1} & \ldots & q_{N-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{0} & 0 & \ldots & 0 & 0
\end{array}\right)
$$

(vii) $\tilde{Q}_{m}=Q_{m} \otimes I_{\bar{W}}, m=\overline{1,3}, \bar{W}=W+1$;
(viii) $\widehat{Q}=\widetilde{Q}_{1}+\gamma \widetilde{Q}_{2}+(1-\gamma) \int_{0}^{\infty}\left(d F(t) \otimes e^{D_{0} t}\right) \widetilde{Q}_{3}, Q=Q_{1}+\gamma Q_{2}+(1-\gamma) E Q_{3} ;$

$$
E=\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0  \tag{2.2}\\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0
\end{array}\right), \widehat{I}=\left(\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right), \tilde{I}=I-\widehat{I}, \widehat{\mathbf{e}}=(1,0, \ldots, 0)
$$

## 3. The Stationary Distribution of the Embedded Markov Chain

Let $t_{n}$ denote the time of the $n$th service completion at Station 1 . Consider the process $\xi_{n}=$ $\left\{i_{n}, r_{n}, v_{n}\right\}, n \geq 1$, where $i_{n}, i_{n} \geq 0$, is the number of customers at Station 1 (not counting the blocked customer, if any) at epoch $t_{n}+0 ; r_{n}, r_{n}=\overline{0, N}$, is the number of busy servers at Station 2 at epoch $t_{n}-0 ; v_{n}, v_{n}=\overline{0, W}$, is the state of the BMAP at epoch $t_{n}$.

It is easy to verify that the process $\xi_{n}=\left\{i_{n}, r_{n}, v_{n}\right\}, n \geq 1$, is a Markov chain. Enumerating the states of this Markov chain in lexicographic order, and denoting by $P_{l, k}, l, k \geq 0$, the square matrix of order $(W+1)(N+1)$ governing the transition probabilities of the chain from the set of states $\{l, \cdot, \cdot\}$ to the set $\{k, \cdot, \cdot\}$, the following lemma gives the entries of the transition probability matrix of the Markov chain $\xi_{n}$.

Lemma 3.1. The transition probability matrix of the chain $\xi_{n}, n \geq 1$, has the following block structure:

$$
P=\left(P_{l, k}\right)_{l, k \geq 0}=\left(\begin{array}{ccccc}
C_{0} & C_{1} & C_{2} & C_{3} & \cdots  \tag{3.1}\\
Y_{0} & \Upsilon_{1} & \Upsilon_{2} & \Upsilon_{3} & \cdots \\
0 & \Upsilon_{0} & \Upsilon_{1} & \Upsilon_{2} & \cdots \\
0 & 0 & \Upsilon_{0} & \Upsilon_{1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right),
$$

where

$$
\begin{gather*}
C_{i}=\sum_{k=1}^{i+1}\left[-\widehat{Q}\left(\Delta \oplus D_{0}\right)^{-1} \tilde{D}_{k}+(1-\gamma) F_{k} \widetilde{Q}_{3}\right] \Omega_{i-k+1,} \\
Y_{i}=\left(\widetilde{Q}_{1}+\gamma \widetilde{Q}_{2}\right) \Omega_{i}+(1-\gamma) \sum_{k=0}^{i} F_{k} \widetilde{Q}_{3} \Omega_{i-k} \\
\Omega_{j}=\int_{0}^{\infty} e^{\Delta t} \otimes P(j, t) d B(t), \quad F_{j}=\int_{0}^{\infty} d F(t) \otimes P(j, t), \quad j \geq 0,  \tag{3.2}\\
\Delta=\left(\begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
\mu & -\mu & 0 & \cdots & 0 & 0 \\
0 & 2 \mu & -2 \mu & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & N \mu & -N \mu
\end{array}\right) .
\end{gather*}
$$

Proof. First, we write the transition probability matrices $C_{i}, Y_{i}$ in block forms as $C_{i}=$ $\left(C_{i}^{\left(r, r^{\prime}\right)}\right)_{r, r^{\prime}=\overline{0, N}}, Y_{i}=\left(Y_{i}^{\left(r, r^{\prime}\right)}\right)_{r, r^{\prime}=\overline{0, N}}$, where the blocks $C_{i}^{\left(r, r^{\prime}\right)}, Y_{i}^{\left(r, r^{\prime}\right)}$ correspond to the transitions of the number of busy servers from $r$ to $r^{\prime}$ at Station 2.

Denote by $\delta_{r, r^{\prime}}(t)$ the probability that during the time interval of the length $t$ the number of busy servers at Station 2 decreases from $r$ to $r^{\prime}$ conditioned on the fact that none arrive from Station 1.

For use in the sequel, we register the following probabilistic interpretations of the matrices.

The $\left(v, v^{\prime}\right)$ th entry of the matrix $P(j, t)$ gives the probability that $j$ customers arrive in the BMAP during the interval $(0, t]$ and the state of the BMAP at epoch $t$ is $v^{\prime}$ given $v_{0}=v$.

The $\left(v, v^{\prime}\right)$ th entry of the matrix $\int_{0}^{\infty} \delta_{r, r^{\prime}}(t) P(j, t) d B(t)$ gives the probability that during the service time of a customer at Station 1, exactly $j$ customers arrive, the number of busy servers at Station 2 decreases from $r$ to $r^{\prime}$ and the BMAP has changed from $v$ to $\mathcal{v}^{\prime}$.

The $\left(\mathcal{v}, \nu^{\prime}\right)$ th entry of the matrix $\int_{0}^{\infty} \delta_{r, r^{\prime}}(t) e^{D_{0} t} D_{k} d t$ gives the probability that at an arbitrary time instant with the number of busy servers at Station 2 equal to $r$, and the BMAP in state $\nu$, the first batch to arrive is of size $k$; soon after that instant, the number of busy servers at Station 2 equals $r^{\prime}$ and the BMAP is in state $\nu^{\prime}$.
$F_{r, r^{\prime}}(t)$ is the distribution function of the time interval during which the number of busy servers at Station 2 decreases from $r$ to $r^{\prime}$ conditioned on the fact that none arrive to this station. Then the $\left(v, v^{\prime}\right)$ th entry of the matrix $\int_{0}^{\infty} P(j, t) d F_{r, r^{\prime}}(t)$ defines the probability that exactly $j$ customers arrive with the BMAP moving from $v$ to $v^{\prime}$ and that the number of busy servers at Station 2 decreases from $r$ to $r^{\prime}$ during that time interval.

From the above probabilistic interpretations, analyzing the one-step transitions of the chain $\xi_{n}$ with a careful analysis of the service completion epochs, in which the customers may get lost due to lack of servers or wait (and thus block Station 1) until enough servers are available, we obtain the following expressions for the matrices $C_{i}^{\left(r, r^{\prime}\right)}, Y_{i}^{\left(r, r^{\prime}\right)}, i \geq 0$ :

$$
\begin{align*}
C_{i}^{\left(r, r^{\prime}\right)}= & \sum_{m=0}^{N-r} q_{m} \sum_{l=r^{\prime}}^{r+m} \int_{0}^{\infty} \delta_{r+m, l}(t) e^{D_{0} t} d t \sum_{k=1}^{i+1} D_{k} \int_{0}^{\infty} \delta_{l, r^{\prime}}(t) P(i-k+1, t) d B(t) \\
& +\gamma \sum_{m=N-r+1}^{N} q_{m} \sum_{l=r^{\prime}}^{r} \int_{0}^{\infty} \delta_{r, l}(t) e^{D_{0} t} d t \sum_{k=1}^{i+1} D_{k} \int_{0}^{\infty} \delta_{l, r^{\prime}}(t) P(i-k+1, t) d B(t) \\
& +(1-\gamma) \sum_{m=N-r+1}^{N} q_{m}\left[\int_{0}^{\infty} e^{D_{0} t} d F_{r, N-m}(t) \sum_{l=r^{\prime}}^{N} \int_{0}^{\infty} \delta_{N, l}(t) e^{D_{0} t} d t\right. \\
& \times \sum_{k=1}^{i+1} D_{k} \int_{0}^{\infty} \delta_{l, r^{\prime}}(t) P(i-k+1, t) d B(t)+\sum_{k=1}^{i+1} \int_{0}^{\infty} P(k, t) d F_{r, N-m}(t) \\
Y_{i}^{\left(r, r^{\prime}\right)}= & \sum_{m=0}^{N-r} q_{m} \int_{0}^{\infty} P(i, t) \delta_{r+m, r^{\prime}}^{\infty}(t) d B(t) \\
& \left.+\sum_{0, r^{\prime}}(t) P(i-k+1, t) d B(t)\right], \\
& \sum_{m=N-r+1}^{N} q_{m}\left[\gamma \int_{0}^{\infty} \delta_{r, r^{\prime}}(t) P(i, t) d B(t)\right. \\
& \left.+(1-\gamma) \sum_{k=0}^{i} \int_{0}^{\infty} P(k, t) d F_{r, N-m}(t) \int_{0}^{\infty} \delta_{N, r^{\prime}}(t) P(i-k, t) d B(t)\right] . \tag{3.3}
\end{align*}
$$

In order to arrive at equations (3.2) from (3.3), we use the matrix notations introduced above and the relation: $\left(\delta_{r, r^{\prime}}(t)\right)_{r, r^{\prime}=\overline{0, N}}=e^{\Delta t}$, which follows from the fact that under the case when none arrive to Station 2 , the process $r_{t}$ governing the number of busy servers at this station is Markovian with generator $\Delta$.

It is easy to see that the Markov chain $\xi_{n}$ belongs to the class of $M / G / 1$ type Markov chains, see [16]. We can use this fact to derive the ergodicity condition and calculate the stationary distribution of the chain.

Let $C(z)=\sum_{i=0}^{\infty} C_{i} z^{i}, Y(z)=\sum_{i=0}^{\infty} Y_{i} z^{i},|z| \leq 1$, be the generating functions of the transition probability matrices $C_{i}$ and $Y_{i, i} \geq 0$.

Corollary 3.2. The matrix generating functions $C(z), Y(z)$ can be written as

$$
\begin{gather*}
C(z)=\frac{1}{z}\left[-\widehat{Q}\left(\Delta \oplus D_{0}\right)^{-1}\left(\tilde{D}(z)-\tilde{D}_{0}\right)+(1-\gamma)\left(F(z)-F_{0}\right) \tilde{Q}_{3}\right] \Omega(z)  \tag{3.4}\\
Y(z)=\left[\tilde{Q}_{1}+\gamma \tilde{Q}_{2}+(1-\gamma) F(z) \tilde{Q}_{3}\right] \Omega(z) \tag{3.5}
\end{gather*}
$$

where

$$
\begin{equation*}
\Omega(z)=\sum_{n=0}^{\infty} \Omega_{n} z^{n}=\int_{0}^{\infty} e^{\Delta t} \otimes e^{D(z) t} d B(t), \quad F(z)=\sum_{n=0}^{\infty} F_{n} z^{n}=\int_{0}^{\infty} d F(t) \otimes e^{D(z) t} \tag{3.6}
\end{equation*}
$$

Theorem 3.3. The necessary and sufficient condition for ergodicity of the Markov chain $\xi_{n}, n \geq 1$, is the fulfillment of the inequality

$$
\begin{equation*}
\rho=\lambda\left[b_{1}+(1-\gamma) \sum_{r=1}^{N} \vartheta_{r} \sum_{m=N-r+1}^{N} q_{m} \sum_{l=N-m+1}^{r}(l \mu)^{-1}\right]<1 . \tag{3.7}
\end{equation*}
$$

Here $\left(\vartheta_{1}, \ldots, \vartheta_{N}\right)$ is a part of the vector $\vartheta=\left(\vartheta_{0}, \ldots, \vartheta_{N}\right)$, which is the unique solution to the system

$$
\begin{equation*}
\vartheta Q B^{*}(0)=\vartheta, \quad \vartheta \mathbf{e}=1, \tag{3.8}
\end{equation*}
$$

where $B^{*}(s)=\int_{0}^{\infty} e^{-s t} e^{\Delta t} d B(t)$.
Proof. It can be verified that the matrix $Y(1)$ is irreducible. Hence, from [16], the necessary and sufficient condition for ergodicity of the chain $\xi_{n}$ is the fulfillment of the inequality

$$
\begin{equation*}
\mathbf{x} Y^{\prime}(1) \mathbf{e}<1 \tag{3.9}
\end{equation*}
$$

where the vector $\mathbf{x}$ is the unique solution of the system

$$
\begin{equation*}
\mathbf{x} Y(1)=\mathbf{x}, \quad \mathbf{x e}=1 \tag{3.10}
\end{equation*}
$$

The theorem will be proven if we show that inequality (3.9) is equivalent to inequality (3.7).

Let the vector $\mathbf{x}$ be of the form

$$
\begin{equation*}
\mathbf{x}=\vartheta \otimes \boldsymbol{\theta} \tag{3.11}
\end{equation*}
$$

By the direct substitution into the system (3.10), where $Y(1)$ is calculated using (3.5), we verify that such a vector provides the unique solution of this system. Differentiating (3.5) at

Table 1: The value of the system load $\rho$ for different value of the mean service time and service time variation.

|  | $c_{\text {var }}=0$ | $c_{\text {var }}=1$ | $c_{\text {var }}=5$ | $c_{\text {var }}=9.95$ |
| :--- | :---: | :---: | :---: | :---: |
| $b_{1}=0.1$ | 0.42754 | 0.43016 | 0.45557 | 0.47940 |
| $b_{1}=0.2$ | 0.47460 | 0.48228 | 0.53677 | 0.57784 |
| $b_{1}=0.3$ | 0.53173 | 0.54515 | 0.62243 | 0.67725 |
| $b_{1}=0.4$ | 0.59698 | 0.61590 | 0.71010 | 0.77695 |
| $b_{1}=0.5$ | 0.66879 | 0.69247 | 0.79902 | 0.87678 |
| $b_{1}=0.6$ | 0.74579 | 0.77339 | 0.88883 | 0.97667 |
| $b_{1}=0.7$ | 0.82687 | 0.85763 | 0.97937 | $\mathbf{1 . 0 7 6 6 3}$ |
| $b_{1}=0.8$ | 0.9116 | 0.94440 | $\mathbf{1 . 0 7 0 5 1}$ | $\mathbf{1 . 1 7 6 5 5}$ |

the point $z=1$ and substituting the resulting expression for $Y^{\prime}(1)$ and the vector $\mathbf{x}$ of form (3.11) into the inequality (3.9), we get

$$
\begin{equation*}
\rho=\lambda\left[b_{1}+(1-\gamma) \vartheta \int_{0}^{\infty} t d F(t) Q_{3} \mathbf{e}\right]<1 \tag{3.12}
\end{equation*}
$$

The stated expression in (3.7) follows from (3.12) and the expression for $\int_{0}^{\infty} t d F(t)$ given in (C.3)-(C.4) (see Appendix C).

Remark 3.4. The inequality (3.7) is intuitively clear on noting that the vector $\vartheta$ gives the stationary distribution of the number of busy servers at Station 2 at the service completion epochs at Station 1 given the latter station works non-stop. Then $(1-\gamma) \sum_{r=1}^{N} \vartheta_{r}$ $\sum_{m=N-r+1}^{N} q_{m} \sum_{l=N-m+1}^{r}(l \mu)^{-1}$ defines the average blocking time of Station 1 under overload condition and $\rho$ is the system load.

Remark 3.5. In a majority of queueing systems, the system load depends only on the first moment of the service time distribution. In the model under study, the value of $\rho$ depends not only on the first moment $b_{1}$ of the service time distribution at Station 1, but also on the shape of this distribution. In particular, $\rho$ depends on variance of the service time. This fact is illustrated in Table 1 in Section 5.

In what follows, we assume that the inequality (3.7) holds true.
Denote the stationary state probabilities of the Markov chain $\xi_{n}=\left\{i_{n}, r_{n}, v_{n}\right\}$ by $\pi(i, r, v), i \geq 0, r=\overline{0, N}, v=\overline{0, W}$. Introduce the notation for the row vectors of these probabilities

$$
\begin{equation*}
\pi(i, r)=(\pi(i, r, 0), \pi(i, r, 1), \ldots, \pi(i, r, W)), \quad \pi_{i}=(\pi(i, 0), \pi(i, 1), \ldots, \pi(i, N)), \quad i \geq 0 \tag{3.13}
\end{equation*}
$$

Let also $\Pi(z)=\sum_{i=0}^{\infty} \pi_{i} z^{i},|z| \leq 1$, be the vector generating function of vectors $\pi_{i}, i \geq 0$. To compute these vectors as well as the vectors $\Pi(1)$ and $\Pi^{\prime}(1)$, known algorithms, see, for example, [16], can be applied.

Once the stationary distribution has been computed, we can calculate some key performance measures of the system as follows.
(i) The mean number of customers at Station 1 at the service completion epochs $L=$ $\Pi^{\prime}(1) \mathbf{e}$.
(ii) The vector of the stationary distribution of the number of busy servers at Station 2 at the service completion epoch at Station $1: \mathbf{r}=\Pi(1)\left(I_{N+1} \otimes \mathbf{e}_{\bar{W}}\right)$.
(iii) The mean number of busy servers at Station 2 at the service completion epoch at Station 1

$$
\begin{equation*}
N_{\text {busy }}=\mathbf{r} \operatorname{diag}\{r, r=\overline{0, N}\} \mathbf{e} . \tag{3.14}
\end{equation*}
$$

(iv) The probability that an arbitrary customer leaves the system or causes the blocking of the server at Station 1

$$
\begin{equation*}
P_{\text {loss }}=\gamma \Pi(1) \tilde{Q}_{2} \mathbf{e}, \quad P_{\text {block }}=(1-\gamma) \Pi(1) \tilde{Q}_{2} \mathbf{e} . \tag{3.15}
\end{equation*}
$$

(v) The probability that the server of Station 1 is idle at an arbitrary time $p_{\text {idle }}=$ $\tau^{-1} \boldsymbol{\pi}_{0} \widehat{Q}\left(-\widetilde{D}_{0}\right)^{-1} \mathbf{e}$, where $\tau$ is the mean interdeparture time at Station 1,

$$
\begin{equation*}
\tau=b_{1}+\pi_{0} \widehat{Q}\left(-\tilde{D}_{0}\right)^{-1} \mathbf{e}-(1-\gamma) \Pi(1)\left(I_{N+1} \otimes \mathbf{e}\right) F^{(1)} Q_{3} \mathbf{e} \tag{3.16}
\end{equation*}
$$

the matrix $F^{(1)}$ is defined by formula (C.3) below.
(vi) The probability that the server of Station 1 processes a customer at an arbitrary time $p_{\text {serve }}=\tau^{-1} b_{1}$.
(vii) The probability that the server of Station 1 is blocked at an arbitrary time $p_{\text {block }}=$ $1-p_{\text {idle }}-p_{\text {serve }}$.

## 4. Stationary Distribution of the Sojourn Time

### 4.1. The Virtual Sojourn Time

The virtual sojourn time in the system consists of the virtual sojourn time at Station 1 and the sojourn time at Station 2. We assume that customers are served according to FIFO (first-in-first-out) discipline.

For use in the sequel, we define the generalized service time of an arbitrary customer as the service time of this customer by the first server and the possible blocking time of the server by the previous customer.

### 4.1.1. The Virtual Sojourn Time at Station 1

The virtual sojourn time at Station 1 consists of (i) the residual time from an arbitrary time instant (associated with virtual customer arrival) to the next service completion epoch at Station 1 (ii) the generalized service times of customers staying in the queue at an arbitrary time, and (iii) the generalized service time of the virtual customer.

First, we study the residual time. To this end, we consider the process $X_{t}=\left\{i_{t}, m_{t}, r_{t}\right.$, $\left.v_{t}, \tilde{v}_{t}\right\}, t \geq 0$, whose components are defined as follows: $i_{t}$ is the number of customers in Station 1 (including the blocked customer, if any), $m_{t}$ takes values $0,1,2$, respectively, based on the server at Station 1 is idle, busy, or blocked at time $t, v_{t}$ is the state of the BMAP, $r_{t}$ is the number of busy servers at Station 2 just before the service completion epoch following the time $t, \tilde{v}_{t}$ is the residual time from $t$ to that service completion epoch.

Using the definition of semiregenerative processes given in [17], it can be verified that the process $\chi_{t}$ is a semi-regenerative one with the embedded Markov renewal process $\left\{\xi_{n}, t_{n}\right\}$, $n \geq 1$. Let

$$
\begin{align*}
& \tilde{V}(i, m, r, v, x) \\
& =\lim _{t \rightarrow \infty} P\left\{i_{t}=i, m_{t}=m, r_{t}=r, v_{t}=v, \tilde{v}_{t}<x\right\}, \quad i \geq 0, m=\overline{0,2}, r=\overline{0, N}, v=\overline{0, W}, x \geq 0, \tag{4.1}
\end{align*}
$$

be the stationary distribution of the process $X_{t}, t \geq 0$.
From [17], the limits in (4.1) exist if the process $\left\{\xi_{n}, t_{n}\right\}, n \geq 1$, is irreducible aperiodic recurrent and the value $\tau$ of the mean inter-departure time at Station 1 (given by (3.16)) is finite. All these conditions hold if inequality (3.7) is satisfied.

Let $\tilde{\mathbf{V}}(i, m, x)$ be the row vector of the steady state probabilities $\tilde{V}(i, m, r, v, x)$ arranged according to the lexicographic order of the components $(r, v)$, and let $\tilde{\mathbf{v}}(i, m, s)$ be the corresponding vector of the Laplace-Stieltjes transforms, that is, let $\tilde{\mathbf{v}}(i, m, s)=\int_{0}^{\infty} e^{-s x} d \tilde{\mathbf{V}}(i$, $m, x), i \geq 0, m=\overline{0,2}$.

Lemma 4.1. The vector Laplace-Stieltjes transforms $\tilde{\mathbf{v}}(i, m, s)$ are calculated by

$$
\begin{gather*}
\tilde{\mathbf{v}}(0,1, s)=\mathbf{0}, \tilde{\mathbf{v}}(i, 0, s)=0, \quad i>0, \quad \tilde{\mathbf{v}}(0,2, s)=\mathbf{0}  \tag{4.2}\\
\tilde{\mathbf{v}}(0,0, s)=-\tau^{-1} \boldsymbol{\pi}_{0} \widehat{Q}\left(\Delta \oplus D_{0}\right)^{-1}\left[B^{*}(s) \otimes I_{\bar{W}}\right]  \tag{4.3}\\
\widetilde{\mathbf{v}}(i, 1, s)=\tau^{-1}\left\{\pi_{0} \sum_{k=1}^{i}\left[-\widehat{Q}\left(\Delta \oplus D_{0}\right)^{-1} \tilde{D}_{k}+(1-\gamma) F_{k} \tilde{Q}_{3}\right]\right. \\
\times \int_{0}^{\infty}\left(e^{\Delta u} \otimes I_{\bar{W}}\right) \int_{0}^{u} I_{N+1} \otimes P(i-k, y) e^{-s(u-y)} d y d B(u) \\
+\sum_{j=1}^{i} \pi_{j}\left[\left(\widetilde{Q}_{1}+\gamma \tilde{Q}_{2}\right) \int_{0}^{\infty}\left(e^{\Delta u} \otimes I_{\bar{W}}\right) \int_{0}^{u} I_{N+1} \otimes P(i-j, y) e^{-s(u-y)} d y d B(u)\right. \\
\left.\left.+(1-\gamma) \sum_{k=0}^{i-j} F_{k} \widetilde{Q}_{3} \int_{0}^{\infty} e^{\Delta u} \otimes P(i-k-j, y) e^{-s(u-y)} d y d B(u)\right]\right\} \tag{4.4}
\end{gather*}
$$

$$
\begin{align*}
\tilde{\mathbf{v}}(i, 2, s)= & \tau^{-1}(1-\gamma) \sum_{j=0}^{i-1} \pi_{j}\left(\int_{0}^{\infty} d F(u) \otimes I_{\bar{W}}\right) \tilde{Q}_{3}  \tag{4.5}\\
& \times \int_{0}^{u} e^{-s(u-y)}\left[I_{N+1} \otimes P(i-j-1, y)\right] d y\left[B^{*}(s) \otimes I_{\bar{W}}\right], \quad i>0 .
\end{align*}
$$

Proof. Let $\kappa_{j}^{(i, m, x, t)}\left(r, v ; r^{\prime}, v^{\prime}\right)$ denote the conditional probability that, given time 0 is an instant of the service completion at Station 1 and the embedded Markov chain $\xi_{n}$ is in the state $(j, r, v)$ at that time, the next service completion epoch at Station 1 occurs later than $t$, the discrete components of the process $X_{t}$ take values $\left(i, m, r^{\prime}, v^{\prime}\right)$ at time $t$ and the continuous-time component $\tilde{v}_{t}<x$.

Let us arrange the probabilities $\kappa_{j}^{(i, m, x, t)}\left(r, v ; r^{\prime}, \nu^{\prime}\right)$, for fixed values $i, j, m$, according to the lexicographic order of the states $\left(r, v ; r^{\prime}, \nu^{\prime}\right)$ and form the square matrices

$$
\begin{equation*}
\tilde{K}_{j}(i, m, x, t)=\left(\kappa_{j}^{(i, m, x, t)}\left(r, v ; r^{\prime}, v^{\prime}\right)\right)_{v, v^{\prime}=\overline{0, W} ; r, r^{\prime}=\overline{0, N}} . \tag{4.6}
\end{equation*}
$$

Then, using the ergodic theorem for semi-regenerative processes, (see [17, Theorem 6.12$]$ ), the probability vectors $\tilde{\mathbf{V}}(i, m, x)$ can be related to the stationary distribution $\boldsymbol{\pi}_{j}$, $j \geq 0$, of the embedded Markov chain $\xi_{n}, n \geq 1$, by

$$
\begin{equation*}
\tilde{\mathbf{V}}(i, m, x)=\tau^{-1} \sum_{j=0}^{\infty} \pi_{j} \int_{0}^{\infty} \tilde{K}_{j}(i, m, x, t) d t, \quad i \geq 0, m=\overline{0,2} \tag{4.7}
\end{equation*}
$$

The corresponding vector Laplace-Stieltjes transforms $\tilde{\mathbf{v}}(i, m, s)$ are defined by

$$
\begin{equation*}
\tilde{\mathbf{v}}(i, m, s)=\tau^{-1} \sum_{j=0}^{\infty} \pi_{j} \int_{0}^{\infty} \tilde{K}_{j}^{*}(i, m, s, t) d t, \quad i \geq 0, m=\overline{0,2} \tag{4.8}
\end{equation*}
$$

where $\tilde{K}_{j}^{*}(i, m, s, t)=\int_{0}^{\infty} e^{-s x} d \tilde{K}_{j}(i, m, x, t)$.
From (4.8), formulas (4.2) follow immediately when we note that $\widetilde{K}_{j}^{*}(i, m, s, t)=0$ for the range of arguments $\{j \geq 0, i=0, m=1,2\}$ and $\{j \geq 0, i>0, m=0\}$.

Let $m=1$. The lengthy but straightforward expressions for the matrices $\widetilde{K}_{j}^{*}(i, 1, s, t)$, $i>0$, are presented in Appendix A. Substituting these expressions into (4.8) and after routine algebraic manipulations including rearranging the order of integration, we get formula (4.4) for the vectors $\widetilde{\mathbf{v}}(i, 1, s), i>0$. Similar calculations yield (4.3) and (4.5).

Further, we study the generalized service time distribution at Station 1.
Let $\widehat{B}(x)$ be the matrix distribution function of generalized service time. More specifically, let $\widehat{B}(x)=\left(\widehat{B}(x)_{r, r^{\prime}}\right)_{r, r^{\prime}=0, N}$, where $\widehat{B}(x)_{r, r^{\prime}}=P\left\{t_{n+1}-t_{n}<x, r_{n+1}=r^{\prime} \mid r_{n}=r, i_{n} \neq 0\right\}$. Denote $B(s)=\int_{0}^{\infty} e^{-s t} d \widehat{B}(t)$.

Lemma 4.2. The matrix Laplace-Stieltjes transform of the generalized service time distribution at Station 1 is calculated as

$$
\begin{equation*}
\mathcal{B}(s)=\left[Q_{1}+\gamma Q_{2}+(1-\gamma) F^{*}(s) Q_{3}\right] B^{*}(s), \tag{4.9}
\end{equation*}
$$

where $F^{*}(s)=\int_{0}^{\infty} e^{-s t} d F(t)$.
Proof. To prove, we need to analyze the structure of the generalized service time. The generalized service time of a tagged customer is just the service time of the customer at Station 1 if the previous customer did not block the server of this station. In this case, the matrix Lap-lace-Stieltjes transform of the generalized service time distribution is calculated by ( $Q_{1}+$ $\left.r Q_{2}\right) B^{*}(s)$. However, when blocking occurs, the generalized service time consists of the time during which the server is blocked by the previous customer and the service time of the tagged customer. The corresponding Laplace-Stieltjes transform is defined by ( $1-\gamma$ ) $\int_{0}^{\infty} e^{-s t} d F(t) Q_{3} B^{*}(s)$. The stated result (4.9) follows immediately.

Now we are ready to derive the equation for the vector Laplace-Stieltjes transform $\mathbf{v}_{1}(s)$ of the distribution of the virtual sojourn time at Station 1 . Let $v_{1}(r, v, x)$ be the probability that, at an arbitrary epoch, the BMAP is in state $v$, the virtual sojourn time at Station 1 is less than $x$, and the number of busy servers at Station 2 just before the end of the virtual sojourn time is $r$. Then $\mathbf{v}_{1}(s)$ is defined as a vector of Laplace-Stieltjes transforms $v_{1}(r, v, s)=$ $\int_{0}^{\infty} e^{-s x} d v_{1}(r, v, x)$ written in lexicographic order.

Theorem 4.3. The vector Laplace-Stieltjes transform $\mathbf{v}_{1}(s)$ satisfies the equation

$$
\begin{equation*}
\mathbf{v}_{1}(s) A(s)=\boldsymbol{\pi}_{0} \Phi(s), \tag{4.10}
\end{equation*}
$$

where

$$
\begin{equation*}
A(s)=s I+\sum_{r=0}^{\infty} B^{r}(s) \otimes D_{r}, \quad \Phi(s)=\tau^{-1} \widehat{Q}\left(\Delta \oplus D_{0}\right)^{-1}\left(\Delta \otimes I_{\bar{W}}-s I\right)\left[B^{*}(s) \otimes I_{\bar{W}}\right] . \tag{4.11}
\end{equation*}
$$

Proof. As mentioned above, the virtual sojourn time at Station 1 consists of the residual time from an arbitrary time $t$ to the next service completion epoch, the generalized service times of customers that await for a service at time $t$, and the generalized service time of the virtual customer.

Taking into account the structure of the virtual sojourn time and using the law of total probability, we express the vector Laplace-Stieltjes transform $\mathbf{v}_{1}(s)$ as follows:

$$
\begin{equation*}
\mathbf{v}_{1}(s)=\widetilde{\mathbf{v}}(0,0, s)+\sum_{i=1}^{\infty} \widetilde{\mathbf{v}}(i, 1, s)\left[\mathbb{B}^{i}(s) \otimes I_{\bar{W}}\right]+\sum_{i=1}^{\infty} \widetilde{\mathbf{v}}(i, 2, s)\left[\mathbb{B}^{i-1}(s) \otimes I_{\bar{W}}\right] . \tag{4.12}
\end{equation*}
$$

Further, we multiply (4.12) by the matrix $s I+\sum_{r=0}^{\infty} B^{r}(s) \otimes D_{r}$, and, after some laborious calculations, get

$$
\begin{align*}
& \mathbf{v}_{1}(s)\left(s I+\sum_{r=0}^{\infty} B^{r}(s) \otimes D_{r}\right) \\
& =\tau^{-1}\left\{\pi_{0} \sum_{i=0}^{\infty}\left[C_{i}+\sum_{j=1}^{i+1} \pi_{j} Y_{i-j+1}\right]\left[\mathcal{B}^{i+1}(s) \otimes I_{\bar{W}}\right]\right.  \tag{4.13}\\
& \\
& \left.\quad+\pi_{0} \hat{Q}\left[-\left(\Delta \oplus D_{0}\right)^{-1}\left(s I+\tilde{D}_{0}\right)+I\right]\left[B^{*}(s) \otimes I_{\bar{W}}\right]-\sum_{j=0}^{\infty} \pi_{j}\left[\mathcal{B}^{j+1}(s) \otimes I_{\bar{W}}\right]\right\} .
\end{align*}
$$

Multiplying the balance equations for stationary probability vectors $\pi_{i}$ of the form

$$
\begin{equation*}
\pi_{i}=\pi_{0} C_{i}+\sum_{l=1}^{i+1} \pi_{l} Y_{i-l+1}, \quad i \geq 0 \tag{4.14}
\end{equation*}
$$

by $乃^{i+1}(s) \otimes I_{\bar{W}}$ and summing over $i$ we obtain

$$
\begin{equation*}
\sum_{i=0}^{\infty} \boldsymbol{\pi}_{i}\left[\mathbb{B}^{i+1}(s) \otimes I_{\bar{W}}\right]=\boldsymbol{\pi}_{0} \sum_{i=0}^{\infty} C_{i}\left[\mathbb{B}^{i+1}(s) \otimes I_{\bar{W}}\right]+\sum_{i=0}^{\infty} \sum_{j=1}^{i+1} \boldsymbol{\pi}_{j} Y_{i-j+1}\left[\mathbb{B}^{i+1}(s) \otimes I_{\bar{W}}\right] \tag{4.15}
\end{equation*}
$$

Using (4.15) to simplify equation (4.13), we obtain (4.10).

### 4.1.2. The Sojourn Time at Station 2

Let $\mathbf{v}_{2}(s)$ be the column vector of the Laplace-Stieltjes transforms of the conditional sojourn time distributions at Station 2. The $r$ th entry of this vector is the Laplace-Stieltjes transform of the sojourn time distribution of a customer at Station 2 given that the number of busy servers is equal to $r$ just before the end of the sojourn time of this customer at Station 1.

Lemma 4.4. The vector Laplace-Stieltjes transform of the sojourn time distribution at Station 2 is given by

$$
\begin{equation*}
\mathbf{v}_{2}(s)=\left[Q_{4}\left(F^{*}(s)+I\right) \widehat{I}+\gamma Q_{2}+(1-\gamma) F^{*}(s) \operatorname{diag}\left\{f_{r, 0}(s), r=N, N-1, \ldots, 0\right\} Q_{3}\right] \mathbf{e} \tag{4.16}
\end{equation*}
$$

Proof. The sojourn time of a customer who requires $m$ servers at Station 2 consists of:
(i) the service time of the customer when at least $m$ servers are available at the time of the request;
(ii) zero time, if the required number of servers is not available and the customer leaves the system;
(iii) the blocking time and the service time of a customer when the required number of servers is not available and the customer awaits the release of sufficiently many servers.
Note that we assume that the service of type $m$ is performed by $m$ servers independently of each other and finishes when all $m$ servers complete the service. The distribution of this service time is defined by the Laplace-Stieltjes transform $f_{m, 0}(s), m=\overline{1, N}$.

Taking into account this fact together with (i)-(iii) and using the matrix notation, we obtain expression (4.16) for the vector $\mathbf{v}_{2}(s)$. In the expression, the first summand corresponds to the case (i), the second and the third summands give the Laplace-Stieltjes transform under study in the cases (ii) and (iii), respectively.

### 4.1.3. The Virtual Sojourn Time in the System

Theorem 4.5. The Laplace-Stieltjes transform of the virtual sojourn time distribution in the system is given by

$$
\begin{equation*}
v(s)=\mathbf{v}_{1}(s)\left(I_{N+1} \otimes \mathbf{e}_{\bar{W}}\right) \mathbf{v}_{2}(s), \tag{4.17}
\end{equation*}
$$

where the vectors $\mathbf{v}_{1}(s)$ and $\mathbf{v}_{2}(s)$, respectively, are as given in (4.10) and (4.16).
Proof. Formula (4.17) readily follows from the structure of the virtual sojourn time in the system which consists of the virtual sojourn time at Station 1 and the sojourn time at Station 2.

### 4.2. The Actual Sojourn Time

Let $v_{1}^{(a)}(s)$ and $v^{(a)}(s)$ be the Laplace-Stieltjes transforms of the distribution of the actual sojourn time at Station 1 and in the whole system.

Theorem 4.6. The Laplace-Stieltjes transform of the actual sojourn time distribution at Station 1 is calculated as follows:

$$
\begin{equation*}
v_{1}^{(a)}(s)=\lambda^{-1} \mathbf{v}_{1}(s) \sum_{k=0}^{\infty}\left[\mathbb{B}^{k}(s)(\mathbb{B}(s)-I)^{-1} \otimes D_{k}\right] \mathbf{e} . \tag{4.18}
\end{equation*}
$$

Proof. The actual sojourn time at Station 1 of an arbitrary-tagged customer, who arrived in a group of size $k$ and placed at the $j$ th position within the group, consists of (a) the actual sojourn time at Station 1 of the first customer in the group, which coincides with the virtual sojourn time at Station 1 ; (b) the generalized service times at Station 1 of the $j-2$ customers of the group who arrived with the tagged customer; (c) the generalized service time of the tagged customer at Station 1.

The vector Laplace-Stieltjes transform of the sojourn time distribution at Station 1 of the first customer of the $k$-size group that contains the tagged customer is evidently given by the vector $\mathbf{v}_{1}(s)\left(I_{N+1} \otimes k D_{k} \mathbf{e} / \lambda\right)$.

Assuming that an arbitrary customer arriving in a group of size $k$ is placed on the $j$ th position with probability $1 / k$ and using the law of total probability, we immediately obtain
the following expression:

$$
\begin{equation*}
v_{1}^{(a)}(s)=\sum_{k=1}^{\infty} \mathbf{v}_{1}(s)\left(I_{N+1} \otimes \frac{k D_{k} \mathbf{e}}{\lambda}\right) \sum_{j=1}^{k} \frac{1}{k} 乃^{j-1}(s) \mathbf{e} . \tag{4.19}
\end{equation*}
$$

After some algebraic manipulations (4.19) is reduced to (4.18).
Corollary 4.7. The Laplace-Stieltjes transform of the actual sojourn time distribution in the whole system is calculated as follows:

$$
\begin{equation*}
v^{(a)}(s)=\lambda^{-1} \mathbf{v}_{1}(s) \sum_{k=0}^{\infty}\left[\mathbb{B}^{k}(s)(\mathbb{B}(s)-I)^{-1} \otimes D_{k}\right]\left(I_{N+1} \otimes \mathbf{e}_{\bar{W}}\right) \mathbf{v}_{2}(s) \tag{4.20}
\end{equation*}
$$

### 4.3. Moments of the Sojourn Time Distribution

The formulas for the moments of the virtual sojourn time distribution can be obtained by differentiating the expression in (4.10) at the point $s=0$. This requires the calculation of the derivatives of $\mathbf{v}_{1}(s)$ at the point $s=0$. However, the matrix $A(s)$ in (4.10) is singular at the point $s=0$ and calculation of the derivatives at this point is the nontrivial task. The results given below allow one to develop a procedure for calculating the required derivatives.

We will use notation $\mathbf{v}_{1}^{(m)}(s)$ for the $m$ th derivative of the vector $\mathbf{v}_{1}(s), m \geq 1$, and set $\mathbf{v}_{1}^{(0)}(s)=\mathbf{v}_{1}(s)$. Similar notations will be used for other functions of $s$.

Theorem 4.8. Let $\int_{0}^{\infty} t^{m} d B(t)<\infty, m=\overline{1, M+1}$, where $M$ is an arbitrary positive integer. Then the vectors $\mathbf{v}_{1}^{(m)}(0), m=\overline{1, M}$, are computed recursively by

$$
\begin{align*}
\mathbf{v}_{1}^{(m)}(0)= & {\left[\left(\pi_{0} \Phi^{(m)}(0)-\sum_{l=0}^{m-1}\binom{m}{l} \mathbf{v}_{1}^{(l)}(0) A^{(m-l)}(0)\right) \tilde{I}\right.} \\
& \left.+\frac{1}{m+1}\left(\boldsymbol{\pi}_{0} \Phi^{(m+1)}(0)-\sum_{l=0}^{m-1}\binom{m+1}{l} \mathbf{v}_{1}^{(l)}(0) A^{(m+1-l)}(0)\right) \mathbf{e} \widehat{\mathbf{e}}\right] \tilde{A}^{-1}, \tag{4.21}
\end{align*}
$$

with the initial condition

$$
\begin{equation*}
\mathbf{v}_{1}^{(0)}(0)=\mathbf{v}_{1}(0)=\left[\tau^{-1} \boldsymbol{\pi}_{0} \widehat{Q}\left(\Delta \oplus D_{0}\right)^{-1}\left(\Delta B^{*}(0) \otimes I_{\bar{W}}\right) \tilde{I}+p_{\text {idle }} \widehat{\mathbf{e}}\right] \tilde{A}^{-1} \tag{4.22}
\end{equation*}
$$

where $\tilde{A}=A(0) \tilde{I}+A^{\prime}(0) \mathbf{e} \widehat{\text { e. }}$
Proof of the theorem is presented in Appendix B. In what follows we assume that $\int_{0}^{\infty} t^{k} d B(t)<\infty, k=\overline{1,2}$.

Corollary 4.9. The mean virtual sojourn time at Station 1 is given by

$$
\begin{align*}
\bar{v}_{1}=\{ & {\left[\tau^{-1} \boldsymbol{\pi}_{0} \widehat{Q}\left(\Delta \oplus D_{0}\right)^{-1}\left(\left(B^{*}(0)-\Delta B^{*^{\prime}}(0)\right) \otimes I_{\bar{W}}\right)+\mathbf{v}_{1}(0) A^{\prime}(0)\right] \tilde{I} } \\
& \left.+\left[p_{\text {idle }} b_{1}+\frac{1}{2} \mathbf{v}_{1}(0) A^{\prime \prime}(0) \mathbf{e}\right] \widehat{\mathbf{e}}\right\} \tilde{A}^{-1} \mathbf{e} \tag{4.23}
\end{align*}
$$

where the vector $\mathbf{v}_{1}(0)$ is defined by formula (4.22).
Proof follows from the relation $\bar{v}_{1}=-\mathbf{v}_{1}^{\prime}(0) \mathbf{e}$ and formula (4.21).
Theorem 4.10. The mean virtual sojourn time in the system is given by

$$
\begin{equation*}
\bar{v}=\bar{v}_{1}+\mathbf{v}_{1}(0)\left(I_{N+1} \otimes \mathbf{e}_{\bar{W}}\right) \overline{\mathbf{v}}_{2} \tag{4.24}
\end{equation*}
$$

where $\mathbf{v}_{1}(0)$ and $\bar{v}_{1}$ are given in (4.22) and (4.23), respectively, and $\overline{\mathbf{v}}_{2}$ is a vector of conditional means of the sojourn time at Station 2,

$$
\begin{equation*}
\overline{\mathbf{v}}_{2}=-\left\{Q_{4} F^{(1)} \widehat{\mathbf{e}}^{T}+(1-\gamma)\left[F^{(1)}-E \operatorname{diag}\left\{\sum_{l=1}^{r}(l \mu)^{-1}, r=N, N-1, \ldots, 0\right\}\right] Q_{3} \mathbf{e}\right\} \tag{4.25}
\end{equation*}
$$

where $F^{(1)}$ is given by formula (C.3) below.
Proof. To calculate the value $\bar{v}$ we differentiate the expression in (4.17). Setting $s=0$ and replacing the sign, we have that

$$
\begin{equation*}
\bar{v}=-\mathbf{v}_{1}^{\prime}(0)\left(I_{N+1} \otimes \mathbf{e}_{\bar{W}}\right) \mathbf{v}_{2}(0)-\mathbf{v}_{1}(0)\left(I_{N+1} \otimes \mathbf{e}_{\bar{W}}\right) \mathbf{v}_{2}^{\prime}(0) \tag{4.26}
\end{equation*}
$$

Putting $s=0$ in (4.16) we get $\mathbf{v}_{2}(0)=\left[Q_{4}(E+I) \hat{I}+\gamma Q_{2}+(1-\gamma) E Q_{3}\right] \mathbf{e}=\mathbf{e}$. This implies that the first term in the right-hand side of (4.26) is equal to $-\mathbf{v}_{1}^{\prime}(0) \mathbf{e}=\bar{v}_{1}$. Using the relation $\overline{\mathbf{v}}_{2}=-\mathbf{v}_{2}^{\prime}(0)$ and differentiating (4.16) at the point $s=0$, we readily verify that $\overline{\mathbf{v}}_{2}$ has form (4.25). This completes the proof.

Theorem 4.11. The mean actual sojourn time at Station 1 is given by

$$
\begin{equation*}
\bar{v}_{1}^{(a)}=-\lambda^{-1}\left\{\mathbf{v}_{1}^{\prime}(0)\left(\mathbf{e} \otimes \sum_{k=1}^{\infty} k D_{k} \mathbf{e}\right)+\mathbf{v}_{1}(0) \sum_{k=1}^{\infty}\left(I_{N+1} \otimes D_{k} \mathbf{e}\right)\left[\sum_{n=1}^{k-1} \sum_{l=0}^{n-1} \boldsymbol{B}^{l}(0) B^{\prime}(0) \mathbf{e}\right]\right\} \tag{4.27}
\end{equation*}
$$

Proof of the theorem follows from the relation $\bar{v}_{1}^{(a)}=-d v_{1}^{(a)}(s) /\left.d s\right|_{s=0}$ and formula (4.19).

Corollary 4.12. The mean actual sojourn time in the system is given by

$$
\begin{equation*}
\bar{v}_{a}=\bar{v}_{1}^{(a)}+\lambda^{-1} \mathbf{v}_{1}(0) \sum_{k=1}^{\infty}\left(I_{N+1} \otimes D_{k} \mathbf{e}\right) \sum_{n=0}^{k-1} \mathcal{B}^{n}(0) \overline{\mathbf{v}}_{2} \tag{4.28}
\end{equation*}
$$

## 5. Numerical Examples

In this section, we demonstrate feasibility of the algorithms developed here and show numerically some interesting features of the system under consideration.

Experiment 1. In this experiment, we investigate the impact of coefficient of variation in the service process at Station 1 on the main performance measures of the system.

To this end, we consider four service processes with the same mean service time $b_{1}=$ 0.1 , but different values for the coefficient of variation, $c_{\mathrm{var}}$. The first process is coded as $D$ and corresponds to the deterministic service time distribution. The second process is coded as $M$ and corresponds to the exponential service time. The third and the fourth service processes are coded as $H M_{2}^{(1)}, H M_{2}^{(2)}$ and correspond to hyperexponential service time distributions of order 2 . These distributions are defined by the mixing probabilities $(0.05,0.95)$ and the intensities $0.62025,48.9998$ in the case of $H M_{2}^{(1)}$ and $(0.98,0.02)$ and the intensities 10000, 0.2 in the case of $H M_{2}^{(2)}$. The coefficients of variation of processes $D, M, H M_{2}^{(1)}$, and $H M_{2}^{(2)}$ are, respectively, equal to $0,1,5,9.95$.

The input process is defined by the matrices

$$
D_{0}=\left(\begin{array}{ccc}
-15.7327 & 0.6062 & 0.5924  \tag{5.1}\\
0.5178 & -2.2897 & 0.4679 \\
0.5971 & 0.5653 & -1.9597
\end{array}\right), \quad D=\left(\begin{array}{ccc}
14.1502 & 0.3021 & 0.0818 \\
0.1071 & 1.032 & 0.1646 \\
0.0858 & 0.1979 & 0.5136
\end{array}\right)
$$

The matrices $D_{k}, k=\overline{1,5}$, are calculated as follows: $D_{k}=D h^{k-1}(1-h) /\left(1-h^{5}\right)$, where $h=0.8$. Then we normalize the matrices $D_{k}, k=\overline{0,5}$, so as to get the arrival rate $\lambda=1$. This BMAP has coefficient of correlation $c_{\text {cor }}=0.2$. The other parameters of the system are as follows: $N=5, \mu=0.8, \gamma=0.5, q_{0}=0.1, q_{1}=q_{2}=0.3, q_{3}=q_{4}=q_{5}=0.1$.

