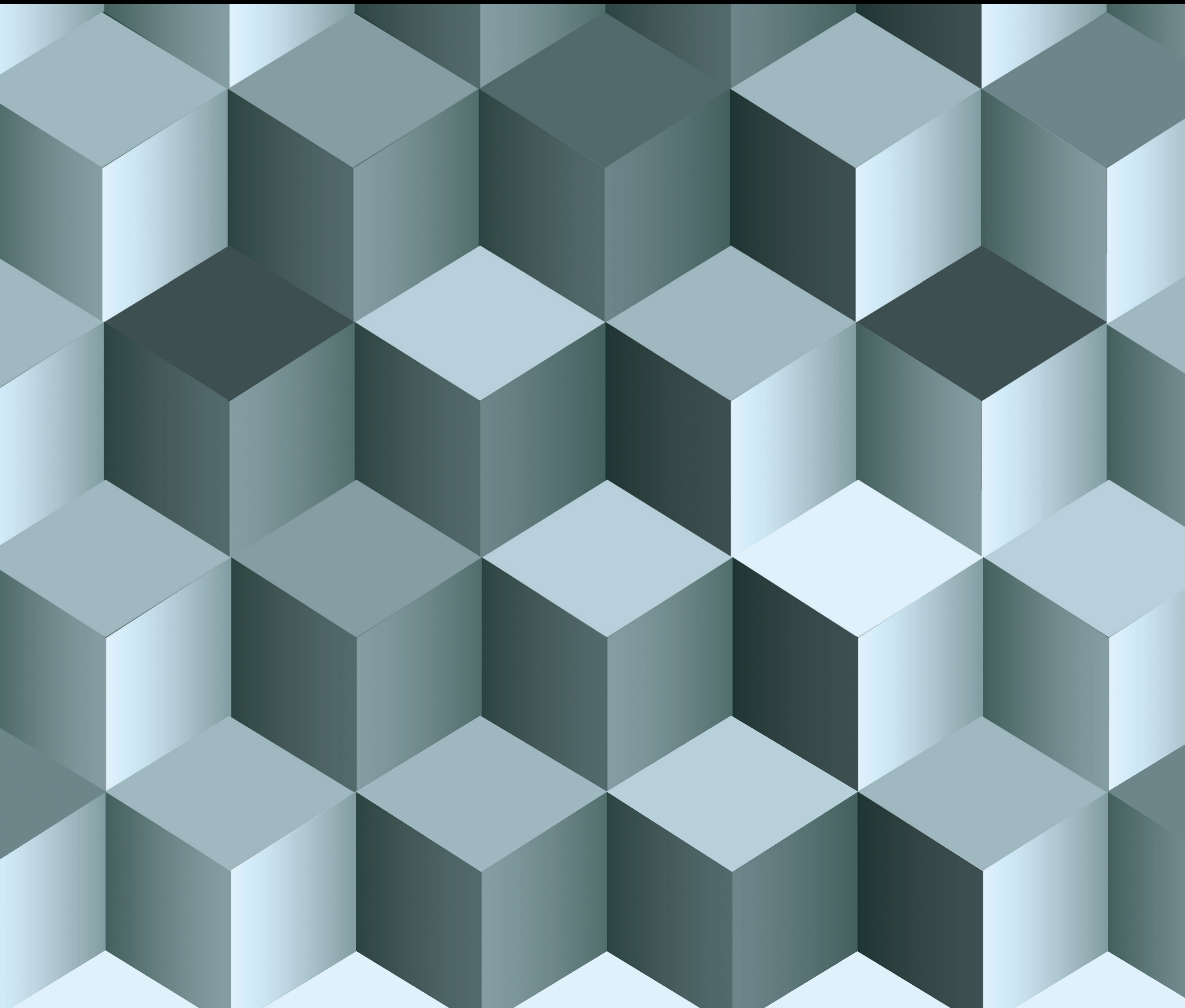
Journal of Function Spaces

Unique and Non-Unique Fixed Points and their Applications 2022

Lead Guest Editor: Anita Tomar
Guest Editors: Santosh Kumar and Cristian Chifu



Unique and Non-Unique Fixed Points and their Applications 2022

Journal of Function Spaces

Unique and Non-Unique Fixed Points and their Applications 2022

Lead Guest Editor: Anita Tomar
Guest Editors: Santosh Kumar and Cristian Chifu

Contents

Hindawi
Journal of Function Spaces
Volume 2023, Article ID 2877019, 12 pages
https://doi.org/10.1155/2023/2877019

Research Article

Sehgal-Guseman-Type Fixed Point Theorems in Rectangular b-Metric Spaces and Solvability of Nonlinear Integral Equation

Hongyan Guan, Chen Lang, and Yan Hao

School of Mathematics and Systems Science, Shenyang Normal University, Shenyang 110034, China

Correspondence should be addressed to Yan Hao; haoyan8012@163.com

Received 1 July 2022; Revised 27 October 2022; Accepted 12 May 2023; Published 26 May 2023

Academic Editor: Cristian Chifu

Firstly, the concept of a new triangular α-orbital admissible condition is introduced, and two fixed point theorems for Sehgal-Guseman-type mappings are investigated in the framework of rectangular b-metric spaces. Secondly, some examples are presented to illustrate the availability of our results. At the same time, we furnished the existence and uniqueness of solution of an integral equation.

1. Introduction

In nonlinear analysis, the most famous result is the Banach contraction principle, which is established by Banach [1] in 1922. After that, there are a large number of excellent results for fixed point in metric spaces. On recent development on fixed point theory in metric spaces, one can consult [2] the related references involved. Branciari [3] introduced a new concept, that is, the definition of rectangular metric spaces, and established an analogue of the Banach fixed point theorem in such a space. Then, a lot of fixed point theorems for a wide range of contractions on rectangular metric spaces had emerged in a blowout manner. In such type space, Lakzian and Samet [4] gave some results involving (ψ, ϕ) weakly contraction. Furthermore, several common fixed point results about (ψ, ϕ)-weakly contractions were obtained by Bari and Vetro [5]. In [6], George and Rajagopalan considered common fixed points of a new class of (ψ, ϕ) contractions. By use of C-functions, Budhia et al. furnished several fixed point results in [7].

In [8], Czerwik put forward firstly the definition of b-metric space, an extension of a metric space. Since then, this result has been extended in different angles. In a b-metric space, in [9], Mitrovic provided a new method to prove Czerwik's fixed point theorem. By using of increased range of the Lipschitzian constants, Hussain et al. [10] provided a proof of the Fisher contraction theorem. Mustafa et al. [11] gave several fixed point theorems for some new classes of T-Chatterjea-contraction and T-Kannan-contraction. Recently, also in this type spaces, Mitrovic et al. [12] presented some new versions of existing theorems. Savanović et al. [13] constructed some new results for multivalued quasicontraction. Furthermore, in [14], Aydi et al. obtained the existence of fixed point for α-β_E-Geraghty contractions. In [15], several fixed point theorems of set valued interpolative Hardy-Rogers type contractions were studied. In [16], George et al. put forward the concept of rectangular b-metric mapping. Meanwhile, they gave some fixed point theorems. Lately, Gulyaz-Ozyurt [17], Zheng et al. [18], and Guan et al. [19] also studied fixed point theory in such spaces and obtained some excellent results. In 2021, Hussain [20] presented some fractional symmetric α-η-contractions and built up some new fixed point theorems for these types of contractions in F-metric spaces. Recently, Arif et al. [21] introduced an ordered implicit relation and investigated the existence of the fixed points of contractive mapping dealing with implicit relation in a cone b-metric space. Lately, in [22], some fixed point theorems of two new classes of multivalued almost contractions in a partial b-metric spaces were established by Anwar et al.

On the other hand, in 1969, Sehgal [23] formulated an inequality that can be considered an extension of the renowned Banach fixed point theorem in a metric space. Matkowski [24] generalized some previous results of Khazanchi [25] and Iseki [26]. In 2012, the definition of α-admissible mappings was given by Samet et al. [27]. Later, the notion of triangular α-admissible mappings was introduced by Popescu [28]. Recently, Lang and Guan [29] studied the common fixed point theory of $\alpha_{i,j}$-$\varphi_{E_{M,N}}$-Geraghty contraction and $\alpha_{i,j}$-φ_{E_N}-Geraghty contractions in a b-metric space.

In this paper, inspired by [30], we established two fixed point theorems for Sehgal-Guseman-type mappings in a rectangular b-metric space. Also, we present two examples to illustrate the usability of established results.

2. Preliminaries

Definition 1 (see [8]). Suppose $\mathbb{G}$ is a nonempty set and $\varsigma : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$. We call ς a b -metric if

(i) $\varsigma(\epsilon, \varpi) = 0 \Leftrightarrow \epsilon = \varpi, \forall \epsilon, \varpi \in \mathbb{G}$

(ii) $\varsigma(\epsilon, \varpi) = \varsigma(\varpi, \epsilon), \forall \epsilon, \varpi \in \mathbb{G}$

(iii) $\varsigma(\epsilon, \varpi) \le s[\varsigma(\epsilon, \gamma) + \varsigma(\gamma, \varpi)], \forall \epsilon, \varpi, \gamma \in \mathbb{G}$

where $s \ge 1$ is constant.

It is usual that $(\mathbb{G}, \varsigma)$ is called a b-metric space with parameter $s \ge 1$.

Definition 2 (see [3]). Suppose $\mathbb{G}$ is a nonempty set and $\tau : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$. We call τ a triangular metric if

(i) $\tau(\epsilon, \varpi) = 0 \Leftrightarrow \epsilon = \varpi, \forall \epsilon, \varpi \in \mathbb{G}$

(ii) $\tau(\epsilon, \varpi) = \tau(\varpi, \epsilon), \forall \epsilon, \varpi \in \mathbb{G}$

(iii) $\tau(\epsilon, \varpi) \le \tau(\epsilon, \gamma) + \tau(\gamma, \epsilon) + \tau(\epsilon, \varpi), \forall \epsilon, \varpi \in \mathbb{G}, \gamma, \epsilon \in \mathbb{G} - \{\epsilon, \varpi\}$

Usually, $(\mathbb{G}, \tau)$ is called a rectangular metric space.

Definition 3 (see [16]). Suppose $\mathbb{G}$ is a nonempty set and $\upsilon : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty)$. We call υ a rectangular b -metric if

(i) $\upsilon(\epsilon, \varpi) = 0 \Leftrightarrow \epsilon = \varpi, \forall \epsilon, \varpi \in \mathbb{G}$

(ii) $\upsilon(\epsilon, \varpi) = \upsilon(\varpi, \epsilon), \forall \epsilon, \varpi \in \mathbb{G}$

(iii) $\upsilon(\epsilon, \varpi) \le s[\upsilon(\epsilon, \gamma) + \upsilon(\gamma, \varepsilon) + \upsilon(\varepsilon, \varpi)], \forall \epsilon, \varpi \in \mathbb{G}, \gamma, \varepsilon \in \mathbb{G} - \{\epsilon, \varpi\}$

where $s \ge 1$ is constant.

In general, $(\mathbb{G}, \upsilon)$ is called a rectangular b-metric space with parameter $s \ge 1$.

Remark 4. A rectangular metric space is a rectangular b-metric space, so is a b -metric space. Moreover, the converse is not true.

Example 1. Suppose $\mathbb{G} = A \cup B$, where $A = \{0, 2/41, 3/61, 4/81\}$ and $B = \{1/2, 1/3, \cdots, 1/i, \cdots\}$. For $\epsilon, \varpi \in \mathbb{G}$, define $\upsilon : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ with $\upsilon(\epsilon, \varpi) = \upsilon(\varpi, \epsilon)$ and

$$\begin{cases} \upsilon\left(0, \frac{2}{41}\right) = \upsilon\left(\frac{2}{41}, \frac{3}{61}\right) = \upsilon\left(\frac{3}{61}, \frac{4}{81}\right) = 0.05, \\ \upsilon\left(0, \frac{3}{61}\right) = \upsilon\left(\frac{2}{41}, \frac{4}{81}\right) = 0.08, \\ \upsilon\left(0, \frac{4}{81}\right) = 0.3, \\ \upsilon(\epsilon, \varpi) = \max\{\epsilon, \varpi\}, \text{otherwise}. \end{cases} \tag{1}$$

Thus, $(\mathbb{G}, \upsilon)$ is a rectangular b-metric space with $s = 2$. Furthermore, one can obtain the following:

(1) υ is not a b-metric with $s = 2$, since

$$\begin{aligned} \upsilon\left(0, \frac{4}{81}\right) &= 0.3 > 0.26 = 2 \times 0.13 \\ &= 2 \times \left(\upsilon\left(0, \frac{2}{41}\right) + \upsilon\left(\frac{2}{41}, \frac{4}{81}\right)\right). \end{aligned} \tag{2}$$

(2) υ is not a rectangular metric, since

$$\begin{aligned} \upsilon\left(0, \frac{4}{81}\right) = 0.3 > 0.15 = \upsilon\left(0, \frac{2}{41}\right) \\ + \upsilon\left(\frac{2}{41}, \frac{3}{61}\right) + \upsilon\left(\frac{3}{61}, \frac{4}{81}\right). \end{aligned} \tag{3}$$

(3) υ is not a metric, since

$$\upsilon\left(0, \frac{4}{81}\right) = 0.3 > 0.13 = \upsilon\left(0, \frac{2}{41}\right) + \upsilon\left(\frac{2}{41}, \frac{4}{81}\right). \tag{4}$$

Definition 5 (see [16]). Suppose $(\mathbb{G}, \upsilon)$ is a rectangular b -metric space with $s \ge 1$. Assume that $\{\varpi_n\}$ in $\mathbb{G}$ is a sequence and $\varpi \in \mathbb{G}$

(i) $\{\varpi_n\}$ is convergent to ϖ iff $\lim_{n \longrightarrow +\infty} \upsilon(\varpi_n, \varpi) = 0$

(ii) $\{\varpi_n\}$ is Cauchy iff $\upsilon(\varpi_i, \varpi_j) \longrightarrow 0$ as $i, j \longrightarrow +\infty$

(iii) $(\mathbb{G}, \upsilon)$ is complete iff each Cauchy sequence is convergent

Remark 6. In a rectangular b-metric space, a convergent sequence does not possess unique limit and a convergent sequence is not necessarily a Cauchy sequence. However, one can find that the limit of a Cauchy sequence is unique.

In fact, suppose the sequence $\{\varpi_n\}$ is Cauchy and converges to ϖ^* and ϖ^{**} with $\varpi^* \neq \varpi^{**}$. It follows that

$$v(\varpi^*,\varpi^{**}) \leq s\left[v(\varpi^*, \varpi_n) + v\left(\varpi_n, \varpi_{n+p}\right) + v\left(\varpi_{n+p},\varpi^{**}\right)\right], \quad (5)$$

for all $p > 0$. Let $n \longrightarrow \infty$; we get that $v(\varpi^*,\varpi^{**}) = 0$. Hence, $\varpi^* = \varpi^{**}$, a contradiction.

Example 2 (see [16]). Let $\mathbb{G} = A \cup B$, where $A = \{1/n : n \in \mathbb{N}\}$ and $B = \mathbb{N}$. Define $v : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ with $v(\epsilon, \varpi) = v(\varpi, \epsilon)$ and

$$v(\epsilon, \varpi) = \begin{cases} 0, & \text{if } \epsilon = \varpi, \\ 2\alpha, & \text{if } \epsilon, \varpi \in A, \\ \dfrac{\alpha}{2n}, & \text{if } \epsilon \in A \text{ and } \varpi \in \{2, 3\}, \\ \alpha, & \text{otherwise.} \end{cases} \quad (6)$$

Here, α is a positive number. Thus, v is a rectangular b-metric with $s = 2$. However, we have that $\{1/n\}$ is convergent to 2 and 3. Moreover, $\lim_{n\longrightarrow\infty} v(1/n, 1/(n+p)) = 2\alpha \neq 0$; therefore, $\{1/n\}$ is not a Cauchy sequence.

Definition 7 (see [28]). Suppose $\mathbb{G}$ is a nonempty set and $T : \mathbb{G} \longrightarrow \mathbb{G}$ and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}$ are two mappings. We call $T\alpha$-orbital admissible mapping if

$$\forall \varpi \in \mathbb{G}, \alpha(\varpi, T\varpi) \geq 1 \Rightarrow \alpha\left(T\varpi, T^2\varpi\right) \geq 1. \quad (7)$$

Definition 8 (see [28]). Assume that $T : \mathbb{G} \longrightarrow \mathbb{G}$ and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}$. We call T a triangular α-orbital admissible mapping if

(i) $\alpha(\epsilon, \varpi) \geq 1$ and $\alpha(\varpi, T\varpi) \geq 1$ imply $\alpha(\epsilon, T\varpi) \geq 1$, $\forall \epsilon, \varpi \in \mathbb{G}$

(ii) T is α-orbital admissible

Lemma 9 (see [24]). *Assume $\Theta : [0,+\infty) \longrightarrow [0,+\infty)$ is an increasing mapping. Then, $\forall t > 0, \lim_{n\longrightarrow\infty} \Theta^n(t) = 0 \Rightarrow \Theta(t) < t$.*

3. Main Results

In this part, two fixed point results of injective mappings will be presented on rectangular b-metric spaces.

Definition 10. Suppose $\mathbb{G}$ is a nonempty set, $s \geq 1$ and $p > 0$ are two constants, and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$, $T : \mathbb{G} \longrightarrow \mathbb{G}$. We call $T\alpha_{s^p}$ orbital admissible mapping if

$$\forall \varpi \in \mathbb{G}, \alpha(\varpi, T\varpi) \geq s^p \Rightarrow \alpha\left(T\varpi, T^2\varpi\right) \geq s^p. \quad (8)$$

Definition 11. Suppose $\mathbb{G}$ is a nonempty set, $s \geq 1$ and $p > 0$ are two constants, and $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$, $T : \mathbb{G} \longrightarrow \mathbb{G}$. We call T triangular α_{s^p} orbital admissible mapping if

(i) $\alpha(\epsilon, \varpi) \geq s^p$ and $\alpha(\varpi, T\varpi) \geq s^p$ imply $\alpha(\epsilon, T\varpi) \geq s^p$, $\forall \epsilon, \varpi \in \mathbb{G}$

(ii) T is α_{s^p} orbital admissible

Lemma 12. *Suppose $\mathbb{G}$ is a nonempty set and $T : \mathbb{G} \longrightarrow \mathbb{G}$, $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ are mappings satisfying T which is triangular α_{s^p} orbital admissible, $s \geq 1, p > 0$. Suppose there has a $\varpi_0 \in \mathbb{G}$ with $\alpha(\varpi_0, T\varpi_0) \geq s^p$. Define $\{\varpi_n\}$ in $\mathbb{G}$ by $\varpi_1 = T^{n(\varpi_0)}\varpi_0, \cdots, \varpi_{n+1} = T^{n(\varpi_n)}\varpi_n, \cdots$. Then, $\forall m \in \mathbb{N} \cup \{0\}$, $\alpha(\varpi_m, T^k\varpi_m) \geq s^p, k = 0, 1, 2, \cdots$.*

Proof. Since $\alpha(\varpi_0, T\varpi_0) \geq s^p$ and T is triangular α_{s^p} orbital admissible, we have

$$\begin{aligned} \alpha(\varpi_0, T\varpi_0) \geq s^p \text{ implies } \alpha\left(T\varpi_0, T^2\varpi_0\right) \\ \geq s^p \text{ and } \alpha\left(\varpi_0, T^2\varpi_0\right) \geq s^p. \end{aligned} \quad (9)$$

Similarly, since $\alpha(T\varpi_0, T^2\varpi_0) \geq s^p$, we get

$$\alpha\left(T^2\varpi_0, T^3\varpi_0\right) \geq s^p, \quad (10)$$

$$\alpha\left(\varpi_0, T^3\varpi_0\right) \geq s^p. \quad (11)$$

Applying the above argument repeatedly, one can deduce that $\alpha(\varpi_0, T^k\varpi_0) \geq s^p$ for all $k \in \mathbb{N} \cup \{0\}$. Since $\alpha(\varpi_0, T\varpi_0) \geq s^p$ implies $\alpha(T\varpi_0, T^2\varpi_0) \geq s^p$ and $\alpha(T\varpi_0, T^2\varpi_0) \geq s^p$ implies $\alpha(T^2\varpi_0, T^3\varpi_0) \geq s^p, \cdots$, we can obtain $\alpha(T^{n(\varpi_0)}\varpi_0, T^{n(\varpi_0)+1}\varpi_0) = \alpha(\varpi_1, T\varpi_1) \geq s^p$. Based on this conclusion, we deduce that $\alpha(\varpi_1, T^k\varpi_1) \geq s^p, k = 0, 1, 2, \cdots$. Repeatedly using the above discussion, we have $\alpha(\varpi_m, T^k\varpi_m) \geq s^p, k = 0, 1, 2, \cdots$ for all $m \in \mathbb{N} \cup \{0\}$. □

Define $\Theta = \{\Phi : \mathbb{R}^{+3} \longrightarrow \mathbb{R}^+$ is increasing and continuous in each coordinate variable$\}$. That is, if $\kappa_1^{(1)}, \kappa_2^{(1)}, \kappa_1^{(2)}, \kappa_2^{(2)}, \kappa_1^{(3)}, \kappa_2^{(3)} \in \mathbb{R}^+$ with $\kappa_1^{(1)} \leq \kappa_2^{(1)}, \kappa_1^{(2)} \leq \kappa_2^{(2)}, \kappa_1^{(3)} \leq \kappa_2^{(3)}$, we have

$$\begin{aligned} \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right) &\leq \Phi\left(\kappa_2^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right), \\ \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right) &\leq \Phi\left(\kappa_1^{(1)}, \kappa_2^{(2)}, \kappa_1^{(3)}\right), \\ \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}\right) &\leq \Phi\left(\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_2^{(3)}\right). \end{aligned} \quad (12)$$

Furthermore, we set $\Phi(\epsilon, \epsilon, \epsilon) = \varphi(\epsilon)$ for $\varepsilon \in \mathbb{R}^+$.

Theorem 13. *Suppose $(\mathbb{G}, v)$ is a complete rectangular b-metric space with $s \geq 1$. Suppose $T : \mathbb{G} \longrightarrow \mathbb{G}$ is a continuous injectivity, $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty)$ and $p > 0$. Assume that for any $\epsilon \in \mathbb{G}$, there is a positive number $n(\epsilon)$ satisfying*

$$\begin{aligned}\forall \varpi \in \mathbb{G}, \alpha(\epsilon, \varpi) \geq s^p \Rightarrow \alpha(\epsilon, \varpi) \upsilon\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \\ \leq \Phi\left(\upsilon(\epsilon, \varpi), \upsilon\left(\epsilon, T^{n(\epsilon)}\epsilon\right), \upsilon\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right),\end{aligned} \quad (13)$$

where $\Phi \in \Theta$ *and*

(1) $\lim_{\epsilon \longrightarrow \infty}(\epsilon - s\varphi(\epsilon)) = \infty$

(2) $\forall \epsilon > 0, \lim_{m \longrightarrow \infty}\varphi^m(\epsilon) = 0$

Suppose that

(i) there has a ϵ_0 *in* $\mathbb{G}$ *such that* $\alpha(\epsilon_0, T\epsilon_0) \geq s^p$

(ii) T *is triangular* α_{s^p} *orbital admissible*

(iii) if $\{\varpi_n\}$ *in* $\mathbb{G}$ *satisfies* $\alpha(\varpi_n, \varpi_{n+1}) \geq s^p (\forall n \in \mathbb{N})$ *and* $\varpi_n \longrightarrow \varpi \in \mathbb{G}(n \longrightarrow \infty)$*, then one can choose a subsequence* $\{\varpi_{n_k}\}$ *of* $\{\varpi_n\}$ *with* $\alpha(\varpi_{n_k}, \varpi) \geq s^p, \forall k \in \mathbb{N}$

(iv) $\forall \epsilon \in \mathbb{G}$ *with* $T^{n(\epsilon)}\epsilon = \epsilon$*, we have* $\alpha(\epsilon, \varpi) \geq s^p$ *for any* $\varpi \in \mathbb{G}$

Then, T *possesses a unique fixed point* $\epsilon^* \in \mathbb{G}$*. Further, for each* $\epsilon \in \mathbb{G}$*, the iteration* $\{T^n\epsilon\}$ *converges to* ϵ^*

Proof. By condition (i), one can choose an $\epsilon_0 \in \mathbb{G}$ such that $\alpha(\epsilon_0, T\epsilon_0) \geq s^p$. If ϵ_0 is a fixed point of T and ϖ_0 is the other one, then $\epsilon_0 = T\epsilon_0 = \cdots = T^{n(\epsilon_0)}\epsilon_0 = \cdots$ and $\varpi_0 = T\varpi_0 = \cdots = T^{n(\epsilon_0)}\varpi_0 = \cdots$. From condition (iv), we have $\alpha(\epsilon_0, \varpi_0) \geq s^p$. It follows from (13) that

$$\begin{aligned}\upsilon(\epsilon_0, \varpi_0) &\leq \alpha(\epsilon_0, \varpi_0)\upsilon\left(T^{n(\epsilon_0)}\epsilon_0, T^{n(\epsilon_0)}\varpi_0\right) \\ &\leq \Phi\left(\upsilon(\epsilon_0, \varpi_0), \upsilon\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right), \upsilon\left(\epsilon_0, T^{n(\epsilon_0)}\varpi_0\right)\right) \\ &\leq \varphi(\upsilon(\epsilon_0, \varpi_0)).\end{aligned} \quad (14)$$

From Lemma 9, we have $\varphi(\upsilon(\epsilon_0, \varpi_0)) < \upsilon(\epsilon_0, \varpi_0)$. Thus,

$$\upsilon(\epsilon_0, \varpi_0) \leq \varphi(\upsilon(\epsilon_0, \varpi_0)) < \upsilon(\epsilon_0, \varpi_0), \quad (15)$$

which is contradiction. From this, we get that ϵ_0 is the unique fixed point. After that, in the subsequent discussion, we assume that $T\epsilon_0 \neq \epsilon_0$. Now we define $\{\epsilon_n\}$ in $\mathbb{G}$ by $\epsilon_1 = T^{n(\epsilon_0)}\epsilon_0, \cdots, \epsilon_{n+1} = T^{n(\epsilon_n)}\epsilon_n$.

First, we shall show that the orbit $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded. For this purpose, we fix an integer $\ell, 0 \leq \ell < n(\epsilon_0)$. Let

$$u_j = \upsilon\left(\epsilon_0, T^{jn(\epsilon_0)+\ell}\epsilon_0\right), j = 0, 1, 2, \cdots, \quad (16)$$

$$\begin{aligned}h = \max\Big\{u_0, \upsilon\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right), \upsilon\left(\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right), \\ \upsilon\left(T^{n(\epsilon_0)}\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right)\Big\}.\end{aligned} \quad (17)$$

Since $\lim_{\epsilon \longrightarrow \infty}(\epsilon - s\varphi(\epsilon)) = \infty$, there has $c > h$ such that $\epsilon - s\varphi(\epsilon) > 2sh, \epsilon \geq c$. It is obvious that $u_0 \leq h < c$. Assume that there has a positive number j_0 with $u_{j_0} \geq c$. Evidently, one may suppose that $u_i < c, \forall i < j_0$. Let $\epsilon_0, T^{n(\epsilon_0)}\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$ be different from each other. Otherwise, we consider six cases.

Case 1. $\epsilon_0 = T^{n(\epsilon_0)}\epsilon_0$. One can get that

$$\epsilon_0 = T^{n(\epsilon_0)}\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0 = T^{3n(\epsilon_0)}\epsilon_0 = \cdots. \quad (18)$$

It follows that $u_j = \upsilon(\epsilon_0, T^{\ell}\epsilon_0)$ is a constant which implies that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

Case 2. $\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0$. We deduce that

$$\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0 = T^{4n(\epsilon_0)}\epsilon_0 = T^{6n(\epsilon_0)}\epsilon_0 = \cdots, \quad (19)$$

$$T^{n(\epsilon_0)}\epsilon_0 = T^{3n(\epsilon_0)}\epsilon_0 = T^{5n(\epsilon_0)}\epsilon_0 = \cdots. \quad (20)$$

Hence,

$$u_j = \begin{cases} \upsilon\left(\epsilon_0, T^{n(\epsilon_0)+\ell}\epsilon_0\right), & j \text{ is odd}, \\ \upsilon\left(\epsilon_0, T^{\ell}\epsilon_0\right), & j \text{ is even}. \end{cases} \quad (21)$$

It follows that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

Case 3. $T^{n(\epsilon_0)}\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0$. Obviously,

$$T^{n(\epsilon_0)}\epsilon_0 = T^{2n(\epsilon_0)}\epsilon_0 = T^{3n(\epsilon_0)}\epsilon_0 = T^{4n(\epsilon_0)}\epsilon_0 = \cdots. \quad (22)$$

As the argument of Case 1, we get that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

Case 4. $\epsilon_0 = T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$. In this case, we obtain that $u_{j_0} = 0$, a contradiction.

Case 5. $T^{n(\epsilon_0)}\epsilon_0 = T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$. It follows that

$$u_{j_0} = \upsilon\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right) = \upsilon\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right) \leq h < c. \quad (23)$$

It is a contradiction.

Case 6. $T^{2n(\epsilon_0)}\epsilon_0 = T^{j_0 n(\epsilon_0)+\ell}\epsilon_0$. It is obvious that

$$u_{j_0} = \upsilon\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right) = \upsilon\left(\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right) \leq h < c, \quad (24)$$

a contradiction.

It is easy to get $\alpha(\epsilon_0, T^k\epsilon_0) \geq s^p, \forall k \in \mathbb{N}$ from Lemma 12. By using triangle inequality and (16), we have

$$\begin{aligned}
&\upsilon\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right)\\
&\leq s\left[\upsilon\left(\epsilon_0, T^{2n(\epsilon_0)}\epsilon_0\right) + \upsilon\left(T^{2n(\epsilon_0)}\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right)\right.\\
&\quad\left. + \upsilon\left(T^{n(\epsilon_0)}\epsilon_0, T^{j_0(\epsilon_0)+\ell}\epsilon_0\right]\right.\\
&\leq 2sh + s\alpha\left(\epsilon_0, T^{(j_0-1)n(\epsilon_0)+\ell}\epsilon_0\right)\upsilon\left(T^{n(\epsilon_0)}\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right)\\
&\leq 2sh + s\Phi\left(\upsilon\left(\epsilon_0, T^{(j_0-1)n(\epsilon_0)+\ell}\epsilon_0\right),\right.\\
&\quad\left.\upsilon\left(\epsilon_0, T^{n(\epsilon_0)}\epsilon_0\right), \upsilon\left(\epsilon_0, T^{j_0 n(\epsilon_0)+\ell}\epsilon_0\right)\right)\\
&\leq 2sh + s\Phi\left(u_{j_0}, u_{j_0}, u_{j_0}\right) = 2sh + s\varphi\left(u_{j_0}\right).
\end{aligned} \tag{25}$$

That is, $u_{j_0} - s\varphi(u_{j_0}) \leq 2sh$, which is impossible. Therefore, $u_j < c$ for $j = 0, 1, 2, \cdots$. It follows that $\{T^i\epsilon_0\}_{i=0}^{\infty}$ is bounded.

If there exists some $n_0 \in \mathbb{N}$ satisfying $\epsilon_{n_0} = \epsilon_{n_0+1} = T^{n(\epsilon_{n_0})}\epsilon_{n_0}$, then ϵ_{n_0} is a fixed point of $T^{n(\epsilon_{n_0})}$. Assume there is $\varpi \in \mathbb{G}$ such that $\varpi = T^{n(\epsilon_{n_0})}\varpi$ and $\varpi \neq \epsilon_{n_0}$, by condition (iv), we have $\alpha(\epsilon_{n_0}, \varpi) \geq s^p$ and

$$\begin{aligned}
\upsilon(\epsilon_{n_0}, \varpi) &\leq \alpha(\epsilon_{n_0}, \varpi)\upsilon\left(T^{n(\epsilon_{n_0})}\epsilon_{n_0}, T^{n(\epsilon_{n_0})}\varpi\right)\\
&\leq \Phi\left(\upsilon(\epsilon_{n_0}, \varpi), \upsilon\left(\epsilon_{n_0}, T^{n(\epsilon_{n_0})}\epsilon_{n_0}\right), \upsilon\left(\epsilon_{n_0}, T^{n(\epsilon_{n_0})}\varpi\right)\right)\\
&\leq \varphi(\upsilon(\epsilon_{n_0}, \varpi)) < \upsilon(\epsilon_{n_0}, \varpi),
\end{aligned} \tag{26}$$

which is contradiction. From this, $T^{n(\epsilon_{n_0})}$ possesses the unique fixed point ϵ_{n_0}. Since $T\epsilon_{n_0} = TT^{n(\epsilon_{n_0})}\epsilon_{n_0} = T^{n(\epsilon_{n_0})}T\epsilon_{n_0}$, we have $T\epsilon_{n_0} = \epsilon_{n_0}$ because of the uniqueness of $T^{n(\epsilon_{n_0})}$. Subsequently, we assume that $\epsilon_n \neq \epsilon_{n+1}, \forall n \in \mathbb{N}$.

Next, we show that $\{\epsilon_n\}$ is Cauchy. Suppose n and i are two positive numbers. It is obvious that $\alpha(\epsilon_{n-1}, T^k\epsilon_{n-1}) \geq s^p, \forall k \in \mathbb{N}$. Then,

$$\begin{aligned}
\upsilon(\epsilon_n, \epsilon_{n+i}) &\leq \alpha\left(\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+n(\epsilon_{n+i-2})+\cdots+n(\epsilon_n)}\epsilon_{n-1}\right)\\
&\quad\cdot \upsilon\left(T^{n(\epsilon_{n-1})}\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+\cdots+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\\
&\leq \Phi\left(\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+n(\epsilon_{n+i-2})+\cdots+n(\epsilon_n)}\epsilon_{n-1}\right),\right.\\
&\quad\left.\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right), \upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n+i-1})+\cdots+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\right)\\
&\leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}\right\}\right).
\end{aligned} \tag{27}$$

For each $q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}$, we have

$$\begin{aligned}
\upsilon(\epsilon_{n-1}, q) &= \upsilon(\epsilon_{n-1}, T^m\epsilon_{n-1})\\
&\leq \alpha(\epsilon_{n-2}, T^m\epsilon_{n-2})\upsilon\left(T^{n(\epsilon_{n-2})}\epsilon_{n-2}, T^{m+n(\epsilon_{n-2})}\epsilon_{n-2}\right)\\
&\leq \Phi\left(\upsilon(\epsilon_{n-2}, T^m\epsilon_{n-2}), \upsilon\left(\epsilon_{n-2}, T^{n(\epsilon_{n-2})}\epsilon_{n-2}\right),\right.\\
&\quad\left.\upsilon\left(\epsilon_{n-2}, T^{n(\epsilon_{n-2})+m}\epsilon_{n-2}\right)\right)\\
&\leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-2}, q) | q \in \{T^m\epsilon_{n-2}\}_{m=0}^{\infty}\right\}\right.
\end{aligned} \tag{28}$$

According to (27) and (28), we deduce

$$\begin{aligned}
\upsilon(\epsilon_n, \epsilon_{n+i}) &\leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}\right)\right.\\
&\leq \cdots \leq \varphi^n\left(\sup\left\{\upsilon(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=0}^{\infty}\right\}\right) \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{29}$$

That is, $\{\epsilon_n\}$ is Cauchy. In light of the completeness of $(\mathbb{G}, \upsilon)$, one can find an $\epsilon^* \in \mathbb{G}$ with $\lim_{n\longrightarrow\infty}\epsilon_n = \epsilon^*$. We might as well let $\epsilon_n \neq \epsilon^*$ and $\epsilon_n \neq T^{n(\epsilon^*)}\epsilon_n$. Otherwise, we have $\epsilon^* = T^{n(\epsilon^*)}\epsilon^*$ according to the continuity of T. In view of triangle inequality, one deduce

$$\begin{aligned}
&\upsilon\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right)\\
&\leq s\left[\upsilon(\epsilon^*, \epsilon_n) + \upsilon\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) + \upsilon\left(T^{n(\epsilon^*)}\epsilon_n, T^{n(\epsilon^*)}\epsilon^*\right)\right].
\end{aligned} \tag{30}$$

On the other hand,

$$\begin{aligned}
&\upsilon\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right)\\
&\leq \alpha\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right)\upsilon\left(T^{n(\epsilon_{n-1})}\epsilon_{n-1}, T^{n(\epsilon^*)+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\\
&\leq \Phi\left(\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right), \upsilon\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right),\right.\\
&\quad\left.\upsilon\left(\epsilon_{n-1}, T^{n(\epsilon^*)+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\right)\\
&\leq \varphi\left(\sup\left\{\upsilon(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=0}^{\infty}\right\}\right)\\
&\leq \cdots \leq \varphi^n\left(\sup\left\{\upsilon(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=0}^{\infty}\right\}\right) \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{31}$$

From the continuity of T, $\lim_{n\longrightarrow\infty}\upsilon(T^{n(\epsilon^*)}\epsilon_n, T^{n(\epsilon^*)}\epsilon^*) = 0$. Thereupon, by the use of (30) and (31), one can obtain $\upsilon(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*) = 0$ as $n \longrightarrow \infty$. Assume there exists $\varpi^* \neq \epsilon^*$ satisfying $\varpi^* = T^{n(\epsilon^*)}\varpi^*$ and we have $\alpha(\epsilon^*, \varpi^*) \geq s^p$ according to the condition (iv). Then,

$$\begin{aligned}
\upsilon(\epsilon^*, \varpi^*) &\leq \alpha(\epsilon^*, \varpi^*)\upsilon\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right)\\
&\leq \Phi\left(\upsilon(\epsilon^*, \varpi^*), \upsilon\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), \upsilon\left(\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right)\right)\\
&\leq \varphi(\upsilon(\epsilon^*, \varpi^*)) < \upsilon(\epsilon^*, \varpi^*),
\end{aligned} \tag{32}$$

impossible. After that, $T^{n(\epsilon^*)}$ has the unique fixed point ϵ^*. Since $T\epsilon^* = TT^{n(\epsilon^*)}\epsilon^* = T^{n(\epsilon^*)}T\epsilon^*$, we deduce $T\epsilon^* = \epsilon^*$. That is, T has a fixed point.

Now we show that if condition (iv) is met. So T possesses a unique fixed point. Assume ϖ^* is another one; from condition (iv), one can obtain $\alpha(\epsilon^*, \varpi^*) \geq s^p$. In view of (13), we have

$$\begin{aligned} v(\epsilon^*, \varpi^*) &\leq \alpha(\epsilon^*, \varpi^*) v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right) \\ &\leq \Phi\left(v(\epsilon^*, \varpi^*), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right)\right) \\ &\leq \varphi(v(\epsilon^*, \varpi^*)). \end{aligned} \tag{33}$$

Lemma 9 ensures that $\varphi(v(\epsilon^*, \varpi^*)) < v(\epsilon^*, \varpi^*)$. Thus,

$$v(\epsilon^*, \varpi^*) \leq \varphi(v(\epsilon^*, \varpi^*)) < v(\epsilon^*, \varpi^*), \tag{34}$$

which is impossible. It follows that ϵ^* is the unique fixed point of T.

Finally, we prove the last part. To show this statement, we fix an integer ℓ, $0 \leq \ell < n(\epsilon^*)$, and let $v_k = v(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon)$, $k = 0, 1, 2, \cdots$ for $\epsilon \in \mathbb{G}$. If there exists $k \in \mathbb{N}$ satisfying $v_k = 0$, we have

$$\begin{aligned} v_{k+1} &= v\left(\epsilon^*, T^{(k+1)n(\epsilon^*)+\ell}\epsilon\right) \\ &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}T^{kn(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \alpha\left(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right) v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}T^{kn(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \Phi(v_k, 0, v_{k+1}). \end{aligned} \tag{35}$$

If $v_{k+1} > 0$, one can obtain that $v_{k+1} \leq \Phi(v_{k+1}, v_{k+1}, v_{k+1}) = \varphi(v_{k+1}) < v_{k+1}$, which is a contradiction. Hence, $v_{k+1} = 0$. It follows that $v_{k+2} = v_{k+3} = \cdots = 0$.

Now we suppose that $v_k \neq 0$, $\forall n \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} v\left(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right) &\leq \alpha\left(\epsilon^*, T^{(k-1)n(\epsilon^*)+\ell}\epsilon\right) v\left(T^{n(\epsilon^*)}\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \Phi\left(v\left(\epsilon^*, T^{(k-1)n(\epsilon^*)+\ell}\epsilon\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), v\left(\epsilon^*, T^{kn(\epsilon^*)+\ell}\epsilon\right)\right) \\ &= \Phi(v_{k-1}, 0, v_k). \end{aligned} \tag{36}$$

If for some $k \in \mathbb{N}$, $v_k \geq v_{k-1}$, we deduce $v_k \leq \Phi(v_k, v_k, v_k) = \varphi(v_k) < v_k$, which is a contradiction. Hence, we get $v_k \leq \varphi(v_{k-1}) \leq \cdots \leq \varphi^k(v_0) \longrightarrow 0 \quad (k \longrightarrow \infty)$. That is, for ℓ, the sequence $\{T^{kn(\epsilon^*)+\ell}\epsilon\}$ converges to ϵ^* for any $\epsilon \in \mathbb{G}$. Consequently, one can obtain that the sequences $\{T^{kn(\epsilon^*)}\epsilon\}$, $\{T^{kn(\epsilon^*)+1}\epsilon\}, \{T^{kn(\epsilon^*)+2}\epsilon\}, \cdots, \{T^{kn(\epsilon^*)+n(\epsilon^*)-1}\epsilon\}$ are convergent to the point ϵ^*. It follows that we get $\{T^n\epsilon\}$ converges to the point ϵ^* for $\epsilon \in \mathbb{G}$. □

Example 3. Let $(\mathbb{G}, v)$ be the same as it is in Example 1. Define $T : \mathbb{G} \longrightarrow \mathbb{G}$ as

$$T\epsilon = \begin{cases} 0, & \epsilon = 0, \\ \dfrac{2}{41}, & \epsilon = \dfrac{1}{2}, \\ \dfrac{3}{61}, & \epsilon = \dfrac{1}{3}, \\ \dfrac{4}{81}, & \epsilon = \dfrac{1}{4}, \\ \dfrac{1}{2^2 \cdot 2}, & \epsilon = \dfrac{2}{41}, \\ \dfrac{1}{2^2 \cdot 3}, & \epsilon = \dfrac{3}{61}, \\ \dfrac{1}{2^2 \cdot 4}, & \epsilon = \dfrac{4}{81}, \\ \dfrac{1}{2^2 \cdot \chi}, & \epsilon = \dfrac{1}{\chi}, \chi \geq 5. \end{cases} \tag{37}$$

Define mapping $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ by

$$\alpha(\epsilon, \varpi) = \begin{cases} s^p, & \epsilon, \varpi \in \{0\} \cup \left\{\dfrac{1}{\chi}, \chi \geq 5\right\}, \\ 0, & \text{otherwise}. \end{cases} \tag{38}$$

Define $\Phi(\kappa_1, \kappa_2, \kappa_3) = (1/12)(\kappa_1 + \kappa_2 + \kappa_3)$ for all $\kappa_i \in [0, +\infty)(i = 1, 2, 3)$, and it follows that $\varphi(t) = (1/4)t$. Let $n(\epsilon) = 3$ for all $\epsilon \in \mathbb{G}$. For $\epsilon, \varpi \in \mathbb{G}$ such that $\alpha(\epsilon, \varpi) \geq s^p$, we get that $\epsilon, \varpi \in \{0\} \cup \{1/\chi, \chi \geq 5\}$. It follows that we consider the following two cases:

(i) $\epsilon = 0$ and $\varpi \in \{1/\chi, \chi \geq 5\}$

$$\begin{aligned} &\alpha(\epsilon, \varpi) v\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \\ &\quad = 4 \cdot v\left(T^3(0), T^3\left(\frac{1}{\chi}\right)\right) = \frac{1}{16\chi}, \\ &\Phi\left(v(\epsilon, \varpi), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right) \\ &\quad = \frac{1}{12} \cdot \left[v\left(0, \frac{1}{\chi}\right) + v\left(0, T^3(0)\right) + v\left(0, T^3\left(\frac{1}{\chi}\right)\right)\right] \\ &\quad = \frac{1}{12} \cdot \left(\frac{1}{\chi} + \frac{1}{64\chi}\right) > \frac{1}{12\chi}. \end{aligned} \tag{39}$$

That is, $\alpha(\epsilon, \varpi) v(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi) \leq \Phi(v(\epsilon, \varpi), v(\epsilon, T^{n(\epsilon)}\epsilon), v(\epsilon, T^{n(\epsilon)}\varpi))$.

(ii) $\epsilon, \varpi \in \{1/\chi, r \geq 5\}$. Let $\epsilon = 1/\chi$ and $\varpi = 1/l$ with $l \geq \chi$. One can obtain that

$$\begin{aligned}
&\alpha(\epsilon, \varpi)v\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \\
&\quad = 4 \cdot v\left(T^3\left(\frac{1}{\chi}\right), T^3\left(\frac{1}{l}\right)\right) = \frac{1}{16\chi}, \\
&\Phi\left(v(\epsilon, \varpi), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right) \\
&\quad = \frac{1}{12} \cdot \left[v\left(\frac{1}{\chi}, \frac{1}{l}\right) + v\left(\frac{1}{\chi}, T^3\left(\frac{1}{\chi}\right)\right)\right. \\
&\qquad \left. + v\left(\frac{1}{\chi}, T^3\left(\frac{1}{l}\right)\right)\right] = \frac{1}{4\chi}.
\end{aligned} \tag{40}$$

The above inequalities imply that

$$\begin{aligned}
&\alpha(\epsilon, \varpi)v\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \\
&\quad \leq \Phi\left(v(\epsilon, \varpi), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right).
\end{aligned} \tag{41}$$

Thus, all conditions of Theorem 13 are fulfilled with $p = s = 2$. As a result, T possesses a unique fixed point 0. Meanwhile, for each $\epsilon \in \mathbb{G}$, $\{T^n\epsilon\}$ converges to the point 0.

Remark 14.

(1) Since rectangular metric spaces can be seen as rectangular b -metric spaces with parameter $s = 1$, one can get the corresponding conclusions of Sehgal-Guseman-type mappings in rectangular metric spaces

(2) Since b-metric spaces with parameter s can be seen as rectangular b-metric spaces with parameter s^2, one can obtain the corresponding conclusions of Sehgal-Guseman-type mappings in b-metric spaces

(3) If $\alpha(x, y) = s^p$, one can get the generalized Φ-Sehgal-Guseman-type contractive mappings in rectangular b-metric spaces

Theorem 15. *Suppose $(\mathbb{G}, v)$ is a complete rectangular b -metric space with $s \geq 1$. Suppose $T : \mathbb{G} \longrightarrow \mathbb{G}$ is a continuous injectivity and $\psi : [0, +\infty) \longrightarrow [0, 1/2s)$ satisfying that for any $\epsilon \in \mathbb{G}$; there is a positive number $n(\epsilon)$ satisfying*

$$v\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) \leq \psi(M(\epsilon, \varpi))M(\epsilon, \varpi), \forall \varpi \in \mathbb{G}, \tag{42}$$

where

$$M(\epsilon, \varpi) = \max\left\{v(\epsilon, \varpi), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\varpi\right)\right\}. \tag{43}$$

Then, T possesses a unique fixed point ϵ^*. Furthermore, for each $\epsilon \in \mathbb{G}$, the iteration $\{T^n\epsilon\}$ is convergent to ϵ^*.

Proof. Let $\epsilon_0 \in \mathbb{G}$. Consider a sequence $\{\epsilon_n\}$ in $\mathbb{G}$ by $\epsilon_1 = T^{n(\epsilon_0)}\epsilon_0, \cdots, \epsilon_{n+1} = T^{n(\epsilon_n)}\epsilon_n$. If $\epsilon_{n_0} = \epsilon_{n_0+1} = T^{n(\epsilon_{n_0})}\epsilon_{n_0}$ for an $n_0 \in \mathbb{N}$, then ϵ_{n_0} becomes to a fixed point of $T^{n(\epsilon_{n_0})}$. Assume there exists $\varpi \in \mathbb{G}$ with $\varpi = T^{n(\epsilon_{n_0})}\varpi$ and $\varpi \neq \epsilon_{n_0}$; then,

$$v\left(\epsilon_{n_0}, \varpi\right) = v\left(T^{n\left(\epsilon_{n_0}\right)}\epsilon_{n_0}, T^{n\left(\epsilon_{n_0}\right)}\varpi\right) \leq \psi\left(M\left(\epsilon_{n_0}, \varpi\right)\right)M\left(\epsilon_{n_0}, \varpi\right), \tag{44}$$

where

$$\begin{aligned}
M\left(\epsilon_{n_0}, \varpi\right) = \max &\left\{v\left(\epsilon_{n_0}, \varpi\right), v\left(\epsilon_{n_0}, T^{n\left(\epsilon_{n_0}\right)}\epsilon_{n_0}\right),\right. \\
&\left. v\left(\epsilon_{n_0}, T^{n\left(\epsilon_{n_0}\right)}\varpi\right)\right\} = v\left(\epsilon_{n_0}, \varpi\right) > 0.
\end{aligned} \tag{45}$$

From this, we get $v(\epsilon_{n_0}, \varpi) < (1/2s)v(\epsilon_{n_0}, \varpi)$ which is impossible. Therefore, ϵ_{n_0} is the unique fixed point of $T^{n(\epsilon_{n_0})}$. Since $T\epsilon_{n_0} = T^{n(\epsilon_{n_0})}T\epsilon_{n_0}$, we have $T\epsilon_{n_0} = \epsilon_{n_0}$ because of the uniqueness of $T^{n(\epsilon_{n_0})}$. Subsequently, we assume that $\epsilon_n \neq \epsilon_{n+1}$, $\forall n \in \mathbb{N}$.

For $\epsilon \in \mathbb{G}$, set $z(\epsilon) = \max\{v(\epsilon, T^k\epsilon), k = 1, 2, \cdots, n(\epsilon), n(\epsilon) + 1, \cdots, 2n(\epsilon)\}$. We first prove that $r(\epsilon) = \sup v(\epsilon, T^n\epsilon) < \infty$ for all $n \in \mathbb{N}$. Assume $n > n(\epsilon)$ is a positive number satisfying $n = rn(\epsilon) + \ell, r \geq 1, 0 \leq \ell < n(\epsilon)$ and $\delta_r(\epsilon) = v(\epsilon, T^{rn(\epsilon)+\ell}\epsilon), r = 0, 1, 2, \cdots$. We suppose that $\epsilon, T^{n(\epsilon)}\epsilon, T^{2n(\epsilon)}\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon$ are four distinct elements. Otherwise, the conclusion is true. Thus,

$$\begin{aligned}
v(\epsilon, T^n\epsilon) &= v\left(\epsilon, T^{rn(\epsilon)+\ell}\epsilon\right) \\
&\leq s\left[v\left(\epsilon, T^{2n(\epsilon)}\epsilon\right) + v\left(T^{2n(\epsilon)}\epsilon, T^{n(\epsilon)}\epsilon\right)\right. \\
&\quad \left. + v\left(T^{n(\epsilon)}\epsilon, T^{rn(\epsilon)+\ell}\epsilon\right)\right] \\
&\leq s\left[z(\epsilon) + \psi\left(M\left(\epsilon, T^{n(\epsilon)}\epsilon\right)\right)M\left(\epsilon, T^{n(\epsilon)}\epsilon\right)\right. \\
&\quad \left. + \psi\left(M\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right)\right)M\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right)\right],
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&M\left(\epsilon, T^{n(\epsilon)}\epsilon\right) \\
&\quad = \max\left\{v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{2n(\epsilon)}\epsilon\right)\right\} = z(\epsilon),
\end{aligned} \tag{47}$$

$$\begin{aligned}
&M\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right) \\
&\quad = \max\left\{v\left(\epsilon, T^{(r-1)n(\epsilon)+\ell}\epsilon\right), v\left(\epsilon, T^{n(\epsilon)}\epsilon\right), v\left(\epsilon, T^{rn(\epsilon)+\ell}\epsilon\right)\right\} \\
&\quad \leq \max\left\{\delta_{r-1}(\epsilon), z(\epsilon), \delta_r(\epsilon)\right\}.
\end{aligned} \tag{48}$$

By (46), (47), and (48), we deduce

$$\delta_r(\epsilon) \le s\left[z(\epsilon) + \frac{1}{2s}z(\epsilon) + \frac{1}{2s}\max\{\delta_{r-1}(\epsilon), z(\epsilon), \delta_r(\epsilon)\}\right]. \tag{49}$$

Hence, one can conclude that $(1/(1+2s))\delta_r(\epsilon) \le z(\epsilon)$ by induction. Indeed, when $r=1$, we have $\delta_1(\epsilon) \le ((1+2s)/2)z(\epsilon) + (1/2)\max\{z(\epsilon), \delta_1(\epsilon)\}$. If $\delta_1(\epsilon) \ge z(\epsilon)$, we get $\delta_1(\epsilon) \le (1+2s)z(\epsilon)$. If $\delta_1(\epsilon) < z(\epsilon)$, we get $\delta_1(\epsilon) \le (1+s)z(\epsilon) < (1+2s)z(\epsilon)$. We assume $\delta_r(\epsilon) \le (1+2s)z(\epsilon)$; then, $\delta_{r+1}(\epsilon) \le ((1+2s)/2)z(\epsilon) + (1/2)\max\{(1+2s)z(\epsilon), z(\epsilon), \delta_{r+1}(\epsilon)\} \le (1+2s)z(\epsilon)$. Hence, $r(\epsilon) = \sup d(T^n\epsilon, \epsilon) < \infty$.

Next, we prove that $\lim_{n\longrightarrow\infty} v(\epsilon_n, \epsilon_{n+1}) = 0$. By contractive condition (42), we have

$$\begin{aligned} v(\epsilon_n, \epsilon_{n+1}) &= v\left(T^{n(\epsilon_{n-1})}\epsilon_{n-1}, T^{n(\epsilon_n)+n(\epsilon_{n-1})}\epsilon_{n-1}\right) \\ &\le \psi\left(M\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right)\right)M\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right), \end{aligned} \tag{50}$$

where

$$\begin{aligned} &M\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right) \\ &\quad = \max\left\{v\left(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}\right), v\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right), \right. \\ &\qquad \left. v\left(\epsilon_{n-1}, T^{n(\epsilon_n)+n(\epsilon_{n-1})}\epsilon_{n-1}\right)\right\} \\ &\quad \le \sup\left\{v(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\}. \end{aligned} \tag{51}$$

It is obvious that $M(\epsilon_{n-1}, T^{n(\epsilon_n)}\epsilon_{n-1}) > 0$, so

$$v(\epsilon_n, \epsilon_{n+1}) < \frac{1}{2s}\sup\left\{v(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\}. \tag{52}$$

For each $q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}$, we have

$$\begin{aligned} v(\epsilon_{n-1}, q) &= v(\epsilon_{n-1}, T^m\epsilon_{n-1}) \\ &= v\left(T^{n(\epsilon_{n-2})}\epsilon_{n-2}, T^{m+n(\epsilon_{n-2})}\epsilon_{n-2}\right) \\ &\le \psi(M(\epsilon_{n-2}, T^m\epsilon_{n-2}))M(\epsilon_{n-2}, T^m\epsilon_{n-2}), \end{aligned} \tag{53}$$

where

$$\begin{aligned} &M(\epsilon_{n-2}, T^m\epsilon_{n-2}) \\ &\quad = \max\left\{v(\epsilon_{n-2}, T^m\epsilon_{n-2}), v\left(\epsilon_{n-2}, T^{n(\epsilon_{n-2})}\epsilon_{n-2}\right), \right. \\ &\qquad \left. v\left(\epsilon_{n-2}, T^{m+n(\epsilon_{n-2})}\epsilon_{n-2}\right)\right\} \\ &\quad \le \sup\left\{v(\epsilon_{n-2}, q) | q \in \{T^m\epsilon_{n-2}\}_{m=1}^{\infty}\right\} > 0. \end{aligned} \tag{54}$$

It means $v(\epsilon_{n-1}, q) < (1/2s)\sup\{v(\epsilon_{n-2}, q) | q \in \{T^m\epsilon_{n-2}\}_{m=1}^{\infty}\}$. So we deduce

$$\begin{aligned} v(\epsilon_n, \epsilon_{n+1}) &< \frac{1}{2s}\sup\left\{v(\epsilon_{n-1}, q) | q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\} \\ &< \cdots < \frac{1}{(2s)^n}\sup\left\{v(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=1}^{\infty}\right\} \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{55}$$

That is, $\lim_{n\longrightarrow\infty} v(\epsilon_n, \epsilon_{n+1}) = 0$.

For the sequence $\{\epsilon_n\}$, we consider $v(\epsilon_n, \epsilon_{n+p})$ by the following cases. For the sake of convenience, set $r_0 = \sup\{v(\epsilon_0, q) | q \in \{T^m\epsilon_0\}_{m=1}^{\infty}\}$.

If p is odd, assume $p = 2m+1$,

$$\begin{aligned} &v(\epsilon_n, \epsilon_{n+2m+1}) \\ &\quad \le s[v(\epsilon_n, \epsilon_{n+1}) + v(\epsilon_{n+1}, \epsilon_{n+2}) + v(\epsilon_{n+2}, \epsilon_{n+2m+1})] \\ &\quad < s\left[\frac{1}{(2s)^n}r_0 + \frac{1}{(2s)^{n+1}}r_0\right] + s^2[v(\epsilon_{n+2}, \epsilon_{n+3}) \\ &\qquad + v(\epsilon_{n+3}, \epsilon_{n+4}) + v(\epsilon_{n+4}, \epsilon_{n+2m+1})] \\ &\quad < \cdots < s\frac{1}{(2s)^n}r_0 + s\frac{1}{(2s)^{n+1}}r_0 + s^2\frac{1}{(2s)^{n+2}}r_0 \\ &\qquad + s^2\frac{1}{(2s)^{n+3}}r_0 + \cdots + s^m\frac{1}{(2s)^{n+2m}}r_0 \\ &\quad \le \frac{s}{(2s)^n}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 + s\frac{1}{(2s)^{n+1}}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 \\ &\quad \le \frac{s}{(2s)^n}\cdot\frac{1+(1/2s)}{1-(1/4s)}r_0 \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{56}$$

If p is even, assume $p = 2m$,

$$\begin{aligned} v(\epsilon_n, \epsilon_{n+2m}) &\le s[v(\epsilon_n, \epsilon_{n+1}) + v(\epsilon_{n+1}, \epsilon_{n+2}) + v(\epsilon_{n+2}, \epsilon_{n+2m})] \\ &< s\left[\frac{1}{(2s)^n}r_0 + \frac{1}{(2s)^{n+1}}r_0\right] + s^2\left[\frac{1}{(2s)^{n+2}}r_0 + \frac{1}{(2s)^{n+3}}r_0\right] \\ &\quad + \cdots + s^{m-1}\left[\frac{1}{(2s)^{n+2m-4}}r_0 + \frac{1}{(2s)^{n+2m-3}}r_0\right] \\ &\quad + s^{m-1}v(\epsilon_{n+2m-2}, \epsilon_{n+2m}) \\ &\le s\frac{1}{(2s)^n}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 + s\frac{1}{(2s)^{n+1}}\left[1 + s\frac{1}{(2s)^2} + \cdots\right]r_0 \\ &\quad + s^{m-1}\frac{1}{(2s)^{n+2m-2}}r_0 \\ &\le s\frac{1}{(2s)^n}\cdot\frac{1}{2m}\frac{1}{(2s)^{n-2}}r_0 \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{57}$$

In view of (56) and (57), one can get that $\{\epsilon_n\}$ is Cauchy. By the completeness of $(\mathbb{G}, v)$, one can choose a point $\epsilon^* \in \mathbb{G}$ with $\lim_{n\longrightarrow\infty}\epsilon_n = \epsilon^*$. We might as well let $\epsilon_n \ne \epsilon^*$ and ϵ_n

$\neq T^{n(\epsilon^*)}\epsilon_n$. Otherwise, we have $\epsilon^* = T^{n(\epsilon^*)}\epsilon^*$ according to the continuity of T. And from that, one can deduce

$$\begin{aligned} v\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) &= v\left(T^{\epsilon_{n-1}}\epsilon_{n-1}, T^{n(\epsilon^*)+n(\epsilon_{n-1})}\epsilon_{n-1}\right) \\ &\leq \psi\left(M\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right)\right)M\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right), \end{aligned} \tag{58}$$

where

$$\begin{aligned} M\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right) = \max\Big\{& v\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_{n-1}\right), \\ & v\left(\epsilon_{n-1}, T^{n(\epsilon_{n-1})}\epsilon_{n-1}\right), v\left(\epsilon_{n-1}, T^{n(\epsilon^*)}\epsilon_n\right)\Big\} > 0. \end{aligned} \tag{59}$$

It follows that

$$\begin{aligned} v\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) &< \frac{1}{2s}\sup\left\{v(\epsilon_{n-1}, q) \mid q \in \{T^m\epsilon_{n-1}\}_{m=1}^{\infty}\right\} \\ &< \cdots < \frac{1}{(2s)^n}\sup\left\{v(\epsilon_0, q) \mid q \in \{T^m\epsilon_0\}_{m=1}^{\infty}\right\} \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{60}$$

Since T is a continuous mapping, $\lim_{n\longrightarrow\infty} d(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\epsilon_n) = 0$. Therefore,

$$\begin{aligned} v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right) \leq s\Big[& v(\epsilon^*, \epsilon_n) + v\left(\epsilon_n, T^{n(\epsilon^*)}\epsilon_n\right) \\ & + v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\epsilon_n\right)\Big] \longrightarrow 0 \ (n \longrightarrow \infty). \end{aligned} \tag{61}$$

This means that $\epsilon^* = T^{n(\epsilon^*)\epsilon^*}$. Now,

$$\begin{aligned} v(\epsilon^*, T\epsilon^*) &= v\left(T^{n(\epsilon^*)}\epsilon^*, TT^{n(\epsilon^*)}\epsilon^*\right) \\ &\leq \psi(M(\epsilon^*, T\epsilon^*))M(\epsilon^*, T\epsilon^*), \end{aligned} \tag{62}$$

where

$$\begin{aligned} M(\epsilon^*, T\epsilon^*) = \max\Big\{& v(\epsilon^*, T\epsilon^*), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), \\ & v\left(\epsilon^*, T^{n(\epsilon^*)}T\epsilon^*\right)\Big\} = v(\epsilon^*, T\epsilon^*). \end{aligned} \tag{63}$$

Hence, we get $v(\epsilon^*, T\epsilon^*) \leq (1/2s)v(\epsilon^*, T\epsilon^*)$, i.e., $\epsilon^* = T\epsilon^*$. Assume there has a ϖ^* satisfying $\varpi^* = T\varpi^*$ and $\epsilon^* \neq \varpi^*$; then, $\varpi^* = T\varpi^* = \cdots = T^{n(\epsilon^*)}\varpi^*$ and

$$\begin{aligned} v(\epsilon^*, \varpi^*) &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{n(\epsilon^*)}\varpi^*\right) \\ &\leq \psi(M(\epsilon^*, \varpi^*))M(\epsilon^*, \varpi^*) < \frac{1}{2s}d(\epsilon^*, \varpi^*), \end{aligned} \tag{64}$$

which is impossible. So T possesses the unique fixed point ε^*.

At the end, we prove the last part. To do this, we fix an integer ℓ, $0 \leq \ell < n(\epsilon^*)$, and $\forall n > n(\epsilon^*)$; we put $n = in(\epsilon^*) + \ell$, $i \geq 1$. Then, $\forall \epsilon \in \mathbb{G}$; we have

$$\begin{aligned} v(\epsilon^*, T^n\epsilon) &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{in(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \psi\left(M\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right)\right)M\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), \end{aligned} \tag{65}$$

where

$$\begin{aligned} & M\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) \\ & \quad = \max\left\{v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), v(\epsilon^*, T^n\epsilon)\right\}. \end{aligned} \tag{66}$$

If $v(\epsilon^*, T^n\epsilon) \geq v(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon)$, then $M(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon) = v(\epsilon^*, T^n\epsilon)$. According to (65), we have

$$v(\epsilon^*, T^n\epsilon) \leq \frac{1}{2s}v(\epsilon^*, T^n\epsilon), \textit{i.e.,}\epsilon^* = T^n\epsilon. \tag{67}$$

It follows that $T^n\epsilon \longrightarrow \epsilon^*$ as $n \longrightarrow \infty$. If $v(\epsilon^*, T^n\epsilon) < v(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon)$, one can get that

$$v(\epsilon^*, T^n\epsilon) \leq \frac{1}{2s}v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right). \tag{68}$$

Similarly,

$$\begin{aligned} v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) &= v\left(T^{n(\epsilon^*)}\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) \\ &\leq \psi\left(M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right)\right)M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right), \end{aligned} \tag{69}$$

where

$$\begin{aligned} & M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right) \\ & \quad = \max\Big\{v\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right), v\left(\epsilon^*, T^{n(\epsilon^*)}\epsilon^*\right), \\ & \qquad v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right)\Big\}. \end{aligned} \tag{70}$$

If $v(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon) \geq v(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon)$, then

$$M\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right) = v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), \tag{71}$$

that is,

$$\begin{aligned} v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) \leq \frac{1}{2s}v\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right), \text{i.e.,}\epsilon^* \\ = T^{(i-1)n(\epsilon^*)+\ell}\epsilon. \end{aligned} \tag{72}$$

Since ϵ^* is a fixed point of T, one get $\epsilon^* = T^{n(\epsilon^*)}\epsilon^* = T^{n(\epsilon^*)}T^{(i-1)n(\epsilon^*)+\ell}\epsilon$. Consequently, $T^n\epsilon \longrightarrow \epsilon^*$ as $n \longrightarrow \infty$.

If $\upsilon(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon) < \upsilon(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon)$, then

$$\upsilon\left(\epsilon^*, T^{(i-1)n(\epsilon^*)+\ell}\epsilon\right) \le \frac{1}{2s}\upsilon\left(\epsilon^*, T^{(i-2)n(\epsilon^*)+\ell}\epsilon\right). \quad (73)$$

We continue to calculate according to this method; if there exists $i_0 \le i$ satisfying $\epsilon^* = T^{(i-i_0)n(\epsilon^*)+\ell}\epsilon$, then $T^n\epsilon \longrightarrow \epsilon^*$ as $n \longrightarrow \infty$. Otherwise, one can conclude that

$$\upsilon(\epsilon^*, T^n\epsilon) \le \cdots \le \frac{1}{(2s)^i}\upsilon\left(\epsilon^*, T^\ell\epsilon\right) \longrightarrow 0(i \longrightarrow \infty). \quad (74)$$

Therefore, for each $\epsilon \in \mathbb{G}$, the iteration $\{T^n\epsilon\}$ is convergent to ϵ^*. □

Example 4. Let $\mathbb{G} = [0,+\infty)$ and $\upsilon(\epsilon, \varpi) = (\epsilon - \varpi)^2$. Obviously, $(\mathbb{G}, \upsilon)$ is a complete rectangular b-metric space with $s = 3$. Define $T : \mathbb{G} \longrightarrow \mathbb{G}$ with

$$T\epsilon = \frac{\epsilon}{2}, \quad \epsilon \in [0,+\infty). \quad (75)$$

Define mappings $\psi(\epsilon) = 1/3s$ and $n(\epsilon) = 3$, $\forall\epsilon \in [0,+\infty)$. One has

$$\upsilon\left(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi\right) = \upsilon\left(T^3\epsilon, T^3\varpi\right) = \frac{1}{64}(\epsilon - \varpi)^2, \quad (76)$$

$$\begin{aligned}&\psi(M(\epsilon, \varpi))M(\epsilon, \varpi)\\ &\quad= \frac{1}{9}\max\left\{\upsilon(\epsilon, \varpi), \upsilon\left(\epsilon, T^3\epsilon\right), \upsilon\left(\epsilon, T^3\varpi\right)\right\}\\ &\quad\ge \frac{1}{9}\upsilon(\epsilon, \varpi) = \frac{1}{9}(\epsilon - \varpi)^2.\end{aligned} \quad (77)$$

That is, $\upsilon(T^{n(\epsilon)}\epsilon, T^{n(\epsilon)}\varpi) \le \psi(M(\epsilon, \varpi))M(\epsilon, \varpi)$.

Thus, all hypotheses of Theorem 15 are fulfilled. So T possesses the unique common fixed point 0. Furthermore, for each $\epsilon \in \mathbb{G}$, the iteration $\{T^n\epsilon\}$ is convergent to 0.

4. Application

In this part, we will prove the solvability of this initial value problem:

$$\begin{cases} m\dfrac{d^2\epsilon}{d\varepsilon^2} + c\dfrac{d\epsilon}{d\varepsilon} - mF(\varepsilon, \epsilon(\varepsilon)) = 0,\\ \epsilon(0) = 0,\\ \epsilon'(0) = 0,\end{cases} \quad (78)$$

where m and $c > 0$ are constants and $F : [0, H] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous mapping.

Obviously, problem (78) is related to the integral equation:

$$\epsilon(\varepsilon) = \int_0^H Y(\varepsilon, \nu)F(\nu, \epsilon(\nu))d\nu, \varepsilon \in [0, H], \quad (79)$$

where $Y(\varepsilon, r)$ is defined as

$$Y(\varepsilon, \rho) = \begin{cases} \dfrac{1 - e^{\omega(\varepsilon-\nu)}}{\omega}, & 0 \le \varrho \le \varepsilon \le H,\\ 0, & 0 \le \varepsilon \le \varrho \le H,\end{cases} \quad (80)$$

where $\omega = c/m$ is a constant.

Next, by using Theorem 13 and Theorem 15, we shall present the solvability of the integral equation:

$$\epsilon(\varepsilon) = \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho))d\varrho. \quad (81)$$

Let $\mathbb{G} = C([0, H])$. For $p \ge 2, \varepsilon, \varpi \in \mathbb{G}$, define

$$\upsilon(\epsilon, \varpi) = \sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - \varpi(\varepsilon)|^p. \quad (82)$$

Hence, $(\mathbb{G}, \upsilon)$ is a complete rectangular b-metric space with $s = 3^{p-1}$.

In the following, define $T : \mathbb{G} \longrightarrow \mathbb{G}$ by

$$T\epsilon(\varepsilon) = \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho))d\varrho. \quad (83)$$

Suppose $\Xi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function that satisfies the following condition:

$$\begin{aligned}\Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \ge 0 \text{ and } \Xi(\varpi(\varepsilon), T\varpi(\varepsilon))\\ \ge 0 \text{ implies } \Xi(\epsilon(\varepsilon), T\varpi(\varepsilon))\\ \ge 0, \forall\epsilon, \varpi \in \mathbb{G}.\end{aligned} \quad (84)$$

Theorem 16. *Assume that*

(i) $\Gamma : [0, H] \times [0, H] \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is continuous

(ii) there has an $\epsilon_0 \in \mathbb{G}$ satisfying $\Xi(\epsilon_0(\varepsilon), T\epsilon_0(\varepsilon)) \ge 0$ for all $\varepsilon \in [0, H]$

(iii) $\forall\varepsilon \in [0, H]$ and $\epsilon, y \in \mathbb{G}$, $\Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \ge 0$ imply $\Xi(T\epsilon(\varepsilon), T\varpi(\varepsilon)) \ge 0$

(iv) if $\{\epsilon_n\} \subset \mathbb{G}$ satisfies $\Xi(\epsilon_n(\varepsilon), \epsilon_{n+1}(\varepsilon)) \ge 0$, $\forall n \in \mathbb{N}$, and $\lim_{n\longrightarrow\infty}\epsilon_n = \epsilon$, then we can choose a subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ such that $\Xi(\epsilon_{n_k}(\varepsilon), \epsilon(\varepsilon)) \ge 0$, $\forall k \in \mathbb{N}$

(v) for each $\epsilon \in \mathbb{G}$ with $T^{n(\epsilon)}\epsilon = \epsilon$, we have $\Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \ge 0$ for any $\varpi \in \mathbb{G}$

(vi) there is a continuous mapping $Y : [0, H] \times [0, H] \longrightarrow \mathbb{R}^+$ satisfying

$$\sup_{\varepsilon\in[0,H]}\int_0^H Y(\varepsilon, \varrho)d\varrho \le \sqrt[p]{\frac{1}{3^{p^2+1}}}, \quad (85)$$

$$|\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) - \Gamma(\varepsilon, \rho, \varpi(\varrho))| \leq Y(\varepsilon, \varrho)|\epsilon(\varrho) - \varpi(\varrho)|. \tag{86}$$

Then, (81) possesses a unique solution $\epsilon \in \mathbb{G}$.

Proof. Set $\alpha : \mathbb{G} \times \mathbb{G} \longrightarrow [0,+\infty)$ by

$$\alpha(\epsilon, \varpi) = \begin{cases} s^p, & \text{if } \Xi(\epsilon(\varepsilon), \varpi(\varepsilon)) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{87}$$

One can check that T is triangular α_{s^p} orbital admissible. In view of (i)-(vi), for $\epsilon, \varpi \in \mathbb{G}$, we obtain

$$\begin{aligned} s^p \upsilon(T\epsilon(\varepsilon), T\varpi(\varepsilon)) &= s^p \sup_{\varepsilon\in[0,H]} |T\epsilon(\varepsilon) - T\varpi(\varepsilon)|^p \\ &= s^p \sup_{\varepsilon\in[0,H]} \left| \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho - \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right|^p \\ &\leq s^p \sup_{\varepsilon\in[0,H]} \left(\int_0^H |\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) - \Gamma(\varepsilon, \varrho, \varpi(\varrho))| d\varrho \right)^p \\ &\leq s^p \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho)|\epsilon(\varrho) - \varpi(\varrho)| d\varrho \right)^p \\ &\leq s^p \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) d\varrho \right)^p \sup_{\varepsilon\in[0,H]} |\epsilon(t) - \varpi(\varepsilon)|^p \\ &\leq s^p \cdot \frac{1}{3^{p^2+1}} \sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - \varpi(\varepsilon)|^p \\ &\leq \frac{\upsilon(\epsilon(\varepsilon), \varpi(\varepsilon))}{3^{p+1}}, \end{aligned} \tag{88}$$

which implies that

$$\begin{aligned} &\alpha(\epsilon(\varepsilon), \varpi(\varepsilon)) \upsilon\left(T^{n(\epsilon)}\epsilon(\varepsilon), T^{n(\epsilon)}\varpi(\varepsilon)\right) \\ &\quad \leq \Phi\left(\upsilon(\epsilon(\varepsilon), \varpi(\varepsilon)), \upsilon\left(\epsilon(\varepsilon), T^{n(\epsilon)}\epsilon(\varepsilon)\right), \upsilon\left(\epsilon(\varepsilon), T^{n(\epsilon)}\varpi(\varepsilon)\right)\right), \end{aligned} \tag{89}$$

where $\Phi(\epsilon_1, \epsilon_2, \epsilon_3) = (\epsilon_1 + \epsilon_2 + \epsilon_3)/3^{p+1}$, $s = 3^{p-1}$, and $n(\epsilon) = 1$. After that, all hypotheses of Theorem 13 are fulfilled. Hence, T has a unique fixed point $\epsilon \in \mathbb{G}$. That is, ϵ is the unique solution of integral equation (81). □

Remark 17. If $\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) = Y(\varepsilon, \varrho)F(\varrho, \epsilon(\varrho))$, $|F(\varrho, \epsilon(\varrho)) - F(\varrho, \varpi(\varrho))| \leq |\epsilon(\varrho) - \varpi(\varrho)|$; then, (78) has a unique solution by Theorem 16.

Theorem 18. *Suppose that*

(i) $\Gamma : [0, H] \times [0, H] \times \mathbb{R} \longrightarrow \mathbb{R}^+$ *is continuous*

(ii) *there is a continuous mapping* $Y : [0, H] \times [0, H] \longrightarrow \mathbb{R}^+$ *satisfying*

$$\begin{aligned} &|\Gamma(\varepsilon, \varrho, \epsilon(\varrho)) - \Gamma(\varepsilon, \varrho, \varpi(\varrho))| \\ &\quad \leq Y(\varepsilon, \varrho) \left| \epsilon(\varepsilon) + \varpi(\varepsilon) - \left(\int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho + \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right) \right|, \end{aligned} \tag{90}$$

$$\sup_{\varepsilon\in[0,H]} \int_0^H Y(\varepsilon, \varrho) d\varrho \leq \frac{1}{3^2}. \tag{91}$$

Then, (81) possesses a unique solution $\epsilon \in \mathbb{G}$.

Proof. For $\epsilon, \varpi \in \mathbb{G}$, according to the conditions (i)-(ii), one can get

$$\begin{aligned} &\upsilon(T\epsilon(\varepsilon), T\varpi(\varepsilon)) \\ &\quad = \sup_{\varepsilon\in[0,H]} |T\epsilon(\varepsilon) - T\varpi(\varepsilon)|^p \\ &\quad = \sup_{\varepsilon\in[0,H]} \left| \int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho - \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right|^p \\ &\quad \leq \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) \middle| \epsilon(\varepsilon) + \varpi(\varepsilon) \right. \\ &\qquad \left. - \left(\int_0^H \Gamma(\varepsilon, \varrho, \epsilon(\varrho)) d\varrho + \int_0^H \Gamma(\varepsilon, \varrho, \varpi(\varrho)) d\varrho \right) \middle| d\varrho \right)^p \\ &\quad \leq \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho)(|\epsilon(\varepsilon) - T\varpi(\varepsilon)| + |\varpi(\varepsilon) - T\epsilon(\varepsilon)|) d\varrho \right)^p \\ &\quad \leq \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho)(|\epsilon(\varepsilon) - T\varpi(\varepsilon)| + |\varpi(\varepsilon) - \epsilon(\varepsilon)| + |\epsilon(\varepsilon) - T\epsilon(\varepsilon)|) d\varrho \right)^p \\ &\quad \leq \sup_{\varepsilon\in[0,H]} \left(\int_0^H Y(\varepsilon, \varrho) d\varrho \right)^p \cdot \sup_{\varepsilon\in[0,H]} (|\epsilon(\varepsilon) - T\varpi(\varepsilon)| + |\varpi(\varepsilon) - \epsilon(\varepsilon)| + |\epsilon(\varepsilon) - T\epsilon(\varepsilon)|)^p \\ &\quad \leq \frac{1}{3^{2p}} \cdot 3^p \cdot \frac{\sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - T\varpi(\varepsilon)|^p + \sup_{\varepsilon\in[0,H]} |\varpi(\varepsilon) - \epsilon(\varepsilon)|^p + \sup_{\varepsilon\in[0,H]} |\epsilon(\varepsilon) - T\epsilon(\varepsilon)|^p}{3} \\ &\quad \leq \frac{1}{3s} M(\epsilon, \varpi), \end{aligned} \tag{92}$$

where $M(\varepsilon, \varpi)$ is the same as in Theorem 15. Thus, all the hypotheses of Theorem 15 are fulfilled with $\psi(\varepsilon) = 1/3s$ and $n(\varepsilon) = 1$. It follows that T possesses a unique fixed point $\epsilon \in \mathbb{G}$, and so is a solution of (81). □

5. Conclusions

In rectangular b-metric spaces, we introduced a new triangular α-orbital admissible condition and established two fixed point results for mappings with a contractive iterate at a point. Further, we provided two examples that elaborated the usability of presented results. At the same time, we proved the existence and uniqueness of solution of an integral equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

No potential conflicts of interest are declared by the authors.

Authors' Contributions

All authors contributed equally in writing this article. All authors approved the final manuscript.

Acknowledgments

This work was financially supported by the Science and Research Project Foundation of Liaoning Province Education Department (No. LJC202003).

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 3, pp. 133–181, 1922.

[2] P. Agarwal, M. Jleli, and B. Samet, *Fixed Point Theory in Metric Spaces*, Springer, Berlin, 2018.

[3] A. Branciari, "A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematiques*, vol. 57, no. 1-2, pp. 31–37, 2000.

[4] H. Lakzian and B. Samet, "Fixed points for(ψ, φ)-weakly contractive mappings in generalized metric spaces," *Applied Mathematics Letters*, vol. 25, no. 5, pp. 902–906, 2012.

[5] C. D. Bari and P. Vetro, "Common fixed points in generalized metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7322–7325, 2012.

[6] R. George and R. Rajagopalan, "Common fixed point results for $\psi - \varphi$ contractions in rectangular metric spaces," *Bulletin of Mathematical Analysis and Applications*, vol. 5, no. 1, pp. 44–52, 2013.

[7] L. B. Budhia, H. Aydi, A. H. Ansari, and D. Gopal, "Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations," *Nonlinear Analysis: Modelling and Control*, vol. 25, no. 4, pp. 580–597, 2020.

[8] S. Czerwik, "Contraction mappings in b – metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[9] Z. D. Mitrovic, "Fixed point results in b – metric space," *Fixed Point Theory*, vol. 20, no. 2, pp. 559–566, 2019.

[10] N. Hussain, Z. D. Mitrović, and S. Radenović, "A common fixed point theorem of Fisher in b-metric spaces," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 2, pp. 949–956, 2019.

[11] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Fixed point theorems for weakly T – Chatterjea and weakly T – Kannan contractions in b – metric spaces," *Journal of Inequalities and Applications*, vol. 2014, no. 1, Article ID 46, 2014.

[12] Z. D. Mitrovic, A. Bodaghi, A. Aloqaily, N. Mlaiki, and R. George, "New versions of some results on fixed points in b -metric spaces," *Mathematics*, vol. 11, no. 5, p. 1118, 2023.

[13] N. Savanović, I. D. Aranđelović, and Z. D. Mitrović, "The results on coincidence and common fixed points for a new type multivalued mappings in b-metric spaces," *Mathematics*, vol. 10, no. 6, p. 856, 2022.

[14] H. Aydi, A. Felhi, E. Karapinar, H. AlRubaish, and M. AlShammari, "Fixed points for $\alpha - \beta_E$ Geraghty contractions on b – metric spaces and applications to matrix equations," *Filomat*, vol. 33, no. 12, pp. 3737–3750, 2019.

[15] M. U. Ali, H. Aydi, and M. Alansari, "New generalizations of set valued interpolative Hardy-Rogers type contractions in b – metric spaces," *Journal of Function Spaces*, vol. 2021, Article ID 6641342, 8 pages, 2021.

[16] R. George, S. Radenovic, and K. P. Reshma, "Rectangular b-metric space and contraction principles," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1005–1013, 2015.

[17] S. Gulyaz-Ozyurt, "On some alpha-admissible contraction mappings on Branciari b-metric spaces," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 1, no. 1, pp. 1–13, 2017.

[18] D. Zheng, G. Ye, and D. Liu, "Sehgal–Guseman-type fixed point theorem in b-rectangular metric spaces," *Mathematics*, vol. 9, no. 24, p. 3149, 2021.

[19] H. Guan, J. Li, and Y. Hao, "Common fixed point theorems for weakly contractions in rectangular b – metric spaces with supportive applications," *Journal of Function Spaces*, vol. 2022, Article ID 8476040, 16 pages, 2022.

[20] A. Hussain, "Solution of fractional differential equations utilizing symmetric contraction," *Journal of Mathematics*, vol. 2021, Article ID 5510971, 17 pages, 2021.

[21] A. Arif, M. Nazam, A. Hussain, and M. Abbas, "The ordered implicit relations and related fixed point problems in the cone b – metric spaces," *AIMS Mathematics*, vol. 7, no. 4, pp. 5199–5219, 2022.

[22] S. Anwar, M. Nazam, H. H. Al Sulami, A. Hussain, K. Javed, and M. Arshad, "Existence fixed-point theorems in the partial b – metric spaces and an application to the boundary value problem," *AIMS Mathematics*, vol. 7, no. 5, pp. 8188–8205, 2022.

[23] V. M. Sehgal, "A fixed point theorem for mappings with a contractive iterate," *Proceedings of the American Mathematical Society*, vol. 23, no. 3, pp. 631–634, 1969.

[24] J. Matkowski, "Fixed point theorems for mappings with a contractive iterate at a point," *Proceedings of the American Mathematical Society*, vol. 62, no. 2, pp. 344–348, 1977.

[25] L. Khazanchi, "Results on fixed points in complete metric space," *Mathematica Japonica*, vol. 19, no. 3, pp. 283–289, 1974.

[26] K. Iseki, "A generalization of Sehgal-Khazanchi's fixed point theorems," *Mathematics Seminar Notes*, vol. 2, pp. 89–95, 1974.

[27] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for $\alpha - \psi -$ contractive type mappings," *Nonlinear Analysis: Theory Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[28] O. Popescu, "Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, Article ID 190, 12 pages, 2014.

[29] C. Lang and H. Guan, "Common fixed point and coincidence point results for generalized $\alpha - \varphi E$-Geraghty contraction mappings in *b*-metric spaces," *AIMS Mathematics*, vol. 7, no. 8, pp. 14513–14531, 2022.

[30] M. Ming, J. C. Saut, and P. Zhang, "Long-time existence of solutions to Boussinesq systems," *SIAM Journal on Mathematical Analysis*, vol. 44, no. 6, pp. 4078–4100, 2012.

Hindawi
Journal of Function Spaces
Volume 2023, Article ID 7714095, 7 pages
https://doi.org/10.1155/2023/7714095

Research Article

A Special Mutation Operator in the Genetic Algorithm for Fixed Point Problems

Mohammad Jalali Varnamkhasti[1] and Masoumeh Vali[2]

[1] *Department of Science, Isfahan Branch (Khorasgan), Islamic Azad University, Isfahan, Iran*
[2] *Department of Industrial Management, Persian Gulf University, Bushehr, Iran*

Correspondence should be addressed to Mohammad Jalali Varnamkhasti; jalali.mo.var@gmail.com

Received 11 July 2022; Accepted 29 September 2022; Published 17 April 2023

Academic Editor: Santosh Kumar

Over the past century, the fixed point theory has emerged as a very useful and efficient tool in the study of nonlinear problems. This study introduced a progressed genetic algorithm (GA) based on a particular mutation operator applying on a subdivided search space where integer label and relative coordinates are used. This algorithm eventually categorizes each fixed point as its solution in appropriate set. Extensive computational experiments are conducted to assess the performance of the proposed technique with a standard GA for solving some nonlinear numerical functions from the literature.

1. Introduction

The fixed point theory was introduced scientifically in the 20th century. The basis of this theory is the principle of the Picard-Banach-Caccioppoli, which led to important lines of research and applications of this theory [1]. Fixed point theory is used and is important in various theoretical and practical fields. Theoretical fields such as variable and linear inequalities, theory of approximation, nonlinear analysis, equations, integrals, and differential components, theory of dynamic systems, fractals theory, financial mathematic, and game theory and applied fields such as biology, chemistry, management and economics, engineering in various disciplines, computer science, physics, geometry, astronomy, fluid mechanics, and image processing.

Riehl et al. [2] considered fixed points of discrete systems in large networks and optimized them. In this study, the equilibrium fixed points of discrete systems in distributed networks were considered; and by using appropriate partitions, they recursively decompose the main problem into a set of smaller and simpler problems and combine their solutions to gain a set of fixed points. The results showed the proposed algorithm with examples in two areas of calculating the number of fixed points in brain networks and finding the minimum energy combinations of network-based protein folding models.

Lael et al. [3] introduced a method for the Caristi-Kirk fixed point result for single mappings in conical metric spaces with a simple yet complete argument. The results of this research showed that the Caristi-Kirk fixed point in conical metric spaces turns into a result similar to traditional methods in reduced metric spaces. Bakery and Mohamed [4] proposed a new definition of the variable exponent of the Cesàro complex function space using the official power series. In this space, by utilizing s-numbers produced prequasi-ideal and then presented the topological and geometric structures of this class of ideal.

Metric space developed with the introduction of the Banach contraction principle and found more applications. One of the concepts presented in this space was the concept of F-metric [5]. Asif et al. [6] considered f-metric and create common fixed point results of Reich-type contraction. The results of this definition and its development showed that a unique common fixed point can be obtained if the contraction conditions are limited to only one closed ball subset of the total F-metric space. In addition, some significant implications are exploited from the significant results that characterize the fixed point outcomes for a single mapping. Among

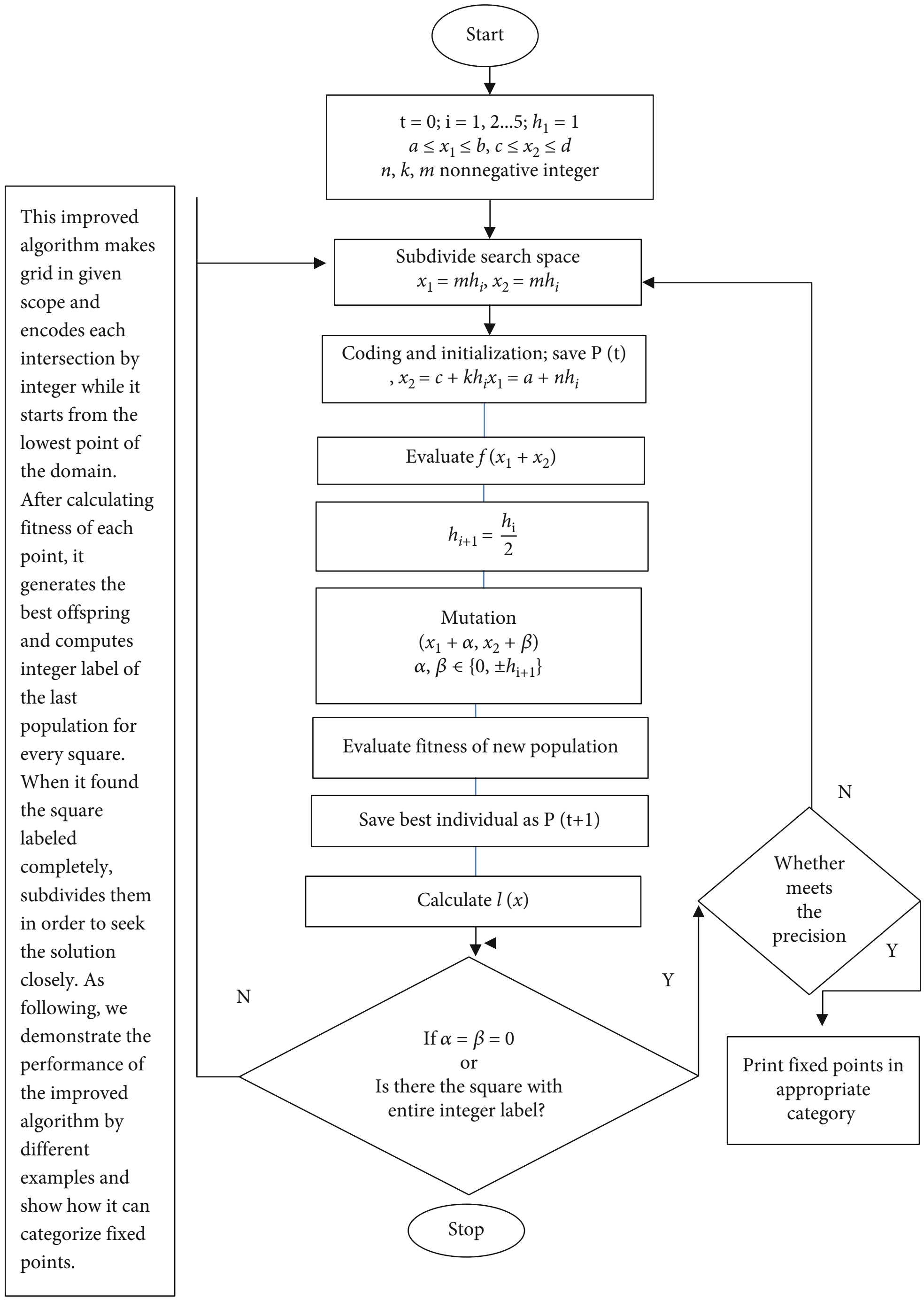


Figure 1: Improvement genetic algorithm flow chart.

nonlinear maps, nonexpansion maps are of particular importance. Expansion maps are maps that have a Lipschitz constant equal to one. Shukla and Panickar [7] assumed a nonexpansion map and they gained a number of fixed point theorems for these maps in geodetic spaces.

When we consider different optimization methods and compare them with the genetic algorithm, we find that the genetic algorithm (GA) by simulating the evolutionary process in organisms can provide an effective solution to find the optimal point in most cases [8, 9]. Mutation is used for avoiding of premature convergence and consequently escaping from local optimal. The GAs have been very successful in handling combinatorial optimization problems which are difficult [10].

Tang et al. [11], in order to prevent premature convergence in the GA, utilized the idea of flight behavior in the bird swarm algorithm to maintain population diversity and reduce the probability of falling to the local optimal. Mutation and the mutation probability (p_m) are important parameters in GAs. The mutation operator generates a new string by altering one or more bits of a string. By applying the mutation operator to a string, muting each bit of the string independently from the other bits is considered. So,

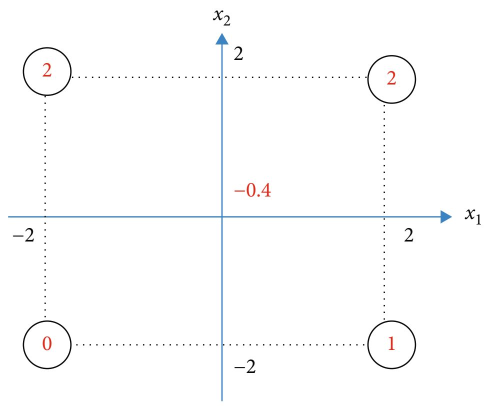


Figure 2: Initial population of f_1.

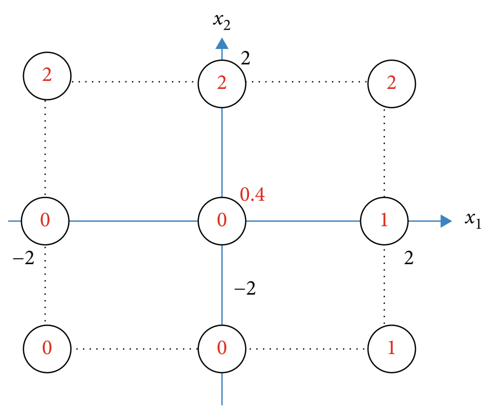


Figure 3: First generation of f_1.

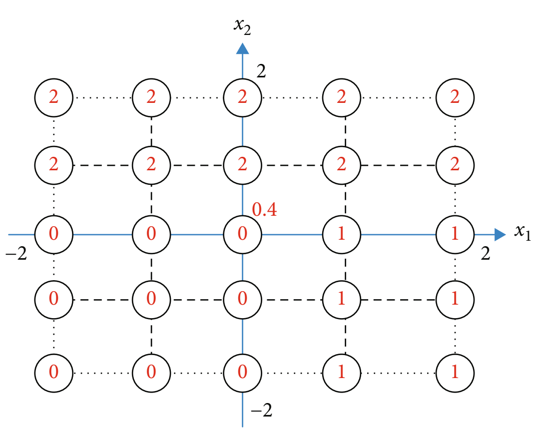


Figure 4: Second generation of f_1.

the mutation operator is more likely to significantly disrupt the allocation of trials to high order schemata than to low order ones. The efficiency of the mutation operator as a means of exploring the search space is questionable. A GA using mutation as the only genetic operator would be a random search that is biased toward sampling good hyper planes rather than poor ones [12].

The relationship between the genetic algorithm and the fixed points is a two-way relationship. In this sense, in some

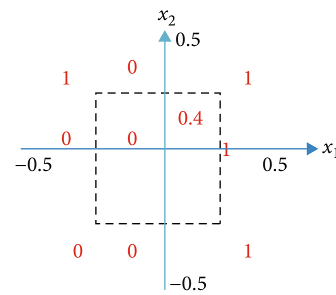


Figure 5: Third generation of f_1.

studies, fixed point properties have been used to improve the performance of genetic algorithms [13–18], and in some studies, updated models of genetic algorithms have been used to solve fixed point problems [19–22].

The concepts of fixed point and subdivision theory are used in some researches for improving GA. Gao et al. [13] introduced a GA based on fixed point algorithm and subdivision theory of continuous self-mapping in Euclidean space. They used subdivision of searching space and generate the integer labels and then these labels utilized for operators in GA. Pop [14] introduced a new developed GA based on the fixed point theorem and triangulation technique. Researcher utilized the crossover and mutation and increased the dimension genetic operators to avoid of premature convergence. Also, they utilized a custom increase dimension operator that expressively increases the total fitness.

Wolfram [23] used GA for controlling fixed point optimization. The researcher considers the floating point and fixed point display error in the optimization. Since both methods allow weighing between the theoretical and actual simulation, error occurred. Due to the script features of the simulation system, this can be easily automated. Zhang et al. [15] introduced triangulation theory into GA by the virtue of the concept of relative coordinate genetic coding and designed corresponding crossover and mutation operator. Hayes and Gedeon [17] considered the infinite population model for GA where the generation of the algorithm corresponds to a generation of a map. They showed that for a typical mixing operator, all the fixed points are hyperbolic.

Ren et al. [24] introduced the fixed point theory in PSO optimization and proposed an improved FP-PSO (fixed point PSO) algorithm. In the FP-PSO algorithm, the objective function is converted to a set of fixed point equations and the set of solutions obtained by the simple algorithm (SA) is used as the initial population of the PSO algorithm. Therefore, the remaining parameters can be obtained based on this choice of the classical PSO algorithm. Zhang et al. [16] introduced a GA that the population of individual is regarded as the triangulation of the point. They used the vertex label information of the individual simplex of individual to design selection operator, crossover, and mutation operators.

Zhang and Shang [25] proposed an improved multiobjective genetic algorithm based on Pareto front and fixed

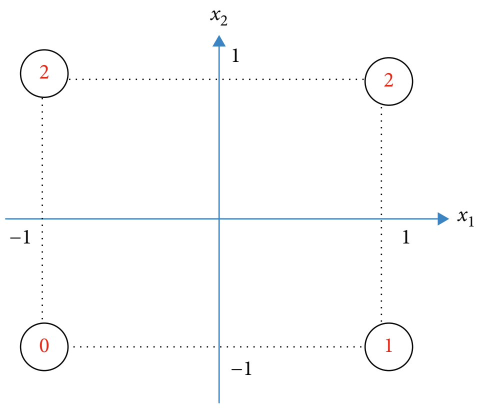


Figure 6: Initial population of f_2.

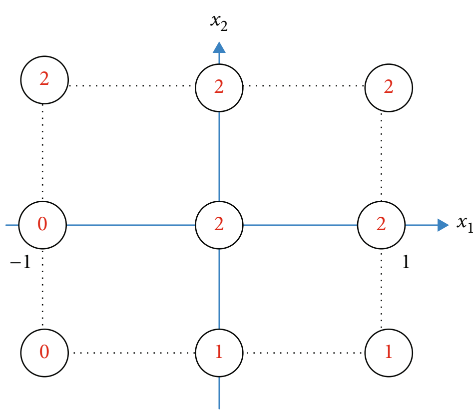


Figure 7: First generation of f_2.

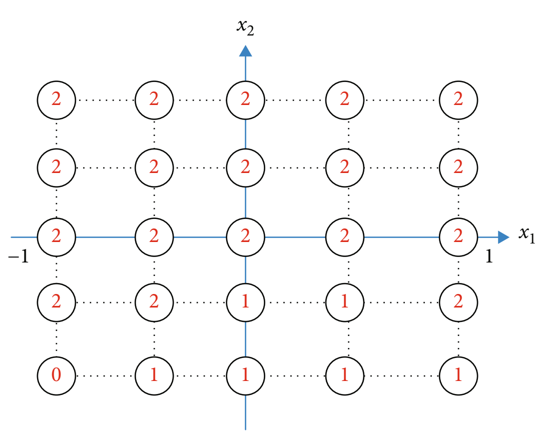


Figure 8: Second generation of f_2.

point theory. In this algorithm, the fixed point theory is introduced to a multiobjective optimization questions, and K1 triangulation is carried on to solutions for the weighting function constructed by all subfunctions, so the optimal problems are transferred to fixed point problems. Yang et al. [11] introduced the van der Laan-Talman algorithm to the GA to design convergence criteria objectively and to solve the convergence problem in the later period. The par-

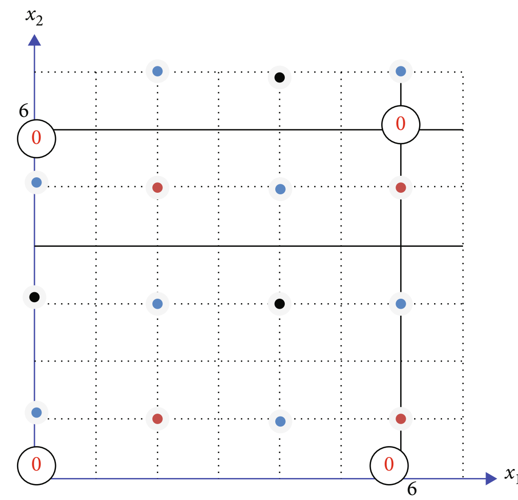


Figure 9: First generation of f_3.

allel GA of multibody model vehicle suspension optimization implemented through establishing the interface between ADAMS software and the GA. Wright et al. [26] developed a dynamical system model of a GA that uses gene pool crossover, proportional selection, and mutation. They introduced the concept of bistability for GA and they showed that it is possible for a GA to have two stable fixed points on a single-peak fitness landscape. These can correspond to a metastable finite populations.

Gedeon et al. [27] showed that for an arbitrary selection mechanism and a typical mixing operator, their composition has finitely many fixed points. Qian et al. [28] proposed a GA to treat with such constrained integer programming problem for the sake of efficiency. Then the fixed point evolved (E)-UTRA PRACH detector presented, which further underlines the feasibility and convenience of applying this methodology to practice. Wright et al. [29] considered the dynamic system model of Wright and Vose [18] and showed that with the increase of mutation percentage, the hyperbolic asymptotic fixed points are directed towards the simplex, and the hyperbolic unstable asymptotic fixed points are directed out of the simplex.

Thianwan [30] introduced a new iteration scheme of mixed type for two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive non-self-mappings. After introducing this method, some convergence theorems based on the proposed iterative scheme in uniformly convex Banach spaces have been presented, proved, and compared with previous results on some problems. A new mixed type iteration process for approximating a common fixed point from two asymptotic self-expansion mappings and two nonasymptotic self-expansion mappings was introduced by Thianwan [31]. In the continuation of this research, a convergence theorem was proposed in a uniform convex hyperbolic space and using the introduced method, the presented results showed that the presented model has better results than the previous models.

This paper investigates the concepts of fixed point and square labels with a special mutation operator for improving

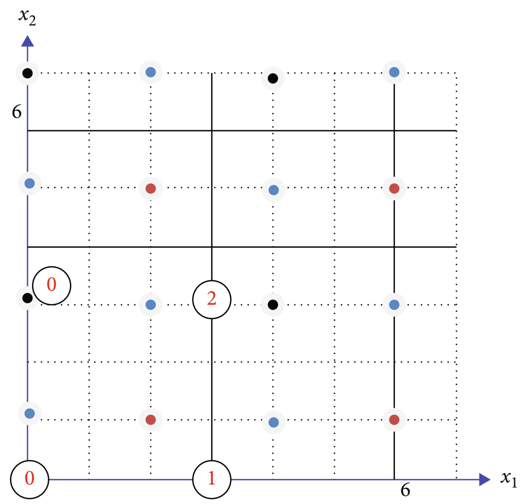


FIGURE 10: Second generation of f_3.

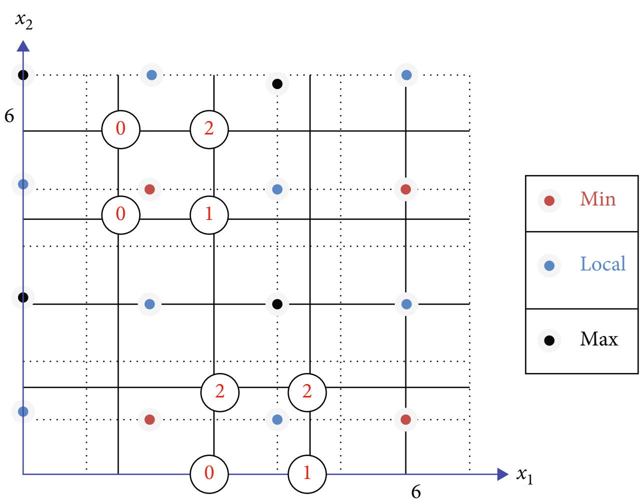


FIGURE 11: Third generation of f_3.

performance of the GA. The performance of proposed algorithm on some nonlinear numerical optimization problems shows this algorithm converge to a reasonable results in a few numbers of generations.

2. Construing of Optimal Problems as Fixed Point Problem

In genetic algorithm like other evolutionary algorithm, its optimal solutions are points that the algorithm improves, keeps, or returns to them after a certain number of iterations because these points meet the required criteria of the algorithm. When infinite population is used in GA, the algorithm must converge, and the average population fitness increase from one generation to the next. The consequence for a finite population simple genetic algorithm (SGA) is that the expected population fitness increases from one generation to the next. Moreover, the only stable fixed point of the expected next population operator corresponds to the population consisting entirely of the optimal string. This result is then extended by way of a perturbation argument to allow nonzero mutation. Supposing that algorithm is searching a point x, which can make continuous function of f to achieve its minimum. The necessary and sufficient condition of extreme point is that this point gradient is 0, that is, $\nabla f(x) = 0$.

For self-mapping $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we say, $x \in \mathbb{R}^n$ is a fixed point of g if $g(x) = x$, then we can convert the solution of zero point problems to fixed point ones of function $g(x) = x + \nabla f(x)$.

3. Subdivision and Relative Coordinates

Supposing that definition domain of $f(x_1, x_2)$ is that $a \le x_1 \le b, c \le x_2 \le d$ and dividing the domain into many squares with two groups of straight lines of $\{x_1 = mh_i\}, \{x_2 = mh_i\}$ in which m is a not negative integer and h_i is a positive quantity relating to precision of the problem; as a result, we can code each point of intersection as $x_1 = a + nh_i$, $x_2 = c + kh_i$ where n, k are not negative integers, so (n, k) is called the relative coordinates of points. Consequently, by changing n, k relative coordinates of each point in search space is determined.

Supposing that x is a vertex of a square that will be labeled as the following [23]:

$$l(x) = \begin{cases} 0, & g(x_1) - x_1 \ge 0, g(x_2) - x_2 \ge 0, \\ 1, & g(x_1) - x_1 < 0, g(x_2) - x_2 \ge 0, \\ 2, & g(x_2) - x_2 < 0. \end{cases} \quad (1)$$

The square with all different kinds of integer label is called a completely labeled unite, when $h_i \longrightarrow 0$ within iteration stages, vertices of that square approximately converge to one point which is a fixed point.

4. Mutation Operator

For each point coded (n, k), the GA is trying to improve it to reach optimal solution by mutation operator searching all points surrounding it in certain step determined by h_{i+1}. Thus, mutation probability $p_m = 1$.

For instance, (n, k) in $P(0)$, initial population, addressing $(x_1 + nh_i, x_2 + kh_i)$ will be changed as $(x_1 + \alpha, x_2 + \beta)$, $\alpha, \beta \in \{0, \pm h_{i+1}\}$. Subsequently, the algorithm saves the best-mutated individual among all possible offspring. Therefore, this operator produces new population located on intersection of the next grid. Because of this, coming squares are specified to evaluate and label. Furthermore, the next generation is producing from the previous one. For instance, in example 1, we show that the operator mutates (-2, 2) to (-2, 0), (2, 0), and (0, 0) in the given scope, then (0, 0) is selected as the best offspring.

5. The Improved Genetic Algorithm

This improved algorithm makes grid in given scope and encodes each intersection by integer while it starts from the lowest point of the domain. After calculating fitness of each point, it generates the best offspring and computes integer label of the last population for every square. When it found the square labeled completely, it subdivides them in order to seek the solution closely (the process of this method is shown in Figure 1). As following, we demonstrate the performance of the improved algorithm by different examples and show how it can categorize fixed points.

6. Computational Experiments

In this section, we present the computational results of the proposed algorithm for solving some nonlinear numerical functions.

6.1. Test Problem 1. This function is a continuous and unimodal function taken from [32]. The optimization problem is

$$\min f(x_1, x_2) = {x_1}^2 + (x_2 - 0.4)^2 - 2 < x_i < 2, i = 1, 2. \quad (2)$$

The function achieves the minimum when $x_1 = 0$ and $x_2 = 0.4$. In this example, $h_i \in \{4, 2, 1, 0.5, 0.25\}$, mutation probability $p_m = 1$. The completely label square obtains through the iteration, the search scope for both x_1 and x_2 are (-2, 2), (0, 2), (0, 1), and finally (0, 0.5), respectively (as show from Figures 2–5). During iterations, squares are contracting to (0, 0.5) gradually, if we started from $h_1 = 1$, we got closer answer, i.e., (0, 0.4).

6.2. Test Problem 2. The optimization problem considered here is also a nonlinear function problem taken from [32]. The problem is

$$\min f(x_1, x_2) = {x_1}^3 + {x_2}^3 - 1 < x_i < 1, i = 1, 2. \quad (3)$$

The best obtained solution is $x_1 = -1$ and $x_2 = -1$ with $f(x_1, x_2) = -2$. In this example, $h_i \in \{2, 1, 0.5\}$ mutation probability $p_m = 1$. The completely label square obtains through the iteration, the search scope for both x_1 and x_2 are (-1, 1), (-1, 0), and (-1, -0.5).

During iterations, squares are contracting to (-1, -1) gradually, which is a boundary point for this increasing function (as show from Figures 6–8).

6.3. Test Problem 3. In this problem, we choose a nonlinear optimization problem with two continuous variables. It was also taken from [32].

$$\min f(x_1, x_2) = \cos \frac{\pi}{2} x_1 - \sin \frac{\pi}{2} x_2 - 7 < x_i < 7, i = 1, 2. \quad (4)$$

This multimodal function has many local optimal in its domain. The GA keeps each local and global optimal one found in squares labeled completely. In this example, for $h_i \in \{6, 3, 1.5, 0.75\}$ while mutation probability $p_m = 1$, as shown in figure 7, these points can be gotten. Three following generation have been shown in the first quarter of the coordinates system (see Figures 9–11).

7. Conclusion

In this paper, we show that labeling technique and the mutation operator producing later generation on the next gridding points have some advantages. First of all, making network on search space provides integer-coding system that simplifies locating of all individuals in the future and present generation, so we can easily label each vertex of square and investigate the possibility of finding every optimal solution. Moreover, the algorithm is capable of starting from a fixed point located in domain boundary; hence, it overcomes weakness of man-made initial point. Second, finding square completely labeled avoids missing local answers because the algorithm focuses on such squares when it is trying to seek global minimum inside of not entirely labeled squares or in other completely ones. Third, this mutation operator works systematically in order to estimate better solution. In other words, it does not work so randomly that loses possible fixed points in an area as it is clear in Figure 3. In addition, the algorithm moves toward obtaining the best solution among likely offspring. Consequently, it performs more quickly and effectively because it eliminates unneeded iterations and calculations. Finally, it categorizes different fixed points at the end of its run.

Data Availability

Specific data has not been used for this research and only a few numerical functions whose references are given in the text have been used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Bucur, “About applications of the fixed point theory,” *Scientific Bulletin*, vol. 22, no. 1, pp. 13–17, 2017.

[2] J. R. Riehl, M. Zimmerman, M. F. Singh, G. Bowman, and S. Ching, “Computing and optimizing over all fixed-points of discrete systems on large networks,” *Journal of the Royal Society Interface*, vol. 17, no. 170, pp. 1–14, 2020.

[3] F. Lael, N. Saleem, and R. George, “Caristi’s fixed point theorem in cone metric space,” *Journal of Function Spaces*, vol. 2022, Article ID 7523333, 6 pages, 2022.

[4] A. A. Bakery and E. A. E. Mohamed, “Fixed point property of variable exponent Cesàro complex function space of formal power series under premodular,” *Journal of Function Spaces*, vol. 2022, Article ID 3811326, 22 pages, 2022.

[5] M. Jleli and B. Samet, “On a new generalization of metric spaces,” *Journal of Fixed Point Theory and Applications*, vol. 20, no. 3, p. 128, 2018.

[6] A. Asif, N. Hussain, H. Al-sulami, and M. Arshad, “Some fixed point results in function weighted metric spaces,” *Journal of Mathematics*, vol. 2021, Article ID 6636504, 9 pages, 2021.

[7] R. Shukla and R. Panickar, “Approximating fixed points of enriched nonexpansive mappings in geodesic spaces,” *Journal of Function Spaces*, vol. 2022, Article ID 6161839, 8 pages, 2022.

[8] J. H. Holland, *Adaptation in natural and artificial systems: an introductory analysis with applications to biology, control, and artificial intelligence*, University of Michigan Press, Ann Arbor, Mich, USA, 1975.

[9] M. Vali, "Sub-dividing genetic method for optimization problems," 2013, https://arxiv.org/abs/1307.5679.

[10] M. Jalali Varnamkhasti and L. S. Lee, "A fuzzy genetic algorithm based on binary encoding for solving 0/1 knapsack problems," *Journal of Applied Mathematics*, vol. 2012, Article ID 703601, 24 pages, 2012.

[11] Y. Tang, C. Li, S. Li, B. Cao, and C. Chen, "A fusion crossover mutation sparrow search algorithm," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9952606, 17 pages, 2021.

[12] M. Jalali Varnamkhasti and L. S. Lee, "A fuzzy genetic algorithm based on binary encoding for solving multidimensional knapsack problems," *Journal of Applied Mathematics*, vol. 2012, Article ID 703601, 23 pages, 2012.

[13] R. Gao, J. Zhang, Y. Shang, and Y. Dong, "An improve genetic algorithm based on fixed point algorithms," *Journal Of Computer*, vol. 7, no. 5, pp. 1109–1115, 2012.

[14] L. Pop, "A novel improved genetic algorithm based on the fixed point theorem and triangulation method," *Journal of Computer Science & Systems Biology*, vol. 9, pp. 105–111, 2016.

[15] J. Zhang, Y. Dong, R. Gao, and Y. Shang, "An improved genetic algorithm based on fixed point theory for function optimization," in *2009 International Conference on Computer Engineering and Technology*, Singapore, 2009.

[16] J. Zhang, H. Wang, and R. Gao, "Study of an improved genetic algorithm based on fixed point theory and hJ1 triangulation in Euclidean space," *Journal of Computers*, vol. 6, no. 10, pp. 2173–2179, 2011.

[17] C. Hayes and T. Gedeon, "Hyperbolicity of the fixed point set for the simple genetic algorithm," *Theoretical Computer Science*, vol. 411, no. 25, pp. 2368–2383, 2010.

[18] A. H. Wright and M. D. Vose, "Finiteness of the fixed point set for the simple genetic algorithm," *Evolutionary Computation*, vol. 3, no. 3, pp. 299–309, 1995.

[19] O. Abu Arqub, Z. Abo Hammour, S. Momani, and N. Shawagfeh, "Solving singular two-point boundary value problems using continuous genetic algorithm," *Abstract and Applied Analysis*, vol. 2012, Article ID 205391, 25 pages, 2012.

[20] M. Li, X. Kao, and H. Che, "A simultaneous iteration algorithm for solving extended split equality fixed point problem," *Mathematical Problems in Engineering*, vol. 2017, Article ID 9737062, 9 pages, 2017.

[21] M. Saeed, M. Bazazzadeh, and A. Mostofizadeh, "Finocyl grain design using the genetic algorithm in combination with adaptive basis function construction," *Hindawi International Journal of Aerospace Engineering*, vol. 2019, article 3060173, 12 pages, 2019.

[22] M. Eberlein, *The GA Heuristic Generically Has Hyperbolic Fixed Points [PhD Thesis]*, The University of Tennessee, 1996.

[23] H. Wolfram, "Controller fixed-point optimization with genetic algorithms," in *International Conference on Applied Electronics, (AE)*, Pilsen, Czech Republic, October 2015.

[24] M. Ren, X. Huang, X. Zhu, and L. Shao, "Optimized PSO algorithm based on the simplicial algorithm of fixed point theory," *Applied Intelligence*, vol. 50, no. 7, pp. 2009–2024, 2020.

[25] J. Zhang and Y. Shang, "An improved multi-objective genetic algorithm based on Pareto front and fixed point theory," in *2009 International Workshop on Intelligent Systems and Applications*, pp. 1–5, Wuhan, China, 2009.

[26] A. H. Wright, J. E. Rowe, C. R. Stephens, and R. Poli, "Bistability in a Gene Pool GA with Mutation," in *Foundations of Genetic Algorithms-7*, Morgan Kaufmann, San Mateo, 2003.

[27] T. Gedeon, C. Hayes, and R. Swanson, "Genericity of the fixed point set for the infinite population genetic algorithm," *Foundations of Genetic Algorithms Lecture Notes in Computer Science*, vol. 4436, pp. 97–109, 2007.

[28] R. Qian, T. Peng, Y. Qi, and W. Wang, "Genetic algorithm-aided fixed point design of E-UTRA PRACH detector on multi-core DSP," in *the Proceeding of 17th European Signal Processing Conference*, pp. 1007–1011, Glasgow, Scotland, 2009.

[29] A. H. Wright, T. Gedeon, and J. N. Richter, "On the movement of vertex fixed points in the simple genetic algorithm," in *the Proceedings of the 11th workshop proceedings on Foundations of genetic algorithms*, pp. 193–208, Schwarzenberg, Austria, 2011.

[30] T. Thianwan, "Convergence theorems for a new iteration scheme for mixed-type asymptotically nonexpansive mappings," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 4, p. 145, 2018.

[31] T. Thianwan, "Mixed type algorithms for asymptotically nonexpansive mappings in hyperbolic spaces," *European Journal of Pure and Applied Mathematics*, vol. 14, no. 3, pp. 650–665, 2021.

[32] J. G. Digalakis and K. G. Margaritis, "On benchmarking functions for genetic algorithms," *International Journal of Computer Mathematic*, vol. 77, no. 4, pp. 481–506, 2001.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3021457, 9 pages
https://doi.org/10.1155/2022/3021457

Research Article

Characterization and Stability of Multi-Euler-Lagrange Quadratic Functional Equations

Abasalt Bodaghi,[1] **Hossein Moshtagh**,[2] **and Amir Mousivand**[3]

[1]*Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran*
[2]*Department of Computer Science, University of Garmsar, Garmsar, Iran*
[3]*Department of Mathematics, West Tehran Branch, Islamic Azad University, Tehran, Iran*

Correspondence should be addressed to Abasalt Bodaghi; abasalt.bodaghi@gmail.com

Received 22 March 2022; Revised 22 August 2022; Accepted 22 September 2022; Published 10 October 2022

Academic Editor: Cristian Chifu

The aim of the current article is to characterize and to prove the stability of multi-Euler-Lagrange quadratic mappings. In other words, it reduces a system of equations defining the multi-Euler-Lagrange quadratic mappings to an equation, say, the multi-Euler-Lagrange quadratic functional equation. Moreover, some results corresponding to known stability (Hyers, Rassias, and Găvruta) outcomes regarding the multi-Euler-Lagrange quadratic functional equation are presented in quasi-β-normed and Banach spaces by using the fixed point methods. Lastly, an example for the nonstable multi-Euler-Lagrange quadratic functional equation is indicated.

