# Recent Advances in Analytical Methods in Mathematical Physics 

Guest Editors: Teoman Özer, Vladimir B. Taranov, Roman G. Smirnov, Thomas Klemas, Prakash Thamburaja, Sanith Wijesinghe, and Burak Polat

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## Contents

Recent Advances in Analytical Methods in Mathematical Physics, Teoman Özer, Vladimir B. Taranov, Roman G. Smirnov, Thomas Klemas, Prakash Thamburaja, Sanith Wijesinghe, and Burak Polat Volume 2012, Article ID 843204, 3 pages

The Asymptotic Synchronization Analysis for Two Kinds of Complex Dynamical Networks, Ze Tang and Jianwen Feng
Volume 2012, Article ID 309289, 14 pages
Existence and Linear Stability of Equilibrium Points in the Robes Restricted Three-Body Problem with Oblateness, Jagadish Singh and Abubakar Umar Sandah
Volume 2012, Article ID 679063, 18 pages
Relativistic Double Barrier Problem with Three Transmission Resonance Regions, A. D. Alhaidari, H. Bahlouli, and A. Jellal

Volume 2012, Article ID 762908, 13 pages
Nonlinear Effects of Electromagnetic TM Wave Propagation in Anisotropic Layer with Kerr Nonlinearity, Yu G. Smirnov and D. V. Valovik
Volume 2012, Article ID 609765, 21 pages
Combination Mode of Internal Waves Generated by Surface Wave Propagating over Two Muddy Sea Beds, Ray-Yeng Yang and Hwung Hweng Hwung
Volume 2012, Article ID 183503, 9 pages
Spacetime Junctions and the Collapse to Black Holes in Higher Dimensions, Filipe C. Mena Volume 2012, Article ID 638726, 14 pages

Mixed Initial-Boundary Value Problem for Telegraph Equation in Domain with Variable Borders, V. A. Ostapenko

Volume 2012, Article ID 831012, 17 pages
A Study on the Convergence of Series Solution of Non-Newtonian Third Grade Fluid with Variable Viscosity: By Means of Homotopy Analysis Method, R. Ellahi
Volume 2012, Article ID 634925, 11 pages
Peristaltic Transport of a Jeffrey Fluid with Variable Viscosity through a Porous Medium in an Asymmetric Channel, A. Afsar Khan, R. Ellahi, and K. Vafai
Volume 2012, Article ID 169642, 15 pages

## Editorial

## Recent Advances in Analytical Methods in Mathematical Physics

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## 1. Introduction to the Special Issue

This special issue of the journal Advances in Mathematical Physics was planned to focus on the most recent advances in analytical techniques of particular use to researchers in the field of mathematical physics that covers a very wide area of topics and has a key role in interdisciplinary studies including mathematics, mechanics, and physics. In this special issue, we were particularly interested in receiving novel contributions detailing analytical methods together with appropriate formulations applied to address problems in mathematical physics. We aimed to compile contributions across a variety of disciplines in mathematical physics such as integrability and chaos in dynamical systems, nonlinear partial differential equations, nonlinear problems in mechanics, control theory, geometrical methods, plasma physics, Lie algebras and representation theory, and statistical mechanics.

From different areas of mathematical physics mentioned previously, we have received manuscripts, above thirty, from different countries for consideration in this special issue. After the strict peer-review process the acceptance rate for these manuscripts was $28 \%$. The
brief descriptions for each accepted manuscript provided by corresponding academic editors are given below.

In their study "Existence and linear stability of equilibrium points in the Robe's restricted three-body problem with oblateness," J. Singh and A. U. Sandah have investigated the positions and linear stability of an infinitesimal body around the equilibrium points in Robe's circular restricted three-body problem when the hydrostatic equilibrium figure of the fluid of the first primary is an oblate spheroid and the second one is an oblate spheroid as well. The equations of motion and the existence of the equilibrium points as well as their linear stability conditions are demonstrated in detail.

In their study "Relativistic double barrier problem with three transmission resonance regions," A. D. Alhaidari et al. have obtained exact scattering solutions of the Dirac equation in $1+1$ dimensions for a double square barrier vector potential. Their main findings are two additional subbarrier transmission resonance regions below the conventional ones. The authors plan to pursue the current line of inquiry to investigate the transport properties of graphene.

In their study "Nonlinear effects of electromagnetic TM wave propagation in anisotropic layer with kerr nonlinearity," Y. G. Smirnov and D. V. Valovik have investigated new eigenvalues and new eigenwaves for the physically important problem of the electromagnetic TM wave propagation through a layer with Kerr nonlinearity. Numerical investigations were performed and the obtained dispersion relation was applied, for example, to the nonlinear metamaterials. The results were compared to the linear theory of the problem under consideration. The approach is restricted to the layer between half spaces with constant permittivity.

In their studies "Combination mode of internal waves generated by surface wave propagating over two muddy sea beds," R.-Y. Yang and H. H. Hwung have investigated the nonlinear response of an initially flat sea bed, with two muddy sections, to a monochromatic surface progressive wave. They showed how resonance of internal waves on a sediment bed can lead to sediment suspension. These results are obtained using a standard perturbation analysis of the weekly nonlinear wavefield system.

In his comprehensive review "Spacetime junctions and the collapse to black holes in higher dimensions," F. C. Mena has focused on recent results of the modeling of gravitational collapse to black holes in higher dimensions with emphasis on cases which involve spacetime junctions with no shell, and in particular nonspherical spacetimes containing a nonzero cosmological constant. The investigation continues with a review of the interesting case of a model of radiating gravitational collapse, in particular, the anisotropic Bizon-ChmajSchmidt (BCS) solution in $4+1$ dimensions, which is compared to data for the SchwarzschildTangherlini solution.

In his study "Mixed initial-boundary value problem for telegraph equation in domain with variable borders," V. A. Ostapenko has provided a novel contribution to exact solutions for a class of mixed initial-boundary value problems for the telegraph equation in an arbitrary domain with time variable borders. While the investigation focuses on the calculation of stress fields in ropes of elevating devices, the developed techniques can be applied to many other physical problems that are represented by similar mixed initial-boundary value problems of hyperbolic type.

In his study "A study on the convergence of series solution of non-Newtonian third grade fluid with variable viscosity: by means of homotopy analysis method," R. Ellahi studied the series solutions of the third-grade non-Newtonian flow. It is mentioned that this flow has a variable viscosity. Since the governing equations of the problem are coupled and highly nonlinear,
it not possible to find analytical solutions to the problem and hence the homotopy analysis method was applied. Based on this method, the series solutions for the problem are obtained and the convergence of these series solutions and constant and variable viscosity for a thirdgrade flow is presented.

In their studies "Peristaltic transport of a Jeffrey fluid with variable viscosity through a porous medium in an asymmetric channel," A. A. Khan et al. have derived analytic solutions for stream function, velocity, pressure gradient, and pressure rise for peristaltic transport of a Jeffrey fluid with variable viscosity through a porous medium in an asymmetric channel. The variation in flow characteristics are presented graphically as functions of viscosity, Daray number, porosity, amplitude ratio, and Jeffrey fluid parameters for future experimental verification.

In their study "The asymptotic synchronization analysis for two kinds of complex dynamical networks," Z. Tang and J. Feng studied a class of complex networks and obtained sufficient criteria for both time delay-independent and time delay-dependent asymptotic synchronization by using the Lyapunov-Krasovskii stability theorem and linear matrix inequality.

## Acknowledgments

As the guest editors of this special issue, we would like to thank all authors who sent their studies and all referees who spent time in the review process. We wish to thank them again for their contributions and efforts to the success of this special issue.

Teoman Özer<br>Vladimir B. Taranov<br>Roman G. Smirnov<br>Thomas Klemas<br>Prakash Thamburaja<br>Sanith Wijesinghe<br>Burak Polat

## Research Article

# The Asymptotic Synchronization Analysis for Two Kinds of Complex Dynamical Networks 

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We consider a class of complex networks with both delayed and nondelayed coupling. In particular, we consider the situation for both time delay-independent and time delay-dependent complex dynamical networks and obtain sufficient conditions for their asymptotic synchronization by using the Lyapunov-Krasovskii stability theorem and the linear matrix inequality (LMI). We also present some simulation results to support the validity of the theories.

## 1. Introduction

A complex dynamical network is a large set of interconnected nodes that represent the individual elements of the system and their mutual relationships. Owing to their immense potential for applications to various fields, complex networks have been intensively investigated in the past decade in areas as diverse as mathematics, physics, biology, engineering, and even the social sciences [1-3]. The synchronization problem for complex networks was first posed by Saber and Murray $[4,5]$ who also introduced a theoretical framework for their investigation by viewing them as the adjustments of the rhythms of their interaction states [6] and many different kinds of synchronization phenomena and models have also been discovered such as complete synchronization, phase synchronization, lag synchronization, antisynchronization, impulsive synchronization, and projective synchronization.

Time delays are an important consideration for complex networks although these were usually ignored in early investigations of synchronization and control problems [6-11]. To make up for this deficiency, uniformly distributed time delays have recently been incorporated into network models [12-25] and Wang et al. [18] even considered networks with both delayed and nondelayed couplings and obtained sufficient conditions for asymptotic stability. Similarly, Wu and Lu [19] investigated the exponential synchronization
of general weighted delay and nondelay coupled complex dynamical networks with different topological structures. There remains, however, much room for improvement in both the scope of the systems considered by Wang and Xu as well as in their methods of proofs.

The main contributions of this paper are two-fold. Firstly, we present a more general model for networks with both delayed and nondelayed couplings and derive criteria for their asymptotical synchronization. Secondly, we apply the Lyapunov-Krasovskii theorem and the LMIs to ensure the inevitable attainment of the required synchronization.

The rest of the paper is organized as follows. In Section 2, we present the general complex dynamical network model under consideration and state some preliminary definitions and results. In Section 3, we present the main results of this paper. In particular, we consider the situation for both time delay-independent and time delay-dependent complex dynamical networks and derive sufficient conditions for their asymptotic synchronization by using the Lyapunov-Krasovskii stability theorem and the linear matrix inequality (LMI). In Section 4, we present some numerical simulation results that verify our theoretical results. The paper concludes in Section 5.

## 2. Preliminaries and Model Description

In general, a linearly coupled ordinary differential equation system (LCODES) can be described as follows:

$$
\begin{equation*}
\frac{d x_{i}(t)}{d t}=f\left(x_{i}(t)\right)+c_{1} \sum_{j \neq i, j=1}^{N} b_{i j} A x_{j}(t)+c_{2} \sum_{j \neq i, j=1}^{N} b_{i j}^{\prime} A^{\prime} x_{j}(t-\tau) \tag{2.1}
\end{equation*}
$$

Since $x_{i}-x_{i}=0$ for all $i=1, \ldots, N$, we can choose any values for $a_{i i}$ in the above equations. Hence, letting $b_{i i}=-\sum_{j \neq i, j=1}^{N} b_{i j}$ and $b_{i i}^{\prime}=-\sum_{j \neq i, j=1}^{N} b_{i j}^{\prime}$, the above equations can be rewritten as follows:

$$
\begin{equation*}
\frac{d x_{i}(t)}{d t}=f\left(x_{i}(t)\right)+c_{1} \sum_{j=1}^{N} b_{i j} A x_{j}(t)+c_{2} \sum_{j=1}^{N} b_{i j}^{\prime} A^{\prime} x_{j}(t-\tau), \tag{2.2}
\end{equation*}
$$

where $N$ is the number of nodes, $x^{i}(t)=\left(x_{i 1}, x_{i 2}, \ldots, x_{i N}\right)^{T} \in R^{n}$ are the state variables of the $i$ th node, $t \in[0,+\infty)$ and $f: R^{n} \rightarrow R^{n}$ is a continuously differentiable function. The constants $c_{1}$ and $c_{2}$ (possibly distinct) are the coupling strengths, $b_{i j} \geq 0, b_{i j}^{\prime}>0($ for $i, j=1, \ldots, N)$, $A, A^{\prime} \in R^{n \times n}$ are inner-coupled matrices, $B, B^{\prime} \in R^{n \times n}$ are coupled matrices with zero-sum rows with $b_{i j}, b_{i j}^{\prime} \geq 0$ for $i \neq j$ that determines the topological structure of the network. We assume that $B$ and $B^{\prime}$ are symmetric and irreducible matrices so that there are no isolated nodes in the system.

If all the eigenvalues of a matrix $A \in R^{n \times n}$ are real, then we denote its $i$ th eigenvalue by $\lambda_{i}(A)$ and sort them by $\lambda_{1}(A) \leq \lambda_{2}(A) \leq \cdots \leq \lambda_{n}(A)$. A symmetric real matrix $A$ is positive definite (semidefinite) if $x^{T} A x>0(\geq 0)$ for all nonzero $x$ and denoted by $A>0(A \geq 0)$. Finally, $I$ stands for the identity matrix and the dimensions of all vectors and matrices should be clear in the context.

Definition 2.1. A complex network with delayed and nondelayed coupling (2.2) is said to achieve asymptotic synchronization if

$$
\begin{equation*}
x_{1}(t)=x_{2}(t)=\cdots=x_{N}(t)=s(t), \quad t \longrightarrow+\infty \tag{2.3}
\end{equation*}
$$

where $s(t)$ is a solution of the local dynamics of an isolated node satisfying $s \dot{(t)}=f(s(t))$.
Definition 2.2. A matrix $L=\left(l_{i j}\right)_{i, j=1}^{N}$ is said to belong to the class $A 1$, denoted by $L \in A 1$ if
(1) $l_{i j} \leq 0, i \neq j, l_{i i}=-\sum_{j=1, j \neq i}^{N} l_{i j}, i=1,2, \ldots, N$,
(2) $L$ is irreducible.

If $L \in A 1$ is symmetrical, then we say that $L$ belongs to the class $A 2$, denoted by $L \in A 2$.
Lemma 2.3 (see [26]). If $L \in A 1$, then $\operatorname{rank}(L)=N-1$, that is, 0 is an eigenvalue of $L$ with multiplicity 1 , and all the nonzero eigenvalues of $L$ have positive real part.

Lemma 2.4 (Wang and Chen [11]). If $G=\left(g_{i j}\right)_{N \times N}$ satisfies the above conditions, then there exists a unitary matrix $\Phi=\left(\phi_{1}, \ldots, \phi_{N}\right)$ such that

$$
\begin{equation*}
G^{T} \phi_{k}=\lambda_{k} \phi_{k}, \quad k=1,2, \ldots, N \tag{2.4}
\end{equation*}
$$

where $\lambda_{i}, i=1,2, \ldots, N$, are the eigenvalues of $G$.
Lemma 2.5 (Schur complement [22]). The linear matrix inequality (LMI)

$$
\left(\begin{array}{cc}
Q(x) & S(x)  \tag{2.5}\\
S(x)^{T} & R(x)
\end{array}\right)>0
$$

where $Q(x)$ and $R(x)$ are symmetric matrices and $S(x)$ is a matrix with suitable dimensions is equivalent to one of the following conditions:
(i) $Q(x)>0, R(x)-S(x)^{T} Q(x)^{-1} S(x)>0$;
(ii) $R(x)>0, Q(x)-S(x) R(x)^{-1} S(x)>0$.

Lemma 2.6 ((the Lyapunov-Krasovskii stability theorem). (Kolmanovskii and Myshkis, Hale and Verduyn Lunel [16])). Consider the delayed differential equation

$$
\begin{equation*}
x(t)=\dot{f}(t, x(t)) \tag{2.6}
\end{equation*}
$$

where $f: R \times C \rightarrow R^{n}$ is continuous and takes $R \times$ (bounded subsets of $C$ ) into bounded subsets of $R^{n}$, and let $u, v, w: R^{+} \rightarrow R^{+}$be continuous and strictly monotonically nondecreasing functions with $u(s), v(s), w(s)$ being positive for $s>0$ and $u(0)=v(0)=0$. If there exists a continuous functional $V: R \times C \rightarrow R$ such that

$$
\begin{gather*}
u(\|x\|) \leq V(t, x) \leq v(\|x\|)  \tag{2.7}\\
\dot{V}(t, x(t, x(t))) \leq-w(\|x(t)\|)
\end{gather*}
$$

where $\dot{V}$ is the derivative of $V$ along the solutions of the above delayed differential equation, then the solution $x=0$ of this equation is uniformly asymptotically stable.

Remark 2.7. The functional $V$ is called a Lyapunov-Krasovskii functional.
Lemma 2.8 (Moon et al. [22]). Let $a(\cdot) \in R^{n_{a}}, b(\cdot) \in R^{n_{b}}$ and $M(\cdot) \in R^{n_{a} \times n_{b}}$ be defined on an interval $\Omega$. Then, for any matrices $X \in R^{n_{a} \times n_{a}}, Y \in R^{n_{a} \times n_{b}}$, and $Z \in R^{n_{b} \times n_{b}}$, one has

$$
-2 \int_{\Omega} a(x)^{T} M b(x) d x \leq \int_{\Omega}\left[\begin{array}{l}
a(x)  \tag{2.8}\\
b(x)
\end{array}\right]^{T}\left[\begin{array}{cc}
X & Y-M \\
Y^{T}-M^{T} & Z
\end{array}\right]\left[\begin{array}{l}
a(x) \\
b(x)
\end{array}\right] d x
$$

where

$$
\left[\begin{array}{cc}
X & Y  \tag{2.9}\\
Y^{T} & Z
\end{array}\right] \geq 0
$$

Lemma 2.9. For all positive-definite matrices $P$ and vectors $x$ and $y$, one has

$$
\begin{equation*}
-2 x^{T} y \leq \inf _{P>0}\left\{x^{T} P x+y^{T} P^{-1} y\right\} \tag{2.10}
\end{equation*}
$$

Lemma 2.10 (see [16]). Consider the delayed dynamical network (2.2). Let

$$
\begin{align*}
& 0=\lambda_{1}>\lambda_{2} \geq \lambda_{3} \geq \cdots \geq \lambda_{N}  \tag{2.11}\\
& 0=\mu_{1}>\mu_{2} \geq \mu_{3} \geq \cdots \geq \mu_{N}
\end{align*}
$$

be the eigenvalues of the outer-coupling matrices $B$ and $B^{\prime \prime}$ respectively. If the n-dimensional linear time-delayed and nontime delayed system

$$
\begin{equation*}
\dot{w}_{i}(t)=J(t) w_{i}(t)+c_{1} \lambda_{i} A w_{i}(t)+c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau), \quad k=2,3, \ldots, N \tag{2.12}
\end{equation*}
$$

of $N-1$ differential equations is asymptotically stable about their zero solutions for some Jacobian matrix $J(t) \in R^{n \times n}$ of $f(x(t))$ at $s(t)$, then the synchronized states (2.3) are asymptotically stable.

## 3. The Criteria for Asymptotic Synchronization

In this section, we derive the conditions for the asymptotic synchronization of time-delayed coupled dynamical networks when they are either time-dependent or time-independent.

### 3.1. Case 1: The Time Delay-Independent Stability Criterion

Theorem 3.1. Consider the general time delayed and non-time delayed complex dynamical network (2.2). If there exist two positive definite matrices $P$ and $Q>0$ such that

$$
\left[\begin{array}{cc}
J(t)^{T} P+P J(t)+2 c_{1} \lambda_{i} A+Q & c_{2} \mu_{N} P A^{\prime}  \tag{3.1}\\
c_{2} \mu_{N} A^{\prime} P & -Q
\end{array}\right]>0,
$$

then the synchronization manifold (2.3) of network (2.2) can be asymptotically synchronized for all fixed time delay $\tau>0$.

Proof. For each fixed $i=1,2, \ldots, N$, choose the Lyapunov-Krasovskii functional

$$
\begin{equation*}
V_{i}(t)=w_{i}(t)^{T} P w_{i}(t)+\int_{t-\tau}^{t} w_{i}(s)^{T} Q w_{i}(s) d s \tag{3.2}
\end{equation*}
$$

for some matrices $P>0$ and $Q>0$ to be determined. Then the derivative of $V_{i}(t)$ along the trajectories of (3.2) is

$$
\begin{equation*}
\frac{d V_{i}(t)}{d t}=\dot{w}_{i}(t)^{T} P w_{i}(t)+w_{i}(t)^{T} P \dot{w}_{i}(t)+w_{i}(t)^{T} Q w_{i}(t)-w_{i}(t-\tau)^{T} Q w_{i}(t-\tau) \tag{3.3}
\end{equation*}
$$

which, upon substitution of (2.12), gives

$$
\begin{align*}
\dot{V}_{i}(t)= & {\left[J(t) w_{i}(t)+c_{1} \lambda_{i} A w_{i}(t)+c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau)\right]^{T} P w_{i}(t)+w_{i}(t)^{T} P } \\
& \times\left[J(t) w_{i}(t)+c_{1} \lambda_{i} A w_{i}(t)+c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau)\right]+w_{i}(t)^{T} Q w_{i}(t)-w_{i}(t-\tau)^{T} Q w_{i}(t-\tau) \\
= & w_{i}(t)^{T} J(t)^{T} w_{i}(t)+w_{i}(t)^{T} \lambda_{i} A^{T} P w_{i}(t)+w_{i}(t-\tau)^{T} \mu_{i} A^{\prime} P w_{i}(t) \\
& +w_{i}(t)^{T} P J(t) w_{i}(t)+w_{i}(t)^{T} c_{1} \lambda_{i} A w_{i}(t)+w_{i}(t)^{T} P c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau) \\
& +w_{i}(t)^{T} Q w_{i}(t)-w_{i}(t-\tau)^{T} Q w_{i}(t-\tau) \\
= & w_{i}(t)^{T}\left[J(t)^{T} P+P J(t)+2 c_{1} \lambda_{i} A+Q\right] w_{i}(t)+2 w_{i}(t)^{T} c_{2} \mu_{i} P A^{\prime} w_{i}(t-\tau) \\
& -w_{i}(t-\tau)^{T} Q w_{i}(t-\tau) . \tag{3.4}
\end{align*}
$$

Now, by using the inequality in Lemma 2.9, we have

$$
\begin{equation*}
2 w_{i}(t)^{T} c_{2} \mu_{i} P A^{\prime} w_{i}(t-\tau) \leq w_{i}(t-\tau)^{T} Q w_{i}(t-\tau)+w_{i}(t)^{T} c_{2}^{2} \mu_{i}^{2} P A^{\prime} Q A^{\prime} P w_{i}(t) \tag{3.5}
\end{equation*}
$$

which, upon substituting (3.5) into (3.2), gives

$$
\begin{equation*}
\dot{V}_{i}(t) \leq w_{i}(t)^{T}\left[J(t)^{T} P+P J(t)+2 c_{1} \lambda_{i} A+c_{2} \mu_{i}^{2} w_{i}(t)^{T} P A^{\prime} Q A^{\prime} A w(t)+Q\right] w_{i}(t) \tag{3.6}
\end{equation*}
$$

It therefore follows from the Schur complement (Lemma 2.5) and the linear matrix inequality (3.1) that $\dot{V}_{i}(t)<0$ for all the $N-1$ equations in the general time delayed and non-time delayed system (2.12) and hence the system (2.12) is asymptotically synchronized by the LyapunovKrasovskii stability theorem. So, by Theorem 3.1, the synchronization manifold (2.3) of the network (2.2) is asymptotically synchronized. This completes the proof of the theorem.

The following corollaries follow immediately from the above theorem.
Corollary 3.2. Consider the general non-time delayed complex dynamical network

$$
\begin{equation*}
\dot{x}_{i}(t)=f\left(x_{i}(t)\right)+c_{1} \sum_{j=1}^{N} b_{i j} A x_{j}(t) . \tag{3.7}
\end{equation*}
$$

If there exists a positive definite matrix $P>0$ such that

$$
\begin{equation*}
J(t)^{T} P+c_{1} \lambda_{i} A P<0 \tag{3.8}
\end{equation*}
$$

then the synchronization manifold (2.3) of network (3.7) can be asymptotically synchronized.
Proof. From Lemma 2.10, we have

$$
\begin{equation*}
\dot{w}_{i}(t)=J(t) w_{i}(t)+c_{1} \lambda_{i} A w_{i}(t) \tag{3.9}
\end{equation*}
$$

and the result follows by choosing the Lyapunov functional $V_{i}(t)=(1 / 2) w_{i}(t)^{T} P w_{i}(t)$.
Corollary 3.3. Consider the general time delayed complex dynamical network

$$
\begin{equation*}
\dot{x}_{i}(t)=f\left(x_{i}(t)\right)+c_{2} \sum_{j=1}^{N} b_{i j}^{\prime} A^{\prime} x_{j}(t-\tau) . \tag{3.10}
\end{equation*}
$$

If there exist two positive definite matrices $P>0$ and $Q>0$ such that

$$
\left[\begin{array}{cc}
J(t)^{T} P+P J(t)+Q & c_{2} \mu_{N} P A^{\prime}  \tag{3.11}\\
c_{2} \mu_{N} A^{\prime} P & -Q
\end{array}\right]<0
$$

then the synchronization manifold (2.3) of network (3.10) can be asymptotically synchronized.
Remark 3.4. The results of [16] are obtainable as particular cases of Theorem 3.1.
Remark 3.5. The above analysis is applicable to a general system with arbitrary time delays. A simpler synchronization scheme, however, could be applied to systems with time delays that are already known and are small in value.

### 3.2. Case 2: The Criterion for Time Delay-Dependent Stability

Theorem 3.6. Consider the general time delayed and non-time delayed complex dynamical network (2.2) with a fixed time delay $\tau \in(0, h]$ for some small $h$. If there exist three positive definite matrices $P, Q, Z>0$, such that

$$
\left[\begin{array}{cc}
(1,1) & P c_{2} \mu_{i} A^{\prime}-Y+h\left(J(t)+c_{1} \lambda_{i} A\right)^{T} Z c_{2} \mu_{i} A^{\prime}  \tag{3.12}\\
c_{2} \mu_{i} A^{\prime T} P-Y^{T}+h c_{2} \mu_{i} A^{\prime T} Z\left(J(t)+c_{1} \lambda_{i} A\right) & h c_{2} \mu_{i}^{2} A^{\prime T} Z A^{\prime}-Q
\end{array}\right]<0
$$

with

$$
\begin{gather*}
(1,1)=\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)^{T} P+P\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)+h X \\
+Y^{T}+Y+Q+h\left(J(t)+c_{1} \lambda_{i} A\right)^{T}\left(J(t)+c_{1} \lambda_{i} A\right)  \tag{3.13}\\
{\left[\begin{array}{cc}
X & Y \\
Y^{T} & Z
\end{array}\right] \geq 0,}
\end{gather*}
$$

then the synchronization manifold (2.3) of network (2.2) can be asymptotically synchronized.
Proof. For each fixed $i=1,2, \ldots, N$, choose the Lyapunov-Krasovskii functional

$$
\begin{equation*}
V_{i}(t)=w_{i}(t)^{T} P w_{i}(t)+\int_{t-\tau}^{t} w_{i}(s)^{T} Q w_{i}(s)+\int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) d s d \beta \tag{3.14}
\end{equation*}
$$

for some matrices $P, Q, Z>0$ to be determined and let

$$
\begin{gather*}
V_{1}=\dot{w}_{i}(t)^{T} P w_{i}(t), \quad V_{2}=\int_{t-\tau}^{t} w_{i}(s)^{T} Q w_{i}(s), \\
V_{3}=\int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) d s d \beta . \tag{3.15}
\end{gather*}
$$

Then, $V_{i}(t)=V_{1}+V_{2}+V_{3}$ and it follows from the Newton-Leibniz equation that

$$
\begin{equation*}
\int_{t-\tau}^{t} \dot{w}_{i}(\xi) d \xi=w_{i}(t)-w_{i}(t-\tau) \tag{3.16}
\end{equation*}
$$

so that (2.12) can be transformed into

$$
\begin{equation*}
\dot{w}_{i}(t)=\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right) w_{i}(t)-c_{2} \mu_{i} A^{\prime} \int_{t-\tau}^{t} \dot{w}_{i}(s) d s \tag{3.17}
\end{equation*}
$$

Hence

$$
\begin{align*}
V_{1}= & \dot{w}_{i}(t)^{T} P w_{i}(t)+w_{i}(t)^{T} P \dot{w}_{i}(t) \\
= & w_{i}(t)^{T}\left[\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)^{T} P+P\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)\right] w_{i}(t)  \tag{3.18}\\
& -2 w_{i}(t)^{T} P c_{2} \mu_{i} A^{\prime} \int_{t-\tau}^{t} \dot{w}_{i}(s) d s
\end{align*}
$$

and so, by Lemma 2.9, we have

$$
\begin{align*}
&-2 w_{i}(t)^{T} P c_{2} \mu_{i} A^{\prime} \int_{t-\tau}^{t} \dot{w}_{i}(s) d s \\
&=-2 \int_{t-\tau}^{t} w_{i}(t)^{T}\left(P c_{2} \mu_{i} A^{\prime}\right) \dot{w}_{i}(s) d s \\
& \leq \int_{t-\tau}^{t}\left[\begin{array}{c}
w_{i}(t) \\
\dot{w}_{i}(s)
\end{array}\right]^{T}\left[\begin{array}{cc}
X & Y-P c_{2} \mu_{i} A^{\prime} \\
Y^{T}-A^{\prime T} c_{2} \mu_{i} P & Z
\end{array}\right]\left[\begin{array}{l}
w_{i}(t) \\
\dot{w}_{i}(s)
\end{array}\right] d x \\
&= \int_{t-\tau}^{t} w_{i}(t)^{T} X w_{i}(t) d s+\int_{t-\tau}^{t} \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) d s+2 \int_{t-\tau}^{t} w_{i}(t)^{T}\left(Y-P c_{2} \mu_{i} A^{\prime}\right) \dot{w}_{i}(s) d s \\
&= \tau w_{i}(t)^{T} X w_{i}(t)+2 w_{i}(t)^{T}\left(Y-P c_{2} \mu_{i} A^{\prime}\right) \int_{t-\tau}^{t} \dot{w}_{i}(s) d s+\int_{t-\tau}^{t} \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) \\
&= \tau w_{i}(t)^{T} X w_{i}(t)+2 w_{i}(t)^{T}\left(Y-P c_{2} \mu_{i} A^{\prime}\right) w_{i}(t)-2 w_{i}(t)^{T}\left(Y-P c_{2} \mu_{i} A^{\prime}\right) w_{i}(t-\tau) \\
&+\int_{t-\tau}^{t} \dot{w}_{i}(s) Z \dot{w}_{i}(s) \tag{3.19}
\end{align*}
$$

and so

$$
\begin{align*}
V_{1} \leq & w_{i}(t)^{T}\left[\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)^{T} P+P\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)\right] w_{i}(t) \\
& +2 w_{i}(t)^{T}\left(P c_{2} \mu_{i} A^{\prime}-Y\right) w_{i}(t-\tau)+\int_{t-\tau}^{t} \dot{w}_{i}(s)^{T} Z w_{i}(s) d s \tag{3.20}
\end{align*}
$$

Similarly, we have

$$
\begin{align*}
V_{2}= & w_{i}(t)^{T} Q w_{i}(t)-w_{i}(t-\tau)^{T} Q w_{i}(t-\tau), \\
V_{3}= & \tau \dot{w}_{i}(t)^{T} Z \dot{w}_{i}(t)-\int_{t-\tau}^{t} \tau \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) d s \leq h\left[\left(J(t)+c_{1} \lambda_{i} A\right) w_{i}(t)+c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau)\right]^{T} \\
& \times Z\left[\left(J(t)+c_{1} \lambda_{i} A\right) w_{i}(t)+c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau)\right]-\int_{t-\tau}^{t} \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) d s \\
= & h w_{i}(t)^{T}\left(J(t)+c_{1} \lambda_{i} A\right)^{T} Z\left(J(t)+c_{1} \lambda_{i} A\right) w_{i}(t)+h w_{i}(t)^{T} \\
& \times\left(J(t) t+c_{1} \lambda_{i} A\right)^{T} Z c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau)+h w_{i}(t-\tau)^{T} c_{2} \mu_{i} A^{\prime T} Z^{T} \\
& \times\left(J(t)+c_{1} \lambda_{i} A\right) w_{i}(t)+h w_{i}(t-\tau)^{T} \mu_{i}^{2} A^{\prime T} Z A^{\prime} w_{i}(t-\tau)-\int_{t-\tau}^{t} \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) d s \\
= & h w_{i}(t)^{T}\left(J(t)+c_{1} \lambda_{i} A\right)^{T} Z\left(J(t)+c_{1} \lambda_{i} A\right) w_{i}(t)+2 h w_{i}(t)^{T} \\
& \times\left(J(t)+c_{1} \lambda_{i} A\right)^{T} Z c_{2} \mu_{i} A^{\prime} w_{i}(t-\tau)+h w_{i}(t-\tau)^{T} \mu_{i}^{2} A^{T} Z A^{\prime} w_{i}(t-\tau) \\
& -\int_{t-\tau}^{t} \dot{w}_{i}(s)^{T} Z \dot{w}_{i}(s) d s \tag{3.21}
\end{align*}
$$

and so

$$
\begin{align*}
\dot{V}_{i}(t)= & V_{1}+V_{2}+ \\
\leq & V_{3} \\
\leq & w_{i}(t)^{T}\left[\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)^{T} P+P\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)\right.  \tag{3.22}\\
& \left.+h X+Y^{T}+Y+Q+h\left(J(t)+c_{1} \lambda_{i} A\right)^{T}\left(J(t)+c_{1} \lambda_{i} A\right)\right] w_{i}(t) \\
& +w_{i}(t-\tau)^{T}\left[h \mu_{i}^{2} A^{\prime T} Z A^{\prime}-Q\right] w_{i}(t-\tau) \\
& +2 w_{i}(t)^{T}\left[\left(P c_{2} \mu_{i} A^{\prime}-Y\right)+h\left(J(t)+c_{1} \lambda_{i}\right)^{T} Z c_{2} \mu_{i} A^{\prime}\right] w_{i}(t-\tau)
\end{align*}
$$

Finally, we have

$$
\dot{V}_{i}(t) \leq\left[\begin{array}{c}
w_{i}(t)  \tag{3.23}\\
w_{i}(t-\tau)
\end{array}\right]^{T}\left[\begin{array}{cc}
(1,1) & (1,2) \\
(2,1) & h c_{2} \mu^{2} A^{\prime T} Z A_{Q}^{\prime}
\end{array}\right]\left[\begin{array}{c}
w_{i}(t) \\
w_{i}(t-\tau)
\end{array}\right]
$$

where

$$
\begin{align*}
&(1,1)=\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)^{T} P+P\left(J(t)+c_{1} \lambda_{i} A+c_{2} \mu_{i} A^{\prime}\right)+h X \\
&+Y^{T}+Y+Q+h\left(J(t)+c_{1} \lambda_{i} A\right)^{T}\left(J(t)+c_{1} \lambda_{i} A\right) \\
&(1,2)= P c_{2} \mu_{i} A^{\prime}-Y+h\left(J(t)+c_{1} \lambda_{i} A\right)^{T} Z c_{2} \mu_{i} A^{\prime}  \tag{3.24}\\
&(2,1)= c_{2} \mu_{i} A^{\prime T} P-Y^{T}+h c_{2} \mu_{i} A^{\prime T} Z\left(J(t)+c_{1} \lambda_{i} A\right) \\
& {\left[\begin{array}{cc}
X & Y \\
Y^{T} & Z
\end{array}\right] \geq 0 }
\end{align*}
$$

It now follows from Lemma 2.5 that the conditions of the theorem are equivalent to $\dot{V}_{i}(t)<0$ and that by the Lyapunov-Krasovskii Stability Theorem all the nodes of the system (2.12) are asymptotically stable when (3.12) and (3.13) hold for $i=1,2, \ldots, N$. This completes the proof of Theorem 3.6.

Corollary 3.7. Consider the general time delayed complex dynamical network (3.10) with a fixed time delay $\tau \in(0, h]$

$$
\begin{equation*}
\dot{x}_{i}(t)=f\left(x_{i}(t)\right)+c_{2} \sum_{j=1}^{N} b_{i j}^{\prime} A^{\prime} x_{j}(t-\tau) \tag{3.25}
\end{equation*}
$$

for some $h<+\infty$. If there exist two positive definite matrices, $P, Q>0, X, Y$, and $Z$ such that

$$
\left[\begin{array}{cc}
(1,1) & c_{2} \mu_{i} P A^{\prime}-Y+h J(t)^{T} Z c_{2} \mu_{i} A^{\prime}  \tag{3.26}\\
c_{2} \mu_{A}^{\prime T} P-Y^{T}+h c_{2} \mu_{i} A^{\prime T} Z J(t) & h c_{2}{ }^{2} \mu_{i}{ }^{2} A^{\prime T} Z A^{\prime}-Q
\end{array}\right]<0
$$

where $(1,1)=P J(t)+J(t)^{T}+h X+Y^{T}+Y+Q+h J\left(t^{T} Z J(t)\right)$, then the synchronization manifold (2.3) of network (3.10) is asymptotic synchronization.

Remark 3.8. The proof can be found in [16]. Those are the two results of general complex dynamical network with fixed time-invariant delay $\tau \in(0, h]$ for some $h<+\infty$; the conclusions are less conservative than the time-independent delay. The delay-dependent stability is another method applying to the delayed system. And it could provide a useful and meaningful upper bound of the delay $h$, which could ensure the delayed system achieves asymptotic synchronization only if the time delay is less than $h$.

## 4. Numerical Simulations

The above criteracould be applied to networks with different topologies and different size. We put two examples to illustrate the validity of the theories.

Example 4.1. We use a three-dimensional stable nonlinear system as an example to illustrate the main results, Theorem 3.1, of our paper; this is the time delay-independent situation. The model could be described as follows:

$$
\left[\begin{array}{c}
\dot{x}_{i 1}  \tag{4.1}\\
\dot{x}_{i 2} \\
\dot{x}_{i 3}
\end{array}\right]=\left[\begin{array}{c}
-x_{i 1}+x_{i 2}^{2} \\
-2 x_{i 2} \\
-3 x_{i 3}+x_{i 2} x_{i 3}
\end{array}\right], \quad i=1,2,3 .
$$

The solution of the 3-dimensional stable nonlinear system equations can be written as

$$
\begin{equation*}
x_{i 1}=c_{1} e^{-t}-c e^{-4 t}, \quad x_{i 2}=c_{2} e^{-2 t}, \quad x_{i 3}=\frac{c_{3} e^{-3 t}-c_{2} e^{2 t}}{2} \tag{4.2}
\end{equation*}
$$

which is asymptotically stable at the equilibrium point of the system $s(t)=0$, where $c=-c_{2}^{2} / 3$ and $c_{1}, c_{2}, c_{3}$ are all constants. It is easy to see that the Jacobian matrix is $J=\operatorname{diag}\{-1,-2,-3\}$. We assume the inner-coupling matrices $A, A^{\prime}$ are all identity matrices, namely, $A=A^{\prime}=$ $\operatorname{diag}\{1,1,1\}$, and the outer coupling configuration matrices

$$
B=B^{\prime}=\left[\begin{array}{cccc}
-2 & 1 & 0 & 1  \tag{4.3}\\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 \\
1 & 0 & 1 & 2
\end{array}\right]
$$

The eigenvalues of the coupling matrices are $\lambda(B)=\lambda\left(B^{\prime}\right)=\{0,-1.5,-1.5\}$. We choose the coupling strength $c_{1}=0.5, c_{2}=1$. By using Theorem 3.1 and the LMI Toolbox in MATLAB, we obtained the following common two positive-definite matrices:

$$
\begin{equation*}
P=\operatorname{diag}\{1.1204,12.3091,10.165\}, \quad Q=\operatorname{diag}\{2.3710,22.5713,26.0849\} \tag{4.4}
\end{equation*}
$$

According to the conditions in Theorem 3.1, we know the synchronized state $s(t)$ is global asymptotically stable for any fixed delay. The quantity

$$
\begin{equation*}
E(t)=\sqrt{\left(\frac{\sum_{i=1}^{N}\left|x_{i}(t)-s(t)\right|^{2}}{N}\right)} \tag{4.5}
\end{equation*}
$$

is used to measure the quality of the synchronization process. We plot the evolution of $E(t)$ in the upper part in Figure 1. For the time delay here we choose $\tau=0.1$. The lower subplot indicates the synchronization results of the network.

Example 4.2. We use a 4-nodes networks model as another example to illustrate the Theorem 3.6; this is the time delay-dependent situation. The model could be described as follows:

$$
\left[\begin{array}{c}
\dot{x}_{i 1}  \tag{4.6}\\
\dot{x}_{i 2} \\
\dot{x}_{i 3}
\end{array}\right]=\left[\begin{array}{c}
-x_{i 1} \\
-2 x_{i 2}+x_{i 3}^{2} \\
-3 x_{i 3}+x_{i 2} x_{i 3}
\end{array}\right], \quad i=1,2,3,4 .
$$



Figure 1: Synchronization evolution $E(t)$ for the delay-independent network with $c_{1}=0.5, c_{2}=1$, and $\tau=0.1$.

We choose the same coupling strength $c_{1}=0.5, c_{2}=1$; the eigenvalues of the coupling matrices are $\lambda(B)=\lambda\left(B^{\prime}\right)=\{0,-2,-2,-4\}$. By using Theorem 3.6 and the LMI Toolbox in MATLAB, we obtained the following matrices:

$$
\begin{array}{ll}
P=\operatorname{diag}\{1.4078,1.4057,1.4054\}, & Q=\{1.4157,1.4157,1.4157\}, \\
Z=\{-1.4218,-1.4303,-1.4367\}, & X=\{18.3024,19.6808,21.0815\},  \tag{4.7}\\
Y=\{0.0221,0.0208,0.0196\} .
\end{array}
$$

By using Theorem 3.6 in this paper, it is found that the maximum delay bound for the complex dynamical network to form asymptotic synchronization is $h=1$. $E(t)$ are defined the same as in the example. We plot the evolution of $E(t)$ in upper part in Figure 2. The lower subplot indicates the synchronization results of the network. It can be seen from the figures that the network in this example can achieve asymptotic synchronization.

## 5. Conclusion

This paper considered a class of complex networks with both time delayed and non-time delayed coupling. We derived, respectively, a sufficient criterion for time delay-dependent and time delay-independent asymptotic synchronization which are more general than those obtained in previous works. These asymptotic synchronization results were obtained by using the Lyapunov-Krasovskii stability theorem and the linear matrix inequality. Two simple examples were also used to validate the theoretical analysis.


Figure 2: Synchronization evolution $E(t)$ for the delay-dependent network with $c_{1}=0.5, c_{2}=1$, and $\tau=0.1$.