We vary the mean service time $b_{1}$ for all considered service processes in the interval [0.1, 0.95 ] by scaling appropriately. The coefficients of variation do not change under such scaling. Note that, as it was mentioned in Remark 3.5 above, the system load $\rho$ depends not only on the mean service time, but also on the variance of the service time. In Table 1, the value of $\rho$ is given as function of $b_{1}$ and $c_{\text {var }}$. The values of $\rho$ that exceed 1 are printed in bold face. The tandem queueing system is not stable for these values.

Figures 1 and 2 show the dependence of the main performance measures of the system on the value of the mean service time $b_{1}$ for service processes with different service time distribution. From these figures, it is very clear that the key performance measures of the system are very sensitive with respect to the service time variance. We also ran other examples, besides the one presented here, involving Erlangian and uniform distributions for the service time distribution. Since all these distributions have coefficient of variation in the range $(0,1)$, the corresponding curves, as expected, were located between the two lower curves in Figures 1 and 2.

Experiment 2. Here we solve numerically the following optimization problem. Find an optimal choice for the number $N$ of servers at Station 2 that will minimize the expected total


Figure 1: The mean virtual and actual sojourn time as functions of the mean service time for different service time distributions.


Figure 2: The loss probability and the mean number of busy servers at Station 2 as functions of the mean service time for different service time distributions.
cost per unit of time:

$$
\begin{equation*}
J=J(N)=a N+c_{1} \lambda P_{\mathrm{loss}}+c_{2} \bar{v}^{(a)} \tag{5.2}
\end{equation*}
$$

where $a$ is the cost of utilization per unit time of a server at Station 2 (maintenance cost), $c_{1}$ is the cost of a customer leaving the system after a service at Station 1 due to lack of required servers, and $c_{2}$ is the cost per unit of time of holding (the sojourn time) an arbitrary customer in the system (holding cost).

Using the MAP characterized by the matrices

$$
D_{0}=\left(\begin{array}{cc}
-6.74538 & 5.45412 \times 10^{-6}  \tag{5.3}\\
5.45412 \times 10^{-6} & -0.219455
\end{array}\right), \quad D=\left(\begin{array}{cc}
6.700685 & 0.044695 \\
0.122427 & 0.097023
\end{array}\right)
$$

we construct the BMAP with the matrices $D_{k}, k=\overline{0,5}$, similar to Experiment 1 and normalize so as to have $\lambda=3$. The BMAP has the coefficient of correlation $c_{\text {cor }}=0.2$ and the coefficient of variation $c_{\mathrm{var}}=3.5$.

Table 2: The value of the objective function for different number of servers and different service rate at Station 2.

|  | $\mu=1$ | $\mu=2$ | $\mu=3$ | $\mu=4$ | $\mu=5$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $N=1$ | $\infty$ | 457.0346 | 74.9351 | 51.0311 | 42.5146 |
| $N=2$ | 353.7378 | 48.6349 | 34.1130 | 27.0826 | 22.9120 |
| $N=3$ | 71.0480 | 34.6565 | 25.8441 | 22.2656 | 20.6657 |
| $N=4$ | 51.7822 | 30.4068 | 25.8202 | 24.6008 | 24.2393 |
| $N=5$ | 45.2556 | 31.2080 | 29.3925 | 29.1141 | 29.0630 |
| $N=6$ | 43.0750 | 34.6437 | 34.0980 | 34.0535 | 34.0486 |
| $N=7$ | 43.6867 | 39.0715 | 39.0533 | 39.0504 | 39.0476 |
| $N=8$ | 47.2564 | 44.0527 | 44.0514 | 44.0495 | 44.0474 |
| $N=9$ | 51.9854 | 49.0521 | 49.0511 | 49.0491 | 49.0473 |
| $N=10$ | 57.0056 | 55.0458 | 54.6001 | 54.0489 | 54.0472 |



Figure 3: The objective function as a function of the number of servers in Station 2 for different service rates.

The service time distribution at Station 1 is assumed to be Erlang of order 3 with parameter 20. The probability $\gamma$ is taken to be 0.5 . The components of the vector $\mathbf{q}$ are: $q_{0}=$ $0.1, q_{1}=0.9, q_{m}=0, m=\overline{2, N}$. The various costs are taken as follows: $a=5, c_{1}=50, c_{2}=3$.

The objective function, $J$, as a function of the number $N$ of servers under different service rates $\mu$ is plotted in Figure 3. Table 2 contains the values of the objective function.

The optimal values $J^{*}$ of the objective function for each of the five service rates are displayed in bold face. It is seen from Figure 3 and Table 2 that as the service rate decreases from 5 to 1 , the optimal number of servers $N^{*}$ increases from 3 to 6 . The relative gain of the optimal configuration in comparison to a system with an arbitrary number $N$ of the servers at Station 2 is defined as $R_{\text {rel }}(N)=\left(\left(J(N)-J^{*}\right) / J^{*}\right) 100 \%$.

We now focus on the result of optimal value in the case $\mu=5$. It is seen from Figure 3 and Table 2 that the optimal value of the objective function $J^{*}$ is 20.6657 and the optimal number of servers $N^{*}=3$. It should be also noted that for the case under consideration the minimal relative gain is more than $10 \%$ if we install the optimal number of servers $N^{*}=3$ instead of 2 servers and maximal relative gain is more than $161 \%$ if we use $N^{*}=3$ servers instead of 10 servers.

Experiment 3. In this experiment, we show that the correlation in the input flow has a great impact on the performance measures of the system. In addition to the BMAP defined in the first experiment and having the coefficient of correlation $c_{\text {cor }}=0.2$, let us consider two another BMAPs, having the same mean arrival rate, but different coefficients of correlation. These


Figure 4: The mean virtual and actual sojourn time as functions of the system load for the BMAPs with different correlation.

BMAPs are defined by the matrices $D_{0}$ and $D_{1}=D$, from which the matrices $D_{k}, k=\overline{0,5}$, are defined in the same way as in Experiment 1.

The BMAP having the coefficient of correlation $c_{\text {cor }}=0.1$ is characterized by the matrices

$$
D_{0}=\left(\begin{array}{ccc}
-13.3346 & 0.5886 & 0.6173  \tag{5.4}\\
0.6927 & -2.4466 & 0.4229 \\
0.6823 & 0.4144 & -1.6354
\end{array}\right), \quad D=\left(\begin{array}{ccc}
11.5469 & 0.3631 & 0.2187 \\
0.3842 & 0.8659 & 0.0809 \\
0.2852 & 0.0425 & 0.2111
\end{array}\right) .
$$

The BMAP having the coefficient of correlation $c_{\text {cor }}=0.3$ is defined by the matrices

$$
D_{0}=\left(\begin{array}{ccc}
-25.5398 & 0.3933 & 0.3612  \tag{5.5}\\
0.1452 & -2.2322 & 0.2000 \\
0.2960 & 0.3874 & -1.7526
\end{array}\right), \quad D=\left(\begin{array}{ccc}
24.2421 & 0.4669 & 0.0763 \\
0.0341 & 1.6668 & 0.1861 \\
0.0090 & 0.2555 & 0.8047
\end{array}\right)
$$

All the three defined BMAPs have the coefficient of variation $c_{\mathrm{var}}=2$.
In addition, we consider the BMAP which is a group Poisson process. It has the same mean arrival rate as three other BMAPs, coefficient of correlation $c_{\text {cor }}=0$ and the coefficient of variation $c_{\text {var }}=1$.

The service time at Station 1 has the Erlangian distribution of order 3 with parameter 20. The mean service time $b_{1}=3 / 20$ and the squared coefficient of variation $c_{\mathrm{var}}^{2}=1 / 3$.

The number of servers at Station $2 N=5$. Service rate $\mu$ is equal to 5. Probability $\gamma$ that a customer will await the release of servers is equal to 0.5 . The components of the vector $q$, which defines the type of the customer, are $q_{0}=0.1, q_{1}=q_{2}=0.3$, and $q_{3}=q_{4}=q_{5}=0.1$.

Figures 4 and 5 illustrate the dependence of the mean virtual sojourn time $\bar{v}$, the mean actual sojourn time $\bar{v}_{a}$, the loss probability $P_{\text {loss, }}$, and the mean number $N_{\text {busy }}$ of busy servers at Station 2 on the system load $\rho$. The load $\rho$ varies by means of scaling the fundamental rate $\lambda$. Note that the coefficients of correlation and variation of the BMAP do not change under such scaling.

Figures 4 and 5 confirm the fact that values of $\bar{v}, \bar{v}_{a}, P_{\text {loss }}$, and $N_{\text {busy }}$ increase when the system load, $\rho$, increases. We also note that, under the same scenario for the system load


Figure 5: The loss probability and the mean number of busy servers at Station 2 as functions of the system load for the BMAPs with different correlation.
the correlation of the interarrival times shows a strong (negative) impact on the system performance characteristics.

## 6. Conclusion

In this paper, the BMAP/G/1 $\rightarrow \bullet / M / N / 0$ tandem queue with heterogeneous customers is studied. The system is studied by looking at selected embedded epochs. The condition for the existence of the stationary distribution is derived. Expressions for the loss probability, blocking probability, and some other performance characteristics of the system are obtained. The Laplace-Stieltjes transforms of the distribution of the virtual and the actual sojourn time at both stations as well as at the whole system are derived. Although the required analytical derivations are very complicated and cumbersome, the resulting formulas have very simple forms. The procedure for calculating the moments of the virtual and the actual sojourn time distribution is elaborated. Illustrative numerical results highlight the important role played by the variance of the service time. An optimization problem to illustrate the usefulness of such problems in practice involving tandem queues is discussed. The results of this paper can be applied to areas such as capacity planning, performance evaluation, and optimization of real-world tandem queues and two-node networks.

## Appendices

A. Expressions for the Matrices $\tilde{K}_{j}^{*}(i, 1, s, t), j>0, i>0$

$$
\begin{aligned}
\tilde{K}_{0}^{*}(i, 1, s, t)= & \left(\tilde{Q}_{1}+\gamma \tilde{Q}_{2}\right) \int_{0}^{t}\left[I_{N+1} \otimes e^{D_{0} x} \sum_{k=1}^{i} D_{k} d x P(i-k, t-x)\right] \\
& \times \int_{0}^{\infty} e^{-s u} e^{\Delta(t+u)} \otimes I_{\bar{W}} d B(t-x+u)+(1-\gamma)
\end{aligned}
$$

$$
\begin{align*}
& \times\left\{\int_{0}^{t}\left[d F(x) \otimes \sum_{k=1}^{i} P(k, x) P(i-k, t-x)\right] \tilde{Q}_{3}\right. \\
& \times \int_{0}^{\infty} e^{-s u} e^{\Delta(t-x+u)} \otimes I_{\bar{W}} d B(t-x+u) \\
&+\int_{0}^{t} \int_{0}^{y}\left[d F(x) \otimes\left(e^{D_{0} y} \sum_{k=1}^{i} D_{k} d y P(i-k, t-y)\right)\right] \tilde{Q}_{3} \\
&\left.\times \int_{0}^{\infty} e^{-s u} e^{\Delta(t-x+u)} \otimes I_{\bar{W}} d B(t-y+u)\right\}, \quad i>0, \\
& \widetilde{K}_{j}^{*}(i, 1, s, t)=\left(\widetilde{Q}_{1}+\gamma \widetilde{Q}_{2}\right)\left[I_{N+1} \otimes P(i-j, t)\right] \int_{0}^{\infty} e^{-s u} e^{\Delta(t+u)} \otimes I_{\bar{W}} d B(t+u) \\
&+(1-\gamma) \int_{0}^{t}\left[d F(x) \otimes \sum_{k=0}^{i-j} P(k, x) P(i-k-j, t-x)\right] \tilde{Q}_{3} \\
& \times \int_{0}^{\infty} e^{-s u} e^{\Delta(t-x+u)} \otimes I_{\bar{W}} d B(t-x+u) . \tag{A.1}
\end{align*}
$$

## B. Proof of Theorem 4.8.

Proof. Successively differentiating the expression in (4.10), we get

$$
\begin{equation*}
\mathbf{v}_{1}^{(m)}(0) A(0)=\pi_{0} \Phi^{(m)}(0)-\sum_{l=0}^{m-1}\binom{m}{l} \mathbf{v}_{1}^{(l)}(0) A^{(m-l)}(0), \quad m \geq 0 \tag{B.1}
\end{equation*}
$$

It follows from (4.11) that $A(0)=\sum_{r=0}^{\infty} \mathcal{B}^{r}(0) \otimes D_{r}$, where $\mathcal{B}(0)$ is an irreducible stochastic matrix. This implies that $A(0)$ is an irreducible infinitesimal generator, and hence $A(0)$ is a singular matrix. Thus, it is not possible to develop a recursive scheme for computing the vectors $\mathbf{v}_{1}^{(m)}(0), m \geq 0$, directly from (B.1). We will now modify the system (B.1) to get the system with a nonsingular matrix. To this end, we postmultiply the expression for $m+1$ in (B.1) on both sides with e. Taking into account that $A(0) \mathbf{e}=\mathbf{0}^{T}$, we get

$$
\begin{equation*}
\mathbf{v}_{1}^{(m)} A^{\prime}(0) \mathbf{e}=\frac{1}{m+1}\left[\pi_{0} \Phi^{(m+1)}(0)-\sum_{l=0}^{m-1}\binom{m+1}{l} \mathbf{v}_{1}^{(l)}(0) A^{(m+1-l)}(0)\right] \mathbf{e} . \tag{B.2}
\end{equation*}
$$

It can be shown that the right-hand side of (B.2) is not equal to zero. It is positive if $m=2 k$ and it is negative if $m=2 k+1, k \geq 0$. Thus, replacing one of the equations in the system (B.1) (without loss of generality, we replace the first equation) with equation (B.2), we get the
following (inhomogeneous) system of linear algebraic equations for the entries of the vector $\mathbf{v}_{1}^{(m)}(0)$ :

$$
\begin{align*}
\mathbf{v}_{1}^{(m)}(0) \tilde{A}= & {\left[\pi_{0} \Phi^{(m)}(0)-\sum_{l=0}^{m-1}\binom{m}{l} \mathbf{v}_{1}^{(l)}(0) A^{(m-l)}(0)\right] \tilde{I} }  \tag{B.3}\\
& +\frac{1}{m+1}\left[\pi_{0} \Phi^{(m+1)}(0)-\sum_{l=0}^{m-1}\binom{m+1}{l} \mathbf{v}_{1}^{(l)}(0) A^{(m+1-l)}(0)\right] \mathbf{e} \widehat{\mathbf{e}}, \quad m \geq 0
\end{align*}
$$

The above system has the unique solution if $\tilde{A}$ is non-singular. We prove this by showing that $\operatorname{det} \widetilde{A} \neq 0$.

Let us calculate $\operatorname{det} \tilde{A}$ as

$$
\begin{equation*}
\operatorname{det} \tilde{A}=\nabla A^{\prime}(0) \mathbf{e} \tag{B.4}
\end{equation*}
$$

where $\nabla$ is a vector of algebraic cofactors of the first column of the matrix $A(0)$. Since $A(0)$ is irreducible, the vector $\nabla$ is proportional to any solution of the system

$$
\begin{equation*}
\mathbf{x} A(0)=\mathbf{0} \tag{B.5}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\nabla=c \mathbf{x} \tag{B.6}
\end{equation*}
$$

where the scalar $c \neq 0$.
Let the vector $\vartheta$ be the unique solution to the system

$$
\begin{equation*}
\vartheta B(0)=\vartheta, \quad \vartheta \mathbf{e}=1 . \tag{B.7}
\end{equation*}
$$

Then, by the direct substitution it can be verified that the vector $\mathbf{x}=\boldsymbol{\theta} \otimes \boldsymbol{\theta}$ is a solution to the system (B.5).

From (B.6), $\nabla=c(\boldsymbol{\vartheta} \otimes \boldsymbol{\theta})$. Substituting the vector $\nabla$ into (B.4), we obtain

$$
\begin{align*}
\operatorname{det} \tilde{A} & =c(\boldsymbol{\vartheta} \otimes \boldsymbol{\theta}) A^{\prime}(0) \mathbf{e}=\left.c(\boldsymbol{\vartheta} \otimes \boldsymbol{\theta})\left(s I+\sum_{r=0}^{\infty} \mathbb{B}^{r}(s) \otimes D_{r}\right)^{\prime}\right|_{s=0} \mathbf{e}  \tag{B.8}\\
& =c+c \sum_{r=1}^{\infty} r \vartheta \boldsymbol{B}^{\prime}(0) \mathbf{e} \otimes \boldsymbol{\theta} D_{r} \mathbf{e}=c\left(1+\lambda \boldsymbol{\vartheta} \boldsymbol{B}^{\prime}(0) \mathbf{e}\right) .
\end{align*}
$$

In further evaluation of $\operatorname{det} \tilde{A}$, we use the ergodicity condition given in Theorem 3.3.
Setting $s=0$ in (4.9) and noting that $Q_{1}+\gamma Q_{2}+(1-\gamma) F^{*}(0) Q_{3}=Q$, we see that $B(0)=Q B^{*}(0)$ and that the vector $\vartheta$ defined by (B.7) is the unique solution of system (3.8). It
can be easily verified that

$$
\begin{equation*}
\vartheta B^{\prime}(0) \mathbf{e}=-\left[b_{1}+(1-\gamma) \vartheta \int_{0}^{\infty} t d F(t) Q_{3} \mathbf{e}\right] . \tag{B.9}
\end{equation*}
$$

Multiplying (B.9) by $\lambda$ and comparing the obtained equation with the equality in (3.12) we see that

$$
\begin{equation*}
\lambda \vartheta B^{\prime}(0) \mathbf{e}=-\rho . \tag{B.10}
\end{equation*}
$$

It follows from (B.8) and (B.10) that $\operatorname{det} \tilde{A}=c(1-\rho)$. Since the stability condition implies that $\rho<1$ and since $c \neq 0$, we have $\operatorname{det} \tilde{A} \neq 0$. This competes the proof of the theorem.

## C. Formulas for Calculation of the Mean Virtual Sojourn Time

The expressions (4.21)-(4.22) for the mean virtual sojourn time involve the matrices $\widetilde{A}, A^{\prime}(0)$ and the vector $A^{\prime \prime}(0) \mathbf{e}$ for which we derive explicit expressions below.

$$
\begin{gather*}
\tilde{A}=\left[\sum_{r=0}^{\infty} \mathbb{B}^{r}(0) \otimes D_{r}\right] \tilde{I}+\left[I+\sum_{r=1}^{\infty} \sum_{n=0}^{r-1} \mathbb{B}^{n}(0) \mathbb{B}^{\prime}(0) \otimes D_{r}\right] \mathbf{e} \widehat{\mathbf{e}}, \\
A^{\prime}(0)=I+\sum_{r=1}^{\infty} \sum_{n=0}^{r-1}\left(\mathbb{B}^{n}(0) \mathbb{B}^{\prime}(0) \mathbb{B}^{r-n-1}(0) \otimes D_{r}\right)  \tag{C.1}\\
A^{\prime \prime}(0) \mathbf{e}=\left\{\sum_{r=1}^{\infty} \sum_{n=1}^{r-1}\left[2 \sum_{l=0}^{n-1} \mathbb{B}^{l}(0) \mathbb{B}^{\prime}(0) \mathbb{B}^{n-l-1}(0) \mathbb{B}^{\prime}(0)+\mathbb{B}^{n}(0) \mathbb{B}^{\prime \prime}(0)\right] \otimes D_{r}\right\} \mathbf{e},
\end{gather*}
$$

where

$$
\begin{gather*}
B(0)=Q B^{*(0)}, \quad B^{\prime}(0)=Q B^{*(1)}+(1-r) F^{(1)} Q_{3} B^{*(0)}, \\
B^{\prime \prime}(0)=Q B^{*(2)}-2(1-\gamma) F^{(1)} Q_{3} B^{*(1)}+(1-\gamma) F^{(2)} Q_{3} B^{*(0)},  \tag{C.2}\\
F^{(m)}=(-1)^{m} \int_{0}^{\infty} t^{m} d F(t)=\left(f_{r, r^{\prime}}^{(m)}\right)_{r, r^{\prime}=0, N^{\prime}} \quad m=1,2,  \tag{С.3}\\
f_{r, r^{\prime}}^{(1)}= \begin{cases}0, & r \leq r^{\prime}, \\
-\sum_{l=r^{\prime}+1}^{r}(l \mu)^{-1}, & r>r^{\prime},\end{cases} \tag{C.4}
\end{gather*}
$$

$$
\begin{gather*}
f_{r, r^{\prime}}^{(2)}= \begin{cases}0, & r \leq r^{\prime}, \\
2 \sum_{l=r^{\prime}+1}^{r}(-1)^{l-r^{\prime}+1}\binom{r-r^{\prime}}{l-r^{\prime}}(l \mu)^{-2}, & r>r^{\prime},\end{cases} \\
B^{*(m)}=(-1)^{m} \int_{0}^{\infty} t^{m} e^{\Delta t} d B(t)=\left(\beta_{r, r^{\prime}}^{*(m)}\right)_{r, r^{\prime}=\overline{0, N^{\prime}}} \quad m=\overline{0,2},  \tag{C.5}\\
\beta_{r, r^{\prime}}^{*(m)}=\left\{\begin{array}{c}
0, \\
\binom{r}{r^{\prime}} \begin{cases}\left.\sum_{i=0}^{r-r^{\prime}}(-1)^{i}\binom{r-r^{\prime}}{i} \beta^{(m)}\left(\mu\left(r^{\prime}+i\right)\right)\right\}, & r \geq r^{\prime} .\end{cases}
\end{array} . \begin{array}{l}
\quad m
\end{array}\right.
\end{gather*}
$$

Remark C.1. Formulas for $\tilde{A}, A^{\prime}(0), A^{\prime \prime}(0)$ contain infinite sums. However, the calculations of these should not create any difficulty as for overwhelming majority of interesting and useful queueing models; the parameter matrices, $D_{k}$, of the BMAP process are equal to zero for $k$ greater than some threshold, say, $K$. Thus, all sums become finite. In alternative case, some analytical formula for computing the infinite sequence of matrices $D_{k}, k \geq 1$, should be given. If this sequence is generated such as $D_{k}=D(1-\sigma) \sigma^{k-1}, k \geq 1$, where $D$ and $0<\sigma<1$ are given parameters, the infinite sums can be computed explicitly.

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Research Article

# Adaptive State-Feedback Stabilization for High-Order Stochastic Nonlinear Systems Driven by Noise of Unknown Covariance 

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This paper further considers more general high-order stochastic nonlinear system driven by noise of unknown covariance and its adaptive state-feedback stabilization problem. A smooth statefeedback controller is designed to guarantee that the origin of the closed-loop system is globally stable in probability.

## 1. Introduction

In this paper, we consider the following high-order stochastic nonlinear system:

$$
\begin{gather*}
d x_{1}=x_{2}^{p} d t+f_{1}\left(x_{1}\right) d t+g_{1}\left(x_{1}\right) \Sigma d \omega, \\
d x_{2}=x_{3}^{p} d t+f_{2}\left(\bar{x}_{2}\right) d t+g_{2}\left(\bar{x}_{2}\right) \Sigma d \omega,  \tag{1.1}\\
\vdots \\
d x_{n}=u^{p} d t+f_{n}\left(\bar{x}_{n}\right) d t+g_{n}\left(\bar{x}_{n}\right) \Sigma d \omega,
\end{gather*}
$$

where $x=\left(x_{1}, \ldots, x_{n}\right) \in R^{n}, u \in R$, are the state and control input, respectively. $\bar{x}_{i}=\left(x_{1}, \ldots\right.$, $\left.x_{i}\right), i=1, \ldots, n . p \geq 1$ is odd integer. $w$ is an $r$-dimensional standard Wiener process defined in a complete probability space $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \geq 0}, P\right)$ with $\Omega$ being a sample space, $\mathcal{F}$ being a $\sigma$ field, $\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$ being a filtration, and $P$ being the probability measure. $\Sigma: R_{+} \rightarrow R^{r \times r}$ is an
unknown bounded nonnegative definite Borel measurable matrix function and $\Sigma \Sigma^{T}$ denotes the infinitesimal covariance function of the driving noise $\Sigma d \omega . f_{i}: R^{i} \rightarrow R$ and $g_{i}: R^{i} \rightarrow R^{r}$, $i=1, \ldots, n$, are assumed to be smooth with $f_{i}(0)=0$ and $g_{i}(0)=0$.

When $p=1$, system (1.1) reduced to the well-known normal form whose study on feedback control problem has achieved great development in recent years. According to the difference of selected Lyapunov functions, the existing literature on controller design can be mainly divided into two types. One type is based on quadratic Lyapunov functions which are multiplied by different weighting functions, see, for example, [1-5] and the references therein. Another essential improvement belongs to Krstić and Deng. By introducing the quartic Lyapunov function, $[6,7]$ presented asymptotical stabilization control in the large under the assumption that the nonlinearities equal to zero at the equilibrium point of the open-loop system. Subsequently, for several classes of stochastic nonlinear systems with unmodeled dynamics and uncertain nonlinear functions, by combining Krstić and Deng's method with stochastic small-gain theorem [8], and with dynamic signal and changing supply function $[9,10]$, different adaptive output-feedback control schemes are studied.

When $p>1$, some intrinsic features of (1.1), such as that its Jacobian linearization is neither controllable nor feedback linearizable, lead to the existing design tools hardly applicable to this kind of systems. Motivated by the fruitful deterministic results in [11, 12] and the related papers and based on stochastic stability theory in [13-15], and so forth, [16] firstly considered, this class of systems with stochastic inverse dynamics. Subsequently, [17-21] considered respectively, the state-feedback stabilization problem for more general systems with different structures. [22,23] solved the output-feedback stabilization, and [24] addressed the inverse optimal stabilization.

All the papers mentioned above, however, only consider the case of $\Sigma \Sigma^{T}=I$. In this paper, we will further consider more general high-order stochastic nonlinear system driven by noise of unknown covariance and its adaptive state-feedback stabilization problem. A smooth state-feedback controller is designed to guarantee that the origin of the closed-loop system is globally stable in probability. A simulation example verifies the effectiveness of the control scheme.

The paper is organized as follows. Section 2 provides some preliminary results. Section 3 gives the state-feedback controller design and stability analysis, following a simulation example in Section 4. In Section 5, we conclude the paper.

## 2. Preliminary Results

The following notations definitions and lemmas are to be used throughout the paper.
$R_{+}$stands for the set of all nonnegative real numbers, $R^{n}$ is the $n$-dimensional Euclidean space, $R^{n \times m}$ is the space of real $n \times m$-matrixes. $\mathcal{C}^{2}$ denotes the family of all the functions with continuous second partial derivatives. $|x|$ is the usual Euclidean norm of a vector $x .\|X\|=\left(\operatorname{Tr}\left\{X X^{T}\right\}\right)^{1 / 2}$, where $\operatorname{Tr}\{X\}$ is its trace when $X$ is a square matrix and $X^{T}$ denotes the transpose of $X$. $\mathcal{K}$ denotes the set of all functions: $R_{+} \rightarrow R_{+}$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_{\infty}$ is the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{K} £$ denotes the set of all functions $\beta(s, t): R_{+} \times R_{+} \rightarrow R_{+}$, which are of class $\mathcal{K}$ for each fixed $t$ and decrease to zero as $t \rightarrow \infty$ for each fixed $s$. To simplify the procedure, we sometimes denote $\chi(t)$ by $x$ for any variable $\chi(t)$.

Consider the nonlinear stochastic system

$$
\begin{equation*}
d x=f(x) d t+g(x) d \omega \tag{2.1}
\end{equation*}
$$

where $x \in R^{n}$ is the state, $w$ is an $r$-dimensional independent Wiener process with incremental covariance $\Sigma \Sigma^{T} d t$, that is, $E\left\{d \omega d \omega^{T}\right\}=\Sigma \Sigma^{T} d t$, where $\Sigma$ is a bounded function taking values in the set of nonnegative definite matrices, $f: R^{n} \rightarrow R^{n}$ and $g: R^{n} \rightarrow R^{n \times r}$ are locally Lipschitz functions.

Definition 2.1 (see [13]). For any given $V(x) \in \mathcal{C}^{2}$ associated with stochastic system (2.1), the differential operator $\mathcal{L}$ is defined as

$$
\begin{equation*}
\varrho V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x)+\frac{1}{2} \operatorname{Tr}\left\{g^{T}(x) \frac{\partial^{2} V(x)}{\partial x^{2}} g(x)\right\} . \tag{2.2}
\end{equation*}
$$

Definition 2.2 (see [25]). For the stochastic system (2.1) with $f(0)=0, g(0)=0$, the equilibrium $x(t)=0$ is globally asymptotically stable (GAS) in probability if for any $\xi>0$, there exists a class $\mathcal{K} \perp$ function $\beta(\cdot, \cdot)$ such that

$$
\begin{equation*}
P\left\{|x(t)|<\beta\left(\left|x_{0}\right|, t\right)\right\} \geq 1-\xi, \quad t \geq 0, \quad \forall x_{0} \in R^{n} \backslash\{0\} . \tag{2.3}
\end{equation*}
$$

Lemma 2.3 (see [14]). Consider the stochastic system (2.1). If there exist a $\mathcal{C}^{2}$ function $V(x)$, class $\mathcal{K}_{\infty}$ function $\alpha_{1}$ and $\alpha_{2}$, constants $c_{1}>0$ and $c_{2} \geq 0$, and a nonnegative function $W(x)$ such that for all $x \in R^{n}, t \geq 0$

$$
\begin{equation*}
\alpha_{1}(|x|) \leq V(x) \leq \alpha_{2}(|x|), \quad \varrho V \leq-c_{1} W(x)+c_{2} \tag{2.4}
\end{equation*}
$$

then,
(a) there exists an almost surely unique solution on $[0, \infty)$ for each $x_{0} \in R^{n}$,
(b) when $c_{2}=0, f(0)=0, g(0)=0$, and $W(x)$ is continuous, then the equilibrium $x=0$ is globally stable in probability and the solution $x(t)$ satisfies $P\left\{\lim _{t \rightarrow \infty} W(x(t))=0\right\}=1$.

Lemma 2.4 (see [12]). Let $x, y$ be real variables, for any positive integers $m, n$, positive real number $b$ and nonnegative continuous function $a(\cdot)$, then

$$
\begin{equation*}
a(\cdot) x^{m} y^{n} \leq b|x|^{m+n}+\frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-m / n}(a(\cdot))^{(m+n) / n} b^{-m / n}|y|^{m+n} \tag{2.5}
\end{equation*}
$$

when $a(\cdot)=1, b=(m /(m+n)) d$, $d$ is a positive constant, then the above inequality is

$$
\begin{equation*}
x^{m} y^{n} \leq \frac{m}{m+n} d|x|^{m+n}+\frac{n}{m+n} d^{-m / n}|y|^{m+n} \tag{2.6}
\end{equation*}
$$

Lemma 2.5 (see [12]). Let $x, y$ and $z_{i}, i=1, \ldots, p$, be real variables and let $l_{1}(\cdot)$ and $l_{2}(\cdot)$ be smooth mappings. Then for any positive integers $m, n$ and real number $N>0$, there exist two nonnegative smooth functions $h_{1}(\cdot)$ and $h_{2}(\cdot)$ such that the following inequalities hold:
(i) $\left|x^{m}\left[\left(y+x l_{1}(x)\right)^{n}-\left(x l_{1}(x)\right)^{n}\right]\right| \leq|x|^{m+n} / N+|y|^{m+n} h_{1}(x, y)$,
(ii) $\left|y^{m}\left(z_{1}^{n}+\cdots+z_{p}^{n}+y^{n}\right) l_{2}\left(z_{1}, \ldots, z_{p}, y\right)\right| \leq(1 / N) \sum_{k=1}^{p}\left|z_{k}\right|^{m+n}+|y|^{m+n} h_{2}\left(z_{1}, \ldots, z_{p}, y\right)$.

Lemma 2.6 (see [12]). Let $x_{1}, \ldots, x_{n}, p$, be positive real variables, then

$$
\begin{equation*}
\left(x_{1}+\cdots+x_{n}\right)^{p} \leq \max \left\{n^{p-1}, 1\right\}\left(x_{1}^{p}+\cdots+x_{n}^{p}\right) \tag{2.7}
\end{equation*}
$$

## 3. Controller Design and Stability Analysis

The objective of this paper is to design a smooth state-feedback controller for system (1.1), such that the solution of the closed-loop system is GAS in probability. To achieve the control objective, we need the following assumption.

Assumption 3.1. There are nonnegative smooth functions $f_{i 1}, g_{i 1}, i=1, \ldots, n$, such that

$$
\begin{equation*}
\left|f_{i}\left(\bar{x}_{i}\right)\right| \leq\left(\sum_{j=1}^{i}\left|x_{j}\right|^{p}\right) f_{i 1}\left(\bar{x}_{i}\right), \quad\left|g_{i}\left(\bar{x}_{i}\right)\right| \leq\left(\sum_{j=1}^{i}\left|x_{j}\right|^{p}\right) g_{i 1}\left(\bar{x}_{i}\right) \tag{3.1}
\end{equation*}
$$

### 3.1. Controller Design

Now, we give the controller design procedure by using the backstepping method.
First, we introduce the following coordinate change:

$$
\begin{equation*}
z_{1}=x_{1}, \quad z_{i}=x_{i}-\alpha_{i-1}\left(\bar{x}_{i-1}, \hat{\theta}\right), \quad i=2, \ldots, n \tag{3.2}
\end{equation*}
$$

where $\alpha_{i-1}\left(\bar{x}_{i-1}, \widehat{\theta}\right), i=2, \ldots, n$, are smooth virtual controllers which will be designed later, $\widehat{\theta}$ is the estimation of $\theta$, and

$$
\begin{equation*}
\theta \triangleq \max _{t \geq 0}\left\{\left\|\Sigma \Sigma^{T}\right\|^{(p+3) / 2},\left\|\Sigma \Sigma^{T}\right\|^{(p+3) / 3},\left\|\Sigma \Sigma^{T}\right\|\right\} . \tag{3.3}
\end{equation*}
$$

Then, by Itô's differentiation rule, one has

$$
\begin{align*}
d z_{i}= & d\left(x_{i}-\alpha_{i-1}\left(\bar{x}_{i-1}, \hat{\theta}\right)\right) \\
= & \left(x_{i+1}^{p}+F_{i}\left(\bar{x}_{i}\right)-\sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{l}} x_{l+1}^{p}-\frac{1}{2} \sum_{k, m=1}^{i-1} \frac{\partial^{2} \alpha_{i-1}}{\partial x_{k} \partial x_{m}} g_{k}\left(\bar{x}_{k}\right) \Sigma \Sigma^{T} g_{m}^{T}\left(\bar{x}_{m}\right)-\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \hat{\hat{\theta}}\right) d t  \tag{3.4}\\
& +G_{i}\left(\bar{x}_{i}\right) \Sigma d \omega
\end{align*}
$$

where

$$
\begin{equation*}
F_{i}\left(\bar{x}_{i}\right)=f_{i}\left(\bar{x}_{i}\right)-\sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{l}} f_{l}\left(\bar{x}_{l}\right), \quad G_{i}\left(\bar{x}_{i}\right)=g_{i}\left(\bar{x}_{i}\right)-\sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{l}} g_{l}\left(\bar{x}_{l}\right) \tag{3.5}
\end{equation*}
$$

Step 1. Define the first Lyapunov function

$$
\begin{equation*}
V_{1}\left(z_{1}, \hat{\theta}\right)=\frac{1}{4} z_{1}^{4}+\frac{1}{2 \Gamma} \tilde{\theta}^{2} \tag{3.6}
\end{equation*}
$$

where $\Gamma$ is a positive constant, $\tilde{\theta}=\theta-\widehat{\theta}$ is the parameter estimation error. By (3.2)-(3.4) and Assumption 3.1, there exist nonnegative smooth functions $\mu_{11}$ and $\mu_{15}$ such that

$$
\begin{align*}
£ V_{1} & =z_{1}^{3} x_{2}^{p}+z_{1}^{3} f_{1}\left(x_{1}\right)+\frac{3}{2} z_{1}^{2} g_{1}\left(x_{1}\right) \Sigma \Sigma^{T} g_{1}^{T}\left(x_{1}\right)-\frac{\tilde{\theta}}{\Gamma} \dot{\hat{\theta}} \\
& \leq z_{1}^{3}\left(x_{2}^{p}-\alpha_{1}^{p}\right)+z_{1}^{3} \alpha_{1}^{p}+z_{1}^{p+3} \mu_{11}\left(z_{1}\right)+z_{1}^{p+3} \mu_{15}\left(z_{1}\right) \theta-\frac{\tilde{\theta}}{\Gamma} \dot{\hat{\theta}} \\
& \leq z_{1}^{3}\left(x_{2}^{p}-\alpha_{1}^{p}\right)+z_{1}^{3} \alpha_{1}^{p}+z_{1}^{p+3} \mu_{11}\left(z_{1}\right)+z_{1}^{p+3} \mu_{15}\left(z_{1}\right) \sqrt{1+\hat{\theta}^{2}}-\frac{\tilde{\theta}}{\Gamma}\left(\dot{\hat{\theta}}-\Gamma z_{1}^{p+3} \mu_{15}\left(z_{1}\right)\right) . \tag{3.7}
\end{align*}
$$

Choosing the first smooth virtual controller

$$
\begin{equation*}
\alpha_{1}\left(x_{1}, \widehat{\theta}\right)=-z_{1} \beta_{1}\left(z_{1}, \widehat{\theta}\right), \quad \beta_{1}\left(z_{1}, \widehat{\theta}\right)=\left(c_{11}+\mu_{11}\left(z_{1}\right)+\mu_{15}\left(z_{1}\right) \sqrt{1+\hat{\theta}^{2}}\right)^{1 / p} \tag{3.8}
\end{equation*}
$$

and the tuning function

$$
\begin{equation*}
\tau_{1}\left(z_{1}\right)=\Gamma z_{1}^{p+3} \mu_{15}\left(z_{1}\right) \tag{3.9}
\end{equation*}
$$

one has

$$
\begin{equation*}
\varrho V_{1} \leq-c_{11} z_{1}^{p+3}+z_{1}^{3}\left(x_{2}^{p}-\alpha_{1}^{p}\right)-\frac{\tilde{\theta}}{\Gamma}\left(\dot{\hat{\theta}}-\tau_{1}\right) \tag{3.10}
\end{equation*}
$$

where $c_{11}>0$ is a design parameter.
Step $i(2 \leq i \leq n)$. For notational coherence, denote $u=x_{n+1}$. Assuming that at step $i-1$, one has

$$
\begin{equation*}
\varrho V_{i-1} \leq-\sum_{j=1}^{i-1} c_{j, i-1} z_{j}^{p+3}-\left(\frac{\tilde{\theta}}{\Gamma}+\sum_{j=2}^{i-1} z_{j}^{3} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\right)\left(\dot{\hat{\theta}}-\tau_{i-1}\right)+z_{i-1}^{3}\left(x_{i}^{p}-\alpha_{i-1}^{p}\right) \tag{3.11}
\end{equation*}
$$

where $\tau_{i-1}=\tau_{i-2}+\Gamma z_{i-1}^{p+3}\left(\mu_{i-1,4}+\mu_{i-1,5}\right)$. In the sequel, we will prove that (3.11) still holds for the $i$ th Lyapunov function $V_{i}\left(\bar{z}_{i}, \widehat{\theta}\right)=V_{i-1}\left(\bar{z}_{i-1}, \widehat{\theta}\right)+(1 / 4) z_{i}^{4}$. By (3.4) and (3.11), one has

$$
\begin{align*}
\mathscr{L} V_{i} \leq & \mathcal{L} V_{i-1}+z_{i}^{3}\left(x_{i+1}^{p}+F_{i}\left(\bar{x}_{i}\right)-\sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{l}} x_{l+1}^{p}-\frac{1}{2} \sum_{k, m=1}^{i-1} \frac{\partial^{2} \alpha_{i-1}}{\partial x_{k} \partial x_{m}} g_{k}\left(\bar{x}_{k}\right) \Sigma \Sigma^{T} g_{m}^{T}\left(\bar{x}_{m}\right)\right) \\
& -z_{i}^{3} \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \dot{\hat{\theta}}+\frac{3}{2} z_{i}^{2} \operatorname{Tr}\left\{\Sigma^{T} G_{i}^{T}\left(\bar{x}_{i}\right) G_{i}\left(\bar{x}_{i}\right) \Sigma\right\} \\
\leq & -\sum_{j=1}^{i-1} c_{j, i-1} z_{j}^{p+3}-\left(\frac{\tilde{\theta}}{\Gamma}+\sum_{j=2}^{i-1} z_{j}^{3} \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}}\right)\left(\dot{\hat{\theta}}-\tau_{i-1}\right)+z_{i-1}^{3}\left(x_{i}^{p}-\alpha_{i-1}^{p}\right)  \tag{3.12}\\
& +z_{i}^{3}\left(x_{i+1}^{p}+F_{i}\left(\bar{x}_{i}\right)-\sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{l}} x_{l+1}^{p}-\frac{1}{2} \sum_{k, m=1}^{i-1} \frac{\partial^{2} \alpha_{i-1}}{\partial x_{k} \partial x_{m}} g_{k}\left(\bar{x}_{k}\right) \Sigma \Sigma^{T} g_{m}^{T}\left(\bar{x}_{m}\right)\right) \\
& -z_{i}^{3} \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \hat{\theta}+\frac{3}{2} z_{i}^{2} \operatorname{Tr}\left\{\Sigma^{T} G_{i}^{T}\left(\bar{x}_{i}\right) G_{i}\left(\bar{x}_{i}\right) \Sigma\right\}+z_{i}^{3} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \tau_{i-1}-z_{i}^{3} \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \tau_{i-1} .
\end{align*}
$$

To proceed further, an estimate for each term in the right-hand side of (3.12) is needed. Using Itô's differentiation rule, Lemmas 2.4-2.6, (3.2), and (3.3), it follows that

$$
\begin{aligned}
z_{i-1}^{3}\left(x_{i}^{p}-\alpha_{i-1}^{p}\right) & =z_{i-1}^{3}\left(\left(z_{i}+\alpha_{i-1}\right)^{p}-\alpha_{i-1}^{p}\right) \\
& \leq \xi_{i 1} z_{i-1}^{p+3}+\mu_{i 1}\left(\bar{z}_{i}, \hat{\theta}\right) z_{i}^{p+3}, \\
z_{i}^{3} F_{i}\left(\bar{x}_{i}\right) & \leq\left|z_{i}\right|^{3} \sum_{j=1}^{i}\left|z_{j}\right|^{p} \rho_{i 2 j}\left(\bar{z}_{i}, \hat{\theta}\right) \\
& \leq \sum_{j=1}^{i-1} \xi_{i 2 j} z_{j}^{p+3}+\mu_{i 2}\left(\bar{z}_{i}, \widehat{\theta}\right) z_{i}^{p+3}, \\
-z_{i}^{3} \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{l}} x_{l+1}^{p} & \leq\left|z_{i}\right|^{3} \sum_{j=1}^{i-1}\left|z_{j}\right|^{p} \rho_{i 3 j}\left(\bar{z}_{i}, \hat{\theta}\right) \\
& \leq \sum_{j=1}^{i-1} \xi_{i 3 j} z_{j}^{p+3}+\mu_{i 3}\left(\bar{z}_{i}, \hat{\theta}\right) z_{i}^{p+3}, \\
-\frac{1}{2} z_{i}^{3} \sum_{k, m=1}^{i-1} \frac{\partial^{2} \alpha_{i-1}}{\partial x_{k} \partial x_{m}} g_{k}\left(\bar{x}_{k}\right) \sum^{T} g_{m}^{T}\left(\bar{x}_{m}\right) & \leq\left|z_{i}\right|^{3} \sum_{j=1}^{i-1}\left|z_{j}\right|^{2 p} \rho_{i 4 j}\left(\bar{z}_{i}, \hat{\theta}\right)\left\|\Sigma^{T}\right\|
\end{aligned}
$$

$$
\begin{align*}
& \leq \sum_{j=1}^{i-1} \xi_{i 5 j} z_{j}^{p+3}+\mu_{i 5}\left(\bar{z}_{i}, \widehat{\theta}\right) z_{i}^{p+3} \theta \\
-z_{i}^{3} \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \tau_{i-1} & \leq\left|z_{i}\right|^{3} \sum_{j=1}^{i-1}\left|z_{j}\right|^{p} \rho_{i 6 j}\left(\bar{z}_{i}, \widehat{\theta}\right) \\
& \leq \sum_{j=1}^{i-1} \xi_{i 6 j} z_{j}^{p+3}+\mu_{i 6}\left(\bar{z}_{i}, \widehat{\theta}\right) z_{i}^{p+3} \tag{3.13}
\end{align*}
$$

where $\xi_{i 1}, \xi_{i 2 j}, \xi_{i 3 j}, \xi_{i 4 j}, \xi_{i 5 j}, \xi_{i 6 j}, j=1, \ldots, i-1$, are positive constants and $\mu_{i 1}, \mu_{i 2}, \mu_{i 3}, \mu_{i 4}, \mu_{i 5}$, $\mu_{i 6}, \rho_{i 2 j}, \rho_{i 3 j}, \rho_{i 4 j}, \rho_{i 5 j}, \rho_{i 6 j}, j=1, \ldots, i$, are nonnegative smooth functions. Substituting (3.13) into (3.12), one has

$$
\begin{align*}
\mathscr{L} V_{i} \leq & -\sum_{j=1}^{i-1} c_{j, i-1} z_{j}^{p+3}+\xi_{i 1} z_{i-1}^{p+3}-\left(\frac{\tilde{\theta}}{\Gamma}+\sum_{j=2}^{i-1} z_{j}^{3} \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}}\right)\left(\dot{\hat{\theta}}-\tau_{i-1}\right)+z_{i}^{3} \alpha_{i}^{p} \\
& +z_{i}^{p+3}\left(\mu_{i 1}+\mu_{i 2}+\mu_{i 3}+\mu_{i 6}+\left(\mu_{i 4}+\mu_{i 5}\right) \sqrt{1+\widehat{\theta}^{2}}\right)+\frac{\tilde{\theta}}{\Gamma}\left(\mu_{i 4}+\mu_{i 5}\right) \Gamma z_{i}^{p+3}  \tag{3.14}\\
& +\sum_{j=1}^{i-1}\left(\xi_{i 2 j}+\xi_{i 3 j}+\xi_{i 4 j}+\xi_{i 5 j}+\xi_{i 6 j}\right) z_{j}^{p+3}+z_{i}^{3}\left(x_{i+1}^{p}-\alpha_{i}^{p}\right) .
\end{align*}
$$

Choosing the $i$ th smooth virtual controller $\alpha_{i}$

$$
\begin{align*}
& \alpha_{i}\left(\bar{x}_{i}, \widehat{\theta}\right)=-z_{i} \beta_{i}\left(\bar{z}_{i}, \widehat{\theta}\right) \\
& \beta_{i}\left(\bar{z}_{i}, \hat{\theta}\right)=\left(c_{i i}+\mu_{i 1}+\mu_{i 2}+\mu_{i 3}+\mu_{i 6}+\left(\mu_{i 4}+\mu_{i 5}\right)\left(\sqrt{1+\hat{\theta}^{2}}+\sum_{j=2}^{i} z_{j}^{3} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma\right)\right)^{1 / p}, \tag{3.15}
\end{align*}
$$

and tuning function $\tau_{i}$

$$
\begin{equation*}
\tau_{i}\left(\bar{z}_{i}\right)=\tau_{i-1}\left(\bar{z}_{i-1}\right)+\Gamma z_{i}^{p+3}\left(\mu_{i 4}+\mu_{i 5}\right) \tag{3.16}
\end{equation*}
$$

and substituting (3.15) and (3.16) into (3.14), it follows that

$$
\begin{equation*}
\varrho V_{i}\left(\bar{z}_{i}, \widehat{\theta}\right) \leq-\sum_{j=1}^{i} c_{j i} z_{j}^{p+3}-\left(\frac{\tilde{\theta}}{\Gamma}+\sum_{j=2}^{i} z_{j}^{3} \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}}\right)\left(\dot{\hat{\theta}}-\tau_{i}\right)+z_{i}^{3}\left(x_{i+1}^{p}-\alpha_{i}^{p}\right) \tag{3.17}
\end{equation*}
$$

where $c_{j i}=c_{j j}-\xi_{j+1,1}-\sum_{k=2}^{6} \xi_{i k j}, j=1, \ldots, i-1$.