1. Introduction

The celebrated Ulam challenge [1] arises from this question that how we can find an exact solution near to an approximate solution of an equation. This phenomenon of mathematics is called the *stability* of functional equations which has many applications in nonlinear analysis. The mentioned question has been partially solved by Hyers [2], Aoki [3], and Rassias [4] for the linear, additive, and linear (unbounded Cauchy difference) mappings, respectively. Next, many Hyers-Ulam stability problems for miscellaneous functional equations were studied by authors in the spirit of Rassias approach (see for instance [5–14] and other resources).

During the last two decades, stability problems for multivariable mappings were studied and extended by a number of authors. One of the mappings is the multiquadratic mapping, studied, for example, in [15–17]. Recall that a multivariable mapping $f : V^n \longrightarrow W$ is said to be *multiquadratic* [11] if it fulfills the famous quadratic equation

$$Q(u+v) + Q(u-v) = 2Q(u) + 2Q(v), \tag{1}$$

in each component. Note that equation (1) is a suitable tool for obtaining some characterizations in the setting of inner product spaces and in fact plays a prominent role. In other words, any square norm on an inner product space fulfills

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + \|v\|^2, \tag{2}$$

which is called the *parallelogram equality*. However, some functional equations have been applied to characterize inner product spaces and are available in [18, 19] and references therein. In addition, the quadratic functional equation was used to characterize inner product spaces in [20, 21].

A lot of information about equation (1) and some equations which are equivalent to it (in particular, about their solutions and stability) and more applications can be found for instance in [22–24]. Park was the first author who studied the stability of multiquadratic in the setting of Banach algebras [16]. After that, some authors introduced various versions of multiquadratic mappings and investigated the Hyers-Ulam stability of such mappings in Banach spaces and non-Archimedean spaces; these results are available for instance in [15, 25–29]. As for an unification of the multiquadratic mappings, Zhao et al. [17] were the first authors who described the structure of multiquadratic mappings, and in fact, they showed that $f : V^n \longrightarrow W$ is a multiquadratic mapping if and only if the equation

$$\sum_{t\in\{-1,1\}^n} f(v_1 + tv_2) = 2^n \sum_{i_1,\cdots,i_n\in\{1,2\}} f\left(v_{1i_1}, \cdots, v_{ni_n}\right) \quad (3)$$

holds, where $v_i = (x_{1i}, \cdots, v_{ni}) \in V^n$ and $i \in \{1, 2\}$.

Rassias [30] introduced the following notion of a generalized Euler-Lagrange-type quadratic mapping and investigated its generalized stability.

Definition 1. Suppose that V and W are linear spaces. A nonlinear mapping $\mathfrak{Q} : V \longrightarrow W$ satisfying the functional equation

$$\mathfrak{Q}(au + bv) + \mathfrak{Q}(bu - av) = \left(a^2 + b^2\right)[\mathfrak{Q}(u) + \mathfrak{Q}(v)] \quad (4)$$

is called 2-dimensional quadratic, where $u, v \in V$ and a, b are the fixed reals with $a^2 + b^2 > 1$.

It is easily seen that the Euler-Lagrange equality

$$(au + bv)^2 + (bu - av)^2 = \left(a^2 + b^2\right)\left(u^2 + v^2\right) \quad (5)$$

is valid for $\mathfrak{Q}$, defined in Definition 1 with any fixed reals a, b, and hence, (4) is also called Euler-Lagrange quadratic functional equation; we refer to [31] for Euler-Lagrange type cubic functional equation and its stability. Note that equation (4) is a general form of (1) in the case that $a = b = 1$, and so the function $\mathfrak{Q}(v) = v^2$ satisfies (4). Next, Xu [32] extended the definition above to several variable mappings and presented the next definition.

Definition 2. Let V and W be vector spaces. A mapping $f : V^n \longrightarrow W$ is said to be the n -Euler-Lagrange quadratic or multi-Euler-Lagrange quadratic if the mapping

$$v \mapsto f(v_1, \cdots, v_{i-1}, v, v_{i+1}, \cdots, v_n) \quad (6)$$

satisfies (4), for all $i \in \{1, \cdots, n\}$ and all $v_i \in V$.

In this article, we include a characterization of multi-Euler-Lagrange quadratic mappings and show that every multi-Euler-Lagrange quadratic mapping can be described as an equation (namely, the multi-Euler-Lagrange quadratic equation). Under the quadratic condition (2-power condition) in each variable, every multivariable mappings satisfying the mentioned earlier equation is multi-Euler-Lagrange quadratic (Theorem 5). Furthermore, we bring two Hyers-Ulam stability results for multi-Euler-Lagrange quadratic functional equations in quasi-β-normed and Banach spaces which their proof is based according to some known fixed point methods; see [33, 34] for more stability results in quasi-β-Banach spaces setting. Finally, we indicate an example to show that the multi-Euler-Lagrange quadratic functional equation is nonstable in the case of singularity.

2. Characterization of Multi-Euler-Lagrange Quadratic Mappings

Throughout, we consider the following known notations:

(i) $\mathbb{N}$=the set of all natural numbers

(ii) $\mathbb{Z}$= the set of all integer numbers

(iii) $\mathbb{Q}$= the set of all rational numbers

(iv) $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$

(v) $\mathbb{R}_+ \coloneqq [0,\infty)$

Let V be a linear space over $\mathbb{Q}$. Given $n \in \mathbb{N}$, $p \in \mathbb{N}_0$, $s = (s_1, \cdots, s_n) \in \mathbb{Q}^n$, and $v = (v_1, \cdots, v_n) \in V^n$. We write $sv \coloneqq (s_1 v_1, \cdots, s_n v_n)$ and $pv \coloneqq (pv_1, \cdots, pv_n)$ which belong to V^n. Here and subsequently, V is linear space over $\mathbb{Q}$ and $v_i^n = (v_{i1}, v_{i2}, \cdots, v_{in}) \in V^n$, in which $i \in \{1, 2\}$. Furthermore, for given the fixed elements $a_i^n = (a_{i1}, a_{i2}, \cdots, a_{in}) \in \mathbb{Z}^n$ such that $a_{ij} \neq 0, \pm1$, where $i = 1, 2$ and $j = 1, \cdots, n$ (here and the rest of the paper). We will write a_i^n and v_i^n simply a_i and v_i, respectively, when no confusion can arise.

For $v_1, v_2 \in V^n$ and $a_1, a_2 \in \mathbb{Z}^n$, set

$$\begin{aligned} A_j^{+1} &= \sum_{i=1}^{2} a_{ij} v_{ij}, \\ A_j^{-1} &= \sum_{i=1}^{2} (-1)^{i+1} a_{3-i,j} v_{ij}, (j \in \{1, \cdots, n\}). \end{aligned} \quad (7)$$

In continuation, we show that the equation

$$\begin{aligned} &\sum_{t_1,\cdots t_n\in\{-1,+1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) \\ &\quad = \prod_{j=1}^{n}\left(a_{1j}^2 + a_{2j}^2\right) \sum_{l_1,\cdots,l_n\in\{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right) \end{aligned} \quad (8)$$

is a general form of (4) for the multivariable case. In other words, we prove that every multi-Euler-Lagrange quadratic mapping fulfills (1) and vice versa. For doing this, we need some definitions and the upcoming lemma.

Definition 3. Let V and W be vector spaces over $\mathbb{Q}$ and $f : V^n \longrightarrow W$ be a multivariable mapping.

(i) We say f satisfies (has) the 2-power (quadratic) condition in the jth component if

$$\begin{aligned} &f\left(x_1, \cdots, x_{j-1}, a^* x_j, x_{j+1}, \cdots, x_n\right) \\ &\quad = (a^*)^2 f\left(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_n\right), \end{aligned} \quad (9)$$

for all $x_1, \cdots, x_n \in V$, where $a^* \in \{a_{1j}, a_{2j}\}$ for all $j \in \{1, \cdots, n\}$

(ii) If $f(x_1, \cdots, x_n) = 0$ when the fixed x_j is zero, then we say that f has zero functional equation in the

jth variable. Moreover, if $f(x_1, \cdots, x_n) = 0$ for any $(x_1, \cdots, x_n) \in V^n$ with at least one x_j is zero, we say f has zero functional equation

We consider two hypotheses as follows:
(H1) f has the quadratic condition in all variables.
(H2) f has zero functional equation.

Remark 4. It is clear that if a mapping $f : V^n \longrightarrow W$ satisfies the quadratic condition in the jth variable, then it has zero functional equation in the same variable. Therefore, if f fulfills (H1), then it satisfies (H2).

Theorem 5. *For a mapping $f : V^n \longrightarrow W$, the following assertions are equivalent:*

(i) f is multi-Euler-Lagrange quadratic

(ii) f fulfills (8) and H1

Proof. (i) ⇒ (ii) In view of [30], one can show that f satisfies H1. By induction on n, we now proceed the rest of this implication so that f satisfies equation (8). Obviously, f satisfies equation (4) for $n = 1$. The induction hypothesis is

$$\sum_{t_1,\cdots t_n \in \{-1,+1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) = \prod_{j=1}^{n} \left(a_{1j}^2 + a_{2j}^2\right) \sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right). \tag{10}$$

Then

$$\begin{aligned}
\sum_{t_1,\cdots,t_{n+1} \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_{n+1}^{t_{n+1}}\right)
&= \sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_{n+1}^{+1}\right) + \sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_{n+1}^{-1}\right) \\
&= \left(a_{1,n+1}^2 + a_{2,n+1}^2\right) \left(\sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}, v_{1,n+1}\right) + \sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}, v_{2,n+1}\right) \right) \\
&= \left(a_{1,n+1}^2 + a_{2,n+1}^2\right) \prod_{j=1}^{n} \left(a_{1j}^2 + a_{2j}^2\right) \cdot \left(\sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}, v_{1,n+1}\right) + \sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}, v_{2,n+1}\right) \right) \\
&= \prod_{j=1}^{n+1} \left(a_{1j}^2 + a_{2j}^2\right) \sum_{l_1,\cdots,l_{n+1} \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_{n+1},n+1}\right).
\end{aligned} \tag{11}$$

(ii) ⇒ (i) Let $j \in \{1,\cdots,n\}$ be arbitrary and fixed. Taking $v_{2k} = 0$ for all $k \in \{1, \cdots, n\} \setminus \{j\}$ in (8) and applying Remark 4, the left side will be as follows:

$$\begin{aligned}
& f\left(a_{11}v_{11}, \cdots, a_{1,j-1}v_{1,j-1}, A_j^{+1}, a_{1,j+1}v_{1,j+1}, \cdots, a_{1n}v_{1n}\right) \\
&\quad + f\left(a_{21}v_{11}, \cdots, a_{2,j-1}v_{1,j-1}, A_j^{+1}, a_{2,j+1}v_{1,j+1}, \cdots, a_{2n}v_{1n}\right) \\
&\quad + f\left(a_{11}v_{11}, \cdots, a_{1,j-1}v_{1,j-1}, A_j^{-1}, a_{1,j+1}v_{1,j+1}, \cdots, a_{1n}v_{1n}\right) \\
&\quad + f\left(a_{21}v_{11}, \cdots, a_{2,j-1}v_{1,j-1}, A_j^{-1}, a_{2,j+1}v_{1,j+1}, \cdots, a_{2n}v_{1n}\right) \\
&= a_{11}^2 a_{21}^2 a_{12}^2 a_{22}^2 \cdots a_{1,j-1}^2 a_{2,j-1}^2 a_{1,j+1}^2 a_{2,j+1}^2 \cdots a_{1n}^2 a_{2n}^2 \\
&\quad \cdot \left[f\left(v_{11}, \cdots, v_{1,j-1}, A_j^{+1}, v_{1,j+1}, \cdots, v_{1n}\right) + f\left(v_{11}, \cdots, v_{1,j-1}, A_j^{-1}, v_{1,j+1}, \cdots, v_{1n}\right) \right].
\end{aligned} \tag{12}$$

Once again, the same replacements convert the right side of (8) to

$$\prod_{\substack{k=1 \\ k \neq j}}^{n-1} \left(a_{1k}^2 + a_{2k}^2\right) \left(a_{1j}^2 + a_{2j}^2\right) \left[f\left(v_{11}, \cdots, v_{1,j-1}, v_{1j}, v_{1,j+1}, \cdots, v_{1n}\right) + f\left(v_{11}, \cdots, v_{1,j-1}, v_{2j}, v_{1,j+1}, \cdots, v_{1n}\right) \right]. \tag{13}$$

It follows from (12) and (13) that f is Euler-Lagrange (a_{1j}, a_{2j})-quadratic in the jth component, and this completes the proof. □

We should note that Theorem 5 necessitates that the mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined through $f(x_1, \cdots, x_n) = C\prod_{j=1}^{n} x_j^2$ fulfills equation (8). Hence, this equation can be called the multi-Euler-Lagrange quadratic functional equation.

3. Stability and Nonstability Results

The goals of this section are to prove miscellaneous result stability of multi-Euler-Lagrange quadratic equation (14) such as Hyers and Găvruta stability. Here, we mention a special case of equation (8) in which $a_1 = (a, \cdots, a)$ and $a_2 = (b, \cdots, b)$, and so (8) converts to

$$\sum_{t_1,\cdots t_n \in \{(a,b),(b,a)\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) = \left(a^2 + b^2\right)^n \sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right), \tag{14}$$

in which

$$A_j^{(a,b)} = av_{1j} + bv_{2j}, \text{and } A_j^{(b,a)} = bv_{1j} - av_{2j}, \tag{15}$$

and $m = a^2 + b^2$ (used here and from now on) for all $j \in \{1, \cdots, n\}$.

For a set E, a function $d : E \times E \longrightarrow [0,\infty]$ is said to be a generalized metric on E provided that d fulfills the statements below, for all $u, v, w \in E$.

(i) $d(u, v) = 0$ if and only if $u = v$

(ii) $d(u, v) = d(v, u)$

(iii) $d(u, w) \leq d(u, v) + d(v, w)$

The next theorem from [35] is one of fundamental results in fixed point theory and useful to achieve our first purpose in this section.

Theorem 6. *Suppose that (Ω, d) is a complete generalized metric space and $\mathcal{J} : \Omega \longrightarrow \Omega$ is a mapping such that its Lipschitz constant is $L < 1$. Then, for each element $x \in \Omega$, one of following cases can be happen:*

(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$for all $n \geq 0$ or

(ii) There is an $n_0 \in \mathbb{N}$ such that $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$, and the sequence $\{\mathcal{J}^n x\}$ is convergent to a fixed point x^* of $\mathcal{J}$ which belongs to the set $\Lambda = \{x \in \Omega : d(\mathcal{J}^{n_0} x, x) < \infty\}$. Moreover, $d(x, x^*) \leq (1/(1 - L)) d(x, \mathcal{J} x)$ for all $x \in \Lambda$

In the sequel, for any mapping $f : V^n \longrightarrow W$, we define the operator $\mathbf{D}f : V^n \times V^n \longrightarrow W$ via

$$\mathbf{D}f(v_1, v_2) \coloneqq \sum_{t_1, \cdots t_n \in \{(a,b),(b,a)\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) - m^n \sum_{l_1, \cdots, l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right), \tag{16}$$

for the fixed nonzero integers a, b where $A_j^{(a,b)}$ and $A_j^{(b,a)}$ are defined in (15) for all $j = 1, \cdots, n$.

In the incoming stability result for equation (14), $\|\mathbf{D}f(v_1, v_2)\|$ is controlled by a small positive number ε. We recall that for $i = 1, 2$, we consider $v_i = (v_{i1}, \cdots, v_{in}) \in V^n$.

Theorem 7. *Given $\varepsilon > 0$. Let V and W be a linear space and a complete normed space, respectively. Suppose that a mapping $f : V^n \longrightarrow W$ fulfilling H2 and*

$$\|\mathbf{D}f(v_1, v_2)\| \leq \varepsilon, \tag{17}$$

for all $v_1, v_2 \in V^n$. Then, there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (14) such that

$$\|f(v) - \mathcal{Q}(v)\| \leq \frac{m^n + 1}{m^{2n} - 1} \varepsilon, \tag{18}$$

for all $v \in V^n$. In addition,

$$\mathcal{Q}(v) = \lim_{l \longrightarrow \infty} \left(\frac{1}{m^{2n}}\right)^l f\left(m^l v\right), \tag{19}$$

for all $v \in V^n$.

Proof. Putting $v_2 = 0$ in (17) and using the assumption H2, we have

$$\left\|\tilde{f}(v_1) - m^n f(v_1)\right\| \leq \varepsilon, \tag{20}$$

for all $v_1 \in V^n$, where

$$\tilde{f}(v_1) = \sum_{a_{l_1 1}, \cdots, a_{l_n n} \in \{a,b\}} f\left(a_{l_1 1} v_{11}, \cdots, a_{l_n n} v_{1n}\right). \tag{21}$$

Set $v_1 = v$ for simply and for the rest of the proof, all the equations and inequalities are valid for all $v \in V^n$. Once more, by replacing (v_1, v_2) instead of $(av_1, bv_1) = (av, bv)$ in (17), we get

$$\left\|f(mv) - m^n \tilde{f}(v)\right\| \leq \varepsilon. \tag{22}$$

Multiplying both sides of (20) by m^n and plugging to (22), we obtain

$$\begin{aligned} \|f(mv) - m^{2n} f(v)\| &\leq \left\|f(mv) - m^n \tilde{f}(v)\right\| \\ &\quad + \left\|m^n \tilde{f}(v) - m^{2n} f(v)\right\| \\ &\leq (m^n + 1)\varepsilon, \end{aligned} \tag{23}$$

and thus

$$\|f(mv) - m^{2n} f(v)\| \leq (m^n + 1)\varepsilon. \tag{24}$$

Let $\Omega \coloneqq \{f : V^n \longrightarrow W | f \text{ satisfies } (H2)\}$. For each $f, g \in \Omega$, we define the function d on Ω as follows:

$$d(g, h) \coloneqq \inf\left\{C \in [0,\infty] : \|g(v) - h(v)\| \leq C_{g,h} \varepsilon, \text{for all } v \in V^n\right\}. \tag{25}$$

Similar to the proof of ([36], Theorem 2.2), it is seen that (Ω, d) is a complete generalized metric space. Define $\mathcal{J} : \Omega \longrightarrow \Omega$ through

$$\mathcal{J} f(v) \coloneqq \frac{1}{m^{2n}} f(mv), \tag{26}$$

for all $v \in V^n$. Take $g, h \in \Omega$ and $C_{g,h} \in [0,\infty]$ with $d(g, h) \leq C_{g,h}$. Then, $\|g(v) - h(v)\| \leq C_{g,h} \varepsilon$, and hence

$$\|\mathcal{J} g(v) - \mathcal{J} h(v)\| \leq \frac{1}{m^{2n}} \|g(mv) - h(mv)\| \leq \frac{1}{m^{2n}} C_{g,h} \varepsilon. \tag{27}$$

Therefore, $d(\mathcal{J}g, \mathcal{J}h) \le (1/m^{2n})C_{g,h}$. This shows that $d(\mathcal{J}g, \mathcal{J}h) \le (1/m^{2n})d(g,h)$ and in fact $\mathcal{J}$ is a strictly contractive operator such that its Lipschitz is $1/m^{2n}$. It concludes from (24) that

$$\|\mathcal{J}f(v) - f(v)\| \le \left\| \frac{1}{m^{2n}} f(mv) - f(v) \right\| \le \frac{m^n + 1}{m^{2n}} \varepsilon. \quad (28)$$

Hence,

$$d(\mathcal{J}f, f) \le \frac{m^n + 1}{m^{2n}} < \infty. \quad (29)$$

An application of Theorem 6 for the space (Ω, d), the operator $\mathcal{J}$, $n_0 = 0$, and $x = f$, shows that the sequence $(\mathcal{J}^l f)_{l\in\mathbb{N}}$ is convergent in (Ω, d) and its limit; $\mathcal{Q}$ is a fixed point of $\mathcal{J}$. Indeed, $\mathcal{Q}(v) = \lim_{l\longrightarrow\infty} \mathcal{J}^l f(v)$, and

$$\mathcal{Q}(v) = \frac{1}{m^{2n}} \mathcal{Q}(mv), (v \in V^n). \quad (30)$$

In other words, by induction on l, it is easily verified that for each $v \in V^n$, we have

$$\mathcal{J}^l f(v) := \left(\frac{1}{m^{2n}}\right)^l f\left(m^l v\right), \quad (31)$$

and (19) follows. Note that clearly $f \in \Lambda$, and hence, part (iii) of Theorem 6 and (29) necessitate that

$$d(f, \mathcal{Q}) \le \frac{1}{1 - (1/m^{2n})} d(\mathcal{J}f, f) \le \frac{m^n + 1}{m^{2n} - 1}, \quad (32)$$

which proves (18). In addition,

$$\begin{aligned} \|D\mathcal{Q}(v_1, v_2)\| &= \lim_{l\longrightarrow\infty} \left(\frac{1}{m^{2n}}\right)^l \left\| Df\left(m^l v_1, m^l v_2\right) \right\| \\ &\le \lim_{l\longrightarrow\infty} \left(\frac{1}{m^{2n}}\right)^l \varepsilon = 0, \end{aligned} \quad (33)$$

for all $v_1, v_2 \in V^n$. The last relation shows that $D\mathcal{Q}(v_1, v_2) = 0$ for all $v_1, v_2 \in V^n$ and means that $\mathcal{Q}$ fulfills (14). Let us finally suppose that $\mathfrak{Q} : V^n \longrightarrow W$ is another solution of equation (14) satisfies H2 such that inequality (18) holds. Then, $\mathfrak{Q}$ satisfies (30), and so it is a fixed point of $\mathcal{J}$. Furthermore, by (18), we get

$$d(f, \mathfrak{Q}) \le \frac{m^n + 1}{m^{2n} - 1} < \infty, \quad (34)$$

and consequently, $\mathfrak{Q} \in \Lambda$. It now follows from part (ii) of Theorem 6 that $\mathfrak{Q} = \mathcal{Q}$. This finishes the proof. □

Remark 8. In the proof of Theorem 7, if we put $v_1 = 0$, we can not reach to (20) unless it is assumed that f is even in each component. Recall from [33] that $f : V^n \longrightarrow W$ is even in the kth component if

$$f(x_1, \cdots, x_{k-1}, -x_k, x_{k+1}, \cdots, x_n) = f(x_1, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n). \quad (35)$$

In other words, this condition is redundant, and we do not need it.

Hereafter, we concentrate our mind on the quasi-β-normed spaces.

Definition 9. Let β be a fix real number with $0 < \beta < 1$ and $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Suppose that E is a vector space over $\mathbb{K}$. A quasi-β-norm is a real-valued function on E fulfilling the next conditions for all $x, y \in E$ and $t \in \mathbb{K}$.

(i) $\|x\| \ge 0$ and moreover $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|tx\| = |t|^\beta \|x\|$

(iii) There exists a real number $M \ge 1$ such that $\|x + y\| \le M(\|x\| + \|y\|)$

When $\beta = 1$, the norm above is a quasinorm. Recall that M is the modulus of concavity of the norm $\|\cdot\|$. Moreover, if $\|\cdot\|$ is a quasi-β-norm on E, the pair $(E, \|\cdot\|)$ is said to be a *quasi- β -normed space*. Similar to normed spaces, a complete quasi-β-normed space is called a *quasi-β-Banach space*. For $0 < p \le 1$, if $\|x + y\|^p \le \|x\|^p + \|y\|^p$, for all $x, y \in E$, then the quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm. In this case, every quasi-β-Banach space is said to be a (β, p)-Banach space. A result of the Aoki-Rolewicz theorem [37] shows that every quasinorm can be equivalent to a p-norm, for some p.

A main tool of this section is the upcoming fixed point lemma which has been proved in ([38], Lemma 3.1).

Lemma 10. *Given the fixed $j \in \{-1, 1\}$ and $a, t \in \mathbb{N}$ with $a \ge 2$. Suppose that V is a linear space and W is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_W$. If $\phi : V \longrightarrow [0,\infty)$ is a function such that there exists an $L < 1$ with $\phi(a^j v) < L a^{jt\beta} \phi(v)$ for all $v \in V$ and $g : V \longrightarrow W$ is a mapping satisfying*

$$\left\| g(av) - a^t g(v) \right\|_W \le \phi(v), \quad (36)$$

for all $v \in V$, then there exists a uniquely determined mapping $G : V \longrightarrow W$ such that $G(av) = a^t G(v)$ and

$$\|g(v) - G(v)\|_W \le \frac{1}{a^{t\beta} \left|1 - L^j\right|} \phi(v), (v \in V). \quad (37)$$

Furthermore, for each $v \in V$, we have $G(v) = \lim_{l\longrightarrow\infty} (g(a^{jl} v)/a^{jlt})$.

In the next theorem, we prove the Găvruta stability of (14) in quasi-β-normed spaces.

Theorem 11. *Given $j \in \{-1, 1\}$. Let V be a vector space over $\mathbb{Q}$ and W be a (β, p) -Banach space. Assume that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a function such that $\varphi(m^j v_1, m^j v_2) \le m^{2nj\beta} L \varphi(v_1, v_2)$ for all $v_1, v_2 \in V^n$, where $0 < L < 1$. If a mapping $f : V^n \longrightarrow W$ satisfying H2 and*

$$\|Df(v_1, v_2)\|_W \le \varphi(v_1, v_2), (v_1, v_2 \in V^n), \tag{38}$$

then there is a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (14) so that

$$\|f(v) - \mathcal{Q}(v)\|_W \le \frac{1}{|1 - L^j|} \frac{1}{m^{2n\beta}} \tilde{\varphi}(v), (v \in V^n), \tag{39}$$

where

$$\tilde{\varphi}(v) = M\left[m^{n\beta}\varphi(v, 0) + \varphi(av, bv)\right], \tag{40}$$

whereas M is the modulus of concavity of the norm $\|\cdot\|_W$.

Proof. Setting $v_2 = 0$ in (38) and applying H2, we have

$$\left\|\tilde{f}(v) - m^n f(v)\right\|_W \le \varphi(v, 0), \tag{41}$$

for all $v_1 \coloneqq v \in V^n$, where $\tilde{f}(v) = \tilde{f}(v_1)$ is defined in (21). Interchanging (v_1, v_2) into $(av_1, bv_1) = (av, bv)$ in (38), we obtain

$$\left\|f(mv) - m^n \tilde{f}(v)\right\|_W \le \varphi(av, bv), \tag{42}$$

for all $v \in V^n$. Multiplying both sides of (41) by $m^{n\beta}$, we get

$$\left\|m^n \tilde{f}(v) - m^{2n} f(v)\right\|_W \le m^{n\beta}\varphi(v, 0), \tag{43}$$

for all $v \in V^n$. It follows from (42), (43), and part (iii) of Definition 9 that

$$\left\|f(mv) - m^{2n} f(v)\right\|_W \le \tilde{\varphi}(v), \tag{44}$$

for all $v \in V^n$, where $\tilde{\varphi}(v)$ is defined in (40). By Lemma 10, there exists a mapping $\mathcal{Q} : V^n \longrightarrow W$ which is unique such that $\mathcal{Q}(mv) = m^{2n}\mathcal{Q}(v)$ and

$$\|f(v) - \mathcal{Q}(v)\|_W \le \frac{1}{|1 - L^j|} \frac{1}{m^{2n\beta}} \tilde{\varphi}(v), (v \in V^n). \tag{45}$$

Lastly, we show that $\mathcal{Q}$ fulfilling (14). Note that Lemma 10 implies that for each $v \in V^n$, $\mathcal{Q}(v) = \lim_{l \longrightarrow \infty}(f(m^{jl}v)/m^{2njl})$. For each $v_1, v_2 \in V^n$ and $l \in \mathbb{N}$, by (38), we find

$$\begin{aligned} \left\|\frac{Df\left(m^{jl}v_1, m^{jl}v_2\right)}{m^{2njl}}\right\|_W &\le m^{-2njl\beta}\varphi\left(m^{jl}v_1, m^{jl}v_2\right) \\ &\le m^{-2njl\beta}\left(m^{2nj\beta}L\right)^l \varphi(v_1, v_2) \\ &= L^l \varphi(v_1, v_2). \end{aligned} \tag{46}$$

Taking $l \longrightarrow \infty$ in the last relation, we observe that $D\mathcal{Q}(v_1, v_2) = 0$ for all $v_1, v_2 \in V^n$, and therefore, $\mathcal{Q}$ fulfills (14). □

The following corollary is a consequence of Theorem 11 when the norm of $\|Df(v_1, v_2)\|$ is controlled by sum of variable norms of v_1 and v_2 with positive powers.

Corollary 12. *Let V be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_V$, and W be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_W$. Let θ and λ be positive numbers with $\lambda \ne 2n(\beta/\alpha)$. If a mapping $f : V^n \longrightarrow W$ satisfying*

$$\|Df(v_1, v_2)\|_W \le \theta \sum_{k=1}^{2}\sum_{l=1}^{n} \|v_{kl}\|_V^\lambda, \tag{47}$$

for all $v_1, v_2 \in V^n$, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (14) such that

$$\|f(v) - \mathcal{Q}(v)\|_W \le \begin{cases} \dfrac{\theta\Lambda}{m^{2n\beta} - m^{\alpha\lambda}} \displaystyle\sum_{l=1}^{n} \|v_{1l}\|_V^\lambda, & \lambda \in \left(0, 2n\dfrac{\beta}{\alpha}\right), \\ \dfrac{m^{\alpha\lambda}\Lambda\theta}{m^{2n\beta}\left(m^{\alpha\lambda} - m^{2n\beta}\right)} \displaystyle\sum_{l=1}^{n} \|v_{1l}\|_V^\lambda, & \lambda \in \left(2n\dfrac{\beta}{\alpha}, \infty\right), \end{cases} \tag{48}$$

for all $v = v_1 \in V^n$, where $\Lambda = M[m^{n\beta} + |a|^{\alpha\lambda} + |b|^{\alpha\lambda}]$.

Proof. Taking $\varphi(v_1, v_2) = \theta\sum_{k=1}^{2}\sum_{l=1}^{n}\|v_{kl}\|_V^\lambda$, the result concludes from Theorem 11. □

We bring an elementary lemma without proof as follows.

Lemma 13. *If a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous and satisfies (1), then it has the form $g(x) = cx^2$, for all $x \in \mathbb{R}$, where $c = g(1)$.*

It is easily seen that when $a = b = 1$ in (14), then this equation and (3) are the same. In the upcoming result, we extend Lemma 13 for multivariable functions. In fact, we use it to make a counterexample.

Proposition 14. *Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous which satisfies (3). Then, f has the form*

$$f(r_1, \cdots, r_n) = cr_1^2 \cdots r_n^2, (r_1, \cdots, r_n \in \mathbb{R}), \tag{49}$$

where c is a constant in $\mathbb{R}$.

Proof. We first recall from Theorem 2 in [17] that f is a n-quadratic mapping. By induction on n, we proceed the proof. For $n = 1$, (49) holds by Lemma 13. Assume that (49) is valid for a $n \in \mathbb{N}$, and $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is a continuous $(n + 1)$-quadratic function. Fix the variables $r_1, \cdots, r_n$ in $\mathbb{R}$.

Then, the function $r \mapsto f(r_1, \cdots, r_n, r)$ is quadratic and continuous, and hence, by Lemma 13, f has the form

$$f(r_1, \cdots, r_n, r) = cr^2, \ (r \in \mathbb{R}), \tag{50}$$

where c is a constant in $\mathbb{R}$. One should note that c depends on $r_1, \cdots, r_n$, and hence

$$c = c(r_1, \cdots, r_n). \tag{51}$$

Letting $r = 1$ in (50) and applying (51), we have

$$c = c(r_1, \cdots, r_n) = f(r_1, \cdots, r_n, 1). \tag{52}$$

It is known that f is $(n+1)$-quadratic and c is an n-quadratic function. Therefore, by the induction assumption, there exists a real number c_0 so that

$$c = c(r_1, \cdots, r_n) = c_0 r_1^2 \cdots r_n^2. \tag{53}$$

It now follows from (50) and (53) that (49) holds for $n+1$.

Here, we present a nonstable example for the multi-quadratic mappings on $\mathbb{R}^n$ (see [8]). Indeed, for the case $\alpha = \beta = a = b = 1$, we show that the assumption $\lambda \neq 2n$ can not be eliminated in Corollary 12. □

Example 1. Given $n \in \mathbb{N}$ and $\delta > 0$. Set $\mu := ((2^{2n} - 1)/2^{4n}(2^n + 4^n))\delta$. The function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined via

$$\psi(r_1, \cdots, r_n) = \begin{cases} \mu \prod_{j=1}^{n} r_j^2, \text{for all } r_j \text{ with } |r_j| < 1, \\ \mu, \text{otherwise}. \end{cases} \tag{54}$$

Consider $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as a function defined by

$$f(r_1, \cdots, r_n) = \sum_{l=0}^{\infty} \frac{\psi(2^l r_1, \cdots, 2^l r_n)}{2^{2nl}}, \ (r_j \in \mathbb{R}). \tag{55}$$

Obviously, f is a nonnegative function and moreover is an even function in all components. Additionally, ψ is bounded by μ and continuous. Since f is a uniformly convergent series of continuous functions, it is continuous and bounded. In other words, we get $f(r_1, \cdots, r_n) \leq (2^{2n}/(2^{2n} - 1))\mu$ for all $(r_1, \cdots, r_n) \in \mathbb{R}^n$. For $i \in \{1, 2\}$, take $x_i = (x_{i1}, \cdots, x_{in})$. We shall prove that

$$|Df(x_1, x_2)| \leq \delta \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}, \tag{56}$$

for all $x_1, x_2 \in \mathbb{R}^n$. Clearly, (56) holds for $x_1 = x_2 = 0$. Let $x_1, x_2 \in \mathbb{R}^n$ with

$$\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} < \frac{1}{2^{2n}}. \tag{57}$$

Inequality (57) necessitates that there is $N \in \mathbb{N}$ such that

$$\frac{1}{2^{2n(N+1)}} < \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} < \frac{1}{2^{2nN}}, \tag{58}$$

and so $x_{ij}^{2n} < \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} < 1/2^{2nN}$. It follows the last relation that $2^N |x_{ij}| < 1$ for all $i = 1, 2$ and $j = 1, \cdots, n$. Hence, $2^{N-1} |x_{ij}| < 1$. Let $y_1, y_2 \in \{x_{ij} | i = 1, 2, j = 1, \cdots, n\}$. Then $2^{N-1} |y_1 \pm y_2| < 1$. It is known that ψ is multiquadratic function on $(-1, 1)^n$, and hence, $D\psi(2^l x_1, 2^l x_2) = 0$ for all $l \in \{0, 1, 2, \cdots, N-1\}$. Now, the last equality and relation (58) imply that

$$\begin{aligned} \frac{|Df(2^l x_1, 2^l x_2)|}{\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} &\leq \sum_{l=N}^{\infty} \frac{|D\psi(2^l x_1, 2^l x_2)|}{2^{2nl} \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} \\ &\leq \sum_{l=0}^{\infty} \frac{\mu(2^n + 4^n)}{2^{2n(l+N)} \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} \\ &\leq \mu(2^n + 4^n) \sum_{l=0}^{\infty} \frac{1}{2^{2nl}} \\ &\leq \mu(2^n + 4^n) 2^{2n} \frac{2^{2n}}{2^{2n} - 1} \\ &= \mu(2^n + 4^n) \frac{2^{4n}}{2^{2n} - 1} = \delta, \end{aligned} \tag{59}$$

for all $x_1, x_2 \in \mathbb{R}^n$. Hence, (56) is valid for case (57). If $\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} \geq 1/2^{2n}$, then

$$\frac{|Df(2^l x_1, 2^l x_2)|}{\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} \leq 2^{2n} \frac{2^{2n}}{2^{2n} - 1} \mu(2^n + 4^n) = \delta. \tag{60}$$

Therefore, f satisfies in (56) for all $x_1, x_2 \in \mathbb{R}^n$. Assume that there exists a number $b \in [0, \infty)$ and a multiquadratic function $\mathcal{Q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ for which the inequality $|f(r_1, \cdots, r_n) - \mathcal{Q}(r_1, \cdots, r_n)| < b \prod_{j=1}^{n} r_j^2$ is valid for all $(r_1, \cdots, r_n) \in \mathbb{R}^n$. An application of Proposition 14 shows that there is a constant $c \in \mathbb{R}$ such that $\mathcal{Q}(r_1, \cdots, r_n) = c \prod_{j=1}^{n} r_j^2$, and hence

$$f(r_1, \cdots, r_n) \leq (|c| + b) \prod_{j=1}^{n} r_j^2, \ ((r_1, \cdots, r_n) \in \mathbb{R}^n). \tag{61}$$

Furthermore, choose $N \in \mathbb{N}$ such that $N\mu > |c| + b$. Take $r = (r_1, \cdots, r_n) \in \mathbb{R}^n$ in which $r_j \in (0, 1/2^{N-1})$ for all $j \in \{1, \cdots, n\}$, then $2^l r_j \in (0, 1)$ for all $l = 0, 1, \cdots, N-1$. Therefore

$$\begin{aligned} f(r_1, \cdots, r_n) &= \sum_{l=0}^{\infty} \frac{\psi(2^l r_1, \cdots, 2^l r_2)}{2^{2nl}} \geq \sum_{l=0}^{N-1} \frac{\mu 2^{2nl} \prod_{j=1}^{n} r_j^2}{2^{2nl}} \\ &= N\mu \prod_{j=1}^{n} r_j^2 > (|c| + b) \prod_{j=1}^{n} r_j^2, \end{aligned} \tag{62}$$

which is a contradiction with (61).

We close the paper by an alternative stability result for equation (14) as follows.

Corollary 15. *Let V be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_V$ and W be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_W$. Suppose $\lambda_{il}>0$ for $i\in\{1,2\}$ and $l\in\{1,\cdots,n\}$ with $\lambda=\lambda^*+\lambda^\bullet\neq 2n(\beta/\alpha)$, where $\lambda^*=\sum_{l=1}^n\lambda_{1l}$ and $\lambda^\bullet=\sum_{l=1}^n\lambda_{2l}$. If a mapping $f: V^n\longrightarrow W$ fulfilling the inequality*

$$\|Df(v_1,v_2)\|_W\leq\theta\prod_{i=1}^{2}\prod_{l=1}^{n}\|v_{il}\|_V^{\lambda_{il}}, \tag{63}$$

for all $v_1,v_2\in V^n$, then there exists a unique solution $\mathcal{Q}: V^n\longrightarrow W$ of (14) so that

$$\|f(v)-\mathcal{Q}(v)\|_W\leq\begin{cases}\dfrac{\theta\Omega}{m^{2n\beta}-m^{\alpha\lambda}}\displaystyle\prod_{l=1}^{n}\|v_{1l}\|_V^{2\lambda_{1l}}, & \lambda\in\left(0,2n\dfrac{\beta}{\alpha}\right),\\ \dfrac{m^{\alpha\lambda}\Omega\theta}{m^{2n\beta}\left(m^{\alpha\lambda}-m^{2n\beta}\right)}\displaystyle\prod_{l=1}^{n}\|v_{1l}\|_V^{2\lambda_{1l}}, & \lambda\in\left(2n\dfrac{\beta}{\alpha},\infty\right),\end{cases} \tag{64}$$

for all $v=v_1\in V^n$, where $\Omega=M|a|^{\alpha\lambda^*}|b|^{\alpha\lambda^\bullet}$.

Proof. Setting $\varphi(v_1,v_2)=\theta\prod_{i=1}^2\prod_{l=1}^n\|v_{il}\|_V^{\lambda_{il}}$ in Theorem 11, one can obtain the desired results. □

4. Conclusion

In this paper, by using Euler-Lagrange type quadratic functional equations, we have defined the multi-Euler-Lagrange quadratic mappings and have studied the structure of such mappings. Indeed, we have described the multi-Euler-Lagrange quadratic mapping as an equation. In continuation, we have shown that some fixed point theorems can be applied to prove the Hyers-Ulam stability version of multi-Euler-Lagrange quadratic functional equations in the setting of quasi-β-normed and Banach spaces. In the last part, we have brought an example which shows that such functional equations can be nonstable in the some cases.

The current work provides guidelines for further research and proposals for new directions and open problems with relevant discussions. Here, we give some questions and information on the connections between the fixed point theory and the Hyers-Ulam stability.

(1) Which equation can describe the multi-Euler-Lagrange cubic mappings defined in [31]? Are these mappings stable on various Banach spaces? Can the known fixed point methods be useful to prove their Hyers-Ulam stability?

(2) Definition of the multiadditive-quartic mappings by using [14] as a system of n functional equations. The characterization of such mappings and discussion about their stability via a fixed point approach

(3) Applying the functional equations indicated in [5, 12, 13, 34], we can generalize such mappings and equations to multiple variables

Data Availability

All results are obtained without any software and found by manual computations. In other words, the manuscript is in the pure mathematics (mathematical analysis) category.

Conflicts of Interest

There do not exist any competing interests regarding this article.

Authors' Contributions

A.B proposed the topic. H.M and A.M prepared the first draft. Lastly, A.B edited and finalized the manuscript.

References

[1] S. M. Ulam, *Problems in Modern Mathematic, Science Editions*, John Wiley & Sons, Inc., New York, 1964.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, no. 1-2, pp. 64–66, 1950.

[4] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] L. Aiemsomboon and W. Sintunavarat, "Hyperstability results for generalized p-radical functional equations in non-Archimedean Banach spaces with the secret key in the client-server environment," *Mathematics in Engineering, Science & Aerospace (MESA)*, vol. 11, no. 2, pp. 467–479, 2020.

[6] A. Bodaghi and I. A. Alias, "Approximate ternary quadratic derivations on ternary Banach algebras and C^*-ternary rings," *Advances in Difference Equations*, vol. 2012, Article ID 11, 2012.

[7] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, no. 1, pp. 59–64, 1992.

[8] Z. Gajda, "On stability of additive mappings," *Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[9] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *International Journal of Mathematics and Mathematical*, vol. 184, pp. 431–436, 1994.

[10] C. Park and M. T. Rassias, "Additive functional equations and partial multipliers in C^*-algebras," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas*, vol. 113, pp. 2261–2275, 2019.

[11] F. Skof, "Proprieta locali e approssimazione di operatori," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, no. 1, pp. 113–129, 1983.

[12] W. Suriyacharoen and W. Sintunavarat, "On additive ρ-functional equations arising from Cauchy-Jensen functional equations and their stability," *Applied Mathematics & Information Sciences*, vol. 11, no. 2, pp. 277–285, 2022.

[13] A. Thanyacharoen and W. Sintunavarat, "On new stability results for composite functional equations in quasi-β-normed spaces," *Demonstratio Mathematica*, vol. 54, no. 1, pp. 68–84, 2021.

[14] A. Thanyacharoen and W. Sintunavarat, "The stability of an additive-quartic functional equations in quasi-β-normed spaces with the fixed point alternative," *Thai Journal of Mathematics*, vol. 18, no. 2, pp. 577–592, 2020.

[15] K. Ciepliński, "On the generalized Hyers-Ulam stability of multi-quadratic mappings," *Computers & Mathematcs with Applications*, vol. 62, no. 9, pp. 3418–3426, 2011.

[16] C.-G. Park, "Multi-quadratic mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 131, pp. 2501–2504, 2003.

[17] X. Zhao, X. Yang, and C.-T. Pang, "Solution and stability of the multiquadratic functional equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 415053, 8 pages, 2013.

[18] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 2011.

[19] D. Amir, *Characterizations of Inner Product Spaces*, Dirkhiiuser-Verlag, Basel, 1986.

[20] M. M. Day, "Some characterizations of inner-product spaces," *Transactions of the American Mathematical Society*, vol. 62, no. 2, pp. 320–337, 1947.

[21] P. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.

[22] N. J. Daras and T. M. Rassias, *Approximation and Computation in Science and Engineering, Series Springer Optimization and Its Applications (SOIA)*, vol. 180, Springer, 2022.

[23] P. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, 2009.

[24] P. K. Sahoo and P. Kannappan, *Introduction to Functional Equations*, CRC Press, Boca Raton, FL, 2011.

[25] A. Bodaghi, H. Moshtagh, and H. Dutta, "Characterization and stability analysis of advanced multi-quadratic functional equations," *Advances in Difference Equations*, vol. 2021, no. 1, Article ID 380, 2021.

[26] A. Bodaghi, C. Park, S. Yun, 1 Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran, 2 Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea, and 3 Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea, "*Almost multi-quadratic mappings in non-Archimedean spaces*," *AIMS Mathematics*, vol. 5, no. 5, pp. 5230–5239, 2020.

[27] A. Bodaghi, S. Salimi, and G. Abbasi, "Characterization and stability of multi-quadratic functional equations in non-Archimedean spaces," *Annals of the University of Craiova-Mathematics and Computer Science Series*, vol. 48, no. 1, pp. 88–97, 2021.

[28] S. Salimi and A. Bodaghi, "A fixed point application for the stability and hyperstability of multi-Jensen-quadratic mappings," *Journal of Fixed Point Theory and Applications*, vol. 22, no. 1, p. 9, 2020.

[29] A. Bodaghi, "Functional inequalities for generalized multi-quadratic mappings," *Journal of Inequalities and Applications*, vol. 2021, no. 1, Article ID 145, 2021.

[30] J. M. Rassias, "On the stability of the general Euler-Lagrange functional equation," *Demonstratio Mathematica*, vol. 29, no. 4, pp. 755–766, 1996.

[31] A. Thanyacharoen, W. Sintunavarat, and N. Dung, "Stability of Euler-Lagrange type cubic functional equations in quasi-Banach spaces," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 44, no. 1, pp. 251–266, 2021.

[32] T.-Z. Xu, "Approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in -Banach spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 648709, 12 pages, 2013.

[33] A. Bodaghi, "Generalized multiquartic mappings, stability, and nonstability," *Journal of Mathematics*, vol. 2022, Article ID 2784111, 9 pages, 2022.

[34] N. V. Dung and W. Sintunavarat, "On positive answer to El-Fassi's question related to hyperstability of the general radical quintic functional equation in quasi-β-Banach spaces," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales Serie A Matematicas*, vol. 115, article 168, 2021.

[35] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, no. 2, pp. 305–309, 1968.

[36] A. Bodaghi, I. A. Alias, and M. H. Ghahramani, "Ulam stability of a quartic functional equation," *Applicable Analysis*, vol. 2012, article 232630, 9 pages, 2012.

[37] S. Rolewicz, *Metric Linear Spaces*, PWN–Polish Scientific Publishers, D. Reidel Publishing Co., Warsaw; Dordrecht, 2nd edition, 1984.

[38] T.-Z. Xu, J. M. Rassias, M. J. Rassias, and W. X. Xu, "A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 423231, 23 pages, 2010.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1689732, 11 pages
https://doi.org/10.1155/2022/1689732

Research Article

Generic Stability of the Weakly Pareto-Nash Equilibrium with Strategy Transformational Barriers

Luping Liu, Wensheng Jia, and Li Zhou

College of Mathematics and Statistics, Guizhou University, 550025 Guiyang, China

Correspondence should be addressed to Wensheng Jia; wsjia@gzu.edu.cn

Received 17 June 2022; Accepted 23 September 2022; Published 6 October 2022

Academic Editor: Santosh Kumar

The object of this paper is to establish a new model with strategy transformational barriers for a class of generalized multileader multifollower multiple objective games (GMLMFMOG) and further deduce some new results of the weakly Pareto-Nash equilibrium (WPNE) with strategy transformational barriers for the GMLMFMOG. First, we investigate the existence of the WPNE with strategy transformational barriers for the GMLMFMOG by using the Kakutani-Fan-Glicksberg fixed point theory. Next, we study the generic stability of the GMLMFMOG with strategy transformational barriers in Hausdorff space. Finally, we obtain that the majority of the WPNE with strategy transformational barriers for the GMLMFMOG are essential on the meaning of Baire's category. In addition, we demonstrate that there is at least an essential component for the GMLMFMOG with strategy transformational barriers.

1. Introduction

Barriers, such as market competition [1], the Lévy risk process [2, 3], the optimal dividend problem [4], and the marketing ethics of medical schemes [5], are common in the field of economics. Transformational barriers, an important aspect of barriers, represent many factors that make the behaviour of shift strategy more difficult or costly for consumers. Furthermore, the payoff function with the strategy transformational barriers may be an abstract partial order rather than a numerical order. Game theory is an important tool for studying the interactions among the decision-making behaviours of players in many fields, such as economics, political science, psychology, and biology. Glicksberg [6] and Mas-Colell [7] provided a maximum element method to analyze the decision-making behaviours of players with the strategy transformational barriers. Therefore, the payoff function with strategy transformational barriers was introduced into game model to further study the decision-making behaviour of players based on the fact that there is a cost for players to change their strategies in practical life.

Fort [8] first presented the essential fixed point in 1950. Wu and Jiang [9] first provided the concept of essential equilibrium for a finite game through using fixed point theory for continuous mapping. Afterwards, Yu and Luo and Yu [10, 11] extended previous work to the general n-person noncooperative game, generalized game, or other games by using entirely different approaches. Recently, Scalzo [12, 13] and Carbonell-Nicolau and Carbonell-Nicolau and Wohl [14, 15] provided some extensions about discontinuous payoffs and further studied the essential stability of discontinuous games. Yang and Zhang [16] proved some existence and essential stability results of cooperative equilibrium for population games. We can also refer to [17–20] for more details on the essential stability. Hence, the essential stability has become one of the important topics in nonlinear analysis and game theory.

The weakly Pareto-Nash equilibrium (WPNE) of the multiple objective game was proposed by Shapley and Rigby [21]. Pang and Fukushima [22] studied the existence of a type of multileader multifollower multiobjective game by using quasivariational inequalities. Sherali [23] obtained the existence and uniqueness results of a WPNE regarding the multileader multifollower game. Kulkarni and Shanbhag [24] considered multileader multifollower game with shared-constraint approach to obtain local Nash equilibrium (NE), Nash B-stationary point, and Nash strong-stationary point. Yu and Wang [25] verified some existence theorems for 2-leader multifollower game in locally convex topological space. Yang and Ju [26] obtained some consequences on existence and stability of solution for multileader multifollower game. Jia et al. [27] provided the existence and stability of a WPNE for the generalized multileader multifollower multiple objective game (GMLMFMOG). Inspired by the above research work, this paper establishes a new generalized multiobjective multileader multifollower model with strategy transformational barriers by considering the influence of strategy transformational barriers and analyzes the strategy selection of the players. The leaders consider multiple objectives when selecting their strategies. The followers also consider multiple objectives when selecting their strategies with complete knowledge and make optimal responses to the leaders' strategies. The goals of all players are to maximize their own incomes. Furthermore, the existence of the WPNE with strategy transformational barriers of a GMLMFMOG is proved, and the generic stability of the GMLMFMOG with strategy transformational barriers is obtained. We prove that the solution set of the GMLMFMOG with the strategy transformational barriers is essential and that there is at least one essential component of the WPNE with the strategy transformational barriers under the meaning of the Baire's category.

This paper is outlined as follows. We present necessary preliminaries and the GMLMFMOG model with strategy transformational barriers in Section 2. In Section 3, we provide the existence of the WPNE with the strategy transformational barriers of the GMLMFMOG. In Section 4, we investigate some generic stability results of the GMLMFMOG with strategy transformational barriers. In Section 5, we show that the majority of WPNE with strategy transformational barriers of the GMLMFMOG are essential, and then there is at least an essential component. Finally, some brief and concise conclusions are given.

2. Preliminaries and Model

2.1. Preliminaries. In this paragraph, we introduce some substantial definitions, lemmas, and game models.

Definition 1 (see [28]). Suppose that $\mathscr{A}$ is not empty subset of Hausdorff topological vector space (HTVS) F, $L \subset F$ is not empty convex cone, and a vector-valued correspondence is denoted by $\mathscr{S}: \mathscr{A} \longrightarrow F$. We define $\mathscr{S}$ is L *-usc* (resp. L *-lsc*) at $a_0 \in \mathscr{A}$ if, for each open neighbourhood V of the 0 element in F, there exists an open neighbourhood $\mathcal{O}(a_0)$ of a_0 such that $\mathscr{S}(a) \in \mathscr{S}(a_0) + V - L$ (resp. $\mathscr{S}(a) \in \mathscr{S}(a_0) + V + L$), $\forall a \in \mathcal{O}(a_0)$. Furthermore, we say $\mathscr{S}$ is L *-usc* (resp. L *-lsc*) on $\mathscr{A}$, if $\mathscr{S}$ is L *-usc* (resp. L *-lsc*) for all $a \in \mathscr{A}$. We call $\mathscr{S}$ is L -continuous on $\mathscr{A}$, if $\mathscr{S}$ is L *-usc* and L *-lsc* on $\mathscr{A}$. $\mathscr{S}$ is closed if $\mathrm{Graph}(\mathscr{S}) = \{(a, f) \in \mathscr{A} \times F | f \in \mathscr{S}(a)\}$ is closed on $\mathscr{A} \times F$.

Definition 2 (see [29]). Let $\mathscr{A}$ and $\mathscr{B}$ be two HTVSs, $L \subset \mathscr{B}$ be a closed convex pointed cone, $\mathrm{int}\, L \neq \varnothing$, $D \subset \mathscr{A}$ be not empty convex subset, and $\mathscr{S}: D \longrightarrow \mathscr{B}$ be a vector-valued correspondence. If, $\forall a_1, a_2 \in D$ and $\theta \in (0,1)$, $\mathscr{S}(\theta a_1 + (1-\theta) a_2) - [\theta \mathscr{S}(a_1) + (1-\theta)\mathscr{S}(a_2)] \notin -\mathrm{int}\, L$ holds, then $\mathscr{S}$ is *L-concave*, and $-\mathscr{S}$ is *L- convex*. If, $\forall a_1, a_2 \in D$, $b \in \mathscr{B}$, and $\theta \in (0,1)$, $\mathscr{S}(a_1) \notin b - \mathrm{int}\, L$, $\mathscr{S}(a_2) \notin b - \mathrm{int}\, L$ such that $\mathscr{S}(\theta a_1 + (1-\theta) a_2) \notin b - \mathrm{int}\, L$, then $\mathscr{S}$ is *L- quasiconcave-like*, and $-\mathscr{S}$ is *L- quasiconvex-like*.

Remark 3. For $\mathscr{B} = (-\infty, +\infty)$, $L = [0, +\infty)$, if $\mathscr{S}$ is *L- quasiconcave-like*, then $\mathscr{S}$ is obviously *quasiconcave*. However, $D = [0,1]$, $\mathscr{B} = (-\infty, +\infty) \times (-\infty, +\infty)$, $L = [0, +\infty) \times [0, +\infty)$, and $f = (f_1, f_2) = (a, -a)$, $g = (g_1, g_2) = (a^2, a^2)$. We know that f is *L- concave* but not *L- quasiconcave-like*, and g is *L- quasiconcave-like* but not *L- concave*. Thus, *L- quasiconcave-like* and *L- concave* do not include each other.

Definition 4 (maximal element theorem, see [30]). Let $\mathscr{A}$ be not empty compact convex subset (NECCS) of HTVS F and $\mathscr{S}: \mathscr{A} \longrightarrow 2^{\mathscr{A}}$ with the following conditions, where $2^{\mathscr{A}}$ denotes all nonempty subsets of $\mathscr{A}$:

(1) $\forall a \in \mathscr{A}$, $a \notin \mathrm{conv}\mathscr{S}(a)$, where $\mathrm{conv}\mathscr{S}(a)$ denotes the convex hull of $\mathscr{S}(a)$

(2) $\forall b \in \mathscr{A}$, $\mathscr{S}^{-1}(b) = \{a \in \mathscr{A} | b \in \mathscr{S}(a)\}$ is open in $\mathscr{A}$

Then, there is $a^* = (a_1^*, a_2^*, \cdots, a_n^*) \in \mathscr{A}$ such that $\mathscr{S}(a^*) = \varnothing$.

2.2. Model. A model of the GMLMFMOG with strategy transformational barrier is denoted by a tuple $\{\mathbb{N}, \mathbb{M}, \mathscr{A}, \mathscr{B}, \mathscr{V}, P\}$, where

(i) $\mathbb{N} = \{1, \cdots, n\}$ and $\mathbb{M} = \{1, \cdots, m\}$ indicate the index set of leaders and followers, respectively

(ii) $\forall i \in \mathbb{N}$, $\forall j \in \mathbb{M}$, $\mathscr{A}_i$, and $\mathscr{B}_j$ denote the strategy set of the ith leader and the jth follower, separately. The leaders' strategy represents $a = (a_i, a_{-i}) \in \mathscr{A}$, where $\mathscr{A} = \prod_{i \in \mathbb{N}} \mathscr{A}_i$, $\mathscr{A}_{-i} = \prod_{l \in \{\mathbb{N} \setminus i\}} \mathscr{A}_l$. Meanwhile, the strategy of the followers denotes $b = (b_j, b_{-j}) \in \mathscr{B}$, where $\mathscr{B} = \prod_{j \in \mathbb{M}} \mathscr{B}_j$, $\mathscr{B}_{-j} = \prod_{k \in \{\mathbb{M} \setminus j\}} \mathscr{B}_k$

(iii) $\forall i \in \mathbb{N}$, $\mathscr{U}_i = \mathscr{B}$, $\mathscr{U} = \prod_{i \in \mathbb{N}} \mathscr{U}_i$, and $\mathscr{U}_{-i} = \prod_{k \in \{\mathbb{N} \setminus i\}} \mathscr{U}_k$. Let $Y_i = \{\chi_1^i, \cdots, \chi_l^i\}$: $\mathscr{A}_i \times \mathscr{A}_{-i} \times \mathscr{U}_i \longrightarrow R_+^l$ be the payoff function of the ith leader. Let $\Psi_j = \{\psi_1^j, \cdots, \psi_k^j\}$: $\mathscr{A} \times \mathscr{B}_j \times \mathscr{B}_{-j} \longrightarrow R^k$, $\forall j \in \mathbb{M}$ be a payoff function of the jth follower and $G_j: \mathscr{A} \times \mathscr{B}_{-j} \longrightarrow 2^{\mathscr{B}_j}$ be a constraint correspondence of the jth follower

(iv) Let $\mathscr{V}_i: \mathscr{A}_i \times \mathscr{A}_i \longrightarrow R_+^l$ be the strategy transformational barrier function of the leader i. $\forall i \in \mathbb{N}$, there exists $a_i \in \mathscr{A}_i$ such that

$$\mathscr{V}_i\left(a_i, a_i'\right) > 0, \forall a_i' \in \mathscr{A}_i, \tag{1}$$

where $\mathscr{V}_i(a_i, a'_i)$ denotes the strategy transformational barriers of the leader i changing from strategy a_i to strategy a'_i. In particular, $\mathscr{V}_i(a_i, a'_i) = 0$ denotes that the ith leader has no transformational strategy

(v) The followers are a generalized constraint multiobjective with the strategy parametric game (PGCMOG) after fixing the strategy $a \in \mathscr{A}$ of the leaders. Let $P : \mathscr{A}_i \times \mathscr{A}_{-i} \longrightarrow 2^{\mathscr{B}}$ be the solution mapping of the WPNE with strategy transformational barriers for the PGCMOG. Particularly, $\forall b^* \in P(a_i, a_{-i})$ such that there is $b_j^* \in G_j(a, b_{-j}^*), \forall j \in \mathbb{M}$, and we have $\Psi_j(a, b_j, b_{-j}^*) - \Psi_j(a, b_j^*, b_{-j}^*) \notin \operatorname{int} R_+^k$, $\forall b_j \in G_j(a, b_{-j}^*)$. Furthermore, if there is $u_i^* \in \mathscr{U}_i$ such that $u_i^* \in P(a_i^*, a_{-i}^*), \forall i \in \mathbb{N}$, satisfying

$$Y_i\left(a'_i, a_{-i}^*, u_i\right) - Y_i(a_{-i}^*, a_{-i}^*, u_i^*) - \mathscr{V}_i\left(a_{-i}^*, a'_i\right) \notin \operatorname{int} R_+^l, \forall \left(a'_i, u_i\right) \in \mathscr{A}_i \times P\left(a'_i, a_{-i}^*\right), \tag{2}$$

then $a^* = (a_{-i}^*, a_{-i}^*) \in \mathscr{A}$ is called a WPNE with strategy transformational barriers of the GMLMFMOG, where $\operatorname{int} R_+^l = \{(a_1, a_2, \cdots, a_l) \in R^l : a_i > 0, i = 1, \cdots, l\}$, $\mathscr{V}_i(a_{-i}^*, a'_i)$ denotes the leader i's cost changing from strategy a_{-i}^* to strategy a'_i

Let i, a_{-i}, and u_{-i} be elements in $\mathbb{N}$, $\mathscr{A}_{-i}$, and $\mathscr{U}_{-i}$, respectively. By Definition 4, then we have the best response of the ith leader with strategy transformation barriers to the other players, i.e.,

$$B_i(a_{-i}, u_{-i}) = \left\{a_i \in \mathscr{A}_i, u_i \in P(a_i, a_{-i}) \middle| Y_i\left(a'_i, a_{-i}, v_i\right) - Y_i(a_i, a_{-i}, u_i) - \mathscr{V}_i\left(a_i, a'_i\right) \notin \operatorname{int} R_+^l\right\}, \forall \left(a'_i, v_i\right) \in \mathscr{A}_i \times P\left(a'_i, a_{-i}\right), \tag{3}$$

where B_i is independent of $u_{-i} \in \mathscr{U}_i$.

Fixing $a_{-i} \in \mathscr{A}_{-i}$, we know that the player's set-valued mapping B_i provides the order relation "$\underset{a_{-i}}{\geq}$" as follows:

$$(w_i, u_i) \underset{a_{-i}}{\geq} (a_i, u_i) \Leftrightarrow (w_i, u_i) \in B_i(a_{-i}, u_{-i}). \tag{4}$$

In general, the order relation is not transitive, and we give a sufficient condition for the transitivity of the order relation "$\underset{a_{-i}}{\geq}$" with the following propositions.

Proposition 5. *Let $\{\mathbb{N}, \mathbb{M}, \mathscr{A}, \mathscr{B}, \mathscr{V}, P\}$ be a GMLMFMOG with strategy transformational barriers, if, for any $w_i, a_i, z_i \in \mathscr{A}_i$, and*

$$\mathscr{V}_i(z_i, w_i) + \mathscr{V}_i(w_i, a_i) \leq \mathscr{V}_i(z_i, a_i) (i.e., \mathscr{V}_i \textit{ has negative subadditivity}). \tag{5}$$

Then, the order relation "$\underset{a_{-i}}{\geq}$" has transitivity.

Proof. Setting w_i, a_i, z_i which are three elements in $\mathscr{A}_i$ and $u_{-i}, v_i \in \mathscr{U}_{-i}$ such that $(z_i, u_i) \underset{a_{-i}}{\geq} (w_i, u_i) \underset{a_{-i}}{\geq} (a_i, u_i)$ holds, we obtain

$$\begin{aligned} &Y_i(w_i, a_{-i}, v_i) - Y_i(z_i, a_{-i}, u_i) - \mathscr{V}_i(z_i, w_i) \notin \operatorname{int} R_+^l, \\ &Y_i(a_i, a_{-i}, v_i) - Y_i(w_i, a_{-i}, u_i) - \mathscr{V}_i(w_i, a_i) \notin \operatorname{int} R_+^l, \end{aligned} \tag{6}$$

by $(z_i, u_i) \in B_i(a_{-i}, u_{-i})$, $(w_i, u_i) \in B_i(a_{-i}, u_{-i})$, and the definition of best response mapping $B_i(a_{-i}, u_{-i})$. Then, we attain

$$\begin{aligned} &Y_i(a_i, a_{-i}, v_i) - Y_i(z_i, a_{-i}, u_i) - \mathscr{V}_i(z_i, w_i) - \mathscr{V}_i(w_i, a_i) \\ &\quad - Y_i(w_i, a_{-i}, u_i) + Y_i(w_i, a_{-i}, v_i) \notin \operatorname{int} R_+^l \Rightarrow Y_i(a_i, a_{-i}, v_i) \\ &\quad - Y_i(z_i, a_{-i}, u_i) - (\mathscr{V}_i(z_i, w_i) + \mathscr{V}_i(w_i, a_i)) \\ &\quad - (Y_i(w_i, a_{-i}, u_i) - Y_i(w_i, a_{-i}, v_i)) \notin \operatorname{int} R_+^l \Rightarrow Y_i(a_i, a_{-i}, v_i) \\ &\quad - Y_i(z_i, a_{-i}, u_i) - \mathscr{V}_i(z_i, a_i) \notin \operatorname{int} R_+^l. \end{aligned} \tag{7}$$

Since B_i is not dependent on $u_{-i} \in \mathscr{U}_i$, we can see that $Y_i(w_i, a_{-i}, u_i) - Y_i(w_i, a_{-i}, v_i)$ is equal to zero element of R_+^l. Therefore, $(z_i, u_i) \in B_i(a_{-i}, u_{-i}) \Leftrightarrow (z_i, u_i) \underset{a_{-i}}{\geq} (a_i, u_i)$; then, the order relation "$\underset{a_{-i}}{\geq}$" has transitivity. □

Example 1. Considering the Hotelling model [31], the influence of the strategy transformational barrier function can be added. Assume that consumers are evenly distributed on a street and that businessmen ($\mathbb{N} = 1, 2$) choose their shop location on the street. Suppose that the street can be abstracted to a line segment with a length of 1, namely, [0, 1]. Meanwhile, $c \in [0, 1]$ and $d \in [0, 1]$ represent the positions of the two businessmen. The strategy set of the businessmen is $[0, 1]$, and the payoff functions $f_1, f_2 : [0, 1] \longrightarrow R$ are expressed as

$$\begin{aligned} f_1 &= \begin{cases} \dfrac{c+d}{2}, c < d, \\ 1 - \dfrac{c+d}{2}, c > d, \\ \dfrac{1}{2}, c = d, \end{cases} \\ f_2 &= \begin{cases} \dfrac{c+d}{2}, d < c, \\ 1 - \dfrac{c+d}{2}, d > c, \\ \dfrac{1}{2}, d = c. \end{cases} \end{aligned} \tag{8}$$

It is a well-known fact that $(c, d) = (1/2, 1/2)$ is the unique NE point of the Hotelling game [31], which can better explain the phenomenon of shop centralization. However, it is worth noting that shops may not be concentrated in the centre because of the influence of relocation costs and other factors. In reality, the distribution of shop locations corresponds to a WPNE with strategy transformational barriers, which means a state of equilibrium under weaker conditions.

Suppose that the strategy transformational barrier functions are $\mathscr{V}(c_1, c_2)$ and $\mathscr{V}(d_1, d_2)$, respectively. If $\mathscr{V}(c_1, c_2)$ and $\mathscr{V}(d_1, d_2)$ are

$$\begin{aligned}\mathscr{V}(c_1, c_2) &= \alpha_1|c_1 - c_2| + \beta_1, \forall c_1, c_2 \in [0, 1],\\ \mathscr{V}(d_1, d_2) &= \alpha_2|d_1 - d_2| + \beta_2, \forall d_1, d_2 \in [0, 1].\end{aligned} \tag{9}$$

Setting businessmen 1 taking $d = 1/2$, $c_1 = 1/4$, $c_2 = 3/4$, and $c_3 = 3/8$, we have

$$\begin{aligned}f_1(c_1, d) &= f_1\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{3}{8},\\ f_1(c_2, d) &= f_1\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{3}{8},\\ f_1(c_3, d) &= f_1\left(\frac{3}{8}, \frac{1}{2}\right) = \frac{7}{16}.\end{aligned} \tag{10}$$

If $\alpha_1 = 1/3$, $\beta_1 = 0$, then

$$\begin{aligned}f_1(c_1, d) = \frac{3}{8} \geq f_1(c_2, d) - \mathscr{V}(c_1, c_2) = \frac{3}{8} - \frac{1}{3}|c_1 - c_2| = \frac{5}{24}, \text{i.e.,} c_1 \underset{b}{\geq} c_2,\\ f_1(c_2, d) = \frac{3}{8} \geq f_1(c_3, d) - \mathscr{V}(c_2, c_3) = \frac{7}{16} - \frac{1}{3}|c_2 - c_3| = \frac{5}{16}, \text{i.e.,} c_2 \underset{b}{\geq} c_3,\end{aligned} \tag{11}$$

but

$$f_1(c_1, d) = \frac{3}{8} \leq f_1(c_3, d) - \mathscr{V}(c_1, c_3) = \frac{7}{16} - \frac{1}{3}|c_1 - c_3| = \frac{7}{16} - \frac{1}{24} = \frac{19}{48}. \tag{12}$$

Furthermore,

$$\begin{aligned}\mathscr{V}(c_1, c_2) &= \frac{1}{3}|c_1 - c_2| = \frac{1}{3} \times \left|\frac{1}{4} - \frac{3}{4}\right| = \frac{1}{6},\\ \mathscr{V}(c_2, c_3) &= \frac{1}{3}|c_2 - c_3| = \frac{1}{3} \times \left|\frac{3}{4} - \frac{3}{8}\right| = \frac{1}{8},\\ \mathscr{V}(c_1, c_3) &= \frac{1}{3}|c_3 - c_1| = \frac{1}{3} \times \left|\frac{3}{8} - \frac{1}{4}\right| = \frac{1}{24}.\end{aligned} \tag{13}$$

Since $\mathscr{V}(c_1, c_2) + \mathscr{V}(c_2, c_3) \nleq \mathscr{V}(c_1, c_3)$, "$c_1 \underset{d}{\geq} c_3$" has no negative subadditivity. Then, "$c_1 \underset{d}{\geq} c_3$" does not hold; thus, the order relation "$\geq$" is not satisfied to transitive.

Remark 6. When the strategy transformational barrier function does not have negative subadditivity, the order relationship "$\underset{a_{-i}}{\geq}$" does not have transitivity. Furthermore, the game with a strategy transformational barrier function may not have a numerical payoff function since the strategy transformational barrier function often possesses subadditivity rather than negative subadditivity.

3. Existence

In this paragraph, the existence of the WPNE with the strategy transformational barriers of the GMLMFMOG is demonstrated.

Lemma 7 (Kakutani-Fan-Glicksberg, see [6]). *Assume that $\mathscr{A}$ is a NECCS of locally convex Hausdorff space F, $\mathscr{S} : \mathscr{A} \longrightarrow 2^{\mathscr{A}}$ is a set-valued mapping, $\forall a \in \mathscr{A}$, $\mathscr{S}(a)$ is a nonempty, convex, compact set, and $\mathscr{S}(a)$ is usc on $\mathscr{A}$. Then, there exists $a^* \in \mathscr{A}$ such that $a^* \in \mathscr{S}(a^*)$.*

Lemma 8 (see [17]). *Assume that $\mathscr{A}$ is a nonempty subset of Hausdorff space F and $Y : \mathscr{A} \longrightarrow R^l_+$ is a vector value correspondence, where $Y = \{\chi_1, \cdots, \chi_l\}$. In that case, Y is R^l_+-continuous if $\chi_i (\forall i = 1, \cdots, l)$ is continuous.*

Lemma 9 (see [28]). *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are two Hausdorff spaces and $\mathscr{B}$ is compact. If a set-valued correspondence $\mathscr{S} : \mathscr{A} \longrightarrow 2^{\mathscr{B}}$ is closed, then $\mathscr{S}$ is usc.*

Lemma 10 (see [29]). *Assume that $\mathscr{A}$ and $\mathscr{B}$ are two NECCSs of locally convex Hausdorff space F and H, respectively. $Y : \mathscr{A} \times \mathscr{B} \longrightarrow R^l_+$ is continuous correspondence; $\mathscr{W} : \mathscr{B} \longrightarrow 2^{\mathscr{A}}$ is a continuous set-valued correspondence on $\mathscr{B}$, $\forall b \in \mathscr{B}$, $\mathscr{W}(b)$ is not empty and compact subset of $\mathscr{A}$, as well as $\mathscr{W}(b) = \{a \in \mathscr{W}(b) : Y(a', b) - Y(a, b) \notin \operatorname{int} R^l_+, \forall a' \in \mathscr{W}(b)\}$. Then, we obtain that $\mathscr{W}(b)$ is a compact, nonempty set as well as $\mathscr{W} : \mathscr{B} \longrightarrow 2^{\mathscr{A}}$ is usc on $\mathscr{B}$.*

Theorem 11 (Fort theorem, see [8]). *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are Hausdorff and metric spaces, respectively. Given a set-valued mapping $\mathscr{S} : \mathscr{A} \longrightarrow 2^{\mathscr{B}}$ is usc on $\mathscr{A}$ with nonempty compact value (briefly, usco), then there is a residual subset Q in $\mathscr{A}$ such that $\mathscr{S}$ is lsc on Q.*

Remark 12 (see [29]). If $\mathscr{A}$ is Baire space, then the residual set in $\mathscr{A}$ is dense.

Theorem 13. *Suppose that $\mathscr{A}_i (i \in \mathbb{N})$ and $\mathscr{B}_j (j \in \mathbb{M})$ are two NECCSs of locally convex Hausdorff space F_i and H_j, respectively. If $\{\mathbb{N}, \mathbb{M}, \mathscr{A}, \mathscr{B}, \mathscr{V}, P\}$ satisfies the following conditions.*

(1) $\forall i \in \mathbb{N}$, $Y_i = \{\chi^i_1, \cdots, \chi^i_l\}$: $\mathscr{A}_i \times \mathscr{A}_{-i} \times \mathscr{U}_i \longrightarrow R^l_+$ is R^l_+-continuous

(2) $\forall i \in \mathbb{N}$, $\mathscr{V}_i : \mathscr{A}_i \times \mathscr{A}_i \longrightarrow R^l_+$ is R^l_+-continuous, $\forall a'_i \in \mathscr{A}_i$, $a_i \longrightarrow \mathscr{V}(a_i, a'_i)$ is convex

(3) $\forall i \in \mathbb{N}$, $\forall a_{-i} \in \mathscr{A}_{-i}$, $(a_i, u_i) \longrightarrow Y_i(a_i, a_{-i}, u_i)$ *is* R^l_+-*quasiconcave-like*

(4) $\forall i \in \mathbb{N}$, $P : \mathscr{A}_i \times \mathscr{A}_{-i} \longrightarrow 2^{\mathscr{B}}$ *is continuous, and* $\forall a = (a_i, a_{-i}) \in \mathscr{A}$, $P(a_i, a_{-i})$ *is a nonempty and compact subset of* $\mathscr{B}$

(5) $\forall a_{-i} \in \mathscr{A}_{-i}$, *the set-valued correspondence* $a_i \longrightarrow P(a_i, a_{-i})$ *is convex (i.e.,* $\forall \theta \in (0,1)$, $a^1_i, a^2_i \in \mathscr{A}_i$, $P(\theta a^1_i + (1-\theta)a^2_i, a_{-i}) \subset \theta P(a^1_i, a_{-i}) + (1-\theta)P(a^2_i, a_{-i})$*)*

Then, the GMLMFMOG with strategy transformational barriers contains at least a point $(a^*_{-i}, a^*_{-i}, u^*_i) \in \mathscr{A}_i \times \mathscr{A}_{-i} \times \mathscr{U}_i$ *such that* $u^*_i \in P(a^*_{-i}, a^*_{-i}), \forall i \in \mathbb{N}$, *satisfying*

$$Y_i(a_i, a^*_{-i}, u_i) - Y_i(a^*_{-i}, a^*_{-i}, u^*_i) - \mathscr{V}_i(a^*_{-i}, a_i) \notin \operatorname{int} R^l_+, \forall (a_i, u_i) \in \mathscr{A}_i \times P(a_i, a^*_{-i}). \tag{14}$$

Proof. $\forall i \in \mathbb{N}$, the set-valued correspondence $\mathscr{T}_i : \mathscr{A}_{-i} \times \mathscr{U}_{-i} \longrightarrow 2^{\mathscr{A}_i \times \mathscr{U}_i}$ is defined, $\forall a_{-i} \in \mathscr{A}_{-i}$, $u_{-i} \in \mathscr{U}_{-i}$, we have

$$\begin{aligned} \mathscr{T}_i(a_{-i}, u_{-i}) = \Big\{ & a_i \in \mathscr{A}_i, u_i \in P(a_i, a_{-i}) | Y_i\left(a'_i, a_{-i}, v_i\right) \\ & - Y_i(a_i, a_{-i}, u_i) - \mathscr{V}_i\left(a_i, a'_i\right) \notin \operatorname{int} R^l_+ \Big\}, \forall \left(a'_i, v_i\right) \in \mathscr{A}_i \\ & \times P\left(a'_i, a_{-i}\right), \end{aligned} \tag{15}$$

where $\mathscr{T}_i$ is independent of $u_i \in \mathscr{U}_{-i}$.

By Lemma 7, we only need to prove that the set-valued mapping $\mathscr{T}_i$ is *usc* mapping with nonempty convex compact value.

(1) $\mathscr{T}_i(a_{-i}, u_{-i}) \neq \varnothing$. Because $\mathscr{A}_i$ is compact and P is a continuous correspondence with compact value, $\{P(w_i, a_{-i}): w_i \in \mathscr{A}_i, \forall i \in \mathbb{N}\}$ is compact. Y is R^l_+-continuous from Lemma 8; then, $\forall i = 1, \cdots, l$, Y_i is R^l_+-continuous and $\mathscr{V}_i$ is also R^l_+-continuous. Thus, $\mathscr{T}_i(a_{-i}, u_{-i}) \neq \varnothing$ from Lemma 7

(2) $\mathscr{T}_i(a_{-i}, u_{-i})$ is convex. $\forall (a^1_i, u^1_i) \in \mathscr{T}_i(a_{-i}, u_{-i}), (a^2_i, u^2_i) \in \mathscr{T}_i(a_{-i}, u_{-i})$, i.e., $a^1_i \in \mathscr{A}_i$, $u^1_i \in P(a^1_i, a_{-i})$, $a^2_i \in \mathscr{A}_i$, $u^2_i \in P(a^2_i, a_{-i})$, and $\forall i \in \mathbb{N}$, we obtain

$$\begin{gathered} Y_i\left(a'_i, a_{-i}, v_i\right) - Y_i\left(a^1_i, a_{-i}, u^1_i\right) - \mathscr{V}_i\left(a^1_i, a'_i\right) \notin \operatorname{int} R^l_+, \\ Y_i\left(a'_i, a_{-i}, v_i\right) - Y_i\left(a^2_i, a_{-i}, u^2_i\right) - \mathscr{V}_i\left(a^2_i, a'_i\right) \notin \operatorname{int} R^l_+, \\ \forall \left(a'_i, v_i\right) \in \mathscr{A}_i \times P\left(a'_i, a_{-i}\right), \end{gathered} \tag{16}$$

i.e., we have

$$\begin{gathered} Y_i\left(a^1_i, a_{-i}, u^1_i\right) \notin Y_i\left(a'_i, a_{-i}, v_i\right) - \mathscr{V}_i\left(a^1_i, a'_i\right) - \operatorname{int} R^l_+, \\ Y_i\left(a^2_i, a_{-i}, u^2_i\right) \notin Y_i\left(a'_i, a_{-i}, v_i\right) - \mathscr{V}_i\left(a^2_i, a'_i\right) - \operatorname{int} R^l_+, \\ \forall \left(a'_i, v_i\right) \in \mathscr{A}_i \times P\left(a'_i, a_{-i}\right) \end{gathered} \tag{17}$$

Since $\mathscr{A}_i$ is convex, $\theta a^1_i + (1-\theta)a^2_i \in \mathscr{A}_i$, $\forall \theta \in (0,1)$, and $\forall a_{-i} \in \mathscr{A}_{-i}$ by Theorem 13 (5), we have $\theta a^1_i + (1-\theta)a^2_i \in P(\theta a^1_i + (1-\theta)a^2_i, a_{-i}) \subset \theta P(a^1_i, a_{-i}) + (1-\theta)P(a^2_i, a_{-i})$.

Since $\forall a_{-i} \in \mathscr{A}_{-i}$, $(a_i, u_i) \longrightarrow Y_i(a_i, a_{-i}, u_i)$ is R^l_+- *quasi-concave-like*, and $\forall a'_i \in \mathscr{A}_i a_i \longrightarrow \mathscr{V}(a_i, a'_i)$ is convex, we obtain

$$\begin{aligned} Y_i\left(\theta a^1_i + (1-\theta)a^2_i, a_{-i}, \theta u^1_i + (1-\theta)u^2_i\right) & \notin Y_i\left(a'_i, a_{-i}, v_i\right) \\ - \mathscr{V}_i\left(\theta a^1_i + (1-\theta)a^2_i, a'_i\right) & - \operatorname{int} R^l_+, \end{aligned} \tag{18}$$

i.e.,

$$\begin{aligned} Y_i\left(a'_i, a_{-i}, v_i\right) & - Y_i\left(\theta a^1_i + (1-\theta)a^2_i, a_{-i}, \theta u^1_i + (1-\theta)u^2_i\right) \\ & - \mathscr{V}\left(\theta a^1_i + (1-\theta)a^2_i, a'_i\right) \notin \operatorname{int} R^l_+. \end{aligned} \tag{19}$$

Thus, $(\theta a^1_i + (1-\theta)a^2_i, a_{-i}, \theta u^1_i + (1-\theta)u^2_i) \in \mathscr{T}_i(a_{-i}, u_{-i})$, $\mathscr{T}_i(a_{-i}, u_{-i})$ is convex.

(3) $\mathscr{T}_i(a_{-i}, u_{-i})$ is a *usc* mapping. According to Lemma 9, we just need to verify that Graph($\mathscr{T}_i$) is closed. Thus, we next demonstrate that the set-valued correspondence $C(a_{-i}) = \{(a'_i, v_i) \in \mathscr{A}_i \times \mathscr{U}_i : a'_i \in \mathscr{A}_i, v_i \in P(a'_i, v_i)\}$ is continuous

Suppose that $\{a^\alpha_{-i} : \alpha \in \mathscr{K}\}$ is any net on $\mathscr{A}_i$, and $a^\alpha_{-i} \longrightarrow a_{-i}$, $\forall ({a'}^\alpha_i, v^\alpha_i) \in C(a^\alpha_{-i})$, $({a'}^\alpha_i, v^\alpha_i) \longrightarrow (a'_i, v_i) \in \mathscr{A}_i \times \mathscr{U}_i$. Because P is a *usc* mapping with compact value and ${a'}^\alpha_i \longrightarrow a'_i$, $v^\alpha_i \in P({a'}^\alpha, a_{-i})$, $v^\alpha_i \longrightarrow v_i$ from Theorem 16.17 in [32], we attain $v_i \in P(a'_i, a_{-i})$. Therefore, $(a'_i, v_i) \in C(a_{-i})$, C is closed. Since $\mathscr{A}_i \times \mathscr{U}_i$ is compact from Lemma 9, C is *usc* on $\mathscr{A}_{-i}$.

Meanwhile, assume that $\{a^\alpha_{-i} : \alpha \in \mathscr{K}\}$ is any net on $\mathscr{A}_i$, $a^\alpha_{-i} \longrightarrow a_{-i}$, $\forall (a'_i, v_i) \in C(a_{-i})$, then $a'_i \in \mathscr{A}_i$, $v_i \in P(a'_i, v_i)$. For any $\alpha \in \mathscr{K}$, we set ${a'}^\alpha_i = a'_i$, since P is continuous, from Theorem 16.19 in [32] if there is some $v^\alpha_i \in P({a'}^\alpha_i, a^\alpha_{-i}) = P(a'_i, a^\alpha_{-i})$, $v^\alpha_i \longrightarrow v_i$, $(a'_i, v^\alpha_i) \in C(a^\alpha_{-i})$, and $(a'_i, v^\alpha_i) \longrightarrow (a'_i, v_i)$ hold. Thus, C is *lsc* on $\mathscr{A}_{-i}$.

Hence, we have proved that C is continuous with compact values. $\mathscr{T}_i(a_{-i}, u_{-i})$ is compact and $\mathscr{T}_i$ is a *usc* mapping from Lemma 10. On the basis of the above proof, we know that $\mathscr{T}_i$ is a *usco* correspondence.

A set-valued correspondence $\mathcal{S} : \mathcal{A} \times \mathcal{U} \longrightarrow 2^{\mathcal{A}\times\mathcal{U}}$ is defined, and $\forall (a, u) \in (\mathcal{A}, \mathcal{U})$ contains $\mathcal{S}(a, u) = \mathcal{T}_1(a_{-1}, u_{-1}) \times \cdots \times \mathcal{T}_n(a_{-n}, u_{-n}) \subset \mathcal{A} \times \mathcal{U}$.

Because $\mathcal{A} \times \mathcal{U}$ is a NECCS of locally convex Hausdorff space, $\mathcal{S}$ is a *usco* mapping and Lemma 7, if there is $(a^*, u^*) \in (\mathcal{A}, \mathcal{U})$, then $(a^*, u^*) \in \mathcal{S}(a^*, u^*)$ holds. We obtain $(a^*_{-i}, a^*_{-i}, u^*_i) \in \mathcal{T}_i(a^*_{-i}, u^*_{-i}), \forall i \in \mathbb{N}$. Consequently, there is $(a^*_{-i}, a^*_{-i}, u^*_i) \in \mathcal{A}_i \times \mathcal{A}_{-i} \times \mathcal{U}_i$ such that $\forall i \in \mathbb{N}$, $u^*_i \in P(a^*_{-i}, a^*_{-i})$, $Y_i(a'_i, a^*_{-i}, u_i) - Y_i(a^*_{-i}, a^*_{-i}, u^*_i) - \mathcal{V}_i(a^*_{-i}, a'_i) \notin \operatorname{int} R^l_+, \forall (a'_i, u_i) \in \mathcal{A}_i \times P(a'_i, a^*_{-i})$. This concludes the proof. □

Remark 14. In this paper, the WPNE with strategy transformational barriers are more broadly concepts than the WPNE in literature [27] in practical life, which means that the player needs to consider the impact of other some factors, such as the cost of changing strategies. In particular, if the leaders have no transformational strategy barriers, then the WPNE can be considered as the WPNE with strategy transformational barriers.

4. Generic Stability

In this paragraph, we prove the generic stability of the WPNE with the strategy transformational barriers of the GMLMFMOG.

Let $\mathcal{A}_i (i \in \mathbb{N})$ and $\mathcal{B}_j (j \in \mathbb{M})$ be two NECCSs of Banach space F and H, respectively, and $\Omega = \{\phi = Y_1, \cdots, Y_n, \mathcal{V}_1, \cdots, \mathcal{V}_n, P|$ for any $i \in \mathbb{N}$, Y_i, $\mathcal{V}_i$ and P satisfy all conditions provided in Theorem 13.

For $\phi^1 = (Y^1_1, \cdots, Y^1_n, \mathcal{V}^1_1, \cdots, \mathcal{V}^1_n, P^1)$ and $\phi^2 = (Y^2_1, \cdots, Y^2_n, \mathcal{V}^2_1, \cdots, \mathcal{V}^2_n, P^2) \in \Omega$, the distance on Ω is defined as follows:

$$\begin{aligned}\varpi(\phi^1, \phi^2) = &\sup_{(a_i, u_i) \in \mathcal{A}_i \times \mathcal{U}_i} \sum_{i=1}^{n} \left\| Y^1_i(a_i, a_{-i}, u_i) - Y^2_i(a_i, a_{-i}, u_i) \right\| \\ &+ \sup_{(a_i, a'_i) \in \mathcal{A}_i \times \mathcal{A}_i} \sum_{i=1}^{n} \left\| \mathcal{V}^1_i(a_i, a'_i) - \mathcal{V}^2_i(a_i, a'_i) \right\| \\ &+ \sup_{(a_i, a_{-i}) \in \mathcal{A}_i \times \mathcal{A}_{-i}} \mathcal{H}\left(P^1(a_i, a_{-i}), P^2(a_i, a_{-i})\right),\end{aligned} \tag{20}$$

where $\mathcal{H}(P^1(a_i, a_{-i}), P^2(a_i, a_{-i}))$ is the Hausdorff distance between $P^1(a_i, a_{-i})$ and $P^2(a_i, a_{-i})$ on $\mathcal{A}$.

Theorem 15. *(Ω, ϖ) is a complete metric space.*

Proof. It is easy to see that (Ω, ϖ) serves as a metric space. Then, we just need to check that (Ω, ϖ) is complete.

Setting $\phi^\alpha = (Y^\alpha_1, \cdots, Y^\alpha_n, \mathcal{V}^\alpha_1, \cdots, \mathcal{V}^\alpha_n, P^\alpha) \in \Omega$, $(Y^\alpha_1, \cdots, Y^\alpha_n, \mathcal{V}^\alpha_1, \cdots, \mathcal{V}^\alpha_n, P^\alpha) \longrightarrow (Y_1, \cdots, Y_n, \mathcal{V}_1, \cdots, \mathcal{V}_n, P)$, we need to prove $\phi = (Y_1, \cdots, Y_n, \mathcal{V}_1, \cdots, \mathcal{V}_n, P) \in \Omega$.

(1) Let $\phi^\alpha = (Y^\alpha_1, \cdots, Y^\alpha_n, \mathcal{V}^\alpha_1, \cdots, \mathcal{V}^\alpha_n, P^\alpha)$ be any Cauchy sequence in Ω. $\forall \varepsilon > 0$, there is a positive whole number $N(\varepsilon)$ such that $\varpi(\phi^\alpha, \phi^{\tilde{\alpha}}) < \varepsilon, \forall \alpha, \tilde{\alpha} \ge N(\varepsilon)$. On the one hand, $\forall i \in \mathbb{N}$, $\varepsilon > 0$ and $\tilde{\alpha} > 0$, when $\tilde{\alpha} > \alpha$, $\sup_{(a_i, u_i) \in \mathcal{A}_i \times \mathcal{U}_i} \| Y^\alpha_i(a_i, a_{-i}, u_i) - Y_i(a_i, a_{-i}, u_i) \| < \varepsilon/3$, thus $\sup_{(a'_i, u_i) \in \mathcal{A}_i \times \mathcal{U}_i} \| Y^{\tilde{\alpha}}_i(a'_i, a_{-i}, u_i) - Y_i(a'_i, a_{-i}, u_i) \| < \varepsilon/3$. We know that $Y^{\tilde{\alpha}}_i$ is R^l_+-continuous by means of Theorem 13 (1); then, there is $\delta > 0$, $\forall a_i, a'_i \in \mathcal{A}$; when $\| a_i - a'_i \| < \delta$, we obtain $\| Y^{\tilde{\alpha}}_i(a_i, a_{-i}, u_i) - Y^{\tilde{\alpha}}_i(a'_i, a_{-i}, u_i) \| < \varepsilon/3$. Similarly,

$$\begin{aligned}&\left\| Y_i(a_i, a_{-i}, u_i) - Y_i\left(a'_i, a_{-i}, u_i\right) \right\| \\ &= \left\| Y_i(a_i, a_{-i}, u_i) - Y^{\tilde{\alpha}}_i(a_i, a_{-i}, u_i) + Y^{\tilde{\alpha}}_i(a_i, a_{-i}, u_i) \right. \\ &\quad \left. - Y^{\tilde{\alpha}}_i\left(a'_i, a_{-i}, u_i\right) + Y^{\tilde{\alpha}}_i\left(a'_i, a_{-i}, u_i\right) - Y_i\left(a'_i, a_{-i}, u_i\right) \right\| \\ &\le \left\| Y_i(a_i, a_{-i}, u_i) - Y^{\tilde{\alpha}}_i(a_i, a_{-i}, u_i) \right\| + \left\| Y^{\tilde{\alpha}}_i(a_i, a_{-i}, u_i) \right. \\ &\quad \left. - Y^{\tilde{\alpha}}_i\left(a'_i, a_{-i}, u_i\right) \right\| + \left\| Y^{\tilde{\alpha}}_i\left(a'_i, a_{-i}, u_i\right) - Y_i\left(a'_i, a_{-i}, u_i\right) \right\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned} \tag{21}$$

Thus, Y_i is R^l_+-continuous on $\mathcal{A}$, $\forall (a_i, a'_i) \in \mathcal{A}_i \times \mathcal{A}_i$ and $\mathcal{V}_i$ is also R^l_+-continuous by proving the same method on $\mathcal{A}$. Meanwhile, $\forall i \in \mathbb{N}$ and $\varepsilon > 0$, there is a positive integer $N(\varepsilon)$ and $\forall \alpha, \tilde{\alpha} \ge N(\varepsilon)$, we obtain

$$\sup_{(a_i, a_{-i}) \in \mathcal{A}_i \times \mathcal{A}_{-i}} \mathcal{H}\left(P^\alpha(a_i, a_{-i}), P^{\tilde{\alpha}}(a_i, a_{-i})\right) < \varepsilon. \tag{22}$$

Then, $\forall i \in \mathbb{N}$, there is $P : \mathcal{A}_i \times \mathcal{A}_{-i} \longrightarrow 2^{\mathcal{B}}$ such that $\lim_{\tilde{\alpha} \longrightarrow \infty} P^{\tilde{\alpha}}(a_i, a_{-i}) = P(a_i, a_{-i})$, and $\forall \alpha \ge N(\varepsilon)$, we have

$$\sup_{(a_i, a_{-i}) \in \mathcal{A}_i \times \mathcal{A}_{-i}} \mathcal{H}(P^\alpha(a_i, a_{-i}), P(a_i, a_{-i})) \le \varepsilon. \tag{23}$$

Since the set-valued correspondence P^α is continuous on $\mathcal{A}$, it is easy to know that P is continuous on $\mathcal{A}$.

(2) Since $(a_i, u_i) \longrightarrow Y^\alpha_i(a_i, a_{-i}, u_i)$ is R^l_+- *quasiconcave-like*, $a_i \longrightarrow \mathcal{V}^\alpha(a_i, a'_i)$ is convex, fixing $a^1_i, a^2_i \in \mathcal{A}_i$ and $u^1_i, u^2_i, \nu_i \in \mathcal{U}_{-i}$, if $\forall \theta \in (0, 1)$, $\theta u^1_i + (1 - \theta) u^2_i \in \mathcal{U}_i$ holds, then $\forall i \in \mathbb{N}$, $\theta u^1_i + (1 - \theta) u^2_i \in P(\theta a^1_i + (1 - \theta) a^2_i, a_{-i})$, we have

$$\begin{aligned}&Y^\alpha_i\left(\theta a^1_i + (1 - \theta) a^2_i, a_{-i}, \theta u^1_i + (1 - \theta) u^2_i\right) \notin Y^\alpha_i\left(a'_i, a_{-i}, \nu_i\right) \\ &\quad - \mathcal{V}^\alpha_i\left(\theta a^1_i + (1 - \theta) a^2_i, a'_i\right) - \operatorname{int} R^l_+\end{aligned} \tag{24}$$

Since $Y^\alpha_i(a, u) \longrightarrow Y_i(a, u), \mathcal{V}^\alpha_i(a, a') \longrightarrow \mathcal{V}_i(a, a')(\alpha \longrightarrow \infty)$, $\forall a \in \mathcal{A}$, $\forall u \in \mathcal{U}$ and the strategy space is closed, we conclude that

$$Y_i\left(\theta a_i^1+(1-\theta)a_i^2, a_{-i}, \theta u_i^1+(1-\theta)u_i^2\right) \notin Y_i\left(a_i', a_{-i}, v_i\right) - \mathscr{V}_i\left(\theta a_i^1+(1-\theta)a_i^2, a_i'\right) - \operatorname{int} R_+^l. \tag{25}$$

This indicates that $\forall a_{-i} \in \mathscr{A}_{-i}$, $(a_i, u_i) \longrightarrow Y_i(a_i, a_{-i}, u_i)$ is R_+^l- *quasiconcave-like*and $a_i \longrightarrow \mathscr{V}(a_i, a'_i)$ is convex

(3) Since $\forall a_{-i} \in \mathscr{A}_{-i}, a_i \longrightarrow P^\alpha(a_i, a_{-i})$ is convex, $\forall a_i^1, a_i^2 \in \mathscr{A}_i$, $\theta \in (0,1)$, and $\varepsilon > 0$, we have

$$P^\alpha\left(\theta a_i^1+(1-\theta)a_i^2, a_{-i}\right)+\varepsilon \subset \theta P^\alpha\left(a_i^1, a_{-i}\right) + (1-\theta)P^\alpha\left(a_i^2, a_{-i}\right)+\varepsilon \tag{26}$$

When α is sufficiently large number, we have

$$\begin{aligned} P\left(\theta a_i^1+(1-\theta)a_i^2, a_{-i}\right) &\subset P^\alpha\left(\theta a_i^1+(1-\theta)a_i^2, a_{-i}\right)+\varepsilon, \\ \theta P^\alpha\left(a_i^1, a_{-i}\right)+(1-\theta)P^\alpha\left(a_i^2, a_{-i}\right)+\varepsilon &\subset \theta P\left(a_i^1, a_{-i}\right) + (1-\theta)P\left(a_{-i}^2, a_{-i}\right)+2\varepsilon. \end{aligned} \tag{27}$$

Thus,

$$P\left(\theta a_i^1+(1-\theta)a_i^2, a_{-i}\right) \subset \theta P\left(a_i^1, a_{-i}\right)+(1-\theta)P\left(a_i^2, a_{-i}\right)+2\varepsilon. \tag{28}$$

We take $\varepsilon \longrightarrow 0$ because ε is arbitrary, and we can obtain $P(\theta a_i^1+(1-\theta)a_i^2, a_{-i}) \subset \theta P(a_i^1, a_{-i})+(1-\theta)P(a_i^2, a_{-i})$. Hence, $\forall a_{-i} \in \mathscr{A}_{-i}, a_i \longrightarrow P(a_i, a_{-i})$ is convex on $\mathscr{A}$. In conclusion, $\phi = (Y_1, \cdots, Y_n, \mathscr{V}_1, \cdots, \mathscr{V}_n, P) \in \Omega$, and (Ω, ϖ) is a complete metric space. □

$\forall \phi \in \Omega$, we define $\Gamma : \Omega \longrightarrow 2^{\mathscr{A}_1\times\mathscr{U}_1\times\cdots\times\mathscr{A}_n\times\mathscr{U}_n}$, where $\Gamma(\phi) = \{(a_{-i}^*, a_{-i}^*, u_i^*) \in \mathscr{A}_i \times \mathscr{A}_{-i} \times \mathscr{U}_i : \forall i \in \mathbb{N}, u_i^* \in P(a_{-i}^*, a_{-i}^*)$, $Y_i(a'_i, a_{-i}^*, u_i) - Y_i(a_{-i}^*, a_{-i}^*, u_i^*) - \mathscr{V}_i(a_{-i}^*, a'_i) \notin \operatorname{int} R_+^l, \forall (a'_i, u_i) \in \mathscr{A}_i \times P(a'_i, a_{-i}^*)\}$. By Theorem 13, there is $(a_{-i}^*, a_{-i}^*, u_i^*) \in \mathscr{A}_i \times \mathscr{A}_{-i} \times \mathscr{U}_i$ such that $\Gamma(\phi) \neq \varnothing$. Then, Γ is also called an equilibrium mapping.

Next, we denote to verify the generic stability result of the WPNE with the strategy transformational barriers of the GMLMFMOG.

Lemma 16. *An equilibrium mapping $\Gamma : \Omega \longrightarrow 2^{\mathscr{A}_1\times\mathscr{U}_1\times\cdots\times\mathscr{A}_n\times\mathscr{U}_n}$ is a usco correspondence.*

Proof. By means of the compactness of $\mathscr{A}$ and Lemma 9, we need to demonstrate that the Γ is closed. In other words, if $\forall \phi^\beta = (Y_1^\beta, \cdots, Y_n^\beta, \mathscr{V}_1^\beta, \cdots, \mathscr{V}_n^\beta, P^\beta) \in \Omega$, $\phi^\beta \longrightarrow \phi = (Y_1, \cdots, Y_n, \mathscr{V}_1, \cdots, \mathscr{V}_n, P)$, $\forall (a_1^\beta, u_1^\beta, \cdots, a_n^\beta, u_n^\beta) \in \Gamma(\phi^\beta)$, $(a_1^\beta, u_1^\beta, \cdots, a_n^\beta, u_n^\beta) \longrightarrow (a_1^*, u_1^*, \cdots, a_n^*, u_n^*)$, then we only need to prove

$$(a_1^*, u_1^*, \cdots, a_n^*, u_n^*) \in \Gamma(\phi). \tag{29}$$

(1) Since $\mathscr{A}_i$ is compact, we assume that $a_i^\beta \longrightarrow a_{-i}^* \in \mathscr{A}_i$, P is continuous, $P(a_i^\beta, a_{-i}^\beta) \longrightarrow P(a_{-i}^*, a_{-i}^*)$, $u_i^\beta \in P^\beta(a_i^\beta, a_{-i}^\beta)$. Let d be the distance on $\mathscr{U}_i$; since $\phi^\beta \longrightarrow \phi$, $P^\beta \longrightarrow P$, and $u_i^\beta \longrightarrow u_i^*$, we have $d(u_i^*, P(a_{-i}^*, a_{-i}^*)) \le d(u_i^*, u_i^\beta) + d(u_i^\beta, P^\beta(a_i^\beta, a_{-i}^\beta)) + \mathscr{H}(P^\beta(a_i^\beta, a_{-i}^\beta), P(a_i^\beta, a_{-i}^\beta)) + \mathscr{H}(P(a_i^\beta, a_{-i}^\beta), P(a_{-i}^*, a_{-i}^*)) \longrightarrow 0$. Thus, $u_i^* \in P(a_{-i}^*, a_{-i}^*), \forall i \in \mathbb{N}$

(2) We verify that $\forall i \in \mathbb{N}$, $u_i^* \in P(a_{-i}^*, a_{-i}^*)$, and we have

$$Y_i\left(a_i', a_{-i}^*, u_i\right) - Y_i(a_{-i}^*, a_{-i}^*, u_i^*) - \mathscr{V}_i\left(a_{-i}^*, a_i'\right) \notin \operatorname{int} R_+^l, \forall\left(a_i', u_i\right) \in \mathscr{A}_i \times P\left(a_i', a_{-i}^*\right) \tag{30}$$

By contradiction, suppose that formula (30) is not true, then there is some $i \in \mathbb{N}$ such that $(a_i', u_i) \in \mathscr{A}_i \times P(a_i', a_{-i}^*)$, $Y_i(a_i', a_{-i}^*, u_i) - Y_i(a_{-i}^*, a_{-i}^*, u_i^*) - \mathscr{V}_i(a_{-i}^*, a_i') \in \operatorname{int} R_+^l$. Therefore, there exists some open neighbourhood V of the 0 element of R_+^l satisfying

$$Y_i\left(a_i', a_{-i}^*, u_i\right) - Y_i(a_{-i}^*, a_{-i}^*, u_i^*) - \mathscr{V}_i\left(a_{-i}^*, a_i'\right) + V \subset \operatorname{int} R_+^l. \tag{31}$$

Because $Y_i^\beta \longrightarrow Y_i$, there is a positive integer β_1 such that $\forall \beta \ge \beta_1$,

$$\left[Y_i^\beta\left(a_i, a_{-i}^\beta, v_i\right) - Y_i^\beta\left(a_i^\beta, a_{-i}^\beta, u_i^\beta\right) - \mathscr{V}_i^\beta\left(a_i^\beta, a_i\right)\right] - \left[Y_i\left(a_i, a_{-i}^\beta, v_i\right) - Y_i\left(a_i^\beta, a_{-i}^\beta, u_i^\beta\right) - \mathscr{V}_i^\beta\left(a_i^\beta, a_i\right)\right] \in \frac{1}{2}V. \tag{32}$$

Furthermore, since $(a_i', u_i) \in \mathscr{A}_i \times P(a_i', a_{-i}^*)$, $Y_i(a_i', a_{-i}^*, u_i) - Y_i(a_{-i}^*, a_{-i}^*, u_i^*) - \mathscr{V}(a_{-i}^*, a_i')$ is *lsc* at (a_i, a_{-i}, u_i) with $(a_i^\beta, a_{-i}^\beta) \longrightarrow (a_i^*, a_{-i}^*)$, there is a positive integer β_2 and $\beta_2 \ge \beta_1$ such that $\forall \beta \ge \beta_2$,

$$Y_i\left(a_i, a_{-i}^\beta, u_i\right) - Y_i\left(a_i^\beta, a_{-i}^\beta, u_i^\beta\right) - \mathscr{V}_i\left(a_i^\beta, a_i\right) \in Y_i(a_i, a_{-i}^*, u_i) - Y_i(a_i^*, a_{-i}^*, u_i^*) - \mathscr{V}_i(a_i^*, a_i) + \frac{1}{2}V + R_+^l. \tag{33}$$

Then, $\forall \beta \geq \beta_2$, and we can obtain that

$$
\begin{aligned}
&Y_i^\beta\left(a_i, a_{-i}^\beta, u_i\right) - Y_i^\beta\left(a_i^\beta, a_{-i}^\beta, u_i^\beta\right) - \mathscr{V}_i^\beta\left(a_i^\beta, a_i\right) \\
&= \left[Y_i^\beta\left(a_i, a_{-i}^\beta, u_i\right) - Y_i^\beta\left(a_i^\beta, a_{-i}^\beta, u_i^\beta\right) - \mathscr{V}_i^\beta\left(a_i^\beta, a_i\right)\right] \\
&\quad - \left[Y_i\left(a_i, a_{-i}^\beta, u_i\right) - Y_i\left(a_i^\beta, a_{-i}^\beta, u_i^\beta\right) - \mathscr{V}_i\left(a_i^\beta, a_i\right)\right] \\
&\quad + \left[Y_i\left(a_i, a_{-i}^\beta, u_i\right) - Y_i\left(a_i^\beta, a_{-i}^\beta, u_i^\beta\right) - \mathscr{V}_i\left(a_i^\beta, a_i\right)\right] \in \frac{1}{2}V \\
&\quad + Y_i(a_i, a_{-i}^*, u_i) - Y_i(a_i^*, a_{-i}^*, u_i^*) - \mathscr{V}_i(a_i^*, a_i) + \frac{1}{2}V + R_+^l \\
&= Y_i(a_i, a_{-i}^*, u_i) - Y_i(a_i^*, a_{-i}^*, u_i^*) - \mathscr{V}_i(a_i^*, a_i) + V \\
&\quad + R_+^l \subset \operatorname{int} R_+^l + R_+^l \subset \operatorname{int} R_+^l.
\end{aligned}
\tag{34}
$$

It is a contradiction with $(a_1^\beta, u_1^\beta, \cdots, a_n^\beta, u_n^\beta) \in \Gamma(\phi^\beta)$. Thus, we can obtain $(a_1^*, u_1^*, \cdots, a_n^*, u_n^*) \in \Gamma(\phi)$; i.e., Γ is a closed correspondence and Γ is a *usco* correspondence on Ω by means of Lemma 9.

Next, we define a set-valued map $\mathscr{T} : \mathscr{A}_1 \times \mathscr{U}_1 \times \cdots \times \mathscr{A}_n \times \mathscr{U}_n \longrightarrow \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$, wherein $\mathscr{T}(a_1, u_1, \cdots, a_n, u_n) = (a_1, \cdots, a_n) \in \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$, $\forall (a_1, u_1, \cdots, a_n, u_n) \in \mathscr{A}_1 \times \mathscr{U}_1 \times \cdots \times \mathscr{A}_n \times \mathscr{U}_n$. It is obvious that $\mathscr{T}$ is continuous on $\mathscr{A}_1 \times \mathscr{U}_1 \times \cdots \times \mathscr{A}_n \times \mathscr{U}_n$.

Finally, we define a set-valued mapping $\mathscr{F} = \mathscr{T}(\Gamma)$: $\Omega \longrightarrow 2^{\mathscr{A}_1 \times \cdots \times \mathscr{A}_n}$, where $\forall \phi \in \Omega$, $\mathscr{F}(\phi) = \mathscr{T}(\Gamma)(\phi)$ represents the set of WPNE with strategy transformational barriers for the GMLMFMOG. According to Theorem 13, $\Gamma(\phi) \neq \varnothing$, then $\mathscr{F}(\phi) = \mathscr{T}(\Gamma(\phi)) \neq \varnothing$. □

Lemma 17. *A set-valued mapping $\mathscr{F} = \mathscr{T}(\Gamma)$: $\Omega \longrightarrow 2^{\mathscr{A}_1 \times \cdots \times \mathscr{A}_n}$ is a usco correspondence.*

Proof. According to Lemma 16, $\Gamma : \Omega \longrightarrow 2^{\mathscr{A}_1 \times \mathscr{U}_1 \times \cdots \times \mathscr{A}_n \times \mathscr{U}_n}$ is *usc* on Ω, and $\Gamma(\phi)$ is compact $\forall \phi \in \Omega$. Since $\mathscr{T}$ is continuous on $\mathscr{A}_1 \times \mathscr{U}_1 \times \cdots \times \mathscr{A}_n \times \mathscr{U}_n$, it is obvious to check that $\mathscr{F} = \mathscr{T}(\Gamma)$: $\Omega \longrightarrow 2^{\mathscr{A}_1 \times \cdots \times \mathscr{A}_n}$ is also a *usco* correspondence on Ω. □

Definition 18.

(1) An equilibrium point $a \in \mathscr{A}$ of the game $\phi \in \Omega$ is referred to essential if for every $\mathscr{O}(a)$ of a, there is one $\mathscr{O}(\phi)$ of ϕ such that $\forall \phi' \in \mathscr{O}(\phi)$, and there exists at least an equilibrium point a' of ϕ' with $a' \in \mathscr{O}(a)$. If all equilibria points of the game $\phi \in \Omega$ are essential, then the game ϕ is an essential game

(2) A set $\tilde{m}(\phi)$ of the game $\phi \in \Omega$ is referred to essential set if for each open set O of $\mathscr{A}$ is associated with $\tilde{m}(\phi) \subset O$, and there is an $\varepsilon > 0$ satisfying $\forall \phi' \in \Omega$, $\varpi(\phi, \phi') < \varepsilon$, and $\mathscr{F}(\phi') \cap O = \varnothing$. Given that $\tilde{m}(\phi)$ is one minimal element in total essential sets of $\mathscr{F}(\phi)$ which are ordered by inclusion relations, then $\tilde{m}(\phi)$ is a minimal essential set

(3) $\forall \phi \in \Omega$, $\mathscr{F}(\phi)$ is composed of the union of the pairing of disjoint connected subsets [33], i.e.,

$$
\mathscr{F}(\phi) = \bigcup_{\kappa \in \mathscr{K}} C^\kappa(\phi), \tag{35}
$$

wherein $\mathscr{K}$ signifies one index set. Given a component $C^\kappa(\phi)$ of $\mathscr{F}(\phi)$ is essential, then $C^\kappa(\phi)$ is one essential set

Theorem 19. *$\forall \phi \in \Omega$, there is a dense Q in Ω such that Ω is essential.*

Proof. (Ω, ϖ) is complete by using Theorem 15, and $\mathscr{F} : \Omega \longrightarrow 2^{\mathscr{A}_1 \times \cdots \times \mathscr{A}_n}$ is a *usco* correspondence by means of Lemma 17. By Theorem 11 and Remark 12, $\mathscr{F}$ serves as *lsc* on one dense Q of Ω such that Ω is essential. □

Remark 20. By Theorem 19, we proved that most of $\phi \in \Omega$ have a stable solution set in the dense Q of Ω on the meaning of Baire's category.

5. Essential Component

In this paragraph, we derive the essential component results of the WPNE with the strategy transformation barrier solution sets of the GMLMFMOG.

Theorem 21. *$\mathscr{F}(\phi)$ encompasses at least one minimal essential set $\forall \phi \in \Omega$, where $\mathscr{F} : \Omega \longrightarrow 2^{\mathscr{A}_1 \times \cdots \times \mathscr{A}_n}$.*

Proof. For $\phi \in \Omega$, $\mathscr{F} : \Omega \longrightarrow 2^{\mathscr{A}_1 \times \cdots \times \mathscr{A}_n}$ is *usco* mapping by Lemma 17, and then, $\mathscr{F}(\phi)$ is one essential set of itself. Suppose that $\mathbb{E}$ is the collections of all essential sets of $\mathscr{F}(\phi)$, which is defined by the set inclusion order relation, we obtain $\mathbb{E} \neq \varnothing$. Assume that any total order subset be $\{e_\gamma(\phi) : \gamma \in \mathscr{K}\}$ on $\mathscr{A}$, where $\mathscr{K}$ denote the index set. Let $e(\phi) = \bigcap_{\gamma \in \mathscr{K}} e_\gamma(\phi)$, then $e(\phi)$ serves as compact. If $e(\phi) = \varnothing$, then $\mathscr{F}(\phi) = \mathscr{F}(\phi) \setminus e(\phi) = \bigcup_{\gamma \in \mathscr{K}} [\mathscr{F}(\phi) \setminus e_\gamma(\phi)]$. Note that $\mathscr{F}(\phi) \setminus e_\gamma(\phi)$ is one open set as well as $\mathscr{F}(\phi)$ is compact, then there are $e_1(\phi), e_2(\phi), \cdots, e_n(\phi)$ such that $\mathscr{F}(\phi) = \bigcup_{i=1}^n [\mathscr{F}(\phi) \setminus e_i(\phi)]$ by using the open covering theorem. It is obvious that $\bigcap_{i=1}^n e_i(\phi) = \varnothing$ from $\mathscr{F}(\phi) = \bigcup_{i=1}^n [\mathscr{F}(\phi) \setminus e_i(\phi)] = \mathscr{F}(\phi) \setminus \bigcap_{i=1}^n e_i(\phi)$. It means that $\bigcap_{i=1}^n e_i(\phi) = \varnothing$ is in contradiction with $\bigcap_{i=1}^n e_i(\phi) \neq \varnothing$. Thus, $e(\phi) \neq \varnothing$. Given any open set O with $e(\phi) \subset O$, if $\forall \gamma \in \mathscr{K}$, there exists $a_\gamma \in e_\gamma(\phi) \subset \mathscr{F}(\phi)$ with $a_\gamma \notin O$; then, we can assume that $a_\gamma \longrightarrow a \in \mathscr{F}(\phi)$. Because $\forall \gamma \in \mathscr{K}, e_\gamma(\phi)$ is compact and $\{e_\gamma(\phi)\}_{\gamma \in \mathscr{K}}$ is totally order set, then $a_{\gamma_1} \in e_\gamma(\phi)$ when $\gamma_1 > \gamma$ and $a \in e_\gamma(\phi), \forall \gamma \in \mathscr{K}$. Hence, $a \in \bigcap_{\gamma \in \mathscr{K}} e_\gamma(\phi) = e(\phi) \subset O$, which contradicts with $a_\gamma \longrightarrow a$ and $a_\gamma \notin O, \forall \gamma \in \mathscr{K}$. Therefore, there exists $a_{\gamma_0} \in \mathscr{K}$ such that $e_{\gamma_0}(\phi) \subset O$. Since $e_{\gamma_0}(\phi)$ is an essential set of $\mathscr{F}(\phi)$, $\forall \varepsilon > 0$, there is $\delta > 0$ such that $\phi^1 \in \Omega$ with $\varpi(\phi, \phi^1) < \delta$, $\mathscr{F}(a') \bigcap O \neq \varnothing$ with $\|a - a'\| < \varepsilon$, $\forall a' \in \mathscr{F}(\phi^1)$. Thus, $e(\phi)$ is essential, and there must be a lower bound of $\{e_\gamma(\phi$

): $\gamma \in \mathscr{K}\}$ in $\mathbb{E}$. According to Zorn's lemma, there is one minimal element $\tilde{m}(\phi)$ in $\mathbb{E}$ such that $\mathscr{F}(\phi)$ includes at least one minimal essential set $\tilde{m}(\phi)$. □

Theorem 22. *$\forall \phi \in \Omega$, each minimal essential set of $\mathscr{F}(\phi)$ is connected.*

Proof. Let $\tilde{m}(\phi)$ be a minimum essential set of $\mathscr{F}(\phi)$. By contradiction, we assume that $\tilde{m}(\phi)$ is disconnected. There are two not empty closed sets $\tilde{c}_1(\phi)$ and $\tilde{c}_2(\phi)$ with $\tilde{m}(\phi) = \tilde{c}_1(\phi) \cap \tilde{c}_2(\phi)$, as well as two disjoint open sets O_1 and O_2 with $O_1 \cap O_2 = \varnothing$ such that $\tilde{c}_1(\phi) \subset O_1$ and $\tilde{c}_2(\phi) \subset O_2$.

Since $\tilde{m}(\phi)$ is the minimum essential set, $\tilde{c}_1(\phi)$ and $\tilde{c}_2(\phi)$ are not essential set for $\mathscr{F}(\phi)$. Therefore, there exist two open sets, namely, D_1 and D_2, with $\tilde{c}_1(\phi) \subset D_1$ and $\tilde{c}_2(\phi) \subset D_2$ such that $\forall \delta > 0$, $\phi^1, \phi^2 \in \Omega$; we obtain $\varpi(\phi, \phi^1) < \delta$ and $\varpi(\phi, \phi^2) < \delta$, but $\mathscr{F}(\phi) \cap D_1 = \varnothing$, $\mathscr{F}(\phi) \bigcap D_2 = \varnothing$. Suppose that $U_1 = O_1 \cap D_1$ and $U_2 = O_2 \cap D_2$ are open sets and that $\tilde{c}_1(\phi) \subset U_1$ and $\tilde{c}_2(\phi) \subset U_2$. Beacuse $\tilde{c}_1(\phi)$ and $\tilde{c}_2(\phi)$ are compact, there are two open sets, namely, Z_1 and Z_2, such that $\tilde{c}_1(\phi) \subset Z_1 \subset \bar{Z}_1 \subset U_1$ and $\tilde{c}_2(\phi) \subset Z_2 \subset \bar{Z}_2 \subset U_2$. Since $\tilde{m}(\phi)$ is one essential set of $\mathscr{F}(\phi)$ and $\tilde{m}(\phi) \subset Z_1 \cup Z_2$, there is $\delta' > 0$ such that $\forall \bar{\phi} \in \Omega$ with $\varpi(\phi, \bar{\phi}) < \delta'$, and

$$\mathscr{F}(\bar{\phi}) \cap (Z_1 \cup Z_2) \neq \varnothing. \tag{36}$$

Since $Z_1 \subset O_1$ and $Z_2 \subset O_2$, there exist $\psi^1 \in \Omega$ and $\psi^2 \in \Omega$ such that $\varpi(\phi, \psi^1) < \delta'/2$ and $\varpi(\phi, \psi^2) < \delta'/2$ with $\mathscr{F}(\psi^1) \cap Z_1 = \varnothing$ and $\mathscr{F}(\psi^2) \cap Z_2 = \varnothing$.