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Research Article

# Existence and Linear Stability of Equilibrium Points in the Robe's Restricted Three-Body Problem with Oblateness 

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#### Abstract

This paper investigates the positions and linear stability of an infinitesimal body around the equilibrium points in the framework of the Robe's circular restricted three-body problem, with assumptions that the hydrostatic equilibrium figure of the first primary is an oblate spheroid and the second primary is an oblate body as well. It is found that equilibrium point exists near the centre of the first primary. Further, there can be one more equilibrium point on the line joining the centers of both primaries. Points on the circle within the first primary are also equilibrium points under certain conditions and the existence of two out-of-plane points is also observed. The linear stability of this configuration is examined and it is found that points near the center of the first primary are conditionally stable, while the circular and out of plane equilibrium points are unstable.


## 1. Introduction

Robe [1] considered a new kind of restricted three-body problem in which, one of the primaries of mass $m_{1}$ is a rigid spherical shell, filled with homogenous, incompressible fluid of density $\rho_{1}$; the second one is a point mass $m_{2}$ located outside the shell and moving around the mass $m_{1}$ in a Keplerian orbit; the infinitesimal mass $m_{3}$ is a small sphere of density $\rho_{3}$, moving inside the shell and is subject to the attraction of $m_{2}$ and the buoyancy force due to the fluid of the first primary. Further, he discussed the linear stability of an equilibrium point obtained in two cases. In the first case, the orbit of $m_{2}$ around $m_{1}$ is circular and in the second case, the orbit is elliptic, but the shell is empty (there is no fluid inside it) or densities of $m_{1}$ and $m_{3}$ are equal. Since then various studies (e.g., [2-4]) under different assumptions have been carried out.


Figure 1: The Robe's CRTBP with oblate primaries.

In his study, Robe [1] assumed that the pressure field of the fluid $\rho_{1}$ has a spherical symmetry around the center of the shell and he took into account only one out of the three components of the pressure field which is due to the own gravitational field of the fluid $\rho_{1}$. He did not consider the other two components arising from the attraction of $m_{2}$ and the centrifugal force. Taking care of all these three components of the pressure field, A. R. Plastino and A. Plastino [5] reanalyzed the Robe's. But in their study, they assumed the hydrostatic equilibrium figure of the first primary as Roche's ellipsoid (see Figure 1). They found that when the density parameter $D$ is taken as zero, every point inside the fluid is an equilibrium point; otherwise the center of the ellipsoid is the only equilibrium point and it is linearly stable.

Hallan and Rana [3] investigated the existence of all equilibrium point and their stability in the Robe's [1] restricted three-body problem. It was seen that the Robe's elliptic restricted three-body problem has only one equilibrium point for all values of the density parameter $K$ and the mass parameter $\mu$, while the Robe's circular restricted three-body problem can have two, three, or infinite numbers of equilibrium points. As regards to the stability of these equilibria, they confirmed the stability result given by Robe [1] of the equilibrium point $(-\mu, 0,0)$, whereas triangular and circular points are always unstable. The equilibrium point collinear with the center of the shell and the second primary was found to be stable under some conditions.

Hallan and Mangang [4] studied the Robe's [1] restricted three-body problem by considering the full buoyancy force as in A. R. Plastino and A. Plastino [5] and assuming the hydrostatic equilibrium figure of the first primary as an oblate spheroid. They derived the pertinent equations of motion and discussed the existence of equilibrium point and their linear stability.

The participating bodies in the classical restricted three-body problem are strictly spherical in shape, but in actual situations several heavenly bodies, such as Saturn and Jupiter, are sufficiently oblate. The minor planets and meteoroids have irregular shape. The lack of sphericity, or the oblateness, of the planet causes large perturbations from a two-body orbit. The motions of artificial Earth satellites are examples of this. Global studies of problems with oblateness have been carried out by many researchers (e.g., [6-9]).

Therefore, our effort in this paper aims at investigating the equilibrium points and their stability in the Robe's circular restricted three-body problem when the hydrostatic equilibrium figure of the fluid of the first primary is an oblate spheroid and the second one is an oblate spheroid as well. The model of this study can be used to study the small oscillation of the Earth's inner core taking into account the Moon's attraction.

This paper is organized as follows; Section 2 represents the equations of motion; the existence of the equilibrium points is mentioned in Section 3, while Section 4 investigates their linear stability; Section 5 discusses the results obtained; the conclusion is drawn in Section 6.

## 2. Equation of Motion

Let the first primary $m_{1}$ be a fluid of density $\rho_{1}$ in the shape of an oblate spheroid as assumed by Hallan and Mangang [4]; let the second primary $m_{2}$ be an oblate body too as Sharma and Subba Rao [6] assumed, which describes a circular orbit around $m_{1}$.

We adopt a uniformly rotating coordinate system $O x_{1} x_{2} x_{3}$ with origin at the center of mass $m_{1}, O x_{1}$ pointing towards $m_{2}$, with $O x_{1} x_{2}$ being the orbital plane of $m_{2}$ coinciding with the equatorial plane of $m_{1}$. Then, the equations of motion of the infinitesimal body of density $\rho_{3}$ in the coordinate system take the form [4, 6]:

$$
\begin{equation*}
\ddot{x}_{1}-2 n \dot{x}_{2}=\frac{\partial U}{\partial x_{1}}, \quad \ddot{x}_{2}+2 n \dot{x}_{1}=\frac{\partial U}{\partial x_{2}}, \quad \ddot{x}_{3}=\frac{\partial U}{\partial x_{3}}, \tag{2.1}
\end{equation*}
$$

where

$$
\begin{gather*}
U=V+\frac{n^{2}\left\{\left(x_{1}-\left(m_{2} /\left(m_{1}+m_{2}\right)\right) R\right)^{2}+x_{2}^{2}\right\}}{2}, \\
V=B+B^{\prime}-\frac{\rho_{1}}{\rho_{3}}\left[B+B^{\prime}+\frac{n^{2}\left\{\left(x_{1}-\left(m_{2} /\left(m_{1}+m_{2}\right)\right) R\right)^{2}+x_{2}^{2}\right\}}{2}\right], \\
B=\pi G \rho_{1}\left[I-A_{1} x_{1}^{2}-A_{1} x_{2}^{2}-A_{2} x_{3}^{2}\right], \\
B^{\prime}=\frac{G m_{2} \alpha_{2}}{\left[\left(R-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{1 / 2}}+\frac{}{2\left[\left(R-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{3 / 2}}-\frac{3 G m_{2} \alpha_{2} x_{3}^{2}}{2\left[\left(R-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{5 / 2}},  \tag{2.2}\\
A_{1}=a_{1}^{2} a_{2} \int_{0}^{\infty} \frac{d u}{\Delta\left(a_{1}^{2}+u\right)^{\prime}}, A_{2}=a_{1}^{2} a_{2} \int_{0}^{\infty} \frac{d u}{\Delta\left(a_{2}^{2}+u\right)}, \\
\Delta^{2}=\left(a_{1}^{2}+u\right)^{2}\left(a_{2}^{2}+u\right), \\
n^{2}=\frac{G\left(m_{1}+m_{2}\right)}{R^{2}}\left(1+\frac{3}{2} \alpha_{1}+\frac{3}{2} \alpha_{2}\right) ; \quad \alpha_{1}=\frac{a_{1}^{2}-a_{3}^{2}}{5 R^{2}}, \alpha_{2}=\frac{a_{2}^{2}-a_{4}^{2}}{5 R^{2}}, \\
D=1-\frac{\rho_{1}}{\rho_{3}} .
\end{gather*}
$$

Here $V$ is the potential that explains the combined forces upon the infinitesimal mass, $B$ denotes the potential due to the fluid mass of the first primary, $B^{\prime}$ stands for the potential due to the second primary, $R$ is the distance between the primaries, and $G$ is the gravitational constant. $n$ is the mean motion. $a_{1}, a_{2}$ and $a_{3}, a_{4}$ are the equatorial and polar radii of the first and second primary, respectively. $I$ stands for the polar moment of inertia, while $A_{i}(i=$ $1,2)$ are the index symbols. $\alpha_{1}$ and $\alpha_{2}$ are the oblateness coefficients of the first and second primaries, respectively.

We choose the unit of mass such that the sum of the masses of the primaries is taken as unity, thus we take $m_{2}=\mu, 0<\mu=m_{2} /\left(m_{1}+m_{2}\right)<1$. For the unit of length, we take the distance between the primaries as unity, that is, $R=1$ and the unit of time is also selected such that $G=1$. With these units and substituting the expression for the potential $B$ due to the fluid in the first primary and the potential $B^{\prime}$ due to the second oblate primary, the equations of motion (2.1) are recast to the form:

$$
\begin{equation*}
\ddot{x}_{1}-2 n \dot{x}_{2}=\frac{\partial U}{\partial x_{1}}, \quad \ddot{x}_{2}+2 n \dot{x}_{1}=\frac{\partial U}{\partial x_{2}}, \quad \ddot{x}_{3}=\frac{\partial U}{\partial x_{3}}, \tag{2.3}
\end{equation*}
$$

where

$$
\begin{align*}
& \begin{array}{l}
U=D\left[\pi \rho_{1}\left\{I-A_{1}\left(x_{1}^{2}+x_{2}^{2}\right)-A_{2} x_{3}^{2}\right\}+\frac{\mu}{\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{1 / 2}}\right. \\
\\
\left.\quad+\frac{\mu \alpha_{2}}{\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{3 / 2}}-\frac{3 \mu \alpha_{2} x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{5 / 2}}+\frac{n^{2}\left\{\left(x_{1}-\mu\right)^{2}+x_{2}^{2}\right\}}{2}\right], \\
n^{2}=\left(1+\frac{3}{2} \alpha_{1}+\frac{3}{2} \alpha_{2}\right) .
\end{array},
\end{align*}
$$

These above equations of motion of the infinitesimal mass $m_{3}$ under the framework of the Robe's circular restricted three-body problem have been obtained by taking into account the shapes of the primaries, the full buoyancy force, the forces due to the gravitational attraction of the second primary, and the gravitational force exerted by the fluid of density $\rho_{1}$. In the case when the second primary is not an oblate spheroid (i.e., $\alpha_{2}=0$ ), the equations are the same as those of Hallan and Mangang [4].

## 3. Position of Equilibrium Points

The equilibrium points are the solutions of the equations:

$$
\begin{equation*}
U_{x_{1}}=U_{x_{2}}=U_{x_{3}}=0 \tag{3.1}
\end{equation*}
$$

That is,

$$
\begin{align*}
& U_{x_{1}}=D\left[-2 \pi \rho_{1} x_{1} A_{1}+\frac{\mu\left(1-x_{1}\right)}{\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{3 / 2}}\right.  \tag{3.2}\\
& \left.+\frac{3 \mu \alpha_{2}\left(1-x_{1}\right)}{2\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{5 / 2}}-\frac{15 \mu \alpha_{2}\left(1-x_{1}\right) x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{7 / 2}}+n^{2}\left(x_{1}-\mu\right)\right]=0, \\
& U_{x_{2}}=D x_{2}\left[-2 \pi \rho_{1} A_{1}-\frac{\mu}{\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{3 / 2}}\right.  \tag{3.3}\\
& \left.-\frac{3 \mu \alpha_{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{5 / 2}}+\frac{15 \mu \alpha_{2} x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{7 / 2}}+n^{2}\right]=0, \\
& U_{x_{3}}=D x_{3}\left[-2 \pi \rho_{1} A_{2}-\frac{\mu}{\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{3 / 2}}-\frac{3 \mu \alpha_{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{5 / 2}}\right. \\
& \left.-\frac{3 \mu \alpha_{2}}{\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{5 / 2}}+\frac{15 \mu \alpha_{2} x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{2}^{2}+x_{3}^{2}\right]^{7 / 2}}\right]=0 . \tag{3.4}
\end{align*}
$$

### 3.1. Equilibrium Points Near the Centre of the First Primary

The positions of the equilibrium points near the first primary are the solutions of (3.2) when $U_{x_{1}}=0, x_{1} \neq 0, x_{2}=x_{3}=0, D \neq 0$, and $n^{2}=1+(3 / 2)\left(\alpha_{1}+\alpha_{2}\right)$. The $x_{1}$ coordinate of the equilibrium points are then the roots of the equation:

$$
\begin{equation*}
-2 \pi \rho_{1} x_{1} A_{1}+\frac{\mu\left(1-x_{1}\right)}{\left|1-x_{1}\right|^{3}}+\frac{3 \mu \alpha_{2}\left(1-x_{1}\right)}{2\left|1-x_{1}\right|^{5}}+\left(1+\frac{3}{2} \alpha_{1}+\frac{3}{2} \alpha_{2}\right)\left(x_{1}-\mu\right)=0 \tag{3.5}
\end{equation*}
$$

We first determine the roots of (3.5) in the absence of oblateness, that is, the case when the primaries are spherical. In this case, the roots are [4]

$$
\begin{equation*}
x_{11}=1+\frac{\mu+\sqrt{\mu^{2}+8 \mu \pi \rho_{1} A_{1}-4 \mu}}{2\left(1-2 \pi \rho_{1} A_{1}\right)}, \quad x_{12}=1+\frac{\mu-\sqrt{\mu^{2}+8 \mu \pi \rho_{1} A_{1}-4 \mu}}{2\left(1-2 \pi \rho_{1} A_{1}\right)} \tag{3.6}
\end{equation*}
$$

The term $A_{1}$ which appears in (3.2) and is due to the fluid mass affects these roots. Therefore, these roots will be real if the discriminant is nonnegative, that is if

$$
\begin{equation*}
\mu+8 \pi \rho_{1} A_{1}-4 \geq 0 \tag{3.7}
\end{equation*}
$$

When $(1 / 4) \mu \geq 1-2 \pi \rho_{1} A_{1}>0$, both roots are greater than unity and we reject them because they lie outside the first primary. Now, if $1-2 \pi \rho_{1} A_{1}<0$, we have $x_{12}>1$ and $x_{11}<1$. Further, we see that $x_{11}>-1$ when $1-2 \pi \rho_{1} A_{1}<-(3 / 4) \mu$. Thus, in the case when $1-2 \pi \rho_{1} A_{1}<-(3 / 4) \mu$, the point $\left(x_{11}, 0,0\right)$ lies within the first primary if $\left|x_{11}\right|<a_{1}$. When $1-2 \pi \rho_{1} A_{1}<-(3 / 4) \mu$, $\left|x_{11}\right|<a_{1} ; x_{11}$ is a root of (3.5). Hence for $1-2 \pi \rho_{1} A_{1}=0$, the only root is $x_{11}=2$ which lies outside the first primary and we neglect it. Hence, for $\alpha_{1}=0, \alpha_{2}=0, x_{11}=0$ is always a root of (3.5) and $x_{1}=x_{11}$ is also a root provided $1-2 \pi \rho_{1} A_{1}<-(3 / 4) \mu,\left|x_{11}\right|<a_{1}$.

Now, we find the roots of (3.5) when oblateness of both primaries is considered (i.e., $\alpha_{1} \neq 0, \alpha_{2} \neq 0$ ).

Let the roots be such that

$$
\begin{align*}
& x_{1}=0+p_{1}, \quad\left|p_{1}\right| \ll 1  \tag{3.8}\\
& x_{1}=x_{11}+p_{2}, \quad\left|p_{2}\right| \ll 1
\end{align*}
$$

Putting these values in (3.5), multiplying throughout by $\left(1-p_{1}\right)^{4}$, expanding and neglecting second and higher powers of $p_{1}, \alpha_{1}, \alpha_{2}$, as they are very small quantities, we have

$$
\begin{equation*}
p_{1} \cong-\frac{3 \alpha_{1}}{2} \frac{\mu}{2 \pi \rho_{1} A_{1}-(1+2 \mu)} \tag{3.9}
\end{equation*}
$$

Similarly, putting $x_{1}=x_{11}+p_{2}$ in (3.5) and then simplifying it, we get

$$
\begin{align*}
& \left(1-x_{11}\right)^{4}\left[\left(x_{11}-\mu\right)\left(1+\frac{3}{2} \alpha_{1}+\frac{3}{2} \alpha_{2}\right)-2 \pi \rho_{1} A_{1} x_{11}\right] \\
& \quad+p_{2}\left(1-x_{11}\right)^{3}\left[\left(1-3 x_{11}\right)\left(1-2 \pi \rho_{1} A_{1}\right)+2 \mu\right]  \tag{3.10}\\
& \quad+\mu\left(1-x_{11}\right)^{2}-2 p_{2}\left(1-x_{11}\right)\left[\left(1-x_{11}\right)^{2}\left\{x_{11}-2 \pi \rho_{1} A_{1} x_{11}-\mu\right\}+\mu\right]=-\frac{3 \mu \alpha_{2}}{2} .
\end{align*}
$$

Multiplying (3.8) by $\left(1-x_{11}\right)^{2}$, simplifying and then using it in (3.10), we get

$$
\begin{equation*}
p_{2} \cong-\frac{\left(1-x_{11}\right)\left[\left(x_{11}-\mu\right)\left((3 / 2) \alpha_{1}+(3 / 2) \alpha_{2}\right)\right]}{\left[\left(1-3 x_{11}\right)\left(1-2 \pi \rho_{1} A_{1}\right)+2 \mu\right]}-\frac{3 \mu \alpha_{2}}{2\left(1-x_{11}\right)^{3}\left[\left(1-3 x_{11}\right)\left(1-2 \pi \rho_{1} A_{1}\right)+2 \mu\right]} \tag{3.11}
\end{equation*}
$$

A substitution of (3.11) in the second equation of (3.8) at once gives the position of the other equilibrium point near the center of the first primary.

### 3.1.1. Positions of Circular Points

The positions of the circular points are sought using the first two equations of system (3.1) with the conditions $x_{1} \neq 0, x_{2} \neq 0, x_{3}=0$; that is, they are the solutions of

$$
\begin{align*}
& x_{1}\left[-2 \pi \rho_{1} A_{1}-\frac{\mu}{\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{3 / 2}}-\frac{3 \mu \alpha_{2}}{2\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{5 / 2}}+1+\frac{3}{2} \alpha_{1}+\frac{3}{2} \alpha_{2}\right]  \tag{3.12}\\
& +\frac{\mu}{\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{3 / 2}}+\frac{3 \mu \alpha_{2}}{2\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{5 / 2}}-\mu\left(1+\frac{3}{2} \alpha_{1}+\frac{3}{2} \alpha_{2}\right)=0 \\
& -2 \pi \rho_{1} A_{1}-\frac{\mu}{\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{3 / 2}}-\frac{3 \mu \alpha_{2}}{2\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{5 / 2}}+1+\frac{3}{2} \alpha_{1}+\frac{3}{2} \alpha_{2}=0 \tag{3.13}
\end{align*}
$$

Solving the above equations and knowing that $\mu \neq 0$, we get

$$
\begin{equation*}
\frac{1}{\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{3 / 2}}+\frac{3 \alpha_{2}}{2\left\{\left(1-x_{1}\right)^{2}+x_{2}^{2}\right\}^{5 / 2}}-n^{2}=0 \tag{3.14}
\end{equation*}
$$

We let

$$
\begin{equation*}
\left(1-x_{1}\right)^{2}+x_{2}^{2}=r^{2} \tag{3.15}
\end{equation*}
$$

Substituting (3.15) in (3.14), and simplifying, we get

$$
\begin{equation*}
n^{2} r^{5}-r^{2}-\frac{3}{2} \alpha_{2}=0 \tag{3.16}
\end{equation*}
$$

Now, we let

$$
\begin{equation*}
r=1+\varepsilon, \quad \varepsilon \ll 1 \tag{3.17}
\end{equation*}
$$

Substituting (3.16) in (3.15), neglecting second and higher powers of $\varepsilon$, we get

$$
\begin{equation*}
\varepsilon=-\frac{1}{2} \alpha_{1} . \tag{3.18}
\end{equation*}
$$

Therefore, (3.17) is now expressed as

$$
\begin{equation*}
r \cong 1-\frac{1}{2} \alpha_{1} \tag{3.19}
\end{equation*}
$$

A substitution of (3.14) in (3.13) yields

$$
\begin{equation*}
2 \pi \rho_{1} A_{1}=n^{2}(1-\mu) \tag{3.20}
\end{equation*}
$$

Therefore, when $2 \pi \rho_{1} A_{1}=n^{2}(1-\mu)$, the points on the circle given by (3.15) with $x_{3}=0$ and $r=1-(1 / 2) \alpha_{1}$ lying within the first primary are also equilibrium points. The general coordinates of these circular points are given by $(1+r \cos \theta, r \sin \theta, 0)$, where $\theta$ is a parameter. When $y=0$, the circular points coalesce to those lying on the line joining the primaries.

### 3.1.2. Positions of Out-of-Plane Equilibrium Points

The out-of-plane points have no analogy in the classical restricted three-body problem. However the investigation concerning these points in the photogravitational restricted threebody problem was first carried out by Radzievskii [10]. Afterwards, other researchers, for instance Douskos and Markellos [8], Singh and Leke [11], and so forth, have worked on the out-of-plane points. In this section, we locate these points for our study, as it has remained an open problem to date.

The positions of the out-of-plane equilibrium points of the Robe's problem with oblate primaries are the solutions of the first and last equations of (3.1) with $x_{2}=0, D \neq 0$; that is,

$$
\begin{gather*}
x_{1}\left[-2 \pi \rho_{1} A_{1}-\frac{\mu}{\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{3 / 2}}-\frac{3 \mu \alpha_{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{5 / 2}}+\frac{15 \mu \alpha_{2} x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{7 / 2}}+n^{2}\right] \\
+\frac{\mu}{\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{3 / 2}}+\frac{3 \mu \alpha_{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{5 / 2}}-\frac{15 \mu \alpha_{2} x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{7 / 2}}-n^{2} \mu=0,  \tag{3.21}\\
x_{3}\left[-2 \pi \rho_{1} A_{2}-\frac{\mu}{\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{3 / 2}}-\frac{9 \mu \alpha_{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{5 / 2}}+\frac{15 \mu \alpha_{2} x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{7 / 2}}\right]=0 . \tag{3.22}
\end{gather*}
$$

From (3.22), since $x_{3} \neq 0$, we have

$$
\begin{equation*}
-2 \pi \rho_{1} A_{2}=\frac{\mu}{\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{3 / 2}}+\frac{9 \mu \alpha_{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{5 / 2}}-\frac{15 \mu \alpha_{2} x_{3}^{2}}{2\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]^{7 / 2}} \tag{3.23}
\end{equation*}
$$

Let

$$
\begin{equation*}
l^{2}=\left(1-x_{1}\right)^{2}+x_{3}^{2} . \tag{3.24}
\end{equation*}
$$

Then, (3.23) and (3.21) may be written respectively:

$$
\begin{gather*}
\frac{15 \mu \alpha_{2} x_{3}^{2}}{2 l^{7}}=2 \pi \rho_{1} A_{2}+\frac{\mu}{l^{3}}+\frac{9 \mu \alpha_{2}}{2 l^{5}} \\
x_{1}\left[-2 \pi \rho_{1} A_{1}-\frac{\mu}{l^{3}}-\frac{3 \mu \alpha_{2}}{2 l^{5}}+\frac{15 \mu \alpha_{2} x_{3}^{2}}{2 l^{7}}+n^{2}\right]+\frac{\mu}{l^{3}}+\frac{3 \mu \alpha_{2}}{2 l^{5}}-\frac{15 \mu \alpha_{2} x_{3}^{2}}{2 l^{7}}-n^{2} \mu=0 . \tag{3.25}
\end{gather*}
$$

Now, from first equations (3.25), we get

$$
\begin{equation*}
x_{3}= \pm \frac{l}{\sqrt{15 \mu \alpha_{2}}}\left[\mu\left(9 \alpha_{2}+2 l^{2}\right)+4 \pi \rho_{1} A_{2} l^{5}\right]^{1 / 2} \tag{3.26}
\end{equation*}
$$

The use of (3.24) in second equation of (3.25) yields

$$
\begin{equation*}
x_{1}=\frac{\left(2 \pi \rho_{1} A_{2}+n^{2} \mu\right) l^{5}+3 \mu \alpha_{2}}{\left[2 \pi \rho_{1}\left(A_{2}-A_{1}\right)+n^{2}\right] l^{5}+3 \mu \alpha_{2}} \tag{3.27}
\end{equation*}
$$

We use the software package Mathematica (Wolfram 2004) to compute the coordinates of the out-of-plane equilibrium points denoted by $L_{6}$ and $L_{7}$ starting with the initial values $x_{1}=1-\mu$ and $x_{3}=\sqrt{3} \sqrt{\alpha_{2}}$ in the case where we have kept up to first order terms in both the numerator and the denominator; we then get

$$
\begin{align*}
x_{1}= & \frac{2 \mu+3 \mu \alpha_{1}+4 A_{2} \pi \rho_{1}}{2+3 \alpha_{1}-4 A_{1} \pi \rho_{1}+4 A_{2} \pi \rho_{1}} \\
& -\frac{3\left\{\mu A_{1}+(1-\mu) A_{2}\right\} \alpha_{2} \pi \rho_{1}}{\left[1+3 \alpha_{1}\left(1+2 A_{2} \pi \rho_{1}-2 A_{1} \pi \rho_{1}\right)+4 \pi \rho_{1}\left(A_{2}-A_{1}+A_{1}^{2} \pi \rho_{1}+A_{2}^{2} \pi \rho_{1}-2 A_{1} A_{2} \pi \rho_{1}\right)\right]} \\
x_{3}= & \frac{\sqrt{2 / 15} \mu \sqrt{\mu^{3}\left(1+2 \pi A_{2} \mu^{2} \rho_{1}\right)}}{\sqrt{\mu \alpha_{2}}}+\frac{7 \sqrt{3 \mu / 10}\left(\mu+2 \pi A_{2} \mu^{3} \rho_{1}\right) \sqrt{\alpha_{2}}}{2 \sqrt{\mu^{3}\left(1+2 \pi A_{2} \mu^{2} \rho_{1}\right)}} \tag{3.28}
\end{align*}
$$

The location of the out-of-plane equilibrium points can be obtained by solving numerically equations (3.26) and (3.27) using (3.24).

Now, from the expression for the density parameter

$$
\begin{equation*}
D=\left(1-\frac{\rho_{1}}{\rho_{3}}\right) \tag{3.29}
\end{equation*}
$$

We assume that $\rho_{1} \neq \rho_{3}$, then $D>$ or $<0$. In the case when the density parameter is positive, we have

$$
\begin{equation*}
\rho_{1}<\rho_{3} \tag{3.30}
\end{equation*}
$$

Hence, numerically we choose

$$
\begin{equation*}
a_{1}^{2}=0.94, \quad a_{2}^{2}=0.9, \quad a_{3}^{2}=0.82, \quad a_{4}^{2}=0.8, \quad \mu=0.01, \quad \pi=3.14 \tag{3.31}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\alpha_{1}=0.024, \quad \alpha_{2}=0.02, \quad \rho_{1}=0.236 \tag{3.32}
\end{equation*}
$$

Now, we perform a numerical exploration of computing the out-of-plane points in the case of the Earth-Moon system. To do this, we arbitrarily choose values for the $A_{i}(i=1,2)$. We found that when $A_{1}=2.5076$ and $A_{2}=2.555$, the positions of the out-of-plane points $\left(x_{1}, 0, \pm x_{3}\right)$ :

$$
\begin{equation*}
x_{1}=3.3527, \quad x_{3}=0.271418 \tag{3.33}
\end{equation*}
$$

The abscissae of the out-of-plane point is outside the possible region of motion of the infinitesimal mass and so we neglect it. However, in the case when the $A_{i}(i=1,2)$ are chosen such that

$$
\begin{equation*}
\left|A_{1}-A_{2}\right| \ll 1, \quad A_{1}=0.7, A_{2}=0.68(\text { say }) \tag{3.34}
\end{equation*}
$$

$A_{i} \in(0,0.7]$, the point $L_{6}$, and $L_{7}$ are, respectively,

$$
\begin{equation*}
x_{1}=0.98611, \quad x_{3}=0.271381 \tag{3.35}
\end{equation*}
$$

and lies within the fluid.

## 4. Linear Stability of the Equilibrium Points

In order to study the linear stability of any equilibrium point $\left(x_{10}, x_{20}, x_{30}\right)$ of an infinitesimal body, we displace it to the position $\left(x_{1}, x_{2}, x_{3}\right)$ such that

$$
\begin{equation*}
\left(x_{10}+\xi, x_{20}+\eta, x_{30}+\zeta\right) \tag{4.1}
\end{equation*}
$$

where $\xi, \eta, \zeta$ are small displacements, and then linearize equation (2.3) to obtain the equations:

$$
\begin{gather*}
\ddot{\xi}-2 n \dot{\eta}=\left(U_{x_{1} x_{1}}^{0}\right) \xi+\left(U_{x_{1} x_{2}}^{0}\right) \eta+\left(U_{x_{1} x_{3}}^{0}\right) \zeta \\
\ddot{\eta}+2 n \dot{\xi}=\left(U_{x_{1} x_{2}}^{0}\right) \xi+\left(U_{x_{2} x_{2}}^{0}\right) \eta+\left(U_{x_{2} x_{3}}^{0}\right) \zeta  \tag{4.2}\\
\ddot{\zeta}=\left(U_{x_{1} x_{3}}^{0}\right) \xi+\left(U_{x_{2} x_{3}}^{0}\right) \eta+\left(U_{x_{3} x_{3}}^{0}\right) \zeta
\end{gather*}
$$

where the partial derivatives are evaluated at the equilibrium points.

### 4.1. Equilibrium Points Near the Center of the First Primary

In order to consider the motion near any equilibrium point in the $x_{1} x_{2}$-plane, we let solutions of the first two equations of (4.2) be

$$
\begin{equation*}
\xi=A \exp (\lambda t), \quad \eta=B \exp (\lambda t) \tag{4.3}
\end{equation*}
$$

where $A, B$, and $\lambda$ are constants.
Taking first and second derivatives of the above, substituting them into the first two equations of system (4.2) and has a non-zero solution when

$$
\left|\begin{array}{cc}
\left(\lambda^{2}-U_{x_{1} x_{1}}^{0}\right) & \left(2 n \lambda+U_{x_{1} x_{2}}^{0}\right)  \tag{4.4}\\
\left(2 n \lambda-U_{x_{1} x_{2}}^{0}\right) & \left(\lambda^{2}-U_{x_{2} x_{2}}^{0}\right)
\end{array}\right|=0
$$

Expanding the determinant, we have

$$
\begin{equation*}
\lambda^{4}-\left(U_{x_{1} x_{1}}^{0}+U_{x_{2} x_{2}}^{0}-4 n^{2}\right) \lambda^{2}+U_{x_{1} x_{1}}^{0} U_{x_{2} x_{2}}^{0}-\left(U_{x_{1} x_{2}}^{0}\right)^{2}=0 \tag{4.5}
\end{equation*}
$$

Equation (4.5) is the characteristic equation corresponding to the variational equations (4.2) in the case when motion is considered in the $x_{1}, x_{2}$-plane.

Now, the values of the second-order partial derivatives of the equilibrium point $\left(x_{L}, 0,0\right)$, where $x_{L}=p_{1}$ stands for the first equilibrium and $x_{L}=x_{11}+p_{2}$ for the second one, are given as

$$
\begin{align*}
& U_{x_{1} x_{1}}^{0}=D \mu\left[\frac{-\left(1-x_{L}\right)^{3}-\left(3 \alpha_{2} / 2\right)\left(1-x_{L}\right)+2 x_{L}\left(1-x_{L}\right)^{2}+6 x_{L} \alpha_{2}+n^{2}\left(1-x_{L}\right)^{5}}{x_{L}\left(1-x_{L}\right)^{5}}\right], \\
& U_{x_{2} x_{2}}^{0}=D \mu\left[\frac{-\left(1-x_{L}\right)^{3}-(3 / 2) \alpha_{2}\left(1-x_{L}\right)-x_{L}\left(1-x_{L}\right)^{2}-(3 / 2) \alpha_{2} x_{L}+n^{2}\left(1-x_{L}\right)^{5}}{x_{L}\left(1-x_{L}\right)^{5}}\right],  \tag{4.6}\\
& U_{x_{3} x_{3}}^{0}=-D\left[2 \pi \rho_{1} A_{2}+\frac{\mu}{\left(1-x_{L}\right)^{3}}+\frac{9 \mu \alpha_{2}}{2\left(1-x_{L}\right)^{5}}\right], \quad U_{x_{1} x_{2}}^{0}=0=U_{x_{2} x_{3}}^{0}=U_{x_{1} x_{3}}^{0} .
\end{align*}
$$

Substituting these in (4.2), we at once have the variational equations:

$$
\begin{gather*}
\ddot{\xi}-2 n \dot{\eta}=U_{x_{1} x_{1}}^{0} \xi \\
\ddot{\eta}+2 n \dot{\xi}=U_{x_{2} x_{2}}^{0} \eta  \tag{4.7}\\
\ddot{\zeta}=-D\left[\frac{\mu}{\left(1-x_{L}\right)^{3}}+\frac{9 \mu \alpha_{2}}{2\left(1-x_{L}\right)^{5}}+2 \pi \rho_{1} A_{2}\right] \zeta \tag{4.8}
\end{gather*}
$$

where the partial derivatives have been computed at each equilibrium point $x_{L}$.

Now, (4.8) is independent of (4.7), the solution being a periodic function is bounded and therefore, the motion of the infinitesimal body in the $x_{3}$ direction is stable.

Now, the characteristic equation of the equilibrium points $\left(x_{L}, 0,0\right)$ corresponding to the system (4.7) is

$$
\begin{equation*}
\lambda^{4}-\left(U_{x_{1} x_{1}}^{0}+U_{x_{2} x_{2}}^{0}-4 n^{2}\right) \lambda^{2}+U_{x_{1} x_{1}}^{0} U_{x_{2} x_{2}}^{0}=0 \tag{4.9}
\end{equation*}
$$

where

$$
\begin{align*}
& U_{x_{1} x_{1}}^{0}=3 D \mu\left(\frac{\alpha_{1}}{2 x_{L}}\right)  \tag{4.10}\\
& U_{x_{2} x_{2}}^{0}=3 D \mu\left(-1+\frac{\alpha_{1}}{2 x_{L}}\right) . \tag{4.11}
\end{align*}
$$

These equations have been obtained using binomial expansion and ignoring terms with second and higher power in $p_{1}, p_{2}, \alpha_{2}$, and their product.

Now, let $\lambda_{1}^{2}$ and $\lambda_{2}^{2}$ be the roots of (4.9), then, the equilibrium point is stable if both the roots are real and negative. This means that their sum must be negative and their product must be positive. Hence, the points $\left(x_{L}, 0,0\right)$ will be stable if the following two conditions hold:

$$
\begin{gather*}
\lambda_{1}^{2}+\lambda_{2}^{2}=U_{x_{1} x_{1}}^{0}+U_{x_{2} x_{2}}^{0}-4 n^{2}<0,  \tag{4.12}\\
\lambda_{1}^{2} \lambda_{2}^{2}=U_{x_{1} x_{1}}^{0} U_{x_{2} x_{2}}^{0}>0 . \tag{4.13}
\end{gather*}
$$

Now, in the case of the first equilibrium point $x_{L}=p_{1}$, if we suppose in (4.10) that $p_{1}<0$ then, $U_{x_{1} x_{1}}^{0}<0$ since $0<\mu<1, D>0,0<\alpha_{1} \ll 1$ and when $p_{1}>0$, we have $U_{x_{1} x_{1}}^{0}>0$.

Similarly, in (4.11), if we suppose $p_{1}<0$ then $U_{x_{2} x_{2}}^{0}<0$.
For the case $p_{1}>0$, we will have $U_{x_{2} x_{2}}^{0}>0$ when $\alpha_{1}>2\left|p_{1}\right|$ which is not possible, hence $U_{x_{2} x_{2}}^{0}<0$.

In the case $0<p_{1}<\alpha_{1} / 2, U_{x_{1} x_{1}}^{0}>0$, and $U_{x_{2} x_{2}}^{0}>0$.
Also, if $0<\alpha_{1} / 2<p_{1}$, we see that $U_{x_{1} x_{1}}^{0}>0$ and $U_{x_{2} x_{2}}^{0}<0$. Thus, for the case $p_{1}<0$, the equilibrium point is stable. For $0<p_{1}<\alpha_{1} / 2$, the equilibrium point is stable if the condition (4.12) holds. When $0<\alpha_{1} / 2<p_{1}$, the equilibrium point is unstable.

Next, for the other equilibrium point positioned at $x_{L}=x_{11}+p_{2}$, when $x_{11}>0$, then $x_{11}^{\prime}>0$ since $\left|p_{2}\right| \ll 1$ and the equilibrium point is stable if the conditions (4.12) and (4.13) are satisfied. If $x_{11}<0$ then $x_{11}^{\prime}<0$; it makes $U_{x_{1} x_{1}}^{0}<0, U_{x_{2} x_{2}}^{0}<0$. Therefore, when $x_{11}<0$, both the conditions (4.12) and (4.13) are fulfilled and the equilibrium point is stable.

### 4.2. Circular Points

At circular points $(1+r \cos \theta, r \sin \theta, 0)$, the values of the second partial derivatives with the use of (3.14) and neglecting the product $\alpha_{1} \alpha_{2}$ are

$$
\begin{align*}
& U_{x_{1} x_{1}}^{0}=3 D \mu \cos ^{2} \theta\left(n^{2}+\alpha_{2}\right) \\
& U_{x_{1} x_{2}}^{0}=3 D \mu \cos \theta \sin \theta\left(n^{2}+\alpha_{2}\right)  \tag{4.14}\\
& U_{x_{2} x_{2}}^{0}=3 D \mu \sin ^{2} \theta\left(n^{2}+\alpha_{2}\right) \\
& U_{x_{3} x_{3}}^{0}=-D\left[2 \pi \rho_{1} A_{2}+\mu\left(n^{2}+3 \alpha_{2}\right)\right]
\end{align*}
$$

Substituting these values in the variational equations (4.2), we get

$$
\begin{gather*}
\ddot{\xi}-2 n \dot{\eta}=3 D \mu \cos ^{2} \theta\left(n^{2}+\alpha_{2}\right) \xi+3 D \mu \cos \theta \sin \theta\left(n^{2}+\alpha_{2}\right) \eta+(0) \zeta  \tag{4.15}\\
\ddot{\eta}+2 n \dot{\xi}=3 D \mu \cos \theta \sin \theta\left(n^{2}+\alpha_{2}\right) \xi+3 D \mu \sin ^{2} \theta\left(n^{2}+\alpha_{2}\right) \eta+(0) \zeta \\
\ddot{\zeta}=-D\left[2 \pi \rho_{1} A_{2}+\mu\left(n^{2}+3 \alpha_{2}\right)\right] \zeta \tag{4.16}
\end{gather*}
$$

Equation (4.16) is independent of (4.15), it shows that the motion of the infinitesimal mass along the $x_{3}$-direction is stable.

Now, a substitution of these partial derivatives in the characteristic equation (4.5) yields

$$
\begin{equation*}
\lambda^{4}-\left[3 D \mu\left(n^{2}+\alpha_{2}\right)-4 n^{2}\right] \lambda^{2}=0 \tag{4.17}
\end{equation*}
$$

Let $\lambda^{2}=\Lambda$ in (4.17) then, we have

$$
\begin{equation*}
\Lambda\left[\Lambda-\left\{3 D \mu\left(n^{2}+\alpha_{2}\right)-4 n^{2}\right\}\right]=0 \tag{4.18}
\end{equation*}
$$

Hence, either

$$
\begin{equation*}
\Lambda=0 \quad \text { or } \quad \Lambda=3 D \mu\left(n^{2}+\alpha_{2}\right)-4 n^{2} \tag{4.19}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\lambda=0 \text { twice, } \quad \text { or } \quad \lambda= \pm\left[3 D \mu\left(n^{2}+\alpha_{2}\right)-4 n^{2}\right]^{1 / 2} \tag{4.20}
\end{equation*}
$$

Therefore, (4.20) gives the roots of the characteristic equation (4.17). Hence we conclude that the circular points are unstable due to the presence of multiple roots.