Hence at step $n$, the smooth adaptive state-feedback controller

$$
\begin{gather*}
u=\alpha_{n}\left(\bar{x}_{n}, \hat{\theta}\right)=-z_{n} \beta_{n}\left(\bar{z}_{n}, \hat{\theta}\right), \quad \dot{\hat{\theta}}=\tau_{n}\left(\bar{z}_{n}\right), \\
\beta_{n}\left(\bar{z}_{n}, \hat{\theta}\right)=\left(c_{n n}+\mu_{n 1}+\mu_{n 2}+\mu_{n 3}+\mu_{n 6}+\left(\mu_{n 4}+\mu_{n 5}\right)\left(\sqrt{1+\hat{\theta}^{2}}+\sum_{j=2}^{n} z_{j}^{3} \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}} \Gamma\right)\right)^{1 / p}, \\
\tau_{n}\left(\bar{z}_{n}\right)=\Gamma z_{1}^{p+3} \mu_{15}\left(z_{1}\right)+\sum_{j=2}^{n} \Gamma z_{j}^{p+3}\left(\mu_{j 4}+\mu_{j 5}\right), \tag{3.18}
\end{gather*}
$$

such that the $n$th Lyapunov function

$$
\begin{equation*}
V_{n}\left(\bar{z}_{n}, \widehat{\theta}\right)=\frac{1}{4} \sum_{j=1}^{n} z_{j}^{4}+\frac{1}{2 \Gamma} \widetilde{\theta}^{2} \tag{3.19}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
\varrho V_{n} \leq-\sum_{j=1}^{n} c_{j n} z_{j}^{p+3} \tag{3.20}
\end{equation*}
$$

where $\mu_{n l}, l=1, \ldots, 6$, are nonnegative smooth functions, $c_{j n}, j=1, \ldots, n$, are constants, and

$$
\begin{equation*}
c_{j n}=c_{j j}-\xi_{j+1,1}-\sum_{k=2}^{6} \xi_{n k j}, \quad j=1, \ldots, n-1 . \tag{3.21}
\end{equation*}
$$

### 3.2. Stability Analysis

Theorem 3.2. If Assumption 3.1 holds for the high-order stochastic nonlinear system (1.1), under the state-feedback controller (3.18), then
(i) the closed-loop system consisting of (1.1), (3.2), (3.8), (3.9), (3.15), (3.16), and (3.18) has an almost surely unique solution on $[0, \infty)$ for each $\left(x_{0}, \tilde{\theta}(0)\right) \in R^{n+1}$,
(ii) the origin of the closed-loop system is globally stable in probability,
(iii) $P\left\{\lim _{t \rightarrow \infty}|x(t)|=0\right\}=1$ and $P\left\{\lim _{t \rightarrow \infty} \widehat{\theta}(t)\right.$ exists and is finite $\}=1$.

Proof. It is easy to verify that $V_{n}\left(\bar{z}_{n}, \widehat{\theta}\right)$ is $\mathcal{C}^{2}$ on $\bar{z}_{n}$ and $\hat{\theta}$. For $j=1, \ldots, n-1$, choose the design parameter $c_{j j}>\xi_{j+1,1}+\sum_{k=2}^{6} \xi_{n k j}, c_{n n}>0$, then by $(3.21), c_{j n}>0, j=1, \ldots, n-1$. Since $V_{n}\left(\bar{z}_{n}, \hat{\theta}\right)$ is continuous, positive, and radially bounded, by (3.20), (3.21), and Lemma 4.3 in [25], there exist two class $\not_{\infty}$ functions $\alpha_{1}$ and $\alpha_{2}$ such that $\alpha_{1}(|x|,|\widetilde{\theta}|) \leq V_{n}\left(\bar{z}_{n}, \widehat{\theta}\right) \leq \alpha_{2}(|x|,|\widetilde{\theta}|)$. Hence, the condition of Lemma 2.3 holds.

By Lemma 2.3, it follows that conclusion (i), (ii) hold, and $P\left\{\lim _{t \rightarrow \infty}|z(t)|=0\right\}=1$. In view of $\alpha_{i}(0, \widehat{\theta})=0$ and $x_{i}=z_{i}+\alpha_{i-1}\left(\bar{x}_{i-1}, \widehat{\theta}\right)$, one has $P\left\{\lim _{t \rightarrow \infty}|x(t)|=0\right\}=1$. By (3.20)
and the definition of $V_{n}\left(\bar{z}_{n}, \widehat{\theta}\right)$ in (3.19), it holds that $\tilde{\theta}(t)$ converges a.s. to a finite limit $\tilde{\theta}_{\infty}$ as $t \rightarrow \infty$, therefore $\hat{\theta}(t)$ converges a.s. to a finite limit as $t \rightarrow \infty$.

## 4. A Simulation Example

Consider a two-order nonlinear stochastic system

$$
\begin{align*}
& d x_{1}=x_{2}^{3} d t+f_{1}\left(x_{1}\right) d t+x_{1}^{3} \Sigma d \omega, \\
& d x_{2}=u^{3} d t+f_{2}\left(\bar{x}_{2}\right) d t+x_{2}^{3} \Sigma d \omega, \tag{4.1}
\end{align*}
$$

where $f_{1}\left(x_{1}\right)=x_{1}^{3}, f_{2}\left(\bar{x}_{2}\right)=x_{1} x_{2}^{2}$. By Lemma 2.4, one gets $\left|f_{1}\left(x_{1}\right)\right| \leq\left|x_{1}\right|^{3},\left|g_{1}\left(x_{1}\right)\right| \leq\left|x_{1}\right|^{3}$, $\left|f_{2}\left(\bar{x}_{2}\right)\right| \leq(1 / 3)\left|x_{1}\right|^{3}+(2 / 3)\left|x_{2}\right|^{3}, g_{2}\left(\bar{x}_{2}\right) \leq\left|x_{2}\right|^{3}$. We choose $f_{11}\left(x_{1}\right)=1, g_{11}\left(x_{1}\right)=1, f_{21}\left(\bar{x}_{2}\right)=$ $2 / 3, g_{21}\left(\bar{x}_{2}\right)=1$, Assumption 3.1 is satisfied.

We now give the design of state-feedback controller for system (4.1).
Step 1. Define $z_{1}=x_{1}, V_{1}\left(z_{1}, \widehat{\theta}\right)=(1 / 4) z_{1}^{4}+(1 / 2 \Gamma) \tilde{\theta}^{2}$. A smooth virtual controller

$$
\begin{align*}
& \alpha_{1}\left(x_{1}, \hat{\theta}\right)=-z_{1} \beta_{1}\left(z_{1}, \hat{\theta}\right) \\
& \beta_{1}\left(z_{1}, \hat{\theta}\right)=\left(c_{11}+1+\mu_{15}\left(z_{1}\right) \sqrt{1+\hat{\theta}^{2}}\right)^{1 / 3} \tag{4.2}
\end{align*}
$$

and the tuning function

$$
\begin{equation*}
\tau_{1}\left(z_{1}\right)=\Gamma z_{1}^{6} \mu_{15}\left(z_{1}\right) \tag{4.3}
\end{equation*}
$$

yield $\mathcal{L} V_{1}\left(z_{1}, \widehat{\theta}\right) \leq-c_{11} z_{1}^{6}+z_{1}^{3}\left(x_{2}^{3}-\alpha_{1}^{3}\right)+(\tilde{\theta} / \Gamma)\left(\dot{\widehat{\theta}}-\tau_{1}\right)$, where

$$
\begin{equation*}
\mu_{15}\left(z_{1}\right)=\frac{3}{2} z_{1}^{2}, \quad \theta(t)=\max _{t \geq 0}\left\{\left\|\Sigma(t) \Sigma(t)^{T}\right\|^{3},\left\|\Sigma(t) \Sigma(t)^{T}\right\|^{2},\left\|\Sigma(t) \Sigma(t)^{T}\right\|\right\} . \tag{4.4}
\end{equation*}
$$

Step 2. Defining $z_{2}=x_{2}-\alpha_{1}\left(x_{1}, \widehat{\theta}\right), V_{2}\left(\bar{z}_{2}, \widehat{\theta}\right)=V_{1}\left(z_{1}, \hat{\theta}\right)+(1 / 4) z_{2}^{4}$, by (3.12), one has

$$
\begin{align*}
\varrho V_{2}\left(z_{1}, \hat{\theta}\right) \leq & -c_{11} z_{1}^{6}+z_{1}^{3}\left(x_{2}^{3}-\alpha_{1}^{3}\right)+\frac{\tilde{\theta}}{\Gamma}\left(\dot{\hat{\theta}}-\tau_{1}\right) \\
& +z_{2}^{3}\left(u^{3}+F_{2}\left(\bar{x}_{2}\right)-\frac{\partial \alpha_{1}}{\partial x_{1}} x_{2}^{3}-\frac{1}{2} \frac{\partial^{2} \alpha_{1}}{\partial x_{1}^{2}} g_{1}\left(\bar{x}_{1}\right) \Sigma \Sigma^{T} g_{1}^{T}\left(\bar{x}_{1}\right)\right)  \tag{4.5}\\
& -z_{2}^{3} \frac{\partial \alpha_{1}}{\partial \widehat{\theta}} \dot{\theta}+\frac{3}{2} z_{2}^{2} \operatorname{Tr}\left\{\Sigma^{T} G_{2}^{T}\left(\bar{x}_{2}\right) G_{2}\left(\bar{x}_{2}\right) \Sigma\right\},
\end{align*}
$$

where $F_{2}\left(\bar{x}_{2}\right)=f_{2}\left(\bar{x}_{2}\right)-\left(\partial \alpha_{1} / \partial x_{1}\right) f_{1}\left(x_{1}\right), G_{2}\left(\bar{x}_{2}\right)=g_{2}\left(\bar{x}_{2}\right)-\left(\partial \alpha_{1} / \partial x_{1}\right) g_{1}\left(x_{1}\right)$. By Lemma 2.4, the definition of $z_{2}$, and (4.2), one can obtain

$$
\begin{align*}
z_{1}^{3}\left(x_{2}^{3}-\alpha_{1}^{3}\right) & \leq \frac{1}{2} d_{11} z_{1}^{6}+\frac{1}{2} d_{11}^{-1} z_{2}^{6}+3\left(\frac{2}{3} d_{12} z_{1}^{6}+\frac{1}{3} d_{12}^{-1} \beta_{1}^{3} z_{2}^{6}+\frac{5}{6} d_{13} z_{1}^{6}+\frac{1}{6} d_{13}^{-1} \beta_{1}^{12} z_{2}^{6}\right) \\
& =\xi_{21} z_{1}^{6}+\mu_{21}\left(z_{1}, \widehat{\theta}\right) z_{2}^{6} \\
z_{2}^{3} F_{2}\left(\bar{x}_{2}\right) & \leq 2\left|z_{2}\right|^{3}\left(\frac{1}{3}\left|z_{1}\right|^{3}+\frac{2}{3}\left|z_{2}\right|^{3}+\left|z_{1}\right|^{3} \beta_{1}^{2}-\frac{\partial \alpha_{1}}{\partial x_{1}} z_{1}^{3}\right) \\
& =\xi_{221} z_{1}^{6}+\mu_{22}\left(\bar{z}_{2}, \widehat{\theta}\right), \\
-z_{2}^{3} \frac{\partial \alpha_{1}}{\partial x_{1}} x_{2}^{3} & \leq\left|z_{2}\right|^{3}\left|\frac{\partial \alpha_{1}}{\partial x_{1}}\right|\left(z_{2}^{3}-3 z_{2}^{2} z_{1} \beta_{1}+3 z_{2} z_{1} \beta_{1}-z_{1}^{3} \beta_{1}^{3}\right) \\
& \leq \xi_{231} z_{1}^{6}+\mu_{23}\left(\bar{z}_{2}, \widehat{\theta}\right) z_{2}^{6} \\
-\frac{1}{2} z_{2}^{3} \frac{\partial^{2} \alpha_{1}}{\partial x_{1}^{2}} g_{1}^{3} \Sigma \Sigma^{T} g_{1}^{3} & \leq\left|z_{2}\right|^{3} \frac{\partial^{2} \alpha_{1}}{\partial x_{1}^{2}} z_{1}^{6}\left\|\Sigma \Sigma^{T}\right\| \\
& \leq \xi_{241} z_{1}^{6}+\mu_{24}\left(\bar{z}_{2}, \widehat{\theta}\right) z_{2}^{6} \theta ; \tag{4.6}
\end{align*}
$$

by (4.3), Lemmas $2.4,2.6$, and the definitions of $z_{2}$ and $G_{2}\left(\bar{x}_{2}\right)$, one has

$$
\begin{align*}
-z_{2}^{3} \frac{\partial \alpha_{1}}{\partial \widehat{\theta}} \tau_{1} & \leq\left|z_{2}\right|^{3}\left|z_{1}\right|^{3}\left(\frac{3}{2} \frac{\partial \alpha_{1}}{\partial \widehat{\theta}} \Gamma\left|z_{1}\right|^{5}\right) \\
& \leq \xi_{261} z_{1}^{6}+\mu_{26}\left(\bar{z}_{2}, \widehat{\theta}\right) z_{2}^{6} \\
\frac{3}{2} z_{2}^{2} \operatorname{Tr}\left\{\Sigma^{T} G_{2}^{T}\left(\bar{x}_{2}\right) G_{2}\left(\bar{x}_{2}\right) \Sigma\right\} & \leq \frac{3}{2} z_{2}^{2}\left(\left(z_{2}-z_{1} \beta_{1}\right)^{3}-\frac{\partial \alpha_{1}}{\partial x_{1}} z_{1}^{3}\right)^{2}\left\|\Sigma \Sigma^{T}\right\|  \tag{4.7}\\
& \leq z_{2}^{2}\left(\left(3 \cdot 2^{5} \beta_{1}^{6}+3 \cdot\left(\frac{\partial \alpha_{1}}{\partial x_{1}}\right)^{2}\right) z_{1}^{6}+2^{5} z_{2}^{6}\right)\left\|\Sigma \Sigma^{T}\right\| \\
& \leq \xi_{251} z_{1}^{6}+\mu_{25}\left(\bar{z}_{2}, \widehat{\theta}\right) z_{2}^{6} \theta
\end{align*}
$$

where $\xi_{21}=(1 / 2) d_{11}+2 d_{12}+(5 / 2) d_{13}, \xi_{221}=(1 / 3) d_{21}+d_{22}+d_{23}, \xi_{231}=(1 / 2) d_{31}+(1 / 2) d_{32}$, $\xi_{241}=(1 / 2) d_{41}, \xi_{251}=(2 / 3) d_{51}, \xi_{261}=(1 / 2) d_{61}, \mu_{21}\left(z_{1}, \widehat{\theta}\right)=(1 / 2) d_{11}^{-1}+d_{12}^{-1} \beta_{1}^{3}\left(z_{1}, \widehat{\theta}\right)+$ $(1 / 2) d_{13}^{-5} \beta_{1}^{12}\left(z_{1}, \widehat{\theta}\right), \mu_{22}\left(\bar{z}_{2}, \widehat{\theta}\right)=(1 / 3) d_{21}^{-1}+(4 / 3)+d_{22}^{-1} \beta_{1}^{4}+d_{23}^{-1}\left(\partial \alpha_{1} / \partial x_{1}\right)^{2}, \mu_{23}\left(\bar{z}_{2}, \widehat{\theta}\right)=\left(\partial \alpha_{1} /\right.$ $\left.\partial x_{1}\right)\left(1-2 \beta_{1}^{3 / 2}+\beta_{1}^{6}\right)+\left(\partial \alpha_{1} / \partial x_{1}\right)^{2}\left((1 / 2) d_{31}^{-1}+(1 / 2) d_{32}^{-1}\right), \mu_{24}\left(\bar{z}_{2}, \widehat{\theta}\right)=(1 / 2) d_{41}^{-1}\left(\partial^{2} \alpha_{1} / \partial x_{1}^{2}\right)^{2} z_{1}^{6}$, $\mu_{25}\left(\bar{z}_{2}, \widehat{\theta}\right)=32 z_{2}^{2}+d_{51}^{-1}\left(128 \beta_{1}^{18}+4\left(\partial \alpha_{1} / \partial x_{1}\right)^{6}\right) z_{1}^{6}, \mu_{26}\left(\bar{z}_{2}, \widehat{\theta}\right)=(9 / 8) d_{61}^{-1}\left(\partial \alpha_{1} / \partial \widehat{\theta}\right)^{2} \Gamma^{2} z_{1}^{10}, d_{11}, d_{12}$, $d_{13}, d_{21}, d_{22}, d_{23}, d_{31}, d_{32}, d_{41}, d_{51}, d_{61}$ are positive constants.

$-x_{1}$

$\qquad$

$u$
(b)

(d)

Figure 1: The responses of closed-loop system (4.1)-(4.3), (4.8).

Choosing the smooth adaptive controller

$$
\begin{gather*}
u=-z_{2} \beta_{2}\left(\bar{z}_{2}, \hat{\theta}\right), \quad \dot{\hat{\theta}}=\tau_{2}\left(\bar{z}_{2}\right), \\
\tau_{2}\left(\bar{z}_{2}\right)=\Gamma z_{1}^{6} \mu_{15}\left(z_{1}\right)+\Gamma z_{2}^{6}\left(\mu_{24}\left(\bar{z}_{2}, \hat{\theta}\right)+\mu_{25}\left(\bar{z}_{2}, \hat{\theta}\right)\right),  \tag{4.8}\\
\beta_{2}\left(\bar{z}_{2}, \hat{\theta}\right)=\left(c_{22}+\mu_{21}+\mu_{22}+\mu_{23}+\mu_{26}+\left(\mu_{24}+\mu_{25}\right)\left(\sqrt{1+\hat{\theta}^{2}}+\Gamma z_{2}^{3} \frac{\partial \alpha_{1}}{\partial \widehat{\theta}}\right)\right)^{1 / 3},
\end{gather*}
$$

and substituting (4.6)-(4.8) into (4.5), one has

$$
\begin{equation*}
\varrho V_{2} \leq-c_{12} z_{1}^{6}-c_{22} z_{2}^{6} \tag{4.9}
\end{equation*}
$$

where $c_{12}=c_{11}-\xi_{21}-\xi_{221}-\xi_{231}-\xi_{241}-\xi_{251}-\xi_{261}>0$.

In simulation, we choose $\Sigma(t) \equiv 1$, the parameters $\Gamma=1, c_{11}=11, c_{22}=1, d_{11}=1$, $d_{12}=1, d_{13}=1, d_{21}=1, d_{22}=1, d_{23}=1, d_{31}=0.01, d_{32}=1, d_{41}=1, d_{51}=1, d_{61}=1$, the initial values $\theta(0)=0, x_{1}(0)=0, x_{2}(0)=-0.5$, the sampling period $=0.01$. Figure 1 verifies the effectiveness of the control scheme.

## 5. Conclusion

In this paper, we further consider more general high-order stochastic nonlinear system driven by noise of unknown covariance and its adaptive state-feedback stabilization problem.

There is a still remaining problem to be investigated: under current investigation, how to design an output feedback controller for system (1.1) with Assumption 3.1?

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Research Article

# The Use of Geographically Weighted Regression for the Relationship among Extreme Climate Indices in China 

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The changing frequency of extreme climate events generally has profound impacts on our living environment and decision-makers. Based on the daily temperature and precipitation data collected from 753 stations in China during 1961-2005, the geographically weighted regression (GWR) model is used to investigate the relationship between the index of frequency of extreme precipitation (FEP) and other climate extreme indices including frequency of warm days (FWD), frequency of warm nights (FWN), frequency of cold days (FCD), and frequency of cold nights (FCN). Assisted by some statistical tests, it is found that the regression relationship has significant spatial nonstationarity and the influence of each explanatory variable (namely, FWD, FWN, FCD, and FCN) on FEP also exhibits significant spatial inconsistency. Furthermore, some meaningful regional characteristics for the relationship between the studied extreme climate indices are obtained.

## 1. Introduction

There is a general agreement that changes in the frequency or intensity of extreme climate events are likely to exert a much greater impact on nature and humanity than shifts in the mean value [1]. Starting from IPCC (1996) [2], many scientists have stressed the importance to study extreme climate events [3-5]. In the research field of extreme temperature and precipitation events, indices that are based on either fixed thresholds [6] or relative thresholds [7] are commonly used. To the best of our knowledge, most of the previous studies of climate
extremes mainly focus on some individual extreme climate index; however, the investigation of the relationship between them is relatively rare.

As for the relationship between some extreme climate indices, researchers generally assume that it is stationary over space and use an ordinary linear regression (OLR) model to analyze it. Nevertheless, it is known that an OLR model can only represent global relationship and it hardly takes into consideration the variations in relationships over space, in other words, the explicit incorporation of space or location has not been that commonly considered. In this context, there has been recently a surge focusing on the inclusion of spatial effects in climate models. A geographically weighted regression (GWR) model, which extends the traditional regression framework by allowing regression coefficients to vary with individual locations (spatial nonstationarity), is an effective method of utilizing spatial information to improve this issue [8-13]. Hence, GWR produces locally linear regression estimates for every point in space. For this purpose, weighted least squares methodology is used, with weights based on the distance between observations $i$ and all the others in the sample. GWR allows the exploration of variation of the parameters as well as the testing of the significance of this variation. It is of great appeal to apply GWR technique to analyze spatial data in a number of areas such as geography econometrics, epidemiology, and environmental science [14-16].

China is strongly influenced by the East Asian monsoon [17]. During the winter half year, the climate is mostly cold and dry. Cold days and strong winds accompanied by dust storms are the major climate features particularly observed in northern China [18]. During the summer period, the rain belt moves gradually from south to north with the hot and humid climate in eastern China [19]. The regional characteristics of extreme climate are particularly prominent in China. The purpose of this paper is to analyze the spatially varying impacts of some temperature extreme indices on one precipitation extreme index in China. In this paper, relative thresholds based on the 1961-1990 base period were firstly used to build some extreme indices, namely, FEP (frequency of extreme precipitation), FWD (frequency of warm days), FWN (frequency of warm nights), FCD (frequency of cold days), and FCN (frequency of cold nights). The spatial distributions of these indices were then analyzed. In order to investigate the relationship among these indices, a GWR model was utilized to study how FEP was affected by the other indices. Moreover, two statistical tests were carried out to confirm some of our guesswork and some promising results were obtained.

The rest of the paper is organized as follows. Section 2 presents the data source, gives the definitions of extreme climate indices used in this paper, and briefly outlines the method of GWR. Results for annual mean extreme climate indices over China are displayed in Section 3. Section 4 provides a conclusion.

## 2. Data and Method

### 2.1. Experimental Data

The experimental data sets used in this paper consist of daily maximum and minimum temperatures and daily precipitation observed at 753 meteorological stations in China from January 1, 1961 to December 31, 2005, which were offered by National Meteorological Information Center in China Meteorological Administration. Because the study must rely on reliable data, the missing data in each month should be no more than three days. Therefore, the data collected from the 504 stations (Figure 1) which comply with this requirement


Figure 1: Stations for which data were available in China. (•) Stations used in this paper; (+) stations omitted due to excessive missing data.
were utilized in this work. With respect to the missing values in these 504 stations, a linear interpolation method was adopted to impute them.

### 2.2. Extreme Climate Indices

Numerous temperature indices have been used in previous studies of climate events. Some indices involved arbitrary thresholds, such as the number of hot days exceeding $35^{\circ} \mathrm{C}$ and summer days exceeding $25^{\circ} \mathrm{C}$. As indicated by Manton et al. [5], these are suitable for regions with little spatial variability in climate, but arbitrary thresholds are inappropriate for regions spanning a broad range of climates. In China, climates vary widely from monsoon region in the eastern part to the westerly region in the northwestern part of the country, so there is no single temperature threshold that would be considered an event in all regions. For this reason, some studies have used weather and climate indices based on statistical quantities such as the 10th (5th) or 90th (95th) percentile [20,21]; detailed information can be found from the European Climate Assessment \& Dataset (ECA\&D) Indices List (http://www.knmi.nl/). Upper and lower percentiles of temperature indices are used in all regions, but vary in absolute magnitude from site to site. A regional climate study in the Caribbean region using the same indices can also be found in [21].

As this study covers a broad region in China, climate indices chosen are based on the 10th and 90th percentiles. The extreme climate indices studied in this paper include FEP, FWD, FWN, FCD, and FCN whose definitions are described in detail in Table 1. As for the experimental data of these extreme indices based on the 1961-1990 base period, the relative values of them were calculated. For each station, the values for FEP, FWD, FWN, FCD, and FCN are their respective values averaged over the period 1961-2005, which are still denoted as FEP, FWD, FWN, FCD, and FCN in order to facilitate the following discussions.

Table 1: Five extreme climate indices calculated based on daily temperature and precipitation data.

| Indicator <br> name | Indicator definition (unit: days) |
| :--- | :--- |
| FEP | Let $T p_{i j}$ be the daily precipitation on day $i$ of year $j$, and let $T p_{i n} 90$ be the calendar day <br> 900 th percentile centered on a 5-day window for the base period 1961-1990. Frequency of <br> extreme precipitation (FEP) in year $j$ is the annual count of days when $T p_{i j}>T p_{i n} 90$. |
| FWD | Let $T x_{i j}$ be the daily maximum temperature on day $i$ of year $j$, and let $T x_{i n} 90$ be the <br> calendar day 90th percentile centered on a 5-day window for the base period 1961-1990. <br> Frequency of warm days (FWD) in year $j$ is the annual count of days when $T x_{i j}>T x_{i n} 90$. |
| FWN | Let $T n_{i j}$ be the daily minimum temperature on day $i$ of year $j$, and let $T n_{i n} 90$ be the <br> calendar day 90th percentile centered on a 5-day window for the base period 1961-1990. <br> Frequency of warm nights (FWN) in year $j$ is the annual count of days when $T n_{i j}>T n_{i n} 90$. |
| FCD | Let $T x_{i j}$ be the daily maximum temperature on day $i$ of year $j$, and let $T x_{i n} 10$ be the <br> calendar day 10th percentile centered on a 5-day window for the base period 1961-1990. <br> Frequency of cold days (FCD) in year $j$ is the annual count of days when $T x_{i j}<T x_{i n} 10$. |
| FCN | Let $T n_{i j}$ be the daily minimum temperature on day $i$ of year $j$, and let $T n_{i n} 10$ be the <br> calendar day 10th percentile centered on a 5-day window for the base period $1961-1990$. <br> Frequency of cold nights (FCN) in year $j$ is the annual count of days when $T n_{i j}<T n_{i n} 10$. |

### 2.3. Geographically Weighted Regression (GWR)

The technique of linear regression estimates a parameter $\beta$ that links the explanatory variables to the response variable. However, when this technique is applied to spatial data, some issues concerning the stationarity of these parameters over the space come out. In "normal" regression, it is generally assumed that the modeling relationship holds everywhere in the study area-that is, the regression parameters are "whole-map" statistics. In many situations this is not the case, however, as mapping the residuals (the difference between the observed and predicted data) may reveal. The realization in the statistical and geographical sciences that a relationship between an explanatory variable and a response variable in a linear regression model is not always constant across a study area has led to the development of regression models allowing for spatially varying coefficients. Many different solutions have been proposed for dealing with spatial variation in the relationship. One of them, developed by Brunsdon et al. [8], has been labelled geographically weighted regression (GWR), which provides an elegant and easily grasped means of modeling such relationships by subtly incorporating the spatial characteristics of data via allowing regression coefficients to depend on some covariates such as longitude and latitude of the meteorological stations. Specifically, it is a nonparametric model of spatial drift that relies on a sequence of locally linear regressions to produce estimates for every point in space by using a subsample of data information from nearby observations. That is to say, this technique allows the modeling of relationships that vary over space by introducing distance-based weights to provide estimates $\beta_{k i}$ for each variable $k$ and each geographical location $i$. Thus the spatial variation of regression relationship can be effectively analyzed and the inherent disciplines of spatial data by the estimated coefficients over different locations can be better understood.

An ordinary linear regression (OLR) model can be expressed by

$$
\begin{equation*}
y_{i}=\beta_{0}+\sum_{j=1}^{p} \beta_{j} x_{i j}+\varepsilon_{i}, \quad i=1,2, \ldots, n, \tag{2.1}
\end{equation*}
$$

where $y_{i}, i=1,2, \ldots, n$, are the observation of the response variable $y, \beta_{j}(j=1,2, \ldots, p)$ represents the regression coefficients, $x_{i j}$ is the $i$ th value of the explanatory variable $x_{j}$, and $\varepsilon_{i}$ are normally distributed error terms with zero mean and constant variance.

In GWR model, the global regression coefficients are replaced by local parameters

$$
\begin{equation*}
y_{i}=\beta_{0}\left(u_{i}, v_{i}\right)+\sum_{j=1}^{p} \beta_{j}\left(u_{i}, v_{i}\right) x_{i j}+\varepsilon_{i}, \quad i=1,2, \ldots, n \tag{2.2}
\end{equation*}
$$

where $\left(u_{i}, v_{i}\right)$ denotes the longitude and latitude coordinates of the $i$ th meteorological station, $\left(y_{i} ; x_{i 1}, x_{i 2}, \ldots, x_{i p}\right)$ represent the observed value of the response $Y$ and explanatory variables $X_{1}, X_{2}, \ldots, X_{p}$ at $\left(u_{i}, v_{i}\right), \beta_{0}\left(u_{i}, v_{i}\right)$ is the intercept, and $\beta_{j}\left(u_{i}, v_{i}\right)(j=1,2, \ldots, p)$ are $p$ unknown coefficient functions of spatial locations, which represent the strength and type of relationship that the $j$ th explanatory variable $X_{j}$ has to the response variable $Y$. Additionally, $\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{n}$ are error terms which are generally assumed to be independent and identically distributed variables with mean 0 and common variance $\sigma^{2}$. It is worth noticing that the OLR model is actually a special case of the GWR model where $\beta_{j}\left(u_{i}, v_{i}\right)$ are constant for all $i=1,2, \ldots, n$.

The coefficient function vector $\widehat{\boldsymbol{\beta}}\left(u_{i}, v_{i}\right)$ for the $i$ th observation in GWR can be estimated via the locally weighted least square procedure [22] as

$$
\begin{gather*}
\widehat{\boldsymbol{\beta}}\left(u_{i}, v_{i}\right)=\left(\widehat{\beta}_{0}\left(u_{i}, v_{i}\right), \widehat{\beta}_{1}\left(u_{i}, v_{i}\right), \ldots, \widehat{\beta}_{p}\left(u_{i}, v_{i}\right)\right)^{T} \\
=\left(X^{T} W(i) X\right)^{-1} X^{T} W(i) Y, \quad i=1,2, \ldots, n,  \tag{2.3}\\
\widehat{\boldsymbol{\beta}}_{j}(u, v)=\left(\widehat{\beta}_{j}\left(u_{1}, v_{1}\right), \widehat{\beta}_{j}\left(u_{2}, v_{2}\right), \ldots, \widehat{\beta}_{j}\left(u_{n}, v_{n}\right)\right)^{T}, \quad j=0,1,2, \ldots, p,
\end{gather*}
$$

where

$$
\begin{gather*}
X=\left(\begin{array}{c}
x_{1}^{T} \\
x_{2}^{T} \\
\vdots \\
x_{n}^{T}
\end{array}\right)=\left(\begin{array}{cccc}
1 & x_{11} & \cdots & x_{1 p} \\
1 & x_{21} & \cdots & x_{2 p} \\
\vdots & \vdots & & \vdots \\
1 & x_{n 1} & \cdots & x_{n p}
\end{array}\right), \quad Y=\left(\begin{array}{c}
y_{1} \\
y_{2} \\
\vdots \\
y_{n}
\end{array}\right),  \tag{2.4}\\
W(i)=\operatorname{diag}\left[K_{h}\left(d_{i 1}\right), K_{h}\left(d_{i 2}\right), \ldots, K_{h}\left(d_{i n}\right)\right] \tag{2.5}
\end{gather*}
$$

is a diagonal weight matrix, ensuring that observations near to the location have greater influence than those far away. Here, $d_{i j}$ denotes the distance between two observed locations $\left(u_{i}, v_{i}\right)$ and $\left(u_{j}, v_{j}\right)$, which can be calculated as

$$
\begin{equation*}
d_{i j}=R \arccos \left(\sin v_{i} \sin v_{j}+\cos v_{i} \cos v_{j} \cos \left(u_{i}-u_{j}\right)\right), \tag{2.6}
\end{equation*}
$$

where $R$ is the earth radius, namely, 6371 kilometers. In (2.5), $K_{h}(\cdot)=1 / h K(\cdot / h)$ with $K(\cdot)$ being Gaussian kernel function

$$
\begin{equation*}
K(t)=\frac{1}{\sqrt{2 \pi}} \exp \left(-\frac{1}{2} t^{2}\right) \tag{2.7}
\end{equation*}
$$

and $h$ being the bandwidth which can be estimated by some data-driven procedures such as the cross-validation (CV) method [23], the generalized cross-validation (GCV) procedure [13], or the corrected Akaike information criterion ( $\mathrm{AIC}_{\mathrm{c}}$ ) [24]. In this paper, the CV method utilized by [23] was employed to select the optimal $h$ which was chosen to minimize

$$
\begin{equation*}
\mathrm{CV}(h)=\sum_{i=1}^{n}\left(y_{i}-\widehat{y}_{(i)}(h)\right)^{2} \tag{2.8}
\end{equation*}
$$

where $\widehat{y}_{(i)}(h)$ is the fitted value of $y_{i}$ under bandwidth $h$ with the observation at location $\left(u_{i}, v_{i}\right)$ omitted from the fitting process.

Although GWR is very appealing in analyzing spatial nonstationarity, from the statistical viewpoint, two critical questions still remain. One is the goodness-of-fit test, that is, a OLR model is compared to a GWR model to see which one provides the best fit. Usually, a GWR model can fit a given data set better than an OLR model. However, the simpler a model, the easier it can be applied and interpreted in practice. If a GWR model does not perform significantly better than an OLR model, it means that there is no significant drift in any of the model parameters. Thus, we will prefer an OLR model in practice. On the other hand, if a GWR model significantly outperforms an OLR model, we will be concerned with the second question, that is, whether each coefficient function estimate $\widehat{\beta}_{j}(u, v)(j=$ $1,2, \ldots, p)$ exhibits significant spatial variation over the studied area [11, 25]. If the answer to this question is positive, the characteristics of the data will be investigated in more details.

To compare the goodness-of-fit of a GWR model and an OLR model, a simplified procedure is summarized as follows.
(1) Formulate the hypothesis

$$
\begin{equation*}
\binom{H_{0}: Y=\beta_{0}+\beta_{1} X_{1}+\cdots+\beta_{p} X_{p}+\varepsilon}{H_{1}: Y=\beta_{0}(u, v)+\beta_{1}(u, v) X_{1}+\cdots+\beta_{p}(u, v) X_{p}+\varepsilon} \tag{2.9}
\end{equation*}
$$

(2) Construct the test statistic

$$
\begin{equation*}
F=\frac{\Upsilon^{T}(I-H) Y-\Upsilon^{T}(I-L)^{T}(I-L) Y}{Y^{T}(I-L)^{T}(I-L) Y} \tag{2.10}
\end{equation*}
$$

Here, $H=X\left(X^{T} X\right)^{-1} X^{T}, I$ is an identity matrix of order $n$, and

$$
L=\left(\begin{array}{c}
x_{1}^{T}\left(X^{T} W(1) X\right)^{-1} X^{T} W(1)  \tag{2.11}\\
x_{2}^{T}\left(X^{T} W(2) X\right)^{-1} X^{T} W(2) \\
\vdots \\
x_{n}^{T}\left(X^{T} W(n) X\right)^{-1} X^{T} W(n)
\end{array}\right)
$$

is an $n \times n$ matrix. If $H_{0}$ is true, the test statistic $F$ is to be

$$
\begin{equation*}
F=\frac{\varepsilon^{T}\left((I-H)-(I-L)^{T}(I-L)\right) \varepsilon}{\varepsilon^{T}(I-L)^{T}(I-L) \varepsilon} \tag{2.12}
\end{equation*}
$$

(3) Test the hypothesis. The $p$ value should be calculated as

$$
\begin{equation*}
p_{0}=P_{H_{0}}\left(F>F_{0}\right)=P_{H_{0}}\left(\varepsilon^{T}\left((I-H)-\left(1+F_{0}\right)(I-L)^{T}(I-L)\right) \varepsilon>0\right) \tag{2.13}
\end{equation*}
$$

where $F_{0}$ is the observed value of $F$ in (2.12). Since it is difficult to derive the null distribution of $F$ theoretically, the three-moment $X^{2}$ approximation procedure [26,27] devoted to approximate the distribution of normal variable quadratic form such as $\boldsymbol{\varepsilon}^{T}\left((I-H)-\left(1+F_{0}\right)(I-L)^{T}(I-L)\right) \varepsilon$ was used to compute the $p$ value defined in (2.13). Given a significance level $\alpha$, if $p_{0}<\alpha$, the null hypothesis should be rejected. Otherwise, we may conclude that the GWR model cannot improve the fitness significantly in comparison with the OLR model.

In order to test whether each coefficient function estimate $\widehat{\beta}_{j}(u, v)(j=1,2, \ldots, p)$ exhibits significant variation over the studied area, we employed the method developed by [12] to achieve the goal. The main steps of it are summarized as follows.
(a) For a given $k(k=0,1,2, \ldots, p)$, formulate the hypothesis

$$
\begin{equation*}
\binom{H_{0 k}: \beta_{k}\left(u_{1}, v_{1}\right)=\beta_{k}\left(u_{2}, v_{2}\right)=\cdots=\beta_{k}\left(u_{n}, v_{n}\right)=\beta_{k}}{H_{1 k}: \text { not all } \beta_{k}\left(u_{i}, v_{i}\right)(i=1,2, \ldots, n) \text { are equal }} \tag{2.14}
\end{equation*}
$$

(b) Construct the test statistic

$$
\begin{equation*}
T_{k}=\frac{Y^{T} B^{T}\left(I-(1 / n) 11^{T}\right) B Y}{Y^{T}(I-L)^{T}(I-L) Y} \tag{2.15}
\end{equation*}
$$

Here, 1 is an $n \times 1$ column vector with unity for each element, and

$$
B=\left(\begin{array}{c}
e_{k+1}^{T}\left(X^{T} W(1) X\right)^{-1} X^{T} W(1)  \tag{2.16}\\
e_{k+1}^{T}\left(X^{T} W(2) X\right)^{-1} X^{T} W(2) \\
\vdots \\
e_{k+1}^{T}\left(X^{T} W(n) X\right)^{-1} X^{T} W(n)
\end{array}\right)
$$

$e_{k+1}$ is an $n \times 1$ column vector which takes value 1 for the $(k+1)$ th element and zero for the other elements. Under the null hypothesis $H_{0 k}$, the test statistic $T_{k}$ is simplified as

$$
\begin{equation*}
T_{k}=\frac{\varepsilon^{T} B^{T}\left(I-(1 / n) 11^{T}\right) B \varepsilon}{\varepsilon^{T}(I-L)^{T}(I-L) \varepsilon} \tag{2.17}
\end{equation*}
$$

(c) Test the hypothesis. The $p$ value is

$$
\begin{equation*}
p_{k}=P_{H_{0 k}}\left(T_{k}>T_{0 k}\right)=P_{H_{0 k}}\left(\varepsilon^{T}\left(B^{T}\left(I-\frac{1}{n} 11^{T}\right) B-T_{0 k}(I-L)^{T}(I-L)\right) \varepsilon>0\right) \tag{2.18}
\end{equation*}
$$

where $T_{0 k}$ is the observed value of $T_{k}$ in (2.17). Similar to the goodness-of-fit test, the three-moment $X^{2}$ approximation procedure was used to derive the $p$ value defined in (2.18). Given a significance level $\alpha$, if $p_{k}<\alpha$, reject $H_{0 k}$; accept $H_{0 k}$ otherwise.

## 3. Analysis of Results

In this part, we will carry out numerical experiments for the OLR model and GWR model. All programs are written in Matlab.

### 3.1. Spatial Distributions of Extreme Climate Indices

Based on the values of FWD, FWN, FCD, FCN, and FEP, Figure 2 presents the spatial distributions for each of them over the 504 stations in China.

As shown in Figure 2, FWD, FWN, FCD, FCN, and FEP exhibit some regional features. Generally, there are 16 to 29 times per year for FWD and the larger values for FWD are mainly located in the north as well as the east of China. There are 18-35 times per year for FWN. If using the Yangtze River as the boundary, FWN values in the north are generally larger than those in the south. As for FCD, there are 14 to 26 times per year. Specially, FCD has small values about $14-18$ times per year in most parts of northwest China. With regard to FCN, it is about 13-28 times per year and it has small values in southern China. Furthermore, FEP values are between 9 and 33 times per year. In most of the country, its value varies from 25


Figure 2: Spatial distributions of the considered extreme climate indices ((a) FWD, (b) FWN, (c) FCD, (d) FCN, and (e) FEP) over the 504 stations in China.

Table 2: Correlation coefficients of the independent variables, that is, FWD, FWN, FCD, and FCN.

|  | FWD | FWN | FCD | FCN |
| :--- | :---: | :---: | :---: | :---: |
| FWD | 1.0000 | 0.3862 | 0.3453 | 0.1836 |
| FWN |  | 1.0000 | 0.1318 | 0.1329 |
| FCD |  | 1.0000 | 0.4174 |  |
| FCN |  |  |  | 1.0000 |

Table 3: Correlation coefficients of the GWR coefficient estimates, that is, $\widehat{\beta}_{0}(u, v), \widehat{\beta}_{1}(u, v), \widehat{\beta}_{2}(u, v), \widehat{\beta}_{3}(u, v)$, and $\widehat{\beta}_{4}(u, v)$.

|  | $\widehat{\beta}_{0}(u, v)$ | $\widehat{\beta}_{1}(u, v)$ | $\widehat{\beta}_{2}(u, v)$ | $\widehat{\beta}_{3}(u, v)$ | $\widehat{\beta}_{4}(u, v)$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $\widehat{\beta}_{0}(u, v)$ | 1.0000 | -0.4068 | -0.1734 | -0.5044 | 0.0828 |
| $\widehat{\beta}_{1}(u, v)$ |  | 1.0000 | 0.0491 | -0.3281 | -0.0366 |
| $\widehat{\beta}_{2}(u, v)$ |  |  | 1.0000 | -0.4295 | 0.6309 |
| $\widehat{\beta}_{3}(u, v)$ |  |  | 1.0000 | -0.6375 |  |
| $\widehat{\beta}_{4}(u, v)$ |  |  |  | 1.0000 |  |

to 33 times per year, and only in some stations in southern Xinjiang and Tibet, its values lie between 9 and 17 times per year.

### 3.2. The Fitted Geographically Weighted Regression Model

In order to make clear the relationship among these extreme climate indices in 504 stations in China so that some useful information can be provided to decision-makers to help them to deduce the disaster caused by extreme weather, a GWR model was fitted by considering FEP as the response variable $Y$ and FWD, FWN, FCD, and FCN as the explanatory variables $\left(X_{1}, X_{2}, X_{3}, X_{4}\right)$, respectively. Letting $n$ be equal to 504 and $p$ equal to 4 and letting ( $y_{i}, x_{i 1}, x_{i 2}, x_{i 3}, x_{i 4}$ ) be the observations of the variables $\left(Y, X_{1}, X_{2}, X_{3}, X_{4}\right)$ at the location ( $u_{i}, v_{i}$ ), the model (2.2) can be expressed as

$$
\begin{equation*}
y_{i}=\beta_{0}\left(u_{i}, v_{i}\right)+\sum_{j=1}^{4} \beta_{j}\left(u_{i}, v_{i}\right) x_{i j}+\varepsilon_{i}, \quad i=1,2, \ldots, 504 \tag{3.1}
\end{equation*}
$$

based on the data collected from the 504 stations.
When we apply a fixed Gaussian function, the minimum score of (2.8) is obtained when the bandwidth $h$ equals approximately 240 km . Thus, the weighting matrix $W(i)$ is estimated, where $w_{i j}=(1 /(240 * \sqrt{2 \pi})) \exp \left(-d_{i j}^{2} /\left(2 * 240^{2}\right)\right)$. Based on $(2.3), \widehat{\beta}_{j}(u, v)(j=$ $0,1,2,3,4)$ are calculated by the locally weighted least square approach. Hence, the strength and type of relationship that FWD (FWN, FCD, FCN) has with FEP over 504 stations in China can be studied.

Because Wheeler [28-30] raised the multicollinearity issues, correlation coefficients of the independent variables as well as that of the GWR coefficient estimates were presented in Tables 2 and 3, respectively.


Figure 3: Prediction error (PE) of the responsible variable, FEP, for ordinary linear regression (OLR) and geographically weighted regression (GWR) over the 504 stations in China.

As shown in Tables 2 and 3, correlation coefficients of the independent variables as well as that of the GWR coefficient estimates are all not large, except for that between $\widehat{\beta}_{2}(u, v)$ and $\widehat{\beta}_{4}(u, v)$, as well as $\widehat{\beta}_{3}(u, v)$ and $\widehat{\beta}_{4}(u, v)$, whose absolute values are more than 0.5 . It indicates that $\widehat{\beta}_{4}(u, v)$ has a positive correlation with $\widehat{\beta}_{2}(u, v)$, while it has a negative correlation with $\widehat{\beta}_{3}(u, v)$. We ignore the correlation between the independent variables in this paper.

After conducting the goodness-of-fit test, the computed $p$ value is smaller than the significance level 0.05 . Thus, the GWR model can describe the regression relationship significantly better than the OLR model and it indicates that the relationship between FEP and FWD, FWN, FCD, and FCN has spatial nonstationarity. Define

$$
\begin{equation*}
R^{2}=\frac{\sum_{i=1}^{504}\left(\widehat{y}_{i}-\bar{y}\right)^{2}}{\sum_{i=1}^{504}\left(y_{i}-\bar{y}\right)^{2}} \tag{3.2}
\end{equation*}
$$

to measure the goodness of fit of the regression relationship on the given data set. The $R^{2}$ values for the OLR and GWR model are 0.3953 and 0.7750 , respectively, which indicates that the GWR model can capture a larger amount $(77.50 \%$ ) of variance of FEP based on the climate indices FWD, FWN, FCD, and FCN, than the OLR model. The prediction errors (i.e., residual errors) for the OLR and GWR model are presented in Figure 3, which shows the prediction error of the GWR model and its standard error are both lower than that of the OLR model.

Furthermore, the statistical significance tests for the variations of the coefficient functions are carried out. The obtained results show that all the regression coefficient estimates

Table 4: $p$ value of relevant tests for the GWR model (3.1).

| Global <br> stationarity for <br> regression <br> relationship | Significance for <br> $\hat{\beta}_{0}(u, v)$ | Significance for <br> $\hat{\beta}_{1}(u, v)$ | Significance for <br> $\widehat{\beta}_{2}(u, v)$ | Significance for <br> $\widehat{\beta}_{3}(u, v)$ | Significance for <br> $\widehat{\beta}_{4}(u, v)$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 0 | $1.4816 * 10^{-6}$ | 0 | 0.0018 | 0 | $1.5184 * 10^{-6}$ |

$\widehat{\beta}_{j}(u, v)(j=0,1,2,3,4)$ vary significantly with the locations, that is, the influence of each explanatory variable (viz., FWD, FWN, FCD, and FCN) on the response variable FEP has spatial inconsistency. All $p$ values of relevant tests for the GWR model (3.1) are presented in Table 4.

In order to visualize these spatial inconsistencies, Figure 4 shows geographic distributions of the estimated GWR coefficient functions in China. As there is not much meaning of $\widehat{\beta}_{0}\left(u_{i}, v_{i}\right)$, the plot of it is omitted here.