We define a GMLMFMOG with strategy transformational barrier $\psi = (\mathscr{A}_i^3, Y_i^3, \mathscr{V}_i^3, P^3)_{i \in \mathbb{N}}$ by a linear combination function between $\psi^1 = (\mathscr{A}_i^1, Y_i^1, \mathscr{V}_i^1, P^1)_{i \in \mathbb{N}}$ and $\psi^1 = (\mathscr{A}_i^2, Y_i^2, \mathscr{V}_i^2, P^2)_{i \in \mathbb{N}}$ as follows:

$$\begin{aligned} Y_i^3(a, u) &= v(a) Y_i^1(a, u) + u(a) Y_i^2(a, u), \\ \mathscr{V}_i^3\left(a_i, a_i'\right) &= v(a) \mathscr{V}_i^1\left(a_i, a_i'\right) + u(a) \mathscr{V}_i^2\left(a_i, a_i'\right), \\ P^3(a_i, a_{-i}) &= v(a) \mathscr{H}\left(P^1(a_i, a_{-i}), P^2(a_i, a_{-i})\right) \\ &\quad + u(a) \mathscr{H}\left(P^2(a_i, a_{-i}), P^2(a_i, a_{-i})\right), \end{aligned} \tag{37}$$

where

$$\begin{aligned} v(a) &= \frac{\hbar(a, \bar{Z}_2)}{\hbar(a, \bar{Z}_1) + \hbar(a, \bar{Z}_2)}, \\ u(a) &= \frac{\hbar(a, \bar{Z}_1)}{\hbar(a, \bar{Z}_1) + \hbar(a, \bar{Z}_2)}, \end{aligned} \tag{38}$$

and $\hbar$ represents the distance function on $\mathscr{A}$. Note that $v(a)$ and $u(a)$ are continuous and nonnegative; furthermore, $v(a) + u(a) = 1, \forall a \in \mathscr{A}$.

We can check that $\psi = (\mathscr{A}_i^3, Y_i^3, \mathscr{V}_i^3, P^3)_{i \in \mathbb{N}} \in \Omega$. Noting that

$$\begin{aligned} \varpi(\phi, \psi) &= \sup_{(a, u_i) \in \mathscr{A} \times \mathscr{U}_i} \sum_{i=1}^{n} \left\| Y_i(a, u_i) - Y_i^3(a, u_i) \right\| \\ &\quad + \sup_{\left(a_i, a_i'\right) \in \mathscr{A}_i \times \mathscr{A}_i} \sum_{i=1}^{n} \left\| \mathscr{V}_i\left(a_i, a_i'\right) - \mathscr{V}_i^3\left(a_i, a_i'\right) \right\| \\ &\quad + \sup_{(a_i, a_{-i}) \in \mathscr{A}_i \times \mathscr{A}_{-i}} \mathscr{H}\left(P(a_i, a_{-i}), P^3(a_i, a_{-i})\right), \\ &= \sup_{(a, u_i) \in \mathscr{A} \times \mathscr{U}_i} \sum_{i=1}^{n} \| v(a) Y_i(a, u_i) \\ &\quad + u(a) Y_i(a, u_i) \quad - v(a) Y_i^1(a, u_i) - u(a) Y_i^2(a, u_i) \| \\ &\quad + \sup_{\left(a_i, a_i'\right) \in \mathscr{A}_i \times \mathscr{A}_i} \sum_{i=1}^{n} \Big\| v(a) \mathscr{V}_i\left(a_i, a_i'\right) + u(a) \mathscr{V}_i\left(a_i, a_i'\right) \\ &\quad - v(a) \mathscr{V}_i^1\left(a_i, a_i'\right) - u(a) \mathscr{V}_i^2\left(a_i, a_i'\right) \Big\| \\ &\quad + \sup_{(a_i, a_{-i}) \in \mathscr{A}_i \times \mathscr{A}_{-i}} \left(v(a) \mathscr{H}\left(P(a_i, a_{-i}), P^1(a_i, a_{-i})\right) \right. \\ &\quad \left. + u(a) \mathscr{H}\left(P(a_i, a_{-i}), P^2(a_i, a_{-i})\right) \right) \\ &\leq \varpi\left(w, \psi^1\right) + \varpi\left(w, \psi^2\right) < \frac{\delta'}{2} + \frac{\delta'}{2} = \delta', \end{aligned} \tag{39}$$

we obtain $\mathscr{F}(\psi) \cap (Z_1 \cup Z_2) \neq \varnothing$ since $\varpi(\phi, \psi) < \delta'$. Next, we assume that $\mathscr{F}(\psi) \cap Z_1 \neq \varnothing$; then there exists $a' \in \mathscr{F}(\psi) \cap Z_1$. By $a' \in Z_1$, we attain $w(a') = 1$, $u(a') = 0$, $Y_i^3(a, u_i) = Y_i^1(a, u_i)$, $\mathscr{V}_i^3(a_i, {a'}_i) = \mathscr{V}_i^1(a_i, {a'}_i)$, and $P^3(a_i, a_{-i}) = P^1(a_i, a_{-i})$. Then, we obtain $a' \in \mathscr{F}(\psi)$, which implies

$$Y_i^3\left(a_i', a_{-i}^*, u_i\right) - Y_i^3(a_{-i}^*, a_{-i}^*, u_i^*) - \mathscr{V}_i\left(a_{-i}^*, a_i'\right) \notin \operatorname{int} R_+^l, \forall \left(a_i', u_i\right) \in \mathscr{A}_i \times P\left(a_i', a_{-i}^*\right). \tag{40}$$

Thus, $a' \in \mathscr{F}(\psi^1)$. This contradicts the fact that $\mathscr{F}(\psi^1) \cap Z_1 = \varnothing$. Then, $\tilde{m}(\phi)$ is connected. □

Theorem 23. *$\forall \phi \in \Omega$, if there exists $\mathscr{F}(\phi) = \{a\}$ (single point set), then ϕ is essential.*

Theorem 24. *$\forall \phi \in \Omega$, there is at least an essential connected component of $\mathscr{F}(\phi)$.*

Proof. According to Theorems 21 and 22, $\mathscr{F}(\phi)$ encompasses at least a minimum essential set $\tilde{m}(\phi)$ and $\tilde{m}(\phi)$ is connected. Aiming at a component $C^\kappa(\phi)$ of $\mathscr{F}(\phi)$ as well as $\tilde{m}(\phi) \subset C^\kappa(\phi)$, we obtain that $C^\kappa(\phi)$ is one essential connected component of $\mathscr{F}(\phi)$ by Definition 18 (3). □

6. Summaries

In this paper, we have investigated a new generalized multileader multifollower multiple objective game (GMLMFMOG) model with strategy transformational barriers and obtained some new stability results of the WPNE with the strategy transformational barriers for the GMLMFMOG. Furthermore, we have proved the existence of the WPNE with the strategy transformational barriers of the GMLMFMOG and studied its generic stability. In fact, we have obtained that most of the WPNE with the strategy transformational barriers of the GMLMFMOG serve as essential on the meaning of Baire's category. In addition, we have demonstrated that there is at least an essential connected component of the GMLMFMOG with the strategy transformational barriers. These results extend the corresponding results obtained in reference [27] by introducing strategy transformational barrier function into the decision-making behaviour of players.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nos. 12061020 and 71961003), the Science and Technology Foundation of Guizhou Province (Grant Nos. 20201Y284, 20205016, 2021088, and 20215640), and the Foundation of Guizhou University (Grant Nos. 201405 and 201811). The authors acknowledge these supports.

References

[1] M. A. Jones, D. L. Mothersbaugh, and S. E. Beatty, "Switching barriers and repurchase intentions in services," *Journal of Retailing*, vol. 76, no. 2, pp. 259–274, 2000.

[2] C. C. Yin, C. Y. Kam, and S. Ying, "Convexity of ruin probability and optimal dividend strategies for a general Lévy process," *The Scientific World Journal*, vol. 2014, Article ID 354129, 9 pages, 2014.

[3] J. L. Pérez and K. Yamazaki, "On the optimality of periodic barrier strategies for a spectrally positive Lévy process," *Insurance: Mathematics and Economics*, vol. 77, no. 11, pp. 1–13, 2017.

[4] K. Noba, "On the optimality of double barrier strategies for Lévy processes," *Stochastic Processes and their Applications*, vol. 131, pp. 73–102, 2021.

[5] T. E. Mofokeng, "Switching costs, customer satisfaction, and their impact on marketing ethics of medical schemes in South Africa: an enlightened marketing perspective," *Cogent Business and Management*, vol. 7, no. 1, article 1811000, 2020.

[6] I. L. Glicksberg, "A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points," *Proceeding of the American Mathematical Society*, vol. 3, no. 1, pp. 170–174, 1952.

[7] A. Mas-Colell, "An equilibrium existence theorem without complete or transitive preferences," *Journal of Mathematical Economics*, vol. 1, no. 3, pp. 237–246, 1974.

[8] M. K. Fort, "Essential and non essential fixed points," *American Journal of Mathematics*, vol. 72, no. 2, pp. 315–322, 1950.

[9] W. T. Wu and J. H. Jiang, "Essential equilibrium points of n -person noncooperative games," *Science in China Series A*, vol. 10, pp. 7–22, 1962.

[10] J. Yu and Q. Luo, "On essential components of the solution set of generalized games," *Journal of Mathematical Analysis and Applications*, vol. 230, no. 2, pp. 303–310, 1999.

[11] J. Yu, "Essential equilibria of n-person noncooperative games," *Journal of Mathematical Economics*, vol. 31, no. 3, pp. 361–372, 1999.

[12] V. Scalzo, "Essential equilibria of discontinuous games," *Economic Theory*, vol. 54, no. 1, pp. 27–44, 2013.

[13] V. Scalzo, "Remarks on the existence and stability of some relaxed Nash equilibrium in strategic form games," *Economic Theory*, vol. 61, no. 3, pp. 571–586, 2016.

[14] O. Carbonell-Nicolau, "Further results on essential Nash equilibria in normal-form games," *Economic Theory*, vol. 59, no. 2, pp. 277–300, 2015.

[15] O. Carbonell-Nicolau and N. Wohl, "Essential equilibrium in normal-form games with perturbed actions and payoffs," *Journal of Mathematical Economics*, vol. 75, no. 75, pp. 108–115, 2018.

[16] Z. Yang and H. J. Zhang, "Essential stability of cooperative equilibria for population games," *Optimization Letters*, vol. 13, no. 7, pp. 1573–1582, 2019.

[17] H. Yang and J. Yu, "Essential components of the set of weakly Pareto-Nash equilibrium points," *Applied Mathematics Letters*, vol. 15, no. 5, pp. 553–560, 2002.

[18] Z. Lin, "Essential components of the set of weakly Pareto-Nash equilibrium points for multi-objective generalized games in two different topological spaces," *Journal of Optimization Theory and Applications*, vol. 124, no. 2, pp. 387–405, 2005.

[19] J. C. Chen and X. H. Gong, "The stability of set of solutions for symmetric vector quasi-equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 136, no. 3, pp. 359–374, 2008.

[20] S. Xiang and Y. Zhou, "On essential sets and essential components of efficient solutions for vector optimization problems," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 1, pp. 317–326, 2006.

[21] L. S. Shapley and F. D. Rigby, "Equilibrium points in games with vector payoffs," *Naval Research Logistics*, vol. 6, no. 1, pp. 57–61, 1959.

[22] J. S. Pang and M. Fukushima, "Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games," *Computational Management Science*, vol. 2, no. 1, pp. 21–56, 2005.

[23] H. D. Sherali, "A multiple leader Stackelberg model and analysis," *Operations Research*, vol. 32, no. 2, pp. 390–404, 1984.

[24] A. A. Kulkarni and U. V. Shanbhag, "A shared-constraint approach to multi-leader multi-follower games," *Set-valued and Variational Analysis*, vol. 22, no. 4, pp. 691–720, 2014.

[25] J. Yu and H. L. Wang, "An existence theorem for equilibrium points for multi-leader-follower games," *Nonlinear Analysis*, vol. 69, no. 5-6, pp. 1775–1777, 2008.

[26] Z. Yang and Y. Ju, "Existence and generic stability of cooperative equilibria for multi-leader-multi-follower games," *Journal of Global Optimization*, vol. 65, no. 3, pp. 563–573, 2016.

[27] W. S. Jia, S. W. Xiang, J. H. He, and Y. L. Yang, "Existence and stability of weakly Pareto-Nash equilibrium for generalized multiobjective multi-leader-follower games," *Journal of Global Optimization*, vol. 61, no. 2, pp. 397–405, 2015.

[28] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley Sons Inc, New York, 1984.

[29] J. Yu, *Fifteen Lectures on Game Theory*, Science Press, Beijing, 2020.

[30] P. Deguire, K. K. Tan, and X. Z. Yuan, "The study of maximal elements, fixed points for L_s-majorized mapping and their applications to minimax and variational inequalities in the product topological spaces," *Nonlinear Analysis Theory Methods and Applications*, vol. 37, pp. 933–951, 1999.

[31] J. D. Michler and B. M. Gramig, "Differentiation in a two-dimensional market with endogenous sequential entry," 2012, https://arxiv.org/abs/2103.11051.

[32] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, Springer, Berlin Heidelberg New York, 2006.

[33] S. Willard, "General topology," in *The Mathematical Gazette*, Dover Publications, Inc., Mineola, New York, 1954.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 4483348, 17 pages
https://doi.org/10.1155/2022/4483348

Research Article

An Existence Study on the Fractional Coupled Nonlinear q-Difference Systems via Quantum Operators along with Ulam–Hyers and Ulam–Hyers–Rassias Stability

Shahram Rezapour,[1,2,3] **Chatthai Thaiprayoon**,[4] **Sina Etemad**,[5] **Weerawat Sudsutad**,[6] **Chernet Tuge Deressa**,[7] **and Akbar Zada**[8]

[1]*Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam*
[2]*Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Vietnam*
[3]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[4]*Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131, Thailand*
[5]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[6]*Theoretical and Applied Data Integration Innovations Group, Department of Statistics, Faculty of Science, Ramkhamhaeng University, Bangkok 10240, Thailand*
[7]*College of Natural Sciences, Department of Mathematics, Jimma University, Jimma, Ethiopia*
[8]*Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan*

Correspondence should be addressed to Chernet Tuge Deressa; chernet.deressa@ju.edu.et

Received 1 June 2022; Accepted 29 August 2022; Published 24 September 2022

Academic Editor: Anita Tomar

In this paper, we study the existence of solutions and their uniqueness and different kinds of Ulam–Hyers stability for a new class of nonlinear Caputo quantum boundary value problems. Also, we investigate such properties for the relevant generalized coupled q-system involving fractional quantum operators. By using the Banach contraction principle and Leray-Schauder's fixed–point theorem, we prove the existence and uniqueness of solutions for the suggested fractional quantum problems. The Ulam–Hyers stability of solutions in different forms are studied. Finally, some examples are provided for both q-problem and coupled q-system to show the validity of the main results.

1. Introduction

Fractional calculus is one of the most important fields in applied mathematics. In recent years, many researchers have studied different branches of this theory and conducted numerous analyses analytically and numerically. Particularly, in recent decades, we can see some papers on the applications of fixed-point theorems to prove the existence of solutions of fractional boundary value problems [1–4]. Because of the quick developments in fractional calculus, many mathematicians discussed on the theory of q-calculus that is an equivalent of traditional calculus without defining the concept of limit, and also the parameter q refers to quantum. This theory was originally developed by Jackson [5, 6], and it includes many practical aspects in the fields of hypergeometric series, theory of relativity, particle physics, discrete mathematics, quantum mechanics, combinatorics, and complex analysis. For a fundamental introduction of the basic notions of q-calculus, one can refer to [7–9]. In the early years, for finding positive solutions of given q-difference equations in the nonlinear settings, we lead you to study a work published by both El-Shahed and Al-Askar [10] and also a manuscript by Graef and Kong [11].

So later, various mathematical q-difference fractional models of IVPs and BVPs have been presented in which different methods like the lower-upper solutions technique, fixed-point results, and iterative methods have been implemented. For instance, we see q-intego-equation on time scales in [12], q-delay equations in [13], q-integro-equations under the q-integral conditions in [14], singular q-equations in [15], q-sequential symmetric BVPs in [16], q-difference equations having p-Laplacian in [17], four-point q-BVP with different orders in [18], oscillation on q-difference inclusions in [19], etc.

Here, we apply similar techniques to discuss the existence property of solutions for given q-integro-sum-difference FBVPs depending on the quantum operators. This shows an application of fixed-point theory in relation to q-difference theory. This specifies the main contribution of the present reseach.

In 2014, Ahmad et al. [20] studied a q-sequential equation in the nonlinear case via four-point q-integral conditions given by

$$\begin{cases} {}_q^C\mathbb{D}_{0^+}^{k_1}\left({}_q^C\mathbb{D}_{0^+}^{k_2}+\sigma\right)u(r)=G(r,u(r)), & (r\in[0,1], q\in(0,1)), \\ u(0)=e_{1q}\mathbb{I}_{0^+}^{s-1}u(b_1), & u(1)=e_{2q}\mathbb{I}_{0^+}^{s-1}u(b_2), \end{cases} \tag{1}$$

so that $k_1,k_2\in(0,1)$, $b_1,b_2\in(0,1)$, $s>2$, and $\sigma,e_1,e_2\in\mathbb{R}$. As well as, $G:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous, and ${}_q\mathbb{I}_{0^+}^{s-1}$ indicates the q-RL-integral. These mathematicians extracted different qualitative aspects of solutions for the above q-FBVP by means of the classical methods which are available in fixed-point theory.

In 2015, Etemad et al. [21] focused on the new four-point three-term q-difference FBVP

$$\begin{aligned} &\left({}_q^C\mathbb{D}_{0^+}^{\rho}u\right)(r)=G\left(r,u(r),{}_q^C\mathbb{D}_{0^+}^{1}u(r)\right), 0<q<1, \\ &c_1u(0)+d_{1q}{}^C\mathbb{D}_{0^+}^{1}u(0)=b_{1q}\mathbb{I}_{0^+}^{\alpha}u(k_1)=b_1\int_0^{k_1}\frac{(k_1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)}u(z)d_qz, \\ &c_2u(1)+d_{2q}{}^C\mathbb{D}_{0^+}^{1}u(1)=b_{2q}\mathbb{I}_{0^+}^{\alpha}u(k_2)=b_2\int_0^{k_2}\frac{(k_2-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)}u(z)d_qz, \end{aligned} \tag{2}$$

where $0\le r\le1$, $1<\rho\le2$, $\alpha\in(0,2]$, $c_1,c_2,d_1,d_2,b_1,b_2\in\mathbb{R}$, and $k_1,k_2\in(0,1)$ with $k_1<k_2$.

In 2019, two mathematicians named Ntouyas and Samei [22] devoted their attention to investigate the existence property about the multiterm q-integro-difference FBVP

$$\begin{aligned} &{}_q^C\mathbb{D}_{0^+}^{\rho}u(r)=G\left(r,u(r),(\hbar_1u)(r),(\hbar_2u)(r),{}_q^C\mathbb{D}_{0^+}^{\rho_1}u(r),{}_q^C\mathbb{D}_{0^+}^{\rho_2}u(r),\cdots,{}_q^C\mathbb{D}_{0^+}^{\rho_m}u(r)\right), \\ &u(0)+b_1u(1)=0, u'(0)+b_2u'(1)=0 \end{aligned} \tag{3}$$

where $r\in[0,1]$, $q\in(0,1)$, $\rho\in(1,2)$, $\rho_i\in(0,1)$ with $i=1,2,\cdots,m$, $b_1,b_2\neq-1$, $\hbar_j$ are formulated as

$$(\hbar_ju)(r)=\int_0^r v_j(r,z)u(z)\,\mathrm{d}_qz, \tag{4}$$

for $j=1,2$ and $G:[0,1]\times\mathbb{R}^{m+3}\longrightarrow\mathbb{R}$ is continuous with respect to all variables [22].

In 2020, Phuong et al. [23] formulated a novel extended configuration of the Caputo q-multi-integro-difference equation with two nonlinearity under q-multi-order-integrals conditions

$$\begin{aligned} &\left(m_q^C\mathbb{D}_{0^+}^{\rho}-(m+1)_q\mathbb{I}_{0^+}^{k_1}-(m+2)_q\mathbb{I}_{0^+}^{k_2}\right)u(r) \\ &\quad=b_{1q}\mathbb{I}_{0^+}^{k_3}G_1(r,u(r))+b_{2q}\mathbb{I}_{0^+}^{k_4}G_2(r,u(r)), \\ &u(0)=0, n_q\mathbb{I}_{0^+}^{p_1}u(1)+(n+1)_q\mathbb{I}_{0^+}^{p_2}u(1)+(n+2)_q\mathbb{I}_{0^+}^{p_3}u(1)=0, \end{aligned} \tag{5}$$

where $r\in[0,1]$, $\rho\in(1,2)$, $k_1,k_2,k_3,k_4\in(0,1)$, $p_1,p_2,p_3,m,n>0$, and $b_1,b_2\in\mathbb{R}^{\ge0}$.

In this paper, inspired by above q-problems, we analyze a structure of the nonlinear Caputo quantum difference fractional boundary problem (or q-CFBVP) in the form

$$\begin{aligned} &{}_q^C\mathfrak{D}_{0^+}^{\varsigma}\mu(r)=G\left(r,\mu(r),{}_q^R\mathfrak{J}_{0^+}^{\omega}\mu(r)\right)\coloneqq\mathcal{G}_\mu(r), \quad (r\in\mathcal{O}\coloneqq[0,1], q\in(0,1)), \\ &\mu(0)+\mu(\zeta)=\sum_{j=1}^k\alpha_{jq}{}^R\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \\ &{}_q^C\mathfrak{D}_{0^+}^{\varrho}\mu(0)+{}_q^C\mathfrak{D}_{0^+}^{\varrho}\mu(\zeta)=\sum_{j=1}^k\beta_{jq}{}^R\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \\ &{}_q^C\mathfrak{D}_{0^+}^{2}\mu(0)+{}_q^C\mathfrak{D}_{0^+}^{2}\mu(\zeta)=\sum_{j=1}^k\gamma_{jq}{}^R\mathfrak{J}_{0^+}^{\varrho}\left[{}_q^C\mathfrak{D}_{0^+}^{2}\mu(1)\right], \end{aligned} \tag{6}$$

where $\varsigma\in(2,3)$, $\varrho\in(1,2)$, $\zeta\in(0,1)$, $\alpha_j,\beta_j,\gamma_j\in\mathbb{R}^{>0}$, $\omega,\sigma_j>0$ for $j=1,2,\cdots,k$, and $G:\mathcal{O}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous. As the same way, the operators ${}_q^C\mathfrak{D}_{0^+}^{(\cdot)}$ and ${}_q\mathfrak{J}_{0^+}^{(\cdot)}$ denote the q-Caputo derivative and the q-RL integral, respectively. In the direction of the above problem, we consider a coupled system of nonlinear q-CFBVPs with the same q-boundary conditions. In other words, the mentioned fractional q-system is organized as

$$\begin{aligned} &{}_q^C\mathfrak{D}_{0^+}^{\varsigma_1}\mu(r)=G_1\left(r,\vartheta(r),{}_q^R\mathfrak{J}_{0^+}^{\omega_1}\vartheta(r)\right): \mathcal{U}_\vartheta(r), \quad (r\in\mathcal{O}, q\in(0,1)), \\ &{}_q^C\mathfrak{D}_{0^+}^{\varsigma_2}\vartheta(r)=G_2\left(r,\mu(r),{}_q^R\mathfrak{J}_{0^+}^{\omega_2}\mu(r)\right)\coloneqq\mathcal{V}_\mu(r), \\ &\mu(0)+\mu(\zeta)=\sum_{j=1}^k\alpha_{jq}{}^R\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \\ &\vartheta(0)+\vartheta(\zeta)=\sum_{j=1}^k\phi_{jq}{}^R\mathfrak{J}_{0^+}^{\delta_j}\vartheta(1), \\ &{}_q^C\mathfrak{D}_{0^+}^{\varrho}\mu(0)+{}_q^C\mathfrak{D}_{0^+}^{\varrho}\mu(\zeta)=\sum_{j=1}^k\beta_{jq}{}^R\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \end{aligned}$$

$$
\begin{aligned}
&{}^{C}_{q}\mathfrak{D}^{\rho}_{0^+}\vartheta(0) + {}^{C}_{q}\mathfrak{D}^{\rho}_{0^+}\vartheta(\zeta) = \sum_{j=1}^{k}\varphi_{j\,q}{}^{R}\mathfrak{J}^{\delta_j}_{0^+}\vartheta(1),\\
&{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta) = \sum_{j=1}^{k}\gamma_{j\,q}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\right],\\
&{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\vartheta(0) + {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\vartheta(\zeta) = \sum_{j=1}^{k}\eta_{j\,q}{}^{R}\mathfrak{J}^{\delta_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\vartheta(1)\right],
\end{aligned}
\tag{7}
$$

where $\varsigma_1, \varsigma_2 \in (2,3)$, $\varrho, \rho \in (1,2)$, $\zeta \in (0,1)$, $\alpha_j, \beta_j, \gamma_j, \phi_j, \varphi_j, \eta_j \in \mathbb{R}^{>0}$, $\omega_1, \omega_2, \sigma_j, \delta_j > 0$ for $j = 1, 2, \cdots, k$, and $G_1, G_2 : \mathcal{O} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous.

In other words, we extend our q-CFBVP to a coupled q-difference system and derive the existence and stability results on such a generalized coupled q-CFBVP system. In fact, a large number of researchers have devoted their concentration to the discussion on various categories of Ulam-Hyers stabilities for standard systems of FDEs (or refer to [24, 25]), while a few articles can be found in the literature in which the researchers developed the relevant existence and stability theory in relation to nonlinear fractional q-difference systems.

The present work is assembled as follows: In Section 2, we state some basic materials required to prove our theoretical results. In both Section 3 and Section 4, several criteria and conditions are presented for the desired uniqueness-existence results, along with different classes of stabilities in relation to the proposed q-CFBVPs (6) and (7), respectively, with the help of some known fixed–point theorems. A simulative example, to represent the consistency of our results, is given with each suggested q-CFBVP in the relevant section. We give Section 6 to the presentation of the conclusion of this research work.

2. Preliminaries

The basic notions of q-calculus are collected in this section by assuming $q \in (0,1)$. The q-analogue of $(a_1 - a_2)^k$ is given by

$$
(a_1 - a_2)^{(0)} = 1, (a_1 - a_2)^{(k)} = \prod_{j=0}^{k-1}\left(a_1 - a_2 q^j\right), (a_1, a_2 \in \mathbb{R}, k \in \mathbb{N}_0 := \{0, 1, 2, \cdots\})
\tag{8}
$$

Rajkovic et al. [26]. Now, if $k = \varsigma \in \mathbb{R}$, then

$$
(a_1 - a_2)^{(\varsigma)} = a_1^{\varsigma}\prod_{k=0}^{\infty}\frac{1 - (a_2/a_1)q^k}{1 - (a_2/a_1)q^{\varsigma+k}}, (a_1 \neq 0).
\tag{9}
$$

On the other side, by taking $a_2 = 0$, we have $a_1^{(\varsigma)} = a_1^{\varsigma}$ [26]. A q-number $[a_1]_q$ for $a_1 \in \mathbb{R}$ is defined by

$$
[a_1]_q = \frac{1 - q^{a_1}}{1 - q} = q^{a_1 - 1} + \cdots + q + 1.
\tag{10}
$$

Accordingly, the Gamma function in the quantum settings is defined by

$$
\Gamma_q(r) = \frac{(1-q)^{(r-1)}}{(1-q)^{r-1}}, (r \in \mathbb{R} \setminus (\mathbb{Z}^- \cup \{0\})),
\tag{11}
$$

and $\Gamma_q(r+1) = [r]_q \Gamma_q(r)$ [5, 26].

Definition 1 (see [27]). The q-difference-derivative of the given function μ is defined by

$$
\left({}_{q}\mathfrak{D}_{0^+}\mu\right)(r) = \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)_q \mu(r) = \frac{\mu(r) - \mu(qr)}{(1-q)r},
\tag{12}
$$

where $({}_{q}\mathfrak{D}_{0^+}\mu)(0) = \lim_{r \longrightarrow 0}({}_{q}\mathfrak{D}_{0^+}\mu)(r)$.

Clearly, we have $({}_{q}\mathfrak{D}^{k}_{0^+}\mu)(r) = {}_{q}\mathfrak{D}_{0^+}({}_{q}\mathfrak{D}^{k-1}_{0^+}\mu)(r)$ for all $k \in \mathbb{N}$ and $({}_{q}\mathfrak{D}^{0}_{0^+}\mu)(r) = \mu(r)$ [27].

Definition 2 (see [27]). The q-integral of the supposed function $\mu \in C([0, m_2], \mathbb{R})$ is defined as

$$
\left({}_{q}\mathfrak{J}_{0^+}\mu\right)(r) = \int_0^r \mu(v)\,\mathrm{d}_q v = r(1-q)\sum_{j=0}^{\infty}\mu\left(rq^j\right)q^j,
\tag{13}
$$

if the series is absolutely convergent.

Similarly, $({}_{q}\mathfrak{J}^{k}_{0^+}\mu)(r) = {}_{q}\mathfrak{J}_{0^+}({}_{q}\mathfrak{J}^{k-1}_{0^+}\mu)(r)$ for all $k \geq 1$ and $({}_{q}\mathfrak{J}^{0}_{0^+}\mu)(r) = \mu(r)$ [27].

Definition 3 (see [27]). By letting $a_1 \in [0, a_2]$, the definite q-integral of the given function $\mu \in C([0, a_2], \mathbb{R})$ is defined by

$$
\begin{aligned}
\int_{a_1}^{a_2}\mu(v)\mathrm{d}_q v &= {}_{q}\mathfrak{J}_{0^+}\mu(a_2) - {}_{q}\mathfrak{J}_{0^+}\mu(a_1)\\
&= \int_0^{a_2}\mu(v)\mathrm{d}_q v - \int_0^{a_1}\mu(v)\mathrm{d}_q v\\
&= (1-q)\sum_{j=0}^{\infty}\left[a_2\mu\left(a_2 q^j\right) - a_1\mu\left(a_1 q^j\right)\right]q^j,
\end{aligned}
\tag{14}
$$

if the series exists.

By considering μ as a continuous function at $r = 0$, then $({}_{q}\mathfrak{J}_{0^+ q}\mathfrak{D}_{0^+}\mu)(r) = \mu(r) - \mu(0)$ [27]. Furthermore, $({}_{q}\mathfrak{D}_{0^+ q}\mathfrak{J}_{0^+}\mu)(r) = \mu(r)$ for all r.

Definition 4 (see [11, 28]). The ς^{th}-RL-q-integral of $\mu \in \mathscr{C}_{\mathbb{R}}([0,+\infty))$ is defined by

$$ {}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mu(r) = \begin{cases} \dfrac{1}{\Gamma_q(\varsigma)}\displaystyle\int_0^r (r-qv)^{(\varsigma-1)}\mu(v)\,\mathrm{d}_q v, & \varsigma > 0, \\ \mu(r), & \varsigma = 0, \end{cases} \tag{15} $$

if integral exists.

One can simply see that the q-semi-group property satisfies as ${}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}({}_q^R\mathfrak{J}_{0^+}^{\varsigma_2}\mu)(r) = {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\varsigma_2}\mu(r)$ for $\varsigma_1, \varsigma_2 \geq 0$ [28]. Also, for $\zeta > -1$, we have

$$ \begin{aligned} {}_q^R\mathfrak{J}_{0^+}^{\varsigma} r^{\zeta} &= \frac{\Gamma_q(\zeta+1)}{\Gamma_q(\zeta+\varsigma+1)} r^{\zeta+\varsigma}, \\ {}_q^R\mathfrak{J}_{0^+}^{\varsigma} 1(r) &= \frac{1}{\Gamma_q(\varsigma+1)} r^{\varsigma}, (r>0). \end{aligned} \tag{16} $$

Definition 5 (see [11, 28]). Let $\ell - 1 < \varsigma < \ell$, i.e., $\ell = [\varsigma] + 1$. The ς^{th}-Caputo q-derivative of $\mu \in \mathscr{C}_{\mathbb{R}}^{(\ell)}([0,+\infty))$ is defined as

$$ {}_q^C\mathfrak{D}_{0^+}^{\varsigma}\mu(r) = \frac{1}{\Gamma_q(\ell-\varsigma)}\int_0^r (r-qv)^{(\ell-\varsigma-1)}\, {}_q\mathfrak{D}_{0^+}^{\ell}\mu(v)\,\mathrm{d}_q v, \tag{17} $$

if the integral exists.

Note that for $\zeta > -1$, we have

$$ \begin{aligned} {}_q^C\mathfrak{D}_{0^+}^{\varsigma} r^{\iota} &= \frac{\Gamma_q(\iota+1)}{\Gamma_q(\iota-\varsigma+1)} r^{\iota-\varsigma}, \\ {}_q^C\mathfrak{D}_{0^+}^{\varsigma} 1(r) &= 0, (r>0). \end{aligned} \tag{18} $$

Lemma 6 (see [10]). *Let $\ell - 1 < \varsigma < \ell$. Then,*

$$ \left({}_q^C\mathfrak{J}_{0^+}^{\varsigma}\, {}_q^C\mathfrak{D}_{0^+}^{\varsigma}\mu\right)(r) = \mu(r) - \sum_{j=0}^{\ell-1} \frac{r^j}{\Gamma_q(j+1)}\left({}_q\mathfrak{D}_{0^+}^{j}\mu\right)(0). \tag{19} $$

By Lemma 6, the general series solution of q-difference FDE ${}_q^C\mathfrak{D}_{0^+}^{\varsigma}\mu(r) = 0$ is given as $\mu(r) = \tilde{c}_0 + \tilde{c}_1 r + \tilde{c}_2 r^2 + \cdots + \tilde{c}_{\ell-1} r^{\ell-1}$ with $\tilde{c}_0, \cdots, \tilde{c}_{\ell-1} \in \mathbb{R}$ and $\ell = [\varsigma] + 1$ [10]. In this case, we get

$$ \left({}_q^R\mathfrak{J}_{0^+}^{\varsigma}\, {}_q^C\mathfrak{D}_{0^+}^{\varsigma}\mu\right)(r) = \mu(r) + \tilde{c}_0 + \tilde{c}_1 r + \tilde{c}_2 r^2 + \cdots + \tilde{c}_{\ell-1} r^{\ell-1}. \tag{20} $$

3. Analysis of the Cap-q-Difference FBVP (6)

Let $\mathfrak{A} = \mathscr{C}_{\mathbb{R}}(\mathscr{O})$ be the space of all real-valued continuous functions on $\mathscr{O} = [0, 1]$. Clearly, $\mathfrak{A}$ is a Banach space under the norm $\|\mu\|_{\mathfrak{A}} = \mathrm{Sup}_{r\in\mathscr{O}}|\mu(r)|$ for all members $\mu \in \mathfrak{A}$. In the first step, we provide the following fundamental lemma which presents a characterization of the structure of solutions for the proposed Cap-q-difference FBVP (6)

Remark 7. For convenience, we consider the following nonzero constants:

$$ \begin{aligned} W_1 &= 2 - \sum_{j=1}^{k} \frac{\alpha_j}{\Gamma_q(\sigma_j+1)}, \\ W_2 &= \zeta - \sum_{j=1}^{k} \frac{\alpha_j}{\Gamma_q(\sigma_j+2)}, \\ W_3 &= \zeta^2 - \sum_{j=1}^{k} \frac{\alpha_j(1+q)}{\Gamma_q(\sigma_j+3)}, \end{aligned} \tag{21} $$

$$ \begin{aligned} W_4 &= -\sum_{j=1}^{k} \frac{\beta_j}{\Gamma_q(\sigma_j+1)}, \\ W_5 &= -\sum_{j=1}^{k} \frac{\beta_j}{\Gamma_q(\sigma_j+2)}, \\ W_6 &= \frac{2\zeta^{2-\rho}}{\Gamma_q(3-\rho)} - \sum_{j=1}^{k} \frac{\beta_j(1+q)}{\Gamma_q(\sigma_j+3)}, \end{aligned} \tag{22} $$

$$ \begin{aligned} W_7 &= 2(1+q) - \sum_{j=1}^{k} \frac{\gamma_j(1+q)}{\Gamma_q(\sigma_j+1)}, \\ W_8 &= W_2 W_4 - W_1 W_5, \\ W_9 &= W_3 W_4 - W_1 W_6, \end{aligned} \tag{23} $$

$$ \begin{aligned} W_{10} &= W_8 - W_2 W_4, \\ W_{11} &= W_3 W_8 - W_2 W_9. \end{aligned} \tag{24} $$

Lemma 8. *Let $\phi_* \in \mathfrak{A}$, $\varsigma \in (2,3)$, $\rho \in (1,2)$, $\zeta \in (0,1)$, $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}^{>0}$, and $\sigma_j > 0$ for $j = 1, 2, \cdots, k$. The solution of the linear Cap-q-difference FBVP*

$$ \begin{gathered} {}_q^C\mathfrak{D}_{0^+}^{\varsigma}\mu(r) = \phi_*(r), \quad (r \in \mathscr{O}, q \in (0,1)), \\ \mu(0) + \mu(\zeta) = \sum_{j=1}^{k} \alpha_{jq}{}^R\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \\ {}_q^C\mathfrak{D}_{0^+}^{\varrho}\mu(0) + {}_q^C\mathfrak{D}_{0^+}^{\varrho}\mu(\zeta) = \sum_{j=1}^{k} \beta_{jq}{}^R\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \\ {}_q^C\mathfrak{D}_{0^+}^{2}\mu(0) + {}_q^C\mathfrak{D}_{0^+}^{2}\mu(\zeta) = \sum_{j=1}^{k} \gamma_{jq}{}^R\mathfrak{J}_{0^+}^{\sigma_j}\left[{}_q^C\mathfrak{D}_{0^+}^{2}\mu(1)\right] \end{gathered} \tag{25} $$

is given by

$$\mu(r)=\int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,d_qv-\frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,d_qv+\frac{\Theta_2(r)}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\rho-1)}}{\Gamma_q(\varsigma-\rho)}\phi_*(v)\,d_qv-\frac{\Theta_3(r)}{W_1W_7W_8}\cdot\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,d_qv+\frac{\Theta_1(r)}{W_1W_8}\cdot\sum_{j=1}^k\alpha_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,d_qv-\frac{\Theta_2(r)}{W_8}\cdot\sum_{j=1}^k\beta_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,d_qv+\frac{\Theta_3(r)}{W_1W_7W_8}\cdot\sum_{j=1}^k\gamma_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,d_qv, \quad (26)$$

where

$$\begin{aligned}\Theta_1(r)&=rW_1W_4+W_{10},\\ \Theta_2(r)&=rW_1-W_2,\\ \Theta_3(r)&=r^2W_1W_8-rW_1W_9-W_{11},\end{aligned}\quad (27)$$

and W_i are defined in (24).

Proof. Let μ satisfies the linear Cap-q-difference FBVP (25). Then ${}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)=\phi_*(r)$. By virtue of $\varsigma\in(2,3)$ and taking ς^{th}-RL-q-integral, we have

$$\mu(r)=\frac{1}{\Gamma_q(\varsigma)}\int_0^r (r-qv)^{(\varsigma-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_0+\tilde{c}_1r+\tilde{c}_2r^2, \quad (28)$$

where $\tilde{c}_0,\tilde{c}_1,\tilde{c}_2\in\mathbb{R}$ are unknown coefficients that we have to explore them. It is immediately computed that

$${}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma-2)}\int_0^r (r-qv)^{(\varsigma-3)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_2(1+q), \quad (29)$$

$${}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma-\varrho)}\int_0^r (r-qv)^{(\varsigma-\varrho-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_2\frac{2}{\Gamma_q(3-\varrho)}r^{2-\varrho}, \quad (30)$$

$${}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma+\sigma_j)}\int_0^r (r-qv)^{(\varsigma+\sigma_j-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_0\frac{1}{\Gamma_q(\sigma_j+1)}r^{\sigma_j}+\tilde{c}_1\frac{1}{\Gamma_q(\sigma_j+2)}r^{\sigma_j+1}+\tilde{c}_2\frac{1+q}{\Gamma_q(\sigma_j+3)}r^{\sigma_j+2}, \quad (31)$$

$${}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(r)\right]=\frac{1}{\Gamma_q(\varsigma+\sigma_j-2)}\int_0^r (r-qv)^{(\varsigma+\sigma_j-3)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_2\frac{1+q}{\Gamma_q(\sigma_j+1)}r^{\sigma_j}. \quad (32)$$

By considering the constants $W_1,\cdots,W_{11}$ given by (24) and by virtue the given boundary conditions implemented on (29)–(32) and by some straightforward computations, we obtain the following coefficients

$$\begin{aligned}\tilde{c}_0=&\frac{W_2}{W_8}\sum_{j=1}^k\beta_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_2}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\rho-1)}}{\Gamma_q(\varsigma-\rho)}\phi_*(v)\,\mathrm{d}_qv\\&+\frac{W_{10}}{W_1W_8}\sum_{j=1}^k\alpha_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_{10}}{W_1W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv\\&+\frac{W_{11}}{W_1W_7W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_{11}}{W_1W_7W_8}\sum_{j=1}^k\gamma_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv,\end{aligned} \quad (33)$$

$$\begin{aligned}\tilde{c}_1=&\frac{W_4}{W_8}\sum_{j=1}^k\alpha_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_4}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv\\&+\frac{W_1}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\rho-1)}}{\Gamma_q(\varsigma-\rho)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_1}{W_8}\sum_{j=1}^k\beta_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv\\&+\frac{W_9}{W_7W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_9}{W_7W_8}\sum_{j=1}^k\gamma_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv,\end{aligned} \quad (34)$$

$$\tilde{c}_2=\frac{1}{W_7}\sum_{j=1}^k\gamma_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv-\frac{1}{W_7}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv. \quad (35)$$

By inserting (33), (34), and (35) into (28), we derive equation (26) which is the same desired q-integral solution of the linear Cap-q-difference FBVP (25). The proof is completed. □

Now, consider the following estimates:

$$\begin{aligned}
\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_1(r)| &\le \mathrm{Sup}_{r\in\mathcal{O}}(|rW_1W_4|+|W_{10}|)\\
&\le |W_1W_4|+|W_{10}| \coloneqq \Theta_1^* > 0,\\
\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_2(r)| &\le \mathrm{Sup}_{r\in\mathcal{O}}(|rW_1|+|W_2|)\\
&\le |W_1|+|W_2| \coloneqq \Theta_2^* > 0,\\
\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_3(r)| &\le \mathrm{Sup}_{r\in\mathcal{O}}\left(\left|r^2W_1W_8\right|+|rW_1W_9|+|W_{11}|\right)\\
&\le |W_1W_8|+|W_1W_9|+|W_{11}| \coloneqq \Theta_3^* > 0.
\end{aligned} \tag{36}$$

In this paper, for convenience in computation, we set

$${}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathcal{G}_\mu(\nu)(r) = \frac{1}{\Gamma_q(\varsigma)}\int_0^r (r-q\nu)^{(\varsigma-1)}\mathcal{G}_\mu(\nu)\,\mathrm{d}_q\nu. \tag{37}$$

According to Lemma 8, we define the operator $\mathcal{F}:\mathfrak{A}\longrightarrow\mathfrak{A}$ as

$$\begin{aligned}
(\mathcal{F}\mu)(r) &= {}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathcal{G}_\mu(\nu)(r) + \frac{\Theta_1(r)}{W_1W_8}\\
&\cdot\left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathcal{G}_\mu(\nu)(\zeta) + \sum_{j=1}^k \alpha_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\mathcal{G}_\mu(\nu)(1)\right]\\
&+ \frac{\Theta_2(r)}{W_8}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}\mathcal{G}_\mu(\nu)(\zeta) - \sum_{j=1}^k \beta_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\mathcal{G}_\mu(\nu)(1)\right]\\
&+ \frac{\Theta_3(r)}{W_1W_7W_8}\left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}\mathcal{G}_\mu(\nu)(\zeta) + \sum_{j=1}^k \gamma_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}\mathcal{G}_\mu(\nu)(1)\right].
\end{aligned} \tag{38}$$

Notice that the Cap-q-difference FBVP (6) has solutions if and only if $\mathcal{F}$ has fixed points.

To simplify the computations, we set the following notation and the constants

$$\begin{aligned}
\Lambda &= \frac{1}{\Gamma_q(\varsigma+1)} + \frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^\varsigma}{\Gamma_q(\varsigma+1)} + \sum_{j=1}^k \frac{|\alpha_j|}{\Gamma_q(\varsigma+\sigma_j+1)}\right)\\
&+ \frac{\Theta_2^*}{|W_8|}\left(\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\rho+1)} + \sum_{j=1}^k \frac{|\beta_j|}{\Gamma_q(\varsigma+\sigma_j+1)}\right)\\
&+ \frac{\Theta_3^*}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)} + \sum_{j=1}^k \frac{|\gamma_j|}{\Gamma_q(\varsigma+\sigma_j-1)}\right).
\end{aligned} \tag{39}$$

3.1. Uniqueness Result. The uniqueness result for the Cap-q-difference FBVP (6) is proved by using the Banach's fixed-point theorem.

Theorem 9. *Assume that $G\in\mathcal{C}(\mathcal{O}\times\mathbb{R}^2,\mathbb{R})$ satisfies the following assumptions.*

($\mathcal{H}_1$) There are $\mathbb{L}_1, \mathbb{L}_2 > 0$ such that

$$|G(r,u_1,v_1) - G(r,u_2,v_2)| \le \mathbb{L}_1|u_1-u_2| + \mathbb{L}_2|v_1-v_2|, \tag{40}$$

for every u_i, $v_i\in\mathbb{R}$, $i=1,2$, and $r\in\mathcal{O}$.

If

$$\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda < 1, \tag{41}$$

where Λ is given in (39), and then the Cap-q-difference FBVP (6) has a unique solution μ in $\mathfrak{A}$.

Proof. We convert the Cap-q-difference FBVP (6) into $\mu=\mathcal{F}\mu$, where $\mathcal{F}$ is defined by (38). By the Banach's contraction principle, we shall guarantee that $\mathcal{F}$ has exactly one fixed point.

At first, we define a bounded, closed convex subset $\mathbb{B}_{Y_1} \coloneqq \{\mu\in\mathfrak{A}: \|\mu\|_{\mathfrak{A}} \le Y_1\} \ne \emptyset$ with

$$Y_1 \ge \frac{\Lambda\mathbb{G}}{1-\left(\mathbb{L}_1 + \left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda}, \tag{42}$$

where Λ is defined by (39).

Let $\sup_{r\in\mathcal{O}}|\mathcal{G}(r,0,0)| \coloneqq \mathbb{G} < \infty$. The proof will be completed in two steps:

Step 1. $\mathcal{F}\mathbb{B}_{Y_1} \subset \mathbb{B}_{Y_1}$.

Let $\mu\in\mathbb{B}_{Y_1}$ and $r\in\mathcal{O}$. Estimate

$$\begin{aligned}
|(\mathcal{F}\mu)(r)| &\le {}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathcal{G}_\mu(\nu)|(r) + \frac{\Theta_1(r)}{|W_1W_8|}\\
&\cdot\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathcal{G}_\mu(\nu)|(\zeta) + \sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathcal{G}_\mu(\nu)|(1)\right]\\
&+ \frac{\Theta_2(r)}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}|\mathcal{G}_\mu(\nu)|(\zeta) + \sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathcal{G}_\mu(\nu)|(1)\right]\\
&+ \frac{\Theta_3(r)}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}|\mathcal{G}_\mu(\nu)|(\zeta) + \sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}|\mathcal{G}_\mu(\nu)|(1)\right].
\end{aligned} \tag{43}$$

By using the property of integral (16), we get

$${}_q^R\mathfrak{J}_{0^+}^{\omega}|\mu(\nu)|(r) = \frac{1}{\Gamma_q(\omega)}\int_0^r (r-q\nu)^{(\omega-1)}|\mu(\nu)|\mathrm{d}_q\nu \le \frac{r^\omega\|\mu\|_{\mathfrak{A}}}{\Gamma_q(\omega+1)}. \tag{44}$$

From the assumptions ($\mathcal{H}_1$) and (44), we can estimate

$$\begin{aligned}
|\mathcal{G}_\mu(r)| &\le \left|g\left(r,\mu(r),{}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right) - g(r,0,0)\right| + |g(r,0,0,0)|\\
&\le \mathbb{L}_1|\mu(r)| + \mathbb{L}_2\left|{}_q^r\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right| + \mathbb{G} \le \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}.
\end{aligned} \tag{45}$$

From (45) and by the property of integral (16), we obtain

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathscr{G}_{\mu}(\nu)\right|(r) \le \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right]\frac{r^{\varsigma}}{\Gamma_q(\varsigma+1)}, \tag{46} $$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathscr{G}_{\mu}(\nu)\right|(\zeta) \le \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right]\frac{\zeta^{\varsigma}}{\Gamma_q(\varsigma+1)}, \tag{47} $$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-\varrho}\left|\mathscr{G}_{\mu}(\nu)\right|(\zeta) \le \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right]\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\varrho+1)}, \tag{48} $$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-2}\left|\mathscr{G}_{\mu}(\nu)\right|(\zeta) \le \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right]\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)}, \tag{49} $$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\left|\mathscr{G}_{\mu}(\nu)\right|(1) \le \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right]\frac{1}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}, \tag{50} $$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}\left|\mathscr{G}_{\mu}(\nu)\right|(1) \le \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right]\frac{1}{\Gamma_q\left(\varsigma+\sigma_j-1\right)}. \tag{51} $$

Substituting (46)–(51) into (43), we obtain

$$ \begin{aligned} |(\mathscr{F}\mu)(r)| \le{}& \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right] \\ &\cdot \left[\frac{r^{\varsigma}}{\Gamma_q(\varsigma+1)} + \frac{\Theta_1(r)}{|W_1W_8|}\left(\frac{\zeta^{\varsigma}}{\Gamma_q(\varsigma+1)} + \sum_{j=1}^{k}\frac{|\alpha_j|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right)\right. \\ &+ \frac{\Theta_2(r)}{|W_8|}\left(\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\varrho+1)} + \sum_{j=1}^{k}\frac{\left|\beta_j\right|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right) \\ &\left.+ \frac{\Theta_3(r)}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)} + \sum_{j=1}^{k}\frac{\left|\gamma_j\right|}{\Gamma_q\left(\varsigma+\sigma_j-1\right)}\right)\right]. \end{aligned} \tag{52} $$

Then,

$$ |(\mathscr{F}\mu)(r)| \le \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}\right]\Lambda, \tag{53} $$

which implies that $\|\mathscr{F}\mu\|_{\mathfrak{A}} \le Y_1$. Thus, $\mathscr{F}\mathbb{B}_{Y_1} \subset \mathbb{B}_{Y_1}$.

Step 2. $\mathscr{F}: \mathfrak{A} \longrightarrow \mathfrak{A}$ is a contraction.

Let $\mu, \vartheta \in \mathfrak{A}$. For each $r \in \mathscr{O}$, we have

$$ \begin{aligned} |(\mathscr{F}\mu)(r) - (\mathscr{F}\vartheta)(r)| \le{}& \frac{\Theta_1(r)}{|W_1W_8|} \\ &\cdot \left[{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathscr{G}_{\mu}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\left|\mathscr{G}_{\mu}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right|(1)\right] \\ &+ \frac{\Theta_2(r)}{|W_8|}\left[{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-\varrho}\left|\mathscr{G}_{\vartheta}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\left|\mathscr{G}_{\mu}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right|(1)\right] \\ &+ \frac{\Theta_3(r)}{|W_1W_7W_8|}\left[{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-2}\left|\mathscr{G}_{\mu}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}\left|\mathscr{G}_{\mu}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right|(1)\right] \\ &+ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathscr{G}_{\mu}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right|(r). \end{aligned} \tag{54} $$

By $(\mathscr{H}_1)$, it follows that

$$ \begin{aligned} \left|\mathscr{G}_{\mu}(\nu) - \mathscr{G}_{\vartheta}(\nu)\right| &\le \left|g\left(r, \mu(r), {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right) - g\left(r, \vartheta(r), {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\vartheta(r)\right)\right| \\ &\le \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu - \vartheta\|_{\mathfrak{A}}. \end{aligned} \tag{55} $$

Hence, by inserting (55) into (54) and using the property of integral (16), we get

$$ \begin{aligned} |(\mathscr{F}\mu)(r) - (\mathscr{F}\vartheta)(r)| \le{}& \left[\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu - \vartheta\|_{\mathfrak{A}}\right] \\ &\cdot \left[\frac{r^{\varsigma}}{\Gamma_q(\varsigma+1)} + \frac{\Theta_1(r)}{|W_1W_8|}\left(\frac{\zeta^{\varsigma}}{\Gamma_q(\varsigma+1)} + \sum_{j=1}^{k}\frac{|\alpha_j|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right)\right. \\ &+ \frac{\Theta_2(r)}{|W_8|}\left(\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\varrho+1)} + \sum_{j=1}^{k}\frac{\left|\beta_j\right|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right) \\ &\left.+ \frac{\Theta_3(r)}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)} + \sum_{j=1}^{k}\frac{\left|\gamma_j\right|}{\Gamma_q\left(\varsigma+\sigma_j-1\right)}\right)\right], \end{aligned} \tag{56} $$

which implies that $\|\mathscr{F}\mu - \mathscr{F}\vartheta\|_{\mathfrak{A}} \le (\mathbb{L}_1 + (\mathbb{L}_2/\Gamma_q(\omega+1)))\Lambda\,\|\mu - \vartheta\|_{\mathfrak{A}}$.

In view of (41), $(\mathbb{L}_1 + (\mathbb{L}_2/\Gamma_q(\omega+1)))\Lambda < 1$, and we conclude that $\mathscr{F}$ is a contraction. Hence, in accordance with the Banach's contraction principle, the Cap-q-difference FBVP (6) has a unique solution $\mu \in \mathfrak{A}$. □

3.2. Existence Result. The second result is based on the Leray-Schauder's nonlinear alternative theorem.

Lemma 10 (Leray-Schauder's nonlinear alternative theorem [29]). *Let M be a Banach space, C be its closed convex subset, and X be an open set in C such that $0 \in X$. Let $G: \bar{X} \longrightarrow C$ be a continuous and compact function. Then either (i) there is $\mu \in \bar{X}$ such that $\mu = G(\mu)$ or (ii) there are $\mu \in \partial X$ and $\varrho \in (0, 1)$ such that $\mu = \varrho G(\mu)$.*

Theorem 11. *Let $G \in \mathscr{C}(\mathcal{O} \times \mathbb{R}^2, \mathbb{R})$ satisfies the following assumptions:*

($\mathscr{H}_2$) There is continuous nondecreasing functions $\mathbb{Y}: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, p_1, $p_2 \in \mathscr{C}(\mathscr{J}, \mathbb{R}^+)$ such that

$$|G(r, u, v)| \le p_1(r)\mathbb{Y}(|u|) + p_2(r)|v|, \forall (r, u, v) \in \mathcal{O} \times \mathbb{R}^2, \tag{57}$$

where $\bar{p}_i = \sup_{r\in \mathscr{J}}\{p_i(r)\}$, $i = 1, 2$.

($\mathscr{H}_3$) There is $\mathbb{M}^* > 0$ such that

$$\frac{\left(1 - \left(\Lambda \bar{p}_2/\Gamma_q(\omega + 1)\right)\right)\mathbb{M}^*}{\Lambda_1 \bar{p}_1 \mathbb{Y}(\mathbb{M}^*)} > 1. \tag{58}$$

Then the Cap-q-difference FBVP (6) has at least one solution μ in $\mathfrak{A}$.

Proof. Consider $\mathscr{F}$ as (38). In the first step, we will prove that $\mathscr{F}$ corresponds bounded sets (balls) to bounded ones in $\mathfrak{A}$. For each positive real constant Y_2, $\mathbb{B}_{Y_2} \coloneqq \{\mu \in \mathfrak{A} : \|\mu\| \le Y_2\}$ is a bounded set (ball) in $\mathfrak{A}$. Let $\mu \in \mathbb{B}_{Y_2}$. We have

$$\begin{aligned} |(\mathscr{F}\mu)(r)| \le{}& {}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}|\mathscr{G}_\mu(v)|(r) + \frac{\Theta_1(r)}{|W_1 W_8|} \\ &\cdot \left[{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{\Theta_2(r)}{|W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\varrho}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{\Theta_3(r)}{|W_1 W_7 W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}|\mathscr{G}_\mu(v)|(1)\right]. \end{aligned} \tag{59}$$

From ($\mathscr{H}_2$) and (44) in Theorem 9, we obtain

$$\begin{aligned} \left|G\left(r, \mu(r), {}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mu(r)\right)\right| &\le p_1(r)\mathbb{Y}(|\mu|) + p_2(r)\left|{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mu(r)\right| \\ &\le \bar{p}_1\mathbb{Y}(Y_2) + \frac{\bar{p}_2 Y_2}{\Gamma_q(\omega+1)} \coloneqq \bar{g}. \end{aligned} \tag{60}$$

By the same process in Theorem 9, we can estimate

$$\|(\mathscr{F}\mu)(r)\|_{\mathfrak{A}} \le \Lambda \bar{g}. \tag{61}$$

Further, it will be investigated that $\mathscr{F}$ corresponds bounded sets to equicontinuous sets of $\mathfrak{A}$.

Let r_1, $r_2 \in \mathcal{O}$ with $r_1 < r_2$ and $\mu \in \mathbb{B}_{Y_2}$, where $\mathbb{B}_{Y_2}$ is a bounded set in $\mathfrak{A}$. Also, we obtain

$$\begin{aligned} |(\mathscr{F}\mu)(r_2) - (\mathscr{F}\mu)(r_1)| \le{}& \left|{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mathscr{G}_\mu(v)(r_2) - {}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mathscr{G}_\mu(v)\right|(r_1)\Big| \\ &+ \frac{|\Theta_1(r_2) - \Theta_1(r_1)|}{|W_1 W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{|\Theta_2(r_2) - \Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\varrho}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{|\Theta_3(r_2) - \Theta_3(r_1)|}{|W_1 W_7 W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ \le{}& \left|\frac{1}{\Gamma_q(\varsigma)}\int_{r_1}^{r_2}(r_2 - qv)^{(\varsigma-1)}G_\mu(v)d_q v\right| \\ &+ \left|\frac{1}{\Gamma_q(\varsigma)}\int_{0}^{r_1}\left[(r_2 - qv)^{(\varsigma-1)} - (r_1 - qv)^{(\varsigma-1)}\right]G_\mu(v)d_q v\right| \\ &+ \frac{|\Theta_1(r_2) - \Theta_1(r_1)|}{|W_1 W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{|\Theta_2(r_2) - \Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\varrho}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{|\Theta_3(r_2) - \Theta_3(r_1)|}{|W_1 W_7 W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ \le{}& \frac{\bar{g}}{\Gamma_q(\varsigma)}\left[\left|\int_{r_1}^{r_2}(r_2 - qv)^{(\varsigma-1)}d_q v\right| + \left|\int_{0}^{r_1}\left[(r_2 - qv)^{(\varsigma-1)} - (r_1 - qv)^{(\varsigma-1)}\right]d_q v\right|\right] \\ &+ \frac{|\Theta_1(r_2) - \Theta_1(r_1)|}{|W_1 W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{|\Theta_2(r_2) - \Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\rho}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}|\mathscr{G}_\mu(v)|(1)\right] \\ &+ \frac{|\Theta_3(r_2) - \Theta_3(r_1)|}{|W_1 W_7 W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}|\mathscr{G}_\mu(v)|(1)\right]. \end{aligned} \tag{62}$$

Obviously, the above inequality goes to zero as $r_2 - r_1 \longrightarrow 0$, independent of $\mu \in \mathbb{B}_{Y_2}$. Hence, by helping the Arzelá-Ascoli theorem, $\mathscr{F} : \mathfrak{A} \longrightarrow \mathfrak{A}$ is completely continuous.

Now, we prove that there is an open set $\mathscr{D} \subset \mathfrak{A}$ such that $\mu \ne \kappa\mathscr{F}(\mu)$ for $\kappa \in (0, 1)$ and $x\mu \in \partial\mathscr{D}$.

Let $\mu \in \mathfrak{A}$ satisfies $\mu = \kappa\mathscr{F}\mu$ for each $\kappa \in (0, 1)$. So, for $r \in \mathcal{O}$, by following the calculations applied in proving the boundedness of $\mathscr{F}$, we have

$$|\mu(r)| = |\kappa(\mathscr{F}\mu)(r)| \le \Lambda\left[\bar{p}_1\left(\|\mu\|_{\mathfrak{A}}\right) + \frac{\bar{p}_2\|\mu\|_{\mathfrak{A}}}{\Gamma_q(\omega+1)}\right]. \tag{63}$$

It yields that

$$\|\mu\|_{\mathfrak{A}} \le \bar{p}_1\Lambda\mathbb{Y}\left(\|\mu\|_{\mathfrak{A}}\right) + \frac{\bar{p}_2\Lambda\|\mu\|_{\mathfrak{A}}}{\Gamma_q(\omega+1)}. \tag{64}$$

Consequently, we obtain

$$\frac{\left[\Gamma_q(\omega+1)-\bar{p}_2\Lambda\right]\|\mu\|_{\mathfrak{A}}}{\bar{p}_1\Lambda\Gamma_q(\omega+1)\mathbb{Y}\left(\|\mu\|_{\mathfrak{A}}\right)}\leq 1. \tag{65}$$

From $(\mathscr{H}_3)$, there is $\mathbb{M}^*>0$ such that $\|\mu\|_{\mathfrak{A}}\neq\mathbb{M}^*$. Let

$$\mathscr{D}:=\{\mu\in\mathfrak{A}:\|\mu\|\leq\mathbb{M}^*+1\},\quad \mathscr{U}=\mathscr{D}\cup\mathbb{B}_{\mathrm{Y}_2}. \tag{66}$$

Notice that $\mathscr{F}:\bar{\mathscr{U}}\longrightarrow\mathfrak{A}$ is completely continuous. For the sake of the choice of $\mathscr{D}$, $\nexists x\in\partial\mathscr{D}$ such that $\mu=\kappa\mathscr{F}\mu$ for some $\kappa\in(0,1)$. Therefore, by Lemma 10, we find out that $\mathscr{F}$ has the fixed point $\mu\in\bar{\mathscr{U}}$ which implies that the Cap-q-difference FBVP (6) has at least one solution on $\mathscr{O}$. □

3.3. On the Stability Property for (6). Stability analysis is one of the most important parts of each research in the field of existence of solution of fractional boundary value problems. For instances, we can mention to such a stability analysis in some newly published works including [24, 25, 30–32]. In this subsection, we introduce some concepts of stabilities for the Cap-q-difference FBVP (6). These definitions were extracted from [33].

Let $\epsilon>0$, $G:\mathscr{O}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous and $\theta:\mathscr{O}\longrightarrow\mathbb{R}^+$ be a nondecreasing mapping. Assume that

$$\left|{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)-G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)\right|\leq\varepsilon, \tag{67}$$

$$\left|{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)-G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)\right|\leq\theta(r), \tag{68}$$

$$\left|{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)-G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)\right|\leq\varepsilon\theta(r). \tag{69}$$

Definition 12. The Cap-q-difference FBVP (6) is called Ulam-Hyers stable if $\exists C_G\in\mathbb{R}^+$ s.t. $\forall\varepsilon>0$ and every solution $\mu\in\mathfrak{U}$ of (67), a solution $\kappa\in\mathfrak{U}$ of (6) exists s.t.

$$|\mu(r)-\kappa(r)|\leq C_G\epsilon,\ r\in\mathscr{O}. \tag{70}$$

Definition 13. The Cap-q-difference FBVP (6) is called generalized Ulam-Hyers stable if $\exists P\in\mathscr{C}(\mathbb{R}^+,\mathbb{R}^+)$, $P(0)=0$ s.t. $\forall\mu\in\mathfrak{U}$ fulfilling (67), a solution $\kappa\in\mathfrak{U}$ of (6) exists s.t.

$$|\mu(r)-\kappa(r)|\leq P(\varepsilon),\ r\in\mathscr{O}. \tag{71}$$

Definition 14. The Cap-q-difference FBVP (6) is Ulam-Hyers-Rassias stable w.r.t. θ if $\exists C_\theta\in\mathbb{R}^+$ s.t. $\forall\varepsilon>0$ and every solution $\mu\in\mathfrak{U}$ of (69), $\exists$ a solution $\kappa\in\mathfrak{U}$ of (6) s.t.

$$|\mu(r)-\kappa(r)|\leq C_\theta\theta(r)\varepsilon,\ r\in\mathscr{O}. \tag{72}$$

Definition 15. The Cap-q-difference FBVP (6) is termed generalized Ulam-Hyers-Rassias stable w.r.t. θ if $\exists C_\theta\in\mathbb{R}^+$ s.t. for every solution $\mu\in\mathfrak{U}$ of (68), $\exists$ a solution $\kappa\in\mathfrak{U}$ of (6) s.t.

$$|\mu(r)-\kappa(r)|\leq C_\theta\theta(r),\ r\in\mathscr{O}. \tag{73}$$

Remark 16. $\mu\in\mathfrak{U}$ is a solution of (67) if $\exists\omega_\varsigma\in\mathfrak{U}$ (dependent on μ) s.t.

$$\begin{aligned}&(b_1)\,{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)=G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)+\omega_\varsigma(r),\ r\in\mathscr{O},\\&(b_2)\left|\omega_\varsigma(r)\right|\leq\varepsilon.\end{aligned} \tag{74}$$

Lemma 17. *If $\mu\in\mathfrak{U}$ satisfies (67), then*

$$|\mu(r)-\lambda(r)|\leq\Lambda\varepsilon, \tag{75}$$

where Λ is given as in (39) and $\lambda(r)$ is introduced in the proof.

Proof. Let μ satisfie (67). By (b_1) of Remark 16, there is $\omega_\varsigma\in\mathfrak{U}$ (dependent on μ) such that

$$\begin{aligned}&{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)=G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)+\omega_\varsigma(r),\quad(r\in\mathscr{O},q\in(0,1)),\\&\mu(0)+\mu(\zeta)=\sum_{j=1}^{k}\alpha_{jq}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1),\\&{}^{C}_{q}\mathfrak{D}^{\mathrm{Q}}_{0^+}\mu(0)+{}^{C}_{q}\mathfrak{D}^{\mathrm{Q}}_{0^+}\mu(\zeta)=\sum_{j=1}^{k}\beta_{jq}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1),\\&{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0)+{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta)=\sum_{j=1}^{k}\gamma_{jq}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\right].\end{aligned} \tag{76}$$

Then, the solution of (76) is given as

$$\begin{aligned}\mu(r)={}&{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mathscr{G}_\mu(v)(r)+\frac{\Theta_1(r)}{W_1W_8}\\&\cdot\left[-{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mathscr{G}_\mu(v)(\zeta)+\sum_{j=1}^{k}\alpha_{jq}{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\mathscr{G}_\mu(v)(1)\right]\\&+\frac{\Theta_2(r)}{W_8}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\mathrm{Q}}_{0^+}\mathscr{G}_\mu(v)(\zeta)-\sum_{j=1}^{k}\beta_{jq}{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\mathscr{G}_\mu(v)(1)\right]\\&+\frac{\Theta_3(r)}{W_1W_7W_8}\left[-{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}\mathscr{G}_\mu(v)(\zeta)+\sum_{j=1}^{k}\gamma_{jq}{}^{R}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}\mathscr{G}_\mu(v)(1)\right]\\&+{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\omega_\varsigma(r)+\frac{\Theta_1(r)}{W_1W_8}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\omega_\varsigma(\zeta)+\sum_{j=1}^{k}\alpha_{jq}{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\omega_\varsigma(1)\right]\\&+\frac{\Theta_2(r)}{W_8}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\mathrm{Q}}_{0^+}\omega_\varsigma(\zeta)-\sum_{j=1}^{k}\beta_{jq}{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\omega_\varsigma(1)\right]\\&+\frac{\Theta_3(r)}{W_1W_7W_8}\left[-{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}\omega_\varsigma(\zeta)+\sum_{j=1}^{k}\gamma_{jq}{}^{R}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}\omega_\varsigma(1)\right].\end{aligned} \tag{77}$$

For convenience, consider $\lambda(r)$ for the terms that are independent of $\omega_\varsigma(r)$. That is,

$$\begin{aligned}\lambda(r) = {}^R_q\mathfrak{I}^{\varsigma}_{0^+}\mathcal{G}_\mu(\nu)(r) + \frac{\Theta_1(r)}{W_1W_8}\left[-{}^R_q\mathfrak{I}^{\varsigma}_{0^+}\mathcal{G}_\mu(\nu)(\zeta) + \sum_{j=1}^{k}\alpha_{jq}{}^R\mathfrak{I}^{\varsigma+\sigma_j}_{0^+}\mathcal{G}_\mu(\nu)(1)\right] \\ + \frac{\Theta_2(r)}{W_8}\left[{}^R_q\mathfrak{I}^{\varsigma-\varrho}_{0^+}\mathcal{G}_\mu(\nu)(\zeta) - \sum_{j=1}^{k}\beta_{jq}{}^R\mathfrak{I}^{\varsigma+\sigma_j}_{0^+}\mathcal{G}_\mu(\nu)(1)\right] \\ + \frac{\Theta_3(r)}{W_1W_7W_8}\left[-{}^R_q\mathfrak{I}^{\varsigma-2}_{0^+}\mathcal{G}_\mu(\nu)(\zeta) + \sum_{j=1}^{k}\gamma_{jq}{}^R\mathfrak{I}^{\varsigma+\sigma_j-2}_{0^+}\mathcal{G}_\mu(\nu)(1)\right].\end{aligned} \tag{78}$$

Therefore, (77) can be rewritten and by using (b_2) of Remark 16, we have

$$\begin{aligned}|\mu(r)-\lambda(r)| \le {}^R_q\mathfrak{I}^{\varsigma}_{0^+}|\omega_\varsigma(r)| + \frac{\Theta_1(r)}{|W_1W_8|} \\ \cdot\left[{}^R_q\mathfrak{I}^{\varsigma}_{0^+}|\omega_\varsigma(\zeta)| + \sum_{j=1}^{k}|\alpha_j|{}^R_q\mathfrak{I}^{\varsigma+\sigma_j}_{0^+}|\omega_\varsigma(1)|\right] \\ + \frac{\Theta_2(r)}{|W_8|}\left[{}^R_q\mathfrak{I}^{\varsigma-\varrho}_{0^+}|\omega_\varsigma(\zeta)| + \sum_{j=1}^{k}|\beta_j|{}^R_q\mathfrak{I}^{\varsigma+\sigma_j}_{0^+}|\omega_\varsigma(1)|\right] \\ + \frac{\Theta_3(r)}{|W_1W_7W_8|}\left[{}^R_q\mathfrak{I}^{\varsigma-2}_{0^+}|\omega_\varsigma(\zeta)| + \sum_{j=1}^{k}|\gamma_j|{}^R_q\mathfrak{I}^{\varsigma+\sigma_j-2}_{0^+}|\omega_\varsigma(1)|\right] \le \Lambda\varepsilon.\end{aligned} \tag{79}$$

This inequality completes the proof. □

Theorem 18. *Let* $(\mathcal{H}_1)$ *and*

$$\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda < 1, \tag{80}$$

to be held. Then, the Cap-q-difference FBVP (6) is both Ulam-Hyers and generalized Ulam-Hyers stable.

Proof. Let $\mu \in \mathfrak{U}$ satisfies (67) and κ fulfills the Cap-q-difference FBVP (6) given as

$$\begin{aligned}{}^C_q\mathfrak{D}^{\varsigma}_{0^+}\kappa(r) = G\left(r, \kappa(r), {}^R_q\mathfrak{I}^{\omega}_{0^+}\kappa(r)\right), \quad (r \in \mathcal{O}, q \in (0,1)), \\ \kappa(0) + \kappa(\zeta) = \sum_{j=1}^{k}\alpha_{jq}{}^R\mathfrak{I}^{\sigma_j}_{0^+}\kappa(1), \\ {}^C_q\mathfrak{D}^{\varrho}_{0^+}\kappa(0) + {}^C_q\mathfrak{D}^{\varrho}_{0^+}\kappa(\zeta) = \sum_{j=1}^{k}\beta_{jq}{}^R\mathfrak{I}^{\sigma_j}_{0^+}\kappa(1), \\ {}^C_q\mathfrak{D}^{2}_{0^+}\kappa(0) + {}^C_q\mathfrak{D}^{2}_{0^+}\kappa(\zeta) = \sum_{j=1}^{k}\gamma_{jq}{}^R\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^C_q\mathfrak{D}^{2}_{0^+}\kappa(1)\right].\end{aligned} \tag{81}$$

By the previous lemma, let

$$|\mu(r)-\kappa(r)| \le |\mu(r)-\lambda(r)| + |\lambda(r)-\kappa(r)|. \tag{82}$$

By using Lemma 17 in (82), we have

$$|\mu(r)-\kappa(r)| \le \Lambda\epsilon + \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda|\mu(r)-\kappa(r)|. \tag{83}$$

For $r \in \mathcal{O}$, we have

$$\|\mu-\kappa\|_{\mathfrak{U}} \le \Lambda\epsilon + \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda\|\mu-\kappa\|_{\mathfrak{U}}. \tag{84}$$

After simplification, we get

$$\|\mu-\kappa\|_{\mathfrak{U}} \le \left(\frac{\Lambda}{1-\left(\mathbb{L}_1 + \left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda}\right)\epsilon. \tag{85}$$

Thus

$$|\mu(r)-\kappa(r)| \le C_G\epsilon, \tag{86}$$

where

$$C_G = \frac{\Lambda}{1-\left(\mathbb{L}_1 + \left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda}. \tag{87}$$

Thus, the Cap-q-difference FBVP (6) is Ulam-Hyers stable.

In the sequel, the function $P(\epsilon) = C_G\epsilon$ implies that the Cap-q-difference FBVP (6) is generalized Ulam-Hyers stable and $P(0) = 0$.

Now, we add another condition.

$(\mathcal{A}_1)$ Consider an increasing map $\pi_\varsigma \in \mathcal{C}(\mathcal{O}, \mathbb{R}^+)$. Then, there is $\xi_{\pi_\varsigma} > 0$ such that

$${}^R_q\mathfrak{I}^{\varsigma}_{0^+}\pi_\varsigma(r) \le \xi_{\pi_\varsigma}\pi_\varsigma(r). \tag{88}$$

□

Remark 19. Under the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{A}_1)$ and (80), the Cap-q-difference FBVP (6) is the Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stable.

4. Analysis of the Cap-q-Difference System (7)

Here, we continue to discuss the existence and uniqueness results for the proposed system (7). In view of the assumptions of Section 3 for the Banach space $\mathfrak{A}$, the norm considered on the product space $\mathfrak{A}\times\mathfrak{A}$ is $\|(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} = \|\mu\|_{\mathfrak{A}} + \|\vartheta\|_{\mathfrak{A}}$ which implies that $(\mathfrak{A}\times\mathfrak{A}, \|(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}})$ is a Banach space.

Remark 20. For convenience, and based on the given parameters in (7), we have nonzero constants:

$$
\begin{aligned}
\bar{W}_1 &= 2 - \sum_{j=1}^{k} \frac{\phi_j}{\Gamma_q(\delta_j + 1)}, \\
\bar{W}_2 &= \zeta - \sum_{j=1}^{k} \frac{\phi_j}{\Gamma_q(\delta_j + 2)}, \\
\bar{W}_3 &= \zeta^2 - \sum_{j=1}^{k} \frac{\phi_j(1+q)}{\Gamma_q(\delta_j + 3)}, \\
\bar{W}_4 &= -\sum_{j=1}^{k} \frac{\varphi_j}{\Gamma_q(\delta_j + 1)}, \\
\bar{W}_5 &= -\sum_{j=1}^{k} \frac{\varphi_j}{\Gamma_q(\delta_j + 2)}, \\
\bar{W}_6 &= \frac{2\zeta^{2-\rho}}{\Gamma_q(3-\rho)} - \sum_{j=1}^{k} \frac{\varphi_j(1+q)}{\Gamma_q(\delta_j + 3)}, \\
\bar{W}_7 &= 2(1+q) - \sum_{j=1}^{k} \frac{\eta_j(1+q)}{\Gamma_q(\delta_j + 1)}, \\
\bar{W}_8 &= \bar{W}_2 \bar{W}_4 - \bar{W}_1 \bar{W}_5, \\
\bar{W}_9 &= \bar{W}_3 \bar{W}_4 - \bar{W}_1 \bar{W}_6, \\
\bar{W}_{10} &= \bar{W}_8 - \bar{W}_2 \bar{W}_4, \\
\bar{W}_{11} &= \bar{W}_3 \bar{W}_8 - \bar{W}_2 \bar{W}_9, \\
\bar{\Theta}_1(r) &= r\bar{W}_1 \bar{W}_4 + \bar{W}_{10}, \\
\bar{\Theta}_2(r) &= r\bar{W}_1 - \bar{W}_2, \\
\bar{\Theta}_3(r) &= r^2 \bar{W}_1 \bar{W}_8 - r\bar{W}_1 \bar{W}_9 - \bar{W}_{11}.
\end{aligned}
\tag{89}
$$

Keeping in mind Lemma 8, consider the operator $\mathcal{S} : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A} \times \mathfrak{A}$ as

$$
\mathcal{S}(x,y)(r) := (\mathcal{S}_1(\mu,\vartheta)(r), \mathcal{S}_2(\mu,\vartheta)(r)), \tag{90}
$$

where

$$
\begin{aligned}
\mathcal{S}_1(\mu,\vartheta)(r) = {}&{}^{R}_{q}\mathfrak{J}^{\varsigma_1}_{0^+}\mathcal{U}_{\vartheta}(\nu)(r) + \frac{\Theta_1(r)}{W_1 W_8} \\
&\cdot \left[-{}^{R}_{q}\mathfrak{J}^{\varsigma_1}_{0^+}\mathcal{U}_{\vartheta}(\nu)(\zeta) + \sum_{j=1}^{k} \alpha_{jq}{}^{R}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}\mathcal{U}_{\vartheta}(\nu)(1) \right] \\
&+ \frac{\Theta_2(r)}{W_8} \left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1-\varrho}_{0^+}\mathcal{U}_{\vartheta}(\nu)(\zeta) - \sum_{j=1}^{k} \beta_{jq}{}^{R}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}\mathcal{U}_{\vartheta}(\nu)(1) \right] \\
&+ \frac{\Theta_3(r)}{W_1 W_7 W_8} \left[-{}^{R}_{q}\mathfrak{J}^{\varsigma_1-2}_{0^+}\mathcal{U}_{\vartheta}(\nu)(\zeta) + \sum_{j=1}^{k} \gamma_{jq}{}^{R}\mathfrak{J}^{\varsigma_1+\sigma_j-2}_{0^+}\mathcal{U}_{\vartheta}(\nu)(1) \right],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_2(\mu,\vartheta)(r) = {}&{}^{R}_{q}\mathfrak{J}^{\varsigma_2}_{0^+}\mathcal{V}_{\mu}(\nu)(r) + \frac{\bar{\Theta}_1(r)}{\bar{W}_1 \bar{W}_8} \\
&\cdot \left[-{}^{R}_{q}\mathfrak{J}^{\varsigma_2}_{0^+}\mathcal{V}_{\mu}(\nu)(\zeta) + \sum_{j=1}^{k} \phi_{jq}{}^{R}\mathfrak{J}^{\varsigma_2+\delta_j}_{0^+}\mathcal{V}_{\mu}(\nu)(1) \right] \\
&+ \frac{\bar{\Theta}_2(r)}{\bar{W}_8} \left[{}^{R}_{q}\mathfrak{J}^{\varsigma_2-\rho}_{0^+}\mathcal{V}_{\mu}(\nu)(\zeta) - \sum_{j=1}^{k} \varphi_{jq}{}^{R}\mathfrak{J}^{\varsigma_2+\delta_j}_{0^+}\mathcal{V}_{\mu}(\nu)(1) \right] \\
&+ \frac{\bar{\Theta}_3(r)}{\bar{W}_1 \bar{W}_7 \bar{W}_8} \left[-{}^{R}_{q}\mathfrak{J}^{\varsigma_2-2}_{0^+}\mathcal{V}_{\mu}(\nu)(\zeta) + \sum_{j=1}^{k} \eta_{jq}{}^{R}\mathfrak{J}^{\varsigma_2+\delta_j-2}_{0^+}\mathcal{V}_{\mu}(\nu)(1) \right].
\end{aligned}
\tag{91}
$$

Before proceeding, consider the following estimates

$$
\begin{aligned}
\mathrm{Sup}_{r\in\mathcal{O}} \left|\bar{\Theta}_1(r)\right| &:= \bar{\Theta}_1^*, \\
\mathrm{Sup}_{r\in\mathcal{O}} \left|\Theta_2(r)\right| &:= \bar{\Theta}_2^*, \\
\mathrm{Sup}_{r\in\mathcal{O}} \left|\Theta_3(r)\right| &:= \bar{\Theta}_3^*.
\end{aligned}
\tag{92}
$$

To simplify, we also set the following notation and the constants

$$
\begin{aligned}
\Lambda_1 = {}& \frac{1}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1 W_8|} \left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^{k} \frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)} \right) \\
&+ \frac{\Theta_2^*}{|W_8|} \left(\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^{k} \frac{|\beta_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)} \right) \\
&+ \frac{\Theta_3^*}{|W_1 W_7 W_8|} \left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^{k} \frac{|\gamma_j|}{\Gamma_q(\varsigma_1+\sigma_j-1)} \right), \\
\Lambda_2 = {}& \frac{1}{\Gamma_q(\varsigma_2+1)} + \frac{\bar{\Theta}_1^*}{|\bar{W}_1 \bar{W}_8|} \left(\frac{\zeta^{\varsigma_2}}{\Gamma_q(\varsigma_2+1)} + \sum_{j=1}^{k} \frac{|\phi_j|}{\Gamma_q(\varsigma_2+\delta_j+1)} \right) \\
&+ \frac{\bar{\Theta}_2^*}{|\bar{W}_8|} \left(\frac{\zeta^{\varsigma_2-\rho}}{\Gamma_q(\varsigma_2-\rho+1)} + \sum_{j=1}^{k} \frac{|\varphi_j|}{\Gamma_q(\varsigma_2+\delta_j+1)} \right) \\
&+ \frac{\bar{\Theta}_3^*}{|\bar{W}_1 \bar{W}_7 \bar{W}_8|} \left(\frac{\zeta^{\varsigma_2-2}}{\Gamma_q(\varsigma_2-1)} + \sum_{j=1}^{k} \frac{|\eta_j|}{\Gamma_q(\varsigma_2+\delta_j-1)} \right).
\end{aligned}
\tag{93}
$$

4.1. Uniqueness Result. In this step, we shall establish the existence of a unique solution to the coupled system of nonlinear q-CFBVPs (7), by the Banach's contraction principle.

Theorem 21. *Let $G_1, G_2 : \mathcal{O} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous. Assume that*

($\mathcal{H}_4$) There exist positive constants $\mathcal{L}_i, \mathcal{K}_i, i = 1, 2$ such that for each $r \in [0, 1]$ and $u_i, v_i, \bar{u}_i, \bar{v}_i \in \mathbb{R}$, and for $i = 1, 2$

$$
\begin{aligned}
\left|G_1(r, u_1, v_1) - G_1(r, u_2, v_2)\right| &\le \mathcal{L}_1 |u_1 - u_2| + \mathcal{L}_2 |v_1 - v_2|, \\
\left|G_2(r, \bar{u}_1, \bar{v}_1) - G_2(r, \bar{u}_2, \bar{v}_2)\right| &\le \mathcal{K}_1 |\bar{u}_1 - \bar{u}_2| + \mathcal{K}_2 |\bar{v}_1 - \bar{v}_2|.
\end{aligned}
\tag{94}
$$

Then the coupled system of nonlinear q-CFBVPs (7) has a solution on $\mathcal{O}$ provided that

$$\Omega := \max\left\{\left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1, \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\right\} < 1. \quad (95)$$

Proof. We transform the coupled system of nonlinear q-CFBVPs (7) into a fixed-point problem $(\mu, \vartheta)(r) = \mathcal{S}(\mu, \vartheta)(r)$, where $\mathcal{S}$ is an operator as (90).

Let $\sup_{r\in\mathcal{O}}|G_1(r,0,0)| := \mathbb{M}_{\mathcal{U}} < \infty$ and $\sup_{r\in\mathcal{O}}|G_2(r,0,0)| := \mathbb{M}_{\mathcal{V}} < \infty$. Next, we set $\mathbb{B}_{Y_3} := \{(\mu,\vartheta) \in \mathfrak{A}\times\mathfrak{A} : \|\mu,\vartheta\|_{\mathfrak{A}\times\mathfrak{A}} \le Y_3\}$ with

$$Y_3 \ge \frac{\mathbb{M}_{\mathcal{U}}\Lambda_1 + \mathbb{M}_{\mathcal{V}}\Lambda_2}{1-\Omega}. \quad (96)$$

Note that $\mathbb{B}_{Y_3}$ is a bounded convex closed set in $\mathfrak{A}$.

Step 1. $\mathcal{S}\mathbb{B}_{Y_3} \subset \mathbb{B}_{Y_3}$.

For each $(\mu,\vartheta) \in \mathbb{B}_{Y_3}$ and $r \in \mathcal{O}$, and by using the condition $(\mathcal{H}_4)$ and (44), we have

$$\begin{aligned}
|\mathcal{U}_\vartheta(r)| &\le \left|G_1\left(r, \vartheta(r), {}_q^R\mathfrak{J}_{0^+}^{\omega_1}\vartheta(r)\right) - G_1(r,0,0)\right| + |G_1(r,0,0)| \\
&\le \mathcal{L}_1|\vartheta(r)| + \mathcal{L}_2\left|{}_q^R\mathfrak{J}_{0^+}^{\omega_1}\vartheta(r)\right| \\
&\quad + \mathbb{M}_{\mathcal{U}} \le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}, \\
|\mathcal{V}_\mu(r)| &\le \left|G_2\left(r, \mu(r), {}_q^R\mathfrak{J}_{0^+}^{\omega_2}\mu(r)\right) - G_2(r,0,0)\right| + |G_2(r,0,0)| \\
&\le \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}.
\end{aligned} \quad (97)$$

Then, we get

$$\begin{aligned}
|\mathcal{S}_1(\mu,\vartheta)(r)| &\le {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(r) + \frac{|\Theta_1(r)|}{|W_1W_8|} \\
&\quad \cdot \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right] \\
&\quad + \frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-\varrho}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right] \\
&\quad + \frac{|\Theta_3(r)|}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-2}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_\vartheta(v)|(1)\right], \\
&\le \left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right. \\
&\quad + \frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right] + \frac{\Theta_3^*}{|W_1W_7W_8|} \\
&\quad \cdot\left.\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right] \\
&\quad \times\left[\left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}\right].
\end{aligned} \quad (98)$$

Hence

$$\|\mathcal{S}_1(\mu,\vartheta)\|_{\mathfrak{A}} \le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}\Lambda_1. \quad (99)$$

Similarly, we find that

$$\|\mathcal{S}_2(\mu,\vartheta)\|_{\mathfrak{A}} \le \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}\Lambda_2. \quad (100)$$

Consequently, we have

$$\begin{aligned}
\|\mathcal{S}(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} &\le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}\Lambda_1 \\
&\quad + \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}\Lambda_2 \\
&\le \Omega Y_3 + \mathbb{M}_{\mathcal{U}}\Lambda_1 + \mathbb{M}_{\mathcal{V}}\Lambda_2 \le Y_3,
\end{aligned} \quad (101)$$

which implies that $\mathcal{S}\mathbb{B}_{Y_3} \subset \mathbb{B}_{Y_3}$.

Step 2. We show that $\mathcal{S} : \mathfrak{A}\times\mathfrak{A} \longrightarrow \mathfrak{A}\times\mathfrak{A}$ is a contraction.

Using condition $(\mathcal{H}_4)$, for any $(\mu_1,\vartheta_1), (\mu_2,\vartheta_2) \in \mathfrak{A}\times\mathfrak{A}$ and for each $r \in \mathcal{O}$, we have

$$\begin{aligned}
&|\mathcal{S}_1(\mu_1,\vartheta_1)(r) - \mathcal{S}_1(\mu_2,\vartheta_2)(r)| \\
&\le \frac{|\Theta_1(r)|}{|W_1W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(\zeta) + \sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(1)\right] \\
&\quad + \frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-\varrho}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(\zeta) + \sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(1)\right] \\
&\quad + \frac{|\Theta_3(r)|}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-2}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(\zeta) + \sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(1)\right] \\
&\quad + {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(r) \\
&\le \left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right. \\
&\quad + \frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right] \\
&\quad + \left.\frac{\Theta_3^*}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right] \\
&\quad \times\left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\|\vartheta_1 - \vartheta_2\|_{\mathfrak{A}},
\end{aligned} \quad (102)$$

and therefore

$$\|\mathcal{S}_1(\mu_1,\vartheta_1) - \mathcal{S}_1(\mu_2,\vartheta_2)\|_{\mathfrak{A}} \le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1\|\vartheta_1 - \vartheta_2\|_{\mathfrak{A}}. \quad (103)$$

Similarly, we get

$$\|\mathcal{S}_2(\mu_1, \vartheta_1) - \mathcal{S}_2(\mu_2, \vartheta_2)\|_{\mathfrak{A}} \le \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2 + 1)}\right)\Lambda_2\|\mu_1 - \mu_2\|_{\mathfrak{A}}. \tag{104}$$

From (103) and (104), it yields

$$\|\mathcal{S}(\mu_1, \vartheta_1) - \mathcal{S}(\mu_2, \vartheta_2)\|_{\mathfrak{A}\times\mathfrak{A}} \le \Omega\left(\|\vartheta_1 - \vartheta_2\|_{\mathfrak{A}} + \|\mu_1 - \mu_2\|_{\mathfrak{A}}\right). \tag{105}$$

As $\Omega < 1$, by (95), the operator $\mathcal{S}$ is a contraction. The Banach's contraction principle implies the existence of unique solution for the coupled system of nonlinear q-CFBVPs (7) on $[0.1]$. □

4.2. Existence Result. We get help from Lemma 10 to complete the main result of this subsection.

Theorem 22. *Let G_1, $G_2 : \mathcal{O} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous. Assume that*

($\mathcal{H}_4$) There exist nonnegative continuous maps $x_i(r), y_i(r) \in C(\mathcal{O}, \mathbb{R}^+ \cup \{0\})$, for $i = 1, 2, 3$ such that

$$\begin{aligned} |G_1(r, u, v)| &\le x_1(r) + x_2(r)|u| + x_3(r)|v|, (r, u, v) \in \left(\mathcal{O}, \mathbb{R}^2\right), \\ |G_2(r, \bar{u}, \bar{v})| &\le y_1(r) + y_2(r)|\bar{u}| + y_3(r)|\bar{v}|, (r, \bar{u}, \bar{v}) \in \left(\mathcal{O}, \mathbb{R}^2\right), \end{aligned} \tag{106}$$

with $x_i^* = \sup_{r\in\mathcal{O}} \{x_i(t)\}$ and $y_i^* = \sup_{r\in\mathcal{O}} \{y_i(t)\}$.

Then the coupled system of nonlinear q-CFBVPs (7) has at least one solution on $\mathcal{O}$.

Proof. Here, the process of the proof will be continued during four steps as follows.

Step 1. $\mathcal{S}$ is continuous.

Let μ_n and ϑ_n be two sequences such that $\mu_n \longrightarrow \mu$ and $\vartheta_n \longrightarrow \vartheta$ in $\mathfrak{A}$. Then for each $r \in \mathcal{O}$, we get

$$\begin{aligned} |\mathcal{S}_1(\mu_n, \vartheta_n)(r) - \mathcal{S}_1(\mu, \vartheta)(r)| &\le \frac{|\Theta_1(r)|}{|W_1 W_8|} \\ &\cdot \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_n}(v) - \mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_n}(v) - \mathcal{U}_\vartheta(v)|(1)\right] \\ &+ \frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-\varrho}|\mathcal{U}_{\vartheta_n}(v) - \mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_n}(v) - \mathcal{U}_\vartheta(v)|(1)\right] \\ &+ \frac{|\Theta_3(r)|}{|W_1 W_7 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-2}|\mathcal{U}_{\vartheta_n}(v) - \mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_{\vartheta_n}(v) - \mathcal{U}_\vartheta(v)|(1)\right], \\ &+ {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_n}(v) - \mathcal{U}_\vartheta(v)|(r) \le \left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1 W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right. \\ &+ \frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right] \\ &+ \left.\frac{\Theta_3^*}{|W_1 W_7 W_8|}\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right]\|\mathcal{U}_{\vartheta_n} - \mathcal{U}_\vartheta\|_{\mathfrak{A}}, \end{aligned} \tag{107}$$

and therefore

$$\|\mathcal{S}_1(\mu_n, \vartheta_n) - \mathcal{S}_1(\mu, \vartheta)\|_{\mathfrak{A}} \le \Lambda_1\left\|\mathcal{U}_{\vartheta_n} - \mathcal{U}_\vartheta\right\|_{\mathfrak{A}}. \tag{108}$$

Similarly, we get

$$\|\mathcal{S}_2(\mu_n, \vartheta_n) - \mathcal{S}_2(\mu, \vartheta)\|_{\mathfrak{A}} \le \Lambda_2\left\|\mathcal{V}_{\mu_n} - \mathcal{V}_\mu\right\|_{\mathfrak{A}}. \tag{109}$$

From (108) and (109), it yields

$$\|\mathcal{S}(\mu_n, \vartheta_n) - \mathcal{S}(\mu, \vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} \le \Lambda_1\left\|\mathcal{U}_{\vartheta_n} - \mathcal{U}_\vartheta\right\|_{\mathfrak{A}} + \Lambda_2\left\|\mathcal{V}_{\mu_n} - \mathcal{V}_\mu\right\|_{\mathfrak{A}}. \tag{110}$$

Since the continuity of G_1 and G_2 imply that of $\mathcal{U}_\vartheta$, $\mathcal{V}_\mu$, so we have $\|\mathcal{U}_{\vartheta_n} - \mathcal{U}_\vartheta\|_{\mathfrak{A}} \longrightarrow 0$ and $\|\mathcal{V}_{\mu_n} - \mathcal{V}_\mu\|_{\mathfrak{A}} \longrightarrow 0$ as $n \longrightarrow \infty$; and $\mathcal{S}$ is continuous.

Step 2. $\mathcal{S}$ is uniformly bounded.

We prove that for $Y_4 > 0$, there exists $\mathcal{N}_{\mathcal{S}} > 0$ such that for every $(\mu, \vartheta) \in \mathbb{B}_{Y_4}$, where

$$\mathbb{B}_{Y_4} = \left\{(\mu, \vartheta) \in \mathfrak{A} \times \mathfrak{A} : \|(x, y)\|_{\mathfrak{A}\times\mathfrak{A}} < Y_4\right\}, \tag{111}$$

we get $\|S(\mu, \vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} \le \mathcal{N}_{\mathcal{S}}$.

Using the condition $(\mathcal{H}_5)$ and (16), we have

$$\begin{aligned} |\mathcal{U}_\vartheta(r)| &= \left|G_1\left(r, \vartheta(r), {}_q^R\mathfrak{J}_{0^+}^{\omega_1}\vartheta(r)\right)\right| \le x_1(t) + x_2(t)|\vartheta(r)| \\ &+ x_3(t)\left|{}_q^R\mathfrak{J}_{0^+}^{\omega_1}\vartheta(r)\right| \le x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}, \\ |\mathcal{V}_\mu(r)| &= \left|G_2\left(r, \mu(r), {}_q^R\mathfrak{J}_{0^+}^{\omega_2}\mu(r)\right)\right| \le y_1^* + \left(y_2^* + \frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}. \end{aligned} \tag{112}$$

Then, we get

$$\begin{aligned} |\mathcal{S}_1(\mu, \vartheta)(r)| &\le {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(r) + \frac{|\Theta_1(r)|}{|W_1 W_8|} \\ &\cdot \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right] \\ &+ \frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-\rho}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right] \\ &+ \frac{|\Theta_3(r)|}{|W_1 W_7 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-2}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_\vartheta(v)|(1)\right] \\ &\le \left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1 W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right. \\ &+ \frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right] \\ &+ \left.\frac{\Theta_3^*}{|W_1 W_7 W_8|}\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right] \\ &\times \left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right]. \end{aligned} \tag{113}$$

Hence

$$\|\mathcal{S}_1(\mu,\vartheta)\|_{\mathfrak{A}} \le \Lambda_1\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right]. \quad (114)$$

Similarly, we find that

$$\|\mathcal{S}_2(\mu,\vartheta)\|_{\mathfrak{A}} \le \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}\Lambda_2. \quad (115)$$

Consequently, we have

$$\begin{aligned}\|\mathcal{S}(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} &\le \Lambda_1\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right]\\ &\quad + \Lambda_2\left[y_1^* + \left(y_2^* + \frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}\right] \coloneqq \mathcal{N}_{\mathcal{S}}.\end{aligned} \quad (116)$$

Then, $\mathcal{S}$ is uniformly bounded.

Step 3. $\mathcal{S}$ maps bounded sets into equi-continuous sets of $\mathfrak{A}$. Let $r_1, r_2 \in \mathcal{O}$ such that $r_1 < r_2$ and $(\mu,\vartheta) \in \mathbb{B}_{Y_4}$ where $\mathbb{B}_{Y_4}$ is defined as in Step 2. Then we have

$$\begin{aligned}
&|\mathcal{S}_1(\mu,\vartheta)(r_2) - \mathcal{S}_1(\mu,\vartheta)(r_1)|\\
&\le \left|{}^{R}_{q}\mathfrak{I}^{\varsigma_1}_{0^+}\mathcal{U}_\vartheta(v)(r_2) - {}^{R}_{q}\mathfrak{I}^{\varsigma_1}_{0^+}\mathcal{U}_\vartheta(v)(r_1)\right| + \frac{|\Theta_1(r_2)-\Theta_1(r_2)|}{|W_1W_8|}\\
&\quad\cdot\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\quad+\frac{|\Theta_2(r_2)-\Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1-\varrho}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\quad+\frac{|\Theta_3(r_2)-\Theta_3(r_1)|}{|W_1W_7W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1-2}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j-2}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\le \frac{1}{\Gamma_q(\varsigma_1)}\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right]\\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-qv)^{(\varsigma_1-1)}d_qv\right| + \left|\int_0^{r_1}\left[(r_2-qv)^{(\varsigma_1-1)}-(r_1-qv)^{(\varsigma_1-1)}\right]d_qv\right|\right]\\
&\quad+\frac{|\Theta_1(r_2)-\Theta_1(r_2)|}{|W_1W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\quad+\frac{|\Theta_2(r_2)-\Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1-\varrho}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\quad+\frac{|\Theta_3(r_2)-\Theta_3(r_1)|}{|W_1W_7W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1-2}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j-2}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right],\\
&|\mathcal{S}_2(\mu,\vartheta)(r_2) - \mathcal{S}_2(\mu,\vartheta)(r_1)|\\
&\le \frac{1}{\Gamma_q(\varsigma_2)}\left[y_1^* + \left(y_2^* + \frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}\right]\\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-qv)^{(\varsigma_2-1)}d_qv\right| + \left|\int_0^{r_1}\left[(r_2-qv)^{(\varsigma_2-1)}-(r_1-qv)^{(\varsigma_2-1)}\right]d_qv\right|\right]\\
&\quad+\frac{\left|\bar{\Theta}_1(r_2)-\bar{\Theta}_1(r_1)\right|}{\left|\bar{W}_1\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_2}_{0^+}\left|\mathcal{V}_\mu(v)\right|(\zeta) + \sum_{j=1}^{k}\left|\phi_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(v)\right|(1)\right]\\
&\quad+\frac{\left|\bar{\Theta}_2(r_2)-\bar{\Theta}_2(r_1)\right|}{\left|\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_2-\rho}_{0^+}\left|\mathcal{V}_\mu\right|(v)(\zeta) + \sum_{j=1}^{k}\left|\varphi_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(v)\right|(1)\right]\\
&\quad+\frac{\left|\bar{\Theta}_3(r_2)-\bar{\Theta}_3(r_1)\right|}{\left|\bar{W}_1\bar{W}_7\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_2-2}_{0^+}\left|\mathcal{V}_\mu(v)\right|(\zeta) + \sum_{j=1}^{k}\left|\eta_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_2+\delta_j-2}_{0^+}\left|\mathcal{V}_\mu(v)\right|(1)\right],
\end{aligned} \quad (117)$$

which implies that

$$\begin{aligned}
&|\mathcal{S}(\mu,\vartheta)(r_2) - \mathcal{S}(\mu,\vartheta)(r_1)| \le \frac{1}{\Gamma_q(\varsigma_1)}\\
&\quad\cdot\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right]\\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-qv)^{(\varsigma_1-1)}d_qv\right| + \left|\int_0^{r_1}\left[(r_2-qv)^{(\varsigma_1-1)}-(r_1-qv)^{(\varsigma_1-1)}\right]d_qv\right|\right]\\
&\quad+\frac{1}{\Gamma_q(\varsigma_2)}\left[y_1^* + \left(y_2^* + \frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}\right]\\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-qv)^{(\varsigma_2-1)}d_qv\right| + \left|\int_0^{r_1}\left[(r_2-qv)^{(\varsigma_2-1)}-(r_1-qv)^{(\varsigma_2-1)}\right]d_qv\right|\right]\\
&\quad+\frac{|\Theta_1(r_2)-\Theta_1(r_2)|}{|W_1W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\quad+\frac{\left|\bar{\Theta}_1(r_2)-\bar{\Theta}_1(r_1)\right|}{\left|\bar{W}_1\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_2}_{0^+}\left|\mathcal{V}_\mu(v)\right|(\zeta) + \sum_{j=1}^{k}\left|\phi_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(v)\right|(1)\right]\\
&\quad+\frac{|\Theta_2(r_2)-\Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1-\varrho}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\quad+\frac{\left|\bar{\Theta}_2(r_2)-\bar{\Theta}_2(r_1)\right|}{\left|\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_2-\rho}_{0^+}\left|\mathcal{V}_\mu\right|(v)(\zeta) + \sum_{j=1}^{k}\left|\varphi_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(v)\right|(1)\right]\\
&\quad+\frac{|\Theta_3(r_2)-\Theta_3(r_1)|}{|W_1W_7W_8|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_1-2}_{0^+}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_1+\sigma_j-2}_{0^+}|\mathcal{U}_\vartheta(v)|(1)\right]\\
&\quad+\frac{\left|\bar{\Theta}_3(r_2)-\bar{\Theta}_3(r_1)\right|}{\left|\bar{W}_1\bar{W}_7\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{I}^{\varsigma_2-2}_{0^+}\left|\mathcal{V}_\mu(v)\right|(\zeta) + \sum_{j=1}^{k}\left|\eta_j\right|{}^{R}_{q}\mathfrak{I}^{\varsigma_2+\delta_j-2}_{0^+}\left|\mathcal{V}_\mu(v)\right|(1)\right].
\end{aligned} \quad (118)$$

The right-hand side tends to 0 as $r_2 \longrightarrow r_1$, which is independent of $(\mu,\vartheta) \in \mathbb{B}_{Y_4}$. By helping the Arzelá-Ascoli theorem, $S : \mathfrak{A} \longrightarrow \mathfrak{A}$ is completely continuous.

Step 4. The set $\mathfrak{B} = \{(\mu,\vartheta) \in \mathfrak{A}\times\mathfrak{A} : (\mu,\vartheta) = \kappa S(\mu,\vartheta), \kappa \in (0,1]\}$ is bounded.

Let $(\mu,\vartheta) \in \mathfrak{B}$. Then $(\mu,\vartheta) = \kappa S(\mu,\vartheta)$ for some $\kappa \in (0,1]$. Thus, for any $r \in \mathcal{O}$, by using the computations of Step 2, we have

$$\|\mathcal{S}(\mu,\vartheta)(r)\|_{\mathfrak{A}\times\mathfrak{A}} \le \mathcal{N}_{\mathcal{S}}. \quad (119)$$

This means that $\mathfrak{B}$ is bounded. Consequently, by Lemma 10, $\mathcal{S}$ has a fixed point and so a solution to the coupled system of nonlinear q-CFBVPs (7). □

5. Numerical Examples

In this section, we provide some illustrative examples of the exactness and applicability of our main results.

Example 1. (i) Consider the Cap-q-difference FBVP of the form

$${}^{C}_{0.8}\mathfrak{D}^{2.5}_{0^+}\mu(r) = G\left(r, \mu(r), {}^{R}_{0.8}\mathfrak{I}^{3.8}_{0^+}\mu(r)\right), \quad (r \in \mathcal{O}, q \in (0,1)),$$

$$\mu(0) + \mu(0.4) = \sum_{j=1}^{2}\left(\frac{12-4j}{10}\right){}^{R}_{0.8}\mathfrak{I}^{3j/10}_{0^+}\mu(1),$$

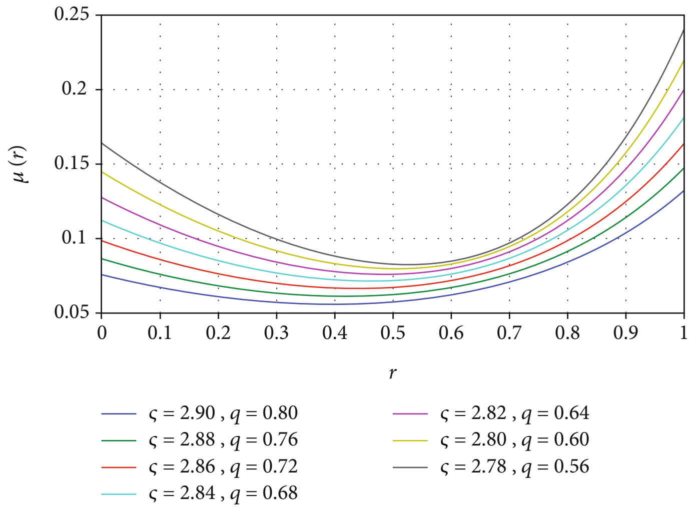


FIGURE 1: The exact solution $\mu(r)$ of (120) for $r \in [0,1]$.

$$
\begin{aligned}
&{}^{C}_{0.8}\mathfrak{D}^{1.2}_{0^+}\mu(0) + {}^{C}_{0.8}\mathfrak{D}^{1.2}_{0^+}\mu(0.4) = \sum_{j=1}^{2}\left(\frac{2j+3}{10}\right){}^{R}_{0.8}\mathfrak{J}^{3j/10}_{0^+}\mu(1),\\
&{}^{C}_{0.8}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{0.8}\mathfrak{D}^{2}_{0^+}\mu(0.4) = \sum_{j=1}^{2}\left(\frac{12-5j}{10}\right){}^{R}_{0.8}\mathfrak{J}^{3j/10}_{0^+}\left[{}^{C}_{0.8}\mathfrak{D}^{2}_{0^+}\mu(1)\right].
\end{aligned}
\tag{120}
$$

Here $\varsigma = 2.5$, $q = 0.8$, $\omega = 3.8$, $\zeta = 0.4$, $\varrho = 1.2$, $\alpha_j = (2j+3)/10$, $\beta_j = (12-5j)/10$, $\gamma_j = (12-4j)/10$, $\sigma_j = 3j/10$, and $j = 1, 2$. From the given data, we obtain $W_1 \approx 0.676686276 \neq 0$, $W_7 \approx 1.814092676 \neq 0$, and $W_8 \approx 1.431872331 \neq 0$. We consider the functions

$$
G\left(r, \mu(r), {}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)\right) = \frac{4r-1}{re^{2r}+4} + \frac{9\cos(\pi/3)}{2e^{r}+6}\cdot\frac{|\mu(r)|}{|\mu(r)|+3} + \frac{10\sin(\pi/6)}{(2r+3)^2+2e^{3r}}\cdot\frac{\left|{}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)\right|}{\left|{}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)\right|+2}.
\tag{121}
$$

For u_i, $v_i \in \mathbb{R}$, and $r \in \mathcal{O}$, we can find that

$$
|G(r, u_1, v_1) - G(r, u_2, v_2)| \leq \frac{3}{8}|u_1 - u_2| + \frac{5}{11}|v_1 - v_2|.
\tag{122}
$$

The assumption $(\mathscr{H}_1)$ is satisfied under the values $\mathbb{L}_1 = 3/8$ and $\mathbb{L}_2 = 5/11$. Thus,

$$
\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda \approx 0.8324696807 < 1.
\tag{123}
$$

All assumptions of Theorem 9 are valid. Then the Cap-q-difference FBVP (120) has a unique solution on $[0,1]$. Moreover,

$$
C_G = \frac{\Lambda}{1-\left(\mathbb{L}_1 + \left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda} \approx 11.85782552 > 0.
\tag{124}
$$

By the conclusions of Theorem 18, the Cap-q-difference FBVP (120) is both Ulam–Hyers and also generalized Ulam-Hyers stable on $[0,1]$. (ii) Set $G(r, \mu(r), {}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)) = r^{\lambda}$.

By using the property of integral (16) and setting $\lambda = 2.8$, the implicit solution of the Cap-q-difference FBVP (120) is given by

$$
\begin{aligned}
\mu(r) ={}& \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+1)}r^{\lambda+\varsigma} + \frac{\Theta_1(r)}{W_1 W_8}\\
&\cdot\left[-\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+1)}\zeta^{\lambda+\varsigma} + \sum_{j=1}^{k}\frac{\alpha_j\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+\sigma_j+1)}\right]\\
&+\frac{\Theta_2(r)}{W_8}\left[\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma-\rho+1)}\zeta^{\lambda+\varsigma-\varrho} - \sum_{j=1}^{k}\frac{\beta_j\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+\sigma_j+1)}\right]\\
&+\frac{\Theta_3(r)}{W_1W_7W_8}\left[-\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma-1)}\zeta^{\lambda+\varsigma-2} + \sum_{j=1}^{k}\frac{\gamma_j\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+\sigma_j-1)}\right].
\end{aligned}
\tag{125}
$$

Figure 1 displays the solution of the Cap-q-difference FBVP (120) involving various values of $\varsigma = 2.78, 2.80, \cdots, 2.90$ and $q = 0.56, 0.60, \cdots, 0.80$.

Example 2. Consider the coupled system of nonlinear Cap-q-difference FBVP under the conditions

$$
\begin{aligned}
&{}^{C}_{0.7}\mathfrak{D}^{2.8}_{0^+}\mu(r) = G_1\left(r, \vartheta(r), {}^{R}_{0.7}\mathfrak{J}^{1.7}_{0^+}\vartheta(r)\right), \quad (r \in \mathcal{O}),\\
&{}^{C}_{0.7}\mathfrak{D}^{2.9}_{0^+}\vartheta(r) = G_2\left(r, \mu(r), {}^{R}_{0.7}\mathfrak{J}^{2.3}_{0^+}\mu(r)\right),\\
&\mu(0) + \mu(0.3) = \sum_{j=1}^{2}\left(\frac{4j}{10}\right){}^{R}_{0.7}\mathfrak{J}^{5j-3/10}_{0^+}\mu(1),
\end{aligned}
$$

$$\vartheta(0) + \vartheta(0.3) = \sum_{j=1}^{k} \left(\frac{12-5j}{10}\right) {}^{R}_{0.7}\mathfrak{J}^{3j-1/10}_{0^+}\vartheta(1),$$

$${}^{C}_{0.7}\mathfrak{D}^{1.8}_{0^+}\mu(0) + {}^{C}_{0.7}\mathfrak{D}^{1.8}_{0^+}\mu(0.3) = \sum_{j=1}^{2} \left(\frac{7-2j}{10}\right) {}^{R}_{0.7}\mathfrak{J}^{5j-3/10}_{0^+}\mu(1),$$

$${}^{C}_{0.7}\mathfrak{D}^{1.4}_{0^+}\vartheta(0) + {}^{C}_{0.7}\mathfrak{D}^{1.4}_{0^+}\vartheta(0.3) = \sum_{j=1}^{2} \left(\frac{10-4j}{10}\right) {}^{R}_{0.7}\mathfrak{J}^{3j-1/10}_{0^+}\vartheta(1),$$

$${}^{C}_{0.7}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{0.7}\mathfrak{D}^{2}_{0^+}\mu(0.3) = \sum_{j=1}^{2} \left(\frac{10-3j}{10}\right) {}^{R}_{0.7}\mathfrak{J}^{5j-3/10}_{0^+}\left[{}^{C}_{0.7}\mathfrak{D}^{2}_{0^+}\mu(1)\right],$$

$${}^{C}_{0.7}\mathfrak{D}^{2}_{0^+}\vartheta(0) + {}^{C}_{0.7}\mathfrak{D}^{2}_{0^+}\vartheta(0.3) = \sum_{j=1}^{2} \left(\frac{8-3j}{10}\right) {}^{R}_{0.7}\mathfrak{J}^{3j-1/10}_{0^+}\left[{}^{C}_{0.7}\mathfrak{D}^{2}_{0^+}\vartheta(1)\right]. \tag{126}$$

Here $\varsigma_1 = 2.8$, $\varsigma_2 = 2.9$, $q = 0.7$, $\omega_1 = 1.7$, $\omega_2 = 2.3$, $\zeta = 0.3$, $\rho = 1.8$, $\rho = 1.4$, $\alpha_j = 4j/10$, $\beta_j = (7-2j)/10$, $\gamma_j = (10-3j)/10$, $\phi_j = (12-5j)/10$, $\varphi_j = (10-4j)/10$, $\eta_j = (8-3j)/10$, $\sigma_j = (5j-3)/10$, $\delta_j = (3j-1)/10$, and $j = 1, 2$. From all the given data, we obtain $W_1 \approx 0.705064917 \neq 0$, $W_7 \approx 1.385967560 \neq 0$, $W_8 \approx 1.029770834 \neq 0$, $\bar{W}_1 \approx 1.026846802 \neq 0$, $\bar{W}_7 \approx 2.110974612 \neq 0$, and $\bar{W}_8 \approx 1.174518052 \neq 0$. We consider the functions

$$\begin{aligned} G_1\left(r, \vartheta(r), {}^{R}_{0.7}\mathfrak{J}^{1.7}_{0^+}\vartheta(r)\right) &= 3r^2 - 2r + 1 + \frac{r+1}{\sin^2(r)+6} \\ &\cdot \frac{|\vartheta(r)|}{|\vartheta(r)|+3} + \frac{2\cos(r)}{(3r+4)^2} \\ &\cdot \frac{\left|{}^{R}_{0.7}\mathfrak{J}^{1.7}_{0^+}\vartheta(r)\right|}{\left|{}^{R}_{0.7}\mathfrak{J}^{1.7}_{0^+}\vartheta(r)\right|+1}, \\ G_2\left(r, \mu(r), {}^{R}_{0.7}\mathfrak{J}^{2.3}_{0^+}\mu(r)\right) &= re^{2r} - 3r + \frac{(2r+\sin(r))}{3e^r+4} \\ &\cdot \frac{|\mu(r)|}{|\mu(r)|+1} + \frac{r}{\ln(2r+1)+3} \\ &\cdot \frac{\left|{}^{R}_{0.7}\mathfrak{J}^{2.3}_{0^+}\mu(r)\right|}{\left|{}^{R}_{0.7}\mathfrak{J}^{2.3}_{0^+}\mu(r)\right|+2}. \end{aligned} \tag{127}$$

For $u_i, v_i, \bar{u}_i, \bar{v}_i \in \mathbb{R}$, and $r \in \mathcal{O}$, we can find that

$$\begin{aligned} |G_1(r, u_1, v_1) - G(r, u_2, v_2)| &\leq \frac{1}{9}|u_1 - u_2| + \frac{1}{8}|v_1 - v_2|, \\ |G_2(r, \bar{u}_1, \bar{v}_1) - G(r, \bar{u}_2, \bar{v}_2)| &\leq \frac{3}{7}|\bar{u}_1 - \bar{u}_2| + \frac{1}{3}|\bar{v}_1 - \bar{v}_2|. \end{aligned} \tag{128}$$

The assumption $(\mathcal{H}_4)$ is satisfied with $\mathcal{L}_1 = 1/9$, $\mathcal{L}_2 = 1/9$, $\mathcal{K}_1 = 3/7$, and $\mathcal{K}_2 = 1/3$. Hence, $(\mathcal{L}_1 + (\mathcal{L}_2/\Gamma_q(\omega_1 + 1)))\Lambda_1 \approx 0.6937912556 < 1$ and $(\mathcal{K}_1 + (\mathcal{K}_2/\Gamma_q(\omega_2 + 1)))\Lambda_1 \approx 0.8947974715 < 1$. All assumptions of Theorem 21 are satisfied. Then the coupled system of nonlinear Cap-q-difference FBVPs (126) has a unique solution on $[0, 1]$.

6. Conclusion

In this paper, a new category of nonlinear Caputo quantum boundary problems and its relevant generalized coupled q-system involving fractional quantum operators was discussed. We presented new q-difference equations and system in which we dealt with q-integro-sum-difference bundary conditions. Some qualitative aspects of solutions such as the existence, uniqueness, and different classes of stabilities of Ulam-Hyers type were investigated for both given q-Cap-difference problems. The results were examined with some examples. As a new idea in the next papers, we aim to extend our method for similar generalized coupled systems under the newly introduced generalized (p, q)-operators (postquantum operators).

Data Availability

No data were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Acknowledgments

The first and third authors would like to thank Azarbaijan Shahid Madani University.

References

[1] K. Shah, M. Sher, A. Ali et al., "On degree theory for non-monotone type fractional order delay differential equations," *AIMS Mathematics*, vol. 7, no. 5, pp. 9479–9492, 2022.

[2] S. W. Ahmad, M. Sarwar, K. Shah, Eiman, and T. Abdeljawad, "Study of a coupled system with sub-strip and multi-valued boundary conditions via topological degree theory on an infinite domain," *Symmetry*, vol. 14, no. 5, p. 841, 2022.

[3] K. Shah, M. Arfan, A. Ullah, Q. Al-Mdallal, K. J. Ansari, and T. Abdeljawad, "Computational study on the dynamics of fractional order differential equations with applications," *Chaos, Solitons & Fractals*, vol. 157, article 111955, 2022.

[4] Y. Rahmani, M. M. Alizadeh, H. Schuh, J. Wickert, and L. C. Tsai, "Probing vertical coupling effects of thunderstorms on lower ionosphere using GNSS data," *Advances in Space Research*, vol. 66, no. 8, pp. 1967–1976, 2020.

[5] F. H. Jackson, "On q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[6] F. H. Jackson, “XI.—On q-functions and a certain difference operator,” *Transactions of the Royal Society Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.

[7] B. Ahmad, S. K. Ntouyas, and J. Tariboon, *Quantum Calculus: New Concepts, Impulsive IVPs and BVPs, Inequalities*, World Scientific, Singapore, 2016.