### 4.3. Out-of-Plane Points

To determine the stability of the out-of-plane equilibrium points, we consider the following partial derivatives:

$$
\begin{align*}
& U_{x_{1} x_{1}}=D\left[-2 \pi \rho_{1} A_{1}+\mu\left\{\frac{\left[2\left(1-x_{1}\right)^{2}-x_{3}^{2}\right]}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{5 / 2}}\right\}\right. \\
& \left.+\frac{3}{2} \mu \alpha_{2}\left\{\frac{\left[4\left(1-x_{1}\right)^{2}-x_{3}^{2}\right]}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{7 / 2}}\right\}-\frac{15}{2} \mu \alpha_{2} x_{3}^{2}\left\{\frac{\left[6\left(1-x_{1}\right)^{2}-x_{3}^{2}\right]}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{9 / 2}}\right\}+n^{2}\right], \\
& U_{x_{1} x_{3}}=D\left[\frac{-3 \mu\left(1-x_{1}\right) x_{3}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{5 / 2}}-\frac{15}{2} \frac{\mu \alpha_{2}\left(1-x_{1}\right) x_{3}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{7 / 2}}\right. \\
& \left.-\frac{15 \mu \alpha_{2}\left(1-x_{1}\right)}{2} \frac{2 x_{3}\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]-7 x_{3}^{3}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{9 / 2}}\right],  \tag{4.21}\\
& U_{x_{2} x_{2}}=D\left[-2 \pi \rho_{1} A_{1}-\mu\left\{\frac{\left(1-x_{1}\right)^{2}+x_{3}^{2}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{5 / 2}}\right\}\right. \\
& \left.-\frac{3 \mu \alpha_{2}}{2}\left\{\frac{\left(1-x_{1}\right)^{2}+x_{3}^{2}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{7 / 2}}\right\}+\frac{15 \mu \alpha_{2} x_{3}^{2}}{2}\left\{\frac{\left(1-x_{1}\right)^{2}+x_{3}^{2}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{9 / 2}}\right\}+n^{2}\right], \\
& U_{x_{3} x_{3}}=D\left[-2 \pi \rho_{1} A_{2}-\mu\left\{\frac{\left(1-x_{1}\right)^{2}-2 x_{3}^{2}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{5 / 2}}\right\}\right. \\
& \left.-\frac{9 \mu \alpha_{2}}{2}\left\{\frac{\left(1-x_{1}\right)^{2}-4 x_{3}^{2}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{7 / 2}}\right\}+\frac{15 \mu \alpha_{2}}{2}\left\{\frac{3 x_{3}^{2}\left[\left(1-x_{1}\right)^{2}+x_{3}^{2}\right]-7 x_{3}^{4}}{\left\{\left(1-x_{1}\right)^{2}+x_{3}^{2}\right\}^{9 / 2}}\right\}\right] \text {. }
\end{align*}
$$

Since $x_{2}=0$, therefore the partial derivatives to be computed at the out-of-plane equilibrium points are

$$
\begin{align*}
& U_{x_{1} x_{2}}^{0}=0=U_{x_{2} x_{1}}^{0}=U_{x_{2} x_{3}}^{0}=U_{x_{3} x_{2}}^{0} \\
& U_{x_{1} x_{1}}^{0}=D\left[-2 \pi \rho_{1} A_{1}-2 \pi \rho_{1} A_{2}+\frac{\mu}{l^{3}}-\frac{3 \mu x_{3}^{2}}{l^{5}}+\frac{3 \mu \alpha_{2}}{2 l^{5}}-\frac{9 \mu \alpha_{2}}{2 l^{5}}-\frac{45 \mu \alpha_{2} x_{3}^{2}}{l^{7}}+\frac{105 \mu \alpha_{2} x_{3}^{4}}{2 l^{9}}+n^{2}\right], \\
& U_{x_{2} x_{2}}^{0}=D\left[-2 \pi \rho_{1} A_{1}+2 \pi \rho_{1} A_{2}+\frac{3 \mu \alpha_{2}}{l^{5}}+n^{2}\right], \\
& U_{x_{1} x_{3}}^{0}=-3\left(1-x_{1}\right) x_{3} D\left[\frac{\mu}{l^{5}}+\frac{15 \mu \alpha_{2}}{2 l^{7}}-\frac{35 \mu \alpha_{2} x_{3}^{2}}{2 l^{9}}\right], \\
& U_{x_{3} x_{3}}^{0}=D\left[\frac{3 \mu x_{3}^{2}}{l^{5}}+\frac{75 \mu \alpha_{2} x_{3}^{2}}{2 l^{7}}-\frac{105 \mu \alpha_{2} x_{3}^{4}}{2 l^{9}}\right] . \tag{4.22}
\end{align*}
$$

Now, we let,

$$
\begin{array}{ll}
U_{x_{1} x_{1}}^{0}=U_{11}, & U_{x_{2} x_{2}}^{0}=U_{22}, \\
U_{x_{1} x_{3}}^{0}=U_{13}, & U_{x_{3} x_{3}}^{0}=U_{33},  \tag{4.23}\\
U_{x_{1} x_{2}}^{0}=U_{12}, & U_{x_{2} x_{3}}^{0}=U_{23} .
\end{array}
$$

Using (4.23), the variational equation can be recast in the form:

$$
\begin{gather*}
\ddot{\xi}-2 n \dot{\eta}=U_{11} \xi+U_{13} \zeta, \\
\ddot{\eta}+2 n \dot{\xi}=U_{22} \eta+U_{23} \zeta,  \tag{4.24}\\
\ddot{\zeta}=U_{13} \xi+U_{33} \zeta .
\end{gather*}
$$

In order to consider the motion of the out-of-plane points, we let solution of the system (4.24) be

$$
\begin{equation*}
\xi=A \exp (\lambda t), \quad \eta=B \exp (\lambda t), \quad \zeta=C \exp (\lambda t), \tag{4.25}
\end{equation*}
$$

where $A, B, C$, and $\lambda$ are constants. $\xi, \eta$, and $\zeta$ are the small displacements in the coordinates of the infinitesimal body.

Now, the characteristic equation corresponding to the variational equations (4.24) in the case of the out-of-plane point may be expressed as

$$
\begin{equation*}
\lambda^{6}-a_{1} \lambda^{4}+a_{2} \lambda^{2}+a_{3}=0, \tag{4.26}
\end{equation*}
$$

where the coefficients of the characteristic equation (4.26) are such that

$$
\begin{align*}
& a_{1}=D[-4 \pi\left.\rho_{1} A_{1}-2 \pi \rho_{1} A_{2}+2 n^{2}\right] \\
& \begin{aligned}
a_{2}=\frac{1}{4 l^{16}}[D[ & -9 \mu^{2} D x_{3}^{2}\left(-1+x_{1}\right)^{2}\left[2 l^{4}+5 \alpha_{2}\left(3 l^{2}-7 x_{3}^{2}\right)\right] \\
& +3 l^{4} \mu x_{3}^{2}\left[2 l^{4}+5 \alpha_{2}\left(5 l^{2}-7 \mu x_{3}^{2}\right)\right] \\
& \times\left[3 D \mu \alpha_{2}+2 l^{5}\left\{(-4+D) n^{2}-2 D A_{1} \pi \rho_{1}+2 D A_{2} \pi \rho_{1}\right\}\right] \\
& +D\left[6 l^{4} \mu x_{3}^{2}+3 \alpha_{2}\left\{l^{4} \mu-25 l^{2}-35 \mu x_{3}^{4}\right\}+2 l^{9}\left(n^{2}-2 A_{1} \pi \rho_{1}+2 A_{2} \pi \rho_{1}\right)\right] \\
& \left.\left.\times\left[3 \alpha_{2}\left\{l^{4} \mu-2 l^{2}\left(15+l^{2}\right) x_{3}^{2}+35 \mu x_{3}^{4}\right\}+2 l^{6}\left(l^{3} n^{2}+\mu-2 l^{3} A_{1} \pi \rho_{1}-2 l^{3} A_{2} \pi \rho_{1}\right)\right]\right]\right] \\
a_{3}=\frac{3 D^{3}}{4 l^{18}} \mu x_{3}^{2} & {\left[2 l^{4}+5 \alpha_{2}\left(5 l^{2}-7 \mu x_{3}^{2}\right)\right]\left[n^{2}+\frac{3 \mu \alpha_{2}}{2 l^{5}}-2 A_{1} \pi \rho_{1}+2 A_{2} \pi \rho_{1}\right] } \\
& \times\left[3 \alpha_{2}\left\{l^{4} \mu-2 l^{2}\left(15+l^{2}\right) \mu x_{3}^{2}+35 \mu x_{3}^{4}\right\}+2 l^{6}\left(l^{3} n^{2}+\mu-2 l^{3} A_{1} \pi \rho_{1}-2 l^{3} A_{2} \pi \rho_{1}\right)\right],
\end{aligned}
\end{align*}
$$

where $l^{2}=\left(1-x_{1}\right)^{2}+x_{3}^{2}$.
These computations have been done using the software package Mathematica.
For the stability analysis of the out-of-plane equilibrium point, we compute numerically the partial derivatives calculated at the out-of-plane points with the use of (3.28) and the following numerical values:

$$
\begin{array}{cccc}
\mu=0.01, & \pi=3.14, & \alpha_{1}=0.024, & \alpha_{2}=0.02  \tag{4.28}\\
A_{1}=0.7, & A_{2}=0.68, & \rho_{1}=0.236, & D=0.2133
\end{array}
$$

Now, substituting the above values in the characteristic equation (4.27), we get

$$
\begin{equation*}
\lambda^{6}-0.192437 \lambda^{4}+0.248034 \lambda^{2}-0.0000197624=0 \tag{4.29}
\end{equation*}
$$

Its roots are:

$$
\begin{align*}
& \lambda_{1,2}=-0.545066 \pm 0.448239 i \\
& \lambda_{3,4}= \pm 0.00892642  \tag{4.30}\\
& \lambda_{5,6}=0.545066 \pm 0.448239 i
\end{align*}
$$

The positive root and the positive real part of the complex roots induce instability at the out-of-plane point. Hence, the motion of the infinitesimal mass around the out-of-plane equilibrium points is unstable for the specific numerical example given here. However, fuller discussion of their stability remains a theme for future research.

## 5. Discussion

The equation of motion (2.3) is different from those of Hallan and Mangang [4] due to oblateness of the second primary. If we assume that the second primary is not oblate (i.e., $\alpha_{2}=0$ ), then these equations will fully coincide with those of Hallan and Mangang [4].

Equation (3.9) gives the equilibrium position of the point $\left(P_{1}, 0,0\right)$ near the center of the first primary and fully coincides with that of Hallan and Mangang [4]. It shows that the position of this equilibrium point does not depend on oblateness of the second primary, while the other equilibrium point $\left(x_{11}+P_{2}, 0,0\right)$ given by (3.11) is different from that of Hallan and Mangang [4] due to the appearance of oblateness of the second primary. When $2 \pi \rho_{1} A_{1}=$ $n^{2}(1-\mu)$, points on the circle $\left(1-x_{1}\right)^{2}+x_{2}^{2}=r^{2}, x_{3}=0$ lying within the first primary are also equilibrium points. These points are affected by oblateness of both primaries. Equations (3.28) give the positions of the out-of-plane points when only linear terms in oblateness of the second primary are retained. We have been able to show that the oblateness of the primaries allows the existence of the out-of-plane equilibrium points in the $x_{1} x_{3}$-plane within the first primary. These points have no analogy in the previous studies of the Robe's restricted threebody problem.

The linear stability analysis of the equilibrium solutions of the problem is investigated with the help of characteristic roots. The characteristic equation (4.9) in the case of the equilibrium point $x_{L}=p_{1}$ near the center of the first primary is the same as that of Hallan and Mangang [4], while that of the other point $x_{L}=x_{11}+p_{2}$ near the center differs from that of Hallan and Mangang [4] due to oblateness of the second primary. The characteristic equation of the circular case (4.17) also differs from that of Hallan and Mangang [4] due to oblateness of the second primary. The stability in the first approximation of this configuration shows that points near the centre of the first primary are conditionally stable; the circular points are unstable. This confirms the earlier results of Hallan and Rana [3], Hallan and Mangang [4]. A numerical exploration shows that the out-of-plane equilibrium points are also unstable. This outcome validates the earlier results of Douskos and Markellos [8] and Singh and Leke [11] that the points are unstable.

## 6. Conclusion

We have derived the equations of motion and established the positions of the equilibrium points of the infinitesimal body in the Robe's [1] restricted three-body problem with oblateness. The term "oblateness" is used in the sense that both primaries are considered as oblate spheroids under the effects of the full buoyancy force exerted by the fluid on the infinitesimal mass.

We have obtained one equilibrium point $\left(P_{1}, 0,0\right)$ near the centre of the first primary which will be on the left or right of the centre of the first primary accordingly as $2 \pi \rho_{1} A_{1}-$ $2 \mu><1$. This point is the same as that of Hallan and Mangang [4]. In addition to this, another equilibrium point $\left(x_{11}+P_{2}, 0,0\right)$ is found within the first primary on the line joining the center of the primaries when $1-2 \pi \rho_{1} A_{1}<-3 \mu / 4$ and $\left|x_{11}\right|<a_{1}$. When $2 \pi \rho_{1} A_{1}=n^{2}(1-\mu)$, points on the circle $\left(1-x_{1}\right)^{2}+x_{2}^{2}=r^{2}, x_{3}=0$ lying within the first primary are also equilibrium points. We call them circular points. Finally, we have been able to show that the oblateness of the primaries allows the existence of the out-of-plane equilibrium points in the $x_{1} x_{3}$-plane within the first primary.

The result of this paper can be summarized as follows. The restricted three-body problem under the framework of the Robe's [1] problem with oblate primaries has the equilibrium points of the type: points near the center of the first primary, points on the circle (circular points), and two out-of-plane points $L_{6,7}$. It is seen that points near the first primary are conditionally stable, the circular points are unstable, while the out-of-plane equilibrium points are unstable for the specific numerical example given here. The effect of drag forces as considered by Giordano et al. [2] under the present context, particularly as regards the analysis of the properties of the equilibrium points located inside the first primary, will be interesting.

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Research Article

# Relativistic Double Barrier Problem with Three Transmission Resonance Regions 

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#### Abstract

We obtain exact scattering solutions of the Dirac equation in $1+1$ dimensions for a double square barrier vector potential. The potential bottom between the two barriers is chosen to be higher than $2 m c^{2}$, whereas the top of the barriers is at least $2 m c^{2}$ above the bottom. The relativistic version of the conventional double barrier transmission resonances is obtained for energies within $\pm m c^{2}$ from the height of the barriers. However, due to our judicious choice of potential configuration we also find two more (subbarrier) transmission resonance regions below the conventional one. Both are located within the two Klein energy zones and characterized by resonances that are broader than the conventional ones. The design of our double barrier so as to enable us to establish these two new subbarrier transmission resonance regions is our main finding.


## 1. Introduction

The basic equation of relativistic quantum mechanics was formulated more than 80 years ago by Paul Dirac [1-3]. It describes the state of electrons in a way consistent with special relativity, requiring that electrons have spin $1 / 2$ and predicting the existence of an antiparticle partner to the electron (the positron). The physics and mathematics of the Dirac equation is very rich, illuminating, and provides a theoretical framework for different physical phenomena that are not present in the nonrelativistic regime such as the Klein paradox, supercriticality (supercritical transmission of a relativistic particle through a potential barrier) [17], and the anomalous quantum Hall effect in graphene [8,9]. It is well known that the Dirac equation has positive as well as negative energy solutions [1-3]. The positive and negative energy subspaces are completely disconnected. This is a general feature of the solution space
of the Dirac equation, which is sometimes overlooked. Since the equation is linear, then the complete solution must be a linear combination of the two. Incorporating the missing part of the negative energy solution, which is not taken into account in the traditional solution, one of the authors (A. D. Alhaidari) gave a new approach to the resolution of the famous Klein paradox within relativistic quantum mechanics [10]. This approach leads to the correct physical and mathematical interpretations of this phenomenon without resort to quantum field theory.

On the other hand, tunneling phenomena played an important role in nonrelativistic quantum mechanics due to its important application in electronic devices [11, 12]. It was Esaki who discovered a characteristic called negative differential resistance (NDR) whereby, for PN junction diodes, the current voltage characteristics has a sharp peak at a certain voltage associated with resonant tunneling. This constituted the first important confirmation that this phenomenon is due to the quantum mechanical tunneling effect of electrons [13-15]. Tunneling is a purely quantum phenomenon that happens in the classically forbidden region; its experimental observation constituted a very important support to the quantum theory. On the other hand, the study of tunneling of relativistic particles through one-dimensional potentials has been restricted to some simple configurations such as $\delta$-potentials and square barriers, mainly, in the study of the possible relativistic corrections to mesoscopic conduction [16] and the analysis of resonant tunneling through multibarrier systems [17]. Relativistic studies of the quark and Dirac particles in 1D periodic potential were also reported separately in $[18,19]$.

Very recently, electron transport through electrostatic barriers in single and bilayer graphene has been studied using the Dirac equation, and barrier penetration effects analogous to the Klein paradox were noted [20]. The study of transmission resonances in relativistic wave equations in external potentials has been discussed extensively in the literature [21, 22]. In this case, for given values of the energy and shape of the barrier, the probability of transmission reaches unity even if the potential strength is larger than the energy of the particle, a phenomenon that is not present in the nonrelativistic case. The relation between low momentum resonances and supercritical transmission has been established by Dombey and Calogeracos [7] and Kennedy [23]. Some results on the scattering of Dirac particles by a one-dimensional potential exhibiting resonant behavior have also been reported [22-24].

However, recent studies have shown that inducing a finite bandgap in graphene by epitaxially growing it on a substrate is possible [25] and, therefore, its energy dispersion relation is no longer linear in momentum. This process generated gaps in graphene energy spectrum and resulted in a finite effective mass for its charge carriers and opened up nanoelectronic opportunities for graphene. This 2D massless system can then be mapped into an effectively massive 1D one [26] and, consequently, the problem of Klein tunneling of Dirac fermions across a potential can be put on the test, because the potential barriers can be seen as n-p-n junctions of graphene if they are high enough. Under these circumstances, it is interesting to characterize the system behavior by determining the full expressions of the corresponding reflection and transmission coefficients.

Motivated by the above progress, we study in this work the resonant transmission of a beam of relativistic particles through two separated square barriers with elevated potential bottom in between and investigate transmission resonance in this structure [1-3]. Under special conditions, dictated by our judicious choice of potential bottom elevations and barrier heights, we demonstrate the occurrence of three (subbarrier) regions of transmission resonances. One of them is the relativistic extension of the conventional nonrelativistic double


Figure 1: The potential configuration of the relativistic double barrier problem with $V_{-}>2 m$ and $V_{+}>V_{-}+$ $2 m$. Oscillatory solutions are in the grey regions, whereas exponential solutions are in the white regions. The oscillatory positive/negative energy solutions are located in the light/dark grey areas.
barrier transmission resonances for energies within $\pm m c^{2}$ from the height of the barriers. The other two are located within the two Klein energy zones where only positive and negative energy oscillatory solutions coexist at the same energy. The latter resonances are broader than the conventional ones.

## 2. Scattering Solution of the Dirac Equation

The physical configuration associated with the double barrier problem in our study is shown in Figure 1. In the relativistic units $\hbar=c=1$, the one-dimensional stationary Dirac equation with vector potential coupling can be written as [1-3]

$$
\left(\begin{array}{cc}
m+V(x)-E & -\frac{d}{d x}  \tag{2.1}\\
+\frac{d}{d x} & -m+V(x)-E
\end{array}\right)\binom{\psi^{+}(x)}{\psi^{-}(x)}=0,
$$

where $V(x)$ is the time component of the vector potential whose space component vanishes (i.e., gauged away due to gauge invariance). The potential $V(x)$ is defined by

$$
V(x)= \begin{cases}0 & |x| \geq a_{+}+a_{-},  \tag{2.2}\\ V_{+} & a_{-}<|x|<a_{+}+a_{-}, \\ V_{-} & |x| \leq a_{-},\end{cases}
$$

where $V_{ \pm}$and $a_{ \pm}$are positive potential parameters (see Figure 1) such that $V_{-}>2 m$ and $V_{+}>$ $V_{-}+2 m$. We divide configuration space according to the piece-wise constant potential sections into three regions numbered 0 and $\pm$ corresponding to $V=0$ and $V=V_{ \pm}$, respectively. In
regions 0 , where the potential vanishes, the equation becomes the free Dirac equation that relates the two spinor components as follows:

$$
\begin{equation*}
\psi^{\mp}(x)=\frac{1}{m \pm E} \frac{d}{d x} \psi^{ \pm}(x) \tag{2.3}
\end{equation*}
$$

This relationship is valid for $E \neq \mp m$. Since the problem is linear and because $E=\mp m$ belongs to the 干tive energy spectrum, then (2.3) with the top/bottom sign is valid only for positive/negative energy, respectively. After choosing a sign in (2.3) then the other spinor component obeys the following Schrödinger-like second order differential equation:

$$
\begin{equation*}
\left(\frac{d^{2}}{d x^{2}}+E^{2}-m^{2}\right) \psi^{ \pm}(x)=0 \tag{2.4}
\end{equation*}
$$

We should emphasize that (2.4) does not give the two components of the spinor that belong to the same energy subspace. One has to choose one sign in (2.4) to obtain only one of the two components then substitute that into (2.3) with the corresponding sign to obtain the other component. Now, within the double barrier (the $V_{ \pm}$regions) the same analysis follows but with the substitution $E \rightarrow E-V_{ \pm}$giving

$$
\begin{gather*}
\psi^{\mp}(x)=\frac{1}{m \pm(E-V)} \frac{d}{d x} \psi^{ \pm}(x) \\
{\left[\frac{d^{2}}{d x^{2}}+(E-V)^{2}-m^{2}\right] \psi^{ \pm}(x)=0} \tag{2.5}
\end{gather*}
$$

where $V$ stands for either of the two potentials $V_{ \pm}$. Generally, in any region of constant potential $V$, positive/negative energy solutions occur for relativistic energies larger/smaller than $V$. Of these, the oscillatory solutions of the form $e^{ \pm i k x}$ hold for $|E-V|>m$, where $k^{2}=\left|(E-V)^{2}-m^{2}\right|$. On the other hand, the exponential solutions of the form $e^{ \pm k x}$ hold for $|E-V|<m$.

The scattering solution, which is the subject of this work, pertains to energies $E>m$. It is straightforward to write down the positive and negative energy solutions of (2.3)-(2.5). First, we write the wave vector associated with regions of space in which the potential equals to zero, $V_{+}$, and $V_{-}$, as

$$
\begin{equation*}
k_{\mu}(E)=\sqrt{m^{2}-\left(E-U_{\mu}\right)^{2}} \tag{2.6}
\end{equation*}
$$

where $\mu=0,+,-$ and $U_{\mu}=\left\{0, V_{+}, V_{-}\right\}$. This results in oscillatory solutions if $k_{\mu}$ is pure imaginary which happens when $E>U_{\mu}+m$ or $E<U_{\mu}-m$ (i.e., in the two grey regions of Figure 1). Otherwise, these solutions are exponentials (i.e., in the white areas of the figure). The oscillatory positive/negative energy solutions are located in the light/dark grey areas of Figure 1, respectively. We divide configuration space from left to right into five regions
indexed by $v=1,2, \ldots, 5$. The general positive energy solution in these regions (both oscillatory and exponentials) can be written as

$$
\begin{equation*}
\psi_{\mu, v}(x)=\frac{A_{v}}{\sqrt{1+\left|\alpha_{\mu}\right|^{2}}}\binom{1}{\alpha_{\mu}} e^{k_{\mu} x}+\frac{B_{v}}{\sqrt{1+\left|\alpha_{\mu}\right|^{2}}}\binom{1}{-\alpha_{\mu}} e^{-k_{\mu} x} \tag{2.7a}
\end{equation*}
$$

whereas the negative energy solutions are of the form

$$
\begin{equation*}
\psi_{\mu, v}(x)=\frac{A_{v}}{\sqrt{1+\left|\beta_{\mu}\right|^{2}}}\binom{-\beta_{\mu}}{1} e^{-k_{\mu} x}+\frac{B_{v}}{\sqrt{1+\left|\beta_{\mu}\right|^{2}}}\binom{\beta_{\mu}}{1} e^{k_{\mu} x} \tag{2.7b}
\end{equation*}
$$

$A_{v}$ and $B_{v}$ are constants (the complex amplitudes) associated with right and left "traveling" solutions in the $v$ th region, respectively. The energy parameters $\alpha_{\mu}$ and $\beta_{\mu}$ are defined by

$$
\begin{align*}
& \alpha_{\mu}=\sqrt{\frac{\left(m-E+U_{\mu}\right)}{\left(m+E-U_{\mu}\right)}},  \tag{2.8a}\\
& \beta_{\mu}=\sqrt{\frac{\left(m+E-U_{\mu}\right)}{\left(m-E+U_{\mu}\right)}} \tag{2.8b}
\end{align*}
$$

Note that $\beta_{\mu}= \pm 1 / \alpha_{\mu}$ for real/imaginary values, respectively. The complex constant amplitudes $\left\{A_{v}, B_{v}\right\}$ will be determined by the boundary conditions. We should note that the oscillatory solutions, $e^{ \pm i k x}$, in (2.7a) and ( 2.7 b ) represent a wave traveling in the $\pm x$ direction for positive energy solutions and in the $\mp x$ direction for negative energy solutions. The solution of the Dirac equation to the right of the double barrier consists of positive energy plane-wave solutions traveling in the $\pm x$ directions. However, the physical boundary conditions of the problem allow only transmitted waves traveling to the right after passing through the double barrier (i.e., $B_{5}=0$ ). Moreover, and without loss of generality, we can normalize the incident beam to unit amplitude (i.e., $A_{1}=1$ ).

Matching the spinor wavefunctions at the four boundaries defined by $|x|=a_{-}$and $|x|=a_{+}+a_{-}$gives relations between $\left(A_{v}, B_{v}\right)$ in $v$ th region and those in the neighboring region. We prefer to express these relationships in terms of $2 \times 2$ transfer matrices between different regions, $\left\{M_{n}\right\}$, with $\binom{A_{n}}{B_{n}}=M_{n}\binom{A_{n+1}}{B_{n+1}}$. Finally, we obtain the full transfer matrix over the whole double barrier which can be written, in an obvious notation, as follows:

$$
\begin{equation*}
\binom{1}{R}=\left(\prod_{n=1}^{4} M_{n}\right)\binom{T}{0}=M(E)\binom{T}{0} \tag{2.9}
\end{equation*}
$$

where $M(E)=M_{1} M_{2} M_{3} M_{4}$ and we have set $R=B_{1}$ and $T=A_{5} ; R$ and $T$ being the reflection and transmission amplitudes, respectively. We have assumed an incident wave from left normalized to unit amplitude (i.e., $A_{1}=1$ and $B_{5}=0$ ). The explicit form of the transfer matrices $M_{n}$ depends on the specific energy range. There are three such ranges for all $E>m$.


Figure 2: The transmission coefficient as a function of energy associated with the potential configuration of Figure 1 for $V_{+}=8 m, V_{-}=4 m, a_{+}=3 / m$, and $2 a_{-}=5 / m$. Evident are the three subbarrier transmissionresonance regions. The lowest two are within the two Klein energy zones and the highest one with sharp resonances is bounded within the energy range $V_{+} \pm m$.

These ranges are (i) $m<E<V_{-}$, (ii) $V_{-}<E<V_{+}$, and (iii) $E>V_{+}$. Therefore, we end up with the full set of twelve transfer matrices given in the Appendix. Equation (2.9) leads to

$$
\begin{equation*}
T(E)=\frac{1}{M_{11}(E)}, \quad R(E)=\frac{M_{21}(E)}{M_{11}(E)} \tag{2.10}
\end{equation*}
$$

Time reversal invariance and the relevant conservation laws dictate that the transfer matrix $M(E)$ has a unit determinant along with the following symmetry properties $M_{11}(E)=$ $M_{22}(E)^{*}$ and $M_{12}(E)=M_{21}(E)^{*}$, where ${ }^{*}$ stands for complex conjugation. These can easily be checked using the explicit forms given in the Appendix. Thus, symmetry considerations impose strong conditions on the structure of the transfer matrix. Using these properties in (2.10) gives the expected flux conservation $|T|^{2}+|R|^{2}=1$. Moreover, from (2.10) we see that full transmission or resonance transmission occurs at energies where the condition $\left|M_{11}(E)\right|=1$ is satisfied (equivalently, $\left|M_{21}(E)\right|=0$ ).

## 3. Results and Discussion

The physical content of particle scattering through the double barrier depends on the energy of the incoming particle, which can assume any value larger than $m$. In order to allow for supercritical transmission through the potential, we need to impose certain conditions on the heights of the potential barriers. Our study concentrates on Klein energy zones where full transmission can take place. The situation of interest to our study concerns two Klein energy zones, which arises when $V_{-}>2 m$ and $V_{+}-V_{-}>2 m$. As an example, we calculate the transmission coefficient as a function of energy for a given set of potential parameters. The result is shown in Figures 2 and 3. We combine Figures 1 and 2 together in such a way that Figure 2 is turned 90 degrees such that the energy axis of Figures 1 and 2 coincide. In this way, one can clearly see how the different transmission regions correspond to the different potential regions. In addition to the expected above-barrier full transmission for some values of energies larger than $V_{+}+m$, one can clearly identify three subbarrier regions where transmission resonances occur. These are
(i) the lower Klein energy zone ( $m<E<V_{-}-m$ ): a region of seven resonances,
(ii) the higher Klein energy zone $\left(V_{-}+m<E<V_{+}-m\right)$ : a region of four resonances,


Figure 3: We combine Figures 1 and 2 together in such a way that Figure 2 is turned 90 degrees such that the energy axis of Figures 1 and 2 coincide. In this way, one can clearly see how the different transmission regions correspond to the different potential regions.
(iii) the relativistic version of the conventional nonrelativistic double barrier transmission resonance $\left(V_{+}-m<E<V_{+}+m\right)$ : a region of four resonances.

It is also clear that resonances in the conventional energy zone are very sharp (or very narrow) whereas those in the two Klein energy zones are broad (or wide). This means that resonance states corresponding to the former decay are much slower and have longer tunneling time than the latter. In Table 1, we list the resonance energies for this potential configuration to an accuracy of 10 decimal places. These results were obtained as solution of the equation $M_{21}(E)=0$. As further insight into the dynamics of this relativistic model, Figure 4 shows an animation of Figure 2 as the distance between the two barriers, $2 a_{-}$, varies from $2 / m$ to $6 / m$. The animation shows the following.
(i) The density of resonances in each of the three subbarrier regions increases with $a_{-}$, that is, the energy separation between resonances decreases with $a_{-}$.

Table 1: Transmission resonance energies (in units of $m c^{2}$ ) for the potential configuration associated with Figure $2\left(V_{+}=8 m, V_{-}=4 m, a_{+}=3 / m, a_{-}=2.5 / m\right)$. These values were obtained as solutions to the equation $M_{21}(E)=0$.

| Level | Lower Klein <br> energy zone | Higher Klein <br> energy zone | Conventional <br> energy zone | Above-barrier <br> energy zone |
| :--- | :---: | :---: | :---: | :---: |
| 0 | 1.1913921248 | 5.1824247690 | 7.2022544582 | 9.1265979020 |
| 1 | 1.4523858967 | 5.5378483868 | 7.7033320458 | 9.4112532302 |
| 2 | 1.7708714661 | 6.1348174089 | 8.2103665633 | 9.4794638424 |
| 3 | 2.0643517404 | 6.7590893689 | 8.7007206203 | 9.7910774141 |
| 4 | 2.2185949080 |  |  | 10.1475989665 |
| 5 | 2.4744005714 |  |  | 10.3256144827 |
| 6 | 2.7966547987 |  |  | 10.5446769142 |
| 7 |  |  | 10.9095057090 |  |



Figure 4: Animation of Figure 2 as the distance between the two barriers, $2 a_{-}$, varies from $2 / \mathrm{m}$ to $6 / \mathrm{m}$ (2.4 MB MPG).
(ii) As $a_{-}$increases, resonance energies drop down (fall or dive) from the above-barrier region into the conventional resonance region then into the higher Klein energy zone.
(iii) Additionally, as $a_{-}$increases, resonance energies are created at the bottom of the spectrum (at $E \approx m$ ) then move up into the lower Klein energy zone.

Figure 4 gives another animation of Figure 2 as the width of the barriers, $a_{+}$, varies from $1 / m$ to $3 / m$. The animation shows that all resonances get sharper with an increase in $a_{+}$ but the number of resonances in the conventional region does not change (i.e., the population density of resonances in this region is independent of $a_{+}$). We would like to mention a related recent work by Villalba and Gonzalez-Arraga [27] who considered the resonant tunneling through a double square barrier and double cusp potentials. Our problem differs from that in [27] by the choice of an elevated bottom of the potential well, which gives rise to two Klein energy zones of resonance. This potential design gave rise to a peculiar energy dependence of the transmission with three resonance regions; one is due to the conventional quantum tunneling and two others are due to Klein tunneling. Another animation of Figure 2 is given in Figure 5.

For completeness, it is also interesting to consider the potential barrier with the configuration shown in Figure 6, where $V_{+}>2 m$ and $V_{-}>V_{+}+2 m$. The associated transmission coefficient as a function of energy is shown in Figure 7 for the given potential


Figure 5: Animation of Figure 2 as the width of the barriers, $a_{+}$, varies from $1 / m$ to $3 / m$ ( 2.4 MB MPG).


Figure 6: The potential configuration with $V_{+}>2 m$ and $V_{-}>V_{+}+2 m$.


Figure 7: The transmission coefficient associated with the potential configuration of Figure 6 as a function of energy for $V_{+}=4 m, V_{-}=8 m, a_{+}=3 / \mathrm{m}$, and $2 a_{-}=5 / \mathrm{m}$.
parameters where the values of $V_{ \pm}$were interchanged. In Figure 8, we combine Figures 6 and 7 together in such a way that Figure 7 is turned 90 degrees such that the energy axis of Figures 6 and 7 coincide. In this way, one can clearly see how the different transmission regions correspond to the different potential regions. It looks as if the energy region [ $V_{-}-m, V_{+}+m$ ] of Figure 2 was flipped and placed in the energy region $\left[V_{+}-m, V_{-}+m\right.$ ] of Figure 7 . Consequently, the sharp resonance region became sandwiched between the two Klein energy zones while the transmission structure of the higher Klein energy zone was reversed.


Figure 8: We combine Figures 6 and 7 together in such a way that Figure 7 is turned 90 degrees such that the energy axis of Figures 6 and 7 coincide. In this way, one can clearly see how the different transmission regions correspond to the different potential regions.

Finally, we note that the present work will not remain at this stage but will be pursued further. We plan to use the results obtained so far to deal with different issues related to transport properties in graphene. One of the main characteristics of Dirac fermions in graphene is the accuracy with which we can model their behavior by having extremely small mass (in fact, even massless). This implies that at any finite energy the model should be treated relativistically. This endows fermions in graphene with the ability to tunnel through a single potential barrier with probability one [20,28-31]. It is then natural to extend that analysis to our two-barrier problem case and investigate the basic features of such a system. However, we would like to mention that graphene is a two-dimensional (2D) system, and 2D carrier tunneling through 1D barriers can be very complicated and direction dependent. Our present results correspond to the transmission of 2D carriers only when the carriers move perpendicular to the potential barriers.

## Appendix

## Transfer Matrices

If we define $a=a_{+}+a_{-}, \sigma_{0}=e^{a k_{0}}, \sigma_{+}=e^{a k_{+}}$, and $\gamma_{ \pm}=e^{a_{-} k_{ \pm}}$, then in the first energy interval, $m<E<V_{-}$, the four transfer matrices at the boundaries $|x|=a_{-}$and $|x|=a$ are given by

$$
\begin{align*}
& M_{1}=\frac{1}{2}\left(\begin{array}{cc}
\sigma_{0} \sigma_{+}\left(\frac{1}{\alpha_{0}}-\beta_{+}\right) & \frac{\sigma_{0}}{\sigma_{+}}\left(\frac{1}{\alpha_{0}}+\beta_{+}\right) \\
-\frac{\sigma_{+}}{\sigma_{0}}\left(\frac{1}{\alpha_{0}}+\beta_{+}\right) & \frac{1}{\sigma_{0} \sigma_{+}}\left(\beta_{+}-\frac{1}{\alpha_{0}}\right)
\end{array}\right), \quad M_{2}=\frac{1}{2}\left(\begin{array}{cc}
\frac{\gamma_{-}}{\gamma_{+}}\left(1+\frac{\beta_{-}}{\beta_{+}}\right) & \frac{1}{\gamma_{+} \gamma_{-}}\left(1-\frac{\beta_{-}}{\beta_{+}}\right) \\
\gamma_{+}\left(1-\frac{\gamma_{-}}{\beta_{+}}\right) & \frac{\gamma_{+}}{\gamma_{-}}\left(1+\frac{\beta_{-}}{\beta_{+}}\right)
\end{array}\right), \\
& M_{3}=\frac{1}{2}\left(\begin{array}{cc}
\frac{\gamma_{-}}{\gamma_{+}}\left(1+\frac{\beta_{+}}{\beta_{-}}\right) & \gamma_{+} \gamma_{-}\left(1-\frac{\beta_{+}}{\beta_{-}}\right) \\
\frac{1}{\gamma_{+} \gamma_{-}}\left(1-\frac{\beta_{+}}{\beta_{-}}\right) & \frac{\gamma_{+}}{\gamma_{-}}\left(1+\frac{\beta_{+}}{\beta_{-}}\right)
\end{array}\right), \quad M_{4}=\frac{1}{2}\left(\begin{array}{cc}
\sigma_{0} \sigma_{+}\left(\alpha_{0}-\frac{1}{\beta_{+}}\right) & -\frac{\sigma_{+}}{\sigma_{0}}\left(\alpha_{0}+\frac{1}{\beta_{+}}\right) \\
\frac{\sigma_{0}}{\sigma_{+}}\left(\alpha_{0}+\frac{1}{\beta_{+}}\right) & \frac{1}{\sigma_{0} \sigma_{+}}\left(\frac{1}{\beta_{+}}-\alpha_{0}\right)
\end{array}\right) . \tag{A.1}
\end{align*}
$$

Then, in the second energy range, $V_{-}<E<V_{+}$, we have

$$
\begin{align*}
& M_{1}=\frac{1}{2}\left(\begin{array}{cc}
\sigma_{0} \sigma_{+}\left(\frac{1}{\alpha_{0}}-\beta_{+}\right) & \frac{\sigma_{0}}{\sigma_{+}}\left(\frac{1}{\alpha_{0}}+\beta_{+}\right) \\
-\frac{\sigma_{+}}{\sigma_{0}}\left(\frac{1}{\alpha_{0}}+\beta_{+}\right) & \frac{1}{\sigma_{0} \sigma_{+}}\left(\beta_{+}-\frac{1}{\alpha_{0}}\right)
\end{array}\right), \quad M_{2}=\frac{1}{2}\left(\begin{array}{cc}
\frac{1}{\gamma_{+} \gamma_{-}}\left(\alpha_{-}-\frac{1}{\beta_{+}}\right) & -\frac{\gamma_{-}}{\gamma_{+}}\left(\alpha_{-}+\frac{1}{\beta_{+}}\right) \\
\frac{\gamma_{-}}{\gamma_{-}}\left(\alpha_{-}+\frac{1}{\beta_{+}}\right) & \gamma_{+} \gamma_{-}\left(\frac{1}{\beta_{+}}-\alpha_{-}\right)
\end{array}\right), \\
& M_{3}=\frac{1}{2}\left(\begin{array}{cc}
\frac{1}{\gamma_{+}}\left(\frac{1}{\alpha_{-}}-\beta_{+}\right) & \frac{\gamma_{+}}{\gamma_{-}}\left(\frac{1}{\alpha_{-}}+\beta_{+}\right) \\
-\frac{\gamma_{-}}{\gamma_{+}}\left(\frac{1}{\alpha_{-}}+\beta_{+}\right) & \gamma_{+} \gamma_{-}\left(\beta_{+}-\frac{1}{\alpha_{-}}\right)
\end{array}\right), \quad M_{4}=\frac{1}{2}\left(\begin{array}{cc}
\sigma_{0} \sigma_{+}\left(\alpha_{0}-\frac{1}{\beta_{+}}\right) & -\frac{\sigma_{+}}{\sigma_{0}}\left(\alpha_{0}+\frac{1}{\beta_{+}}\right) \\
\frac{\sigma_{0}}{\sigma_{+}}\left(\alpha_{0}+\frac{1}{\beta_{+}}\right) & \frac{1}{\sigma_{0} \sigma_{+}}\left(\frac{1}{\beta_{+}}-\alpha_{0}\right)
\end{array}\right) . \tag{A.2}
\end{align*}
$$

Note that $M_{1}$ and $M_{4}$ have the same form as in (A.1). Finally, in the energy range, $E>V_{+}$, we obtain

$$
\begin{array}{ll}
M_{1}=\frac{1}{2}\left(\begin{array}{cc}
\frac{\sigma_{0}}{\sigma_{+}}\left(1+\frac{\alpha_{+}}{\alpha_{0}}\right) & \sigma_{0} \sigma_{+}\left(1-\frac{\alpha_{+}}{\alpha_{0}}\right) \\
\frac{1}{\sigma_{0} \sigma_{+}}\left(1-\frac{\alpha_{+}}{\alpha_{0}}\right) & \frac{\sigma_{+}}{\sigma_{0}}\left(1+\frac{\alpha_{+}}{\alpha_{0}}\right)
\end{array}\right), & M_{2}=\frac{1}{2}\left(\begin{array}{cc}
\frac{\gamma_{+}}{\gamma_{-}}\left(1+\frac{\alpha_{-}}{\alpha+}\right) & \gamma_{+} \gamma_{-}\left(1-\frac{\alpha_{-}}{\alpha+}\right) \\
\frac{1}{\gamma_{+} \gamma_{-}}\left(1-\frac{\alpha_{-}}{\alpha+}\right) & \frac{\gamma_{-}}{\gamma_{+}}\left(1+\frac{\alpha_{-}}{\alpha+}\right)
\end{array}\right), \\
M_{3}=\frac{1}{2}\left(\begin{array}{cc}
\frac{\gamma_{+}}{\gamma_{-}}\left(1+\frac{\alpha_{+}}{\alpha_{-}}\right) & \frac{1}{\gamma_{+} \gamma_{-}}\left(1-\frac{\alpha_{+}}{\alpha_{-}}\right) \\
\gamma_{+}\left(1-\frac{\alpha_{+}}{\alpha_{-}}\right) & \frac{\gamma_{-}}{\gamma_{+}}\left(1+\frac{\alpha_{+}}{\alpha_{-}}\right)
\end{array}\right), & M_{4}=\frac{1}{2}\left(\begin{array}{cc}
\frac{\sigma_{0}}{\sigma_{+}}\left(1+\frac{\alpha_{0}}{\alpha_{+}}\right) & \frac{1}{\sigma_{0} \sigma_{+}}\left(1-\frac{\alpha_{0}}{\alpha_{+}}\right) \\
\sigma_{0} \sigma_{+}\left(1-\frac{\alpha_{0}}{\alpha_{+}}\right) & \frac{\sigma_{+}}{\sigma_{0}}\left(1+\frac{\alpha_{0}}{\alpha_{+}}\right)
\end{array}\right) . \tag{A.3}
\end{array}
$$

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## Research Article

# Nonlinear Effects of Electromagnetic TM Wave Propagation in Anisotropic Layer with OKerr Nonlinearity 

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#### Abstract

The problem of electromagnetic TM wave propagation through a layer with Kerr nonlinearity is considered. The layer is located between two half-spaces with constant permittivities. This electromagnetic problem is reduced to the nonlinear boundary eigenvalue problem for ordinary differential equations. It is necessary to find eigenvalues of the problem (propagation constants of an electromagnetic wave). The dispersion equation (DE) for the eigenvalues is derived. The DE is applied to nonlinear metamaterial as well. Comparison with a linear case is also made. In the nonlinear problem there are new eigenvalues and new eigenwaves. Numerical results are presented.


## 1. Introduction

Problems of electromagnetic wave propagation in nonlinear waveguide structures are intensively investigated during several decades. First known studies about nonlinear optics' problems are given in the monographs [1,2]. Propagation of electromagnetic wave in a layer and a circle cylindrical waveguide are among such problems. Phenomena of electromagnetic wave propagation in nonlinear media have original importance and also find a lot of applications, for example, in plasma physics, microelectronics, optics, and laser technology. There are a lot of different nonlinear phenomena in media when an electromagnetic wave propagates, such as self-focusing, defocusing, and self-channeling [1-5].

Investigation of nonlinear phenomena leads us to solve nonlinear differential equations. In some cases it is necessary to solve nonlinear boundary eigenvalue problems (NBEPs), which rarely can be solved analytically. One of the important nonlinear phenomenan is the case when the permittivity of the sample depends on electric field intensity. And one of the simplest nonlinearities is a Kerr nonlinearity [4, 6, 7]. When we
speak about NBEPs we mean that differential equations and boundary conditions nonlinearly depend on the spectral parameter and also the differential equations nonlinearly depend on the sought-for functions. These facts do not allow to apply well-known methods of spectral problems' investigation.