Figure $4(\mathrm{a})$ shows that the values of $\widehat{\beta}_{1}(u, v)$ are between -3.5 and 2.6. Negative values of $\widehat{\beta}_{1}(u, v)$ can be observed in most of mainland China, and the most largest absolute values are located in the northern and western parts of the Xinjiang region. Few stations with positive values of $\widehat{\beta}_{1}(u, v)$ are concentrated in the southern part of Tibet, Gansu, Chongqing, and the eastern part of north China and east China.

As Figure $4(\mathrm{~b})$ manifests, the values of $\widehat{\beta}_{2}(u, v)$ are between -1.3 and 0.28 , and some stations with positive values of $\widehat{\beta}_{2}(u, v)$ are concentrated in Jilin, northern inner Mongolia, eastern coast and Hainan. However, for China as a whole, it is obvious that many areas show negative values, especially in the Xinjiang, Tibet region as well as the middle Yellow River valley and the southern part of Northeast China.

From Figure 4(c), it can be seen that the values of $\widehat{\beta}_{3}(u, v)$ are between -1.3 and 4.6. Its value is positive in most parts of the country, and it is larger in western China than in eastern China. Scattered stations with negative values can be found in the northern part of inner Mongolia and south China, especially concentrated in Yunnan and Guangdong Province.

As for $\widehat{\beta}_{4}\left(u_{i}, v_{i}\right)$, it can be found in Figure $4(\mathrm{~d})$ that its values are between -2.3 and 0.57. Negative values occur in the western China and center China, while in the north of the northeast China, north of north China and south China, positive values can be found.

On the basis of the above analysis, some regional characteristics for the relationship between the studied extreme climate indices can be observed. In western China, FEP increases with the increase of FCD, while it decreases with the increase of FWD, FWN, and FCN. In southern China, FEP increases with the increase of FCN, while it decreases with the increase of FWD, FWN, and FCD. In the northern part of northeast China, FEP increases with the increase of FCD and FCN, while it decreases with the increase of FWD and FWN. The impacts of FCN and FCD on the FEP are roughly the opposite over almost all China.

## 4. Conclusions

Based on the Chinese daily temperature and precipitation data collected at 753 meteorological stations from 1961 to 2005, the relationship among the numbers of days that experience extreme temperature or precipitation events (i.e., FEP, FWD, FWN, FCD, and FCN) is


$$
\begin{array}{ll}
\circ[-3.5,-1.7) & \cdot[0,1.3) \\
\bullet[-1.7,0) & \bullet[1.3,2.6]
\end{array}
$$

(a)


$$
\left.\begin{array}{l}
\text { - }[-1.3,0) \\
\cdot \\
\cdot
\end{array} 0,1.5\right)
$$

- $[1.5,3)$
- $[3,4.6]$
(c)


$$
\begin{array}{ll}
\circ[-1.3,-0.86) & \circ[-0.43,0) \\
\circ[-0.86,-0.43) & \cdot[0,0.28]
\end{array}
$$

(b)


$$
\circ[-2.3,-1.5)
$$

$$
\circ[-1.5,-0.74)
$$

$$
\text { - }[-0.74,0)
$$

(d)

Figure 4: Geographic distributions of the estimated GWR coefficient functions ((a) $\widehat{\beta}_{1}(u, v)$, (b) $\widehat{\beta}_{2}(u, v)$, (c) $\widehat{\beta}_{3}(u, v)$, and (d) $\left.\widehat{\beta}_{4}(u, v)\right)$ over the 504 stations in China.
investigated by a GWR model and their spatial distributions in China. The main conclusions can be summarized as follows.
(1) FWD, FWN, FCD, FCN and FEP exhibit different spatial variations. There are larger values about 24-29 times per year for FWD mainly in northeast China. In the north of the Yangtze River, FWN has larger values of 24-35 times per year. FCD has larger values about 18-26 times per year in most part of China but northwest China. As for FCN, most of China has larger values about 18-28 times except for the south. Except in some stations in southern Xinjiang and Tibet, FEP has larger values of 17-33 times per year.
(2) With respect to how FWD, FWN, FCD, and FCN affect FEP, the GWR model is significantly superior to the OLR model at the significance level 0.05 . Furthermore,
the statistical tests indicate that the influence of each explanatory variable (viz., FWD, FWN, FCD, and FCN) on FEP has spatial inconsistency.
(3) Some regional features are detected for the relationship between the studied extreme climate indices. In western China, FCD has a positive effect on FEP, which is contrary to that of FWD, FWN, and FCN. However, it is just the opposite in southern China. The effects of FCD as well as FCN on FEP are positive in the northern part of Northeast China, while those of FWD and FWN are negative. Meanwhile, FCN and FCD have the opposite influence on FEP over most of China.

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Research Article

# Robust $H_{\infty}$ Filtering for General Nonlinear Stochastic State-Delayed Systems 

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This paper studies the robust $H_{\infty}$ filtering problem of nonlinear stochastic systems with time delay appearing in state equation, measurement, and controlled output, where the state is governed by a stochastic Itô-type equation. Based on a nonlinear stochastic bounded real lemma and an exponential estimate formula, an exponential (asymptotic) mean square $H_{\infty}$ filtering design of nonlinear stochastic time-delay systems is presented via solving a Hamilton-Jacobi inequality. As one corollary, for linear stochastic time-delay systems, a Luenberger-type filter is obtained by solving a linear matrix inequality. Two simulation examples are finally given to show the effectiveness of our results.

## 1. Introduction

Robust $H_{\infty}$ filtering has been studied extensively for more than two decades, which is very useful in signal processing and engineering applications; see [1-7] and the references therein. Compared with classical Kalman filter, one does not need to know the exact statistic information about the external disturbance in the $H_{\infty}$ filtering design. $H_{\infty}$ filtering requires one to design a filter such that the $\mathcal{L}_{2}$-gain from the external disturbance to the estimation error is below a prescribed level $\gamma>0$. Stochastic Itô modelling has become more and more important in both theory and practical applications such as in mathematical finance and population models [8]. In recent years, the study of stochastic $H_{\infty}$ filtering for the systems governed by stochastic Itô-type equations has attracted a great deal of attention, for example, [2,5,9]. References [2,5] presented approaches to linear stochastic delay-free and time
delayed $H_{\infty}$ filtering design via linear matrix inequalities (LMIs), respectively. Reference [9] first solved the nonlinear stochastic delay-free $H_{\infty}$ filtering problem by means of a stochastic bounded real lemma derived in [10]. References [11, 12], respectively, solve the $H_{\infty}$ filtering and control of nonlinear stochastic time delayed systems, where time delay only happens in the state equation.

It is well known that time delay phenomena are often encountered in many engineering applications such as network control and communication, and a study of time delay systems has been a popular research topic for a long time [13]. Stochastic time delay systems are ideal models in mathematical finance and population growth theory [8]. Recently, $[14,15]$ investigated the Kalman filter problem of linear stochastic time delay systems. Reference [5] presented an approach to stochastic $H_{\infty}$ filtering design for linear uncertain time delay systems via LMIs. Reference [11] first studied the $H_{\infty}$ design issue for a class of nonlinear stochastic time delayed systems under a stronger assumption (assumption 2.1 of [11]), for which only the state equation contains a time delay. Because, in practice, time delay often exists not only in a state equation but also in a measurement equation and a controlled output, it is necessary to study such a nonlinear stochastic $H_{\infty}$ filtering design.

To our best knowledge, few works on $H_{\infty}$ filtering have been reported for general nonlinear stochastic time delay systems. The aim of this paper is to study the robust $H_{\infty}$ filtering design for nonlinear stochastic state-delayed systems, where the time delay appears in the state equation, measurement equation, and controlled output. Similar to Proposition 1 of [9], a nonlinear stochastic bounded real lemma for time delay systems is obtained, and then an exponential estimate formula is also presented. Finally, based on our developed nonlinear stochastic bounded real lemma and exponential estimate formula, we present a sufficient condition for exponential and asymptotic mean square $H_{\infty}$ filtering synthesis of nonlinear stochastic time delay systems via solving a constrained Hamilton-Jacobi inequality (HJI), respectively. Compared with the delay-free $H_{\infty}$ filtering [9], the current HJI depends on more variables due to the appearance of time delays. A key procedure to derive an exponential mean square $H_{\infty}$ filtering is to develop an exponential estimate formula (Lemma 2.3), which is very useful in its own right. In particular, in the case of linear time-invariant-delayed systems, if a quadratic Lyapunov function is chosen, then the HJI reduces to an LMI, which may be easily solved by the existing Matlab control toolbox [16].

For convenience, we adopt the following traditional notations: $A^{\prime}$ : transpose of the matrix $A ; A \geq 0(A>0)$ : $A$ is a positive semidefinite (positive definite) matrix; $I$ : identity matrix. $\|x\|$ : Euclidean 2-norm of $n$-dimensional real vector $x ; \mathcal{L}_{\mp}^{2}\left(\mathcal{R}_{+}, \boldsymbol{R}^{l}\right)$ : the space of nonanticipative stochastic processes $y(t)$ with respect to filtration $\mathcal{F}_{t}$ satisfying $\|y(t)\|_{L_{2}}^{2}:=$ $E \int_{0}^{\infty}\|y(t)\|^{2} d t<\infty ; C^{2,1}(U, T)$ : class of functions $V(x, t)$ twice continuously differentiable with respect to $x \in U$ and once continuously differentiable with respect to $t \in T$ except possibly at $x=0 ; V_{t}(x, t):=(\partial V(x, t)) / \partial t ; V_{x}(x, t):=\left(\partial V(x, t) / \partial x_{i}\right)_{n \times 1} ; V_{x x}(x, t):=$ $\left(\partial^{2} V(x, t) / \partial x_{i} \partial x_{j}\right)_{n \times n} ; C\left([-\tau, 0], \mathcal{R}^{n}\right)$ : a vector space of all continuous $\mathcal{R}^{n}$-valued functions defined on $[-\tau, 0]$.

## 2. Preliminaries

Consider the following nonlinear stochastic time delay system:

$$
\begin{aligned}
d x(t)= & (f(x(t), x(t-\tau), t)+g(x(t), x(t-\tau), t) v(t)) d t \\
& +(h(x(t), x(t-\tau), t)+s(x(t), x(t-\tau), t) v(t)) d W(t)
\end{aligned}
$$

$$
\begin{align*}
& y(t)=l(x(t), x(t-\tau), t)+k(x(t), x(t-\tau), t) v(t), \\
& z(t)=m(x(t), x(t-\tau), t) \\
& x(t)=\phi(t) \in C_{母_{0}}^{b}\left([-\tau, 0] ; \mathcal{R}^{n}\right), \tag{2.1}
\end{align*}
$$

where $x(t) \in \mathcal{R}^{n}$ is called the system state, $y(t) \in \mathcal{R}^{r}$ is the measurement, $W(\cdot)$ is a standard one-dimensional Wiener process defined on a complete filtered space $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \in \mathcal{R}_{+}}, D\right)$ with a filtration $\left\{\mathcal{F}_{t}\right\}_{t \in \mathcal{R}_{+}}$satisfying usual conditions, $z(t) \in \mathcal{R}^{m}$ is the state combination to be estimated, $v \in \mathcal{L}_{母}^{2}\left(\mathcal{R}_{+}, \mathcal{R}^{n_{v}}\right)$ stands for the exogenous disturbance signal, which is a square integrable, $\mathcal{F}_{t}$-adapted stochastic process, and $C_{\mathscr{F}_{0}}^{b}\left([-\tau, 0] ; \mathcal{R}^{n}\right)$ denotes all $\mathcal{F}_{0}$-measurable bounded $C\left([-\tau, 0], R^{n}\right)$-valued random variable $\xi(s)$ with $s \in[-\tau, 0]$. We assume that $f, h$ : $\mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}_{+} \mapsto \mathcal{R}^{n}, g, s: \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}_{+} \mapsto \mathcal{R}^{n \times n_{v}}, l: \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}_{+} \mapsto \mathcal{R}^{r}, k: \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}_{+} \mapsto \mathcal{R}^{r \times n_{v}}$, and $m: \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}_{+} \mapsto \mathcal{R}^{n_{z}}$ satisfy the local Lipschitz condition and the linear growth condition, which guarantee that the system (2.1) admits a unique strong solution; see [8]. In addition, suppose that $f(0,0, t)=h(0,0, t)=l(0,0, t) \equiv 0$, so $x \equiv 0$ is an equilibrium point of (2.1).

Since this paper deals with the infinite horizon stochastic $H_{\infty}$ filtering problem, it is inevitably related to stochastic stability. Hence, we first present the following definition.

Definition 2.1. The nonlinear stochastic time delayed system

$$
\begin{gather*}
d x(t)=f(x(t), x(t-\tau), t) d t+h(x(t), x(t-\tau), t) d W(t), \\
x(t)=\phi(t) \in C_{\Psi_{0}}^{b}\left([-\tau, 0] ; R^{n}\right), \tag{2.2}
\end{gather*}
$$

is called exponentially mean square stable if there are positive constants $A$ and $\alpha$ such that

$$
\begin{equation*}
E\|x(t)\|^{2} \leq A\|\phi\|^{2} e^{-\alpha t} \tag{2.3}
\end{equation*}
$$

where $\|\phi\|^{2}=E \max _{-\tau \leq t \leq 0}\|\phi(t)\|^{2}$.
Associated with (2.1) and $V: \mathcal{R}^{n} \times \mathcal{R}_{+} \mapsto \mathcal{R}_{+}$, we define an operator $\mathfrak{L}_{1} V: \mathcal{R}^{n} \times \mathcal{R}^{n} \times$ $\mathcal{R}_{+} \mapsto \boldsymbol{R}$ as follows:

$$
\begin{align*}
\perp_{1} V(x, y, t)= & V_{t}(x, t)+V_{x}^{\prime}(x, t)[f(x, y, t)+g(x, y, t) v(t)] \\
& +\frac{1}{2}[h(x, y, t)+s(x, y, t) v(t)]^{\prime} V_{x x}(x, t)[h(x, y, t)+s(x, y, t) v(t)] \tag{2.4}
\end{align*}
$$

The following lemma is a generalized version of Proposition 1 in [9], which may be viewed as a nonlinear stochastic bounded real lemma for time delayed systems.

Lemma 2.2. Consider the following input-output system:

$$
\begin{align*}
d x(t)= & (f(x(t), x(t-\tau), t)+g(x(t), x(t-\tau), t) v(t)) d t \\
& +(h(x(t), x(t-\tau), t)+s(x(t), x(t-\tau), t) v(t)) d W(t)  \tag{2.5}\\
z(t)= & m(x(t), x(t-\tau), t), \quad x(t)=\phi(t) \in C_{母_{0}}^{b}\left([-\tau, 0] ; R^{n}\right)
\end{align*}
$$

If there exists a positive definite Lyapunov function $V(x, t) \in C^{2,1}\left(\boldsymbol{R}^{n}, \boldsymbol{R}_{+}\right)$solving the following HJI:

$$
\begin{align*}
& \Gamma(x, y, t):= V_{t}(x, t)+V_{x}^{\prime}(x, t) f(x, y, t) \\
&+ \frac{1}{2}\left(V_{x}^{\prime}(x, t) g(x, y, t)+h^{\prime}(x, y, t) V_{x x}(x, t) s(x, y, t)\right) \\
& \times\left(r^{2} I-s^{\prime}(x, y, t) V_{x x}(x, t) s(x, y, t)\right)^{-1} \\
& \times\left(g^{\prime}(x, y, t) V_{x}(x, t)+s^{\prime}(x, y, t) V_{x x}(x, t) h(x, y, t)\right)  \tag{2.6}\\
&+\frac{1}{2}\|z(t)\|^{2}+\frac{1}{2} h^{\prime}(x, y, t) V_{x x}(x, t) h(x, y, t)<0 \\
& r^{2} I-s^{\prime}(x, y, t) V_{x x}(x, t) s(x, y, t)>0, \quad \forall(x, y, t) \in \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}_{+}, \\
& V(0,0)=0
\end{align*}
$$

for some $\gamma>0$, then the following inequality:

$$
\begin{equation*}
\|z(t)\|_{L_{2}}^{2} \leq r^{2}\|v(t)\|_{L_{2}}^{2} \quad \forall v \in \mathcal{R}_{\Psi}^{2}\left(\mathcal{R}^{+}, \mathcal{R}^{n_{v}}\right), v \neq 0 \tag{2.7}
\end{equation*}
$$

holds with initial state $x(s)=0$, a.s., for all, $s \in[-\tau, 0]$.
Proof. See Appendix A.
Lemma 2.3. Consider the unforced system

$$
\begin{gather*}
d x(t)=f(x(t), x(t-\tau), t) d t+h(x(t), x(t-\tau), t) d W(t), \\
x(t)=\phi(t) \in C_{\Psi_{0}}^{b}\left([-\tau, 0] ; \mathcal{R}^{n}\right) . \tag{2.8}
\end{gather*}
$$

If there exists a positive definite Lyapunov function $V(x, t) \in C^{2,1}\left(R^{n},[-\tau, \infty)\right), c_{1}, c_{2}, c_{3}, c_{4}>0$ with $c_{1} c_{3}>c_{2} c_{4}$ satisfying the following conditions:
(i) $c_{1}\|x\|^{2} \leq V(x, t) \leq c_{2}\|x\|^{2}$, for all $(x, t) \in R^{n} \times[-\tau, \infty)$,
(ii) $\left.\mathfrak{L}_{1} V(x, y, t)\right|_{v=0} \leq-c_{3}\|x\|^{2}+c_{4}\|y\|^{2}$, for all $t>0$,
then

$$
E\|x(t)\|^{2} \leq \begin{cases}\frac{\left(c_{4} c_{2} / c_{1}\right) \tau+c_{2}}{c_{1}}\|\phi\|^{2} e^{-\left(c_{3} / c_{2}\right) t}, & 0 \leq t \leq \tau  \tag{2.9}\\ \frac{\left(c_{4} c_{2} / c_{1}\right) \tau+c_{2}}{c_{1}}\|\phi\|^{2} e^{-\left(\left(c_{3} / c_{2}\right)-\left(c_{4} / c_{1}\right)\right) t}, & t>\tau\end{cases}
$$

that is, (2.8) is exponentially mean square stable.
Proof. See Appendix B.

In what follows, we construct the following filtering equation for the estimation of $z(t):$

$$
\begin{gather*}
d \widehat{x}(t)=\widehat{f}(\widehat{x}(t), \widehat{x}(t-\tau), t) d t+\widehat{G}(\widehat{x}(t), \widehat{x}(t-\tau), t) y(t) d t  \tag{2.10}\\
\widehat{z}(t)=\widehat{m}(\widehat{x}(t), \widehat{x}(t-\tau), t), \quad \widehat{x}(0)=0,
\end{gather*}
$$

where $\widehat{f}, \widehat{G}$, and $\widehat{m}$ that are to be determined are matrices of appropriate dimensions. One may find that (2.10) is more general, which includes the following Luenberger-type filtering as a special form:

$$
\begin{gather*}
d \widehat{x}(t)=f(\widehat{x}(t), \widehat{x}(t-\tau), t) d t+G(\widehat{x}(t), \widehat{x}(t-\tau), t)(y(t)-l(\widehat{x}(t), \widehat{x}(t-\tau), t)) d t  \tag{2.11}\\
\widehat{z}(t)=m(\widehat{x}(t), \widehat{x}(t-\tau), t), \quad \widehat{x}(0)=0 .
\end{gather*}
$$

Set $\eta(t)=\left[x^{\prime}(t) \widehat{x}^{\prime}(t)\right]^{\prime}$, and let

$$
\begin{equation*}
\widetilde{z}(t)=z(t)-\widehat{z}(t)=m(x(t), x(t-\tau), t)-\widehat{m}(\widehat{x}(t), \widehat{x}(t-\tau), t) \tag{2.12}
\end{equation*}
$$

denote the estimation error; then we get the following augmented system:

$$
\begin{gather*}
d \eta(t)=\left(f_{e}(\eta(t))+g_{e}(\eta(t)) v(t)\right) d t+\left(h_{e}(\eta(t))+s_{e}(\eta(t)) v(t)\right) d W(t), \\
\tilde{z}(t)=z(t)-\widehat{z}(t)=m(x(t), x(t-\tau), t)-\widehat{m}(\widehat{x}(t), \widehat{x}(t-\tau), t)  \tag{2.13}\\
\eta(t)=\left[\begin{array}{c}
\phi(t) \\
0
\end{array}\right], \quad \phi(t) \in C_{\Psi_{0}}^{b}\left([-\tau, 0] ; \mathcal{R}^{n}\right), \quad \forall t \in[-\tau, 0]
\end{gather*}
$$

where

$$
\left.\begin{array}{rl}
f_{e}(\eta(t))= & f(x(t), x(t-\tau), t) \\
\widehat{f}(\widehat{x}(t), \widehat{x}(t-\tau), t)+\widehat{G}(\widehat{x}(t), \widehat{x}(t-\tau), t) l(x(t), x(t-\tau), t)
\end{array}\right], ~ \begin{gathered}
g(x(t), x(t-\tau), t)  \tag{2.14}\\
\\
g_{e}(\eta(t))=\left[\begin{array}{c} 
\\
\widehat{G}(\widehat{x}(t), \widehat{x}(t-\tau), t) k(x(t), x(t-\tau), t)
\end{array}\right], \\
h_{e}(\eta(t))=\left[\begin{array}{c}
h(x(t), x(t-\tau), t) \\
0
\end{array}\right], \quad s_{e}(\eta(t))=\left[\begin{array}{c}
s(x(t), x(t-\tau), t) \\
0
\end{array}\right] .
\end{gathered}
$$

In Section 3, we let $\perp_{\eta}$ denote the infinitesimal operator of system (2.13). According to different requirements for internal stability, we are in a position to define various of $H_{\infty}$ filters as follows.

Definition 2.4 (exponential mean square $H_{\infty}$ filtering). Find the matrices $\widehat{f}, \widehat{G}$, and $\widehat{m}$ in (2.10), such that
(i) the equilibrium point $\eta \equiv 0$ of the augmented system (2.13) is exponentially mean square stable in the case $v=0$,
(ii) for a given disturbance attenuation level $\gamma>0$, the following $H_{\infty}$ performance holds for $x(t) \equiv 0$ on $t \in[-\tau, 0]$ :

$$
\begin{equation*}
\|\tilde{z}\|_{L_{2}}^{2} \leq r^{2}\|v\|_{L_{2}}^{2}, \quad \forall v \in \mathcal{L}_{\dot{F}}^{2}\left(\mathcal{R}^{+}, \mathcal{R}^{n_{v}}\right), v \neq 0 \tag{2.15}
\end{equation*}
$$

Definition 2.5 (asymptotic mean square $H_{\infty}$ filtering). If in (i) of Definition 2.4 the equilibrium point $\eta \equiv 0$ of the augmented system (2.13) is asymptotically mean square stable, that is,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} E\|\eta(t)\|^{2}=0 \tag{2.16}
\end{equation*}
$$

and (2.15) holds, then (2.10) is called an asymptotic mean square $H_{\infty}$ filter.

## 3. Main Results

Our first main result is about exponential mean square $H_{\infty}$ filter.
Theorem 3.1. Suppose that there exists a positive Lyapunov function $V(\eta, t)=V(x, \widehat{x}, t) \in$ $C^{2,1}\left(\mathcal{R}^{2 n} \times[-\tau, \infty)\right), c_{1}, c_{2}, c_{3}, c_{4}>0$ with $c_{1} c_{3}>c_{2} c_{4}$, such that

$$
\begin{align*}
& c_{1}\left(\|x\|^{2}+\|\widehat{x}\|^{2}\right) \leq V(x, \widehat{x}, t) \leq c_{2}\left(\|x\|^{2}+\|\widehat{x}\|^{2}\right), \quad \forall(x, \widehat{x}, t) \in R^{2 n} \times[-\tau, \infty) \\
& -\frac{1}{2}\|m(x, y, t)-\widehat{m}(\widehat{x}, \widehat{y}, t)\|^{2} \leq-c_{3}\left(\|x\|^{2}+\|\widehat{x}\|^{2}\right)+c_{4}\left(\|y\|^{2}+\|\widehat{y}\|^{2}\right), \quad \forall t>0 \tag{3.1}
\end{align*}
$$

For given disturbance attenuation level $\gamma>0$, if $V(\eta, t)$ solves the following HJI:

$$
\begin{align*}
& \Gamma(x, y, \widehat{x}, \widehat{y}):= V_{t}+V_{x}^{\prime} f(x, y, t)+V_{\hat{x}}^{\prime}(\widehat{f}(\widehat{x}, \widehat{y}, t)+\widehat{G}(\widehat{x}, \widehat{y}, t) l(x, y, t)) \\
&+\frac{1}{2} \Theta^{\prime}(x, \widehat{x}, y, \widehat{y}, t)\left(\gamma^{2} I-s^{\prime}(x, y, t) V_{x x} s(x, y, t)\right)^{-1} \Theta(x, \widehat{x}, y, \widehat{y}, t) \\
&+\frac{1}{2}\|m(x, y, t)-\widehat{m}(\widehat{x}, \widehat{y}, t)\|^{2}+\frac{1}{2} h^{\prime}(x, y, t) V_{x x} h(x, y, t)<0,  \tag{3.2}\\
& r^{2} I-s^{\prime}(x, y, t) V_{x x} s(x, y, t)>0, \quad \forall(x, y, \widehat{x}, \widehat{y}, t) \in \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}^{n} \times \mathcal{R}_{+} \\
& V(0,0)=0
\end{align*}
$$

for some matrices $\widehat{f}, \widehat{G}$, and $\widehat{m}$ of suitable dimensions, then the exponential mean square $H_{\infty}$ filtering is obtained by (2.10), where

$$
\begin{equation*}
\Theta^{\prime}(x, \widehat{x}, y, \widehat{y}, t)=V_{x}^{\prime} g(x, y, t)+V_{\hat{x}}^{\prime} \widehat{G}(\widehat{x}, \widehat{y}, t) k(x, y, t)+h^{\prime}(x, y, t) V_{x x} s(x, y, t) \tag{3.3}
\end{equation*}
$$

Proof of Theorem 3.1. In Lemma 2.2, we substitute $V(x, \widehat{x}, t), \tilde{z}=m(x, y, t)-\widehat{m}(\widehat{x}, \widehat{y}, t)$,

$$
\begin{gather*}
f_{e}=\left[\begin{array}{c}
f(x, y, t) \\
\widehat{f}(\widehat{x}, \widehat{y}, t)+\widehat{G}(\widehat{x}, \widehat{y}, t) l(x, y, t)
\end{array}\right], \quad g_{e}=\left[\begin{array}{c}
g(x, y, t) \\
\widehat{G}(\widehat{x}, \widehat{y}, t) k(x, y, t)
\end{array}\right], \\
h_{e}=\left[\begin{array}{c}
h(x, y, t) \\
0
\end{array}\right], \quad s_{e}=\left[\begin{array}{c}
s(x, y, t) \\
0
\end{array}\right], \tag{3.4}
\end{gather*}
$$

for $V(x, t), z, f, g, h$, and $s$, respectively; then, by a series of simple computations, (2.15) is obtained.

Next, we show the augmented system (2.13) to be exponential mean square stable for $v \equiv 0$. Set

$$
\begin{equation*}
\mathscr{L}_{\eta}^{v=0} V(x, \widehat{x}, t):=V_{t}+V_{\eta}^{\prime} f_{e}+\frac{1}{2} h_{e}^{\prime} V_{\eta \eta} h_{e} . \tag{3.5}
\end{equation*}
$$

By (3.2),

$$
\begin{align*}
\mathfrak{L}_{\eta}^{v=0} V(x, \widehat{x}, t)< & -\frac{1}{2}\|m(x, y, t)-\widehat{m}(\widehat{x}, \widehat{y}, t)\|^{2} \\
& -\frac{1}{2} \Theta^{\prime}(x, \widehat{x}, y, \widehat{y}, t)\left(r^{2} I-s^{\prime}(x, y, t) V_{x x} s(x, y, t)\right)^{-1} \Theta(x, \widehat{x}, y, \widehat{y}, t)  \tag{3.6}\\
\leq & -\frac{1}{2}\|m(x, y, t)-\widehat{m}(\widehat{x}, \widehat{y}, t)\|^{2} \\
\leq & -c_{3}\left(\|x\|^{2}+\|\widehat{x}\|^{2}\right)+c_{4}\left(\|y\|^{2}+\|\hat{y}\|^{2}\right)
\end{align*}
$$

Applying Lemma 2.3, we know that (2.13) is internally stable in exponential mean square sense. The proof of Theorem 3.1 is ended.

Inequality (3.2) is a constrained HJI, which is not easily tested in practice. However, if in (2.1) , $s \equiv 0$, that is, only the state depends on noise, then the constraint condition $\gamma^{2} I-$ $s^{\prime}(x, y, t) V_{x x} s(x, y, t)>0$ holds automatically, and HJI (3.2) becomes an unconstrained one.

The following theorem is about asymptotic mean square $H_{\infty}$ filter, which is weaker than the exponential mean square $H_{\infty}$ filter.

Theorem 3.2. Assume that $V(\eta, t) \in C^{2,1}\left(\boldsymbol{R}^{2 n}, \boldsymbol{R}_{+}\right)$has an infinitesimal upper limit, that is,

$$
\begin{equation*}
\lim _{\|\eta\| \rightarrow \infty} \inf _{t>0} V(\eta, t)=\infty \tag{3.7}
\end{equation*}
$$

Additionally, one assume that $V(\eta, t)>c\|\eta\|^{2}$ for some $c>0$. If $V(\eta, t)$ solves HJI (3.2), then (2.10) is an asymptotic mean square $H_{\infty}$ filter.

Proof. Obviously, it only needs to show that (2.13) is asymptotically mean square stable while $v=0$. From (3.6), $\mathfrak{L}_{\eta}^{v=0} V(x, \widehat{x}, t)<0$, so (2.13) is globally asymptotically stable in probability 1 according to the result of [17].

By Itô's formula and the property of stochastic integration, we have

$$
\begin{align*}
E V(\eta(t), t) & =E V(\eta(0), 0)+\left.E \int_{0}^{t} \curvearrowleft_{\eta} V(\eta(s), s)\right|_{v=0} d s+E \int_{0}^{t} h_{e}^{\prime}(\eta(s), s) V_{\eta}(\eta(s), s) d W(s) \\
& =E V(\eta(0), 0)+\left.E \int_{0}^{t} \curvearrowleft_{\eta} V(\eta(s), s)\right|_{v=0} d s \\
& \leq E V(\eta(0), 0)-\frac{1}{2} E \int_{0}^{t}\|m(x(s), x(s-\tau), s)-\widehat{m}(\widehat{x}(s), \widehat{x}(s-\tau), s)\|^{2} d s \\
& \leq E V(\eta(0), 0)<\infty \tag{3.8}
\end{align*}
$$

Set $\tilde{\mathscr{F}}_{t}=\mathcal{F}_{t} \cup \sigma(y(s), 0 \leq s \leq t)$; then (3.8) yields

$$
\begin{equation*}
E\left[V(\eta(t), t) \mid \tilde{\mathcal{F}}_{s}\right] \leq V(\eta(s), s) \quad \text { a.s. } \tag{3.9}
\end{equation*}
$$

which says that $\left\{V(\eta(t), t), \tilde{\mathscr{F}}_{t}, 0 \leq s \leq t\right\}$ is a nonnegative supermartingale with respect to $\left\{\tilde{\mathscr{F}}_{t}\right\}_{t \geq 0}$. By Doob's convergence theorem [18] and the fact that $\lim _{t \rightarrow \infty} \eta(t)=0$ a.s., it immediately yields $V(\eta(\infty), \infty)=\lim _{t \rightarrow \infty} V(\eta(t), t)=0$ a.s. Moreover, $\lim _{t \rightarrow \infty} E V(\eta(t), t)=$ $E V(\eta(\infty), \infty)=E V(0, \infty)=0$. Because $V(\eta, t) \geq c\|\eta\|^{2}$ for some $c>0$, it follows that $\lim _{t \rightarrow \infty} E\|\eta(t)\|^{2}=0$. This theorem is proved.

As one application of Theorem 3.2, we concentrate our attention on linear stochastic time delay $H_{\infty}$ filtering design. Consider the following linear time-invariant stochastic time delay system:

$$
\begin{gather*}
d x(t)=\left(A_{0} x(t)+A_{1} x(t-\tau)+B v(t)\right) d t+\left(C_{0} x(t)+C_{1} x(t-\tau)+D v(t)\right) d W(t), \\
y(t)=l_{0} x(t)+l_{1} x(t-\tau)+K v(t), \\
z(t)=m_{0} x(t)+m_{1} x(t-\tau),  \tag{3.10}\\
x(t)=\phi(t) \in C_{母_{0}}^{b}\left([-\tau, 0] ; \mathcal{R}^{n}\right),
\end{gather*}
$$

where, in (3.10), all coefficient matrices are assumed to be constant. Consider the following Luenberger-type filtering equation:

$$
\begin{gather*}
d \widehat{x}(t)=A_{0} \widehat{x}(t)+A_{1} \widehat{x}(t-\tau) d t+G\left(y(t)-l_{0} \widehat{x}(t)-l_{1} \widehat{x}(t-\tau)\right) d t, \\
\widehat{z}(t)=m_{0} \widehat{x}(t)+m_{1} \widehat{x}(t-\tau), \quad \widehat{x}(0)=0, \tag{3.11}
\end{gather*}
$$

with $G$ a constant matrix to be determined later. In this case,

$$
\begin{equation*}
\widehat{f}(\widehat{x}(t), \widehat{x}(t-\tau), t)=A_{0} \widehat{x}(t)+A_{1} \widehat{x}(t-\tau)-G\left(l_{0} \widehat{x}(t)+l_{1} \widehat{x}(t-\tau)\right), \quad \widehat{G}=G \tag{3.12}
\end{equation*}
$$

Set

$$
\begin{equation*}
V(x, \widehat{x}, t)=x^{\prime}(t) P x(t)+\int_{t-\tau}^{t} x^{\prime}(\theta) P_{1} x(\theta) d \theta+\widehat{x}^{\prime}(t) Q \widehat{x}(t)+\int_{t-\tau}^{t} \widehat{x}^{\prime}(\theta) Q_{1} \widehat{x}(\theta) d \theta \tag{3.13}
\end{equation*}
$$

where $P>0, P_{1}>0, Q>0$, and $Q_{1}>0$ are to be determined. Then by a series of computations, we have from HJI (3.2) that

$$
\begin{align*}
& V_{t}=x^{\prime}(t) P_{1} x(t)-x^{\prime}(t-\tau) P_{1} x(t-\tau)+\widehat{x}^{\prime}(t) Q_{1} \widehat{x}(t)-\widehat{x}^{\prime}(t-\tau) Q_{1} \widehat{x}(t-\tau), \\
& V_{x}^{\prime} f(x, y, t)=\left[\begin{array}{llll}
x^{\prime} & y^{\prime} & \widehat{x}^{\prime} & \widehat{y}^{\prime}
\end{array}\right]\left[\begin{array}{cccc}
P A_{0}+A_{0}^{\prime} P & \star & 0 & 0 \\
A_{1}^{\prime} P & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
\hat{x} \\
\hat{y}
\end{array}\right] \text {, } \\
& V_{\hat{x}}^{\prime} G l(x, y, t)=\left[\begin{array}{llll}
x^{\prime} & y^{\prime} & \widehat{x}^{\prime} & \widehat{y}^{\prime}
\end{array}\right]\left[\begin{array}{cccc}
0 & 0 & \star & 0 \\
0 & 0 & \star & 0 \\
Q G l_{0} & Q G l_{1} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
\widehat{x} \\
\hat{y}
\end{array}\right], \\
& V_{\hat{x}}^{\prime} \widehat{f}(\widehat{x}, \widehat{y}, t)=\left[\begin{array}{llll}
x^{\prime} & y^{\prime} & \widehat{x}^{\prime} & \widehat{y}^{\prime}
\end{array}\right]\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & Q\left(A_{0}-G l_{0}\right)+\left(A_{0}-l_{0}^{\prime} G^{\prime}\right) Q & \star \\
0 & 0 & \left(A_{1}-G l_{1}\right)^{\prime} Q & 0
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
\widehat{x} \\
\widehat{y}
\end{array}\right], \\
& \frac{1}{2} h^{\prime}(x, y, t) V_{x x}^{\prime} h(x, y, t)=\left[\begin{array}{llll}
x^{\prime} & y^{\prime} & \widehat{x}^{\prime} & \widehat{y}^{\prime}
\end{array}\right]\left[\begin{array}{cccc}
C_{0}^{\prime} P C_{0} & \star & 0 & 0 \\
C_{1}^{\prime} P C_{0} & C_{1}^{\prime} P C_{1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
\widehat{x} \\
\hat{y}
\end{array}\right], \\
& \frac{1}{2}\|m(x, y, t)-m(\widehat{x}, \widehat{y}, t)\|^{2}=\left[\begin{array}{lll}
x^{\prime} & y^{\prime} & \widehat{x}^{\prime} \\
\widehat{y}^{\prime}
\end{array}\right] \frac{1}{2}\left[\begin{array}{cccc}
m_{0}^{\prime} m_{0} & \star & \star & \star \\
m_{1}^{\prime} m_{0} & m_{1}^{\prime} m_{1} & \star & \star \\
-m_{0}^{\prime} m_{0} & -m_{0}^{\prime} m_{1} & m_{0}^{\prime} m_{0} & \star \\
-m_{1}^{\prime} m_{0} & -m_{1}^{\prime} m_{1} & m_{1}^{\prime} m_{0} & m_{1}^{\prime} m_{1}
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
\hat{x} \\
\widehat{y}
\end{array}\right], \\
& \frac{1}{2} \Theta^{\prime}(x, \widehat{x}, y, \hat{y}, t)\left(r^{2} I-s^{\prime}(x, y, t) V_{x x} s(x, y, t)\right)^{-1} \Theta(x, \widehat{x}, y, \widehat{y}, t) \\
& =\left[\begin{array}{llll}
x^{\prime} & y^{\prime} & \widehat{x}^{\prime} & \widehat{y}^{\prime}
\end{array}\right]\left[\begin{array}{c}
C_{0}^{\prime} P D+2 P B \\
C_{1}^{\prime} P D \\
2 Q G K \\
0
\end{array}\right] \frac{1}{2}\left[r^{2} I-2 D^{\prime} P D\right]^{-1}\left[\begin{array}{c}
C_{0}^{\prime} P D+2 P B \\
C_{1}^{\prime} P D \\
2 Q G K \\
0
\end{array}\right]^{\prime}\left[\begin{array}{l}
x \\
y \\
\hat{x} \\
\hat{y}
\end{array}\right], \tag{3.14}
\end{align*}
$$

where $\star$ is derived by symmetry. Hence, HJI (3.2) is equivalent to

$$
\begin{align*}
& {\left[\begin{array}{cccc}
A_{11} & \star & \star & \star \\
A_{21} & A_{22} & \star & A_{33} \\
Q G l_{0}-\frac{1}{2} m_{0}^{\prime} m_{0} & Q G l_{1}-\frac{1}{2} m_{0}^{\prime} m_{1} & & \star \\
-\frac{1}{2} m_{1}^{\prime} m_{0} & -\frac{1}{2} m_{1}^{\prime} m_{1} & \left(A_{1}-G l_{1}\right)^{\prime} Q+\frac{1}{2} m_{1}^{\prime} m_{0} & \frac{1}{2} m_{1}^{\prime} m_{1}-Q_{1}
\end{array}\right]} \\
& +\left[\begin{array}{c}
C_{0}^{\prime} P D+2 P B \\
C_{1}^{\prime} P D \\
2 Q G K \\
0
\end{array}\right] \frac{1}{2}\left[r^{2} I-2 D^{\prime} P D\right]^{-1}\left[\begin{array}{c}
C_{0}^{\prime} P D+2 P B \\
C_{1}^{\prime} P D \\
2 Q G K \\
0
\end{array}\right]^{\prime}<0,  \tag{3.15}\\
& r^{2} I-2 D^{\prime} P D>0
\end{align*}
$$

with

$$
\begin{gather*}
A_{11}=P A_{0}+A_{0}^{\prime} P+C_{0}^{\prime} P C_{0}+P_{1}+\frac{1}{2} m_{0}^{\prime} m_{0} \\
A_{21}=A_{1}^{\prime} P+C_{1}^{\prime} P C_{0}+\frac{1}{2} m_{1}^{\prime} m_{0}, \quad A_{22}=-P_{1}+C_{1}^{\prime} P C_{1}+\frac{1}{2} m_{1}^{\prime} m_{1}  \tag{3.16}\\
A_{33}=Q\left(A_{0}-G l_{0}\right)+\left(A_{0}-G l_{0}\right)^{\prime} Q+Q_{1}+\frac{1}{2} m_{0}^{\prime} m_{0} .
\end{gather*}
$$

By Schur's complement, (3.15) are equivalent to

$$
\left[\begin{array}{ccccc}
A_{11} & \star & \star & \star & \star  \tag{3.17}\\
A_{21} & A_{22} & \star & \star & \star \\
G_{1} l_{0}-\frac{1}{2} m_{0}^{\prime} m_{0} & G_{1} l_{1}-\frac{1}{2} m_{0}^{\prime} m_{1} & A_{33} & \star & \star \\
-\frac{1}{2} m_{1}^{\prime} m_{0} & -\frac{1}{2} m_{1}^{\prime} m_{1} & A_{1} Q-l_{1}^{\prime} G_{1}^{\prime}+\frac{1}{2} m_{1}^{\prime} m_{0} & \frac{1}{2} m_{1}^{\prime} m_{1}-Q_{1} & 0 \\
2 B^{\prime} P+D^{\prime} P C_{0} & D^{\prime} P C_{1} & 2 K^{\prime} G_{1}^{\prime} & 0 & -2 \gamma^{2} I+4 D^{\prime} P D
\end{array}\right]<0
$$

with $Q G=G_{1}$. Obviously, (3.17) is an LMI on $P, P_{1}, Q, Q_{1}, G_{1}$. By Theorem 3.2, we immediately obtain the following corollary.

Corollary 3.3. If (3.17) is feasible with solutions $P>0, P_{1}>0, Q>0, Q_{1}>0$, and $G_{1}$, then (3.11) is an asymptotic mean square $H_{\infty}$ filter with the filtering gain $G=Q^{-1} G_{1}$.


Figure 1: Simulation results for Example 4.1.

## 4. Illustrative Examples

Below, we give two examples to illustrate the validity of our developed theory in the above section.

Example 4.1 (one-dimensional exponential mean square $H_{\infty}$ filtering). Suppose that a stochastic signal $z$ is generated by the following nonlinear stochastic system driven by a standard Wiener process and corrupted by a stochastic external disturbance $v$, where the power of $v$ is 0.05 . We construct an $H_{\infty}$ filter to estimate $z$ from the measurement signal $y$ :

$$
\begin{gather*}
d x(t)=\left[\left(-10 x(t)-x(t) x^{2}(t-\tau)\right)+x(t-\tau) v(t)\right] d t+x(t) d W(t) \\
x(t)=\phi(t) \in C_{\not_{0}}^{b}([-\tau, 0] ; \mathcal{R}), \\
y(t)=-\frac{25}{2} x(t)-2 x(t) x(t-\tau)+v(t),  \tag{4.1}\\
z(t)=5 x(t) .
\end{gather*}
$$

For given disturbance attenuation level $\gamma=1$, according to Theorem 3.1, in order to determine the filtering parameters $\widehat{f}, \widehat{\mathrm{G}}$, and $\widehat{m}$, we must solve HJI (3.2). Set $V(x, \widehat{x})=x^{2}+\widehat{x}^{2}, \widehat{m}=-5 \widehat{x}$; then (3.1) hold obviously. In addition, we can easily test that $\Gamma(x, y, \widehat{x}, \hat{y})=-6.5 x^{2}-13.5 \hat{x}^{2}<0$ when we take $\widehat{f}=-14 \widehat{x}, \widehat{G}=1, \widehat{m}=5 \widehat{x}$. So the exponential mean square $H_{\infty}$ filter is given as

$$
\begin{equation*}
d \widehat{x}(t)=-14 \widehat{x}(t) d t+y(t) d t, \quad \widehat{z}(t)=-5 \widehat{x}(t) . \tag{4.2}
\end{equation*}
$$

Because there may be more than one triple ( $\widehat{f}, \widehat{G}, \widehat{m}$ ) solving HJI (3.2), $H_{\infty}$ filtering is in general not unique. The simulation result can be seen in Figures 1(a) and 1(b).


Figure 2: Simulation results for Example 4.2.

Example 4.2 (linear mean square $H_{\infty}$ filtering). In (3.10), we take the power of $v$ to be 0.01 , and

$$
\begin{gather*}
A_{0}=\left[\begin{array}{cc}
-2.6 & -0.2 \\
0.4 & -1.8
\end{array}\right], \quad A_{1}=\left[\begin{array}{cc}
-1.8 & 0.2 \\
-0.7 & -0.9
\end{array}\right], \quad B=\left[\begin{array}{c}
0.7 \\
0.94
\end{array}\right], \\
C_{0}=\left[\begin{array}{cc}
-0.8 & 0 \\
0 & -0.9
\end{array}\right], \quad C_{1}=\left[\begin{array}{cc}
-0.3 & 0.4 \\
0.21 & -1.05
\end{array}\right], \quad D=\left[\begin{array}{c}
0.2 \\
0.3
\end{array}\right],  \tag{4.3}\\
l_{0}=\left[\begin{array}{ll}
1.3 & 0.8
\end{array}\right], \quad l_{1}=\left[\begin{array}{ll}
1.2 & 3
\end{array}\right], \quad K=0.5, \\
m_{0}=\left[\begin{array}{lll}
-0.11 & 0.3
\end{array}\right], \quad m_{1}=\left[\begin{array}{ll}
0.28 & 0.63
\end{array}\right] .
\end{gather*}
$$

Obviously, substituting the above data into (3.17) with $\gamma=2$ and solving LMI (3.17), we have

$$
\begin{gather*}
P=\left[\begin{array}{cc}
1.6095 & -0.0293 \\
-0.0293 & 0.7909
\end{array}\right]>0, \quad P_{1}=\left[\begin{array}{cc}
3.8622 & -0.5054 \\
-0.5054 & 1.6277
\end{array}\right]>0, \\
Q=\left[\begin{array}{cc}
1.0009 & 0.0275 \\
0.0275 & 1.3260
\end{array}\right]>0, \quad Q_{1}=\left[\begin{array}{cc}
3.6487 & 0.1333 \\
0.1333 & 3.6199
\end{array}\right]>0,  \tag{4.4}\\
G_{1}=\left[\begin{array}{c}
-0.0772 \\
0.0235
\end{array}\right], \quad G=Q^{-1} G_{1}=\left[\begin{array}{c}
-0.0777 \\
0.0194
\end{array}\right] .
\end{gather*}
$$

The simulation result can be found in Figures 2(a) and 2(b).

## 5. Conclusions

This paper presents an approach to the design of $H_{\infty}$ filtering for general nonlinear stochastic time delay systems via solving HJI (3.2). Although it is difficult to solve the general HJI (3.2), under some special cases such as linear time delay systems, HJI (3.2) reduces to LMIs, which can be easily solved. How to solve HJI (3.2) is a very valuable research topic, which deserves further study. In addition, in order to avoid solving HJI (3.2), a possible scheme is to adopt a fuzzy linearized method for the original system (2.1) as done in [19].