[8] T. Ernst, *A Comprehensive Treatment of* q*-Calculus*, Springer, Basel, Switzerland, 2012.

[9] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[10] M. El-Shahed and F. M. Al-Askar, “Positive solutions for boundary value problem of nonlinear fractional q-difference equation,” *International Scholarly Research Notices*, vol. 2011, Article ID 385459, 12 pages, 2011.

[11] J. R. Graef and L. Kong, “Positive solutions for a class of higher order boundary value problems with fractional q-derivatives,” *Applied Mathematics and Computation*, vol. 218, no. 19, pp. 9682–9689, 2012.

[12] J. Alzabut, B. Mohammadaliee, and M. E. Samei, “Solutions of two fractional q-integro-differential equations under sum and integral boundary value conditions on a time scale,” *Applied Mathematics and Computation*, vol. 2020, no. 1, p. 304, 2020.

[13] R. I. Butt, T. Abdeljawad, M. A. Alqudah, and M. ur Rehman, “Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach,” *Journal of Inequalities and Applications*, vol. 2019, Article ID 305, 3 pages, 2019.

[14] S. Etemad, S. K. Ntouyas, and B. Ahmad, “Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders,” *Mathematics*, vol. 7, no. 8, p. 659, 2019.

[15] S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, and Y. M. Chu, “On multi-step methods for singular fractional q-integro-differential equations,” *Open Mathematics*, vol. 19, no. 1, pp. 1378–1405, 2021.

[16] R. Ouncharoen, N. Patanarapeelert, and T. Sitthiwirattham, “Nonlocal q-symmetric integral boundary value problem for sequential q-symmetric integro-difference equations,” *Mathematics*, vol. 6, no. 11, p. 218, 2018.

[17] J. Wang, C. Yu, B. Zhang, and S. Wang, “Positive solutions for eigenvalue problems of fractional q-difference equation with ϕ -Laplacian,” *Advances in Difference Equations*, vol. 2021, Article ID 499, 5 pages, 2021.

[18] N. Patanarapeelert and T. Sitthiwirattham, “On four-point fractional q-integro-difference boundary value problems involving separate nonlinearity and arbitrary fraction order,” *Boundary Value Problems*, vol. 2018, Article ID 41, 20 pages, 2018.

[19] S. Abbas, M. Benchohra, and J. R. Graef, “Oscillation and non-oscillation results for the Caputo fractional q-difference equations and inclusions,” *Journal of Mathematical Sciences*, vol. 258, no. 5, pp. 577–593, 2021.

[20] B. Ahmad, J. J. Nieto, A. Alsaedi, and H. Al-Hutami, “Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions,” *Filomat*, vol. 28, no. 8, pp. 1719–1736, 2014.

[21] S. Etemad, M. Ettefagh, and S. Rezapour, “On the existence of solutions for nonlinear fractional q-difference equations with q -integral boundary conditions,” *Journal of Advanced Mathematical Studies*, vol. 8, no. 2, pp. 265–285, 2015.

[22] S. K. Ntouyas and M. E. Samei, “Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus,” *Advances in Difference Equations*, vol. 2019, Article ID 475, 20 pages, 2019.

[23] N. D. Phuong, F. M. Sakar, S. Etemad, and S. Rezapour, “A novel fractional structure of a multi-order quantum multi-integro-differential problem,” *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 633, p. 23, 2020.

[24] Z. Ali, A. Zada, and K. Shah, “Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem,” *Boundary Value Problems*, vol. 2018, Article ID 175, 6 pages, 2018.

[25] A. Khan, K. Shah, Y. Li, and T. S. Khan, “Ulam type stability for a coupled systems of boundary value problems of nonlinear fractional differential equations,” *Journal of Function Spaces*, vol. 2017, Article ID 3046013, 8 pages, 2017.

[26] P. M. Rajkovic, S. D. Marinkovic, and M. S. Stankovic, “Fractional integrals and derivatives in q-calculus,” *Applicable Analysis and Discrete Mathematics*, vol. 1, no. 1, pp. 311–323, 2007.

[27] C. R. Adams, “The general theory of a class of linear partial q -difference equations,” *Transactions of the American Mathematical Society*, vol. 26, no. 3, pp. 283–312, 1924.

[28] R. A. C. Ferreira, “Positive solutions for a class of boundary value problems with fractional q-differences,” *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 367–373, 2011.

[29] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, USA, 2003.

[30] S. Li, L. Shu, X. B. Shu, and F. Xu, “Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays,” *Stochastics*, vol. 91, no. 6, pp. 857–872, 2019.

[31] Y. Guo, X.-B. Shu, Y. Li, and F. Xu, “The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$,” *Boundary Value Problems*, vol. 2019, Article ID 59, 8 pages, 2019.

[32] Y. Guo, M. Chen, X. B. Shu, and F. Xu, “The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm,” *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.

[33] I. A. Rus, “Ulam stabilities of ordinary differential equations in a Banach space,” *Carpathian Journal of Mathmatics*, vol. 26, no. 1, pp. 103–107, 2010.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2132957, 9 pages
https://doi.org/10.1155/2022/2132957

Research Article

Unique Fixed Point Results and Its Applications in Complex-Valued Fuzzy b-Metric Spaces

Humaira,[1] Muhammad Sarwar ,[1] and Nabil Mlaiki[2]

[1]*Department of Mathematics, University of Malakand, Chakdara Dir (L), Pakistan*
[2]*Department of Mathematics and Sciences, Prince Sultan University, Saudi Arabia*

Correspondence should be addressed to Muhammad Sarwar; sarwarswati@gmail.com

Received 25 May 2022; Revised 23 July 2022; Accepted 9 August 2022; Published 1 September 2022

Academic Editor: Santosh Kumar

The goal of this paper is to extend the concept of complex-valued fuzzy metric space to complex-valued fuzzy b-metric spaces and to discuss various existence results for fixed points to ensure their existence and uniqueness. To demonstrate the viability of the proposed strategies, a nontrivial example is used. Finally, applications to integral equations and initial value problems in mechanical engineering are discussed to demonstrate the superiority of the obtained results.

1. Introduction and Preliminaries

Fixed point theory combines topology, geometry, and analysis in an amazing way. Fixed point theory has emerged as a powerful tool in the study of nonlinear analysis in recent years. In fixed point theory and many other mathematical subjects, multiple separate objects are considered. As a result, mathematics is not only about numbers and shapes but also about prepositions, fluid flows, vector connections, and chemical interactions, among other things. Many researchers investigated the significance of various features of symmetry and demonstrated how they might be applied to many types of mathematical problems [1, 2]. There are several generalizations of the concept of metric spaces in the literature. Azam et al. developed the idea of complex-valued metric space and discovered that the Banach contraction principle may be applied to complex-valued metric spaces [3]. They studied its applications to complex integral equations. After that, fixed point theorems have been studied by many authors in complex-valued metric spaces [4–8].

The concept of b-metric spaces has been introduced by Bakhtin and Czerwik [9, 10]. Later on, many authors studied fixed point theorems for single and multivalued mappings in b-metric spaces for instance [11, 12]. In [13], the author generalized the concept of b-metric spaces by introducing the setting of complex-valued b-metric spaces. Many other researchers worked on complex-valued b-metric, and they extended generalized fixed point theorems in the sense of complex-valued b-metric spaces (see [14, 15] and the references therein).

The concept of fuzzy sets was given by Zadeh [2] and opened the door of new direction in mathematical research. Pao-Ming and Ying-Ming established the notion of fuzzy metric spaces [16]. Afterwards, George and Veeramani improved the settings of fuzzy metric spaces [17]. Heilpern introduced the concept of fuzzy mapping and obtained fixed point results for fuzzy mappings [18]. Heilpern's work was further extended by many authors, for instance, see [19–21]. Shukla et al. worked on the neighborhood structure of fuzzy fixed point [22]. Several other researchers worked on fuzzy metric spaces and obtained the generalizations of related results [23, 24].

George and Veeramani generalized the concept of fuzzy metric to the context of complex-valued fuzzy metric and obtained the complex-valued fuzzy version of Banach contraction mapping result in different forms [17]. Also, they obtain some related fixed point results with valid examples.

In this paper, we introduce the setting of complex-valued fuzzy b-metric spaces to generalize the setting of complex-valued b-metric space and establish the complex-valued fuzzy version of the Banach contraction principle. We also provide examples to back up our findings. The paper concludes with an application to integral and differential equation.

All over the manuscript we have symbolized the set of complex numbers by C. We mark some shortcut representation used in this manuscript, as t_c-norm for a complex-valued continuous triangular norm, CF b-metric for complex-valued fuzzy b-metric, and s.t. for such that.

Let $\mathscr{P} = \{(\xi, \rho): 0 \le \xi < \infty, 0 \le \rho < \infty\} \subset C$. The elements $(0,0), (1,1) \in \mathscr{P}$ are denoted by ϑ and ℓ, respectively. The set $\mathscr{P}_\vartheta = \{(\xi, \rho): 0 < \xi < \infty, 0 < \rho < \infty\}$. Clearly for $\varphi, \xi \in C$, $\xi \preceq \varphi$ iff $\xi - \varphi \in \mathscr{P}_\vartheta$. Let the unit closed complex interval be symbolized by $\mathscr{I} = \{(\xi, \rho): 0 \le \xi \le 1, 0 \le \rho \le 1\}$ and the open unit complex interval by $\mathscr{I}_0 = \{(\xi, \rho): 0 \le \xi < 1, 0 \le \rho < 1\}$.

Definition 1 (see [17]). Define an ordered relation $\preceq$ on C by $\varsigma_1 \preceq \varsigma_2$ if and only if $\varsigma_2 - \varsigma_1 \in \mathscr{P}$. The relations $\varsigma_1 \preceq \varsigma_2$ and $\varsigma_1 \prec \varsigma_2$ indicate that $\mathrm{Re}(\varsigma_1) \le \mathrm{Re}(\varsigma_2), \mathrm{Im}(\varsigma_1) \le \mathrm{Im}(\varsigma_2)$ and $\mathrm{Re}(\varsigma_1) < \mathrm{Re}(\varsigma_2), \mathrm{Im}(\varsigma_1) < \mathrm{Im}(\varsigma_2)$, respectively.

Let $B \subset C$. If there exists inf B such that it i the lower bound of B, that is, $\inf B \preceq a \forall a \in B$ and $v \preceq \inf B$ for every lower bound v of B, then inf B is called the greatest lower bound of B.

Definition 2 (see [25]). Let X be a nonempty set. A complex fuzzy set M is characterized by a mapping such that domain is X and the range in the closed unit complex interval $\mathscr{I}$.

Definition 3 (see [17]). A binary equation $\star : \mathscr{I} \times \mathscr{I} \longrightarrow \mathscr{I}$ is said to be complex-valued t-norm if the following conditions hold:

(1) $\xi_1 \star \xi_2 = \xi_2 \star \xi_1$

(2) $\xi_1 \star \xi_2 \preceq \xi_3 \star \xi_4$ whenever $\xi_1 \preceq \xi_3, \xi_2 \preceq \xi_4$

(3) $\xi_1 \star (\xi_2 \star \xi_3) = (\xi_1 \star \xi_2) \star \xi_3$

(4) $\xi \star \vartheta = \vartheta, \xi \star \ell = \xi$

for all $\xi, \xi_1, \xi_2, \xi_3, \xi_4 \in \mathscr{I}$.

Some fundamental examples of a t_c-norm are as follows:

(1) $\xi_1 \star_a \xi_2 = \{e_1 e_2, e_3, e_4\}$, for all $\xi_1 = (e_1, e_3), \xi_2 = (e_2, e_4) \in \mathscr{I}$

(2) $\xi_1 \star_b \xi_2 = \{\min\{e_1, e_2\}, \min\{e_3, e_4\}\}$, for all $\xi_1 = (e_1, e_3), \xi_2 = (e_2, e_4) \in \mathscr{I}$

(3) $\xi_1 \star_c \xi_2 = \{\max\{e_1 + e_2 - 1, 0\}, \max\{e_3 + e_4 - 1, 0\}\}$, for all $\xi_1 = (e_1, e_3), \xi_2 = (e_2, e_4) \in \mathscr{I}$

Definition 4 (see [17]). Let $(\mathscr{X}, M, \star)$ be a complex-valued fuzzy metric space. A sequence $\{\varphi_q\}$ in $\mathscr{X}$ is known as a Cauchy sequence if

$$\lim_{q \longrightarrow \infty} \inf_{d > q} M\left(\varphi_q, \varphi_d, t\right) = \ell \forall t \in \mathscr{P}_\vartheta. \tag{1}$$

The complex-valued fuzzy metric space $(\mathscr{X}, M, \star)$ is complete if every Cauchy sequence is convergent in $\mathscr{X}$.

Definition 5 (see [17]). A sequence is monotonic with respect to $\preceq$ if either $\varsigma_b \preceq \varsigma_{b+1}$ or $\varsigma_{b+1} \preceq \varsigma_b \forall b \in N$.

Lemma 6 (see [17]). *Let $(\mathscr{X}, M, \star)$ be a complex-valued fuzzy metric space. If $t, t' \in \mathscr{P}\vartheta$ and $t \preceq t'$, then $M(\varphi, u, t) \preceq M(\varphi, u, t') \forall \varphi, u \in \mathscr{X}$.*

Lemma 7 (see [17]). *Let $(\mathscr{X}, M, \star)$ be complex-valued fuzzy metric space. A sequence $\{\varphi_q\}$ in $\mathscr{X}$ converges to $v \in \mathscr{X}$ iff $\lim_{q \longrightarrow \infty} M(\varphi_q, v, t) = \ell$ holds $\forall t \in \mathscr{P}_\vartheta$.*

Remark 8 (see [17]). Let $\varphi_q \in \mathscr{P} \forall n \in N$ then:

(a) If the sequence $\{\varphi_q\}$ is monotonic with respect to $\preceq$ and there exist $\gamma, \eta \in \mathscr{P}$ with $\gamma^\circ \varphi_q \preceq \eta, \forall q \in \mathrm{N}$, then there exists $\varphi \in \mathscr{P}$ such that $\lim_{q \longrightarrow \infty} \varphi_q = \varphi$

(b) Although the partial ordering $\preceq$ is not a linear order on C, the pair $(C, \preceq)$ is a lattice

(c) If $\mathscr{X} \subset C$ and there exists $\gamma, \eta \in C$ with $\gamma \preceq s \preceq \eta \forall s \in \mathscr{X}$, then inf $\mathscr{X}$ and sup $\mathscr{X}$ both exist

Remark 9 (see [17]). Let $\varphi_q, \varphi'_q, \xi \in \mathscr{P}, \forall q \in N$, then

(a) If $\varphi_q \preceq \varphi'_q \preceq \ell \forall q \in N$ and $\lim_{q \longrightarrow \infty} \varphi_q = \ell$, then $\lim_{b \longrightarrow \infty} \varphi'_q = \ell$

(b) If $\varphi_q \preceq \xi \forall q \in N$ and $\lim_{b \longrightarrow \infty} \varphi_q = \varphi$, then $\varphi \preceq \xi$

(c) If $\xi \preceq \varphi_q \forall q \in N$ and $\lim_{q \longrightarrow \infty} \varphi_q = \varphi$, then $\xi \preceq \varphi$

Definition 10 (see [15]). Let $\mathscr{X}$ be a nonempty set and let $b \ge 1$ be a given real number. A function $\mathscr{D} : \mathscr{X} \times \mathscr{X} \longrightarrow C$ is called a complex-valued b-metric on $\mathscr{X}$ if, for all $\xi, \varphi, v \in C$, the following conditions are satisfied:

(i) $D(\xi, \varphi) \succeq 0$

(ii) $D(\xi, \varphi) = 0$ if and only if $\xi = \varphi$

(iii) $D(\xi, \varphi) = D(\varphi, \xi)$

(iv) $b[D(\xi, v) + D(v, \varphi)] \succeq D(\xi, \varphi)$

The pair $(\mathscr{X}, D)$ is called a complex-valued b-metric space.

Example 1 (see [15]). Let $\mathscr{X} = C$. Define the mapping $D : C \times C \longrightarrow C$ by $D(\xi, \varphi) = |\xi - \varphi|^2 + i|\xi - \varphi|^2$ for all $\xi, \varphi, v \in C$. Then, $(C, \mathscr{X})$ is complex-valued b-metric space with $b = 2$.

Definition 11 (see [17]). Let $\mathscr{X}$ be a nonempty set, $\star$ a continuous complex-valued t_C-norm, and M a complex fuzzy set on $\mathscr{X} \times \mathscr{X} \times \mathscr{P}_\theta \longrightarrow \mathscr{I}$ satisfying conditions:

(1) $0 \preceq M(\xi, \varphi, t)$

(2) $M(\xi, \varphi, t) = \ell$ for every $t \in \mathscr{P}_\vartheta$ if and only if $\xi = \varphi$

(3) $M(\xi, \varphi, t) = M(\varphi, \xi, t)$

(4) $M(\xi, \varphi, t) \star M(\varphi, \rho, t') \preceq M(\xi, \rho, t + t')$

(5) $M(\xi, \varphi, \star): \mathscr{P}_\vartheta \longrightarrow \mathscr{I}$ is continuous for all $\xi, \varphi, \rho \in \mathscr{X}$ and $t, t' \in \mathscr{P}_\vartheta$

Then, the triplet $(\mathscr{X}, M, \star)$ is said to be a complex-valued fuzzy metric space, and M is called a complex-valued fuzzy metric on $\mathscr{X}$. The functions $M(\xi, \varphi, t)$ denote the degree of nearness and the degree of nonnearness between ξ and φ with respect to the complex parameter t, respectively.

Example 2 (see [17]). Let $\mathscr{X} = \aleph$. Define $\star$ by $\varsigma' \star \varsigma'' = (s's'', u'u'')$ for all $\varsigma' = (s', u'), \varsigma'' = (s'', u'') \in \mathscr{I}$. Define complex fuzzy set M as

$$M(\xi, \varphi, t) = \begin{cases} \dfrac{\xi}{\varphi} \ell \text{ if } \xi \leq \varphi, \\ \dfrac{\varphi}{\xi} \ell \text{ if } \varphi \leq \xi, \end{cases} \tag{2}$$

for each $\xi, \varphi \in \mathscr{X}, \varsigma \in \mathscr{P}_\theta$. Then, $(\mathscr{X}, M, \star)$ is complex-valued fuzzy metric spaces.

2. Fixed Point Results in Complex-Valued Fuzzy b-Metric Spaces

We start this section with the following definition.

Definition 12. $(\mathscr{X}, M, \star, b)$ is said a complex-valued fuzzy b-metric space if $\mathscr{X}$ is an arbitrary set, $\star$ is a t_C-norm, and M is a fuzzy set on $\mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{P}$ meeting the points below for all $\xi, \varphi \in \mathscr{X}, t, s > \vartheta$ and provided a number $b \pm 1$:

(1) $0 \preceq M(\xi, \varphi, t)$

(2) $M(\xi, \varphi, t) = \ell$ for every $t \in \mathscr{P}_\vartheta$ if and only if $\xi = \varphi$

(3) $M(\xi, \varphi, t) = M(\varphi, \xi, t)$

(4) $M(\xi, \varphi, t/b) \star M(\varphi, \rho, t'/b) \preceq M(\xi, \rho, (t + t'))$

(5) $M(\xi, \varphi, \star): \mathscr{P}_\vartheta \longrightarrow \mathscr{I}$ is continuous for all $\xi, \varphi, \rho \in \mathscr{X}$ and $t, t' \in \mathscr{P}_\vartheta$

Then, the triplet $(\mathscr{X}, M, \star)$ is said to be a complex-valued fuzzy metric space, and M is called a complex-valued fuzzy metric on $\mathscr{X}$.

Example 3. Let $M(\xi, \varphi, t)$ be a complex-valued fuzzy metric defined by $e(-|\varphi - \xi|^r / t)\ell$ such that $t > 1$ be a real number. Then, M is CF b-matric space with $b = 2^{r-1}$.

Proof. (1), (2), (3), and (5) are obvious. Here, we prove (4). For an arbitrary integer b, we have

$$\begin{aligned} |\xi - \rho| &\preceq \frac{b\left(t + t'\right)}{t} |\xi - \varphi| + \frac{b\left(t + t'\right)}{t'} |\varphi - \rho| \frac{|\xi - \rho|}{t + t'} \\ &\preceq \frac{b}{t} |\xi - \varphi| + \frac{b}{t'} |\varphi - \rho| \preceq \frac{|\xi - \varphi|}{t/b} + \frac{|\varphi - \rho|}{t'/b}. \end{aligned} \tag{3}$$

Since e^ξ is an increasing function for ξ, one can write

$$e^{|\xi - \rho|/t + t'} \preceq e^{|\xi - \varphi|/t/b} + e^{|\varphi - \rho|/t'/b}. \tag{4}$$

Thus, we have

$$\begin{aligned} &e^{-|\xi - \rho|/t + t'} \ell \succeq e^{-|\xi - \varphi|/t/b} + e^{-|\varphi - \rho|/t'/b} \ell, \\ &M\left(\xi, \rho, \left(t + t'\right)\right) \succeq M\left(\xi, \varphi, \frac{t}{b}\right) \star M\left(\varphi, \rho, \frac{t'}{b}\right). \end{aligned} \tag{5}$$

□

Remark 13. CF b -metric is the generalization of complex-valued fuzzy metric space. It is obvious from example that is every CF b -metric is complex-valued fuzzy metric for $b = 1$. Similarly, some important results like Lemmas 6 and 7 and definitions of convergence and Cauchy presented in Section 1 can also be defined in the same manner in CF b -metric space as mentioned in complex-valued fuzzy metric space.

Theorem 14. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b -metric space and let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be mapping enjoying the following condition:*

$$\frac{\ell}{M(\varsigma\xi, \varsigma\rho, t)} - \ell \preceq q\left[\frac{\ell}{M(\xi, \rho, t)} - \ell\right], \tag{6}$$

for all $\xi, \rho \in \mathscr{X}$ and $q \in [0, 1)$. Then, ς has a unique fixed point $\tau\mathscr{X}$, for all $\tau \in \mathscr{P}_\vartheta$.

Proof. Let $\varphi_0 \in \mathscr{X}$. Define a sequence $\{\varphi_r\}$ in $\mathscr{X}$ by

$$\varphi_r = \varsigma\varphi_{r-1} \text{ for all } r \in N. \tag{7}$$

If $\varphi_0 = \varphi_{r-1}$ for some $r \in N$. Then clearly, ς has a fixed point. Suppose $\varphi_0 = \varphi_{r-1}$ for all $r \in \mathrm{N}$. To show that $\{\varphi_r\}$ is a Cauchy sequence, let define

$$B_r = \left\{M\left(\varphi_i, \varphi_j, t\right): j > i\right\} \subset \mathscr{I}. \tag{8}$$

Since $\vartheta \prec M(\varphi_i, \varphi_j, t)$, by Remark 8, the inf $B_r = \beta_r$ exists. For $j, i \in \mathrm{N}$, using (6), we get

$$\begin{aligned}&\frac{\ell}{M\left(\varphi_{i+1},\varphi_{j+1},t\right)}-\ell\\&=\frac{\ell}{M\left(\varsigma\varphi_i,\varsigma\varphi_j,t\right)}-\ell\preceq q\left[\frac{\ell}{M\left(\varphi_i,\varphi_j,t\right)}-\ell\right]\\&\preceq\frac{\ell}{M\left(\varphi_i,\varphi_j,t\right)}-\ell,\end{aligned}\tag{9}$$

which implies

$$\frac{\ell}{M\left(\varphi_{i+1},\varphi_{j+1},t\right)}\preceq\frac{\ell}{M\left(\varphi_i,\varphi_j,t\right)}.\tag{10}$$

Therefore, by definition, we get

$$\ell\preceq\beta_r\preceq\beta_{r+1}\preceq\vartheta,\text{ for all } r\in\mathrm{N}.\tag{11}$$

Thus, $\{\varphi_r\}$ is monotonic in $\mathscr{P}$. Using Remark 8 and from (11), there exists $\ell^\star\in\mathscr{P}$, with

$$\lim_{r\infty}\beta_r=\ell^\star.\tag{12}$$

From inequality (9), we have

$$\frac{\ell}{M\left(\varphi_{i+1},\varphi_{j+1},t\right)}\preceq\frac{q\ell}{M\left(\varphi_i,\varphi_j,t\right)}+(1-q)\ell,\tag{13}$$

for all i,j and so $\ell/\beta_{i+1}\preceq q\ell/\beta_i+(1-q)\ell$ for every $i\in N$, which yields from (12)

$$(1-q)\ell\preceq(1-q)\ell\star\ell^\star.\tag{14}$$

Since $q\in[0,1)$ and applying Remark 9, we must obtained $\ell=\ell^\star$. Thus,

$$\lim_{r\longrightarrow\infty}\beta_r=\ell.\tag{15}$$

Hence,

$$\lim_{r\longrightarrow\infty}\inf_{j>i}M\left(\varphi_i,\varphi_j,t\right)=\ell,\text{ for all } t\in\mathscr{P}_\vartheta.\tag{16}$$

Therefore, from (16), we have that $\{\varphi_r\}$ is a Cauchy sequence. From the completeness of $\mathscr{X}$ and Lemma 7, we get that there exists $\tau\in\mathscr{X}$ such that

$$\lim_{r\longrightarrow\infty}M(\varphi_r,\tau,t)=\ell,\text{ for all } t\in\mathscr{P}_\vartheta.\tag{17}$$

Now for $t\in\mathscr{P}_\vartheta$ and $r\in\mathrm{R}$, it yields from (6) that

$$\frac{\ell}{M(\varsigma\varphi_r,\varsigma\tau,t)}-\ell\preceq q\left[\frac{\ell}{M(\varphi_r,\tau,t)}-\ell\right],\tag{18}$$

that is

$$M(\varsigma\varphi_r,\varsigma\tau,t)\succeq\frac{1}{(q/M(\varphi_r,\tau,t))+(1-q)}.\tag{19}$$

Now, for any $t\in\mathscr{P}_\vartheta$,

$$\begin{aligned}M(\tau,\varsigma\tau,t)&\succeq M\left(\tau,\varphi_{r+1},\frac{t}{2b}\right)\star M\left(\varphi_{r+1},\varsigma\varphi_\tau,\frac{t}{2b}\right)\\&=M\left(\tau,\varphi_{r+1},\frac{t}{2b}\right)\star M\left(\varsigma\varphi_r,\varsigma\varphi_\tau,\frac{t}{2b}\right).\end{aligned}\tag{20}$$

Taking $r\longrightarrow\infty$ and using (17), (19), and Remark 9, we get that $M(\tau,\varsigma\tau,t)=\ell$ for all $t\in\mathscr{P}_\vartheta$; that is, $\varsigma\tau=\tau$.

Now, we have to show the uniqueness of fixed point τ of ς. On contrary, suppose ν be another fixed point of ς. Then, there exists $t\in\mathscr{P}_\vartheta$ such that $M(\tau,\nu,t)<\ell$, than from (6) we have

$$\frac{\ell}{M(\tau,\nu,t)}-\ell=\frac{\ell}{M(\varsigma\tau,\varsigma\nu,t)}-\ell\preceq q\left[\frac{\ell}{M(\tau,\nu,t)}\ell\right],\tag{21}$$

which is a contradiction. Therefore, we must obtain $M(\tau,\nu,t)=\ell$ for all $t\in\mathscr{P}_\vartheta$. Hence, $\tau=\nu$. □

Corollary 15. *Let $(\mathscr{X},M,\star,b)$ be a complete CF b -metric space and let $\varsigma:\mathscr{X}\longrightarrow\mathscr{X}$ be mapping enjoying the following condition:*

$$\frac{\ell}{M(\varsigma^r\xi,\varsigma^r\rho,t)}-\ell\preceq q\left[\frac{\ell}{M(\xi,\rho,t)}-\ell\right],\tag{22}$$

for all $\xi,\rho\in\mathscr{X}$ and $q\in[0,1)$. Then, ς has a unique fixed point $\tau\mathscr{X}$, for all $t\in\mathscr{P}_\vartheta$.

Proof. By the use of Theorem 14, ς^r has a fixed point τ as ς^r observes all conditions. But $\varsigma^r\varsigma\tau=\varsigma\varsigma^r\tau\varsigma\tau$, implies that $\varsigma\tau$ is another fixed point of ς^r. By uniqueness of fixed point, we have $\varsigma\tau=\tau$.As fixed point of ς is also a fixed point ofς. Thus, ς has a unique fixed point. □

Corollary 16. *Let $(\mathscr{X},M,\star,b)$ be a complete CF b-metric space and let $\varsigma:\mathscr{X}\longrightarrow\mathscr{X}$ be mapping enjoying the following condition:*

$$\frac{\ell}{M(\varsigma^r\xi,\varsigma^r\rho,t)}-\ell\preceq q(t)\left[\frac{\ell}{M(\xi,\rho,t)}-\ell\right],\tag{23}$$

for all $\xi,\rho\in\mathscr{X}$ and $q:\mathscr{P}_\vartheta\longrightarrow[0,1)$. Then, ς has a unique fixed point $\tau\mathscr{X}$, for all $t\in\mathscr{P}_\vartheta$.

Example 4. Let $\mathscr{X}=[0,\infty)$ and t-norm be defined by $c_1\star c_2=c_1c_2$ for all $c_1=(a_1,a_2),c_2=(a_1,a_2)\in\mathscr{I}$. Define M as

$$M(\xi,\rho,t)=\left[\exp^{(\xi-\rho)^2/t}\right]^{-1}\ell\text{ for all }\xi,\rho\in\mathscr{X},t\in\mathscr{P}_\vartheta.\tag{24}$$

Then, $(\mathscr{X}, M, \star)$ is a CF b-metric space. Define $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ as

$$\varsigma(\xi)=\begin{cases} 0, \text{if } \xi = m, \\ \dfrac{\xi}{4}, \text{if } \xi \in (0, m), \\ \dfrac{\xi}{8}, \text{if } \xi \in (m,\infty). \end{cases} \quad (25)$$

Then, we have the following cases.

Case 1. If $\xi, \rho = m$, then $\varsigma\xi, \varsigma\rho = 0$.

Case 2. If $\xi = m$ and $\rho \in (0, m)$, then $\varsigma\xi = 0$ and $\varsigma\rho = \rho/4$.

Case 3. If $\xi = m$ and $\rho \in (m,\infty)$, then $\varsigma\xi = 0$ and $\varsigma\rho = \rho/8$.

Case 4. If $\xi \in [0, m)$ and $\rho \in (m,\infty)$, then $\varsigma\xi = \xi/4$ and $\varsigma\rho = \rho/8$.

Case 5. If $\xi \in [0, m)$ and $\rho \in [0, m)$, then $\varsigma\xi = \xi/4$ and $\varsigma\rho = \rho/4$.

Case 6. If $\xi \in [0, m)$ and $\rho = m$, then $\varsigma\xi = \xi/4$ and $\varsigma\rho = 0$.

Case 7. If $\xi \in (m,\infty)$ and $\rho = m$, then $\varsigma\xi = \xi/8$ and $\varsigma\rho = 0$.

Case 8. If $\xi \in (m,\infty)$ and $\rho \in (m,\infty)$, then $\varsigma\xi = \xi/8$ and $\varsigma\rho = \rho/8$.

The above-mentioned cases observe all conditions of Theorem 14 with $q \in [1/2, 1)$. Thus, the fuzzy contractive mapping ς has a unique fixed point, which is $(0, 0)$.

Theorem 17. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b -metric space with $t \preceq t \star t$ for $t \in \mathscr{I}_\vartheta$. Let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be mapping enjoying the following conditions:*

(i) There exists $\varphi_0 \in \mathscr{X}$ and $\varepsilon \in \mathscr{I}_\vartheta$ such that $\ell - \varepsilon \preceq M(\varphi_0, \varsigma\varphi_0, t)$ for all $t \in \mathscr{P}_\vartheta$

(ii) There exists $q \in [0, 1)$ such that for all $\xi, \rho \in \mathscr{B}[\varphi_0, \varepsilon, t]$,

$$\frac{\ell}{M(\varsigma\xi, \varsigma\rho, t)} - \ell \preceq q\left[\frac{\ell}{M(\xi, \rho, t)} - \ell\right]. \quad (26)$$

Then, ς has a unique fixed point in $\mathscr{B}[\varphi_0, \varepsilon, t]$.

Proof. It is enough to proof that $\mathscr{B}[\varphi_0, \varepsilon, t]$ is complete and $\varsigma\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$ for all $\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$. Let $\{\varphi_r\}$ be a Cauchy sequence in $\mathscr{B}[\varphi_0, \varepsilon, t]$. Since $\mathscr{X}$ is complete thus by the use of Lemma 7, there exists $u \in \mathscr{X}$ such that

$$\lim_{r\longrightarrow\infty} \mathrm{M}(\varphi_r, u, t) = \ell, \quad (27)$$

for all $t \in \mathscr{P}_\vartheta$. Now for all $i, r \in \mathrm{N}$,

$$M\left(\varphi_0, u, t + \frac{t}{i}\right) \succeq M\left(\varphi_0, \varphi_r, \frac{t}{b}\right) \star M\left(\varphi_0, \varphi_r, \frac{t}{ib}\right). \quad (28)$$

Since $\varphi_r \in \mathscr{B}[\varphi_0, \varepsilon, t]$ for every $r \in \mathrm{N}$, also $\lim_{r\longrightarrow\infty} M(\varphi_r, u, t) = \ell$. By using the properties of t-norm and Remark 9, we obtain

$$M\left(\varphi_0, u, t + \frac{t}{i}\right) \succeq (\ell - r) \star \ell = \ell - r, \text{forever } i \in \mathrm{N}. \quad (29)$$

Taking $\lim_{i\longrightarrow\infty}$ and using Remark 9, we get $M(\varphi_0, u, t) \pm \ell - r$. Therefore, $u \in \mathscr{B}[\varphi_0, \varepsilon, t]$.

For every $\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$, it yields from (26)

$$\frac{\ell}{M(\varsigma\varphi_0, \varsigma\varphi, t)} - \ell \preceq q\left[\frac{\ell}{M(\varphi_0, \varphi, t)} - \ell\right], \quad (30)$$

that is

$$M(\varsigma\varphi_0, \varsigma\varphi, t) \succeq \frac{1}{(q/M(\varphi_0, \varphi, t)) + (1 - q)}. \quad (31)$$

Thus, for all $i \in \mathrm{N}$, we get

$$\begin{aligned} M\left(\varphi_0, \varsigma\varphi, t + \frac{t}{i}\right) &\pm M\left(\varphi_0, \varsigma\varphi_0, \frac{t}{ib}\right) \\ &\succeq M\left(\varsigma\varphi_0, \varsigma\varphi, \frac{t}{b}\right) \succeq (\ell - \varepsilon) \star \left[\frac{1}{(q/M(\varphi_0, \varphi, t/b)) + (1 - q)}\right] \\ &\succeq (\ell - \varepsilon) \star \left[\frac{1}{(q/(\ell - \varepsilon)) + (1 - q)}\right] \succeq (\ell - \varepsilon) \star (\ell - \varepsilon). \end{aligned} \quad (32)$$

Taking $\lim_{i\longrightarrow\infty}$ and using Remark 9, we have

$$M(\varphi_0, \varsigma\varphi, t) \succeq (\ell - \varepsilon). \quad (33)$$

Therefore, $\varsigma\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$. □

Theorem 18. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b -metric space such that for any sequence $\{t_r\} \in \mathscr{P}_\vartheta$ with $\lim_{r\longrightarrow\infty}\{t_r\} = \infty$, we get $\lim_{r\longrightarrow\infty} \inf_{\rho\in\mathscr{X}} M(\xi, \rho, \{t_r\}) = \ell$, for all $\xi \in \mathscr{X}$. Let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be a mapping observing that*

$$M(\varsigma\xi, \varsigma\rho, \delta t) \succeq M(\xi, \rho, t), \quad (34)$$

for all $t \in \mathscr{P}_\vartheta$, where $0 < \delta < 1$. Then, ς has a unique fixed point in $\mathscr{X}$.

Proof. Let $\varphi_0 \in \mathscr{X}$. Define a sequence $\{\varphi_r\}$ in $\mathscr{X}$ by

$$\varphi_r = \varsigma\varphi_{r-1} \text{ for all } r \in \mathrm{N}. \quad (35)$$

If $\varphi_0 = \varphi_{r-1}\xi$ for some $r \in \mathrm{N}$. Then clearly, ς has a fixed point. Suppose $\varphi_0 \neq$ for all $r \in \mathrm{N}$. To show that $\{\varphi_r\}$ is a

Cauchy sequence, let define

$$B_r = \{M(\varphi_r, \varphi_s, t): s > r\} \subset \mathcal{J}. \quad (36)$$

Since $\vartheta \prec M(\varphi_r, \varphi_s, t)$, by Remark 8, the inf $B_r = \beta_r$ exists. For $s, r \in \mathrm{N}$, by the use of (??) and Lemma 6, we get

$$\begin{aligned} M(\varphi_{r+1}, \varphi_{s+1}, t) &\succeq M(\varphi_{r+1}, \varphi_{s+1}, \delta t) \succeq M(\varsigma\varphi_r, \varsigma\varphi_s, \delta t) \\ &\succeq M(\varsigma\varphi_r, \varsigma\varphi_s, t), \end{aligned} \quad (37)$$

which yields

$$M(\varsigma\varphi_r, \varsigma\varphi_s, t) \preceq M(\varphi_{r+1}, \varphi_{s+1}, t) \text{ for } s > r. \quad (38)$$

Therefore, by definition, we obtain

$$\vartheta \preceq \beta_r \preceq \beta_{r+1} \preceq \ell, \text{ for all } r \in \mathrm{N}. \quad (39)$$

Hence, $\{\beta_r\}$ is monotonic in $\mathscr{P}$, and by the use of Remark 8 and (39), there exists ℓ^* such that

$$\lim_{r \longrightarrow \infty} \beta_r = \ell^*. \quad (40)$$

For $t \in \mathscr{P}_\vartheta$, once again from (34), we have

$$\begin{aligned} \beta_{r+1} &= \inf_{s>r} M(\varphi_{r+1}, \varphi_{s+1}, t) \succeq \inf_{s>r} M\left(\varphi_r, \varphi_s, \frac{t}{\delta}\right) \\ &= \inf_{s>r} M\left(\varsigma\varphi_r, \varsigma\varphi_s, \frac{t}{\delta}\right) \succeq \inf_{s>r} M\left(\varphi_{r-1}, s_{r-1}, \frac{t}{\delta^2}\right) \\ &= \inf_{s>r} M\left(\varsigma\varphi_{r-2}, \varsigma\varphi_{s-2}, \frac{t}{\delta^2}\right) \succeq \inf_{s>r} M\left(\varsigma\varphi_{r-2}, \varsigma\varphi_{r-2}, \frac{t}{\delta^3}\right) \\ &\succeq \cdots \succeq \inf_{s>r} M\left(\varphi_0, \varphi_{s-r}, \frac{t}{\delta^{r+1}}\right), \end{aligned} \quad (41)$$

for all $r \in \mathrm{N}$ and $t \in \mathscr{P}_\vartheta$, we have

$$\begin{aligned} \beta_{r+1} &= \inf_{s>r} M(\varphi_{r+1}, \varphi_{s+1}, t) \succeq \inf_{s>r} M\left(\varphi_0, \varphi_{s-r}, \frac{t}{\delta^{r+1}}\right) \\ &\succeq \inf_{s>r} \mathrm{M}\left(\varphi_0, \rho, \frac{t}{\delta^{r+1}}\right). \end{aligned} \quad (42)$$

As $\lim_{r \longrightarrow \infty} t/\delta^{r+1} = \infty$, using (40) and assumption, we get

$$\ell^* \pm \lim_{r \longrightarrow \infty} \inf_{\rho \in \mathscr{X}} \mathrm{M}(\varphi_r, \varphi_s, t) \succeq = \ell. \quad (43)$$

From (40) and (43)

$$\lim_{r \longrightarrow \infty} \beta_r = \ell. \quad (44)$$

Thus, $\{\varphi_r\}$ is a Cauchy sequence in $\mathscr{X}$. Since $\mathscr{X}$ is complete, by Lemma 7, there exists $u \in \mathscr{X}$ such that

$$\lim_{r \longrightarrow \infty} M(\varphi_r, u, t) = \ell. \quad (45)$$

For $t \in \mathscr{P}_\vartheta$, (34) yields that

$$\begin{aligned} M(u, \varsigma u, t) &\succeq M\left(u, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varphi_{r+1}, \varsigma u, \frac{t}{2b}\right) \\ &\succeq M\left(u, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varsigma\varphi_r, \varsigma u, \frac{t}{2b}\right) \\ &\succeq M\left(u, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varphi_r, u, \frac{t}{2b\delta}\right). \end{aligned} \quad (46)$$

Taking $\lim_{r \longrightarrow \infty}$ and by (45) and Remark 9, we have $M(u, \varsigma u, t) = \ell$; that is, $\varsigma u = u$.

Now to investigate the uniqueness of fixed point, let on contrary that $v \in \mathscr{X}$ be any other fixed point of ς. So there exist $t \in \mathscr{P}_\vartheta$ with $M(u, v, t) = \ell$; then, (34) yields

$$M(u, v, t) = M(\varsigma u, \varsigma v, t) \succeq M\left(u, v, \frac{t}{\delta}\right). \quad (47)$$

Continuing this way, we obtain

$$M(u, v, t) \succeq M\left(u, v, \frac{t}{\delta^r}\right) \succeq \inf_{\rho \in \mathscr{X}} \mathrm{M}\left(u, v, \frac{t}{\delta^r}\right). \quad (48)$$

Using $\lim_{r \longrightarrow \infty} t/\delta^r = \infty$, it follows that $M(u, v, t) \succeq \ell$, which is contradiction. Thus, $M(u, v, t) = \ell$; that is, $u = v$. □

Example 5. Let $\mathscr{X} = [0, 1]$ and t-norm be defined by $c_1 \star c_2 = c_1 c_2$ for all $c_1 = (a_1, a_2), c_2 = (a_1, a_2) \in \mathscr{J}$. Define M as

$$M(\xi, \rho, t) = \exp^{-|\xi-\rho|/t} \ell \text{ for all } \xi, \rho \in \mathscr{X}, t \in \mathscr{P}_\vartheta. \quad (49)$$

Then, $(\mathscr{X}, \mathrm{M}, \star)$ is a CF b-metric space. Define $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ as

$$\varsigma(\xi) = \begin{cases} 0, \text{ if } \xi \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{\xi}{14}, \text{ if } \xi \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \quad (50)$$

For $\lim_{t \longrightarrow \infty} \mathrm{M}(\xi, \rho, t) = \lim_{t \longrightarrow \infty} \exp^{-|\xi-\rho|/t} \ell = \ell$, we obtain that for all values of $\mathscr{X}$ we have $M(\varsigma\xi, \varsigma\rho, \delta t) \pm M(\varsigma\xi, \varsigma\rho, t)$, and for only 0, we have $\lim_{t \longrightarrow \infty} \inf_{\rho \in \mathscr{X}} M(\xi, \rho, t_r) = \exp^0 \ell = \ell$. Thus, all conditions of Theorem 18 are satisfied so, $(0, 0)$ is a unique fixed point of ς.

Example 6. Let $\mathscr{X} = \mathscr{C}([1, 3], R)$, $A > 0$ and for every $\xi, \rho \in \mathscr{X}$ let

$$\mathrm{M}(\xi, \rho, t) = \exp^{-|\xi-\rho|/t} \ell. \quad (51)$$

Let define $\varsigma : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$\varsigma(\xi(\tau)) = 4 + \int_1^\tau (\xi(v) + \rho(v))e^{v-1}dv, t \in [1,3]. \tag{52}$$

For every $\xi, \rho \in \mathcal{X}$

$$\begin{aligned} M(\varsigma\xi, \varsigma\rho, t) &= \exp^{-|\varsigma\xi(\tau)-\varsigma\rho(\tau)|/t}\ell = \exp^{-\int_1^\tau \max_{\tau\in[1,3]}|\varsigma\xi(\tau)-\varsigma\rho(\tau)|/t}\ell \\ &\geq \exp^{-\int_1^\tau \max_{\tau\in[1,3]}|\varsigma\xi(v)-\varsigma\rho(v)|e^2/t}\ell \geq 2e^2 M(\xi, \rho, t). \end{aligned} \tag{53}$$

Similarly

$$M(\varsigma^r\xi, \varsigma^r\rho, t) \geq \frac{2^r}{r!}e^{2r}M(\xi, \rho, t). \tag{54}$$

Note that

$$e^{2r}\frac{2^r}{r!} = \begin{cases} 537.9 \text{ if } r = 3, \\ 5{,}873.7 \text{ if } r = 5, \\ 1.31 \text{ if } r = 37, \\ 0.202 \text{ if } r = 39. \end{cases} \tag{55}$$

Thus, all conditions of Corollary 15 are satisfied for $q = 0.202$ and $r = 39$, so ς has a fixed point which is a solution of the integral equation

$$\xi(\tau) = 4 + \int_1^\tau (\xi(v) + \rho(v))e^{v-1}dv, t \in [1,3], \tag{56}$$

or the differential equation

$$\xi'(\tau) = \left(\xi + \tau^2\right)e^{\tau-1}\tau \in [1,3], \xi(1) = 4. \tag{57}$$

3. Application

Integral equations have plenty applications in many scientific fields. It is a ripely rising field in abstract theory. One of its significant approach in the study of integral equations is to apply fixed point results to the function defined by the right-hand side of the equation or to develop homotopy methods, which are highly considered in fixed point theory to find the approximate solution. In this section, firstly, we study application of our main Theorem 14 the existence of unique solution to Fredholm integral equation.

Theorem 19. *Let $\Xi = \mathscr{C}([0,m], R)$ be the spaces of continuous real valued functions defined on interval $[0,m]$, where $m > 0$. The Fredholm integral equation is*

$$z(t) = \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta. \tag{58}$$

Let $\Xi = \mathscr{C}[0, m, R]$ and $M : \Xi \times \Xi \times \mathcal{J} \longrightarrow \mathcal{J}$ be a CF b -metric defined as follows:

$$M(y, z, c) = \frac{c}{c + |y - z|^2}\ell, y, z \in \mathcal{X}, c > 0. \tag{59}$$

If there exists $q \in (0,1)$ with

$$\Theta(y, z)(t) \succeq \frac{1}{q}\Lambda(y, z)(t), \tag{60}$$

where

$$\begin{aligned} \Theta(y, z)(t) &= \frac{c}{c + \left|\int_0^m \mathscr{K}(t, \delta, y(\delta))d\delta - \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta\right|^2}\ell, \\ \Lambda(y, z)(t) &= \frac{c}{c + |y(t) - z(t)|^2}\ell, \end{aligned} \tag{61}$$

holds. Then, (58) has a unique solution in $\mathcal{X}$.

Proof. Let $\Gamma : \Xi \longrightarrow \Xi$ define as

$$\Gamma z(t) = \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta. \tag{62}$$

Then

$$|\Gamma y - \Gamma z|^2 = \left|\int_0^m \mathscr{K}(t, \delta, y(\delta))d\delta - \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta\right|^2. \tag{63}$$

For all $y, z \in \mathcal{X}$, we have

$$\frac{\ell}{\Theta(y, z)(t)} \preceq \frac{q\ell}{\Lambda(y, z)(t)}, \tag{64}$$

so,

$$\frac{\ell}{\Theta(y, z)(t)} - \ell \preceq \frac{q\ell}{\Lambda(y, z)(t)} - \ell \preceq q\left(\frac{\ell}{\Lambda(y, z)(t)} - \ell\right), \tag{65}$$

which implies that

$$\begin{aligned} &\frac{\ell}{c/c + \left|\int_0^m \mathscr{K}(t, \delta, y(\delta))\delta - \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta\right|^2} - \ell \\ &\preceq q\left(\frac{\ell}{c/c + |y(t) - z(t)|^2} - \ell\right). \end{aligned} \tag{66}$$

Therefore,

$$\frac{\ell}{\mathrm{M}(\Gamma y, \Gamma z, c)} - \ell \preceq q\left(\frac{\ell}{\mathrm{M}(y, z, c)} - \ell\right). \tag{67}$$

Since all conditions of Theorem 14 are satisfied, thus (58) has a unique solution in $\mathscr{X}$. □

Next, we study the application of Theorem 18, in mechanical engineering, since the system of auto mobile suspension is an achievable application for the system of spring mass in the field of engineering. We are going to study the motion of an auto mobile spring when its motion is upon a craggy and cleft road, where the forcing term is the craggy road and bumps noticed provide the absorbing. Tension, gravity, and earth quick are the possible external forces acting on the system. We express spring mass by κ and the external force acting on it by Θ. Then the following initial value problem represents the damped motion of the spring mass system under the action of external force Θ.

$$\begin{cases} \kappa \dfrac{d^2\bar{y}}{dt^2} + \pi \dfrac{d\bar{y}}{dt} = \Theta(t, \bar{y}(t)) = 0, \\ \bar{y}(0) = 0, \\ \bar{y}'(0) = 0, \end{cases} \tag{68}$$

where $\pi > 0$ express the damping constant and $\Theta : [0, \phi] \times \bar{R}^+ \longrightarrow \bar{R}$ is a continuous mapping. Clearly, the problem (68) is equivalent to the following integral equation

$$\bar{y}(t) = \int_0^\phi \Lambda(t, \delta)\Theta(\delta, \bar{y}(\delta))d\delta, \text{with } t, \delta \in [0, \phi], \tag{69}$$

where $\Lambda(t, \delta)$ represents the corresponding Green's function and defined as

$$\Lambda(t, \delta) = \begin{cases} \dfrac{1 - e^{\rho(t-\delta)}}{\rho}, \text{for } 0 \le \delta \le t \le \phi, \\ 0 \text{ for } 0 \le t \le \delta, \end{cases} \tag{70}$$

where $\rho = \pi/\kappa$ is a constant ratio. Consider the set of real valued functions $\bar{Y} = \mathscr{C}([0, \phi], \mathrm{R})$. For $b > 1$, consider CF b-mertic space defined by

$$M(y, z, c) = e^{-\sup_{n\in[0,1]} |\bar{y}(t) - \bar{z}(t)|^2/c}, \tag{71}$$

for all $y, z \in \bar{Y}$. WE have to show that problem (68) has a solution iff there exists $\bar{y}^*$ in $\bar{Y}$, a solution of the integral equation (69).

Theorem 20. *Consider problem (68), suppose the following conditions are satisfied:*

(i) $|\Theta(\delta, \bar{y}(\delta)) - \Theta(\delta, \bar{z}(\delta))|^2 \le |\bar{y}(\delta), \bar{z}(\delta)|^2$

(ii) $\int_0^\phi \Lambda(t, \delta) \le 1$

Then, the integral equation (69) has a unique solution in $\bar{Y}$.

Proof. Let define an operator $\Gamma : \bar{Y} \longrightarrow \bar{Y}$

$$\Gamma\bar{y}(t) = \int_0^\phi \Lambda(t, \delta)\Theta(\delta, \bar{y}(\delta))d\delta, \text{with } t, \delta \in [0, \phi]. \tag{72}$$

Now,

$$\begin{aligned} e^{-\sup_{n\in[0,1]} |\Gamma\bar{y}(t)-\Gamma\bar{z}(t)|^2/\lambda c} &\ge e^{-\sup_{n\in[0,1]} \int_0^\phi \Lambda(t,\delta)|\Theta(\delta,\bar{y}(\delta)-\Theta(\delta,\bar{y}(\delta))|^2 d\delta)/\lambda c} \\ &\ge e^{-\sup_{n\in[0,1]} |\Theta(\delta,\bar{y}(\delta)-\Theta(\delta,\bar{y}(\delta))|^2 d\delta)/\lambda c} \\ &\ge e^{-\sup_{n\in[0,1]} |\bar{y}(\delta),\bar{z}(\delta)|^2/\lambda c}, \end{aligned} \tag{73}$$

this yields that

$$e^{-\sup_{n\in[0,1]} |\Gamma\bar{y}(t)-\Gamma\bar{z}(t)|^2/\lambda c} \ell \ge e^{-\sup_{n\in[0,1]} |\bar{y}(\delta),\bar{z}(\delta)|^2/\lambda c} \ell. \tag{74}$$

Consequently, we get

$$M(\Gamma\bar{y}, \Gamma\bar{z}, \lambda c) \succeq M(\bar{y}, \bar{z}, c). \tag{75}$$

Thus, by Theorem 18, we obtained the existence of unique solution to integral equation (69). □

4. Conclusion

In this article, we presented the generalization of CF b -metric space and successfully obtained the generalization of Banach contraction principle to the new established setting herein. In support of our obtained results, we have constructed some examples, and with the help of derived result, we guaranteed the existence of unique solution to integral equation, which makes it possible for more integral equations to be verified in such conditions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Acknowledgments

The last author N. Mlaiki would like to thank Prince Sultan University for paying APC and for the support through the TAS Research Lab.

References

[1] P. Hilton and J. Pedersen, "Symmetry in mathematics," *Computers & Mathematcs with Applications*, vol. 12, no. 1-2, pp. 315–328, 1986.

[2] L. A. Zadeh, "Fuzzy sets," *Control*, vol. 8, no. 3, pp. 338–353, 1965.

[3] A. Azam, B. Fisher, and M. Khan, "Common fixed point theorems in complex valued metric spaces," *Analysis and Applications*, vol. 32, no. 3, pp. 243–253, 2011.

[4] M. S. Abdullahi and A. Azam, "Multi-valued fixed points results viarational type contractive conditions in complex valued metric spaces," *Journal of the International Mathematical Virtual Institute*, vol. 7, no. 8, pp. 119–146, 2017.

[5] A. Azam, J. Ahmad, and P. Kumam, "Common fixed point theorems for multi-valued mappings in complex-valued metric spaces," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[6] J. Ahmad, C. Klin-Eam, and A. Azam, "Common fixed points for multivalued mappings in complex valued metric spaces with applications," *Abstract and Applied Analysis*, vol. 2013, Article ID 854965, 12 pages, 2013.

[7] B. S. Choudhury, N. Metiya, and P. Konar, "A discussion on best proximity point and coupled best proximity point in partially ordered metric spaces," *Bulletin of International Mathematical Virtual Institute*, vol. 2015, no. 1, pp. 73–80, 2015.

[8] C. Klin-eam and C. Suanoom, "Some common fixed-point theorems for generalized-contractive-type mappings on complex-valued metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 604215, 6 pages, 2013.

[9] I. A. Bakhtin, "The contraction mapping principle in quasi metric spaces," *Functional analysis*, vol. 30, pp. 26–37, 1989.

[10] S. Czerwik, "Contraction mappings in *b*-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[11] N. Hussain and M. H. Shah, "KKM mappings in cone b-metric spaces," *Computers & Mathematcs with Applications*, vol. 62, no. 4, pp. 1677–1684, 2011.

[12] Z. Kadelburg, S. Radenovic, and M. Sarwar, "Remarks on the paper``coupled fixed point theorems for single-valued operators in b-metric spaces''," *Mathematics Interdisciplinary Research*, vol. 2, pp. 1–8, 2017.

[13] A. K. Dubey, R. Shukla, and R. P. Dubey, "Some fixed point theorems in complex valued b-metric spaces," *Journal of Complex Systems*, vol. 2015, Article ID 832467, 7 pages, 2015.

[14] A. H. Ansari, O. Ege, and S. Radenović, "Some fixed point theorem in complex valued G_b metric space," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 112, no. 2, pp. 463–472, 2018.

[15] D. Hasanah, "Fixed point theorems in complex valued B-metric spaces," *CAUCHY*, vol. 4, no. 4, pp. 138–145, 2017.

[16] P. Pao-Ming and L. Ying-Ming, "Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence," *Journal of Mathematical Analysis and Applications*, vol. 76, no. 2, pp. 571–599, 1980.

[17] A. George and P. Veeremani, "On some results of analysis for fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 90, pp. 365–368, 1997.

[18] S. Heilpern, "Fuzzy mappings and fixed point theorem," *Journal of Mathematical Analysis and Applications*, vol. 83, no. 2, pp. 566–569, 1981.

[19] S. C. Arora and V. Sharma, "Fixed point theorems for fuzzy mappings," *Fuzzy Sets System*, vol. 110, no. 1, pp. 127–130, 2000.

[20] V. D. Estruch and A. Vidal, "A note on fixed point for fuzzy mappings," *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, vol. 32, pp. 39–45, 2001.

[21] A. Green and J. Pastor, *Fixed Point Theorem for Fuzzy Contraction Mapping*, Rendiconti dell'Istituto di Matematica dell'Università di Trieste, 1999.

[22] S. Shukla, R. Rodrguez-López, and M. Abbas, "Fixed point results for contractive mappings in complex valued fuzzy metric spaces," *Fixed Point Theory*, vol. 19, pp. 1–22, 2011.

[23] A. Azam and I. Beg, "Common fuzzy fixed points for fuzzy mappings," *Fixed Point Theory and Applications for Function Spaces*, vol. 2013, no. 1, pp. 1–11, 2013.

[24] T. Došenovic, D. Rakic, B. Caric, and S. Radenovic, "Multivalued generalizations of fixed point results in fuzzy metric spaces," *Nonlinear Analysis: Modelling and Control*, vol. 21, no. 2, pp. 211–222, 2016.

[25] I. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces," *Kybernetica*, vol. 11, pp. 336–344, 1975.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 4123949, 13 pages
https://doi.org/10.1155/2022/4123949

Research Article

Existence Results of Fuzzy Delay Impulsive Fractional Differential Equation by Fixed Point Theory Approach

Aziz Khan,[1] **Ramsha Shafqat**,[2] **and Azmat Ullah Khan Niazi**[2]

[1]*Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, 11586 Riyadh, Saudi Arabia*
[2]*Department of Mathematics and Statistics, University of Lahore, Sargodha, Pakistan*

Correspondence should be addressed to Ramsha Shafqat; ramshawarriach@gmail.com

Received 18 May 2022; Accepted 29 July 2022; Published 31 August 2022

Academic Editor: Santosh Kumar

The main aim of this article is to study controllability and existence of solution of fuzzy delay impulsive fractional nonlocal integro-differential equation in the sense of Caputo operator. The existence and uniqueness of the solution have been carried out with the help of the Banach fixed point theorem. Moreover, for fuzzy fractional differential equations (FFDEs) driven by the Liu process, this present work introduced a concept of stability in credibility space. Finally, efficient examples are presented to demonstrate the main theoretical findings.

1. Introduction

Fractional-order dynamical equations can be used to model a huge spectrum of physical processes in modern-world observations [1]. Due to its wide range application in various areas of sciences such as physics, chemistry, biology, electronics, thermal systems, electrical engineering, mechanics, signal processing, weapon systems, electrohydraulics, population modeling, robotics, and control, the concept of fuzzy sets continues to catch the attention of researchers [2]. As a result, in recent years, scholars have been increasingly interested in it. As a concept of describing a set with uncertain boundary, the fuzzy set was developed by Zadeh et al. [3]. The concept of possibility measure was studied by Zadeh [4] in 1978. Fuzzy set theory is a very useful technique for simulating uncertain problems. In fuzzy calculus, therefore, the concept of the fractional derivative is essential. Although the possibility measure provides the theoretical basis for the measurement of fuzzy events, it does not satisfy self-duality. Liu B. and Liu Y. [5] studied the concept of credibility measure in 2002, and a sufficient and necessary condition for credibility measure was derived by Li and Liu [6] in 2006. Fractional differential equations (FDEs) are differential equations with fractional derivatives. It is known from the research on fractional derivatives that they originate uniformly from major mathematical reasons. Different types of derivatives exist, such as Caputo and RL [7]. In 1965, Zadeh used the membership function to propose the concept of fuzzy sets for the first time. The FFDE is the most fascinating field. They are useful for understanding phenomena that have an underlying effect. Kwun et al. [8] and Lee et al. [9] investigated the solution of uniqueness-existence for FDEs. Controlled processes have been explored by several researchers. In the case of the fuzzy system, Kwun et al. [10] for the impulsive semilinear FDEs, controllability in n-dimension fuzzy vector space was demonstrated. Park et al. [11] controllability of semilinear fuzzy integro-differential equations with nonlocal conditions was investigated. Park et al. [12] established controllability of impulsive semilinear fuzzy integro-differential equations. Phu and Dung [13] studied stability analysis and controllability of fuzzy control set differential equations. According to Lee et al. [14], in the n-dimensional fuzzy space $\mathbf{E_N}^n$ of a nonlinear fuzzy control system, controllability with nonlocal initial conditions was examined.

Balasubramaniam and Dauer [15] examined the controllability of stochastic systems in Hilbert space of quasilinear stochastic evolution equations, while Feng [16] explored the controllability of stochastic with control systems associated with time-variant coefficients. Arapostathis et al. [17] analyzed the controllability of stochastic differential systems of equations with linear-controlled diffusion affected by

Lipschitz nonlinearity that is limited, smooth, and uniform. Stochastic differential equations given by Brownian motion are a well-known and well-studied area of modern mathematics. A new type of FDE was created using the Liu technique [18], which was described as follows:

$$dX_\nu = f(X_\nu, \nu)d\nu + g(X_\nu, \nu)dC_\nu, \quad (1)$$

where C_ν denotes Liu operation and f and g are functions that have been assigned to it. This class of equations is solved using a fuzzy technique. For homogeneous FDEs, Chen and Qin [19] studied solutions of existence-uniqueness of few special FDEs. Liu [20] investigated an approximate method for solving unknown differential equations. Abbas et al. [21, 22] worked on a partial differential equation. Niazi et al. [23, 24], Iqbal et al. [25], Shafqat et al. [26], Abuasbeh et al. [27], and Alnahdi [28] existence-uniqueness of the FFEE were investigated. Arjunan et al. [29–32] worked on the fractional differential inclusions.

Using conclusions of Liu [20], Jeong et al. [33] focused on exact controllability in credibility space for FDEs. Abstract FDEs' complete controllability in credibility space is as follows:

$$dx(\nu, \varpi) = Ax(\nu, \varpi)d\nu + f(\nu, x(\nu, \varpi))d\mathscr{C}_\nu + Bu(\nu), \nu \in [0, \mathfrak{J}],$$
$$x(0) = x_0. \quad (2)$$

We used the Caputo derivative to prove controllability for the fuzzy delay impulsive fractional integro-evolution equation in credibility space with nonlocal condition; as a result of the above research,

$$\begin{aligned}{}_0^{\mathscr{C}}D_\nu^\beta u(\nu, \zeta) &= \mathfrak{g}_i(\nu, u(\nu)) + Au(\nu, \zeta) \\ &\quad + \int_0^\nu f\left((\nu, u(\nu, \zeta)), \int_0^s k(s, u(\nu, \zeta))\right) d\mathscr{C}_\nu \\ &\quad + Bx(\nu)\mathscr{C}x(\nu)d\nu, \nu \in (0, \nu_i], i = 1, 2, \cdots, N, \\ u(0) &= u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)),\end{aligned} \quad (3)$$

where $U(\subset \mathbf{E_N})$ and $V(\subset \mathbf{E_N})$ are two bounded spaces. $\mathbf{E_N}$ is denoted for the set of numbers; all upper semicontinuously convex fuzzy on $\mathbf{R^m}$, and $(\Theta_1, \mathbf{P^m}, \mathscr{C}_r)$, is the credibility space.

The fuzzy coefficient is defined by the state function $u : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow U$. $f : [0, \mathfrak{J}] \times U \longrightarrow U$ is a fuzzy process. $x : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow V$ is regular fuzzy function, $x : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow V$ is control function, and $\mathscr{B}$ is linear bounded operator on V to U. The initial value is $u_0 \in \mathbf{E_N}$, and $\mathscr{C}_\nu$ denotes the Liu process.

The goal of this work is to investigate the existence and stability of results to FDEs and the exact controllability driven by the Liu process, in order to deal with a fuzzy process. Some scholars discovered FDE results in the literature, although the vast majority of them were differential equations of the first order. We discovered the results for Caputo derivatives of order $(0, 1)$ in our research. Stability, as a part of differential equation theory, is vital in both theory and application. As a result, stability is a key subject of study for researchers, and research papers on stability for FDE have been published in the last two decades, for example, essential conditions for solution stability and asymptotic stability of FDEs. We use fuzzy delay impulsive fractional integro-evolution equations with the nonlocal condition. The theory of fuzzy sets continues to gain scholars' attention because of its huge range of applications in different fields of sciences such as engineering, robotics, mechanics, control, thermal systems, electrical, and signal processing.

In Section 2, we go over some basic notions relating to Liu's processes and fuzzy sets. Section 3 demonstrates the existence of solutions of FDE and shows that FDE is precisely controllable. The concept of credibility stability for FDEs driven by the Liu process was developed in Section 4. Finally, in Section 5, several theorems for FDEs driven by the Liu process that is stable in credibility space were demonstrated.

2. Preliminary

If $M_k(\mathbf{R^m})$ be the family of all nonempty compact convex subsets of $\mathbf{R^m}$, then addition and scalar multiplication are commonly defined as $M_k(\mathbf{R^m})$. Consider two nonempty bounded subsets of $\mathbf{R^m}$, A_1 and B_1. The distance between A_1 and B_1 is measured using the Hausdorff metric as

$$d(A_i, B_i) = \max\left\{\sup_{a_i \in A_i} \inf_{b_i \in B_i} \|a_i - b_i\|, \sup_{b_i \in B_i} \inf_{a_i \in A_i} \|a_i - b_i\|\right\}, \quad (4)$$

where $\|\cdot\|$ indicates the usual Euclidean norm in $\mathbf{R^m}$. It follows that $(M_k(\mathbf{R^m}), d)$ is a separable and complete metric space [20]. Satisfy the below condition:

$$\mathbf{E^m} = \{j : \mathbf{R^m} \longrightarrow [0, 1] | j \text{ satisfies(a)} - \text{(b)below}\}, \quad (5)$$

where

(a) j is normal; there exists an $j_0 \in \mathbf{R^m}$ such that $j(j_0) = 1$.

(b) j is fuzzy convex, such that is $j(\lambda\nu + (1-\lambda)s) \geq 1$.

(c) j is upper semicontinuous function on $\mathbf{R^m}$, that is, $j(\nu_0) \geq \lim_{k \longrightarrow \infty} j(\bar{\nu}_k)$ for any $\nu_k \in \mathbf{R^m}(k = 0, 1, 2, \cdots)$, $\nu_k \longrightarrow \nu_0$.

(d) $[j]^0 = cl\{u \in \mathbf{R^m} | j(\nu) > 0\}$ is compact.

In $\mathbf{R^m}$ [34], for $0 < \beta < 1$, denote $[j]^\beta = \{\nu \in \mathbf{R^m} | u(\nu) \geq \beta\}$ and $[u]^0$ are nonempty compact convex sets. Then from (a) to (b), it concludes that β-level set $[j]^\beta \nu \in M_k(\mathbf{R^m})$ for all $0 < \beta < 1$. Using Zadeh's extension principle, we can have scalar multiplication and addition in fuzzy number space $\mathbf{E^m}$ as follows:

$$[j \oplus \wp]^\beta = [j]^\beta \oplus [\wp]^\beta, [kj]^\beta = k[\wp]^\beta, \quad (6)$$

where $j, \wp \in \mathbf{E^m}$, $k \in \mathbf{R^m}$ and $0 < \beta < 1$. Assume $\mathbf{E_N}$ denotes a set of all numbers upper semicontinuously convex fuzzy on $\mathbf{R^m}$.

Definition 1 (see [35]). Given a complete metric D_L by

$$D_L(j,y)=\sup_{0<\beta<1} d_L\left\{[j]^\beta,[\wp]^\beta\right\} = \sup_{0<\beta<1}\max\left\{\left|j_l^\beta-\wp_l^\beta\right|,\left|j_l^\beta-\wp_r^\beta\right|\right\}, \tag{7}$$

for any $u, v\in \mathbf{E_N}$, which satisfies $D_L(j+z,\wp+z)=D_L(j,\wp)$ for each $z\in\mathbf{E_N}$ and $[j]^\alpha=[j_l^\beta,u_r^\beta]$, for each $\beta\in(j,\wp)$ where χ_l^β, $u_r^\beta\in\mathbf{R^m}$ with $j_l^\beta\le u_r^\beta$.

Definition 2 (see [36]). The fractional derivative of RL is stated as

$${}_aD_\nu^\lambda f(\nu)=\left(\frac{d}{d\nu}\right)^{n+1}\int_a^\nu(\nu-\tau)^{n-\lambda}f(\tau)d\tau,\text{where }(n\le\lambda\le n+1). \tag{8}$$

Definition 3 (see [37]). The fractional derivatives in the sense of Caputo ${}_a^{\mathscr{C}}D_\nu^\sigma f(\nu)$ of order $\alpha\in\mathbf{R}^{m^+}$ are described by

$${}_a^{\mathscr{C}}D_\nu^\sigma f(\nu)={}_aD_\nu^\sigma\left(f(\nu)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(\nu-a)^k\right), \tag{9}$$

where $n=[\sigma]+1$ for $\sigma\notin N_0$; $n=\sigma$ for $\sigma\in N_0$.

Definition 4 (see [37]). The Wright function ψ_σ is defined by

$$\psi_\sigma(\varpi)=\sum_{n=0}^{\infty}\frac{(-\varpi)^n}{n!\Gamma(-\sigma n+1-\sigma)} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-\varpi)^n}{(n-1)!}\Gamma(n\sigma)\sin(n\pi\sigma), \tag{10}$$

where $\varpi\in\mathbb{C}$ with $0<\sigma<1$.

Definition 5 (see [38]). For any $j,\wp\in\mathscr{C}([0,T],E_N)$, metric $H_1(\chi,\wp)$ on $\mathscr{C}([0,T],E_N)$ is defined by

$$H_1(j,\wp)=\sup_{0\le\nu\le T}D_L(j(\nu),\wp(\nu)). \tag{11}$$

Consider that Θ_1 is a nonempty set and $\mathbf{P^m}$ denotes power set on Θ_1. A case is a label given to each element of $\mathbf{P^m}$. To present an axiomatic credibility, an idea based on the consideration of A_i will occur. To validate that the number $\mathscr{C}_r\{A_i\}$ is applied to each A_i event, representing the probability of A_i happens. We accept the four main axioms to ensure that the number $\mathscr{C}_r\{A_i\}$ has certain mathematical features that we predict:

(a) Normality property $\mathscr{C}_r\{\Theta_1\}=1$,

(b) Monotonicity property $\mathscr{C}_r\{A_i\}\le\mathscr{C}_r\{B_i\}$, whenever $A_i\subset B_i$,

(c) Self-duality property $\mathscr{C}_r\{A_i\}+\mathscr{C}_r\{A_i^c\}=1$ for any event A_i,

(d) Maximality property $\mathscr{C}_r\{\cup_iA_i\}=\sup_i\mathscr{C}_r\{A_i\}$ for any events $\{A_i\}$ with $\sup_i\mathscr{C}_r\{A_i\}<0.5$.

Definition 6 (see [39]). Take Θ be the nonempty set, P^m be the power set of Θ_1, and $\mathscr{C}_r$ be the credibility measure. After that, the triplet $(\Theta_1,P^m,\mathscr{C}_r)$ is assigned to the set of real numbers.

Definition 7 (see [39]). A fuzzy variable is a function that is generated from a set of real numbers $(\Theta_1,P^m,\mathscr{C}_r)$ to credibility space $(\Theta_1,P^m,\mathscr{C}_r)$.

Definition 8 (see [39]). If $(\Theta_1,P^m,\mathscr{C}_r)$ be credibility space and $(\Theta_1,P^m,\mathscr{C}_r)$ be an index set, a fuzzy process is a function that takes a set of real numbers and multiplies them by $T\times(\Theta_1,P^m,\mathscr{C}_r)$.

It is a fuzzy method. $u(\nu,\zeta)$ is a two-variable function in which $u(\nu,\zeta^*)$ represents a fuzzy variable for each ν^*. For each fixed ζ^*, the function $u(\nu,\zeta)$ is termed a sample path of fuzzy process. The fuzzy process $u(\nu,\zeta)$ is said to be sample continuous if sample ping is continuous for almost all ζ. Alternately of $u(\nu,\zeta)$, we frequently use the notation u_ν.

Definition 9 (see [39]). $(\Theta_1,P^m,\mathscr{C}_r)$ is the symbol of a credibility space. The β-level set is applied for the fuzzy random variable u_ν in credibility space for each $\beta\in(0,1)$.

$$[u_\nu]^\beta=\left[(u_\nu)_l^\beta,(u_\nu)_r^\beta\right], \tag{12}$$

is defined by

$$(u_\nu)_l^\beta=\inf\,(u_\nu)^\beta=\inf\,\{a\in\mathbf{R^m}\,;u_\nu(a)\ge\beta\}, \quad (u_\nu)_r^\beta=\sup\,(u_\nu)^\beta=\inf\,\{a\in\mathbf{R^m}\,;u_\nu(a)\ge\beta\}, \tag{13}$$

where $(u_\nu)_l^\beta,(u_\nu)_r^\beta\in\mathbf{R^m}$ with $(u_\nu)_l^\beta\le(u_\nu)_r^\beta$ when $\beta\in(0,1)$.

Definition 10 (see [5]). Suppose that ϖ is a fuzzy variable and that r is a real number. Then, ϖ's expected value is defined:

$$E\varpi=\int_0^{+\infty}C_r\{\varpi\ge r\}dr-\int_{-\infty}^0\mathscr{C}_r\{\varpi\le r\}dr, \tag{14}$$

if at least one of the integrals is finite.

Lemma 11 (see [5]). *If ϖ is a fuzzy vector, then the following are properties of expected value operator E:*

(a) If $f\le\mathfrak{g}$, $\mathbf{E}[f(\varpi)]\le\mathbf{E}[\mathfrak{g}(\varpi)]$

(b) $\mathbf{E}[-f(\varpi)]=-\mathbf{E}[f(\varpi)]$

(c) If f and $\mathfrak{g}$ are comonotonic, we have for any nonnegative real numbers a_i and b_i,

(a)

$$\mathbf{E}[a_i f(\varpi) + b_i \mathfrak{g}(\varpi)] = a_1 \mathbf{E}[f(\varpi)] + b_1 \mathbf{E}[\mathfrak{g}(\varpi)], \tag{15}$$

where $f(\varpi)$ and $\mathfrak{g}(\varpi)$ are fuzzy variables, respectively.

Definition 12 (see [5]). A fuzzy process $\mathscr{C}_\nu$ is Liu process, if

(a) $\mathscr{C}_0 = 0$,

(b) the $\mathscr{C}_\nu$ has independent and stationary increments,

(c) any increment $\mathscr{C}_{\nu+s} - \mathscr{C}_s$ is normally distributed fuzzy variable with expected value $e\nu$ and variance $\phi^2\nu^2$, with membership function.

$$\xi(u) = 2\left(1 + \exp\left(\frac{\pi|u - e\nu|}{\sqrt{6}\phi\nu}\right)\right)^{-1}, u \in \mathbf{R}^\mathbf{m} \tag{16}$$

The parameters ϕ and e represent the diffusion and drift coefficients, respectively. If $e = 0$ and $\phi = 1$, the Liu process is standard.

Definition 13 (see [40]). Suppose that $\mathscr{C}_\nu$ is a standard Liu process and u_ν is a fuzzy process. The mesh is fixed as $c = \nu_0 < \cdots < \nu_n = d$ for any partition of the closed interval $[c, d]$ with $c = \nu_0 < \cdots < \nu_n = d$,

$$\Delta = \max_{1 \le i \le n}(\nu_i - \nu_{i-1}). \tag{17}$$

After that, the fuzzy integral of u_ν with regard to $\mathscr{C}_\nu$ is calculated:

$$\int_c^d u_\nu d\mathscr{C}_\nu = \lim_{\Delta \longrightarrow 0} \sum_{i=1}^{n} \mu(\nu_{i-1})\left(\mathscr{C}_{\nu_i} - \mathscr{C}_{\nu_{i-1}}\right), \tag{18}$$

determined by the limit exists almost positively and is a fuzzy variable.

Lemma 14 (see [40]). *Consider that $\mathscr{C}_\nu$ represent the standard Liu process with $\mathscr{C}_r\{\zeta\} > 0$, and the direction $\mathscr{C}_\nu$ is Lipschitz continuous, employing the below inequality:*

$$\left|\mathscr{C}_{\nu_1} - \mathscr{C}_{\nu_2}\right| < \mathscr{K}(\zeta)|\nu_1 - \nu_2|, \tag{19}$$

where $\mathscr{K}(\zeta)$ is Lipschitz, which is a fuzzy variable described by

$$\mathscr{K}(\zeta) = \begin{cases} \sup\limits_{0 \le s \le \nu} \dfrac{|\mathscr{C}_\nu - \mathscr{C}_s|}{\nu} - s, & \mathscr{C}_r\{\zeta\} > 1, \\ \infty, & \text{otherwise}, \end{cases} \tag{20}$$

and $E[\mathscr{K}^p] < \infty$ for all $p > 1$.

Lemma 15 (see [40]). *Assume that $h(\nu\,;c)$ is a continuously differentiable function and that $\mathscr{C}_\nu$ is a standard Liu process. The function is defined as $u_\nu = h(\nu\,;\mathscr{C}_\nu)$. Then, there is the chain rule, which is as follows:*

$$du_\nu = \frac{\partial h(\nu\,;\mathscr{C}_\nu)}{\partial \nu} d\nu + \frac{\partial h(\nu\,;\mathscr{C}_\nu)}{\partial \mathscr{C}} d\mathscr{C}_\nu. \tag{21}$$

Lemma 16 (see [40]). *The fuzzy integral inequality exists if $f(nu)$ is a continuous fuzzy process:*

$$\left|\int_c^d f(\nu) d\mathscr{C}_\nu\right| \le \mathscr{K} \int_c^d |f(\nu)| d\nu. \tag{22}$$

In Lemma 14, the term $\mathscr{K} = \mathscr{K}(\zeta)$ is defined.

3. Existence of Solutions

This part applies the symbol u_ν instead of the lengthy notation $u(\nu, \zeta)$, as defined by Definition 8. The existence-uniqueness of solutions to FDE 1 ($x \equiv 0$) has been investigated.

$$\begin{cases} {}^{\mathscr{C}}_{0}D^{\beta}_{\nu} u_\nu = \mathfrak{g}_i u_\nu + A u_\nu + \displaystyle\int_0^\nu f\left((\nu, u_\nu) + \int_0^s \mathscr{K}(s, u_\nu)\right) d\mathscr{C}_\nu, & \beta \in (0, 1), \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)), & \in E_N, \end{cases} \tag{23}$$

where u_ν is state that includes values from the $U(\subset \mathbf{E_N})$ set of values. The set of all upper semicontinuously convex fuzzy numbers on $\mathbf{R}^\mathbf{m}$ is called $\mathbf{E_N}$, credibility space is $(\Theta_1, \mathbf{P}^\mathbf{m}, \mathscr{C}_r)$, fuzzy coefficient is A, and state function $u : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P}^\mathbf{m}, \mathscr{C}_r) \longrightarrow U$ is fuzzy process, $f : [0, \mathfrak{J}] \times U \longrightarrow U$ is regular fuzzy function, $\mathscr{C}_\nu$ is standard Liu process, and $u_0 \in \mathbf{E_N}$ is initial value.

Lemma 17. *If $u(\nu)$ is the solution of equation (3) for $u(0) = u_0 + \mathfrak{g}(\nu_1, \nu_2, \cdots, \nu_p, u(.)$, then $u(\nu)$ is given by*

$$\begin{aligned} u(\nu) = {} & \nu^{\beta-1}\left(u_0 + \mathfrak{g}\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right) + \frac{1}{\sqrt{q}}\left[\int_0^\nu (\nu - s)^{\beta-1} \mathfrak{g}_i(s, x(s)) ds\right.\right. \\ & + \int_0^\nu (\nu - s)^{\beta-1}\left[Au(s, \zeta) + \int_0^\nu f\right. \\ & \left.\left.\cdot \left(s, u(s, \zeta), \int_0^s \mathscr{K}(s, u(s, \zeta)) d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\right]\right] ds, \end{aligned} \tag{24}$$

holds, and then,

$$\begin{aligned} u(\nu) = {} & \nu^{\beta-1} P_\beta(\nu)\left(u_0 + \mathfrak{g}\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)\right) \\ & + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_i(s, x(s)) ds \\ & + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s)[Au(s, \zeta) \\ & + \int_0^\nu f\left(s, u(s, \zeta), \int_0^s \mathscr{K}(s, u(s, \zeta)) d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\Bigg] ds, \end{aligned} \tag{25}$$

where

$$P_q(\nu) = \int_0^{\infty} q\zeta M_q(\zeta) Q(\nu^q \zeta) d\zeta. \quad (26)$$

Suppose that the statements below are correct:

(J_1) For $u_\nu, v_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U), \nu \in [0, \mathfrak{T}]$. There exist positive number m that is

$$d_L\left([f(\nu, u_\nu)]^\beta, [f(\nu, v_\nu)]^\beta\right) \le m d_L\left([u_\nu]^\beta, [v_\nu]^\beta\right)$$
$$f\left(0, X_{\{0\}}(0)\right) \equiv 0. \quad (27)$$

(J_2) $2cm\mathscr{K}\mathfrak{T} \le 1$. Because of Lemma 17, (23) has the solution u_ν. As a result, we establish in Theorem 18 that the solution to (23) is unique.

Theorem 18. *For $(u_0 + \mathfrak{g}(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \in E_N$, if ($J_1$) and ($J_2$) are hold, (23) has an unique solution $u_\nu \in \mathscr{C}([0, \mathfrak{T}]) \times (\Theta_1, P^m, \mathscr{C}_r), U)$.*

Proof. For all $\varpi_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U), \nu \in [0, \mathfrak{T}]$, define

$$\begin{aligned} \phi\varpi_\nu = \nu^{\beta-1} P_\beta(\nu)\big(u_0 + h\big(\nu_1, \nu_2, \cdots, \nu_p, u(.)\big) \\ + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s)\mathfrak{g}_i(s, \varpi_s)\Big) ds \\ + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \\ \cdot \left[A\varpi_s + \int_0^\nu f\left(s, \varpi_s, \int_0^s K(s, \varpi_s) d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\right] ds. \end{aligned} \quad (28)$$

□

As a result, the $\phi\varpi : [0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow ([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U)$ can be established as

$$\phi : \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U) \longrightarrow \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U). \quad (29)$$

For equation (23), ϕ is a fixed point which is likewise an obvious solution. $\varpi_\nu, \mu_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U)$, according to hypothesis (J_1) and Lemma 16.

$$\begin{aligned} & d_L\left([\phi\varpi_\nu]^\beta, [\phi\mu_\nu]^\beta\right) \\ &= d_L\Bigg(\Bigg[\Bigg[\int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s)\mathfrak{g}_i(s, \varpi_s) + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \\ &\quad \cdot \left[A(s, \varpi_s) + f\left((s, \varpi_s), \int_0^s \mathscr{K}(s, \varpi_s) d\mathscr{C}_s\right)\right]\Bigg]\Bigg]^\beta, \\ &\quad \cdot \left[\int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s)\mathfrak{g}_i(s, \mu_s) + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s)\right. \\ &\quad \left. \cdot \left[A\mu_s + f\left((s, \mu_s), \int_0^s \mathscr{K}(s, \mu_s) dC_s\right)\right]^\beta\right) \\ &\le cm\mathscr{K} \int_0^\nu d_L\left([\theta_s]^\beta, [\mu_s]^\beta\right) ds. \end{aligned} \quad (30)$$

Therefore, we obtain

$$\begin{aligned} D_L(\phi\varpi_\nu, \phi\mu_\nu) &= \sup_{\beta \in (0,1)} d_L\left([\phi\varpi_\nu]^\beta, [\phi\mu_\nu]^\beta\right) \\ &\le \mathrm{cm}\mathscr{K} \int_0^\nu \sup_{\beta \in (0,1)} d_L\left([\varpi_\nu]^\beta, [\mu_\nu]^\beta\right) ds \\ &= \mathrm{cm}\mathscr{K} \int_0^\nu D_L(\varpi_s, \mu_s) ds. \end{aligned} \quad (31)$$

As a result, according to Lemma 11, for a.s. $\varpi \in \Theta_1$,

$$\begin{aligned} E(H_1(\phi\varpi, \phi\mu)) &= E\left(\sup_{\nu \in (0,T]} D_L(\phi\varpi_\nu, \phi\mu_\nu)\right) \\ &\le E\left(cm\mathscr{K} \sup_{\nu \in (0,\mathfrak{T}]} \int_0^\nu D_L(\varpi_\nu, \mu_\nu)\right) \\ &\le cm\mathscr{K}\mathfrak{T} E(H_1(\varpi, \mu)). \end{aligned} \quad (32)$$

A contraction mapping is ϕ according to hypothesis (J_2). The Banach fixed point theorem equation (23) has unique fixed point $x_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U)$.

3.1. Exact Controllability. In this section, we will study exact controllability for differential equation in the context of Caputo operator (3). We investigate a solution for equation (3) x in $V(\subset \mathbf{E_N})$.

$$\begin{cases} \phi\varpi_\nu = \nu^{\beta-1} P_\beta(\nu)(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s)\mathfrak{g}_i(s, u_s)) ds + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s)\left[Au_s + \int_0^\nu f\left(s, u_s, \int_0^s \mathscr{K}(s, u_s) d\mathscr{C}_s\right) + Bu_s\mathscr{C}u_s\right] ds, \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)), \end{cases} \quad (33)$$

where $\mathbf{S}(\nu)$ continuous, such that $\mathbf{S}(0)=I=\mathbf{S}'(0)$ and $|\mathbf{S}(\nu)|\le c, c>0, \nu\in[0,\mathfrak{T}]$. The term of controllability is defined for Caputo fuzzy differential equations.

Definition 19. Equation (3) is called a controllable on $[0,\mathfrak{T}]$, if there is control $u_\nu \in V$ for every $u_0\in E_N$ where the solution u of (3) satisfies the condition $u_\nu = u^{-1}\in U$, a.s. ζ, that is, $[u_\nu]^\beta=[u^1]^\beta$.

Given fuzzy $\tilde{G}:\tilde{P}(\mathbf{R^m})\longrightarrow U$ mapping such that

$$\tilde{G}^\beta(\nu)=\begin{cases}\int_0^T(\nu-s)^{\beta-1}P_\beta(\nu-s)B\nu_s\mathscr{C}\nu_s ds, & \wp\subset\bar{\Gamma}_j,\\ 0, & otherwise,\end{cases} \tag{34}$$

where $\bar{\Gamma}_x$ is closure of support x and a nonempty fuzzy subset $\tilde{P}(\mathbf{R^m})$ of $\mathbf{R^m}$.

After that, there is a $\tilde{G}_i^\beta (i=m,n)$,

$$\begin{aligned}\tilde{G}_m^\beta(\wp_m)&=\int_0^T(\nu-s)^{q-1}P_m^\beta(\nu-s)B(\wp_s)_m\mathscr{C}(\wp_s)_m ds, (\wp_s)_m\in\left[(\wp_s)_m^\beta,(\wp_s)^1\right],\\ \tilde{G}_n^\beta(\wp_n)&=\int_0^T(\nu-s)^{q-1}P_n^\beta(\nu-s)B(\wp_s)_n\mathscr{C}(\wp_s)_n ds, (\wp_s)_n\in\left[(\wp_s)^1,(\wp_s)_n^\beta\right].\end{aligned} \tag{35}$$

We assume that $\tilde{G}_m^\beta, \tilde{G}_n^\beta$ are bijective functions. A β-level set of x_s can be presented as below:

$$\begin{aligned}[x_s]^\beta&=\left[(x_s)_m^\beta,(x_s)_n^\beta\right]\\ &=\left[\left(\tilde{G}_m^\beta\right)^{-1}\left\{\left(u^1\right)_m^\beta-\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_m^\beta\right.\right.\right.\\ &\quad\left.-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{im}^\beta(s,u_s)\right)ds\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+\int_0^\nu f_m^\beta\left(s,u_s,\int_0^s\mathscr{K}_m^\beta(s,u_s)d\mathscr{C}_s\right)\right.\\ &\quad\left.\left.+B_m^\beta(u_s)\mathscr{C}_m^\beta(u_s)ds\right]ds\right\},\left(\tilde{G}_n^\beta\right)^{-1}\\ &\quad\cdot\left\{-\left(u^1\right)_n^\beta-\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_n^\beta\right.\right.\\ &\quad\left.-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{\text{in}}^\beta(s,u_s)\right)ds\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\\ &\quad\left.\left.\cdot\left[Au_s+\int_0^\nu f_n^\beta\left(s,u_s,\int_0^s\mathscr{K}_n^\beta(s,u_s)d\mathscr{C}_s\right)+B_n^\beta(u_s)\mathscr{C}_n^\beta(u_s)ds\right]\right\}\right].\end{aligned} \tag{36}$$

This expression is substituted into (33) to get the β-level of x_ν.

$$\begin{aligned}[u_\nu]^\beta&=\left[\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right.\right.\\ &\quad\left.+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_s\right)\right.\\ &\quad\left.+Bu_s\mathscr{C}u_s]ds\right]^\beta\\ &=\left[\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_m^\beta\right.\right.\\ &\quad\left.+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{mi}^\beta(s,u_s)\right)ds\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\\ &\quad\cdot\left[Au_s+f_m^\beta\left(s,u_s,\int_0^s\mathscr{K}_m^\beta(s,u_s)d\mathscr{C}_s\right)\right]\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)B\left(\tilde{G}_m^\beta\right)^{-1}\\ &\quad\cdot\left\{\left(u^1\right)_m^\beta-\nu^{\beta-1}P_\beta(\nu)\left(u_0+\mathfrak{g}\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_m^\beta\right.\right.\\ &\quad\left.-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{mi}^\beta(s,u_s)\right)ds\\ &\quad-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s-f_m^\beta\left(s,u_s,\int_0^s\mathscr{K}_m^\beta(s,u_s)d\mathscr{C}_s\right)\right.\\ &\quad\left.\left.-Bu_s\mathscr{C}u_s\right]\right\}ds,\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_n^\beta\right.\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{ni}^\beta(s,u_s))ds\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+f_n^\beta\left(s,u_s,\int_0^s\mathscr{K}_n^\beta(s,u_s)d\mathscr{C}_s\right)\right]\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)B\left(\tilde{G}_n^\beta\right)^{-1}\\ &\quad\cdot\left\{\left(u^1\right)_n^\beta-\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_n^\beta\right.\right.\\ &\quad-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{ni}^\beta(s,u_s))ds\\ &\quad-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\\ &\quad\left.\left.\cdot\left[Au_s-f_n^\beta\left(s,u_s,\int_0^s\mathscr{K}_n^\beta(s,u_s)d\mathscr{C}_s\right)-Bu_s\mathscr{C}u_s\right]\right\}ds\right]\\ &=\left[\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_m^\beta\right.\right.\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{mi}^\beta(s,u_s))ds\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+f_m^\beta\left(s,u_s,\int_0^s\mathscr{K}_m^\beta(s,u_s)d\mathscr{C}_s\right)\right]\\ &\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\tilde{G}_n^\beta\left(\tilde{G}_n^\beta\right)^{-1}\\ &\quad\cdot\left\{\left(u^1\right)_m^\beta-\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_m^\beta\right.\right.\\ &\quad-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{mi}^\beta(s,u_s))ds\\ &\quad-\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s-f_m^\beta\left(s,u_s,\int_0^s\mathscr{K}_m^\beta(s,u_s)d\mathscr{C}_s\right)\right.\\ &\quad\left.-Bu_s\mathscr{C}u_s]\}ds,\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)_n^\beta\right.\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{ni}^{\beta}(s,u_s))ds\\
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+f_n^{\beta}\left(s,u_s,\int_0^s\mathscr{K}_n^{\beta}(s,u_s)d\mathscr{C}_s\right)\right]\\
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\tilde{G}_n^{\beta}\left(\tilde{G}_n^{\beta}\right)^{-1}\\
&\cdot\left\{\left(u^1\right)_n^{\beta}-\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right)_n^{\beta}\right.\\
&-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{ni}^{\beta}(s,u_s))ds\\
&-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\\
&\cdot\left[Au_s-f_n^{\beta}\left(s,u_s,\int_0^s\mathscr{K}_n^{\beta}(s,u_s)d\mathscr{C}_s\right)Bu_s\mathscr{C}u_s\right]\}ds]\\
&=\left[\left(u^1\right)_m^{\beta},\left(u^1\right)_n^{\beta}\right]=\left[u^1\right]^{\alpha}.
\end{aligned}
\tag{37}
$$

Hence, this control x_ν satisfies $u_\nu=u^1$, a.s. ζ.
We now set

$$
\begin{aligned}
\psi u_\nu={}&\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right.\\
&+\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds+\int_0^{\nu}(\nu-s)^{\beta-1}\\
&\cdot P_\beta(\nu-s)\left[Au_s+f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_s\right)\right]\\
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\left(\tilde{G}\right)^{-1}\left\{\left(u^1\right)-\nu^{\beta-1}P_\beta(\nu)\right.\\
&\cdot\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\right.\\
&\cdot\left.\mathfrak{g}_i(s,u_s)\right)ds-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s-f\right.\\
&\cdot\left.\left.\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_s\right)-Bu_s\mathscr{C}u_s\right]\right\}ds.
\end{aligned}
\tag{38}
$$

Fuzzy mappings $\tilde{G}^{-1}$ holds the above equation.

$$
\begin{aligned}
&d_L\left([\psi u_\nu]^{\beta},[\psi \nu_\nu]^{\beta}\right]\right)\\
&=d_L\left(\left[\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right.\right.\right.\\
&+\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds\\
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_s\right)\right]\\
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\left(\tilde{G}\right)^{-1}\left\{\left(u^1\right)-\nu^{\beta-1}P_\beta(\nu)\right.\\
&\cdot\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds\\
&-\left.\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s-f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_s\right)-Bu_s\mathscr{C}u_s\right]\right\}ds\right]^{\beta},\\
&\nu^{\beta-1}P_\beta(\nu)\left(\nu_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,\nu(.)\right)+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)\right)ds\\
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[A\nu_s+f\left(s,\nu_s,\int_0^s\mathscr{K}(s,\nu_s)d\mathscr{C}_s\right)\right]\\
&+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\tilde{G}^{-1}\left\{\left(\nu^1\right)-\nu^{\beta-1}P_\beta(\nu)\right.\\
&\cdot\left(\nu_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,\nu(.)\right)-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)\right)ds\\
&-\left.\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[A\nu_s-f\left(s,\nu_s,\int_0^s\mathscr{K}(s,\nu_s)dC_s\right)-B\nu_s\mathscr{C}\nu_s\right]\right\}ds\right)\\
&\le d_L\left(\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)ds+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\right.\right.\right.\\
&\cdot\left.\left.\left[Au_s+f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_s\right)\right]\right]\right]^{\beta},\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)ds\right.\right.\\
&+\left.\left.\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[A\nu_s+f\left(s,\nu_s,\int_0^s\mathscr{K}(s,\nu_s)d\mathscr{C}_\tau(s)\right)\right]\right]\right]^{\beta}\right)\\
&+d_L\left(\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\tilde{G}^{-1}\times\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds\right.\right.\\
&-\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_\tau(s)ds\right]\right.,\\
&\cdot\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)ds+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\\
&\cdot\left.\left.\left[A\nu_s+f\left(s,\nu_s,\int_0^s\mathscr{K}(s,\nu_s)d\mathscr{C}_s\right)\right]\right]^{\beta}\right)\\
&+d_L\left(\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\tilde{G}^{-1}\times\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds\right.\right.\\
&-\left.\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_\tau(s)ds\right)\right]^{\beta}\right)\\
&\le cm\mathscr{K}\int_0^{\nu}d_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds\\
&+d_L\left(\left[\tilde{G}\tilde{G}^{-1}\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)ds\right.\right.\right.\\
&+\left.\left.\int_0^{\nu}f\left(s,u_s,\int_0^s\mathscr{K}(s,u_s)d\mathscr{C}_s\right)\right]\right]^{\beta}\left[\tilde{G}\tilde{G}^{-1}\left[\int_0^{\nu}(\nu-s)^{\beta-1}\right.\right.\\
&\cdot\left.\left.\left.P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)ds+\int_0^{\nu}f\left(s,\nu_s,\int_0^s\mathscr{K}(s,\nu_s)d\mathscr{C}_s\right)\right]\right]^{\beta}\right)\\
&\le cm\mathscr{K}\int_0^{\nu}d_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds+cm\mathscr{K}\int_0^{\nu}d_L\left([f(s,u_s)]^{\beta},[f(s,\nu_s)]^{\beta}\right)ds\\
&\le 2cm\mathscr{K}\int_0^{\nu}d_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds.
\end{aligned}
\tag{39}
$$

Theorem 20. *If Lemma 16 and hypotheses* (J_1) *and* (J_2) *are hold, then equation (3) is controllable on* $[0,\mathfrak{I}]$.

Proof. From $\mathscr{C}([0,\mathfrak{I}]\times(\Theta_1,\mathbf{P^m},U)$ to $\mathscr{C}([0,\mathfrak{I}]$, we can clearly see that ψ is continuous. We have Lemma 16 and hypotheses (J_1) and (J_2) for any given ζ with $\mathscr{C}_r\{\zeta\}>0$, x_ν, $\wp_\nu\in\mathscr{C}([0,\mathfrak{I}]\times(\Theta_1,\mathbf{P^m},\mathscr{C}_r),U)$. ☐

Hence, by Lemma 11,

$$
\begin{aligned}
E(H_1(\psi u,\psi\nu))&=E\left(\sup_{\nu\in[0,\mathfrak{I}]}D_L(\psi u_\nu,\psi\nu_\nu)\right)=E\left(\sup_{\nu\in[0,\mathfrak{I}]}\sup_{0<\beta\le1}D_L\left(|\psi u_\nu|^{\beta},|\psi\nu_\nu|^{\beta}\right)ds\right)\\
&\le E\left(\sup_{\nu\in[0,\mathfrak{I}]}\sup_{0<\beta\le1}2cm\mathscr{K}\int_0^{\nu}D_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds\right)\\
&\le E\left(\sup_{\nu\in[0,\mathfrak{I}]}2cm\mathscr{K}\int_0^{\nu}D_L(u_s,\nu_s)ds\right)\le 2cm\mathscr{K}\mathfrak{I}F(H_1(u,\nu)).
\end{aligned}
\tag{40}
$$

As a consequence, $(2\text{cm}\mathscr{K}\mathfrak{I}) < 1$ is a Ã, sufficient $\mathfrak{I}$. As a result, ψ stands for contraction. The Banach fixed point theorem is now being applied to show that (33) has a single fixed point. $[0, \mathfrak{I}]$ can be used to control (3).

Example 1. We investigate FFDE in credibility space:

$$\begin{cases} {}_0^{\mathscr{C}}D_\nu^\beta u(\nu,\zeta) = \mathfrak{g}_i(\nu, u(\nu)) + Au(\nu,\zeta) + \int_0^\nu f\left((\nu, u(\nu,\zeta)) + \int_0^s k(s, u(\nu,\zeta))\right) d\mathscr{C}_\nu + Bx(\nu)\mathscr{C}x(\nu)d\nu, \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)), \qquad \in \boldsymbol{E}_N, \end{cases}, \tag{41}$$

where states consider values from $U(\subset E_N)$ and space V $(\subset E_N)$ two bounded spaces. The set of all, upper semicontinuously convex, fuzzy numbers on $\mathbf{R^m}$ is $\mathbf{E_N}$ and $(\Theta_1, \mathbf{P^m}, \mathscr{C}_r)$ denotes credibility space.

The state function $u : [0, \mathfrak{I}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow U$ is fuzzy coefficient. Fuzzy process $f : [0, \mathfrak{I}] \times U \longrightarrow U$. $x : [0, \mathfrak{I}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow V$ is a regular fuzzy function, $x : [0, \mathfrak{I}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow V$ is a control function, and B is a V to U linear bounded operator. $u_0 \in \mathbf{E_N}$ is an initial value, and $\mathscr{C}_\nu$ is standard Liu process.

Assume $f(\nu, u_\nu) = \tilde{2}\nu u_\nu$, $\mathbf{S}^{-1}(\nu) = e^{-\tilde{2}\nu}$, defining $w_\nu = \mathbf{S}^{-1}(\nu)u_\nu$. Then, the equations of balance become

$$\begin{cases} u_\nu = \nu^{\beta-1}P_\beta(\nu)(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.))) + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s, x(s))ds + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au(s,\zeta) + \int_0^\nu f\left(s, u(s,\zeta), \int_0^s \mathscr{K}(s, u(s,\zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\right]ds, \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.) \in \mathbf{E_N}. \end{cases} \tag{42}$$

Therefore, Lemma 17 is satisfied.

$[2]^\beta = [\beta+1, 3-\beta]$ is the β-level, set of fuzzy, number $\tilde{2}$, for all $\beta \in (0,1)$. β-level set of $f(\nu, u_\nu)$ is

$$[f(\nu, u_\nu)]^\beta = \nu\left[(\beta+1)(u_\nu)_m^\beta, (3-\beta)(u_\nu)_m^\beta\right]. \tag{43}$$

Further, we have

$$\begin{aligned} & d_L\left([f(\nu, u_\nu)]^\beta, [f(\nu, \nu_\nu)]^\beta\right) \\ & \quad = d_L\left(\nu\left[(\beta+1)(u_\nu)_m^\beta, (3-\beta)(u_\nu)_n^\beta\right], \nu \right. \\ & \qquad \left. \cdot\left[(\beta+1)_m^\beta, (3-\beta)(\nu_\nu)_n^\beta\right]\right) \\ & \quad = \nu \max\left\{(\beta+1)\left|(u_\nu)_m^\beta - (\nu_\nu)_m^\beta\right|,\right. \\ & \qquad \left. \cdot(3-\beta)\left|(u_\nu)_n^\beta - (\nu_\nu)_n^\beta\right|\right\} \\ & \quad \le 3\mathfrak{I} \max\left\{\left|(u_\nu)_m^\beta - (\nu_\nu)_m^\beta\right|, \left|(u_\nu)_n^\beta - (\nu_\nu)_n^\beta\right|\right\} \\ & \quad = md_L\left([u_\nu]^\beta, [\nu_\nu]^\beta\right), \end{aligned} \tag{44}$$

where $m = 3\mathfrak{I}$ satisfies an inequality in the (J1) and (J_2) hypotheses. All conditions given in Theorem 18 are fulfilled. Assume that $\tilde{1}$ is the initial value for u_0. The plan set $u^1 = \tilde{2}$. $\tilde{1}$ is $[\tilde{1}] = [\beta-1, 1-\beta]$, $\beta \in (0,1)$ is β-level set of fuzzy numbers $\tilde{1}$. The x_s of (41)'s β-level set is presented.

$$\begin{aligned} [x_s] &= \left[(x_s)_m^\beta, (x_s)_n^\beta\right] \\ &= \left[\left(\tilde{G}_m^\beta\right)^{-1}\left\{(\beta+1) - S_m^\beta(\mathfrak{I}-s)(\beta-1)\right.\right. \\ &\qquad \left. - \int_0^{\mathfrak{I}} S_m^\beta(\mathfrak{I}-s)s(\beta+1)(u_s)_m^\beta d\mathscr{C}_s\right\}, \left(\tilde{G}_n^\beta\right)^{-1} \\ &\qquad \cdot\left\{(3-\beta) - S_n^\beta(\mathfrak{I})(3-\beta)\right. \\ &\qquad \left.\left. - \int_0^{\mathfrak{I}} S_n^\beta(\mathfrak{I}-s)s(3-\beta)(u_s)_n^\beta d\mathscr{C}_s\right\}\right]. \end{aligned} \tag{45}$$

This expression is then substituted into (42) to get the β-level of u_ν:

$$
\begin{aligned}
[u_\nu]^\beta = \Bigg[& S_m^\beta(\mathfrak{I})(\beta-1) + \int_0^{\mathfrak{I}} S_m^\beta(\mathfrak{I}-s)s(\beta+1)(u_s)_m^\beta d\mathscr{C}_s \\
& + \int_0^{\mathfrak{I}} S_m^\beta(\mathfrak{I}-s)B\left(\tilde{G}_m^\beta\right)^{-1}\Big\{(\beta+1) - S_m^\beta(\mathfrak{I})(\beta-1) \\
& - \int_0^{\mathfrak{I}} S_m^\beta(\mathfrak{I}-s)s(\beta+1)(u_s)_m^\beta d\mathscr{C}_s\Big\} ds, S_n^\beta(\mathfrak{I})(1-\beta) \\
& + \int_0^{\mathfrak{I}} S_n^\beta(\mathfrak{I}-s)s(1-\beta)(u_s)_n^\beta d\mathscr{C}_s \\
& + \int_0^{\mathfrak{I}} S_n^\beta(\mathfrak{I}-s)B\left(\tilde{G}_n^\beta\right)^{-1}\Big\{(3-\beta) - S_r^\beta(\mathfrak{I})(1-\beta) \\
& - \int_0^{\mathfrak{I}} S_n^\beta(\mathfrak{I}-s)s(3-\beta)(u_s)_n^\beta d\mathscr{C}_s\Big\} ds\Bigg] \\
= {} & [(\beta+1), (3,-\beta)] = [\tilde{2}]^\beta.
\end{aligned} \tag{46}
$$

Following that, conditions in Theorem 20 have been fulfilled. As a result, (41) on $[0, T]$ can be controlled.

4. Definition of Stability in Credibility

We shall provide a concept of credibility stability for FFDEs driven by the Liu process in this part.

Definition 21. The FDE 1 is said to be stability in credibility if for, any two, solutions u_ν and v_ν corresponding to different initial values $u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)$ and $v_0 + h(\nu_1, \nu_2, \cdots, \nu_p, v(.)$, we have

$$
\lim_{|u_0 - v_0| \longrightarrow 0} \mathscr{C}_r\{|u_\nu - v_\nu| < \varepsilon\} = 1, \text{for } all\ \nu \geq 0, \tag{47}
$$

where ε is any given number and $\varepsilon > 0$.

Example 2. Take the FFDE to better understand the concept of credibility stability.

$$
\begin{aligned}
u_\nu = {} & \nu^{\beta-1}P_\beta(\nu)\left(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)\right) \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s, x(s))ds \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\Bigg[Au(s,\zeta) \\
& + \int_0^\nu f\left(s, u(s,\zeta), \int_0^s \mathscr{K}(s, u(s,\zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\Bigg]ds,
\end{aligned}
$$

$$
\begin{aligned}
v_\nu = {} & \nu^{\beta-1}P_\beta(\nu)\left(v_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, v(.)\right)\right) \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s, x(s))ds \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\Bigg[Av(s,\zeta) \\
& + \int_0^\nu f\left(s, v(s,\zeta), \int_0^s \mathscr{K}(s, v(s,\zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\Bigg]ds,
\end{aligned} \tag{48}
$$

respectively. Then, we have

$$
\begin{aligned}
|u_\nu - v_\nu| = {} & \Big|\left(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)\right) \\
& - \left(v_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, v(.)\right)\right)\Big|.
\end{aligned} \tag{49}
$$

Deduce to, for any given $\varepsilon > 0$, we always have

$$
\begin{aligned}
& \lim_{\left|\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right)-\left(v_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,v(.)\right)\right)\right|\longrightarrow 0} \mathscr{C}_r\{|u_\nu - v_\nu| < \varepsilon\} \\
& = \lim_{\left|\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right)-\left(v_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,v(.)\right)\right)\right|\longrightarrow 0} \mathscr{C}_r\left\{\left|\left(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)\right) - \left(v_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, v(.)\right)\right)\right| < \varepsilon\right\} = 1, \forall \nu \geq 0.
\end{aligned} \tag{50}
$$

As a result, the credibility of FFDE is stable.

Definition 22. The n-dimensional FDE 1 is called stable in credibility, if for any two solutions u_ν and v_ν corresponding to different initial values $u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)$ and $v_0 + h(\nu_1, \nu_2, \cdots, \nu_p, v(.)$, we have

$$
\lim_{\left\|\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right)-\left(v_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,v(.)\right)\right)\right\|\longrightarrow 0} \mathscr{C}_r\{|u_\nu - v_\nu| < \varepsilon\} = 1, \forall \nu \geq 0. \tag{51}
$$

Example 3. Take an m-dimensional FFDE:

$$
\begin{aligned}
{}_0^{\mathscr{C}}D_\nu^\beta u(\nu,\zeta) = {} & \mathfrak{g}_i(\nu, u(\nu)) + Au(\nu,\zeta) \\
& + \int_0^\nu f\left((\nu, u(\nu,\zeta)), \int_0^s k(s, u(\nu,\zeta))\right) d\mathscr{C}_\nu \\
& + Bx(\nu)\mathscr{C}x(\nu)d\nu.
\end{aligned} \tag{52}
$$

The two solutions corresponding to different initial values are

$$
\begin{aligned}
u_\nu &= \nu^{\beta-1}P_\beta(\nu)(u_0+\mathfrak{g}(\nu_1,\nu_2,\cdots,\nu_p,u(.))\\
&\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)g_i(s,x(s))ds\\
&\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)[Au(s,\zeta)\\
&\quad+\int_0^\nu f\left(s,u(s,\zeta),\int_0^s\mathscr{K}(s,u(s,\zeta))d\mathscr{C}_s\right)+B(s)\mathscr{C}(s)\bigg]ds\\
\nu_\nu &= \nu^{\beta-1}P_\beta(\nu)(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))\\
&\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,x(s))ds\\
&\quad+\int_0^\nu(\nu-s)^{\beta-1}P_\beta(\nu-s)[A\nu(s,\zeta)\\
&\quad+\int_0^\nu f\left(s,\nu(s,\zeta),\int_0^s\mathscr{K}(s,\nu(s,\zeta))d\mathscr{C}_s\right)\\
&\quad+B(s)\mathscr{C}(s)]ds,
\end{aligned}
\tag{53}
$$

respectively. Then, we have

$$
\|u_\nu-\nu_\nu\|=\left\|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))\right\|. \tag{54}
$$

As a result, we always have

$$
\begin{aligned}
&\lim_{(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))|\longrightarrow 0}\mathscr{C}_r\{|u_\nu-\nu_\nu|<\varepsilon\}\\
&=\lim_{|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))|\longrightarrow 0}\mathscr{C}_r\left\{\left|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))\right|<\varepsilon\right\}=1,\forall\nu\ge 0.
\end{aligned}
\tag{55}
$$

Thus, m-dimensional FFDE is stability in credibility.

Note that some fuzzy differential equations driven by the Liu process are not stable in credibility. It will be demonstrated in the following example.

5. Theorems of Stability in Credibility

In this part, we will discuss the necessary criteria for a FFDE driven by the Liu process to achieve credibility stability.

Theorem 23. *Assume the FFDE 1 for each initial value has a unique solution. Then, it is stable in credibility space, if coefficients $f(\nu,u)$ and $\mathfrak{g}(\nu,u)$ satisfy strongly Lipschitz condition*

$$
\begin{aligned}
&D(f(\nu,u)-f(\nu,\nu))+(\mathfrak{g}(\nu,u)+\mathfrak{g}(\nu,\nu))\\
&\le L(\nu)D(u-\nu),\forall u,\nu\in\mathbf{R^m},\nu\ge 0,
\end{aligned}
\tag{56}
$$

for some integrable function $L(\nu)$ on $[0,+\infty)$.

Proof. Let u_ν and ν_ν be two solutions corresponding to differential initial values $(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))$ and $(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))$, respectively. Then, for each $\vartheta\in\Theta_1$,

$$
\begin{aligned}
&D(u_\nu-\nu_\nu)\\
&=D(f(\nu,u_t)d\nu-f(\nu,\nu_\nu)d\nu+D(\mathfrak{g}(\nu,u_\nu)d\mathscr{C}_\nu-g(\nu,\nu_\nu)d\mathscr{C}_\nu)\\
&=D((f(\nu,u_\nu)-f(\nu,\nu_\nu))d\nu+D((g(\nu,u_\nu)-\mathfrak{g}(\nu,\nu_\nu))d\mathscr{C}_\nu)\\
&\le D((f(\nu,u_\nu)-f(\nu,\nu_\nu))d\nu)+D((\mathfrak{g}(\nu,u_\nu)-g(\nu,\nu_\nu))d\mathscr{C}_\nu)\\
&\le L(\nu)D(u_\nu-\nu_\nu)d\nu+DL(t)(u_t-\nu_\nu)d\mathscr{C}_\nu\\
&\le L(\nu)D(u_t-\nu_\nu)d\nu+DL(\nu)|\mathscr{K}(\vartheta)|(u_\nu-\nu_\nu)d\nu\\
&=L(t)(1+|K(\vartheta)|)D(u(\nu)-\nu(\nu)),
\end{aligned}
\tag{57}
$$

where $\mathscr{K}(\vartheta)$ is the Lipschitz constant of the Liu process. When we take integral on both sides of equation (57),

$$
\begin{aligned}
D(u_\nu-\nu_\nu)&\le D((u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))\\
&\quad-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.)))\exp\\
&\quad\cdot\left(1+|\mathscr{K}(\vartheta)|\int_0^\nu L(s)ds\right).
\end{aligned}
\tag{58}
$$

□

For any given $\varepsilon>0$, we always have

$$
\begin{aligned}
&\mathscr{C}_r\{|u_\nu-\nu_\nu|<\varepsilon\}\\
&\ge\left\{\left|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))\right.\right.\\
&\quad\left.-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))\right|\exp\left(1+|\mathscr{K}(\vartheta)|\int_0^\nu L(s)ds\right)<\varepsilon\}.
\end{aligned}
\tag{59}
$$

Since

$$
\begin{aligned}
&\mathscr{C}_r\left\{\left|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))\right.\right.\\
&\quad-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))|\exp\\
&\quad\left.\cdot\left(1+|\mathscr{K}(\vartheta)|\int_0^\nu L(s)ds\right)<\varepsilon\right\}\longrightarrow 1,
\end{aligned}
\tag{60}
$$

as $|u_0-\nu_0|\longrightarrow 0$, we obtain

$$
\lim_{|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))|\longrightarrow 0}\mathscr{C}_r\{|u_\nu-\nu_\nu|<\varepsilon\}=1. \tag{61}
$$

Hence, the FFDE is stability in credibility. If it is not easy to determine whether or not $f(\nu,u)$ and $\mathfrak{g}(\nu,u)$ satisfy strong

Lipschitz condition, the following corollary can be used to determine whether the FFDE is stable in credibility space.

Corollary 24. *Assume $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$ be bounded real value functions on $[0, +\infty)$. If $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$ have derivatives with respect to u and satisfy*

$$\left|f_u'(\nu, u)\right| + \left|\mathfrak{g}_u'(\nu, u)\right| \leq L(\nu), \forall \geq 0, \tag{62}$$

for some integrable function $L(\nu)$ on $[0, +\infty)$, then FFDE 1 is stability in credibility.

Proof. For the bounded real valued functions $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$,

$$|f(\nu, u)| + |\mathfrak{g}(\nu, u)| < \mathscr{K}(1 + |u|), \tag{63}$$

where $\mathscr{K}$ is constant which satisfy $|f(\nu, u)| + |\mathfrak{g}(\nu, u)| < \mathscr{K}$. We can derive from the mean value theorem that

$$\begin{aligned} &\left|f\left(\nu, u'\right) - f\left(\nu, u''\right)\right| + \left|g\left(\nu, u'\right) - \mathfrak{g}\left(\nu, u''\right)\right| \\ &\quad = f_u'(\nu, \xi)\left|u' - u''\right| + \mathfrak{g}_u'(\nu, \eta)\left|u' - u''\right| \\ &\quad \leq L(\nu)\left|u' - u''\right| + L(\nu)\left|u' - u''\right| = 2L(\nu)\left|u' - u''\right|, \end{aligned} \tag{64}$$

where $\xi, \eta \in (u' - u)$ existence-uniqueness theorem demonstrates that FFDE has a unique solution. We can deduce from Theorem 23 that FFDE is stable in credibility. Different from Theorem 23 and Corollary 24, we have below corollary when FFDE is general linear FFDE driven by the Liu process. □

Corollary 25. *Suppose that $u_{1\nu}$, $u_{2\nu}$, $\nu_{1\nu}$, and $\nu_{2\nu}$ are bounded functions, with respect to ν on $[0, +\infty)$. If $u_{1\nu}$ and $\nu_{1\nu}$ are integrable, on $[0, +\infty)$, then linear FDE driven by Liu process*

$$du_\nu = (u_{1\nu}u_\nu + u_{2\nu})d\nu + (\nu_{1\nu}u_\nu + \nu_{2\nu})d\mathscr{C}_\nu, \tag{65}$$

is stability in credibility.

Proof. For the linear FFDE 7, we have $f(\nu, x) = u_{1\nu}x + u_{2\nu}$ and $\mathfrak{g}(\nu, x) = \nu_{1\nu}x + \nu_{2\nu}$, since

$$\begin{aligned} &|u_{1\nu}u_\nu + u_{2\nu}| + |\nu_{1\nu}\nu_\nu + \nu_{2\nu}| \\ &\quad \leq |u_{1\nu}||u_\nu| + |u_{2\nu}| + |\nu_{1\nu}||u_\nu| + |\nu_{2\nu}| \\ &\quad < \mathscr{K}|u_\nu| + \mathscr{K} + \mathscr{K}|u_\nu| + \mathscr{K} = 2\mathscr{K}(|u_\nu| + 1), \\ &|(u_{1\nu}u_\nu + u_{2\nu}) - (u_{1\nu}\nu_\nu + u_{2\nu})| + |(\nu_{1\nu}u_\nu + \nu_{2\nu}) - (\nu_{1\nu}\nu_\nu + \nu_{2\nu})| \\ &\quad = |u_{1\nu}(u_\nu - \nu_\nu)| + |\nu_{1\nu}(u_\nu + \nu_\nu)| \\ &\quad \leq |u_{1\nu}||u_\nu - \nu_\nu| + |\nu_{1\nu}||u_\nu + \nu_\nu| \\ &\quad = (|u_{1\nu}| + |\nu_{1\nu}|)|(u_\nu - \nu_\nu)| \leq 2\mathscr{K}(u_\nu - \nu_\nu), \end{aligned} \tag{66}$$

where $\mathscr{K}$ is a constant which make $u_{1\nu} < \mathscr{K}, u_{2\nu} < \mathscr{K}, \nu_{1\nu} < \mathscr{K}, \nu_{2\nu} < \mathscr{K}$ hold. The existence-uniqueness theorem shows that FDE 7 has a unique solution. Since $L(\nu) = |u_{1\nu}| + |\nu_{1\nu}|$ is integrable function on $[0, +\infty)$, from Theorem 23, the credibility of FFDE can be determined. □

According to Definition 22, Theorem 23 can be used to n -dimensional FFDEs driven by the Liu process.