Here we consider electromagnetic TM wave propagation in a layer with Kerr nonlinearity. Perhaps, the papers $[6,7]$ were the first studies where some problems of electromagnetic wave propagation are considered in a strong electromagnetic statement. Propagation of polarized electromagnetic waves in a layer and in a circle cylindrical waveguide with Kerr nonlinearity is considered in this paper. When one says that the permittivity $\varepsilon$ is described by Kerr law this means that (for an isotropic material) $\varepsilon=$ $\varepsilon_{\text {const }}+\alpha|E|^{2}, \varepsilon_{\text {const }}$ is the constant part of the permittivity $\varepsilon ; \alpha$ is the nonlinearity coefficient; $|\mathrm{E}|^{2}=E_{x}^{2}+E_{y}^{2}+E_{z}^{2}$, where $\mathbf{E}=\left(E_{x}, E_{y}, E_{z}\right)$ is an electric field. Below we consider an anisotropic case (some results were presented in [8]). The first approximation for eigenvalues of the problem is presented in [9].

Problems of electromagnetic wave propagation in a linear layer (with constant permittivity) and in a linear circle cylindrical waveguide were deeply studied many years ago, see, for example, [10]. Such problems are formulated as boundary eigenvalue problems for ordinary differential equations. Indeed, the main interest in this problem is the value of the spectral parameter (eigenvalues) which corresponds to the propagating wave. If an eigenvalue is known it is easy to solve differential equations numerically. Otherwise numerical methods cannot be successfully applied. However, in nonlinear cases it is often paid more attention to solve the differential equations (see, e.g., [11-13]). Though the first problem is to find eigenvalues therefore to find Des, from the mathematical standpoint the DE is an equation with respect to the spectral parameter. Analysis of this equation allows us to make conclusions about problem's solvability, eigenvalues' localization, and so forth. In most cases the equations of the problem cannot be integrated in an explicit form. Of course, if one has the explicit solutions of the differential equations it is easy to derive the DE. Therefore, when the equations cannot be integrated things do not go to a DE. However, in some cases the DE can be found in an explicit form and it is not necessary to have explicit solutions of differential equations.

Let us discuss in detail the case of Kerr nonlinearity. The work [4] contains a wide range of details of third-order nonlinear electromagnetic TE and TM guided waves. Problems of surface wave propagation along the interface between two semi-infinite linear or/and nonlinear media were studied completely (see the results in [4]). At the same time we should notice that problems of wave propagation in a nonlinear layer that is located between two semi-infinite linear or/and nonlinear media are much more difficult than (and cannot be reduced to) the problems where surface waves are considered only at the interface between two semi-infinite linear or/and nonlinear media. Propagation of TE waves was more studied. The work [14] is devoted to the problem of electromagnetic wave propagation in a nonlinear dielectric layer with absorption and the case of Kerr nonlinearity is considered separately. One of the most interesting works about propagation of TE waves in a layered structure with Kerr nonlinearity is the paper [15]. Also the reader can see the work [16], where a layer with Kerr nonlinearity without absorption is considered.

The case of TM wave propagation in a nonlinear medium is more complicated. This is due to the fact that two components of the electric field make the analysis much harder [17].

In the work [18] a linear dielectric layer is considered. The layer is located between two half-spaces. The half-spaces are filled by nonlinear medium with Kerr nonlinearity. This problem for TE waves is solved analytically [19, 20]. For the TM case in [18] obtained DE is
an algebraic equation. It should be noticed that in [18] authors simplify the problem. Earlier in [21] the DE is obtained with other simplifying assumption (authors take into account only one component $E_{x}$ of the electric field). Later in [22] it is proved that the dominating nonlinear contribution in the permittivity is proportional to the transversal component $E_{z}$. In the works [11] propagation of TM waves in a nonlinear half-space with Kerr nonlinearity is considered. Formal solutions of differential equations in quadratures are obtained. In the paper [11] DEs are presented for isotropic and anisotropic media in a half-space with nonlinear permittivity. The DEs are rational functions with respect to the value of field's components at the interface. Authors found the first integral of the system of differential equations (so called a conservation law). This is also very interesting work to study, another way to simplify the problem pointed out in [23].

In the case of TE wave you can see the papers [24-26]. Propagation of TM wave in terms of the magnetic component is studied in [12, 13]. The paper [21] is devoted to the question (from physical standpoint) why it is possible to take into account only one component of the electric field in the expression for permittivity in the case of TM waves in a nonlinear layer. The results are compared with the case of TE waves.

The most important results about TM wave propagation in a layer with Kerr nonlinearity (system of differential equations, first integral) and a circle cylindrical waveguide (system of differential equations) were obtained in [6, 7]. In some papers (e.g., [12]) polarized wave propagation in a layer with arbitrary nonlinearity is considered. However, DEs were not obtained and no results about solvability of the boundary eigenvalue problem were obtained as well. The problem of TM wave propagation in a layer with Kerr nonlinearity is solved at first for a thin layer and then for a layer of arbitrary thickness [27-29]. Theorems of existence and localization of eigenvalues are proved in [30,31]. Some numerical results are shown in $[8,9]$.

In this paper the DE is an equation with additional conditions. Only for linear media (when permittivity is a constant) in a layer or in a circle cylindrical waveguide the DEs are sufficiently simple (but even for these cases the DEs are transcendental equations). For a nonlinear layer the DE is quite complicated nonlinear integral equation, where the integrand is defined by implicit algebraic function. It should be stressed that in spite of the fact that the DE is complicated it can be rather easily solved numerically.

This DE allows to study both nonlinear materials and nonlinear metamaterials. It should be noticed that in this paper materials with nonlinear permittivity and constant positive permeability are studied. But it is not difficult to take into account the sign of the permeability.

Problems of propagation of TE and TM waves in a nonlinear circle cylindrical waveguide are also close to the problem considered here. These problems are more complicated in comparison with corresponding problems in nonlinear layers. And even in the case of Kerr nonlinearity the results are not so complete as in the case in layers [30,32,33].

## 2. Statement of the Problem

Let us consider electromagnetic wave propagation through a homogeneous anisotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x<0$ and $x>h$ in Cartesian coordinate system Oxyz. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_{1} \geq \varepsilon_{0}$ and $\varepsilon_{3} \geq \varepsilon_{0}$, respectively, where $\varepsilon_{0}$ is the permittivity of free space. Assume that everywhere $\mu=\mu_{0}$ is the permeability of free space (see Figure 1).


Figure 1: The geometry of the problem.

It should be noticed that conditions $\varepsilon_{1} \geq \varepsilon_{0}, \varepsilon_{3} \geq \varepsilon_{0}$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

The electromagnetic field depends on time harmonically [6]:

$$
\begin{align*}
\tilde{\mathbf{E}}(x, y, z, t) & =\mathbf{E}_{+}(x, y, z) \cos \omega t+\mathbf{E}_{-}(x, y, z) \sin \omega t  \tag{2.1}\\
\tilde{\mathbf{H}}(x, y, z, t) & =\mathbf{H}_{+}(x, y, z) \cos \omega t+\mathbf{H}_{-}(x, y, z) \sin \omega t
\end{align*}
$$

where $\omega$ is the circular frequency; $\widetilde{\mathbf{E}}, \mathbf{E}_{+}, \mathbf{E}_{-}, \widetilde{\mathbf{H}}, \mathbf{H}_{+}, \mathbf{H}_{-}$are real functions. Everywhere below the time multipliers are omitted.

Form complex amplitudes of the electromagnetic field

$$
\begin{equation*}
\mathbf{E}=\mathbf{E}_{+}+i \mathbf{E}_{-}, \quad \mathbf{H}=\mathbf{H}_{+}+i \mathbf{H}_{-}, \tag{2.2}
\end{equation*}
$$

where $\mathbf{E}=\left(E_{x}, E_{y}, E_{z}\right)^{T}, \mathbf{H}=\left(H_{x}, H_{y}, H_{z}\right)^{T}$, and $(\cdot)^{T}$ denotes the operation of transposition, and each component of the fields is a function of three spatial variables.

Electromagnetic field ( $\mathbf{E}, \mathbf{H}$ ) satisfies the Maxwell equations

$$
\begin{equation*}
\operatorname{rot} \mathbf{H}=-i \omega \varepsilon \mathbf{E}, \quad \operatorname{rot} \mathbf{E}=i \omega \mu \mathbf{H} \tag{2.3}
\end{equation*}
$$

the continuity condition for the tangential field components on the media interfaces $x=0$, $x=h$, and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \rightarrow \infty$ in the domains $x<0$ and $x>h$.

The permittivity inside the layer is described by the diagonal tensor

$$
\widehat{\varepsilon}=\left(\begin{array}{ccc}
\varepsilon_{x x} & 0 & 0  \tag{2.4}\\
0 & \varepsilon_{y y} & 0 \\
0 & 0 & \varepsilon_{z z}
\end{array}\right)
$$

where

$$
\begin{equation*}
\varepsilon_{x x}=\varepsilon_{2 x}+b\left|E_{x}\right|^{2}+a\left|E_{z}\right|^{2}, \quad \varepsilon_{z z}=\varepsilon_{2 z}+a\left|E_{x}\right|^{2}+b\left|E_{z}\right|^{2} \tag{2.5}
\end{equation*}
$$

and $a, b, \varepsilon_{2}>\max \left(\varepsilon_{1}, \varepsilon_{3}\right)$ are positive constants (below the solutions are sought under more general conditions). It does not matter what a form $\varepsilon_{y y}$ has. Since $\varepsilon_{y y}$ is not contained in the equations below for the TM case, it should be noticed that $\widehat{\varepsilon}$ describes tensor Kerr nonlinearity. When $a=b$ we obtain scalar Kerr nonlinearity. Moreover, chosen nonlinearity satisfies
the condition $\partial \varepsilon x x / \partial E_{z}^{2}=\partial \varepsilon z z / \partial E_{x}^{2}$. This equation is satisfied by almost every known nonlinear Kerr mechanism, such as electronic distortion, molecular orientation, electrostriction, and Kerr nonlinearities described within the uniaxial approximation mentioned in the paper [11]. The case when $\varepsilon_{2 x}=\varepsilon_{2 z}$ is studied in [8]. Pay heed to the fact that the problem considered here is not studied in [31].

The solutions to the Maxwell equations are sought in the entire space.

## 3. TM Waves

Let us consider TM waves

$$
\begin{equation*}
\mathbf{E}=\left(E_{x}, 0, E_{z}\right)^{T}, \quad \mathbf{H}=\left(0, H_{y}, 0\right)^{T} \tag{3.1}
\end{equation*}
$$

and $E_{x}, E_{z}, H_{y}$ are functions of three spatial variables. It is easy to show that the components of the fields do not depend on $y$. Waves propagating along medium interface $z$ depend on $z$ harmonically. This means that the fields components have the form

$$
\begin{equation*}
E_{x}=E_{x}(x) e^{i \gamma z}, \quad E_{z}=E_{z}(x) e^{i \gamma z}, \quad H_{y}=H_{y}(x) e^{i \gamma z} \tag{3.2}
\end{equation*}
$$

where $\gamma$ is the spectral parameter of the problem.
So we obtain from system (2.3) [6]

$$
\begin{align*}
i \gamma E_{x}(x)-E_{z}^{\prime}(x) & =i \omega \mu H_{y}(x) \\
H_{y}^{\prime}(x) & =-i \omega \varepsilon_{z z} E_{z}(x)  \tag{3.3}\\
i \gamma H_{y}(x) & =i \omega \varepsilon_{x x} E_{x}(x)
\end{align*}
$$

where ( $\cdot)^{\prime} \equiv d / d x$.
The following equation can be easily derived from the previous system:

$$
\begin{equation*}
H_{y}(x)=\frac{1}{i \omega \mu}\left(i \gamma E_{x}(x)-E_{z}^{\prime}(x)\right) \tag{3.4}
\end{equation*}
$$

Differentiating (3.4) and using the second and the third equations of system (3.3) we obtain

$$
\begin{gather*}
\gamma\left(i E_{x}(x)\right)^{\prime}-E_{z}^{\prime \prime}(x)=\omega^{2} \mu \varepsilon_{z z} E_{z}(x), \\
\gamma^{2}\left(i E_{x}(x)\right)-\gamma E_{z}^{\prime}(x)=\omega^{2} \mu \varepsilon_{x x}\left(i E_{x}(x)\right) \tag{3.5}
\end{gather*}
$$

Let us denote $k_{0}^{2}:=\omega^{2} \mu_{0} \varepsilon_{0}$ and perform the normalization according to the formulas $\tilde{x}=k_{0} x, d / d x=k_{0}(d / d \tilde{x}), \tilde{\gamma}=\gamma / k_{0}, \tilde{\varepsilon}_{j}=\varepsilon_{j} / \varepsilon_{0}(j=1,2,3), \tilde{a}=a / \varepsilon_{0}, \tilde{b}=b / \varepsilon_{0}$. Denoting by
$Z(\tilde{x}):=E_{z}, X(\tilde{x}):=i E_{x}$ and omitting the tilde symbol, from system (3.5) we obtain

$$
\begin{gather*}
-Z^{\prime \prime}+\gamma X^{\prime}=\varepsilon_{z z} Z \\
-Z^{\prime}+\gamma X=\gamma^{-1} \varepsilon_{x x} X \tag{3.6}
\end{gather*}
$$

It is necessary to find eigenvalues $\gamma$ of the problem that correspond to surface waves propagating along boundaries of the layer $0<x<h$. We seek the real values of the spectral parameter $\gamma$ such that real solutions $X(x)$ and $Z(x)$ to system (3.6) exist. Indeed, in this case $|\mathbf{E}|^{2}$ does not depend on $z$. Since $\mathbf{E}=\left(E_{x}(x) e^{i \gamma z}, 0, E_{z}(x) e^{i \gamma z}\right)=e^{i \gamma z}\left(E_{x}, 0, E_{z}\right)$, therefore, $|\mathbf{E}|=$ $\left|e^{i \gamma z}\right| \cdot \sqrt{\left|E_{x}\right|^{2}+\left|E_{z}\right|^{2}}$. It is known that $\left|e^{i \gamma z}\right|=1$ as $\operatorname{Im} \gamma=0$. Let $\gamma=\gamma^{\prime}+i \gamma^{\prime \prime}$. Then, we obtain $\left|e^{i \gamma z}\right|=\left|e^{i \gamma^{\prime} z}\right| \cdot\left|e^{-\gamma^{\prime \prime} z}\right|=e^{-\gamma^{\prime \prime} z}$. If $\gamma^{\prime \prime} \neq 0$, then $e^{-\gamma^{\prime \prime} z}$ is a function on $z$. In this case the components $E_{x}$ and $E_{z}$ depend on $z$, but it contradicts to the choice of $E_{x}(x)$ and $E_{z}(x)$. So we can consider only real values of $\gamma$.

We consider that

$$
\varepsilon= \begin{cases}\varepsilon_{1}, & x<0  \tag{3.7}\\ \widehat{\varepsilon}, & 0<x<h \\ \varepsilon_{3}, & x>h\end{cases}
$$

Also we assume that $\max \left(\varepsilon_{1}, \varepsilon_{3}\right)<\gamma^{2}<\min \left(\varepsilon_{2 x}, \varepsilon_{2 z}\right)$. This two-sided inequality naturally appears for the problem in a layer with a constant permittivity tensor.

Functions $X, Z$ are supposed to be sufficiently smooth due to physical nature of the problem

$$
\begin{gather*}
X(x) \in C(-\infty, 0] \cap C[0, h] \cap C[h,+\infty) \cap C^{1}(-\infty, 0] \cap C^{1}[0, h] \cap C^{1}[h,+\infty) \\
Z(x) \in C(-\infty,+\infty) \cap C^{1}(-\infty, 0] \cap C^{1}[0, h] \cap C^{1}[h,+\infty) \cap C^{2}(-\infty, 0) \cap C^{2}(0, h) \cap C^{2}(h,+\infty) \tag{3.8}
\end{gather*}
$$

It is clear that system (3.6) is an autonomous one. System (3.6) can be rewritten in a normal form. This system in the normal form can be considered as a dynamical system with analytical with respect to $X$ and $Z$ right-hand sides. Of course, in the domain where these right-hand sides are analytical with respect to $X$ and $Z$, it is well known (see, e.g., [34]) that the solutions $X$ and $Z$ of such a system are analytical functions with respect to the independent variable as well. This is an important fact for DEs' derivation.

We consider that $\gamma^{2}>\max \left(\varepsilon_{1}, \varepsilon_{3}\right)$.

## 4. Differential Equations of the Problem

In the domain $x<0$ we have $\varepsilon=\varepsilon_{1}$. From system (3.6) we obtain $X^{\prime}=\gamma Z, Z^{\prime}=\gamma^{-1}\left(\gamma^{2}-\varepsilon_{1}\right) X$. In accordance with the radiation condition we obtain

$$
\begin{gather*}
X(x)=A e^{x \sqrt{r^{2}-\varepsilon_{1}}} \\
Z(x)=-A \gamma^{-1} \sqrt{\gamma^{2}-\varepsilon_{1}} e^{x \sqrt{r^{2}-\varepsilon_{1}}} \tag{4.1}
\end{gather*}
$$

We assume that $\gamma^{2}-\varepsilon_{1}>0$; otherwise it will be impossible to satisfy the radiation condition.

In the domain $x>h$ we have $\varepsilon=\varepsilon_{3}$. From system (3.6) we obtain $X^{\prime}=\gamma Z, Z^{\prime}=$ $\gamma^{-1}\left(\gamma^{2}-\varepsilon_{3}\right) X$. In accordance with the radiation condition we obtain

$$
\begin{gather*}
X(x)=B e^{-(x-h) \sqrt{\gamma^{2}-\varepsilon_{3}}} \\
Z(x)=-B \gamma^{-1} \sqrt{\gamma^{2}-\varepsilon_{3}} e^{-(x-h) \sqrt{\gamma^{2}-\varepsilon_{3}}} \tag{4.2}
\end{gather*}
$$

Here for the same reason as above we consider that $\gamma^{2}-\varepsilon_{3}>0$.
Constants $A$ and $B$ in (4.1) and (4.2) are defined by transmission conditions and initial conditions.

Inside the layer $0<x<h$ system (3.6) takes the form

$$
\begin{align*}
& -\frac{d^{2} Z}{d x^{2}}+r \frac{d X}{d x}=\left[\varepsilon_{2 z}+a X^{2}+b Z^{2}\right] Z  \tag{4.3}\\
& -\frac{d Z}{d x}+r X=r^{-1}\left[\varepsilon_{2 x}+b X^{2}+a Z^{2}\right] X
\end{align*}
$$

Differentiating the second equation and substituting its right-hand side instead of lefthand side into the first equation we can rewritten system (4.3) in the following form:

$$
\begin{gather*}
\frac{d X}{d x}=\frac{2 a}{r} \frac{\varepsilon_{2 x}-r^{2}+b X^{2}+a Z^{2}}{\varepsilon_{2 x}+3 b X^{2}+a Z^{2}} X^{2} Z+r \frac{\varepsilon_{2 z}+a X^{2}+b Z^{2}}{\varepsilon_{2 x}+3 b X^{2}+a Z^{2}} Z  \tag{4.4}\\
\frac{d Z}{d x}=-r^{-1}\left[\varepsilon_{2 x}-r^{2}+b X^{2}+a Z^{2}\right] X
\end{gather*}
$$

Now system (4.4) is written in a normal form. If the right-hand sides are analytic functions with respect to $X$ and $Z$, then the solutions are analytic functions with respect to its independent variable.

Dividing the first equation in system (4.4) to the second one we obtain the ordinary differential equation

$$
\begin{equation*}
-\left(\varepsilon_{2 x}+3 b X^{2}+a Z^{2}\right) \frac{d X}{d Z}=2 a X Z+r^{2} \frac{\varepsilon_{2 z}+a X^{2}+b Z^{2}}{\varepsilon_{2 x}-\gamma^{2}+b X^{2}+a Z^{2}} \frac{Z}{X} \tag{4.5}
\end{equation*}
$$

Equation (4.5) can be transformed into a total differential equation.

Its solution (first integral of system (4.4)) can be easily found and be written in the following form:

$$
\begin{align*}
C= & b^{2} X^{6}+2 a b X^{4} Z^{2}+a^{2} X^{2} Z^{4}+\frac{1}{2}\left(4 \varepsilon_{2 x}-3 \gamma^{2}\right) b X^{4}+\left(2 \varepsilon_{2 x}-r^{2}\right) a X^{2} Z^{2}+\frac{1}{2} \gamma^{2} b Z^{4}  \tag{4.6}\\
& +\gamma^{2}\left(\varepsilon_{2 x}-\gamma^{2}\right) X^{2}+\left(\varepsilon_{2 x}-\gamma^{2}\right)^{2} X^{2}+\gamma^{2} \varepsilon_{2 z} Z^{2}
\end{align*}
$$

where $C$ is a constant of integration.

## 5. Transmission Conditions and the Transmission Problem

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are $H_{y}$ and $E_{z}$. Hence, we obtain

$$
\begin{align*}
H_{y}(h+0)=H_{y}(h-0), & H_{y}(0-0)=H_{y}(0+0)  \tag{5.1}\\
E_{z}(h+0)=E_{z}(h-0), & E_{z}(0-0)=E_{z}(0+0)
\end{align*}
$$

From the continuity conditions for the tangential components of the fields $\mathbf{E}$ and $\mathbf{H}$ and using (3.4) we obtain

$$
\begin{align*}
\gamma X(h)-Z^{\prime}(h)=H_{y}^{(h)}, & \gamma X(0)-Z^{\prime}(0)=H_{y}^{(0)} \\
Z(h)=E_{z}(h+0)=E_{z}^{(h)}, & Z(0)=E_{z}(0-0)=E_{z}^{(0)} \tag{5.2}
\end{align*}
$$

where $H_{y}^{(h)}:=i\left(\sqrt{\mu} / \sqrt{\varepsilon_{0}}\right) H_{y}(h+0), H_{y}^{(0)}:=i\left(\sqrt{\mu} / \sqrt{\varepsilon_{0}}\right) H_{y}(0-0)$.
The constant $E_{z}^{(h)}:=E_{z}(h+0)$ is supposed to be known (initial condition). Let us denote $X_{0}:=X(0), X_{h}:=X(h), Z_{0}:=Z(0)$, and $Z_{h}:=Z(h)$. So we obtain that $A=\left(\gamma / \sqrt{\gamma^{2}-\varepsilon_{1}}\right) Z_{0}$, $B=\left(\gamma / \sqrt{\gamma^{2}-\varepsilon_{3}}\right) Z_{h}$.

Then from conditions (5.2) we obtain

$$
\begin{equation*}
H_{y}^{(h)}=-Z_{h} \frac{\varepsilon_{3}}{\sqrt{\gamma^{2}-\varepsilon_{3}}}, \quad H_{y}^{(0)}=Z_{0} \frac{\varepsilon_{1}}{\sqrt{r^{2}-\varepsilon_{1}}} \tag{5.3}
\end{equation*}
$$

In accordance with (3.6), (3.7) inside the layer

$$
\begin{equation*}
-Z^{\prime}(x)+\gamma X(x)=\gamma^{-1}\left(\varepsilon_{2 x}+b X^{2}(x)+a Z^{2}(x)\right) X(x) \tag{5.4}
\end{equation*}
$$

Then for $x=h$, using (5.2), we obtain from (5.4)

$$
\begin{equation*}
r^{-1}\left[\varepsilon_{2 x}+b X_{h}^{2}+a Z_{h}^{2}\right] X_{h}=H_{y}^{(h)} \tag{5.5}
\end{equation*}
$$

From (5.5) we obtain the equation with respect to $X_{h}$ :

$$
\begin{equation*}
X_{h}^{3}+b^{-1}\left(\varepsilon_{2 x}+a Z_{h}^{2}\right) X_{h}-b^{-1} \gamma H_{y}^{(h)}=0 \tag{5.6}
\end{equation*}
$$

Under taken assumptions (in regard to $\varepsilon_{2}$ and $a$ ) the value $a^{-1}\left(\varepsilon_{2}+a Z_{h}^{2}\right)>0$. Hence, this equation has at least one real root, which is considered (the root can be find explicitly by using Cardanus-Ferrari formula [35].

Using first integral (4.6) at $x=h$, we find the value $C_{h}^{X}:=\left.C\right|_{x=h}$ from the equation

$$
\begin{align*}
C_{h}^{X}= & b^{2} X_{h}^{6}+2 a b X_{h}^{4} Z_{h}^{2}+a^{2} X_{h}^{2} Z_{h}^{4}+2^{-1}\left(4 \varepsilon_{2 x}-3 \gamma^{2}\right) b X_{h}^{4}+\left(2 \varepsilon_{2 x}-r^{2}\right) a X_{h}^{2} Z_{h}^{2}+2^{-1} r^{2} b Z_{h}^{4} \\
& +r^{2}\left(\varepsilon_{2 x}-\gamma^{2}\right) X_{h}^{2}+\left(\varepsilon_{2 x}-r^{2}\right)^{2} X_{h}^{2}+\gamma^{2} \varepsilon_{2 z} Z_{h}^{2} \tag{5.7}
\end{align*}
$$

In order to find the values $X_{0}$ and $Z_{0}$ it is necessary to solve the following system (this system is obtained using formula (5.4) at $x=0$ and the first integral at the same point):

$$
\begin{align*}
\gamma \varepsilon_{1} Z_{0}= & \sqrt{\gamma^{2}-\varepsilon_{1}}\left[\varepsilon_{2 x}+b X_{0}^{2}+a Z_{0}^{2}\right] X_{0} \\
C_{h}^{X}= & b^{2} X_{0}^{6}+2 a b X_{0}^{4} Z_{0}^{2}+a^{2} X_{0}^{2} Z_{0}^{4}+2^{-1}\left(4 \varepsilon_{2 x}-3 \gamma^{2}\right) b X_{0}^{4}  \tag{5.8}\\
& +\left(2 \varepsilon_{2 x}-r^{2}\right) a X_{0}^{2} Z_{0}^{2}+2^{-1} r^{2} b Z_{0}^{4}+r^{2}\left(\varepsilon_{2 x}-r^{2}\right) X_{0}^{2}+\left(\varepsilon_{2 x}-r^{2}\right)^{2} X_{0}^{2}+r^{2} \varepsilon_{2 z} Z_{0}^{2}
\end{align*}
$$

It is easy to see from the second equation of this system that the values $X_{0}$ and $Z_{0}$ can have arbitrary signs. At the same time from the first equation of this system we can see that $X_{0}$ and $Z_{0}$ must be positive or negative simultaneously.

Normal components of electromagnetic field are known to be discontinues at media interfaces. And it is the discontinuity of the first kind. In this case the normal component is $E_{x}$. It is also known that the value $\varepsilon E_{x}$ is continuous at media interfaces. From the above and from the continuity of the tangential component $E_{z}$ it follows that the transmission conditions for the functions $\varepsilon X$ and $Z$ are

$$
\begin{equation*}
[\varepsilon X]_{x=0}=0, \quad[\varepsilon X]_{x=h}=0, \quad[Z]_{x=0}=0, \quad[Z]_{x=h}=0 \tag{5.9}
\end{equation*}
$$

where $[f]_{x=x_{0}}=\lim _{x \rightarrow x_{0}-0} f(x)-\lim _{x \rightarrow x_{0}+0} f(x)$ denotes a jump of the function $f$ at the interface.

We also suppose that functions $X(x)$ and $Z(x)$ satisfy the condition

$$
\begin{equation*}
X(x)=O\left(\frac{1}{|x|}\right), \quad Z(x)=O\left(\frac{1}{|x|}\right) \quad \text { as }|x| \longrightarrow \infty \tag{5.10}
\end{equation*}
$$

## 6. Dispersion Equation

Introduce the new variables

$$
\begin{equation*}
\tau(x)=\frac{\varepsilon_{2 x}+b X^{2}(x)+a Z^{2}(x)}{r^{2}}, \quad \eta(x)=\gamma \frac{X(x)}{Z(x)} \tau(x) \tag{6.1}
\end{equation*}
$$

Using new variables rewrite system (4.4),

$$
\begin{align*}
& \frac{d \tau}{d x}=2 \frac{\tau \eta\left(\gamma^{2} \tau-\varepsilon_{2 x}\right)}{b \eta^{2}+a \gamma^{2} \tau^{2}} \times \frac{\left(b \eta^{2}+a \gamma^{2} \tau^{2}\right)\left(b \varepsilon_{2 z}-a \gamma^{2} \tau(\tau-1)\right)+\left(a \eta^{2}+b \gamma^{2} \tau^{2}\right)\left(\gamma^{2} \tau-\varepsilon_{2 x}\right)}{\gamma^{2} \tau\left(b \eta^{2}+a \gamma^{2} \tau^{2}\right)+2 b \eta^{2}\left(\gamma^{2} \tau-\varepsilon_{2 x}\right)} \\
& \frac{d \eta}{d x}=\frac{\tau-1}{\tau} \eta^{2}+\varepsilon_{2 z}+\left(\gamma^{2} \tau-\varepsilon_{2 x}\right) \frac{a \eta^{2}+b \gamma^{2} \tau^{2}}{b \eta^{2}+a \gamma^{2} \tau^{2}} \tag{6.2}
\end{align*}
$$

and (4.6)

$$
\begin{gather*}
\frac{r^{2} \tau-\varepsilon_{2 x}}{b \eta^{2}+a \gamma^{2} \tau^{2}}\left[\eta^{2}\left(\left(r^{2} \tau-\varepsilon_{2 x}\right)^{2}+\varepsilon_{2 x}\left(\varepsilon_{2 x}-\gamma^{2}\right)\right)+r^{4} \varepsilon_{2 z} \tau^{2}\right]+\frac{\left(\gamma^{2} \tau-\varepsilon_{2 x}\right)^{2}}{2\left(b \eta^{2}+a \gamma^{2} \tau^{2}\right)^{2}}  \tag{6.3}\\
\times\left[\left(4 \varepsilon_{2 x}-3 r^{2}\right) b \eta^{4}+2\left(2 \varepsilon_{2 x}-\gamma^{2}\right) a r^{2} \tau^{2} \eta^{2}+r^{6} b \tau^{4}\right]=C
\end{gather*}
$$

where constant $C$ is equal to the constant $C$ in (4.6).
In order to obtain the DE for the propagation constants it is necessary to find the values $\eta(0), \eta(h)$.

It is clear that $\eta(0)=\gamma(X(0) / Z(0)) \tau(0), \eta(h)=\gamma(X(h) / Z(h)) \tau(h)$. Taking into account that $\gamma^{2} X(x) \tau(x)=\varepsilon X(x)$ and using formulas (5.2), (5.3), it is easy to obtain that

$$
\begin{equation*}
\eta(0)=\frac{\varepsilon_{1}}{\sqrt{\gamma^{2}-\varepsilon_{1}}}>0, \quad \eta(h)=-\frac{\varepsilon_{3}}{\sqrt{\gamma^{2}-\varepsilon_{3}}}<0 . \tag{6.4}
\end{equation*}
$$

It is easy to see that the right-hand side of the second equation of system (VI) is strictly positive. This means that the function $\eta(x)$ monotonically increases on interval ( $0, h$ ). Taking into account (6.4) we obtain that the function $\eta(x)$ cannot be differentiable on the entire interval $(0, h)$. This means that the function $\eta(x)$ has a break point. Let $x^{*} \in(0, h)$ be the break point. From (6.3) it is obvious that $x^{*}$ is such that $\tau^{*}=\tau\left(x^{*}\right)$ is a root of the equation $C_{h}^{\tau}+3\left(\tau^{*}\right)^{2}-2\left(\tau^{*}\right)^{3}-2 \tau_{0}\left(2-\tau^{*}\right) \tau^{*}=0$. In addition $\eta\left(x^{*}-0\right) \rightarrow+\infty$ and $\eta\left(x^{*}+0\right) \rightarrow-\infty$.

It is natural to suppose that the function $\eta(x)$ on interval $(0, h)$ has several break points $x_{0}, x_{1}, \ldots, x_{N}$. The properties of function $\eta(x)$ imply

$$
\begin{equation*}
\eta\left(x_{i}-0\right)=+\infty, \quad \eta\left(x_{i}+0\right)=-\infty, \quad \text { where } i=\overline{0, N} \tag{6.5}
\end{equation*}
$$

Let

$$
\begin{equation*}
\frac{1}{w}:=\frac{\tau-1}{\tau} \eta^{2}+\varepsilon_{2 z}+\left(\gamma^{2} \tau-\varepsilon_{2 x}\right) \frac{a \eta^{2}+b \gamma^{2} \tau^{2}}{b \eta^{2}+a \gamma^{2} \tau^{2}} \tag{6.6}
\end{equation*}
$$

where $w=w(\eta) ; \tau=\tau(\eta)$ is expressed from (5.4).
Taking into account our hypothesis we will seek the solutions on each interval $\left[0, x_{0}\right),\left(x_{0}, x_{1}\right), \ldots,\left(x_{N}, h\right]:$

$$
\begin{gather*}
-\int_{\eta(x)}^{\eta\left(x_{0}\right)} w d \eta=x+c_{0} \\
\quad \int_{\eta\left(x_{i}\right)}^{\eta(x)} w d \eta=x+c_{i}  \tag{6.7}\\
\quad \int_{\eta\left(x_{N}\right)}^{\eta(x)} w d \eta=x+c_{N}
\end{gather*}
$$

where $0 \leq x \leq x_{0}, x_{i} \leq x \leq x_{i+1}$, and $x_{N} \leq x \leq h$, respectively, and $i=\overline{0, N-1}$.
Substituting $x=0, x=x_{i+1}$, and $x=x_{N}$ into equations in (6.7) (into the first, the second, and the third, resp., ) and taking into account (6.5), we find constants $c_{1}, c_{2}, \ldots, c_{N+1}$ :

$$
\begin{align*}
c_{0} & =-\int_{\eta(0)}^{+\infty} w d \eta, \\
c_{i+1} & =\int_{-\infty}^{+\infty} w d \eta-x_{i+1},  \tag{6.8}\\
c_{N+1} & =\int_{-\infty}^{\eta(h)} w d \eta-h,
\end{align*}
$$

where $i=\overline{0, N-1}$.
Using (6.8) we can rewrite (6.7) in the following form:

$$
\begin{align*}
& \int_{\eta(x)}^{\eta\left(x_{0}\right)} w d \eta=-x+\int_{\eta(0)}^{+\infty} w d \eta \\
& \int_{\eta\left(x_{i}\right)}^{\eta(x)} w d \eta=x+\int_{-\infty}^{+\infty} w d \eta-x_{i+1}  \tag{6.9}\\
& \int_{\eta\left(x_{N}\right)}^{\eta(x)} w d \eta=x+\int_{-\infty}^{\eta(h)} w d \eta-h
\end{align*}
$$

where $0 \leq x \leq x_{0}, x_{i} \leq x \leq x_{i+1}, x_{N} \leq x \leq h$, respectively, and $i=\overline{0, N-1}$.
Introduce the notation $T:=\int_{-\infty}^{+\infty} w d \eta$. It follows from formula (6.9) that $0<x_{i+1}-x_{i}=$ $T<h$, where $i=\overline{0, N-1}$. This implies the convergence of the improper integral (it will be
proved in other way below). Now consider $x$ in (6.9) such that all the integrals on the left side vanish (i.e., $x=x_{0}, x=x_{i}$, and $x=x_{N}$ ), and sum all equations in (6.9). We obtain

$$
\begin{equation*}
0=-x_{0}+\int_{\eta(0)}^{+\infty} w d \eta+x_{0}+T-x_{1}+\cdots+x_{N-1}+T-x_{N}+x_{N}+\int_{-\infty}^{\eta(h)} w d \eta-h \tag{6.10}
\end{equation*}
$$

Finally we obtain

$$
\begin{equation*}
-\int_{\eta(h)}^{\eta(0)} w d \eta+(N+1) T=h \tag{6.11}
\end{equation*}
$$

where $\eta(0), \eta(h)$ are defined by formulas (6.4).
Expression (6.11) is the DE, which holds for any finite $h$. Let $\gamma$ be a solution of DE (6.11) and an eigenvalue of the problem. Then, there are eigenfunctions $X$ and $Z$, which correspond to the eigenvalue $\gamma$. The eigenfunction $Z$ has $N+1$ zeros on the interval $(0, h)$.

Notice that improper integrals in DE (6.11) converge. Indeed, function $\tau=\tau(\eta)$ is bounded as $\eta \rightarrow \infty$ since $\tau=\gamma^{-2}\left(\varepsilon_{2 x}+b X^{2}+a Z^{2}\right)$, and $X, Z$ are bounded.

Then

$$
\begin{equation*}
|w| \leq \frac{1}{\alpha \eta^{2}+\beta^{\prime}} \tag{6.12}
\end{equation*}
$$

where $\alpha>0, \beta>0$ are constants. It is obvious that improper integral $\int_{-\infty}^{+\infty}\left(d \eta /\left(\alpha \eta^{2}+\beta\right)\right)$ converges. Convergence of the improper integrals in (6.11) in inner points results from the requirement that the right-hand side of the second equation of system (VI) is positive.

The first equation of system (VI) jointly with the first integral can be integrated in hyperelliptic functions. The solution is expressed in implicit form by means of hyperelliptic integrals. This is the simple example of Abelian integrals. The inversion of these integrals is hyperelliptic functions and they are solutions of system (VI). Hyperelliptic functions are Abelian functions, which are meromorphic and periodic functions. Since function $\eta$ is expressed algebraically through $\tau$, therefore, $\eta$ is a meromorphic periodic function. This means that the break point $x^{*}$ is a pole of function $\eta$.

## 7. Generalized Dispersion Equation

Here we derive the generalized DE , which holds for any real values $\varepsilon_{2}$. In addition the sign of the right-hand side of the second equation in system (VI) and condition $\max \left(\varepsilon_{1}, \varepsilon_{3}\right)<\gamma^{2}<\varepsilon_{2}$ are not taken into account. These conditions appear in the case of a linear layer and are used for derivation of $D E$ (6.11). Though on the nonlinear case it is not necessary to limit the value $r^{2}$ from the right side, at the same time it is clear that $\gamma$ is limited from the left side, since this limit appears from the solutions in the half-spaces.

Now we assume that $\gamma$ satisfies the following two-sided inequality:

$$
\begin{equation*}
\max \left(\varepsilon_{1}, \varepsilon_{3}\right)<\gamma^{2}<+\infty \tag{7.1}
\end{equation*}
$$

Using first integral (6.3) it is possible to integrate formally any of the equations of system (VI). As earlier we integrate the second equation, we cannot obtain the solution on the entire interval $(0, h)$, since function $\eta(x)$ can have break points, which belong to $(0, h)$. It is known that function $\eta(x)$ has break points only of the second kind ( $\eta$ is an analytical function).

Assume that function $\eta(x)$ on interval $(0, h)$ has $N+1$ break points $x_{0}, \quad x_{1}, \ldots, x_{N}$. It should be noticed that

$$
\begin{equation*}
\eta\left(x_{i}-0\right)= \pm \infty \quad \eta\left(x_{i}+0\right)= \pm \infty, \tag{7.2}
\end{equation*}
$$

where $i=\overline{0, N}$, and signs $\pm$ are independent and unknown.
Taking into account the previous, solutions are sought on each interval $\left[0, x_{0}\right),\left(x_{0}, x_{1}\right), \ldots,\left(x_{N}, h\right]:$

$$
\begin{align*}
- & \int_{\eta(x)}^{\eta\left(x_{0}-0\right)} w d \eta=x+c_{0} \\
& \int_{\eta\left(x_{i}+0\right)}^{\eta(x)} w d \eta=x+c_{i+1}  \tag{7.3}\\
& \int_{\eta\left(x_{N}+0\right)}^{\eta(x)} w d \eta=x+c_{N+1}
\end{align*}
$$

where $0 \leq x \leq x_{0}, x_{i} \leq x \leq x_{i+1}$, and $x_{N} \leq x \leq h$, respectively, and $i=\overline{0, N-1}$.
From (7.3), substituting $x=0, x=x_{i+1}$, and $x=x_{N}$ into the first, the second, and the third equations in (7.3), respectively, we find required constants $c_{1}, c_{2}, \ldots, c_{N+1}$ :

$$
\begin{align*}
c_{0} & =-\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta \\
c_{i+1} & =\int_{\eta\left(x_{i}+0\right)}^{\eta\left(x_{i+1}-0\right)} w d \eta-x_{i+1},  \tag{7.4}\\
c_{N+1} & =\int_{\eta\left(x_{N}+0\right)}^{\eta(h)} w d \eta-h,
\end{align*}
$$

where $i=\overline{0, N-1}$.

Using (7.4), (7.3) take the form

$$
\begin{align*}
& \int_{\eta(x)}^{\eta\left(x_{0}-0\right)} w d \eta=-x+\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta \\
& \int_{\eta\left(x_{i}+0\right)}^{\eta(x)} w d \eta=x+\int_{\eta\left(x_{i}+0\right)}^{\eta\left(x_{i+1}-0\right)} w d \eta-x_{i+1}  \tag{7.5}\\
& \int_{\eta\left(x_{N}+0\right)}^{\eta(x)} w d \eta=x+\int_{\eta\left(x_{N}+0\right)}^{\eta(h)} w d \eta-h
\end{align*}
$$

where $0 \leq x \leq x_{0}, x_{i} \leq x \leq x_{i+1}$, and $x_{N} \leq x \leq h$, respectively, and $i=\overline{0, N-1}$.
From formulas (7.5) we obtain that

$$
\begin{equation*}
x_{i+1}-x_{i}=\int_{\eta\left(x_{i}+0\right)}^{\eta\left(x_{i+1}-0\right)} w d \eta, \quad i=\overline{0, N-1} \tag{7.6}
\end{equation*}
$$

Expressions $0<x_{i+1}-x_{i}<h<\infty$ imply that under the assumption about the break point existence the integral on the right side converges and $\int_{\eta\left(x_{i}+0\right)}^{\eta\left(x_{i+1}-0\right)} w d \eta>0$. In the same way, from the first and the last equations of (7.5) we obtain that $x_{0}=\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta$ and $0<x_{0}<h$ then

$$
\begin{equation*}
0<\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta<h<\infty \tag{7.7}
\end{equation*}
$$

and $h-x_{N}=\int_{\eta\left(x_{N}+0\right)}^{\eta(h)} w d \eta$ and $0<h-x_{N}<h$ then

$$
\begin{equation*}
0<\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta<h<\infty \tag{7.8}
\end{equation*}
$$

These considerations yield that the function $w(\eta)$ has no nonintegrable singularities for $\eta \in(-\infty, \infty)$. And also this proves that the assumption about finite number break points is true.