## Appendices

## A. Proof of Lemma 2.2

As done in [9], applying the completing squares technique and considering (2.6), it is easy to obtain

$$
\begin{equation*}
\complement_{1} V(x, y, t) \leq \frac{1}{2} \gamma^{2} v^{\prime}(t) v(t)-\frac{1}{2} z^{\prime}(t) z(t) \tag{A.1}
\end{equation*}
$$

In addition, by Itô's formula, for any $T>0$, we have

$$
\begin{align*}
E V(x(T), T) & =E V(x(0), 0)+E \int_{0}^{T} d V(x(s), s) \\
& =E V(x(0), 0)+E \int_{0}^{T} \bumpeq V(x(t), t) d t  \tag{A.2}\\
& =E V(x(0), 0)+E \int_{0}^{T} \rho_{1} V(x(t), x(t-\tau, t)) d t \\
& \leq E V(x(0), 0)+\frac{1}{2} E \int_{0}^{T}\left(r^{2}\|v(t)\|^{2}-\|z(t)\|^{2}\right) d t
\end{align*}
$$

where, in (A.2), $£$ is the so-called infinitesimal operator of (2.5), which is defined by

$$
\begin{align*}
\mathscr{\perp} V(x(t), t)= & V_{t}(x(t), t)+V_{x}^{\prime}(x(t), t)[f(x(t), x(t-\tau), t)+g(x(t), x(t-\tau), t) v(t)] \\
& +\frac{1}{2}[h(x(t), x(t-\tau), t)+s(x(t), x(t-\tau), t) v(t)]^{\prime} V_{x x}(x(t), t)  \tag{A.3}\\
& \cdot[h(x(t), x(t-\tau), t)+s(x(t), x(t-\tau), t) v(t)]
\end{align*}
$$

In view of $V$ being positive and $V(0,0)=0$, it follows that for the zero initial condition $x(s) \equiv 0$, for all $s \in[-\tau, 0]$,

$$
\begin{equation*}
E \int_{0}^{T}\|z(t)\|^{2} d t \leq E \int_{0}^{T}\|v(t)\|^{2} d t \tag{A.4}
\end{equation*}
$$

which proves Lemma 2.2.

## B. Proof of Lemma 2.3

By (A.2), we know that, for any $t>0$,

$$
\begin{equation*}
E V(x(t), t)-E V(x(0), 0)=\left.\int_{0}^{t} E \mathfrak{L}_{1} V(x(s), x(s-\tau), s)\right|_{v=0} d s \tag{B.1}
\end{equation*}
$$

By given conditions (i) and (ii), (B.1) yields

$$
\begin{align*}
E V(x(t), t)-E V(x(0), 0) & \leq-c_{3} \int_{0}^{t} E\|x(s)\|^{2} d s+c_{4} \int_{0}^{t} E\|x(s-\tau)\|^{2} d s \\
& \leq-\frac{c_{3}}{c_{2}} \int_{0}^{t} E V(x(s), s) d s+\frac{c_{4}}{c_{1}} \int_{0}^{t} E V(x(s-\tau), s-\tau) d s \tag{B.2}
\end{align*}
$$

When $0 \leq t \leq \tau$, we have

$$
\begin{equation*}
E V(x(t), t) \leq\left(\frac{c_{4} c_{2}}{c_{1}} \tau+c_{2}\right)\|\phi\|^{2}-\frac{c_{3}}{c_{2}} \int_{0}^{t} E V(x(s), s) d s \tag{B.3}
\end{equation*}
$$

Applying Gronwall's inequality, it follows that $E V(x(t), t) \leq\left(\left(c_{4} c_{2} / c_{1}\right) \tau+c_{2}\right)\|\phi\|^{2} e^{-\left(c_{3} / c_{2}\right) t}$. Again, using condition (i),

$$
\begin{equation*}
E\|x(t)\|^{2} \leq \frac{\left(\left(c_{4} c_{2} / c_{1}\right) \tau+c_{2}\right)}{c_{1}}\|\phi\|^{2} e^{-\left(c_{3} / c_{2}\right) t} \tag{B.4}
\end{equation*}
$$

When $t>\tau>0$, letting $\mu=s-\tau$, (B.2) yields

$$
\begin{align*}
E V(x(t), t) \leq & c_{2}\|\phi\|^{2}-\frac{c_{3}}{c_{2}} \int_{0}^{t} E V(x(s), s) d s+\frac{c_{4}}{c_{1}} \int_{-\tau}^{t-\tau} E V(x(\mu), \mu) d t \mu \\
\leq & c_{2}\|\phi\|^{2}-\frac{c_{3}}{c_{2}} \int_{0}^{t} E V(x(s), s) d s \\
& +\frac{c_{4}}{c_{1}} \int_{-\tau}^{0} E V(x(\mu), \mu) d \mu+\frac{c_{4}}{c_{1}} \int_{0}^{t} E V(x(\mu), \mu) d \mu  \tag{B.5}\\
= & \left(\frac{c_{4} c_{2}}{c_{1}} \tau+c_{2}\right)\|\phi\|^{2}-\left(\frac{c_{3}}{c_{2}}-\frac{c_{4}}{c_{1}}\right) \int_{0}^{t} E V(x(s), s) d s
\end{align*}
$$

Repeating the same procedure as above, we have

$$
\begin{equation*}
E\|x(t)\|^{2} \leq \frac{\left(c_{4} c_{2} / c_{1}\right) \tau+c_{2}}{c_{1}}\|\phi\|^{2} e^{-\left(\left(c_{3} / c_{2}\right)-\left(c_{4} / c_{1}\right)\right) t} \tag{B.6}
\end{equation*}
$$

Lemma 2.3 is hence proved.

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Research Article

# Master-Slave Synchronization of Stochastic Neural Networks with Mixed Time-Varying Delays 

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This paper investigates the problem on master-salve synchronization for stochastic neural networks with both time-varying and distributed time-varying delays. Together with the driveresponse concept, LMI approach, and generalized convex combination, one novel synchronization criterion is obtained in terms of LMIs and the condition heavily depends on the upper and lower bounds of state delay and distributed one. Moreover, the addressed systems can include some famous network models as its special cases, which means that our methods extend those present ones. Finally, two numerical examples are given to demonstrate the effectiveness of the presented scheme.

## 1. Introduction

In the past decade, synchronization of chaotic systems has attracted considerable attention since the pioneering works of Pecora and Carroll [1], in which it shows when some conditions are satisfied, a chaotic system (the slave/response system) may become synchronized to another identical chaotic system (the master/drive system) if the master system sends some driving signals to the slave one. Now, it is widely known that there exist many benefits of having synchronization or chaos synchronization in various engineering fields, such as secure communication [2], image processing [3], and harmonic oscillation generation. Meanwhile, there exists synchronization in language development, which comes up with a common vocabulary, while agents' synchronization in organization management will improve their work efficiency. Recently, chaos synchronization has been widely investigated due to its
great potential applications. Especially, since artificial neural network model can exhibit the chaotic behaviors $[4,5]$, the synchronization has become an important area of study, see [623] and references therein. As special complex networks, delayed neural networks have been also found to exhibit some complex and unpredictable behaviors including stable equilibria, periodic oscillations, bifurcation, and chaotic attractors [24-27]. Presently, many literatures dealing with chaos synchronization phenomena in delayed neural networks have appeared. Together with various techniques such as LMI tool, M-matrix, and Jensen's inequalities, some elegant results have been derived for global synchronization of various delayed neural networks including discrete-time ones in [6-14]. Moreover, some authors have considered the problems on adaptive synchronization and $H_{\infty}$ synchronization in [15, 16].

Meanwhile, it is worth noting that, like time-delay and parameter uncertainties, noises are ubiquitous in both nature and man-made systems and the stochastic effects on neural networks have drawn much particular attention. Thus a large number of elegant results concerning dynamics of stochastic neural networks have already been presented in [17-23, 28,29 ]. Since noise can induce stability and instability oscillations to the system, by virtue of the stability theory for stochastic differential equations, there has been an increasing interest in the study of synchronization for delayed neural networks with stochastic perturbations [17-23]. Based on LMI technique, in [17-19], some novel results have been derived on the global synchronization as the addressed networks were involved in distributed delay or neutral type. Also the works [20-23] have considered the adaptive synchronization and lag synchronization for stochastic delayed neural networks. However, the control schemes in [17-19] cannot tackle the cases as the upper bound of delay's derivative is not less than 1 , and the presented results in [20-23] are not formulated in terms of LMIs, which makes them checked inconveniently by most recently developed algorithms. Meanwhile, in order to implement the practical point of view better, distributed delay should be taken into consideration and thus, some researchers have began to give some preliminary discussions in $[9-11,19]$. It is worth pointing out that the range of time delays considered in [1723] is from 0 to an upper bound. In practice, the range of delay may vary in a range for which the lower bound is not restricted to be 0 . Thus the criteria in the above literature can be more conservative because they have not considered the information on the lower bound of delay. Meanwhile, it has been verified that the convex combination idea was more efficient than some previous techniques when tackling time-varying delay, and furthermore, the novel idea needs some improvements since it has not taken distributed delay into consideration altogether [30]. Yet, few authors have employed improved convex combination to consider the stochastic neural networks with both variable and distributed variable delays and proposed some less conservative and easy-to-test control scheme for the exponential synchronization, which constitutes the main focus of the presented work.

Motivated by the above-mentioned discussion, this paper focuses on the exponential synchronization for a broad class of stochastic neural networks with mixed time-varying delays, in which two involved delays belong to the intervals. The form of addressed networks can include several well-known neural network models as the special cases. Together with the drive-response concept and Lyapunov stability theorem, a memory control law is proposed which guarantees the exponential synchronization of the drive system and response one. Finally, two illustrative examples are given to illustrate that the obtained results can improve some earlier reported works.

Notation 1. For symmetric matrix $X, X>0$ (resp., $X \geq 0)$ means that $X>0(X \geq 0)$ is a positive-definite (resp., positive-semidefinite) matrix; $A^{T}, A^{-T}$ represent the transposes of
matrices $A$ and $A^{-1}$, respectively. For $\tau>0, \mathcal{C}\left([-\tau, 0] ; \mathbf{R}^{n}\right)$ denotes the family of continuous functions $\varphi$ from $[-\tau, 0]$ to $\mathbf{R}^{n}$ with the norm $\|\varphi\|=\sup _{-\tau \leq \theta \leq 0}|\varphi|$. Let $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions; $L_{\mathscr{F}_{0}}^{p}\left([-\tau, 0] ; \mathbf{R}^{n}\right)$ is the family of all $\mathcal{F}_{0}$-measurable $\mathcal{C}\left([-\tau, 0] ; \mathbf{R}^{n}\right)$-valued random variables $\xi=\{\xi(\theta):-\tau \leq \theta \leq 0\}$ such that $\sup _{-\tau \leq \theta \leq 0} \mathbf{E}|\xi(\theta)|^{p}<\infty$, where $E\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $P$; I denotes the identity matrix with an appropriate dimension and $\left[\begin{array}{cc}X & Y \\ \gamma^{T} & Z\end{array}\right]=\left[\begin{array}{ll}X & Y \\ * & Z\end{array}\right]$ with $*$ denoting the symmetric term in a symmetric matrix.

## 2. Problem Formulations

Consider the following stochastic neural networks with time-varying delays described by

$$
\begin{equation*}
d z(t)=\left[-b(z(t))+A g(z(t))+B g(z(t-\tau(t)))+D \int_{t-\rho(t)}^{t} g(z(s)) d s+\mathrm{I}\right] d t \tag{2.1}
\end{equation*}
$$

where $z(t)=\left[z_{1}(t), \ldots, z_{n}(t)\right]^{T} \in \mathbf{R}^{n}$ is the neuron state vector, $g(z(\cdot))=$ $\left[g_{1}\left(z_{1}(\cdot)\right), \ldots, g_{n}\left(z_{n}(\cdot)\right)\right]^{T} \in \mathbf{R}$ represents the neuron activation function, $\mathrm{I} \in \mathbf{R}^{n}$ is a constant external input vector, and $A, B, D$ are the connection weight matrix, the delayed weight matrix, and the distributively delayed connection weight one, respectively.

In the paper, we consider the system (2.1) as the master system and the slave system as follows:

$$
\begin{align*}
d y(t)= & {\left[-b(y(t))+A g(y(t))+B g(y(t-\tau(t)))+D \int_{t-\rho(t)}^{t} g(y(s)) d s+\mathrm{I}+u(t)\right] d t }  \tag{2.2}\\
& +\sigma(t, \varepsilon(t), \varepsilon(t-\tau(t))) d w(t)
\end{align*}
$$

with $\varepsilon(t)=\left[\varepsilon_{1}(t), \ldots, \varepsilon_{n}(t)\right]^{T}=y(t)-z(t)$, where $A, B, D$ are constant matrices similar to the relevant ones (2.1) and $u(t)$ is the appropriate control input that will be designed in order to obtain a certain control objective. In practical situations, the output signals of the drive system (2.1) can be received by the response one (2.2).

The following assumptions are imposed on systems (2.1) and (2.2) throughout the paper.
(A1) Here $\tau(t)$ and $\rho(t)$ denote the time-varying delay and the distributed one satisfying

$$
\begin{equation*}
0 \leq \tau_{0} \leq \tau(t) \leq \tau_{m}, \quad \dot{\tau}(t) \leq \mu, 0 \leq \varrho_{0} \leq \rho(t) \leq \varrho_{m} \tag{2.3}
\end{equation*}
$$

and we introduce $\bar{\tau}_{m}=\tau_{m}-\tau_{0}, \bar{\varphi}_{m}=\varrho_{m}-\varrho_{0}$, and $\tau_{\max }=\max \left\{\tau_{m}, \varrho_{m}\right\}$.
(A2) Each function $b_{i}(\cdot): \mathbf{R} \rightarrow \mathbf{R}$ is locally Lipschitz, and there exist positive scalars $\pi_{i}$ and $\gamma_{i}(i=1,2, \ldots, n)$ such that $\pi_{i} \geq \dot{b}_{i}(z) \geq \gamma_{i}>0$ for all $z \in \mathbf{R}$. Here, we denote $\Pi=\operatorname{diag}\left\{\pi_{1}, \ldots, \pi_{n}\right\}$ and $\Gamma=\operatorname{diag}\left\{\gamma_{1}, \ldots, r_{n}\right\}$.
(A3) For the constants $\sigma_{i}^{+}, \sigma_{i}^{-}$, the neuron activation functions in (2.1) are bounded and satisfy

$$
\begin{equation*}
\sigma_{i}^{-} \leq \frac{g_{i}(x)-g_{i}(y)}{x-y} \leq \sigma_{i}^{+}, \quad \forall x, y \in \mathbf{R}, x \neq y, i=1,2, \ldots, n . \tag{2.4}
\end{equation*}
$$

(A4) In system (2.2), the function $\sigma(t, \cdot, \cdot): \mathbf{R}^{+} \times \mathbf{R}^{n} \times \mathbf{R}^{n} \rightarrow \mathbf{R}^{n \times m}(\sigma(t, 0,0)=0)$ is locally Lipschitz continuous and satisfies the linear growth condition as well. Moreover, $\sigma(t, \cdot, \cdot)$ satisfies the following condition:

$$
\begin{equation*}
\operatorname{trace}\left[\sigma^{T}(t, x, y) \sigma(t, x, y)\right] \leq x^{T} \Pi_{1}^{T} \Pi_{1} x+y^{T} \Pi_{2}^{T} \Pi_{2} y, \quad \forall x, y \in \mathbf{R}^{n} \tag{2.5}
\end{equation*}
$$

where $\Pi_{i}(i=1,2)$ are the known constant matrices of appropriate dimensions.
Let $\varepsilon(t)$ be the error state and subtract (2.1) from (2.2); it yields the synchronization error dynamical systems as follows:

$$
\begin{align*}
d \varepsilon(t)= & {\left[-\beta(\varepsilon(t))+A f(\varepsilon(t))+B f(\varepsilon(t-\tau(t)))+D \int_{t-\rho(t)}^{t} f(\varepsilon(s)) d s+u(t)\right] d t }  \tag{2.6}\\
& +\sigma(t, \varepsilon(t), \varepsilon(t-\tau(t))) d w(t),
\end{align*}
$$

where $f(\varepsilon(\cdot))=g(y(\cdot))-g(z(\cdot))$. One can check that the function $f_{i}(\cdot)$ satisfies $f_{i}(0)=0$, and

$$
\begin{equation*}
\sigma_{i}^{-} \leq \frac{f_{i}(x)-f_{i}(y)}{x-y} \leq \sigma_{i}^{+}, \quad \forall x, y \in \mathbf{R}, x \neq y, i=1,2, \ldots, n . \tag{2.7}
\end{equation*}
$$

Moreover, we denote $\bar{\Sigma}=\operatorname{diag}\left\{\sigma_{1}^{+}, \ldots, \sigma_{n}^{+}\right\}, \Sigma=\operatorname{diag}\left\{\sigma_{1}^{-}, \ldots, \sigma_{n}^{-}\right\}$, and

$$
\begin{equation*}
\Sigma_{1}=\operatorname{diag}\left\{\sigma_{1}^{+} \sigma_{1}^{-}, \ldots, \sigma_{n}^{+} \sigma_{n}^{-}\right\}, \quad \Sigma_{2}=\operatorname{diag}\left\{\frac{\sigma_{1}^{+}+\sigma_{1}^{-}}{2}, \ldots, \frac{\sigma_{n}^{+}+\sigma_{n}^{-}}{2}\right\} . \tag{2.8}
\end{equation*}
$$

In the paper, we adopt the following definition.
Definition 2.1 (see [18]). For the system (2.6) and every initial condition $\varphi=\phi-\psi \in$ $L_{q}^{2}\left(\left[-2 \tau_{\text {max }}, 0\right] ; \mathbf{R}^{n}\right)$, the trivial solution is globally exponentially stable in the mean square, if there exist two positive scalars $\mu, k$ such that

$$
\begin{equation*}
\mathrm{E}\|\varepsilon(t ; \varphi)\|^{2} \leq \mu \sup _{-\tau_{\max } \leq s \leq 0} \mathrm{E}\|\phi(s)-\psi(s)\|^{2} e^{-k t}, \quad \forall t \geq 0, \tag{2.9}
\end{equation*}
$$

where E stands for the mathematical expectation and $\phi, \psi$ are the initial conditions of systems (2.1) and (2.2), respectively.

In many real applications, we are interested in designing a memoryless state-feedback controller $u(t)=K \varepsilon(t)$, where $K \in \mathbf{R}^{n \times n}$ is a constant gain matrix. In the paper, for a special case that the information on the size of $\tau(t)$ is available, we consider the delayed feedback controller of the following form:

$$
\begin{equation*}
u(t)=K \varepsilon(t)+K_{1} \varepsilon(t-\tau(t)) \tag{2.10}
\end{equation*}
$$

then replacing $u(t)$ into system (2.6) yields

$$
\begin{align*}
d \varepsilon(t)= & {\left[-\beta(\varepsilon(t))+K \varepsilon(t)+K_{1} \varepsilon(t-\tau(t))+A f(\varepsilon(t))+B f(\varepsilon(t-\tau(t)))+D \int_{t-\rho(t)}^{t} f(\varepsilon(s)) d s\right] d t } \\
& +\sigma(t, \varepsilon(t), \varepsilon(t-\tau(t))) d w(t) . \tag{2.11}
\end{align*}
$$

Then the purpose of the paper is to design a controller $u(t)$ in (2.10) to let the slave system (2.2) synchronize with the master one (2.1).

## 3. Main Results

In this section, some lemmas are introduced firstly.
Lemma 3.1 (see [18]). For any symmetric matrix $W \in \mathbf{R}^{n \times n}, W=W^{T} \geq 0$, scalar $h>0$, vector function $\omega:[0, h] \rightarrow \mathbf{R}^{n}$ such that the integrations concerned are well defined, then $\left(\int_{0}^{h} \omega(s) d s\right)^{T} W\left(\int_{0}^{h} \omega(s) d s\right) \leq h \int_{0}^{h} \omega^{T}(s) W \omega(s) d s$.

Lemma 3.2 (see [19]). Given constant matrices $P, Q, R$, where $P^{T}=P, Q^{T}=Q$, then the linear matrix inequality (LMI) $\left[\begin{array}{rr}P & R \\ R^{T} & -Q\end{array}\right]<0$ is equivalent to the condition: $Q>0, P+R Q^{-1} R^{T}<0$.

Lemma 3.3 (see [31]). Suppose that $\Omega, \Xi_{1 i}, \Xi_{2 i}, i=1,2$ are the constant matrices of the appropriate dimensions, $\alpha \in[0,1]$, and $\beta \in[0,1]$, then the inequality $\Omega+\left[\alpha \Xi_{11}+(1-\alpha) \Xi_{12}\right]+\left[\beta \Xi_{21}+\right.$ $(1-\beta) \Xi_{22}$ ] 0 holds, if the four inequalities $\Omega+\Xi_{11}+\Xi_{21}<0, \Omega+\Xi_{11}+\Xi_{22}<0, \Omega+\Xi_{12}+\Xi_{21}<$ $0, \Omega+\Xi_{12}+\Xi_{22}<0$ hold simultaneously.

Then, a novel criterion is presented for the exponential stability for system (2.11) which can guarantee the master system (2.1) to synchronize the slave one (2.2).

Theorem 3.4. Supposing that assumptions (A1)-(A4) hold, then system (2.11) has one equilibrium point and is globally exponentially stable in the mean square, if there exist $n \times n$ matrices $P>0, Q_{j}>$ $0, R_{j}>0(j=1,2,3), Z_{i}>0, S_{i}>0, T_{i}>0, P_{i}(i=1,2), n \times n$ diagonal matrices $L>0, Q>$ $0, H>0, U>0, V>0, W>0, R>0, E>0,13 n \times n$ matrices $M, N, G$, and one scalar $\lambda \geq 0$ such that the matrix inequalities (3.1)-(3.2) hold:

$$
\begin{gather*}
-\lambda I+P+(L+H)(\bar{\Sigma}-\Sigma)+Q(\Pi-\Gamma)+\bar{\tau}_{m} Z_{2}+\tau_{0} S_{2} \leq 0  \tag{3.1}\\
{\left[\begin{array}{cc}
\Omega+\$+\$^{T}-\mathrm{I}_{i} T_{2} \mathrm{I}_{i}^{T} & \Xi_{1} \\
* & \Phi
\end{array}\right]<0, \quad\left[\begin{array}{cc}
\Omega+\$+\$^{T}-\mathrm{I}_{i} T_{2} \mathrm{I}_{i}^{T} & \Xi_{2} \\
* & \Phi
\end{array}\right]<0, \quad i=1,2,} \tag{3.2}
\end{gather*}
$$

where $\mathrm{I}_{1}=\left[\begin{array}{lll}0_{n \cdot 10 n} & I_{n} & 0_{n \cdot 2 n}\end{array}\right], I_{2}=\left[\begin{array}{lll}0_{n \cdot 11 n} & I_{n} & 0_{n \cdot n}\end{array}\right]$ and

$$
\begin{align*}
& \Omega=\left[\begin{array}{ccccccccccccc}
\Omega_{11} & 0 & 0 & P_{1}^{T} A+U \Sigma_{2} & 0 & 0 & \Omega_{17} & P_{1}^{T} K_{1} & P_{1}^{T} B & P_{1}^{T} D & P_{1}^{T} D & 0 & \Omega_{1,13} \\
* & \Omega_{22} & 0 & 0 & W \Sigma_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Omega_{33} & 0 & 0 & R \Sigma_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Omega_{44} & 0 & 0 & \Omega_{47} & 0 & 0 & 0 & 0 & 0 & A^{T} Q \\
* & * & * & * & \Omega_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Omega_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Omega_{77} & P_{2}^{T} K_{1} & P_{2}^{T} B & P_{2}^{T} D & P_{2}^{T} D & 0 & -P_{2}^{T} \\
* & * & * & * & * & * & * & \Omega_{88} & V \Sigma_{2} & 0 & 0 & 0 & K_{1}^{T} Q \\
* & * & * & * & * & * & * & * & \Omega_{99} & 0 & 0 & 0 & B^{T} Q \\
* & * & * & * & * & * & * & * & * & -T_{1} & 0 & 0 & D^{T} Q \\
* & * & * & * & * & * & * & * & * & * & -T_{2} & 0 & D^{T} Q \\
* & * & * & * & * & * & * & * & * & * & * & -T_{2} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -Q-Q^{T}
\end{array}\right], \\
& \left.\$=\left[\begin{array}{llllll}
M & -M+N & -G & 0_{13 n \cdot 4 n}-N+G & 0_{13 n \cdot 5 n}
\end{array}\right], \quad \Xi \begin{array}{lllll}
\sqrt{\tau_{0}} M & \sqrt{\bar{\tau}_{m}} N & M & N
\end{array}\right] \text {, } \\
& \Xi_{2}=\left[\begin{array}{llll}
\sqrt{\tau}_{0} M & \overline{\bar{\tau}}_{m} G & M & N G
\end{array}\right], \quad \Phi=-\operatorname{diag}\left\{S_{1}, Z_{1}, S_{2}, Z_{2}, Z_{2}\right\}, \tag{3.3}
\end{align*}
$$

With

$$
\begin{gather*}
\Omega_{11}=P_{1}^{T} K+K^{T} P_{1}+Q_{2}-U \Sigma_{1}-2 \Gamma E+\lambda \Pi_{1}^{T} \Pi_{1}, \\
\Omega_{17}=K^{T} P_{2}+P-P_{1}^{T}+\bar{\Sigma} H-\Sigma L-\Gamma Q, \\
\Omega_{1,13}=-P_{1}^{T}+K^{T} Q+E, \quad \Omega_{22}=-W \Sigma_{1}+Q_{1}+Q_{3}-Q_{2}, \\
\Omega_{33}=-Q_{3}-R \Sigma_{1}, \quad \Omega_{44}=-U+R_{2}+\varrho_{0}^{2} T_{1}+\bar{\varrho}_{m}^{2} T_{2},  \tag{3.4}\\
\Omega_{47}=L-H+A^{T} P_{2}, \quad \Omega_{55}=-W+R_{1}+R_{3}-R_{2}, \quad \Omega_{66}=-R-R_{3}, \\
\Omega_{77}=-P_{2}^{T}-P_{2}+\bar{\tau}_{m} Z_{1}+\tau_{0} S_{1}, \quad \Omega_{88}=-(1-\mu) Q_{1}-V \Sigma_{1}+\lambda \Pi_{2}^{T} \Pi_{2}, \\
\Omega_{99}=-(1-\mu) R_{1}-V .
\end{gather*}
$$

Proof. Denoting $\sigma(t)=\sigma(t, \varepsilon(t), \varepsilon(t-\tau(t)))$, we represent system (2.11) as the following equivalent form:

$$
\begin{align*}
d \varepsilon(t)= & v(t) d t+\sigma(t) d w(t), \\
v(t)= & -\beta(\varepsilon(t))+K \varepsilon(t)+K_{1} \varepsilon(t-\tau(t))+A f(\varepsilon(t))+B f(\varepsilon(t-\tau(t)))  \tag{3.5}\\
& +D \int_{t-\rho(t)}^{t} f(\varepsilon(s)) d s .
\end{align*}
$$

Now, together with assumptions (A1) and (A2), we construct the following LyapunovKrasovskii functional:

$$
\begin{equation*}
V\left(\varepsilon_{t}\right)=V_{1}\left(\varepsilon_{t}\right)+V_{2}\left(\varepsilon_{t}\right)+V_{3}\left(\varepsilon_{t}\right)+V_{4}\left(\varepsilon_{t}\right) \tag{3.6}
\end{equation*}
$$

where

$$
\begin{align*}
V_{1}\left(\varepsilon_{t}\right)= & \varepsilon^{T}(t) P \varepsilon(t)+2 \sum_{j=1}^{n} l_{j} \int_{0}^{\varepsilon_{j}}\left[f_{j}(s)-\sigma_{j}^{-} s\right] d s+2 \sum_{j=1}^{n} h_{j} \int_{0}^{\varepsilon_{j}}\left[\sigma_{j}^{+} s-f_{j}(s)\right] d s \\
& +2 \sum_{j=1}^{n} q_{j} \int_{0}^{\varepsilon_{j}}\left[\beta_{j}(s)-\gamma_{j} s\right] d s, \\
V_{2}\left(\varepsilon_{t}\right)= & \int_{t-\tau(t)}^{t-\tau_{0}}\left[\varepsilon^{T}(s) Q_{1} \varepsilon(s)+f^{T}(\varepsilon(s)) R_{1} f(\varepsilon(s))\right] d s \\
& +\int_{t-\tau_{0}}^{t}\left[\varepsilon^{T}(s) Q_{2} \varepsilon(s)+f^{T}(\varepsilon(s)) R_{2} f(\varepsilon(s))\right] d s \\
& +\int_{t-\tau_{m}}^{t-\tau_{0}}\left[\varepsilon^{T}(s) Q_{3} \varepsilon(s)+f^{T}(\varepsilon(s)) R_{3} f(\varepsilon(s))\right] d s, \\
V_{3}\left(\varepsilon_{t}\right)= & \int_{-\tau_{m}}^{-\tau_{0}} \int_{t+\theta}^{t} v^{T}(s) Z_{1} v(s) d s d \theta+\int_{-\tau_{m}}^{-\tau_{0}} \int_{t+\theta}^{t} \operatorname{trace}\left(\sigma^{T}(s) Z_{2} \sigma(s)\right) d s d \theta \\
& +\int_{-\tau_{0}}^{0} \int_{t+\theta}^{t} v^{T}(s) S_{1} v(s) d s d \theta+\int_{-\tau_{0}}^{0} \int_{t+\theta}^{t} \operatorname{trace}\left(\sigma^{T}(s) S_{2} \sigma(s)\right) d s d \theta, \\
V_{4}\left(\varepsilon_{t}\right)= & \varrho_{0} \int_{-\varrho_{0}}^{0} \int_{t+\theta}^{t} f^{T}(\varepsilon(s)) T_{1} f(\varepsilon(s)) d s d \theta+\bar{\varrho}_{m} \int_{-Q_{m}}^{-\varrho_{0}} \int_{t+\theta}^{t} f^{T}(\varepsilon(s)) T_{2} f(\varepsilon(s)) d s d \theta \tag{3.7}
\end{align*}
$$

with setting $L=\operatorname{diag}\left\{l_{1}, \ldots, l_{n}\right\}>0, H=\operatorname{diag}\left\{h_{1}, \ldots, h_{n}\right\}>0$, and $Q=\operatorname{diag}\left\{q_{1}, \ldots, q_{n}\right\}>0$. In the following, the weak infinitesimal operator $\mathcal{L}$ of the stochastic process $\left\{\varepsilon_{t}, t \geq 0\right\}$ is given in [32].

By employing (A1) and (A2) and directly computing $\rho V_{i}\left(\varepsilon_{t}\right)(i=1,2,3,4)$, it follows from any $n \times n$ matrices $P_{1}, P_{2}$ that

$$
\begin{aligned}
& \mathscr{L} V_{1}\left(\varepsilon_{t}\right) \\
& \leq 2 \varepsilon^{T}(t) P v(t)+2\left[f^{T}(\varepsilon(t))-\varepsilon^{T}(t) \Sigma\right] L v(t)+2\left[\varepsilon^{T}(t) \bar{\Sigma}-f^{T}(\varepsilon(t))\right] H v(t)+2 \beta^{T}(\varepsilon(t)) Q \\
& \quad \times\left[-\beta(\varepsilon(t))+K \varepsilon(t)+K_{1} \varepsilon(t-\tau(t))+A f(\varepsilon(t))+B f(\varepsilon(t-\tau(t)))+D \int_{t-\rho(t)}^{t} f(\varepsilon(s)) d s\right] \\
& \quad-2 \varepsilon^{T}(t) \Gamma^{T} Q v(t)+\operatorname{trace}\left[\sigma^{T}(t)[P+(L+H)(\bar{\Sigma}-\Sigma)+Q(\Pi-\Gamma)] \sigma(t)\right]+2\left[\varepsilon^{T}(t) P_{1}^{T}+v(t) P_{2}^{T}\right]
\end{aligned}
$$

$$
\begin{align*}
& \times[-v(t)- \beta(\varepsilon(t))+K \varepsilon(t)+K_{1} \varepsilon(t-\tau(t))+A f(\varepsilon(t))+B f(\varepsilon(t-\tau(t))) \\
&\left.+D \int_{t-\rho(t)}^{t} f(\varepsilon(s)) d s\right]  \tag{3.8}\\
& \varrho \\
& \leq V_{2}\left(\varepsilon_{t}\right) \\
& \leq {\left[\varepsilon^{T}\left(t-\tau_{0}\right) Q_{1} \varepsilon\left(t-\tau_{0}\right)+f^{T}\left(\varepsilon\left(t-\tau_{0}\right)\right) R_{1} f\left(\varepsilon\left(t-\tau_{0}\right)\right)\right] } \\
&-(1-\mu)\left[\varepsilon^{T}(t-\tau(t)) Q_{1} \varepsilon(t-\tau(t))+f^{T}(\varepsilon(t-\tau(t))) R_{1} f(\varepsilon(t-\tau(t)))\right]  \tag{3.9}\\
&+ {\left[\varepsilon^{T}(t) Q_{2} \varepsilon(t)+f^{T}(\varepsilon(t)) R_{2} f(\varepsilon(t))\right] } \\
&+ {\left[\varepsilon^{T}\left(t-\tau_{0}\right) \times\left(Q_{3}-Q_{2}\right) \varepsilon\left(t-\tau_{0}\right)+f^{T}\left(\varepsilon\left(t-\tau_{0}\right)\right)\left(R_{3}-R_{2}\right) f\left(\varepsilon\left(t-\tau_{0}\right)\right)\right] } \\
&- {\left[\varepsilon^{T}\left(t-\tau_{m}\right) Q_{3} \varepsilon\left(t-\tau_{m}\right)+f^{T}\left(\varepsilon\left(t-\tau_{m}\right)\right) R_{3} f\left(\varepsilon\left(t-\tau_{m}\right)\right)\right] }
\end{align*}
$$

$\varrho V_{3}\left(\varepsilon_{t}\right)$

$$
\begin{align*}
= & \bar{\tau}_{m} v^{T}(t) Z_{1} v(t)-\int_{t-\tau_{m}}^{t-\tau_{0}} v^{T}(s) Z_{1} v(s) d s+\bar{\tau}_{m} \operatorname{trace}\left[\sigma^{T}(t) Z_{2} \sigma(t)\right]-\int_{t-\tau_{m}}^{t-\tau_{0}} \operatorname{trace}\left[\sigma^{T}(s) Z_{2} \sigma(s)\right] d s \\
& +\tau_{0} v^{T}(t) S_{1} v(t)-\int_{t-\tau_{0}}^{t} v^{T}(s) S_{1} v(s) d s+\tau_{0} \operatorname{trace}\left[\sigma^{T}(t) S_{2} \sigma(t)\right]-\int_{t-\tau_{0}}^{t} \operatorname{trace}\left[\sigma^{T}(s) S_{2} \sigma(s)\right] d s \tag{3.10}
\end{align*}
$$

$\varrho V_{4}\left(\varepsilon_{t}\right)$

$$
\begin{align*}
= & f^{T}(\varepsilon(t))\left(\rho_{0}^{2} T_{1}+\bar{\varrho}_{m}^{2} T_{2}\right) f(\varepsilon(t))-\int_{t-\rho_{0}}^{t} \varrho_{0} f^{T}(\varepsilon(s)) T_{1} f(\varepsilon(s)) d s-\int_{t-\rho_{m}}^{t-\rho_{0}} \bar{\varrho}_{m} f^{T}(\varepsilon(s)) T_{2} f(\varepsilon(s)) d s \\
\leq & f^{T}(\varepsilon(t))\left(\rho_{0}^{2} T_{1}+\bar{\varrho}_{m}^{2} T_{2}\right) f(\varepsilon(t))-\left[\int_{t-\rho_{0}}^{t} f(\varepsilon(s)) d s\right]^{T} T_{1}\left[\int_{t-\rho_{0}}^{t} f(\varepsilon(s)) d s\right]-\left(1+\mu_{1}\right) \\
& \times\left[\int_{t-\rho(t)}^{t-\rho_{0}} f(\varepsilon(s)) d s\right]^{T} T_{2}\left[\int_{t-\rho(t)}^{t-\rho_{0}} f(\varepsilon(s)) d s\right] \\
& -\left(1+\mu_{2}\right)\left[\int_{t-\rho_{m}}^{t-\rho(t)} f^{T}(\varepsilon(s)) d s\right] T_{2}\left[\int_{t-\rho_{m}(t)}^{t-\rho(t)} f(\varepsilon(s)) d s\right], \tag{3.11}
\end{align*}
$$

where $\mu_{1}=\left(\rho_{m}-\rho(t)\right) / \bar{\rho}_{m}$ and $\mu_{2}=\left(\rho(t)-\rho_{0}\right) / \bar{\rho}_{m}$.

Now adding the terms on the right side of (3.8)-(3.11) to $£ V\left(\varepsilon_{t}\right)$ and employing (2.5), (3.1), it is easy to obtain

$$
\begin{align*}
& \bumpeq V\left(\varepsilon_{t}\right) \leq 2 \varepsilon^{T}(t) P \mathcal{v}(t)+2\left[f^{T}(\varepsilon(t))-\varepsilon^{T}(t) \Sigma\right] L \mathcal{v}(t)+2\left[\varepsilon^{T}(t) \bar{\Sigma}-f^{T}(\varepsilon(t))\right] H \mathcal{v}(t) \\
& +2\left[\varepsilon^{T}(t) P_{1}^{T}+v(t) P_{2}^{T}+\beta^{T}(\varepsilon(t)) Q\right] \\
& \times\left[-\beta(\varepsilon(t))+K \varepsilon(t)+K_{1} \varepsilon(t-\tau(t))+A f(\varepsilon(t))+B f(\varepsilon(t-\tau(t)))\right. \\
& \left.+D\left[\int_{t-\rho_{0}}^{t} f(\varepsilon(s)) d s+\int_{t-\rho(t)}^{t-\rho_{0}} f(\varepsilon(s)) d s\right]\right] \\
& -2\left[\varepsilon^{T}(t) P_{1}^{T}+\mathcal{v}(t) P_{2}^{T}\right] \mathcal{v}(t)-2 \varepsilon^{T}(t) \Gamma^{T} Q \mathcal{v}(t) \\
& +\left[\varepsilon^{T}\left(t-\tau_{0}\right) Q_{1} \varepsilon\left(t-\tau_{0}\right)+f^{T}\left(\varepsilon\left(t-\tau_{0}\right)\right) R_{1} f\left(\varepsilon\left(t-\tau_{0}\right)\right)\right] \\
& -(1-\mu)\left[\varepsilon^{T}(t-\tau(t)) Q_{1} \varepsilon(t-\tau(t))+f^{T}(\varepsilon(t-\tau(t))) R_{1} f(\varepsilon(t-\tau(t)))\right] \\
& +\left[\varepsilon^{T}(t) Q_{2} \varepsilon(t)+f^{T}(\varepsilon(t)) R_{2} f(\varepsilon(t))\right] \\
& +\left[\varepsilon^{T}\left(t-\tau_{0}\right)\left(Q_{3}-Q_{2}\right) \varepsilon\left(t-\tau_{0}\right)+f^{T}\left(\varepsilon\left(t-\tau_{0}\right)\right)\left(R_{3}-R_{2}\right) f\left(\varepsilon\left(t-\tau_{0}\right)\right)\right] \\
& -\left[\varepsilon^{T}\left(t-\tau_{m}\right) Q_{3} \varepsilon\left(t-\tau_{m}\right)+f^{T}\left(\varepsilon\left(t-\tau_{m}\right)\right) R_{3} f\left(\varepsilon\left(t-\tau_{m}\right)\right)\right]+v^{T}(t)\left(\bar{\tau}_{m} Z_{1}+\tau_{0} S_{1}\right) v(t) \\
& -\int_{t-\tau_{m}}^{t-\tau_{0}} v^{T}(s) Z_{1} v(s) d s-\int_{t-\tau_{0}}^{t} v^{T}(s) S_{1} v(s) d s-\int_{t-\tau_{m}}^{t-\tau_{0}} \operatorname{trace}\left[\sigma^{T}(s) Z_{2} \sigma(s)\right] d s \\
& -\int_{t-\tau_{0}}^{t} \operatorname{trace}\left[\sigma^{T}(s) S_{2} \sigma(s)\right] d s+\lambda\left[\varepsilon^{T}(t) \Pi_{1}^{T} \Pi_{1} \varepsilon(t)+\varepsilon^{T}(t-\tau(t)) \Pi_{2}^{T} \Pi_{2} \varepsilon(t-\tau(t))\right] \\
& +f^{T}(\varepsilon(t))\left(\varrho_{0}^{2} T_{1}+\bar{\varrho}_{m}^{2} T_{2}\right) f(\varepsilon(t))-\left[\int_{t-\varrho_{0}}^{t} f(\varepsilon(s)) d s\right]^{T} T_{1}\left[\int_{t-\varrho_{0}}^{t} f(\varepsilon(s)) d s\right]-\left(1+\mu_{1}\right) \\
& \times\left[\int_{t-\rho(t)}^{t-\rho_{0}} f(\varepsilon(s)) d s\right]^{T} T_{2}\left[\int_{t-\rho(t)}^{t-\rho_{0}} f(\varepsilon(s)) d s\right] \\
& -\left(1+\mu_{2}\right)\left[\int_{t-\varrho_{m}}^{t-\rho(t)} f^{T}(\varepsilon(s)) d s\right] T_{2}\left[\int_{t-\varrho_{m}}^{t-\rho(t)} f(\varepsilon(s)) d s\right] \text {. } \tag{3.12}
\end{align*}
$$

Based on methods in [33] and (2.7), for any $n \times n$ diagonal matrices $U>0, V>0, W>$ $0, R>0$, the following inequality can be achieved:

$$
\begin{align*}
0 \leq & -\left[x^{T}(t) U \Sigma_{1} x(t)-2 x^{T}(t) U \Sigma_{2} f(x(t))+f^{T}(x(t)) U f(x(t))\right] \\
& -\left[x^{T}(t-\tau(t)) V \Sigma_{1} x(t-\tau(t))-2 x^{T}(t-\tau(t)) V \Sigma_{2} f(x(t-\tau(t)))\right. \\
& \left.+f^{T}(x(t-\tau(t))) V f(x(t-\tau(t)))\right] \\
- & {\left[x^{T}\left(t-\tau_{0}\right) W \Sigma_{1} x\left(t-\tau_{0}\right)-2 x^{T}\left(t-\tau_{0}\right) W \Sigma_{2} f\left(x\left(t-\tau_{0}\right)\right)+f^{T}\left(x\left(t-\tau_{0}\right)\right) W f\left(x\left(t-\tau_{0}\right)\right)\right] } \\
- & {\left[x^{T}\left(t-\tau_{m}\right) R \Sigma_{1} x\left(t-\tau_{m}\right)-2 x^{T}\left(t-\tau_{m}\right) R \Sigma_{2} f\left(x\left(t-\tau_{m}\right)\right)+f^{T}\left(x\left(t-\tau_{m}\right)\right) R f\left(x\left(t-\tau_{m}\right)\right)\right] . } \tag{3.13}
\end{align*}
$$

From (A1), for any $n \times n$ diagonal matrix $E$, one can yield

$$
\begin{equation*}
0 \leq 2[\beta(\varepsilon(t))-\Gamma \varepsilon(t)]^{T} E \varepsilon(t) . \tag{3.14}
\end{equation*}
$$

Furthermore, for any $13 n \times n$ constant matrices $M, N, G$, we can obtain

$$
\begin{align*}
0= & 2 \zeta^{T}(t) M\left[\varepsilon(t)-\varepsilon\left(t-\tau_{0}\right)-\int_{t-\tau_{0}}^{t} v(s) d s-\int_{t-\tau_{0}}^{t} \sigma(s) d \omega(s)\right] \\
& +2 \zeta^{T}(t) N\left[\varepsilon\left(t-\tau_{0}\right)-\varepsilon(t-\tau(t))-\int_{t-\tau(t)}^{t-\tau_{0}} v(s) d s-\int_{t-\tau(t)}^{t-\tau_{0}} \sigma(s) d \omega(s)\right]  \tag{3.15}\\
& +2 \zeta^{T}(t) G\left[\varepsilon(t-\tau(t))-\varepsilon\left(t-\tau_{m}\right)-\int_{t-\tau_{m}}^{t-\tau(t)} v(s) d s-\int_{t-\tau_{m}}^{t-\tau(t)} \sigma(s) d \omega(s)\right],
\end{align*}
$$

where

$$
\begin{align*}
\zeta^{T}(t)= & {\left[\varepsilon^{T}(t) \varepsilon^{T}\left(t-\tau_{0}\right) \varepsilon^{T}\left(t-\tau_{m}\right) f^{T}(\varepsilon(t)) f^{T}\left(\varepsilon\left(t-\tau_{0}\right)\right) f^{T}\left(\varepsilon\left(t-\tau_{m}\right)\right)\right.} \\
& v^{T}(t) \varepsilon^{T}(t-\tau(t)) f^{T}(\varepsilon(t-\tau(t)))\left[\int_{t-\rho_{0}}^{t} f(\varepsilon(s) d s)\right]^{T}  \tag{3.16}\\
& {\left.\left[\int_{t-\rho(t)}^{t-\rho_{0}} f(\varepsilon(s) d s)\right]^{T}\left[\int_{t-\rho_{m}}^{t-\rho(t)} f(\varepsilon(s) d s)\right]^{T} \beta^{T}(\varepsilon(t))\right] . }
\end{align*}
$$

Then together with the methods in $[28,29]$, combining (3.12)-(3.15) yields

$$
\begin{align*}
\rho V\left(\varepsilon_{t}\right) \leq & \zeta^{T}(t)\left[\Omega+\$+\$^{T}+\tau_{0} M S_{1}^{-1} M^{T}+\left[\tau(t)-\tau_{0}\right] N Z_{1}^{-1} N^{T}+\left[\tau_{m}-\tau(t)\right] G Z_{1}^{-1} G^{T}\right. \\
& \left.-\mu_{1} \mathrm{I}_{1} T_{2} \mathrm{I}_{1}^{T}-\mu_{2} \mathrm{I}_{2} T_{2} \mathrm{I}_{2}^{T}+M S_{2}^{-1} M^{T}+N Z_{2}^{-1} N^{T}+G Z_{2}^{-1} G^{T}\right]  \tag{3.17}\\
& \times \zeta(t)+h(t):=\zeta^{T}(t) \Delta(t) \zeta(t)+h(t)
\end{align*}
$$

where $\Omega, \$$ are presented in (3.2) and

$$
\begin{align*}
h(t)= & {\left[\int_{t-\tau_{0}}^{t} \sigma(s) d \omega(s)\right]^{T} S_{2}\left[\int_{t-\tau_{0}}^{t} \sigma(s) d \omega(s)\right]+\left[\int_{t-\tau(t)}^{t-\tau_{0}} \sigma(s) d \omega(s)\right]^{T} Z_{2}\left[\int_{t-\tau(t)}^{t-\tau_{0}} \sigma(s) d \omega(s)\right] } \\
& +\left[\int_{t-\tau_{m}}^{t-\tau(t)} \sigma(s) d \omega(s)\right]^{T} Z_{2}\left[\int_{t-\tau_{m}}^{t-\tau(t)} \sigma(s) d \omega(s)\right]-\int_{t-\tau_{0}}^{t} \operatorname{trace}\left[\sigma^{T}(s) S_{2} \sigma(s)\right] d s \\
& -\int_{t-\tau(t)}^{t-\tau_{0}} \operatorname{trace}\left[\sigma^{T}(s) Z_{2} \sigma(s)\right] d s-\int_{t-\tau_{m}}^{t-\tau(t)} \operatorname{trace}\left[\sigma^{T}(s) Z_{2} \sigma(s)\right] d s \tag{3.18}
\end{align*}
$$

Together with Lemmas 3.2 and 3.3, the nonlinear matrix inequalities in (3.2) can guarantee $\Delta(t)<0$ to be true. Therefore, there must exist a negative scalar $\mathcal{X}<0$ such that

$$
\begin{equation*}
\mathcal{L} V\left(\varepsilon_{t}\right) \leq \zeta^{T}(t) \Delta(t) \zeta(t)+h(t) \leq x\left[\|\varepsilon(t)\|^{2}+\|\varepsilon(t-\tau(t))\|^{2}\right]+h(t) \tag{3.19}
\end{equation*}
$$

Taking the mathematic expectation of (3.19), we can deduce $\mathrm{E} h(t)=0, \mathrm{E} \mathcal{\rho} V\left(\varepsilon_{t}\right) \leq$ $\chi \mathbf{E}\left[\|\varepsilon(t)\|^{2}+\|\varepsilon(t-\tau(t))\|^{2}\right]$, which indicates that the dynamics of the system (2.11) is globally asymptotically stable in the mean square. Based on $V\left(\varepsilon_{t}\right)$ in (3.6) and directly computing, there must exist three positive scalars $\Theta_{i}>0, i=1,2,3$ such that

$$
\begin{equation*}
V\left(\varepsilon_{t}\right) \leq \Theta_{1}\|\varepsilon(t)\|^{2}+\Theta_{2} \int_{t-\tau_{\max }}^{t}\|\varepsilon(v)\|^{2} d v+\Theta_{3} \int_{t-\tau_{m}}^{t}\|\varepsilon(v-\tau(v))\|^{2} d v \tag{3.20}
\end{equation*}
$$

Letting $\bar{V}\left(\varepsilon_{t}\right)=e^{k t} V\left(\varepsilon_{t}\right)$, we can deduce

$$
\begin{align*}
\mathbf{E} \bar{V}\left(\varepsilon_{t}\right)-\mathbf{E} \bar{V}\left(\varepsilon_{0}\right)= & \mathbf{E} \int_{0}^{t} £\left(e^{k s} V\left(\varepsilon_{s}\right)\right) d s \\
\leq & \mathbf{E} \int_{0}^{t} e^{k s}\left\{k\left[\Theta_{1}\|\varepsilon(s)\|^{2}+\Theta_{2} \int_{s-\tau_{\max }}^{s}\|\varepsilon(v)\|^{2} d v+\Theta_{3} \int_{s-\tau_{m}}^{s}\|\varepsilon(v-\tau(v))\|^{2} d v\right]\right. \\
& \left.+\chi\left[\|\varepsilon(s)\|^{2}+\|\varepsilon(s-\tau(s))\|^{2}\right]\right\} d s \tag{3.21}
\end{align*}
$$

By changing the integration sequence, it can be deduced that

$$
\begin{align*}
& \int_{0}^{t} e^{k s} \int_{s-\tau_{\max }}^{s}\|\varepsilon(v)\|^{2} d v d s \\
& \quad \leq \int_{-\tau_{\max }}^{t} \int_{v}^{v+\tau_{\max }} e^{k s}\|\varepsilon(v)\|^{2} d s d v  \tag{3.22}\\
& \quad \leq \tau_{\max } e^{k \tau_{\max }} \int_{-\tau_{\max }}^{t}\|\varepsilon(v)\|^{2} e^{k v} d v, \\
& \int_{0}^{t} e^{k s} \int_{s-\tau_{m}}^{s}\|\varepsilon(v-\tau(v))\|^{2} d v d s \leq \tau_{m} e^{k \tau_{m}} \int_{-\tau_{m}}^{t}\|\varepsilon(v-\tau(v))\|^{2} e^{k v} d v .
\end{align*}
$$

Substituting the terms (3.22) into the relevant ones in (3.21), it is easy to have

$$
\begin{align*}
\mathbf{E} \bar{V}\left(\varepsilon_{t}\right) \leq \mathbf{E} \bar{V}\left(\varepsilon_{0}\right)+\mathbf{E}\{ & {\left[k \Theta_{1}+k \Theta_{2} \tau_{\max } e^{k \tau_{\max }}+x\right] \int_{0}^{t}\|\varepsilon(v)\|^{2} e^{k v} d v }  \tag{3.23}\\
& \left.+\left[k \Theta_{3} \tau_{m} e^{k \tau_{m}}+x\right] \int_{0}^{t}\|\varepsilon(v-\tau(v))\|^{2} e^{k v} d v+h_{0}(k)\right\},
\end{align*}
$$

where $h_{0}(k)=k \Theta_{2} \tau_{\max } e^{k \tau_{\max }} \int_{-\tau_{\max }}^{0}\|\varepsilon(v)\|^{2} e^{k v} d v+k \Theta_{3} \tau_{m} e^{k \tau_{m}} \int_{-\tau_{m}}^{0}\|\varepsilon(v-\tau(v))\|^{2} e^{k v} d v$. Choose one sufficiently small scalar $k_{0}>0$ such that $k_{0} \Theta_{1}+k_{0} \Theta_{2} \tau_{\max } e^{k_{0} \tau_{\max }}+\chi \leq 0, k_{0} \Theta_{3} \tau_{m} e^{k_{0} \tau_{m}}+\chi \leq$ 0 . Then, $\overline{\mathrm{V}}\left(\varepsilon_{t}\right) \leq \mathrm{E} h_{0}\left(k_{0}\right)+\mathrm{E} \bar{V}\left(\varepsilon_{0}\right)$. Through directly computing, there must exist a positive scalar $\Upsilon>0$ such that

$$
\begin{equation*}
\mathrm{E} \bar{V}\left(\varepsilon_{0}\right)+\mathrm{E} h_{0}\left(k_{0}\right) \leq \Upsilon \sup _{-2 \tau_{\max } \leq s \leq 0} \mathrm{E}\|\varphi(s)\|^{2} \tag{3.24}
\end{equation*}
$$

Meanwhile, $\mathbf{E} \bar{V}\left(\varepsilon_{t}\right) \geq \lambda_{\text {min }}(P) e^{k_{0} t} \mathbf{E}\|\varepsilon(t)\|^{2}$. Thus with (3.24), one can obtain

$$
\begin{equation*}
\mathbf{E}\|\varepsilon(t)\|^{2} \leq \lambda_{\min }^{-1}(P) \Upsilon \sup _{-2 \tau_{\max } \leq s \leq 0} \mathrm{E}\|\varphi(s)\|^{2} e^{-k_{0} t}, \quad \forall t \geq 0, \tag{3.25}
\end{equation*}
$$

which indicates that system (2.11) is globally exponentially stable in the mean square, and the proof is completed.