Theorem 26. *Assume that each initial value of the n -dimensional FFDE 1 has a unique solution. If coefficients $f(\nu, u)$ and $g(\nu, u)$ satisfy Lipschitz's strong condition,then it is stable in credibility space:*

$$\begin{aligned} &\|f(\nu, u) - f(\nu, \nu)\| + \|\mathfrak{g}(\nu, u) - \mathfrak{g}(\nu, \nu)\| \\ &\quad \leq L(\nu)\|u - \nu\|, for \forall u, \nu \in \mathbf{R}^{\mathbf{m}}, \nu \geq 0, \end{aligned} \tag{67}$$

for some integrable function $L(\nu)$ on $[0, +\infty)$.

6. Conclusion

Accurate controllability for FFDEs can be used as a standard when analyzing controllability for semilinear integro-differential equations in the credibility space and fuzzy delay integro-differential equations. Therefore, the research's theoretical conclusions can be applied to construct stochastic extensions on credibility space. The FFDEs driven by the Liu process have an important role in both theory and practice as a technique for dealing with dynamic systems in a fuzzy environment. There have been some proposed stability approaches for FFDEs driven by the Liu process up until now. This is a rewarding field with numerous research projects that can lead to a variety of applications and theories. We hope to learn more about fuzzy fractional evolution problems in future projects. We can discover uniqueness and existence with uncertainty using the Caputo derivative. Future work could include expanding on the mission concept, including observability, and generalizing other activities. This is an interesting area with a lot of study going on that could lead to a lot of different applications and theories. This is a path in which we intend to invest significant resources.

Data Availability

Data is original and references are given where required.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The author Aziz Khan would like to thank the Prince Sultan University for paying the APC and the support through TAS Research Lab.

References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, Hoboken, 1993.

[2] S. S. Mansouri, M. Gachpazan, and O. S. Fard, "Existence, uniqueness and stability of fuzzy fractional differential equations with local Lipschitz and linear growth conditions," *Advances in Difference Equations*, vol. 2017, no. 1, p. 13, 2017.

[3] L. A. Zadeh, G. J. Klir, and B. Yuan, *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers*, vol. 6, World Scientific, 1996.

[4] L. A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 3–28, 1978.

[5] B. Liu and Y. K. Liu, "Expected value of fuzzy variable and fuzzy expected value models," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 4, pp. 445–450, 2002.

[6] X. Li and B. Liu, "A sufficient and necessary condition for credibility measures," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 14, no. 5, pp. 527–535, 2006.

[7] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.

[8] Y. C. Kwun, J. S. Kim, J. S. Hwang, and J. H. Park, "Existence of SOLUTIONS for the impulsive semilinear fuzzy intergrodifferential equations with nonlocal conditions and forcing term with memory in n-dimensional fuzzy vector space(ENn, dε)," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 11, no. 1, pp. 25–32, 2011.

[9] B. Y. Lee, Y. C. Kwun, Y. C. Ahn, and J. H. Park, "The existence and uniqueness of fuzzy solutions for semilinear fuzzy integrodifferential equations using integral contractor," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 9, no. 4, pp. 339–342, 2009.

[10] Y. C. Kwun, M. J. Park, J. S. Kim, J. S. Park, and J. H. Park, "Controllability for the impulsive semilinear fuzzy differential equation in n-dimension fuzzy vector space," *In international conference on fuzzy systems and knowledge discovery*, vol. 19, no. 1, pp. 45–48, 2009.

[11] J. H. Park, J. S. Park, and Y. C. Kwun, "Controllability for the semilinear fuzzy integrodifferential equations with nonlocal conditions," in *International Conference on Fuzzy Systems and Knowledge Discovery*, pp. 221–230, Berlin, Heidelberg, 2006.

[12] J. H. Park, J. S. Park, Y. C. Ahn, and Y. C. Kwun, "Controllability for the impulsive semilinear fuzzy integrodifferential equations," in *Fuzzy Information and Engineering*, pp. 704–713, Springer, Berlin, Heidelberg, 2007.

[13] N. D. Phu and L. Q. Dung, "On the stability and controllability of fuzzy control set differential equations," *International Journal of Reliability and Safety*, vol. 5, no. 3/4, pp. 320–335, 2011.

[14] B. Y. Lee, D. G. Park, G. T. Choi, and Y. C. Kwun, "Controllability for the nonlinear fuzzy control system with nonlocal initial condition in EnN," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 6, no. 1, pp. 15–20, 2006.

[15] P. Balasubramaniam and J. P. Dauer, "Controllability of semilinear stochastic evolution equations in Hilbert space," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 14, no. 4, pp. 329–339, 2001.

[16] Y. Feng, "Convergence theorems for fuzzy random variables and fuzzy martingales," *Fuzzy Sets and Systems*, vol. 103, no. 3, pp. 435–441, 1999.

[17] A. Arapostathis, R. K. George, and M. K. Ghosh, "On the controllability of a class of nonlinear stochastic systems," *Systems & Control Letters*, vol. 44, no. 1, pp. 25–34, 2001.

[18] B. Liu, "Fuzzy process, hybrid process and uncertain process," *Journal of Uncertain systems*, vol. 2, no. 1, pp. 3–16, 2008.

[19] X. Chen and Z. Qin, "A new existence and uniqueness theorem for fuzzy differential equations," *International Journal of Fuzzy Systems*, vol. 13, no. 2, 2011.

[20] Y. Liu, "An analytic method for solving uncertain differential equations," *Journal of Uncertain Systems*, vol. 6, no. 4, pp. 244–249, 2012.

[21] A. Abbas, R. Shafqat, M. B. Jeelani, and N. H. Alharthi, "Significance of chemical reaction and Lorentz force on third-grade fluid flow and heat transfer with Darcy–Forchheimer law over an inclined exponentially stretching sheet embedded in a porous medium," *Symmetry*, vol. 14, no. 4, p. 779, 2022.

[22] A. Abbas, R. Shafqat, M. B. Jeelani, and N. H. Alharthi, "Convective heat and mass transfer in third-grade fluid with Darcy–Forchheimer relation in the presence of thermal-diffusion and diffusion-thermo effects over an exponentially inclined stretching sheet surrounded by a porous medium: a CFD study," *Processes*, vol. 10, no. 4, p. 776, 2022.

[23] A. U. K. Niazi, J. He, R. Shafqat, and B. Ahmed, "Existence, uniqueness, and Eq–Ulam-type stability of fuzzy fractional differential equation," *Fractal and Fractional*, vol. 5, no. 3, p. 66, 2021.

[24] A. U. K. Niazi, N. Iqbal, R. Shah, F. Wannalookkhee, and K. Nonlaopon, "Controllability for fuzzy fractional evolution equations in credibility space," *Fractal and fractional*, vol. 5, no. 3, p. 112, 2021.

[25] N. Iqbal, A. U. K. Niazi, R. Shafqat, and S. Zaland, "Existence and uniqueness of mild solution for fractional-order controlled fuzzy evolution equation," *Journal of Function Spaces*, vol. 2021, Article ID 5795065, 8 pages, 2021.

[26] R. Shafqat, A. U. K. Niazi, M. B. Jeelani, and N. H. Alharthi, "Existence and uniqueness of mild solution where $\alpha \in (1, 2)$ for fuzzy fractional evolution equations with uncertainty," *Fractal and Fractional*, vol. 6, no. 2, p. 65, 2022.

[27] K. Abuasbeh, R. Shafqat, A. U. K. Niazi, and M. Awadalla, "Local and global existence and uniqueness of solution for time-fractional fuzzy Navier–Stokes equations," *Fractal and Fractional*, vol. 6, no. 6, p. 330, 2022.

[28] A. S. Alnahdi, R. Shafqat, A. U. K. Niazi, and M. B. Jeelani, "Pattern formation induced by fuzzy fractional-order model of COVID-19," *Axioms*, vol. 11, no. 7, p. 313, 2022.

[29] M. M. Arjunan, T. Abdeljawad, V. Kavitha, and A. Yousef, "On a new class of Atangana-Baleanu fractional Volterra-Fredholm integro-differential inclusions with non-instantaneous impulses," *Chaos, Solitons & Fractals*, vol. 148, article 111075, 2021.

[30] M. M. Arjunan, A. Hamiaz, and V. Kavitha, "Existence results for Atangana-Baleanu fractional neutral integro-differential systems with infinite delay through sectorial operators," *Chaos, Solitons & Fractals*, vol. 149, article 111042, 2021.

[31] M. M. Arjunan, V. Kavitha, and D. Baleanu, "A new existence results on fractional differential inclusions with state-dependent delay and Mittag-Leffler kernel in Banach space," *Analele ÅŸtiinÅ£ifice ale UniversitÄƒÅ£ii" Ovidius" ConstanÅ£a. Seria MatematicÄƒ*, vol. 30, no. 2, pp. 5–24, 2022.

[32] M. M. Arjunan, P. Anbalagan, and Q. Al-Mdallal, "Robust uniform stability criteria for fractional-order gene regulatory networks with leakage delays," *Mathematical Methods in the Applied Sciences*, pp. 1–18, 2022.

[33] J. H. Jeong, J. S. Kim, H. E. Youm, and J. H. Park, "Exact controllability for fuzzy differential equations using extremal solutions," *Journal of Computational Analysis and Applications*, vol. 23, pp. 1056–1069, 2017.

[34] P. Diamand and P. E. Kloeden, *Metric Space of Fuzzy Sets*, World Scientific, 1994.

[35] G. Wang, Y. Li, and C. Wen, "On fuzzy n-cell numbers and n-dimension fuzzy vectors," *Fuzzy Sets and Systems*, vol. 158, no. 1, pp. 71–84, 2007.

[36] I. Podlubny, "Fractional differential equations," *Mathematics in Science and Engineering*, vol. 198, pp. 41–119, 1999.

[37] F. Mainardi, P. Paradisi, and R. Gorenflo, "Probability distributions generated by fractional diffusion equations," 2007, http://arxiv.org/abs/0704.0320.

[38] W. Fei, "Uniqueness of solutions to fuzzy differential equations driven by Liu's process with non-Lipschitz coefficients," in *2009 Sixth International Conference on Fuzzy Systems and Knowledge Discovery*, vol. 6, pp. 565–569, 2009.

[39] C. C. Travis and G. F. Webb, "Cosine families and abstract nonlinear second order differential equations," *Acta Mathematica Hungarica*, vol. 32, no. 1-2, pp. 75–96, 1978.

[40] Y. Zhou and J. W. He, "New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$," *Evolution Equations & Control Theory*, vol. 10, no. 3, p. 491, 2021.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9011506, 24 pages
https://doi.org/10.1155/2022/9011506

Research Article

Decision-Making on the Solution of a Stochastic Nonlinear Dynamical System of Kannan-Type in New Sequence Space of Soft Functions

Meshayil M. Alsolmi [1] **and Awad A. Bakery** [2]

[1] University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Cairo 11566, Abbassia, Egypt

Correspondence should be addressed to Awad A. Bakery; awad_bakery@yahoo.com

Received 30 June 2022; Accepted 1 August 2022; Published 28 August 2022

Academic Editor: Santosh Kumar

In this paper, we construct and investigate the space of weighted Gamma matrix of order r in Nakano sequence space of soft functions. The idealization of the mappings has been achieved through the use of extended s-soft functions and this sequence space of soft functions. This new space's topological and geometric properties, the multiplication mappings that stand in on it, and the mappings' ideal that correspond to them are discussed. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated prequasi ideal. Interestingly, several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of nonlinear difference equations of soft functions are introduced.

1. Introduction

Probability theory, fuzzy set theory, soft sets, and rough sets have all contributed substantially to the study of uncertainty. But there are drawbacks to these theories that must be considered. For more information and real-world examples, please refer to [1–10]. Numerous mathematicians have investigated potential expansions to the theorem and its applications in various contexts since the publication of the book [11] on the Banach fixed point theorem. The Banach contraction principle is an important part of nonlinear analysis, which uses it as a powerful tool [12–15]. Kannan [16] presented a collection of mappings with the same actions at fixed places as contractions. However, this collection is discontinuous. In Reference [17], an explanation of Kannan operators in modular vector spaces was once tried. Only this one try was ever made as [18–23] show that much attention has been paid to the s-number mapping ideal and the multiplication operator hypothesis in functional analysis. Bakery and Mohamed [24] offered the idea of a prequasi norm on the Nakano sequence space with a variable exponent that fell somewhere in the range $(0, 1]$. They talked about the conditions that must be met to generate prequasi Banach and closed space when it is endowed with a specified prequasi norm and the Fatou property of various prequasi norms on it. They also determined a fixed point for Kannan prequasi norm contraction mappings on it, in addition to the ideal of prequasi Banach mappings derived from s-numbers in this sequence space. Both of these ideals were established. In addition, several fixed point findings of Kannan nonexpansive mappings on generalized Cesàro backward difference sequence space of a nonabsolute type were discovered in [25]. Assume that $\mathfrak{R}$ is the set of real numbers and $\mathcal{N}$ is the set of nonnegative integers. We denote the collection of all nonempty bounded subsets of $\mathcal{R}$ by $\mathfrak{B}(\mathcal{R})$, and E is the set of parameters. By $\mathcal{R}(A)^*$ and $\mathcal{R}(A)$, we indicate the set of nonnegative and all soft real numbers (corresponding to A), where $A \subset E$. The additive identity and multiplicative identity in $\mathcal{R}(A)$ are denoted by $\tilde{0}$ and $\tilde{1}$, respectively. For more details on the arithmetic operations on $\mathcal{R}(A)$, see [26]. Let $\mu : \mathcal{R}(A) \times \mathcal{R}(A) \longrightarrow \mathcal{R}(A)^*$, where $\mu(\tilde{f}, \tilde{g}) = |\tilde{f} - \tilde{g}|$, for all $\tilde{f}, \tilde{g} \in \mathscr{R}(A)$. Assume $\tilde{\rho} : \mathscr{R}(A) \times \mathscr{R}(A) \longrightarrow \mathfrak{R}^+$ is defined by

$$\tilde{\rho}\left(\tilde{f}, \tilde{g}\right) = \max_{\lambda \in A} \mu\left(\tilde{f}, \tilde{g}\right)(\lambda). \tag{1}$$

Given that the proof of many fixed point theorems in a given space requires either growing the space itself or expanding the self-mapping that acts on it, both of these options are viable; we have constructed the space, $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$, which is the domain of weighted Gamma matrix of order r in Nakano soft sequence space since it is constructed by the domain of weighted Gamma matrix of order r defined in $\ell^{\mathfrak{S}}_{((v_l))}$, where the weighted Gamma matrix of order r, $W\Gamma_r = (\lambda^r_{lz}(q))$, is defined as

$$\lambda^r_{lz}(q) = \begin{cases} \dfrac{\begin{bmatrix} r+z-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+l \\ l \end{bmatrix}}, & 0 \le z \le l, \\ 0, & z > l, \end{cases} \tag{2}$$

where r is a positive integer, $q_z \in (0,\infty)$, for all $z \in \mathscr{N}$ and

$$\begin{bmatrix} r+z-1 \\ z \end{bmatrix} = \frac{(r+z-1)!}{z!(r-1)!}. \tag{3}$$

In [27], Roopaei and Basar studied the Gamma spaces, including the spaces of absolutely p-summable, null, convergent, and bounded sequences.

In this article, we have introduced a new general space called $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$ and the mappings' ideal space of solutions for many stochastic nonlinear and matrix systems of Kannan contraction type. We have offered some geometric and topological structures of the soft function space, $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$, multiplication operator acting on it, and its operators' ideal. A fixed point of the Kannan contraction operator exists in this space, and its prequasi operator ideal is confirmed. Finally, we discuss many applications of solutions to nonlinear stochastic dynamical systems and illustrative examples of our findings.

2. Properties of $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$ and Its Operators' Ideal

Some geometric and topological structures of the soft function space, $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$, and its operators' ideals are presented in this section.

By c_0, ℓ_∞, and ℓ_r, we denote the space of null, bounded, and r-absolutely summable sequences of reals. We indicate the space of all bounded, finite rank linear mappings from an infinite-dimensional Banach space $\mathscr{G}$ into an infinite-dimensional Banach space $\mathscr{V}$ by $\mathbb{D}(\mathscr{G}, \mathscr{V})$ and $\mathbb{F}(\mathscr{G}, \mathscr{V})$, and if $\mathscr{G} = \mathscr{V}$, we write $\mathbb{D}(\mathscr{G})$ and $\mathbb{F}(\mathscr{G})$. The space of approximable and compact bounded linear operators from $\mathscr{G}$ into $\mathscr{V}$ will be marked by $\mathscr{A}(\mathscr{G}, \mathscr{V})$ and $\mathscr{K}(\mathscr{G}, \mathscr{V})$, respectively. The ideal of bounded, approximable, and compact mappings between every two infinite-dimensional Banach spaces will be denoted by $\mathbb{D}$, $\mathscr{A}$, and $\mathscr{K}$, respectively. Suppose $\omega^{\mathfrak{S}}$ is the class of all sequence spaces of soft reals.

Definition 1. If $(v_l) \in \mathfrak{R}^{+\mathscr{N}}$, $\mathfrak{R}^{+\mathscr{N}}$ is the space of all sequences of positive reals. The sequence space $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$ with the function τ is defined by

$$\left(\Gamma_r^{\mathfrak{S}}(q, v)\right)_\tau = \left\{\tilde{h} = \left(\widetilde{h_m}\right) \in \omega^{\mathfrak{S}} : \tau\left(\delta\tilde{h}\right) < \infty, \text{for some } \varepsilon > 0\right\},$$
$$\text{where } \tau\left(\tilde{h}\right) = \sum_{m=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{h_z}, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}} \right)^{v_m}. \tag{4}$$

Lemma 2 (see [28]). *If $v_b > 0$ and $x_b, z_b \in \mathfrak{R}$, for all $b \in \mathscr{N}$, and $\hbar = \max\{1, \sup_b v_b\}$, then*

$$|x_b + z_b|^{v_b} \le 2^{\hbar - 1}\left(|x_b|^{v_b} + |z_b|^{v_b}\right). \tag{5}$$

Theorem 3. *Suppose $(v_l) \in \ell_\infty \cap \mathfrak{R}^{+\mathscr{N}}$, then*

$$\left(\Gamma_r^{\mathfrak{S}}(q, v)\right)_\tau = \left\{\tilde{h} = \left(\widetilde{h_b}\right) \in \omega^{\mathfrak{S}} : \tau\left(\delta\tilde{h}\right) < \infty, \textit{for all } \delta > 0\right\}. \tag{6}$$

Proof. Obviously, (v_l) is a bounded sequence. □

Theorem 4. *The space $(\Gamma_r^{\mathfrak{S}}(q, v))_\tau$ is a nonabsolute type, whenever $(v_l) \in [1,\infty)^{\mathscr{N}} \cap \ell_\infty$.*

Proof. Clearly, since

$$\tau\left(\tilde{1}, -\tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots\right) = (q_0)^{v_0} + \left(\frac{|q_0 - rq_1|}{1+r}\right)^{v_1} + \left(\frac{|q_0 - rq_1|}{\begin{bmatrix} r+2 \\ 2 \end{bmatrix}}\right)^{v_2}$$
$$+ \cdots \neq (q_0)^{v_0} + \left(\frac{q_0 + rq_1}{1+r}\right)^{v_1} + \left(\frac{q_0 + rq_1}{\begin{bmatrix} r+2 \\ 2 \end{bmatrix}}\right)^{v_2}$$
$$+ \cdots = \tau\left(\tilde{1}, \tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots\right). \tag{7}$$

□

Definition 5. Assume $(v_b)\in\mathfrak{R}^{+\mathcal{N}}$ and $v_b\ge 1$, for all $b\in\mathcal{N}$:

$$\left(\left|\Gamma_r^{\mathfrak{S}}\right|(q,v)\right)_{\varphi}:=\left\{\tilde{h}=\left(\tilde{h_b}\right)\in\omega^{\mathfrak{S}}:\varphi(\delta f)<\infty,\text{for some}\,\delta>0\right\},\tag{8}$$

where

$$\varphi\left(\tilde{h}\right)=\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\left|\tilde{h_z}\right|,\tilde{0}\right)}{\begin{bmatrix}r+b\\ b\end{bmatrix}}\right)^{v_b}.\tag{9}$$

Theorem 6. *Suppose $(v_l)\in(1,\infty)^{\mathcal{N}}\cap\ell_\infty$ with*

$$\left(\frac{l+1}{\begin{bmatrix}r+l\\ l\end{bmatrix}}\right)\notin\ell_{(v_l)},\tag{10}$$

hence $(|\Gamma_r^{\mathfrak{S}}|(q,v))_{\varphi}\subsetneq(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$.

Proof. Assume $\tilde{f}\in(|\Gamma_r^{\mathfrak{S}}|(q,v))_{\varphi}$, as

$$\begin{aligned}&\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\tilde{f_z},\tilde{0}\right)}{\begin{bmatrix}r+b\\ b\end{bmatrix}}\right)^{v_b}\\ &\le\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\left|\tilde{f_z}\right|,\tilde{0}\right)}{\begin{bmatrix}r+b\\ b\end{bmatrix}}\right)^{v_b}<\infty.\end{aligned}\tag{11}$$

Then $\tilde{f}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$. If we choose

$$\tilde{g}=\left(\frac{\left(-\tilde{1}\right)^z}{\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z}\right)_{z\in\mathcal{N}},\tag{12}$$

one gets $\tilde{g}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$ and $\tilde{g}\notin(|\Gamma_r^{\mathfrak{S}}|(q,v))_{\varphi}$.

Suppose $\mathscr{E}^{\mathfrak{S}}$ is a linear space of sequences of soft functions, and $[p]$ describes an integral part of the real number p. □

Definition 7. The space $\mathscr{E}^{\mathfrak{S}}$ is said to be a private sequence space of soft functions ($\mathfrak{pssf}$) if it satisfies the next setups:

(a1) For all $b\in\mathcal{N}$, then $\tilde{e_b}\in\mathscr{E}^{\mathfrak{S}}$, where $\tilde{e_b}=(\tilde{0},\tilde{0},\cdots,\tilde{1},\tilde{0},\tilde{0},\cdots)$, while $\tilde{1}$ displays at the b^{th} place

(a2) If $\tilde{f}=(\tilde{f_b})\in\omega^{\mathfrak{S}}$, $|\tilde{g}|=(|\tilde{g_b}|)\in\mathscr{E}^{\mathfrak{S}}$ and $|\tilde{f_b}|\le|\tilde{g_b}|$, with $b\in\mathcal{N}$, then $|\tilde{f}|\in\mathscr{E}^{\mathfrak{S}}$

(a3) $(|\widetilde{h_{[b/2]}}|)_{b=0}^{\infty}\in\mathscr{E}^{\mathfrak{S}}$, whenever $(|\tilde{h_b}|)_{b=0}^{\infty}\in\mathscr{E}^{\mathfrak{S}}$

Definition 8 (see [29]). An s-number is a function $s:\mathbb{D}(\mathscr{G},\mathscr{V})\longrightarrow\mathfrak{R}^{+\mathcal{N}}$ that gives all $V\in\mathbb{D}(\mathscr{G},\mathscr{V})$ a $(s_d(V))_{d=0}^{\infty}$ holds the following conditions:

(1) $\|V\|=s_0(V)\ge s_1(V)\ge s_2(V)\ge\cdots\ge 0$, for all $V\in\mathbb{D}(\mathscr{G},\mathscr{V})$

(2) $s_d(VYW)\le\|V\|s_d(Y)\|W\|$, for every $W\in\mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $Y\in\mathbb{D}(\mathscr{G},\mathscr{V})$ and $V\in\mathbb{D}(\mathscr{V},\mathscr{V}_0)$, where $\mathscr{G}_0$ and $\mathscr{V}_0$ are arbitrary Banach spaces

(3) $s_{l+d-1}(V_1+V_2)\le s_l(V_1)+s_d(V_2)$, for every $V_1,V_2\in\mathbb{D}(\mathscr{G},\mathscr{V})$ and $l,d\in\mathcal{N}$

(4) Assume $V\in\mathbb{D}(\mathscr{G},\mathscr{V})$ and $\gamma\in\mathfrak{R}$, then $s_d(\gamma V)=|\gamma|s_d(V)$

(5) If rank $(V)\le d$, then $s_d(V)=0$, for all $V\in\mathbb{D}(\mathscr{G},\mathscr{V})$

(6) $s_{l\ge a}(I_a)=0$ or $s_{l<a}(I_a)=1$, where I_a indicates the unit mapping on the a-dimensional Hilbert space ℓ_2^a

Some examples of s-numbers:

(a) The bth approximation number is defined as $\alpha_b(X)=\inf\{\|X-Y\|: Y\in\mathbb{D}(\mathscr{G},\mathscr{V})\text{ and rank }(Y)\le b\}$

(b) The bth Kolmogorov number is defined as $d_b(X)=\inf_{\dim J\le b}\sup_{\|f\|\le 1}\inf_{g\in J}\|Xf-g\|$

Notation 9 (see [30]).

$$\begin{aligned}\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}&:=\left\{\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})\right\},\text{where}\,\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})\\&:=\left\{V\in D(\mathscr{G},\mathscr{V}):\left(\left(\widetilde{s_j(V)}\right)_{j=0}^{\infty}\in\mathscr{E}^{\mathfrak{S}}\right\},\end{aligned}$$

$$\begin{aligned}\widetilde{D^\alpha}_{\mathscr{E}^{\mathfrak{S}}}&:=\left\{\widetilde{D^\alpha}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})\right\},\text{where}\,\widetilde{D^\alpha}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})\\&:=\left\{V\in D(\mathscr{G},\mathscr{V}):\left(\left(\widetilde{\alpha_j(V)}\right)_{j=0}^{\infty}\in\mathscr{E}^{\mathfrak{S}}\right\},\end{aligned}$$

$$\begin{aligned}\widetilde{D^d}_{\mathscr{E}^{\mathfrak{S}}}&:=\left\{\widetilde{D^d}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})\right\},\text{where}\,\widetilde{D^d}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})\\&:=\left\{V\in D(\mathscr{G},\mathscr{V}):\left(\left(\widetilde{d_j(V)}\right)_{j=0}^{\infty}\in\mathscr{E}^{\mathfrak{S}}\right\},\end{aligned}$$

$$\left(\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}\right)^{\gamma} := \left\{\left(\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}\right)^{\gamma}(\mathscr{G},\mathscr{V})\right\}, \text{ where } \left(\widetilde{D^s}_{\mathscr{E}^{\mathfrak{S}}}\right)^{\gamma}(\mathscr{G},\mathscr{V}) := \left\{V \in D(\mathscr{G},\mathscr{V}) : \left(\left(\widetilde{\gamma_b(V)}\right)_{b=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}} \text{ and } \cdot \left\| V - \tilde{\rho}\left(\widetilde{\gamma_b(V)}, \tilde{0}\right) I \right\| = 0, \text{ for all } b \in \mathscr{N}\right\}. \tag{13}$$

Theorem 10. *Assume the linear sequence space $\mathscr{E}^{\mathfrak{S}}$ is a $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, then $\widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}$ is an operator ideal.*

Proof.

(i) Assume $V \in \mathbb{F}(\mathscr{G},\mathscr{V})$ and rank $(V) = n$ with $n \in \mathscr{N}$, as $\widetilde{e_i} \in \mathscr{E}^{\mathfrak{S}}$ for all $i \in \mathscr{N}$ and $\mathscr{E}^{\mathfrak{S}}$ is a linear space, one has $(\widetilde{s_i(V)})_{i=0}^{\infty} = (\widetilde{s_0(V)}, \widetilde{s_1(V)}, \cdots, \widetilde{s_{n-1}(V)}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) = \sum_{i=0}^{n-1} \widetilde{s_i(V)}\widetilde{e_i} \in \mathscr{E}^{\mathfrak{S}}$, for that $V \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ then $\mathbb{F}(\mathscr{G},\mathscr{V}) \subseteq \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$

(ii) Suppose $V_1, V_2 \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ and $\beta_1, \beta_2 \in \mathfrak{R}$ then by Definition 7 condition (iii), one has $(\widetilde{s_{[i/2]}(V_1)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$ and $(\widetilde{s_{[i/2]}(V_1)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$, as $i \geq 2[i/2]$, by the definition of $\tilde{s}$-numbers and $\widetilde{s_i(P)}$ is a decreasing sequence, we have

$$\widetilde{s_i(\beta_1 V_1 + \beta_2 V_2)} \leq \widetilde{s_{2[i/2]}(\beta_1 V_1 + \beta_2 V_2)} \leq \widetilde{s_{[i/2]}(\beta_1 V_1)} + \widetilde{s_{[i/2]}(\beta_2 V_2)} = |\beta_1|\widetilde{s_{[i/2]}(V_1)} + |\beta_2|\widetilde{s_{[i/2]}(V_2)}, \tag{14}$$

for each $i \in \mathscr{N}$. In view of Definition 7 condition (ii) and $\mathscr{E}^{\mathfrak{S}}$ is a linear space, one obtains $(\widetilde{s_i(\beta_1 V_1 + \beta_2 V_2)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$, then $\beta_1 V_1 + \beta_2 V_2 \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$

(iii) If $P \in \mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $T \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$, and $R \in \mathbb{D}(\mathscr{V},\mathscr{V}_0)$, one has $(\widetilde{s_i(T)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$ and as $\widetilde{s_i(RTP)} \leq \|R\| \widetilde{s_i(T)} \|P\|$, by Definition 7 conditions (i) and (ii), one gets $(\widetilde{s_i(RTP)})_{i=0}^{\infty} \in \mathscr{E}^{\mathfrak{S}}$, hence $RTP \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(\mathscr{G}_0,\mathscr{V}_0)$

Assume $\widetilde{\theta} = (\tilde{0}, \tilde{0}, \tilde{0}, \cdots)$ and $\mathscr{F}$ is the space of finite sequences of soft numbers. □

Definition 11. A subspace of the $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$ is called a premodular $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, if there is a function $\tau : \mathscr{E}^{\mathfrak{S}} \longrightarrow [0,\infty)$ satisfies the next setups:

(i) If $\tilde{h} \in \mathscr{E}^{\mathfrak{S}}$, $\tilde{h} = \widetilde{\theta} \Leftrightarrow \tau(|\tilde{h}|) = 0$, and $\tau(\tilde{h}) \geq 0$

(ii) Assume $\tilde{h} \in \mathscr{E}^{\mathfrak{S}}$ and $\varepsilon \in \mathfrak{R}$, one has $E_0 \geq 1$ so that $\tau(\varepsilon\tilde{h}) \leq |\varepsilon| E_0 \tau(\tilde{h})$

(iii) There are $G_0 \geq 1$ so that $\tau(\tilde{f} + \tilde{g}) \leq G_0(\tau(\tilde{f}) + \tau(\tilde{g}))$, for all $\tilde{f}, \tilde{g} \in \mathscr{E}^{\mathfrak{S}}$

(iv) Assume $|\widetilde{f_b}| \leq |\widetilde{g_b}|$, for all $b \in \mathscr{N}$, then $\tau(|\widetilde{f_b}|) \leq \tau(|\widetilde{g_b}|)$

(v) One gets $D_0 \geq 1$ such that $\tau(|\tilde{f}|) \leq \tau(|\widetilde{f_{[\cdot]}}|) \leq D_0 \tau(|\tilde{f}|)$

(vi) The closure of $\mathscr{F} = \mathscr{E}_{\tau}^{\mathfrak{S}}$

(vii) There are $\varepsilon > 0$ with $\tau(\tilde{v}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) \geq \varepsilon |v| \tau(\tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots)$

Definition 12. The $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}\mathscr{E}_{\tau}^{\mathfrak{S}}$ is said to be a prequasi normed $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, if τ confirms the setups (i)-(iii) of Definition 11. The space $\mathscr{E}_{\tau}^{\mathfrak{S}}$ is called a prequasi Banach $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, whenever $\mathscr{E}^{\mathfrak{S}}$ is complete under τ.

Theorem 13. *The space $\mathscr{E}_{\tau}^{\mathfrak{S}}$ is a prequasi normed $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, whenever it is premodular $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$. By $\uparrow$ and $\downarrow$, we denote the space of all monotonic increasing and decreasing sequences of positive reals, respectively.*

Theorem 14. $(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$ *is a prequasi Banach $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{f}$, if the next setups are confirmed:*

(f1) $(v_l) \in \uparrow \cap \ell_{\infty}$ *with* $v_0 > 1/r$

(f2) $\left(\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b\right)_{b=0}^{\infty} \in \downarrow$ *or* $\left(\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b\right)_{b=0}^{\infty} \in \uparrow \cap \ell_{\infty}$ *and there exists $C \geq 1$ such that*

$$\begin{bmatrix} 2b+r \\ 2b+1 \end{bmatrix} q_{2b+1} \leq C \begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b \tag{15}$$

Proof. First, we have to show that $(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$ is a premodular $\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathfrak{s}$.

(i) Obviously, $\tau(|\tilde{h}|) = 0 \Leftrightarrow \tilde{h} = \widetilde{\theta}$ and $\tau(\tilde{h}) \geq 0$

(a1) and (iii) If $\tilde{f}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$, then

$$\tau\left(\tilde{f} + \tilde{g}\right) = \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left(\widetilde{f_z} + \widetilde{g_z}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \leq 2^{\hbar-1} \left(\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{f_z}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} + \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{g_z}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \right) = 2^{\hbar-1}\left(\tau\left(\tilde{f}\right) + \tau(\tilde{g})\right) < \infty, \tag{16}$$

hence $\tilde{f} + \tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau}$

(ii) Next, suppose $\lambda \in \mathfrak{R}$, $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and as $(v_l) \in \uparrow \cap \ell_\infty$, we get

$$\begin{aligned}\tau\left(\lambda\tilde{f}\right) &= \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \lambda \widetilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le \sup_m |\lambda|^{v_m} \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le E_0|\lambda|\tau\left(\tilde{f}\right) < \infty,\end{aligned} \tag{17}$$

where $E_0 = \max\{1, \sup_l |\lambda|^{v_l - 1}\} \ge 1$. So, $\lambda\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$.

As $(v_l) \in \uparrow \cap \ell_\infty$ and $v_0 > 1/r$, one obtains

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{(e_b)}_z, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} &= \sum_{m=b}^{\infty}\left(\frac{\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le \sup_{m=b}^{\infty}\left(\begin{bmatrix} b+r-1 \\ b \end{bmatrix} q_b\right)^{v_m} \sum_{m=b}^{\infty}\left(\frac{1}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} < \infty.\end{aligned} \tag{18}$$

Therefore, $\widetilde{e_b} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, for every $b \in \mathcal{N}$.

(a2) and (iv) If $|\widetilde{f_m}| \le |\widetilde{g_m}|$, for all $m \in \mathcal{N}$ and $|\tilde{g}| \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, then

$$\begin{aligned}\tau\left(\left|\tilde{f}\right|\right) &= \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\le \sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z |\widetilde{g}_z|, \tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &= \tau(|\tilde{g}|) < \infty,\end{aligned} \tag{19}$$

hence $|\tilde{f}| \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$

(a3) and (v) Assume $(|\widetilde{f_z}|) \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, with $(v_l) \in \uparrow \cap \ell_\infty$ and

$$\left(\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right)_{z=0}^{\infty} \in \downarrow, \tag{20}$$

we get

$$\begin{aligned}\tau\left(\left|\widetilde{f_{[z/2]}}\right|\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} = \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+2l \\ 2l \end{bmatrix}}\right)^{v_{2l}} \\ &+ \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l+1}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+2l+1 \\ 2l+1 \end{bmatrix}}\right)^{v_{2l+1}} \le \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &+ \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l+1}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f_{[z/2]}}\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &\le \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\begin{bmatrix} 2l+r-1 \\ 2l \end{bmatrix} q_{2l}\left|\widetilde{f}_l\right| + \sum_{z=0}^{l}\left(\begin{bmatrix} 2z+r-1 \\ z \end{bmatrix} q_{2z} + \begin{bmatrix} 2z+r \\ 2z+1 \end{bmatrix} q_{2z+1}\right)\left|\widetilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &+ \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\left(\begin{bmatrix} 2z+r-1 \\ z \end{bmatrix} q_{2z} + \begin{bmatrix} 2z+r \\ 2z+1 \end{bmatrix} q_{2z+1}\right)\left|\widetilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\ &\le 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} + \sum_{l=0}^{\infty}\left(\frac{2\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right) \\ &+ \sum_{l=0}^{\infty}\left(\frac{2\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \left|\widetilde{f}_z\right|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \le D_0\tau\left(\left|\tilde{f}\right|\right) < \infty,\end{aligned} \tag{21}$$

where $D_0 \ge (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}) \ge 1$. Hence, $(|\widetilde{f_{[z/2]}}|) \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$

(vi) It is clear that the closure of $\mathcal{F} = \Gamma_r^{\mathfrak{S}}(q,v)$

(vii)There are $0 < \delta \le \sup_l |\lambda|^{v_l-1}$ so that $\tau(\tilde{\lambda}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) \ge \delta|\lambda|\tau(\tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots)$, for all $\lambda \ne 0$ and $\delta > 0$, if $\lambda = 0$

By Theorem 13, the space $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ is a prequasi normed $\mathfrak{pSSSf}$. Second, to prove that $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ is a Banach space, suppose $\widetilde{h^i} = (\widetilde{h_k^i})_{k=0}^{\infty}$ is a Cauchy sequence in $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, hence for every $\gamma \in (0,1)$, one has $i_0 \in \mathcal{N}$ with $i, j \ge i_0$, we have

$$\tau\left(\widetilde{h^i} - \widetilde{h^j}\right) = \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\widetilde{h_z^i} - \widetilde{f_z^j}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} < \gamma^{\hbar}. \tag{22}$$

That implies

$$\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\widetilde{h_z^i}-\widetilde{h_z^j}\right),\tilde{0}\right)<\gamma. \tag{23}$$

As $(\mathscr{R}(A),\tilde{\rho})$ is a complete metric space. Therefore, $(\widetilde{h_k^j})$ is a Cauchy sequence in $\mathscr{R}(A)$, for constant $k\in\mathscr{N}$. So, it is convergent to $\widetilde{h_k^0}\in\mathscr{R}(A)$. This implies $\tau(\widetilde{h^i}-\widetilde{h^0})<\gamma^{\hbar}$, for every $i\geq i_0$. Clearly, from condition (iii) that $\widetilde{h^0}\in(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$.

In view of Theorems 10 and 14, we have the next theorem. □

Theorem 15. *The space $\widetilde{\mathbb{D}^s}_{\Gamma_r^{\mathfrak{S}}(q,\nu)}$ is an operator ideal, if the conditions of Theorem 14 are verified.*

Theorem 16. *If s-type $\mathscr{E}_\tau^{\mathfrak{S}}:=\{\tilde{h}=(\widetilde{s_j(H)})\in\mathfrak{R}^{\mathscr{N}}: H\in D(\mathscr{G},\mathscr{V})$ and $\tau(\tilde{h})<\infty\}$. Assume $\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau}$ is an operator ideal, one has the next setups:*

(a) s-type $\mathscr{E}_\tau^{\mathfrak{S}}\supset\mathscr{F}$

(b) Suppose $(\widetilde{s_j(H_1)})_{j=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$ and $(\widetilde{s_j(H_2)})_{j=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$, then $(\widetilde{s_j(H_1+H_2)})_{j=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$

(c) If $\varepsilon\in\mathfrak{R}$ and $(\widetilde{s_j(H)})_{j=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$, one has $|\varepsilon|(\widetilde{s_j(H)})_{j=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$

(d) Suppose $(\widetilde{s_j(U)})_{j=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$ and $\widetilde{s_j(T)}\leq\widetilde{s_j(U)}$, for all $j\in\mathscr{N}$ and $T,U\in\mathbb{D}(\mathscr{G},\mathscr{V})$, one gets $(\widetilde{s_j(T)})_{j=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$, i.e., $\mathscr{E}_\tau^{\mathfrak{S}}$ is a solid space

Proof. If $\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau}$ is a mappings' ideal.

(a)We have $\mathbb{F}(\mathscr{G},\mathscr{V})\subset\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$. Hence, for all $X\in\mathbb{F}(\mathscr{G},\mathscr{V})$, we have $(\widetilde{s_r(X)})_{r=0}^\infty\in\mathscr{F}$. This gives $(\widetilde{s_r(X)})_{r=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$. Hence, $\mathscr{F}\subset$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$

(b) and (c)The space $\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ is linear over $\mathfrak{R}$. Hence, for each $\lambda\in\mathfrak{R}$ and $X_1,X_2\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$, we have $X_1+X_2\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$ and $\lambda X_1\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$. That implies

$$\begin{gathered}\left(\widetilde{s_r(X_1)}\right)_{r=0}^\infty\in\text{s-type}\,\mathscr{E}_\tau^{\mathfrak{S}}\quad\text{and}\quad\left(\widetilde{s_r(X_2)}\right)_{r=0}^\infty\in\text{s-type}\,\mathscr{E}_\tau^{\mathfrak{S}}\Rightarrow\left(\widetilde{s_r(X_1+X_2)}\right)_{r=0}^\infty\in\text{s-type}\,\mathscr{E}_\tau^{\mathfrak{S}},\\ \lambda\in\mathfrak{R}\quad\text{and}\quad\left(\widetilde{s_r(X_1)}\right)_{r=0}^\infty\in\text{s-type}\,\mathscr{E}_\tau^{\mathfrak{S}}\Rightarrow|\lambda|\left(\widetilde{s_r(X_1)}\right)_{r=0}^\infty\in\text{s-type}\,\mathscr{E}_\tau^{\mathfrak{S}}\end{gathered} \tag{24}$$

(d)If $A\in\mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $B\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G},\mathscr{V})$, and $D\in\mathbb{D}(\mathscr{V},\mathscr{V}_0)$, then $DBA\in\widetilde{\mathbb{D}^s}_{\mathscr{E}_\tau^{\mathfrak{S}}}(\mathscr{G}_0,\mathscr{V}_0)$. Therefore, since $(\widetilde{s_r(B)})_{r=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$, then $(\widetilde{s_r(DBA)})_{r=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$. Since $\widetilde{s_r(DBA)}\leq\|D\|\widetilde{s_r(B)}\|A\|$. By using condition (c), if $(\|D\|\|A\|\widetilde{s_r(B)})_{r=0}^\infty\in\mathscr{E}_\tau^{\mathfrak{S}}$, we have $(\widetilde{s_r(DBA)})_{r=0}^\infty\in$ s-type $\mathscr{E}_\tau^{\mathfrak{S}}$. This means s-type $\mathscr{E}_\tau^{\mathfrak{S}}$ is solid

Some properties of s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$ are presented in the next theorem according to Theorems 16 and 15. □

Theorem 17.

(a) s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau\supset\mathscr{F}$

(b) If $(\widetilde{s_n(X_1)})_{n=0}^\infty\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$ and $(\widetilde{s_n(X_2)})_{n=0}^\infty\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$, then $(s_n(X_1+X_2))_{n=0}^\infty\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$

(c) Assume $\lambda\in\mathfrak{R}$ and $(\widetilde{s_n(X)})_{n=0}^\infty\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$, hence $|\lambda|(\widetilde{s_n(X)})_{n=0}^\infty\in$ s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$

(d) s-type $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$ is a solid space

Definition 18 (see [31]). A subclass $\mathscr{U}$ of $\mathbb{D}$ is said to be a mappings' ideal, if every $\mathscr{U}(\mathscr{G},\mathscr{V})=\mathscr{U}\cap\mathbb{D}(\mathscr{G},\mathscr{V})$ satisfies the following setups:

(i) $I_\Gamma\in\mathscr{U}$, where Γ indicates Banach space of one dimension

(ii) The space $\mathscr{U}(\mathscr{G},\mathscr{V})$ is linear over $\mathfrak{R}$

(iii) If $W\in\mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $X\in\mathscr{U}(\mathscr{G},\mathscr{V})$, and $Y\in\mathbb{D}(\mathscr{V},\mathscr{V}_0)$, then $YXW\in\mathscr{U}(\mathscr{G},\mathscr{V}_0)$

Definition 19 (see [32]). A function $H\in[0,\infty)^{\mathscr{U}}$ is said to be a prequasi norm on the ideal $\mathscr{U}$ if the following conditions hold:

(1) Assume $V\in\mathscr{U}(\mathscr{G},\mathscr{V})$, $H(V)\geq0$ and $H(V)=0$, if and only if, $V=0$

(2) One has $Q\geq1$ with $H(\alpha V)\leq D|\alpha|H(V)$, for all $V\in\mathscr{U}(\mathscr{G},\mathscr{V})$ and $\alpha\in\mathfrak{R}$

(3) There are $P\geq1$ such that $H(V_1+V_2)\leq P[H(V_1)+H(V_2)]$, for all $V_1,V_2\in\mathscr{U}(\mathscr{G},\mathscr{V})$

(4) There are $\sigma\geq1$ so that if $V\in\mathbb{D}(\mathscr{G}_0,\mathscr{G})$, $X\in\mathscr{U}(\mathscr{G},\mathscr{V})$, and $Y\in\mathbb{D}(\mathscr{V},\mathscr{V}_0)$, then $H(YXV)\leq\sigma\|Y\|H(X)\|V\|$

Theorem 20 (see [32]). *Every quasi norm on the ideal $\mathscr{U}$ is a prequasi norm.*

We have discussed some properties of the ideal constructed by our soft space and extended s-numbers, supposing that the conditions of Theorem 14 are verified.

Theorem 21. *The conditions of Theorem 14 are sufficient only for $\mathbb{D}^\alpha_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau}(\mathscr{G},\mathscr{V})=$ the closure of $\mathbb{F}(\mathscr{G},\mathscr{V})$.*

Proof. Clearly, the closure of $\mathbb{F}(\mathscr{G},\mathscr{V}) \subseteq \widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$ from the linearity of the space $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ and $\widetilde{e_m} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, for all $m \in \mathscr{N}$. Next, to show that $\widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V}) \subseteq$ the closure of $\mathbb{F}(\mathscr{G},\mathscr{V})$. If $H \in \widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$, one has $(\widetilde{\alpha_l(H)})_{m=0}^{\infty} \in (\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. As $\tau(\widetilde{\alpha_m(H)})_{m=0}^{\infty} < \infty$, assume $\gamma \in (0,1)$, we have $l_0 \in \mathscr{N} - \{0\}$ so that $\tau((\widetilde{\alpha_m(H)})_{m=l_0}^{\infty}) < \gamma/2^{\hbar+3}\delta j$, for some $j \geq 1$ and

$$\delta = \max\left\{1, \sum_{l=l_0}^{\infty}\left(\frac{1}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right\}. \tag{25}$$

Since $\widetilde{\alpha_l(H)} \in \mathfrak{I}_{\searrow}^{\mathfrak{S}}$, we get

$$\begin{aligned}
&\sum_{l=l_0+1}^{2l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{\alpha_{2l_0}(H)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq \sum_{l=l_0+1}^{2l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{\alpha_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq \sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{\alpha_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} < \frac{\gamma}{2^{\hbar+3}\delta j}.
\end{aligned} \tag{26}$$

We get $U \in \mathbb{F}_{2l_0}(\mathscr{G},\mathscr{V})$ with rank $(U) \leq 2l_0$ and

$$\begin{aligned}
&\sum_{l=2l_0+1}^{3l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq \sum_{l=l_0+1}^{2l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&< \frac{\gamma}{2^{\hbar+3}\delta j},
\end{aligned} \tag{27}$$

since $(v_l) \in \uparrow \cap \ell_\infty$, we have

$$\sup_{l=l_0}^{\infty} \tilde{\rho}^{v_l}\left(\sum_{z=0}^{l_0}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right) < \frac{\gamma}{2^{2\hbar+2}\delta}. \tag{28}$$

Therefore, one has

$$\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} < \frac{\gamma}{2^{\hbar+3}\delta j}. \tag{29}$$

Because of inequalities (5), (26), (27), (28), and (29), one gets

$$\begin{aligned}
d(H,U) &= \tau\left(\alpha_l(\widetilde{H-U})\right)_{l=0}^{\infty} = \sum_{l=0}^{3l_0-1}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&+ \sum_{l=3l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right) \leq \sum_{l=0}^{3l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&+ \sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l+2l_0}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l+2l_0 \\ l+2l_0 \end{bmatrix}}\right)^{v_{l+2l_0}\, v_l} \leq \sum_{l=0}^{3l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&+ \sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l+2l_0}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&+ \sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l_0-1}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}) + \sum_{z=2l_0}^{l+2l_0}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} + 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l_0-1}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=2l_0}^{l+2l_0}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{2l_0-1}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \alpha_z(\widetilde{Z-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+2l_0+r-1 \\ z+2l_0 \end{bmatrix} q_{z+2l_0} \alpha_{z+2l_0}(\widetilde{H-U}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} + 2^{2\hbar-1}\sup_{l=l_0}^{\infty} \tilde{\rho}^{v_l}\left(\sum_{z=0}^{l_0}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \|\widetilde{H-U}\|, \tilde{0}\right) \\
&\cdot \sum_{l=l_0}^{\infty}\left(\frac{1}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} + 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{\alpha_z(H)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} < \gamma.
\end{aligned} \tag{30}$$

On the other hand, one has a negative example as $I_2 \in \widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$, where $z+r-1zq_z = 1$, for all $z \in \mathscr{N}$ and

$v=(0,-1,2,2,2)$, but $(v_l)\notin\uparrow$. This gives a negative answer to the Rhoades [33] open problem about the linearity of s-type $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$ spaces. $\square$

Theorem 22. *The class $(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau},\Xi)$ is a prequasi Banach ideal, where $\Xi(H)=\tau((\widetilde{s_b(H)})_{b=0}^{\infty})$.*

Proof. Evidently, Ξ is a prequasi norm on $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}$ since τ is a prequasi norm on $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. Assume $(H_m)_{m\in\mathcal{N}}$ is a Cauchy sequence in $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$. Since $\mathbb{D}(\mathcal{G},\mathcal{V})\supseteq\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$, we have

$$\Xi(H_j-H_m)=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z s_z\left(\widetilde{H_j-H_m}\right),\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l}\geq\left(\|H_j-H_m\|\right)^{v_0},\tag{31}$$

then $(H_m)_{m\in\mathcal{N}}$ is a Cauchy sequence in $\mathbb{D}(\mathcal{G},\mathcal{V})$. As $\mathbb{D}(\mathcal{G},\mathcal{V})$ is a Banach space, one has $H\in\mathbb{D}(\mathcal{G},\mathcal{V})$ so that $\lim_{m\longrightarrow\infty}\|H_m-H\|=0$. As $(s_l(\widetilde{H_m}))_{l=0}^{\infty}\in(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, for all $m\in\mathcal{N}$. By Definition 11 conditions (ii), (iii), and (v), we have

$$\begin{aligned}\Xi(H)&=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z\widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l}\\&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z s_{[z/2]}\left(\widetilde{H-H_m}\right),\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l}\\&\quad+2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z s_{[z/2]}\left(\widetilde{H_m}\right),\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l}\\&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z\|\widetilde{H-H_m}\|,\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l}\\&\quad+2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z s_z\left(\widetilde{H_m}\right),\tilde{0}\right)}{\begin{bmatrix}r+l\\l\end{bmatrix}}\right)^{v_l}<\infty.\end{aligned}\tag{32}$$

Hence, $(\widetilde{s_b(H)})_{b=0}^{\infty}\in(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, so $H\in\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathcal{V})$. $\square$

Theorem 23. *If $1<v_b^{(1)}<v_b^{(2)}$, and $0<q_b^{(2)}\leq q_b^{(1)}$, for every $b\in\mathcal{N}$, then*

$$\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_b^{(1)}\right),\left(v_b^{(1)}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V})\subsetneqq\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(q_b^{(2)}\right),\left(v_b^{(2)}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V})\subsetneqq\mathbb{D}(\mathcal{G},\mathcal{V}).\tag{33}$$

Proof. Let $H\in\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_b^{(1)}),(v_b^{(1)})))_\tau}(\mathcal{G},\mathcal{V})$, then $(\widetilde{s_b(H)})\in(\Gamma_r^{\mathfrak{S}}((q_b^{(1)}),(v_b^{(1)})))_\tau$. One obtains

$$\begin{aligned}&\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(2)}\widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix}r+b\\b\end{bmatrix}}\right)^{v_b^{(2)}}\\&\quad<\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(1)}\widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix}r+b\\b\end{bmatrix}}\right)^{v_b^{(1)}}<\infty,\end{aligned}\tag{34}$$

then $H\in\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathcal{G},\mathcal{V})$. Take $(\widetilde{s_b(H)})_{b=0}^{\infty}$ with

$$\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(1)}\widetilde{s_z(H)},\tilde{0}\right)=\frac{\begin{bmatrix}r+b\\b\end{bmatrix}}{\sqrt[vb1]{b+1}},\tag{35}$$

we have $H\in\mathbb{D}(\mathcal{G},\mathcal{V})$ with

$$\begin{aligned}&\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(1)}\widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix}r+b\\b\end{bmatrix}}\right)^{v_b^{(1)}}=\sum_{b=0}^{\infty}\frac{1}{b+1}=\infty,\\&\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(2)}\widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix}r+b\\b\end{bmatrix}}\right)^{v_b^{(2)}}\\&\leq\sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix}z+r-1\\z\end{bmatrix}q_z^{(1)}\widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix}r+b\\b\end{bmatrix}}\right)^{v_b^{(2)}}=\sum_{b=0}^{\infty}\left(\frac{1}{b+1}\right)^{v_b^{(2)}/v_b^{(1)}}<\infty.\end{aligned}\tag{36}$$

Hence, $H \notin \widetilde{\mathbb{D}^{s}}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(1)}),(v_b^{(1)})))_\tau}(\mathscr{G},\mathscr{V})$ and $H \in \widetilde{\mathbb{D}^{s}}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathscr{G},\mathscr{V})$.

Clearly, $\widetilde{\mathbb{D}^{s}}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathscr{G},\mathscr{V}) \subset \mathbb{D}(\mathscr{G},\mathscr{V})$. Take $(\widetilde{s_b(H)})_{b=0}^{\infty}$ with

$$\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(2)}\widetilde{s_z(H)},\tilde{0}\right) = \frac{\begin{bmatrix} r+b \\ b \end{bmatrix}}{\sqrt[vb2]{b+1}}. \tag{37}$$

Then $H \in \mathbb{D}(\mathscr{G},\mathscr{V})$ and $H \notin \widetilde{\mathbb{D}^{s}}_{(\Gamma_r^{\mathfrak{C}}((q_b^{(2)}),(v_b^{(2)})))_\tau}(\mathscr{G},\mathscr{V})$.

Recall that if $\mathscr{G}$ and $\mathscr{V}$ are infinite-dimensional, by Dvoretzky's theorem [34], there are $\mathscr{G}/Y_j$ and $M_j \subseteq \mathscr{V}$ operated onto ℓ_2^j through isomorphisms V_j and X_j such that $\|V_j\|\|V_j^{-1}\| \le 2$ and $\|X_j\|\|X_j^{-1}\| \le 2$, for all $j \in \mathscr{N}$. Assume T_j is the quotient mapping from $\mathscr{G}$ onto $\mathscr{G}/Y_j$, I_j is the identity operator on ℓ_2^j and J_j is the natural embedding operator from M_j into $\mathscr{V}$. Assume m_j is the Bernstein numbers [18]. □

Theorem 24. *The class $\widetilde{\mathbb{D}^{\alpha}}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}$ is minimum, whenever*

$$\left(\frac{\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)_{l=0}^{\infty} \notin \ell_{((v_l))}. \tag{38}$$

Proof. Assume $\widetilde{\mathbb{D}^{\alpha}}_{\Gamma_r^{\mathfrak{C}}(q,v)}(\mathscr{G},\mathscr{V}) = \mathbb{D}(\mathscr{G},\mathscr{V})$, one has $\gamma > 0$ so that $\Xi(H) \le \gamma\|H\|$, for all $H \in \mathbb{D}(\mathscr{G},\mathscr{V})$ and

$$\Xi(H) = \sum_{b=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{b}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\widetilde{\alpha_z(H)},\tilde{0}\right)}{\begin{bmatrix} r+b \\ b \end{bmatrix}}\right)^{v_b}. \tag{39}$$

We have

$$\begin{aligned} 1 &= m_z(I_j) = m_z\left(X_jX_j^{-1}I_jV_jV_j^{-1}\right) \\ &\le \|X_j\|m_z\left(X_j^{-1}I_jV_j\right)\left\|V_j^{-1}\right\| = \|X_j\|m_z\left(J_jX_j^{-1}I_jV_j\right)\left\|V_j^{-1}\right\| \\ &\le \|X_j\|d_z\left(J_jX_j^{-1}I_jV_j\right)\left\|V_j^{-1}\right\| = \|X_j\|d_z\left(J_jX_j^{-1}I_jV_jT_j\right) \\ &\quad\cdot\left\|V_j^{-1}\right\| \le \|X_j\|\alpha_z\left(J_jX_j^{-1}I_jV_jT_j\right)\left\|V_j^{-1}\right\|. \end{aligned} \tag{40}$$

Take $0 \le m \le j$, one has

$$\begin{aligned} &\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \le \tilde{\rho}\left(\sum_{z=0}^{m}\|X_j\|\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\alpha_z\left(\widetilde{J_jX_j^{-1}I_jV_jT_j}\right)\left\|V_j^{-1}\right\|,\tilde{0}\right) \Rightarrow \\ &\cdot\left(\frac{\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \le \left(\|X_j\|\left\|V_j^{-1}\right\|\right)^{v_m} \\ &\cdot\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\alpha_z\left(\widetilde{J_jX_j^{-1}I_jV_jT_j}\right),\tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m}. \end{aligned} \tag{41}$$

Therefore, for some $\lambda \ge 1$, we obtain

$$\begin{aligned} &\sum_{m=0}^{j}\left(\frac{\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \le \lambda\|X_j\|\left\|V_j^{-1}\right\|\sum_{m=0}^{j} \\ &\cdot\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\alpha_z\left(\widetilde{J_jX_j^{-1}I_jV_jT_j}\right),\tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \Rightarrow \sum_{m=0}^{j} \\ &\cdot\left(\frac{\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \le \lambda\|X_j\|\left\|V_j^{-1}\right\|\Xi\left(J_jX_j^{-1}I_jV_jT_j\right) \\ &\le \lambda\gamma\|X_j\|\left\|V_j^{-1}\right\|\left\|J_jX_j^{-1}I_jV_jT_j\right\| \le 4\lambda\gamma. \end{aligned} \tag{42}$$

When $j \longrightarrow \infty$, one has a contradiction. So, $\mathscr{G}$ and $\mathscr{V}$ both cannot be infinite-dimensional when $\widetilde{\mathbb{D}^{\alpha}}_{\Gamma_r^{\mathfrak{C}}(q,v)}(\mathscr{G},\mathscr{V}) = \mathbb{D}(\mathscr{G},\mathscr{V})$. □

Theorem 25. *The class $\widetilde{\mathbb{D}^{d}}_{\Gamma_r^{\mathfrak{C}}(q,v)}$ is minimum, whenever*

$$\left(\frac{\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)_{l=0}^{\infty} \notin \ell_{((v_l))}. \tag{43}$$

Lemma 26 (see [19]). *Suppose $W \in \mathbb{D}(\mathscr{G},\mathscr{V})$ and $W \notin \mathscr{A}(\mathscr{G},\mathscr{V})$, one has $P \in \mathbb{D}(\mathscr{G})$ and $A \in \mathbb{D}(\mathscr{V})$ with $AWPe_j = e_j$, for every $j \in \mathscr{N}$.*

Theorem 27 (see [19]). *If $\mathscr{E}^{\mathfrak{C}}$ is an infinite-dimensional Banach space, then*

$$\mathbb{F}\left(\mathscr{E}^{\mathfrak{C}}\right) \subsetneqq \mathscr{A}\left(\mathscr{E}^{\mathfrak{C}}\right) \subsetneqq \mathscr{K}\left(\mathscr{E}^{\mathfrak{C}}\right) \subsetneqq \mathbb{D}\left(\mathscr{E}^{\mathfrak{C}}\right). \tag{44}$$

Theorem 28. *If $1 < v_l^{(1)} < v_l^{(2)}$ and $0 < q_l^{(2)} \le q_l^{(1)}$, for every $l \in \mathscr{N}$, then*

$$\begin{aligned} &\mathbb{D}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}), \widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right) \\ &\quad = \mathscr{A}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}), \widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right). \end{aligned} \tag{45}$$

Proof. Let $X \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}), \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$ and $X \notin \mathscr{A}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}), \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$. In view of Lemma 26, there are $Y \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}))$ and $Z \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathscr{N}$, one has

$$\begin{aligned} &\|I_b\|_{\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(1)}\right),\left(v_l^{(1)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})} \\ &\quad = \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(1)} \widetilde{s_z(I_b)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l^{(1)}} \\ &\quad \le \|ZXY\|\|I_b\|_{\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})} \\ &\quad \le \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z^{(2)} \widetilde{s_z(I_b)}, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l^{(2)}}. \end{aligned} \tag{46}$$

This contradicts Theorem 23; hence, $X \in \mathscr{A}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(2)}),(v_l^{(2)})))_\tau}(\mathscr{G},\mathscr{V}), \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}((q_l^{(1)}),(v_l^{(1)})))_\tau}(\mathscr{G},\mathscr{V}))$. □

Corollary 29. *Suppose $1 < v_l^{(1)} < v_l^{(2)}$, and $0 < q_l^{(2)} \le q_l^{(1)}$, for every $l \in \mathscr{N}$, then*

$$\begin{aligned} &\mathbb{D}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}), \widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(1)}\right),\left(v_l^{(1)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right) \\ &\quad = \mathscr{K}\left(\widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(2)}\right),\left(v_l^{(2)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V}), \widetilde{\mathbb{D}^s}_{\left(\Gamma_r^{\mathfrak{C}}\left(\left(q_l^{(1)}\right),\left(v_l^{(1)}\right)\right)\right)_\tau}(\mathscr{G},\mathscr{V})\right). \end{aligned} \tag{47}$$

Proof. Evidently, as $\mathscr{A} \subset \mathscr{K}$. □

Definition 30 (see [19]). A Banach space $\mathscr{E}^{\mathfrak{C}}$ *is said to be simple when $\mathbb{D}(\mathscr{E}^{\mathfrak{C}})$ has a unique nontrivial closed ideal.*

Theorem 31. *The class $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}$ is simple.*

Proof. Let the closed ideal $\mathscr{K}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$ contain a mapping $H \notin \mathscr{A}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$. By Lemma 26, there are $P, A \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$ so that $AHPI_j = I_j$. Therefore, $I_{\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V})} \in \mathscr{K}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$. Hence, $\mathbb{D}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V})) = \mathscr{K}(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V}))$. Therefore, $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}$ is a simple Banach space. □

Theorem 32. *Assume*

$$\inf_l\left(\frac{\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} > 0, \tag{48}$$

then $(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau})^{\gamma}(\mathscr{G},\mathscr{V}) = \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$.

Proof. Let $H \in (\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau})^{\gamma}(\mathscr{G},\mathscr{V})$, then $(\widetilde{\gamma_m(H)})_{m=0}^{\infty} \in (\Gamma_r^{\mathfrak{C}}(q,v))_\tau$ and $\|H - \tilde{\rho}(\widetilde{\gamma_m(H)},\tilde{0})I\| = 0$, for every $m \in \mathscr{N}$. One has $H = \tilde{\rho}(\widetilde{\gamma_m(H)},\tilde{0})I$, for all $m \in \mathscr{N}$, then

$$\tilde{\rho}\left(\widetilde{s_m(H)},\tilde{0}\right) = \tilde{\rho}\left(\widetilde{s_m(\tilde{\rho}\left(\widetilde{\gamma_m(H)},\tilde{0}\right)I)},\tilde{0}\right) = \tilde{\rho}\left(\widetilde{\gamma_m(H)},\tilde{0}\right), \tag{49}$$

for all $m \in \mathscr{N}$. Hence $(\widetilde{s_m(H)})_{m=0}^{\infty} \in (\Gamma_r^{\mathfrak{C}}(q,v))_\tau$, so $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$.

Next, assume $H \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{C}}(q,v))_\tau}(\mathscr{G},\mathscr{V})$. Hence, $(\widetilde{s_m(H)})_{m=0}^{\infty} \in (\Gamma_r^{\mathfrak{C}}(q,v))_\tau$. Therefore, one has

$$\begin{aligned} &\sum_{m=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{s_z(H)},\tilde{0}\right)}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \\ &\quad \ge \inf_m\left(\frac{\sum_{z=0}^{m}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}{\begin{bmatrix} r+m \\ m \end{bmatrix}}\right)^{v_m} \sum_{m=0}^{\infty}\left[\tilde{\rho}\left(\widetilde{s_m(H)},\tilde{0}\right)\right]^{v_m}. \end{aligned} \tag{50}$$

Hence, $\lim_{m\longrightarrow\infty}\widetilde{s_m(H)} = \tilde{0}$. If $\|H - \tilde{\rho}(\widetilde{s_m(H)},\tilde{0})I\|^{-1}$

exists, for all $m \in \mathcal{N}$. Then $\|H - \tilde{\rho}(\widetilde{s_m(H)}, \tilde{0})I\|^{-1}$ exists and bounded, for all $m \in \mathcal{N}$. So, $\lim_{m \longrightarrow \infty} \|H - \tilde{\rho}(\widetilde{s_m(H)}, \tilde{0})I\|^{-1} = \|H\|^{-1}$ exists and bounded. Since $(\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau}, \Xi)$ is a pre-prequasi ideal, one obtains

$$I = HH^{-1} \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau}(\mathcal{G}, \mathcal{V}) \Rightarrow \left(\widetilde{s_m(I)}\right)_{m=0}^{\infty} \in \Gamma_r^{\mathfrak{S}}(q, \nu)$$
$$\Rightarrow \lim_{m \longrightarrow \infty} \widetilde{s_m(I)} = \tilde{0}. \tag{51}$$

One has a contradiction, as $\lim_{m \longrightarrow \infty} \widetilde{s_m(I)} = \tilde{1}$. Then, $\|H - \tilde{\rho}(\widetilde{s_m(H)}, \tilde{0})I\| = 0$, for all $m \in \mathcal{N}$. So, $\|H - \tilde{\rho}(\widetilde{\gamma_m(H)}, \tilde{0})I\| = 0$, for all $m \in \mathcal{N}$. Therefore, $H \in (\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau})^{\gamma}(\mathcal{G}, \mathcal{V})$. □

3. Multiplication Mappings on $(\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$

Under the conditions of Theorem 14, we have presented in this section some properties of the multiplication mapping acting on $(\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$.

Let $(\text{Range}(V))^c$ indicate the complement of Range(V). Let $\mathfrak{I}$ be the space of all sets with a finite number of elements. Assume $\ell_\infty^{\mathfrak{S}}$ is the space of bounded sequences of soft functions.

Definition 33. Suppose $\mathcal{E}_\tau^{\mathfrak{S}}$ is a prequasi normed $\mathfrak{pssf}$ and $\lambda = (\lambda_k) \in \mathfrak{R}^{\mathcal{N}}$ *. The mapping $H_\lambda : \mathcal{E}_\tau^{\mathfrak{S}} \longrightarrow \mathcal{E}_\tau^{\mathfrak{S}}$ is said to be a multiplication mapping on $\mathcal{E}_\tau^{\mathfrak{S}}$, if $H_\lambda \tilde{f} = (\lambda_b \tilde{f}_b) \in \mathcal{E}_\tau^{\mathfrak{S}}$*, for all $f \in \mathcal{E}_\tau^{\mathfrak{S}}$. The multiplication mapping is called constructed by λ, if $H_\lambda \in \mathbb{D}(\mathcal{E}_\tau^{\mathfrak{S}})$.

Definition 34 (see [35]). A mapping $V \in \mathbb{D}(\mathcal{E})$ is said to be Fredholm if dim $(\text{Range}(V))^c < \infty$, Range$(V)$ is closed and dim $(\ker(V)) < \infty$.

Theorem 35.

(1) $\lambda \in \ell_\infty \Leftrightarrow H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$

(2) $|\lambda_a| = 1$, for every $a \in \mathcal{N}$, if and only if, H_λ is an isometry

(3) $H_\lambda \in \mathcal{A}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau) \Leftrightarrow (\lambda_a)_{a=0}^{\infty} \in c_0$

(4) $H_\lambda \in \mathcal{K}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau) \Leftrightarrow (\lambda_b)_{b=0}^{\infty} \in c_0$

(5) $\mathcal{K}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau) \subsetneqq \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$

(6) $0 < \alpha < |\lambda_a| < \eta$, for every $a \in (\ker(\lambda))^c$, if and only if, Range(H_λ) is closed

(7) $0 < \alpha < |\lambda_a| < \eta$, for all $a \in \mathcal{N}$, if and only if, $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$ is invertible

(8) H_λ is Fredholm operator, if and only if (g1) $\ker(\lambda) \subsetneqq \mathcal{N} \cap \mathfrak{I}$ and (g2) $|\lambda_a| \geq \rho$, for all $a \in (\ker(\lambda))^c$

Proof.

(1) Suppose $\lambda \in \ell_\infty$, one has $\nu > 0$ with $|\lambda_a| \leq \nu$, for all $a \in \mathcal{N}$. If $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$, we have

$$\tau\left(H_\lambda \tilde{f}\right) = \tau\left(\lambda \tilde{f}\right) = \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \lambda_z \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{f_z}, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l}$$
$$\leq \sup_l \nu^{\nu_l} \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{f_z}, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l}$$
$$= \sup_l \nu^{\nu_l} \tau\left(\tilde{f}\right). \tag{52}$$

Therefore, $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$.

Next, if $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$ and $\lambda \notin \ell_\infty$. One has $x_b \in \mathcal{N}$, for every $b \in \mathcal{N}$ with $\lambda_{x_b} > b$. Then,

$$\tau(H_\lambda \widetilde{e_{x_b}}) = \tau(\lambda \widetilde{e_{x_b}}) = \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \lambda_z \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{(e_{x_b})_z}, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l}$$
$$= \sum_{l=x_b}^{\infty} \left(\frac{\lambda_{(x_b)} \begin{bmatrix} x_b+r-1 \\ x_b \end{bmatrix} q_{x_b}}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} > \sum_{l=x_b}^{\infty} \left(\frac{b \begin{bmatrix} x_b+r-1 \\ x_b \end{bmatrix} q_{x_b}}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} > b^{\nu_0} \tau(\widetilde{e_{x_b}}). \tag{53}$$

Hence, $H_\lambda \notin \mathbb{D}((\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau)$. So, $\lambda \in \ell_\infty$.

(2) Let $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q, \nu))_\tau$ and $|\lambda_b| = 1$, for every $b \in \mathcal{N}$. One obtains

$$\tau\left(H_\lambda \tilde{f}\right) = \tau\left(\lambda \tilde{f}\right) = \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \lambda_z \widetilde{f_z}, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l}$$
$$= \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{f_z}, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{\nu_l} = \tau\left(\tilde{f}\right), \tag{54}$$

then H_λ is an isometry.

Next, if for some $b=b_0$ that $|\lambda_b|<1$, one has

$$\tau(H_\lambda\widetilde{e_{b_0}})=\tau(\lambda\widetilde{e_{b_0}})=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\lambda_z\left(\widetilde{e_{b_0}}\right)_z,\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\sum_{l=b_0}^{\infty}\left(\frac{|\lambda_{b_0}|\begin{bmatrix} b_0+r-1\\ b_0\end{bmatrix}q_{b_0}}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}<\sum_{l=b_0}^{\infty}\left(\frac{\begin{bmatrix} b_0+r-1\\ b_0\end{bmatrix}q_{b_0}}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}=\tau\left(\widetilde{e_{b_0}}\right). \tag{55}$$

When $|\lambda_{b_0}|>1$, so $\tau(H_\lambda\widetilde{e_{b_0}})>\tau(\widetilde{e_{b_0}})$. Hence, $|\lambda_a|=1$, for every $a\in\mathcal{N}$.

(3) Assume $H_\lambda\in\mathscr{A}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$, so $H_\lambda\in\mathscr{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$. If $\lim_{b\longrightarrow\infty}\lambda_b\neq 0$. One has $\rho>0$ with $K_\varrho=\{a\in\mathcal{N}:|\lambda_a|\geq\rho\}\subsetneqq\mathfrak{I}$. Let $\{\alpha_a\}_{a\in\mathcal{N}}\subset K_\rho$. We have $\{\widetilde{e_{\alpha_a}}:\alpha_a\in K_\varrho\}\in\ell_\infty^{\mathfrak{S}}$ be an infinite set in $(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$. For all $\alpha_a,\alpha_b\in K_\rho$, one gets

$$\tau\left(H_\lambda\widetilde{e_{\alpha_a}}-H_\lambda\widetilde{e_{\alpha_b}}\right)=\tau\left(\lambda\widetilde{e_{\alpha_a}}-\lambda\widetilde{e_{\alpha_b}}\right)$$
$$=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\lambda_z\left(\left(\widetilde{e_{\alpha_a}}\right)_z-\left(\widetilde{e_{\alpha_b}}\right)_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$\geq\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\rho\left(\left(\widetilde{e_{\alpha_a}}\right)_z-\left(\widetilde{e_{\alpha_b}}\right)_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$\geq\inf_l\rho^{v_l}\tau\left(\widetilde{e_{\alpha_a}}-\widetilde{e_{\alpha_b}}\right). \tag{56}$$

Hence, $\{\widetilde{e_{\alpha_b}}:\alpha_b\in K_\rho\}\in\ell_\infty^{\mathfrak{S}}$ has not a convergent subsequence under H_λ. So, $H_\lambda\notin\mathscr{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$. Therefore, $H_\lambda\notin\mathscr{A}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$; this is a contradiction. So, $\lim_{b\longrightarrow\infty}\lambda_b=0$. Next, let $\lim_{a\longrightarrow\infty}\lambda_a=0$. Hence, for every $\rho>0$, we have $K_\rho=\{b\in\mathcal{N}:|\lambda_b|\geq\rho\}\subset\mathfrak{I}$. Therefore, for all $\rho>0$, one gets $\dim(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_\rho})=\dim(\mathfrak{R}^{K_\rho})<\infty$. So, $H_\lambda\in\mathbb{F}(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_\rho})$. If $\lambda_a\in\mathfrak{R}^{\mathcal{N}}$, for all $a\in\mathcal{N}$, where

$$(\lambda_a)_b=\begin{cases}\lambda_b, & b\in K_{1/a+1},\\ 0, & \text{otherwise.}\end{cases} \tag{57}$$

Obviously, $H_{\lambda_a}\in\mathbb{F}(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_{1/a+1}})$, since $\dim(((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)_{K_{1/a+1}})<\infty$, for all $a\in\mathcal{N}$. According to $(v_l)\in\uparrow\cap\ell_\infty$ with $v_0>1/r$, we have

$$\tau\left(\left(H_\lambda-H_{\lambda_a}\right)\tilde{f}\right)=\tau\left(\left(\left(\lambda_b-(\lambda_a)_b\right)\widetilde{f_b}\right)_{b=0}^{\infty}\right)$$
$$=\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\left(\lambda_z-(\lambda_a)_z\right)\widetilde{f_z},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\sum_{l=0,l\in K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\left(\lambda_z-(\lambda_a)_z\right)\widetilde{f_z},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$+\sum_{l=0,l\notin K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\left(\lambda_z-(\lambda_a)_z\right)\widetilde{f_z},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\sum_{l=0,l\notin K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\lambda_z\widetilde{f_z},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$\leq\frac{1}{(a+1)^{v_0}}\sum_{l=0,l\notin K_{1/a+1}}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\widetilde{f_z},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$<\frac{1}{(a+1)^{v_0}}\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1\\ z\end{bmatrix}q_z\widetilde{f_z},\tilde{0}\right)}{\begin{bmatrix} r+l\\ l\end{bmatrix}}\right)^{v_l}$$
$$=\frac{1}{(a+1)^{v_0}}\tau\left(\tilde{f}\right). \tag{58}$$

Therefore, $\|H_\lambda-H_{\lambda_a}\|\leq 1/(a+1)^{v_0}$. This implies H_λ is a limit of finite rank mappings.

(4) As $\mathscr{A}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)\subsetneqq\mathscr{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$, the proof follows

(5) Since $I=I_\lambda$, where $\lambda=(1,1,)$, one has $I\notin\mathscr{K}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$ and $I\in\mathbb{D}((\Gamma_r^{\mathfrak{S}}(q,v))_\tau)$

(6) Let the sufficient setups be verified. One has $\rho>0$ with $|\lambda_a|\geq\rho$, for every $a\in(\ker(\lambda))^c$. We have to show that Range(H_λ) is closed; let $\tilde{g}$ be a limit point of Range(H_λ). One has $H_\lambda\widetilde{f_b}\in(\Gamma_r^{\mathfrak{S}}(q,v))_\tau$, for all b

$\in \mathcal{N}$ with $\lim_{b\longrightarrow\infty} H_\lambda \widetilde{f}_b = \tilde{g}$. Clearly, $H_\lambda \widetilde{f}_b$ is a Cauchy sequence. Since $(v_l) \in \uparrow \cap \ell_\infty$, we have

$$\begin{aligned}
\tau\left(H_\lambda \widetilde{f}_a - H_\lambda \widetilde{f}_b\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\lambda_z\left(\widetilde{f_a}\right)_z - \lambda_z\left(\widetilde{f_b}\right)_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&= \sum_{l=0, l\in(\ker(\lambda))^c}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\lambda_z\left(\widetilde{f_a}\right)_z - \lambda_z\left(\widetilde{f_b}\right)_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\quad + \sum_{l=0, l\notin(\ker(\lambda))^c}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\lambda_z\left(\widetilde{f_a}\right)_z - \lambda_z\left(\widetilde{f_b}\right)_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\geq \sum_{l=0, l\in(\ker(\lambda))^c}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\lambda_z\left(\widetilde{f_a}\right)_z - \lambda_z\left(\widetilde{f_b}\right)_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\lambda_z\left(\widetilde{u_a}\right)_z - \lambda_z\left(\widetilde{u_b}\right)_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&> \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\rho\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\left(\widetilde{u_a}\right)_z - \left(\widetilde{u_b}\right)_z\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\geq \inf_l \rho^{v_l} \tau\left(\widetilde{u_a} - \widetilde{u_b}\right),
\end{aligned} \tag{59}$$

where

$$\left(\widetilde{u_a}\right)_k = \begin{cases} \left(\widetilde{f_a}\right)_k, & k \in (\ker(\lambda))^c, \\ 0, & k \notin (\ker(\lambda))^c. \end{cases} \tag{60}$$

Therefore, $\{\widetilde{u_a}\}$ is a Cauchy sequence in $(\Gamma_r^{\mathfrak{G}}(q,v))_\tau$. Since $(\Gamma_r^{\mathfrak{G}}(q,v))_\tau$ is complete. One has $\tilde{f} \in (\Gamma_r^{\mathfrak{G}}(q,v))_\tau$ with $\lim_{b\longrightarrow\infty} \widetilde{u_b} = \tilde{f}$. As $H_\lambda \in \mathbb{D}((\Gamma_r^{\mathfrak{G}}(q,v))_\tau)$, we have $\lim_{b\longrightarrow\infty} H_\lambda \widetilde{u_b} = H_\lambda \tilde{f}$. As $\lim_{b\longrightarrow\infty} H_\lambda \widetilde{u_b} = \lim_{b\longrightarrow\infty} H_\lambda \widetilde{f}_b = \tilde{g}$. So, $H_\lambda \tilde{f} = \tilde{g}$. Then, $\tilde{g} \in \mathrm{Range}(H_\lambda)$, i.e., $\mathrm{Range}(H_\lambda)$ is closed. Next, suppose the necessary condition is satisfied. One has $\rho > 0$ with $\tau(H_\lambda \tilde{f}) \geq \rho\tau(\tilde{f})$ and $\tilde{f} \in ((\Gamma_r^{\mathfrak{G}}(q,v))_\tau)_{(\ker(\lambda))^c}$. Let $K = \{b \in (\ker(\lambda))^c : |\lambda_b| < \rho\} \neq \varnothing$, then for $a_0 \in K$, we have

$$\begin{aligned}
\tau\left(H_\lambda \widetilde{e_{a_0}}\right) &= \tau\left(\left(\lambda_b\left(\widetilde{e_{a_0}}\right)_b\right)_{b=0}^{\infty}\right) \\
&= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \lambda_z\left(\widetilde{e_{a_0}}\right)_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&< \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\rho\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\widetilde{e_{a_0}}\right)_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l} \\
&\leq \sup_l \rho^{v_l} \tau\left(\widetilde{e_{a_0}}\right),
\end{aligned} \tag{61}$$

which introduces a contradiction. So $K = \phi$, we have $|\lambda_a| \geq \rho$, for all $a \in (\ker(\lambda))^c$.

(7) First, assume $\kappa \in \mathfrak{R}^{\mathcal{N}}$ so that $\kappa_a = 1/\lambda_a$. By Theorem 35 part (1), we have $H_\lambda, H_\kappa \in \mathbb{D}((\Gamma_r^{\mathfrak{G}}(q,v))_\tau)$. One has $H_\lambda \cdot H_\kappa = H_\kappa \cdot H_\lambda = I$. So, $H_\kappa = H_\lambda^{-1}$. Second, if H_λ is invertible. Then $\mathrm{Range}(H_\lambda) = ((\Gamma_r^{\mathfrak{G}}(q,v))_\tau)_{\mathcal{N}}$. Therefore, $\mathrm{Range}(H_\lambda)$ is closed. From Theorem 35 part (5), one has $\alpha > 0$ with $|\lambda_a| \geq \alpha$, for all $a \in (\ker(\lambda))^c$. Then, $\ker(\lambda) = \varnothing$, when $\lambda_{a_0} = 0$, where $a_0 \in \mathcal{N}$; this implies $e_{a_0} \in \ker(H_\lambda)$, which is a contradiction, since $\ker(H_\lambda)$ is trivial. Then, $|\lambda_a| \geq \alpha$, for all $a \in \mathcal{N}$. As $H_\lambda \in \ell_\infty$. From Theorem 35 part (1), one has $\eta > 0$ with $|\lambda_a| \leq \eta$, for all $a \in \mathcal{N}$. So $\alpha \leq |\lambda_a| \leq \eta$, for all $a \in \mathcal{N}$

(8) First, if $\ker(\lambda) \subsetneqq \mathcal{N}$ and $\ker(\lambda) \notin \mathfrak{I}$, one has $\widetilde{e_a} \in \ker(H_\lambda)$, for all $a \in \ker(\lambda)$. As $\widetilde{e_a}$'s are linearly independent, we have $\dim(\ker(H_\lambda)) = \infty$; this is a contradiction. Therefore, $\ker(\lambda) \subsetneqq \mathcal{N} \in \mathfrak{I}$. The condition (g2) comes from Theorem 35 part (6). Next, assume the setups (g1) and (g2) are satisfied. According to Theorem 35 part (6), the setup (g2) gives that $\mathrm{Range}(H_\lambda)$ is closed. The condition (g1) implies that $\dim((\mathrm{Range}(H_\lambda))^c) < \infty$ and $\dim(\ker(H_\lambda)) < \infty$. Therefore, H_λ is Fredholm

□

4. Fixed Points of Kannan Contraction Type

In this section, we offer the existence of a fixed point of Kannan contraction mapping acting on this new space under the conditions of Theorem 14 and its associated prequasi ideal. Interestingly, several numerical experiments are presented to illustrate our results.

Definition 36. A prequasi normed $\mathfrak{pssst}\tau$ on $\mathscr{E}^{\mathfrak{G}}$ confirms the Fatou property, if for every sequence $\{\widetilde{h^b}\} \subseteq \mathscr{E}_\tau^{\mathfrak{G}}$ so that $\lim_{b\longrightarrow\infty} \tau(\widetilde{h^b} - \tilde{h}) = \tilde{0}$ and every $\tilde{g} \in \mathscr{E}_\tau^{\mathfrak{G}}$, one has $\tau(\tilde{g} - \tilde{h}) \leq \sup_p \inf_{b\geq p} \tau(\tilde{g} - \widetilde{h^b})$.

Throughout the next part of this article, we will use the two functions τ_1 and τ_2 as

$$\begin{aligned}
\tau_1\left(\tilde{f}\right) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar}, \\
\tau_2\left(\tilde{f}\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{f}_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l},
\end{aligned} \tag{62}$$

for all $\tilde{f} \in \Gamma_r^{\mathfrak{G}}(q,v)$.

Theorem 37. *The function τ_1 satisfies the Fatou property.*

Proof. Assume $\{\widetilde{g^b}\} \subseteq (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ so that $\lim_{b\longrightarrow\infty}\tau_1(\widetilde{g^b} - \tilde{g}) = 0$. Clearly, $\tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$. For every $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$, one has

$$\begin{aligned}
\tau_1\left(\tilde{f}-\tilde{g}\right) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\
&\leq \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\widetilde{g_z^b}\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\
&\quad + \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\widetilde{g_z^b}-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right]^{1/\hbar} \\
&\leq \sup_j \inf_{b\geq j}\tau_1\left(\tilde{f}-\widetilde{g^b}\right).
\end{aligned} \tag{63}$$

□

Theorem 38. *Suppose $\nu_0 > 1$, then τ_2 does not verify the Fatou property.*

Proof. If $\{\widetilde{g^b}\} \subseteq (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_2}$ so that $\lim_{b\longrightarrow\infty}\tau_2(\widetilde{g^b} - \tilde{g}) = 0$. Clearly, $\tilde{g} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_2}$. For every $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_2}$, one has

$$\begin{aligned}
\tau_2\left(\tilde{f}-\tilde{g}\right) &= \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l} \\
&\leq 2^{\hbar-1}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\tilde{f}_z-\widetilde{g_z^b}\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right. \\
&\quad \left. + \sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z\left(\widetilde{g_z^b}-\tilde{g}_z\right),\tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{\nu_l}\right] \\
&\leq 2^{\hbar-1}\sup_j \inf_{b\geq j}\tau_2\left(\tilde{f}-\widetilde{g^b}\right).
\end{aligned} \tag{64}$$

□

Hence, τ_2 does not satisfy the Fatou property.

Definition 39 (see [30]). A mapping $G : \mathscr{E}_\tau^{\mathfrak{S}} \longrightarrow \mathscr{E}_\tau^{\mathfrak{S}}$ is called a Kannan τ-contraction, if one has $\zeta \in [0, 1/2)$, with $\tau(G\tilde{g} - G\tilde{h}) \leq \zeta(\tau(G\tilde{g} - \tilde{g}) + \tau(G\tilde{h} - \tilde{h}))$, for all $\tilde{g}, \tilde{h} \in \mathscr{E}_\tau^{\mathfrak{S}}$. When $G(\tilde{g}) = \tilde{g}$, then $\tilde{g} \in \mathscr{E}_\tau^{\mathfrak{S}}$ is called a fixed point of G.

Theorem 40. *Suppose $G : (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ is Kannan τ_1-contraction operator, then G has a unique fixed point.*

Proof. If $\tilde{h} \in \Gamma_r^{\mathfrak{S}}(q,\nu)$, one has $G^m\tilde{h} \in \Gamma_r^{\mathfrak{S}}(q,\nu)$. As G is a Kannan τ_1-contraction, one has

$$\begin{aligned}
\tau_1\left(G^{m+1}\tilde{h} - G^m\tilde{h}\right) &\leq \zeta\left(\tau_1\left(G^{m+1}\tilde{h} - G^m\tilde{h}\right) + \tau_1\left(G^m\tilde{h} - G^{m-1}\tilde{h}\right)\right) \\
&\Rightarrow \tau_1\left(G^{m+1}\tilde{h} - G^m\tilde{h}\right) \leq \frac{\zeta}{1-\zeta}\tau_1\left(G^m\tilde{h} - G^{m-1}\tilde{h}\right) \\
&\leq \left(\frac{\zeta}{1-\zeta}\right)^2\tau_1\left(G^{m-1}\tilde{h} - G^{m-2}\tilde{h}\right) \\
&\leq \leq\left(\frac{\zeta}{1-\zeta}\right)^m\tau_1\left(G\tilde{h} - \tilde{h}\right).
\end{aligned} \tag{65}$$

We get for all $m, n \in \mathscr{N}$ so that $n > m$ that

$$\begin{aligned}
\tau_1\left(G^m\tilde{h} - G^n\tilde{h}\right) &\leq \zeta\left(\tau_1\left(G^m\tilde{h} - G^{m-1}\tilde{h}\right) + \tau_1\left(G^n\tilde{h} - G^{n-1}\tilde{h}\right)\right) \\
&\leq \zeta\left(\left(\frac{\zeta}{1-\zeta}\right)^{m-1} + \left(\frac{\zeta}{1-\zeta}\right)^{n-1}\right)\tau_1\left(G\tilde{h} - \tilde{h}\right).
\end{aligned} \tag{66}$$

□

Therefore, $\{G^m\tilde{h}\}$ is a Cauchy sequence in $(\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$. As $(\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ is prequasi Banach space. One has $\tilde{J} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ with $\lim_{m\longrightarrow\infty}G^m\tilde{h} = \tilde{J}$. To show that $G(\tilde{J}) = \tilde{J}$. Since τ_1 satisfies the Fatou property, one can see

$$\begin{aligned}
\tau_1\left(G\tilde{J} - \tilde{J}\right) &\leq \sup_i \inf_{m\geq i}\tau_1\left(G^{m+1}\tilde{h} - G^m\tilde{h}\right) \\
&\leq \sup_i \inf_{m\geq i}\left(\frac{\zeta}{1-\zeta}\right)^m\tau_1\left(G\tilde{h} - \tilde{h}\right) = 0,
\end{aligned} \tag{67}$$

then $G(\tilde{J}) = \tilde{J}$. Therefore, $\tilde{J}$ is a fixed point of G. To indicate the uniqueness of the fixed point. Let us have two different fixed points $\tilde{f}, \tilde{J} \in (\Gamma_r^{\mathfrak{S}}(q,\nu))_{\tau_1}$ of G. We have

$$\tau_1\left(\tilde{f} - \tilde{J}\right) \leq \tau_1\left(G\tilde{f} - G\tilde{J}\right) \leq \zeta\left(\tau_1\left(G\tilde{f} - \tilde{f}\right) + \tau_1\left(G\tilde{J} - \tilde{J}\right)\right) = 0. \tag{68}$$

Therefore, $\tilde{f} = \tilde{J}$.

Corollary 41. *If $G:(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}\longrightarrow(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}$ is Kannan τ_1-contraction, then G has a unique fixed point $\tilde{J}$ so that $\tau_1(G^m\tilde{h}-\tilde{J})\leq\zeta(\zeta/1-\zeta)^{m-1}\tau_1(G\tilde{h}-\tilde{h})$.*

Proof. By Theorem 40, one has a unique fixed point $\tilde{J}$ of G. Hence,

$$\begin{aligned}\tau_1\left(G^m\tilde{h}-\tilde{J}\right)&=\tau_1\left(G^m\tilde{h}-G\tilde{J}\right)\\&\leq\zeta\left(\tau_1\left(G^m\tilde{h}-G^{m-1}\tilde{h}\right)+\tau_1\left(G\tilde{J}-\tilde{J}\right)\right)\\&=\zeta\left(\frac{\zeta}{1-\zeta}\right)^{m-1}\tau_1\left(G\tilde{h}-\tilde{h}\right).\end{aligned}\tag{69}$$

□

Definition 42. If $\mathscr{E}_\tau^{\mathfrak{S}}$ is a prequasi normed $\mathfrak{pssf}$, $G:\mathscr{E}_\tau^{\mathfrak{S}}\longrightarrow\mathscr{E}_\tau^{\mathfrak{S}}$ and $\tilde{j}\in\mathscr{E}_\tau^{\mathfrak{S}}$. The mapping G is called τ-sequentially continuous at $\tilde{j}$, if and only if, when $\lim_{i\longrightarrow\infty}\tau(\tilde{g}_i-\tilde{j})=0$, then $\lim_{i\longrightarrow\infty}\tau(G\tilde{g}_i-G\tilde{j})=0$.

Theorem 43. *If $v_0>1$, and $G:(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}\longrightarrow(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$. The element $\tilde{h}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$ is the unique fixed point of G, when the following conditions are confirmed:*

(i) G is Kannan τ_2-contraction

(ii) G is τ_2-sequentially continuous at $\tilde{h}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$

(iii) One has $\tilde{j}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$ with $\{G^m\tilde{j}\}$ has $\{G^{m_i}\tilde{j}\}$ converges to $\tilde{h}$

Proof. Assume $\tilde{h}$ is not a fixed point of G, one has $G\tilde{h}\neq\tilde{h}$. According to conditions (ii) and (iii), we have

$$\begin{aligned}\lim_{m_i\longrightarrow\infty}\tau_2\left(G^{m_i}\tilde{j}-\tilde{h}\right)&=0,\\\lim_{m_i\longrightarrow\infty}\tau_2\left(G^{m_i+1}\tilde{j}-G\tilde{h}\right)&=0.\end{aligned}\tag{70}$$

As G is Kannan τ_2-contraction, one has

$$\begin{aligned}0<\tau_2\left(G\tilde{h}-\tilde{h}\right)&=\tau_2\left(\left(G\tilde{h}-G^{m_i+1}\tilde{j}\right)+\left(G^{m_i}\tilde{j}-\tilde{h}\right)\right.\\&\quad\left.+\left(G^{m_i+1}\tilde{j}-G^{m_i}\tilde{j}\right)\right)\leq2^{2\hbar-2}\tau_2\\&\quad\cdot\left(G^{m_i+1}\tilde{j}-G\tilde{h}\right)+2^{2\hbar-2}\tau_2\left(G^{m_i}\tilde{j}-\tilde{h}\right)\\&\quad+2^{\hbar-1}\zeta\left(\frac{\zeta}{1-\zeta}\right)^{m_i-1}\tau_2\left(G\tilde{j}-\tilde{j}\right).\end{aligned}\tag{71}$$

Take $m_i\longrightarrow\infty$, one obtains a contradiction. Therefore, $\tilde{h}$ is a fixed point of G. To explain the uniqueness of $\tilde{h}$. Suppose we have two different fixed points $\tilde{h},\tilde{g}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$ of G. Then

$$\tau_2\left(\tilde{h}-\tilde{g}\right)\leq\tau_2\left(G\tilde{h}-G\tilde{g}\right)\leq\zeta\left(\tau_2\left(G\tilde{h}-\tilde{h}\right)+\tau_2\left(G\tilde{g}-\tilde{g}\right)\right)=0.\tag{72}$$

So $\tilde{h}=\tilde{g}$. □

Example 44. If $T:(\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^\infty,(2l+3/l+2)_{l=0}^\infty))_{\tau_1}\longrightarrow(\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^\infty,(2l+3/l+2)_{l=0}^\infty))_{\tau_1}$ and

$$T\left(\tilde{f}\right)=\begin{cases}\dfrac{\tilde{f}}{4}, & \tau_1\left(\tilde{f}\right)\in[0,1),\\[2mm]\dfrac{\tilde{f}}{5}, & \tau_1\left(\tilde{f}\right)\in[1,\infty).\end{cases}\tag{73}$$

For all $\tilde{f},\tilde{g}\in(\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^\infty,(2l+3/l+2)_{l=0}^\infty))_{\tau_1}$. If $\tau_1(\tilde{f}),\tau_1(\tilde{g})\in[0,1)$, we have

$$\begin{aligned}\tau_1\left(T\tilde{f}-T\tilde{g}\right)&=\tau_1\left(\frac{\tilde{f}}{4}-\frac{\tilde{g}}{4}\right)\leq\frac{1}{\sqrt[4]{27}}\left(\tau_1\left(\frac{3\tilde{f}}{4}\right)+\tau_1\left(\frac{3\tilde{g}}{4}\right)\right)\\&=\frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f}-\tilde{f}\right)+\tau_1\left(T\tilde{g}-\tilde{g}\right)\right).\end{aligned}\tag{74}$$

For every $\tau_1(\tilde{f}),\tau_1(\tilde{g})\in[1,\infty)$, we have

$$\begin{aligned}\tau_1\left(T\tilde{f}-T\tilde{g}\right)&=\tau_1\left(\frac{\tilde{f}}{5}-\frac{\tilde{g}}{5}\right)\leq\frac{1}{\sqrt[4]{64}}\left(\tau_1\left(\frac{4\tilde{f}}{5}\right)+\tau_1\left(\frac{4\tilde{g}}{5}\right)\right)\\&=\frac{1}{\sqrt[4]{64}}\left(\tau_1\left(T\tilde{f}-\tilde{f}\right)+\tau_1\left(T\tilde{g}-\tilde{g}\right)\right).\end{aligned}\tag{75}$$

For every $\tau_1(\tilde{f})\in[0,1)$ and $\tau_1(\tilde{g})\in[1,\infty)$, one has

$$\begin{aligned}\tau_1\left(T\tilde{f}-T\tilde{g}\right)&=\tau_1\left(\frac{\tilde{f}}{4}-\frac{\tilde{g}}{5}\right)\leq\frac{1}{\sqrt[4]{27}}\tau_1\left(\frac{3\tilde{f}}{4}\right)+\frac{1}{\sqrt[4]{64}}\tau_1\\&\quad\cdot\left(\frac{4\tilde{g}}{5}\right)\leq\frac{1}{\sqrt[4]{27}}\left(\tau_1\left(\frac{3\tilde{f}}{4}\right)+\tau_1\left(\frac{4\tilde{g}}{5}\right)\right)\\&=\frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f}-\tilde{f}\right)+\tau_1\left(T\tilde{g}-\tilde{g}\right)\right).\end{aligned}\tag{76}$$

Hence, T is Kannan τ_1-contraction, as τ_1 satisfies the Fatou property. By Theorem 40, T has a unique fixed point $\widetilde{\theta}$. Assume

$$\left\{\widetilde{h^{(k)}}\right\} \subseteq \left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+5)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}, \tag{77}$$

so that $\lim_{k\longrightarrow\infty}\tau_1(\widetilde{h^{(k)}} - \widetilde{h^{(0)}}) = 0$, where

$$\widetilde{h^{(0)}} \in \left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+5)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}, \tag{78}$$

such that $\tau_1(\widetilde{h^{(0)}}) = 1$. As τ_1 is continuous, one has

$$\begin{aligned} \lim_{k\longrightarrow\infty}\tau_1\left(T\widetilde{h^{(k)}} - T\widetilde{h^{(0)}}\right) &= \lim_{k\longrightarrow\infty}\tau_1\left(\frac{\widetilde{h^{(k)}}}{4} - \frac{\widetilde{h^{(0)}}}{5}\right) \\ &= \tau_1\left(\frac{\widetilde{h^{(0)}}}{20}\right) > 0. \end{aligned} \tag{79}$$

So T is not τ_1-sequentially continuous at $\widetilde{h^{(0)}}$. This implies T is not continuous at $\widetilde{h^{(0)}}$.

For every $\tilde{f}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}))_{\tau_2}$. If $\tau_2(\tilde{f}), \tau_2(\tilde{g}) \in [0,1)$, one has

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{\tilde{f}}{4} - \frac{\tilde{g}}{4}\right) \le \frac{2}{\sqrt{27}}\left(\tau_2\left(\frac{3\tilde{f}}{4}\right) + \tau_2\left(\frac{3\tilde{g}}{4}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{80}$$

Let $\tau_2(\tilde{f}), \tau_2(\tilde{g}) \in [1,\infty)$, one has

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{\tilde{f}}{5} - \frac{\tilde{g}}{5}\right) \le \frac{1}{4}\left(\tau_2\left(\frac{4\tilde{f}}{5}\right) + \tau_2\left(\frac{4\tilde{g}}{5}\right)\right) \\ &= \frac{1}{4}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{81}$$

For every $\tau_2(\tilde{f}) \in [0,1)$ and $\tau_2(\tilde{g}) \in [1,\infty)$, one has

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{\tilde{f}}{4} - \frac{\tilde{g}}{5}\right) \le \frac{2}{\sqrt{27}}\tau_2\left(\frac{3\tilde{f}}{4}\right) + \frac{1}{4}\tau_2\left(\frac{4\tilde{g}}{5}\right) \\ &\le \frac{2}{\sqrt{27}}\left(\tau_2\left(\frac{3\tilde{f}}{4}\right) + \tau_2\left(\frac{4\tilde{g}}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{82}$$

Hence, T is Kannan τ_2-contraction and

$$T^m\left(\tilde{f}\right) = \begin{cases} \dfrac{\tilde{f}}{4^m}, & \tau_2\left(\tilde{f}\right) \in [0,1), \\ \dfrac{\tilde{f}}{5^m}, & \tau_2\left(\tilde{f}\right) \in [1,\infty). \end{cases} \tag{83}$$

Evidently, T is τ_2-sequentially continuous at $\tilde{\theta}$ and $\{T^m\tilde{f}\}$ has a subsequence $\{T^{m_j}\tilde{f}\}$ converges to $\tilde{\theta}$. According to Theorem 43, the element $\tilde{\theta}$ is the only fixed point of T.

Example 45. Let $T : (\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}))_{\tau_2} \longrightarrow (\Gamma_r^{\mathfrak{S}}((1/(l+5)l+r-1l)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}))_{\tau_2}$ and

$$T\left(\tilde{f}\right) = \begin{cases} \dfrac{1}{4}\left(\tilde{e_1} + \tilde{f}\right), & \tilde{f_0}(t) \in \left[0, \dfrac{1}{3}\right), \\ \dfrac{1}{3}\tilde{e_1}, & \tilde{f_0}(t) = \dfrac{1}{3}, \\ \dfrac{1}{4}\tilde{e_1}, & \tilde{f_0}(t) \in \left(\dfrac{1}{3}, 1\right]. \end{cases} \tag{84}$$

As $\tilde{f_0}(t), \tilde{g_0}(t) \in [0, 1/3)$, we get

$$\begin{aligned} \tau_2\left(T\tilde{f} - T\tilde{g}\right) &= \tau_2\left(\frac{1}{4}\left(\tilde{f_0} - \tilde{g_0}, \tilde{f_1} - \tilde{g_1}, \tilde{f_2} - \tilde{g_2}, \cdots\right)\right) \\ &\le \frac{2}{\sqrt{27}}\left(\tau_2\left(\frac{3\tilde{f}}{4}\right) + \tau_2\left(\frac{3\tilde{g}}{4}\right)\right) \\ &\le \frac{2}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \end{aligned} \tag{85}$$

For all $\tilde{f_0}(t), \tilde{g_0}(t) \in (1/3, 1]$, hence for all $\varepsilon > 0$, we have

$$\tau_2\left(T\tilde{f} - T\tilde{g}\right) = 0 \le \varepsilon\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \tag{86}$$

For all $\widetilde{f_0}(t) \in [0, 1/3)$ and $\widetilde{g_0}(t) \in (1/3, 1]$, one has

$$\tau_2\left(T\tilde{f} - T\tilde{g}\right) = \tau_2\left(\frac{\tilde{f}}{4}\right) \le \frac{1}{\sqrt{27}}\tau_2\left(\frac{3\tilde{f}}{4}\right) = \frac{1}{\sqrt{27}}\tau_2\left(T\tilde{f} - \tilde{f}\right) \le \frac{1}{\sqrt{27}}\left(\tau_2\left(T\tilde{f} - \tilde{f}\right) + \tau_2(T\tilde{g} - \tilde{g})\right). \tag{87}$$

Hence, T is Kannan τ_2-contraction. Obviously, T is τ_2-sequentially continuous at $1/3\widetilde{e_1}$, and there is $\tilde{f} \in (\Gamma_r^{\mathfrak{S}}((1/(l+5)l + r - 1l)_{l=0}^{\infty}, (2l + 3/l + 2)_{l=0}^{\infty}))_{\tau_2}$ with $\widetilde{f_0}(t) \in [0, 1/3)$ such that the sequence of iterates $\{T^m\tilde{f}\} = \{\sum_{a=1}^{m} 1/4^a\widetilde{e_1} + 1/4^m\tilde{f}\}$ includes a subsequence $\{T^{m_j}\tilde{f}\} = \{\sum_{a=1}^{m_j} 1/4^a \widetilde{e_1} + 1/4^{m_j}\tilde{f}\}$ converges to $1/3\widetilde{e_1}$. In view of Theorem 43, the operator T has one fixed point $1/3\widetilde{e_1}$. Note that T is not continuous at $1/3\widetilde{e_1}$.

For all $\tilde{f}, \tilde{g} \in (\Gamma_r^{\mathfrak{S}}((1/(l+5)l + r - 1l)_{l=0}^{\infty}, (2l + 3/l + 2)_{l=0}^{\infty}))_{\tau_1}$. If $\widetilde{f_0}(t), \widetilde{g_0}(t) \in [0, 1/3)$, we have

$$\begin{aligned}\tau_1\left(T\tilde{f} - T\tilde{g}\right) &= \tau_1\left(\frac{1}{4}\left(\widetilde{f_0} - \widetilde{g_0}, \widetilde{f_1} - \widetilde{g_1}, \widetilde{f_2} - \widetilde{g_2}, \cdots\right)\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(\frac{3\tilde{f}}{4}\right) + \tau_1\left(\frac{3\tilde{g}}{4}\right)\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1(T\tilde{g} - \tilde{g})\right).\end{aligned} \tag{88}$$

For all $\widetilde{f_0}(t), \widetilde{g_0}(t) \in (1/3, 1]$, hence for all $\varepsilon > 0$, one has

$$\tau_1\left(T\tilde{f} - T\tilde{g}\right) = 0 \le \varepsilon\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1(T\tilde{g} - \tilde{g})\right). \tag{89}$$

For all $\widetilde{f_0}(t) \in [0, 1/3)$ and $\widetilde{g_0}(t) \in (1/3, 1]$, we have

$$\tau_1\left(T\tilde{f} - T\tilde{g}\right) = \tau_1\left(\frac{\tilde{f}}{4}\right) \le \frac{1}{\sqrt[4]{27}}\tau_1\left(\frac{3\tilde{f}}{4}\right) = \frac{1}{\sqrt[4]{27}}\tau_1\left(T\tilde{f} - \tilde{f}\right) \le \frac{1}{\sqrt[4]{27}}\left(\tau_1\left(T\tilde{f} - \tilde{f}\right) + \tau_1(T\tilde{g} - \tilde{g})\right). \tag{90}$$

Therefore, the operator T is Kannan τ_1-contraction. Since τ_1 confirms the Fatou property. By Theorem 40, the operator T has a unique fixed point $1/3\widetilde{e_1}$.

In this part, we will use

$$\Xi(V) = \tau\left(\left(s_b(\widetilde{V})\right)_{b=0}^{\infty}\right) = \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z s_z(\widetilde{V}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l}\right]^{1/\hbar}, \tag{91}$$

for every $V \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$.

Definition 46. A function Ξ on $\widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}$ satisfies the Fatou property if for all $\{V_b\}_{b\in\mathcal{N}} \subseteq \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$ so that $\lim_{b\longrightarrow\infty}\Xi(V_b - V) = 0$ and all $T \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$, one has $\Xi(T - V) \le \sup_b \inf_{j\ge b}\Xi(T - V_j)$.

Theorem 47. *The function Ξ does not verify the Fatou property.*

Proof. Assume $\{W_m\}_{m\in\mathcal{N}} \subseteq \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$ so that $\lim_{m\longrightarrow\infty}\Xi(W_m - W) = 0$. Clearly, $W \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$. Hence, for every $V \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$, we have

$$\begin{aligned}\Xi(V - W) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z s_z(\widetilde{V - W}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\le \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z s_{[z/2]}(\widetilde{V - W_i}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\quad + \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z s_{[z/2]}(\widetilde{W - W_i}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\le \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{1/\hbar}\sup_m \inf_{i\ge m} \\ &\quad \cdot \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z\end{bmatrix} q_z s_z(\widetilde{V - W_i}), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l\end{bmatrix}}\right)^{v_l}\right]^{1/\hbar}.\end{aligned} \tag{92}$$

□

Therefore, Ξ does not satisfy the Fatou property.

Definition 48 (see [30]). A mapping $W : \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M) \longrightarrow \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$ is said to be a Kannan Ξ-contraction, assume there is $\zeta \in [0, 1/2)$ with $\Xi(WV - WT) \le \zeta(\Xi(WV - V) + \Xi(WT - T))$, for all $V, T \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$.

Definition 49. Assume $G : \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M) \longrightarrow \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$ and $B \in \widetilde{\mathbb{D}^s}_{\mathscr{E}^{\mathfrak{S}}}(Z, M)$. The mapping G is called Ξ-sequentially continuous at B, if and only if, when $\lim_{m\longrightarrow\infty}\Xi(W_m - B) = 0$, one has $\lim_{m\longrightarrow\infty}\Xi(GW_m - GB) = 0$.

Theorem 50. *If $G : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$. The operator $A \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$ is the only fixed point of G, when the following conditions are confirmed:*

(i) G is Kannan Ξ-contraction

(ii) G is Ξ-sequentially continuous at $A \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$

(iii) One has $B \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$ with $\{G^m B\}$ has $\{G^{m_i} B\}$ converges to A

Proof. Suppose A is not a fixed point of G, then $GA \neq A$. By conditions (ii) and (iii), one has

$$\begin{aligned} &\lim_{m_i \longrightarrow \infty} \Xi(G^{m_i}B - A) = 0, \\ &\lim_{m_i \longrightarrow \infty} \Xi\left(G^{m_i+1}B - GA\right) = 0. \end{aligned} \tag{93}$$

As G is Kannan Ξ-contraction operator, we get

$$\begin{aligned} 0 < \Xi(GA - A) &= \Xi\left(\left(GA - G^{m_i+1}B\right) + \left(G^{m_i}B - A\right) + \left(G^{m_i+1}B - G^{m_i}B\right)\right) \\ &\leq \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{1/\hbar} \Xi\left(G^{m_i+1}B - GA\right) \\ &\quad + \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{2/\hbar} \Xi\left(G^{m_i}B - A\right) \\ &\quad + \left(2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}\right)^{2/\hbar} \zeta\left(\frac{\zeta}{1-\zeta}\right)^{m_i-1} \Xi(GB - B). \end{aligned} \tag{94}$$

By $m_i \longrightarrow \infty$, we have a contradiction. Then, A is a fixed point of G. To show the uniqueness of the fixed point A. If one has two different fixed points $A, D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathscr{G}, \mathscr{V})$ of G. So

$$\Xi(A - D) \leq \Xi(GA - GD) \leq \zeta(\Xi(GA - A) + \Xi(GD - D)) = 0. \tag{95}$$

Therefore, $A = D$. □

Example 51. Assume

$$\begin{aligned} M : &S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)_\tau} \\ &(\mathscr{G}, \mathscr{V}) \longrightarrow S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)_\tau}(\mathscr{G}, \mathscr{V}), \end{aligned} \tag{96}$$

$$M(H) = \begin{cases} \dfrac{H}{6}, & \Xi(H) \in [0, 1), \\ \dfrac{H}{7}, & \Xi(H) \in [1, \infty). \end{cases} \tag{97}$$

For all

$$H_1, H_2 \in S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)_\tau}. \tag{98}$$

If $\Xi(H_1), \Xi(H_2) \in [0, 1)$, we have

$$\begin{aligned} \Xi(MH_1 - MH_2) &= \Xi\left(\frac{H_1}{6} - \frac{H_2}{6}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Xi\left(\frac{5H_1}{6}\right) + \Xi\left(\frac{5H_2}{6}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{125}}(\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)). \end{aligned} \tag{99}$$

Suppose $\Xi(H_1), \Xi(H_2) \in [1, \infty)$, one has

$$\begin{aligned} \Xi(MH_1 - MH_2) &= \Xi\left(\frac{H_1}{7} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Xi\left(\frac{6H_1}{7}\right) + \Xi\left(\frac{6H_2}{7}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}(\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)). \end{aligned} \tag{100}$$

Assume $\Xi(H_1) \in [0, 1)$ and $\Xi(H_2) \in [1, \infty)$, one gets

$$\begin{aligned} \Xi(MH_1 - MH_2) &= \Xi\left(\frac{H_1}{6} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}}\Xi\left(\frac{5H_1}{6}\right) \\ &\quad + \frac{\sqrt{2}}{\sqrt[4]{216}}\Xi\left(\frac{6H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}} \\ &\quad \cdot (\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)). \end{aligned} \tag{101}$$

Hence, M is Kannan Ξ-contraction and

$$M^m(H) = \begin{cases} \dfrac{H}{6^m}, & \Xi(H) \in [0, 1), \\ \dfrac{H}{7^m}, & \Xi(H) \in [1, \infty). \end{cases} \tag{102}$$

Evidently, M is Ξ-sequentially continuous at the zero operator Θ and $\{M^m H\}$ has a subsequence $\{M^{m_j} H\}$ converges to Θ. According to Theorem 50, the zero operator is the only fixed point of M.

If

$$\left\{H^{(a)}\right\} \subseteq S_{\left(\Gamma_r^{\mathfrak{S}}\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)_\tau}, \tag{103}$$

with $\lim_{a\longrightarrow\infty}\Xi(H^{(a)} - H^{(0)}) = 0$, where

$$H^{(0)} \in S\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(1/(l+4)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}\right)_{l=0}^{\infty}, (2l+3/l+2)_{l=0}^{\infty}\right)\right)\right)_{\tau}, \tag{104}$$

so that $\Xi(H^{(0)}) = 1$. As Ξ is continuous, one has

$$\begin{aligned} \lim_{a\longrightarrow\infty} \Xi\left(MH^{(a)} - MH^{(0)}\right) &= \lim_{a\longrightarrow\infty} \Xi\left(\frac{H^{(0)}}{6} - \frac{H^{(0)}}{7}\right) \\ &= \Xi\left(\frac{H^{(0)}}{42}\right) > 0. \end{aligned} \tag{105}$$

Therefore, M is not Ξ-sequentially continuous at $H^{(0)}$. This implies M is not continuous at $H^{(0)}$.

5. Applications on Stochastic Nonlinear Dynamical System

We investigate in this section a solution in $(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}$ to stochastic nonlinear dynamical system (106) under the conditions of Theorem 14. For every $\tilde{f} \in \Gamma_r^{\mathfrak{S}}(q,v)$.

Consider the stochastic nonlinear dynamical system [36]:

$$\widetilde{f_z} = \widetilde{y_z} + \sum_{m=0}^{\infty} \Pi(z,m) g\left(m, \widetilde{f_m}\right), \tag{106}$$

and assume $W : (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}$ is constructed by

$$W\left(\widetilde{f_z}\right)_{z\in\mathcal{N}} = \left(\widetilde{y_z} + \sum_{m=0}^{\infty} \Pi(z,m) g\left(m, \widetilde{f_m}\right)\right)_{z\in\mathcal{N}}. \tag{107}$$

Theorem 52. *The stochastic nonlinear dynamical system (106) has one and only one solution in* $(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}$, *if* $\Pi : \mathcal{N}^2 \longrightarrow \mathfrak{R}, g : \mathcal{N} \times \mathcal{R}(A) \longrightarrow \mathcal{R}(A), \tilde{f} : \mathcal{N} \longrightarrow \mathcal{R}(A)$, $\tilde{y} : \mathcal{N} \longrightarrow \mathcal{R}(A), \tilde{\eta} : \mathcal{N} \longrightarrow \mathcal{R}(A)$, *one has* $\lambda \in \mathfrak{R}$ *with* $\sup_l |\lambda|^{v_l/\hbar} \in [0, 1/2)$ *and for every* $l \in \mathcal{N}$, *one obtains*

$$\begin{aligned} &\left|\sum_{z=0}^{l}\left(\sum_{m\in\mathcal{N}} \Pi(z,m)\left[g\left(m,\widetilde{f_m}\right) - g(m,\widetilde{\eta_m})\right]\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\ &\tilde{\leq} |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y_z} - \widetilde{f_z} + \sum_{m=0}^{\infty}\Pi(z,m) g\left(m,\widetilde{f_m}\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right| \\ &\quad + |\lambda| \left|\sum_{z=0}^{l}\left(\widetilde{y_z} - \widetilde{\eta_z} + \sum_{m=0}^{\infty}\Pi(z,m) g(m,\widetilde{\eta_m})\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\right|. \end{aligned} \tag{108}$$

Proof. Let the conditions be established. Assume the mapping $W : (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1} \longrightarrow (\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}$ is defined by equation (11). Hence,

$$\begin{aligned} \tau_1\left(W\tilde{f} - W\tilde{\eta}\right) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(W\widetilde{f_z} - W\widetilde{\eta_z}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\left(\sum_{m\in\mathcal{N}}\Pi(z,m)\left[g\left(m,\widetilde{f_m}\right) - g(m,\widetilde{\eta_m})\right]\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\leq \sup_l |\lambda|^{v_l/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\left(\widetilde{y_z} - \widetilde{f_z} + \sum_{m=0}^{\infty}\Pi(z,m) g\left(m,\widetilde{f_m}\right)\right)\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\quad + \sup_l |\lambda|^{v_l/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}(\widetilde{y_z} - \widetilde{\eta_z} + \sum_{m=0}^{\infty}\Pi(z,m) g(m,\widetilde{\eta_m}))\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &= \sup_l |\lambda|^{v_l/\hbar}\left(\tau_1\left(W\tilde{f} - \tilde{f}\right) + \tau_1(W\tilde{\eta} - \tilde{\eta})\right). \end{aligned} \tag{109}$$

□

From Theorem 40, one has one and only one solution of (106) in $(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_1}$.