Now, setting $x=x_{0}, x=x_{i}$, and $x=x_{N}$ into the first, the second, and the third equations in (7.5), respectively, we have that all the integrals on the left sides vanish. We add all the equations in (7.5) to obtain

$$
\begin{align*}
0= & -x_{0}+\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta+x_{0}+\int_{\eta\left(x_{0}+0\right)}^{\eta\left(x_{1}-0\right)} w d \eta-x_{1}+\cdots+x_{N-1}  \tag{7.9}\\
& +\int_{\eta\left(x_{N-1}+0\right)}^{\eta\left(x_{N}-0\right)} w d \eta-x_{N}+x_{N}+\int_{\eta\left(x_{N}+0\right)}^{\eta(h)} w d \eta-h .
\end{align*}
$$

From (7.9) we obtain

$$
\begin{equation*}
\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta+\int_{\eta\left(x_{N}+0\right)}^{\eta(h)} w d \eta+\sum_{i=0}^{N-1} \int_{\eta\left(x_{i}+0\right)}^{\eta\left(x_{i+1}-0\right)} w d \eta=h . \tag{7.10}
\end{equation*}
$$

It follows from formulas (7.6) that

$$
\begin{equation*}
\eta\left(x_{i}+0\right)= \pm \infty, \quad \eta\left(x_{i}-0\right)=\mp \infty, \quad \text { where } i=\overline{0, N} \tag{7.11}
\end{equation*}
$$

and it is necessary to choose the infinities of different signs.
Thus we obtain that

$$
\begin{equation*}
\int_{\eta\left(x_{0}+0\right)}^{\eta\left(x_{1}-0\right)} w d \eta=\cdots=\int_{\eta\left(x_{N-1}+0\right)}^{\eta\left(x_{N}-0\right)} w d \eta=: T^{\prime} . \tag{7.12}
\end{equation*}
$$

Hence $x_{1}-x_{0}=\cdots=x_{N}-x_{N-1}$.
Now we can rewrite (7.10) in the following form:

$$
\begin{equation*}
\int_{\eta(0)}^{\eta\left(x_{0}-0\right)} w d \eta+\int_{\eta\left(x_{N}+0\right)}^{\eta(h)} f d \eta+N T^{\prime}=h \tag{7.13}
\end{equation*}
$$

Let $T \equiv \int_{-\infty}^{+\infty} w d \eta$; then we finally obtain

$$
\begin{equation*}
-\int_{\eta(h)}^{\eta(0)} w d \eta \pm(N+1) T=h \tag{7.14}
\end{equation*}
$$

where $\eta(0), \eta(h)$ are defined by formulas (6.4).
Expression (7.14) is the DE, which holds for any finite $h$. Let $\gamma$ be a solution of DE (7.14) and an eigenvalue of the problem. Then, there are eigenfunctions $X$ and $Z$, which correspond to the eigenvalue $\gamma$. The eigenfunction $Z$ has $N+1$ zeros on the interval $(0, h)$. It should be noticed that for every number $N+1$ it is necessary to solve two DEs: for $N+1$ and for $-(N+1)$.

Note 1. If there is a certain value $\gamma_{*}^{2}$, such that some of the integrals in DEs (6.11) or (7.14) diverge at certain inner points this simply means that the value $\gamma_{*}^{2}$ is not a solution of chosen DE and the value $\gamma_{*}^{2}$ is not an eigenvalue of the problem.

Note 2. It is necessary to emphasize that this boundary eigenvalue (transmission) problem essentially depends on the initial condition $Z_{h}$. The transmission problem for a linear layer does not depend on the initial condition. If the nonlinearity function is a specific one, then in some cases it will be possible to normalize the Maxwell equations in such a way that the transmission problem does not depend on initial condition $Z_{h}$ explicitly (it is possible e.g., for Kerr nonlinearity in a layer and in a circle cylindrical waveguide). Once more we stress the fact that the opportunity of such normalization is an exceptional case. What is more, in spite of the fact that in certain cases this normalization is possible it does not mean that the normalized transmission problem is independent of the initial condition. In this case one of the problem's parameter depends on the initial condition.

## 8. Passage to the Limit in the Generalized Dispersion Equation

In this section we assume that $\varepsilon_{2 x}=\varepsilon_{2 z}=\varepsilon_{2}$ and $b=a$. Now consider the passage to the limit as $a \rightarrow 0$. The value $a=0$ corresponds to the case of a linear medium in the layer. Two cases are possible:
(a) $\varepsilon_{2}>0$,
(b) $\varepsilon_{2}<0$ (metamaterial case).

Let us examine case (a). The DE for a linear case is well known [10] and has the form

$$
\begin{equation*}
\operatorname{tg}\left(h \sqrt{\varepsilon_{2}-\gamma^{2}}\right)=\frac{\varepsilon_{2} \sqrt{\varepsilon_{2}-\gamma^{2}}\left(\varepsilon_{1} \sqrt{\gamma^{2}-\varepsilon_{3}}+\varepsilon_{3} \sqrt{\gamma^{2}-\varepsilon_{1}}\right)}{\varepsilon_{1} \varepsilon_{3}\left(\varepsilon_{2}-\gamma^{2}\right)-\varepsilon_{2}^{2} \sqrt{\gamma^{2}-\varepsilon_{3}} \sqrt{\gamma^{2}-\varepsilon_{1}}} \tag{8.1}
\end{equation*}
$$

Let

$$
\begin{equation*}
f=\frac{\tau}{r^{2} \tau^{2}+\eta^{2}(\tau-1)}, \quad f_{1}=\frac{\varepsilon_{2}}{\varepsilon_{2}-\gamma^{2}} \frac{1}{\varepsilon_{2}^{2} /\left(\varepsilon_{2}-r^{2}\right)+\eta^{2}} \tag{8.2}
\end{equation*}
$$

Using passage to the limit as $a \rightarrow 0$ we obtain the function $f_{1}$ from the function $f$. We seek bounded solutions $X(x)$ and $Z(x)$. This implies that the denominator of the function $f_{1}$ cannot vanish. What is more, the function $f$ as $a \rightarrow 0$ tends to the function $f_{1}$ uniformly on $x \in[0, h]$. It is possible to pass to the limit under integral sign as $a \rightarrow 0$ in (7.14) using results of classical analysis

$$
\begin{equation*}
h=-\frac{\varepsilon_{2}}{\varepsilon_{2}-\gamma^{2}} \int_{\eta(h)}^{\eta(0)} \frac{1}{\varepsilon_{2}^{2} /\left(\varepsilon_{2}-\gamma^{2}\right)+\eta^{2}} d \eta+\frac{\varepsilon_{2}}{\varepsilon_{2}-\gamma^{2}}(N+1) \int_{-\infty}^{+\infty} \frac{1}{\varepsilon_{2}^{2} /\left(\varepsilon_{2}-r^{2}\right)+\eta^{2}} d \eta \tag{8.3}
\end{equation*}
$$

where $\eta(0), \eta(h)$ are defined by formulas (6.4).
The integrals in (8.3) are calculated analytically. Calculating these integrals we obtain

$$
\begin{equation*}
h \sqrt{\varepsilon_{2}-r^{2}}=\operatorname{arctg} \frac{\varepsilon_{2} \sqrt{\varepsilon_{2}-r^{2}}\left(\varepsilon_{1} \sqrt{\gamma^{2}-\varepsilon_{3}}+\varepsilon_{3} \sqrt{r^{2}-\varepsilon_{1}}\right)}{\varepsilon_{1} \varepsilon_{3}\left(\varepsilon_{2}-r^{2}\right)-\varepsilon_{2}^{2} \sqrt{r^{2}-\varepsilon_{3}} \sqrt{r^{2}-\varepsilon_{1}}}+(N+1) \pi \tag{8.4}
\end{equation*}
$$

Expression (8.4) can be easily transformed into expression (8.1).
Let us examine (b) case. We have $\varepsilon_{2}<0$ (metamaterial) and the DE for the linear case has the form [31]

$$
\begin{equation*}
e^{2 h \sqrt{\gamma^{2}-\varepsilon_{2}}}=\frac{\varepsilon_{1} \sqrt{\gamma^{2}-\varepsilon_{2}}-\varepsilon_{2} \sqrt{\gamma^{2}-\varepsilon_{1}}}{\varepsilon_{1} \sqrt{\gamma^{2}-\varepsilon_{2}}+\varepsilon_{2} \sqrt{\gamma^{2}-\varepsilon_{1}}} \frac{\varepsilon_{3} \sqrt{\gamma^{2}-\varepsilon_{2}}-\varepsilon_{2} \sqrt{\gamma^{2}-\varepsilon_{3}}}{\varepsilon_{3} \sqrt{\gamma^{2}-\varepsilon_{2}}+\varepsilon_{2} \sqrt{\gamma^{2}-\varepsilon_{3}}} \tag{8.5}
\end{equation*}
$$

where $\gamma^{2}-\varepsilon_{1}>0, \gamma^{2}-\varepsilon_{2}>0$, and $\gamma^{2}-\varepsilon_{3}>0$.


Figure 2: Plot of $\gamma(h)$. The first few dispersion curves are shown. Solid curves for the nonlinear case (solutions of (7.14)); dashed curves for the linear case (solutions of (8.1)). The following parameters are used for both cases: $\varepsilon_{1}=1.44, \varepsilon_{2}=9, \varepsilon_{3}=1$, and for the nonlinear case $a=0.1$, and $E_{z}^{(h)}=1$. Dashed lines are described by formulas: $h^{*}=3.206$ (thickness of the layer), $\gamma=1.2$ (lower bound for $\gamma$ ), and $\gamma=3$ (upper bound for $\gamma$ in the case of linear medium in the layer).

In the same way as above, passing to the limit in the function $f$ as $a \rightarrow 0$ we obtain $f_{2}=\left|\varepsilon_{2}\right| /\left(\gamma^{2}-\varepsilon_{2}\right)\left(1 / \eta^{2}-\varepsilon_{2}^{2} /\left(\gamma^{2}-\varepsilon_{2}\right)\right)$. Passing to the limit in equation (7.14) as $a \rightarrow 0$ and integrating the function $f_{2}$, after simple calculations, we obtain formula(8.5)

The results here show that it is possible to pass to the limit as $a \rightarrow 0$. DE in (7.14) for the nonlinear case turns into (8.1) or (8.5) for the linear case as $a \rightarrow 0$.

## 9. Numerical Results

The way of solution to the DE in (7.14) is the following: we choose the segment on $\gamma$ then cut this segment into $p$ pieces with nods $\gamma_{i}, i=0,1, \ldots, p$. Then for each $\gamma_{i}$ we can calculate all necessary values in (7.14). The integral in (7.14) can be calculated using any method of numerical integration. In order to calculate the value of the integrand at the point $\gamma_{i}$ it is necessary to use first integral (6.3). For each value $\gamma_{i}$ we calculate value $h_{i}$, so we obtain the grid $\left\{\gamma_{i}, h_{i}\right\}, i=0,1, \ldots, p$. Choosing reasonably dense grid on $\gamma$ we can plot dependence $\gamma$ on $h$, as it is done below.

Dispersion curves (DC) calculated from (7.14), (8.1), and (8.5) are shown in Figures 2 and 4. Eigenmodes for eigenvalues indicated in Figures 3 and 5 are shown in Figures 3 and 5.

As it is known and it is shown in Figure 2, the line $\gamma=3$ is an asymptote for DCs in the linear case. It should be noticed that in the linear case there are no DCs in the region $\gamma^{2} \geq \varepsilon_{2}$. It can be proved that function $h \equiv h(\gamma)$ defined from equation (7.14) is continuous at the neighborhood $\gamma^{2}=\varepsilon_{2}$ when $a \neq 0$ (see Figure 2). This is the important distinction between linear and nonlinear cases.


Figure 3: Eigenfunctions (fields) for the nonlinear problem are shown. Solid curves for $X$; dashed curves for $Z$. The same parameters as in Figure 2 are used. For (a), $\gamma=2.994$; for (b), $\gamma=3.892$ : for (c), $\gamma=8.657$, and $h=3.206$ is used for all three cases. The eigenvalues are marked in Figure 2.


Figure 4: Plot of $\gamma(h)$. The first few dispersion curves are shown. Solid curves for the nonlinear case (solutions of (7.14)); dashed curve for the linear case (solutions of (8.5)). The following parameters are used for both cases: $\varepsilon_{1}=1, \varepsilon_{2}=-1.5, \varepsilon_{3}=1$, and for the nonlinear case $a=5.2$ and $E_{z}^{(h)}=1$. Dashed lines are described by formulas: $h^{*}=1.71$ (thickness of the layer), $\gamma=1$ (lower bound for $\gamma$ ) and $\gamma=3$ (upper bound for $\gamma$ in the case of linear medium in the layer).

Further, it can be proved that function $h \equiv h(\gamma)$ defined from equation (7.14) when $a \neq 0$ has the following property:

$$
\begin{equation*}
\lim _{\gamma^{2} \rightarrow+\infty} h(\gamma)=0 \tag{9.1}
\end{equation*}
$$



Figure 5: Eigenfunctions (fields) for the nonlinear problem are shown. Solid curves for $X$; dashed curves for $Z$. The same parameters as in Figure 4 are used. For (a), $\gamma=2.620$, and $a=0$ (see (8.5)); for (b), $r=1.565$; for (c), $\gamma=3.481$, and $h=1.71$ is used for all three cases. The eigenvalues are marked in Figure 4.


Figure 6: Plot of $\gamma(h)$ for the different values of $a: 1-a=1 ; 2-a=0.1 ; 3-a=0.01 ; 4-a=0.001$; $5-$ $a=0.0001 ; 6-a=0$ (linear case). The following parameters are used for both cases: $\varepsilon_{1}=4, \varepsilon_{2}=9, \varepsilon_{3}=1$, and for the nonlinear case $E_{z}^{(h)}=1$. Dashed lines are described by formulas: $\gamma=2$ (lower bound for $\gamma$ ), $\gamma=3$ (upper bound for $\gamma$ in the case of linear medium in the layer). Curves (solid) $1-5$ are solutions of (7.14), and curve 6 (dashed) is solutions of (8.1).

In Figure 2 for $h=3.206$ in the case of a linear layer there are 3 eigenvalues (black dots where the line $h=3.206$ intersects with DCs). These eigenvalues correspond to 3 eigenmodes. In the case of a nonlinear layer in Figure 2 are shown 5 eigenvalues (uncolored dots). These eigenvalues correspond to 5 eigenmodes. Taking into account the last paragraph's statement it is clear that in this case there is infinite number of eigenvalues.

It is easy to see in Figure 6 that the less nonlinearity coefficient $a$ is the more stretched DCs in the nonlinear case. The maximum points of the curves $h(\gamma)$ (in Figure 6 they are marked by asterisks) move to the right. The parts of the DCs that locate below the maximum points asymptotically tend to the DCs for the linear case as $a \rightarrow 0$.

It should be noticed that in the case of Kerr nonlinearity in a layer and TE waves there are strong constraints on the value $a$ depending on the value $\varepsilon_{2}$ (for details see [16]). It is natural to suppose that there are constraints of the kind in the case under consideration.

As far as we know experiments to observe the new nonlinear eigenmodes were not carried out. So the question if the modes corresponding to the new eigenvalues exist (in an experiment) stays open! We should like to emphasize that it is interesting to observe purely nonlinear modes that do not arise in a linear limiting case. If to see Figure 2 then points (b) and (c) correspond to these purely nonlinear modes.

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Research Article

# Combination Mode of Internal Waves Generated by Surface Wave Propagating over Two Muddy Sea Beds 

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#### Abstract

When surface wave propagating over the two layer system usually induces internal wave in three different modes: they are external, internal and combination. In the present study, the nonlinear response of an initially flat sea bed, with two muddy sections, to a monochromatic surface progressive wave was investigated. From this theoretical result, it shows that a surface water wave progressing over two different muddy sections, the surface wave will excite two opposite-traveling short interfacial waves, forming a nearly standing wave at the interface of the fresh water and the muddy layer. Meanwhile, two opposite-outgoing "mud" waves each with very long wavelength will be simultaneously induced at the interface of two muddy sections. As a result, the amplitudes of the two short internal waves are found to grow exponentially in time. Furthermore, it will be much difficult to excite the internal waves when surface water wave progressing over two muddy sections with the large density gap.


## 1. Introduction

Generally, wave propagating over the two layer system usually induces internal wave in three different modes: they are external, internal, and combination, as shown in Figure 1. These internal waves dissipate rapidly with imaginary and real wave numbers of a similar magnitude and remove energy from the surface waves. Also, nonlinear damping mechanisms have been proposed based on wave-wave instabilities and interactions.

The present work is motivated by recent studies on the interaction between a progressive surface wave and the nearly standing subharmonic internal waves in a twolayer system. It is well known that the loading of progressive surface waves, a silty sediment


Figure 1: Schematic diagram of the two layer system showing (a) the external wave mode with the same frequency and wavelength for both the air-water and water-mud interface waves, (b) the internal wave mode with the same frequency, but much shorter wavelength for the water-mud interface waves, and (c) the combination of the two. The arrows in panel (c) indicate shear across the mud water interface, which could lead to a shear instability mechanism to generate the internal mode waves in each half cycle of the surface wave.
bed was repeatedly and extensively fluidized. The broad-based interest in understanding this phenomenon induces from the application to studies in sediment transport, wave attenuation, and the design of marine structures. The nonlinear response of an initially flat sea bed to a monochromatic surface progressive wave was studied by Wen [1] for using the multiple scale perturbation method. She found that two opposite-traveling subliminal internal "mud" waves are triggered and form a resonant triad with the surface wave. The resonant generation of internal waves on sediment bed was presented as a new mechanism of sediment suspension. Nonlinear wave interactions are considered to be as an important aspect of the dynamics of the oceans and the atmosphere. Of particular interest are resonant interactions, which are important in the redistribution of energy among wave modes with different spatial and temporal scales. Ball [2] used a second-order nonlinear resonance theory to analyze that linear growth of an internal wave could result from the interactions between two finite-amplitude surface waves. Watson et al. [3] extended the investigation by analyzing how a spectrum of surface waves could generate a corresponding spectrum of internal waves. Wen's [1] work was followed by Hill and Foda [4], who treated the problem in two dimensions for both an inviscid and a viscous lower layer. Hill and Foda [5] and Jamali [6] have presented theoretical and experimental studies of the resonant interaction between a surface wave and two oblique interfacial waves. Despite many similarities between the findings, there is one seemingly major difference. Hill and Foda's [5] analysis predicts only narrow bands of frequency, density ratio, and direction angle within which growth is possible. However, Jamali [6] predicted and observed wave growth over wide ranges of frequency and direction angle, and for all the density ratios that he investigated. Therefore, Jamali et al. [7] presented the study to investigate the contradictory results between the findings of Hill and Foda [5] and Jamali [6]. From their result, it is showed that the crucial difference between the two studies is in the dynamic interfacial boundary condition. The boundary condition used by Hill and Foda is missing a term proportional to the time derivative of the square of the velocity shear across the interface. When this missing term is included in the analysis, the theoretical predictions are consistent with the results of Jamali's [6] laboratory experiments. Both of the Hill and Foda's [5] and Jamali's [6] study found that the interfacial waves are short, have a frequency of nearly half that of the surface wave, and propagate in nearly opposite directions. The nonlinear response of an initially flat sea bed, with two muddy sections, to a monochromatic surface progressive wave was investigated in the present study. Based on an analysis similar to that of Hill and Foda's paper [5], the multiple-scale perturbation method was adopted, and the boundary value problem was


Figure 2: Configuration of the problem in the two-layer system with two muddy sections.
expanded in a power series of the surface-wave steepness. The linear harmonics and the conditions for resonance were obtained by the leading order, while the temporal evolution equations for the internal-wave amplitudes were investigated by a second-order analysis. It was found that result for equal density of two muddy sections is similar to that of Hill and Foda's paper [5]. Two opposite-traveling internal "mud" waves are selectively excited and form a resonant triad with the progressive surface wave. However, for a surface water wave progressing over two different muddy sections, the surface wave will also excite only two opposite-traveling short interfacial waves, forming a nearly standing wave at the interface of the fresh water and the muddy layer. Meanwhile, two opposite-outgoing "mud" waves each with very long wavelength will be simultaneously induced at the interface of two muddy sections. As a result, the amplitudes of the two short internal waves are found to grow exponentially in time.

## 2. Formulation

As shown in Figure 2, the origin of a two-dimensional Cartesian coordinate system is placed on the undisturbed interface between a surface layer of depth $H$ and density $\rho$ and a lower depth $h$ at the interface between two different density sections, $\rho$ and $\rho^{\prime \prime}$. The y-coordinate is defined as pointing vertically upward, and the density rations, $\gamma=\rho / \rho$ and $\gamma=\rho / \rho^{\prime \prime}$, are assumed to be less than unity. To the leading order, the wave field is assumed to be made up of a linear progressive surface wave of amplitude $A$, wave number $k$, and frequency $\omega$, propagating in the positive $x$-direction. Firstly, we assumed that the perturbation internal waves at left muddy section have amplitudes $a_{1}$ and $a_{2}$, wave numbers $\lambda_{1}$ and $\lambda_{2}$, frequencies $\sigma_{1}$ and $\sigma_{2}$, and propagate in the positive and negative $x$-directions, respectively. Then for the right muddy section, there are also the perturbation internal waves with amplitudes $a_{3}$ and $a_{4}$, wave numbers $\lambda_{3}$ and $\lambda_{4}$, frequencies $\sigma_{3}$ and $\sigma_{4}$, and move in the positive and negative $x$-directions, respectively. It is noted that $a_{1}, a_{2}, a_{3}, a_{4}$, and $A$ are taken to be complex and $\lambda_{1} \sim \lambda_{4}$ and $\sigma_{1} \sim \sigma_{4}$ are all defined to have positive real values. For resonant interactions to occur in two muddy sections, the following resonance conditions are imposed on the fourhanded wave numbers and frequencies:

$$
\begin{align*}
& \lambda_{1}-\lambda_{2}-\lambda_{4}=k \\
& \lambda_{1}+\lambda_{3}-\lambda_{4}=k  \tag{2.1}\\
& \sigma_{1}-\sigma_{2}-\sigma_{4}=\omega \\
& \sigma_{1}+\sigma_{3}-\sigma_{4}=\omega
\end{align*}
$$

Expressing the flow-field in the two-layer inviscid system in terms of a velocity potential $\Phi$, we assume that $\Phi$ satisfies Laplace's equation throughout the depth of the fluid

$$
\begin{equation*}
\nabla^{2} \Phi=0, \quad-h \leq y \leq H+\xi \tag{2.2}
\end{equation*}
$$

The displacement $\xi$ of the free surface from its static elevation $y=H$ is given by

$$
\begin{equation*}
\xi=A e^{i \theta_{0}} \tag{2.3}
\end{equation*}
$$

in which the phase function $\theta_{0} \equiv(k x-\omega t)$. The free surface will oscillate synchronously with $\xi$, with an amplitude $A$ in response to the passage of the surface wave.

At the free surface, the usual kinematic and dynamic conditions are given by

$$
\begin{gather*}
\frac{D \xi}{D t}=\Phi y, \quad y=H+\xi \\
\Phi_{t}+g \xi+\frac{1}{2} \nabla \Phi \cdot \nabla \Phi=0, \quad y=H+\xi \tag{2.4}
\end{gather*}
$$

At the interface between the water and the slurry, there are similar conditions of continuity of pressure and vertical velocity. Therefore, on the left muddy section we have

$$
\begin{gather*}
\rho\left(\Phi_{t}^{l}+g \eta^{\prime}+\frac{1}{2} \nabla \Phi^{l} \cdot \nabla \Phi^{l}\right)^{+}=\rho^{\prime}\left(\Phi_{t}^{l}+g \eta^{\prime}+\frac{1}{2} \nabla \Phi^{l} \cdot \nabla \Phi^{l}\right)^{-}, \quad y=\eta^{\prime}  \tag{2.5}\\
\frac{D \eta^{\prime}}{D t}=\Phi_{y}^{l+}=\Phi_{y}^{l-}, \quad y=\eta^{\prime}
\end{gather*}
$$

Similar conditions of continuity of normal velocities and traction stresses are imposed at the disturbed interface on the right muddy section

$$
\begin{gather*}
\rho\left(\Phi_{t}^{r}+g \eta^{\prime \prime}+\frac{1}{2} \nabla \Phi^{r} \cdot \nabla \Phi^{r}\right)^{+}=\rho^{\prime \prime}\left(\Phi_{t}^{l}+g \eta^{\prime \prime}+\frac{1}{2} \nabla \Phi^{r} \cdot \nabla \Phi^{r}\right)^{-}, \quad y=\eta^{\prime \prime}  \tag{2.6}\\
\frac{D \eta^{\prime \prime}}{D t}=\Phi_{y}^{r+}=\Phi_{y}^{r-}, \quad y=\eta^{\prime \prime}
\end{gather*}
$$

Meanwhile at the interface between two muddy mass of sections, conditions of continuity of pressure and continuity of water and mud are shown as followed

$$
\begin{gather*}
{\left[\rho\left(\Phi_{l}\right)_{x}\right]_{y=\eta^{\prime}}^{y=H+\xi}=\left[\rho\left(\Phi_{r}\right)_{x}\right]_{y=\eta^{\prime \prime}}^{y=H, \xi}, \quad \text { at } x=0, \text { (forwater), }}  \tag{2.7}\\
{\left[\rho^{\prime}\left(\Phi_{l}\right)_{x}\right]_{y=0}^{y=\eta^{\prime}}=\left[\rho^{\prime \prime}\left(\Phi_{r}\right)_{x}\right]_{y=0}^{y=\eta^{\prime \prime}}, \quad \text { at } x=0, \text { (formud) }}
\end{gather*}
$$

Note that the $l$ and $r$ superscripts denote the tiny interface between left muddy section and right muddy section, and + and - superscripts show evaluation just above and just below the interface between the water and the fluid-mud, respectively. And finally, at the bottom of mud layer, we have the no flux condition

$$
\begin{equation*}
\Phi_{y}=0, \quad y=-h \tag{2.8}
\end{equation*}
$$

With the system for the interface and free surface being assumed to be only weakly nonlinear interaction, Taylor's expansion at the interface and free surface is used to eliminate $\xi, \eta^{\prime}$, and $\eta^{\prime \prime}$. And by using the method of successive approximations, we have three free wave harmonics in the velocity potential $\Phi^{l}$ for left side $(x<0)$, that is we assume the following expansion for $\Phi^{l}$ :

$$
\begin{equation*}
\Phi^{l}=\varepsilon \varphi(y) e^{i(k x+\omega t)}+\varepsilon^{2}\left\{\psi(y) e^{i\left(\lambda_{1} x+\sigma_{1} t\right)}+\chi(y) e^{i\left(\lambda_{2} x-\sigma_{2} t\right)}\right\}+\Phi_{n . l .}^{l}+\text { c.c. } \tag{2.9}
\end{equation*}
$$

The first three terms of the above expansion represent the three free harmonics; first term is the surface wave, and the following two terms are for the internal waves, $\Phi_{n . l .}^{l}$ represents the harmonics, and c.c. denotes complex conjugate. The expansion parameter is taken to be the steepness of the surface wave, $\varepsilon=k A$. Our analysis is restricted to the case of small internal waves, only so the internal wave harmonics appear at $O\left(\varepsilon^{2}\right)$ and not at $O(\varepsilon)$.

Similarly, the velocity potential $\Phi^{r}$ for fight side $(x>0)$ is expanded as follows:

$$
\begin{equation*}
\Phi^{r}=\varepsilon \varphi(y) e^{i(k x-\omega t)}+\varepsilon^{2}\left\{\psi(y) e^{i\left(\lambda_{3} x-\sigma_{3} t\right)}+\chi(y) e^{i\left(\lambda_{4} x+\sigma_{4} t\right)}\right\}+\Phi_{n . l .}^{r}+\text { c.c. } \tag{2.10}
\end{equation*}
$$

## 3. Perturbation Solution

For the solution procedure, a standard perturbation analysis for a weakly nonlinear wavefield system is used. The solution procedure involves solving the above boundary value problem in an ordered sequence, by separating terms in the governing equations and the boundary conditions according to their order in $\varepsilon$ and their phase. Substituting (2.9) and (2.10) in the governing equations and collecting the leading $O(\varepsilon)$ order, the approximate linear solutions for the interacting harmonics are obtained. For the dispersion relationship of surface wave,

$$
\begin{align*}
& \omega^{4}\left\{\left(r^{\prime} \gamma^{\prime \prime}\right)^{1 / 2} \operatorname{coth}(k h) \operatorname{coth}(k H)+1\right\}-\omega^{2}\left(r^{\prime} r^{\prime \prime}\right)^{1 / 2} \\
& \quad \times\{\operatorname{coth}(k H)+\operatorname{coth}(k h)\} g k+\left\{\left(r^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}-1\right\} g^{2} k^{2}=0 \tag{3.1}
\end{align*}
$$

For the dispersion relationship for internal waves of two muddy sections, the following is obtained

$$
\begin{gather*}
\sigma_{1}^{2}=\frac{g\left\{1-\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}\right\}}{\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}+\operatorname{coth}\left(\lambda_{1} h\right)}, \quad x<0, \\
\sigma_{2}^{2}=\frac{g\left\{1-\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}\right\}}{\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}+\operatorname{coth}\left|\lambda_{2} h+\lambda_{4} h\right|}, \quad x<0, \\
\sigma_{3}^{2}=\frac{g\left\{1-\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}\right\}}{\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}+\operatorname{coth}\left(\lambda_{1} h+\lambda_{3} h\right)}, \quad x>0,  \tag{3.2}\\
\sigma_{4}^{2}=\frac{g\left\{1-\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}\right\}}{\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2}+\operatorname{coth}\left|\left(\lambda_{4} h\right)\right|}, \quad x>0 .
\end{gather*}
$$

Quadratic interactions between the above linear harmonics are analyzed at the second order, $O\left(\varepsilon^{2}\right)$. Since the homogeneous version of the boundary value problem had a nontrivial solution, the inhomogeneous problem has a solution only if the forcing terms are orthogonal to the homogeneous solution. Invoking solvability, through the use of Green's theorem, the desired evolution equations via the internal wave amplitudes are obtained. For simplicity, the amplitude equations-via solvability-may be straightforwardly shown as follows:

$$
\begin{align*}
& \frac{d a_{l}}{d t}=i \alpha_{l} a_{r} A  \tag{3.3}\\
& \frac{d a_{r}}{d t}=i \alpha_{r} a_{l} A^{*}
\end{align*}
$$

in which $a_{l}=a_{1}+a_{2}+a_{4}$ and $a_{r}=a_{1}+a_{3}+a_{4} . \alpha_{l}$ and $\alpha_{r}$ are the interaction coefficients. Taking cross differentiation of (3.3), the growth for amplitudes of the internal waves is governed by exponentials:

$$
\begin{gather*}
a_{l}, a_{r} \alpha \exp \left\{ \pm \sqrt{-\alpha_{l} \alpha_{r}} A t\right\} \\
\alpha=\sqrt{-\alpha_{l} \alpha_{r}}=  \tag{3.4}\\
\frac{1}{2}|A|\left\{\frac{-g}{H}\left(1-2\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2} \frac{\sigma_{2}+\sigma_{4}}{\sigma_{1}+\sigma_{3}}+\frac{k}{\lambda_{1}+\lambda_{3}}\right)\right. \\
\left.\times\left(1-2\left(\gamma^{\prime} \gamma^{\prime \prime}\right)^{1 / 2} \frac{\sigma_{1}+\sigma_{3}}{\sigma_{2}+\sigma_{4}}-\frac{k}{\lambda_{2}+\lambda_{4}}\right)\left(\lambda_{1}+\lambda_{3}\right)\left(\lambda_{2}+\lambda_{4}\right)\right\}^{1 / 2} .
\end{gather*}
$$

Under the condition of $\alpha$ being purely imaginary, then the amplitudes $a_{l}$ and $a_{r}$ are not able to grow with time. Equivalently, the surface water wave will be stable to these internal wave perturbations. Only when $\alpha$ is real, the growth of the internal wave amplitudes will occur.


Figure 3: Two internal wave frequencies as functions of surface wave frequency $\omega, H=4 \mathrm{~m}, \gamma^{\prime}=\gamma^{\prime \prime}=0.83$, and $h=0.1 \mathrm{~m}$.


Figure 4: Two internal wave frequencies as functions of surface wave frequency $\omega, H=4 \mathrm{~m}, \gamma^{\prime}=\gamma^{\prime \prime}=0.83$, and $h=0.3 \mathrm{~m}$.

## 4. Results and Discussion

In an effort to develop a mechanism of the generation for internal waves of two muddy sections by surface wave, an inviscid second-order nonlinear resonant interaction has been analyzed. An example of the same density for two muddy sections, the internal-wave parameters as functions of $\omega$ for the cases of $\gamma^{\prime}=\gamma^{\prime \prime}=0.83, H=4 \mathrm{~m}, h=0.1 \mathrm{~m}$ and $r^{\prime}=r^{\prime \prime}=0.83, H=4 \mathrm{~m}, h=0.3 \mathrm{~m}$ is shown in Figures 3 and 4 . The result of the present study is similar to that of Hill and Foda's paper [5]. It is obvious that there is a surfacewave frequency, a cut-off frequency, below which resonant triads do not exist. The cut-off frequency is a function of $H, h, \gamma^{\prime}$, and $\gamma^{\prime \prime}$. Furthermore, the critical frequency corresponds to these cases in which the internal waves in muddy layer are nearly subharmonic to the surface wave, that is, $\sigma_{1} \approx \sigma_{4} \approx \omega / 2$, and are propagating in the same and opposite directions to the surface wave. Figure 5 shows the growth rate of internal waves as a function of the surface wave frequency for $r^{\prime \prime}=0.83, H=4 \mathrm{~m}$, and $h=0.1 \mathrm{~m}$ and 0.3 m . The growth rate of internal waves will be suppressed when the depth of mud layer increases. That is because if the mud layer depth being too thick, the interaction among surface and interfacial internal waves weakens. Internal waves are also subdued. However, for a surface water wave progressing over two different muddy sections $\left(\gamma^{\prime} \neq \gamma^{\prime \prime}\right)$, the surface wave will also excite only two opposite-traveling interfacial short waves, forming a nearly standing wave at the interface of the fresh water and the muddy layer. Meanwhile, two opposite-outgoing


Figure 5: Theoretical determined internal wave growth rates as functions of surface wave frequency $\omega$, $H=4 \mathrm{~m}, \gamma^{\prime}=\gamma^{\prime \prime}=0.83$, (a) $h^{\prime}=0.1 \mathrm{~m}$, and (b) $h=0.3 \mathrm{~m}$.


Figure 6: Two internal wave frequencies as functions of surface wave frequency $\omega, H=4 \mathrm{~m}, h=0.1 \mathrm{~m}$ (a) $\gamma^{\prime}=\gamma^{\prime \prime}=0.83$, (b) $\gamma^{\prime}=0.83, \gamma^{\prime \prime}=0.77$, and (c) $\gamma^{\prime}=0.83, \gamma^{\prime \prime}=0.71$.
"mud" waves each with very long wavelength will be simultaneously induced at the interface between two muddy sections. The wave numbers of these two long internal waves will be of the order $10^{-5} \sim 10^{-4}$ and have opposite value; $\lambda_{2}=-\lambda_{3} \approx 10^{-5} \sim 10^{-4}$ for the cases of different mud layer depth. Figure 6 shows that if the density of the right muddy section becomes increasingly larger than that of left side, then the right resonant internal wave frequency will also have increasingly higher value than the left. Meanwhile if the density difference between two muddy sections increases, the growth rate of the two resonant short internal waves will be suppressed. This result can be seen in Figure 7. The result means that it will


Figure 7: Growth rate of internal short waves for different density ratios of the right muddy section ( $\gamma^{\prime \prime}=$ $0.83,0.78$ ), water depth $H=4 \mathrm{~m}$, mud layer depth $h=0.1 \mathrm{~m}$, and density ratio of the left muddy section $r^{\prime}=0.83$.
be much difficult to excite the internal waves when surface water wave progressing over two muddy sections with the large density gap.

## 5. Conclusions

The resonant generation of internal waves on a sediment bed was presented as a new mechanism of sediment suspension. In this study, the corresponding theoretical analysis for equal density of two muddy sections is similar to that of the previous studies. A progressive surface wave simultaneously generates two opposite-traveling short internal waves. However, for a surface water wave progressing over two different muddy sections, the surface wave will also excite only two opposite-traveling interfacial short waves, forming the triad resonance with the surface wave. Meanwhile, two opposite-outgoing long internal waves will be also triggered at the interface between two muddy sections. Furthermore, it will be much difficult to excite the internal waves when surface water wave is progressing over two muddy sections with the large density gap.

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## Review Article

# Spacetime Junctions and the Collapse to Black Holes in Higher Dimensions 

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#### Abstract

We review recent results about the modelling of gravitational collapse to black holes in higher dimensions. The models are constructed through the junction of two exact solutions of the Einstein field equations: an interior collapsing fluid solution and a vacuum exterior solution. The vacuum exterior solutions are either static or containing gravitational waves. We then review the global geometrical properties of the matched solutions which, besides black holes, may include the existence of naked singularities and wormholes. In the case of radiating exteriors, we show that the data at the boundary can be chosen to be, in some sense, arbitrarily close to the data for the Schwarzschild-Tangherlini solution.


## 1. Introduction

Higher dimensional black holes have recently been the subject of an increasing number of research works (see [1] for a review) and play an important role in theoretical physics, particularly in String Theory. However, the study of the actual dynamical process of collapse resulting in black hole formation has deserved little attention from the mathematical point of view.

Mathematically, black hole formation can be analysed by constructing appropriate matched spacetimes which settle down through gravitational collapse to black hole solutions. The junction (or matching) of two spacetimes requires the equality of the respective first and second fundamental forms at some matching boundary hypersurface. These conditions amount to a set of differential equations that the spacetime metric functions need to satisfy at that matching boundary. This is often not an easy problem as the metric functions are also required to satisfy the respective Einstein field equations (EFEs) on both sides of the matching surface. In what follows, we review some results about models of black hole formation with
emphasis on models resulting from the spacetime matching with an exterior which is either vacuum or a cosmological background.

For a zero cosmological constant $\Lambda$, there are well known spherically symmetric examples, one of the simplest being the Oppenheimer-Snyder model which results from the matching of a Friedman-Lemaître-Robertson-Walker (FLRW) dust metric to a Schwarzschild exterior. For $\Lambda \neq 0$ spherical cases, the results are less well-known: the matching of $\Lambda$-FLRW with the $\Lambda$-Schwarzschild solution (also called Kottler solution) has been investigated by Balbinot et al. [2], Nakao [3], and Markovic and Shapiro [4] and its $\Lambda$-Lemaître-TolmanBondi counterpart by Lake [5]. The matching of the dust $\Lambda$-Szekeres solution with Kottler was recently studied by Debnath et al. [6] (following a work of Bonnor [7] for $\Lambda=0$ ), while the matching of a collapsing fluid with tangential pressure and $\Lambda \neq 0$ to Kottler has been investigated by Madhav et. al. [8]. The above examples give rise to black hole solutions with $\Lambda \neq 0$ in spherical symmetry.

The collapse to nonspherical black holes has been less studied. Smith and Mann [9] have shown that one can match a collapsing $k=-1$ FLRW spacetime to an asymptotically anti-de Sitter (AdS) exterior, as a model of gravitational collapse to higher genus asymptotically AdS black holes. In a related work, Lemos [10] matched a flat FLRW metric to a radiating Vaidya exterior whose collapse results in a toroidal black hole. Subsequently, there has been considerable interest in models of toroidal and higher genus black holes (also called topological black holes), shells, and horizons, see, for example, [11-14], partly due to the existence of a "landscape" of vacua states in String theory with $\Lambda$ positive, negative, and zero (see, e.g., [15]). More recently, [16] generalised the earlier work and constructed models for the collapse of inhomogeneous and anisotropic fluids to topological asymptotically AdS black holes in planar and hyperbolic symmetry.

There is also a vast literature about the formation of black holes in cosmological models. There are, essentially, three types of models which are geometrically different, namely, models with (i) junctions with no shells where two metrics are involved, for example, swiss-cheese-type models (see, e.g., [17] and references therein); (ii) junctions with shells (these can be thin or thick), for example, models with domain walls and bubbles mentioned in the previous paragraph; (iii) no junctions, where a single metric is involved in the problem, for example, models of collapsing over-densities resulting from perturbations on cosmological backgrounds.

In the context of primordial black hole (PBH) formation, the gravitational collapse due to first-order density perturbations in FLRW models of the early universe radiative phase was first studied by Zeldovich and Novikov [18] as well as Hawking [19]. These works were then generalised [20,21], in particular, to include black hole formation from the collapse of cosmic strings produced in phase transitions with symmetry breaking [22]. The collision of bubble walls formed at phase transitions was also considered as sources of PBHs in [23-25]. Different scenarios were subsequently studied and a review is in [26]. More recently, cosmological black hole formation due to QCD and electroweak phase transitions were studied by, for example, [27] following earlier works of [28] and bubble collision has been investigated in detail (for various potentials and using planar, hyperbolic, and spherical symmetry in both dS and AdS phases) by, for example, [11, 29]. Furthermore, PBH formation in cosmological models with $\Lambda \neq 0$ has very recently been analysed in [30], which contains an interesting review of numerical results up to the year 2011 (for a review on the $\Lambda=0$ case see [31]). In turn, the collapse to black holes in FLRW models have also been investigated using nonlinear scalar metric perturbations on radiative [32] and scalar field backgrounds [33],
with those recent works establishing interesting observational bounds on the cosmological spectra nongaussianity.

In this paper, we will be mostly interested in cases which involve spacetime junctions with no shells and, in particular, we will focus on nonspherical spacetimes containing a nonzero $\Lambda$. We start by reviewing the theory of spacetime junctions and we then summarise recent results about the gravitational collapse to black holes in higher dimensions [34] which generalise the results of [16]. We start by considering a family of solutions to the $\Lambda$-vacuum Einstein equations in $n+2$ dimensions which contains black hole solutions. We find some possible interior collapsing solutions with dust as source and study the corresponding matching problem. We then analyse the global geometrical properties of the matched spacetime and find conditions for which the spacetimes have no naked singularities.