Remark 3.5. As for systems (2.1) and (2.2), many present literatures have much attention to $\beta(z(t))=C z(t)$ with $C$ positive-definite diagonal matrix, which can be checked as one special case of assumption (A3). Also in Theorem 3.4, it can be checked that $\Delta(t)<0$ in (3.17) was not simply enlarged by $\Omega+\$+\$^{T}+\tau_{0} M S_{1}^{-1} M^{T}+\bar{\tau}_{m} N Z_{1}^{-1} N^{T}+\bar{\tau}_{m} G Z_{1}^{-1} G^{T}+M S_{2}^{-1} M^{T}+$ $N Z_{2}^{-1} N^{T}+G Z_{2}^{-1} G^{T}<0$, but equivalently guaranteed by utilizing two matrix inequalities (3.2) and Lemma 3.3, which can be more effective than these techniques employed in [18, 28, 29]. Moreover, we compute and estimate $£ V_{5}\left(\varepsilon_{t}\right)$ in (3.11) more efficiently than those present ones owing to that some previously ignored terms have been taken into consideration.

In order to show the design of the estimate gain matrices $K$ and $K_{1}$, a simple transformation is made to obtain the following theorem.

Theorem 3.6. Supposing that assumptions (A1)-(A4) hold and setting $\epsilon_{1}, \epsilon_{2}>0$, then the system (2.1) and system (2.2) can exponentially achieve the master-slave synchronization in the mean square, if there exist $n \times n$ matrices $P>0, Q_{j}>0, R_{j}>0(j=1,2,3), Z_{i}>0, S_{i}>0, T_{i}>0(i=$ 1,2), $F, F_{1} n \times n$ diagonal matrices $L>0, Q>0, H>0, U>0, V>0, W>0, R>0, E>$ $0,13 n \times n$ matrices $M, N, G$, and one scalar $\lambda \geq 0$ such that the LMIs in (3.26)-(3.27) hold

$$
\begin{gather*}
-\lambda I+P+(L+H)(\bar{\Sigma}-\Sigma)+Q(\Pi-\Gamma)+\bar{\tau}_{m} Z_{2}+\tau_{0} S_{2} \leq 0  \tag{3.26}\\
{\left[\begin{array}{cc}
\Xi+\$+\$^{T}-\mathrm{I}_{i} T_{2} \mathrm{I}_{i}^{T} & \Xi_{1} \\
* & \Phi
\end{array}\right]<0, \quad\left[\begin{array}{cc}
\Xi+\$+\$^{T}-\mathrm{I}_{i} T_{2} \mathrm{I}_{i}^{T} & \Xi_{2} \\
* & \Phi
\end{array}\right]<0, \quad i=1,2} \tag{3.27}
\end{gather*}
$$

where $\mathrm{I}_{i}, \Xi_{i}(i=1,2), \$ \Phi$ are similar to the relevant ones in (3.2), and

$$
\Xi=\left[\begin{array}{ccccccccccccc}
\Xi_{11} & 0 & 0 & \Xi_{14} & 0 & 0 & \Xi_{17} & \epsilon_{1} F_{1} & \epsilon_{1} Q B & \epsilon_{1} Q D & \epsilon_{1} Q D & 0 & \Xi_{1,13}  \tag{3.28}\\
* & \Xi_{22} & 0 & 0 & W \Sigma_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Xi_{33} & 0 & 0 & R \Sigma_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & 0 & 0 & 0 & 0 & 0 & A^{T} Q \\
* & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -R-R_{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Xi_{77} & \epsilon_{2} F_{1} & \epsilon_{2} Q B & \epsilon_{2} Q D & \epsilon_{2} Q D & 0 & -\epsilon_{2} Q \\
* & * & * & * & * & * & * & \Xi_{88} & V \Sigma_{2} & 0 & 0 & 0 & F_{1}^{T} \\
* & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 & 0 & B^{T} Q \\
* & * & * & * & * & * & * & * & * & -T_{1} & 0 & 0 & D^{T} Q \\
* & * & * & * & * & * & * & * & * & * & -T_{2} & 0 & D^{T} Q \\
* & * & * & * & * & * & * & * & * & * & * & -T_{2} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -Q-Q^{T}
\end{array}\right],
$$

with

$$
\begin{gather*}
\Xi_{11}=\epsilon_{1} F+\epsilon_{1} F^{T}+Q_{2}-U \Sigma_{1}-2 \Gamma E+\lambda \Pi_{1}^{T} \Pi_{1}, \quad \Xi_{14}=\epsilon_{1} Q A+U \Sigma_{2}, \\
\Xi_{17}=\epsilon_{2} F^{T}+P-\epsilon_{1} Q+\bar{\Sigma} H-\Sigma L-\Gamma Q, \quad \Xi_{1,13}=-\epsilon_{1} Q+F^{T}+E, \\
\Xi_{22}=-W \Sigma_{1}+Q_{1}+Q_{3}-Q_{2}, \\
\Xi_{33}=-Q_{3}-R \Sigma_{1}, \quad \Xi_{44}=-U+R_{2}+\varrho_{0}^{2} T_{1}+\bar{Q}_{m}^{2} T_{2},  \tag{3.29}\\
\Xi_{47}=L-H+\epsilon_{2} A^{T} Q, \quad \Xi_{55}=-W+R_{1}+R_{3}-R_{2}, \\
\Xi_{77}=-\epsilon_{2} Q-\epsilon_{2} Q+\bar{\tau}_{m} Z_{1}+\tau_{0} S_{1}, \quad \Xi_{88}=-(1-\mu) Q_{1}-V \Sigma_{1}+\lambda \Pi_{2}^{T} \Pi_{2}, \\
\Xi_{99}=-(1-\mu) R_{1}-V .
\end{gather*}
$$

Moreover, the estimation gains $K=Q^{-T} F$ and $K_{1}=Q^{-T} F_{1}$.

Proof. Letting $P_{1}=\epsilon_{1} Q, P_{2}=\epsilon_{2} Q$ and setting $F=Q^{T} K, F_{1}=Q^{T} K_{1}$ in (3.2) of Theorem 3.4, it is easy to derive the result and the detailed proof is omitted here.

Remark 3.7. Theorem 3.6 presents one novel delay-dependent criterion guaranteeing the systems (2.1) and (2.2) to achieve the master-slave synchronization in an exponential way. The method is presented in terms of LMIs, therefore, by using LMI in MATLAB Toolbox, it is straightforward and convenient to check the feasibility of the proposed results without tuning any parameters. Moreover, the systems addressed in this paper can include some famous networks in $[17,19-21,23]$ as its special cases or $\tau(t)$ is not differentiable.

Remark 3.8. Through setting $Q_{1}=R_{1}=0$ in (3.6) and employing similar methods, Theorems 3.4 and 3.6 can be applicable without taking the upper bound on derivative of $\tau(t)$ into consideration, which means that Theorems 3.4 and 3.6 can be true even as $\mu$ is unknown.

Remark 3.9. As we all know, most of $n \times n$ free-weighting matrices of $M, N, G$ in Theorems 3.4 and 3.6 cannot help reduce the conservatism but only result in computational complexity. Thus we can choose the simplified slack matrices $M, N, G$ as follows:

$$
\begin{align*}
& M=\left[\begin{array}{lll}
M_{1} & M_{2} & 0_{n \cdot 11 n}
\end{array}\right]^{T}, \quad N=\left[\begin{array}{lllll}
0_{n \cdot n} & N_{1} & 0_{n \cdot 5 n} & N_{2} & 0_{n \cdot 5 n}
\end{array}\right]^{T},  \tag{3.30}\\
& G=\left[\begin{array}{lllll}
0_{n \cdot 2 n} & G_{1} & 0_{n \cdot 4 n} & G_{2} & 0_{n \cdot 5 n}
\end{array}\right]^{T},
\end{align*}
$$

with $n \times n$ matrices $M_{i}, N_{i}, G_{i}(i=1,2)$. Though the number of $n \times n$ matrix variables in (3.30) is much smaller than the one in (3.2) and (3.27), the numerical examples given in the paper still demonstrate that the simplified criteria can reduce the conservatism as effectively as Theorems 3.4 and 3.6 do.

## 4. Numerical Examples

In this section, two numerical examples will be given to illustrate the effectiveness of the proposed results.

Example 4.1. Consider the drive system (2.1) and response one (2.2) of delayed neural networks as follows:

$$
\begin{gather*}
b(z)=\left[\begin{array}{l}
0.7 z_{1}+0.5 \tanh \left(z_{1}\right) \\
0.7 z_{2}+0.5 \tanh \left(z_{2}\right)
\end{array}\right], \quad A=\left[\begin{array}{cc}
0.2 & -0.4 \\
-0.4 & 0.2
\end{array}\right], \quad B=\left[\begin{array}{ll}
0.2 & 0.2 \\
0.2 & 0.2
\end{array}\right], \quad D=\left[\begin{array}{cc}
0.2 & 0.3 \\
0.1 & 0.21
\end{array}\right], \\
g(z)=\left[\begin{array}{c}
0.25\left(\left|z_{1}+1\right|-\left|z_{1}-1\right|\right) \\
0.25\left(\left|z_{2}+1\right|-\left|z_{2}-1\right|\right)
\end{array}\right], \quad \sigma(t, \varepsilon(t), \varepsilon(t-\tau(t)))=0.1 \times\left[\begin{array}{cc}
\|\varepsilon(t)\| & 0 \\
0 & \|\varepsilon(t-\tau(t))\|
\end{array}\right], \\
\tau(t)=1.0 \sin ^{2}(10 t)+0.5, \quad \varrho(t)=2 \cos ^{2} t+0.5 . \tag{4.1}
\end{gather*}
$$

Then it is easy to check that $\tau_{0}=0.5, \tau_{m}=1.5, \varphi_{0}=0.5, \varphi_{m}=2.5, \mu=10$, and

$$
\begin{gather*}
\Gamma=\left[\begin{array}{cc}
0.7 & 0 \\
0 & 0.7
\end{array}\right], \quad \Pi=\left[\begin{array}{cc}
1.2 & 0 \\
0 & 1.2
\end{array}\right], \quad \Sigma=\left[\begin{array}{cc}
-0.5 & 0 \\
0 & -0.5
\end{array}\right], \quad \bar{\Sigma}=\left[\begin{array}{cc}
0.5 & 0 \\
0 & 0.5
\end{array}\right],  \tag{4.2}\\
\Pi_{1}=\Pi_{2}=\left[\begin{array}{cc}
0.1 & 0 \\
0 & 0.1
\end{array}\right] .
\end{gather*}
$$

By setting $\epsilon_{1}=0.05, \epsilon_{2}=0.01$ and utilizing Theorem 3.6, then the estimator gain matrices $K$ and $K_{1}$ in (2.10) can be worked out

$$
K=Q^{-T} F=\left[\begin{array}{rr}
-2.3981 & -0.0575  \tag{4.3}\\
-0.0575 & -2.3994
\end{array}\right], \quad K_{1}=Q^{-T} F_{1}=\left[\begin{array}{cc}
-0.2175 & 0.3347 \\
0.3347 & -0.2175
\end{array}\right] .
$$

Furthermore, as for $\tau(t)=|\sin (20 t)|+0.5, \rho(t)=2|\cos (6 t)|+0.5$, and setting $\epsilon_{1}=$ $0.05, \epsilon_{2}=0.01$, we can obtain the following estimator gain matrices by using Theorem 3.6 and Remark 3.8:

$$
K=Q^{-T} F=\left[\begin{array}{rr}
-2.5374 & -0.0528  \tag{4.4}\\
-0.0528 & -2.5385
\end{array}\right], \quad K_{1}=Q^{-T} F_{1}=\left[\begin{array}{cc}
-0.3021 & 0.3812 \\
0.3812 & -0.3021
\end{array}\right],
$$

which means that the obtained results still hold as the time delay is not differentiable. However, the methods proposed in [17-19] fail to solve the synchronization problem even without the distributed delay.

Example 4.2. As a special case, we consider the master system (2.1) of delayed stochastic neural networks as follows:

$$
\begin{equation*}
d z(t)=\left[-C z(t)+A g(z(t))+B g(z(t-\tau(t)))+D \int_{t-\rho(t)}^{t} g(z(s)) d s+\mathrm{I}\right] d t \tag{4.5}
\end{equation*}
$$

where $C=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right], A=\left[\begin{array}{cc}1.8 & -0.3 \\ -5.1 & 2.6\end{array}\right], B=\left[\begin{array}{cc}-1.6 & -0.1 \\ -0.3 & -2.5\end{array}\right], D=\left[\begin{array}{ll}2 & 1 \\ 1 & 2\end{array}\right], \mathrm{I}=\left[\begin{array}{l}0 \\ 0\end{array}\right]$, and $\tau(t)=0.95+$ $0.05 \sin ^{2}(40 t), \rho(t)=0.1$. It can be verified that $\tau_{0}=0.95, \tau_{m}=1.0, \mu=2$, and $\rho_{0}=\rho_{m}=0.1$. The activation functions can be taken as $g_{i}(s)=\tanh (s), s \in \mathbf{R}(i=1,2)$. The corresponding slave system can be

$$
\begin{align*}
d y(t)= & {\left[-C y(t)+A g(y(t))+B g(y(t-\tau(t)))+D \int_{t-\rho(t)}^{t} g(y(s)) d s+\mathrm{I}+u(t)\right] d t }  \tag{4.6}\\
& +\sigma(t, \varepsilon(t), \varepsilon(t-\tau(t))) d \omega(t)
\end{align*}
$$



Figure 1: Phase trajectories and state trajectories of drive system, response system and error system.
where $\sigma(t, \varepsilon(t), \varepsilon(t-\tau(t)))=\left[\begin{array}{cc}\|\varepsilon(t)\| & 0 \\ 0 & \|\varepsilon(t-\tau(t))\|\end{array}\right]$. Then together with Theorem 3.6, $\epsilon_{1}=0.05$, and $\epsilon_{2}=0.1$, we can obtain part feasible solution to the LMIs in (3.26) and (3.27) by resorting to the Matlab LMI Toolbox:

$$
Q=\left[\begin{array}{cc}
0.2526 & 0  \tag{4.7}\\
0 & 0.2526
\end{array}\right], \quad F=\left[\begin{array}{cc}
-4.0139 & -0.0421 \\
0.0486 & -4.0182
\end{array}\right], \quad F_{1}=\left[\begin{array}{ll}
0.1133 & 0.0043 \\
0.0168 & 0.1707
\end{array}\right]
$$

Then the estimator gain matrices $K, K_{1}$ can be deduced as follows:

$$
K=Q^{-T} F=\left[\begin{array}{cc}
-15.8892 & -0.1667  \tag{4.8}\\
0.1923 & -15.9062
\end{array}\right], \quad K_{1}=Q^{-T} F_{1}=\left[\begin{array}{ll}
0.4486 & 0.0170 \\
0.0666 & 0.6758
\end{array}\right]
$$

It follows from Theorem 3.6 that the drive system with the initial condition $[0.5,0.4]^{T}$ for $-1 \leq t \leq 0$ synchronizes with the response system when the initial condition is $[0.7,0.6]^{T}$ for
$-1 \leq t \leq 0$. The phase trajectories and state ones of drive system and response one and state trajectories of error system are shown in Figure 1. Therefore, from Figure 1, we can see that the master system synchronizes with the slave system.

## 5. Conclusions

In this paper, we consider the synchronization control of stochastic neural networks with both time-varying and distributed time-varying delays. By using the Lyapunov functional and LMI technique, one sufficient condition has been derived to ensure the global exponential stability for the error system, and thus, the slave system can synchronize the master one. Then, the estimation gains can be obtained. The obtained results are novel since the addressed networks are of more general forms and some good mathematical techniques are employed. Finally, we give two numerical examples to verify the theoretical results.

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## Research Article

# Properties of Recurrent Equations for the Full-Availability Group with BPP Traffic 

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#### Abstract

The paper proposes a formal derivation of recurrent equations describing the occupancy distribution in the full-availability group with multirate Binomial-Poisson-Pascal (BPP) traffic. The paper presents an effective algorithm for determining the occupancy distribution on the basis of derived recurrent equations and for the determination of the blocking probability as well as the loss probability of calls of particular classes of traffic offered to the system. A proof of the convergence of the iterative process of estimating the average number of busy traffic sources of particular classes is also given in the paper.


## 1. Introduction

Dimensioning and optimization of integrated networks, that is, Integrated Services Digital Networks (ISDN) and Broadband ISDN (B-ISDN) as well as wireless multiservice networks (e.g., UMTS), have recently developed an interest in multirate models [1-5]. These models are discrete models in which it is assumed that the resources required by calls of particular traffic classes are expressed as the multiple of the so-called Basic Bandwidth Units (BBUs). The BBU is defined as the greatest common divisor of the resources demanded by all call streams offered to the system $[6,7]$.

Multirate systems can be analysed on the basis of statistical equilibrium equations resulting from the multidimensional Markov process that describe the service process in the considered systems [8-13]. Such an approach, however, is not effective because of the quickly increasing-along with the system's capacity-number of states in which a multidimensional Markov process occurring within the system can take place [14]. Consequently, for an analysis of multirate systems, there are used methods based on the convolution algorithm
[11, 15] and the recurrent methods in which the multidimensional service process-occurring in the considered systems-is approximated by one-dimensional Markov chain [16-21]. The convolution methods allow us to determine exactly the occupancy distribution in the so-called full-availability systems servicing traffic streams with arbitrary distributions (i.e., systems with state-independent admission process and with both state-independent and state-dependent arrival processes). In the case of the systems with state-dependent admission process (i.e., the system in which the admission of a new call is conditioned not only by the sufficient number of free BBUs but also by the structure of the system and the introduced admission policy) the convolution methods lead to elaboration of approximate methods with quite high computational complexity [22,23].

Nowadays, in the analysis and optimization of multirate systems, the recurrent algorithms are usually used. This group of algorithms is based on the approximation of the multidimensional service process in the considered system by the one-dimensional Markov chain. Such approach leads to a determination of the occupancy distribution in systems with state-independent admission process and state-independent arrival process (in teletraffic engineering such system is called the full-availability group with Erlang traffic streams) on the basis of simple Kaufman-Roberts recurrence [24, 25] and its modifications [16-19, 26, 27]. One of them, the so-called Delbrouck recurrence [18], allows us to determine the occupancy distribution in the system with state-independent admission process (the full-availability group) and BPP traffic streams. The research on the full-availability group model, started by Delbrouck, was subsequently continued, for example, in [12, 28-30].

Because of the simplicity of the Kaufman-Roberts equation, in many works the attempts of its modification in order to analyse the systems with BPP traffic were undertaken. In [13] the modified form of the Kaufman-Roberts equation that makes the value of offered traffic dependent on the number of active sources was presented. In [31] the approximation of the number of active sources with their mean values in relation to the total value of occupied resources in particular states of the system was proposed. In [32], on the basis of the method proposed in [31], the Kaufman-Roberts equation was generalized for systems with BPP traffic and state-dependent call admission process. The accuracy of the method for modelling systems with multirate BPP traffic-further on called the Multiple Iteration Method-BPP (MIM-BPP)—proposed in [32] was verified in simulations for systems with both state-independent and state-dependent call admission process. In publications issued so far, no attempt to formally prove the correctness of the MIM-BPP assumptions was taken up.

The aim of this paper is to formally prove that the MIM-BPP algorithm [32], considered earlier as an approximate algorithm, is exact. To this purpose we derive recurrent equations describing the occupancy distribution in the full-availability group with multirate BPP traffic. We are going to demonstrate at the same time that the number of calls of particular Engset and Pascal classes appearing in equations that determine the occupancy distribution is exactly determined with their average values. Additionally, we intend to prove the convergence of the iterative process of estimating the average number of busy traffic sources of particular classes.

The paper is organized as follows. Section 2 presents an analysis of the call admission and the call arrival process in the full-availability group with BPP traffic at the micro- and macrostate level. In Section 3 an iterative method for estimating the average number of busy traffic sources of particular classes is presented, and its convergence is proved. The paper ends with a summary contained in Section 4.


Figure 1: Full-availability group with the Erlang, Engset, and Pascal traffic stream.

## 2. Full-Availability Group with BPP Traffic

### 2.1. Basic Assumptions

Let us consider a model of the full-availability group with the capacity of $V$ BBUs (Figure 1). The group is offered traffic streams of three types: $m_{I}$ Erlang streams (Poisson distribution of call streams) from the set $I=\left\{1, \ldots, i, \ldots, m_{I}\right\}, m_{J}$ Engset streams (binomial distribution of call streams) from the set $J=\left\{1, \ldots, j, \ldots, m_{J}\right\}$, and $m_{K}$ Pascal streams (negative binomial distribution of call stream) from the set $K=\left\{1, \ldots, k, \ldots, m_{K}\right\}$. In the paper it has been adopted that the letter " $i$ " denotes any class of Erlang traffic, letter " $j$ " any class of Engset traffic, and letter " $k$ " any class of Pascal traffic, whereas the letter " $c$ " any traffic class. (In relation to the ITU-T recommendations [11], all types of discussed traffic are defined collectively by the term BPP traffic. Thus, we use the term BPP when we talk about all traffic types cumulatively, whereas when we consider single traffic streams, then, because our study is focused on systems with limited capacity only, we use the terms Erlang, Engset, and Pascal streams.) The number of BBUs demanded by calls of class $c$ is denoted by $t_{c}$.

The call arrival rate for Erlang traffic of class $i$ is equal to $\lambda_{i}$. The parameter $\lambda_{j}\left(y_{j}\right)$ determines the call intensity for the Engset traffic stream of class $j$, whereas the parameter $\lambda_{k}\left(z_{k}\right)$ determines the call intensity for Pascal traffic stream of class $k$. The arrival rates $\lambda_{j}\left(y_{j}\right)$ and $\lambda_{k}\left(z_{k}\right)$ depend on the number of $y_{j}$ and $z_{k}$ of currently serviced calls of class $j$ and $k$. In the case for Engset stream, the arrival rate of class $j$ stream decreases with the number of serviced traffic sources:

$$
\begin{equation*}
\lambda_{j}\left(y_{j}\right)=\left(N_{j}-y_{j}\right) \Lambda_{j}, \tag{2.1}
\end{equation*}
$$

where $N_{j}$ is the number of Engset traffic sources of class $j$, while $\Lambda_{j}$ is the arrival rate of calls generated by a single free source of class $j$. In the case of Pascal stream of class $k$, the arrival rate increases with the number of serviced sources:

$$
\begin{equation*}
\lambda_{k}\left(z_{k}\right)=\left(S_{k}+z_{k}\right) \gamma_{k}, \tag{2.2}
\end{equation*}
$$

where $S_{k}$ is the number of Pascal traffic sources of classes $k$, while $\gamma_{k}$ is the arrival rate of calls generated by a single free source of class $k$.


Figure 2: Fragment of a diagram of Markov process in the full-availability group with BPP traffic.

The total intensity of Erlang traffic of class $i$ offered to the group amounts to

$$
\begin{equation*}
A_{i}=\frac{\lambda_{i}}{\mu_{i}}{ }^{\prime} \tag{2.3}
\end{equation*}
$$

whereas the intensity of Engset traffic $\alpha_{j}$ and Pascal traffic $\beta_{k}$ of class $j$ and $k$, respectively, offered by one free source, is equal to

$$
\begin{equation*}
\alpha_{j}=\frac{\Lambda_{j}}{\mu_{j}}, \quad \beta_{k}=\frac{\gamma_{k}}{\mu_{k}} . \tag{2.4}
\end{equation*}
$$

In (2.3) and (2.4) the parameter $\mu$ is the average service intensity with the exponential distribution.

### 2.2. The Multidimensional Erlang-Engset-Pascal Model at the Microstate Level

Let us consider now a fragment of the multidimensional Markov process in the fullavailability group with the capacity of $V$ BBUs presented in Figure 2. The group is offered traffic streams of three types: Erlang, Engset, and Pascal. Each microstate of the process $\left\{x_{1}, \ldots, x_{i}, \ldots, x_{m_{I}}, y_{1}, \ldots, y_{j}, \ldots, y_{m_{J}}, z_{1}, \ldots, z_{k}, \ldots, z_{m_{K}}\right\}$ is defined by the number of serviced calls of each of the classes of offered traffic, where $x_{i}$ denotes the number of serviced calls of the Poisson stream of class $i$ (Erlang traffic), $y_{j}$ denotes the number of serviced calls of the binomial stream of class $j$ (Engset traffic), whereas $z_{k}$ determines the number of serviced calls of the negative binomial stream of class $k$ (Pascal traffic). To simplify the description, the microstate probability will be denoted by the symbol $p\left(x_{i}, y_{j}, z_{k}\right)$.

The multidimensional service process in the Erlang-Engset-Pascal model is a reversible process. In concordance with Kolmogorov reversibility test considering any cycle for the microstates shown in Figure 2, we always obtain equality in the intensity of transitions (streams) in both directions. The property of reversibility implies the local equilibrium
equations between any of the two neighbouring states of the process. Such equations for the Erlang stream of class $i$, the Engset stream of class $j$, and Pascal stream of class $k$ can be written in the following way (Figure 2):

$$
\begin{gather*}
x_{i} \mu_{i} p\left(x_{i}, y_{j}, z_{k}\right)=\lambda_{i} p\left(x_{i}-1, y_{j}, z_{k}\right),  \tag{2.5}\\
y_{j} \mu_{j} p\left(x_{i}, y_{j}, z_{k}\right)=\left[N_{j}-\left(y_{j}-1\right)\right] \Lambda_{j} p\left(x_{i}, y_{j}-1, z_{k}\right),  \tag{2.6}\\
z_{k} \mu_{k} p\left(x_{i}, y_{j}, z_{k}\right)=\left[S_{k}+\left(z_{k}-1\right)\right] \gamma_{k} p\left(x_{i}, y_{j}, z_{k}-1\right) . \tag{2.7}
\end{gather*}
$$

Since the call streams offered to the group are independent, we can add up, for the microstate $\left\{x_{i}, y_{j}, z_{k}\right\}$, all $m_{I}$ equations of type (2.5) for the Erlang streams, $m_{J}$ equations of type (2.6) for the Engset streams, and $m_{K}$ equations of type (2.7) for the Pascal streams. Additionally, taking into consideration traffic intensity (see (2.3) and (2.4)), we get

$$
\begin{align*}
& p\left(x_{i}, y_{j}, z_{k}\right)\left[\sum_{i=1}^{m_{I}} x_{i} t_{i}+\sum_{j=1}^{m_{J}} y_{j} t_{j}+\sum_{k=1}^{m_{K}} z_{k} t_{k}\right] \\
& \quad=\sum_{i=1}^{m_{I}} A_{i} t_{i} p\left(x_{i}-1, y_{j}, z_{k}\right)+\sum_{j=1}^{m_{J}}\left[N_{j}-\left(y_{j}-1\right)\right] \alpha_{j} t_{j} p\left(x_{i}, y_{j}-1, z_{k}\right)  \tag{2.8}\\
& \quad+\sum_{k=1}^{m_{K}}\left[S_{k}+\left(z_{k}-1\right)\right] \beta_{k} t_{k} p\left(x_{i}, y_{j}, z_{k}-1\right)
\end{align*}
$$

### 2.3. The Full-Availability Group with BPP Traffic at the Macrostate Level

It is convenient to consider the multidimensional process occurring in the considered system at the level of the so-called macrostates. Each macrostate $n$ determines the number of $n$ busy BBUs in the considered group, regardless of the number of serviced calls of particular classes. Therefore, each of the microstates $\left\{x_{i}-1, y_{j}, z_{k} \cdots\right\}$ is associated with such a macrostate in which the number of busy BBUs is decreased by $t_{i}$ BBUs, necessary to set up a connection of class $i$, that is, with such a macrostate in which the number of busy BBUs equals $n-t_{i}$. The following equation is then fulfilled:

$$
\begin{equation*}
\sum_{\substack{c=1 \\ c \neq i}}^{m} x_{c} t_{c}+\left(x_{i}-1\right) t_{i}=\sum_{c=1}^{m} x_{c} t_{c}-t_{i}=\left(n-t_{i}\right) \tag{2.9}
\end{equation*}
$$

where $m$ determines the number of all traffic classes offered to the system, that is, $m=m_{I}+$ $m_{J}+m_{K}$.

The macrostate probability $P(n)$ defines then the occupancy probability of $n$ BBUs of the group and can be expressed as the aggregation of the probabilities of appropriate microstates:

$$
\begin{equation*}
P(n)=\sum_{\Omega(n)} p\left(x_{i}, y_{j}, z_{k}\right) \tag{2.10}
\end{equation*}
$$

where $\Omega(n)$ is a set of all such subsets $\left\{x_{i}, y_{j}, z_{k}\right\}$ that fulfil the following equation:

$$
\begin{equation*}
n=\sum_{i=1}^{m_{I}} x_{i} t_{i}+\sum_{j=1}^{m_{J}} y_{j} t_{j}+\sum_{k=1}^{m_{K}} z_{k} t_{k} \tag{2.11}
\end{equation*}
$$

The definition of the macrostate (2.11) makes it possible to convert (2.8) into the following form:

$$
\begin{align*}
n p\left(x_{i}, y_{j}, z_{k}\right)= & \sum_{i=1}^{m_{I}} A_{i} t_{i} p\left(x_{i}-1, y_{j}, z_{k}\right) \\
& +\sum_{j=1}^{m_{J}}\left[N_{j}-\left(y_{j}-1\right)\right] \alpha_{j} t_{j} p\left(x_{i}, y_{j}-1, z_{k}\right)  \tag{2.12}\\
& +\sum_{k=1}^{m_{K}}\left[S_{k}+\left(z_{k}-1\right)\right] \beta_{k} t_{k} p\left(x_{i}, y_{j}, z_{k}-1\right) .
\end{align*}
$$

Adding on both sides all microstates that belong to the set $\Omega(n)$, we get

$$
\begin{align*}
n \sum_{\Omega(n)} p\left(x_{i}, y_{j}, z_{k}\right)= & \sum_{i=1}^{m_{I}} A_{i} t_{i} \sum_{\Omega(n)} p\left(x_{i}-1, y_{j}, z_{k}\right) \\
& +\sum_{j=1}^{m_{J}}\left[N_{j}-\left(y_{j}-1\right)\right] \alpha_{j} t_{j} \sum_{\Omega(n)} p\left(x_{i}, y_{j}-1, z_{k}\right)  \tag{2.13}\\
& +\sum_{k=1}^{m_{K}}\left[S_{k}+\left(z_{k}-1\right)\right] \beta_{k} t_{k} \sum_{\Omega(n)} p\left(x_{i}, y_{j}, z_{k}-1\right)
\end{align*}
$$

Following the application of the definition of macrostate probability, expressed by (2.10), we are in a position to convert (2.13) as follows:

$$
\begin{aligned}
n P(n)= & \sum_{i=1}^{m_{I}} A_{i} t_{i} P\left(n-t_{i}\right)+\sum_{j=1}^{m_{I}}\left[N_{j}-\left(y_{j}-1\right)\right] \alpha_{j} t_{j} \\
& \times \sum_{\Omega(n)} p\left(x_{i}, y_{j}-1, z_{k}\right)+\sum_{k=1}^{m_{K}}\left[S_{k}+\left(z_{k}-1\right)\right] \beta_{k} t_{k} \\
& \times \sum_{\Omega(n)} p\left(x_{i}, y_{j}, z_{k}-1\right)
\end{aligned}
$$

$$
\begin{align*}
= & \sum_{i=1}^{m_{I}} A_{i} t_{i} P\left(n-t_{i}\right)+\sum_{j=1}^{m_{J}} \alpha_{j} t_{j} \sum_{\Omega(n)}\left[N_{j}-\left(y_{j}-1\right)\right] \frac{p\left(x_{i}, y_{j}-1, z_{k}\right)}{\sum_{\Omega(n)} p\left(x_{i}, y_{j}-1, z_{k}\right)} \\
& \times \sum_{\Omega(n)} p\left(x_{i}, y_{j}-1, z_{k}\right)+\sum_{k=1}^{m_{K}} \beta_{k} t_{k} \sum_{\Omega(n)}\left[S_{k}+\left(z_{k}-1\right)\right] \\
& \times \frac{p\left(x_{i}, y_{j}, z_{k}-1\right)}{\sum_{\Omega(n)} p\left(x_{i}, y_{j}, z_{k}-1\right)} \sum_{\Omega(n)} p\left(x_{i}, y_{j}, z_{k}-1\right), \tag{2.14}
\end{align*}
$$

where $P\left(n-t_{c}\right)=0$, if $n<t_{c}$, and the value $P(0)$ ensues from the normative condition $\sum_{n=0}^{V} P(n)=1$.

In (2.14) the sums

$$
\begin{align*}
& \sum_{\Omega(n)}\left[y_{j}-1\right] \frac{p\left(x_{i}, y_{j}-1, z_{k}\right)}{\sum_{\Omega(n)} p\left(x_{i}, y_{j}-1, z_{k}\right)}=\overline{y_{j}-1}  \tag{2.15}\\
& \sum_{\Omega(n)}\left[z_{k}-1\right] \frac{p\left(x_{i}, y_{j}, z_{k}-1\right)}{\sum_{\Omega(n)} p\left(x_{i}, y_{j}, z_{k}-1\right)}=\overline{z_{k}-1} \tag{2.16}
\end{align*}
$$

determine the value of the average number $\overline{y_{j}-1}, \overline{z_{k}-1}$ of calls of class $j$ and $k$ in occupancy states (macrostates) $n-t_{j}$ and $n-t_{k}$, respectively. In order to determine the relationship between the number of serviced calls of particular traffic classes and the macrostate (for which the average values $\overline{y_{j}-1}$ and $\overline{z_{k}-1}$ are determined), in the subsequent part of the paper we have adopted the following notations:

$$
\begin{align*}
& \overline{y_{j}-1}=r_{j}\left(n-t_{j}\right)  \tag{2.17}\\
& \overline{z_{k}-1}=r_{k}\left(n-t_{k}\right)
\end{align*}
$$

Taking into consideration (2.15) and (2.16), we can rewrite (2.14) in the following way:

$$
\begin{align*}
n P(n)= & \sum_{i=1}^{m_{I}} A_{i} t_{i} P\left(n-t_{i}\right)+\sum_{j=1}^{m_{J}} \alpha_{j} t_{j}\left[N_{j}-\left(\overline{y_{j}-1}\right)\right] P\left(n-t_{j}\right) \\
& +\sum_{k=1}^{m_{K}} \beta_{k} t_{k}\left[S_{k}+\left(\overline{z_{k}-1}\right)\right] P\left(n-t_{k}\right)  \tag{2.18}\\
= & \sum_{i=1}^{m_{I}} A_{i} t_{i} P\left(n-t_{i}\right)+\sum_{j=1}^{m_{J}} \alpha_{j} t_{j}\left[N_{j}-r_{j}\left(n-t_{j}\right)\right] P\left(n-t_{j}\right) \\
& +\sum_{k=1}^{m_{K}} \beta_{k} t_{k}\left[S_{k}+r_{k}\left(n-t_{k}\right)\right] P\left(n-t_{k}\right) .
\end{align*}
$$



Figure 3: A fragment of the one-dimensional Markov process in the full-availability group with multirate traffic, servicing two call streams ( $m=1, t_{1}=1 \mathrm{BBU}, t_{2}=3 \mathrm{BBUs}$ ).

In (2.18) the value of Engset traffic of class $j$ and Pascal traffic of class $k$ depends on the occupancy state of the system. Let us introduce the following notation for the offered traffic intensity in appropriate occupancy states of the group:

$$
\begin{gather*}
A_{i}(n)=A_{i}  \tag{2.19}\\
A_{j}(n)=\alpha_{j}\left[N_{j}-\left(r_{j}(n)\right)\right]  \tag{2.20}\\
A_{k}(n)=\beta_{k}\left[S_{k}+\left(r_{k}(n)\right)\right] \tag{2.21}
\end{gather*}
$$

Formula (2.18) can be now finally rewritten to the following form:

$$
\begin{align*}
n P(n)= & \sum_{i=1}^{m_{I}} A_{i}\left(n-t_{i}\right) t_{i} P\left(n-t_{i}\right)+\sum_{j=1}^{m_{J}} A_{j}\left(n-t_{j}\right) t_{j} P\left(n-t_{j}\right) \\
& +\sum_{k=1}^{m_{K}} A_{k}\left(n-t_{k}\right) t_{k} P\left(n-t_{k}\right)  \tag{2.22}\\
= & \sum_{c=1}^{m} A_{c}\left(n-t_{c}\right) t_{c} P\left(n-t_{c}\right)
\end{align*}
$$

## 3. Modelling the Full-Availability Group

### 3.1. Average Number of Serviced Calls of Class c in State n

In order to determine the average number of calls serviced in particular states of the system, let us consider a fragment of the one-dimensional Markov chain presented in Figure 3 and corresponding to the recurrent determination of the occupancy distribution in the fullavailability group on the basis of (2.22). The diagram presented in Figure 3 shows the service process in the group with two call streams ( $m=2, t_{1}=1 \mathrm{BBU}, t_{2}=3 \mathrm{BBUs}$ ).

Let us notice that each state of the Markov process in the full-availability group (Figure 3) fulfils the following equilibrium equation:

$$
\begin{equation*}
P(n)\left[\sum_{c=1}^{m} A_{c}(n) t_{c}+\sum_{c=1}^{m} t_{c} r_{c}(n)\right]=\sum_{c=1}^{m} A_{c}\left(n-t_{c}\right) t_{c} P\left(n-t_{c}\right)+\sum_{c=1}^{m} t_{c} r_{c}\left(n+t_{c}\right) P\left(n+t_{c}\right) \tag{3.1}
\end{equation*}
$$

where $r_{c}(n)$ is the average number of calls of a given class being serviced in state $n$. From (3.1) it results that the sum of all service streams outgoing from state $n$ towards lower states is equal to $n$ :

$$
\begin{equation*}
n=\sum_{c=1}^{m} t_{c} r_{c}(n) \tag{3.2}
\end{equation*}
$$

On the basis of (2.22) and (3.2), Formula (3.1) can be rewritten in the following form:

$$
\begin{equation*}
\sum_{c=1}^{m} A_{c}(n) t_{c} P(n)=\sum_{c=1}^{m} t_{c} r_{c}\left(n+t_{c}\right) P\left(n+t_{c}\right) . \tag{3.3}
\end{equation*}
$$

Equation (3.3) is a balance equation between the total stream of calls outgoing from state $n$ and the total service stream coming in to state $n$. This equation is fulfilled only when the local equilibrium equations for streams of particular traffic classes are fulfilled:

$$
\begin{equation*}
A_{c}(n) t_{c} P(n)=t_{c} r_{c}\left(n+t_{c}\right) P\left(n+t_{c}\right) . \tag{3.4}
\end{equation*}
$$

On the basis of (3.4), the average number of calls of class $c$ in state $n+t_{c}$ of the group may be finally expressed in the following way:

$$
r_{c}\left(n+t_{c}\right)= \begin{cases}A_{c}(n) \frac{P(n)}{P\left(n+t_{c}\right)}, & \text { for } n+t_{c} \leq V  \tag{3.5}\\ 0, & \text { for } n+t_{c}>V\end{cases}
$$

### 3.2. MIM-BPP Method

Let us notice that, in order to determine the parameter $r_{c}(n)$, it is necessary to determine first the occupancy distribution $P(n)$. Simultaneously, in order to determine the occupancy distribution $P(n)$, it is also necessary to determine the value $r_{c}(n)$. This means that (2.22) and (3.5) form a set of confounding equations that can be solved with the help of iterative methods [32]. Let $P^{(l)}(n)$ denote the occupancy distribution determined in step $l$, and let $r_{c}^{(l)}(n)$ denote the average number of serviced calls of class $c$, determined in step $l$. In order to determine the initial value of the parameter $r_{c}^{(0)}(n)$, it is assumed, according to [32], that the traffic intensities of Engset and Pascal classes do not depend on the state of the system and are equal to the traffic intensity offered by all free Engset sources of class $j$ and Pascal sources of class $k$, respectively: $A_{j}^{(0)}(n)=A_{j}=N_{j} \alpha_{j}, A_{k}^{(0)}(n)=A_{k}=S_{k} \beta_{k}$. When we have the initial values of offered traffic, in the subsequent steps, we are in a position to determine the occupancy distribution, taking into account the dependence of the arrival process on the state of the system. The iteration process finishes when the assumed accuracy is obtained.

On the basis of the reasoning presented above, in [32] the MIM-BPP method for a determination of the occupancy distribution, blocking probability, and the loss probability in the full-availability group with BPP traffic is proposed. The MIM-BPP method can be presented in the form of the following algorithm.

Algorithm 3.1 (MIM-BPP method). Consider the following steps.
(1) Determination of the value of Erlang traffic $A_{i}$ of class $i$ on the basis of (2.3).
(2) Setting the iteration step: $l=0$.
(3) Determination of initial values of the number $r_{j}^{(l)}(n)$ of Engset serviced calls of class $j$ and the number $r_{k}^{(l)}(n)$ of Pascal serviced calls of class $k$ :

$$
\begin{equation*}
\forall_{0 \leq n \leq V} \quad\left(\forall_{1 \leq j \leq m_{J}} r_{j}^{(l)}(n)=0, \forall_{1 \leq k \leq m_{K}} r_{k}^{(l)}(n)=0\right) \tag{3.6}
\end{equation*}
$$

(4) Increase in each iteration step: $l=l+1$.
(5) Determination of the value of Engset traffic $A_{j}^{(l)}(n)$ of class $j$ and Pascal traffic $A_{k}^{(l)}(n)$ of class $k$ on the basis of (2.20) and (2.21):

$$
\begin{align*}
& A_{j}^{(l)}(n)=\alpha_{j}\left[N_{j}-\left(r_{j}^{(l-1)}(n)\right)\right]  \tag{3.7}\\
& A_{k}^{(l)}(n)=\beta_{k}\left[S_{k}+\left(r_{k}^{(l-1)}(n)\right)\right] .
\end{align*}
$$

(6) Determination of the state probabilities $P^{(l)}(n)$ on the basis of (2.22):

$$
\begin{align*}
n P^{(l)}(n)= & \sum_{i=1}^{m_{I}} A_{i} t_{i} P^{(l)}\left(n-t_{i}\right)+\sum_{j=1}^{m_{J}} A_{j}^{(l)}\left(n-t_{j}\right) t_{j} P^{(l)}\left(n-t_{j}\right)  \tag{3.8}\\
& +\sum_{k=1}^{m_{K}} A_{k}^{(l)}\left(n-t_{k}\right) t_{k} P^{(l)}\left(n-t_{k}\right)
\end{align*}
$$

(7) Determination of the average number of serviced calls $r_{j}^{(l)}(n)$ and $r_{k}^{(l)}(n)$ on the basis of (3.5):

$$
r_{c}^{(l)}(n)= \begin{cases}A_{c}^{(l)}\left(n-t_{c}\right) \frac{P^{(l)}\left(n-t_{c}\right)}{P^{(l)}(n)} & \text { for } 0 \leq n \leq V  \tag{3.9}\\ 0 & \text { otherwise }\end{cases}
$$

(8) Repetition of steps (3)-(6) until predefined accuracy $\epsilon$ of the iterative process is achieved:

$$
\begin{equation*}
\forall_{0 \leq n \leq V} \quad\left(\left|\frac{r_{j}^{(l-1)}(n)-r_{j}^{(l)}(n)}{r_{j}^{(l)}(n)}\right| \leq \epsilon,\left|\frac{r_{k}^{(l-1)}(n)-r_{k}^{(l)}(n)}{r_{k}^{(l)}(n)}\right| \leq \epsilon\right) \tag{3.10}
\end{equation*}
$$

(9) Determination of the blocking probability $E_{c}$ for calls of class $c$ and the loss probability $B_{i}$ for Erlang calls of class $i, B_{j}$ for Engset calls of class $j$, and $B_{k}$ for Pascal calls of class $k$,

$$
\begin{align*}
& E_{c}=\sum_{n=V-t_{c}+1}^{V} P^{(l)}(n), \\
& B_{i}=E_{i,} \\
& B_{j}=\frac{\sum_{n=V-t_{j}+1}^{V} P^{(l)}(n)\left[N_{j}-r_{j}^{(l)}(n)\right] \Lambda_{j}}{\sum_{n=0}^{V} P^{(l)}(n)\left[N_{j}-r_{j}^{(l)}(n)\right] \Lambda_{j}},  \tag{3.11}\\
& B_{k}=\frac{\sum_{n=V-t_{k}+1}^{V} P^{(l)}(n)\left[S_{k}+r_{k}^{(l)}(n)\right] r_{k}}{\sum_{n=0}^{V} P(n)\left[S_{k}+r_{k}^{(l)}(n)\right] r_{k}} .
\end{align*}
$$

### 3.3. Convergence of the Iterative Process of Estimation of the Average Number of Serviced Engset Calls

In this section we prove that the process for a determination of the average number of serviced traffic sources proposed in the MIM-BPP method is, in the case of multiservice Engset sources, a convergent process. Thus, the following theorem needs to be proved.