Example 53. Consider

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1}. \tag{110}$$

Suppose the stochastic nonlinear dynamical system:

$$\widetilde{f_z} = e^{\widetilde{-(3z+6)}} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d} + m^2 + 1}}, \tag{111}$$

with $b, d, \widetilde{f_{-2}}(t), \widetilde{f_{-1}}(t) > 0$, for all $t \in A$ and suppose

$$\begin{aligned} W : &\left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1} \\ &\longrightarrow \left(\Gamma_r^{\mathfrak{S}}\left(\left(\left(\frac{1}{(l+1)\begin{bmatrix} l+r-1 \\ l \end{bmatrix}}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)\right)_{\tau_1} \end{aligned} \tag{112}$$

is defined by

$$W\left(\widetilde{f}_z\right)_{z=0}^{\infty}=\left(e^{-\widetilde{(3z+6)}}+\sum_{m=0}^{\infty}(-1)^{z+m}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d}}+\widetilde{m^2+1}}\right)_{z=0}^{\infty}. \tag{113}$$

Evidently, one has $\lambda\in\mathfrak{R}$ with $\sup_l|\lambda|^{2l+3/2l+4}\in[0,1/2)$ and for every $l\in\mathcal{N}$, we have

$$\begin{aligned}&\left|\sum_{z=0}^{l}\left(\sum_{m=0}^{\infty}(-1)^{z}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d}}+\widetilde{m^2+1}}\left((-1)^m-(-1)^m\right)\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|\\&\widetilde{\le}|\lambda|\left|\sum_{z=0}^{l}\left(e^{-\widetilde{(3z+6)}}-f_z+\sum_{m=0}^{\infty}(-1)^{z+m}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d}}+\widetilde{m^2+1}}\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|\\&\quad+|\lambda|\left|\sum_{z=0}^{l}\left(e^{-\widetilde{(3z+6)}}-\widetilde{\eta}_z+\sum_{m=0}^{\infty}(-1)^{z+m}\frac{\widetilde{\eta_{z-2}^{b}}}{\widetilde{\eta_{z-1}^{d}}+\widetilde{m^2+1}}\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|.\end{aligned} \tag{114}$$

From Theorem 52, system (111) has one and only one solution in

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\frac{1}{(l+1)\begin{bmatrix}l+r-1\\ l\end{bmatrix}}\right)_{l=0}^{\infty},\left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\tau_1}. \tag{115}$$

Example 54. Suppose the sequence space

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\frac{1}{(l+1)\begin{bmatrix}l+r-1\\ l\end{bmatrix}}\right)_{l=0}^{\infty},\left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\tau_1}. \tag{116}$$

Assume the stochastic nonlinear dynamical system:

$$\widetilde{f}_z=\widetilde{y}_z+\sum_{m=0}^{\infty}e^{z+m}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d}}+\widetilde{f_{z-1}^{b}}+\widetilde{2}}, \tag{117}$$

with $b,d,\widetilde{f_{-2}}(t),\widetilde{f_{-1}}(t)>0$, for all $t\in A$ and suppose

$$\begin{aligned}W:&\left(\Gamma_r^{\mathfrak{S}}\left(\left(\frac{1}{(l+1)\begin{bmatrix}l+r-1\\ l\end{bmatrix}}\right)_{l=0}^{\infty},\left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\tau_1}\\&\longrightarrow\left(\Gamma_r^{\mathfrak{S}}\left(\left(\frac{1}{(l+1)\begin{bmatrix}l+r-1\\ l\end{bmatrix}}\right)_{l=0}^{\infty},\left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\tau_1}\end{aligned} \tag{118}$$

is defined by

$$W\left(\widetilde{f}_z\right)_{z=0}^{\infty}=\left(\widetilde{y}_z+\sum_{m=0}^{\infty}e^{z+m}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d}}+\widetilde{f_{z-1}^{b}}+\widetilde{2}}\right)_{z=0}^{\infty}. \tag{119}$$

Evidently, there is $\lambda\in\mathfrak{R}$ such that $\sup_l|\lambda|^{2l+3/2l+4}\in[0,1/2)$ and for every $l\in\mathcal{N}$, we have

$$\begin{aligned}&\left|\sum_{z=0}^{l}\left(\sum_{m=0}^{\infty}e^{z}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d}}+\widetilde{f_{z-1}^{b}}+\widetilde{2}}\left(e^m-e^m\right)\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|\\&\widetilde{\le}|\lambda|\left|\sum_{z=0}^{l}\left(\widetilde{y}_z-f_z+\sum_{m=0}^{\infty}e^{z+m}\frac{\widetilde{f_{z-2}^{b}}}{\widetilde{f_{z-1}^{d}}+\widetilde{f_{z-1}^{b}}+\widetilde{2}}\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|\\&\quad+|\lambda|\left|\sum_{z=0}^{l}\left(\widetilde{y}_z-\widetilde{\eta}_z+\sum_{m=0}^{\infty}e^{z+m}\frac{\widetilde{\eta_{z-2}^{b}}}{\widetilde{\eta_{z-1}^{d}}+\widetilde{\eta_{z-1}^{b}}+\widetilde{2}}\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|.\end{aligned} \tag{120}$$

According to Theorem 52, the stochastic nonlinear dynamical system (14) contains a unique solution in

$$\left(\Gamma_r^{\mathfrak{S}}\left(\left(\frac{1}{(l+1)\begin{bmatrix}l+r-1\\ l\end{bmatrix}}\right)_{l=0}^{\infty},\left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\tau_1}. \tag{121}$$

Theorem 55. *If $W:(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}\longrightarrow(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$ is defined by (11) and $v_0>1$. The stochastic nonlinear dynamical system (106) has a unique solution $\widetilde{Z}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$, when the following conditions are satisfied:*

(1) If $\Pi:\mathcal{N}^2\longrightarrow\mathfrak{R}$, $g:\mathcal{N}\times\mathcal{R}(A)\longrightarrow\mathcal{R}(A)$, $\widetilde{f}:\mathcal{N}\longrightarrow\mathcal{R}(A)$, $\widetilde{y}:\mathcal{N}\longrightarrow\mathcal{R}(A)$, $\widetilde{\eta}:\mathcal{N}\longrightarrow\mathcal{R}(A)$, assume there is $\lambda\in\mathfrak{R}$ so that $2^{\hbar-1}\sup_l|\lambda|^{v_l}\in[0,1/2)$ and for every $l\in\mathcal{N}$, one has

$$\begin{aligned}&\left|\sum_{z=0}^{l}\left(\sum_{m\in\mathcal{N}}\Pi(z,m)\left[g\left(m,\widetilde{f_m}\right)-g\left(m,\widetilde{\eta_m}\right)\right]\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|\\&\widetilde{\le}|\lambda|\left|\sum_{z=0}^{l}\left(\widetilde{y}_z-\widetilde{f}_z+\sum_{m=0}^{\infty}\Pi(z,m)g\left(m,\widetilde{f_m}\right)\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|\\&\quad+|\lambda|\left|\sum_{z=0}^{l}\left(\widetilde{y}_z-\widetilde{\eta}_z+\sum_{m=0}^{\infty}\Pi(z,m)g(m,\widetilde{\eta_m})\right)\begin{bmatrix}z+r-1\\ z\end{bmatrix}q_z\right|\end{aligned} \tag{122}$$

(2) W is τ_2-sequentially continuous at $\widetilde{Z}\in(\Gamma_r^{\mathfrak{S}}(q,v))_{\tau_2}$

(3) *There is* $\tilde{Y} \in (\Gamma_r^{\mathfrak{S}}(q, v))_{\tau_2}$ *with* $\{W^m \tilde{Y}\}$ *has* $\{W^{m_j} \tilde{Y}\}$ *converging to* $\tilde{Z}$

Proof. One has

$$\begin{aligned}
\tau_2\left(W\tilde{f} - W\tilde{\eta}\right) &= \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} r+z-1 \\ z \end{bmatrix} q_z \left(W\widetilde{f_z} - W\widetilde{\eta_z} \right), \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \\
&= \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \left(\sum_{m\in\mathcal{N}} \Pi(z,m) \left[g\left(m, \widetilde{f_m}\right) - g(m, \widetilde{\eta_m}) \right] \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \\
&\leq 2^{\hbar-1} \sup_l |\lambda|^{v_l} \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \left(\widetilde{y_z} - \widetilde{f_z} + \sum_{m=0}^{\infty} \Pi(z,m) g\left(m, \widetilde{f_m}\right) \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \\
&\quad + 2^{\hbar-1} \sup_l |\lambda|^{v_l} \sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} (\widetilde{y_z} - \widetilde{\eta_z} + \sum_{m=0}^{\infty} \Pi(z,m) g(m, \widetilde{\eta_m})) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \\
&= 2^{\hbar-1} \sup_l |\lambda|^{v_l} \left(\tau_2\left(W\tilde{f} - \tilde{f} \right) + \tau_2(W\tilde{\eta} - \tilde{\eta}) \right).
\end{aligned} \tag{123}$$

□

By Theorem 43, one gets a unique solution $\tilde{Z} \in (\Gamma_r^{\mathfrak{S}}(q, v))_{\tau_2}$ of equation (106).

Example 56. Suppose the sequence space

$$\left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}. \tag{124}$$

Consider the summable equation (111):
Let

$$\begin{aligned}
W : &\left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2} \\
&\longrightarrow \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}
\end{aligned} \tag{125}$$

defined by (13). Assume W is τ_2-sequentially continuous at

$$\tilde{Z} \in \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}, \tag{126}$$

and there is

$$\tilde{Y} \in \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2} \tag{127}$$

with $\{W^m \tilde{Y}\}$ has $\{W^{m_j} \tilde{Y}\}$ converging to $\tilde{Z}$. Evidently, there is $\lambda \in \mathfrak{R}$ such that $2^{\hbar-1} \sup_l |\lambda|^{2l+3/l+2} \in [0, 1/2)$ and for all $l \in \mathcal{N}$, one has

$$\begin{aligned}
&\left| \sum_{z=0}^{l} \left(\sum_{m=0}^{\infty} (-1)^z \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d} + \widetilde{m^2 + 1}} ((-1)^m - (-1)^m) \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \right| \\
&\leq^{\sim} |\lambda| \left| \sum_{z=0}^{l} \left(e^{-\widetilde{(3z+6)}} - f_z + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{\widetilde{f_{z-2}^b}}{\widetilde{f_{z-1}^d + m^2 + 1}} \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \right| \\
&\quad + |\lambda| \left| \sum_{z=0}^{l} \left(\widetilde{e^{-(3z+6)}} - \widetilde{\eta_z} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{\widetilde{\eta_{z-2}^b}}{\widetilde{\eta_{z-1}^d} + \widetilde{m^2 + 1}} \right) \begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \right|.
\end{aligned} \tag{128}$$

By Theorem 57, the stochastic nonlinear dynamical system (111) has one and only one solution

$$\tilde{Z} \in \left(\Gamma_r^{\mathfrak{S}} \left(\left(\left(\frac{1}{(l+1) \begin{bmatrix} l+r-1 \\ l \end{bmatrix}} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right) \right)_{\tau_2}. \tag{129}$$

In this part, we search for a solution to nonlinear matrix equation (131) at $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$, the conditions of Theorem 14 are satisfied, and

$$\Xi(G) = \left[\sum_{l=0}^{\infty} \left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l} \begin{bmatrix} r+z-1 \\ z \end{bmatrix} q_z \widetilde{s_z(G)}, \tilde{0} \right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}} \right)^{v_l} \right]^{1/\hbar}, \tag{130}$$

for every $G \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G}, \mathcal{V})$. Consider the stochastic nonlinear dynamical system:

$$\widetilde{s_z(G)} = \widetilde{s_z(P)} + \sum_{m=0}^{\infty} \Pi(z,m) f\left(m, \widetilde{s_m(G)}\right), \tag{131}$$

and suppose $W : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V})$ is defined by

$$W(G) = \left(\widetilde{s_z(P)} + \sum_{m=0}^{\infty} \Pi(z,m) f\left(m, \widetilde{s_m(G)}\right)\right) I. \tag{132}$$

Theorem 57. *The stochastic nonlinear dynamical system (131) has one and only one solution $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V})$, if the following conditions are satisfied:*

(1) $\Pi : \mathcal{N}^2 \longrightarrow \mathfrak{R}, f : \mathcal{N} \times \mathscr{R}(A) \longrightarrow \mathscr{R}(A)$, $P \in \mathbb{D}(\mathcal{G},\mathscr{V}), T \in \mathbb{D}(\mathcal{G},\mathscr{V})$, and for every $z \in \mathcal{N}$, there is a positive real number κ so that $\sup_z \kappa^{t_z/\hbar} \in [0, 0.5)$, with

$$\begin{aligned} &\left|\sum_{m\in\mathcal{N}} \Pi(z,m)\left(f\left(m, \widetilde{s_m(G)}\right) - f\left(m, s_m(T)^{\sim}\right)\right)\right| \\ &\quad \tilde{\le} \kappa\left[\left|\widetilde{s_z(P)} - \widetilde{s_z(G)} + \sum_{m\in\mathcal{N}} A(a,m) f\left(m, \widetilde{s_m(G)}\right)\right| \right. \\ &\quad \left. + \left|\widetilde{s_z(P)} - \widetilde{s_z(T)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f\left(m, \widetilde{s_m(T)}\right)\right|\right] \end{aligned} \tag{133}$$

(2) W is Ξ-sequentially continuous at a point $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V})$

(3) There is $B \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V})$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to D

Proof. Suppose the settings are verified. Consider the mapping $W : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V})$ defined by (132). We have

$$\begin{aligned} \Xi(WG - WT) &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\widetilde{s_z(G)} - \widetilde{s_z(T)}\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &= \left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \sum_{m\in\mathcal{N}} A(a,m)\left(f\left(m, \widetilde{s_m(G)}\right) - f\left(m, \widetilde{s_m(T)}\right)\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\le \sup_z \kappa^{t_z/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\widetilde{s_z(P)} - \widetilde{s_z(G)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f\left(m, \widetilde{s_m(G)}\right)\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &\quad + \sup_z \kappa^{t_z/\hbar}\left[\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z\left(\widetilde{s_z(T)} - \widetilde{s_z(G)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f\left(m, \widetilde{s_m(T)}\right)\right), \tilde{0}\right)}{\begin{bmatrix} r+l \\ l \end{bmatrix}}\right)^{v_l}\right]^{1/\hbar} \\ &= \sup_z \kappa^{t_z/\hbar}(\Xi(WG - G) + \Xi(WT - T)). \end{aligned} \tag{134}$$

In view of Theorem 50, one obtains a unique solution of equation (131) at $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}(q,v))_\tau}(\mathcal{G},\mathscr{V})$. □

Example 58. Assume the class $\widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathscr{V})$, where

$$\Xi(G) = \sqrt[2l+3/l+2]{\sum_{l=0}^{\infty}\left(\frac{\tilde{\rho}\left(\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z \widetilde{s_z(G)}, \tilde{0}\right)}{l!\sum_{z=0}^{l}\begin{bmatrix} z+r-1 \\ z \end{bmatrix} q_z}\right)}, \quad \text{for all } G \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathscr{V}).. \tag{135}$$

Consider the stochastic nonlinear dynamical system:

$$\widetilde{s_z(G)} = e^{\widetilde{-(2z+3)}} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b\left|\widetilde{s_{z-2}(G)}\right|}{\sinh^d\left|\widetilde{s_{z-1}(G)}\right| + \widetilde{\sin mz} + \tilde{1}}, \tag{136}$$

where $z \ge 2$ and $b, d > 0$ and let $W : \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathscr{V}) \longrightarrow \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathscr{V})$ be defined as

$$W(G) = \left(e^{\widetilde{-(2z+3)}} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b\left|\widetilde{s_{z-2}(G)}\right|}{\sinh^d\left|\widetilde{s_{z-1}(G)}\right| + \widetilde{\sin mz} + \tilde{1}}\right) I. \tag{137}$$

Suppose W is Ξ-sequentially continuous at a point $D \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathscr{V})$, and there is $B \in \widetilde{\mathbb{D}^s}_{(\Gamma_r^{\mathfrak{S}}((1/l!),(2l+3/l+2)))_\tau}(\mathcal{G},\mathscr{V})$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to D. It is easy to see that

$$\begin{aligned} &\left|\sum_{m=0}^{\infty} \frac{\cosh(3m-z)\cos^b\left|\widetilde{s_{z-2}(G)}\right|}{\sinh^d\left|\widetilde{s_{z-1}(G)}\right| + \widetilde{\sin mz} + \tilde{1}}(\tan(2m+1) - \tan(2m+1))\right| \\ &\tilde{\le} \frac{1}{25}\left|e^{\widetilde{-(2z+3)}} - \widetilde{s_z(G)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b\left|\widetilde{s_{z-2}(G)}\right|}{\sinh^d\left|\widetilde{s_{z-1}(G)}\right| + \widetilde{\sin mz} + \tilde{1}}\right| \\ &\quad + \frac{1}{25}\left|e^{\widetilde{-(2z+3)}} - \widetilde{s_z(T)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-z)\cos^b\left|\widetilde{s_{z-2}(T)}\right|}{\sinh^d\left|\widetilde{s_{z-1}(T)}\right| + \widetilde{\sin mz} + \tilde{1}}\right|. \end{aligned} \tag{138}$$

By Theorem 57, the stochastic nonlinear dynamical system (18) has one solution D.

6. Conclusion

In this article, we introduced a new general space called $(\Gamma_r^{\mathfrak{S}}(q,\nu))_\tau$ and the mappings' ideal space of solutions for many stochastic nonlinear and matrix systems of Kannan contraction type. We have presented some topological and geometric properties of it, of the multiplication operators acting on it, of the mappings' ideal, and of the spectrum of its mappings' ideal. The existence of a fixed point in the Kannan contraction mapping on these spaces is explored. To put our findings to the test, we introduced several numerical experiments. In addition, various effective implementations of the stochastic nonlinear dynamical and matrix system are discussed. The ideal spectrum of mappings, multiplication operators, and the fixed points of any contraction mappings in this new soft functions space are investigated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] L. Guo and Q. Zhu, "Stability analysis for stochastic Volterra–Levin equations with Poisson jumps: fixed point approach," *Journal of Mathematical Physics*, vol. 52, no. 4, article 042702, 2011.

[2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[3] D. Dubois and H. Prade, *Possibility theory: an approach to computerized processing of uncertainty*, Plenum, New York, 1998.

[4] H. Ahmad, M. Younis, and M. E. Koksal, "Double controlled partial metric type spaces and convergence results," *Journal of Mathematics*, vol. 2021, Article ID 7008737, 11 pages, 2021.

[5] W. Mao, Q. Zhu, and X. Mao, "Existence, uniqueness and almost surely asymptotic estimations of the solutions to neutral stochastic functional differential equations driven by pure jumps," *Applied Mathematics and Computation*, vol. 254, pp. 252–265, 2015.

[6] D. Molodtsov, "Soft set theory–first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[7] P. K. Maji, R. Biswas, and A. R. Roy, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[8] X. Yang and Q. Zhu, "Existence, uniqueness, and stability of stochastic neutral functional differential equations of Sobolev-type," *Journal of Mathematical Physics*, vol. 56, no. 12, p. 122701, 2015.

[9] M. Abbas, G. Murtaza, and S. Romaguera, "Soft contraction theorem," *Journal of Nonlinear and Convex Analysis*, vol. 16, pp. 423–435, 2015.

[10] C. M. Chen and I. J. Lin, "Fixed point theory of the soft Meir-Keeler type contractive mappings on a complete soft metric space," *Fixed Point Theory and Applications*, vol. 2015, no. 1, 2015.

[11] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[12] I. Beg, M. Abbas, and M. W. Asghar, "Polytopic fuzzy sets and their applications to multiple-attribute decision-making problems," *International Journal of Fuzzy Systems*, 2022.

[13] M. Younis, S. Deepak, S. Radenovic, and M. Imdad, "Convergence theorems for generalized contractions and applications," *Univerzitet u Nišu*, vol. 34, no. 3, pp. 945–964, 2020.

[14] M. Ruzicka, "Electrorheological fluids. Modeling and mathematical theory," in *In Lecture Notes in Mathematics*, p. 1748, Springer, Berlin, Germany,, 2000.

[15] I. Beg, "Ordered uniform convexity in ordered convex metric spaces with an application to fixed point theory," *Journal of Function Spaces*, vol. 2022, Article ID 8237943, 7 pages, 2022.

[16] R. Kannan, "Some results on fixed points- II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.

[17] S. J. H. Ghoncheh, "Some fixed point theorems for Kannan mapping in the modular spaces," *Ciĕncia eNatura*, vol. 37, no. 6-1, pp. 462–466, 2015.

[18] A. Pietsch, "s-numbers of operators in Banach spaces," *Studia Mathematica*, vol. 51, no. 3, pp. 201–223, 1974.

[19] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford, MR582655 (81j: 47001), 1980.

[20] A. A. Bakery and A. R. Abou Elmatty, "A note on Nakano generalized difference sequence space," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[21] M. İlkhan, S. Demiriz, and E. E. Kara, "Multiplication operators on Cesàro second order function spaces," *Positivity*, vol. 24, no. 3, pp. 605–614, 2020.

[22] M. Mursaleen and A. K. Noman, "On some new sequence spaces of non-absolute type related to the spaces ℓp and ℓ∞ I," *Filomat*, vol. 25, no. 2, pp. 33–51, 2011.

[23] M. Mursaleen and F. Basar, "Domain of Cesàro mean of order one in some spaces of double sequences," *Studia Scientiarum Mathematicarum Hungarica*, vol. 51, no. 3, pp. 335–356, 2014.

[24] A. A. Bakery and O. S. K. Mohamed, "Kannan prequasi contraction maps on Nakano sequence spaces," *Journal of Function Spaces*, vol. 2020, Article ID 8871563, 10 pages, 2020.

[25] A. A. Bakery and O. S. K. Mohamed, "Kannan nonexpansive maps on generalized Cesàro backward difference sequence space of non-absolute type with applications to summable equations," *Journal of Inequalities and Applications*, vol. 2021, no. 1, pp. 1–132, 2021.

[26] S. Das and S. K. Samanta, "Soft metric," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, pp. 77–94, 2013.

[27] H. Roopaei and F. Basar, "On the gamma spaces including the spaces of absolutely p-summable, null, convergent and bounded sequences," *Numerical Functional Analysis and Optimization*, vol. 43, no. 6, pp. 723–754, 2022.

[28] B. Altay and F. Basar, "Generalization of the sequence space ℓ (p) derived by weighted mean," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 174–185, 2007.

[29] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, New York, NY, USA, 1986.

[30] A. A. Bakery and M. M. Mohammed, "Kannan non-expansive mappings on Nakano sequence space of soft reals with some applications," *Journal of function spaces*, vol. 2022, Article ID 2307519, 18 pages, 2022.

[31] A. A. Bakery and O. S. K. Mohamed, "Orlicz generalized difference sequence space and the linked pre-quasi operator ideal," *Journal of Mathematics*, vol. 2020, Article ID 6664996, 9 pages, 2020.

[32] N. Faried and A. A. Bakery, "Small operator ideals formed by s numbers on generalized Cesàro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[33] B. E. Rhoades, "Operators of A-p type," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, vol. 59, no. 3-4, pp. 238–241, 1975.

[34] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag derWissenschaften, Berlin, 1978.

[35] T. Mrowka, *A Brief Introduction to Linear Analysis: Fredholm Operators, Geometry of Manifolds*, Massachusetts Institute of Technology, MIT OpenCouseWare), Fall, 2004.

[36] P. Salimi, A. Latif, and N. Hussain, "Modified α-ψ-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1992684, 18 pages
https://doi.org/10.1155/2022/1992684

Research Article

Kannan Nonexpansive Operators on Variable Exponent Cesàro Sequence Space of Fuzzy Functions

Awad A. Bakery [1,2] **and Mustafa M. Mohammed** [1,3]

[1] *University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Egypt*
[3] *Department of Statistics, Faculty of Science, Sudan University of Science & Technology, Khartoum, Sudan*

Correspondence should be addressed to Mustafa M. Mohammed; mustasta@gmail.com

Received 30 May 2022; Accepted 13 August 2022; Published 27 August 2022

Academic Editor: Santosh Kumar

In general, we have constructed the operator ideal generated by extended s-fuzzy numbers and a certain space of sequences of fuzzy numbers. An investigation into the conditions sufficient for variable exponent Cesàro sequence space of fuzzy functions furnished with the definite function to create pre-quasi-Banach and closed is carried out. The (R) and the normal structural properties of this space are shown. Fixed points for Kannan contraction and nonexpansive mapping have been introduced. Lastly, we explore whether the Kannan contraction mapping has a fixed point in its associated pre-quasioperator ideal. The existence of solutions to nonlinear difference equations is illustrated with a few real-world examples and applications.

1. Introduction

Probability theory, fuzzy set theory, soft sets, and rough sets have contributed substantially to the study of uncertainty. But there are drawbacks to these theories that must be considered. After Zadeh [1] established the concept of fuzzy sets and fuzzy set operations, many researchers adopted the concept of fuzziness in cybernetics and artificial intelligence as well as in expert systems and fuzzy control. For more information and real-world examples, some comparable fixed point results were discussed by Javed et al. [2] to ensure that a fixed point exists and is unique in R-fuzzy b-metric spaces. The viability of the proposed methodologies was demonstrated through a challenging case study. There was no doubt about the superiority of the findings delivered. For the first type of Fredholm-type integral equation, an application was described. In [3], Al-Masarwah and Ahmad defined and investigated the m-Polar (α, β)-Fuzzy Ideals in BCK/BCI-Algebras and explored some pertinent properties. There are many other orthogonal fuzzy metric spaces; however, Javed et al. [4] expanded the orthogonal image fuzzy metric space concept. In the context of the newly specified structure, they displayed some fixed point outcomes. Fuzzy sequence spaces were introduced, and their various features were studied by many workers on sequence spaces and summability theory. Nuray and Savas [5] defined and studied the Nakano sequences of fuzzy numbers, $\ell^F(\tau)$ equipped with the function h. The operator ideal is very important in fixed point theory, Banach space geometry, normal series theory, approximation theory, and ideal transformations. See [6–8] for further proof. Pre-quasioperator ideals are more extensive than quasioperator ideals, according to Faried and Bakery [9]. The learning about the variable exponent Lebesgue spaces obtained impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids (see [10, 11]). There are numerous uses for electrorheological fluids, which include military science, civil engineering, and orthopedic. There have been many developments in mathematics since the Banach fixed point theorem [12] was first published. While contractions have fixed point actions, Kannan [13] cited an example of a type of mapping that is not continuous. In Reference [14], the only attempt was made to explain Kannan operators in modular vector spaces. For more details on Kannan's fixed point theorems, see

[15–20]. Given that the proof of many fixed point theorems in a given space requires either growing the space itself or expanding the self-mapping that acts on it, both options are viable. Hence, we have constructed the Cesàro sequence spaces of fuzzy functions and have presented the solutions of a fuzzy nonlinear dynamical system in this newly created space. This work is aimed at introducing the certain space of sequences of fuzzy numbers, in short (cssf), under a certain function to be pre-quasi (cssf). This space and s-numbers have been used to describe the structure of the ideal operators. We explain the sufficient conditions of variable exponent Cesàro sequence space of fuzzy functions, which is denoted by $C^F_{\tau(.)}$, equipped with the definite function h to be pre-quasi-Banach and closed (cssf). The (R) and the normal structure property of this space are shown. Fixed points for Kannan contraction and nonexpansive mapping have been introduced. Lastly, we explore whether the Kannan contraction mapping has a fixed point in its associated pre-quasioperator ideal. The existence of solutions to nonlinear difference equations is illustrated with a few real-world examples and applications.

2. Definitions and Preliminaries

As a reminder, Matloka [21] presented the notion of ordinary convergence of sequences of fuzzy numbers, where he introduced bounded and convergent fuzzy numbers, explored some of their features, and proved that any convergent fuzzy number sequence is bounded. Nanda [22] studied the sequences of fuzzy numbers and showed the set of all convergent sequences of fuzzy numbers from a complete metric space. Kumar et al. [23] investigated the notion of limit points and cluster points of sequences of fuzzy numbers. Assume Ω is the set of all closed and bounded intervals on the real line $\mathfrak{R}$. For $f = [f_1, f_2]$ and $g = [g_1, g_2]$ in Ω, suppose

$$f \le g, \text{ if and only if } f_1 \le g_1 \text{ and } f_2 \le g_2. \tag{1}$$

Define a metric ρ on Ω by

$$\rho(f, g) = \max\{|f_1 - g_1|, |f_2 - g_2|\}. \tag{2}$$

Matloka [21] showed that ρ is a metric on Ω and (Ω, ρ) is a complete metric space. Also, the relation $\le$ is a partial order on Ω.

Definition 1. A fuzzy number g is a fuzzy subset of $\mathfrak{R}$, i.e., a mapping $g : \mathfrak{R} \longrightarrow [0, 1]$ which verifies the following four settings:

(a) g is fuzzy convex, i.e., for $x, y \in \mathfrak{R}$ and $\alpha \in [0, 1]$, $g(\alpha x + (1-\alpha)y) \ge \min\{g(x), g(y)\}$

(b) g is normal, i.e., there is $y_0 \in \mathfrak{R}$ such that $g(y_0) = 1$

(c) g is an upper semicontinuous, i.e., for all $\alpha > 0$, $g^{-1}([0, x+\alpha))$ for all $x \in [0, 1]$ is open in the usual topology of $\mathfrak{R}$

(d) the closure of $g^0 \coloneqq \{y \in \mathfrak{R} : g(y) > 0\}$ is compact

The β-level set of a fuzzy real number g, $0 < \beta < 1$, indicated by g^β is defined as

$$g^\beta = \{y \in \mathfrak{R} : g(y) \ge \beta\}. \tag{3}$$

The set of every upper semicontinuous, normal, convex fuzzy number, and g^β is compact is denoted by $\mathfrak{R}([0, 1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0, 1])$, if we define $r \in \mathfrak{R}([0, 1])$ by

$$\bar{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \ne r. \end{cases} \tag{4}$$

The additive identity and multiplicative identity in $\mathfrak{R}[0, 1]$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

The arithmetic operations on $\mathfrak{R}[0, 1]$ are defined as follows:

$$\begin{aligned}
(f \oplus g)(y) &= \sup_{y\in\mathfrak{R}} \min\{f(x), g(y-x)\},\\
(f!g)(y) &= \sup_{y\in\mathfrak{R}} \min\{f(x), g(x-y)\},\\
(f \otimes g)(y) &= \sup_{y\in\mathfrak{R}} \min\left\{f(x), g\left(\frac{y}{x}\right)\right\},\\
\left(\frac{f}{g}\right)(y) &= \sup_{y\in\mathfrak{R}} \min\{f(xy), g(x)\},\\
xf(y) &= \begin{cases} f\left(x^{-1}y\right), & x \ne 0, \\ 0, & x = 0. \end{cases}
\end{aligned} \tag{5}$$

The absolute value $|f|$ of $f \in \mathfrak{R}[0, 1]$ is defined by

$$|f|(y) = \begin{cases} \max\{f(y), f(-y)\}, & \text{if } y \ge 0, \\ 0, & \text{if } y < 0. \end{cases} \tag{6}$$

Suppose $f, g \in \mathfrak{R}[0, 1]$ and the β-level sets are $[f]^\beta = [f^\beta_1, f^\beta_2]$, $[g]^\beta = [g^\beta_1, g^\beta_2]$, and $\beta \in [0, 1]$. A partial ordering for any $f, g \in \mathfrak{R}[0, 1]$ as follows: $f^\circ g$, if and only if $f^\beta \le g^\beta$, for all $\beta \in [0, 1]$. Then, the above operations can be defined in terms of β-level sets as follows:

$$\begin{aligned}
[f \oplus g]^\beta &= \left[f^\beta_1 + g^\beta_1, f^\beta_2 + g^\beta_2\right],\\
[f!g]^\beta &= \left[f^\beta_1 - g^\beta_2, f^\beta_2 - g^\beta_1\right],\\
[f \otimes g]^\beta &= \left[\min_{j\in\{1,2\}} f^\beta_j g^\beta_j, \max_{j\in\{1,2\}} f^\beta_j g^\beta_j\right],\\
\left[f^{-1}\right]^\beta &= \left[\left(f^\beta_2\right)^{-1}, \left(f^\beta_1\right)^{-1}\right], f^\beta_j > 0, \text{ for every } \beta \in (0, 1],\\
[xf]^\beta &= \begin{cases} \left[xf^\beta_1, xf^\beta_2\right], & x \ge 0, \\ \left[xf^\beta_2, xf^\beta_1\right], & x < 0. \end{cases}
\end{aligned} \tag{7}$$

Assume $\bar{\rho} : \mathfrak{R}[0,1] \times \mathfrak{R}[0,1] \longrightarrow \mathfrak{R}^+ \cup \{0\}$ is defined by $\bar{\rho}(f,g) = \sup_{0\le\beta\le1} \rho(f^\beta, g^\beta)$.

Recall that

(1) $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space

(2) $\bar{\rho}(f+k, g+k) = \bar{\rho}(f,g)$ for all $f, g, k \in \mathfrak{R}[0,1]$

(3) $\bar{\rho}(f+k, g+l) \le \bar{\rho}(f,g) + \bar{\rho}(k,l)$.

(4) $\bar{\rho}(\xi f, \xi g) = |\xi| \bar{\rho}(f,g)$, for all $\xi \in \mathfrak{R}$.

Definition 2. A sequence $f = (f_j)$ of fuzzy numbers is said to be

(a) bounded if the set $\{f_j : j \in \mathcal{N}\}$ of fuzzy numbers is bounded, i.e., if a sequence (f_j) is bounded, then there are two fuzzy numbers g, l such that $g \le f_j \le l$

(b) convergent to a fuzzy real number f_0 if for every $\varepsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $\bar{\rho}(f_j, f_0) < \varepsilon$, for all $j \ge j_0$

Lemma 3 (see [24]). *Suppose $\tau_a \ge 1$ and $v_a, t_a \in \mathfrak{R}$, for every $a \in \mathcal{N}$, then $|v_a + t_a|^{\tau_a} \le 2^{K-1}(|v_a|^{\tau_a} + |t_a|^{\tau_a})$, where $K = \max\{1, \sup_a \tau_a\}$.*

3. Main Results

3.1. Some Properties of $C^F_{\tau(.)}$. In this section, we have introduced the certain space of sequences of fuzzy numbers or in short (cssf), under the definite function to form pre-quasi (cssf). We explain the sufficient setting of $C^F_{\tau(.)}$ equipped with the definite function h to construct pre-quasi-Banach and closed (cssf). The Fatou property of various pre-quasinorms h on $C^F_{\tau(.)}$ has been investigated. We have presented this space's k-nearly uniformly convex, the property (R), and the h-normal structure-property, which are connected with the fixed point theorem.

By ℓ_∞ and ℓ_r, we denote the spaces of bounded and r-absolutely summable sequences of real numbers, respectively. Let $\omega(F)$ denote the classes of all sequence spaces of fuzzy real numbers. Suppose $\tau = (\tau_a) \in \mathfrak{R}^{+\mathcal{N}}$, where $\mathfrak{R}^{+\mathcal{N}}$ is the space of positive real sequences. The variable exponent Cesàro sequence space of fuzzy functions is denoted by the following: $C^F_{\tau(.)} = \{v = (v_a) \in \omega(F): h(\mu v) < \infty, \text{for some } \mu > 0\}$, when $h(v) = \sum_{a=0}^\infty \left(\sum_{k=0}^a \bar{\rho}(v_k, \bar{0})/a+1\right)^{\tau_a}$. If $(\tau_a) \in \ell_\infty$, then

$$\begin{aligned} C^F_{\tau(.)} &= \{v = (v_a) \in \omega(F): h(\mu v) < \infty, \text{for some } \mu > 0\} \\ &= \left\{ v = (v_a) \in \omega(F): \inf_a |\mu|^{\tau_a} \sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a \bar{\rho}(v_k, \bar{0})}{a+1} \right)^{\tau_a} \right. \\ &\quad \left. \le \sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a \bar{\rho}(\mu v_k, \bar{0})}{a+1} \right)^{\tau_a} < \infty, \text{for some } \mu > 0 \right\} \\ &= \left\{ v = (v_a) \in \omega(F): \sum_{a=0}^\infty \left(\frac{\sum_{k=0}^a \bar{\rho}(v_k, \bar{0})}{a+1} \right)^{\tau_a} < \infty \right\} \\ &= \{v = (v_a) \in \omega(F): h(\mu v) < \infty, \text{for any } \mu > 0\}. \end{aligned} \tag{8}$$

Definition 4 (see [25]). The linear space U is said to be a certain space of sequences of fuzzy numbers (cssf), if

(1) $\{\bar{\mathfrak{b}}_q\}_{q\in\mathcal{N}} \subseteq \mathbf{U}$, where $\bar{\mathfrak{b}}_q = \{\bar{0}, \bar{0}, \cdots, \bar{1}, \bar{0}, \bar{0}, \cdots\}$, while $\bar{1}$ displays at the q^{th} place

(2) suppose $Y = (Y_q) \in \omega(F)$, $Z = (Z_q) \in \mathbf{U}$ and $|Y_q| \le |Z_q|$, for all $q \in \mathcal{N}$, then $Y \in \mathbf{U}$

(3) $(Y_{[q/2]})_{q=0}^\infty \in \mathbf{U}$, where $[q/2]$ marks the integral part of $q/2$, if $(Y_q)_{q=0}^\infty \in \mathbf{U}$

Definition 5 (see [25]). A subclass U_h of U is called a pre-modular (cssf), if there is $h \in [0,\infty)^U$ satisfies the next settings:

(i) If $Y \in \mathbf{U}$, $Y = \bar{\vartheta} \Leftrightarrow h(Y) = 0$ with $h(Y) \ge 0$, where $\bar{\vartheta} = (\bar{0}, \bar{0}, \bar{0},)$

(ii) There is $Q \ge 1$, and the inequality $h(\alpha Y) \le Q|\alpha| h(Y)$ holds, for every $Y \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) There is $P \ge 1$, and the inequality $h(Y+Z) \le P(h(Y) + h(Z))$ holds, for every $Y, Z \in \mathbf{U}$

(iv) If $|Y_q| \le |Z_q|$, for every $q \in \mathcal{N}$, one has $h((Y_q)) \le h((Z_q))$

(v) The inequality $h((Y_q)) \le h((Y_{[q/2]})) \le P_0 h((Y_q))$ holds, for some $P_0 \ge 1$

(vi) Let E be the space of finite sequences of fuzzy numbers; then, the closure of $E = \mathbf{U}_h$

(vii) There is $\sigma > 0$ with $h(\bar{\alpha}, \bar{0}, \bar{0}, \bar{0}, \cdots) \ge \sigma |\alpha| h(\bar{1}, \bar{0}, \bar{0}, \bar{0}, \cdots)$, where

$$\bar{\alpha}(y) = \begin{cases} 1, & y = \alpha, \\ 0, & y \ne \alpha. \end{cases} \tag{9}$$

Definition 6 (see [25]). Suppose U is a (cssf). The function $h \in [0,\infty)^U$ is called a pre-quasinorm on U, if it satisfies the following conditions:

(i) If $Y \in \mathbf{U}$, $Y = \bar{\vartheta} \Leftrightarrow h(Y) = 0$ with $h(Y) \ge 0$, where $\bar{\vartheta} = (\bar{0}, \bar{0}, \bar{0},)$

(ii) There is $Q \ge 1$, and the inequality $h(\alpha Y) \le Q|\alpha| h(Y)$ satisfies, for every $Y \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) There is $P \ge 1$, and the inequality $h(Y+Z) \le P(h(Y) + h(Z))$ holds, for each $Y, Z \in \mathbf{U}$

Clearly, from the last two definitions, we conclude the following two theorems:

Theorem 7 (see [25]). *If U is a premodular (cssf), then it is pre-quasinormed (cssf).*

Theorem 8 (see [25]). *U is a pre-quasinormed (cssf) if it is quasinormed (cssf).*

Definition 9.

(a) The function h on $C^F_{\tau(.)}$ is named h-convex, if

$$h(\alpha Y + (1-\alpha)Z) \leq \alpha h(Y) + (1-\alpha)h(Z), \tag{10}$$

for every $\alpha \in [0, 1]$ and $Y, Z \in C^F_{\tau(.)}$.

(b) $\{Y_q\}_{q\in\mathcal{N}} \subseteq (C^F_{\tau(.)})_h$ is h-convergent to $Y \in (C^F_{\tau(.)})_h$, if and only if $\lim_{q\longrightarrow\infty} h(Y_q - Y) = 0$. When the h-limit exists, then it is unique

(c) $\{Y_q\}_{q\in\mathcal{N}} \subseteq (C^F_{\tau(.)})_h$ is h-Cauchy, if $\lim_{q,r\longrightarrow\infty} h(Y_q - Y_r) = 0$

(d) $\Gamma \subset (C^F_{\tau(.)})_h$ is h-closed, when for all h-converges $\{Y_q\}_{a\in\mathcal{N}} \subset \Gamma$ to Y, then $Y \in \Gamma$

(e) $\Gamma \subset (C^F_{\tau(.)})_h$ is h-bounded, if $\delta_h(\Gamma) = \sup\{h(Y-Z) : Y, Z \in \Gamma\} < \infty$

(f) The h-ball of radius $\varepsilon \geq 0$ and center Y, for every $Y \in (C^F_{\tau(.)})_h$, is described as follows:

$$\mathbf{B}_h(Y, \varepsilon) = \left\{Z \in \left(C^F_{\tau(.)}\right)_h : h(Y-Z) \leq \varepsilon\right\}. \tag{11}$$

(g) A pre-quasinorm h on $C^F_{\tau(.)}$ satisfies the Fatou property, if for every sequence $\{Z^q\} \subseteq (C^F_{\tau(.)})_h$ under $\lim_{q\longrightarrow\infty} h(Z^q - Z) = 0$ and all $Y \in (C^F_{\tau(.)})_h$, one has $h(Y-Z) \leq \sup_r \inf_{q\geq r} h(Y - Z^q)$

Note that the Fatou property implies the h-closed of the h-balls. We will denote the space of all increasing sequences of real numbers by $\mathbf{I}$.

Theorem 10. *$(C^F_{\tau(.)})_h$, where $h(Y) = [\sum_{q=0}^\infty (\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, is a premodular (cssf), when $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$.*

Proof. (i) Evidently, $h(Y) \geq 0$ and $h(Y) = 0 \Leftrightarrow Y = \bar{\vartheta}$

(1-i) Let $Y, Z \in C^F_{\tau(.)}$. One has

$$\begin{aligned} h(Y+Z) &= \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Y_p + Z_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &+ \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Z_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} = h(Y) + h(Z) < \infty, \end{aligned} \tag{12}$$

and then, $Y + Z \in C^F_{\tau(.)}$.

(iii) One gets $P \geq 1$ with $h(Y+Z) \leq P(h(Y) + h(Z))$, for all $Y, Z \in C^F_{\tau(.)}$

(1-ii) Assume $\alpha \in \mathfrak{R}$ and $Y \in C^F_{\tau(.)}$, and we obtain

$$\begin{aligned} h(\alpha Y) &= \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(\alpha Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \sup_q |\alpha|^{\tau_q/K} \\ &\cdot \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq Q|\alpha| h(Y) < \infty. \end{aligned} \tag{13}$$

As $\alpha Y \in C^F_{\tau(.)}$. Hence, from conditions (1-i) and (1-ii), one has $C^F_{\tau(.)}$ is linear. Also, $\bar{\mathfrak{b}}_r \in C^F_{\tau(.)}$, for all $r \in \mathcal{N}$, since $h(\bar{\mathfrak{b}}_r) = [\sum_{q=0}^\infty (\sum_{p=0}^q \bar{\rho}(\bar{\mathfrak{b}}_r, \bar{0})/q+1)^{\tau_q}]^{1/K} \leq [\sum_{q=0}^\infty (1/q+1)^{\tau_0}]^{1/K} < \infty$.

(ii) There is $Q = \max\{1, \sup_q |\alpha|^{\tau_q/K-1}\} \geq 1$ with $h(\alpha Y) \leq Q|\alpha| h(Y)$, for all $Y \in C^F_{\tau(.)}$ and $\alpha \in \mathfrak{R}$

(2) Assume $|Y_q| \leq |Z_q|$, for all $q \in \mathcal{N}$ and $Z \in C^F_{\tau(.)}$. One finds

$$\begin{aligned} h(Y) &= \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Z_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} = h(Z) < \infty, \end{aligned} \tag{14}$$

and then, $Y \in C^F_{\tau(.)}$.

(iv) Obviously, from (2)

(3) Let $(Y_q) \in C^F_{\tau(.)}$, and we get

$$\begin{aligned} h\left(\left(Y_{[q/2]}\right)\right) &= \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}\left(Y_{[p/2]}, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} = \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^{2q} \bar{\rho}\left(Y_{[p/2]}, \bar{0}\right)}{2q+1}\right)^{\tau_{2q}} \right. \\ &\left. + \sum_{q=0}^\infty \left(\frac{\sum_{p=0}^{2q+1} \bar{\rho}\left(Y_{[p/2]}, \bar{0}\right)}{2q+2}\right)^{\tau_{2q+1}}\right]^{1/K} \leq 2^{1/K} \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^\infty \left(\frac{\bar{\rho}(Y_q, \bar{0}) + 2\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q} + \sum_{q=0}^\infty \left(\frac{2\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq 2^{1/K} \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \left[\sum_{q=0}^\infty \left(\frac{3\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q} \right. \\ &\left. + \sum_{q=0}^\infty \left(\frac{2\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \left(3^K + 2^K\right)^{1/K} \left[\sum_{q=0}^\infty \left(\frac{\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &= \left(3^K + 2^K\right)^{1/K} h\left(\left(Y_q\right)\right), \end{aligned} \tag{15}$$

and then, $(Y_{[p/2]}) \in C^F_{\tau(.)}$.

(v) From (4), we obtain $P_0 = (3^K + 2^K)^{1/K} \geq 1$

(vi) Evidently the closure of $E = C^F_{\tau(.)}$

(vii) There is $0 < \sigma \leq \sup_q |\alpha|^{\tau_q/K-1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ with $h(\bar{\alpha}, \bar{0}, \bar{0}, \bar{0}, \cdots) \geq \sigma|\alpha| h(\bar{1}, \bar{0}, \bar{0}, \bar{0}, \cdots)$ ☐

Theorem 11. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, then $(C^F_{\tau(.)})_h$ is a pre-quasi-Banach (cssf), where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{\tau_q}]^{1/K}$, for every $Y \in C^F_{\tau(.)}$.*

Proof. In view of Theorem 10 and Theorem 7, the space $(C^F_{\tau(.)})_h$ is a pre-quasinormed (cssf). Assume $Y^l = (Y^l_q)_{q=0}^{\infty}$ is a Cauchy sequence in $(C^F_{\tau(.)})_h$. Hence, for every $\varepsilon \in (0,1)$, one has $l_0 \in \mathcal{N}$ such that for all $l, m \geq l_0$, one gets

$$h\left(Y^l - Y^m\right) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y^l_p - Y^m_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} < \varepsilon. \tag{16}$$

That implies $\bar{\rho}(Y^l_q - Y^m_q, \bar{0}) < \varepsilon$. As $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space. Then, (Y^m_q) is a Cauchy sequence in $\mathfrak{R}[0,1]$, for fixed $q \in \mathcal{N}$, which implies $\lim_{m\longrightarrow\infty} Y^m_q = Y^0_q$, for constant $q \in \mathcal{N}$. Hence, $h(Y^l - Y^0) < \varepsilon$, for every $l \geq l_0$, since $h(Y^0) = h(Y^0 - Y^l + Y^l) \leq h(Y^l - Y^0) + h(Y^l) < \infty$. So $Y^0 \in C^F_{\tau(.)}$. □

Theorem 12. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, then $(C^F_{\tau(.)})_h$ is a pre-quasiclosed (cssf), where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{\tau_q}]^{1/K}$, for every $Y \in C^F_{\tau(.)}$.*

Proof. In view of Theorem 10 and Theorem 7, the space $(C^F_{\tau(.)})_h$ is a pre-quasinormed (cssf). Assume $Y^l = (Y^l_q)_{q=0}^{\infty} \in (C^F_{\tau(.)})_h$ and $\lim_{l\longrightarrow\infty} h(Y^l - Y^0) = 0$; then, for all $\varepsilon \in (0,1)$, there is $l_0 \in \mathcal{N}$ such that for all $l \geq l_0$, we obtain

$$\varepsilon > h\left(Y^l - Y^0\right) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y^l_p - Y^0_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K}, \tag{17}$$

which implies $\bar{\rho}(Y^l_q - Y^0_q, \bar{0}) < \varepsilon$. As $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space, therefore, (Y^l_q) is a convergent sequence in $\mathfrak{R}[0,1]$, for fixed $q \in \mathcal{N}$. So, $\lim_{l\longrightarrow\infty} Y^l_q = Y^0_q$, for fixed $q \in \mathcal{N}$. Since $h(Y^0) = h(Y^0 - Y^l + Y^l) \leq h(Y^l - Y^0) + h(Y^l) < \infty$, one has $Y^0 \in C^F_{\tau(.)}$. □

Theorem 13. *The function $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{\tau_q}]^{1/K}$ verifies the Fatou property, when $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, for all $Y \in C^F_{\tau(.)}$.*

Proof. Let $\{Z^r\} \subseteq (C^F_{\tau(.)})_h$ such that $\lim_{r\longrightarrow\infty} h(Z^r - Z) = 0$. Since $(C^F_{\tau(.)})_h$ is a pre-quasiclosed space, one has $Z \in (C^F_{\tau(.)})_h$. For all $Y \in (C^F_{\tau(.)})_h$, one gets

$$\begin{aligned} h(Y - Z) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z^r_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Z^r_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \sup_m \inf_{r\geq m} h(Y - Z^r). \end{aligned} \tag{18}$$

□

Theorem 14. *The function $h(Y) = \sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{\tau_q}$ does not satisfy the Fatou property, for all $Y \in C^F_{\tau(.)}$, when $(\tau_q) \in \ell_\infty$ and $\tau_q > 1$, for all $q \in \mathcal{N}$.*

Proof. Let $\{Z^r\} \subseteq (C^F_{\tau(.)})_h$ so that $\lim_{r\longrightarrow\infty} h(Z^r - Z) = 0$. Since $(C^F_{\tau(.)})_h$ is a pre-quasiclosed space, one gets $Z \in (C^F_{\tau(.)})_h$. For every $Z \in (C^F_{\tau(.)})_h$, we obtain

$$\begin{aligned} h(Y - Z) &= \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q} \\ &\leq 2^{K-1}\left(\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z^r_p, \bar{0}\right)}{q+1}\right)^{\tau_q} + \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Z^r_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right) \\ &\leq 2^{K-1}\sup_m \inf_{r\geq m} h(Y - Z^r). \end{aligned} \tag{19}$$

□

Example 1. For $(\tau_q) \in [1,\infty)^{\mathcal{N}}$, the function $h(Y) = \inf\{\alpha > 0 : \sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p/\alpha, \bar{0})/q + 1)^{\tau_q} \leq 1\}$ is a norm on $C^F_{\tau(.)}$.

Example 2. The function $h(Y) = \sqrt[3]{\sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{3q+2/q+1}}$ is a pre-quasinorm (not a norm) on $C^F((3q + 2/q + 1)_{q=0}^{\infty})$.

Example 3. The function $h(Y) = \sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{3q+2/q+1}$ is a pre-quasinorm (not a quasinorm) on $C^F((3q + 2/q + 1)_{q=0}^{\infty})$.

Example 4. The function $h(Y) = \sqrt[d]{\sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{d}}$ is a pre-quasinorm, quasinorm, and not a norm on C^F_d, for $0 < d < 1$.

In the next part of this section, we will use the function h as $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for every $Y \in C^F_{\tau(.)}$.

Definition 15 [26]. The function h is said to be strictly convex, (SC), if for all $Y, Z \in U_h$ such that $h(Y) = h(Z)$ and $h(Y + Z/2) = h(Y) + h(Z)/2$, we get $Y = Z$.

Definition 16 [27]. A sequence $\{Y_p\} \subseteq U$ is said to be ε-separated sequence for some $\varepsilon > 0$, if

$$\operatorname{sep}(Y_p) = \inf \{h(Y_p - Y_q): p \neq q\} > \varepsilon. \tag{20}$$

Definition 17 (see [27]). Let $k \geq 2$ be an integer, and a Banach space U is called k-nearly uniformly convex (k-NUC), if for any $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that for any sequence $\{Y_p\} \subseteq B_h(0, 1)$, with $sep(Y_p) \geq \varepsilon$, there are $p_1, p_2, p_3, \cdots, p_k \in \mathcal{N}$, such that

$$h\left(\frac{Y_{p_1} + Y_{p_2} + Y_{p_3} + \cdots + Y_{p_k}}{k}\right) < 1 - \delta. \tag{21}$$

Definition 18 (see [28]). A function h is said to satisfy the δ_2-condition ($h \in \delta_2$), if for any $\varepsilon > 0$, there exists a constant $k \geq 2$ and $a > 0$ such that $h(2u) \leq kh(u) + \varepsilon$, for each $u \in X_h$, with $h(u) \leq a$.

If h satisfies the δ_2-condition for any $a > 0$ with $k \geq 2$ depending on a, we say that h satisfies the strong δ_2-condition ($\rho \in \delta_2^s$).

The following known results are very important for our consideration.

Theorem 19 (see [28], Lemma 2.1). *If $h \in \delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|h(x + y) - h(x)| < \varepsilon$, where $x, y \in X_h$, with $h(x) \leq L$ and $h(y) \leq \delta$.*

Theorem 20. *Pick an $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$; then, for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|h(x + y) - h(x)| < \varepsilon$, for all $x, y \in (C^F_{\tau(.)})_h$, with $h(x) \leq L$ and $h(y) \leq \delta$.*

Proof. Since (τ_q) is bounded, it is easy to see that $h \in \delta_2^s$. Hence, the proposition is obtained directly from Theorem 19. □

Theorem 21. *Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$; then, $(C^F_{\tau(.)})_h$ is k-NUC, for any integer $k \geq 2$.*

Proof. Let $\varepsilon \in (0, 1)$ and $\{x_n\} \subseteq \mathbf{B}_h(0, 1)$ with $sep(x_n) \geq \varepsilon$, for each $m \in \mathcal{N}$, and let $x_n^m = (0, 0, 0, \cdots, x_n(m), x_n(m + 1), \cdots)$. Since for each $i \in \mathcal{N}$, $(x_n(i))_{n=0}^{\infty}$ is bounded, and by using the diagonal method, we can find a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathcal{N}$, $0 \leq i \leq m$. Therefore, there exists an increasing sequence of positive integers (t_m) such that $sep((x_{n_j}^m)_{j>t_m}) \geq \varepsilon$. Hence, there is a sequence of positive integers $(r_m)_{m=0}^{\infty}$ with $r_0 < r_1 < r_2 < \cdots$, such that

$$h^K\left(x_{r_m}^m\right) \geq \frac{\varepsilon}{2}, \tag{22}$$

for each $m \in \mathcal{N}$. For fixed integer $k \geq 2$, let $\varepsilon_1 = (k^{p_0 - 1} - 1/(k - 1)k^{p_0})(\varepsilon/4)$; then, by Theorem 20, there exists $\delta > 0$ such that

$$\left|h^K(x + y) - h^K(x)\right| < \varepsilon_1, \tag{23}$$

whenever $h^K(x) \leq 1$ and $h^K(y) \leq \delta$. Since $h^K(x_n) \leq 1$, for any $n \in \mathcal{N}$, then there exist positive integers $m_i (i = 0, 1, 2, \cdots, k - 2)$ with $m_0 < m_1 < m_2 < \cdots < m_{k-2}$ such that $h^K(x_i^{m_i}) \leq \delta$. Define $m_{k-1} = m_{k-2} + 1$. By inequality (1), we have $h(x_{r_{m_k}}^{m_k}) \geq \varepsilon/2$. Let $s_i = i$ for $0 \leq i \leq k - 2$ and $s_{k-1} = r_{m_{k-1}}$. Then, in virtue of inequality (1), inequality (2), and convexity of the function $f_n(u) = |u|^{\tau_n}$ for any $n \in \mathcal{N}$, we have

$$\begin{aligned}
&h^K\left(\frac{x_{s_0} + x_{s_1} + x_{s_2} +. \cdots + x_{s_{k-1}}}{k}\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&= \sum_{n=0}^{m_1 - 1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/n+1, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\leq \sum_{n=0}^{m_1 - 1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \varepsilon_1 \leq \sum_{n=0}^{m_1 - 1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i), \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{m_2 - 1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \varepsilon_1 \leq \sum_{n=0}^{m_1 - 1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i), \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{m_2 - 1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_2}(i)+x_{s_3}(i)++x_{s_{k-1}}(i)/k,\bar{0}\right)}{n+1}\right)^{\tau_n}+2\varepsilon_1\\
&\leq\sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_1}^{m_2-1}\frac{1}{k}\sum_{j=1}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_2}^{m_3-1}\frac{1}{k}\sum_{j=2}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&++\sum_{n=m_{k-1}}^{m_k-1}\frac{1}{k}\sum_{j=k-2}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i)/k,\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1\\
&\leq\frac{h^K\left(x_{s_0}+x_{s_1}+x_{s_2}+.\cdots+x_{s_{k-2}}\right)}{k}\\
&+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i)/k,\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1\leq\frac{k-1}{k}\\
&+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\frac{1}{k^{p_0}}\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1\leq 1-\frac{1}{k}\\
&+\frac{1}{k}\left(1-\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\right)+\frac{1}{k^{p_0}}\sum_{n=m_k}^{\infty}\\
&\cdot\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1=1+(k-1)\varepsilon_1\\
&-\left(\frac{k^{p_0-1}-1}{k^{p_0}}\right)\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&\leq 1+(k-1)\varepsilon_1-\left(\frac{k^{p_0-1}-1}{k^{p_0}}\right)\frac{\varepsilon}{2}=1-\left(\frac{k^{p_0-1}-1}{k^{p_0}}\right)\frac{\varepsilon}{4}.
\end{aligned}
\tag{24}
$$

Therefore, $(C^F_{\tau(.)})_h$ is k-NUC. □

Recall that k-NUC implies reflexivity.

Definition 22. The space U_h satisfies the property (R), if and only if, for all decreasing sequence $\{\Gamma_j\}_{j\in\mathcal{N}}$ of h-closed and h-convex nonempty subsets of U_h with $\sup_{j\in\mathcal{N}}\mathfrak{K}_h(Y,\Gamma_j)<\infty$, for some $Y\in U_h$, one has $\bigcap_{j\in\mathcal{N}}\Gamma_j\neq\varnothing$.

By fixing Γ a nonempty h-closed and h-convex subset of $(C^F_{\tau(.)})_h$.

Theorem 23. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, one has the following:*

(i) Suppose $Y\in(C^F_{\tau(.)})_h$ with $\mathfrak{K}_h(Y,\Gamma)=\inf\{h(Y-Z):Z\in\Gamma\}<\infty$. There is a unique $\lambda\in\Gamma$ so that $\mathfrak{K}_h(Y,\Gamma)=h(Y-\lambda)$

(ii) $(C^F_{\tau(.)})_h$ verifies the property (R).

Proof. To prove (i), assume $Y\notin\Gamma$ as Γ is h-closed. One has $C:=\mathfrak{K}_h(Y,\Gamma)>0$. Hence, for all $r\in\mathcal{N}$, one has $Z_r\in\Gamma$ with $h(Y-Z_r)<C(1+1/r)$. If $\{Z_r/2\}$ is not h-Cauchy, one gets a subsequence $\{Z_{g(r)}/2\}$ and $l_0>0$ with $h(Z_{g(r)}-Z_{g(j)}/2)\geq l_0$, for every $r>j\geq 0$, since

$$
\begin{aligned}
&\max\left(h\left(Y-Z_{g(r)}\right),h\left(Y-Z_{g(j)}\right)\right)\leq C\left(1+\frac{1}{g(j)}\right),\\
&h\left(\frac{Z_{g(r)}-Z_{g(j)}}{2}\right)\geq l_0\geq C\left(1+\frac{1}{g(j)}\right)\frac{l_0}{2C},
\end{aligned}
\tag{25}
$$

for every $r>j\geq 0$. Since $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, then the function $f_n(u)=|u|^{\tau_n}$ is strictly convex, for any $n\in\mathcal{N}$. Therefore, the space $(C^F_{\tau(.)})_h$ is strictly convex; hence,

$$
h\left(Y-\frac{Z_{g(r)}+Z_{g(j)}}{2}\right)<C\left(1+\frac{1}{g(j)}\right).\tag{26}
$$

Then,

$$
C=\mathfrak{K}_h(Y,\Gamma)<C\left(1+\frac{1}{g(j)}\right),\tag{27}
$$

for all $j\in\mathcal{N}$. By putting $j\longrightarrow\infty$, one has a contradiction. So $\{Z_r/2\}$ is h-Cauchy. As $(C^F_{\tau(.)})_h$ is h-complete, then $\{Z_r/2\}h$-converges to some Z. For all $j\in\mathcal{N}$, one gets $\{Z_r+Z_j/2\}h$-converges to $Z+Z_j/2$. Since Γ is h-closed and h-convex, then $Z+Z_j/2\in\Gamma$. Since $Z+Z_j/2h$-converges to 2 Z, then $2Z\in\Gamma$. Let $\lambda=2z$, and from Theorem 13, since h satisfies the Fatou property, one has

$$
\begin{aligned}
\mathfrak{K}_h(Y,\Gamma)&\leq h(Y-\lambda)\leq\sup_i\inf_{j\geq i}h\left(Y-\left(Z+\frac{Z_j}{2}\right)\right)\\
&\leq\sup_i\inf_{j\geq i}\sup_i\inf_{r\geq i}h\left(Y-\frac{Z_r+Z_j}{2}\right)\\
&\leq\frac{1}{2}\sup_i\inf_{r\geq i}\sup_i\inf_{r\geq i}\left[h(Y-Z_r)+h(Y-Z_j)\right]\\
&=\mathfrak{K}_h(Y,\Gamma).
\end{aligned}
\tag{28}
$$

Then $h(Y-\lambda)=\mathfrak{K}_h(Y,\Gamma)$. Since h is (SC), this implies the uniqueness of λ. To prove (ii), assume $Y \notin \Gamma_{r_0}$, for some $r_0 \in \mathcal{N}$. Since $(\mathfrak{K}_h(Y,\Gamma_r))_{r\in\mathcal{N}} \in \ell_\infty$ is increasing, put $\lim_{r\longrightarrow\infty}\mathfrak{K}_h(Y,\Gamma_r)=C$, when $C>0$. Otherwise, $Y\in\Gamma_r$, for all $r\in\mathcal{N}$. According to (i), there is one point $Z_r\in\Gamma_r$ with $\mathfrak{K}_h(Y,\Gamma_r)=h(Y-Z_r)$, for every $r\in\mathcal{N}$. A similar proof will prove that $\{Z_r/2\}h$-converges to some $Z\in\left(C^F_{\tau(.)}\right)_h$. As $\{\Gamma_r\}$ is h-convex, decreasing, and h-closed, one has $2Z\in\cap_{r\in\mathcal{N}}\Gamma_r$. □

Definition 24. The space U_h verifies the h-normal structure-property, if and only if, for all nonempty h-bounded, h-convex and h-closed subset Γ of U_h not decreased to one point, and one has $Y\in\Gamma$ with

$$\sup_{Z\in\Gamma} h(Y-Z) < \delta_h(\Gamma) := \sup\{h(Y-Z): Y,Z\in\Gamma\} < \infty. \tag{29}$$

Definition 25 (see [29]). U_h is a real Banach space, and $S(U_h)$ is the unit sphere of U_h. The weakly convergent sequence coefficient of U_h, denoted by $WCS(U_h)$, is defined as follows:

$$\begin{aligned} WCS(\mathbf{U}_h) = \inf\Big\{&A(\{x_n\}): \{x_n\}_{n=1}^{\infty}\subset S(\mathbf{U}_h), A(\{x_n\}) \\ &= A_1(\{x_n\}), x_n \overset{w}{\longrightarrow} 0\Big\}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} A(\{x_n\}) &= \limsup_{n\longrightarrow\infty}\left\{\|x_i-x_j\|: i,j\ge n, i\ne j\right\},\\ A_1(\{x_n\}) &= \liminf_{n\longrightarrow\infty}\left\{\|x_i-x_j\|: i,j\ge n, i\ne j\right\}. \end{aligned} \tag{31}$$

Theorem 26 (see [30]). *A reflexive Banach space U_h with $WCS(U_h)>1$ has normal structure-property.*

Theorem 27. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ with $\tau_0>1$, then $\left(C^F_{\tau(.)}\right)_h$ holds the h-normal structure-property.*

Proof. Take any $\varepsilon>0$ and an asymptotic equidistant sequence $\{x_n\}\subset S\left(\left(C^F_{\tau(.)}\right)_h\right)$ with $x_n \overset{w}{\longrightarrow} 0$ and put $v_1=x_1$. There exists $i_1\in\mathcal{N}$ such that $h\left(\sum_{i=i_1+1}^{\infty}v_1(i)\bar{\mathfrak{b}}_i\right)<\varepsilon$. Since $x_n\longrightarrow 0$ coordinate-wise, there exists $n_2\in\mathcal{N}$ such that $h\left(\sum_{i=1}^{i_1}x_n(i)\bar{\mathfrak{b}}_i\right)<\varepsilon$, whenever $n\ge n_2$. Take $v_2=x_{n_2}$; then, there is $i_2>i_1$ such that $h\left(\sum_{i=i_2+1}^{\infty}v_1(i)\bar{\mathfrak{b}}_i\right)<\varepsilon$. Since $x_n(i)\longrightarrow 0$ coordinate-wise, there exists $n_3\in\mathcal{N}$ such that $h\left(\sum_{i=1}^{i_2}x_n(i)\bar{\mathfrak{b}}_i\right)<\varepsilon$, whenever $n\ge n_3$. Continuing this process in such a way by induction, we get a subsequence $\{v_n\}$ of $\{x_n\}$ such that

$$\begin{aligned} h\left(\sum_{i=i_n+1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right) &< \varepsilon,\\ h\left(\sum_{i=1}^{i_n}v_{n+1}(i)\bar{\mathfrak{b}}_i\right) &< \varepsilon. \end{aligned} \tag{32}$$

Put $z_n=\sum_{i=i_{n-1}+1}^{i_n}v_n(i)\bar{\mathfrak{b}}_i$, for $n=2,3,\cdots$ Then,

$$\begin{aligned} 1\ge h(z_n) &= h\left(\sum_{i=1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=1}^{i_n-1}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=i_n+1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right)\\ &\ge h\left(\sum_{i=1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right) - h\left(\sum_{i=1}^{i_n-1}v_n(i)\bar{\mathfrak{b}}_i\right)\\ &\quad - h\left(\sum_{i=i_n+1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right) > 1-2\varepsilon. \end{aligned} \tag{33}$$

Moreover, for any $n,m\in\mathcal{N}$ with $n\ne m$, we have

$$\begin{aligned} h(v_n-v_m) &= h\left(\sum_{i=1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=1}^{\infty}v_m(i)\bar{\mathfrak{b}}_i\right)\\ &\ge h\left(\sum_{i=i_{n-1}+1}^{i_n}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=i_{m-1}+1}^{i_m}v_m(i)\bar{\mathfrak{b}}_i\right)\\ &\quad - h\left(\sum_{i=1}^{i_{n-1}}v_n(i)\bar{\mathfrak{b}}_i\right) - h\left(\sum_{i=i_n+1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right)\\ &\quad - h\left(\sum_{i=1}^{i_{m-1}}v_m(i)\bar{\mathfrak{b}}_i\right) - h\left(\sum_{i=i_m+1}^{\infty}v_m(i)\bar{\mathfrak{b}}_i\right)\\ &\ge h(z_n-z_m)-4\varepsilon. \end{aligned} \tag{34}$$

This means that $A(\{x_n\})=A(\{v_n\})\ge A(\{z_n\})-4\varepsilon$. Put $u_n=z_n/\|z_n\|$, for $n=2,3,\cdots$ Then,

$$u_n\in S\left(\left(C^F_{\tau(.)}\right)_h\right), \tag{35}$$

$$A(\{x_n\})\ge 1-\varepsilon A(\{u_n\})-4\varepsilon. \tag{36}$$

On the other hand,

$$h(v_n-v_m)\le h(z_n-z_m)+4\varepsilon\le h(u_n-u_m)+4\varepsilon, \tag{37}$$

for any $n,m\in\mathcal{N}$ with $n\ne m$. Therefore,

$$A(\{u_n\})\ge A(\{x_n\})-4\varepsilon. \tag{38}$$

By the arbitrariness of $\varepsilon>0$, we have from the relations (35), (36), and (38) that

$$\mathrm{WCS}\left(\left(C^F_{\tau(.)}\right)_h\right)=\inf\{A(\{u_n\})\}, \tag{39}$$

such that

$$u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)\bar{\mathfrak{b}}_i \in S\left(\left(C^F_{\tau(.)}\right)_h\right), 0 = i_0 < i_1 < \cdots, u_n \overset{w}{\longrightarrow} 0 \text{ and } \{u_n\} \text{ is asymptotic equidistant.} \tag{40}$$

Take $m \in \mathcal{N}$ large enough such that $\sum_{k=i_{m-1}+1}^{\infty}(b/k)^{\tau_k} < \varepsilon$, where $b := \sum_{i=i_{n-1}+1}^{i_n}|u_n(i)|$. We have for $n < m$ that

$$\begin{aligned} h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_n(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k} \\ &\geq \sum_{k=i_{n-1}+1}^{i_{m-1}}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_n(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} \\ &= \sum_{k=i_{n-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_n(i),\bar{0})\right)^{\tau_k} - \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{b}{k}\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} \\ &> 1 - \varepsilon + 1 = 2 - \varepsilon, \end{aligned} \tag{41}$$

that is, $A_n(\{u_n\}) \geq (2-\varepsilon)^{1/K}$. Note that

$$\left[\sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k}\right]^{1/K} \leq \left[\sum_{k=i_{m-1}+1}^{\infty}\left(\frac{b}{k}\right)^{\tau_k}\right]^{1/K} + \left[\sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k}\right]^{1/K} < \varepsilon^{1/K} + 1. \tag{42}$$

Therefore,

$$\begin{aligned} h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k} \\ &\leq \sum_{k=i_{n-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k} \\ &\leq 1 + \left(1 + \varepsilon^{1/K}\right)^K, \end{aligned} \tag{43}$$

for any $n, m \in \mathcal{N}$ with $n \neq m$. Therefore, $A_n(\{u_n\}) \leq \left(1 + \left(1 + \varepsilon^{1/K}\right)^K\right)^{1/K}$, and by the arbitrariness of $\varepsilon > 0$, we obtain $WCS((C^F_{\tau(.)})_h) = 2^{1/K}$. From Theorem 21 and Theorem 26, the sequence space $(C^F_{\tau(.)})_h$ has the h-normal structure-property. □

4. Kannan Contraction Mapping on $C^F_{\tau(.)}$

In this section, we look at how to configure $(C^F_{\tau(.)})_h$ with different h so that there is only one fixed point of Kannan contraction mapping.

Definition 28. An operator $V : U_h \longrightarrow U_h$ is said to be a Kannan h-contraction, if one gets $\alpha \in [0, 1/2)$ with $h(VY - VZ) \leq \alpha(h(VY - Y) + h(VZ - Z))$, for all $Y, Z \in U_h$. The operator V is called Kannan h-nonexpansive, when $\alpha = 1/2$.

An element $Y \in \mathbf{U}_h$ is called a fixed point of V when $V(Y) = Y$.

Theorem 29. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$ is Kannan h-contraction mapping, where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})/q+1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, then V has a unique fixed point.*

Proof. If $Y \in C^F_{\tau(.)}$, one has $V^p Y \in C^F_{\tau(.)}$. As V is a Kannan h-contraction mapping, one gets

$$\begin{aligned} h\left(V^{l+1}Y - V^lY\right) &\leq \alpha\left(h\left(V^{l+1}Y - V^lY\right) + h\left(V^lY - V^{l-1}Y\right)\right) \\ &\Rightarrow h\left(V^{l+1}Y - V^lY\right) \leq \frac{\alpha}{1-\alpha}h\left(V^lY - V^{l-1}Y\right) \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^2 h\left(V^{l-1}Y - V^{l-2}Y\right) \\ &\leq \leq \left(\frac{\alpha}{1-\alpha}\right)^l h(VY - Y). \end{aligned} \tag{44}$$

So for all $l, m \in \mathcal{N}$ with $m > l$, one gets

$$\begin{aligned} h\left(V^lY - V^mY\right) &\leq \alpha\left(h\left(V^lY - V^{l-1}Y\right) + h\left(V^mY - V^{m-1}Y\right)\right) \\ &\leq \alpha\left(\left(\frac{\alpha}{1-\alpha}\right)^{l-1} + \left(\frac{\alpha}{1-\alpha}\right)^{m-1}\right)h(VY - Y). \end{aligned} \tag{45}$$

Then, $\{V^lY\}$ is a Cauchy sequence in $(C^F_{\tau(.)})_h$. As the space $(C^F_{\tau(.)})_h$ is pre-quasi-Banach space, one has $Z \in (C^F_{\tau(.)})_h$ with $\lim_{l\longrightarrow\infty}V^lY = Z$. To prove that $VZ = Z$, since h has the Fatou property, one obtains

$$\begin{aligned} h(VZ - Z) &\leq \sup_i \inf_{l\geq i} h\left(V^{l+1}Y - V^lY\right) \\ &\leq \sup_i \inf_{l\geq i}\left(\frac{\alpha}{1-\alpha}\right)^l h(VY - Y) = 0, \end{aligned} \tag{46}$$

and then, $VZ = Z$. So Z is a fixed point of V. To show the uniqueness. Let $Y, Z \in (C^F_{\tau(.)})_h$ be two not equal fixed points of V. One has

$$h(Y-Z) \le h(VY - VZ) \le \alpha(h(VY-Y) + h(VZ-Z)) = 0. \tag{47}$$

So, $Y = Z$. □

Corollary 30. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$ is Kannan h-contraction mapping, where $h(Y) = [\sum_{q=0}^\infty (\sum_{p=0}^q \bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, one has V has unique fixed point Z so that $h(V^l Y - Z) \le \alpha(a/1-\alpha)^{l-1} h(VY - Y)$.*

Proof. In view of Theorem 29, one has a unique fixed point Z of V. So

$$\begin{aligned} h\left(V^l Y - Z\right) &= h\left(V^l Y - VZ\right) \\ &\le \alpha\left(h\left(V^l Y - V^{l-1} Y\right) + h(VZ - Z)\right) \\ &= \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h(VY - Y). \end{aligned} \tag{48}$$

□

Example 5. Assume $V : (C^F((2q+3/q+2)_{q=0}^\infty))_h \longrightarrow (C^F((2q+3/q+2)_{q=0}^\infty))_h$, where $h(g) = \sqrt{\sum_{q=0}^\infty (\sum_{p=0}^q \bar{\rho}(g_p, \bar{0})/q+1)^{2q+3/q+2}}$, for every $g \in C^F((2q+3/q+2)_{q=0}^\infty)$ and

$$V(g) = \begin{cases} \dfrac{g}{4}, & h(g) \in [0,1), \\ \dfrac{g}{5}, & h(g) \in [1,\infty). \end{cases} \tag{49}$$

As for each $g_1, g_2 \in (C^F((2q+3/q+2)_{q=0}^\infty))_h$ with $h(g_1), h(g_2) \in [0,1)$, one has

$$\begin{aligned} h(Vg_1 - Vg_2) &= h\left(\frac{g_1}{4} - \frac{g_2}{4}\right) \le \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3g_1}{4}\right) + h\left(\frac{3g_2}{4}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(h(Vg_1 - g_1) + h(Vg_2 - g_2)). \end{aligned} \tag{50}$$

For all $g_1, g_2 \in (C^F((2q+3/q+2)_{q=0}^\infty))_h$ with $h(g_1), h(g_2) \in [1,\infty)$, one has

$$\begin{aligned} h(Vg_1 - Vg_2) &= h\left(\frac{g_1}{5} - \frac{g_2}{5}\right) \le \frac{1}{\sqrt[4]{64}}\left(h\left(\frac{4g_1}{5}\right) + h\left(\frac{4g_2}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{64}}(h(Vg_1 - g_1) + h(Vg_2 - g_2)). \end{aligned} \tag{51}$$

For all $g_1, g_2 \in (C^F((2q+3/q+2)_{q=0}^\infty))_h$ with $h(g_1) \in [0,1)$ and $h(g_2) \in [1,\infty)$, we get

$$\begin{aligned} h(Vg_1 - Vg_2) &= h\left(\frac{g_1}{4} - \frac{g_2}{5}\right) \le \frac{1}{\sqrt[4]{27}} h\left(\frac{3g_1}{4}\right) + \frac{1}{\sqrt[4]{64}} h\left(\frac{4g_2}{5}\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3g_1}{4}\right) + h\left(\frac{4g_2}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(h(Vg_1 - g_1) + h(Vg_2 - g_2)). \end{aligned} \tag{52}$$

Hence, V is Kannan h-contraction. As h satisfies the Fatou property, from Theorem 29, one has V holds one fixed point $\bar{\vartheta} \in (C^F((2q+3/q+2)_{q=0}^\infty))_h$.

Definition 31. Pick up U_h be a pre-quasinormed (cssf), $V : U_h \longrightarrow U_h$, and $Z \in U_h$. The operator V is called h-sequentially continuous at Z, if and only if when $\lim_{q\longrightarrow\infty} h(Y_q - Z) = 0$, then $\lim_{q\longrightarrow\infty} h(VY_q - VZ) = 0$.

Example 6. Suppose $V : (C^F((q+1/2q+4)_{q=0}^\infty))_h \longrightarrow (C^F((q+1/2q+4)_{q=0}^\infty))_h$, where $h(Z) = [\sum_{q=0}^\infty (\sum_{p=0}^q \bar{\rho}(Z_p, \bar{0})/q+1)^{q+1/2q+4}]^4$, for every $Z \in C^F((q+1/2q+4)_{q=0}^\infty)$ and

$$V(Z) = \begin{cases} \dfrac{1}{18}(\bar{\mathfrak{b}}_0 + Z), & Z_0(y) \in \left[0, \dfrac{1}{17}\right), \\ \dfrac{1}{17}\bar{\mathfrak{b}}_0, & Z_0(y) = \dfrac{1}{17}, \\ \dfrac{1}{18}\bar{\mathfrak{b}}_0, & Z_0(y) \in \left(\dfrac{1}{17}, 1\right]. \end{cases} \tag{53}$$

V is clearly both h-sequentially continuous and discontinuous at $1/17\bar{\mathfrak{b}}_0 \in (C^F((q+1/2q+4)_{q=0}^\infty))_h$.

Example 7. Assume V is defined as in Example 5. Suppose $\{Z^{(n)}\} \subseteq (C^F((2q+3/q+2)_{q=0}^\infty))_h$ such that $\lim_{n\longrightarrow\infty} h(Z^{(n)} - Z^{(0)}) = 0$, where $Z^{(0)} \in (C^F((2q+3/q+2)_{q=0}^\infty))_h$ with $h(Z^{(0)}) = 1$.

As the pre-quasinorm h is continuous, we have

$$\lim_{n\longrightarrow\infty} h\left(VZ^{(n)} - VZ^{(0)}\right) = \lim_{n\longrightarrow\infty} h\left(\frac{Z^{(n)}}{4} - \frac{Z^{(0)}}{5}\right) = h\left(\frac{Z^{(0)}}{20}\right) > 0. \tag{54}$$

Therefore, V is not h-sequentially continuous at $Z^{(0)}$.

Theorem 32. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$, where $h(Y) = \sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}$, for all $Y \in C^F_{\tau(.)}$. Suppose*

(1) V is Kannan h-contraction mapping

(2) V is h-sequentially continuous at $Z \in (C^F_{\tau(.)})_h$

(3) there is $Y \in (C^F_{\tau(.)})_h$ with $\{V^l Y\}$ has $\{V^{l_j} Y\}$ converging to Z

Then, $Z \in (C^F_{\tau(.)})_h$ is the only fixed point of V.

Proof. Assume Z is not a fixed point of V, and one has $VZ \neq Z$. From parts (2) and (4), we get

$$\lim_{l_j\longrightarrow\infty} h\left(V^{l_j}Y - Z\right) = 0,$$
$$\lim_{l_j\longrightarrow\infty} h\left(V^{l_j+1}Y - VZ\right) = 0. \tag{55}$$

As V is Kannan h-contraction, one obtains

$$\begin{aligned}0 < h(VZ - Z) &= h\Big(\left(VZ - V^{l_j+1}Y\right) + \left(V^{l_j}Y - Z\right) \\ &\quad + \left(V^{l_j+1}Y - V^{l_j}Y\right)\Big) \leq 2^{2\sup_i \tau_i - 2} \\ &\quad \cdot h\left(V^{l_j+1}Y - VZ\right) + 2^{2\sup_i \tau_i - 2} h\left(V^{l_j}Y - Z\right) \\ &\quad + 2^{\sup_i \tau_i - 1} \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l_j - 1} h(VY - Y).\end{aligned} \tag{56}$$

As $l_j \longrightarrow \infty$, one has a contradiction. Then, Z is a fixed point of V. To show the uniqueness, let $Z, Y \in (C^F_{\tau(.)})_h$ be two not equal fixed points of V. One obtains

$$h(Z - Y) \leq h(VZ - VY) \leq \alpha(h(VZ - Z) + h(VY - Y)) = 0. \tag{57}$$

Hence, $Z = Y$. □

Example 8. Assume V is defined as in Example 5. Let $h(Y) = \sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{2q+3/q+2}$, for all $v \in C^F((2q+3/q+2)_{q=0}^{\infty})$. Since for all $Y_1, Y_2 \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$ with $h(Y_1), h(Y_2) \in [0, 1)$, one gets $h(VY_1 - VY_2) = h(Y_1/4 - Y_2/4) \leq 2/\sqrt{27}(h(3Y_1/4) + h(3Y_2/4)) = 2/\sqrt{27}(h(VY_1 - Y_1) + h(VY_2 - Y_2))$. For all $Y_1, Y_2 \in (C^F((2q+3/q\ +2)_{q=0}^{\infty}))_h$ with $h(Y_1), h(Y_2) \in [1,\infty)$, one gets

$$\begin{aligned}h(VY_1 - VY_2) &= h\left(\frac{Y_1}{5} - \frac{Y_2}{5}\right) \leq \frac{1}{4}\left(h\left(\frac{4Y_1}{5}\right) + h\left(\frac{4Y_2}{5}\right)\right) \\ &= \frac{1}{4}(h(VY_1 - Y_1) + h(VY_2 - Y_2)).\end{aligned} \tag{58}$$

For all $Y_1, Y_2 \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$ with $h(Y_1) \in [0, 1)$ and $h(Y_2) \in [1,\infty)$, one gets

$$\begin{aligned}h(VY_1 - VY_2) &= h\left(\frac{Y_1}{4} - \frac{Y_2}{5}\right) \leq \frac{2}{\sqrt{27}}h\left(\frac{3Y_1}{4}\right) + \frac{1}{4}h\left(\frac{4Y_2}{5}\right) \\ &\leq \frac{2}{\sqrt{27}}\left(h\left(\frac{3Y_1}{4}\right) + h\left(\frac{4Y_2}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}(h(VY_1 - Y_1) + h(VY_2 - Y_2)).\end{aligned} \tag{59}$$

So V is Kannan h-contraction and $V^p(Y) = \begin{cases} Y/4^p, & h(Y) \in [0, 1), \\ Y/5^p, & h(Y) \in [1,\infty). \end{cases}$

Obviously, V is h-sequentially continuous at $\bar{\vartheta} \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$, and $\{V^p Y\}$ holds $\{V^{l_j} Y\}$ converges to $\bar{\vartheta}$. By Theorem 32, the point $\bar{\vartheta} \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$ is the only fixed point of V.

5. Kannan Nonexpansive Mapping on $(C^F_{\tau(.)})_h$

We introduce the sufficient conditions of $(C^F_{\tau(.)})_h$, where $h(g) = [\sum_{m=0}^{\infty}\bar{\rho}(g_m, \bar{0})^{\tau_m}]^{1/K}$, for every $g \in C^F_{\tau(.)}$, such that the Kannan nonexpansive mapping on it has a fixed point, by fixing Γ a nonempty h-bounded, h-convex, and h-closed subset of $(C^F_{\tau(.)})_h$.

Lemma 33. *If $(C^F_{\tau(.)})_h$ verifies the (R) property and the h-quasinormal property. Assume $V : \Gamma \longrightarrow \Gamma$ is a Kannan h-nonexpansive mapping. For $t > 0$, let $G_t = \{Y \in \Gamma : h(Y - V(Y)) \leq t\} \neq \varnothing$. Put*

$$\Gamma_t = \bigcap\{\mathbf{B}_h(r, j)\colon V(G_t) \subset \mathbf{B}_h(r, j)\} \cap \Gamma. \tag{60}$$

Then, $\Gamma_t \neq \varnothing$, h-convex, h-closed subset of Γ, and $V(\Gamma_t) \subset \Gamma_t \subset G_t$ and $\delta_h(\Gamma_t) \leq t$.

Proof. Since $V(G_t) \subset \Gamma_t$, then $\Gamma_t \neq \varnothing$. As the h-balls are h-convex and h-closed, then Γ_t is a h-closed and h-convex subset of Γ. To show that $\Gamma_t \subset G_t$, assume $Y \in \Gamma_t$. When $h(Y - V(Y)) = 0$, one has $Y \in G_t$. Else, assume $h(Y - V(Y)) > 0$. Put

$$r = \sup\{h(V(Z) - V(Y))\colon Z \in G_t\}. \tag{61}$$

From the definition of r, one gets $V(G_t) \subset \mathbf{B}_h(V(Y), r)$.

Therefore, $\Gamma_t \subset \mathbf{B}_h(V(Y), r)$, then $h(Y - V(Y)) \le r$. Let $l > 0$. One has $Z \in G_t$ with $r - l \le h(V(Z) - V(Y))$. So

$$\begin{aligned} h(Y - V(Y)) - l \le r - l \le h(V(Z) - V(Y)) \\ \le \frac{1}{2}(h(Y - V(Y)) + h(Z - V(Z))) \\ \le \frac{1}{2}(h(Y - V(Y)) + t). \end{aligned} \tag{62}$$

As l is an arbitrary positive, one obtains $h(Y - V(Y)) \le t$; then, $Y \in G_t$. Since $V(G_t) \subset \Gamma_t$, one gets $V(\Gamma_t) \subset V(G_t) \subset \Gamma_t$, so Γ_t is V-invariant, to show that $\delta_h(\Gamma_t) \le t$, since

$$h(V(Y) - V(Z)) \le \frac{1}{2}(h(Y - V(Y))) + h(Z - V(Z))), \tag{63}$$

for all $Y, Z \in G_t$. Let $Y \in G_t$. Then, $V(G_t) \subset \mathbf{B}_h(V(Y), t)$. The definition of Γ_t gives $\Gamma_t \subset \mathbf{B}_h(V(Y), t)$. Therefore, $V(Y) \in \bigcap_{t\in\Gamma_t} \mathbf{B}_h(Z, t)$. One has $h(Z - Y) \le t$, for all $Z, Y \in \Gamma_t$, so $\delta_h(\Gamma_t) \le t$. □

Theorem 34. *If $(C^F_{\tau(.)})_h$ satisfies the h-quasinormal property and the (R) property, let $V : \Gamma \longrightarrow \Gamma$ be a Kannan h-nonexpansive mapping. Then, V has a fixed point.*

Proof. Let $t_0 = \inf\{h(Y - V(Y)) \colon Y \in \Gamma\}$ and $t_r = t_0 + 1/r$, for every $r \ge 1$. By the definition of t_0, one gets $G_{t_r} = \{Y \in \Gamma : h(Y - V(Y)) \le t_r\} \ne \varnothing$, for every $r \ge 1$. Assume Γ_{t_r} is defined as in Lemma 33. Clearly, $\{\Gamma_{t_r}\}$ is a decreasing sequence of nonempty h-bounded, h-closed, and h-convex subsets of Γ. The property (R) investigates that $\Gamma_\infty = \bigcap_{r\ge1} \Gamma_{t_r} \ne \varnothing$. Let $Y \in \Gamma_\infty$, and one has $h(Y - V(Y)) \le t_r$, for all $r \ge 1$. Suppose $r \longrightarrow \infty$; then, $h(Y - V(Y)) \le t_0$, so $h(Y - V(Y)) = t_0$. Therefore, $G_{t_0} \ne \varnothing$. Then, $t_0 = 0$. Else, $t_0 > 0$; then, V fails to have a fixed point. Let Γ_{t_0} be defined in Lemma 33. As V fails to have a fixed point and Γ_{t_0} is V-invariant, then Γ_{t_0} has more than one point, so $\delta_h(\Gamma_{t_0}) > 0$. By the h-quasinormal property, one has $Y \in \Gamma_{t_0}$ with

$$h(Y - Z) < \delta_h(\Gamma_{t_0}) \le t_0, \tag{64}$$

for all $Z \in \Gamma_{t_0}$. From Lemma 33, we get $\Gamma_{t_0} \subset G_{t_0}$. From definition of Γ_{t_0}, $V(Y) \in G_{t_0} \subset \Gamma_{t_0}$. Then,

$$h(Y - V(Y)) < \delta_h(\Gamma_{t_0}) \le t_0, \tag{65}$$

which contradicts the definition of t_0. Then, $t_0 = 0$ which gives that any point in G_{t_0} is a fixed point of V. □

According to Theorems 23, 27, and 34, we conclude the following:

Corollary 35. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and $V : \Gamma \longrightarrow \Gamma$ is a Kannan h-nonexpansive mapping. Then, V has a fixed point.*

Example 9. Assume $V : \Gamma \longrightarrow \Gamma$ with $V(Y) = \begin{cases} Y/4, & h(Y) \in [0, 1), \\ Y/5, & h(Y) \in [1, \infty), \end{cases}$ where $\Gamma = \{Y \in (C^F((2q + 3/q + 2)_{q=0}^\infty))_h : Y_0 = Y_1 = \bar{0}\}$ and $h(Y) = \sqrt{\sum_{q\in\mathcal{N}} \bar{\rho}(Y_q, \bar{0})^{2q+3/q+2}}$, for every $Y \in (C^F((2q + 3/q + 2)_{q=0}^\infty))_h$. By using Example 8, V is Kannan h-contraction. So it is Kannan h-nonexpansive. By Corollary 35, V has a fixed point $\bar{\vartheta}$ in Γ.

6. Kannan Contraction and Structure of Operator Ideal

The structure of the operator ideal by $(C^F_{\tau(.)})_h$ equipped with the definite function h, where $h(g) = [\sum_{q=0}^\infty (\sum_{p=0}^q \bar{\rho}(g_p, \bar{0})/q + 1)^{\tau_q}]^{1/K}$, for every $g \in C^F_{\tau(.)}$, and s-numbers has been explained. Finally, we examine the idea of Kannan contraction mapping in its associated pre-quasioperator ideal. As well, the existence of a fixed point of Kannan contraction mapping has been introduced. We indicate the space of all bounded, finite rank linear operators from a Banach space Δ into a Banach space Λ by $\mathscr{L}(\Delta, \Lambda)$, and $\mathfrak{F}(\Delta, \Lambda)$, and if $\Delta = \Lambda$, we inscribe $\mathscr{L}(\Delta)$ and $\mathfrak{F}(\Delta)$.

Definition 36 (see [31]). An s-number function is $s : \mathscr{L}(\Delta, \Lambda) \longrightarrow \mathfrak{R}^{+\mathcal{N}}$ which sorts every $V \in \mathscr{L}(\Delta, \Lambda)$ a $(s_d(V))_{d=0}^\infty$ verifies the following settings:

(a) $\|V\| = s_0(V) \ge s_1(V) \ge s_2(V) \ge \cdots \ge 0$, for all $V \in \mathscr{L}(\Delta, \Lambda)$

(b) $s_{l+d-1}(V_1 + V_2) \le s_l(V_1) + s_d(V_2)$, for all $V_1, V_2 \in \mathscr{L}(\Delta, \Lambda)$ and $l, d \in \mathcal{N}$

(c) $s_d(VYW) \le \|V\| s_d(Y) \|W\|$, for all $W \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \mathscr{L}(\Delta, \Lambda)$, and $V \in \mathscr{L}(\Lambda, \Lambda_0)$, where Δ_0 and Λ_0 are arbitrary Banach spaces

(d) If $V \in \mathscr{L}(\Delta, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$

(e) Suppose rank $(V) \le d$, and then, $s_d(V) = 0$, for each $V \in \mathscr{L}(\Delta, \Lambda)$

(f) $s_{l\ge a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a denotes the unit map on the a-dimensional Hilbert space ℓ_2^a

Definition 37 (see [8]).

(i) $\mathscr{L}$ is the class of all bounded linear operators within any two arbitrary Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is said to be an operator ideal, if all $\mathscr{U}(\Delta, \Lambda) = \mathscr{U} \cap \mathscr{L}(\Delta, \Lambda)$ verifies the following conditions: $I_\Gamma \in \mathscr{U}$, where Γ denotes Banach space of one dimension

(ii) The space $\mathscr{U}(\Delta, \Lambda)$ is linear over $\mathfrak{R}$

(iii) Assume $W \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$, then $YXW \in \mathscr{U}(\Delta_0, \Lambda_0)$

Notation 38.

$$\mathbb{F}_{\mathbf{U}} \coloneqq \{\mathbb{F}_{\mathbf{U}}(\Delta, \Lambda)\} \tag{66}$$

,where

$$\mathbb{F}_{\mathbf{U}}(\Delta, \Lambda) \coloneqq \left\{V \in \mathscr{L}(\Delta, \Lambda): \left(\left(s_d\bar{(}V\right)\right)_{d=0}^{\infty} \in \mathbf{U}\right\}, \tag{67}$$

where

$$s_d\bar{(}V)(x) = \begin{cases} 1, & x = s_d(V), \\ 0, & x \neq s_d(V). \end{cases} \tag{68}$$

Theorem 39. *Suppose U is a (cssf); then, $\mathbb{F}_U$ is an operator ideal.*

Proof.

(i) Assume $V \in \mathfrak{F}(\Delta, \Lambda)$ and rank $(V) = n$ for all $n \in \mathscr{N}$; as $\bar{\mathfrak{b}}_i \in \mathbf{U}$ for all $i \in \mathscr{N}$ and $\mathbf{U}$ is a linear space, one has

$(s_i(V)\bar{\ })_{i=0}^{\infty} = (s_0\bar{(}V), s_1\bar{(}V), \cdots, s_{n1}\bar{(}V), \bar{0}, \bar{0}, \bar{0}, \cdots) = \sum_{i=0}^{n-1} s_i\bar{(}V)\bar{\mathfrak{b}}_i \in \mathbf{U}$; for that $V \in \mathbb{F}_{\mathbf{U}}(\Delta, \Lambda)$ then $\mathfrak{F}(\Delta, \Lambda) \subseteq \mathbb{F}_E(\Delta, \Lambda)$.

(ii) Suppose $V_1, V_2 \in \mathbb{F}_{\mathbf{U}}(\Delta, \Lambda)$ and $\beta_1, \beta_2 \in \mathfrak{R}$, then by Definition 4 condition (33), one has $(s_{[i/2]}\bar{(}V_1))_{i=0}^{\infty} \in \mathbf{U}$ and $(s_{[i/2]}\bar{(}V_1))_{i=0}^{\infty} \in \mathbf{U}$, as $i \geq 2[i/2]$; by the definition of s-numbers and $s_i(P)$ is a decreasing sequence, one gets $s_i(\beta_1 V_1 \bar{+} \beta_2 V_2) \leq s_{2[i/2]}\bar{(}\beta_1 V_1 + \beta_2 V_2) \leq s_{[i/2]}(\beta_1 V_1) \bar{+} s_{[i/2]}(\beta_2 V_2) = |\beta_1| s_{[i/2]}\bar{(}V_1) + |\beta_2| s_{[i/2]}\bar{(}V_2)$, for each $i \in \mathscr{N}$. In view of Definition 4 condition (23) and $\mathbf{U}$ is a linear space, one obtains $(s_i(\beta_1 V_1 \bar{+} \beta_2 V_2))_{i=0}^{\infty} \in \mathbf{U}$; hence, $\beta_1 V_1 + \beta_2 V_2 \in \mathbb{F}_{\mathbf{U}}(\Delta, \Lambda)$.

(iii) Suppose $P \in \mathscr{L}(\Delta_0, \Delta)$, $T \in \mathbb{F}_{\mathbf{U}}(\Delta, \Lambda)$, and $R \in \mathscr{L}(\Lambda, \Lambda_0)$, one has $(s_i\bar{(}T))_{i=0}^{\infty} \in \mathbf{U}$, and as $s_i(R\bar{T}P) \leq \|R\| s_i\bar{(}T)\|P\|$, by Definition 4 conditions (22) and (23), one gets

$(s_i(R\bar{T}P))_{i=0}^{\infty} \in \mathbf{U}$, and then, $RTP \in \mathbb{F}_{\mathbf{U}}(\Delta_0, \Lambda_0)$. □

According to Theorems 10 and 39, one concludes the following theorem.

Theorem 40. *Let $(\tau_q)_{q \in \mathscr{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and one has $\mathbb{F}_{(C^F_{\tau(.)})_h}$ is an operator ideal.*

Definition 41 (see [9]). A function $H \in [0,\infty)^{\mathscr{U}}$ is called a pre-quasinorm on the ideal $\mathscr{U}$ if the next conditions hold:

(1) Let $V \in \mathscr{U}(\Delta, \Lambda)$, $H(V) \geq 0$, and $H(V) = 0$, if and only if $V = 0$

(2) We have $Q \geq 1$ so as to $H(\alpha V) \leq D|\alpha| H(V)$, for every $V \in \mathscr{U}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) We have $P \geq 1$ so that $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for each $V_1, V_2 \in \mathscr{U}(\Delta, \Lambda)$

(4) We have $\sigma \geq 1$ for $V \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$; then, $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$.

Theorem 42 (see [9]). *H is a pre-quasinorm on the ideal $\mathscr{U}$ if H is a quasinorm on the ideal $\mathscr{U}$.*

Theorem 43. *If $(\tau_q)_{q \in \mathscr{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, then the function H is a pre-quasinorm on $\mathbb{F}_{(C^F_{\tau(.)})_h}$, with $H(Z) = h(s_q\bar{(}Z))_{q=0}^{\infty}$, for all $Z \in \mathbb{F}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$.*

Proof.

(1) When $X \in \mathbb{F}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$, $H(X) = h(s_q\bar{(}X))_{q=0}^{\infty} \geq 0$ and $H(X) = h(s_q\bar{(}X))_{q=0}^{\infty} = 0$, if and only if $s_q\bar{(}X) = \bar{0}$, for all $q \in \mathscr{N}$, if and only if $X = 0$

(2) There is $Q \geq 1$ with $H(\alpha X) = h(s_q(\bar{\alpha} X))_{q=0}^{\infty} \leq Q|\alpha| H(X)$, for all $X \in \mathbb{F}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) One has $PP_0 \geq 1$ so that for $X_1, X_2 \in \mathbb{F}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$, one can see

$$\begin{aligned} H(X_1 + X_2) &= h\left(s_q(X_1 \bar{+} X_2)\right)_{q=0}^{\infty} \\ &\leq P\left(h\left(s_{[q/2]}\bar{(}X_1)\right)_{q=0}^{\infty} + h\left(s_{[q/2]}\bar{(}X_2)\right)_{q=0}^{\infty}\right) \\ &\leq PP_0\left(h\left(s_q(\bar{X}_1)\right)_{q=0}^{\infty} + h\left(s_q(\bar{X}_2)\right)_{q=0}^{\infty}\right) \end{aligned} \tag{69}$$

(4) We have $\rho \geq 1$, if $X \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \mathbb{F}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$, and $Z \in \mathscr{L}(\Lambda, \Lambda_0)$, and then, $H(ZYX) = h(s_q(\bar{Z}YX))_{q=0}^{\infty} \leq h(\|X\| \|Z\| s_q\bar{(}Y))_{q=0}^{\infty} \leq \rho \|X\| H(Y) \|Z\|$.

□

In the next theorems, we will use the notation $(\mathbb{F}_{(C^F_{\tau(.)})_h}, H)$, where $H(V) = h((s_q\bar{(}V))_{q=0}^{\infty})$, for all $V \in \mathbb{F}_{(C^F_{\tau(.)})_h}$.

Theorem 44. *Suppose $(\tau_q)_{q \in \mathscr{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and one has $(\mathbb{F}_{(C^F_{\tau(.)})_h}, H)$ is a pre-quasi-Banach operator ideal.*

Proof. Suppose $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. As $\mathscr{L}(\Delta,\Lambda)\supseteq S_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, one has

$$\begin{aligned} H(V_r-V_a)&=h\left(\left(s_q(\bar{V_r V_a})\right)_{q=0}^{\infty}\right)\geq h\left(s_0(\bar{V_r V_a}),\bar{0},\bar{0},\bar{0},\cdots\right)\\ &\geq \inf_q \|V_r-V_a\|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{1}{q+1}\right)^{\tau_q}\right]^{1/K}. \end{aligned} \tag{70}$$

Hence, $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\mathscr{L}(\Delta,\Lambda)$. $\mathscr{L}(\Delta,\Lambda)$ is a Banach space, so there exists $V\in\mathscr{L}(\Delta,\Lambda)$ so that $\lim_{a\longrightarrow\infty}\|V_a-V\|=0$ and since $(s_q(\bar{V}_a))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, for all $a\in\mathcal{N}$, and $(C^F_{\tau(.)})_h$ is a premodular (cssf). Hence, one can see

$$\begin{aligned} H(V)&=h\left(\left(s_q(\bar{V})\right)_{q=0}^{\infty}\right)\leq h\left(\left(s_{[q/2]}(\bar{V}V_a)\right)_{q=0}^{\infty}\right)\\ &+h\left(\left(s_{[q/2]}(\bar{V_a})_{q=0}^{\infty}\right)\right)\leq h\left(\left(\|V_a-V\|\bar{1}\right)_{q=0}^{\infty}\right)\\ &+\left(3^K+2^K\right)^{1/K}h\left(\left(s_q(\bar{V}_a)\right)_{q=0}^{\infty}\right)<\varepsilon. \end{aligned} \tag{71}$$

We obtain $(s_q(\bar{V}))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, and hence, $V\in\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. □

Theorem 45. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ with $\tau_0>1$, one has $(\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}, H)$ is a pre-quasiclosed operator ideal.*

Proof. Suppose $V_a\in\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, for all $a\in\mathcal{N}$ and $\lim_{a\longrightarrow\infty}H(V_a-V)=0$. As $\mathscr{L}(\Delta,\Lambda)\supseteq S_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, one has

$$\begin{aligned} H(V_a-V)&=h\left(\left(s_q(\bar{V}_a V)\right)_{q=0}^{\infty}\right)\geq h\left(s_0(\bar{V}_a V),\bar{0},\bar{0},\bar{0},\cdots\right)\\ &\geq \inf_q \|V_a-V\|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{1}{q+1}\right)^{\tau_q}\right]^{1/K}. \end{aligned} \tag{72}$$

So $(V_a)_{a\in\mathcal{N}}$ is convergent in $\mathscr{L}(\Delta,\Lambda)$. i.e., $\lim_{a\longrightarrow\infty}\|V_a-V\|=0$, and since $(s_q(\bar{V}_a))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, for all $q\in\mathcal{N}$ and $(C^F_{\tau(.)})_h$ is a premodular (cssf). Hence, one can see

$$\begin{aligned} H(V)&=h\left(\left(s_q(\bar{V})\right)_{q=0}^{\infty}\right)\leq h\left(\left(s_{[q/2]}(\bar{V}V_a)\right)_{q=0}^{\infty}\right)\\ &+h\left(\left(s_{[q/2]}(\bar{V_a})_{q=0}^{\infty}\right)\right)\leq h\left(\left(\|V_a-V\|\bar{1}\right)_{q=0}^{\infty}\right)\\ &+\left(3^K+2^K\right)^{1/K}h\left(\left(s_q(\bar{V}_a)\right)_{q=0}^{\infty}\right)<\varepsilon. \end{aligned} \tag{73}$$

We obtain $(s_q(\bar{V}))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, and hence, $V\in\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. □

Definition 46. A pre-quasinorm H on the ideal $\bar{\mathfrak{F}}_{U_h}$ verifies the Fatou property if for every $\{T_q\}_{q\in\mathcal{N}}\subseteq\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$ so that $\lim_{q\longrightarrow\infty}H(T_q-T)=0$ and $M\in\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$, one gets

$$H(M-T)\leq\sup_q\inf_{j\geq q}H\left(M-T_j\right). \tag{74}$$

Theorem 47. *Suppose $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ with $\tau_0>1$, then $(\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}, H)$ does not satisfy the Fatou property.*

Proof. Assume $\{T_q\}_{q\in\mathcal{N}}\subseteq\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$ with $\lim_{q\longrightarrow\infty}H(T_q-T)=0$. Since $\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}$ is a pre-quasiclosed ideal, then $T\in\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. So for every $M\in\bar{\mathfrak{F}}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, one has

$$\begin{aligned} H(M-T)&=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(s_p(\bar{M}T),\bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &\leq\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(s_{[p/2]}\left(\bar{M}T_j\right),\bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &+\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(s_{[p/2]}\left(\bar{T}_jT\right),\bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &\leq\left(3^K+2^K\right)^{1/K}\sup_r\inf_{j\geq r}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(s_p\left(\bar{M}T_j\right),\bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K}. \end{aligned} \tag{75}$$

□

Definition 48. An operator $V:\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)\longrightarrow\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$ is said to be a Kannan H-contraction, if one has $\alpha\in[0,1/2)$ with $H(VT-VM)\leq\alpha(H(VT-T)+H(VM-M))$, for all $T,M\in\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$.

Definition 49. An operator $V:\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)\longrightarrow\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$ is said to be H-sequentially continuous at M, where $M\in\bar{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$, if and only if $\lim_{r\longrightarrow\infty}H(T_r-M)=0\Rightarrow\lim_{r\longrightarrow\infty}H(VT_r-VM)=0$.

Example 10. $V:\bar{\mathfrak{F}}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}(\Delta,\Lambda)\longrightarrow\bar{\mathfrak{F}}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}(\Delta,\Lambda)$,

where $H(T)=\sqrt{\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(s_p(\bar{T}),\bar{0})/q+1)^{2q+3/q+2}}$, for every $T\in\bar{\mathfrak{F}}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}(\Delta,\Lambda)$ and

$$V(T)=\begin{cases}\dfrac{T}{6}, & H(T)\in[0,1),\\ \dfrac{T}{7}, & H(T)\in[1,\infty).\end{cases} \tag{76}$$

Evidently, V is H-sequentially continuous at the zero operator $\Theta\in\bar{\mathfrak{F}}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$. Let $\{T^{(j)}\}\subseteq\bar{\mathfrak{F}}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ be such that $\lim_{j\longrightarrow\infty}H(T^{(j)}-T^{(0)})=0$, where $T^{(0)} \in \bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T^{(0)}) = 1$. Since the pre-quasinorm H is continuous, one gets

$$\begin{aligned}\lim_{j\longrightarrow\infty} H\left(VT^{(j)} - VT^{(0)}\right) &= \lim_{j\longrightarrow\infty} H\left(\frac{T^{(0)}}{6} - \frac{T^{(0)}}{7}\right)\\ &= H\left(\frac{T^{(0)}}{42}\right) > 0.\end{aligned} \tag{77}$$

Therefore, V is not H-sequentially continuous at $T^{(0)}$.

Theorem 50. *Pick up $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$ and $V : \bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda) \longrightarrow \bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$. Assume*

(i) V is Kannan H-contraction mapping

(ii) V is H-sequentially continuous at an element $A \in \bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$

(iii) there are $G \in \bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ such that the sequence of iterates $\{V^r G\}$ has a $\{V^{r_m} G\}$ converging to M

Then, $M \in \bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ is the unique fixed point of V.

Proof. Let M be not a fixed point of V; hence, $VM \neq M$. By using parts (ii) and (iii), we get

$$\begin{aligned}&\lim_{r_m\longrightarrow\infty} H(V^{r_m} G - M) = 0,\\ &\lim_{r_m\longrightarrow\infty} H\left(V^{r_m+1} G - VM\right) = 0.\end{aligned} \tag{78}$$

Since V is Kannan H-contraction, one obtains

$$\begin{aligned}0 < H(VM - M) &= H\left(\left(VM - V^{r_m+1}G\right) + (V^{r_m}G\text{minus;}M)\right.\\ &\quad \left. + \left(V^{r_m+1}G - V^{r_m}G\right)\right)\\ &\le \left(3^K + 2^K\right)^{1/K} H\left(V^{r_m+1}G - VM\right)\\ &\quad + \left(3^K + 2^K\right)^{2/K} H(V^{r_m}G - M)\\ &\quad + \left(3^K + 2^K\right)^{2/K} \alpha\left(\frac{\alpha}{1-\alpha}\right)^{r_m-1} H(VG - G).\end{aligned} \tag{79}$$

As $r_m \longrightarrow \infty$, there is a contradiction. Hence, M is a fixed point of V. To prove the uniqueness of the fixed point M, suppose one has two not equal fixed points $M, J \in \bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ of V. So, one gets $H(M - J) \le H(VM - VJ) \le \alpha(H(VM - M) + H(VJ - J)) = 0$. Then, $M = J$. □

Example 11. Given Example 10, since for all $T_1, T_2 \in \bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T_1), H(T_2) \in [0, 1)$, we have

$$\begin{aligned}H(VT_1 - VT_2) &= H\left(\frac{T_1}{6} - \frac{T_2}{6}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}\left(H\left(\frac{5T_1}{6}\right) + H\left(\frac{5T_2}{6}\right)\right)\\ &= \frac{\sqrt{2}}{\sqrt[4]{125}}(H(VT_1 - T_1) + H(VT_2 - T_2)).\end{aligned} \tag{80}$$

For all $T_1, T_2 \in \bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T_1), H(T_2) \in [1,\infty)$, we have

$$\begin{aligned}H(VT_1 - VT_2) &= H\left(\frac{T_1}{7} - \frac{T_2}{7}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{216}}\left(H\left(\frac{6T_1}{7}\right) + H\left(\frac{6T_2}{7}\right)\right)\\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}(H(VT_1 - T_1) + H(VT_2 - T_2)).\end{aligned} \tag{81}$$

For all $T_1, T_2 \in \bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T_1) \in [0, 1)$ and $H(T_2) \in [1,\infty)$, we have

$$\begin{aligned}H(VT_1 - VT_2) &= H\left(\frac{T_1}{6} - \frac{T_2}{7}\right) \le \frac{\sqrt{2}}{\sqrt[4]{125}} H\left(\frac{5T_1}{6}\right)\\ &\quad + \frac{\sqrt{2}}{\sqrt[4]{216}} H\left(\frac{6T_2}{7}\right) \le \frac{\sqrt{2}}{\sqrt[4]{125}}(H(VT_1 - T_1)\\ &\quad + H(VT_2 - T_2)).\end{aligned} \tag{82}$$

Hence, V is Kannan H-contraction and $V^r(T) = \begin{cases} T/6^r, & H(T) \in [0, 1),\\ T/7^r, & H(T) \in [1,\infty).\end{cases}$

Obviously, V is H-sequentially continuous at $\Theta \in \bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$, and $\{V^r T\}$ has a subsequence $\{V^{r_m} T\}$ converges to Θ. By Theorem 50, Θ is the only fixed point of G.

7. Applications

Theorem 51. *Consider the summable equation*

$$Y_p = R_p + \sum_{r=0}^{\infty} D(p, r) m(r, Y_r), \tag{83}$$

which presented by many authors [32, 33, 34], and assume $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$, where $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with

$\tau_0 > 1$ and $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, is defined by

$$V(Y_p)_{p\in\mathcal{N}} = \left(R_p + \sum_{r=0}^{\infty} D(p,r)m(r, Y_r)\right)_{p\in\mathcal{N}}. \quad (84)$$

The summable equation (83) has a unique solution in $(C^F_{\tau(.)})_h$, if $D : \mathcal{N}^2 \longrightarrow \mathfrak{R}$, $m : \mathcal{N} \times \mathfrak{R}[0,1] \longrightarrow \mathfrak{R}[0,1]$, $R : \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$, and $Z : \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$; assume there is $\delta \in \mathfrak{R}$ such that $\sup_q |\delta|^{\tau_q/K} \in [0, 0.5)$, and for all $q \in \mathcal{N}$, let

$$\begin{aligned}\sum_{p=0}^{q}\left[\sum_{r=0}^{\infty} D(p,r)(m(r,Y_r) - m(r,Z_r))\right] &\le |\delta|\left[\sum_{p=0}^{q}\left(R_p - Y_p + \sum_{r=0}^{\infty} D(p,r)m(r,Y_r)\right)\right.\\ &\quad \left.+ \sum_{p=0}^{q}\left(R_p - Z_p + \sum_{r=0}^{\infty} D(p,r)m(r,Z_r)\right)\right].\end{aligned} \quad (85)$$

Proof. One has

$$\begin{aligned}h(VY - VZ) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(VY_p - VZ_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(\sum_{r=0}^{\infty} D(p,r)[m(r,Y_r) - m(r,Z_r)], \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &\le \sup_q |\delta|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(R_p - Y_p + \sum_{r=0}^{\infty} D(p,r)m(r,Y_r), \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &\quad + \sup_q |\delta|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(R_p - Z_p + \sum_{r=0}^{\infty} D(p,r)m(r,Z_r), \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &= \sup_q |\delta|^{\tau_q/K}(h(VY - Y) + h(VZ - Z)).\end{aligned} \quad (86)$$

By Theorem 29, one gets a unique solution of equation (83) in $(C^F_{\tau(.)})_h$. □

Example 12. Suppose $(C^F((2q+3/q+2)_{q=0}^{\infty}))_h$, where $h(Y) = \sqrt{\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{2q+3/q+2}}$, for all $Y \in C^F((2q+3/q+2)_{q=0}^{\infty})$. Consider the summable equation

$$Y_p = R_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Y_p}{p^2 + r^2 + 1}\right)^t, \quad (87)$$

with $p \ge 2$ and $t > 0$. Suppose $\Gamma = \{Y \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h : Y_0 = Y_1 = \bar{0}\}$. Indeed, Γ is a nonempty, h-convex, h-closed, and h-bounded subset of $(C^F((2q+3/q+2)_{q=0}^{\infty}))_h$. Let $V : \Gamma \longrightarrow \Gamma$ be defined by

$$V(Y_p)_{p\ge 2} = \left(R_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Y_p}{p^2 + r^2 + 1}\right)^t\right)_{p\ge 2}. \quad (88)$$

Obviously,

$$\begin{aligned}\sum_{p=0}^{q}\sum_{r=0}^{\infty}(-1)^p\left(\frac{Y_p}{p^2+r^2+1}\right)^t((-1)^r - (-1)^r) &\le \frac{1}{\sqrt{2}}\left[\sum_{p=0}^{q}\left(R_p - Y_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Y_p}{p^2+r^2+1}\right)^t\right)\right.\\ &\quad \left.+ \sum_{p=0}^{q}\left(R_p - Z_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Z_p}{p^2+r^2+1}\right)^t\right)\right].\end{aligned} \quad (89)$$

By Corollary 35 and Theorem 51, the summable equation (87) has a solution in Γ.

Example 13. Suppose $(C^F((2q+3/q+2)_{q=0}^{\infty}))_h$, where $h(Y) = \sqrt{\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{2q+3/q+2}}$, for every $Y \in C^F((2q+3/q+2)_{q=0}^{\infty})$. Consider the following nonlinear difference equation:

$$Y_p = R_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}, \quad (90)$$

with $r, p > 0$, $Y_{-2}(x), Y_{-1}(x) > 0$, for all $x \in \mathfrak{R}$, and assume $V : C^F((2q+3/q+2)_{q=0}^{\infty}) \longrightarrow C^F((2q+3/q+2)_{q=0}^{\infty})$ is defined by

$$V(Y_p)_{p=0}^{\infty} = \left(R_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}\right)_{p=0}^{\infty}. \quad (91)$$

Evidently,

$$\begin{aligned}\sum_{p=0}^{q}\sum_{l=0}^{\infty}(-1)^p\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}\left((-1)^l - (-1)^l\right) &\le \frac{1}{\sqrt{2}}\left[\sum_{p=0}^{q}\left(R_p - Y_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}\right)\right.\\ &\quad \left.+ \sum_{p=0}^{q}\left(R_p - Z_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Z_{p-2}^r}{Z_{p-1}^p + l^2 + 1}\right)\right].\end{aligned} \quad (92)$$

By Theorem 51, the nonlinear difference equation (90) has a unique solution in $C^F((2q+3/q+2)_{q=0}^{\infty})$.

8. Conclusion

Rather than simply referring to a "quasi-normed" place, we used the term "prequasi-normed." It is the concept of a fixed point of the Kannan pre-quasinorm contraction mapping in the pre-quasi-Banach variable exponent Cesàro sequence spaces of fuzzy functions (cssf). Pre-quasinormal structure and (R) are supported. The Kannan nonexpansive mapping's presence of a fixed point was investigated. The presence of a fixed point of Kannan contraction mapping in the pre-quasi-Banach operator ideal produced by variable exponent Cesàro sequence spaces of fuzzy functions (cssf) and s-fuzzy numbers has also been examined. To put our findings to the test, we introduce several numerical experiments. In addition, various effective implementations of the stochastic nonlinear dynamical system are discussed. The fixed points of any Kannan contraction and nonexpansive mappings on this new fuzzy functions space, its associated pre-quasi-ideal, and a new general space of solutions for many stochastic nonlinear dynamical systems are investigated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. UJ-21-DR-75. The authors, therefore, acknowledge with thanks the university for the technical and financial support.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad, "Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application," *Journal of Mathematics*, vol. 2021, Article ID 6663707, 7 pages, 2021.

[3] A. Al-Masarwah and A. Ahmad, "M-Polar (α, β) -Fuzzy Ideals in BCK/BCI-Algebras," *Symmetry*, vol. 11, no. 1, p. 44, 2019.

[4] K. Javed, A. Asif, and E. Savas, "A note on orthogonal fuzzy metric space, its properties, and fixed point theorems," *Journal of Function Spaces*, vol. 2022, Article ID 5863328, 14 pages, 2022.

[5] F. Nuray and E. Savas, "Statistical convergence of sequences of fuzzy numbers," *Mathematica Slovaca*, vol. 45, no. 3, pp. 269–273, 1995.

[6] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.

[7] N. Faried and A. A. Bakery, "Mappings of type Orlicz and generalized Cesáro sequence space," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[8] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980.

[9] N. Faried and A. A. Bakery, "Small operator ideals formed by s numbers on generalized Cesáro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 2018, 2018.

[10] K. Rajagopal and M. Ru⊠iĉka, "On the modeling of electrorheological materials," *Mechanics Research Communications*, vol. 23, no. 4, pp. 401–407, 1996.

[11] M. Ru⊠iĉka, "Electrorheological fluids. Modeling and mathematical theory," in *Lecture Notes in Mathematics*, vol. 1748, Springer, Berlin, Germany, 2000.

[12] S. Banach, "Sur les opérations dans les ensembles abstraits et leurs applications," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[13] R. Kannan, "Some results on fixed points- II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.

[14] S. J. H. Ghoncheh, "Some fixed point theorems for Kannan mapping in the modular spaces," *Ciĕncia eNatura: Santa Maria, CA, USA*, vol. 37, pp. 462–466, 2015.

[15] E. Reich, "Kannan's fixed point theorem," *Bollettino dell'Unione Matematica Italiana*, vol. 4, pp. 1–11, 1971.

[16] U. Aksoy, E. Karapinar, I. M. Erhan, and V. Rakocevic, "Meir-Keeler type contractions on modular metric spaces," *Univerzitet u Nišu*, vol. 32, no. 10, pp. 3697–3707, 2018.

[17] A. A. Bakery and A. R. Abou Elmatty, "Pre-quasi simple Banach operator ideal generated by s-numbers," *Journal of Function Spaces*, vol. 2020, Article ID 9164781, 11 pages, 2020.

[18] A. A. Bakery and O. S. K. Mohamed, "Kannan prequasi contraction maps on Nakano sequence spaces," *Journal of Function Spaces*, vol. 2020, Article ID 9164781, 10 pages, 2020.

[19] A. A. Bakery and O. S. K. Mohamed, "Kannan nonexpansive maps on generalized Cesàro backward difference sequence space of non-absolute type with applications to summable equations," *Journal of Inequalities and Applications*, vol. 2021, 2021.