We also review the interesting case of a model of radiating gravitational collapse. In particular, we take the anisotropic Bizoń-Chmaj-Schmidt (BCS) solution in $4+1$ dimensions and prove that it can be matched to some collapsing dust interiors. The BCS solution is known [35] to settle down via radiation to the Schwarzschild-Tangherlini (SchT) solution. In turn, the latter is known to be stable $[35,36]$ and we show that, for some interiors, the data at the matching surface can be chosen to be arbitrarily close to the data for a SchT exterior [34]. So, the resulting spacetime models the gravitational collapse of a fluid in five dimensions with an exterior emitting gravitational waves which settles down to the SchT solution.

## 2. Geometric Theory of Spacetime Junctions in Brief

Let $\left(M^{+}, g^{+}\right)$and $\left(M^{-}, g^{-}\right)$be two $n$-dimensional $C^{3}$ space times with oriented boundaries $\Omega^{+}$ and $\Omega^{-}$, respectively, such that $\Omega^{+}$and $\Omega^{-}$are diffeomorphic. The matched space time $(M, g)$ is the disjoint union of $M^{ \pm}$with the points in $\Omega^{ \pm}$identified such that the junction conditions are satisfied (Israel [37], Clarke and Dray [38] and Mars and Senovilla [39]). Since $\Omega^{ \pm}$are diffeomorphic, one can then view those boundaries as diffeomorphic to a 3-dimensional oriented manifold $\Omega$ which can be embedded in $M^{+}$and $M^{-}$. Let $\left\{\xi^{\alpha}\right\}$ and $\left\{x^{ \pm i}\right\}$ be coordinate systems on $\Omega$ and $M^{ \pm}$, respectively, where $\alpha, \beta=1,2, \ldots, n-1$ and $i, j=1,2, \ldots, n$. The two embeddings are given by the following $C^{3}$ maps:

$$
\begin{gather*}
\Phi^{ \pm}: \Omega \longrightarrow M^{ \pm},  \tag{2.1}\\
\xi^{\alpha} \longmapsto x^{i^{ \pm}}=\Phi^{i^{ \pm}}\left(\xi^{\alpha}\right), \tag{2.2}
\end{gather*}
$$

such that $\Omega^{ \pm} \equiv \Phi^{ \pm}(\Omega) \subset M^{ \pm}$. The diffeomorphism from $\Omega^{+}$to $\Omega^{-}$is $\Phi^{-} \circ \Phi^{+-1}$.
Given the basis $\left\{\partial /\left.\partial \xi^{\alpha}\right|_{p}\right\}$ of the tangent plane $T_{p} \Omega$ at some $p \in \Omega$, the push forwards $\left.d \Phi^{ \pm}\right|_{p} \operatorname{map}\left\{\partial /\left.\partial \xi^{\alpha}\right|_{p}\right\}$ into three linearly independent vectors at $\Phi^{ \pm}(p)$ represented by $\left.\vec{e}_{\alpha}^{ \pm}\right|_{\Phi^{ \pm}(p)}$ :

$$
\begin{equation*}
d \Phi^{ \pm}\left(\left.\frac{\partial}{\partial \xi^{\alpha}}\right|_{\Omega}\right)=\left.\left.\frac{\partial \Phi^{ \pm i}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x^{ \pm i}}\right|_{\Omega^{ \pm}} \equiv \vec{e}_{\alpha}^{ \pm}\right|_{\Omega^{ \pm}}=\left.e_{\alpha}^{ \pm i} \vec{e}_{\alpha}^{ \pm} \frac{\partial}{\partial x^{ \pm i}}\right|_{\Omega^{ \pm}} \tag{2.3}
\end{equation*}
$$

On the other hand, using the pull backs $\Phi^{ \pm *}$ of the maps $\Phi^{ \pm}$, the metrics $g^{ \pm}$can be mapped to $\Omega$ given two symmetric 2-covariant tensors $\bar{g}^{+}$and $\bar{g}^{-}$whose components in the basis $\left\{d \xi^{\alpha}\right\}$ are

$$
\begin{equation*}
\left.\bar{g}_{\alpha \beta}^{ \pm} \equiv e_{\alpha}^{ \pm i} e_{\beta}^{ \pm j} g_{i j}\right|_{\Omega^{ \pm}}=\left.\left(\vec{e}_{\alpha}^{ \pm} \cdot \vec{e}_{\beta}^{ \pm}\right)\right|_{\Omega^{ \pm}} . \tag{2.4}
\end{equation*}
$$

The first matching conditions are given by the equality of the first fundamental forms (Israel [37])

$$
\begin{equation*}
\bar{g}_{\alpha \beta}^{+}=\bar{g}_{\alpha \beta}^{-} . \tag{2.5}
\end{equation*}
$$

The bases $\left\{\left.\vec{e}_{\alpha}^{+}\right|_{p}\right\}$ and $\left\{\left.\vec{e}_{\alpha}^{-}\right|_{p}\right\}$ can then be identified,

$$
\begin{equation*}
d \Phi^{+}\left(\left.\frac{\partial}{\partial \xi^{\alpha}}\right|_{\Omega}\right)=d \Phi^{-}\left(\left.\frac{\partial}{\partial \xi^{\alpha}}\right|_{\Omega}\right) \tag{2.6}
\end{equation*}
$$

as can the hypersurfaces $\Omega^{+} \equiv \Omega^{-}$, so henceforth we represent both $\Omega^{ \pm}$by $\Omega$.
We now define a 1 -form $n$, normal to the hypersurface $\Omega$, as

$$
\begin{equation*}
n^{ \pm}\left(\vec{e}_{\alpha}^{ \pm}\right)=0 \tag{2.7}
\end{equation*}
$$

The vectors $\left\{\vec{n}^{ \pm}, \vec{e}_{\alpha}^{ \pm}\right\}$constitute a basis on the tangent spaces to $M^{ \pm}$at $\Omega^{ \pm}$. Since the first junction condition allows the identification of $\left\{\vec{e}_{\alpha}^{+}\right\}$with $\left\{\vec{e}_{\alpha}^{-}\right\}$, we only have to ensure that both bases have the same orientation and that $n_{i}^{+} n^{+i} \stackrel{\Omega}{=} n_{i}^{-} n^{-i}$ is satisfied in order to identify the whole 4-dimensional tangent spaces of $M^{ \pm}$at $\Omega,\left\{\vec{n}^{+}, \vec{e}_{\alpha}^{+}\right\} \equiv\left\{\vec{n}^{-}, \vec{e}_{\alpha}^{-}\right\}$.

The second fundamental forms are given by

$$
\begin{equation*}
K_{\alpha \beta}^{ \pm}=-n_{i}^{ \pm} e_{\alpha}^{ \pm j} \nabla_{j}^{ \pm} e_{\beta}^{ \pm i} \tag{2.8}
\end{equation*}
$$

and the second matching conditions, for nonnull surfaces, are the equality of the second fundamental forms

$$
\begin{equation*}
K_{\alpha \beta}^{+}=K_{\alpha \beta}^{-} . \tag{2.9}
\end{equation*}
$$

We note that these matching conditions do not depend on the choice of the normal vectors. The first matching conditions ensure the continuity of the metric across $\Omega$ while the second conditions prevent infinite jump discontinuities in the Riemann tensor so that the Einstein field equations are well defined in the distributional sense.

The theory is also fully developed for the case where the matching boundary $\Omega$ is null or changes character. In that case, the normal vectors $n_{ \pm}{ }^{\alpha}$ are substituted by the so-called rigging vectors, which are vectors nonwhere tangent to $\Omega^{ \pm}$and are used to define generalised second fundamental forms, see [39] for the details.

We are interested in the particular cases for which the matching surface inherits a certain symmetry of the two space times $\left(M^{ \pm}, g^{ \pm}\right)$. Such matching is said to preserve the symmetry. In practice one demands that the matching hypersurface is tangent to the orbits of the symmetry group to be preserved, see [40].

We note that if the exterior is a vacuum spacetime and the interior contains a fluid then the normal pressure of the fluid has to vanish at the boundary $\Omega$. This is a consequence of the fact that the matching conditions imply

$$
\begin{equation*}
n_{-}^{\alpha} T_{\alpha \beta}^{-} \stackrel{\Omega}{=} n_{+}^{\alpha} T_{\alpha \beta}^{+} \tag{2.10}
\end{equation*}
$$

We also note that if the interior contains dust then the boundary $\Omega^{-}$is ruled by geodesics, and we will explore this fact to solve the matching problem in what follows.

## 3. Black Holes in Higher Dimensions

An $(n+2)$-dimensional Lorentzian manifold $\left(M, g_{a b}\right)$ is a solution of the Einstein equations with cosmological constant $\Lambda$ and energy-momentum tensor $T_{a b}$ if its Ricci tensor $R_{a b}$ satisfies

$$
\begin{equation*}
R_{a b}=\Lambda g_{a b}+\kappa\left(T_{a b}-\frac{1}{n} T g_{a b}\right) \tag{3.1}
\end{equation*}
$$

where $\mathcal{\kappa}$ is a constant, $T=T_{a}^{a}$ and the indices $a, b=1,2, \ldots, n+2$. We will consider the family of higher-dimensional black holes given in the following proposition (see also [41, 42]).

Proposition 3.1. Let $\left(N, d \sigma^{2}\right)$ be an n-dimensional Riemannian Einstein manifold with Ricci scalar $n \lambda$, and let

$$
\begin{equation*}
V(r)=\frac{\lambda}{n-1}-\frac{2 m}{r^{n-1}}-\frac{\Lambda r^{2}}{n+1} \tag{3.2}
\end{equation*}
$$

where $m, \Lambda$ are constants. If $J \subset \mathbb{R}$ is an open interval where $V$ is well defined and does not vanish then the $(n+2)$-dimensional Lorentzian manifold $\left(M, d s^{2}\right)$ given by $M=\mathbb{R} \times J \times N$ and

$$
\begin{equation*}
d s^{2}=-V(r) d t^{2}+(V(r))^{-1} d r^{2}+r^{2} d \sigma^{2} \tag{3.3}
\end{equation*}
$$

is a solution of the vacuum Einstein equations with cosmological constant $\Lambda$.
It is easy to see that the above metric generalises the Kottler metric to arbitrary dimensions. Nevertheless, black holes with the above metric do not immediately integrate into the usual intuition of a black hole in four dimensions. For instance, since the metric on the sections of null infinity $\int$ does not need to be a metric of constant curvature, the spacetimes are not asymptotically flat (Nor asymptotically de Sitter (dS), resp., anti de Sitter (AdS), in the case of $\Lambda>0$, resp., $\Lambda<0$.); however, by analysing the equations for null geodesics in $M$ and $N$, one can show that the spacetimes are weakly asymptotically simple [34].

Proposition 3.2. The metrics (3.3) are conformally compactifiable at infinity and are weakly asymp-totically-simple.

Therefore, by comparison with the four-dimensional case [16], the following properties hold.
(i) With $\Lambda=0, \lambda>0$ and $m>0, V$ has a single zero, corresponding to an event horizon, and asymptotes to a positive constant at large $r$. This is a black-hole solution, which can be thought of as generalising the Schwarzschild metric. The (degenerate) metric on the horizon is $\mathrm{d} \sigma^{2}$, which is also the conformal metric on future null infinity $\partial^{+}$.
(ii) With $\Lambda>0, m>0$ and large enough positive $\lambda, V(r)$ is positive between two zeroes, corresponding to a black-hole event horizon and a cosmological event horizon. The solution generalises the asymptotically dS Kottler solution.
(iii) With $\Lambda<0$ and $m>0, V$ again has a single zero, corresponding to an event horizon, and the solution generalises the asymptotically AdS Kottler solution.
(iv) The solutions in the previous class with $\lambda \leq 0$ may have no global symmetries except the staticity Killing vector. This is because compact, negative scalar curvature Einstein manifolds have no global symmetries, nor does, for example, the Ricci-flat metric on $K 3$ (an example with $\lambda=0$ and $n=4$ ).

### 3.1. Generalised Friedman-Lemaître-Robertson-Walker Interiors

We will fill in the solutions of the previous section with (interior) collapsing dust solutions, so that the resulting global spacetime is a model of gravitational collapse with a black hole as the end state. In order to do that, we take the following classes of generalised FLRW spacetimes.

Proposition 3.3. The $(n+2)$-dimensional Lorentzian metric,

$$
\begin{equation*}
d s^{2}=-d \tau^{2}+R^{2}(\tau)\left(d \rho^{2}+f^{2}(\rho) d \sigma^{2}\right) \tag{3.4}
\end{equation*}
$$

is a solution of the Einstein equations with cosmological constant $\Lambda$ and energy-momentum tensor $T_{a b}=\mu u_{a} u_{b}$, corresponding to a dust fluid with density $\mu$ and velocity $u_{a} d x^{a}=d \tau$, if and only if $R(\tau)$ and $\mu(\tau)$ satisfy the conservation equation:

$$
\begin{equation*}
\mu R^{n+1}=\mu_{0} \tag{3.5}
\end{equation*}
$$

for constant $\mu_{0}$, and the Friedman-like equation:

$$
\begin{equation*}
\frac{\dot{R}^{2}}{R^{2}}+\frac{k}{n R^{2}}=\frac{2 \kappa \mu}{n(n+1)}+\frac{\Lambda}{n+1} . \tag{3.6}
\end{equation*}
$$

We consider the matching of an interior metric (3.4) to a static exterior (3.3) at a hypersurface $\Omega$ ruled by radial time-like geodesics in (3.3), that is, a surface of constant $\rho$ (say $\rho=\rho_{0}$ ) in (3.4). We then solve the matching conditions (2.5) and (2.9) together with the EFEs (3.6) to get the following result [34].

Theorem 3.4. The metric (3.3) can be matched to the FLRW-like metric (3.4) at $\rho=\rho_{0}$ provided that $f^{\prime}\left(\rho_{0}\right)>0$ and $m=\kappa \mu_{0} f\left(\rho_{0}\right)^{n+1} / n(n+1)$.

The matching boundary is comoving with the collapsing fluid whose dynamics is given by (3.6). We call the matched spacetime of the previous theorem FLRW-Kottler spacetime. We now summarize some properties of the FLRW-Kottler spacetime in the three cases $\Lambda=0, \Lambda>0$, and $\Lambda<0$ : when the Einstein manifold $N$ is not cobordant to a point (e.g., $\mathbb{C} P^{2}$ ) the solutions we find cannot have a regular origin, though they can be regular with spacetime wormholes or a "cusp" at the origin. When there is a singularity at the origin, it may or may not be visible from infinity. If the singularity is visible it is called naked singularity as it is not hidden by an horizon. The next three propositions exploit these aspects, and Figures 1, 2, and 3 show Penrose diagrams which represent the global structure of the matched spacetimes in the cases $\Lambda=0, \Lambda<0$, and $\Lambda>0$ [34]. In particular, by comparing the conformal lifetime


Figure 1: Penrose diagram for $\Lambda>0$ with the FLRW universe (a) recollapsing; (b) nonrecollapsing, showing the matching surfaces and the horizons.


Figure 2: Penrose diagram for $\Lambda<0$ and (a) $\lambda>0$; (b) $\lambda=0$; (c) $\lambda<0$, showing the matching surfaces and the horizons.
of the FLRW universe with the supremum of the possible values of $\rho_{0}$ we show [34] the following.

Proposition 3.5. If $\Lambda=0$ (hence $\lambda>0)$ and $\left(N, d \sigma^{2}\right)$ is not an $n$-sphere then the locally naked singularity of the FLRW-Kottler spacetime at $\rho=0$ is always visible from future null infinity $\partial^{+}$for $k \leq 0$, but can be hidden if $k>0$ and $n \geq 4$ (space-time dimension $n+2 \geq 6$ ).

Interestingly, [43] investigated higher dimensional dust collapsing spacetimes with $\Lambda=0$ and also found that for dimension $d=n+2 \geq 6$ the final central singularity is always hidden by an horizon, while for $d<6$ naked singularities may appear and thus the cosmic censorship conjecture is violated. By construction, their collapsing interiors are spherical, while here we allow interior spacetimes with other topologies.

Proposition 3.6. If $\Lambda>0$ (hence $\lambda>0$ ) and $\left(N, d \sigma^{2}\right)$ is not an $n$-sphere then the locally naked singularity of the FLRW-Kottler spacetime at $\rho=0$ can be always hidden except if the FLRW universe is recollapsing (hence $k>0$ ) and $n<4$.

The case $\Lambda<0$ has wormhole solutions that may have causal curves which cross from one 2 to the other and may violate causality. The next proposition, in particular, states conditions under which this can be avoided (see also a result of Galloway [44]). Those conditions are obtained by showing that the future (resp., past) horizons hit the matching


Figure 3: Penrose diagram for $\Lambda=0$ and (a) $k \leq 0$; (b) $k>0$, showing the matching surfaces and the horizons.
surfaces at marginally outer trapped (resp., antitrapped) surfaces which are spacelike and, in turn, that these two curves touch at $\dot{R}=\rho=0$.

Proposition 3.7. For $\Lambda<0$ the FLRW-Kottler spacetime satisfies the following.
(1) If $\lambda>0$ and $\left(N, d \sigma^{2}\right)$ is not an $n$-sphere then the locally naked singularity of the FLRWKottler spacetime at $\rho=0$ can always be hidden.
(2) If $\lambda=0$ then the cusp singularity is not locally naked.
(3) If $\lambda<0$ then no causal curve can cross the wormhole from one 3 to the other.

### 3.2. Generalised Lemaître-Tolman-Bondi Interiors

We consider now higher dimensional versions of the inhomogeneous Lemaître-TolmanBondi (LTB) solutions, generalising those of [45].

Proposition 3.8. The $(n+2)$-dimensional Lorentzian metric,

$$
\begin{equation*}
d s^{2}=-d \tau^{2}+A(\tau, \rho)^{2} d \rho^{2}+B(\tau, \rho)^{2} d \sigma^{2} \tag{3.7}
\end{equation*}
$$

is a solution of the Einstein equations with cosmological constant $\Lambda$ and energy-momentum tensor $T_{a b}=\mu u_{a} u_{b}$, corresponding to a dust fluid with density $\mu$ and velocity $u_{a} d x^{a}=d \tau$, if and only if $A(\tau, \rho), B(\tau, \rho)$ and $\mu(\tau, \rho)$ satisfy

$$
\begin{align*}
A & =B^{\prime}(1+w(\rho)),  \tag{3.8}\\
\mu A B^{n} & =M^{\prime}(\rho)(1+w(\rho)),
\end{align*}
$$

for some functions $w(\rho)$ and $M(\rho)$, and

$$
\begin{equation*}
\dot{B}^{2} B^{n-1}+\left(\frac{1}{n-1}-\frac{1}{(1+w(\rho))^{2}}-\frac{\Lambda}{n+1} B^{2}\right) B^{n-1}=\frac{2 \kappa M(\rho)}{n}, \tag{3.9}
\end{equation*}
$$

(where dot and prime denote differentiation with respect to $\tau$ and $\rho$ ).

This metric has three free functions of $\rho$, namely, $w(\rho), M(\rho)$, and $B(0, \rho)$, one of which can be removed by coordinate freedom. We seek to match an interior represented by the metric (3.7) to a static exterior represented by the metric (3.3) at a surface $\Omega$ ruled by radial time-like geodesics in (3.3), that is, a surface of constant $\rho$ (say $\rho=\rho_{0}$ ) in (3.7). The solutions to the matching conditions (2.5) and (2.9) allow us to prove [34].

Theorem 3.9. The metric (3.3) can be matched to the LTB-like metric (3.7) at $\rho=\rho_{0}$ provided that $1+w\left(\rho_{0}\right)>0$ and $m=(\kappa / n) M\left(\rho_{0}\right)$.

Note that the matching boundary is again comoving with the collapsing inhomogeneous fluid and its dynamics is now given by (3.9). We will use the term LTBKottler space-times for these matched solutions. The global properties of the LTB-Kottler spacetime obtained in Theorem 3.9 are much more diverse than what we find in the FLRWKottler spacetimes. For instance, one can easily find examples of black hole formation with wormholes inside the matter with positive $\lambda$ and $\Lambda=0$, see [34] (similar results in 4 dimensions are in [46]).

## 4. Collapse with Radiating Exteriors

In this section, we consider models of gravitational collapse with a gravitational wave exterior, so that the exterior metrics will be time-dependent generalisations of (3.3). This is a specially interesting problem for observational and experimental purposes [36]. We take the Bizoń-Chmaj-Schmidt (BCS) metric in $(4+1)$ dimensions [36], although similar ansetz can be made in other dimensions (see [47]). We then consider three different interiors with this exterior, which are anisotropic generalisations of FLRW models [34]. Consider the metric:

$$
\begin{equation*}
d s^{2+}=-A e^{-2 \delta} d t^{2}+A^{-1} d r^{2}+\frac{r^{2}}{4} e^{2 B}\left(\sigma_{1}^{2}+\sigma_{2}^{2}\right)+\frac{r^{2}}{4} e^{-4 B} \sigma_{3}^{2} \tag{4.1}
\end{equation*}
$$

where $A, \delta$, and $B$ are functions of $t$ and $r$. The one forms $\sigma_{i}$ are left invariant for the standard Lie group structure on $S^{3}$ and can be taken as

$$
\begin{gather*}
\sigma_{1}=\cos \psi d \theta+\sin \theta \sin \psi d \phi, \\
\sigma_{2}=\sin \psi d \theta-\sin \theta \cos \psi d \phi,  \tag{4.2}\\
\sigma_{3}=d \psi+\cos \theta d \phi,
\end{gather*}
$$

where $\theta, \psi, \phi$ are Euler angles on $S^{3}$ with $0<\theta<\pi, 0<\phi<2 \pi$, and $0<\psi<4 \pi$. The spacetime with $B \neq 0$ is interpreted as containing pure gravitational waves with radial symmetry [36] and the Schwarzschild-Tangherlini (SchT) limit is obtained for $B=0$. There is a residual coordinate freedom $t \rightarrow \widehat{t}=f(t) ; \delta \rightarrow \widehat{\delta}=\delta+\log \dot{f}$ in (4.1), which one can use to set $\delta$ arbitrarily along a timelike curve. The $(4+1)$ dimensional vacuum EFEs give [36]

$$
\begin{gather*}
\partial_{r} A=-\frac{2 A}{r}+\frac{1}{3 r}\left(8 e^{-2 B}-2 e^{-8 B}\right)-2 r\left(e^{2 \delta} A^{-1}\left(\partial_{t} B\right)^{2}+A\left(\partial_{r} B\right)^{2}\right)  \tag{4.3}\\
\partial_{t} A=-4 r A\left(\partial_{t} B\right)\left(\partial_{r} B\right) \tag{4.4}
\end{gather*}
$$

$$
\begin{equation*}
\partial_{r} \delta=-2 r\left(e^{2 \delta} A^{-2}\left(\partial_{t} B\right)^{2}+\left(\partial_{r} B\right)^{2}\right) \tag{4.5}
\end{equation*}
$$

together with the quasi-linear wave equation for $B$

$$
\begin{equation*}
\partial_{t}\left(e^{\delta} A^{-1} r^{3}\left(\partial_{t} B\right)\right)-\partial_{r}\left(e^{-\delta} A r^{3}\left(\partial_{r} B\right)\right)+\frac{4}{3} e^{-\delta} r\left(e^{-2 B}-e^{-8 B}\right)=0 \tag{4.6}
\end{equation*}
$$

In [36], the authors solve this system numerically by giving $B$ and $\partial_{t} B$ at $t=0$ with $A(0,0)=$ $0=\delta(t, 0)$. Within the BCS class, they provide numerical evidence that linear perturbations of SchT decay in time so that SchT is linearly stable, in this sense. The nonlinear stability of SchT within the BCS class was later established in [35].

We will give data $A, B$ and the normal derivative $\nabla_{n} B$ at the timelike boundary $\Omega$ of the collapsing interior with the gauge choice $\delta \stackrel{\Omega}{=} 0$. Uniqueness and local existence follow as standard. From [35], one knows that if data close to that for SchT is given on an asymptotically flat hypersurface then the solution will exist forever and stay close to the SchT solution. As we will see, in some of our cases, data on $\Omega$ can be chosen to be arbitrarily close to data for SchT.

### 4.1. Generalised FLRW Interiors

As interior metrics, we will consider three classes of FLRW-like solutions based on Riemannian Bianchi-IX spatial metrics which are, respectively, the Eguchi-Hanson metric (with $R_{i j}=0$ ), the $k$-Eguchi-Hanson metric (with $R_{i j}=k g_{i j}$ excluding the case $k=0$ ), and the $k$-Taub-NUT metric (with $R_{i j}=k g_{i j}$, including, $k=0$ as a particular case).

We summarize now our main result and leave the details of the proof to the next three sections (see also [34]).

Theorem 4.1. In each case, the interior metric gives consistent data for the metric (4.1) at a comoving time-like hypersurface. Local existence of the radiating exterior in the neighbourhood of the matching surface is then guaranteed. In the case of Eguchi-Hanson and $k$-Taub-NUT with $k<0$, the data can be chosen to be close to the data for the Schwarzschild-Tangherlini solution.

### 4.1.1. The Eguchi-Hanson Metric

Eguchi and Hanson found a class of self-dual solutions to the Euclidean Einstein equations with metric given by [48]

$$
\begin{equation*}
h_{\mathrm{EH}}=\left(1-\frac{a^{4}}{\rho^{4}}\right)^{-1} d \rho^{2}+\frac{\rho^{2}}{4}\left(\sigma_{1}^{2}+\sigma_{2}^{2}\right)+\frac{\rho^{2}}{4}\left(1-\frac{a^{4}}{\rho^{4}}\right) \sigma_{3}^{2} \tag{4.7}
\end{equation*}
$$

with $\sigma_{i}$ given by (4.2) and $a$ is a real constant. The generalised FLRW metric built on this Riemannian metric is

$$
\begin{equation*}
d s^{2-}=-d \tau^{2}+R^{2}(\tau) h_{\mathrm{EH}}, \tag{4.8}
\end{equation*}
$$

with the EFEs for a dust source reducing to

$$
\begin{equation*}
\mu R^{4}=\mu_{0}, \quad \dot{R}^{2}=\frac{\kappa \mu_{0}}{6 R^{2}} \tag{4.9}
\end{equation*}
$$

We match at $\rho=\rho_{0}$ so that $\Omega^{-}$is parameterised by $\Omega^{-}=\left\{\tau, \rho=\rho_{0}\right\}$. The equality of the first and second fundamental forms at $\Omega$ then gives

$$
\begin{equation*}
r \stackrel{\Omega}{=} R \rho e^{-B}, \quad e^{-6 B} \stackrel{\Omega}{=} 1-\frac{a^{4}}{\rho^{4}}, \quad \nabla_{n} B \stackrel{\Omega}{=}-\frac{2 a^{4}}{3 R \rho^{5}}\left(1-\frac{a^{4}}{\rho^{4}}\right)^{-1 / 2}, \quad A e^{-\delta} \dot{t} \stackrel{\Omega}{=} e^{2 B}\left(1-\frac{a^{4}}{3 \rho^{4}}\right) \tag{4.10}
\end{equation*}
$$

from which we calculate $A, \partial_{t} B$, and $\partial_{r} B$ on $\Omega$ in terms of quantities from the interior. At this point, we have $B, \nabla_{n} B$, and $A$ on $\Omega$, and we use the gauge freedom to set $\delta=0$ on $\Omega$. By (4.10), we have

$$
\begin{equation*}
B \stackrel{\Omega}{=} O\left(\frac{a^{4}}{\rho^{4}}\right), \quad \nabla_{n} B \stackrel{\Omega}{=}(\rho R)^{-1} O\left(\frac{a^{4}}{\rho^{4}}\right) \tag{4.11}
\end{equation*}
$$

and, to say that the data is close to SchT data, we want these to be small. The first term is small if $\rho \gg a$. The second term will increase without bound as $R$ decreases to zero. If we restrict $R$ by its value when a marginally outer trapped surface forms on $\Omega$ then, from the Friedman equation and with $\rho \gg a$, this happens when $R^{2} \rho^{2} \sim \kappa \mu_{0} \rho^{4}$ so that we control $\nabla_{n} B$ on $\Omega$ by controlling $\mu_{0}$. So, by choice of the location of $\Omega$, at $\rho=\rho_{0}$, and choice of $\mu_{0}$, we can choose data close to SchT.

### 4.1.2. The $k$-Eguchi-Hanson Metric

The $k$-Eguchi-Hanson metric is given by [49]

$$
\begin{equation*}
h_{k \mathrm{EH}}=\Delta^{-1} d \rho^{2}+\frac{\rho^{2}}{4}\left(\sigma_{1}^{2}+\sigma_{2}^{2}\right)+\frac{\rho^{2}}{4} \Delta \sigma_{3}^{2} \tag{4.12}
\end{equation*}
$$

where $\Delta=1-a^{4} / \rho^{4}-k \rho^{2} / 6$. This metric is complete for $k<0$ and $a^{4}=4(p-2)^{2}(p+1) / 3 k^{2}$, $\rho>(-2(p-2) / k)^{1 / 2}$, where $p \geq 3$ in an integer. Since $k$ is related to $a$ for a complete solution, we cannot obtain the previous case from this case by taking $k \rightarrow 0$. However, the matching formulae do formally allow this limit. The generalised FLRW metric built on this metric is $d s^{2-}=-d \tau^{2}+R^{2}(\tau) h_{k E H}$ and the respective EFEs for a dust source reduce to

$$
\begin{equation*}
\mu R^{4}=\mu_{0}, \quad \dot{R}^{2}+\frac{k}{3}=\frac{\kappa \mu_{0}}{6 R^{2}} \tag{4.13}
\end{equation*}
$$

The matching conditions on $\Omega$ give
$r \stackrel{\Omega}{=} R \rho e^{-B}, \quad e^{-6 B} \stackrel{\Omega}{=} \Delta, \quad \nabla_{n} B \stackrel{\Omega}{=}-\frac{\Delta^{-1 / 2}}{3 \rho R}\left(\frac{2 a^{4}}{\rho^{4}}-\frac{k \rho^{2}}{6}\right), \quad A e^{-\delta} \dot{t} \stackrel{\Omega}{=} e^{2 B}\left(1-\frac{a^{4}}{3 \rho^{4}}-\frac{2 k \rho^{2}}{9}\right)$.

We can calculate $A$ in terms of interior data, as before, and check that this is consistent with the EFEs (4.3) and (4.4). However, in this case, it is not clear that we may choose data close to SchT data as the normal derivative of $B$ is

$$
\begin{equation*}
\nabla_{n} B \stackrel{\Omega}{=} \frac{k \rho}{6 R} \tag{4.15}
\end{equation*}
$$

and for this to be small, we would require $R$ to be large on $\Omega$ outside the marginally trapped surface. It is hard to see how to arrange this and so although the solution in the exterior exists locally, we do not have a good reason to think that it will settle down to SchT.

### 4.1.3. $k$-Taub-NUT

We consider the Riemannian Taub-NUT metric with a cosmological constant ( $k$ rather than $\Lambda$ with our conventions) $[50,51]$

$$
\begin{equation*}
h_{\mathrm{TN}}=\frac{1}{4} \Sigma^{-1} d \rho^{2}+\frac{1}{4}\left(\rho^{2}-L^{2}\right)\left(\sigma_{1}^{2}+\sigma_{2}^{2}\right)+L^{2} \Sigma \sigma_{3}^{2} \tag{4.16}
\end{equation*}
$$

where $\Sigma=(\rho-L)(1-(k / 12)(\rho-L)(\rho+3 L)) /(\rho+L)$, and use it to construct the $4+1$ interior:

$$
\begin{equation*}
d s^{2-}=-d \tau^{2}+R^{2}(\tau) h_{\mathrm{TN}} . \tag{4.17}
\end{equation*}
$$

The EFEs for a dust source are again (4.13). From matching the first fundamental forms we get

$$
\begin{equation*}
r \stackrel{\Omega}{=} R\left(\rho^{2}-L^{2}\right)^{1 / 2} e^{-B}, \quad e^{-6 B} \stackrel{\Omega}{=} \frac{4 L^{2} \Sigma}{\rho^{2}-L^{2}} \tag{4.18}
\end{equation*}
$$

and the second matching conditions read

$$
\begin{equation*}
A e^{-\delta} \dot{t} \stackrel{\Omega}{=} \frac{4 R}{3 r} \Sigma^{1 / 2} e^{4 B}\left(\frac{2 L^{2}(2 \rho+L) \Sigma}{\rho^{2}-L^{2}}-\frac{k}{6} L^{2}(\rho-L)\right), \quad \nabla_{n} B \stackrel{\Omega}{=} \frac{1}{3 R}\left(\frac{2 \Sigma^{1 / 2}}{\rho+L}+\frac{k(\rho-L)}{6 \Sigma^{1 / 2}}\right) \tag{4.19}
\end{equation*}
$$

As before, the expression for $A$ on $\Omega$ is consistent with $\dot{A}$ calculated from (4.3) and (4.4). Now note that if $k L^{2}=-3$ then the metric (4.16) is precisely the 4-dimensional hyperbolic metric. In this case, $B$ and $\nabla_{n} B$ vanish on $\Omega$ whatever is the value of $\rho_{0}$, so that the exterior metric is precisely SchT: this is a case from the previous section as the interior is now a standard FLRW cosmology. Consequently, if we take $k L^{2}$ close to -3 we get data close to SchT data. To see this, set $k L^{2}=-3(1+\varepsilon)$. Then $\Sigma=\left(\left(\rho^{2}-L^{2}\right) / 4 L^{2}\right)(1+O(\varepsilon))$, so that

$$
\begin{equation*}
e^{-6 B} \stackrel{\Omega}{=} 1+O(\varepsilon), \quad \nabla_{n} B \stackrel{\Omega}{=} \frac{1}{L R} O(\varepsilon) \tag{4.20}
\end{equation*}
$$

and the data $\left(B, \nabla_{n} B\right)$ can be chosen as small as desired by choosing large $\rho_{0}$ and small $\varepsilon$.

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Research Article

# Mixed Initial-Boundary Value Problem for Telegraph Equation in Domain with Variable Borders 

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#### Abstract

Mixed initial-boundary value problem for telegraph equation in domain with variable borders is considered. On one part of domain's border are the boundary conditions of the first type, on other part of the boundary are set boundary conditions of the second type. Besides, the sizes of area are variable. The solution of such problem demands development of special methods. With the help of consecutive application of procedure of construction waves reflected from borders of domain, it is possible to obtain the solution of this problem in quadratures. In addition, for construction of the waves reflected from mobile border, it is necessary to apply the procedure specially developed for these purposes.


## 1. Introduction

Mixed initial-boundary value problems for telegraph equation in domain with variable borders arise in many applications. In particular, such problem arises in a problem about calculation of a field of stress in ropes of elevating devices. At lifting of load the rope reels up on a drum or reels off a drum at lowering a load. Therefore, the length of that part of a rope, which is reeled up on a drum, changes. If to take into account friction of a rope on a drum, elastic displacements in a rope can be described by the telegraph equation [1]. In [1] it is shown, that it occurs at dry friction. However, it is possible to show, that the telegraph equation describes elastic displacements and at viscous friction as well. Thus, there is a initialboundary value problem for the telegraph equation in domain with variable border. Here the telegraph equation of general view is considered. Regarding a rope which hangs down from a drum, elastic displacements there are described by the wave equation. This circumstance
induces to consider separately problems about elastic displacements to various parts of a rope.

Initial-boundary value problem on elastic displacements to that part of a rope which is reeled on a drum is considered in the present paper. The rope is considered as a flexible string. One end of a rope is attached to a drum and goes together with a drum. The extreme point of contact of a rope with a drum is accepted as the second end of a rope. Elastic stresses are set in this point. Change of length of a rope, which is reeled up on a drum, is taken into account as follows. Portable movement of system is understood as rotation of a drum and a rope as perfectly rigid body. Then the relative movement of a rope will be submitted as motionless in all points of a rope, except for an extreme point of contact of a rope with a drum. This last point in relative movement will make the moving equal $v(t)$, where $v(t)$ is movment of the central axis of a rope together with a drum. In relative movement all points of a rope make only elastic displacements. The axis $x$ is directed lengthways toward conditionally straightened rope and its beginning is located in a point of attaching of a rope to a drum. The initial length of a rope, which is reeled up on a drum, is equal to $l$.

At the solution of initial-boundary value problems for the telegraph equation the various exact and approached methods were used. It is necessary to notice that exact solutions were obtained only for the limited number of boundary problems.

Recently for the solution of boundary problems even for fractional telegraph equations it is actively used differential transformation method [2-4]. This method is an improved version of power series method or its modifications. In this case the same is; and in a power series method, representation of the solution of a problem in the form of type Taylor's series in a neighborhood of some point or a curve is used. Thanks to the developed procedure of use at an early stage of calculations of boundary conditions, calculation of expansion coefficients becomes essentially simpler. However representation of the solution in the form of Taylor's series type demands fulfillment of additional conditions. Following conditions concern them. It is necessary that all factors in the differential equation were analytical functions of the arguments. It is necessary also that the series obtained as the solution of the problem and especially series of derivatives converged uniformly. But the most important feature consists as such expansion of solutions provides good approach only in some neighborhood of an index point or a curve. On the essential distance from such neighborhood the values of higher degree terms become dominating.

Therefore preservation in expansion of final number of the terms can lead to considerable errors. Besides, at a great distance from area of initial or boundary values series can appear divergent. Therefore differential transformation method yields satisfactory results only on small intervals of change of spatial variables and time. Besides, for application of this method it is necessary that all functions included in the equation and boundary conditions supposed the same expansion that is used for solution representation. This condition cannot be executed in general case. Necessity to be limited to calculation of final number of terms of a series leads to that the solution appears approximate. At the same time, if all given problems can be presented in the form of finite series, the solution of such problem can be obtained exactly.

For the telegraph equations, including fractional ones, it is possible to solve some initial-boundary value problems by means of a method of separation variables [5]. For application of this method it is necessary that the domain of search of the solution possessed special symmetry, and initial and boundary functions as well as the right parts of the equations, supposed expansion on eigen functions of a boundary problem. It is clear that these conditions can be executed not always.

Existing methods of the solution of initial-boundary value problems cannot be applied to the telegraph equation in cases when the domain in which the solution is found is a variable. In particular, use for this purpose of a method separation of variables is impossible, as in case of mobile borders the corresponding problem of Sturm-Liouville has only trivial eigenfunctions.

In present paper the method especially developed in [6-12] for solution of problems about movement of waves in domain with mobile borders is used.

## 2. Statement of the Problem

The following initial-boundary value problem is considered: in the domain $0<x<l+v(t)$, $t>0$ to obtain the solution of the telegraph equation

$$
\begin{equation*}
\frac{\partial^{2} u(x, t)}{\partial x^{2}}-\frac{1}{a^{2}} \frac{\partial^{2} u(x, t)}{\partial t^{2}}+D \frac{\partial u(x, t)}{\partial t}+B \frac{\partial u(x, t)}{\partial x}+C u(x, t)=0 \tag{2.1}
\end{equation*}
$$

which satisfies initial conditions

$$
\begin{equation*}
u(x, 0)=0 ; \quad u_{t}(x, 0)=0 \tag{2.2}
\end{equation*}
$$

and mixed boundary conditions

$$
\begin{equation*}
u_{x}(l+v(t), t)=\gamma(t) ; \quad u(0, t)=0, \quad t>0 . \tag{2.3}
\end{equation*}
$$

Concerning function $v(t)$, describing displacement of the bottom end of a rope, it is supposed $v(0)=0$ and from a condition of preservation of integration area of an initial-boundary value problem follows that $v(t)>-l$ at $t>0$.

The solution of the problem put here cannot be carried out by existing methods because domain, in which the solution is found, is a variable. Therefore for the solution of such initial-boundary problems connected with the equations of hyperbolic type in [612] special method developed. This method has three prominent features. The first of them consists in integrated representation of solutions of the telegraph equation in the form of extending waves for an extensive class of boundary conditions [6]. Such representation is obtained by use of Riemann's method.

Use of the given integrated representation of solutions demands performance of continuation of initial and regional functions in the domain of any values of their arguments. This continuation should be carried out taking into account all conditions of statement of a problem. In it the second feature of a method consists.

At last, the third feature consists in working out of a method of construction of the waves reflected from mobile border. The given method reduces a reflexing problem to the solution of an auxiliary initial-boundary value problem with the initial conditions set at the moment of arrival of forward front of the falling wave on mobile border.

This method is used for the solution of stated problem.

## 3. The Solution of the Problem

For the solution of this problem the continuation of function $\gamma(t)$ on all axis $t$ is introduced as

$$
\Gamma(t)= \begin{cases}r(t), & t>0  \tag{3.1}\\ 0, & t<0\end{cases}
$$

and the first boundary condition (2.3) is considered as continued on all axis $t$ :

$$
\begin{equation*}
u_{x}(l+v(t), t)=\Gamma(t) \tag{3.2}
\end{equation*}
$$

At the first stage the solution of the given problem is searched as

$$
\begin{equation*}
u_{0}(x, t)=2 e^{-(B / 2) x} \int_{0}^{t+(x / a)} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta \tag{3.3}
\end{equation*}
$$

with unknown function $\Gamma_{0}$. Here $J_{0}, J_{1}$-Bessel's functions as the zero and first order, respectively,

$$
\begin{gather*}
z=\sqrt{c_{1}\left[x^{2}-a^{2}(t-\eta)^{2}\right]}  \tag{3.4}\\
c_{1}=C+\frac{D^{2} a^{2}}{4}-\frac{B^{2}}{4}
\end{gather*}
$$

In [10] it is shown that function (3.3) satisfies (2.1) at arbitrary function $\Gamma_{0}$. In the same place it is shown that for solution of boundary problems with boundary conditions of the second type it is the most expedient to apply the form of the solution of a kind (3.3). Having substituted the form of the solution (3.3) in a boundary condition (3.2) we obtain

$$
\begin{align*}
& \frac{2}{a} e^{-((B+D a) / 2)(l+v(t))} \Gamma_{0}\left(t+\frac{l+v(t)}{a}\right) \\
& \quad-2 e^{-(B / 2)(l+v(t))} \int_{0}^{t+((l+v(t)) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1}(l+v(t)) \frac{J_{1}(z)}{z}\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta=\Gamma(t) \tag{3.5}
\end{align*}
$$

In (3.5)

$$
\begin{equation*}
z=\sqrt{c_{1}\left[(v(t)+l)^{2}-a^{2}(t-\eta)^{2}\right]} . \tag{3.6}
\end{equation*}
$$

Thus, if function $\Gamma_{0}$ is the solution of the integral equation (3.5), function (3.3) will satisfy the first boundary condition (2.3).