Theorem 3.2. The sequence $\left(r_{j}^{(l)}(n)\right)_{l=0}^{\infty}$ of the average number of serviced class $j$ Engset calls in the system with BPP traffic, where

$$
\begin{gather*}
r_{j}^{(l)}(n)=\frac{\left[N_{j}-r_{j}^{(l-1)}\left(n-t_{j}\right)\right] \alpha_{j} P^{(l-1)}\left(n-t_{j}\right)}{P^{(l-1)}(n)},  \tag{3.12}\\
r_{j}^{(0)}(n)=0 \quad \text { for } \forall_{1 \leq j \leq m_{j}}, \forall_{0 \leq n \leq V}, \tag{3.13}
\end{gather*}
$$

is convergent.
Proof. In order to prove Theorem 3.2, we are going to show first that each succeeding element of sequence (3.12), starting from the first one, could be represented by finite series:

$$
\begin{equation*}
r_{j}^{(l)}(n)=\sum_{s=1}^{l}(-1)^{s+1} N_{j} \alpha_{j}^{s} \prod_{i=1}^{s} \frac{P^{(l-i)}\left(n-i t_{j}\right)}{P^{(l-i)}\left(n-(i-1) t_{j}\right)} \tag{3.14}
\end{equation*}
$$

Since $r_{j}^{(0)}\left(n-t_{j}\right)=0$, then on the basis of (3.12) for $l=1$

$$
\begin{equation*}
r_{j}^{(1)}(n)=N_{j} \alpha_{j} \frac{P^{(0)}\left(n-t_{j}\right)}{P^{(0)}(n)} . \tag{3.15}
\end{equation*}
$$

Now, using (3.15), we can determine the value $r_{j}^{(2)}(n)$ for $l=2$ on the basis of (3.12):

$$
\begin{equation*}
r_{j}^{(2)}(n)=\frac{\left[N_{j}-N_{j} \alpha_{j} P^{(0)}\left(n-2 t_{j}\right) / P^{(0)}\left(n-t_{j}\right)\right] \alpha_{j} P^{(1)}\left(n-t_{j}\right)}{P^{(1)}(n)} \tag{3.16}
\end{equation*}
$$

Rearranging (3.16), we can present it in the following way:

$$
\begin{equation*}
r_{j}^{(2)}(n)=N_{j} \alpha_{j} \frac{P^{(1)}\left(n-t_{j}\right)}{P^{(1)}(n)}-N_{j} \alpha_{j}^{2} \frac{P^{(0)}\left(n-2 t_{j}\right) P^{(1)}\left(n-t_{j}\right)}{P^{(0)}\left(n-t_{j}\right) P^{(1)}(n)} . \tag{3.17}
\end{equation*}
$$

Proceeding in an analogical way for $l=3$, we obtain

$$
\begin{align*}
r_{j}^{(3)}(n)= & N_{j} \alpha_{j} \frac{P^{(2)}\left(n-t_{j}\right)}{P^{(2)}(n)} \\
& -N_{j} \alpha_{j}^{2} \frac{P^{(1)}\left(n-2 t_{j}\right) P^{(2)}\left(n-t_{j}\right)}{P^{(1)}\left(n-t_{j}\right) P^{(2)}(n)}  \tag{3.18}\\
& +N_{j} \alpha_{j}^{3} \frac{P^{(0)}\left(n-3 t_{j}\right) P^{(1)}\left(n-2 t_{j}\right) P^{(2)}\left(n-t_{j}\right)}{P^{(0)}\left(n-2 t_{j}\right) P^{(1)}\left(n-t_{j}\right) P^{(2)}(n)} .
\end{align*}
$$

Generalizing, the value of succeeding element of sequence $\left(r_{j}^{(l)}(n)\right)$ in step $l$ can be expressed by (3.14). Now, setting the limit to infinity $(l \rightarrow \infty)$, we have

$$
\begin{equation*}
\lim _{l \rightarrow \infty} r_{j}^{(l)}(n)=\sum_{s=1}^{\infty}(-1)^{s+1} N_{j} \alpha_{j}^{s} \prod_{i=1}^{s} \frac{P^{(l-i)}\left(n-i t_{j}\right)}{P^{(l-i)}\left(n-(i-1) t_{j}\right)} \tag{3.19}
\end{equation*}
$$

Regardless of the iteration step, for every $n<0$, the probability that system is in a state $n$ is equal to 0 (i.e., $P^{(l)}(n)=0$ ). Thus, we can rewrite (3.19) in the following way:

$$
\begin{equation*}
\lim _{l \rightarrow \infty} r_{j}^{(l)}(n)=\sum_{s=1}^{\left\lfloor n / t_{j}\right\rfloor}(-1)^{s+1} N_{j} \alpha_{j}^{s} \prod_{i=1}^{s} \frac{P^{(l-i)}\left(n-i t_{j}\right)}{P^{(l-i)}\left(n-(i-1) t_{j}\right)} . \tag{3.20}
\end{equation*}
$$

A series appearing on the right side of (3.20) is finite; therefore, there exists a finite limit of sequence $\left(r_{j}^{(l)}(n)\right)_{l=0}^{\infty}$, which was to be proved.

### 3.4. Convergence of the Iterative Process of Estimation of the Average Number of Serviced Pascal Calls

Let us demonstrate now that the process of a determination of the average number of serviced traffic sources proposed in the MIM-BPP method is a convergent process also in the case of multiservice Pascal sources. The following theorem will be then proved.

Theorem 3.3. The sequence $\left(r_{k}^{(l)}(n)\right)_{l=0}^{\infty}$ of the average number of serviced class $k$ Pascal sources in the system with BPP traffic, where

$$
\begin{gather*}
r_{k}^{(l)}(n)=\frac{\left[S_{k}+r_{k}^{(l-1)}\left(n-t_{k}\right)\right] \beta_{k} P^{(l-1)}\left(n-t_{k}\right)}{P^{(l-1)}(n)},  \tag{3.21}\\
r_{k}^{(0)}(n)=0 \quad \text { for } \forall_{1 \leq k \leq m_{J}}, \forall_{0 \leq n \leq V}, \tag{3.22}
\end{gather*}
$$

is convergent.
Proof. Proceeding in the analogical way as we did in the case of sequence (3.12), we can prove that the elements of sequence $\left(r_{k}^{(l)}(n)\right)_{l=0}^{\infty}$ can be expressed by the following expression:

$$
\begin{equation*}
r_{k}^{(l)}(n)=\sum_{j=1}^{l} S_{k} \beta_{k}^{j} \prod_{i=1}^{j} \frac{P^{(l-i)}\left(n-i t_{k}\right)}{P^{(l-i)}\left(n-(i-1) t_{k}\right)} \tag{3.23}
\end{equation*}
$$

Therefore, in order to show that sequence $\left(r_{k}^{(l)}(n)\right)_{l=0}^{\infty}$ is convergent, we only need to prove that for $l \rightarrow \infty$ the series

$$
\begin{equation*}
\sum_{j=1}^{l} S_{k} \beta_{k}^{j} \prod_{i=1}^{j} \frac{P^{(l-i)}\left(n-i t_{k}\right)}{P^{(l-i)}\left(n-(i-1) t_{k}\right)} \tag{3.24}
\end{equation*}
$$

is convergent.
Consider the elements of series (3.24):

$$
\begin{equation*}
b_{j}=S_{k} \beta_{k}^{j} \prod_{i=1}^{j} \frac{P^{(l-i)}\left(n-i t_{k}\right)}{P^{(l-i)}\left(n-(i-1) t_{k}\right)} . \tag{3.25}
\end{equation*}
$$

The elements of series $\left(b_{j}\right)_{j=1}^{\infty}$ are positive, which means that we can use the ratio test ( $\mathrm{d}^{\prime}$ Alembert criterium) for convergence to prove that series is convergent (if in series $\sum_{n=1}^{\infty} u_{n}$ with positive terms beginning from certain place $N$ (this means for all $n \geq N$ ), then the ratio of arbitrary term $u_{n+1}$ to previous term $u_{n}$ is permanently less than number $p$ less than 1 ; i.e, if $u_{n+1} / u_{n} \leq p<1$ for all $n \geq N$, then series $\sum_{n=1}^{\infty} u_{n}$ is convergent [33]). The ratio of two consecutive elements of sequence $\left(b_{j}\right)_{j=1}^{\infty}$ is equal to

$$
\begin{equation*}
\frac{b_{j+1}}{b_{j}}=\beta_{k} \frac{P^{(l-(j+1))}\left(n-(j+1) t_{k}\right)}{P^{(l-(j+1))}\left(n-j t_{k}\right)} . \tag{3.26}
\end{equation*}
$$

For $j \rightarrow \infty$ numerator and denominator of (3.26) converge to 0 . Note also that the numerator converges to 0 faster than the denominator. Hence, $\lim _{j \rightarrow \infty}\left(b_{j+1} / b_{j}\right)$ is equal to 0 , that is, is permanently less than 1 . Therefore, by virtue of the ratio test (d'Alembert criterium) for convergence series (3.24) is convergent. Thus, sequence (3.21) is convergent as well.

### 3.5. Advantages and Possible Applications of MIM-BPP Method

The presented iterative algorithm for systems with state-independent admission process (i.e., the full-availability group) makes it possible to determine exactly the occupancy distribution and the blocking and loss probabilities in systems that service Erlang (Poisson distribution of call streams), Engset (binomial distribution of call streams), and Pascal traffic streams (negative binomial distribution of call stream). The call stream of the types investigated in the paper are typical streams to be considered in traffic theory. They are used for modelling at the call level, where any occupancy of resources of the system, for example, effected by a telephone conversation or by a packet stream with characteristics defined at the packet level, can be treated as a call [11]. In the case of the Integrated Services Digital Networks, resource occupancies were in the main related to voice transmission, whereas nowadays a call is understood to be a packet stream to which appropriate equivalent bandwidth is assigned [3436], and then the demanded resources, as well as the capacity of the system, are discretized [7]. In the case of wired systems, the most important is the Poisson stream and the consequent Erlang traffic stream. This stream assumes stable intensity of generating calls, independent of the number of calls that are already being serviced. In the case of wireless systems, it was soon noticed that, because of the limited number of subscribers serviced within a given area, the application of the Erlang model for certain traffic classes could lead to erroneous estimation of the occupancy distribution. Hence, for certain traffic classes, the application of the Engset model was proposed, initially for single-service (single-rate) systems and then for multiservice (multirate) systems [3, 4]. In general, the Engset distribution is used to model systems with noticeable limitation of the number of users. Currently, the main practical scope for the usage of the Pascal distribution is a simplified modelling of systems with overflow traffic [11]. The presented algorithm makes it then possible to determine traffic characteristics for all three call (traffic) streams considered in traffic theory.

The application of the notion of the basic bandwidth unit (BBU) used in the notation of the presented method makes it possible to obtain high universality for the method. BBU is determined as the highest common divisor of all demands that are offered to the system. Depending on a system under consideration, the basic bandwidth unit can be expressed in bits per second or as the percentage of the occupancy of the radio interface (the so-called interference load) [4,37]. In the presented method for modelling multirate systems with BPP traffic streams, both required resources and the capacity of the system are expressed as the multiplicity of BBU. The method can be thus applied to model both wired broadband integrated services networks as well as wireless networks (UMTS/WCDMA networks in particular).

The algorithm worked out for modelling systems with BPP traffic can be treated as an extension to the Kaufman-Roberts model $[24,25]$ that has been worked out for systems with Poisson traffic streams only. Both the algorithm proposed by Kaufman-Roberts and the algorithm presented in the paper are exact algorithms. Having an exact formula as a base, the algorithm can be extended-analogously as in the case of the Kaufman-Roberts formula for systems with Erlang traffic-into systems with state-dependent call admission process and BPP traffic. In the case of communication system, state dependence in the call admission process results mainly from the introduction of the control policy in allocating resources for calls of individual traffic classes (reservation mechanism [32], threshold mechanism [38]) or a particular structure of the system (e.g., a limited-availability group [5]). An extension of the scope in which the presented algorithm can be applied, including systems with statedependent call admission process, entails only the introduction of the additional transition

Table 1: Relative errors of the number of busy class 3 sources in relation to the number of iterations.

| $a$ | 2 | 3 | 5 | 7 |
| :--- | :---: | :---: | :---: | :---: |
| 0.3 | $3.17 E-02$ | $7.17 E-04$ | $2.97 E-07$ | $1.11 E-10$ |
| 0.4 | $2.89 E-02$ | $3.51 E-04$ | $1.65 E-07$ | $7.68 E-11$ |
| 0.5 | $3.43 E-02$ | $4.86 E-04$ | $1.18 E-07$ | $8.06 E-11$ |
| 0.6 | $3.97 E-02$ | $6.36 E-04$ | $1.93 E-07$ | $1.09 E-10$ |
| 0.7 | $4.52 E-02$ | $8.02 E-04$ | $2.95 E-07$ | $1.42 E-10$ |
| 0.8 | $5.06 E-02$ | $9.87 E-04$ | $4.29 E-07$ | $2.37 E-10$ |
| 0.9 | $5.59 E-02$ | $1.19 E-03$ | $5.98 E-07$ | $3.75 E-10$ |
| 1.0 | $6.11 E-02$ | $1.41 E-03$ | $8.14 E-07$ | $5.68 E-10$ |
| 1.1 | $6.62 E-02$ | $1.64 E-03$ | $1.08 E-06$ | $8.33 E-10$ |
| 1.2 | $7.12 E-02$ | $1.89 E-03$ | $1.40 E-06$ | $1.20 E-09$ |

coefficient [32], without further changes, depending on the considered system. It should be stressed that such a universality cannot be achieved by the convolution algorithm also worked out for systems with state-independent call admission process only.

### 3.6. Numerical Examples

The paper introduces a formula that makes it possible to determine exactly the occupancy distribution in systems with state-independent call admission process. It is then demonstrated that the algorithm for a determination of the average number of serviced traffic sources of particular classes used in the MIM-BPP method is convergent.

In order to present the convergence of the MIM-BPP method (the number of required iterations), in Table 1 the results of relative errors of the number of busy class 3 sources in the full-availability group with the capacity equal to 80 BBUs are contained (with the instance of calls of class 1 and 2 , the number of required iterations is lower than in the case of the presented results for class 3 ). The results are presented depending on the average value of traffic offered to a single bandwidth unit of the group: $a=\left(\sum_{i=1}^{m_{I}} A_{i} t_{i}+\sum_{j=1}^{m_{J}} N_{j} \alpha_{j} t_{j}+\right.$ $\left.\sum_{k=1}^{m_{K}} S_{k} \beta_{k} t_{k}\right) / V$. The group was offered three traffic classes, that is, Erlang traffic class: $t_{1}=$ 1 BBU, Engset traffic class: $t_{2}=4$ BBUs, $N_{2}=60$, and Pascal traffic class: $t_{3}=10$ BBUs, $S_{3}=80$. The results presented in Table 1 indicate that the proposed iterative method converges very quickly.

In this section we limit ourselves to just presenting the results of the convergence of the presented algorithm for one selected system. A comparison of the analytical results for the blocking/loss probability with the results of the simulation is presented in earlier works, for example, [4, 32], in which it was still assumed that the presented analytical method was an approximate method.

## 4. Conclusion

In the paper recurrent equations describing-at the macrostate level-the service process in the full-availability group with multirate BPP traffic were derived. The derived equations made it possible to formulate an exact iterative algorithm for determining the occupancy
distribution, blocking probability, and loss probability of calls of particular classes offered to the system. The convergence of the proposed process of estimating the average number of busy sources of Engset and Pascal traffic was proved.

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Research Article

# Stochastic Stabilization of Nonholonomic Mobile Robot with Heading-Angle-Dependent Disturbance 

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#### Abstract

The problem of exponential stabilization for nonholonomic mobile robot with dependent stochastic disturbance of heading angle is considered in this paper. An integrator backstepping controller based on state-scaling method is designed such that the state of the closed-loop system, starting from a nonzero initial heading angle, is regulated to the origin with exponential rate in almost surely sense. For zero initial heading angle, a controller is designed such that the heading angle is driven away from zero while the position variables are bounded in a neighborhood of the origin. Combing the above two cases results in a switching controller such that for any initial condition the configuration of the robot can be regulated to the origin with exponential rate. The efficiency of the proposed method is demonstrated by a detailed simulation.


## 1. Introduction

In the past decades, there has been increasing attention devoted to the control of nonholonomic systems such as knife edge, rolling disk, tricycle-type robot, and car-like robot with trailers (see, [1,2] and the references therein). From Brockett's necessary condition [3], it is well known that the nonholonomic systems cannot be stabilized to the origin by any static continuous state feedback, so the classical smooth control theory cannot be applied directly. This motivates researchers to seek for novel approaches such as discontinuous feedback and time-varying feedback. The discontinuous feedback uses the state-scaling technique and switching control strategy [4,5], which usually results in an exponential convergence. The time-varying feedback provides smooth controllers, but its convergence rate usually is slow [6, 7]. All the above references considered the nonholonomic systems in the deterministic case, while the nonholonomic systems with stochastic disturbance have rarely been researched up to now.

The purpose of this paper is to consider the posture (including position and direction) adjustment of nonholonomic mobile robot with stochastic disturbance dependent of heading angle. By a state transformation, the mentioned control model can be rewritten as

$$
\begin{gather*}
d x_{1}=v d t+\varphi_{1} d W_{1}, \\
d x_{2}=x_{3} v d t+x_{3} \varphi_{1} d W_{1},  \tag{1.1}\\
d x_{3}=\left(u-x_{2} v\right) d t+\varphi_{2} d W_{2}-x_{2} \varphi_{1} d W_{1},
\end{gather*}
$$

where $u$ is the forward velocity, $v$ is the steering velocity, $\varphi_{1}$ and $\varphi_{2}$ are two smooth functions, and $W_{1}$ and $W_{2}$ are two independent standard Wiener processes. It seems that the stabilization can be achieved by extending the backstepping procedure based on state-scaling technique [5] to the stochastic case.

Our main contribution consists of the following aspects. (i) For nonzero initial value of $x_{1}$, by imposing a reasonable assumption on function $\varphi_{1}$, the state $x_{1}$ can be easily exponentially regulated to zero via control $v$. However, in doing so, $v$ will converge to zero as $t$ goes to infinity. This phenomenon causes serious trouble in controlling $x_{2}$-subsystem via the virtual control $x_{3}$ because, in the limit $\left(\lim _{t \rightarrow \infty} v=0\right.$, a.s.), $x_{2}$-subsystem is uncontrollable. The variable $x_{3}$ appears in the drift term $x_{3} v$ and the diffusion term $x_{3} \varphi_{1}$ of $x_{2}$-subsystem simultaneously. This leads to a new problem that the desirable control $\alpha$ and its square term $\alpha^{2}$ appear in the same procedure of backstepping control (see, (4.11)), which is distinct from the traditional stochastic backstepping method as used in [8]. (ii) For a nonzero $x_{1}\left(t_{0}\right)$, the transformation $z_{1}=x_{2} / x_{1}$ is used in controller design, therefore, it cannot work for systems with initial state whose $x_{1}\left(t_{0}\right)=0$, which motivates us to drive $x_{1}$ away from zero in a small distance during a shorter time interval by designing $v$. (iii) For any initial condition, a switching control is given by combining the above two cases. Different from the usual switching schemes depending only on state, our switching controller depends on a stopping time as well as state. Therefore, the measurement of the stopping time is expected. It is proved that all signals in the closed-loop system converge to zeros with exponential rate. The discontinuous switching function is replaced with a continuous one to eliminate the trembling phenomenon in simulation.

This paper is organized as follows. Section 2 begins with some mathematical preliminaries. The model of a wheeled mobile robot with stochastic disturbance is presented in Section 3. Backstepping stabilizer based on state-scaling technique is investigated for the case of $x_{1}\left(t_{0}\right) \neq 0$ in Section 4. In Section 5, for the case of $x_{1}\left(t_{0}\right)=0$, a controller is designed such that there exists a time interval in which $x_{1}$ is driven away from zero and the other signals are bounded in probability. Section 6 formulates the main stabilization results for any initial condition. A simulation is given in Section 7. Finally, Section 8 draws the conclusion.

The following notations are used throughout the paper: $C^{i}$ denotes the set of all functions with continuous ith partial derivative; for any vector $x$ in $\mathbb{R}^{n},|x|$ means its Euclidean norm and $x^{T}$ is its transpose; for any Matrix $X$ in $R^{m \times n},|X|$ denotes the Frobenius norm defined by $|X|=\left(\operatorname{Tr}\left\{X X^{T}\right\}\right)^{1 / 2}$, where $\operatorname{Tr}(\cdot)$ denotes the matrix trace; $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, which are continuous, strictly increasing and vanish at zero; $\mathcal{K}_{\infty}$ denotes the set of all functions which are of class $\mathbb{K}$ and unbounded.

## 2. Mathematical Preliminaries

Consider the nonlinear stochastic system

$$
\begin{equation*}
d x=f(x, t) d t+g(x, t) d W, \quad x\left(t_{0}\right)=x_{0} \in \mathbb{R}^{n} \tag{2.1}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}$ is the state, $f(0, t)=0, g(0, t)=0$, and $W$ is an $r$-dimensional independent standard Wiener process.

The following notion of boundedness on an interval in probability can be seen as a slight extension from that used in [9].

Definition 2.1. A stochastic process $x(t)$ is said to be bounded on $t \in\left[t_{0}, T\right]$, where $T \leq \infty$, in probability if the random variable $|x(t)|$ satisfies

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \sup _{t \in\left[t_{0}, T\right]} P\{|x(t)|>R\}=0 \tag{2.2}
\end{equation*}
$$

For this notion, a corresponding criterion can be easily obtained following the line of [10].

Lemma 2.2. Consider system (2.1) defined in $\left[t_{0}, T\right]$, where $T \leq \infty$. Assume that there exist a function $V \in C^{2}$, class $\mathcal{K}_{\infty}$ functions $\underline{\alpha}(|x|)$ and $\bar{\alpha}(|x|)$, a positive constant $c$, and a nonnegative constant $d$ such that for all $x_{0} \in \mathbb{R}^{n}$, (i) $\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|)$, and (ii) $\Omega V(x)=V_{x}(x) f(x, t)+$ $(1 / 2) \operatorname{Tr}\left[g^{T}(x, t) V_{x x} g(x, t)\right] \leq-c V(x)+d$, for all $t \in\left[t_{0}, T\right]$, then system (2.1) has a unique solution on $\left[t_{0}, T\right]$, which is bounded on $t \in\left[t_{0}, T\right]$ in probability.

To find condition to let state scaling make sense, the following lemma proved by Mao in [11, pages 51,120 ] is recited as follows.

Lemma 2.3. For system (2.1) defined on $t \in\left[t_{0}, T\right]$, where $T \leq \infty$, assume that there exist two constants $K_{1}$ and $K_{2}$ such that
(i) (lipschitz condition) for all $x, y \in \mathbb{R}^{n}$ and $t \in\left[t_{0}, T\right]$

$$
\begin{equation*}
|f(x, t)-f(y, t)|^{2} \vee|g(x, t)-g(y, t)|^{2} \leq K_{1}|x-y|^{2} \tag{2.3}
\end{equation*}
$$

(ii) (Linear growth condition) for all $(x, t) \in \mathbb{R}^{n} \times\left[t_{0}, T\right]$

$$
\begin{equation*}
|f(x, t)|^{2} \vee|g(x, t)|^{2} \leq K_{2}\left(1+|x|^{2}\right) \tag{2.4}
\end{equation*}
$$

then there exists a unique solution $x(t):=x\left(t_{0}, x_{0}, t\right)$ to system (2.1) and for all $x_{0} \neq 0$ in $\mathbb{R}^{n}$,

$$
\begin{equation*}
P\left(x\left(t, t_{0}, x_{0}\right) \neq 0\right)=1, \quad \forall T \geq t \geq t_{0} \tag{2.5}
\end{equation*}
$$

(i.e., almost all the sample path of any solution starting from a nonzero state will never reach the origin).

The concepts of moment exponential stability and almost surely exponential stability together with their criteria can be found in [12, page 166], which are presented here for selfsufficiency.

Definition 2.4. For $p>0$, system (2.1) is said to be $p$ th moment exponential stable if

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log E\left|x\left(t, t_{0}, x_{0}\right)\right|^{p}<0 \tag{2.6}
\end{equation*}
$$

for each $x_{0} \in \mathbb{R}^{n}$. Moreover, system (2.1) is said to be almost surely exponential stable if

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log \left|x\left(t, t_{0}, x_{0}\right)\right|<0, \quad \text { a.s., } \forall x_{0} \in \mathbb{R}^{n} \tag{2.7}
\end{equation*}
$$

Lemma 2.5. Assume that there exist a function $V \in C^{2}$ and positive constants $c_{1}, c_{2}, c$ and $p$ such that (i) $c_{1}|x|^{p} \leq V(x) \leq c_{2}|x|^{p}$, and (ii) $£ V(x) \leq-c V(x)$, for all $x_{0} \in \mathbb{R}^{n}$ and $t \geq t_{0}$, then system (2.1) has a unique solution on $\left[t_{0}, \infty\right)$, which is pth moment exponential stable. Moreover, if further assume that (iii) there exists a positive constant $K$ such that

$$
\begin{equation*}
|f(x, t)| \vee|g(x, t)| \leq K|x| \tag{2.8}
\end{equation*}
$$

for all $(x, t) \in \mathbb{R}^{n} \times\left[t_{0}, \infty\right)$, then system (2.1) is almost surely exponential stable.

## 3. Problem Formulation

A nonholonomic mobile robot of tricycle type in the presence of stochastic disturbance can be described by

$$
\begin{gather*}
d \theta=v d t+\varphi_{1}(\theta) d W_{1} \\
d x_{c}=\left(u d t+\varphi_{2}(\theta) d W_{2}\right) \cos \theta  \tag{3.1}\\
d y_{c}=\left(u d t+\varphi_{2}(\theta) d W_{2}\right) \sin \theta
\end{gather*}
$$

where $u$ is the forward velocity, $v$ is the steering velocity, $\left(x_{c}, y_{c}\right)$ is the position of the mass center of the robot moving in the plane, $\theta$ is the heading angle from the horizontal axis, $W_{1}$ and $W_{2}$ are two independent standard Wiener processes, and $\varphi_{1}$ and $\varphi_{2}$ are two unknown scaler-valued smooth functions. A tricycle-type robot is described by Figure 1.

Performing the change of coordinate

$$
\begin{equation*}
x_{1}=\theta, \quad x_{2}=x_{c} \sin \theta-y_{c} \cos \theta, \quad x_{3}=x_{c} \cos \theta+y_{c} \sin \theta \tag{3.2}
\end{equation*}
$$

system (3.1) can be transformed into system (1.1). The control objective is to design a state-feedback controller such that all the signals in the closed-loop system are globally exponentially regulated to the origin in probability. For this end, the following assumptions are imposed throughout this paper.


Figure 1: A nonholonomic mobile robot.
(A1) There exists a positive constant $k$ such that

$$
\begin{equation*}
\left|\varphi_{1}\left(x_{1}\right)-\varphi_{1}\left(x_{2}\right)\right| \leq k\left|x_{1}-x_{2}\right|, \quad \varphi_{1}(0)=0 . \tag{3.3}
\end{equation*}
$$

(A2) There exist a positive constant $l$ and a smooth nonnegative function $\phi$ such that

$$
\begin{equation*}
\varphi_{2}^{2}\left(x_{1}\right) \leq l x_{1}^{2} \phi\left(x_{1}\right) . \tag{3.4}
\end{equation*}
$$

Remark 3.1. System (1.1) is similar to the class of systems in strict-feedback form driven by Wiener processes, which motivates us to investigate the backstepping controller design that had been extensively researched by $[8,13]$. Assumptions (A1) and (A2) are given to diffusion terms as same as those imposed to drift terms in [5] in the deterministic case. For nonzero initial value of $x_{1}$, by imposing (A1) on function $\varphi_{1}$, the state $x_{1}$ can be regulated to zero with exponential rate but never reach zero (see the subsequent subsection), which is the key to introduce a state-scaling transformation to deal with other troubles (see Section 1).

## 4. Controller Design for the Case of $x_{1}\left(t_{0}\right) \neq 0$

### 4.1. Design of Controller $v$

It can be seen that the state $x_{1}$ of system (1.1) can be globally exponentially regulated to zero via a static feedback control law. In fact, we can introduce a Lyapunov function

$$
\begin{equation*}
V_{0}=\frac{1}{4} x_{1}^{4} \tag{4.1}
\end{equation*}
$$

whose infinite generator along the first equation of (1.1) satisfies

$$
\begin{equation*}
\varrho V_{0} \leq x_{1}^{3} v+\frac{3}{2} x_{1}^{4} k^{2} \tag{4.2}
\end{equation*}
$$

By choosing the control law $v$ as

$$
\begin{equation*}
v=-\lambda x_{1} \tag{4.3}
\end{equation*}
$$

where $\lambda \geq 2 k^{2}$ is a positive parameter (further requirements for $\lambda$ will be given later), (4.2) becomes

$$
\begin{equation*}
\_V_{0} \leq-\lambda V_{0} \tag{4.4}
\end{equation*}
$$

By substituting (4.3) into the first equation of (1.1), one has

$$
\begin{equation*}
d x_{1}=-\lambda x_{1} d t+\varphi_{1} d W_{1} \tag{4.5}
\end{equation*}
$$

which, together with assumption (A1) and Lemma 2.3, means that there exists a unique solution to (4.5) and that any solution starting from a nonzero state will never reach the origin in almost surely sense. From assumption (A1), (4.1) and (4.4), according to Lemma 2.3, the solution exponentially converges to zero, that is, ${\lim \sup _{t \rightarrow \infty}(1 / t) \log \left|x_{1}(t)\right|<-\lambda \text {, a.s., which }}$ means that

$$
\begin{equation*}
\left|x_{1}(t)\right|<\left|x_{1}\left(t_{0}\right)\right| e^{-\lambda\left(t-t_{0}\right)}, \quad \text { a.s. } \tag{4.6}
\end{equation*}
$$

### 4.2. State-Scaling Transformation

We have designed controller $v$ such that state $x_{1}(t)$ can be globally exponentially regulated to zero. Consequently, $v$ will converge to zero as $t$ goes to $\infty$. This causes trouble in the control of $x_{2}$-subsystem and $x_{3}$-subsystem. To overcome this difficulty, we introduce a state-scaling transformation defined by

$$
\begin{equation*}
z_{1}=\frac{x_{2}}{x_{1}} \tag{4.7}
\end{equation*}
$$

According to the comment in the end of Section 1, the transformation (4.7) makes sense in almost surely sense, for the initial value $x_{1}\left(t_{0}\right) \neq 0$. From (1.1), (4.3), and (4.7), we have

$$
\begin{equation*}
d z_{1}=\left[\lambda z_{1}-\lambda x_{3}-\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} x_{3}+\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}\right] d t+\left[\left(\frac{\varphi_{1}}{x_{1}}\right) x_{3}-\left(\frac{\varphi_{1}}{x_{1}}\right) z_{1}\right] d W_{1} \tag{4.8}
\end{equation*}
$$

### 4.3. Backstepping Controller Design of $u$

In this part, controller $u$ will be constructed, based on backstepping techniques, under the assumption $x_{1}\left(t_{0}\right) \neq 0$.

Step 1. Begin with $z_{1}$-subsystem of (4.8), where $x_{3}$ is regarded as a virtual control. Introducing the transformation

$$
\begin{equation*}
z_{2}=x_{3}-\alpha \tag{4.9}
\end{equation*}
$$

and choosing Lyapunov function

$$
\begin{equation*}
V_{1}=V_{0}+\frac{1}{4} z_{1}^{4} \tag{4.10}
\end{equation*}
$$

it comes from (4.8)-(4.10) that

$$
\begin{align*}
\varrho V_{1}= & z_{1}^{3}\left[\lambda z_{1}-\lambda\left(z_{2}+\alpha\right)-\left(\frac{\varphi_{1}}{x_{1}}\right)^{2}\left(z_{2}+\alpha\right)+\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}\right] \\
& +3 z_{1}^{2}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2}\left(z_{2}+\alpha\right)^{2}+3 z_{1}^{4}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2}+\bumpeq V_{0} \tag{4.11}
\end{align*}
$$

Here, the terms $\alpha$ and $\alpha^{2}$ appear in the same time, which is different from the traditional backstepping procedure. Considering assumption (A1) and the characters of terms of (4.11), the virtual control is chosen as

$$
\begin{equation*}
\alpha=c_{1} z_{1}, \tag{4.12}
\end{equation*}
$$

where $c_{1}>0$ is a design parameter. By the aid of (4.4), (4.12), and $\left(\varphi_{1} / x_{1}\right)^{2} \leq k^{2}$ (that comes from assumption (A1)), (4.11) can be rewritten as

$$
\begin{align*}
\mathscr{\perp} V_{1} \leq & \lambda z_{1}^{4}-\lambda z_{1}^{3} z_{2}-c_{1} \lambda z_{1}^{4}+k^{2}\left|z_{1}^{3}\right|\left|z_{2}\right| c_{1} k^{2} z_{1}^{4} \\
& +k^{2} z_{1}^{4}+6 k^{2} z_{1}^{2} z_{2}^{2}+6 c_{1}^{2} k^{2} z_{1}^{4}+3 k^{2} z_{1}^{4}-\lambda \frac{x_{1}^{4}}{4} \tag{4.13}
\end{align*}
$$

Submitting the inequalities

$$
\begin{gather*}
-\lambda z_{1}^{3} z_{2} \leq \frac{3 d}{4} \lambda z_{1}^{4}+\frac{1}{4 d^{3}} \lambda z_{2}^{4}, \quad k^{2} z_{1}^{3}\left|z_{2}\right| \leq \frac{3}{4} k^{2} z_{1}^{4}+\frac{1}{4} k^{2} z_{2}^{4}  \tag{4.14}\\
6 k^{2} z_{1}^{2} z_{2}^{2} \leq 3 k^{2} z_{1}^{4}+3 k^{2} z_{2}^{4}
\end{gather*}
$$

where $d>0$ is a design parameter, into (4.13) gives

$$
\begin{equation*}
\bumpeq V_{1} \leq\left(\left(1+\frac{3 d}{4}\right) \lambda-c_{1} \lambda+\left(\frac{31}{4}+c_{1}+6 c_{1}^{2}\right) k^{2}\right) z_{1}^{4}+\left(\frac{\lambda}{4 d^{3}}+\frac{k^{2}}{4}+3 k^{2}\right) z_{2}^{4}-\frac{\lambda}{4} x_{1}^{4} \tag{4.15}
\end{equation*}
$$

By selecting parameters $\lambda$ and $c_{1}$ such that $c_{1} \geq(4+3 d) / 4 e, \lambda \geq\left(31+4 c_{1}+24 c_{1}^{2}\right) k^{2} / 2 c_{1}(1-e)$, where $e$ is a design parameter satisfying $0<e<1$, it comes from (4.15) that

$$
\begin{equation*}
\varrho V_{1} \leq-\frac{c_{1} \lambda(1-e)}{2} z_{1}^{4}+\left(\frac{\lambda}{4 d^{3}}+\frac{13 k^{2}}{4}\right) z_{2}^{4}-\frac{\lambda}{4} x_{1}^{4} \tag{4.16}
\end{equation*}
$$

Step 2. In view of (1.1) and (4.9), we have

$$
\begin{align*}
d z_{2}= & {\left[u-x_{2} v-c_{1} \lambda z_{1}+c_{1} \lambda x_{3}+c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} x_{3}-c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}\right] d t }  \tag{4.17}\\
& +\varphi_{2} d W_{2}+\left[-x_{2} \varphi_{1}-c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right) x_{3}+c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right) z_{1}\right] d W_{1}
\end{align*}
$$

Consider the candidate Lyapunov function

$$
\begin{equation*}
V_{2}=V_{1}+\frac{1}{4} z_{2}^{4} \tag{4.18}
\end{equation*}
$$

whose infinite generator along (4.17) satisfies

$$
\begin{align*}
\mathscr{L} V_{2} \leq & z_{2}^{3}\left[u-x_{2} v-c_{1} \lambda z_{1}+c_{1} \lambda x_{3}+c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} x_{3}-c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}\right]+\frac{3}{2} z_{2}^{2} \varphi_{2}^{2}  \tag{4.19}\\
& +\frac{9}{2} z_{2}^{2}\left(x_{2} \varphi_{1}\right)^{2}+\frac{9}{2} z_{2}^{2}\left(c_{1}\right)^{2}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} x_{3}^{2}+\frac{9}{2} z_{2}^{2}\left(c_{1}\right)^{2}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}^{2}+\varrho V_{1}
\end{align*}
$$

By using Young's equality and (A1), (A2), (4.7), and (4.9), it is easy to obtain that

$$
\begin{gather*}
\frac{3}{2} z_{2}^{2} \varphi_{2}^{2} \leq \frac{3}{4} l^{2} z_{2}^{4} \phi^{2}\left(x_{1}\right)+\frac{3}{4} l^{2} x_{1}^{4} \\
\frac{9}{2} z_{2}^{2}\left(x_{2} \varphi_{1}\right)^{2} \leq \frac{9}{4} k^{2} z_{2}^{4} x_{1}^{8}+\frac{9}{4} k^{2} z_{1}^{4} \\
\frac{9}{2} z_{2}^{2}\left(c_{1}\right)^{2}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} x_{3}^{2} \leq 9 c_{1}^{2} k^{2} z_{2}^{4}+\frac{9}{2} c_{1}^{4} k^{2} z_{2}^{4}+\frac{9}{2} c_{1}^{4} k^{2} z_{1}^{4},  \tag{4.20}\\
\frac{9}{2} z_{2}^{2}\left(c_{1}\right)^{2}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}^{2} \leq \frac{9}{4} c_{1}^{2} k^{2} z_{2}^{4}+\frac{9}{4} c_{1}^{2} k^{2} z_{1}^{4}
\end{gather*}
$$

which are submitted into (4.19) to give

$$
\begin{align*}
\left\llcorner V_{2} \leq\right. & z_{2}^{3}\left[u-x_{2} v-c_{1} \lambda z_{1}+c_{1} \lambda x_{3}+c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} x_{3}-c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}+\frac{3}{4} l^{2} \phi^{2}\left(x_{1}\right) z_{2}\right. \\
& \left.+\frac{9}{4} k^{2} x_{1}^{8} z_{2}+9 c_{1}^{2} k^{2} z_{2}+\frac{9}{2} c_{1}^{4} k^{2} z_{2}+\frac{9}{4} c_{1}^{2} k^{2} z_{2}+\frac{1}{4 d^{3}} \lambda z_{2}+\frac{1}{4} k^{2} z_{2}+3 k^{2} z_{2}\right]  \tag{4.21}\\
& +\frac{3}{4} l^{2} x_{1}^{4}+\left(\frac{9}{4} k^{2}+\frac{9}{2} c_{1}^{4} k^{2}+\frac{9}{4} c_{1}^{2} k^{2}-\frac{1}{2} c_{1} \lambda(1-e)\right) z_{1}^{4}-\frac{\lambda}{4} x_{1}^{4}
\end{align*}
$$

By giving a further requirement to the parameter $\lambda \geq \max \left\{\left(9 k^{2}+18 c_{1}^{4} k^{2}+9 c_{1}^{2} k^{2}\right) / c_{1}(1-e), 6 l^{2}\right\}$ and choosing the control

$$
\begin{equation*}
u=u_{1}+u_{2} \tag{4.22}
\end{equation*}
$$

we have

$$
\begin{equation*}
\varrho V_{2} \leq-\frac{\lambda}{8} x_{1}^{4}-\frac{1}{4} c_{1} \lambda(1-e) z_{1}^{4}-c_{2} z_{2}^{4} \leq-c V_{2} \tag{4.23}
\end{equation*}
$$

where $c=\min \left\{\lambda / 2, c_{1} \lambda(1-e), 4 c_{2}\right\}$ and

$$
\begin{align*}
u_{1}= & -c_{2} z_{2}+x_{2} v-c_{1} \lambda x_{3}-\frac{3}{4} l^{2} \phi^{2}\left(x_{1}\right) z_{2} \\
& -\frac{9}{4} k^{2} x_{1}^{8} z_{2}-9 c_{1}^{2} k^{2} z_{2}-\frac{9}{4} c_{1}^{4} k^{2} z_{2}-\frac{1}{d^{3}} 4 \lambda z_{2}-3 k^{2} z_{2}  \tag{4.24}\\
u_{2}= & c_{1} \lambda z_{1}-c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} x_{3}+c_{1}\left(\frac{\varphi_{1}}{x_{1}}\right)^{2} z_{1}-\frac{9}{4} c_{1}^{2} k^{2} z_{2}-\frac{1}{4} k^{2} z_{2} .
\end{align*}
$$

Summing up all the requirements to $\lambda$ leads to

$$
\begin{equation*}
\lambda \geq \max \left\{2 k^{2}, \frac{31 k^{2}+4 c_{1} k^{2}+24 c_{1}^{2} k^{2}}{2 c_{1}(1-e)}, \frac{9 k^{2}+18 c_{1}^{4} k^{2}+9 c_{1}^{2} k^{2}}{c_{1}(1-e)}, 6 l^{2}\right\} \tag{4.25}
\end{equation*}
$$

Remark 4.1. It is noteworthy that the terms in control $u$ is separated into two groups. The terms caused by the state scaling are put in $u_{2}$, in other words, if the transformation $z_{1}=$ $x_{2} / x_{1}$ is replaced with nonscaling one $z_{1}=x_{2}$, the terms in $u_{1}$ will still remain in $u$. This will be used in the subsequent section.

### 4.4. Stability Analysis

It is position to give stability conclusion for the case of $x_{1}\left(t_{0}\right) \neq 0$.

Theorem 4.2. Under assumptions (A1) and (A2), for every $x_{1}\left(t_{0}\right) \neq 0$ and any $x_{2}\left(t_{0}\right), x_{3}\left(t_{0}\right)$, with an appropriate choice of the design parameters $\lambda$ and $c_{1}$, the closed-loop system consists of (3.1), (3.2), (4.3), and (4.22) has a unique solution which is 4 th moment exponential stable.

Proof. The existence and uniqueness of solution comes from (4.18) and (4.23), according to Lemma 2.5. It can also be further concluded that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log E\left|z_{i}\right|^{4}<0 \quad(i=1,2) \tag{4.26}
\end{equation*}
$$

for each $x_{0} \in \mathbb{R}^{n}$. From (4.6), (4.7), and (4.26), we have

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log E\left|x_{2}\right|^{4} \leq \limsup _{t \rightarrow \infty} \frac{1}{t}\left(\log \left|x_{1}(t)\right|-4 \lambda\left(t-t_{0}\right)\right)+\limsup _{t \rightarrow \infty} \frac{1}{t} \log E\left|z_{1}\right|^{4}<0 \tag{4.27}
\end{equation*}
$$

From (4.6), (4.9), (4.12), and (4.26), we have

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log E\left|x_{3}\right|^{4}=\underset{t \rightarrow \infty}{\limsup } \frac{1}{t} \log E\left|z_{2}+c_{1} z_{1}\right|^{4}<0 \tag{4.28}
\end{equation*}
$$

Combining (4.6), (4.27), and (4.28) gives

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log E\left|\left(x_{1}, x_{2}, x_{3}\right)\right|^{4}<0 \tag{4.29}
\end{equation*}
$$

which completes the proof.

## 5. Controller Design to Drive $x_{1}$ Away from Zero

Considering the transformation $z_{1}=x_{2} / x_{1}$, the control $u$ given by (4.22) will escape to infinite for an initial state with element $x_{1}\left(t_{0}\right)=0$. The first thing before the controller (4.22) does work is to drive the state $x_{1}(t)$ away from zero in a small distance denoted by $r$. For a given $r>0$, define a stopping time $\tau_{r}=\inf \left\{t: t \geq t_{0},\left|x_{1}(t)\right| \geq r\right\}$. To let the state $x_{1}$ leave zero, the control $v$ can be chosen as

$$
\begin{equation*}
v=-\lambda \tag{5.1}
\end{equation*}
$$

during $\left[t_{0}, \tau_{r}\right.$ ], where $\lambda$ is the same design parameter used in (4.25) (some explanation will be given latter). In this case, the $x_{1}$-subsystem becomes

$$
\begin{equation*}
d x_{1}(t)=-\lambda d t+\varphi_{1} d W_{1} \tag{5.2}
\end{equation*}
$$

By defining $\bar{\tau}_{r}=\inf \left\{t: t \geq t_{0}, x_{1}(t) \leq-r\right\}$, the expectation of $\tau_{r}$ satisfies $E\left(\tau_{r}-t_{0}\right) \leq E\left(\bar{\tau}_{r}-t_{0}\right)=$ $r / \lambda$, therefore, $P\left(\tau_{r}-t_{0} \geq T\right) \leq r / \lambda T$, which implies that

$$
\begin{equation*}
P\left(\tau_{r}=\infty\right)=0, \quad \forall r>0 \tag{5.3}
\end{equation*}
$$

The existence and uniqueness of solution of $x_{1}$-subsystem in $\left[t_{0}, \tau_{\mathrm{r}}\right]$ comes from assumption (A1) and Lemma 2.3. Since during the interval $\left[t_{0}, \tau_{r}\right]$, the controller (4.22) cannot be used. A new scheme for $u$ is expected to bound the states $x_{2}$ and $x_{3}$ in a neighborhood of the origin in this interval when $x_{1}$ is being driven away from the origin. Substituting $v=-\lambda$ into the last two equations of (1.1) gives

$$
\begin{gather*}
d x_{2}=-\lambda x_{3} d t+x_{3} \varphi_{1} d W_{1} \\
d x_{3}=\left(u+\lambda x_{2}\right) d t+\varphi_{2} d W_{2}-x_{2} \varphi_{1} d W_{1} \tag{5.4}
\end{gather*}
$$

Since (5.4) is a standard strict-feedback form, viewing $x_{1}$ as an external bounded input, backstepping controller can be designed to make the states $x_{2}$ and $x_{3}$ to be bounded in probability in $\left[t_{0}, \tau_{r}\right]$.