[20] A. A. Bakery and O. S. K. Mohamed, "Some fixed point results of Kannan maps on the Nakano sequence space," *Journal of Function Spaces*, vol. 2021, Article ID 2578960, 17 pages, 2021.

[21] M. Matloka, "Sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 28, pp. 28–37, 1986.

[22] S. Nanda, "On sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 123–126, 1989.

[23] V. Kumar, A. Sharma, K. Kumar, and N. Singh, "On limit points and cluster points of sequences of fuzzy numbers," in *International Mathematical Forum. Vol. 2.*, no. 57pp. 2815–2822, Hikari, Ltd., 2007.

[24] B. Altay and F. Basar, "Generalization of the sequence space $\ell(p)$ derived by weighted mean," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 174–185, 2007.

[25] A. A. Bakery and E. A. E. Mohamed, "On the nonlinearity of extended s-type weighted Nakano sequence spaces of fuzzy functions with some applications," *Journal of Function Spaces*, vol. 2022, Article ID 2746942, 20 pages, 2022.

[26] H. Nakano, *Topology of Linear Topological Spaces*, Maruzen Co. Ltd., Tokyo, 1951.

[27] D. Kutzarova, "K-β and k-nearly uniformly convex Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 2, pp. 322–338, 1991.

[28] Y. Cui and H. Hudzik, "On the uniform Opial property in some modular sequence spaces," *Uniwersytet im*, vol. 26, pp. 93–102, 1998.

[29] G. Zhang, "Weakly convergent sequence coeficient of product spaces," *Proceedings of the American Mathematical Society*, vol. 117, no. 3, pp. 637–643, 1993.

[30] W. Bynum, "Normal structure coefficients for Banach spaces," *Pacific Journal of Mathematics*, vol. 86, no. 2, pp. 427–436, 1980.

[31] A. Pietsch, *Eigenvalues and s-Numbers*, Cambridge University Press, New York, NY, USA, 1986.

[32] P. Salimi, A. Latif, and N. Hussain, "Modified α-ψ-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, 2013.

[33] R. P. Agarwal, N. Hussain, and M.-A. Taoudi, "Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 245872, 15 pages, 2012.

[34] N. Hussain, A. R. Khan, and R. P. Agarwal, "Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 3, pp. 475–489, 2010.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 8311587, 14 pages
https://doi.org/10.1155/2022/8311587

Research Article

Analysis of Fractional Differential Inclusion Models for COVID-19 via Fixed Point Results in Metric Space

Monairah Alansari[1] and Mohammed Shehu Shagari[2]

[1]*Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria*

Correspondence should be addressed to Mohammed Shehu Shagari; shagaris@ymail.com

Received 8 April 2022; Revised 17 June 2022; Accepted 28 June 2022; Published 16 July 2022

Academic Editor: Santosh Kumar

We examine in this paper some new problems on coincidence point and fixed point theorems for multivalued mappings in metric space. By applying the characterizations of a modified $\widetilde{\mathscr{M}\mathscr{T}}$-function, under the name $\mathscr{D}$-function, a few novel fixed point results different from the existing fixed point theorems are launched. It is well-known that differential equation of either integer or fractional order is not sufficient to capture ambiguity, since the derivative of a solution to any differential equation inherits all the regularity properties of the mapping involved and of the solution itself. This does not hold in the case of differential inclusions. In particular, fractional-order differential inclusion models are more suitable for describing epidemics. Thus, as a generalization of a newly launched existence result for fractional-order model for COVID-19, using Banach and Shauder fixed point theorems, we investigate solvability criteria of a novel Caputo-type fractional-order differential inclusion model for COVID-19 by applying a standard fixed point theorem of multivalued contraction. Stability analysis of the proposed model in the framework of Ulam-Hyers is also discussed. Nontrivial comparative illustrations are constructed to show that our ideas herein complement, unify and, extend a significant number of existing results in the corresponding literature.

1. Introduction and Preliminaries

Numerous challenges in practical world defined by nonlinear functional equations can be simplified by reconfiguring them to their equivalent fixed point problems. Fixed point theory yields relevant tools for solving problems emanating in various arms of sciences. The fixed point theorem, commonly named as the Banach fixed point theorem (see [1]), came up in clear form in Banach thesis in 1922, where it was availed to study the existence of a solution to an integral equation. Since then, because of its importance, it has gained a number of refinements by many authors. In some modifications of the principle, the inequality is weakened, see, for example [2, 3], and in others, the topology of the ambient space is relaxed, see [4–7] and the references therein. Along the lane, three prominent improvements of the Banach fixed point theorem was presented by Ciric [2], Reich [8], and Rus [9].

Nadler [10] launched a multivalued improvement of the Banach contraction mapping principle. Nadler's contraction mapping principle opened up the concept of metric fixed point theory of multivalued contraction in nonlinear analysis. In line with [10], a number of refinements of fixed point theorems of multivalued contractions have been presented, famously, by Berinde-Berinde [11], Du [12, 13], Mizoguchi and Takahashi [14], Pathak [15], and Reich [16, 17], to cite a few. Fixed point theorems for multivalued mappings are highly advantageous in optimal control theory and have been commonly used to solve several problems in economics, game theory, biomathematics, qualitative physics, viability theory, and many more.

Differential inclusions are found to be of great usefulness in studying dynamical systems and stochastic processes. A few examples include sweeping process, granular systems, nonlinear dynamics of wheeled vehicles, and control problems. In particular, fractional differential inclusions arise in several problems in mathematical physics, biomathematics, control theory, critical point theory for non-smooth energy functionals, differential variational inequalities, fuzzy set arithmetic, traffic theory, etc. Usually, the first most concerned problem in the study of differential inclusion is the conditions for existence of its solutions. In this direction, several authors have applied different fixed point approaches and topological methods to obtain existence results of differential inclusions in abstract spaces. In the current literature, we can find many works on fractional-order models proposing different measures for curbing the novel corona virus (COVID-19) (see, for example, Ali et al. [18], Yu et al. [19], Xu et al. [20], Shaikh et al. [21], and the references therein). Recently, Ahmed et al. [22] constructed a Caputo-type fractional-order model and studied the significance and effect of the lockdown in curbing COVID-19. They ([22]) investigated the existence and uniqueness of solutions of the fractional-order corona virus model by applying the Banach and Schauder fixed point theorems. One of the pioneer results of fixed point theory using fractional-order model was presented by Boccaletti et al. [23]. For some recent results and applications of fraction calculus, we refer [24–26].

Following the above developments, we consider in this paper some problems on coincidence point and fixed point theorems for multivalued mappings. By applying the characterizations of $\mathscr{D}$-function, a few new fixed point results different from the fixed point theorems due to Berinde-Berinde [11], Du [13], Mizoguchi-Takahashi [14], Nadler [10], Reich [17], and Rus [27] are launched. It is a common knowledge that differential equation of either integer or fractional order is not sufficient to capture ambiguity, since the derivative $j'(.)$ of a solution $j(.)$ to the differential equation $j'(t) = g(t, j(t))$ inherits the regularity properties of the mapping g and of the function $j(.)$. This is no longer the case with differential inclusions. In particular, fractional-order differential inclusions models are more suitable for describing epidemics (see, e.g., [28]). Differential inclusions are not only models for handling dynamic processes but also provide powerful analytic tools to prove existence theorems such as in control theory, to derive sufficient conditions of optimality, play a significant role in the theory of control conditions under uncertainty. Thus, as a generalization of the existence theorem presented by Ahmed et al. [22], in the sequel, we investigate solvability conditions of a new Caputo-type fractional differential inclusions model for COVID-19 by applying a fixed point theorem of multivalued contraction. Stability analysis of the proposed model in the context of Ulam-Hyers is also obtained. Our results herein complement, unify, and extend the above-mentioned articles and a few others in the comparable literature. A few nontrivial comparative illustrations are constructed to indicate that our obtained ideas properly advanced corresponding results in the literature.

In what follows, we recall some preliminary concepts that are useful to our main results. Throughout this paper, the set $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ represent the set of real numbers, nonnegative real numbers, and the set of natural numbers, respectively. Let $(\mho, \mu)$ be a metric space. Denote by $\mathscr{N}(\mho)$, $CB(\mho)$, and $\mathscr{K}(\mho)$, the family of nonempty subsets of $\mho$, the collection of all nonempty closed and bounded subsets of $\mho$, and the class of all nonempty compact subsets of $\mho$, respectively. For $A, B \in CB(\mho)$, the mapping $\tilde{H} : CB(\mho) \times CB(\mho) \longrightarrow \mathbb{R}$ is given by

$$\tilde{H}(A, B) = \max \left\{ \sup_{j\in B} \mu(j, A), \sup_{\ell\in A} \mu(\ell, B) \right\}, \quad (1)$$

where $\mu(j, A) = \inf_{\ell\in A} \mu(j, \ell)$ is named the Hausdorff-Pompeiu metric induced by the metric μ. For example, if we consider the set of real numbers endowed with the standard metric, then for any two closed intervals $[a, b]$ and $[c, d]$, we have $\tilde{H}([a, b], [c, d]) = \max \{|a - c|, |b - d|\}$.

Let $\Delta, \Theta, \Lambda : \mho \longrightarrow \mho$ be point-valued mappings and $Y : \mho \longrightarrow \mathscr{N}(\mho)$ be a multivalued mapping. A point u in $\mho$ is a coincidence point of Δ, Θ, Λ and Y if $\Delta u = \Theta u = \Lambda u \in Yu$. If $\Delta = \Theta = \Lambda = I_{\mho}$ is the identity mapping on $\mho$, then $u = \Delta u = \Theta u = \Lambda u \in Yu$ is named a fixed point of Y. We denote the set of fixed points of Y and the set of coincidence point of Δ, Θ, Λ and Y by $\mathscr{F}_{ix}(Y)$ and $\mathscr{COP}(\Delta, \Theta, \Lambda, Y)$, respectively.

Let g be a real-valued function. For $t \in \mathbb{R}$, we recall that

$$\limsup_{r\longrightarrow t} g(r) = \inf_{\varepsilon>0} \sup_{0<|r-t|<\varepsilon} g(r) \text{and} \limsup_{r\longrightarrow t^+} g(r) = \inf_{\varepsilon>0} \sup_{0<r-t<\varepsilon} g(r). \quad (2)$$

Definition 1. (see [12]). $\psi_{\widetilde{\mathscr{MT}}} : (0,\infty) \longrightarrow [0, 1)$ is named an $\widetilde{\mathscr{MT}}$-function if it obeys the Mizoguchi-Takahashi's condition, that is, $\limsup_{r\longrightarrow t^+} \psi_{\widetilde{\mathscr{MT}}}(r) < 1$, for each $t \in \mathbb{R}_+ = [0,\infty)$.

Remark 2. (see [12]).

(i) If $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ is given as $\psi_{\widetilde{\mathscr{MT}}}(t) = \alpha \in [0, 1)$, then $\psi_{\widetilde{\mathscr{MT}}}$ is an $\widetilde{\mathscr{MT}}$-function

(ii) If the function $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ is either increasing or decreasing, then $\psi_{\widetilde{\mathscr{MT}}}$ is an $\widetilde{\mathscr{MT}}$-function

Definition 3. $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ is named a $\mathscr{D}$-function if it obeys the condition: For each $t \in \mathbb{R}_+$, we can find $k \in (1,\infty)$ such that $\limsup_{r\longrightarrow t^+} \psi(r) < 1/k$.

Definition 4. (see [12]). A function $\psi : \mathbb{R}_+ \longrightarrow [0, 1)$ is named a function of contractive factor, if for any strictly decreasing sequence $\{j_n\}_{n\geq 1}$ in $\mathbb{R}_+$, we have $0 \leq \sup_{n\in\mathbb{N}} \psi(j_n) < 1$.

Definition 5. A function $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ is named a function of $1/k$-contractive factor, if for any sequence $\{j_n\}_{n\geq 1}$ in $\mathbb{R}_+$ from and after some fixed terms, it is strictly nonincreasing and $0 \leq \sup_{n\in\mathbb{N}} \psi(j_n) < 1/k$, for some $k \in (1,\infty)$.

The following example recognizes the existence of $\mathscr{D}$-function and function of $1/k$-contractive factor.

Example 6.
Let $\{j_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}_+$ given by

$$j_n = \begin{cases} 3^{2n} - 1, & \text{if } n \leq 7 \\ 3 + \dfrac{1}{2n}, & \text{if } n > 7. \end{cases} \tag{3}$$

Define $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ by

$$\psi(\hat{t}) = \begin{cases} \dfrac{1}{17 + \hat{t}^2}, & \text{if } 0 \leq \hat{t} < 2 \\ \dfrac{1}{3} - \dfrac{\hat{t}}{3^7}, & \text{if } 2 \leq \hat{t} < 50 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then, it is clear that ψ is a $\mathscr{D}$-function, $\{j_n\}_{n\geq 1}$ is a strictly decreasing sequence from and after the eight term and $0 \leq \sup_{n\in\mathbb{N}} \psi(j_n) = 727/2187 < 1/k$ for some $k \in (1,\infty)$. Whence, ψ is also a function of $1/k$-contractive factor. An example which is not a $\mathscr{D}$-function is provided hereunder.

Example 7.
Let $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be given by

$$\psi(\hat{t}) = \begin{cases} \dfrac{\sin \hat{t}}{\hat{t}}, & \text{if } \hat{t} \in \left(0, \dfrac{\pi}{2}\right] \\ \dfrac{1}{\hat{t} + k^2}, & \text{elsewhere.} \end{cases} \tag{5}$$

Since $\limsup_{r\longrightarrow 0^+} \psi(r) = 1$, then ψ is not a $\mathscr{D}$-function.

Remark 8.

(i) Note that if $\psi_{\widetilde{\mathscr{MT}}} = k\psi(\hat{t})$ for all $\hat{t} \in \mathbb{R}_+$ and for some $k \in (1,\infty)$, then $\psi_{\widetilde{\mathscr{MT}}}$ becomes an $\widetilde{\mathscr{MT}}$-function, provided ψ is a $\mathscr{D}$-function

(ii) If we define $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ as $\psi(\hat{t}) = 1/k^n$ for all $n \geq 2$ and $k \in (1,\infty)$, then ψ is a $\mathscr{D}$-function

The following Lemma is in consistent with [16, Lemma 18].

Lemma 9.
Let $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Then $\rho : \mathbb{R}_+ \longrightarrow [0, (1/k))$ given by $\rho(\hat{\hat{t}}) = (\psi(\hat{\hat{t}}) + (1/k))/2$ is also a $\mathscr{D}$-function for each $\hat{\hat{t}} \in \mathbb{R}_+$ and some $k \in (1,\infty)$.

Proof. Obviously, $\psi(\hat{t}) < \rho(\hat{t})$ and $0 < \rho(\hat{t}) < (1/k)$. Let $\hat{t} \in \mathbb{R}_+$ be fixed. Since $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ is a $\mathscr{D}$-function, we can find $\sigma_{\hat{t}} \in [0, (1/k))$ and $\delta_{\hat{t}} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}$ for all $s \in [\hat{t}, \hat{t} + \delta_{\hat{t}})$. Assume that $\eta_{\hat{t}} \coloneqq (\sigma_{\hat{t}} + (1/k))/2 \in [0, (1/k))$. Then, $\rho(s) \leq \eta_{\hat{t}}$ for all $s \in [\hat{t}, \hat{t} + \delta_{\hat{t}})$. Thus, ρ is a $\mathscr{D}$-function. □

The following result due to Nadler [26] is the first metric fixed point theorem for multivalued contractions.

Theorem 10. (see [10]). *Let $(\mho, \mu)$ be a complete metric space and $Y : \mho \longrightarrow CB(\mho)$ be a multivalued λ-contraction, that is, we can find $\lambda \in (0, 1)$ such that*

$$\tilde{H}(Yj, Y\ell) \leq \lambda\mu(j, \ell), \tag{6}$$

for all $j, \ell \in \mho$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

In 2007, Berinde-Berinde [11] presented the following notable fixed point Theorem.

Theorem 11. (see [11]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Assume that we can find $L \geq 0$ such that*

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell) + L\mu(\ell, Yj), \tag{7}$$

for all $j, \ell \in \mho$ with $j \neq \ell$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

Observe that if we take $L = 0$ in Theorem 11, we realize the Mizoguchi-Takahashi fixed point theorem [14] which partially answered the problem posed in Reich [8].

Theorem 12. (see [8]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow \mathscr{K}(\mho)$ be a multivalued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Suppose that*

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell), \tag{8}$$

for all $j, \ell \in \mho$ with $j \neq \ell$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

In [8], Reich raised the question whether Theorem 12 is also valid when $\mathscr{K}(\mho)$ is replaced with $CB(\mho)$. In 1989, Mizoguch-Takahashi [14] responded to this puzzle in affirmative via the following result.

Theorem 13. (see [14]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Suppose that*

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell), \tag{9}$$

for all $j, \ell \in \mho$. Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

Let A be a nonempty subset of $\mho$ and $Y : \mho \longrightarrow \mho$ be a mapping. We recall that the set A is Y-invariant if $Y(A) \subseteq A$. Not long ago, Du [13] obtained the following important fixed point and coincidence point result.

Theorem 14. (see [13]). *Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $g : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi_{\widetilde{\mathscr{MT}}} : \mathbb{R}_+ \longrightarrow [0, 1)$ be an $\widetilde{\mathscr{MT}}$-function. Assume that the following conditions hold:*

(Du_1) Yj is g-invariant for each $j \in \mho$;

(Du_2) we can find a function $h : \mho \longrightarrow \mathbb{R}_+$ such that

$$\tilde{H}(Yj, Y\ell) \leq \psi_{\widetilde{\mathscr{MT}}}(\mu(j, \ell))\mu(j, \ell) + h(g\ell)\mu(g\ell, Yj), \quad (10)$$

for all $j, \ell \in \mho$. Then, $\mathscr{COP}(g, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Notice that Mizoguchi-Takahashi fixed point theorem (13) is an extension of Nadler's fixed point theorem (10), but its original proof is not friendly. Alternative proof presented in [29] is also difficult.

Definition 15. (see [9]). Let $(\mho, \mu)$ be a metric space. A single-valued mapping $Y : \mho \longrightarrow \mho$ is named:

Rus contraction if we can find $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that for all $j, \ell \in \mho$,

$$\mu(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(\ell, Y\ell). \quad (11)$$

Ciric-Reich-Rus contraction if we can find $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that for all $j, \ell \in \mho$,

$$\mu(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell). \quad (12)$$

In [9], it was proved that every Rus and Ciric-Reich-Rus contraction has a unique fixed point. These results have been extended to multivalued mappings in the following manner.

Theorem 16. (see [27]). *Let $(\mho, \mu)$ be a complete metric space and $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping. Assume that we can find $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that for all $j, \ell \in \mho$:*

$$\tilde{H}(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(\ell, Y\ell). \quad (13)$$

Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

Theorem 17. (see [17]). *Let $(\mho, \mu)$ be a complete metric space and $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping. Assume that we can find $a, b \in \mathbb{R}_+$ with $a + b + c < 1$ such that for all $j, \ell \in \mho$:*

$$\tilde{H}(Yj, Y\ell) \leq a\mu(j, \ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell). \quad (14)$$

Then, $\mathscr{F}_{ix}(Y) \neq \varnothing$.

For more variants of fixed point results of multivalued contractions, the interested reader may consult [30–33] and the references therein.

2. Main Results

In line with the characterizations of $\widetilde{\mathscr{MT}}$-function, we begin this section by launching a few characterizations of $\mathscr{D}$-function in Lemma 18. Its proof is a slight adaption of [17, Theorem 2.1].

Lemma 18.

Let $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k)), k \in (1,\infty)$. Then, the following statements are equivalent:

(i) ψ is a $\mathscr{D}$-function

(ii) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(1)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(1)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(1)}$ for all $s \in (\hat{t}, \hat{t} + \delta_{\hat{t}}^{(1)})$

(iii) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(2)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(2)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(2)}$ for all $s \in [\hat{t}, \hat{t} + \delta_{\hat{t}}^{(2)}]$

(iv) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(3)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(3)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(3)}$ for all $s \in (\hat{t}, \hat{t} + \delta_{\hat{t}}^{(3)}]$

(v) For each $\hat{t} \in \mathbb{R}_+$, we can find $\sigma_{\hat{t}}^{(4)} \in [0, (1/k))$ and $\delta_{\hat{t}}^{(4)} > 0$ such that $\psi(s) \leq \sigma_{\hat{t}}^{(4)}$ for all $s \in [\hat{t}, \hat{t} + \delta_{\hat{t}}^{(4)}]$

(vi) For any sequence $\{j_n\}_{n\geq 1}$ in $\mathbb{R}_+$, from and after some fixed term, it is nonincreasing and $0 \leq \sup_{n\in\mathbb{N}} \psi(j_n) < (1/k)$

(vii) ψ is a function of $1/k$-contractive factor, that is, for any sequence $\{j_n\}_{n\geq 1}$ in $\mathbb{R}_+$, from and after some fixed term, it is strictly decreasing and $0 \leq \sup_{n\in\mathbb{N}}\psi(j_n) < (1/k)$

The following existence theorem for coincidence point and fixed point is one of the main results of this paper.

Theorem 19.

Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta, \Theta, \Lambda : \mho \longrightarrow \mho$ be continuous point-valued mappings, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that the following conditions are obeyed:

(ax_1) for each $j \in \mho$, $\{\Delta\ell = \Theta\ell = \Lambda\ell : \ell \in Yj\} \subseteq Yj$;

(ax_2) we can find three mappings $f, g, h : \mho \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}\tilde{H}(Yj, Y\ell) \leq{}& \psi(\mu(j, \ell))[a\mu(j, \ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\ &+ f(\Delta\ell)\mu(\Delta\ell, Yj) + g(\Theta\ell)\mu(\Theta\ell, Yj) \\ &+ h(\Lambda\ell)\mu(\Lambda\ell, Yj),\end{aligned} \quad (15)$$

for all $j, \ell \in \mho$, where $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Proof. By (ax_1), we note that for each $j \in \mho$, $\mu(\Delta\ell, Yj) = \mu(\Theta\ell, Yj) = \mu(\Lambda\ell, Yj) = 0$ for all $\ell \in Yj$. So for each $j \in \mho$, it follows from (ax_2) that for all $\ell \in Yj$,

$$\tilde{H}(Yj, Y\ell) \le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)]. \tag{16}$$

□

Further, for each $\ell \in Yj$, $\mu(\ell, Y\ell) \le \tilde{H}(Yj, Y\ell)$. Whence, for each $j \in \mho$, (16) gives

$$\begin{aligned}\mu(\ell, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)]\\ &\le \frac{\psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj)]}{1 - c\psi(\mu(j,\ell))}\\ &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj)].\end{aligned} \tag{17}$$

Let $j_0 \in \mho$ and choose $j_1 \in Yj_0$. If $\mu(j_0, j_1) = 0$, then $j_0 = j_1 \in Yj_0$, that is, $j_0 \in \mathscr{F}_{ix}(Y)$, and the proof is finished. Otherwise, if $\mu(j_0, j_1) > 0$, then consider a function $\rho : \mathbb{R}_+ \longrightarrow [0, (1/k))$ given by $\rho(t) = ((1/k) + \psi(t))/2$. By Lemma 9, ρ is a $\mathscr{D}$-function and $0 \le \psi(t) < \rho(t) < (1/k)$ for all $t \in \mathbb{R}_+$. From (2.2), it follows that

$$\begin{aligned}\mu(j_1, Yj_1) &\le \psi(\mu(j_0, j_1))[a\mu(j_0, j_1) + b\mu(j_0, Yj_0)]\\ &< \rho(\mu(j_0, j_1))[a\mu(j_0, j_1) + b\mu(j_0, j_1)]\\ &= \rho(\mu(j_0, j_1))[(a+b)\mu(j_0, j_1)].\end{aligned} \tag{18}$$

Since $a + b + c < 1$, then we can find $\eta \in (0,1)$ such that $a + b < \eta = 1 - c < 1$. Thus, (18) can be written as

$$\mu(j_1, Yj_1) < \eta\rho(\mu(j_0, j_1))\mu(j_0, j_1) < \rho(\mu(j_0, j_1))\mu(j_0, j_1). \tag{19}$$

From (19), we claim that we can find $j_2 \in Yj_1$ such that

$$\mu(j_1, j_2) < \rho(\mu(j_0, j_1))\mu(j_0, j_1). \tag{20}$$

Assume that this claim is not true, that is, $\mu(j_1, j_2) \ge \rho(\mu(j_0, j_1))\mu(j_0, j_1)$. Then, we get

$$\mu(j_1, j_2) \ge \inf_{\gamma \in Yj_1} \mu(j_1, \gamma) \ge \rho(\mu(j_0, j_1))\mu(j_0, j_1), \tag{21}$$

that is, $\mu(j_1, Yj_1) \ge \rho(\mu(j_0, j_1))\mu(j_0, j_1)$, contradicting (19). Now, if $\mu(j_1, j_2) = 0$, then $j_1 = j_2 \in Yj_1$ and so $j_1 \in \mathscr{F}_{ix}(Y)$. Otherwise, we can find $j_3 \in Yj_2$ such that

$$\mu(j_2, j_3) < \rho(\mu(j_1, j_2))\mu(j_1, j_2). \tag{22}$$

Let $\tau_n = \mu(j_{n-1}, j_n)$ for each $n \in \mathbb{N}$. Proceeding on similar steps as above, we can construct a sequence $\{j_n\}_{n\in\mathbb{N}}$ in $\mho$ with $j_n \in Yj_{n-1}$ for each $n \in \mathbb{N}$ and

$$\tau_{n+1} < \rho(\tau_n)\tau_n. \tag{23}$$

Given that ψ is a $\mathscr{D}$-function, then by Lemma 18:

$$0 \le \sup_{n\in\mathbb{N}} \psi(\tau_n) < \sup_{n\in\mathbb{N}} \rho(\tau_n) < \frac{1}{k}. \tag{24}$$

Whence,

$$0 < \sup_{n\in\mathbb{N}} \rho(\tau_n) = \left\{\frac{(1/k) + \psi(\tau_n)}{2} : n \in \mathbb{N}, k \in (1,\infty)\right\} < \frac{1}{k} < 1. \tag{25}$$

Take $\xi \coloneqq \sup_{n\in\mathbb{N}} \rho(\tau_n)$, then $0 < \xi < 1$. Since $\rho(t) < (1/k) < 1$ for all $t \in \mathbb{R}_+$, then by (23), $\{\tau_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence of positive real numbers. Therefore, for each $n \in \mathbb{N}$, we have

$$\tau_{n+1} < \rho(\tau_n) \le \xi\tau_n. \tag{26}$$

Whence, it follows from (26) that

$$\mu(j_n, j_{n+1}) = \tau_{n+1} \le \xi\tau_n \le \cdots \le \xi^n\tau_1 = \xi^n d(j_0, j_1). \tag{27}$$

For any $m, n, n_0 \in \mathbb{N}$ with $m > n > n_0$, by (27), we get

$$\begin{aligned}\mu(j_m, j_n) &\le \sum_{j=n}^{m-1} \mu\left(j_j, j_{j+1}\right) \le \sum_{j=n}^{m-1} \xi^j \mu(j_0, j_1) \le \sum_{j=n}^{\infty} \xi^j \mu(j_0, j_1)\\ &\le \frac{\xi^n}{1-\xi}\mu(j_0, j_1) \longrightarrow 0 (\text{as} n \longrightarrow \infty).\end{aligned} \tag{28}$$

Thus, $\limsup_{n\longrightarrow\infty}\{\mu(j_m, j_n): m > n\} = 0$. This proves that $\{j_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mho$. The completeness of $\mho$ implies that we can find $u \in \mho$ such that $j_n \longrightarrow u$ as $n \longrightarrow \infty$. Since $j_n \in Yj_{n-1}$ for each $n \in \mathbb{N}$, it follows from condition (ax_1) that for each $n \in \mathbb{N}$,

$$\Delta j_n = \Theta j_n = \Lambda j_n \in Yj_{n-1}. \tag{29}$$

Using the continuity of the functions Δ, Θ and Λ, we have

$$\begin{aligned}u &= \lim_{n\longrightarrow\infty} \Delta j_n = \lim_{n\longrightarrow\infty} \Theta j_n = \lim_{n\longrightarrow\infty} \Lambda j_n = \lim_{n\longrightarrow\infty} \Delta u\\ &= \lim_{n\longrightarrow\infty} \Theta u = \lim_{n\longrightarrow\infty} \Lambda u.\end{aligned} \tag{30}$$

We claim that $u \in Yu$. Assume contrary so that $\mu(u, Yu) > 0$. Since the function $j \mapsto \mu(j, Yu)$ is continuous, then from condition (ax_2), we realize

$$\begin{aligned}\mu(u, Yu) &= \lim_{n\longrightarrow\infty} \mu(j_n, Yu) \le \lim_{n\longrightarrow\infty} \tilde{H}(Yj_{n-1}, Yu)\\ &\le \lim_{n\longrightarrow\infty} \{\psi(\mu(j_{n-1}, u))[a\mu(j_{n-1}, u) + b\mu(j_{n-1}, Yj_{n-1})\\ &\quad + c\mu(u, Yu)] + f(\Delta u)\mu(\Delta u, Yj_{n-1})\\ &\quad + g(\Theta u)\mu(\Theta u, Yj_{n-1}) + h(\Lambda u)\mu(\Lambda u, Yj_{n-1})\}\\ &< \lim_{n\longrightarrow\infty} \{\rho(\mu(j_{n-1}, u))[a\mu(j_{n-1}, u) + b\mu(j_{n-1}, j_n)\\ &\quad + c\mu(u, Yu)] + f(\Delta u)\mu(\Delta u, j_n)\\ &\quad + g(\Theta u)\mu(\Theta u, j_n) + h(\Lambda u)\mu(\Lambda u, j_n)\}\\ &< \frac{c}{k}(\mu(u, Yu)) < \mu(u, Yu),\end{aligned} \tag{31}$$

a contradiction. Whence, $\mu(u, Yu) = 0$. Since Yu is closed, we have $u \in Yu$. By condition (ax_1), $\Delta u = \Theta u = \Lambda u \in Yu$. Consequently, $u \in \mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y)$.

The following example shows the generality of our Theorem 19 over Theorems 10, 11, 17, and 16 due to Nadler, Berinde-Berinde, Reich, and Rus, respectively.

Example 20.

Let $\mho = \{0, (1/5), 2\}$ and $\mu(j, \ell) = |j - \ell|$ for all $j, \ell \in \mho$. Let $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping and $\Delta, \Theta, \Lambda : \mho \longrightarrow \mho$ be mappings given by

$$Yj = \begin{cases} \{0\}, & \text{if } j = 0 \\ \left\{0, \dfrac{1}{5}\right\}, & \text{if } j = \dfrac{1}{5} \\ \{0, 2\}, & \text{if } j = 2, \end{cases} \tag{32}$$

and $\Delta = \Theta = \Lambda = I_{\mho}$, the identity mapping on $\mho$. Define the function $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ by $\psi(t) = 1/k^2$ for all $t \in \mathbb{R}_+$ and some $k \in (1, \infty)$. Also, define the mappings $f, g, h : \mho \longrightarrow \mathbb{R}_+$ by $f(j) = g(j) = h(j) = 1/3$ for all $j \in \mho$. Then, we realize the following:

(i) for each $j \in \mho$, $\{\Delta\ell = \Theta\ell = \Lambda\ell : \ell \in Yj\} \subseteq Yj$;

(ii) $\mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) = \{0, (1/5), 2\}$;

(iii) Δ, Θ and Λ are continuous

Clearly, $\limsup_{s \longrightarrow t^+} \psi(s) = (1/k^2) < (1/k)$ for all $t \in \mathbb{R}_+$ and some $k \in (1, \infty)$. Whence, ψ is a $\mathscr{D}$-function. Furthermore, it is a routine to verify that condition (ax_2) holds for all $j, \ell \in \mho$.

Now, notice that the mapping Y does not obey the hypotheses of Theorem 10 due to Nadler. To see this, let $j = 0$ and $\ell = 2$, then

$$\tilde{H}(Y0, Y2) = \tilde{H}(\{0\}, \{0, 2\}) = 2 > \lambda\mu(0, 2), \tag{33}$$

for all $\lambda \in (0, 1)$. Moreover, to see that Theorem 11 due to Berinde-Berinde fails in this instance, let $L = 1/9$ and $\psi_{\widetilde{\mathscr{MT}}}(t) = k\psi(t)$ for all $t \in \mathbb{R}_+, k \in (1, \infty)$. Then, for all $\lambda \in (0, 1)$,

$$\tilde{H}(Y0, Y2) = 2 > \lambda\mu(0, 2) + \frac{1}{9}\mu(2, Y0). \tag{34}$$

Moreover, to see that Theorems 17 and 16 of Reich and Rus are also not applicable to this example, again take $j = 0$ and $\ell = 2$. Then, by setting $b = c = 0$ and $a = 0$ in Theorems 1.17 and 1.16, respectively, we have

$$\begin{aligned} &\tilde{H}(Y0, Y2) = 2 > a\mu(0, 2) \text{ for all } a \in (0, 1), \\ &\tilde{H}(Y0, Y2) = 2 > b\mu(2, Y2) \text{ for all } b \in (0, 1). \end{aligned} \tag{35}$$

A slight modification of Example A of Du [13] provided below shows the generality of our Theorem 19 over Mizoguch-Takahash's [14] and Du's [13] fixed point theorems.

Example 21.

Let l^{∞} be the Banach space of all bounded real sequences endowed with the uniform norm $\|.\|_{\infty}$, and let $\{e_n\}$ be the canonical basis of l^{∞}. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers obeying $\tau_1 = \tau_2$ and $\tau_{2n-1} < \tau_n$ for all $n \geq 2$ (for example, take $\tau_1 = 1/9$ and $\tau_n = 1/3^n, n \geq 2$). It follows that $\{\tau_n\}_{n\in\mathbb{N}}$ is convergent. Set $v_n = \tau_n e_n$ for all $n \in \mathbb{N}$, and let $\mho = \{v_n\}_{n\in\mathbb{N}}$ be a bounded and complete subset of l^{∞}. Then, $(\mho, \|.\|_{\infty})$ is a complete metric space and $\|v_n - v_m\|_{\infty} = \tau_n$ if $m > n$.

Let $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping and $\Delta, \Theta, \Lambda : \mho \longrightarrow \mho$ be three mappings, respectively, given by

$$\begin{aligned} &Yv_n = \begin{cases} \{v_1, v_2, v_3\}, & \text{if } n \in \{1, 2, 3\} \\ \{v_{n+1}\}, & \text{if } n > 3, \end{cases} \\ &\Delta v_n = \Theta v_n = \Lambda v_n = \begin{cases} v_2, & \text{if } n \in \{1, 2, 3\} \\ v_{n+1}, & \text{if } n > 3. \end{cases} \end{aligned} \tag{36}$$

Then, we notice that the following results hold:

$$\begin{aligned} &(ax_1) \text{for each } j \in \mho, \{\Delta\ell = \Theta\ell = \Lambda\ell \in Yj\} \subseteq Yj, \\ &(ax_1) \mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) = \{v_1, v_2, v_3\}. \end{aligned} \tag{37}$$

To show that Δ, Θ and Λ are continuous, it is suffices to prove that Δ, Θ and Λ are nonexpansive. So we consider the following six possibilities:

(i) $\|\Delta v_1 - \Delta v_2\|_{\infty} = 0 < \tau_1 = \|v_1 - v_2\|_{\infty}$

(ii) $\|\Delta v_1 - \Delta v_3\|_{\infty} = 0 < \tau_1 = \|v_1 - v_3\|_{\infty}$

(iii) $\|\Delta v_1 - \Delta v_m\|_{\infty} = \tau_2 = \tau_1 = \|v_1 - v_m\|_{\infty}$ for any $m > 3$

(iv) $\|\Delta v_2 - \Delta v_m\|_{\infty} = \tau_2 = \|v_2 - v_m\|_{\infty}$ for any $m > 3$

(v) $\|\Delta v_3 - \Delta v_m\|_{\infty} = \tau_2 = \|v_3 - v_m\|_{\infty}$ for any $m > 3$

(vi) $\|\Delta v_n - \Delta v_m\|_{\infty} = \tau_{n+1} < \tau_n = \|v_n - v_m\|_{\infty}$ for any $m > 3$ and $m > n$

Consequently, Δ is nonexpansive, and, since $\Delta = \Theta = \Lambda$, then Δ, Θ and Λ are continuous.

Next, define the function $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ by

$$\psi(t) = \begin{cases} \dfrac{\tau_{n+2}}{\tau_n}, & \text{if } t = \tau_n \text{ for some } n \in \mathbb{N} \\ 0, & \textit{elsewhere}. \end{cases} \tag{38}$$

Also, define the mappings $f, g, h : \mho \longrightarrow \mho$ by

$$f(\nu_n) = g(\nu_n) = h(\nu_n) = \begin{cases} 0, & \text{if } n \in \{1,2,3\} \\ \tau_1 n, & \text{if } n > 3. \end{cases} \tag{39}$$

Then, we observe that $\limsup_{s\longrightarrow t^+}\psi(s) = 0 < (1/k)$ for all $t \in \mathbb{R}_+$ and some $k \in (1,\infty)$. It follows that ψ is a $\mathscr{D}$-function. Moreover, we claim that

$$\begin{aligned}\tilde{H}_\infty(Yj, Y\ell) \le \psi(\|j-\ell\|_\infty)\left[a\|j-\ell\|_\infty + b\|j - Yj\|_\infty\right.\\ \left.+ c\|\ell - Y\ell\|_\infty\right] + f(\Delta\ell)\|\Delta\ell - Yj\|_\infty\\ + g(\Theta\ell)\|\Theta\ell - Yj\|_\infty + h(\Lambda\ell)\|\Lambda\ell - Yj\|_\infty,\end{aligned} \tag{40}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$, where $\tilde{H}_\infty$ is the Hausdorff metric induced by the norm $\|.\|_\infty$.

To see (40), we consider the following cases:

Case 1. For $n = 1, m = 2$ and $a = 1/2, b = c = 0$, we have

$$\begin{aligned}&\psi(\|\nu_1-\nu_2\|_\infty)(a\|\nu_1-\nu_2\|_\infty + b\|\nu_1 - Y\nu_1\|_\infty\\ &\quad + c\|\nu_2 - Y\nu_2\|_\infty) + f(\Delta\nu_2)\|\Delta\nu_2 - Y\nu_1\|_\infty\\ &\quad + g(\Theta\nu_2)\|\Theta\nu_2 - Y\nu_1\|_\infty + h(\Lambda\nu_2)\|\Lambda\nu_2 - Y\nu_1\|_\infty\\ &= \frac{\tau_3}{2} > 0 = \tilde{H}_\infty(Y\nu_1, Y\nu_2).\end{aligned} \tag{41}$$

Case 2. For $n = 1, m = 3$ and $a = 1/4, b = c = 0$, we have

$$\begin{aligned}&\psi(\|\nu_1-\nu_3\|_\infty)(a\|\nu_1-\nu_3\|_\infty + b\|\nu_1 - Y\nu_1\|_\infty + c\|\nu_3 - Y\nu_3\|_\infty)\\ &\quad + f(\Delta\nu_3)\|\Delta\nu_3 - Y\nu_1\|_\infty + g(\Theta\nu_3)\|\Theta\nu_3 - Y\nu_1\|_\infty\\ &\quad + h(\Lambda\nu_3)\|\Lambda\nu_3 - Y\nu_1\|_\infty\\ &= \frac{\tau_3}{4} > 0 = \tilde{H}_\infty(Y\nu_1, Y\nu_3).\end{aligned} \tag{42}$$

Case 3. For $n = 1, m > 3$ and $a = 1/2, b = c = 0$, we have

$$\begin{aligned}&\psi(\|\nu_1-\nu_m\|_\infty)(a\|\nu_1-\nu_m\|_\infty + b\|\nu_1 - Y\nu_1\|_\infty + c\|\nu_m - Y\nu_m\|_\infty)\\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_1\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_1\|_\infty\\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_1\|_\infty\\ &= \frac{\tau_3}{2}(1 + 6\tau_1(m+1)) > \tau_1 = \tilde{H}_\infty(Y\nu_1, Y\nu_m).\end{aligned} \tag{43}$$

Case 4. For $n = 2, m > 3$ and $a = 1/4, b = c = 0$, we have

$$\begin{aligned}&\psi(\|\nu_2-\nu_m\|_\infty)(a\|\nu_2-\nu_m\|_\infty + b\|\nu_2 - Y\nu_2\|_\infty + c\|\nu_m - Y\nu_m\|_\infty)\\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_2\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_2\|_\infty\\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_2\|_\infty\\ &= \frac{\tau_4}{4}\left(1 + \frac{12\tau_1}{\tau_4}(m+1)\tau_3\right) > \tau_1 = \tilde{H}_\infty(Y\nu_2, Y\nu_m).\end{aligned} \tag{44}$$

Case 5. For $n = 3, m > 3$ and $a = 1/3 = b, c = 0$, we have

$$\begin{aligned}&\psi(\|\nu_3-\nu_m\|_\infty)(a\|\nu_3-\nu_m\|_\infty + b\|\nu_3 - Y\nu_3\|_\infty + c\|\nu_m - Y\nu_m\|_\infty)\\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_3\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_3\|_\infty\\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_3\|_\infty\\ &= \frac{\tau_5}{3}(1 + 9\tau_1(m+1)\tau_3) > \tau_1 = \tilde{H}_\infty(Y\nu_3, Y\nu_m).\end{aligned} \tag{45}$$

Case 6. For $n > 3, m > n$ and $a = 1/2, b = c = 0$, we have

$$\begin{aligned}&\psi(\|\nu_n-\nu_m\|_\infty)(a\|\nu_n-\nu_m\|_\infty + b\|\nu_n - Y\nu_n\|_\infty + c\|\nu_m - Y\nu_m\|_\infty)\\ &\quad + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_n\|_\infty + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_n\|_\infty\\ &\quad + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_n\|_\infty\\ &= \frac{\tau_{n+2}}{2} + 3(m+1)\tau_{n+1} > \tau_{n+1} = \tilde{H}_\infty(Y\nu_n, Y\nu_m).\end{aligned} \tag{46}$$

Therefore, from Cases (1)–(6), we have shown that Condition (40) is obeyed. Consequently, all the assertions of Theorem 19 are obeyed. It follows that $\mathscr{COP}(\Delta, \Theta, \Lambda, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Now, observe that if we take the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ as earlier given, that is, $\tau_1 = \tau_2, \tau_{2n-1} < \tau_n$, where $\tau_n = 1/3^n$ for all $n \ge 2$ and let $\psi_{\widetilde{\mathscr{MT}}}(t) = 2\psi(t)(i.e.k = 2 \in (1,\infty))$ for all $t \in \mathbb{R}_+$, then $\psi_{\widetilde{\mathscr{MT}}}$ is an $\widetilde{\mathscr{MT}}$-function, provided ψ is a $\mathscr{D}$-function. Thus,

(a) for $n = 1$ and any $m > 3$, we have

$$\begin{aligned}\tilde{H}_\infty(Y\nu_1, Y\nu_m) &= \tau_1 > 2\tau_3\\ &= \psi_{\widetilde{\mathscr{MT}}}(\|\nu_1 - \nu_m\|_\infty)\|\nu_1 - \nu_m\|_\infty.\end{aligned} \tag{47}$$

Whence, Mizoguch-Takahashi's Theorem 13 does not hold in this case.

(b) Let the function $f : \mho \longrightarrow \mho$ be given by

$$f(\nu_n) = \begin{cases} 0, & \text{if } n \in \{1,2,3\} \\ \dfrac{\tau_1}{k\tau_n}, & \text{if } n > 3, k \in (1,\infty), \end{cases} \tag{48}$$

and g and h be as given in the above Example. Then, for $n = 1$ and $m > 3$ with $a = 1/2$, $b = c = 0$, the above Case 3 becomes

Case $3'$:

$$\begin{aligned}
&\psi_{\widetilde{\mathscr{MT}}}\left(\|\nu_1 - \nu_m\|_\infty\right)\left(a\|\nu_1 - \nu_m\|_\infty\right) + f(\Delta\nu_m)\|\Delta\nu_m - Y\nu_1\|_\infty \\
&\quad + g(\Theta\nu_m)\|\Theta\nu_m - Y\nu_1\|_\infty + h(\Lambda\nu_m)\|\Lambda\nu_m - Y\nu_1\|_\infty \\
&= \tau_3 + \frac{\tau_1}{k\tau_{m+1}} + 2\tau_1(m+1)\tau_3 > \tau_1 = \tilde{H}_\infty(Y\nu_1, Y\nu_m),
\end{aligned} \tag{49}$$

that is, Case 3 also hold. On the other hand, notice that

$$\begin{aligned}
\tilde{H}_\infty(Y\nu_1, Y\nu_m) &= \tau_1 > \tau_3 + \frac{\tau_1}{k\tau_{m+1}} \\
&= \psi_{\widetilde{\mathscr{MT}}}\left(\|\nu_1 - \nu_m\|_\infty\right)\|\nu_1 - \nu_m\|_\infty \\
&\quad + f(\Delta\nu_m)\|\nu_1 - \nu_m\|_\infty,
\end{aligned} \tag{50}$$

that is, the main result of Du [17, Theorem 19] is not applicable here.

3. Consequences

In this section, we deduce some significant consequences of Theorem 19.

Corollary 2.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that

(i) Yj is Δ-invariant (i.e. $\Delta(Yj) \subseteq Yj$) for each $j \in \mho$

(ii) we can find a mapping $f : \mho \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}
\tilde{H}(Yj, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + f(\Delta\ell)\mu(\Delta\ell, Yj),
\end{aligned} \tag{51}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Proof. Take $g, h : \mho \longrightarrow \mathbb{R}_+$ as $g(j) = h(j) = 0$ for all $j \in \mho$ in Theorem 19. □

The following result is a direct consequence of Corollary 2.

Corollary 23.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that

(i) Yj is Δ-invariant (i.e., $\Delta(Yj) \subseteq Yj$) for each $j \in \mho$

(ii) we can find $\xi \ge 0$ and a mapping $\widehat{f} : \mho \longrightarrow [0, \xi]$ such that

$$\begin{aligned}
\tilde{H}(Yj, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + \widehat{f}(\Delta\ell)\mu(\Delta\ell, Yj),
\end{aligned} \tag{52}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Corollary 24.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, $\Delta : \mho \longrightarrow \mho$ be a continuous point-valued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that

(i) Yj is Δ-invariant (i.e. $\Delta(Yj) \subseteq Yj$) for each $j \in \mho$

(ii) we can find $\xi \ge 0$ such that

$$\begin{aligned}
\tilde{H}(Yj, Yy) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + \xi\mu(\Delta\ell, Yj),
\end{aligned} \tag{53}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathscr{COP}(\Delta, Y) \cap \mathscr{F}_{ix}(Y) \neq \varnothing$.

Proof. Define $\widehat{f} : \mho \longrightarrow [0, \xi]$ as $\widehat{f}(j) = \xi$ for all $j \in \mho$ in Corollary 23. □

By applying Corollary 2, we deduce a generalized version of the primitive Ciric-Reich-Rus fixed point theorem for multivalued mapping as follows.

Corollary 25.
Let $(\mho, \mu)$ be a complete metric space, $Y : \mho \longrightarrow CB(\mho)$ be a multivalued mapping, and $\psi : \mathbb{R}_+ \longrightarrow [0, (1/k))$ be a $\mathscr{D}$-function. Suppose that we can find a mapping $f : \mho \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}
\tilde{H}(Yj, Y\ell) &\le \psi(\mu(j,\ell))[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)] \\
&\quad + f(\ell)\mu(\ell, Yj),
\end{aligned} \tag{54}$$

for all $j, \ell \in \mho$ and $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$.

Then, $\mathcal{F}_{ix}(Y) \neq \varnothing$.

Proof. Take $\Delta \coloneqq I_{\mho}$, the identity mapping on $\mho$ in Corollary 2. □

Remark 26.

(i) If we take $\psi_{\widetilde{\mathcal{MT}}}(t) = ak\psi(t)$, where $a \in (0,1)$, $k \in (1, \infty)$, ψ is a $\mathcal{D}$-function, and set $b = c = 0$, then Corollary 25 reduces to Theorem 13 due to Mizoguchi-Takahashi [14].

(ii) If ψ is a monotonic increasing function such that $0 \leq \psi(t) < (1/k)$ for each $t \in \mathbb{R}_+$ and $k \in (1,\infty)$, then by setting $\psi_{\widetilde{\mathcal{MT}}}(t) = ak\psi(t)$, where $a \in (0,1)$, $k \in (1,\infty)$ and $b = c = 0$, Corollary 24 generalizes [14, Corollary 2.2]. Also, Corollary 24 includes Theorem 1.2 in [29] as a special case, by extending the range of Y from the family of bounded proximal subsets of $\mho$ to $CB(\mho)$.

(iii) If we take $f(j) = 0$ and $\psi(t) = a\mu(j,\ell)/k^2[a\mu(j,\ell) + b\mu(j, Yj) + c\mu(\ell, Y\ell)]$ for all $j, \ell \in \mho$ and $k \in (1,\infty)$, where not all of a, b and c are identically zeros, then Corollary 25 reduces to Theorem 1.10

(iv) If we put $\psi_{\widetilde{\mathcal{MT}}}(t) = ak\psi(t)$, where $a \in (0,1)$, $k \in (1,\infty)$, ψ is a $\mathcal{D}$-function, take $\Delta \coloneqq I_{\mho}$, the identity mapping on $\mho$, and set $b = c = 0$, then Corollary 24 reduces to Theorem 11 due to Berinde-Berinde [11].

(v) If we define the multivalued mapping $Y : \mho \longrightarrow CB(\mho)$ as $Yj = \{\phi j\}$ for all $j \in \mho$, where ϕ is a single-valued mapping on $\mho$, then all the results presented herein can be reduced to their single-valued counterparts

(vi) It is clear that more consequences of our main result can be deduced, but we skip them due to the length of the paper

4. Applications to Caputo-Type Fractional Differential Inclusions Model for COVID-19

Very recently, Ahmed et al. [22] investigated the significance of lockdown in curbing the spread of COVID-19 via the following fractional-order epidemic model:

$$\begin{cases} {}^{C}D^{\nu}_{0^+}\tilde{G}(\hat{t}) = \Lambda^{\nu} - \beta^{\nu}\tilde{G}I - \lambda_1\tilde{G}L - \bar{\mu}^{\nu}\tilde{G} + \gamma_1^{\nu}I + \gamma_2^{\nu}I_L + \theta_1^{\nu}\tilde{G}_L, \\ {}^{C}D^{\nu}_{0^+}\tilde{G}_L(\hat{t}) = \lambda_1^{\nu}\tilde{G}L - \bar{\mu}^{\nu}\tilde{G}_L - \theta_1^{\nu}\tilde{G}_L, \\ {}^{C}D^{\nu}_{0^+}I(\hat{t}) = \beta^{\nu}\tilde{G}I - \gamma_1^{\nu} - \alpha_1^{\nu} - \bar{\mu}^{\nu}I + \lambda_2^{\nu}IL + \theta_2^{\nu}I_L, \\ {}^{C}D^{\nu}_{0^+}I_L(\hat{t}) = \lambda_2^{\nu}IL - \bar{\mu}^{\nu}I_L - \theta_2^{\nu} - \gamma_2^{\nu} - \alpha_2^{\nu}I_L, \\ {}^{C}D^{\nu}_{0^+}L(\hat{t}) = \mu^{\nu}I - \phi^{\nu}L, \end{cases} \tag{55}$$

where the total population under study, $N(\hat{t})$ is divided into four components, namely susceptible population that are not under lockdown $\tilde{G}(\hat{t})$, susceptible population that are under lock-down $\tilde{G}_L(\hat{t})$, infective population that are not under lockdown $I(\hat{t})$, infective population that are under lock-down $I_L(\hat{t})$, and cumulative density of the lockdown program $L(\hat{t})$. For the meaning of the rest parameters and numerical simulations of (55), we refer the reader to [22]. The above model (55) is simplified as follows:

$$\begin{cases} {}^{C}D^{\nu}_{0^+}\tilde{G}(\hat{t}) = \Theta_1\left(\hat{t}, \tilde{G}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D^{\nu}_{0^+}\tilde{G}_L(\hat{t}) = \Theta_2\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D^{\nu}_{0^+}I(\hat{t}) = \Theta_3\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D^{\nu}_{0^+}I_L(\hat{t}) = \Theta_4\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \\ {}^{C}D^{\nu}_{0^+}L(\hat{t}) = \Theta_5\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right), \end{cases} \tag{56}$$

where

$$\begin{cases} \Theta_1\left(\hat{t}, \tilde{G}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \Lambda^{\nu} - \beta^{\nu}\tilde{G}I - \lambda_1\tilde{G}L - \bar{\mu}^{\nu}\tilde{G} + \gamma_1^{\nu}I + \gamma_2^{\nu}I_L + \theta_1^{\nu}\tilde{G}_L, \\ \Theta_2\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \lambda_1^{\nu}\tilde{G}L - \bar{\mu}^{\nu}\tilde{G}_L - \theta_1^{\nu}\tilde{G}_L, \\ \Theta_3\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \beta^{\nu}\tilde{G}I - \gamma_1^{\nu} - \alpha_1^{\nu} - \bar{\mu}^{\nu}I + \lambda_2^{\nu}IL + \theta_2^{\nu}I_L, \\ \Theta_4\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \lambda_2^{\nu}IL - \bar{\mu}^{\nu}I_L - \theta_2^{\nu} - \gamma_2^{\nu} - \alpha_2^{\nu}I_L, \\ \Theta_5\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right) = \mu^{\nu}I - \phi^{\nu}L. \end{cases} \tag{57}$$

Consequently, the model (55) takes the form:

$$\begin{cases} {}^{C}D_0^{\nu} j(\hat{t}) = g(\hat{t}, j(\hat{t})), \hat{t} \in \Omega = [0.b], 0 < \nu < 1 \\ j(0) = j_0 \ge 0, \end{cases} \tag{58}$$

with the condition:

$$\begin{cases} j(\hat{t}) = \left(\tilde{G}, \tilde{G}_L, I, I_L, L\right)^{tr}, \\ j(0) = \left(\tilde{G}_0, \tilde{G}_{L_0}, I_0, I_{L_0}, L_0\right)^{tr}, \\ g(\hat{t}, j(\hat{t})) = \left(\Theta_i\left(\hat{t}, \tilde{G}, \tilde{G}_L, I, I_L, L\right)\right)^{tr}, i = 1, \cdots, 5, \end{cases} \tag{59}$$

where $(.)^{tr}$ denotes the transpose operation.

In this section, we extend problem (55) to its multivalued analogue given by

$$\begin{cases} {}^{C}D_0^{\nu} j(\hat{t}) \in M(\hat{t}, j(\hat{t})), \hat{t} \in \Omega = (0, \delta) \\ j(0) = j_0 \ge 0, \end{cases} \tag{60}$$

where $M : \Omega \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is a multivalued mapping ($P(\mathbb{R})$ is the power set of $\mathbb{R}$). We launch existence criteria for solutions of the inclusion problem (60) for which the right hand side is nonconvex with the aid of standard fixed point theorem for multivalued contraction mapping. First, we outline some preliminary concepts of fractional calculus and multivalued analysis as follows.

Definition 27. (see [34]). Let $\mathrm{v} > 0$ and $\mathrm{f} \in \mathrm{L}'([0, \delta], \mathbb{R})$. Then, the Riemann-Liouville fractional integral order v for a function f is given as

$$I_{0^+}^{\nu} f(\hat{t}) = \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \mu\tau, \hat{t} > 0, \tag{61}$$

where $\Gamma(.)$ is the gamma function given by $\Gamma(\nu) = \int_0^{\infty} \tau^{\nu-1} e^{-\tau} \mu\tau$.

Definition 28. (see [34]). Let $\mathrm{n} - 1 < \mathrm{v} < \mathrm{n}, \mathrm{n} \in \mathbb{N}$, and $\mathrm{f} \in \mathrm{C}^{\mathrm{n}}(0, \delta)$. Then, the Caputo fractional derivative of order v for a function f is given as

$${}^{C}D_{0^+}^{\nu} f(\hat{t}) = \frac{1}{\Gamma(n - \nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{n-\nu-1} f^n(\tau) \mu\tau, \hat{t} > 0. \tag{62}$$

Lemma 29. (see [34]). *Let $\mathfrak{R}(\nu) > 0, n = [\mathfrak{R}(\nu)] + 1$, and $f \in AC^n(0, \delta)$. Then,*

$$\left(I_{0^+}^{\nu}\, {}^{C}D_{0^+}^{\nu} f\right)(\hat{t}) = f(\hat{t}) - \frac{\sum_{k=1}^{m} \left(D_{0^+}^{k} f\right)(0^+)}{k!}. \tag{63}$$

In particular, if $0 < \nu \le 1$, then $(I_{0^+}^{\nu}\, {}^{C}D_{0^+}^{\nu} f)(\hat{t}) = f(\hat{t}) - f(0)$.

In view of Lemma 29, the integral reformulation of problem 16 which is equivalent to the model 13 is given by

$$j(\hat{t}) = j_0 + I_{0^+}^{\nu} g(\hat{t}, j(\hat{t})) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} g(\tau, j(\tau)) \mu\tau. \tag{64}$$

Let $\mho = C(\Omega, \mathbb{R})$ denotes the Banach space of all continuous functions j from Ω to $\mathbb{R}$ equipped with the norm given by

$$\|j\| = \sup \left\{ |j(\hat{t})| : \hat{t} \in \Omega = [0, \delta] \right\}, \tag{65}$$

where

$$|j(\hat{t})| = |\tilde{G}(\hat{t})| + |\tilde{G}_L(\hat{t})| + |I(\hat{t})| + |I_L(\hat{t})| + |L(\hat{t})|, \tag{66}$$

and $\tilde{G}, \tilde{G}_L, I, I_L, L \in \mho$.

Definition 30.

Let $\mho$ be a nonempty set. A single-valued mapping $\mathrm{f} : \mho \longrightarrow \mho$ is named a selection of a multivalued mapping $\mathrm{M} : \mho \longrightarrow \mathrm{P}(\mho)$, if $\mathrm{f}(\mathrm{j}) \in \mathrm{M}(\mathrm{j})$ for each $\mathrm{j} \in \mho$.

For each $j \in \mho$, we define the set of all selections of a multi-valued mapping M by

$$\tilde{G}_{M,j} = \left\{ f \in L'(\Omega, \mathbb{R}) : f(\hat{t}) \in M(\hat{t}, j(\hat{t})) \text{ for } a.e.\hat{t} \in \Omega \right\}. \tag{67}$$

Definition 31. A function $\mathrm{j} \in \mathrm{C}'(\Omega, \mathbb{R})$ is a solution of problem (60) if there is a function $\varphi \in \mathrm{L}'(\Omega, \mathbb{R})$ with $\varphi(\hat{\mathrm{t}}) \in \mathrm{M}(\hat{\mathrm{t}}, \mathrm{j}(\hat{\mathrm{t}}))$a.e. on Ω such that

$$j(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t} - \tau)^{\nu-1} \varphi(\tau) \mu\tau \tag{68}$$

and $j(0) = j_0 \ge 0$.

Definition 32. A multivalued mapping $\mathrm{M} : \Omega \longrightarrow \mathrm{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable, if for every $\varpi \in \mathbb{R}$, the function $\hat{\mathrm{t}} \mapsto \mu(\varpi, \mathrm{M}(\hat{\mathrm{t}})) = \inf \{|\varpi - \zeta| : \zeta \in \mathrm{M}(\hat{\mathrm{t}})\}$ is measurable.

The following is the main result of this section.

Theorem 33. *Assume that the following conditions are obeyed:*

(N_1) $M : \Omega \times \mathbb{R} \longrightarrow \mathscr{K}(\mathbb{R})$ is such that $M(., j) : \Omega \longrightarrow \mathscr{K}(\mathbb{R})$ is measurable for each $j \in \mathbb{R}$

(N_2) We can find a continuous function $h : \Omega \longrightarrow \mathbb{R}_+$ such that for all $j, \ell \in \mathbb{R}$,

$$\tilde{H}(M(\hat{t}, j), M(\hat{t}, \ell)) \le h(\hat{t}) |j - \ell|, \tag{69}$$

for almost all $\hat{t} \in \Omega$ and $\mu(0, M(\hat{t}, 0)) \le h(\hat{t})$ for almost all $\hat{t} \in \Omega$.

Then, the differential inclusion problem (60) has at least one solution on Ω, provided that $\Phi\|h\| < 1$, where $\Phi = b^{\nu}/(\Gamma(\nu+1))$.

Proof. First, we convert the differential inclusions (60) into a fixed point problem. For this, let $\mho = C(\Omega, \mathbb{R})$ and consider the multivalued mapping $Y : \mho \longrightarrow P(\mho)$ given by

$$Y(j) = \left\{ \begin{array}{c} \nabla \in \mho : \\ \nabla(\hat{t}) = j_0 + \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1} \varphi(\tau)\mu\tau, \varphi \in \tilde{G}_{M,j} \end{array} \right\}. \tag{70}$$

□

Clearly, the fixed points of Y are solutions of problem (60). Now, we prove that Y obeys all the conditions of Theorem 10 under the following cases.

Case 1. $Y(j)$ is nonempty and closed for every $\varphi \in \tilde{G}_{M,j}$. Since the multi-valued mapping $M(.,j(.))$ is measurable, by the measurable selection theorem (see, e.g. [35], Theorem III. 6), it admits a measurable selection $\varphi : \Omega \longrightarrow \mathbb{R}$. Furthermore, by condition (N_2), we get $|\varphi(\hat{t})| \leq h(\hat{t}) + h(\hat{t})|j(\hat{t})|$, that is, $\varphi \in L'(\Omega, \mathbb{R})$, and hence M is integrably bounded. Thus, $\tilde{G}_{M,j}$ is nonempty. Now, we show that $Y(j)$ is closed for each $j \in \mho$. Let $\{\varsigma_n\}_{n\in\mathbb{N}} \in Y(j)$ be such that $\varsigma_n \longrightarrow u$ $(n \longrightarrow \infty)$ in $\mho$. Then, $u \in \mho$, and we can find $\varphi_n \in \tilde{G}_{M,j_n}$ such that for each $\hat{t} \in \Omega$,

$$\varsigma_n(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1} \varphi_n(\tau)\mu\tau. \tag{71}$$

Since M has compact values, we pass onto a subsequence to obtain that φ_n converges to $u \in L'(\Omega, \mathbb{R})$. Therefore, $u \in \tilde{G}_{M,j}$ and for each $\hat{t} \in \Omega$, we have

$$\varsigma_n(\hat{t}) \longrightarrow u(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1} \varphi(\tau)\mu\tau. \tag{72}$$

Thus, $u \in Y(j)$.

Case 2. Next, we prove that we can find $a \in (0,1)(a = \Phi\|h\|)$ such that $\tilde{H}(Y(j), Y(\ell)) \leq a\|j-\ell\|$ for each $j, \ell \in \mho$. Let j, $\ell \in \mho$ and $\nabla_1 \in Y(j)$. Then, we can find $\varphi_1(\hat{t}) \in M(\hat{t}, j(\hat{t}))$ such that for each $\hat{t} \in \Omega$,

$$\nabla_1(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} + \int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1} \varphi_1(\tau)\mu\tau. \tag{73}$$

By (N_2), $\tilde{H}(M(\hat{t}, j), M(\hat{t}, \ell)) \leq h(\hat{t})\|j-\ell\|$. Whence, we can find $\rho \in M(\hat{t}, \ell(\hat{t}))$ such that

$$|\nabla_1(\hat{t}) - \rho(\hat{t})| \leq h(\hat{t})|j(\hat{t}) - \ell(\hat{t})|, \hat{t} \in \Omega. \tag{74}$$

Define $\Xi : \Omega \longrightarrow P(\mathbb{R})$ by

$$\Xi(\hat{t}) = \left\{ \hat{t} \in \mathbb{R} : |\nabla_1(\hat{t}) - \rho(\hat{t})| \leq h(\hat{t})|j(\hat{t}) - \ell(\hat{t})| \right\}. \tag{75}$$

Since the multivalued mapping $\Xi(\hat{t}) \cap M(\hat{t}, \ell(\hat{t}))$ is measurable (see ([35], Proposition III.4)), we can find a function φ_2 which is a measurable selection of Ξ. Thus, $\varphi_2(\hat{t}) \in M(\hat{t}, \ell(\hat{t}))$, and for each $\hat{t} \in \Omega$, we have $|\varphi_1(\hat{t}) - \varphi_2(\hat{t})| \leq h(\hat{t})|j(\hat{t}) - \ell(\hat{t})|$. For each $\hat{t} \in \Omega$, take

$$\nabla_2(\hat{t}) = j_0 + \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1} \varphi_2(\tau)\mu\tau. \tag{76}$$

Then, from (73) and (76), we realize

$$\begin{aligned} |\nabla_1(\hat{t}) - \nabla_2(\hat{t})| &\leq \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1} [|\varphi_1(\tau) - \varphi_2(\tau)|]\mu\tau \\ &\leq \frac{1}{\Gamma(\nu)} \int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1} \left[h(\hat{t})|j(\hat{t}) - \ell(\hat{t})|||\right]\mu\tau \\ &\leq \frac{b^{\nu}}{\Gamma(\nu+1)} \|h\|\|j-\ell\| = \Phi\|h\|\|j-\ell\|. \end{aligned} \tag{77}$$

Therefore, $\|\nabla_1 - \nabla_2\| \leq \Phi\|h\|\|j-\ell\|$. On similar steps, interchanging the roles of j and ℓ, we have

$$\tilde{H}(Y(j), Y(\ell)) \leq \Phi\|h\|\|j-\ell\| = a\|j-\ell\|. \tag{78}$$

Note that if we take $f(j) = 0$ and $\psi(\hat{t}) = (\Phi\|h\|\|j-\ell\|)/(k^2[\Phi\|h\|\|j-\ell\| + b\|j - Yj\| + c\|\ell - Y\ell\|])$ for all $j, \ell \in \mho$ and $k \in (1, \infty)$, then (54) coincides with (78). Whence, Corollary 25 can be applied to conclude that the mapping Y has at least one fixed point in $\mho$ which corresponds to the solutions of Problem 4.6.

Example 34. Consider the Caputo-type fractional differential inclusion problem given by

$$\begin{cases} {}^C D_0^{3/5} j(\hat{t}) \in M(\hat{t}, j(\hat{t})), \hat{t} \in \Omega = [0,1], \\ j(0) = 0, \end{cases} \tag{79}$$

where the multivalued mapping $M : [0,1] \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is given as

$$M(\hat{t}, j(\hat{t})) = \left[\frac{1}{50}, \frac{1}{9+10\hat{t}} \left(\frac{\sin^2 j(\hat{t})}{2 - \sin|j(\hat{t})|} \right) + \frac{1}{30} \right]. \tag{80}$$

Obviously, the mapping $j \mapsto [1/50, (1/9 + 10\hat{t})(\sin^2 j(\hat{t})/2 - \sin|j(\hat{t})|) + 1/30]$ is measurable for each $j \in \mathbb{R}$. In this case, we can take $h(\hat{t}) = 1/(9+10\hat{t})$ for all $\hat{t} \in [0,1]$, and thus, $\mu(0, M(\hat{t}, 0)) = 1/30 \le h(\hat{t})$ for almost all $\hat{t} \in [0,1]$. Note that for each $j, \ell \in \mathbb{R}$, we have

$$\begin{aligned}
&\tilde{H}\left(M\left(\hat{t}, j\left(\hat{t}\right)\right), M\left(\hat{t}, \ell\left(\hat{t}\right)\right)\right)\\
&\quad = \left(\left[\frac{1}{50}, \frac{1}{9+10\hat{t}}\left(\frac{\sin^2 j\left(\hat{t}\right)}{2-\sin\left|j\left(\hat{t}\right)\right|}\right) + \frac{1}{30}\right],\right.\\
&\quad\quad \left.\cdot\left[\frac{1}{50}, \frac{1}{9+10\hat{t}}\left(\frac{\sin^2 \ell\left(\hat{t}\right)}{2-\sin\left|\ell\left(\hat{t}\right)\right|}\right) + \frac{1}{30}\right]\right)\\
&\quad \le \frac{1}{9+10\hat{t}}\left|j\left(\hat{t}\right) - \ell\left(\hat{t}\right)\right| = h\left(\hat{t}\right)\left|j\left(\hat{t}\right) - \ell\left(\hat{t}\right)\right|.
\end{aligned} \tag{81}$$

Moreover, $\|h\| = 1/9$. Whence, $\Phi\|h\| \approx 0.124355 < 1$. Consequently, by Theorem 38, Problem (68) has at least one solution on $[0,1]$.

5. Stability Results

Investigated as a type of data dependence, the concept of Ulam stability was initiated by Ulam [36] and developed by Hyers [37], Rassias [38], and later on by many authors. In this section, we study an Ulam-Hyers type stability of the proposed fractional-order model 4.6. In [22], the stability result of the model 4.4 has been obtained in the framework of single-valued mappings. But, it is a known fact that multivalued mappings often have more fixed points than their corresponding single-valued mappings. Whence, the set of fixed points of set-valued mappings becomes more interesting for the study of stability. First, we give some needed definitions as follows.

Let $\varepsilon > 0$ and consider the following inequality:

$$\left|{}^{C}D_{0^+}^{\nu} j^*\left(\hat{t}\right) - j^*\left(\hat{t}\right)\right| \le \varepsilon, \hat{t} \in \Omega a.e. \tag{82}$$

Definition 35. The proposed problem (60) is Ulam-Hyers stable if we can find a real number $\varsigma^* > 0$ such that for every $\varepsilon > 0$ and for each solution $j^* \in C(\Omega, \mathbb{R})$ of the inequality (82), we can find a solution $j \in C(\Omega, \mathbb{R})$ of problem (60) and two functions $\varphi^*, \varphi \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$ and $\varphi(\hat{t}) \in M(\hat{t}, j(\hat{t}))$ a.e. on Ω such that

$$\left\|j^*\left(\hat{t}\right) - j\left(\hat{t}\right)\right\| \le \varsigma^* \varepsilon, \tag{83}$$

for almost all $\hat{t} \in \Omega$, where $\|j\| = \sup\{|j(\hat{t})| : \hat{t} \in \Omega a.e.\}$.

Remark 36. A function $j^* \in C(\Omega, \mathbb{R})$ is a solution of the inequality (82) if and only if we can find a continuous function $m : \Omega \longrightarrow \mathbb{R}$ and $\varphi^* \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$ a.e. on Ω such that the following properties hold:

(i) $|m(\hat{t})| \le \varepsilon$, $m = \max (m_j)^{tr}$, $\hat{t} \in \Omega a.e.$

(ii) ${}^{C}D_{0^+}^{\nu} j^*(\hat{t}) = j^*(\hat{t}) + m(\hat{t}), \hat{t} \in \Omega$ *a.e.*

Lemma 37. *Suppose that $j^* \in C(\Omega, \mathbb{R})$ obeys the inequality (82), then we can find a function $\varphi^* \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$a.e. on Ω such that*

$$\left|j^*\left(\hat{t}\right) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t}-\tau\right)^{\nu-1}\varphi^*(\tau)\mu\tau\right| \le \Phi\varepsilon. \tag{84}$$

Proof. From (ii) of Remark 36, we have ${}^{C}D_{0^+}^{\nu} j^*(\hat{t}) = j^*(\hat{t}) + m(\hat{t})$, and by Lemma 29, we get

$$\begin{aligned}
j^*\left(\hat{t}\right) = j_0^* &+ \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t}-\tau\right)^{\nu-1}\varphi^*(\tau)\mu\tau\\
&+ \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t}-\tau\right)^{\nu-1}m(\tau)\mu\tau.
\end{aligned} \tag{85}$$

Therefore, from (i) of Remark 36, we realize

$$\begin{aligned}
&\left|j^*\left(\hat{t}\right) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t}-\tau\right)^{\nu-1}\varphi^*(\tau)\mu\tau\right|\\
&\quad \le \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}}\left(\hat{t}-\tau\right)^{\nu-1}|m(\tau)|\mu\tau \le \Phi\varepsilon.
\end{aligned} \tag{86}$$

□

Now, we present the main result of this section as follows.

Theorem 38. *Assume that the following conditions are obeyed:*

(i) the multivalued mapping $M(.,j) : \Omega \longrightarrow \mathcal{K}(\mho)$ is measurable for each $j \in \mathbb{R}$

(ii) for all $j, \ell \in \mathbb{R}$, we can find $\varphi(\hat{t}) \in M(\hat{t}, j(\hat{t}))$, $\varphi^*(\hat{t}) \in M(\hat{t}, \ell(\hat{t}))$ a.e. on Ω and a continuous function $h : \Omega \longrightarrow \mathbb{R}_+$ such that for almost all $\hat{t} \in \Omega$,

$$\left|\varphi\left(\hat{t}\right) - \varphi^*\left(\hat{t}\right)\right| \le h\left(\hat{t}\right)\left|j\left(\hat{t}\right) - \ell\left(\hat{t}\right)\right|. \tag{87}$$

(iii) $\|h\| < 1/\Phi$, where $\Phi = b^\nu/(\Gamma(\nu+1))$.

Then the fractional-order inclusion model (60) is Ulam-Hyers stable.

Proof.

Let $j, j^* \in C(\Omega, \mathbb{R})$, where j obeys (82) and j is a solution of problem (60). Then, we can find two functions $\varphi^*, \varphi \in L'(\Omega, \mathbb{R})$ with $\varphi^*(\hat{t}) \in M(\hat{t}, j^*(\hat{t}))$ and $\varphi(\hat{t}) \in M(\hat{t}, j(\hat{t}))$*a.e.*

on Ω such that for every $\varepsilon > 0$, Lemma 37 can be applied to have

$$\begin{aligned}
\left|j^*(\hat{t}) - j(\hat{t})\right| &= \left|j^*(\hat{t}) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1}\varphi(\tau)\mu\tau\right| \\
&= \left|j^*(\hat{t}) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1}\right. \\
&\quad \left.\cdot [\varphi(\tau) - \varphi^*(\tau) + \varphi^*(\tau)]\mu\tau\right| \\
&\le \left|j^*(\hat{t}) - j_0^* - \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}} (\hat{t}-\tau)^{\nu-1}\varphi^*(\tau)\mu\tau\right| \\
&\quad + \frac{1}{\Gamma(\nu)}\int_0^{\hat{t}} (\nu-\tau)^{\nu-1}|\varphi(\tau) - \varphi^*(\tau)|\mu\tau \\
&\le \Phi\varepsilon + \frac{b^\nu}{\Gamma(\nu+1)}\|h\|\|j^* - j\| \\
&= \Phi\varepsilon + \Phi\|h\|\|j^* - j\|,
\end{aligned} \tag{88}$$

that is, $\|j^* - j\| \le \varsigma^*\varepsilon$, where $\varsigma^* = \Phi/(1 - \Phi\|h\|)$. Consequently, the proposed problem (60) is Ulam-Hyers stable. □

6. Conclusions

A new coincidence and fixed point theorem of multivalued mapping defined on a complete metric space has been presented in this work by using the characterizations of a modified $\widetilde{\mathscr{MT}}$-function, named $\mathscr{D}$-function. It has been noted herein that our result is a generalization of the fixed point theorems due Berinde-Berinde [11], Du [13], Mizoguchi-Takahashi [14], Nadler [10], Reich [17], Rus [27], and a few others in the corresponding literature. Though the conjecture raised by Reich [17] has now been proven valid in an almost complete form in [11, 13, 14], however, our main result (Theorem 19) provided a more general affirmative response to this problem. Moreover, from application perspective, we launched an existence theorem for nonlinear fractional-order differential inclusions model for COVID-19 via a standard fixed point theorem of multivalued mapping. Ulam-Hyers stability analysis of the considered model was also discussed. It is interesting to note that more useful analysis and results may be obtained if the metric on the ground set in this context is either quasi or pseudo metric. For better management of uncertainty, and since every fixed point theorem of contractive multivalued mapping has its fuzzy set-valued analogue, the result of this paper can as well be discussed in the framework of fuzzy fixed point theory and related hybrid models of fuzzy mathematics. Furthermore, in order to obtain effective measures for curbing Covid-19, other than observing the significance of lockdown, numerical simulations and better analytic tools of the proposed fractional-order differential inclusions model are another future directions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Conceptualization was made by M. Alansari. Methodology was made by M. S. Shagari. Formal analysis was made by M. S. Shagari. Review and editing was made by M. Alansari. Funding acquisition was made by M. Alansari. Writing, review, and editing was made by M. S. Shagari. In addition, all authors have read and approved the final manuscript for submission and possible publication.

Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under the grant no. G: 234-247-1443. The author, therefore, acknowledges with thanks DSR for technical and financial support.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.
[2] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, no. 2, pp. 267–273, 1974.
[3] W. A. Kirk, "Fixed points of asymptotic contractions," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 645–650, 2003.
[4] C. Di Bari and P. Vetro, "Common fixed points in generalized metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7322–7325, 2012.
[5] M. Jleli and B. Samet, "On a new generalization of metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 3, p. 128, 2018.
[6] S. S. Mohammed, S. Rashid, K. M. Abualnaja, and A. Monairah, "On nonlinear fuzzy set-valued -contraction with applications," *AImetric space Mathematics*, vol. 6, no. 10, pp. 10431–10448, 2021.
[7] S. S. Mohammed, S. Rashid, J. Fahd, and S. M. Mohamed, "Interpolative contractions and intutionistic fuzzy set-valued maps with applications," *AImetric space Mathematics*, vol. 7, no. 6, pp. 10744–10758, 2022.
[8] S. Reich, "Some problems and results in fixed point theory," *Contemporary Mathematics*, vol. 21, pp. 179–187, 1983.
[9] I. A. Rus, "Generalized contractions and applications," *Cluj University Press*, vol. 8, no. 2, pp. 34–41, 2001.
[10] S. B. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.
[11] M. Berinde and V. Berinde, "On a general class of multi-valued weakly Picard mappings," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 772–782, 2007.

[12] W. S. Du, "Some new results and generalizations in metric fixed point theory," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 73, no. 5, pp. 1439–1446, 2010.

[13] W. S. Du, "On coincidence point and fixed point theorems for nonlinear multivalued maps," *Topology and its Applications*, vol. 159, no. 1, pp. 49–56, 2012.

[14] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 177–188, 1989.

[15] H. K. Pathak, R. P. Agarwal, and Y. J. Cho, "Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions," *Journal of Computational and Applied Mathematics*, vol. 283, pp. 201–217, 2015.

[16] S. Reich, "Fixed points of contractive functions," *Bollettino dell'Unione Matematica Italiana*, vol. 5, pp. 26–42, 1972.

[17] S. Reich, "A fixed point theorem for locally contractive multi-valued functions," *Romanian Journal of Pure and Applied Mathematics*, vol. 17, pp. 569–572, 1972.

[18] Z. Ali, F. Rabiei, K. Shah, and T. Khodadadi, "Qualitative analysis of fractal-fractional order COVID-19 mathematical model with case study of Wuhan," *Alexandria Engineering Journal*, vol. 60, no. 1, pp. 477–489, 2021.

[19] Y. Chen, J. Cheng, X. Jiang, and X. Xu, "The reconstruction and prediction algorithm of the fractional TDD for the local outbreak of COVID-19," 2020, http://arxiv.org/abs/2002.10302.

[20] C. Xu, Y. Yu, Q. Yang, and Z. Lu, "Forecast analysis of the epidemics trend of COVID-19 in the United States by a generalized fractional-order SEIR model," 2020, http://arxiv.org/abs/2004.12541.

[21] A. S. Shaikh, I. N. Shaikh, and K. S. Nisar, "A mathematical model of COVID-19 using fractional derivative: outbreak in India with dynamics of transmission and control," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[22] I. Ahmed, I. A. Baba, A. Yusuf, P. Kumam, and W. Kumam, "Analysis of Caputo fractional-order model for COVID-19 with lockdown," *Adv. Difference Equ.*, vol. 2020, no. 1, pp. 1–14, 2020.

[23] S. Boccaletti, W. Ditto, G. Mindlin, and A. Atangana, "Modeling and forecasting of epidemic spreading: the case of Covid-19 and beyond," *Chaos, Solitons, and Fractals*, vol. 135, p. 109794, 2020.

[24] S. Batik and F. Y. Deren, "Analysis of fractional differential systems involving Riemann Liouville fractional derivative," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 69, no. 2, pp. 1345–1355, 2020.

[25] S. Rashid, M. A. Noor, and K. I. Noor, "Some generalize Riemann-Liouville fractional estimates involving functions having exponentially convexity property," *Punjab University Journal of Mathematics*, vol. 51, pp. 1–15, 2019.

[26] S. Rashid, F. Safdar, A. O. Akdemir, M. A. Noor, and K. I. Noor, "Some new fractional integral inequalities for exponentially m-convex functions via extended generalized Mittag-Leffler function," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 17 pages, 2019.

[27] I. A. Rus, "Basic problems of the metric fixed point theory revisited (II)," *Studia Universitatis Babeș-Bolyai*, vol. 36, pp. 81–99, 1991.

[28] V. Krivan, "Differential inclusions as a methodology tool in population biology," in *proceedings of European simulation multiconference ESM*, vol. 95, pp. 544–547, 1995.

[29] P. Z. Daffer and H. Kaneko, "Fixed points of generalized contractive multi-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, pp. 655–666, 1995.

[30] M. Alansari, S. S. Mohammed, A. Azam, and N. Hussain, "On Multivalued Hybrid Contractions with Applications," *Journal of Function Spaces*, vol. 2020, Article ID 8401403, 12 pages, 2020.

[31] I. Altun and G. Minak, "On fixed point theorems for multivalued mappings of Feng-Liu type," *Bulletin of the Korean Mathematical Society*, vol. 52, no. 6, pp. 1901–1910, 2015.

[32] A. Azam and N. Mehmood, "Multivalued fixed point theorems in tvs-cone metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[33] A. Azam and M. Arshad, "Fixed points of a sequence of locally contractive multivalued maps," *Computers and Mathematics with Applications*, vol. 57, no. 1, pp. 96–100, 2009.

[34] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier, 2006.

[35] C. Castaing and M. Valadier, "Measurable multifunctions," in *Convex Analysis and Measurable Multifunctions*, pp. 59–90, Springer, Berlin, Heidelberg, 1977.

[36] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, USA, 1968.

[37] D. Hyers, "on the stability of the linear functional Equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222–224, 1941.

[38] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3550923, 9 pages
https://doi.org/10.1155/2022/3550923

Research Article

Relational Meir-Keeler Contractions and Common Fixed Point Theorems

Faizan Ahmad Khan,[1] **Faruk Sk**,[2] **Maryam Gharamah Alshehri**,[1] **Qamrul Haq Khan**,[2] and **Aftab Alam**[3]

[1]*Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia*
[2]*Department of Mathematics, Aligarh Muslim University, Aligarh, India*
[3]*Department of Mathematics, Jamia Millia Islamia, New Delhi, India*

Correspondence should be addressed to Faruk Sk; sk.faruk.amu@gmail.com

Received 22 March 2022; Revised 15 April 2022; Accepted 9 June 2022; Published 25 June 2022

Academic Editor: Cristian Chifu

In this article, we prove some coincidence and common fixed point theorems under the relation-theoretic Meir-Keeler contractions in a metric space endowed with a locally finitely T-transitive binary relation. Our newly proved results generalize, extend, and sharpen some existing coincidence point as well as fixed point theorems existing in the literature. Moreover, we give some examples to affirm the efficacy of our results.

1. Introduction

Banach [1], a Polish mathematician, established the most successful result in fixed point theory, the Banach contraction principle (in short, BCP), in 1922, which says that a contraction mapping on a complete metric space has a unique fixed point. One of the noted generalizations of BCP comprising the concept of coincidence point (in short, CP) and common fixed point (in short, CFP) theorems was established by Jungck [2] in 1976. In succeeding years, many researchers introduced relatively weaker version of commuting mappings and developed exciting CFP results, see [3, 4].

On the other hand, generalizations of the underlying space have been trending since some decades. One of such important generalizations was initiated by Turinici [5, 6] in 1986, where he proved fixed point results in a partial ordered set. In this continuation, Alam and Imdad [7] generalized the BCP using a binary relation. Since then, many relation-theoretic fixed point theorems are being studied regularly, see [8, 9] and references therein.

Several researchers reported numerous fixed point results employing relatively more generalized contractions. One of such vital contractions was due to Meir and Keeler [10] in 1969, which was further extended by Rao and Rao [11]. In 2013, Patel et al. [12] established some CFP theorems for three and four self-mappings satisfying generalized Meir-Keeler α-contraction in metric spaces. Some generalizations of Meir-Keeler contraction in the framework of different types of spaces have also been reported, see [13–16]. Recently, Sk et al. [17] introduced the Meir-Keeler contraction in relation-theoretic sense and extended relation-theoretic contraction principle to relation-theoretic Meir-Keeler contraction principle.

In this paper, we prove some coincidence and common fixed point theorems using the relation-theoretic Meir-Keeler contraction in a metric space endowed with a locally finitely T-transitive binary relation. We also equip several examples to exhibit the significance of these new findings.

2. Preliminaries

We will go over some basic definitions in this section that will help us to prove our primary results. Throughout the paper, we pertain to $\mathbb{N} \cup \{0\}$ as $\mathscr{K}_0$, and empty set as $\varnothing$.

Definition 1 (see [18]). Let $\mathscr{X} \neq \varnothing$ be a set. A "binary relation" is a subset $\mathfrak{R}$ of $\mathscr{X}^2$. The subsets $\mathscr{X}^2$ and $\varnothing$ of $\mathscr{X}^2$ are called the "universal relation" and "empty relation," respectively.

Definition 2 (see [7]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. If either $(\varrho, \sigma) \in \mathfrak{R}$ or $(\sigma, \varrho) \in \mathfrak{R}$ for $\varrho, \sigma \in \mathscr{X}$, then ϱ and σ are called as "$\mathfrak{R}$-comparative." $[\varrho, \sigma] \in \mathfrak{R}$ is the notion for it.

Definition 3 (see [18–23]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. Then, the relation $\mathfrak{R}$ is called

(a) "amorphous" if $\mathfrak{R}$ has no precise attribute

(b) "reflexive" if $(\varrho, \varrho) \in \mathfrak{R} \forall \varrho \in \mathscr{X}$

(c) "symmetric" if $(\varrho, \sigma) \in \mathfrak{R} (\sigma, \varrho) \in \mathfrak{R}$

(d) "anti-symmetric" if $(\varrho, \sigma) \in \mathfrak{R}$ and $(\sigma, \varrho) \in \mathfrak{R} \varrho = \sigma$

(e) "transitive" if $(\varrho, \sigma) \in \mathfrak{R}$ and $(\sigma, \mathrm{w}) \in \mathfrak{R} (\varrho, \mathrm{w}) \in \mathfrak{R}$

(f) "complete", "connected" or "dichotomous" if $[\varrho, \sigma] \in \mathfrak{R} \forall \varrho, \sigma \in \mathscr{X}$

(g) "partial order" if $\mathfrak{R}$ is "reflexive", "anti-symmetric" and "transitive"

Definition 4 (see [18]). Let $\mathfrak{R}$ be a binary relation on a set $\mathscr{X} \neq \varnothing$. Then,

$$\mathfrak{R}^{-1} = \left\{ (\varrho, \sigma) \in \mathscr{X}^2 : (\sigma, \varrho) \in \mathfrak{R} \right\} \text{ and } \mathfrak{R}^s = \mathfrak{R} \cup \mathfrak{R}^{-1}, \quad (1)$$

are called inverse relation and symmetric closure of $\mathfrak{R}$, respectively.

Proposition 5 (see [7]). *Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. Then, for $\varrho, \sigma \in \mathscr{X}$,*

$$(\varrho, \sigma) \in \mathfrak{R}^s \Longrightarrow [\varrho, \sigma] \in \mathfrak{R}. \quad (2)$$

Definition 6 (see [24]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$ and $\mathscr{S} \subseteq \mathscr{X}$. Then, the set $\mathfrak{R}|_{\mathscr{S}} = \mathfrak{R} \cap \mathscr{S}^2$ is defined as the restriction of $\mathfrak{R}$ to $\mathscr{S}$.

Definition 7 (see [7]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. A sequence $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ is called $\mathfrak{R}$-preserving if

$$(\varrho_{\Bbbk}, \varrho_{\Bbbk+1}) \in \mathfrak{R} \quad \forall \Bbbk \in \mathscr{K}_0. \quad (3)$$

Definition 8 (see [7, 25]). Let T and H be two self-mappings on a set $\mathscr{X} \neq \varnothing$ and $\mathfrak{R}$ a binary relation on $\mathscr{X}$. Then,

(a) $\mathfrak{R}$ is said to be T-closed if

$\forall \varrho, \sigma \in \mathscr{X}, (\rho, \sigma) \in \mathfrak{R} \Longrightarrow (T(\varrho), T(\sigma)) \in \mathfrak{R}$

(b) $\mathfrak{R}$ is said to be (T, H)-closed if

$$\forall \varrho, \sigma \in \mathscr{X}, (H(\varrho), H(\sigma)) \in \mathfrak{R} \Longrightarrow (T(\varrho), T(\sigma)) \in \mathfrak{R} \quad (4)$$

Remark 9. Under $H = I$, the identity mapping on $\mathscr{X}$, the notion of (T, H)-closedness coincides with the notion of T-closedness of $\mathfrak{R}$.

Definition 10 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$. If every $\mathfrak{R}$-preserving Cauchy sequence in $\mathscr{X}$ converges, we say $(\mathscr{X}, \mathrm{d})$ is $\mathfrak{R}$-complete.

Definition 11 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T a self-mapping on $\mathscr{X}$. If for any $\mathfrak{R}$-preserving sequence $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ converging to an element $\varrho \in \mathscr{X}$, we have $T(\varrho_{\Bbbk}) \xrightarrow{\mathrm{d}} T(\varrho)$, then the mapping T is said to be $\mathfrak{R}$-continuous.

Definition 12 (see [2]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T, H two self-mappings on $\mathscr{X}$. Let $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ be a sequence satisfying $\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk})$. Then, the mappings T and H are compatible if $\lim_{\Bbbk \longrightarrow \infty} \mathrm{d}(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0$.

Definition 13 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T, H two self-mappings on $\mathscr{X}$. Let $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ be a sequence such that $\{T(\varrho_{\Bbbk})\}$ and $\{H(\varrho_{\Bbbk})\}$ are $\mathfrak{R}$-preserving sequence satisfying $\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk})$. Then, the mappings T and H are "$\mathfrak{R}$-compatible" if $\lim_{\Bbbk \longrightarrow \infty} d(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0$.

Remark 14 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$. Then, the following relation holds:

$$\begin{aligned} {}'' \text{commutativity} &\Longrightarrow \text{compatibility} \Longrightarrow \mathfrak{R} - \text{compatibility} \\ &\Longrightarrow \text{weak compatibility}''. \end{aligned} \quad (5)$$

Definition 15 (see [7, 25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T, H two self-mappings on $\mathscr{X}$. Consider the $\mathfrak{R}$-preserving sequence $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ such that $\varrho_{\Bbbk} \xrightarrow{d} \varrho$. Then,

(a) $\mathfrak{R}$ is called "d-self-closed" if there exists a subsequence $\{\varrho_{\Bbbk_p}\}$ of $\{\varrho_{\Bbbk}\}$ with $[\varrho_{\Bbbk_p}, \varrho] \in \mathfrak{R} \forall p \in \mathscr{K}_0$

(b) $\mathfrak{R}$ is called "$(H - d)$-self-closed" if there exists a subsequence $\{\varrho_{\Bbbk_p}\}$ of $\{\varrho_{\Bbbk}\}$ with $[H(\varrho_{\Bbbk_p}), H(\varrho)] \in \mathfrak{R}$ $\forall p \in \mathscr{K}_0$

Definition 16 (see [26–29]). Let $\mathscr{X} \neq \varnothing$ be set with a binary relation $\mathfrak{R}$ and T a self-mapping on $\mathscr{X}$

(a) If for any $\varrho, \sigma, \varsigma \in \mathscr{X}$, $(T(\varrho), T(\sigma)) \in \mathfrak{R}$ and $(T(\sigma), T(\varsigma)) \in \mathfrak{R} \Longrightarrow (T(\varrho), T(\varsigma)) \in \mathfrak{R}$, then $\mathfrak{R}$ is called "T-transitive"

(b) If for any $\varrho_0, \varrho_1, \cdots, \varrho_{\mathscr{K}} \in \mathscr{X}$ where $\mathscr{K}$ is a natural number ≥ 2, we have

$$(\varrho_{\ell-1}, \varrho_\ell) \in \mathfrak{R} \text{ for each } \ell (1 \leq \ell \leq \mathscr{K}) \Longrightarrow (\varrho_0, \varrho_{\mathscr{K}}) \in \mathfrak{R}, \quad (6)$$

then $\mathfrak{R}$ is called $\mathscr{K}$-transitive

(c) If for each denumerable subset $\mathscr{S}$ of $\mathscr{X}$, there exists $\mathscr{K} = \mathscr{K}(\mathscr{S}) \geq 2$, such that $\mathfrak{R}|_{\mathscr{S}}$ is $\mathscr{K}$-transitive, then $\mathfrak{R}$ is called "locally finitely transitive"

(d) If for each denumerable subset $\mathscr{S}$ of $T(\mathscr{X})$, there exists $\mathscr{K} = \mathscr{K}(\mathscr{S}) \geq 2$, such that $\mathfrak{R}|_{\mathscr{S}}$ is $\mathscr{K}$-transitive, then $\mathfrak{R}$ is called "locally finitely T-transitive"

Proposition 17 (see [29]). *Let $\mathscr{X}$ be a nonempty set, $\mathfrak{R}$ a binary relation on $\mathscr{X}$ and T a self-mapping on $\mathscr{X}$. Then,*

(a) $\mathfrak{R}$ is "T-transitive" $\Longleftrightarrow \mathfrak{R}|_{T\mathscr{X}}$ is "transitive"

(b) $\mathfrak{R}$ is "locally finitely T-transitive" $\Longleftrightarrow \mathfrak{R}|_{T\mathscr{X}}$ is "locally finitely transitive"

(c) $\mathfrak{R}$ is "transitive" $\Longrightarrow \mathfrak{R}$ is "finitely transitive" $\Longrightarrow \mathfrak{R}$ is "locally finitely transitive" $\Longrightarrow \mathfrak{R}$ is "locally finitely T-transitive"

(d) $\mathfrak{R}$ is "transitive" $\Longrightarrow \mathfrak{R}$ is "T-transitive" $\Longrightarrow \mathfrak{R}$ is "locally finitely T-transitive"

Definition 18 (see [23]). Let $\mathscr{X}$ be a nonempty set and $\mathfrak{R}$ a binary relation on $\mathscr{X}$. A subset $\mathscr{S}$ of $\mathscr{X}$ is called $\mathfrak{R}$-directed if for each $\varrho, \sigma \in \mathscr{S}$, there exists $\varsigma \in \mathscr{X}$ such that $(\varrho, \varsigma) \in \mathfrak{R}$ and $(\sigma, \varsigma) \in \mathfrak{R}$.

Definition 19 (see [24]). Let $\mathfrak{R}$ be a binary relation defined on a nonempty set $\mathscr{X}$. Then, for $\varrho, \sigma \in \mathscr{X}$, a finite sequence $\{\varrho_0, \varrho_1, \cdots, \varrho_p\} \subset \mathscr{X}$ satisfying the following conditions:

$$\begin{gathered}(\varrho_\ell, \varrho_{\ell+1}) \in \mathfrak{R} \text{ for each } \ell (0 \leq \ell \leq p-1), \\ \varrho_0 = \varrho \text{ and } \varrho_p = \sigma,\end{gathered} \quad (7)$$

is said to be a path of length p in $\mathfrak{R}$ from ϱ to σ.

Definition 20 (see [7]). Let $\mathfrak{R}$ be a binary relation on a nonempty set $\mathscr{X}$, and Y a subset of $\mathscr{X}$. If there exists a path in $\mathfrak{R}$ from ρ to σ for each $\varrho, \sigma \in Y$, then Y is called $\mathfrak{R}$-connected.

Lemma 21 (see [28]). *Let $\mathfrak{R}$ be a binary relation on a nonempty set $\mathscr{X}$, and $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ a sequence satisfying $(\varrho_{\Bbbk}, \varrho_{\Bbbk+1}) \in \mathfrak{R}$. Now, if for some natural number $\mathscr{K} \geq 2$, $\mathfrak{R}$ is $\mathscr{K}$-transitive on the set $L = \{\varrho_{\Bbbk} : \Bbbk \in \mathscr{K}_0\}$, then*

$$\left(\varrho_{\Bbbk}, \varrho_{\Bbbk+1+r(\mathscr{K}-1)}\right) \in \mathfrak{R} \text{ for all } \Bbbk, r \in \mathscr{K}_0. \quad (8)$$

3. Main Results

The first result in this section is on the existence of CP for two mapping T and H. For a nonempty set $\mathscr{X}$ and two self-mappings T and H on $\mathscr{X}$, the notations we use herein are as follows:

$$\begin{gathered}\Theta(T, H) = \{\rho \in \mathscr{X} : T(\varrho) = H(\varrho)\}, \\ \bar{\Theta}(T, H) = \{\bar{\varrho} \in \mathscr{X} : \bar{\varrho} = T(\varrho) = H(\varrho), \varrho \in \mathscr{X}\}.\end{gathered} \quad (9)$$

Theorem 22. *Let $\mathscr{X}$ be a nonempty set together with a metric d, $\mathfrak{R}$ a binary relation on $\mathscr{X}$ and T, H two self-mappings on $\mathscr{X}$. Suppose the following conditions hold:*

(a) $T(\mathscr{X}) \subset H(\mathscr{X})$

(b) $(\mathscr{X}, d)$ is $\mathfrak{R}$-complete

(c) there exists $\varrho_0 \in \mathscr{X}$ such that $(H(\varrho_0), T(\varrho_0)) \in \mathfrak{R}$

(d) $\mathfrak{R}$ is (T, H)-closed and locally finitely T-transitive

(e) T and H are $\mathfrak{R}$-compatible

(f) H is $\mathfrak{R}$-continuous

(g) T is $\mathfrak{R}$-continuous or $\mathfrak{R}$ is $(H-d)$-self-closed

(h) for every $\varepsilon > 0$ and $\varrho, \sigma \in \mathscr{X}$, there exists $\delta > 0$ such that

$$(H(\rho), H(\sigma)) \in \mathfrak{R} \text{ and } \varepsilon \leq d(H(\varrho), H(\sigma)) < \varepsilon + \delta \Longrightarrow d(T(\varrho), T(\sigma)) < \varepsilon \quad (10)$$

Then, T and H have a CP.

Proof. Assumption (c) confirms the existence of $\varrho_0 \in \mathscr{X}$ such that $(H(\varrho_0), T(\varrho_0)) \in \mathfrak{R}$. Now, if $H(\varrho_0) = T(\varrho_0)$ then nothing is left to be proved. Otherwise, by assumption (a), we can pick $\varrho_1 \in \mathscr{X}$ such that $T(\varrho_0) = H(\varrho_1)$. Again, there will be $\varrho_2 \in \mathscr{X}$ such that $H(\varrho_2) = T(\varrho_1)$. In this way, we construct a sequence $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ such that

$$H(\varrho_{\Bbbk+1}) = T(\varrho_{\Bbbk}) \quad \forall \Bbbk \in \mathscr{K}_0. \quad (11)$$

Now, we assert that $\{H(\varrho_{\Bbbk})\}$ is $\mathfrak{R}$-preserving, i.e.,

$$(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) \in \mathfrak{R} \quad \forall \Bbbk \in \mathscr{K}_0. \quad (12)$$

We will adopt the induction method to prove this fact. In view of assumption (c), equation (12) holds for $\Bbbk = 0$, i.e.,

$$(H(\varrho_0), H(\varrho_1)) \in \mathfrak{R}. \quad (13)$$

Now, suppose that equation (12) holds for $\Bbbk = p > 0$, i.e.,

$$\left(H(\varrho_p), H(\varrho_{p+1})\right) \in \mathfrak{R}. \quad (14)$$

Then, we have to show that

$$\left(H\left(\varrho_{p+1}\right), H\left(\varrho_{p+2}\right)\right) \in \mathfrak{R}. \tag{15}$$

In view of the fact that $\mathfrak{R}$ is (T, H)-closed, it is clear that

$$\left(H\left(\varrho_{p}\right), H\left(\varrho_{p+1}\right)\right) \in \mathfrak{R}\left(T\left(\varrho_{p}\right), T\left(\varrho_{p+1}\right)\right) \in \mathfrak{R}, \tag{16}$$

implying thereby

$$\left(H\left(\varrho_{p+1}\right), H\left(\varrho_{p+2}\right)\right) \in \mathfrak{R}, \tag{17}$$

which guarantees the fact that equation (2) holds for $\Bbbk = p + 1$. Therefore, $\{H(\varrho_{\Bbbk})\}$ is $\mathfrak{R}$-preserving sequence. Notice that $\{T(\varrho_{\Bbbk})\}$ is also a $\mathfrak{R}$-preserving sequence due to equation (1), i.e.,

$$(T(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) \in \mathfrak{R}. \tag{18}$$

Now, if there exists $n_0 \in \mathscr{K}$ such that $H(\varrho_{n_0}) = H(\varrho_{n_0+1})$, then, in view of equation (1), ϱ_{n_0} turns out to be a CP of T and H. As an alternative, consider that $H(\varrho_{\Bbbk}) \neq H(\varrho_{\Bbbk+1})$ for all $\Bbbk \in \mathscr{K}_0$, i.e., $d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) \neq 0$.

Denote $\mu_{\Bbbk} \coloneqq d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1}))$. Now, in view of assumption (h), we get

$$\mu_{\Bbbk+1} = d(H(\varrho_{\Bbbk+1}), H\varrho_{\Bbbk+2}) = d(T(\varrho_{\Bbbk}), T(\varrho_{\Bbbk+1})) < d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) = \mu_{\Bbbk}, \tag{19}$$

which gives

$$\mu_{\Bbbk+1} < \mu_{\Bbbk}. \tag{20}$$

Therefore, the sequence $\{\mu_{\Bbbk}\}$ is decreasing. As $\{\mu_{\Bbbk}\}$ is also bounded below by 0 (as a lower bound), we can find $r \geq 0$ satisfying

$$\lim_{\Bbbk \longrightarrow \infty} \mu_{\Bbbk} = r = \inf_{\Bbbk \in \mathscr{K}_0} \mu_{\Bbbk}. \tag{21}$$

Now, let us assume that $r > 0$. So, there will always be a $\delta(r) > 0$ such that

$$\begin{gathered}(H(\varrho), H(\sigma)) \in \mathfrak{R}, \\ r \leq d(H(\varrho), H(\sigma)) < r + \delta(r) \Longrightarrow d(T(\varrho), T(\sigma)) < r.\end{gathered} \tag{22}$$

Since $\{\mu_{\Bbbk}\}$ is decreasing sequence converging to r, there exists $p \in \mathscr{K}$ such that

$$r \leq d\left(H\left(\varrho_{p}\right), H\left(\varrho_{p+1}\right)\right) < r + \delta(r). \tag{23}$$

Thus, in view of assumption (h), we have

$$\mu_{p+1} = d\left(H\left(\varrho_{p+1}\right), H\varrho_{p+2}\right) < r, \tag{24}$$

which contradicts the fact that $r = \inf_{\Bbbk \longrightarrow \mathscr{K}_0} \mu_{\Bbbk}$. Hence, we conclude that

$$\lim_{\Bbbk \longrightarrow \infty} d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) = 0. \tag{25}$$

Now, we establish that the sequence $\{H(\varrho_{\Bbbk})\}$ is Cauchy. Utilizing equation (1), since $\{H(\varrho_{\Bbbk})\} \subset T(\mathscr{X})$, we get that the range $\mathscr{S} = \{H(\varrho_{\Bbbk}) \colon \Bbbk \in \mathscr{K}_0\}$ is a denumerable subset of $T(\mathscr{X})$. Hence, in view of assumption (d), there exist $\mathscr{K} = \mathscr{K}(\mathscr{S}) \geq 2$, such that $\mathfrak{R}|_{\mathscr{S}}$ is $\mathscr{K}$-transitive. Let $\varepsilon > 0$ be an arbitrary and fixed real number and let $\delta > 0$ corresponds to ε verifying the assumption (h). WLOG, we may consider that $\delta < \varepsilon$. In view of (2), there exists $n_0(\delta) \in \mathbb{N}$ satisfying

$$d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) < \frac{\delta}{4\mathscr{K}} \quad \forall \Bbbk \geq n_0(\delta). \tag{26}$$

For all $\Bbbk \geq n_0(\delta)$ and for all $p (1 \leq p \leq \mathscr{K})$, using triangular inequality, we get

$$\begin{aligned} d\left(H(\varrho_{\Bbbk}), H\left(\varrho_{\Bbbk+\varrho}\right)\right) &\leq d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) \\ &\quad + d(H(\varrho_{\Bbbk+1}), H\varrho_{\Bbbk+2}) \cdots + d\left(H\varrho_{\Bbbk+p-1}, H\varrho_{\Bbbk+p}\right) \\ &\leq \frac{\delta}{4\mathscr{K}} + \frac{\delta}{4\mathscr{K}} + \cdots + \frac{\delta}{4\mathscr{K}} = \frac{p\delta}{4\mathscr{K}}. \end{aligned} \tag{27}$$

Now, we claim that

$$d\left(H(\varrho_{\Bbbk}), H\varrho_{\Bbbk+p}\right) < \varepsilon + \frac{\delta}{2} \forall \Bbbk \geq n_0(\delta) \text{ and} \forall p \in \mathscr{K}. \tag{28}$$

This is demonstrated herein using the mathematical induction method. From (27), it is clear that (28) holds for all $p \in \{1, 2, 3, \cdots, \mathscr{K}\}$. Suppose that the conclusion holds for all $p \in \{1, 2, 3, \cdots, m\}$, where $m \geq \mathscr{K}$. We have to show that (28) holds for $\Bbbk = m + 1$ also. As $m \geq \mathscr{K}$, so $m - 1 \geq \mathscr{K} - 1 > 0$. By division algorithm, there exists unique integers μ and $\eta (0 \leq \eta \leq \mathscr{K} - 1)$ such that

$$\begin{gathered} m - 1 = (\mathscr{K} - 1)\mu + \eta \\ m = 1 + (\mathscr{K} - 1)\mu + \eta. \end{gathered} \tag{29}$$

Denoting $q =: 1 + (\mathscr{K} - 1)\mu$, the above equation reduces to

$$m = q + \eta, \tag{30}$$

so that

$$2 \leq \mathscr{K} \leq q \leq m = q + \eta. \tag{31}$$

Now, using (27), we get

$$d\left(H\left(\varrho_{\Bbbk+q+1}\right), H(\varrho_{\Bbbk+m+1})\right) = d\left(H\left(\varrho_{\Bbbk+q+1}\right), H\left(\varrho_{\Bbbk+q+\eta+1}\right)\right) \leq \frac{\eta\delta}{4\mathscr{K}}. \tag{32}$$

Now, using Lemma 21, we get

$$\left(H(\varrho_{\Bbbk}), H\left(\varrho_{\Bbbk+q}\right)\right) \in \mathfrak{R}. \tag{33}$$

As $q \in \{\mathscr{K}, \mathscr{K}+1,. \cdots, m\}$, using inductive hypothesis, we get

$$0 < d\left(H(\varrho_{\Bbbk}), H\left(\varrho_{\Bbbk+q}\right)\right) < \varepsilon + \frac{\delta}{2} < \varepsilon + \delta. \tag{34}$$

Using (33) and (34) and applying contractive condition (h), we get

$$d\left(H(\varrho_{\Bbbk+1}), H\left(\varrho_{\Bbbk+q+1}\right)\right) = d\left(T(\varrho_{\Bbbk}), T\left(\varrho_{\Bbbk+q}\right)\right) < \varepsilon. \tag{35}$$

Now, using triangular inequality, (25), (32), and (35), we get

$$\begin{aligned} d(H(\rho_{\Bbbk}), H\rho_{\Bbbk+m+1}) &\leq d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) + d\left(H(\varrho_{\Bbbk+1}), H\left(\varrho_{\Bbbk+q+1}\right)\right) \\ &\quad + d\left(H\left(\varrho_{\Bbbk+q+1}\right), H(\varrho_{\Bbbk+m+1})\right) \\ &< \frac{\delta}{4\mathscr{K}} + \varepsilon + \frac{\eta\delta}{4\mathscr{K}} < \frac{\delta}{4\mathscr{K}} + \varepsilon + \frac{\delta}{4\mathscr{K}}(\mathscr{K}-1)\, as \mathscr{K} \\ &\geq 2 \text{ and } \eta < \mathscr{K} - 1 = \varepsilon + \frac{\delta}{4} < \varepsilon + \frac{\delta}{2}. \end{aligned} \tag{36}$$

Thus, by induction, (28) is verified. From (28), it embraces that the sequence $\{H(\varrho_{\Bbbk})\}$ is Cauchy. Now, the $\mathfrak{R}$-completeness property of $\mathscr{X}$ and $\mathfrak{R}$-preserving property of $\{H(\varrho_{\Bbbk})\}$ confirm the availability of an element $\varsigma \in \mathscr{X}$ such that

$$\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \varsigma. \tag{37}$$

Also, from (11),

$$\lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk}) = \varsigma. \tag{38}$$

Now, by dint of the $\mathfrak{R}$-continuity of H, we acquire

$$\lim_{\Bbbk \longrightarrow \infty} H(H(\varrho_{\Bbbk})) = H\left(\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk})\right) = H(\varsigma). \tag{39}$$

Utilizing (38) and $\mathfrak{R}$-continuity of H,

$$\lim_{\Bbbk \longrightarrow \infty} H(T(\varrho_{\Bbbk})) = H\left(\lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk})\right) = H(\varsigma). \tag{40}$$

Since $\{T(\varrho_{\Bbbk})\}$ and $\{H(\varrho_{\Bbbk})\}$ are $\mathfrak{R}$-preserving and

$$\lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk}) = \lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \varsigma, \tag{41}$$

by assumption (e),

$$\lim_{\Bbbk \longrightarrow \infty} d(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0. \tag{42}$$

The next step is to establish that $\varsigma \in \Theta(T, H)$. From assumption (g), we first consider that T is "$\mathfrak{R}$-continuous." Using (12), (37), and $\mathfrak{R}$-continuity of T,

$$\lim_{\Bbbk \longrightarrow \infty} T(H(\varrho_{\Bbbk})) = T\left(\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk})\right) = T(\varsigma). \tag{43}$$

Applying (40) and (42), we get

$$\begin{aligned} d(H(\varsigma), T(\varsigma)) &= d\left(\lim_{\Bbbk \longrightarrow \infty} HT(\varrho_{\Bbbk}), \lim_{\Bbbk \longrightarrow \infty} TH(\varrho_{\Bbbk})\right) \\ &= \lim_{\Bbbk \longrightarrow \infty} d(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0, \end{aligned} \tag{44}$$

yielding thereby $H(\varsigma) = T(\varsigma)$, which establishes our claim.

Instead of $\mathfrak{R}$-continuity of T, we now suppose that $\mathfrak{R}$ is (H, d)-self-closed, based on assumption (g). Then, $\{H(\varrho_{\Bbbk})\}$ being $\mathfrak{R}$-preserving sequence guarantees the existence of a subsequence $\{H\varrho_{\Bbbk_p}\}$ such that $[H\varrho_{\Bbbk_p}, \varsigma] \in \mathfrak{R}$. If $H\varrho_{\Bbbk_{k_0}} = \varsigma$ for some $k_0 \in \mathscr{K}$, then using (11) and by the $\mathfrak{R}$-preserving property of $\{H(\varrho_{\Bbbk})\}$, we get $H(\varrho_{\Bbbk_{k_0}}) \in \Theta(T, H)$. Otherwise, suppose $H\varrho_{n_p} \neq \varsigma$, i.e., $d(H\varrho_{n_p}, \varsigma) \neq 0$ for all $p \in \mathscr{K}$. In this case, in view of assumption (h), assuming $\varepsilon = d(H\varrho_{\Bbbk_p}, \varsigma)$ and using assumption (h), we get

$$d\left(T\left(H\varrho_{n_p}\right), T(\varsigma)\right) < \varepsilon. \tag{45}$$

Using triangle inequality, we get

$$\begin{aligned} d(H(\varsigma), T(\varsigma)) \leq\, & d\left(H(\varsigma), HT\varrho_{\Bbbk_p}\right) \\ & + d\left(HT\varrho_{\Bbbk_p}, TH\varrho_{\Bbbk_p}\right) \\ & + d\left(TH\varrho_{\Bbbk_p}, T(\varsigma)\right). \end{aligned} \tag{46}$$

Now, using (40), (42), and (45) in the previous equation, we obtain

$$d(H(\varsigma), T(\varsigma)) = 0, \tag{47}$$

which establishes that $T(\varsigma) = H(\varsigma)$. □

It is clear that Theorem 22 solely considers the existence of a CP of T and H. As a result, we must add extra conditions to the hypothesis of Theorem 22 to obtain the uniqueness of point of coincidence, CP and CFPs. This is the purpose of our next theorems.

Theorem 23. *Assume that all of the criteria of Theorem 22 are met. Let the following condition holds additionally:*

(i) $T(\mathscr{X})$ is $\mathfrak{R}^s_{H(\mathscr{X})}$-connected

then T and H have a unique point of coincidence.

Proof. From Theorem 22, we get that $\Theta(T,H)\neq\varnothing$. Consider that $\varrho,\sigma\in\Theta(T,H)$. Then, there exist $\bar{\sigma},\bar{\sigma}\in\mathscr{X}$ such that

$$T(\varrho)=H(\varrho)=\bar{\varrho}\text{ and }T(\sigma)=H(\sigma)=\bar{\sigma}. \tag{48}$$

It is now our goal to prove that $\bar{\varrho}=\bar{\sigma}$. Since $T(\varrho),T(\sigma)\in T(\mathscr{X})\subseteq H(\mathscr{X})$, by assumption (i), there exists a path $\{H(\varsigma_0),H(\varsigma_1),H(\varsigma_2),\cdots,H(\varsigma_p)\}$ of some finite length p in $\mathfrak{R}^s_{H(\mathscr{X})}$ from $T(\rho)$ to $T(\sigma)$. Now, in view of (48), WLOG we can choose $\varsigma_0=\varrho$ and $\varsigma_p=\sigma$. Thus, we have

$$[H(\varsigma_\ell),H(\varsigma_{\ell+1})]\in\mathfrak{R}_{H(\mathscr{X})}\text{ for each }\ell(0\le\ell\le p-1). \tag{49}$$

Define the constant sequences $\varsigma^0_{\Bbbk}=\rho$ and $\varsigma^p_{\Bbbk}$, then in view of equation (48), we have $H(\varsigma^0_{\Bbbk+1})=T(\varsigma^0_{\Bbbk})=\bar{\varrho}$ and $H(\varsigma^p_{\Bbbk+1})=T(\varsigma^p_{\Bbbk})=\bar{\sigma}$ for all $\Bbbk\in\mathscr{K}_0$. Put $\varsigma^1_0=\varsigma_1,\varsigma^2_0=\varsigma_2,\varsigma^3_0=\varsigma_3,\cdots,\varsigma^{p-1}_0=\varsigma_{p-1}$. Now, since $T(\mathscr{X})\subset H(\mathscr{X})$, we can define sequences $\{\varsigma^1_{\Bbbk}\}$, $\{\varsigma^2_{\Bbbk}\}$, ..., $\{\varsigma^{p-1}_{\Bbbk}\}$ such that $H(\varsigma^1_{\Bbbk+1})=T(\varsigma^1_{\Bbbk})$, $H(\varsigma^2_{\Bbbk+1})=T(\varsigma^2_{\Bbbk})$, ..., $H(\varsigma^{p-1}_{\Bbbk+1})=T(\varsigma^{p-1}_{\Bbbk})$ for all $\Bbbk\in\mathscr{K}_0$. Hence, we have

$$H\left(\varsigma^\ell_{\Bbbk+1}\right)=T\left(\varsigma^\ell_{\Bbbk}\right)\forall\Bbbk\in\mathscr{K}_0\text{ and for each }\ell(0\le\ell\le p). \tag{50}$$

Now, we claim that

$$\left[H\left(\varsigma^\ell_{\Bbbk}\right),H\left(\varsigma^{\ell+1}_{\Bbbk}\right)\right]\in\mathfrak{R}\forall\Bbbk\in\mathscr{K}_0\text{ and for each }\ell(0\le\ell\le p-1). \tag{51}$$

This is demonstrated herein using the mathematical induction method. equation (51) holds for $\Bbbk=0$ as a result of (49). Assume that equation (51) is true for $\Bbbk=r$, i.e.,

$$\left[H\left(\varsigma^\ell_r\right),H\left(\varsigma^{\ell+1}_r\right)\right]\in\mathfrak{R}. \tag{52}$$

As $\mathfrak{R}$ is (T,H)-closed, we obtain

$$\left[T\left(\varsigma^\ell_r\right),T\left(\varsigma^{\ell+1}_r\right)\right]\in\mathfrak{R}, \tag{53}$$

which on using (51) gives us that

$$\left[H\left(\varsigma^\ell_{r+1}\right),H\left(\varsigma^{\ell+1}_{r+2}\right)\right]\in\mathfrak{R}\ \Bbbk\in\mathscr{K}_0\text{ and for each }\ell(0\le\ell\le p-1). \tag{54}$$

Therefore, equation (51) holds. Now, for each $\Bbbk\in\mathscr{K}_0$ and for each $(0\le\ell\le p-1)$, define

$$t^\ell_{\Bbbk}=d\left(H\left(\varsigma^\ell_{\Bbbk}\right),H\left(\varsigma^{\ell+1}_{\Bbbk}\right)\right). \tag{55}$$

We show that

$$\lim_{\Bbbk\longrightarrow\infty}t^\ell_{\Bbbk}=0. \tag{56}$$

Now, we look at two scenarios in which ℓ is fixed. Firstly, suppose that

$$t^\ell_{n_0}=d\left(H\left(\varsigma^\ell_{n_0}\right),H\left(\varsigma^{\ell+1}_{n_0}\right)\right)=0\text{ for some }n_0\in\mathscr{K}_0, \tag{57}$$

which gives rise to $H(\varsigma^\ell_{n_0})=H(\varsigma^{\ell+1}_{n_0})$. Now applying (11), we have $t^\ell_{n_0+1}=0$. Continuing this process, we get

$$\varsigma^\ell_{\Bbbk}=0\forall\Bbbk\ge n_0, \tag{58}$$

which establishes that $\lim_{\Bbbk\longrightarrow\infty}\varsigma^\ell_{\Bbbk}=0$.