With the purpose of obtaining an opportunity to represent a function $\Gamma_{0}$ with various arguments we shall introduce into (3.5) such transformation of a variable $t$ :

$$
\begin{equation*}
\tau=t+\frac{v(t)+l}{a} . \tag{3.7}
\end{equation*}
$$

In order that execution of transformation (3.7) in the integral equation (3.5) was possible, it is necessary that the function $t_{0}$ inverse to $\tau$ exists. Here the case is considered when the mobile end moves with subsonic speed. That means the next condition is satisfied

$$
\begin{equation*}
\left|v^{\prime}(t)\right|<a . \tag{3.8}
\end{equation*}
$$

Then from (3.7) follows

$$
\begin{equation*}
\frac{d \tau}{d t}=1+\frac{v^{\prime}(t)}{a}>0 . \tag{3.9}
\end{equation*}
$$

It means that (3.7) will be strictly monotonously growing and consequently there will be an inverse to them, function $t_{0}$, also strictly monotonously growing. Thus as $\tau(0)=l / a$, we obtain that $t_{0}(l / a)=0$. From (3.9) follows that $\tau>l / a$ at $t>0$. As function $v(t)$ is determined only at $t>0$, function $\tau(t)$ is determined also only at $t>0$. Accordingly function $t_{0}(\tau)$ will be determined only at $\tau>l / a$. At the same time during construction of the solution of a considered initial-boundary value problem there is a necessity of knowledge of function $\Gamma_{0}(\tau)$ behavior as well at values of argument $\Gamma_{0}(\tau)$, smaller than $l / a$.

With this purpose it is necessary to execute continuation of function $v(t)$ on all axis $t$. It appears that continuation of function $v(t)$ on all axes $t$ can be executed by arbitrary way, having demanded only existence of a derivative of this continuation on all axes $t$ and performance on all axes $t$ condition (3.8). We shall designate this continuation of function $v(t)$ through $v_{1}(t)$. Then on all axes $t$ such function will be determined:

$$
N(t)=\left\{\begin{align*}
v(t), & t>0 ;  \tag{3.10}\\
v_{1}(t), & t<0 .
\end{align*}\right.
$$

Continued on all axes $t$ of function $\tau(t)$, we shall designate $T(t)$ and we shall determine it by expression

$$
\begin{equation*}
T(t)=t+\frac{N(t)+l}{a} . \tag{3.11}
\end{equation*}
$$

From this expression and (3.10) it is clear that at $t>0, T(t)=\tau(t)$. As function $N(t)$ satisfies to an inequality

$$
\begin{equation*}
\left|N^{\prime}(t)\right|<a \tag{3.12}
\end{equation*}
$$

at all $t$, function $T(t)$ will be strictly monotonously growing and as $\tau(0)=l / a$, at $t<0$ it will be valid such inequality: $T(t)<l / a$.

As function $T(t)$ is strictly monotonous at all $t$, there exists inverse to this the function $T_{0}(T)$, and at $T \geq l / a, T_{0}(T)=T_{0}(\tau)$ and $T_{0}(T)$ will be strictly monotonously growing function. Thus, function $T_{0}(T)$ satisfies a condition

$$
T_{0}(T)= \begin{cases}t_{0}(\tau)>0, & T>\frac{l}{a}  \tag{3.13}\\ 0, & T=\frac{l}{a} \\ <0, & T<\frac{l}{a}\end{cases}
$$

Now after transformation (3.11) integral equation (3.5) will become

$$
\begin{align*}
& \frac{1}{a} e^{-((B+D a) / 2)\left(l+N\left(T_{0}(T)\right)\right)} \Gamma_{0}(T) \\
& \quad-e^{-(B / 2)\left(l+N\left(T_{0}(T)\right)\right)} \int_{0}^{T}\left[\frac{B}{2} J(z)+c_{1}\left(l+N\left(T_{0}(T)\right)\right) \frac{J_{1}(z)}{z}\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta=\frac{1}{2} \Gamma(t) . \tag{3.14}
\end{align*}
$$

In integral equation (3.14) becomes

$$
\begin{equation*}
z=\sqrt{c_{1}\left[\left(l+N\left(T_{0}(T)\right)\right)^{2}-a^{2}\left(T_{0}(T)-\eta\right)^{2}\right]} . \tag{3.15}
\end{equation*}
$$

From the integral equation (3.14), the quality (3.1) of function $\Gamma(t)$ and an equality (3.13) follow, that is

$$
\begin{equation*}
\Gamma_{0}(T)=0, \quad T<\frac{l}{a} \tag{3.16}
\end{equation*}
$$

In turn, from the quality (3.16) of function $\Gamma_{0}(T)$ follows (3.3) satisfing initial conditions (2.2). Really, from the formula (3.3) directly follows that at $t=0$ upper limit of integration becomes equal to $x / a$. But at $t=0, x<l$ and on the basis of the quality (3.16) of function $\Gamma_{0}(T)$ follows that at $t=0$ function (3.3) will be equal to zero. That means it satisfies the first initial condition (2.2).

Having differentiated function (3.3) on $t$, we shall obtain

$$
\begin{align*}
\frac{\partial u_{0}(x, t)}{\partial t}= & \frac{2}{a} e^{-\left(\left(B+D a^{2}\right) / 2\right) x} \Gamma_{0}\left(t+\frac{x}{a}\right) \\
& +2 a^{2} e^{-(B / 2) x} \int_{0}^{t+(x / a)}\left[\frac{D}{2} J_{0}(z)+c_{1}(t-\eta) \frac{J_{1}(z)}{z}\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta \tag{3.17}
\end{align*}
$$

If in the formula (3.17) we set $t=0$, then argument of function $\Gamma_{0}$ and also the upper limit of integration at $x<l$ become smaller than $l / a$. Therefore on the basis of function's $\Gamma_{0}$
quality (3.16) the derivative (3.17) at $t=0$ will equal to zero. And it means that function (3.3) will satisfy also the second initial condition (2.2).

Thus, function (3.3) satisfies all conditions of statement of an initial-boundary value problem, except for the second boundary condition (2.3). With the purpose to check this condition we shall calculate from (3.3)

$$
\begin{equation*}
u_{0}(0, t)=2 \int_{0}^{t} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta \tag{3.18}
\end{equation*}
$$

From the formula (3.18) on the basis of function's $\Gamma_{0}$ quality (3.16) follows the function (3.3) at $t<l / a$ satisfing the second boundary condition (2.3) as well. With the purpose of satisfaction of the second boundary condition (2.3) at $t>l / a$, the solution of an initialboundary value problem we have to search as the sum of two functions is

$$
\begin{equation*}
u(x, t)=u_{0}(x, t)+u_{1}(x, t) \tag{3.19}
\end{equation*}
$$

where

$$
\begin{equation*}
u_{1}(x, t)=-2 e^{-(B / 2) x} \int_{0}^{t-(x / a)} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta . \tag{3.20}
\end{equation*}
$$

Function (3.20) satisfies (2.1) with arbitrary function $\Gamma_{0}$. It is obvious the function (3.19) satisfies the second boundary condition (2.3) at all $t$. In the same way as for function $u_{0}(x, t)$ it is possible to check up that the function (3.19) satisfies initial conditions (2.2).

In order that (3.19) satisfies the first boundary condition (2.3) it is necessary the function $u_{1}(x, t)$ satisfies a boundary condition

$$
\begin{equation*}
u_{1, x}(l+v(t), t)=0, \quad t>0 . \tag{3.21}
\end{equation*}
$$

Having calculated value of function $u_{1, x}(x, t)$ at point $x=l+v(t)$, we obtain

$$
\begin{align*}
& -\frac{2}{a} e^{-((B+D a) / 2)(l+v(t))} \Gamma_{0}\left(t-\frac{l+v(t)}{a}\right) \\
& \quad+2 e^{-(B / 2)(l+v(t))} \int_{0}^{t-((l+v(t)) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1}(l+v(t)) \frac{J_{1}(z)}{z}\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta=0 \tag{3.22}
\end{align*}
$$

In formula (3.22)

$$
\begin{equation*}
z=\sqrt{c_{1}\left[(v(t)+l)^{2}-a^{2}(t-\eta)^{2}\right]} . \tag{3.23}
\end{equation*}
$$

In order that equality (3.22) was valid, on the basis of function's $\Gamma_{0}$ quality (3.16), it is necessary that the argument of this function in equality (3.22) satisfies condition

$$
\begin{equation*}
t-\frac{l+v(t)}{a}<\frac{l}{a} \tag{3.24}
\end{equation*}
$$

Thus, function (3.19) will be the solution of a problem at $t<(2 l+v(t)) / a$. At $t>(2 l+v(t)) / a$ for satisfaction of first boundary condition (2.3) in the solution (3.19) it is necessary to enter the amendment. To enter such amendment by the methods used for domains with motionless borders, as shown in [7-12] for a case of the wave equation, is impossible. Therefore for corrective action the approach developed in [7-12] is used. With this purpose we shall notice that during an interval of time determined by a condition $t=(v(t)+2 l) / a$, forward front of the wave, radiated on the mobile end, starting from the moment of time $t=0$, will reach the end $x=0$, will be reflected from it, and will meet the mobile end. The length of this interval of time will be determined as the less positive root $\tau_{1}$ of equation

$$
\begin{equation*}
a t=v(t)+2 l . \tag{3.25}
\end{equation*}
$$

The left part of (3.25) at $t=0$ is less than the right part. At the same time on the basis of a condition (3.8) at $t>0$ left part of this equation grows faster than right part. Hence, the positive root of $(3.25)$ exists. From the carried out reasoning it is clear also that the condition (3.22) will be valid at $t<\tau_{1}$. Really, at $t=0$ inequality $t<(v(t)+2 l) / a$ is valid, as $v(0)=0$. At the same time $\tau_{1}$ is the less positive number at which this inequality turns into equality (3.25). Therefore correction function $u_{2}(x, t)$, being in essence of a wave reflected from the mobile end, is under construction as the solution of such an auxiliary initial-boundary value problem: in the domain $0<x>l+v(t), t>\tau_{1}$ to obtain the solution of the telegraph equation (2.1) satisfying initial conditions

$$
\begin{equation*}
u\left(x, \tau_{1}\right)=0 ; \quad u_{t}\left(x, \tau_{1}\right)=0 ; \quad 0<x<l+v\left(\tau_{1}\right), \tag{3.26}
\end{equation*}
$$

and boundary conditions

$$
\begin{equation*}
u_{x}(l+v(t), t)=-u_{1, x}(l+v(t), t) ; \quad u(0, t)=0, \quad t>\tau_{1} . \tag{3.27}
\end{equation*}
$$

The solution of this auxiliary initial-boundary value problem is constructed as function satisfying (2.1) at arbitrary function $\Gamma_{2}$ :

$$
\begin{equation*}
u_{2}(x, t)=2 e^{-(B / 2) x} \int_{0}^{t+(x / a)} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2}(\eta) d \eta \tag{3.28}
\end{equation*}
$$

Having substituted function (3.28) in the first boundary condition (3.27), we obtain

$$
\begin{align*}
& \frac{2}{a} \Gamma_{2}\left(t+\frac{v(t)+l}{a}\right) e^{-((D a+B) / 2)(l+v(t))} \\
& \quad-2 e^{-(B / 2)(l+v(t))} \int_{0}^{t+((v(t)+l) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2}(\eta) d \eta  \tag{3.29}\\
& \quad=-u_{1, x}(l+v(t), t) .
\end{align*}
$$

Thus, if function $\Gamma_{2}$ will be the solution of the integral equation (3.29), the function

$$
\begin{equation*}
u(x, t)=u_{0}(x, t)+u_{1}(x, t)+u_{2}(x, t) \tag{3.30}
\end{equation*}
$$

will satisfy the second boundary condition (2.3) at all $t$. Having executed in (3.29) transformation (3.11), we shall obtain

$$
\begin{align*}
& 2 \Gamma_{2}(T) e^{-((D a+B) / 2)\left(l+v\left(T_{0}(T)\right)\right)} \\
& \quad-2 a e^{-(B / 2)\left(l+v\left(T_{0}(T)\right)\right)} \int_{0}^{T}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v\left(T_{0}(T)\right)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)\left(T_{0}(T)-\eta\right)} \Gamma_{2}(\eta) d \eta  \tag{3.31}\\
& \quad=-a u_{1, x}\left(l+v\left(T_{0}(T)\right), T_{0}(T)\right) .
\end{align*}
$$

As the right part of the integral equations (3.29) and (3.31) is equal to zero at $t<\tau_{1}$, function $\Gamma_{2}(T)$ also will be equal to zero at $t<\tau_{1}$. We shall find out, which additional values of $T$ function $\Gamma_{2}(T)$ will be equal to zero. As function $T(t)$ strictly monotonously grows, such inequality will be valid

$$
\begin{equation*}
T(t)<T\left(\tau_{1}\right), \quad \text { at } t<\tau_{1} . \tag{3.32}
\end{equation*}
$$

Using in this inequality the definition (3.11) of function $T(t)$ and value $\tau_{1}$ from (3.25), we shall obtain

$$
\begin{equation*}
T<\tau_{1}+\frac{v\left(\tau_{1}\right)+l}{a}=\frac{3 l+2 v\left(\tau_{1}\right)}{a} . \tag{3.33}
\end{equation*}
$$

Thus, function $\Gamma_{2}(T)$ possesses the following quality:

$$
\begin{equation*}
\Gamma_{2}(T)=0, \quad T<\frac{3 l+2 v\left(\tau_{1}\right)}{a} . \tag{3.34}
\end{equation*}
$$

If now to put in equality (3.28) $t=0$, the upper limit of integration in this formula will accept value $x / a$. Taking into account that at $t=0$ it is valid $x<l$, we shall have that $x / a<l / a$. But as $l>v\left(\tau_{1}\right)$, we shall obtain that

$$
\begin{equation*}
\frac{l}{a}<\frac{3 l+2 v\left(\tau_{1}\right)}{a} \tag{3.35}
\end{equation*}
$$

It means that at $t=0$ function (3.28) will equal to zero, that is, will satisfy the first initial condition (2.2).

Having calculated a derivative of function (3.28) on $t$, we shall obtain

$$
\begin{align*}
\frac{\partial u_{2}(x, t)}{\partial t}= & \frac{2}{a} e^{-\left(\left(B+D a^{2}\right) / 2\right) x} \Gamma_{2}\left(t+\frac{x}{a}\right) \\
& +2 a^{2} e^{-(B / 2) x} \int_{0}^{t+(x / a)}\left[\frac{D}{2} J_{0}(z)+c_{1}(t-\eta) \frac{J_{1}(z)}{z}\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2}(\eta) d \eta \tag{3.36}
\end{align*}
$$

At $t=0$ argument of function $\Gamma_{2}$ and also the upper limit of integration in the formula (3.36) will accept value $x / a$. Hence, as shown above, in the domain of search of the solution at such values of argument, the function $\Gamma_{2}$ will be equal to zero. Therefore, the derivative of function $u_{2}(x, t)$ on $t$ at $t=0$ will be equal to zero. And it means that function $u_{2}(x, t)$ satisfies as well the second initial condition (2.2).

Thus, function (3.30) satisfies all conditions of statement of the basic initial-boundary value problem, except for the second boundary condition (2.3). In order that this boundary condition was carried out, it is necessary that there was valid an equality

$$
\begin{equation*}
u_{2, x}(0, t)=0, \quad t>0 . \tag{3.37}
\end{equation*}
$$

Having substituted function (3.28) in the left part of a boundary condition (3.37), we shall obtain

$$
\begin{equation*}
u_{2, x}(0, t)=\frac{2}{a} \Gamma_{2}(t)-2 \int_{0}^{t} \frac{B}{2} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2}(\eta) d \eta . \tag{3.38}
\end{equation*}
$$

As follows from function's $\Gamma_{2}$ quality (3.34), expression (3.38) will equal zero, that is, will satisfy a boundary condition (3.37) only at validity of an inequality $t<\left(3 l+2 v\left(\tau_{1}\right)\right) / a$. For satisfaction of the second boundary condition at big $t$ in the solution (3.30), the amendment $u_{3}(x, t)$ is entered that is, the solution is represented as

$$
\begin{equation*}
u(x, t)=u_{0}(x, t)+u_{1}(x, t)+u_{2}(x, t)+u_{3}(x, t) \tag{3.39}
\end{equation*}
$$

where

$$
\begin{equation*}
u_{3}(x, t)=-2 e^{-(B / 2) x} \int_{0}^{t-(x / a)} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2}(\eta) d \eta . \tag{3.40}
\end{equation*}
$$

Function $u_{3}(x, t)$ satisfies (2.1) at arbitrary function $\Gamma_{2}$ and should provide performance of a boundary condition

$$
\begin{equation*}
u_{2}(0, t)+u_{3}(0, t)=0, \quad t>0 \tag{3.41}
\end{equation*}
$$

Fact that functions (3.28) and (3.40) satisfy a boundary condition (3.41) is practically obvious.
With the same way, as it is made for function $u_{2}(x, t)$, it is possible to show that function $u_{3}(x, t)$ will satisfy initial conditions (2.2).

Thus, function (3.39) satisfies all conditions of statement of the basic initial-boundary value problem, except for the first boundary condition (2.3). In order that this boundary condition was carried out, it is necessary that there was valid an equality

$$
\begin{equation*}
u_{3, x}(l+v(t), t)=0, \quad t>0 . \tag{3.42}
\end{equation*}
$$

Having calculated value of derivative function (3.40) on $x$ in a point $x=l+v(t)$, we shall obtain

$$
\begin{align*}
u_{3, x}(l+v(t), t)= & \frac{2}{a} e^{((D a-B) / 2)(l+v(t))} \Gamma_{2}\left(t-\frac{v(t)+l}{a}\right) \\
& +2 e^{-(B / 2)(l+v(t))} \int_{0}^{t-((v(t)+l) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2}(\eta) d \eta . \tag{3.43}
\end{align*}
$$

On the basis of function's $\Gamma_{2}$ quality (3.34) one can conclude that the right part of equality (3.43) will be equal to zero if argument of function $\Gamma_{2}$ and the upper limit of integration in the formula (3.43) will satisfy an inequality

$$
\begin{equation*}
t-\frac{l+v(t)}{a}<\frac{3 l+2 v\left(\tau_{1}\right)}{a} \tag{3.44}
\end{equation*}
$$

whence follows

$$
\begin{equation*}
t<\frac{4 l+2 v\left(\tau_{1}\right)+v(t)}{a} \tag{3.45}
\end{equation*}
$$

Hence, as $t$ satisfies an inequality (3.45), the boundary condition (3.42) will be carried out. The inequality (3.45) is inconvenient for use as its left and right parts depend on $t$. With the purpose of more convenient use of this inequality we shall consider the equation

$$
\begin{equation*}
t=\frac{4 l+2 v\left(\tau_{1}\right)+v(t)}{a} . \tag{3.46}
\end{equation*}
$$

Also we shall designate as $\tau_{2}$ the less positive root of this equation. At $t=0$ right part of (3.46) is more than the left part. At the same time by virtue of a condition (3.8) right part of (3.46) grows faster than its left part; therefore, the positive root of (3.46) exists. Hence, number $\tau_{2}$ is the less positive number at which the inequality (3.45) terns into equality. Therefore the inequality (3.45) is equivalent to the inequality

$$
\begin{equation*}
t<\frac{4 l+2 v\left(\tau_{1}\right)+v\left(\tau_{2}\right)}{a}=\tau_{2} . \tag{3.47}
\end{equation*}
$$

Let us notice that on physical sense of an initial-boundary value problem at the moment of time $t=\tau_{2}$ forward front of the wave radiated with the mobile end, having twice reflected from the motionless end and having once reflected from the mobile end, it will meet again the mobile end. At $t>\tau_{2}$ function $u_{3}(x, t)(3.40)$ will not satisfy any more to a boundary condition (3.42). Therefore at $t>\tau_{2}$ solution of the basic initial-boundary value problem is searched as

$$
\begin{equation*}
u(x, t)=u_{0}(x, t)+u_{1}(x, t)+u_{2}(x, t)+u_{3}(x, t)+u_{4}(x, t) \tag{3.48}
\end{equation*}
$$

where function $u_{4}(x, t)$ is under construction as the solution of the following auxiliary initialboundary value problem. In the domain $0<x<l+v(t), t>\tau_{2}$ to obtain the solution of the telegraph equation (2.1) satisfying initial conditions

$$
\begin{equation*}
u\left(x, \tau_{2}\right)=0 ; \quad u_{t}\left(x, \tau_{2}\right)=0 ; \quad 0<x<l+v\left(\tau_{2}\right) \tag{3.49}
\end{equation*}
$$

and boundary conditions

$$
\begin{equation*}
u_{x}(l+v(t), t)=-u_{3, x}(l+v(t), t) ; \quad u_{x}(0, t)=0, \quad t>\tau_{2} . \tag{3.50}
\end{equation*}
$$

The solution of this auxiliary initial-boundary value problem is under construction as the function being the solution of (2.1) at arbitrary function $\Gamma_{4}$ is

$$
\begin{equation*}
u_{4}(x, t)=2 e^{-(B / 2) x} \int_{0}^{t+(x / a)} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{4}(\eta) d \eta \tag{3.51}
\end{equation*}
$$

Having substituted function (3.51) in the first boundary condition (3.50), we shall obtain

$$
\begin{align*}
& \frac{2}{a} \Gamma_{4}\left(t+\frac{v(t)+l}{a}\right) e^{-((D a+B) / 2)(l+v(t))} \\
& \quad-2 e^{-(B / 2)(l+v(t))} \int_{0}^{t+((v(t)+l) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{4}(\eta) d \eta  \tag{3.52}\\
& \quad=-u_{3, x}(l+v(t), t) .
\end{align*}
$$

Thus, if function $\Gamma_{4}$ will be the solution of the integral equation (3.52) the function (3.48) will satisfy the second boundary condition (2.3) at all $t$. Having executed in (3.52) transformation (3.11), we shall obtain

$$
\begin{align*}
& 2 \Gamma_{4}(T) e^{-((D a+B) / 2)\left(l+v\left(T_{0}(T)\right)\right)} \\
& \quad-2 a e^{-(B / 2)\left(l+v\left(T_{0}(T)\right)\right)} \int_{0}^{T}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v\left(T_{0}(T)\right)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)\left(T_{0}(T)-\eta\right)} \Gamma_{4}(\eta) d \eta  \tag{3.53}\\
& \quad=-a u_{3, x}\left(l+v\left(T_{0}(T)\right), T_{0}(T)\right)
\end{align*}
$$

As the right part of the integral equations (3.52) and (3.53) is equal to zero at $t<\tau_{2}$, function $\Gamma_{4}(T)$ also will be equal to zero at $t<\tau_{2}$. We shall find out at what additional values $T$ function $\Gamma_{4}(T)$ will be equal to zero. As function $T(t)$ strictly monotonously grows, the next inequality will be valid

$$
\begin{equation*}
T(t)<T\left(\tau_{2}\right), \quad \text { at } t<\tau_{2} . \tag{3.54}
\end{equation*}
$$

Using in this inequality definition (3.11) of function $T(t)$ and value $\tau_{2}$ from (3.52), we shall obtain

$$
\begin{equation*}
T<\tau_{2}+\frac{v\left(\tau_{2}\right)+l}{a}=\frac{5 l+2 v\left(\tau_{1}\right)+2 v\left(\tau_{2}\right)}{a} \tag{3.55}
\end{equation*}
$$

Thus, function $\Gamma_{4}(T)$ possesses the following quality:

$$
\begin{equation*}
\Gamma_{4}(T)=0, \quad T<\frac{5 l+2 v\left(\tau_{1}\right)+2 v\left(\tau_{1}\right)}{a} . \tag{3.56}
\end{equation*}
$$

Precisely the same as it is made for function $u_{2}(x, t)$, it is possible to show that function $u_{4}(x, t)$ will satisfy initial conditions (2.2). Thus, function (3.48) satisfies all conditions of statement of the basic initial-boundary value problem, except for the second boundary condition (2.3). This boundary condition function $u_{4}(x, t)$ will satisfy only at the some of values of $t>\tau_{2}$. To obtain the solution of the basic initial-boundary value problem at all $t>\tau_{2}$, (3.48) it is necessary to introduce the additional amendment into function (3.48).

Having continued process of corrective actions in the solution, we shall obtain that function

$$
\begin{align*}
u(x, t)= & \sum_{n=0}^{\infty} 2 e^{-(B / 2) x} \int_{0}^{t+(x / a)} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n}(\eta) d \eta \\
& -\sum_{n=0}^{\infty} 2 e^{-(B / 2) x} \int_{0}^{t-(x / a)} J_{0}(z) e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n}(\eta) d \eta \tag{3.57}
\end{align*}
$$

will be the solution of a considered initial-boundary value problem. Here function $\Gamma_{0}$ is the solution of the integral equation (3.14), and other functions $\Gamma_{2 n}$ are solutions of the following integral equations:

$$
\begin{align*}
& 2 \Gamma_{2 n}(T) e^{-((D a+B) / 2)\left(l+v\left(T_{0}(T)\right)\right)} \\
& \qquad-2 a e^{-(B / 2)\left(l+v\left(T_{0}(T)\right)\right)} \int_{0}^{T}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v\left(T_{0}(T)\right)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)\left(T_{0}(T)-\eta\right)} \Gamma_{2 n}(\eta) d \eta  \tag{3.58}\\
& \quad=-a u_{2 n-1, x}\left(l+v\left(T_{0}(T)\right), T_{0}(T)\right)
\end{align*}
$$

Here

$$
\begin{align*}
& u_{2 n-1, x}(l+v(t), t) \\
& =\frac{2}{a} e^{((D a-B) / 2)(l+v(t))} \Gamma_{2 n-2}\left(t-\frac{v(t)+l}{a}\right)  \tag{3.59}\\
& \quad+2 e^{-(B / 2)(l+v(t))} \int_{0}^{t-((v(t)+l) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n-2}(\eta) d \eta .
\end{align*}
$$

Thus functions $\Gamma_{2 n}$ possess the following qualities:

$$
\begin{equation*}
\Gamma_{2 n}(t)=0, \quad t<\frac{(2 n+1) l+2 \sum_{i=1}^{n} v\left(\tau_{i}\right)}{a}, n=0,1, \ldots, \tag{3.60}
\end{equation*}
$$

where $\tau_{n}$ is less positive root of equation

$$
\begin{equation*}
t=\frac{2 n l+2 \sum_{i=1}^{n-1} v\left(\tau_{i}\right)+v(t)}{a} . \tag{3.61}
\end{equation*}
$$

By virtue of these qualities, at everyone fixed $t=H$ in the formula (3.57) will be only final number of terms distinct from zero. Really, in the sums of the formula (3.57) each term under conditions (3.56) becomes equal to zero, if the upper limit of integration is less than the right part of an inequality (3.60). For the first sum of the formula (3.57) such condition at $t=H$ looks like

$$
\begin{equation*}
H+\frac{x}{a}<\frac{(2 n+1) l+2 \sum_{i=1}^{n} v\left(\tau_{i}\right)}{a}, \tag{3.62}
\end{equation*}
$$

whence follows

$$
\begin{equation*}
n>\frac{1}{2 l}\left(H a+x-l-2 \sum_{i-1}^{n} v\left(\tau_{i}\right)\right) . \tag{3.63}
\end{equation*}
$$

And as in the domain of obtaining the solution, next inequality is valid:

$$
\begin{equation*}
0<x<l+v(H), \tag{3.64}
\end{equation*}
$$

and we obtain that at all $n$, satisfying a condition

$$
\begin{equation*}
n>\frac{1}{2 l}\left(H a+v(H)-2 \sum_{i-1}^{n} v\left(\tau_{i}\right)\right), \tag{3.65}
\end{equation*}
$$

all terms in the first sum of formula (3.57) will be equal to zero. Differently, summation in the first sum of the formula (3.57) needs to be made in this case not up to infinity, but up to $N-1$, where $N$ is the less natural number satisfying an inequality (3.65).

For the second sum of the formula (3.57), condition that the upper limit of integration is less than right part of inequality (3.60) at $t=H$ looks like

$$
\begin{equation*}
H-\frac{x}{a}<\frac{(2 n+1) l+2 \sum_{i=1}^{n} v\left(\tau_{i}\right)}{a}, \tag{3.66}
\end{equation*}
$$

from which follows

$$
\begin{equation*}
n>\frac{1}{2 l}\left(H a-x-l-2 \sum_{i=1}^{n} v\left(\tau_{i}\right)\right) \tag{3.67}
\end{equation*}
$$

Therefore on the basis of an inequality (3.64) it is obtained that at all $n$, satisfying a condition

$$
\begin{equation*}
n>\frac{1}{2 l}\left(H a-l-2 \sum_{i=1}^{n} v\left(\tau_{i}\right)\right) \tag{3.68}
\end{equation*}
$$

all terms in the second sum of formula (3.57) will be equal to zero. Differently, summation in the second sum of the formula (3.57) needs to be made in this case not up to infinity, but up to $N_{1}-1$, where $N_{1}$ is the less natural number satisfying inequality (3.68).

All terms in the formula (3.57) are solutions of (2.1). And as for everyone fixed $t$ number of terms in the formula (3.57) is finite, differentiation in the formula (3.57) is possible to carry out term by term. Therefore function (3.57) is the solution of (2.1).

From the formula (3.57) directly follows that at $t=0$ and $0<x<l$ the upper limits of integration of all integrals become smaller, than $l / a$. It means that on the basis of function's $\Gamma_{k}$ qualities, (3.60) from (3.57) follows $u(x, 0)=0$. Thus, function (3.57) satisfies the first initial condition (2.2). Having differentiated function (3.57) on $t$ we obtain

$$
\begin{align*}
\frac{\partial u(x, t)}{\partial t}=\sum_{n=0}^{\infty}\{ & \left\{\frac{2}{a} e^{-\left(\left(B+D a^{2}\right) / 2\right) x} \Gamma_{2 n}\left(t+\frac{x}{a}\right)\right. \\
& \left.+2 a^{2} e^{-(B / 2) x} \int_{0}^{t+(x / a)}\left[\frac{D}{2} J_{0}(z)+c_{1}(t-\eta) \frac{J_{1}(z)}{z}\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n}(\eta) d \eta\right\} \\
+ & \sum_{n=0}^{\infty}\left\{\frac{2}{a} e^{e((D a-B) / 2) x} \Gamma_{2 n}\left(t-\frac{x}{a}\right)\right. \\
& \left.\quad-2 a^{2} e^{-(B / 2) x} \int_{0}^{t-(x / a)}\left[\frac{D}{2} J_{0}(z)+c_{1} \frac{t-\eta}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n}(\eta) d \eta\right\} . \tag{3.69}
\end{align*}
$$

From the formula (3.69) it is obtained that at $t=0$ and $0<x<l$ the upper limits of integration of all integrals and as well arguments of functions $\Gamma_{k}$ become smaller, than $l / a$. It means, on the basis of function's $\Gamma_{k}$ qualities (3.60), that $u_{t}(x, 0)=0$. Thus, function (3.57) satisfies also the second initial condition (2.2).

Having calculated value of derivative of function (3.57) on $x$ in point $x=l+v(t)$, we obtain

$$
\begin{aligned}
& u_{x}(l+v(t), t) \\
& =\sum_{n=0}^{\infty}\left\{\frac{2}{a} \Gamma_{2 n}\left(t+\frac{l+v(t)}{a}\right) e^{-((D a+B) / 2)(l+v(t))}\right. \\
& \left.\quad-2 e^{-(B / 2)(l+v(t))} \int_{0}^{t+((l+v(t)) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n}(\eta) d \eta\right\}
\end{aligned}
$$

$$
\begin{align*}
+\sum_{n=0}^{\infty}\{ & \frac{2}{a} e^{((D a-B) / 2)(l+v(t))} \Gamma_{2 n}\left(t-\frac{l+v(t)}{a}\right) \\
& \left.\quad-2 e^{-(B / 2)(l+v(t))} \int_{0}^{t-((l+v(t)) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n}(\eta) d \eta\right\} \tag{3.70}
\end{align*}
$$

In the formula (3.70) we shall write down the first term of the first sum (at $n=0$ ) separately, and in the second sum we shall replace an index of summation $n$ on $s=n+1$. We shall obtain

$$
\begin{align*}
u_{x}(l+ & v(t), t) \\
= & \frac{2}{a} \Gamma_{0}\left(t+\frac{l+v(t)}{a}\right) e^{-((D a+B) / 2)(l+v(t))} \\
& -2 e^{-(B / 2)(l+v(t))} \int_{0}^{t+((l+v(t)) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{0}(\eta) d \eta \\
+ & \sum_{n=1}^{\infty}\left\{\frac{2}{a} \Gamma_{2 n}\left(t+\frac{l+v(t)}{a}\right) e^{-((D a+B) / 2)(l+v(t))}\right. \\
& \left.-2 e^{-(B / 2)(l+v(t))} \int_{0}^{t+((l+v(t)) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 n}(\eta) d \eta\right\} \\
+ & \sum_{s=1}^{\infty}\left\{\frac{2}{a} e^{((D a-B) / 2)(l+v(t))} \Gamma_{2 s-2}\left(t-\frac{l+v(t)}{a}\right)\right. \\
& \left.-2 e^{-(B / 2)(l+v(t))} \int_{0}^{t-((l+v(t)) / a)}\left[\frac{B}{2} J_{0}(z)+c_{1} \frac{l+v(t)}{z} J_{1}(z)\right] e^{\left(D a^{2} / 2\right)(t-\eta)} \Gamma_{2 s-2}(\eta) d \eta\right\} \tag{3.71}
\end{align*}
$$

First two summands in this formula represent the left part of the integral equation (3.5) and consequently are equal to $\Gamma(t)$. Summands in the first sum of last formula represent the left parts of the integral equations (3.58) divided on $a$. It means they are equal to $-\boldsymbol{u}_{(2 n-1) x}(l+$ $v(t), t)$. Summands in the second sum represent functions $u_{(2 n-1) x}(l+v(t), t)$. Therefore all terms under signs $\Sigma$ in last formula will equal zero. Hence

$$
\begin{equation*}
u_{x}(l+v(t), t)=\Gamma(t)=\gamma(t), \quad t>0 \tag{3.72}
\end{equation*}
$$

And it means that function (3.57) at $t>0$ satisfies the first boundary condition (2.3). The fact that function (3.57) satisfies the second boundary condition (2.3) is obvious.

Thus, it is shown that function (3.57) satisfing all conditions of statement of the basic initial-boundary value problem consequently is its solution.

## 4. Conclusion

The exact solution of the mixed initial-boundary value problem for the telegraph equation in domain with mobile borders is obtained. The solution is obtained as superposition of an initial wave and waves of reflection from the borders of a domain. It is necessary to note that the form of the solution in accuracy corresponds to those natural phenomena which, in particular, occur in a rope during its loading. The solution of a problem represents value of a field of elastic displacements in a rope. To obtain a field of pressure in a rope, it is enough to differentiate a field of displacements on $x$ and to increase this result on the module of elasticity of a rope.

Here, the developed method of the solution of evolutionary initial-boundary value problems with mobile borders of domain is suitable for search of solutions of a wide class of similar problems.

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Research Article

# A Study on the Convergence of Series Solution of Non-Newtonian Third Grade Fluid with Variable Viscosity: By Means of Homotopy Analysis Method 

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#### Abstract

This work is concerned with the series solutions for the flow of third-grade non-Newtonian fluid with variable viscosity. Due to the nonlinear, coupled, and highly complicated nature of partial differential equations, finding an analytical solution is not an easy task. The homotopy analysis method (HAM) is employed for the presentation of series solutions. The HAM is accepted as an elegant tool for effective solutions for complicated nonlinear problems. The solutions of (Hayat et al., 2007) are developed, and their convergence has been discussed explicitly for two different models, namely, constant and variable viscosity. An error analysis is also described. In addition, the obtained results are illustrated graphically to depict the convergence region. The physical features of the pertinent parameters are presented in the form of numerical tables.


## 1. Introduction

During the last few years, there has been substantial progress in the steady and unsteady flows of non-Newtonian fluids. A huge amount of literature is now available on the topic (see some studies [1-6]). All real fluids are diverse in nature. Hence in view of rheological characteristics, all non-Newtonian fluids cannot be explained by employing one constitutive equation. This is the striking difference between viscous and the non-Newtonian fluids. The rheological parameters appearing in the constitutive equations lead to a higher-order and complicated governing equations than the Navier-Stokes equations. The simplest subclass of differential-type fluids is called the second grade. In steady flow such fluids can predict the normal stress and does not show shear thinning and shear thickening behaviors. The thirdgrade fluid puts forward the explanation of shear thinning and shear thickening properties.

Therefore, the present paper aims to study the pipe flow of a third-grade fluid. Some progress on the topic is mentioned in the studies [7,8] and many references therein. In all these studies, variable viscosity is used. Massoudi and Christie [9] numerically examined the pipe flow of a third-grade fluid when viscosity depends upon temperature. Hayat et al. [10] presented the homotopy solution of the problem considered in [10] up to second-order deformation.

In this paper, the motivation comes from a desire to understand the convergence of the problem discussed in [10]. The relevant equations for flow and temperature have been solved analytically by using homotopy analysis method [11-15]. Here the convergence of the obtained solutions is explicitly shown,and that was not previously given in [10].

## 2. Problem

From [10], we have the equations (2.1) to (3.4) in nondimensional and nonlinear coupled partial differential equations of the form

$$
\begin{gather*}
\frac{1}{r} \frac{d}{d r}\left(r \mu(r)\left(\frac{d v}{d r}\right)\right)+\frac{\Lambda}{r} \frac{d}{d r}\left(r\left(\frac{d v}{d r}\right)^{3}\right)=c \\
\frac{d^{2} \theta}{d r^{2}}+\frac{1}{r}\left(\frac{d \theta}{d r}\right)+\Gamma\left(\frac{d v}{d r}\right)^{2}\left(\mu(r)+\Lambda\left(\frac{d v}{d r}\right)^{2}\right)=0 \tag{2.1}
\end{gather*}
$$

subject to boundary conditions

$$
\begin{array}{ll}
v(1)=\theta(1)=0, & \frac{d v(0)}{d r}=\frac{d \theta(0)}{d r}=0 \\
v(1)=\theta(1)=0, & \frac{d v}{d r}(0)=\frac{d \theta}{d r}(0)=0 \tag{2.2}
\end{array}
$$

## 3. Solution of the Problem

Our interest is to carry out the analysis for the homotopy solutions for two cases of viscosity, namely, constant and space-dependent viscous dissipation.