Introduce the transformation

$$
\begin{equation*}
z_{1}=x_{2}, \quad z_{2}=x_{3}-\alpha, \tag{5.5}
\end{equation*}
$$

which implies that

$$
\begin{gather*}
d z_{1}=-\lambda x_{3} d t+\varphi_{1} x_{3} d W_{1} \\
d z_{2}=\left(u-x_{2} v+c_{1} \lambda x_{3}\right) d t+\varphi_{2} d W_{2}+\left(-x_{2} \varphi_{1}-c_{1} \varphi_{1} x_{3}\right) d W_{1} \tag{5.6}
\end{gather*}
$$

where $\alpha=c_{1} z_{1}$ is used as in (4.12) with a design parameter $c_{1}>0$. A careful observation indicates that all the terms in (5.6) have the corresponding terms in (4.8) and (4.17), that is, if $\varphi_{1} / x_{1}$ in the latter is replaced with $\varphi_{1}$, then we can obtain the terms in the former.

In $\left[t_{0}, \tau_{r}\right]$, we have $\left|x_{1}\right| \leq r$. To design controller $u$ to guarantee the boundedness of $x_{2}$ and $x_{3}$, a candidate Lyapunov function is given as follows:

$$
\begin{equation*}
V=\frac{1}{4} z_{1}^{4}+\frac{1}{4} z_{2}^{4} \tag{5.7}
\end{equation*}
$$

Just for simplicity, we will design the controller as consistent as possible with statescaling case in the proceeding section. By selecting $r \leq 1$, according to assumption (A1), we can see that in the nonscaling case, we have $\varphi_{1}^{2} \leq k^{2} r^{2} \leq k^{2}$, which is corresponding to $\left(\varphi_{1} / x_{1}\right)^{2} \leq k^{2}$ used in state-scaling case (4.13). Comparing (5.5)-(5.7) with the corresponding equalities in the proceeding section, it can be found that, in the nonscaling case, some terms used in (4.22) (that are included in $u_{2}$ ) disappear and the others (that are contained in $u_{1}$ ) have the same forms with the same or milder requirements to the parameters $c_{1}$ and $\lambda$. Therefore, by choosing

$$
\begin{equation*}
u=u_{1}, \tag{5.8}
\end{equation*}
$$

we have

$$
\begin{equation*}
\varrho V \leq \frac{3}{4} l^{2} x_{1}^{4}-\frac{1}{4} c_{1} \lambda(1-e) z_{1}^{4}-c_{2} z_{2}^{4} \leq-\bar{c} V+d_{c} \tag{5.9}
\end{equation*}
$$

where $d_{c}=(3 / 4) l^{2} r^{4}$ and $\bar{c}=\min \left\{c_{1} \lambda(1-e), 4 c_{2}\right\}$, which implies that

$$
\begin{equation*}
E V(z(t)) \leq e^{-\bar{c}\left(t-t_{0}\right)} E V\left(z\left(t_{0}\right)\right)-\frac{d_{c}}{\bar{c}} e^{-\bar{c}\left(t-t_{0}\right)}+\frac{d_{c}}{\bar{c}}, \quad t \in\left[t_{0}, \tau_{r}\right] \tag{5.10}
\end{equation*}
$$

The stability analysis before $\tau_{r}$ can be included in the following result.

Theorem 5.1. Under assumptions (A1) and (A2), for every $x_{1}\left(t_{0}\right)=0$ and any $x_{2}\left(t_{0}\right), x_{3}\left(t_{0}\right)$, for any $0<r \leq 1$, with an appropriate choice of the design parameters $\lambda$ and $c_{1}$, the closed-loop system consists of (3.1), (3.2), (5.1), and (5.8) has a unique solution, and all the signals are bounded in probability in the interval $\left[t_{0}, \tau_{r}\right]$.

Proof. According to Lemma 2.2, the existence and uniqueness of $z_{1}$ and $z_{2}$ in $\left[t_{0}, \tau_{r}\right]$ come from (5.7) and (5.9). Noting the existence and uniqueness of $x_{1},(4.7),(4.9)$, and (4.12), the existence and uniqueness of $x_{2}$ and $x_{3}$ can be concluded on $\left[t_{0}, \tau_{r}\right]$. Following the same line, the boundedness of $x_{i}(i=1,2,3)$ on $t \in\left[t_{0}, \tau_{r}\right]$ can be obtained, which complete the proof.

## 6. Design of Switching Controller

Since $\left|x_{1}\left(\tau_{r}\right)\right|=r>0$, at the stochastic moment $t=\tau_{r}$, we switch the control laws $v$ and $u$ from (5.1) and (5.8) to (4.3) and (4.22), respectively. According to Theorem 4.2, the solution of the closed-loop system converges to the origin with exponential rate on $\left[\tau_{r}, \infty\right)$ for any $r>0$. A switching control scheme on $\left[t_{0}, \infty\right)$ can be given as

$$
\begin{gather*}
v=-\lambda\left(1+\left(x_{1}-1\right) s\right), \quad u=u_{1}+u_{2} s \\
z_{1}=\frac{x_{2}}{1+\left(x_{1}-1\right) s}, \quad z_{2}=x_{3}-\alpha, \quad \alpha=c_{1} z_{1}, \tag{6.1}
\end{gather*}
$$

where the switching signal is defined by

$$
s(t)= \begin{cases}0, & t \in\left[0, \tau_{r}\right)  \tag{6.2}\\ 1, & t \in\left[\tau_{r}, \infty\right)\end{cases}
$$

By summing up the above arguments, the main result in this paper can be presented now.

Theorem 6.1. Under assumptions (A1) and (A2), for $x_{1}\left(t_{0}\right)=0$ and any $x_{2}\left(t_{0}\right), x_{3}\left(t_{0}\right)$, with an appropriate choice of the design parameters $\lambda, c_{1}$ and $r$, the closed-loop system consists of (3.1), (3.2), and (6.1) has a unique solution on $\left[t_{0}, \infty\right)$, which is 4 th moment exponential stable.

Proof. For any $0<r \leq 1$, the existence of a.s. finite stopping time $\tau_{r}$ can be concluded from (5.3). The existence and uniqueness of solution of the closed-loop system can be proved by Theorem 5.1 on $\left[t_{0}, \tau_{r}\right.$ ] and by Theorem 4.2 on $\left[\tau_{r}, \infty\right)$, respectively. In the interval $\left[t_{0}, \tau_{r}\right]$, there holds $\left|x_{1}\right| \leq r$, and from (5.10), we have $E V\left(z_{1}^{4}(t)+z_{2}^{4}(t)\right) \leq \bar{d}$, for all $t \in\left[t_{0}, \tau_{r}\right]$, where $\bar{d}=E\left((1 / 4)\left(z_{1}^{4}\left(t_{0}\right)+z_{2}^{4}\left(t_{0}\right)\right)+\left(d_{c} / \bar{c}\right)\right.$, which implies that there exists a constant $\overline{\bar{d}}$ such that

$$
\begin{equation*}
E V\left(x_{1}^{4}(t)+x_{2}^{4}(t)+x_{3}^{4}(t)\right) \leq \overline{\bar{d}}, \quad \forall t \in\left[t_{0}, \tau_{r}\right] \tag{6.3}
\end{equation*}
$$

In the interval $\left[\tau_{r}, \infty\right)$, similar to (4.29), we have

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log E\left|\left(x_{1}, x_{2}, x_{3}\right)\right|^{4}<0 \tag{6.4}
\end{equation*}
$$

Combining (6.3) and (6.4) completes the proof.


Figure 2: The Logic operation of switching.

A new question about the performing of switching signal comes forth. One scheme is presented as follows in a discrete-time form. Suppose that the running time interval is $[0, T]$ and every step equals to $\Delta \ll T$. Initial step: begin with $t=0, \tau_{1}=T$, and $\tau_{2}=0$. Recursive steps: perform the following procedures in turn unless otherwise stated. (a) Write down the value of $x=x(t)$ and let $t=t+\Delta$. (b) If $t>T$, then turn to (g), otherwise, perform the following calculation. (c) If $t \leq \tau_{1}$, then we have $t_{m}=t$, otherwise, we have $t_{m}=\tau_{1}$; we have $s=0, \tau_{1}=\tau_{1}$ and restore $\tau_{2}=t$, otherwise, we have $s=1, \tau_{2}=\tau_{2}$ and $\tau_{1}=t_{m}$. (e) Submitting $s$ into control (6.1) and resolve the response $x(t)$ of closed-loop system. (f) Turn to (a). (g) Output the observed value $\tau_{r}=\tau_{2}$. (h) End the procedure. The procedure is described in Figure 2.

It should be pointed out that the switching strategy will lead to trembling phenomenon. In practice, to eliminate the trembling, the switching signal given by (6.2) can be replaced by a continuous one which depends on the measurement of $\tau_{r}$. The above logic method will be used in the forthcoming simulation.

(a)

-u
$---v$
(b)

Figure 3: The responses of closed-loop system with nonzero initial heading angle.

## 7. Simulation

Consider system (3.1) with $\varphi_{1}=k \theta$ and $\varphi_{2}=l \theta^{2}$. By letting $\phi(\theta)=\theta$, the assumptions (A1) and (A2) can be easily verified. As pointed out by [14, page 63], system (3.1) is an idealization of the following system:

$$
\begin{gather*}
\dot{\theta}=v+\varphi_{1}(\theta) N_{1}, \\
\dot{x}_{c}=\left(u+\varphi_{2}(\theta) N_{2}\right) \cos \theta,  \tag{7.1}\\
\dot{y}_{c}=\left(u+\varphi_{2}(\theta) N_{2}\right) \sin \theta
\end{gather*}
$$

with white noises $N_{1}$ and $N_{2}$, which is formally obtained by replacing " $d W_{i}(t) / d t$ " by $N_{i}(t)$. To give approximate simulation using ordinary differential equation algorithm, system (3.1) is replaced by (7.1), where the power of each $N_{i}$ equals to 1.

The following two cases are to be analyzed: (1) $\theta(0)=-1.5, x_{c}(0)=0.8, y_{c}(0)=-1, k=$ 0.1 and $l=1$; (2) $\theta(0)=0, x_{c}(0)=0.8, y_{c}(0)=-1, k=0.1$ and $l=1$. For the first case, the state-feedback control law is given by (6.1) (not (4.3) and (4.22)) with the design parameters $d=0.8, e=0.7, c_{1}=(1 / e)(1+(3 / 4) d), c_{2}=3, r=1$, and $\lambda$ satisfying the equality of (4.25). Figure 3 demonstrates that the state of the closed-loop system can be regulated to the origin with exponential rate (in almost surely sense) without switching. For the second case, the same control (6.1) with the same design parameters as in the first case is given. From Figure 4,


Figure 4: Responses of closed-loop system with zero initial heading angle by using switching (6.2).
we can see that switching happens at the moment $\tau_{r} \approx 0.1258$. The state of the closed-loop system can be driven to the origin with exponential rate after moment $\tau_{r}$ (in almost surely sense). To eliminate the trembling phenomenon, the switching signal s given by (6.2) can be replaced by a continuous one. Figure 5 describes the responses of the closed-loop system of Case 2 with the following $s(t)$ :

$$
s(t)= \begin{cases}0, & t \in\left[0, \tau_{r}\right)  \tag{7.2}\\ \frac{2}{\pi} \arctan \left(600\left(t-\tau_{r}\right)\right), & t \in\left[\tau_{r}, \infty\right)\end{cases}
$$

Comparison of Figure 4 with Figure 5 indicates that control magnitude in the latter is milder than that in the former.

## 8. Conclusions

A global exponential stabilization controller has been designed for nonholonomic mobile robot with stochastic disturbance by using the integrator backstepping procedure based on the state-scaling technique. There are several interesting problems of the controller design for the same stochastic nonholonomic mobile robot, for example, the tracking control and


$-y_{c}$
$\cdots-\theta$
(a)


(b)

Figure 5: Responses of closed-loop system with zero initial heading angle by using switching (7.2).
the adaptive control, and the further extensions to more general chained-form nonholonomic systems with stochastic disturbance. These directions are all under the current research.

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Research Article

# Wind-Induced Vibration Control of Dalian International Trade Mansion by Tuned Liquid Dampers 

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#### Abstract

This paper focuses on the wind-induced vibration control of the Dalian international trade mansion (DITM) by using the tuned liquid dampers (TLDs). To avoid the intensive computationally demanding problem caused by tens of thousand of degrees of freedom (DOF) of the structure in the numerical analysis, the three-dimension finite element model of the DITM is first simplified to the equivalent series multi-DOF system. The wind loading is subsequently simulated by the Davenport model according to the structural environmental condition where the actual samples of wind speed are measured. Following that, the shallow- and deep-water wave theories are applied to model the liquid sloshing inside TLDs, the tank sizing, and required water depth, and numbers of TLDs are given according to the numerical results of different cases. Comparisons between uncontrolled and controlled displacement and acceleration responses of the DITM under wind forces show that the designed shallow tank has higher efficiency than the deep one, which can effectively reduce the structural response amplitudes and enhance the comfortableness of the mansion. The preliminary TLD design procedure presented in this paper could be applied as a reference to the analysis and design of the wind-induced vibration for high-rise buildings using the TLD.


## 1. Introduction

In recent years, the newly developed construction technologies toward lighter and stronger materials have facilitated the realization of more and more high-rise buildings in urban areas where space usage is demanding [1]. This kind of structures have, in general, low frequencies and damping ratios associated with their fundamental oscillation modes, and when subjected to dynamic loadings, they may experience large amplitudes of bending and torsional oscillation. Especially, if a high-rise building was built in a wind prone area, the
building would experience large deflection due to mean wind and considerable vibration due to aerodynamic effects. Serious vibrations may cause fatigue damage in structural members, and thereby increasing the maintenance cost of the building. In addition, the excessive acceleration magnification will also frequently cause occupants' discomfort [2,3].

The suppression of these oscillations has become one of the major concerns to civil engineers. A number of methods exist for improving the performance of existing structures to meet the requirements. Strengthening of the buildings or the installation of a base isolation system is complicated, difficult, and expensive. Therefore, incorporating control devices such as the active and passive tuned mass dampers (AMD and TMD) $[4,5]$, tuned liquid dampers (TLD) $[6,7]$, and tuned liquid column dampers (TLCD) $[8,9]$ have been proposed to mitigate excessive oscillations. Among many varieties of control devices, the TLD is a good candidate. A TLD consists of one or multiple rigid tanks, partially filled with a liquid (usually water), which is typically located near the top of a building. As the building moves in the severe wind or earthquake attack, the fluid contained within the tank begins to slosh. The fluid thereby absorbs vibrational energy from the structure and transforms it into kinetic and potential energy of the sloshing fluid. The sloshing energy is subsequently dissipated through the fluid's viscosity, or drag produced by flow dampening devices such as baffles, poles, nets or screens [10]. This kind of device is particularly well suited for tall buildings, since they usually contain water storage for potable or emergency use. With the already available water utilized and proper modifications to the existing storage tanks, a TLD can be formed without introducing an unnecessarily large additional mass and only require very low maintenance and operating cost. Furthermore, its natural frequency and damping characteristics can be modified easily by changing the geometry of the tank, the depth of the liquid layer, and the properties of the contained liquid [11].

Since TLD was first proposed by Bauer [12] for suppressing horizontal vibration of building structures, many experimental and numerical research studies were done over the past few years to illustrate the effectiveness of a TLD as a vibration-control device for structures subjected to both harmonic and broad-band excitations. Soong and Dargush proposed the use of a single TLD with a rectangular plan in control of vibration modes along orthogonal two directions [13]. Zhang et al. proposed a liquid damper that can reduce bidirectional response using crossed tube-like liquid container [14]. Tamura et al. reported vibration control effect of cylindrical TSDs obtained for both orthogonal axes of the building plan based on measurement of acceleration response [15]. Research [16] has been done to study the application of rectangular liquid dampers to reduce the vibration of multidegree of freedom structures. Multiple tuned liquid dampers, which consist of a number of tuned liquid dampers whose natural frequencies are distributed over a certain range around the fundamental natural frequency of the structure, were suggested Fujino et al. [17] who referred to the idea of multiple tuned mass dampers [18] for more effectively suppressing dynamic responses, because the modal frequency of structure is not uncertain in practice. Very recently, Samanta and Banerji [19] investigated a modified TLD configuration to improve the effectiveness of TLDs.

This paper presents a practical example for the effectiveness and feasibility of using TLDs on the Dalian international trade mansion (DITM), a super high-rise RC structure, to control wind-induced vibration. The preliminary design procedure for initial TLD sizing design for a high-rise building is summarized and outlined, which could be applied as a reference to the analysis and design of the wind-induced vibration for high-rise buildings using the TLDs.


Figure 1: Dalian international trade mansion.

## 2. Analytical Model

### 2.1. Mansion Outline

The DITM is being built in the center of Dalian city of China, which is of 81 stories (including one-story basement) with the size of 339 m high and 77.7 m long in the east-west direction and 44 m wide in south-north direction. The total building area is $290,000.00 \mathrm{~m}^{2}$. The DITM is the highest building in the northeast of China (shown in Figure 1) [20]. Since the basic wind pressure in the Dalian region is $0.75 \mathrm{KN} / \mathrm{m}^{2}$ and the mansion is slender, that is, the ratio of height over width is 6.7 , it is relatively more flexible to large wind-vibration action in the horizontal direction. Consequently, the water tanks in the building will be designed as the turned liquid dampers to reduce its horizontal displacement and acceleration. Figure 2 shows the 3D finite element (FE) analytical model and the ichnography of the top story. The overall model has 34,308 node elements, 34,791 frame elements, and 29,071 shell elements, considering 36 section types and 11 materials' properties.

### 2.2. Simplified Model

As known, due to inherent nonlinear liquid damping, iteration is generally required in order to obtain the dynamic response of TLD-structure systems. It is no doubt that a solution scheme resorting to iteration requires a great deal of computational effort in searching for optimal parameters of dampers numerically. It would be quite a time-consuming task to carry out a detailed design of the damper. On the other hand, for a large-scale complicated structure like the DITM, whose three-dimension finite element (FE) model may have tens of thousand of DOFs, to facilitate design of the TLDs, a systematic and efficient approach is needed to solve such a computationally demanding problem. Considering that the structural stiffness of the structure in two horizontal directions is obviously different, it could mainly


Figure 2: 3D finite element analytical model.
take into account the structural vibration reduction in the direction of weaker stiffness. Thus, the FE model of the DITM could be simplified as the bending-shear model with 81 degrees of freedom by the use of the approach called as the equivalent rigidity parameter identification method which given by Sun et al. [21].

The damping matrix of structure is expressed as the Rayleigh orthogonal damping formulation, and the damping ratio of the first two modes is chosen as $4 \%$. Here, the diagonal elements of mass, stiffness, and damping matrices are illustrated in Figure 3. The natural frequencies of the first two modes are obtained by the FE model of the original structure and simplified model, and their relative errors to compare the accuracy of two models are calculated by

$$
\begin{equation*}
\frac{\left|T_{f}-T_{s}\right|}{T_{f}} \times 100 \% \tag{2.1}
\end{equation*}
$$

where $T_{f}$ and $T_{s}$ is the periods of finite element model and simplified model, respectively.
The results are listed in Table 1. And the comparisons of first two modes of two models are shown in Figure 4. As known, a TLD generally need to be designed to operate at or near the resonant frequency of the structure in order to maximize the absorbed and dissipated energy, and thus maximize the benefit of the TLD. It can be seen from Table 1 and Figure 4 that the errors of computational results of the two models are quite small so that the simplified model can be applied to calculate the vibration responses of original structure to environmental loadings.


Figure 3: Diagonal elements in mass, stiffness, and damping matrixes of structural model.

Table 1: Comparison of relative errors of periods of two models.

| Mode | Finite element model | Simplified model | Relative error |
| :--- | :---: | :---: | :---: |
| 1st period/s | 6.636 | 6.768 | $1.99 \%$ |
| 2nd period/s | 1.526 | 1.522 | $0.27 \%$ |

## 3. Wind Load

### 3.1. Pulse Wind Load Simulation

In wind engineering, the unitary wind pressure spectrum, suggested by Zhang [22], is the most popularly used one that has the same spectrum value in height

$$
\begin{equation*}
S_{f}(f)=\frac{2 x^{2}}{3 f\left(1+x^{2}\right)^{4 / 3}}, \quad x=1200 \frac{f}{\bar{V}_{10}} \tag{3.1}
\end{equation*}
$$

where $\bar{V}_{10}$ represents the mean wind velocity on the height of 10 meters $(\mathrm{m} / \mathrm{s}), f$ is the pulse frequency (Hz).


Figure 4: Diagrams of the 1st and 2nd mode comparison of two models.

Considering one-dimensional random process $\{f(t)\}$ with $n$ variables and zero-mean value, that is, $f_{1}(t), f_{2}(t), f_{3}(t), \ldots, f_{n}(t)$, their power spectrum density function could be expressed as [23]:

$$
\begin{equation*}
S_{F_{i j}^{*}}(w)=\left[S_{p}\right] S_{f}(w), \quad i, j=1,2,3, \ldots, n, \tag{3.2}
\end{equation*}
$$

in which

$$
\begin{equation*}
\left(S_{p}\right)_{k h}=p_{k h} P_{k} P_{h}, \tag{3.3}
\end{equation*}
$$

where $P_{k}=\mu_{f}\left(Z_{k}\right) \mu_{D}\left(Z_{k}\right) \mu_{Z}\left(Z_{k}\right) \mu_{r} w_{0} \Delta A_{k} / \mu_{\text {, }}$ in which $\mu_{f}, \mu_{D}$, and $\mu_{Z}$ means the pulse coefficient at the $k$ th story, shape coefficient, and varying coefficient of wind pressure along height, $\mu_{r}$ and $\mu$ are the reoccurrence period coefficient and wind factor, $w_{0}$ and $\Delta A_{k}$ represent the basic wind pressure and area of suffering wind at the $k$ th story, and $p_{k h}=$ $\exp \left(-\left|Z_{k}-Z_{h}\right| / 60\right)$ implies the vertical coherence factor of pulse wind.

Based on the Shinozuka theory [24], the random process $\{f(t)\}$ may be simulated by the following equation:

$$
\begin{equation*}
f_{j}(t)=\sqrt{2 \Delta \omega} \sum_{m=1}^{j} \sum_{l=1}^{n}\left|H_{j m}\left(\omega_{m l}\right)\right| \cos \left(\omega_{m l} t-\theta_{j m}\left(\omega_{m l}\right)+\phi_{m l}\right), \quad j=0,1,2,3, \ldots, n, \tag{3.4}
\end{equation*}
$$

where $j$ is the maximum positive integer, $\Delta \omega$ means the frequency increment calculated by $\Delta \omega=\left(\omega_{b}-\omega_{a}\right) / N$ in which $\omega_{a}$ and $\omega_{b}$ are the starting and ending frequency, $\phi_{m l}$ denotes the random phase distributed in the range of $(0,2 \pi), H_{j m}\left(\omega_{m l}\right)$ implies the element in the matrix, $\mathbf{H}(\omega)$ and $\mathbf{H}(\omega)$ is the Cholesky decomposition of matrix $\mathbf{S}(\omega)$; that is,

$$
\begin{equation*}
\mathbf{S}(\omega)=\mathbf{H}(\omega) \mathbf{H}^{*}(\omega)^{T} \tag{3.5}
\end{equation*}
$$

in which, $\omega_{m l}=(l-1) \Delta \omega+\left(\frac{m}{n}\right) \Delta \omega, \theta_{j m}\left(\omega_{l}\right)$ is given by

$$
\begin{equation*}
\theta_{j m}\left(\omega_{l}\right)=\tan ^{-1}\left(\frac{\operatorname{Im}\left\lfloor H_{j m}\left(\omega_{l}\right)\right\rfloor}{\operatorname{Re}\left[H_{j m}\left(\omega_{l}\right)\right]}\right) \tag{3.6}
\end{equation*}
$$

Based on the central limitation theorem, the random process $\left\{f_{j}(t)\right\}$ will gradually tend to be the Gauss process as $N \rightarrow \infty$. And the time interval $\Delta t$ should meet the requirement of $\Delta t \leq 2 \pi / 2 \omega_{b}$ in order to avoid the sample superposition according to the sampling theory.

As mentioned above, the good samples can be acquired from (3.4) only if $\mathbf{S}(\omega)$ is known, and $N, \omega_{b}$ and $\Delta t$ are properly selected. However, since this method would consume more time and energy while calculating, the Fast Fourier Transform algorithm (FFT) is introduced in (3.4) to obtain higher efficiency:

$$
\begin{align*}
& f_{j}(p \Delta t)=\operatorname{Re}\left\{\sum_{m=1}^{j} h_{j m}(q \Delta t) \exp \left[i\left(\frac{m \Delta \omega}{n}\right)(p \Delta t)\right]\right\}  \tag{3.7}\\
& p=0,1,2,3,4, \ldots, 2 N \times n-1 ; \quad j=0,1,2,3,4, \ldots, n
\end{align*}
$$

where

$$
\begin{equation*}
h_{j m}(q \Delta t)=\sum_{l=0}^{2 N-1} g_{j m}(l \Delta \omega) \exp \left[i \frac{\pi d q}{N}\right], \quad q=0,1,2,3,4, \ldots, n-1 \tag{3.8}
\end{equation*}
$$

in which

$$
g_{j m}(l \Delta \omega)= \begin{cases}\sqrt{2 \Delta \omega} H_{j m}\left(l \Delta \omega+\frac{m \Delta \omega}{n}\right) \exp \left(i \phi_{m l}\right), & 0 \leq l<N  \tag{3.9}\\ 0, & N \leq l<2 N\end{cases}
$$

It can be known from (3.8) and (3.9) that $h_{j m}(q \Delta t)$ is the inverse Fourier transform of $g_{j m}(l \Delta \omega)$, which is calculated by FFT. Thus, 81 samples of wind load are calculated by selecting $N=8192, \Delta t=0.05, \omega_{a}=0$, and $\omega_{b}=60$. The wind pressure time history of these samples acting on the top of structure is illustrated in Figure 5 and its theoretical and simulated spectra are shown in Figure 6. It can be seen from these figures that the simulated results are perfect.


Figure 5: Wind load time history on top of building.


Figure 6: Power spectral comparison between simulated and theoretical wind samples.

To verify if the simulated wind load may represent the actual wind action, the wind velocity time histories on the top of the nearby high-rise building ( 200 m in height) were measured by the smart ZDR-1F mode wind velocity instrument (Figure 7), the accuracy of which is as high as $\pm(0.5+0.05 *$ wind velocity $) \mathrm{m} / \mathrm{s}$, the measuring range $1.5 \sim 40 \mathrm{~m} / \mathrm{s}$, recognizing capability rate 0.1 mm , recording interval range from 2 second to 24 hours ( 2 s adopted in the measurement) [25]. The working diagram is illustrated in Figure 8. The measured results give the idea that the average velocity of these strong and stable samples is about $15.36 \mathrm{~m} / \mathrm{s}$, and Figure 9 is the wind velocity time history.


Figure 7: ZDR-1F-type wind velocity instrument.


Figure 8: Working diagram.


Figure 9: Recorded wind velocity time history.


Figure 10: Spectral comparisons between simulated and recorded wind loads.

### 3.2. Comparisons between Simulated and Actual Measured Wind Loads

The spectral comparison between simulated and recorded wind loads is depicted in Figure 10. It is known from Figure 10 that they agree well each other, which indicates the fact that the simulated wind load is enough to represent the actual state and can be applied in the structural design and analysis.

## 4. TLD Principle and Design

### 4.1. TLD Working Principle

The TLD system may be a rectangle or circle tank fixed on the top of the building and may be one large tank or composed of some small tanks, in which the liquid in the tank may be deep or shallow. The TLDs will shake together with the building when suffering from wind load. The dynamic liquid pressure of generating surface wave acting on tank walls will reduce structural vibration. Usually, the TLDs are divided into two categories based on the ratio of liquid depth to the size in shaking direction inside the tank: the TLD of deep liquid if the ratio is larger than $1 / 8$, otherwise, the TLD of shallow liquid. The damping is due to energy dissipation through the internal fluid viscous forces and wave breaking (shallowliquid dampers). In the case of deep-liquid dampers, damping depends on the amplitude of liquid motion and on viscous forces.


Figure 11: Diagram of liquid movement in TLD.

### 4.2. Shallow Liquid Theory

The following basic assumptions in deriving the equations of motion are adopted:
(1) the liquid in the TLDs vibrates in two dimensions,
(2) the liquid in the TLDs is incompressible and gyrating-free current,
(3) the pressure on liquid free surface is invariable,
(4) the friction may be produced only near boundary layers and on solid surface.

The liquid in TLDs may be divided into two parts according to the shallow wave theory, boundary layer and outside boundary layer, as shown in Figure 11. The dynamic viscous damping is generated mainly by liquid inner friction in the boundary layer due to small liquid damping in outside of boundary layer, so the basic equation of motion of liquid in two-dimensional TLDs may be described by continuous equations and Navier-Stokes equation inside and outside boundary layer [26]

$$
\begin{gather*}
\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}=0 \\
\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+w \frac{\partial u}{\partial z}=-\frac{1}{\rho} \frac{\partial p}{\partial x}-\ddot{x}_{g} \quad\left(-\left(h-h_{b}\right) \leq z \leq \eta\right)  \tag{4.1}\\
\frac{\partial w}{\partial t}+u \frac{\partial w}{\partial x}+w \frac{\partial w}{\partial z}=-\frac{1}{\rho} \frac{\partial p}{\partial z}-g \quad\left(-\left(h-h_{b}\right) \leq z \leq \eta\right) .
\end{gather*}
$$

The equation of motion inside boundary layer is

$$
\begin{gather*}
\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+w \frac{\partial u}{\partial z}=-\frac{1}{\rho} \frac{\partial p}{\partial x}+v\left(\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial z^{2}}\right)-\ddot{x}_{g} \quad\left(-h \leq z \leq\left(h-h_{b}\right)\right),  \tag{4.2}\\
\frac{1}{\rho} \frac{\partial p}{\partial z}=-g \quad\left(-h \leq z \leq\left(h-h_{b}\right)\right),
\end{gather*}
$$

where $u$ and $w$ are the velocities of liquid mass in $x$ and $z$ directions, $p$ means the liquid inner pressure, $h_{b}$ denotes the thickness of boundary layer, $\rho$ implies the liquid density, $v$ liquid dynamic viscosity factor, $\ddot{x}_{g}$ represents the acceleration of tank movement, and $g$ is the gravity acceleration.

The boundary conditions are

$$
\begin{array}{ll}
u=0, & (x= \pm a) \\
w=0, & (z=-h) \\
w=\frac{\partial \eta}{\partial t}+u \frac{\partial \eta}{\partial x}, & (z=\eta)  \tag{4.3}\\
p=p_{0}=\text { const, } & (z=\eta) .
\end{array}
$$

The liquid movement outside the boundary layer is treated as potential flow, velocity potential function, $\Phi$, of which could be supposed as follows:

$$
\begin{equation*}
\Phi(x, z, t)=F(x, t) \cosh [k(h+z)] . \tag{4.4}
\end{equation*}
$$

It is derived from potential function that

$$
\begin{align*}
& u=\frac{\partial \Phi}{\partial x}=\frac{\partial F}{\partial x} \cosh (k(h+z))  \tag{4.5}\\
& w=\frac{\partial \Phi}{\partial z}=k F \sinh (k(h+z))
\end{align*}
$$

Substituting (4.5) into (4.1) and (4.2), the basic equations are obtained after conversing by combing boundary condition(4.3)

$$
\begin{gather*}
\frac{\partial \eta}{\partial t}+h \sigma \frac{\partial(\phi u(\eta))}{\partial x}=0  \tag{4.6}\\
\frac{\partial}{\partial t} u(\eta)+\left(1-T_{H}^{2}\right) u(\eta) \frac{\partial}{\partial x} u(\eta)+g \frac{\partial \eta}{\partial x}+g h \sigma \phi \frac{\partial^{2} \eta}{\partial x^{2}} \frac{\partial \eta}{\partial x}=-\lambda u(\eta)-\ddot{x}_{S}
\end{gather*}
$$

where

$$
\begin{gather*}
\sigma=\frac{\tanh (k h)}{k h} \\
\phi=\frac{\tanh (k(h+\eta))}{\tanh (k h)}  \tag{4.7}\\
T_{H}=\tanh [k(h+\eta)] \\
\lambda=\left(\frac{1}{n+h}\right) \frac{8}{3 \pi} \sqrt{\omega v}\left(1+\left(\frac{2 h}{b}\right)+S\right)
\end{gather*}
$$

which could be called as viscous coefficient. in which $k$ is the wave number taken as $\pi / 2$, $b$ means the tank width and $S$ implies the viscous influencing factor on the liquid surface usually taken as 0~2.

To make each parameter dimensionless, it is expressed as

$$
\begin{equation*}
x^{\prime}=\frac{x}{a}, \quad z^{\prime}=\frac{z}{h}, \quad \eta^{\prime}=\frac{\eta}{h}, \quad \varepsilon=\frac{h}{a}, \quad u^{\prime}=\frac{u}{C_{0}}, \quad t^{\prime}=\frac{t}{t_{0}}, \quad k^{\prime}=k a, \quad \ddot{x}_{S}^{\prime}=\frac{t_{0}^{2}}{a} \ddot{x}_{S} \tag{4.8}
\end{equation*}
$$

in which, $C_{0}=\sqrt{g h}, t_{0}=a / C_{0}$.
Equations (4.6) are discredited along the vibration direction of tank as follows:

$$
\begin{gather*}
\frac{d \eta_{i}^{\prime}}{d t^{\prime}}=\frac{\sigma}{\Delta x^{\prime}}\left(\phi_{i} u_{i}^{\prime}-\phi_{i+1} u_{i+1}^{\prime}\right) \quad(i=1 \sim n-1), \\
\frac{d \eta_{0}^{\prime}}{d t^{\prime}}=-\frac{2 \sigma}{\Delta x^{\prime}} \phi_{1} u_{1}^{\prime},  \tag{4.9}\\
\frac{d \eta_{n}^{\prime}}{d t^{\prime}}=\frac{2 \sigma}{\Delta x^{\prime}} \phi_{n} u_{n}^{\prime}, \\
\frac{d u^{\prime}}{d t^{\prime}}=\frac{1}{\Delta x^{\prime}}\left(\eta_{i-1}^{\prime}-\eta_{i}^{\prime}+H_{i}\left(K_{i-1}-K_{i}\right)+C_{i}\left(I_{i-1}-I_{i}\right)\right)-\lambda_{i}^{\prime} u_{i}^{\prime}-\ddot{x}_{S}^{\prime} \quad(i=1 \sim n),
\end{gather*}
$$

where

$$
\begin{gather*}
\Delta x^{\prime}=\frac{2}{n^{\prime}} \\
\phi_{i}=\frac{\tanh \left(k^{\prime} \varepsilon\left(1+\left(\eta_{i-1}^{\prime}+\eta_{i}^{\prime}\right) / 2\right)\right)}{\tanh \left(k^{\prime} \varepsilon\right)} \quad(i=1 \sim n), \\
H_{i}=\frac{\left(1-\left(\phi_{i} \tanh \left(k^{\prime} \varepsilon\right)\right)^{2}\right)}{2} \quad(i=1 \sim n), \\
K_{i}=\left(\frac{\left(u_{i}^{\prime}+u_{i+1}^{\prime}\right)}{2}\right)^{2} \quad(i=1 \sim n-1),  \tag{4.10}\\
I_{i}=\frac{\left(\left(\left(\eta_{i+1}^{\prime}-\eta_{i-1}^{\prime}\right) /\left(2 \Delta x^{\prime}\right)\right)^{2}\right)}{2} \quad(i=1 \sim n-1), \\
\lambda^{\prime}=\frac{1}{1+\left(\eta_{i-1}^{\prime}+\eta_{i}^{\prime}\right) / 2} \frac{8}{3 \pi \varepsilon C_{0}} \sqrt{\omega v}\left(1+\left(\frac{2 h}{b}\right)+S\right) .
\end{gather*}
$$

It is suggested here that the $n$ value may be taken as

$$
\begin{equation*}
n=\frac{\pi}{(2 \arccos (\sqrt{(\tanh (\pi \varepsilon)) /(2 \tanh (\pi \varepsilon / 2)}))} \tag{4.11}
\end{equation*}
$$

and then dynamic liquid pressure can be expressed as follows:

$$
\begin{equation*}
F_{\mathrm{TLD}}=\frac{1}{4} M_{W} \frac{g h}{a}\left(\left(\eta_{n}^{\prime}+1\right)^{2}-\left(\eta_{0}^{\prime}+1\right)^{2}\right) \tag{4.12}
\end{equation*}
$$

where $M_{W}=2 p a b h$ represents liquid weight in the TLD.

### 4.3. TLD Design

It is known that the average participating factor of the 1st mode in structural displacement response is about $92.07 \%$ and the 2 nd mode is $6.67 \%$ by calculations. The two modal participating coefficients in the structural response are illustrated in Figure 12.

As mentioned above, the only first mode response is controlled by the TLDs in design, since it dominates the total structural response in the tall building. Then, the 1st oscillation frequency of liquid in TLDs is tuned to the first modal frequency of structure according to the TLD tuning condition; that is,

$$
\begin{equation*}
\omega_{1}=\sqrt{\frac{\pi g}{2 a} \tanh \left(\frac{\pi h}{2 a}\right)}=0.92836 \tag{4.13}
\end{equation*}
$$

Thus, the sizes of the TLDs, including the length $(R)$ and depth $(H)$, are designed as listed in Table 2.

## 5. Calculation of Wind-Induced Structural Vibration Reduction

### 5.1. Equation of DITM for Structure-TLDs System

The equation of motion for the structure-TLDs system is

$$
\begin{equation*}
[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f(t)\}-[H]\left\{F_{\mathrm{TLD}}\right\} \tag{5.1}
\end{equation*}
$$

where $\{x\},\{\dot{x}\}$, and $\{\ddot{x}\}$ represent the structural displacement, velocity, and acceleration vectors, $[M],[C]$, and $[K]$ imply the structural mass, damping, and rigid matrices, $\{f(t)\}$ denotes the pulse wind load vector acting on structure, $\left\{F_{\mathrm{TLD}}\right\}$ means the control force vector of TLDs, and $[H]$ is the position matrix of TLDs in which its $i$ th column vector $\{H\}_{i}=\left[\begin{array}{llllll}0 & \cdots & 1 & 0 & \cdots\end{array}\right]_{1 \times n}^{T}$ ( 1 is in $j$ th column) is that the $i$ th group TLDs are installed on the $j$ th story.


Figure 12: Participating factors of first two modes participant.

Table 2: Design sizes of TLDs.

| Length/2a | Width/B | Liquid <br> depth/H | Height | Total <br> weight | Number/ $n$ | Liquid <br> depth ratio | Note |
| :--- | :--- | :---: | :--- | :---: | :---: | :---: | :---: | :---: |
| 8.7605 m | 7.47 m | 0.7 m | 1.4 m | 3666 t | 80 | 0.08 | Overlapping <br> with <br> 4 layer |

### 5.2. Effectiveness of Vibration Reduction

In design, the TLDs are placed on the top story of building. Combining (5.1) with (4.9) and (4.12), the structural dynamic response is calculated by the subprogram, ode23 in MATLAB package, to obtain the vibration-reducing rates, Re , as

$$
\begin{equation*}
\operatorname{Re}=\frac{r_{1}-r_{2}}{r_{1}} \times 100 \%, \tag{5.2}
\end{equation*}
$$

where $r_{1}$ and $r_{2}$ are the displacements (or peak accelerations) on the top story without and with TLD control, respectively.

The reduced displacement and acceleration amplitudes by installing the shallow liquid TLD are shown in Table 3.

Figure 13 shows the wave amplitude and time history of liquid oscillation force in the right side of liquid tank. Figures 14 and 15 typically depict the time histories of displacement and acceleration on the top story of structure. It is clear to know from Table 3 and Figures 13~15 that the structural responses can be reduced to different grades by TLDs, and wave amplitude curves indicate that the liquid oscillation is obviously nonlinear, which causes

Table 3: Vibration-reducing rates of displacement and peak acceleration.

| Case | Displacement $(\mathrm{cm})$ | Acceleration $\left(\mathrm{cm} \cdot \mathrm{s}^{-2}\right)$ | Vibration reducing rate (\%) |  |
| :--- | :---: | :---: | :---: | :---: |
| Displacement | Acceleration |  |  |  |
| Without control | 16.00 | 24.09 |  |  |
| With control | 13.27 | 18.33 | 17.06 | 23.856 |



Figure 13: Wave amplitude and water oscillation force in right side of TLD.
partial amplifications in time histories of displacement and acceleration, and the liquid oscillation frequencies corresponding to these amplifications quickly vary. In general, the vibration-reducing effectiveness is very well and ideal and may meet the requirement for the design of wind-induced vibration of DITM.

### 5.3. Comparison with TLD Design of Deep Liquid Theory

The basic assumption of the deep liquid theory is that the liquid wave in TLDs is low and slow and its velocity potential function $\Phi=\Phi(x, y, z, t)$ can be deducted by solving basic equations with boundary conditions. According to the linear Bernoulli equation, the dynamic liquid pressure at any position in a rectangular TLD can be achieved. Thus, the force acting on structure will be the resultant force of dynamic liquid pressure on both left and right sides of rectangular TLDs, that is, [26]

$$
\begin{align*}
F_{\mathrm{TLD}} & =-\int_{-H}^{0} B p_{j}(0, z, t) d z+\int_{-H}^{0} B p_{j}(R, z, t) d z \\
& =-\rho R B H\left[\ddot{x}_{T}(t)+\sum_{n=1,3, \ldots}^{\infty} \ddot{\omega}_{n}(t) \frac{2 b_{n}}{n \pi H} \tanh \frac{n \pi H}{R}\right], \tag{5.3}
\end{align*}
$$

where $H, B$, and $R$ are the liquid depth, width, and length of fluid tank, $p_{j}(0, z, t)$ and $p_{j}(R, z, t)$ represent the dynamic liquid pressure acting on both left and right sides, $\rho$ is the


Figure 14: Displacement time histories with and without TLD control on top story of structure.


Figure 15: Acceleration time history with and without TLD control on top story of structure.
liquid density, $\ddot{x}_{T}(t)$ denotes the story acceleration where TLD is located, and $b_{n}=4 R / n^{2} \pi^{2}$ and $\ddot{\omega}_{n}(t)$ means the acceleration of liquid motion in TLDs.

It is known from the previous section that the first modal response is dominant in the wind-induced vibration of high-rise structures. Therefore, controlling the first modal response is normally enough by TLDs. Thus, the equation of liquid motion in TLD is expressed as follows:

$$
\begin{equation*}
\ddot{\omega}_{1}(t)+2 \tilde{\zeta}_{1} \tilde{\omega}_{1} \dot{\omega}_{1}(t)+\tilde{\omega}_{1}^{2} \omega_{1}(t)=-\ddot{x}_{T}(t), \tag{5.4}
\end{equation*}
$$

and by taking $\tilde{\zeta}_{1}=0.08$ and adjusting $\tilde{\omega}_{1}$ to first mode frequency of structure, (5.3) is rewritten as

$$
\begin{equation*}
F_{\mathrm{TLD}}=-M_{T}\left[\ddot{x}_{T}(t)+a_{1} f_{1} \ddot{\omega}_{1}(t)\right], \tag{5.5}
\end{equation*}
$$

where $M_{T}=\rho R B H, f_{1}=(R / \pi H) \tanh (\pi(H / R))$, and $a_{1}=8 / \pi^{2}$.
According to the tuning relation between frequencies of the TLD and structure, the design sizes of deep liquid TLDs are listed in Table 4, in which the liquid mass is the same as shallow liquid.

The reduced displacement and acceleration amplitude on the top of structure by installing deep liquid TLD are shown in Table 5.

Comparisons of vibration-reducing rates with two kinds of design plans of both deep and shallow liquid theories are summarized in Table 6.

Table 4: Design sizes of deep liquid TLDs.

| Length $/ R$ | Width $/ B$ | Liquid depth $/ H$ | Net weight of liquid $/ M_{T}$ | Mass ratio/ $\mu$ | Number of TLDs $/ n$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 9.95 m | 6.61 m | 3.98 m | 3666 t | $1.0 \%$ | 14 |

Table 5: Vibration-reducing rates of displacement and peak acceleration.

| Case | Displacement $/ \mathrm{cm}$ | Acceleration $/ \mathrm{cm} \cdot \mathrm{s}^{-2}$ | Vibration reducing rate (\%) |  |
| :--- | :---: | :---: | :---: | :---: |
| Displacement | Acceleration |  |  |  |
| Without control | 16.00 | 24.09 |  |  |
| With control | 15.11 | 19.51 | 5.5 | 19.01 |

Table 6: Comparisons of vibration-reducing rates of two design plans.

| Plan | Reducing rate of peak displacement (\%) | Reducing rate of peak acceleration (\%) |
| :--- | :---: | :---: |
| Deep liquid | 5.5 | 19.01 |
| Shallow liquid | 17.06 | 23.85 |

It is clearly seen from Table 6 that the designed shallow tank has higher efficiency than deep one for both displacement and acceleration. Especially, it makes the vibrationreducing rates of displacement increase nearly $12 \%$ and $5 \%$ for acceleration. Thereby, it is recommended to utilize the shallow liquid theory to design the TLD in practical projects because of its higher efficiency.

## 6. Preliminary Design Procedure in Wind-Induced Vibration Control by TLDs

A preliminary design procedure was summarized and outlined for specified target structural response amplitude corresponding to wind-induced serviceability level accelerations. The simple design procedure allows a designer to apply well-known TLD design theory to a TLD equipped with damping screens.

Step 1. After calculating the basic frequency of high-rise structure, the TLD frequency is adjusted to the structural frequency.

Step 2. Combining with the practical situation of engineering projects, the liquid tank sizes for TLDs are designed by the shallow liquid theory.

Step 3. The number of tanks is decided after determining liquid tank mass. Usually, the tank mass is taken as about $1 \%-5 \%$ of the total structural mass.

Step 4. The proper positions are selected to install liquid tanks (usually on equipment floors).
Step 5. The system of structure-tank interactions is calculated to obtain the structural responses by the shallow liquid theory.

Step 6. The design should be going on by returning to Step 3 and adjusting the number of tanks until the required results are achieved if the control effectiveness is not as good as expected, otherwise the design work ends.

## 7. Conclusions

The suppression of the significant oscillations of the high-rise buildings has become an important design consideration in recent years. It is, thus, necessary to find a cost-effective solution for suppressing the vibration. This paper gives a practical application for the effectiveness and feasibility of using the TLDs to the DITM, a super high-rise RC structure, to control wind-induced vibration. From both analytical and numerical inspections, some conclusions with practical significance can be drawn as the following.
(1) It is normally difficult to use the software of current structural analysis to do the 3D dynamic analysis because of more time consuming and more hard disk space needed. Thus, the FE model of original spatial structure should be simplified. On the other hand, the calculation model of the super high-rise building is generally a bending-shear one; that is, its rigidity matrix is full rank. The triple diagonal matrix simplified by traditional methods is not reasonable, while the identification method of equivalent rigidity coefficients is the better one. To facilitate design of the TLDs, the equivalent stiffness approach is adopted here to simplify the model. It has been known by numerical comparisons that two models were in good agreement with each other. Therefore, the simplified model can substitute the spatial FE model in structural dynamic analysis, which could save much more computing time.
(2) The harmonic synthetically method based on the trigonometric series superposition is used to simulate the pulse wind load for the DITM, and the FFT technique is adopted to make the calculation velocity faster. It has been proven by comparing the simulated wind velocity spectra with the on-site measured spectra that the approach adopted here is reliable and reasonable in the structural analysis.
(3) Comparisons between uncontrolled and controlled displacement and acceleration responses of the DITM under wind forces show that the designed shallow tank has higher efficiency than the deep one, which can effectively reduce the structural response amplitudes and enhance the comfortableness of the mansion. According to numerical results, the final designed TLDs on the top of structure suffered pulse wind load could efficiently make the reduction rates be as high as $17 \%$ for structural peak displacement and $23.8 \%$ for peak acceleration.
(4) The preliminary TLD design procedure summarized in this paper could be applied as a reference to the analysis and design of the wind-induced vibration for high-rise buildings using the TLD.

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[^0]:    (b)

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