Case I. For constant viscosity model, we choose

$$
\begin{equation*}
\mu=1 \tag{3.1}
\end{equation*}
$$

For HAM solution, we select

$$
\begin{equation*}
v_{0}(r)=\frac{c}{4}\left(r^{2}-1\right), \quad \theta_{0}=\frac{c^{2} \Gamma\left(1-r^{4}\right)}{64} \tag{3.2}
\end{equation*}
$$

as initial approximations of $v$ and $\theta$, respectively, which satisfy the linear operator and corresponding boundary conditions. We use the method of higher-order differential mapping [16] to choose the linear operator $£$ which is defined by

$$
\begin{equation*}
\left\llcorner_{1}=\frac{d^{2}}{d r^{2}}+\frac{1}{r} \frac{d}{d r}\right. \tag{3.3}
\end{equation*}
$$

such that

$$
\begin{equation*}
\mathfrak{L}_{1}\left(C_{1}+C_{2} \ln r\right)=0 \tag{3.4}
\end{equation*}
$$

where $C_{1}$ and $C_{2}$ are the arbitrary constants.
If the convergence parameter is $\hbar$ and $0 \leq p \leq 1$ is an embedding parameter, then the zeroth-order problems become

$$
\begin{gather*}
(1-p) \mathscr{L}_{1}\left[v^{*}(r, p)-v_{0}(r)\right]=p \hbar \mathcal{N}_{1}\left[v^{*}(r, p), \theta^{*}(r, p)\right], \\
(1-p) \mathscr{L}_{1}\left[\theta^{*}(r, p)-\theta_{0}(r)\right]=p \hbar \mathcal{N}_{2}\left[v^{*}(r, p), \theta^{*}(r, p)\right], \\
v^{*}(1, p)=\theta^{*}(1, p)=0,\left.\quad \frac{\partial v^{*}(r, p)}{\partial r}\right|_{r=0}=\left.\frac{\partial \theta^{*}(r, p)}{\partial r}\right|_{r=0}=0, \tag{3.5}
\end{gather*}
$$

where the nonlinear parameters $\Omega_{1}$ and $\Omega_{2}$ are defined by

$$
\begin{align*}
& \Omega_{1}\left[v^{*}(r, p), \theta^{*}(r, p)\right]=\frac{1}{r} \frac{d v^{*}}{d r}+\frac{d^{2} v^{*}}{d r^{2}}+\frac{\Lambda}{r}\left(\frac{d v^{*}}{d r}\right)^{3}+3 \Lambda\left(\frac{d v^{*}}{d r}\right)^{2} \frac{d^{2} v^{*}}{d r^{2}}-c, \\
& \Omega_{2}\left[v^{*}(r, p), \theta^{*}(r, p)\right]=\frac{1}{r} \frac{d \theta^{*}}{d r}+\frac{d^{2} \theta^{*}}{d r^{2}}+\Gamma\left(\frac{d v^{*}}{d r}\right)^{2}+\Gamma \Lambda\left(\frac{d v^{*}}{d r}\right)^{4} \tag{3.6}
\end{align*}
$$

For $p=0$ and $p=1$, we have

$$
\begin{equation*}
v^{*}(r, 0)=v_{0}(r), \quad \theta^{*}(r, 0)=\theta_{0}(r), \quad v^{*}(r, 1)=v(r), \quad \theta^{*}(r, 1)=\theta(r) \tag{3.7}
\end{equation*}
$$

When $p$ increases from 0 to $1, v^{*}(r, p), \theta^{*}(r, p)$ vary from $v_{0}(r), \theta_{0}(r)$ to $v(r), \theta(r)$, respectively. By Taylor's theorem and (3.7), one can get

$$
\begin{equation*}
v^{*}(r, p)=v_{0}(r)+\sum_{m=1}^{\infty} v_{m}(r) p^{m}, \quad \theta^{*}(r, p)=\theta_{0}(r)+\sum_{m=1}^{\infty} \theta_{m}(r) p^{m} \tag{3.8}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{m}(r)=\left.\frac{1}{m!} \frac{\partial^{m} v^{*}(r, p)}{\partial p^{m}}\right|_{p=0}, \quad \theta_{m}(r)=\left.\frac{1}{m!} \frac{\partial^{m} \theta^{*}(r, p)}{\partial p^{m}}\right|_{p=0} \tag{3.9}
\end{equation*}
$$

The convergence of the series (3.8) depends upon $\hbar$. We choose $\hbar$ in such a way that the series (3.8) is convergent at $p=1$; then, due to (3.7), we get

$$
\begin{equation*}
v(r)=v_{0}(r)+\sum_{m=1}^{\infty} v_{m}(r), \quad \theta(r)=\theta_{0}(r)+\sum_{m=1}^{\infty} \theta_{m}(r) \tag{3.10}
\end{equation*}
$$

The $m$ th-order deformation problems are

$$
\begin{gather*}
\mathcal{L}_{1}\left[v_{m}(r)-X_{m} v_{m-1}(r)\right]=\hbar \mathfrak{R} 1_{m}(r), \\
\mathcal{L}_{1}\left[\theta_{m}(r)-X_{m} \theta_{m-1}(r)\right]=\hbar \Re 2_{m}(r),  \tag{3.11}\\
v_{m}(1)=\theta_{m}(1)=0, \quad v_{m}^{\prime}(0)=\theta_{m}^{\prime}(0)=0,
\end{gather*}
$$

where the recurrence formulae $\mathfrak{R} 1$ and $\mathfrak{R} 2$ are given by

$$
\begin{align*}
\mathfrak{R} 1_{m}(r)= & \frac{1}{r} \frac{d v_{m-1}}{d r}+\frac{d^{2} v_{m-1}}{d r^{2}}+\frac{\Lambda}{r} \sum_{k=0}^{m-1} \sum_{i=0}^{k}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k-i}}{d r} \frac{d v_{i}}{d r} \\
& +3 \Lambda \sum_{k=0}^{m-1} \sum_{i=0}^{k}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k-1}}{d r} \frac{d^{2} v_{i}}{d r^{2}}-\left(1-x_{m}\right) c \\
\mathfrak{R} 2_{m}(r)= & \frac{1}{r} \frac{d \theta_{m-1}}{d r}+\frac{d^{2} \theta_{m-1}}{d r^{2}}+\Gamma \sum_{k=0}^{m-1}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k}}{d r}  \tag{3.12}\\
& +\Lambda \Gamma \sum_{k=0}^{m-1} \sum_{j=0}^{k} \sum_{i=0}^{j}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k-j}}{d r} \frac{d v_{j-i}}{d r} \frac{d v_{i}}{d r}
\end{align*}
$$

in which

$$
X_{m}= \begin{cases}0, & m \leq 1  \tag{3.13}\\ 1, & m>1\end{cases}
$$

For constant viscosity, the velocity and temperature expressions up to second-order deformation are

$$
\begin{gather*}
v(r)=\frac{c}{4}\left(r^{2}-1\right)+\frac{h c^{3} \Lambda(2 h+3)\left(r^{4}-1\right)}{16}+\frac{h^{2} c^{5} \Lambda^{2}\left(r^{6}-1\right)}{32}, \\
\theta(r)=\left[\begin{array}{c}
M_{1}\left(r^{4}-1\right)+M_{2}\left(r^{6}-1\right)+M_{3}\left(r^{8}-1\right)+M_{4}\left(r^{10}-1\right) \\
+M_{5}\left(r^{12}-1\right)+M_{6}\left(r^{14}-1\right)+M_{7}\left(r^{16}-1\right) \\
+M_{8}\left(r^{18}-1\right)+M_{9}\left(r^{20}-1\right)+M_{10}\left(r^{22}-1\right)
\end{array}\right] . \tag{3.14}
\end{gather*}
$$

Case II. For space-dependent viscosity, we take

$$
\begin{equation*}
\mu=r \tag{3.15}
\end{equation*}
$$

For HAM solution, we select

$$
\begin{equation*}
v_{0}(r)=\frac{c}{6}\left(r^{2}-1\right), \quad \theta_{0}=\frac{c^{4} \hbar^{4} \Gamma\left(1-r^{2}\right)}{64} \tag{3.16}
\end{equation*}
$$

As the initial approximation of $v$ and $\theta$. We select

$$
\begin{equation*}
\mathfrak{L}_{2}=\frac{d^{2}}{d r^{2}}+\frac{2}{r} \frac{d}{d r}, \tag{3.17}
\end{equation*}
$$

such that

$$
\begin{equation*}
\mathscr{L}_{2}\left(C_{3}+\frac{C_{4}}{r}\right)=0, \tag{3.18}
\end{equation*}
$$

where $C_{3}$ and $C_{4}$ are arbitrary constants. The zeroth- and $m$ th-order deformation problems are

$$
\begin{gather*}
(1-p) \mathscr{L}_{2}\left[v^{*}(r, p)-v_{0}(r)\right]=p \hbar \mathcal{N}_{3}\left[v^{*}(r, p), \theta^{*}(r, p)\right],  \tag{3.19}\\
(1-p) \mathscr{L}_{2}\left[\theta^{*}(r, p)-\theta_{0}(r)\right]=p \hbar \mathcal{N}_{4}\left[v^{*}(r, p), \theta^{*}(r, p)\right],  \tag{3.20}\\
v^{*}(1, p)=\theta^{*}(1, p)=0,\left.\quad \frac{\partial v^{*}(r, p)}{\partial r}\right|_{r=0}=\left.\frac{\partial \theta^{*}(r, p)}{\partial r}\right|_{r=0}=0, \\
\mathscr{L}_{2}\left[v_{m}(r)-\chi_{m} v_{m-1}(r)\right]=\hbar \Re 3_{m}(r),  \tag{3.21}\\
\mathscr{L}_{2}\left[\theta_{m}(r)-\chi_{m} \theta_{m-1}(r)\right]=\hbar \Re 4_{m}(r), \\
v_{m}(1)=\theta_{m}(1)=0, \quad v_{m}^{\prime}(0)=\theta_{m}^{\prime}(0)=0,
\end{gather*}
$$

where

$$
\begin{align*}
\mathcal{N}_{3}\left[v^{*}(r, p), \theta^{*}(r, p)\right]= & \frac{2}{r} \frac{d v^{*}}{d r}+\frac{d^{2} v^{*}}{d r^{2}}+\frac{\Lambda}{r^{2}}\left(\frac{d v^{*}}{d r}\right)^{3}+\frac{3 \Lambda}{r}\left(\frac{d v^{*}}{d r}\right)^{2} \frac{d^{2} v^{*}}{d r^{2}}-\frac{c}{r}, \\
\mathcal{N}_{4}\left[v^{*}(r, p), \theta^{*}(r, p)\right]= & \frac{1}{r} \frac{d \theta^{*}}{d r}+\frac{d^{2} \theta^{*}}{d r^{2}}+\Gamma\left(\frac{d v^{*}}{d r}\right)^{2}+\Gamma \Lambda\left(\frac{d v^{*}}{d r}\right)^{4}+\Gamma r\left(\frac{d v^{*}}{d r}\right)^{2}, \\
\mathfrak{R} 3_{m}(r)= & 2 r \frac{d v_{m-1}}{d r}+r^{2} \frac{d^{2} v_{m-1}}{d r^{2}}+\Lambda \sum_{k=0}^{m-1} \sum_{i=0}^{k}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k-i}}{d r} \frac{d v_{i}}{d r} \\
& +3 \Lambda r \sum_{k=0}^{m-1} \sum_{i=0}^{k}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k-i}}{d r} \frac{d^{2} v_{i}}{d r^{2}}-\left(1-\chi_{m}\right) c r,  \tag{3.22}\\
\Re 4_{m}(r)= & \frac{1}{r} \frac{d \theta_{m-1}}{d r}+\frac{d^{2} \theta_{m-1}}{d r^{2}}+\Gamma r \sum_{k=0}^{m-1}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k}}{d r} \\
& +\Lambda \Gamma \sum_{k=0}^{m-1} \sum_{j=0}^{k} \sum_{i=0}^{j}\left(\frac{d v_{m-1-k}}{d r}\right) \frac{d v_{k-j}}{d r} \frac{d v_{j-i}}{d r} \frac{d v_{i}}{d r} .
\end{align*}
$$

For variable viscosity, the velocity and temperature expressions up to second-order deformation are

$$
\begin{gather*}
v(r)=\frac{h c}{2}(r-1)+\frac{c(2 h+3)\left(r^{2}-1\right)}{18}+\frac{c^{3} h \Lambda\left(r^{3}-1\right)}{81}  \tag{3.23}\\
\theta(r)=\left[M_{11}\left(r^{2}-1\right)+M_{12}\left(r^{3}-1\right)+M_{13}\left(r^{4}-1\right)+M_{14}\left(r^{5}-1\right)+M_{15}\left(r^{6}-1\right)\right]
\end{gather*}
$$

where the constant coefficients $M_{1}-M_{15}$ can be easily obtained through the routine calculation.

## mth-order solutions

In both cases, for $p=0$ and $p=1$, we have

$$
\begin{align*}
v^{*}(r ; 0)=v_{0}(r), & \theta^{*}(r ; 0)=\theta_{0}(y) \\
v^{*}(r ; 1)=v(r), & \theta^{*}(r ; 1)=\theta(r) \tag{3.24}
\end{align*}
$$

When $p$ increases from 0 to $1, v^{*}(r, p), \theta^{*}(r, p) \phi^{*}(r, p)$ varies from $v_{0}(r), \theta_{0}(r) \phi_{0}(r)$ to $v(r), \theta(r)$ and $\phi(r)$, respectively. By Taylor's theorem and (3.24) the general solutions can be written as

$$
\begin{equation*}
v^{*}(r, p)=v_{0}(r)+\sum_{m=1}^{\infty} v_{m}(r) p^{m}, \quad \theta^{*}(r, p)=\theta_{0}(r)+\sum_{m=1}^{\infty} \theta_{m}(r) p^{m} \tag{3.25}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{m}(r)=\left.\frac{1}{m!} \frac{\partial^{m} v^{*}(r, p)}{\partial p^{m}}\right|_{p=0}, \quad \theta_{m}(r)=\left.\frac{1}{m!} \frac{\partial^{m} \theta^{*}(r, p)}{\partial p^{m}}\right|_{p=0} \tag{3.26}
\end{equation*}
$$

The convergence of (3.25) depends upon $\hbar$; therefore, we choose $\hbar$ in such a way that it should be convergent at $p=1$. In view of (3.24), finally the general form of $m$ th-order solutions is

$$
\begin{equation*}
v(r)=v_{0}(r)+\sum_{m=1}^{\infty} v_{m}(r), \quad \theta(r)=\theta_{0}(r)+\sum_{m=1}^{\infty} \theta_{m}(r) \tag{3.27}
\end{equation*}
$$

## 4. Discussion

It is noticed that the explicit, analytical expressions (3.11), (19), (3.19), and (3.20) contain the auxiliary parameter $\hbar$. As pointed out by Liao [17], the convergence region and rate of approximations given by the HAM are strongly dependent upon $\hbar$. Figures 1 and 2 show the $\hbar$-curves of velocity and temperature profiles, respectively, just to find the range of $\hbar$ for the case of constant viscosity. The range for admissible values of $\hbar$ for velocity is $-2.4 \leq \hbar \leq 0.4$ and for temperature is $-2.2 \leq \hbar \leq 0.5$. Figures 4 and 5 represent the $\hbar$-curves for variable viscosity. The admissible ranges for both velocity and temperature profiles are $-3 \leq \hbar \leq$ 0.4 and $-2.8 \leq \hbar \leq 0.8$, respectively. In Figures 3 and 6, the graphs of residual error are


Figure 1: $\hbar$-curve for velocity in case of constant viscosity at 10th-order approximation.


Figure 2: $\hbar$-curve for temperature in case of constant viscosity at 10th-order approximation.


Figure 3: Residual error curve for constant viscosity.


Figure 4: $\hbar$-curve for velocity in case of variable viscosity at 10th-order approximation.


Figure 5: $\hbar$-curve for temperature in case of variable viscosity at 10th-order approximation.


Figure 6: Residual error curve for variable viscosity.

Table 1: Illustrating the variation of the velocity and temperature with $c$.

| $h$ | $\Lambda$ | $c$ | $V$ | $\theta$ |
| :--- | :---: | :---: | :---: | :---: |
| -0.01 | 1 | -1 | 1.673 | 0.006 |
|  | -2 | 3.191 | 0.068 |  |
|  | -3 | 4.4331 | 0.270 |  |
|  | -4 | 5.339 | 0.661 |  |
|  |  | -5 | 5.924 | 1.205 |

Table 2: Illustrating the variation of the velocity and temperature with $\Lambda$.

| $h$ | $c$ | $\Gamma$ | $\Lambda$ | $V$ | $\theta$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| -0.01 | -1 | 1 | 0 | 1.700 | 0.243 |
|  |  | 5 | 1.571 | 2.002 |  |
|  |  | 10 | 1.455 | 3.209 |  |
|  |  | 15 | 1.353 | 4.011 |  |
|  |  | 20 | 1.263 | 4.520 |  |

plotted for constant and variable viscosity, respectively. The error of norm 2 of two successive approximations over $[0,1]$ with HAM by 10th-order approximations is calculated by

$$
\begin{equation*}
E_{2}=\sqrt{\frac{1}{11} \sum_{i=0}^{10}\left(v_{10}\left(\frac{i}{10}\right)\right)^{2}}=f . \quad \text { (say) } \tag{4.1}
\end{equation*}
$$

It is seen that the error is minimum at $\hbar=-0.01$. These values of $\hbar$ also lie in the admissible range of $\hbar$.

We use the widely applied symbolic computation software MATHEMATICA to see the effects of sundry parameters by Tables 1, 2, and 3.

## 5. Conclusion

In this paper, the convergence of series solution for constant and variable viscosity in a thirdgrade fluid is presented. The steady pipe flow is considered. Convergence values and residual error are also examined in Figures 1 to 6 . To see the effects of emerging parameters for constant and variable viscosity, Tables 1 to 3 have been displayed. In Tables 1 and 2, it is found that the velocity and temperature increase with the decrease in pressure gradient and thirdgrade parameter, respectively, whereas Table 3 explains the variation of viscous dissipation parameter on velocity and temperature distributions. Here, it is revealed that the velocity and temperature decrease by increasing the viscous dissipation. It is observed that the results and figures [10] for important parameters $c, \Lambda$ and $\Gamma$ are correct and remain unchanged.

Table 3: Illustrating the variation of temperature with $\Gamma$.

| $h$ | $c$ | $\Lambda$ | $\Gamma$ | $V$ | $\theta$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| -0.01 | 1 | 1 |  |  |  |
|  |  |  | 0 | 0 | 0 |
|  |  | 10 | 0.075 | 3.242 |  |
|  |  | 15 | 0.158 | 6.484 |  |
|  |  | 20 | 0.249 | 9.726 |  |
|  |  |  | 0.351 | 12.969 |  |

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Research Article

# Peristaltic Transport of a Jeffrey Fluid with Variable Viscosity through a Porous Medium in an Asymmetric Channel 

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The peristaltic flow of a Jeffrey fluid with variable viscosity through a porous medium in an asymmetric channel is investigated. The channel asymmetric is produced by choosing the peristaltic wave train on the wall of different amplitude and phase. The governing nonlinear partial differential equations for the Jeffrey fluid model are derived in Cartesian coordinates system. Analytic solutions for stream function, velocity, pressure gradient, and pressure rise are first developed by regular perturbation method, and then the role of pertinent parameters is illustrated graphically.

## 1. Introduction

Peristalsis is a mechanism to pump the fluid by means of moving contraction on the tubes or channel walls. This process has quite useful applications in many biological systems and industry. It occurs in swallowing food through the esophagus, chyme motion in the gastrointestinal tract, the vasomotion of small blood vessels such as venules, capillaries, and arterioles, urine transport from kidney to bladder, sanitary fluid transport of corrosive fluids, a toxic liquid transport in the nuclear industry, and so forth. In view of such physiological and industrial applications, the peristaltic flows has been studied with great interest by the various researchers for viscous and non-Newtonian fluids [1-9].

In most of the studies which deal with the peristaltic flows, the fluid viscosity is assumed to be constant. This assumption is not valid everywhere. In general the coefficients of viscosity for real fluids are functions of space coordinate, temperature, and pressure. For many liquids such as water, oils, and blood, the variation of viscosity due to space coordinate and temperature change is more dominant than other effects. Therefore, it is highly desirable
to include the effect of variable viscosity instead of considering the viscosity of the fluid to be constant. Some important studies related to the variable viscosity are cited in [10-13].

A porous medium is the matter which contains a number of small holes distributed throughout the matter. Flows through a porous medium occur in filtration of fluids. Several investigations have been published by using generalized Darcy's law where the convective acceleration and viscous stress are taken into account [14-17].

Considering the importance of non-Newtonian fluid in peristalsis and keeping in mind the sensitivity of liquid viscosity, an attempt is made to study the peristaltic transport of Jeffrey having variable viscosity through a porous medium in a two-dimensional asymmetric channel under the assumption of long wave length and the low Reynolds number approximation. A regular perturbation method is used to solve the problem, and the solutions are expanded in a power series of viscosity parameter $\alpha$. The obtained expressions are utilized to discuss the influences of various emerging parameters.

## 2. Mathematical Formulation

We consider an incompressible Jeffrey fluid in an asymmetric channel of width $d_{1}+d_{2}$. A sinusoidal wave propagating with constant speed $c$ on the channel walls induces the flow. The wall surfaces are chosen of the following forms:

$$
\begin{gather*}
H_{1}(X, t)=a_{1}+b_{1} \cos \left[\frac{2 \pi}{\lambda}(X-c t)\right], \quad \text { upper wall, }  \tag{2.1}\\
H_{2}(X, t)=-a_{2}-b_{2} \cos \left[\frac{2 \pi}{\lambda}(X-c t)+\phi\right], \quad \text { lower wall, }
\end{gather*}
$$

where $b_{1}, b_{2}$ are amplitude of the upper and lower waves, $\lambda$ is the wave length, $\phi$ is the phase difference which varies in the range $0 \leq \phi \leq \pi$. Furthermore, $a_{1}, a_{2}, b_{1}, b_{2}$, and $\phi$ should satisfy the following condition

$$
\begin{equation*}
b_{1}^{2}+b_{2}^{2}+2 b_{1} b_{2} \cos \phi \leq\left(a_{1}+a_{2}\right)^{2} \tag{2.2}
\end{equation*}
$$

We assume that the flow becomes steady in the wave frame $(x, y)$ moving with velocity $c$ away from the fixed (laboratory) frame $(X, Y)$. The transformation between these two frames is given by

$$
\begin{equation*}
x=X-c t, \quad y=Y, \quad u=U-c, \quad v=V, \quad p(x)=P(X, t) \tag{2.3}
\end{equation*}
$$

where $u$ and $v$ are the velocity components in the wave frame $(x, y), p$ and $P$ are pressure in wave and fixed frame of reference, respectively. The governing equations in the wave frame of reference are the Brinkman extended Daray equations given by

$$
\begin{gather*}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0  \tag{2.4}\\
\rho\left(u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{1}{\varepsilon} \frac{\partial \tau_{x x}}{\partial x}+\frac{1}{\varepsilon} \frac{\partial \tau_{x y}}{\partial y}-\frac{\mu(y)}{k}(u+1)  \tag{2.5}\\
\rho\left(u \frac{\partial v}{\partial x}+v \frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial \tau_{y x}}{\partial x}+\frac{\partial \tau_{y y}}{\partial y}-\frac{\mu(y)}{k} v \tag{2.6}
\end{gather*}
$$

where

$$
\begin{align*}
& \tau_{x x}=\frac{2 \mu(y)}{\left(1+\lambda_{1}\right)}\left[1+\lambda_{2}\left(u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}\right)\right] \frac{\partial u}{\partial x} \\
& \tau_{x y}=\frac{\mu(y)}{\left(1+\lambda_{1}\right)}\left[1+\lambda_{2}\left(u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}\right)\right]\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)  \tag{2.7}\\
& \tau_{y y}=\frac{2 \mu(y)}{\left(1+\lambda_{1}\right)}\left[1+\lambda_{2}\left(u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}\right)\right] \frac{\partial v}{\partial y}
\end{align*}
$$

where $\lambda_{1}$ is the ratio of relaxation to retardation times, $\lambda_{2}$ is the retardation time, $\rho$ is the density, $k$ is the permeability of the porous medium, and $\varepsilon$ is the porosity of the porous medium.

Introducing the following nondimensional quantities:

$$
\begin{align*}
& \bar{x}=\frac{x}{\lambda}, \quad \bar{y}=\frac{y}{a_{1}}, \quad \bar{u}=\frac{u}{c}, \quad \bar{v}=\frac{v}{c \delta^{\prime}}, \quad h_{1}=\frac{H_{1}}{a_{1}}, \quad h_{2}=\frac{H_{2}}{a_{1}}, \quad \bar{\tau}=\frac{a_{1} \tau}{\mu_{0} c} \\
& \bar{t}=\frac{c t}{\lambda}, \quad \mathrm{Da}=\frac{k}{a_{1}^{2}}, \quad \delta=\frac{a}{\lambda}, \quad \bar{p}=\frac{p a_{1}^{2}}{\mu_{0} c \lambda^{\prime}}, \quad a=\frac{b_{1}}{a_{1}}, \quad b=\frac{b_{2}}{a_{1}}, \quad d=\frac{a_{2}}{a_{1}} . \tag{2.8}
\end{align*}
$$

With the help of (2.8), (2.4) to (2.6) after dropping the bars take the form

$$
\begin{gather*}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0  \tag{2.9}\\
\operatorname{Re} \delta\left(u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\delta}{\varepsilon} \frac{\partial \tau_{x x}}{\partial x}+\frac{1}{\varepsilon} \frac{\partial \tau_{x y}}{\partial y}-\frac{\mu(y)}{\mathrm{Da}}(u+1)  \tag{2.10}\\
\operatorname{Re} \delta^{3}\left(u \frac{\partial v}{\partial x}+v \frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\delta^{2}}{\varepsilon} \frac{\partial \tau_{x y}}{\partial x}+\frac{\delta}{\varepsilon} \frac{\partial \tau_{y y}}{\partial y}-\frac{\delta^{2} \mu(y)}{\mathrm{Da}} v \tag{2.11}
\end{gather*}
$$

where Darcy's number is

$$
\begin{gather*}
\mathrm{Da}=\frac{k}{a_{1}^{2}} \\
\tau_{x x}=\frac{2 \delta \mu(y)}{\left(1+\lambda_{1}\right)}\left[1+\frac{\lambda_{2} \delta c}{a_{1}}\left(u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}\right)\right] \frac{\partial u}{\partial x} \\
\tau_{x y}=\frac{\mu(y)}{\left(1+\lambda_{1}\right)}\left[1+\frac{\lambda_{2} \delta c}{a_{1}}\left(u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}\right)\right]\left(\frac{\partial u}{\partial y}+\delta^{2} \frac{\partial v}{\partial x}\right)  \tag{2.12}\\
\tau_{y y}=\frac{2 \mu(y)}{\left(1+\lambda_{1}\right)}\left[1+\frac{\lambda_{2} \delta c}{a_{1}}\left(u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}\right)\right] \frac{\partial v}{\partial y} .
\end{gather*}
$$

Using the longwave length and small Reynolds number approximation, (2.10) and (2.11) take the form

$$
\begin{gather*}
\frac{\partial p}{\partial x}=\frac{1}{\varepsilon} \frac{\partial}{\partial y}\left[\frac{\mu(y)}{\left(1+\lambda_{1}\right)} \frac{\partial u}{\partial y}\right]-\frac{\mu(y)}{\mathrm{Da}}(u+1)  \tag{2.13}\\
\frac{\partial p}{\partial y}=0 \tag{2.14}
\end{gather*}
$$

The corresponding boundary conditions are

$$
\begin{array}{ll}
u=-1, & \text { at } y=h_{1} \\
u=-1, & \text { at } y=h_{2} \tag{2.15b}
\end{array}
$$

where

$$
\begin{equation*}
h_{1}=1+a \cos 2 \pi x, \quad h_{2}=-d-b \cos (2 \pi x+\phi) \tag{2.15c}
\end{equation*}
$$

Equation (2.14) indicate that $p$ is independent of $y$. Therefore, (2.10) can be written as

$$
\begin{equation*}
\frac{d p}{d x}=\frac{1}{\varepsilon} \frac{\partial}{\partial y}\left[\frac{\mu(y)}{\left(1+\lambda_{1}\right)} \frac{\partial u}{\partial y}\right]-\frac{\mu(y)}{\mathrm{Da}}(u+1) \tag{2.16}
\end{equation*}
$$

where $\mu(y)$ is the viscosity variation on peristaltic flow. For the present analysis, we assume viscosity variation in the dimensionless form [10]:

$$
\begin{equation*}
u(y)=e^{-\alpha y}, \quad u(y)=1-\alpha y+\frac{\alpha y^{2}}{2}, \quad \text { for } \alpha \ll 1 \tag{2.17}
\end{equation*}
$$

The volume flow rate in the wave frame is given by

$$
\begin{equation*}
q=\int_{h_{2}}^{h_{1}} u d y \tag{2.18}
\end{equation*}
$$

The instantaneous flux $Q(x, t)$ in the laboratory frame is defined as

$$
\begin{equation*}
Q(x, t)=\int_{h_{1}}^{h_{2}}(u+1) d y=q+h_{1}-h_{2} \tag{2.19}
\end{equation*}
$$

The average flux over one period $(T=\lambda / c)$ is given by

$$
\begin{equation*}
\bar{Q}=\frac{1}{T} \int_{0}^{T} Q d t=\frac{1}{T} \int_{0}^{T}\left(q+h_{1}-h_{2}\right) d t=q+1+d \tag{2.20}
\end{equation*}
$$

## 3. Perturbation Solution

Equation (2.16) is a nonlinear differential equation so that it is not possible to obtain a closed form solution; so we seek perturbation solution. We expand $u, p$ and $q$ as

$$
\begin{align*}
& u=u_{0}+\alpha u_{1}+\alpha^{2} u_{2}+o\left(\alpha^{3}\right) \\
& p=p_{0}+\alpha p_{1}+\alpha^{2} p_{2}+o\left(\alpha^{3}\right)  \tag{3.1}\\
& q=q_{0}+\alpha q_{1}+\alpha^{2} q_{2}+o\left(\alpha^{3}\right)
\end{align*}
$$

Substituting these equations into (2.15a), (2.15b), (2.15c), and (2.16), we have the following system of equations.

### 3.1. Zeroth-Order Equations $\alpha^{0}$

$$
\begin{equation*}
\frac{\partial^{2} u_{0}}{\partial y^{2}}-N^{2} u_{0}=\varepsilon\left(1+\lambda_{1}\right) \frac{d p_{0}}{d x}+N^{2} \tag{3.2}
\end{equation*}
$$

where

$$
\begin{gather*}
N=\frac{\varepsilon\left(1+\lambda_{1}\right)}{\mathrm{Da}},  \tag{3.3}\\
u_{0}=-1, \quad \text { at } y=h_{1}, h_{2} .
\end{gather*}
$$



Figure 1: The pressure rise versus flow rate when $a=0.2, b=0.6, d=0.8, \varepsilon=0.3, \lambda_{1}=0.8, \mathrm{Da}=0.6$, and $\phi=\pi / 4$.


Figure 2: The pressure rise verses flow rate when $\alpha=0.01, a=0.2, b=0.6, d=0.8, \varepsilon=0.3, \lambda_{1}=0.4$, and $\phi=\pi / 4$.

### 3.2. First-Order Equations $\alpha$

$$
\begin{gather*}
\frac{\partial^{2} u_{1}}{\partial y^{2}}-N^{2} u_{1}=\varepsilon\left(1+\lambda_{1}\right) \frac{d p_{1}}{d x}+\varepsilon y\left(1+\lambda_{1}\right) \frac{d p_{0}}{d x}+\frac{\partial u_{0}}{\partial y},  \tag{3.4}\\
u_{1}=0, \quad \text { at } y=h_{1}, h_{2} . \tag{3.5}
\end{gather*}
$$



Figure 3: The pressure rise verses flow rate when $\alpha=0.01, a=0.2, b=0.6, d=0.8, \lambda_{1}=0.4, \mathrm{Da}=0.5$, and $\phi=\pi / 4$.


Figure 4: The pressure rise verses flow rate when $\alpha=0.01, a=0.2, b=0.6, d=0.8, \varepsilon=0.3, \mathrm{Da}=0.8$, and $\phi=\pi / 4$.

### 3.3. Second-Order Equations $\alpha^{2}$

$$
\begin{gather*}
\frac{\partial^{2} u_{2}}{\partial y^{2}}-N^{2} u_{2}=\varepsilon\left(1+\lambda_{1}\right) \frac{d p_{2}}{d x}+\varepsilon\left(1+\lambda_{1}\right) y \frac{d p_{1}}{d x}+\frac{y^{2}}{2} \varepsilon\left(1+\lambda_{1}\right) \frac{d p_{0}}{d x}+\frac{\partial u_{1}}{\partial y},  \tag{3.6}\\
u_{2}=0 \quad \text { at } y=h_{1}, h_{2} . \tag{3.7}
\end{gather*}
$$

### 3.4. Zeroth-Order Solution

Solving (3.2) and (3.3), we get

$$
\begin{equation*}
u_{0}=\frac{\varepsilon\left(1+\lambda_{1}\right)}{N^{2}} \frac{d p_{0}}{d x}\left[C_{1} \cosh N y+C_{2} \sinh N y-1\right]-1 \tag{3.8}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{1}=\frac{\left(\sinh N h_{1}-\sinh N h_{2}\right)}{\sinh N\left(h_{1}-h_{2}\right)}, \quad C_{2}=\frac{\left(\cosh N h_{2}-\cosh N h_{1}\right)}{\sinh N\left(h_{1}-h_{2}\right)} \tag{3.9}
\end{equation*}
$$

and the volume flow rate $q_{0}$ is given by

$$
\begin{equation*}
q_{0}=\int_{h_{2}}^{h_{1}} u_{0} d y \tag{3.10}
\end{equation*}
$$

From (3.8), we have

$$
\begin{equation*}
\frac{d p_{0}}{d x}=\left(q_{0}+h_{1}-h_{2}\right) A \tag{3.11}
\end{equation*}
$$

where

$$
\begin{equation*}
A=\frac{N^{3} \sinh N\left(h_{1}-h_{2}\right)}{\varepsilon\left(1+\lambda_{1}\right)\left[2 \cosh N\left(h_{1}-h_{2}\right)-2-\left(h_{1}-h_{2}\right) N \sinh N\left(h_{1}-h_{2}\right)\right]} . \tag{3.12}
\end{equation*}
$$

The dimensionless pressure rise at this order is

$$
\begin{equation*}
\Delta P_{0}=\int_{0}^{1} \frac{d p_{0}}{d x} d x \tag{3.13}
\end{equation*}
$$

### 3.5. First-Order Solution

Substituting zeroth order solution (3.8) into (3.4) and then solving the resulting system along with the corresponding boundary conditions, we arrive at

$$
\begin{align*}
u_{1}=\frac{\varepsilon\left(1+\lambda_{1}\right)}{N^{2}} \frac{d p_{1}}{d x}[ & \left.C_{1} \cosh N y+C_{2} \sinh N y-1\right] \\
+\frac{\varepsilon\left(1+\lambda_{1}\right)}{2 N^{2}} \frac{d p_{0}}{d x} & {\left[-2 y+C_{1} y \cosh N y+C_{2} y \sinh N y\right.}  \tag{3.14}\\
& \times \frac{\sinh N y\left(h_{1} \cosh N h_{2}-h_{2} \cosh N h_{1}\right)}{\sinh N\left(h_{1}-h_{2}\right)} \\
& \left.+\frac{\cosh N y\left(h_{2} \sinh N h_{1}-h_{1} \sinh N h_{2}\right)}{\sinh N\left(h_{1}-h_{2}\right)}\right],
\end{align*}
$$



Figure 5: The pressure rise verses flow rate when $\alpha=0.01, a=0.4, b=0.6, d=0.8, \varepsilon=0.4, \lambda_{1}=0.5$, and $\mathrm{Da}=0.5$.


Figure 6: The pressure rise verses flow rate when $\alpha=0.01, b=0.6, d=0.8, \varepsilon=0.3, \lambda_{1}=0.4, \mathrm{Da}=0.5$, and $\phi=\pi / 4$.
and the volume flow rate $q_{1}$ is given by

$$
\begin{equation*}
q_{1}=\int_{h_{2}}^{h_{1}} u_{1} d y \tag{3.15}
\end{equation*}
$$

From (3.14), we get

$$
\begin{equation*}
\frac{d p_{1}}{d x}=A q_{1}+\frac{A \varepsilon\left(1+\lambda_{1}\right)}{2 N^{3}} \frac{d p_{0}}{d x}\left[N^{2}\left(h_{1}^{2}-h_{2}^{2}\right)+\frac{\left(h_{1}+h_{2}\right)\left(1-\cosh N\left(h_{1}-h_{2}\right)\right)}{\sinh N\left(h_{1}-h_{2}\right)}\right] \tag{3.16}
\end{equation*}
$$



Figure 7: The pressure rise verses flow rate when $\alpha=0.01, a=0.2, d=0.8, \varepsilon=0.3, \lambda_{1}=0.4, \mathrm{Da}=0.5$, and $\phi=\pi / 4$.


Figure 8: Axial velocity versus $y$ at $a=0.2, b=0.6, d=0.8, \varepsilon=0.2, \lambda_{1}=1, \mathrm{Da}=1, x=\pi / 6, q=-1$, and $\phi=\pi / 2$.

The dimensionless pressure rise at this order is

$$
\begin{equation*}
\Delta P_{1}=\int_{0}^{1} \frac{d p_{1}}{d x} d x \tag{3.17}
\end{equation*}
$$



Figure 9: Axial velocity versus $y$ at $\alpha=0.05, a=0.2, b=0.6, d=0.8, \varepsilon=0.2, \lambda_{1}=1, \mathrm{Da}=1, x=0$, and $\phi=\pi / 2$.

### 3.6. Second-Order Solution

Solving (3.6) by using (3.8) and (3.14) and the boundary condition (3.5), we obtain

$$
\begin{aligned}
u_{2}= & \frac{\varepsilon\left(1+\lambda_{1}\right)}{N^{2}} \frac{d p_{2}}{d x}[ \\
+\frac{\varepsilon\left(1+\lambda_{1}\right)}{2 N^{2}} \frac{d p_{1}}{d x} & {\left[\frac{\sinh N y\left(h_{1} \cosh N h_{2}-h_{2} \cosh N h_{1}\right)}{\sinh N\left(h_{1}-h_{2}\right)}-2 y\right.} \\
& +C_{1} y \cosh N y+C_{2} y \sinh N y \\
& \left.+\frac{\cosh N y\left(h_{2} \sinh N h_{1}-h_{1} \sinh N h_{2}\right)}{\sinh N\left(h_{1}-h_{2}\right)}\right] \\
+\frac{d p_{0}}{d x} \frac{\varepsilon\left(1+\lambda_{1}\right)}{4 N^{2}}[ & \frac{C_{1}\left(y \sinh N y+N y^{2} \cosh N y\right)+C_{2}\left(y \cosh N y+y^{2} N \sinh N y\right)}{2 N} \\
& -y^{2}+\frac{\left(1-\cosh N\left(h_{1}-h_{2}\right)\right)\left(\sinh N y\left(h_{1} \cosh N h_{2}+h_{2} \cosh N h_{1}\right)\right)}{2 \sinh N\left(h_{1}-h_{2}\right)} \\
& -\frac{\left(1-\cosh N\left(h_{1}-h_{2}\right)\right)\left(\left(h_{1} \sinh N h_{2}+h_{2} \sinh N h_{1}\right) \cosh N y\right)}{2 \sinh N\left(h_{1}-h_{2}\right)} \\
& -\frac{\left(h_{1} \cosh N h_{2}-h_{2} \cosh N h_{1}\right)\left(C_{1} y \cosh N y+C_{2} y \sinh N y\right)}{\left(\cosh N h_{1}-\cosh N h_{2}\right)} \\
& +\frac{\left(h_{2}^{2} \cosh N h_{1}-h_{1}^{2} \cosh N h_{2}\right) \sinh N y}{2 \sinh N\left(h_{1}-h_{2}\right)}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{\left(h_{1}^{2} \sinh N h_{1}-h_{2}^{2} \sinh N h_{2}\right) \cosh N y}{2 \sinh N\left(h_{1}-h_{2}\right)} \\
& \left.\times \frac{8\left(1-C_{1} \cosh N y-C_{2} \sinh N y\right)}{N^{2}}+\frac{\left(h_{1}-h_{2}\right) y \cosh N y}{\left(\cosh N h_{1}-\cosh N h_{2}\right)}\right] \tag{3.18}
\end{align*}
$$

and the volume flow rate $q_{2}$ is given by

$$
\begin{equation*}
q_{2}=\int_{h_{2}}^{h_{1}} u_{2} d y \tag{3.19}
\end{equation*}
$$

From (3.18), we have

$$
\begin{align*}
\frac{d p_{2}}{d x}= & A q_{2}+\frac{A \varepsilon\left(1+\lambda_{1}\right)}{2 N^{3}} \frac{d p_{1}}{d x}\left[\frac{2\left(h_{1}+h_{2}\right)\left(1-\cosh N\left(h_{1}-h_{2}\right)\right)}{\sinh N\left(h_{1}-h_{2}\right)}+N\left(\left(h_{1}^{2}-h_{2}^{2}\right)\right)\right] \\
& -\frac{A \varepsilon\left(1+\lambda_{1}\right)}{4 N^{3}} \frac{d p_{0}}{d x}\left[\frac{8\left(h_{1}-h_{2}\right)}{N}-\frac{3\left(h_{1}-h_{2}\right)}{2 N}-\left(\frac{h_{1}^{3}}{3}-\frac{h_{2}^{3}}{3}\right)\right] \\
& +\frac{\left(h_{1}^{2}+h_{2}^{2}\right)\left(1-\cosh N\left(h_{1}-h_{2}\right)\right)}{2 N \sinh N\left(h_{1}-h_{2}\right)}+\frac{\left(h_{1}-h_{2}\right)\left(h_{1} \sin N h_{1}-h_{2} \sin N h_{2}\right)}{\left(\cosh N h_{1}-\cosh N h_{2}\right)}  \tag{3.20}\\
& +\frac{\left(h_{1}+h_{2}\right)\left(1-\cosh N\left(h_{1}-h_{2}\right)\right)\left(h_{1} \cosh N h_{2}-h_{2} \cosh N h_{1}\right)}{\sinh N\left(h_{1}-h_{2}\right)\left(\cosh N h_{1}-\cosh N h_{2}\right)} \\
& +\frac{\left(h_{1}^{2} N^{2}+h_{2}^{2} N^{2}+2\right)\left(\cosh N\left(h_{1}-h_{2}\right)-1\right)}{2 N^{2} \sinh N\left(h_{1}-h_{2}\right)}+\frac{16\left(1-\cosh N\left(h_{1}-h_{2}\right)\right)}{N^{2} \sinh N\left(h_{1}-h_{2}\right)} .
\end{align*}
$$

The dimensionless pressure rise at this order is

$$
\begin{equation*}
\Delta P_{2}=\int_{0}^{2} \frac{d p_{2}}{d x} d x \tag{3.21}
\end{equation*}
$$

Summarizing the result obtained from (3.11), (3.16), and (3.20), we write

$$
\begin{equation*}
\Delta P=\Delta P_{0}+\alpha \Delta P_{1}+\alpha^{2} \Delta P_{2} \tag{3.22}
\end{equation*}
$$

Corresponding stream functions can be defined as

$$
\begin{equation*}
u=\frac{\partial \Psi}{\partial y}, \quad v=-\delta \frac{\partial \Psi}{\partial x} \tag{3.23}
\end{equation*}
$$



Figure 10: Axial velocity versus $y$ at $\alpha=0.05, a=0.2, b=0.6, d=0.8, \varepsilon=0.2, \lambda_{1}=1, \mathrm{Da}=1, x=\pi / 2$, and $q=-1$.


Figure 11: Axial velocity versus $y$ at $\alpha=0.05, a=0.2, b=0.6, d=0.8, \varepsilon=0.2, \lambda_{1}=1, q=-1, x=\pi / 6$, and $\phi=\pi / 2$.

## 4. Results and Discussion

We have used a regular perturbation series in term of the dimensional viscosity parameter $\alpha$ to obtain analytical solution of the field equations for peristaltic flow of Jeffrey fluid in an asymmetric channel. To study the behavior of solutions, numerical calculations for several values of viscosity parameter $\alpha$, Daray number Da, porosity $\varepsilon$, amplitude ratio $\phi$, Jeffrey fluid parameter $\lambda_{1}, a$ and $b$ have been calculated numerically using MATHEMATICA software.

Figure 1 shows the variation of $\Delta P$ with flow rate $\theta$ for different values of $\alpha$. It is depicted that the time-average flux $\theta$ increase with increasing the viscosity parameter $\alpha$. Figure 2 represents the variation of $\Delta P$ with the flow rate $\theta$ for different values of Da . We observe that an increase in the peristaltic pumping rate pressure rises. Figures 3 and 4 are graphs of pressure rise $\Delta P$ with the flow rate $\theta$ for values of $\varepsilon$ and $\lambda_{1}$. It is observed that the


Figure 12: Axial velocity versus $y$ at $\alpha=0.05, a=0.2, b=0.6, q=-1, \varepsilon=0.2, \lambda_{1}=1, \mathrm{Da}=1, x=\pi / 6$, and $\phi=\pi / 2$.
pumping rate decreases with increase of $\varepsilon$ and $\lambda_{1}$. Figure 5 is the graph of the variation of $\Delta P$ versus the flow rate $\theta$ for different values of phase difference $\phi$. It is observed that the pumping rate decreases with the increase of $\phi$. Figures 6 and 7 plot the relation between pressure rise $\Delta P$ and flow rate $\theta$ for different values of $a$ and $b$, respectively. Figure 8 represents the graph of axial velocity $u$ versus $y$. It can be seen that an increase in $\alpha$ decreases the magnitude of axial velocity $u$. The effects of $q$ on the axial velocity $u$ are seen through Figure 9. It is noticed that an increase in $q$ increase the magnitude of the axial velocity. Figures 10 and 11 illustrate the effect of phase difference $\phi$ and Daray's number Da on the axial velocity $u$. It is observed that the magnitude of axial velocity decreases with the increasing phase difference $\phi$ and Daray's number Da. In Figure 12 the axial velocity $u$ is graphed versus $y$. We note that the magnitude of axial velocity increases as the channel width $d$ increases. It is worth mentioning that in the absence of porosity parameter the solutions of [10] can be derived as special case of the present analysis. This provides the useful check. It may be remarked that the problem for this particular model was not solved earlier even by any traditional perturbation technique. The results presented in this paper will now be available for experimental verification.

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