Alternatively, assume that $\varsigma^\ell_{\Bbbk}>0\forall\Bbbk\in\mathscr{K}_0$. For any $\varepsilon>0$, assume $t^\ell_{\Bbbk}=d(H(\varsigma^\ell_{\Bbbk}),H(\varsigma^{\ell+1}_{\Bbbk}))=\varepsilon$. Then,

$$t^\ell_{\Bbbk+1}=d\left(H\left(\varsigma^\ell_{\Bbbk+1}\right),H\left(\varsigma^{\ell+1}_{\Bbbk+1}\right)\right)=d\left(T\left(\varsigma^\ell_{\Bbbk}\right),T\left(\varsigma^{\ell+1}_{\Bbbk}\right)\right)<\varepsilon=t^\ell_{\Bbbk}, \tag{59}$$

which gives

$$t^\ell_{\Bbbk+1}<t^\ell_{\Bbbk}. \tag{60}$$

As a result, the sequence $\{t^\ell_{\Bbbk}\}$ is decreasing. As $\{t^\ell_{\Bbbk}\}$ is also bounded below by 0 (as a lower bound), there exists $r\ge0$ such that

$$\lim_{\Bbbk\longrightarrow\infty}t^\ell_{\Bbbk}=r=\inf_{\Bbbk\in\mathscr{K}_0}t^\ell_{\Bbbk}. \tag{61}$$

Now, we prove that $r=0$. Assume, on the other hand that $r>0$. So, there will always be a $\delta(r)>0$ such that

$$(H(\varrho),H(\sigma))\in\mathfrak{R}\text{ and }r\le d(H(\varrho),H(\sigma))<r+\delta(r)d(T(\varrho),T(\sigma))<r. \tag{62}$$

Since $\{t^\ell_{\Bbbk}\}$ is decreasing sequence converging to r, there exists $p\in\mathscr{K}$ such that

$$r\le d\left(H\left(\varsigma^\ell_p\right),H\left(\varsigma^{\ell+1}_p\right)\right)<r+\delta(r). \tag{63}$$

Thus, in view of assumption (h), we have

$$t^\ell_{p+1}=d\left(H\left(\varsigma^\ell_{p+1}\right),H\left(\varsigma^{\ell+1}_{p+1}\right)\right)<r, \tag{64}$$

which contradicts the fact that $r=\inf_{\Bbbk\longrightarrow\infty}t^\ell_{\Bbbk}$. Hence, we conclude that

$$\lim_{\Bbbk\longrightarrow\infty}t^\ell_{\Bbbk}=0. \tag{65}$$

Thus, equation (56) holds $\forall\ell(0\le\ell\le p-1)$. Now, in light of equation (56) and triangle inequality, we get

$$d(\bar{\varrho},\bar{\sigma})\le t^0_{\Bbbk}+t^1_{\Bbbk}+\cdots+t^{p-1}_{\Bbbk}\longrightarrow0\ as\ \Bbbk\longrightarrow\infty. \tag{66}$$

Therefore, $\bar{\varrho}=\bar{\sigma}$, which ends the proof. □

Theorem 24. *Assume that all of the criteria of Theorem 22 are met. Let the following condition holds additionally:*

(i) T and H are "weakly compatible"

then T and H have a unique CFP.

Proof. Assume $\rho \in \mathscr{X}$ such that $\varrho \in \Theta(T, H)$. Therefore, there exists $\bar{\rho} \in \mathscr{X}$ such that

$$H(\varrho) = T(\varrho) = \bar{\varrho}. \tag{67}$$

In light of the Remark 14, the concept $\mathfrak{R}$-compatibility coincides with the weak compatibility. Hence, $\bar{\varrho} \in \Theta(T, H)$. Utilizing $\varsigma = \bar{\varrho}$ in Theorem 23, we obtain $H(\varrho) = H(\bar{\varrho})$ yielding thereby

$$\bar{\varrho} = H(\bar{\varrho}) = T(\bar{\varrho}). \tag{68}$$

Hence, $\bar{\varrho}$ is a CFP of T and H.

Now, we assume that ϱ' is another CFP of T and H in order to assert the uniqueness. Applying Theorem 23, we get

$$\varrho' = H\left(\varrho'\right) = H(\bar{\varrho}) = \bar{\varrho}, \tag{69}$$

which finishes the proof. □

Theorem 25. *Assume that all of the criteria of Theorem 22 are met. Suppose either of the mappings T and H is one-to-one. Then, T and H have a unique CP.*

Proof. From Theorem 22, it is evident that $\Theta(T, H) \neq \varnothing$. Let, $\varrho, \sigma \in \Theta(T, H)$. Then, Theorem 23 permits us to write

$$T(\varrho) = H(\varrho) = T(\sigma) = H(\sigma). \tag{70}$$

Now, since T or H is one-to-one, we have, $\varrho = \sigma$ which finishes the proof. □

Theorem 22 has the following implication when we apply Proposition 17.

Corollary 26. *If either of the below conditions:*

(a) $\mathfrak{R}$ is "transitive"

(b) $\mathfrak{R}$ is "T-transitive"

(c) $\mathfrak{R}$ is "finitely transitive"

(d) $\mathfrak{R}$ is "locally finitely transitive"

is utilized in Theorem 22 instead of the locally finitely T-transitivity condition; then, the validity of Theorem 22 remains the same.

Corollary 27. *If either of the below conditions:*
(i'). $T(\mathscr{X})$ is $\mathfrak{R}^s$-directed
(i''). $\mathfrak{R}|_{T(\mathscr{X})}$ is complete

holds in place of condition (i) of Theorem 23, then the validity of Theorem 23 remains the same.

Proof. If condition (i') is satisfied, then, for each $\varrho, \sigma \in T(\mathscr{X})$, we get $\varsigma \in \mathscr{X}$ satisfying $[\rho, \varsigma] \in \mathfrak{R}$ and $[\sigma, \varsigma] \in \mathfrak{R}$. Notice that the sequence $\{\varrho, \varsigma, \sigma\}$ works as a path of length 2 in $\mathfrak{R}^s$ from ρ to σ, which establishes the fact that $T(\mathscr{X})$ is $\mathfrak{R}^s$-connected. Now, applying Theorem 23, we obtain the uniqueness of point of coincidence.

Alternately, from assumption (i''), we get $[\varrho, \sigma] \in \mathfrak{R} \forall \varrho, \sigma \in T(\mathscr{X})$, which assents that $\{\rho, \sigma\}$ constitutes a path of length 1 in $\mathfrak{R}^s$. As a result, $T(\mathscr{X})$ is $\mathfrak{R}^s$-connected, which wrap up the proof when Theorem 23 is applied. □

Under $H = I$, the identity map, we obtain the following result which is proved by Sk et al. [17].

Corollary 28 (see [17]). *Let $(\mathscr{X}, d)$ be a $\mathfrak{R}$-complete metric space endowed with a binary relation $\mathfrak{R}$ on $\mathscr{X}$ and T a self-mapping on $\mathscr{X}$. Suppose that the following conditions hold:*

(a) there exists $\varrho_0 \in \mathscr{X}$ such that $(\varrho_0, T\varrho_0) \in \mathfrak{R}$,

(b) $\mathfrak{R}$ is T-closed and locally finitely T-transitive

(c) either T is $\mathfrak{R}$-continuous or $\mathfrak{R}$ is d-self-closed

(d) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(\varrho, \sigma) \in \mathfrak{R} \text{ and } \varepsilon \leq d(\varrho, \sigma) < \varepsilon + \delta \Longrightarrow d(T(\varrho), T(\sigma)) < \varepsilon \tag{71}$$

Then, T has a fixed point. Further, if we impose an additional hypothesis:

(e) $T(\mathscr{X})$ is $\mathfrak{R}^s$-connected

then T has a unique fixed point.

Remark 29. Under the universal relation $\mathfrak{R}$ and $H = I$, the identity map, Theorem 22, and Theorem 23 reduce to the classical fixed point theorem of Meir and Keeler [10].

Remark 30. Under partial order the relation $\mathbb{R} = {}^\circ$, and $H = I$, the identity map, Theorem 22, and Theorem 23 reduces to fixed point theorem of Harjani et al. [30].

4. Examples

Now, we equip two examples to show how important our results are in comparison to other results in the literature.

Example 1. Let $\mathscr{X} = \{(0, 1), (1, 0), (1, 1), (0, 0)\} \subset \mathbb{R}^2$ together with the usual Euclidean metric d. Consider the following relation endowed with $\mathscr{X}$:

$$\mathfrak{R} = \{((1, 1), (0, 0))\}. \tag{72}$$

Then, $(\mathcal{X}, d)$ is a $\mathfrak{R}$-complete metric space. Now consider that $T, H : \mathcal{X} \longrightarrow \mathcal{X}$ are two mappings defined by

$$\begin{aligned} &T(1,0) = (0,1) \,;\, T(0,1) = (1,0) \,;\, T(1,1) = (1,1) \,;\, T(0,0) = (0,0), \\ &H(0,1) = (1,0) \,;\, H(0,0) = (0,1) \,;\, H(1,1) = (1,1) \,;\, H(1,0) = (0,0). \end{aligned} \tag{73}$$

Notice that for $\varepsilon = d((0,1),(1,0)) = \sqrt{2}$, we have

$$d(T(0,1), T(1,0)) = d((1,0),(0,1)) = \sqrt{2} < \varepsilon, \tag{74}$$

which is absurd. Further, $((1,1),(0,0)) \in \mathfrak{R}$ and $d((1,1),(0,0)) = \sqrt{2}$ but the inequality

$$d(T(1,1), T(0,0)) = d((1,1,),(0,0)) = \sqrt{2} < \varepsilon, \tag{75}$$

does not hold. Hence, the existing theorems cannot be applied for this example. Now, assume that $\varepsilon = d(H(1,1), H(1,0)) = d((1,1),(0,0)) = \sqrt{2}$. Then, the inequality

$$d(T(1,1), T(1,0)) = d((1,1),(0,1)) = 1 < \varepsilon, \tag{76}$$

holds. As a result, assumption (h) of Theorem 22 holds. It can also be seen that all of the conditions of Theorem 22 are met using regular calculation. Therefore, T and H have a CP, namely, $(1,1)$.

Although it does not satisfy Theorem 23, the CP of T and H in Example 1 is unique, proving that condition (i) of Theorem 23 is not a necessary condition for the uniqueness of CPs.

Example 2. Let $\mathcal{X} = \{(0,1),(1,0),(1,1),(0,0)\} \subset \mathbb{R}^2$ together with the usual Euclidean metric d. Consider the following relation endowed with $\mathcal{X}$,

$$\mathfrak{R} = \{(\varrho, \sigma) \colon \varrho, \sigma \in \{(0,1),(1,1)\}\}. \tag{77}$$

Then, $(\mathcal{X}, d)$ is a $\mathfrak{R}$-complete metric space. Now consider that $T, H : \mathcal{X} \longrightarrow \mathcal{X}$ are two mappings defined by

$$\begin{aligned} &T(1,0) = (1,0) \,;\, T(0,1) = (0,1) \,;\, T(1,1) = (1,0) \,;\, T(0,0) = (0,1), \\ &H(1,0) = (1,0) \,;\, H(0,1) = (0,1) \,;\, H(1,1) = (0,1), H(0,0) = (1,1). \end{aligned} \tag{78}$$

Now, for $\varepsilon = d(H(0,1), H(0,0)) = 1$, we have

$$d(T(0,1), T(0,0)) = d((0,1),(0,1)) = 0 < \varepsilon, \tag{79}$$

holds. As a result, assumption (h) of Theorem 22 holds. It can also be seen that all of the conditions of Theorem 22 are met using regular calculation. Therefore, T and H have CPs, namely, $(0,1),(1,0)$. The availability of more than one fixed point certifies the eminence of Theorem 23.

Notice that for $\varepsilon = d((0,1),(1,0)) = \sqrt{2}$, we have

$$d(T(0,1), T(1,0)) = d((1,0),(0,1)) = \sqrt{2} < \varepsilon, \tag{80}$$

which is absurd. Further, $((0,1),(1,1)) \in \mathfrak{R}$ and $d((0,1),(1,1)) = 1$ but the inequality

$$d(T(0,1), T(1,1)) = d((0,1,),(1,0)) = \sqrt{2} < \varepsilon, \tag{81}$$

does not hold. Hence, the existing theorems cannot be applied for this example.

5. Conclusion

In this paper, we have established some coincidence point theorems for two mappings employing the relation-theoretic Meir-Keeler contractions in a metric space endowed with a class of transitive binary relation. Our findings have also led to the deduction of certain related fixed point results. Furthermore, some examples are given to demonstrate the significant progress made in this area.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.

[2] G. Jungck, "Commuting mappings and fixed points," *Amer. Math. Monthly*, vol. 83, no. 4, pp. 261–263, 1976.

[3] D. K. Patel, P. Kumam, and D. Gopal, "Some discussion on the existence of common fixed points for a pair of maps," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[4] D. Gopal and R. K. Bisht, "Metrical common fixed points and commuting type mappings," in *Background and Recent Developments of Metric Fixed Point Theory*, D. Gopal, P. Kumam, and M. Abbas, Eds., CRC Press, 2017.

[5] M. Turinici, "Abstract comparison principles and multivariable Gronwall-Bellman inequalities," *Journal of Mathematical Analysis and Applications*, vol. 117, no. 1, pp. 100–127, 1986.

[6] M. Turinici, "Fixed points for monotone iteratively local contractions," *Demonstratio Mathematica*, vol. 19, no. 1, pp. 171–180, 1986.

[7] A. Alam and M. Imdad, "Relation-theoretic contraction principle," *Journal of Fixed Point Theory and Applications*, vol. 17, no. 4, pp. 693–702, 2015.

[8] A. Alam and M. Imdad, "Nonlinear contractions in metric spaces under locally T-transitive binary relations," *Fixed Point Theory*, vol. 19, no. 1, pp. 13–24, 2018.

[9] F. Sk, A. Hossain, Q. H. Khan, and Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India, "Relation-theoretic metrical coincidence theorems under

weak C-contractions and K-contractions," *AIMS Mathematics*, vol. 6, no. 12, pp. 13072–13091, 2021.

[10] A. Meir and E. Keeler, "A theorem on contraction mappings," *Journal of Mathematical Analysis and Applications*, vol. 28, no. 2, pp. 326–329, 1969.

[11] I. H. N. Rao and K. P. R. Rao, "Generalizations of fixed point theorems of Meir and Keeler type," *Indian Journal of Pure and Applied Mathematics*, vol. 16, pp. 1249–1262, 1985.

[12] D. K. Patel, T. Abdeljawad, and D. Gopal, "Common fixed points of generalized Meir-Keeler α-contractions,," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[13] C.-Y. Li, E. Karapınar, and C.-M. Chen, "A discussion on random Meir-Keeler contractions," *Mathematics*, vol. 8, no. 2, p. 245.

[14] A. Fulga and E. Karapinar, "Revisiting Meir-Keeler type fixed operators on Branciari distance space," *Tbilisi Mathematical Journal*, vol. 12, no. 4, pp. 97–110, 2019.

[15] Ü. Aksoy, E. Karapınar, İ. Erhan, and V. Rakocevic, "Meir-Keeler type contractions on modular metric spaces," *Filomat*, vol. 32, pp. 3697–3707, 2018.

[16] E. Karapinar, B. Samet, and D. Zhang, "Meir–Keeler type contractions on JS-metric spaces and related fixed point theorems," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 2, 2018.

[17] F. Sk, A. Alam, and Q. H. Khan, *Meir-Keeler contractions under a class of transitive binary relation with applications, communicated*.

[18] S. Lipschutz, *Schaum'sOutlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York USA, 1964.

[19] R. D. Maddux, *Relation Algebras, Studies in Logic and the Foundations of Mathematics*, Elsevier, Amsertdam, 2006.

[20] V. Flaška, J. Ježek, T. Kepka, and J. Kortelainen, "Transitive closures of binary relations I," *Acta Univ. Carolin. Math. Phys.*, vol. 48, pp. 55–69, 2007.

[21] H. L. Skala, "Trellis theory," *Algebra universalis*, vol. 1, no. 1, pp. 218–233, 1971.

[22] A. Stouti and A. Maaden, "Fixed points and common fixed points theorems in pseudo-ordered sets," *Proyecciones*, vol. 32, no. 4, pp. 409–418, 2013.

[23] B. Samet and M. Turinici, "Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications," *Communications in Mathematical Analysis*, vol. 13, pp. 82–97, 2012.

[24] B. Kolman, R. C. Busby, and S. Ross, *Discrete Mathematical Structures*, PHI Pvt. Ltd., New Delhi, 3rd edition, 2000.

[25] A. Alam and M. Imdad, "Relation-theoretic metrical coincidence theorems," *Univerzitet u Nišu*, vol. 31, pp. 693–702, 2015.

[26] A.-F. Roldán-López-de-Hierro, E. Karapınar, and M. de la Sen, "Coincidence point theorems in quasi-metric spaces without assuming the mixed monotone property and consequences in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, 2014.

[27] M. Berzig and E. Karapnar, "Fixed point results for ($\alpha\psi$, $\beta\varphi$)-contractive mappings for a generalized altering distance," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 18 pages, 2013.

[28] M. Turinici, "Contractive maps in locally transitive relational metric spaces," *Scientific World Journal*, vol. 2014, article 169358, pp. 1–10, 2014.

[29] A. Alam, M. Arif, and M. Imdad, "Metrical fixed point theorems via locally finitely T-transitive binary relations under certain control functions," *Mathematical Notes*, vol. 20, no. 1, pp. 59–73, 2019.

[30] J. Harjani, B. López, and K. Sadarangini, "A fixed point theorem for Meir-Keeler contractions in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, no. 1, 2011.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2205423, 9 pages
https://doi.org/10.1155/2022/2205423

Research Article

Fixed Point Results of Jaggi-Type Hybrid Contraction in Generalized Metric Space

Jamilu Abubakar Jiddah[1], **Monairah Alansari**[2], **OM Kalthum S. K. Mohamed**[3,4], **Mohammed Shehu Shagari**[1], and **Awad A. Bakery**[3,5]

[1]*Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria*
[2]*Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[3]*College of Science and Arts at Khulis, Department of Mathematics, University of Jeddah, Jeddah, Saudi Arabia*
[4]*Academy of Engineering and Medical Sciences, Department of Mathematics, Khartoum, Sudan*
[5]*Faculty of Science, Department of Mathematics, Ain Shams University, Cairo, Egypt*

Correspondence should be addressed to OM Kalthum S. K. Mohamed; om_kalsoom2020@yahoo.com

Received 7 April 2022; Revised 23 May 2022; Accepted 31 May 2022; Published 18 June 2022

Academic Editor: Santosh Kumar

In this manuscript, a new family of contractions called Jaggi-type hybrid $(G-\phi)$-contraction is introduced and some fixed point results in generalized metric space that are not deducible from their akin in metric space are obtained. The preeminence of this class of contractions is that its contractive inequality can be extended in a variety of manners, depending on the given parameters. Consequently, several corollaries that reduce our result to other well-known results in the literature are highlighted and analyzed. Substantial examples are constructed to validate the assumptions of our obtained theorems and to show their distinction from corresponding results. Additionally, one of our obtained corollaries is applied to set up unprecedented existence conditions for the solution of a family of integral equations.

1. Introduction

The prominent Banach contraction in metric space has laid a solid foundation for fixed point theory in metric space. The applications of fixed point range across inequalities, approximation theory, optimization, and so on. Researchers in this area have introduced several new concepts in metric space and obtained a great deal of fixed point results for linear and nonlinear contractions. Recently, Karapınar and Fulga [1] introduced a new notion of hybrid contraction which is a unification of some existing linear and nonlinear contractions in metric space.

On the other hand, Mustafa [2] pioneered an extension of a metric space by the name, generalized metric space (or more precisely, G-metric space), and proved some fixed point results for Banach-type contraction mappings. This new generalization was brought to spotlight by Mustafa and Sims [3]. Subsequently, Mustafa et al. [4] obtained some engrossing fixed point results for Lipschitzian-type mappings on G-metric space. However, Jleli and Samet [5] as well as Samet et al. [6] noted that most of the fixed point results in G-metric space are direct consequences of existence results in corresponding metric space. Jleli and Samet [5] further observed that if a G-metric is consolidated into a quasimetric, then the resultant fixed point results become the known fixed point results in the setting of quasimetric space. Motivated by the latter observation, many investigators (see for instance, [7, 8]) have established techniques of obtaining fixed point results in G-metric space that are not deducible from their ditto ones in metric space or quasimetric space.

Following the existing literature, we realize that hybrid fixed point results in G-metric space are not adequately investigated. Hence, motivated by the ideas in [1, 7, 8], we introduce a new concept of Jaggi-type hybrid $(G-\phi)$-contraction in G-metric space and prove some related fixed point theorems. An example is constructed to demonstrate that our result is valid, an improvement of existing results and the main ideas obtained herein do not reduce to any existence result in metric space. Some corollaries are presented to show that the concept proposed herein is a generalization and improvement of well-known fixed point results in G-metric space. Finally, one of our obtained corollaries is applied to establish novel existence conditions for solution of a class of integral equations.

2. Preliminaries

In this section, we will present some fundamental notations and results that will be deployed subsequently.

Throughout, every set Φ is considered nonempty, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}_+$ the set of nonnegative real numbers.

Definition 1 (see [3]). Let Φ be a nonempty set and let $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function satisfying

(G_1) $G(r, s, t) = 0$ if $r = s = t$

(G_2) $0 < G(r, r, s)$ for all $r, s \in \Phi$ with $r \neq s$

(G_3) $G(r, r, s) \leq G(r, s, t)$, for all $r, s, t \in \Phi$ with $t \neq s$

(G_4) $G(r, s, t) = G(r, t, s) = G(s, r, t) = \cdots$ (symmetry in all variables)

(G_5) $G(r, s, t) \leq G(r, u, u) + G(u, s, t)$, for all $r, s, t, u \in \Phi$ (rectangle inequality)

Then, the function G is called a generalized metric or, more precisely, a G-metric on Φ, and the pair (Φ, G) is called a G-metric space.

Example 2 (see [4]). Let (Φ, d) be a usual metric space; then, (Φ, G_k) and (Φ, G_m) are G-metric spaces, where

$$\begin{aligned} G_k(r, s, t) &= d(r, s) + d(s, t) + d(r, t) \forall r, s, t \in \Phi, \\ G_m(r, s, t) &= \max \{d(r, s), d(s, t), d(r, t)\} \forall r, s, t \in \Phi. \end{aligned} \quad (1)$$

Definition 3 (see [4]). Let (Φ, G) be a G-metric space and let $\{r_n\}$ be a sequence of points of Φ. Then, $\{r_n\}$ is said to be G-convergent to r if $\lim_{n,m \longrightarrow \infty} G(r, r_n, r_m) = 0$; that is, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(r, r_n, r_m) < \varepsilon$, $\forall n, m \geq n_0$. We refer to r as the limit of the sequence $\{r_n\}$.

Proposition 4 (see [4]). *Let (Φ, G) be a G-metric space. Then, the following are equivalent:*

(i) $\{r_n\}$ is G-convergent to r

(ii) $G(r, r_n, r_m) \longrightarrow 0$, as $n \longrightarrow \infty$

(iii) $G(r_n, r, r) \longrightarrow 0$, as $n \longrightarrow \infty$

(iv) $G(r_n, r_n, r) \longrightarrow 0$, as $n \longrightarrow \infty$

Definition 5 (see [4]). Let (Φ, G) be a G-metric space. A sequence $\{r_n\}$ is called G-Cauchy if for any $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $G(r_n, r_m, r_l) < \varepsilon$, $\forall n, m, l \geq n_0$, that is, $G(r_n, r_m, r_l) \longrightarrow 0$, as $n, m, l \longrightarrow \infty$.

Proposition 6 (see [4]). *If (Φ, G) is a G-metric space, the following statements are equivalent:*

(i) The sequence $\{r_n\}$ is G-Cauchy

(ii) For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(r_n, r_m, r_m) < \varepsilon$, $\forall n, m \geq n_0$

Definition 7 (see [4]). Let (Φ, G) and (Φ', G') be G-metric spaces and $f : (\Phi, G) \longrightarrow (\Phi', G')$ be a function. Then, f is G-continuous at a point $u \in \Phi$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $r, s \in \Phi$; and $G(u, r, s) < \delta$ implies $G'(f(u), f(r), f(s)) < \varepsilon$. A function f is G-continuous on Φ if and only if it is G-continuous at all $u \in \Phi$.

Proposition 8 (see [4]). *Let (Φ, G) and (Φ', G') be G-metric spaces. Then, a function $f : (\Phi, G) \longrightarrow (\Phi', G')$ is said to be G-continuous at a point $r \in \Phi$ if and only if it is G-sequentially continuous at r; that is, whenever $\{r_n\}$ is G-convergent to r, $\{fr_n\}$ is G-convergent to fr.*

Definition 9 (see [4]). A G-metric space (Φ, G) is called symmetric G-metric space if

$$G(r, r, s) = G(s, r, r) \forall r, s \in \Phi. \quad (2)$$

Proposition 10 (see [4]). *Let (Φ, G) be a G-metric space. Then, the function $G(r, s, t)$ is jointly continuous in all variables.*

Proposition 11 (see [4]). *Every G-metric space (Φ, G) defines a metric space (Φ, d_G) by*

$$d_G(r, s) = G(r, s, s) + G(s, r, r) \forall r, s \in \Phi. \quad (3)$$

Note that for a symmetric G-metric space (Φ, G),

$$(\Phi, d_G) = 2G(r, s, s) \forall r, s \in \Phi. \quad (4)$$

On the other hand, if (Φ, G) is not symmetric, then by the G-metric properties,

$$\frac{3}{2} G(r, s, s) \leq d_G(r, s) \leq 3G(r, s, s) \forall r, s \in \Phi, \quad (5)$$

and that in general, these inequalities are sharp.

Definition 12 (see [4]). A G-metric space (Φ, G) is referred to as G-complete (or complete G-metric) if every G-Cauchy sequence in (Φ, G) is G-convergent in (Φ, G).

Proposition 13 (see [4]). *A G-metric space (Φ, G) is G -complete if and only if (Φ, d_G) is a complete metric space.*

Mustafa [2] proved the following result in the framework of G-metric space.

Theorem 14 (see [2]). *Let (Φ, G) be a complete G-metric space, and let $\Gamma : \Phi \longrightarrow \Phi$ be a mapping satisfying the* *following condition:*

$$G(\Gamma r, \Gamma s, \Gamma t) \leq kG(r, s, t), \tag{6}$$

for all $r, s, t \in \Phi$ where $0 \leq k < 1$; then, Γ has a unique fixed point (say u, i.e., $\Gamma u = u$), and Γ is G-continuous at u.

Definition 15 (see [9]). Let Ψ be the set of all functions $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

(i) ϕ is monotone increasing, that is, $t_1 \leq t_2$ implies $\phi(t_1) \leq \phi(t_2)$

(ii) the series $\sum_{n=0}^{\infty} \phi^n(t)$ is convergent for all $t > 0$

Then, ϕ is called a (c)-comparison function.

Remark 16. If $\phi \in \Psi$, then $\phi(t) < t$ for any $t > 0$, $\phi(0) = 0$, and ϕ is continuous at 0.

Karapınar and Fulga [1] gave the following definition of Jaggi-type hybrid contraction in metric space.

Definition 17 (see [1]). Let (Φ, d) be a complete metric space. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called a Jaggi-type hybrid contraction; if there exists $\phi \in \Phi$ such that

$$d(\Gamma r, \Gamma s) \leq \phi(M(r, s)), \tag{7}$$

for all distinct $r, s \in \Phi$, where

$$M(r,s) = \begin{cases} \left[\lambda_1 \left(\dfrac{d(r,\Gamma r)\cdot d(s,\Gamma s)}{d(r,s)}\right)^q + \lambda_2 d(r,s)^q\right]^{1/q}, & \text{for } q > 0, r, s \in \Phi, r \neq s, \\ d(r,\Gamma r)^{\lambda_1} \cdot d(s,\Gamma s)^{\lambda_2}, & \text{for } q = 0, r, s \in \Phi \setminus Fix(\Gamma). \end{cases} \tag{8}$$

$\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ and $Fix(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

3. Main Results

We begin this section by defining the notion of Jaggi-type hybrid $(G - \phi)$-contraction in G-metric space.

Definition 18. Let (Φ, G) be a G-metric space. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called a Jaggi-type hybrid $(G - \phi)$-contraction, if there exists $\phi \in \Phi$ such that

$$G(\Gamma r, \Gamma s, \Gamma^2 s) \leq \phi(M(r, s, \Gamma s)), \tag{9}$$

for all $r, s \in \Phi \setminus Fix(\Gamma)$, where

$$M(r,s,\Gamma s) = \begin{cases} \left[\lambda_1 \left(\dfrac{G(r,\Gamma r,\Gamma^2 r)\cdot G(s,\Gamma s,\Gamma^2 s)}{G(r,s,\Gamma s)}\right)^q + \lambda_2 G(r,s,\Gamma s)^q\right]^{1/q}, & \text{for } q > 0, \\ G(r,\Gamma r,\Gamma^2 r)^{\lambda_1} \cdot G(s,\Gamma s,\Gamma^2 s)^{\lambda_2}, & \text{for } q = 0. \end{cases} \tag{10}$$

$\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ and $Fix(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

We now present the following results.

Theorem 19. *Let (Φ, G) be a complete G-metric space and let $\Gamma : \Phi \longrightarrow \Phi$ be a continuous Jaggi-type hybrid $(G - \phi)$-contraction on (Φ, G). Then, Γ has a fixed point in Φ (say c), and for any $c_0 \in \Phi$, the sequence $\{\Gamma^n c_0\}_{n\in\mathbb{N}}$ converges to c.*

Proof. Let $r_0 \in \Phi$ be an arbitrary point and define a sequence $\{r_n\}_{n\in\mathbb{N}}$ in Φ by $r_n = \Gamma^n r_0$. If there exists some $n \in \mathbb{N}$ such that $\Gamma r_n = r_{n+1} = r_n$, then r_n is a fixed point of Γ, and so the proof is complete. Assume now that $r_n \neq r_{n-1}$ for any $n \in \mathbb{N}$. Since Γ is a Jaggi-type hybrid $(G - \phi)$-contraction, then we have from (9) that

$$G(r_n, r_{n+1}, r_{n+2}) = G(\Gamma r_{n-1}, \Gamma r_n, \Gamma^2 r_n) \leq \phi(M(r_{n-1}, r_n, \Gamma r_n)). \tag{11}$$

□

We then consider the given cases of (10).

Case 1. For $q > 0$, we have

$$\begin{aligned} M(r_{n-1}, r_n, \Gamma r_n) &= \left[\lambda_1 \left(\frac{G(r_{n-1}, \Gamma r_{n-1}, \Gamma^2 r_{n-1}) G(r_n, \Gamma r_n, \Gamma^2 r_n)}{G(r_{n-1}, r_n, \Gamma r_n)}\right)^q + \lambda_2 G(r_{n-1}, r_n, \Gamma r_n)^q\right]^{1/q} \\ &= \left[\lambda_1 \left(\frac{G(r_{n-1}, r_n, r_{n+1}) G(r_n, r_{n+1}, r_{n+2})}{G(r_{n-1}, r_n, r_{n+1})}\right)^q + \lambda_2 G(r_{n-1}, r_n, r_{n+1})^q\right]^{1/q} = [\lambda_1 G(r_n, r_{n+1}, r_{n+2})^q + \lambda_2 G(r_{n-1}, r_n, r_{n+1})^q]^{1/q}. \end{aligned} \tag{12}$$

Since ϕ is nondecreasing, if we assume that

$$G(r_{n-1}, r_n, r_{n+1}) \leq G(r_n, r_{n+1}, r_{n+2}), \tag{13}$$

then (11) becomes

$$\begin{aligned} G(r_n, r_{n+1}, r_{n+2}) &\leq \phi\left(\left[\lambda_1 G(r_n, r_{n+1}, r_{n+2})^q + \lambda_2 G(r_{n-1}, r_n, r_{n+1})^q\right]^{1/q}\right) \\ &\leq \phi\left(\left[\lambda_1 G(r_n, r_{n+1}, r_{n+2})^q + \lambda_2 G(r_n, r_{n+1}, r_{n+2})^q\right]^{1/q}\right) \\ &= \phi\left((\lambda_1 + \lambda_2)^{1/q} G(r_n, r_{n+1}, r_{n+2})\right) \\ &= \phi(G(r_n, r_{n+1}, r_{n+2})) < G(r_n, r_{n+1}, r_{n+2}), \end{aligned} \tag{14}$$

which is a contradiction. Therefore, for every $n \in \mathbb{N}$, we have

$$G(r_n, r_{n+1}, r_{n+2}) < G(r_{n-1}, r_n, r_{n+1}), \tag{15}$$

so that (11) becomes

$$\begin{aligned} G(r_n, r_{n+1}, r_{n+2}) &\leq \phi\left(\left[\lambda_1 G(r_n, r_{n+1}, r_{n+2})^q + \lambda_2 G(r_{n-1}, r_n, r_{n+1})^q\right]^{1/q}\right) \\ &\leq \phi\left((\lambda_1 + \lambda_2)^{1/q} G(r_{n-1}, r_n, r_{n+1})\right) \\ &\leq \phi(G(r_{n-1}, r_n, r_{n+1})). \end{aligned} \tag{16}$$

Continuing inductively, we have

$$G(r_n, r_{n+1}, r_{n+2}) \leq \phi^n(G(r_0, r_1, r_2)). \tag{17}$$

Now, since

$$G(r_n, r_n, r_{n+1}) \leq G(r_n, r_{n+1}, r_{n+2}) \leq \phi^n(G(r_0, r_1, r_2)), \tag{18}$$

for all $n \in \mathbb{N}$ with $r_{n+1} \neq r_{n+2}$, then for any $n, m \in \mathbb{N}$ with $n < m$ and by rectangle inequality, we have

$$\begin{aligned} G(r_n, r_n, r_m) &\leq G(r_n, r_n, r_{n+1}) + G(r_{n+1}, r_{n+1}, r_{n+2}) \\ &\quad + \cdots + G(r_{m-1}, r_{m-1}, r_m) \\ &\leq \left(\phi^n + \phi^{n+1} + \phi^{n+2} + \cdots + \phi^{m-1}\right) G(r_0, r_1, r_2) \\ &= \sum_{i=n}^{m-1} \phi^i(G(r_0, r_1, r_2)) \leq \sum_{i=n}^{\infty} \phi^i(G(r_0, r_1, r_2)). \end{aligned} \tag{19}$$

Since ϕ is a (c)-comparison function, then the series $\sum_{i=0}^{\infty} \phi^i(G(r_0, r_1, r_2))$ is convergent, and so denoting by $S_p = \sum_{i=0}^{\infty} \phi^i(G(r_0, r_1, r_2))$, we have

$$G(r_n, r_n, r_m) \leq S_{m-1} - S_{n-1}. \tag{20}$$

Hence, as $n, m \longrightarrow \infty$, we see that

$$G(r_n, r_n, r_m) \longrightarrow 0. \tag{21}$$

Thus, $\{r_n\}$ is a G-Cauchy sequence in (Φ, G) and so by the completeness of (Φ, G), there exists $c \in \Phi$ such that $\{r_n\}$ is G-convergent to c, that is,

$$\lim_{n \longrightarrow \infty} G(r_n, r_n, c) = 0. \tag{22}$$

We will now show that c is a fixed point of Γ. By the assumption that Γ is continuous, we have

$$\begin{aligned} \lim_{n \longrightarrow \infty} G(c, c, \Gamma c) &= \lim_{n \longrightarrow \infty} G(r_{n+1}, r_{n+1}, \Gamma c) \\ &= \lim_{n \longrightarrow \infty} G(\Gamma r_n, \Gamma r_n, \Gamma c) \\ &= \lim_{n \longrightarrow \infty} G(\Gamma r_n, \Gamma r_n, \Gamma r_n) = 0, \end{aligned} \tag{23}$$

so we get $\Gamma c = c$, that is, c is a fixed point of Γ.

Case 2. For $q = 0$, we have

$$\begin{aligned} M(r_{n-1}, r_n, \Gamma r_n) &= G\left(r_{n-1}, \Gamma r_{n-1}, \Gamma^2 r_{n-1}\right)^{\lambda_1} \cdot G\left(r_n, \Gamma r_n, \Gamma^2 r_n\right)^{\lambda_2} \\ &= G\left(r_{n-1}, r_n, r_{n+1}\right)^{\lambda_1} \cdot G\left(r_n, r_{n+1}, r_{n+2}\right)^{\lambda_2}. \end{aligned} \tag{24}$$

Now, if $G(r_{n-1}, r_n, r_{n+1}) \leq G(r_n, r_{n+1}, r_{n+2})$, then (11) becomes

$$G(r_n, r_{n+1}, r_{n+2}) < G(r_n, r_{n+1}, r_{n+2}), \tag{25}$$

which is a contradiction. Therefore,

$$G(r_n, r_{n+1}, r_{n+2}) < G(r_{n-1}, r_n, r_{n+1}). \tag{26}$$

Hence, by (11) we have

$$\begin{aligned} G(r_n, r_{n+1}, r_{n+2}) &< \phi(G(r_{n-1}, r_n, r_{n+1})) < \phi^2(G(r_{n-1}, r_n, r_{n+1})) \\ &< \cdots < \phi^n(G(r_0, r_1, r_2)). \end{aligned} \tag{27}$$

By similar argument as the case of $q > 0$, we can show that there exists a G-Cauchy sequence $\{r_n\}$ in (Φ, G) and a point c in Φ such that $\lim_{n \longrightarrow \infty} r_n = c$. Similarly, under the assumption that Γ is continuous and by the uniqueness of limit, we have that $\Gamma c = c$, that is, c is a fixed point of Γ.

In the next result, we examine the existence of unique fixed point of Γ under the restriction of continuity of some iterates of Γ.

Theorem 20. *Let (Φ, G) be a complete G-metric space and let $\Gamma : \Phi \longrightarrow \Phi$ be a Jaggi-type hybrid $(G - \phi)$-contraction. If for some integer $i > 2$, we have that Γ^i is continuous, then Γ has a unique fixed point in Φ.*

Proof. In Theorem 19, we have established that there exists a G-Cauchy sequence $\{r_n\}_{n \in \mathbb{N}}$ in (Φ, G) with $r_n = \Gamma r_{n-1}$ such that $r_n \longrightarrow c$ for some c in Φ. Let $\{r_{n_l}\}$ be a subsequence of $\{r_n\}_{n \in \mathbb{N}}$ where $n_l = l \cdot i$ for all $l \in \mathbb{N}$, $i > 2$ fixed. Notice that Γ^0 is an identity self-mapping on Φ so that $r_{n_l} = \Gamma^i r_{n_l - i}$.

Hence, by the continuity of Γ^i, we have

$$G\left(c, \Gamma^i c, \Gamma^{i+1} c\right) = \lim_{l \longrightarrow \infty} G\left(c, \Gamma^i r_{n_l - i}, \Gamma^{i+1} r_{n_l-(i+1)}\right) = \lim_{l \longrightarrow \infty} G\left(c, r_{n_l}, r_{n_l}\right) = G(c, c, c) = 0, \tag{28}$$

that is, c is a fixed point of Γ^i. □

To see that c is a fixed point of Γ, assume contrary that $\Gamma z \neq z$. Then in that case, $\Gamma^{i-j-1} z \neq \Gamma^{i-j} z$ for any $j = 0, 1, \cdots, i-1$. Hence, by (9), we have

$$G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right) \leq \phi\left(M\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right)\right) < M\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right). \tag{29}$$

Considering Case 1, we obtain

$$\begin{aligned} M\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right) &= \left[\lambda_1 \left(\frac{G\left(\Gamma^{i-j-1} c, \Gamma\left(\Gamma^{i-j-1} c\right), \Gamma^2\left(\Gamma^{i-j-1} c\right)\right) G\left(\Gamma^{i-j} c, \Gamma\left(\Gamma^{i-j} c\right), \Gamma^2\left(\Gamma^{i-j} c\right)\right)}{G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma\left(\Gamma^{i-j} c\right)\right)}\right)^q + \lambda_2 G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma\left(\Gamma^{i-j} c\right)\right)^q\right]^{1/q} \\ &= \left[\lambda_1 \left(\frac{G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right) G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right)}{G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right)}\right)^q + \lambda_2 G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right)^q\right]^{1/q} \\ &= \left[\lambda_1 G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right)^q + \lambda_2 G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right)^q\right]^{1/q}, \end{aligned} \tag{30}$$

so that (29) becomes

$$G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right)^q (1 - \lambda_1) < \lambda_2 G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right)^q. \tag{31}$$

Since $\lambda_1 + \lambda_2 = 1$, then for every $j = 0, 1, \cdots, i-1$, we have

$$G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right) < G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right). \tag{32}$$

This clearly implies that for every $l = j, j+1, \cdots, i-1$,

$$G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right) < G\left(\Gamma^{i-j-l-1} c, \Gamma^{i-j-l} c, \Gamma^{i-j-l+1} c\right). \tag{33}$$

In particular, letting $j = 0$ and $l = i-1$, the above inequality becomes

$$G\left(c, \Gamma^i c, \Gamma^{i+1} c\right) = G\left(\Gamma^i c, \Gamma^{i+1} c, \Gamma^{i+2} c\right) < G\left(c, \Gamma c, \Gamma^2 c\right), \tag{34}$$

which is a contradiction. Hence, $\Gamma c = c$.

For Case 2, we have

$$\begin{aligned} M\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right) &= G\left(\Gamma^{i-j-1} c, \Gamma\left(\Gamma^{i-j-1} c\right), \Gamma^2\left(\Gamma^{i-j-1} c\right)\right)^{\lambda_1} \\ &\quad \cdot G\left(\Gamma^{i-j} c, \Gamma\left(\Gamma^{i-j} c\right), \Gamma^2\left(\Gamma^{i-j} c\right)\right)^{\lambda_2} \\ &= G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right)^{\lambda_1} \\ &\quad \cdot G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right)^{\lambda_2}, \end{aligned} \tag{35}$$

so that (29) becomes

$$G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right)^{(1-\lambda_2)} < G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right)^{\lambda_1}, \tag{36}$$

implying that

$$G\left(\Gamma^{i-j} c, \Gamma^{i-j+1} c, \Gamma^{i-j+2} c\right) < G\left(\Gamma^{i-j-1} c, \Gamma^{i-j} c, \Gamma^{i-j+1} c\right). \tag{37}$$

By similar argument as in Case 1, we obtain a contradiction. Hence, $\Gamma c = c$.

Example 21. Let $\Phi = [-1, 1]$ and let $\Gamma : \Phi \longrightarrow \Phi$ be a self-mapping on Φ defined by

$$\Gamma r = \begin{cases} \dfrac{r}{5}, & \text{if } r \in \{-1, 1\}, \\ \dfrac{1}{5}, & \text{if } r \in (-1, 1), \end{cases} \tag{38}$$

for all $r \in \Phi$. Define $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$G(r, s, \Gamma s) = |r - s| + |r - \Gamma s| + |s - \Gamma s| \forall r, s \in \Phi. \tag{39}$$

Then, (Φ, G) is a complete G-metric space and Γ is continuous for all $r \in \Phi$. Define $\phi \in \Psi$ by $\phi(x) = x/2$ for all $x \geq 0$.

To see that Γ is a Jaggi-type hybrid $(G - \phi)$-contraction, notice that $G(\Gamma r, \Gamma s, \Gamma^2 s) = 0$ for all $r, s \in (-1, 1)$. Hence, inequality (9) holds for all $r, s \in (-1, 1)$.

Now, for $r, s \in \{-1, 1\}$, if $r = s = 1$, then $G(\Gamma r, \Gamma s, \Gamma^2 s) = 0$ for all $q \geq 0$. If $r = s = -1$, then letting $\lambda_1 = \lambda_2 = 1/2$ and $q = 1$, we obtain

$$\begin{aligned} G\left(\Gamma r, \Gamma s, \Gamma^2 s\right) &= G\left(\frac{-1}{5}, \frac{-1}{5}, \frac{1}{5}\right) = \frac{4}{5} < \frac{13}{10} = \frac{1}{2}\left(\frac{13}{5}\right) \\ &= \frac{1}{2}\left(M\left(-1,-1, \frac{-1}{5}\right)\right) = \phi(M(r, s, \Gamma s)). \end{aligned} \tag{40}$$

Also, for $q = 0$, we have

$$G\left(\Gamma r, \Gamma s, \Gamma^2 s\right) = \frac{4}{5} < \frac{1}{2}\left(\frac{12}{5}\right) = \phi(M(r, s, \Gamma s)). \tag{41}$$

If $r \neq s$, then letting $\lambda_1 = 2/10$, $\lambda_2 = 4/5$, and $q = 3$, we obtain

$$\begin{aligned} G\left(\Gamma r, \Gamma s, \Gamma^2 s\right) &= G\left(\frac{-1}{5}, \frac{1}{5}, \frac{1}{5}\right) = G\left(\frac{1}{5}, \frac{-1}{5}, \frac{1}{5}\right) \\ &= \frac{4}{5} < \frac{8}{5} = \frac{1}{2}\left(\frac{16}{5}\right) = \frac{1}{2}\left(M\left(-1, 1, \frac{1}{5}\right)\right) \\ &= \frac{1}{2}\left(M\left(1,-1, \frac{-1}{5}\right)\right) = \phi(M(r, s, \Gamma s)). \end{aligned} \tag{42}$$

Also, for $q = 0$, we take $\lambda_1 = \lambda_2 = 1/2$. Then,

$$\begin{aligned} G\left(\Gamma r, \Gamma s, \Gamma^2 s\right) &= G\left(\frac{-1}{5}, \frac{1}{5}, \frac{1}{5}\right) = G\left(\frac{1}{5}, \frac{-1}{5}, \frac{1}{5}\right) \\ &= \frac{4}{5} < \frac{49}{50} = \frac{1}{2}\left(\frac{98}{50}\right) = \frac{1}{2}\left(M\left(-1, 1, \frac{1}{5}\right)\right) \\ &= \frac{1}{2}\left(M\left(1,-1, \frac{-1}{5}\right)\right) = \phi(M(r, s, \Gamma s)). \end{aligned} \tag{43}$$

Hence, inequality (9) is satisfied for all $r, s \in \Phi$. Therefore, Γ is a Jaggi-type hybrid $(G - \phi)$-contraction. Consequently, all the assumptions of Theorem 19 are satisfied, and $r = 1/5$ is the fixed point of Γ.

We now demonstrate that our result is independent of the result of Karapınar and Fulga [1]. Let $d : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be defined by

$$d(r, s) = |r - s| \forall r, s \in \Phi. \tag{44}$$

Consider $r, s \in \{-1, 1\}$ and take for Case 1, $r \neq s$, $\lambda_1 = 3/4$, $\lambda_2 = 1/4$, and $q = 1$. Then, inequality (9) becomes

$$\begin{aligned} G\left(\Gamma r, \Gamma s, \Gamma^2 s\right) &= G\left(\frac{-1}{5}, \frac{1}{5}, \frac{1}{5}\right) = G\left(\frac{1}{5}, \frac{-1}{5}, \frac{1}{5}\right) = \frac{4}{5} < \frac{43}{50} \\ &= \frac{1}{2}\left(\frac{43}{25}\right) = \frac{1}{2}\left(M\left(-1, 1, \frac{1}{5}\right)\right) \\ &= \frac{1}{2}\left(M\left(1,-1, \frac{-1}{5}\right)\right) = \phi(M(r, s, \Gamma s)), \end{aligned} \tag{45}$$

while inequality (7) due to Karapınar and Fulga [1] yields

$$\begin{aligned} d(\Gamma r, \Gamma s) &= d\left(\frac{-1}{5}, \frac{1}{5}\right) = d\left(\frac{1}{5}, \frac{-1}{5}\right) = \frac{2}{5} > \frac{37}{100} = \frac{1}{2}\left(\frac{37}{50}\right) \\ &= \frac{1}{2}(M(-1, 1)) = \frac{1}{2}(M(1,-1)) = \phi(M(r, s)). \end{aligned} \tag{46}$$

Also, Karapınar and Fulga [1] declared in Definition (17) that r and s are distinct, since $M(r, s)$ is undefined for Case 1 if $r = s$. However, our result is valid for all $r, s \in \Phi \setminus Fix(\Gamma)$.

The above comparison is illustrated graphically for all $r, s \in \{-1, 1\}$, using the following Figures 1 and 2.

Therefore, Jaggi-type hybrid $(G - \phi)$-contraction is not Jaggi-type hybrid contraction defined by Karapınar and Fulga [1], and so Theorem 1 due to Karapınar and Fulga [1] is not applicable to this example.

Corollary 22 (see Theorem 14). *Let (Φ, G) be a complete G-metric space, and let $\Gamma : \Phi \longrightarrow \Phi$ be a mapping satisfying the following condition:*

$$G(\Gamma r, \Gamma s, \Gamma t) \leq kG(r, s, t), \tag{47}$$

for all $r, s, t \in \Phi$ where $0 \leq k < 1$; then, Γ has a unique fixed point (say u) and Γ is G-continuous at u.

Proof. Consider Definition (18) and let $\Gamma s = t$, $\lambda_1 = 0$, $\lambda_2 = 1$, $q > 0$, and $\phi(p) = kp$ for all $p \geq 0$ and $k \in [0, 1)$. Clearly, $\phi \in \Psi$ and Γ is a Jaggi-type hybrid $(G - \phi)$-contraction. Hence, (9) coincides with (6) of Theorem 14 due to Mustafa [2]. Therefore, it is easy to see that we can find a unique point u in Φ such that $\Gamma u = u$ and Γ is G-continuous at u. □

Corollary 23 (see [10], Theorem 3.1). *Let (Φ, G) be a complete G-metric space. Suppose the mapping $\Gamma : \Phi \longrightarrow \Phi$ satisfies*

$$G(\Gamma r, \Gamma s, \Gamma t) \leq \phi(G(r, s, t)), \tag{48}$$

for all $r, s, t \in \Phi$. Then, Γ has a unique fixed point (say u) and Γ is G-continuous at u.

Proof. Consider Definition 18 and let $\Gamma s = t$, $\lambda_1 = 0$, $\lambda_2 = 1$ and $q > 0$. Then,

$$M(r, s, t) = G(r, s, t), \tag{49}$$

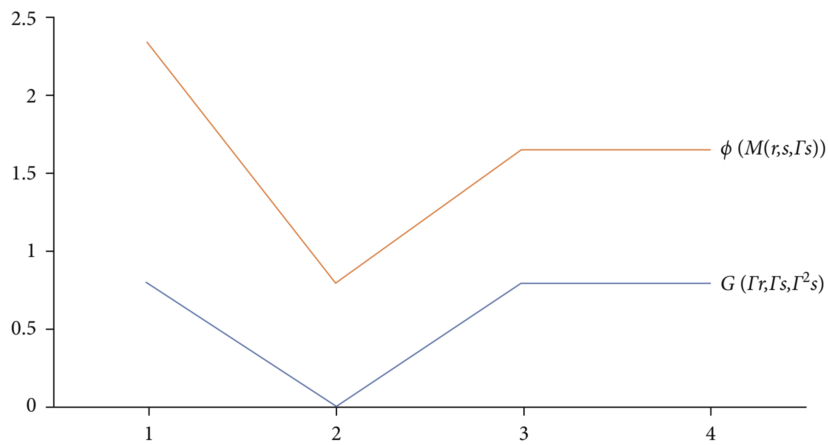


FIGURE 1: Illustration of contractive inequality (9) for all $r, s \in \{-1, 1\}$.

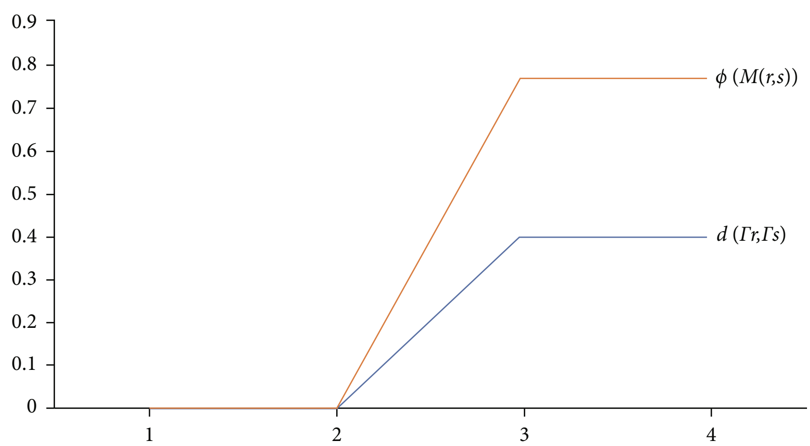


FIGURE 2: Illustration of contractive inequality (7) for all $r, s \in \{-1, 1\}$.

for all $r, s, t \in \Phi$. Hence, inequality (9) becomes

$$G(\Gamma r, \Gamma s, \Gamma t) \leq \phi(G(r, s, t)), \tag{50}$$

for all $r, s, t \in \Phi$ and $\phi \in \Psi$. This coincides with Theorem 3.1 due to Shatanawi [10] and so the proof follows in a similar manner. □

By specializing the parameters $\lambda_i (i = 1, 2)$ and q, as well as letting $\phi(p) = \mu p$ for all $p \geq 0$ and for $\mu \in (0, 1)$, the following result is also a direct consequence of Theorem 19.

Corollary 24. *Let (Φ, G) be a complete G-metric space. If there exists $\mu \in (0, 1)$ such that for all $r, s \in \Phi$, the mapping $\Gamma : \Phi \longrightarrow \Phi$ satisfies*

$$G\left(\Gamma r, \Gamma s, \Gamma^2 s\right) \leq \mu G(r, s, \Gamma s), \tag{51}$$

then Γ has a fixed point in Φ.

4. Applications to Solution of Integral Equation

In this section, Corollary 24 is applied to examine the existence criteria for a solution to a class of integral equations. Ideas in this section are motivated by [7, 11, 12].

Consider the integral equation

$$r(y) = \int_a^b \mathscr{L}(y, x) f(x, r(x)) dx, y \in [a, b]. \tag{52}$$

Let $\Phi = C([a, b], \mathbb{R})$ be the set of all continuous real-valued functions. Define $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$G(r, s, \Gamma s) = \max_{y \in [a,b]} |r(y) - s(y)| + \max_{y \in [a,b]} |r(y) - \Gamma s(y)| + \max_{y \in [a,b]} |s(y) \Gamma s(y)|, \tag{53}$$

$$\forall r, s \in \Phi, y \in [a, b]. \tag{54}$$

Then, (Φ, G) is a complete G-metric space.

Define a function $\Gamma : \Phi \longrightarrow \Phi$ as follows:

$$\Gamma r(y) = \int_a^b \mathscr{L}(y, x) f(x, r(x)) dx, y \in [a, b]. \tag{55}$$

Then, a point u^* is said to be a fixed point of Γ if and only if u^* is a solution to (52).

Now, we study existence conditions of the integral equation (52) under the following hypotheses.

Theorem 25. *Assume that the following conditions are satisfied:*

(C_1) $\mathscr{L} : [a, b] \times [a, b] \longrightarrow \mathbb{R}_+$ *and* $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous*

(C_2) For all $r, s \in \Phi$, $x \in [a, b]$, *we have* $|f(x, r(x)) - f(x, s(x))| \leq |r(x) - s(x)|$

(C_3) $\max_{y\in[a,b]} \int_a^b \mathscr{L}(y, x) dx \leq \mu$ *for some* $\mu < 1$

Then, the integral equation (52) has a solution u^* in Φ.

Proof. Observe that for any $r, s \in \Phi$, using (55) and the above hypotheses, we obtain

$$\begin{aligned} |\Gamma r(y) - \Gamma s(y)| &= \left| \int_a^b [\mathscr{L}(y, x) f(x, r(x)) - \mathscr{L}(y, x) f(x, s(x))] dx \right| \\ &\leq \int_a^b \mathscr{L}(y, x) |f(x, r(x)) - f(x, s(x))| dx \\ &\leq \int_a^b \mathscr{L}(y, x) |r(x) - s(x)| dx \\ &\leq \int_a^b \mathscr{L}(y, x) \max_{x\in[a,b]} |r(x) - s(x)| dx \\ &\leq \mu \max_{y\in[a,b]} |r(y) - s(y)|. \end{aligned} \tag{56}$$

□

Using this in (54), we have

$$\begin{aligned} G(\Gamma r, \Gamma s, \Gamma^2 s) &= \max_{y\in[a,b]} |\Gamma r - \Gamma s| + \max_{y\in[a,b]} |\Gamma r - \Gamma^2 s| \\ &\quad + \max_{y\in[a,b]} |\Gamma s - \Gamma^2 s| \leq \mu \max_{y\in[a,b]} |r - s| \\ &\quad + \mu \max_{y\in[a,b]} |r - \Gamma s| + \mu \max_{y\in[a,b]} |s - \Gamma s| \\ &= \mu \left(\max_{y\in[a,b]} |r - s| + \max_{y\in[a,b]} |r - \Gamma s| + \max_{y\in[a,b]} |s - \Gamma s| \right) \\ &= \mu G(r, s, \Gamma s). \end{aligned} \tag{57}$$

Hence, all the required hypotheses of Corollary 24 are satisfied, implying that there exists a solution u^* in Φ of the integral equation (52).

Conversely, if u^* is a solution of (52), then u^* is also a solution of (55) so that $\Gamma u^* = u^*$, that is, u^* is a fixed point of Γ.

Remark 26.

(i) We can deduce a number of corollaries by particularizing some of the parameters in Definition 18

(ii) None of the results presented in this work can be expressed in the form $G(r, s, s)$ or $G(r, r, s)$. Hence, they cannot be obtained from their corresponding versions in metric space

5. Conclusion

A generalization of metric space was introduced by Mustafa and Sims [3], namely, G-metric space and several fixed point results were studied in that space. However, Jleli and Samet [5] as well as Samet et al. [6] established that most fixed point theorems obtained in G-metric space are direct consequences of their analogues in metric space. Contrary to the above observation, a new family of contractions called Jaggi-type hybrid $(G - \phi)$-contraction is introduced in this manuscript and some fixed point theorems that cannot be deduced from their corresponding ones in metric space are proved. The main distinction of this class of contractions is that its contractive inequality is expressible in a number of ways with respect to multiple parameters. Consequently, some corollaries including recently announced results in the literature are highlighted and analyzed. Nontrivial comparative examples are constructed to validate the assumptions of our obtained theorems. Furthermore, one of our obtained corollaries is applied to set up novel existence conditions for solution of a class of integral equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant no. UJ-21-DR-92. The authors, therefore, acknowledge with thanks the university technical and financial support.

References

[1] E. Karapinar and A. Fulga, "A hybrid contraction that involves Jaggi type," *Symmetry*, vol. 11, no. 5, p. 715, 2019.

[2] Z. Mustafa, *A New Structure for Generalized Metric Spaces-with Applications to Fixed Point Theory, [PhD Thesis]*, The University of Newcastle, Australia, 2005.

[3] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[4] Z. Mustafa, H. Obiedat, and F. Awawdeh, "Some fixed point theorem for mapping on complete G-metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 189870, 2008.

[5] M. Jleli and B. Samet, "Remarks on G-metric spaces and fixed point theorems," *Fixed Point Theory and Applications*, vol. 2012, 2012.

[6] B. Samet, C. Vetro, and F. Vetro, "Remarks on G-metric spaces," *International Journal of Analysis*, vol. 2013, Article ID 917158, 2013.

[7] J. Chen, C. Zhu, L. Zhu, Department of Mathematics, Nanchang University, 330031 Nanchang, China, and Department of Mathematics, Jiangxi Agricultural University, 330045 Nanchang, China, "A note on some fixed point theorems on G-metric spaces," *Journal of Applied Analysis and Computation*, vol. 11, no. 1, pp. 101–112, 2021.

[8] E. Karapinar and R. P. Agarwal, "Further fixed point results on G-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[9] M. Alghamdi and E. Karapinar, "G-β-ψ-contractive type mappings in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[10] W. Shatanawi, "Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 181650, 2010.

[11] Z. Mustafa, M. Arshad, S. U. Khan, J. Ahmad, and M. M. M. Jaradat, "Common fixed points for multivalued mappings in G-metric spaces with applications," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 5, pp. 2550–2564, 2017.

[12] M. Younis, D. Singh, S. Radenovic, and M. Imdad, "Convergence theorems for generalized contractions and applications," *Univerzitet u Nišu*, vol. 34, no. 3, pp. 945–964, 2020.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1934975, 23 pages
https://doi.org/10.1155/2022/1934975

Research Article

Convergence Analysis of New Construction Explicit Methods for Solving Equilibrium Programming and Fixed Point Problems

Chainarong Khunpanuk,[1] **Nuttapol Pakkaranang**,[1] **and Bancha Panyanak**[2,3]

[1]*Mathematics and Computing Science Program, Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun 67000, Thailand*
[2]*Research Group in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[3]*Data Science Research Center, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*

Correspondence should be addressed to Bancha Panyanak; bancha.p@cmu.ac.th

Received 12 March 2022; Revised 8 May 2022; Accepted 24 May 2022; Published 13 June 2022

Academic Editor: Cristian Chifu

In this paper, we present improved iterative methods for evaluating the numerical solution of an equilibrium problem in a Hilbert space with a pseudomonotone and a Lipschitz-type bifunction. The method is built around two computing phases of a proximal-like mapping with inertial terms. Many such simpler step size rules that do not involve line search are examined, allowing the technique to be enforced more effectively without knowledge of the Lipschitz-type constant of the cost bifunction. When the control parameter conditions are properly defined, the iterative sequences converge weakly on a particular solution to the problem. We provide weak convergence theorems without knowing the Lipschitz-type bifunction constants. A few numerical tests were performed, and the results demonstrated the appropriateness and rapid convergence of the new methods over traditional ones.

1. Introduction

Let Π stand for a certain Hilbert space and Ξ stand for a nonempty closed convex subset of Π. The research is about an iterative technique for solving the equilibrium problem ((1), to make it short). Let $\Gamma : \Pi \times \Pi \longrightarrow \mathbb{R}$ be a bifunction with $\Gamma(y_1, y_1) = 0$, for each $y_1 \in \Xi$. An *equilibrium problem* for granted bifunction Γ on Ξ is interpreted this way: find $\hbar^* \in \Xi$ such that

$$\Gamma(\hbar^*, y_1) \geq 0, \quad \forall y_1 \in \Xi. \tag{1}$$

The numerical evaluation of the equilibrium problem under the following conditions is the focus of this study. We will assume that the following conditions have been satisfied:

For $\Gamma 1$, the solution set of a problem (1) is denoted by $\mathrm{sol}(\Gamma, \Xi)$ and it is nonempty.

For $\Gamma 2$, a bifunction Γ is said to be *pseudomonotone* [1, 2], i.e.,

$$\Gamma(y_1, y_2) \geq 0 \Rightarrow \Gamma(y_2, y_1) \leq 0, \quad \forall y_1, y_2 \in \Xi. \tag{2}$$

For $\Gamma 3$, a bifunction Γ is said to be *Lipschitz-type continuous* [3] on Ξ if there exist two constants $c_1, c_2 > 0$, such that

$$\begin{aligned} \Gamma(y_1, y_3) \leq \Gamma(y_1, y_2) + \Gamma(y_2, y_3) + c_1 \|y_1 - y_2\|^2 \\ + c_2 \|y_2 - y_3\|^2, \quad \forall y_1, y_2, y_3 \in \Xi. \end{aligned} \tag{3}$$

For $\Gamma 4$, for any sequence $\{y_k\} \subset \Xi$ satisfying $y_k \rightharpoonup y^*$, then, the following inequality holds:

$$\limsup_{k \longrightarrow +\infty} \Gamma(y_k, y_1) \leq \Gamma(y^*, y_1), \quad \forall y_1 \in \Xi. \tag{4}$$

For ($\varGamma5$), $\varGamma(y_1,\cdot)$ is convex and subdifferentiable on $\varPi$ for each fixed $y_1 \in \varPi$.

Let us represent a problem's solution set as $\mathrm{sol}(\varGamma, \varXi)$, and we will assume in the following text that this solution set is not empty. Researchers are interested in the equilibrium problem because it connects many mathematical problems, including fixed point problems, vector and scalar minimization problems, variational inequalities, complementarity problems, saddle point problems, Nash equilibrium problems in noncooperative games, and inverse optimization problems (see for further information [2, 4–9]). It also has a variety of applications in economics [10], the dynamics of offer and demand [11], and it continues to use the theoretical framework of noncooperative games and Nash's equilibrium models [12, 13]. The phrase "equilibrium problem" was first used in the literature in 1992 by Muu and Oettli [9] and was further investigated by Blum [2]. More precisely, we consider two applications for the problem (1). (i) A *variational inequality problem* for an operator $\mathfrak{I}_1 : \varXi \longrightarrow \varPi$ is stated as follows: find $\hbar^* \in \varXi$ such that

$$\langle \mathfrak{I}_1(\hbar^*) y_1 - \hbar^* \rangle \geq 0, \quad \forall y_1, y_2 \in \varXi. \tag{5}$$

Let us define a bifunction $\varGamma$ as follows:

$$\varGamma(y_1, y_2) \coloneqq \langle \mathfrak{I}_1(y_1), y_2 - y_1 \rangle, \quad \forall y_1, y_2 \in \varXi. \tag{6}$$

Then, the equilibrium problem converts into the problem of variational inequalities defined in (5) and Lipschitz constants of the mapping $\mathfrak{I}_1$ are $L = 2c_1 = 2c_2$. (ii) Letting a mapping $\mathfrak{I}_2 : \varXi \longrightarrow \varXi$ is said to κ-strict pseudocontraction [14] if there exists a constant $\kappa \in (0, 1)$ such that

$$\|\mathfrak{I}_2 y_1 - \mathfrak{I}_2 y_2\|^2 \leq \|y_1 - y_2\|^2 + \kappa \|(y_1 - \mathfrak{I}_2 y_1) - (y_2 - \mathfrak{I}_2 y_2)\|^2, \quad \forall y_1, y_2 \in \varXi. \tag{7}$$

A fixed point problem (FPP) for $\mathfrak{I}_2 : \varXi \longrightarrow \varXi$ is to find $\hbar^* \in \varXi$ such that $\mathfrak{I}_2(\hbar^*) = \hbar^*$. Let us define a bifunction $\varGamma$ as follows:

$$\varGamma(y_1, y_2) = \langle y_1 - \mathfrak{I}_2 y_1, y_2 - y_1 \rangle, \quad \forall y_1, y_2 \in \varXi. \tag{8}$$

It can be easily seen in [15] that expression (8) satisfies the conditions ($\varGamma1$)–($\varGamma5$) as well as the values of Lipschitz constants are $c_1 = c_2 = (3 - 2\kappa)/(2 - 2\kappa)$.

The extragradient method developed by Tran et al. [16] is one useful approach. Take an arbitrary starting point $x_0 \in \varPi$; and the next iteration as follows:

$$\begin{aligned} & x_0 \in \varXi, \\ & y_k = \arg\min_{y \in \varXi} \left\{ \daleth \varGamma(x_k, y) + \frac{1}{2} \|x_k - y\|^2 \right\}, \\ & x_{k+1} = \arg\min_{y \in \varXi} \left\{ \daleth \varGamma(y_k, y) + \frac{1}{2} \|x_k - y\|^2 \right\}, \end{aligned} \tag{9}$$

where $0 < \daleth < \min\{(1/2c_1), (1/2c_2)\}$ and c_1, c_2 are two Lipschitz-type constants.

The main goal is to create an inertial-type technique in the case of [16] that will be designed to increase the convergence rate of the iterative sequence. Such techniques have already been established as a result of the oscillator equation with damping and conservative force restoration. This second-order dynamical system is known as a "heavy friction ball," and it was first proposed by Polyak in [17]. The important feature of this method is that the next iteration is built on the previous two iterations. Numerical results show that inertial terms improve the performance of the approaches in terms of the number of iterations and elapsed time in this context. Inertial-type approaches have been extensively studied in recent years for certain classes of equilibrium problems [18–26] and others in [27–33].

As a result, the following natural question arises: Is it possible to develop new inertial-type weakly convergent extragradient-type methods with monotone and nonmonotone step size rules to solve equilibrium problems?

In our study, we provide a positive answer to this question, namely, that the gradient approach still generates a weak convergence sequence when solving equilibrium problems involving pseudomonotone bifunctions using a novel monotone and nonmonotone variable step size rule. Motivated by the work of Censor et al. [34] and Tran et al. [16], we will describe new inertial extragradient-type approaches to solving problem (1) in the context of an infinite-dimensional real Hilbert space. Our primary contributions to this work are as follows:

(i) We build an inertial subgradient extragradient technique with a novel monotone variable step size rule to solve equilibrium problems in a real Hilbert space and show that the resulting sequence is weakly convergent

(ii) To solve equilibrium problems, we devise another inertial subgradient extragradient technique that leverages a novel variable nonmonotone step size rule that is independent of the Lipschitz constants

(iii) Some results are investigated in order to address different kinds of equilibrium problems in a real Hilbert space

(iv) We offer numerical demonstrations of the suggested methodologies for the verification of theoretical conclusions and compare them to earlier results [22, 35, 36]. Our numerical results indicate that the new approaches are useful and outperform the current ones

The paper is structured as follows: in Section 2, preliminary results were presented. Section 3 gives all new approaches and their convergence analysis. Finally, Section 5 gives some numerical results to explain the practical efficiency of the proposed methods.

2. Preliminaries

In this part, we will go over several fundamental identities as well as crucial lemmas and definitions. A *metric projection* $P_{\Xi}(y_1)$ of $y_1 \in \Pi$ is defined by

$$P_{\Xi}(y_1) = \operatorname{argmin}\{\|y_1 - y_2\|: y_2 \in \Xi\}. \tag{10}$$

The following sections outline the key characteristics of projection mapping.

Lemma 1 (see [37]). *Let $P_{\Xi} : \Pi \longrightarrow \Xi$ be a metric projection. Then, there are the following features:*

$$\|y_1 - P_{\Xi}(y_2)\|^2 + \|P_{\Xi}(y_2) - y_2\|^2 \le \|y_1 - y_2\|^2, \quad y_1 \in \Xi, y_2 \in \Pi,$$

$$y_3 = P_{\Xi}(y_1), \tag{11}$$

if and only if

$$\begin{aligned} &\langle y_1 - y_3, y_2 - y_3\rangle \le 0, \quad \forall y_2 \in \Xi, \\ &\|y_1 - P_{\Xi}(y_1)\| \le \|y_1 - y_2\|, \quad y_2 \in \Xi, y_1 \in \Pi. \end{aligned} \tag{12}$$

Lemma 2 (see [37]). *For any $y_1, y_2 \in \Pi$ and $\ell \in \mathbb{R}$. Then, the following conditions were met:*

$$\|\ell y_1 + (1-\ell)y_2\|^2 = \ell\|y_1\|^2 + (1-\ell)\|y_2\|^2 - \ell(1-\ell)\|y_1 - y_2\|^2,$$

$$\|y_1 + y_2\|^2 \le \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle. \tag{13}$$

A normal cone of Ξ at $y_1 \in \Xi$ is defined by

$$N_{\Xi}(y_1) = \{y_3 \in \Pi : \langle y_3, y_2 - y_1\rangle \le 0, \forall y_2 \in \Xi\}. \tag{14}$$

Assume that $\mho : \Xi \longrightarrow \mathbb{R}$ is a convex function and subdifferential of $\mho$ at $y_1 \in \Xi$ is defined by

$$\partial\mho(y_1) = \{y_3 \in \Pi : \mho(y_2) - \mho(y_1) \ge \langle y_3, y_2 - y_1\rangle, \forall y_2 \in \Xi\}. \tag{15}$$

Lemma 3 (see [38]). *Let $\mho : \Xi \longrightarrow \mathbb{R}$ be a subdifferentiable, convex, and lower semicontinuous function on Ξ. An element $x \in \Xi$ is a minimizer of a function $\mho$ if and only if*

$$0 \in \partial\mho(x) + N_{\Xi}(x), \tag{16}$$

where $\partial\mho(x)$ stands for the subdifferential of $\mho$ at $x \in \Xi$ and $N_{\Xi}(x)$ the normal cone of Ξ at x.

Lemma 4 (see [39]). *Let Ξ be a nonempty subset of Π and $\{x_k\}$ be a sequence in Π satisfying two conditions:*

(i) For each $x \in \Xi$, $\lim_{k\longrightarrow+\infty}\|x_k - x\|$ exists

(ii) Each sequentially weak cluster point of $\{x_k\}$ belongs to Ξ

Then, sequence $\{x_k\}$ weakly converges to an element in Ξ.

Lemma 5 (see [40]). *Suppose that $\{a_k\}$ and $\{t_k\}$ are two sequences of nonnegative real numbers satisfying the inequality*

$$a_{k+1} \le a_k + t_k, \quad \text{for all } k \in \mathbb{N}. \tag{17}$$

If $\sum t_k < +\infty$, then, $\lim_{k\longrightarrow+\infty} a_k$ exists.

3. Main Results

In this section, we present a numerical iterative method for accelerating the rate of convergence of an iterative sequence by combining two strong convex optimization problems with an inertial term. We propose the techniques listed below for solving equilibrium problems.

Remark 6. (i) If $\zeta = 0$ is used in the abovementioned method, then, it is equivalent to the default extragradient method [16] with the updated step size rule. (ii) From the expressions in Algorithm 1, we have

$$\sum_{k=1}^{+\infty} \zeta_k \|x_k - x_{k-1}\| \le \sum_{k=1}^{+\infty} \beta_k \|x_k - x_{k-1}\| < +\infty. \tag{18}$$

It further implies that

$$\lim_{k\longrightarrow+\infty} \beta_k \|x_k - x_{k-1}\| = 0. \tag{19}$$

Lemma 7. *A sequence $\{\daleth_k\}$ is converged to $\daleth$ and*

$$\min\left\{\frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{\max\{2c_1, 2c_2\}}, \daleth_0\right\} \le \daleth \le \daleth_0. \tag{20}$$

Proof. Assume that $\Gamma(v_k, x_{k+1}) - \Gamma(v_k, y_k) - \Gamma(y_k, x_{k+1}) > 0$, such that

$$\begin{aligned} &\frac{\varkappa\left(2-\sqrt{2}-\phi\right)\left(\|v_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2[\Gamma(v_k, x_{k+1}) - \Gamma(v_k, y_k) - \Gamma(y_k, x_{k+1})]} \\ &\quad \ge \frac{\varkappa\left(2-\sqrt{2}-\phi\right)\left(\|v_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\left[c_1\|v_k - y_k\|^2 + c_2\|x_{k+1} - y_k\|^2\right]} \\ &\quad \ge \frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{2\max\{c_1, c_2\}}. \end{aligned} \tag{21}$$

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that $\sum_{k=0}^{+\infty} \psi_k < +\infty$.

Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k/\|x_k - x_{k-1}\|)\} & \text{if } x_k \ne x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y\in\Xi}\{\beth_k \Gamma(\nu_k, y) + 1/2\|\nu_k - y\|^2\} \text{wwhere } \nu_k = x_k + \zeta_k(x_k - x_{k-1}).$$

STEP 2: Given the current iterates x_{k-1}, x_k, y_k. Firstly choose $\omega_k \in \partial_2 \Gamma(\nu_k, y_k)$ satisfying $\nu_k - \beth_k\omega_k - y_k \in N_\Xi(y_k)$ and generate a half-space

$$\Pi_k = \{z \in \Pi : \langle \nu_k - \beth_k\omega_k - y_k, z - y_k\rangle \le 0\}.$$

Compute

$$x_{k+1} = \arg\min_{y\in\Pi_k}\{\beth_k \Gamma(y_k, y) + 1/2\|\nu_k - y\|^2\}.$$

STEP 3: Compute

$$\beth_{k+1} = \begin{cases} \min\{\beth_k, ((2-\sqrt{2}-\phi)\varkappa\|\nu_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2/2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \beth_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = \nu_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Algorithm 1

Thus, we obtain $\lim_{k\longrightarrow+\infty} \beth_k = \beth$ This completes the proof of the lemma. □

Lemma 8. *A sequence $\{\beth_k\}$ is converged to $\beth$ and*

$$\min\left\{\frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{\max\{2c_1, 2c_2\}}, \beth_0\right\} \le \beth \le \beth_0, \tag{22}$$

where $P = \sum_{k=1}^{+\infty} p_k$.

Proof. Assume that $\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0$ such that

$$\begin{aligned} &\frac{\varkappa\left(2-\sqrt{2}-\phi\right)\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})]} \\ &\quad \ge \frac{\varkappa\left(2-\sqrt{2}-\phi\right)\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\left[c_1\|\nu_k - y_k\|^2 + c_2\|x_{k+1} - y_k\|^2\right]} \\ &\quad \ge \frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{2\max\{c_1, c_2\}}. \end{aligned} \tag{23}$$

Applying mathematical induction on the concept of $\beth_{k+1}$, we have

$$\min\left\{\frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{\max\{2c_1, 2c_2\}}, \le \beth_k \le \beth_0 + P\right\}. \tag{24}$$

Suppose that $[\beth_{k+1} - \beth_k]^+ = \max\{0, \beth_{k+1} - \beth_k\}$ and $[\beth_{k+1} - \beth_k]^- = \max\{0, -(\beth_{k+1} - \beth_k)\}$. Due to the definition of $\{\beth_k\}$, we get

$$\sum_{k=1}^{+\infty}(\beth_{k+1} - \beth_k)^+ = \sum_{k=1}^{+\infty}\max\{0, \beth_{k+1} - \beth_k\} \le P < +\infty. \tag{25}$$

That is, the series $\sum_{k=1}^{+\infty}(\beth_{k+1} - \beth_k)^+$ is convergent. The convergence must now be proven of $\sum_{k=1}^{+\infty}(\beth_{k+1} - \beth_k)^-$. Let $\sum_{k=1}^{+\infty}(\beth_{k+1} - \beth_k)^- = +\infty$. Due to the fact that $\beth_{k+1} - \beth_k = (\beth_{k+1} - \beth_k)^+ - (\beth_{k+1} - \beth_k)^-$, we could get

$$\beth_{k+1} - \beth_0 = \sum_{k=0}^{k}(\beth_{k+1} - \beth_k) = \sum_{k=0}^{k}(\beth_{k+1} - \beth_k)^+ - \sum_{k=0}^{k}(\beth_{k+1} - \beth_k)^-. \tag{26}$$

Letting $k \longrightarrow +\infty$ in (26), we have $\beth_k \longrightarrow -\infty$ as $k \longrightarrow +\infty$. This is an absurdity. As a result of the series convergence $\sum_{k=0}^{k}(\beth_{k+1} - \beth_k)^+$ and $\sum_{k=0}^{k}(\beth_{k+1} - \beth_k)^-$ taking $k \longrightarrow +\infty$ in expression (26), we obtain $\lim_{k\longrightarrow+\infty} \beth_k = \beth$. This brings the proof to a conclusion. □

Lemma 9. *The following useful inequality is derived in Algorithm 3.*

$$\beth_k \Gamma(y_k, y) - \beth_k \Gamma(y_k, x_{k+1}) \ge \langle \nu_k - x_{k+1}, y - x_{k+1}\rangle, \quad \forall y \in \Pi_k. \tag{27}$$

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that $\sum_{k=0}^{+\infty}\psi_k < +\infty$.

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty}p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k/\|x_k - x_{k-1}\|)\} & \text{if } x_k \neq x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y\in\Xi}\{\daleth_k \Gamma(\nu_k, y) + 1/2\|\nu_k - y\|^2\} \text{where } \nu_k = x_k + \zeta_k(x_k - x_{k-1}).$$

STEP 2: Given the current iterates x_{k-1}, x_k, y_k. Firstly choose $\omega_k \in \partial_2\Gamma(\nu_k, y_k)$ satisfying $\nu_k - \daleth_k\omega_k - y_k \in N_\Xi(y_k)$ and generate a half-space

$$\Pi_k = \{z \in \Pi : \langle \nu_k - \daleth_k\omega_k - y_k, z - y_k\rangle \le 0\}.$$

Compute

$$x_{k+1} = \arg\min_{y\in\Pi_k}\{\daleth_k\Gamma(y_k, y) + 1/2\|\nu_k - y\|^2\}.$$

STEP 3: Compute

$$\daleth_{k+1} = \begin{cases} \min\{\daleth_k + p_k, ((2-\sqrt{2}-\phi)\varkappa\|\nu_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2/2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \daleth_k + p_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = \nu_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Algorithm 2

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that $\sum_{k=0}^{+\infty}\psi_k < +\infty$.

Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k/\|x_k - x_{k-1}\|)\} & \text{if } x_k \neq x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y\in\Xi}\{\daleth_k \Gamma(\nu_k, y) + 1/2\|\nu_k - y\|^2\}, \text{wwhere } \nu_k = x_k + \zeta_k(x_k - x_{k-1}).$$

STEP 2: Compute

$$x_{k+1} = \arg\min_{y\in\Xi}\{\daleth_k\Gamma(y_k, y) + 1/2\|\nu_k - y\|^2\}.$$

STEP 3: Compute

$$\daleth_{k+1} = \begin{cases} \min\{\daleth_k, ((2-\sqrt{2}-\phi)\varkappa\|\nu_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2/2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \daleth_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = \nu_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Algorithm 3

Proof. By use of Lemma 3, we have

$$0 \in \partial_2\left\{\daleth_k\Gamma(y_k, \cdot) + \frac{1}{2}\|\nu_k - \cdot\|^2\right\}(x_{k+1}) + N_{\Pi_k}(x_{k+1}). \quad (28)$$

Thus, for $\upsilon \in \partial\Gamma(y_k, x_{k+1})$, there exists a vector $\bar{\upsilon} \in N_{\Pi_k}(x_{k+1})$ such that

$$\daleth_k\upsilon + x_{k+1} - \nu_k + \bar{\upsilon} = 0. \quad (29)$$

Thus, we have

$$\langle \nu_k - x_{k+1}, y - x_{k+1}\rangle = \daleth_k\langle \upsilon, y - x_{k+1}\rangle + \langle \bar{\upsilon}, y - x_{k+1}\rangle, \quad \forall y \in \Pi_k. \quad (30)$$

Since $\bar{\upsilon} \in N_{\Pi_k}(x_{k+1})$ implies that $\langle \bar{\upsilon}, y - x_{k+1}\rangle \le 0$ for all $y \in \Pi_k$, thus, we have

$$\langle \nu_k - x_{k+1}, y - x_{k+1}\rangle \le \daleth_k\langle \upsilon, y - x_{k+1}\rangle, \quad \forall y \in \Pi_k. \quad (31)$$

Since $\upsilon \in \partial\Gamma(y_k, x_{k+1})$, we have

$$\Gamma(y_k, y) - \Gamma(y_k, x_{k+1}) \geq \langle \upsilon, y - x_{k+1} \rangle, \quad \forall y \in \Pi. \tag{32}$$

Combining expressions (31) and (32), we have

$$\beth_k \Gamma(y_k, y) - \beth_k \Gamma(y_k, x_{k+1}) \geq \langle \nu_k - x_{k+1}, y - x_{k+1} \rangle, \quad \forall y \in \Pi_k. \tag{33}$$

□

Lemma 10. *In Algorithm 3, we also have the following useful inequality:*

$$\beth_k \Gamma(\nu_k, y) - \beth_k \Gamma(\nu_k, y_k) \geq \langle \nu_k - y_k, y - y_k \rangle, \quad \forall y \in \Xi. \tag{34}$$

Proof. The proof is analogous to the proof of Lemma 9. Next, substituting $y = x_{k+1}$, we have

$$\beth_k \{\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k)\} \geq \langle \nu_k - y_k, x_{k+1} - y_k \rangle. \tag{35}$$

□

Theorem 11. *Let $\{x_k\}$ be a sequence generated by Algorithm 3, and the conditions (Γ1)–(Γ5) are satisfied. Then, the sequence $\{x_k\}$ converges weakly to $\hbar^*$.*

Proof. By substituting $y = \hbar^*$ into Lemma 9, we have

$$\beth_k \Gamma(y_k, \hbar^*) - \beth_k \Gamma(y_k, x_{k+1}) \geq \langle \nu_k - x_{k+1}, \hbar^* - x_{k+1} \rangle. \tag{36}$$

By the use of condition $\Gamma 2$, we obtain

$$\langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle \geq \beth_k \Gamma(y_k, x_{k+1}). \tag{37}$$

From the expression in Algorithm 1, we obtain

$$\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) \leq \frac{\left(2 - \sqrt{2} - \phi\right) \varkappa\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}, \tag{38}$$

which after multiplying both sides by $\beth_k > 0$ implies that

$$\beth_k \Gamma(y_k, x_{k+1}) \geq \beth_k \Gamma(\nu_k, x_{k+1}) - \beth_k \Gamma(\nu_k, y_k) - \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}. \tag{39}$$

Combining expressions (37) and (39), we obtain

$$\langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle \geq \beth_k \{\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k)\} - \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}. \tag{40}$$

By using expression (35), we have

$$\beth_k \{\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k)\} \geq \langle \nu_k - y_k, x_{k+1} - y_k \rangle. \tag{41}$$

Combining expressions (40) and (41), we have

$$\langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle \geq \langle \nu_k - y_k, x_{k+1} - y_k \rangle - \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}. \tag{42}$$

The following facts are available to us:

$$\begin{aligned} 2\langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle &= \|\nu_k - \hbar^*\|^2 - \|x_{k+1} - \nu_k\|^2 - \|x_{k+1} - \hbar^*\|^2, \\ 2\langle y_k - \nu_k, y_k - x_{k+1} \rangle &= \|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2 - \|\nu_k - x_{k+1}\|^2. \end{aligned} \tag{43}$$

Thus, we have

$$\|x_{k+1} - \hbar^*\|^2 \leq \|\nu_k - \hbar^*\|^2 - \|\nu_k - y_k\|^2 - \|x_{k+1} - y_k\|^2 + \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{\beth_{k+1}}. \tag{44}$$

Since $\beth_k \longrightarrow \beth$, thus, there exists a fixed natural number $k_1 \in \mathbb{N}$ such that

$$\lim_{k \longrightarrow +\infty} \frac{\varkappa \beth_k}{\beth_{k+1}} \leq 1. \tag{45}$$

Thus, we have

$$\|x_{k+1} - \hbar^*\|^2 \leq \|\nu_k - \hbar^*\|^2 - \|\nu_k - y_k\|^2 - \|x_{k+1} - y_k\|^2 + \left(2 - \sqrt{2} - \phi\right)\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right). \tag{46}$$

Furthermore, it implies that

$$\|x_{k+1} - \hbar^*\|^2 \leq \|\nu_k - \hbar^*\|^2 - \left(\sqrt{2} - 1\right)\|\nu_k - y_k\|^2 - \left(\sqrt{2} - 1\right)\|x_{k+1} - y_k\|^2 - \phi\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right). \tag{47}$$

From expression (47), we obtain

$$\|x_{k+1} - \hbar^*\|^2 \leq \|\nu_k - \hbar^*\|^2, \quad \forall k \geq k_1. \tag{48}$$

It is possible to write as an expression for every $k \geq k_1$ such that

$$\|x_{k+1} - \hbar^*\| \leq \|x_k + \zeta_k(x_k - x_{k-1}) - \hbar^*\| \leq \|x_k - \hbar^*\| + \zeta_k\|x_k - x_{k-1}\|. \tag{49}$$

Combining expressions (18) and (49) and Lemma 5 implies that

$$\lim_{k\longrightarrow+\infty}\|x_k-\hbar^*\|=l,\quad \text{for some finite } l\geq 0. \tag{50}$$

By using the definition of ν_k, we have

$$\begin{aligned}\|\nu_k-\hbar^*\|^2&=\|x_k+\zeta_k(x_k-x_{k-1})-\hbar^*\|^2=\|(1+\zeta_k)(x_k-\hbar^*)\\&\quad-\zeta_k(x_{k-1}-\hbar^*)\|^2=(1+\zeta_k)\|x_k-\hbar^*\|^2-\zeta_k\|x_{k-1}\\&\quad-\hbar^*\|^2+\zeta_k(1+\zeta_k)\|x_k-x_{k-1}\|^2\leq(1+\zeta_k)\|x_k\\&\quad-\hbar^*\|^2-\zeta_k\|x_{k-1}-\hbar^*\|^2+2\zeta_k\|x_k-x_{k-1}\|^2.\end{aligned} \tag{51}$$

By using expressions (50) and (19) in the abovementioned formula, we may deduce that

$$\lim_{k\longrightarrow+\infty}\|\nu_k-\hbar^*\|=l. \tag{52}$$

Thus, we have

$$\begin{aligned}\|x_{k+1}-\hbar^*\|^2&\leq\|\nu_k-\hbar^*\|^2-\|\nu_k-y_k\|^2-\|x_{k+1}-y_k\|^2\\&\quad+\frac{\left(2-\sqrt{2}-\phi\right)\beth_k\varkappa\left(\|\nu_k-y_k\|^2+\|x_{k+1}-y_k\|^2\right)}{\beth_{k+1}}.\end{aligned} \tag{53}$$

By using expressions (51) and (53), we obtain

$$\begin{aligned}\|x_{k+1}-\hbar^*\|^2&\leq(1+\zeta_k)\|x_k-\hbar^*\|^2-\zeta_k\|x_{k-1}-\hbar^*\|^2+2\zeta_k\|x_k\\&\quad-x_{k-1}\|^2-\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|\nu_k-y_k\|^2\\&\quad-\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|y_k-x_{k+1}\|^2.\end{aligned} \tag{54}$$

Consequently, this implies that

$$\begin{aligned}&\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|\nu_k-y_k\|^2\\&\quad+\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|y_k-x_{k+1}\|^2\leq\|x_k-\hbar^*\|^2\\&\quad-\|x_{k+1}-\hbar^*\|^2+\zeta_k\left(\|x_k-\hbar^*\|^2-\|x_{k-1}-\hbar^*\|^2\right)\\&\quad+2\zeta_k\|x_k-x_{k-1}\|^2.\end{aligned} \tag{55}$$

By taking the limit as $k\longrightarrow+\infty$ in expression (55), we obtain

$$\lim_{k\longrightarrow+\infty}\|\nu_k-y_k\|=\lim_{k\longrightarrow+\infty}\|y_k-x_{k+1}\|=0. \tag{56}$$

Thus, expressions (52) and (56) give that

$$\lim_{k\longrightarrow+\infty}\|y_k-\hbar^*\|=l. \tag{57}$$

By using expressions (50), (52), and (57), so that the sequences $\{x_k\}$, $\{\nu_k\}$, and $\{y_k\}$ are bounded, therefore $\{x_k\}$, $\{\nu_k\}$, and $\{y_k\}$ exist. Thus, $\lim_{k\longrightarrow+\infty}\|x_k-\hbar^*\|^2$, $\lim_{k\longrightarrow+\infty}\|y_k-\hbar^*\|^2$, $\lim_{k\longrightarrow+\infty}\|\nu_k-\hbar^*\|^2$. Following that, we will show that the sequence $\{x_k\}$ weakly converges to $\hbar^*$. As a result, all sequences $\{x_k\}$, $\{\nu_k\}$, and $\{y_k\}$ are bounded. We now demonstrate that each sequential weak cluster point in the sequence $\{x_k\}$ is in $\mathrm{sol}(\Gamma,\Xi)$. Consider that z is a weak cluster point of $\{x_k\}$, which means that there is a subsequence of $\{x_k\}$ that is weakly convergent to z. Then, $z\in\Xi$,$\{y_{k_m}\}$ is also weakly convergent to z.Now let demonstrate that$z\in\mathrm{sol}(\Gamma,\Xi)$. We have obtained the following by combining Lemma 9 with expressions (39) and (35):

$$\begin{aligned}\beth_{k_m}\Gamma\left(y_{k_m},y\right)&\geq\beth_{k_m}\Gamma\left(y_{k_m},x_{k_m+1}\right)+\left\langle\nu_{k_m}-x_{k_m+1},y-x_{k_m+1}\right\rangle\\&\geq\beth_{k_m}\Gamma\left(\nu_{k_m},x_{k_{m+1}}\right)-\beth_{k_m}\Gamma\left(\nu_{k_m},y_{k_m}\right)\\&\quad-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-\nu_{k_m}\right\|^2\\&\quad-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-x_{k_m+1}\right\|^2\\&\quad+\left\langle\nu_{k_m}-x_{k_m+1},y-x_{k_m+1}\right\rangle\geq\left\langle\nu_{k_m}-y_{k_m},x_{k_m+1}-y_{k_m}\right\rangle\\&\quad-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-\nu_{k_m}\right\|^2\\&\quad-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-x_{k_m+1}\right\|^2\\&\quad+\left\langle\nu_{k_m}-x_{k_m+1},y-x_{k_m+1}\right\rangle,\end{aligned} \tag{58}$$

where y is any member of Π_k. The use of expression (56) and the boundedness of the sequence $\{x_k\}$ implies that the right-hand side of the last inequality is convergent to zero. By using the condition $\Gamma 4$ and $y_{k_m}\rightharpoonup z$, we have $\beth_{k_m}\geq\beth>0$ such as

$$0\leq\limsup_{m\longrightarrow+\infty}\left(y_{k_m},y\right)\leq\Gamma(z,y),\quad\forall y\in\Pi_k. \tag{59}$$

Since Ξ is a subset of half-space Π_k, it follows that $\Gamma(z,y)\geq 0,\forall y\in\Xi$. This proves that $z\in\mathrm{sol}(\Gamma,\Xi)$. Thus, Lemma

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that $\sum_{k=0}^{+\infty}⊠\psi_k < +\infty$.

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k / \|x_k - x_{k-1}\|)\} & \text{if } x_k \neq x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y \in \Xi} \{\beth_k \Gamma(v_k, y) + 1/2\|v_k - y\|^2\}, \text{wwhere } v_k = x_k + \zeta_k (x_k - x_{k-1}).$$

STEP 2: Compute

$$x_{k+1} = \arg\min_{y \in \Xi} \{\beth_k \Gamma(y_k, y) + 1/2\|v_k - y\|^2\}.$$

STEP 3: Compute

$$\beth_{k+1} = \begin{cases} \min\{\beth_k + p_k, ((2-\sqrt{2}-\phi)\varkappa\|v_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2 / 2[\Gamma(v_k, x_{k+1}) - \Gamma(v_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(v_k, x_{k+1}) - \Gamma(v_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \beth_k + p_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = v_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Algorithm 4

4 assures that $\{v_k\}, \{x_k\}$, and $\{y_k\}$ converge weakly to $\hbar^*$ as $k \longrightarrow +\infty$. □

We now present two iterative methods based on a monotone and nonmonotone variable step size rule and two strongly convex minimization problems without the need for subgradient methods. The following is a description of the second major result.

4. Results to Solve the Fixed Point Problem and Variational Inequalities

In this section, we solve fixed point problems and variational inequalities using the results from our main results. Expressions (6) and (8) are employed to obtain the following conclusions. All the methods are based on our main findings, which are interpreted as follows.

Corollary 12. *Assume that $\mathfrak{J}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{J}_1, \Xi) \neq \emptyset$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1), \phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{60}$$

Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{61}$$

First, we have to compute

$$\begin{aligned} v_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi(v_k - \hat{u}_k \mathfrak{J}_1(v_k)). \end{aligned} \tag{62}$$

Having x_{k-1}, x_k, y_k with

$$\Pi_k = \{z \in \Pi : \langle v_k - \beth_k \mathfrak{J}_1(v_k) - y_k, z - y_k \rangle \le 0\}, \quad \text{for each } k \ge 0. \tag{63}$$

Compute

$$x_{k+1} = P_{\Pi_k}(v_k - \beth_k \mathfrak{J}_1(y_k)). \tag{64}$$

Update the step size in the following way:

$$_{k+1} = \begin{cases} \min\left\{\beth_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|v_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{J}_1(v_k) - \mathfrak{J}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{J}_1(v_k) - \mathfrak{J}_1(y_k), x_{k+1} - y_k \rangle > 0, \\ \beth_k, \quad \text{otherwise.} \end{cases} \tag{65}$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_1, \Xi)$.

Corollary 13. *Assume that $\mathfrak{I}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L -Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_1, \Xi) \neq \varnothing$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1), \phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0, +\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{66}$$

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{67}$$

First, we have to compute

$$\begin{aligned} v_k &= x_k + \zeta_k(x_k - x_{k-1}), \\ y_k &= P_\Xi(v_k - \hat{u}_k \mathfrak{I}_1(v_k)). \end{aligned} \tag{68}$$

Having x_{k-1}, x_k, y_k with

$$\Pi_k = \{z \in \Pi : \langle v_k - \beth_k \mathfrak{I}_1(v_k) - y_k, z - y_k \rangle \leq 0\}, \quad \text{for each } k \geq 0. \tag{69}$$

Compute

$$x_{k+1} = P_{\Pi_k}(v_k - \beth_k \mathfrak{I}_1(y_k)). \tag{70}$$

Update the step size in the following way:

$${}_{k+1} = \begin{cases} \min\left\{\beth_k + p_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|v_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{I}_1(v_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{I}_1(v_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle > 0, \\ \beth_k + p_k, \quad \text{otherwise.} \end{cases} \tag{71}$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_1, \Xi)$.

Corollary 14. *Assume that $\mathfrak{I}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L -Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_1, \Xi) \neq \varnothing$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1), \phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0, +\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{72}$$

Moreover, choose ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{73}$$

First, we have to compute

$$\begin{aligned} v_k &= x_k + \zeta_k(x_k - x_{k-1}), \\ y_k &= P_\Xi(v_k - \beth_k \mathfrak{I}_1(v_k)), \\ x_{k+1} &= P_\Xi(v_k - \beth_k \mathfrak{I}_1(y_k)). \end{aligned} \tag{74}$$

Update the step size in the following way:

$${}_{k+1} = \begin{cases} \min\left\{\beth_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|v_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{I}_1(v_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{I}_1(v_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle > 0, \\ \beth_k, \quad \text{otherwise.} \end{cases} \tag{75}$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{J}_1, \Xi)$.

Corollary 15. *Assume that $\mathfrak{J}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{J}_1, \Xi) \neq \varnothing$. Choose $\daleth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0, 1), \varkappa \in (0, 1), \phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0, +\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{76}$$

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{77}$$

First, we have to compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k(x_k - x_{k-1}), \\ y_k &= P_\Xi(\nu_k - \daleth_k \mathfrak{J}_1(\nu_k)), \\ x_{k+1} &= P_\Xi(\nu_k - \daleth_k \mathfrak{J}_1(y_k)). \end{aligned} \tag{78}$$

Update the step size in the following way:

$${}_{k+1} = \begin{cases} \min\left\{\daleth_k + p_k, \dfrac{\left(2 - \sqrt{2} - \phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2 - \sqrt{2} - \phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{J}_1(\nu_k) - \mathfrak{J}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{J}_1(\nu_k) - \mathfrak{J}_1(y_k), x_{k+1} - y_k\rangle > 0, \\ \daleth_k + p_k, \quad \text{otherwise.} \end{cases} \tag{79}$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{J}_1, \Xi)$.

Corollary 16. *Assume that $\mathfrak{J}_2 : \Xi \longrightarrow \Pi$ is a κ-strict pseudocontraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{J}_2, \Xi) \neq \varnothing$. Choose $\daleth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0, 1), \varkappa \in (0, 1), \phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0, +\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{80}$$

Moreover, choose ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{81}$$

Compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k(x_k - x_{k-1}), \\ y_k &= P_\Xi[\nu_k - \daleth_k(\nu_k - \mathfrak{J}_2(\nu_k))]. \end{aligned} \tag{82}$$

Having x_{k-1}, x_k, y_k, with

$$\Pi_k = \{z \in \mathscr{E} : \langle (1 - \daleth_k)\nu_k + \daleth_k \mathfrak{J}_2(\nu_k) - y_k, z - y_k\rangle \leq 0\}. \tag{83}$$

Compute

$$x_{k+1} = P_{\Pi_k}[\nu_k - \daleth_k(y_k - \mathfrak{J}_2(y_k))]. \tag{84}$$

The step size rule for the next iteration is evaluated as follows:

$${}_{k+1} = \begin{cases} \min\left\{\daleth_k, \dfrac{\left(2 - \sqrt{2} - \phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2 - \sqrt{2} - \phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle (\nu_k - y_k) - [\mathfrak{J}_2(\nu_k) - \mathfrak{J}_2(y_k)], x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle (\nu_k - y_k) - [\mathfrak{J}_2(\nu_k) - \mathfrak{J}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \daleth_k, \quad \text{otherwise.} \end{cases} \tag{85}$$

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \text{sol}(\mathfrak{F}_1, \Xi)$.

Corollary 17. *Assume that $\mathfrak{F}_2 : \Xi \longrightarrow \Pi$ is a κ -strict pseudocontraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{F}_2, \Xi) \neq \emptyset$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{86}$$

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{87}$$

Compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi[\nu_k - \beth_k(\nu_k - \mathfrak{F}_2(\nu_k))]. \end{aligned} \tag{88}$$

Having x_{k-1}, x_k, y_k, with

$$\Pi_k = \{z \in \mathscr{E} : \langle (1-\beth_k)\nu_k + \beth_k \mathfrak{F}_2(\nu_k) - y_k, z - y_k \rangle \le 0\}. \tag{89}$$

Compute

$$x_{k+1} = P_{\Pi_k}[\nu_k - \beth_k(y_k - \mathfrak{F}_2(y_k))]. \tag{90}$$

The step size rule for the next iteration is evaluated as follows:

$${}_{k+1} = \begin{cases} \min\left\{\beth_k + p_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle(\nu_k - y_k) - [\mathfrak{F}_2(\nu_k) - \mathfrak{F}_2(y_k)], x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle(\nu_k - y_k) - [\mathfrak{F}_2(\nu_k) - \mathfrak{F}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \beth_k + p_k, \quad \text{otherwise.} \end{cases} \tag{91}$$

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \text{sol}(\mathfrak{F}_2, \Xi)$.

Corollary 18. *Assume that $\mathfrak{F}_2 : \Xi \longrightarrow \Pi$ is a κ-strict pseudocontraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{F}_2, \Xi) \neq \emptyset$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1)$, $\varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{92}$$

Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{93}$$

Compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi[\nu_k - \beth_k(\nu_k - \mathfrak{F}_2(\nu_k))], \\ x_{k+1} &= P_\Xi[\nu_k - \beth_k(y_k - \mathfrak{F}_2(y_k))]. \end{aligned} \tag{94}$$

$${}_{k+1} = \begin{cases} \min\left\{\beth_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle(\nu_k - y_k) - [\mathfrak{F}_2(\nu_k) - \mathfrak{F}_2(y_k)], x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle(\nu_k - y_k) - [\mathfrak{F}_2(\nu_k) - \mathfrak{F}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \beth_k, \quad \text{otherwise.} \end{cases} \tag{95}$$

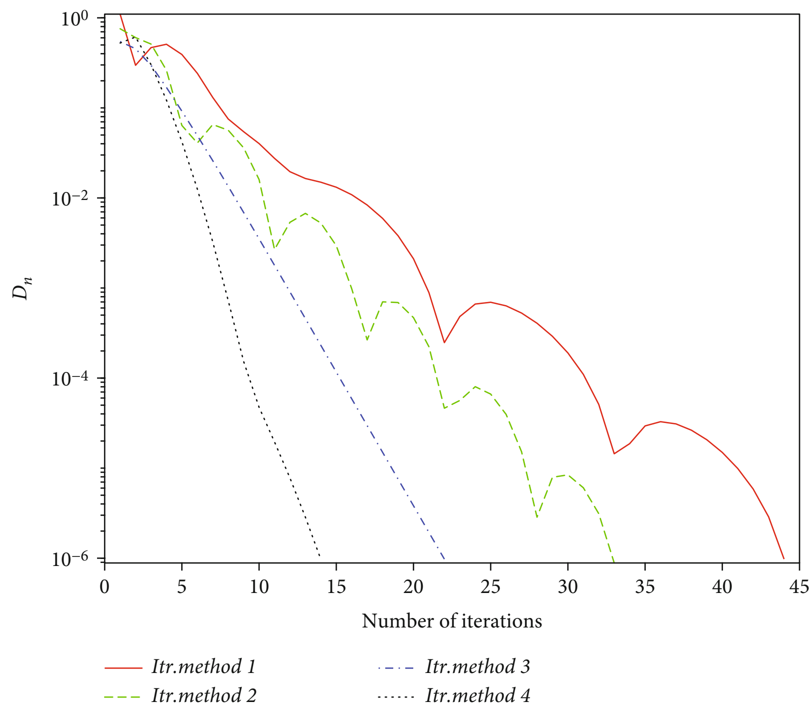


Figure 1: All methods are compared computationally while $x_0 = (0, 0, 0, 0, 0)^T$.

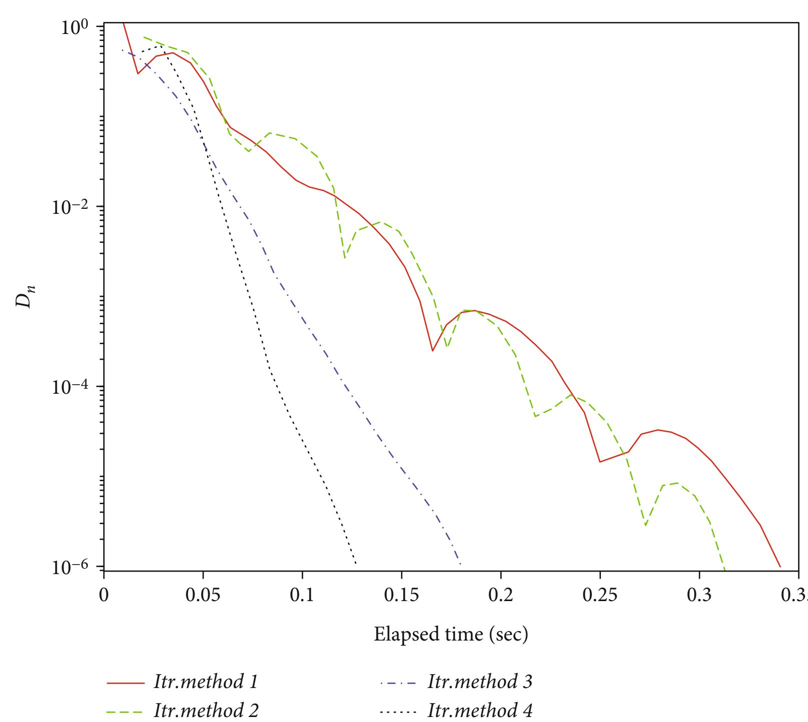


Figure 2: All methods are compared computationally while $x_0 = (0, 0, 0, 0, 0)^T$.

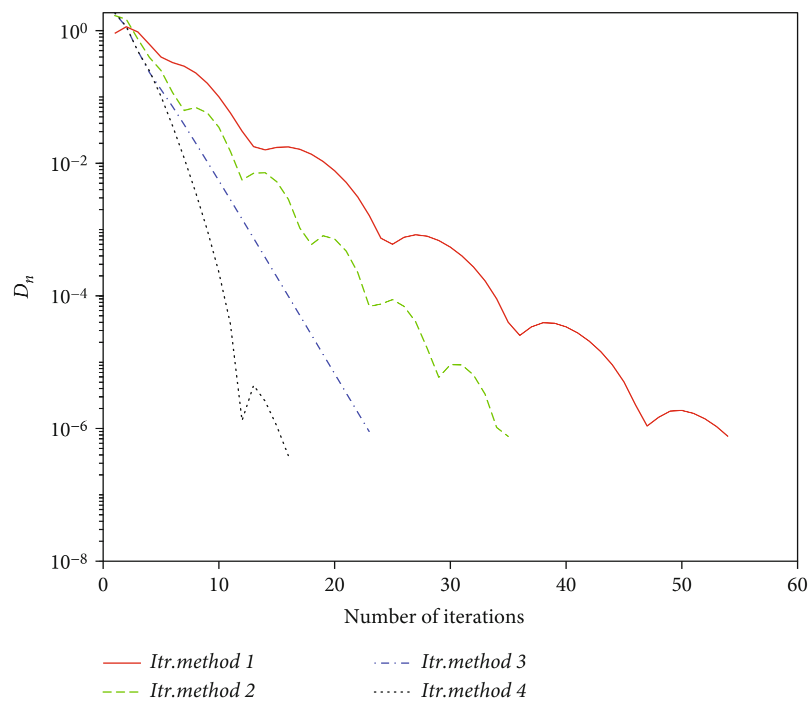


Figure 3: All methods are compared computationally while $x_0 = (1, 2, 1, 2, 1)^T$.

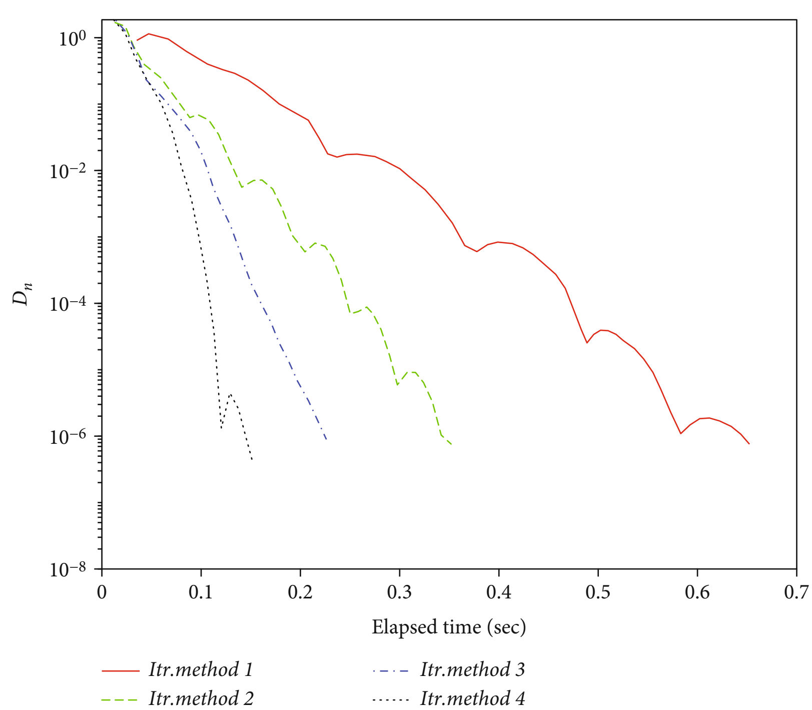


Figure 4: All methods are compared computationally while $x_0 = (1, 2, 1, 2, 1)^T$.

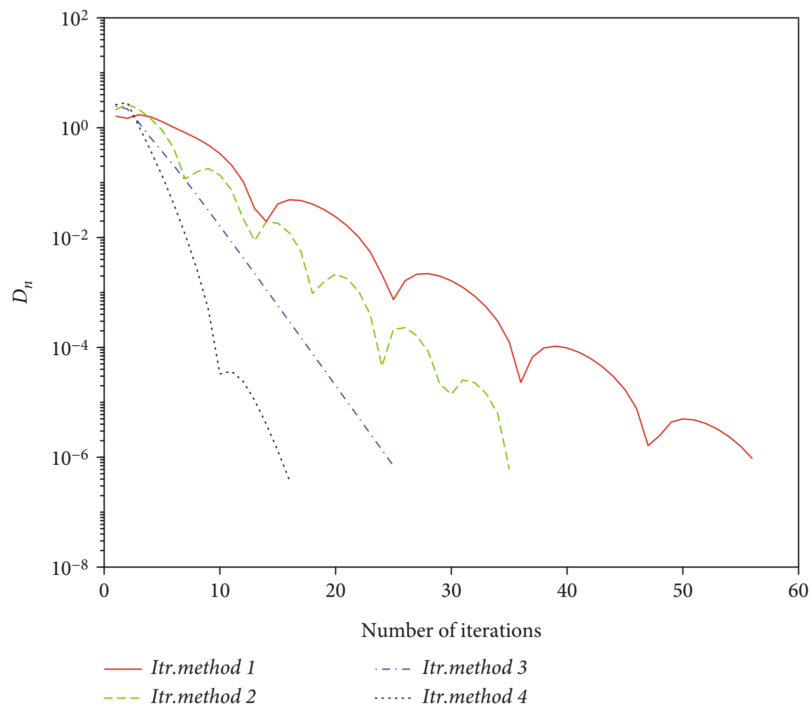


FIGURE 5: All methods are compared computationally while $x_0 = (1, 2, 3, -4, 5)^T$.

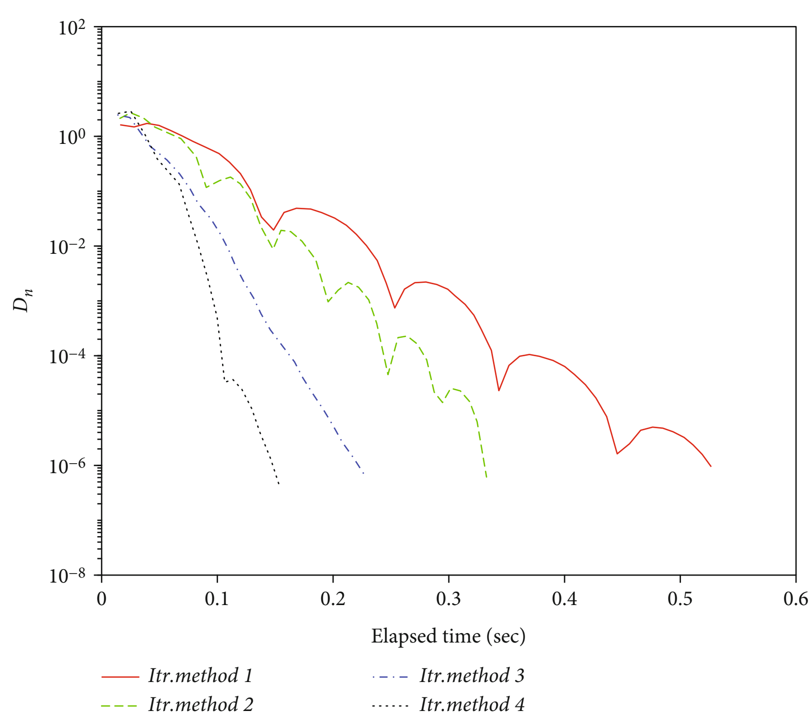


FIGURE 6: All methods are compared computationally while $x_0 = (1, 2, 3, -4, 5)^T$.

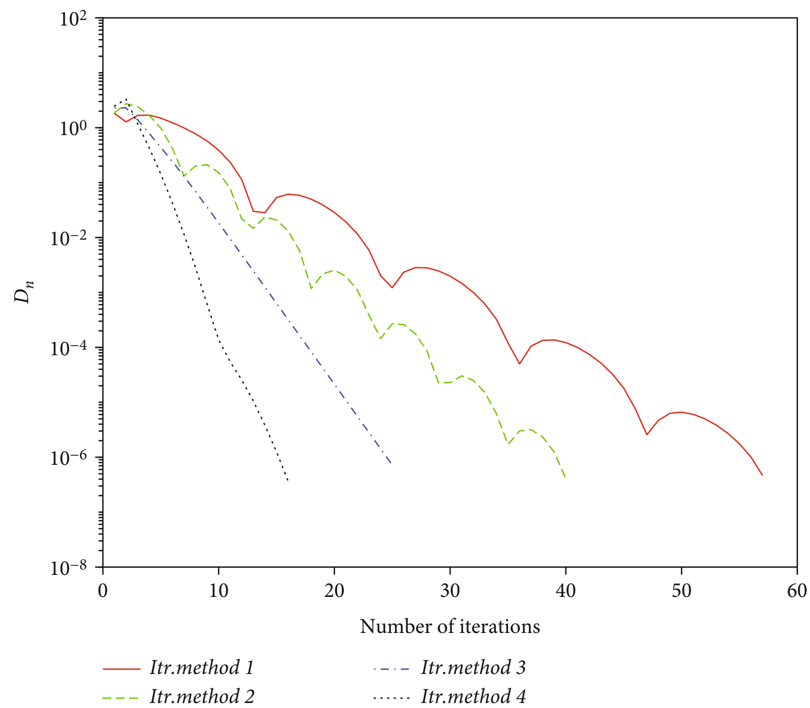


Figure 7: All methods are compared computationally while $x_0 = (2,-1,3,-4,5)^T$.

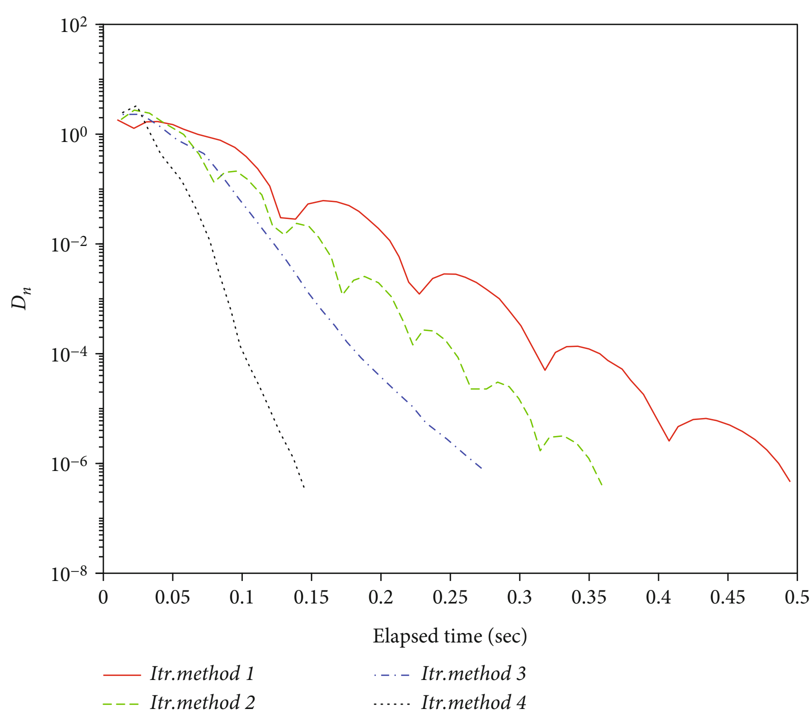


Figure 8: All methods are compared computationally while $x_0 = (2,-1,3,-4,5)^T$.

Table 1: All methods' numerical values for Figures 1–8.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
$(0, 0, 0, 0, 0)^T$	22	14	0.180260200000000	0.127609500000000
$(1, 2, 1, 2, 1)^T$	23	16	0.226162200000000	0.152221400000000
$(1, 2, 3, -4, 5)^T$	25	16	0.226667900000000	0.154296300000000
$(2, -1, 3, -4, 5)^T$	25	16	0.275009100000000	0.144512100000000

Table 2: All methods' numerical values for Figures 1–8.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1 n [22]	Algorithm 2 in [35]	Algorithm 1 in [22]	Algorithm 2 in [35]
$(0, 0, 0, 0, 0)^T$	44	33	0.340814700000000	0.312906600000000
$(1, 2, 1, 2, 1)^T$	54	35	0.652377900000000	0.351818000000000
$(1, 2, 3, -4, 5)^T$	56	35	0.526694900000000	0.332574400000000
$(2, -1, 3, -4, 5)^T$	57	40	0.494837300000000	0.359039600000000

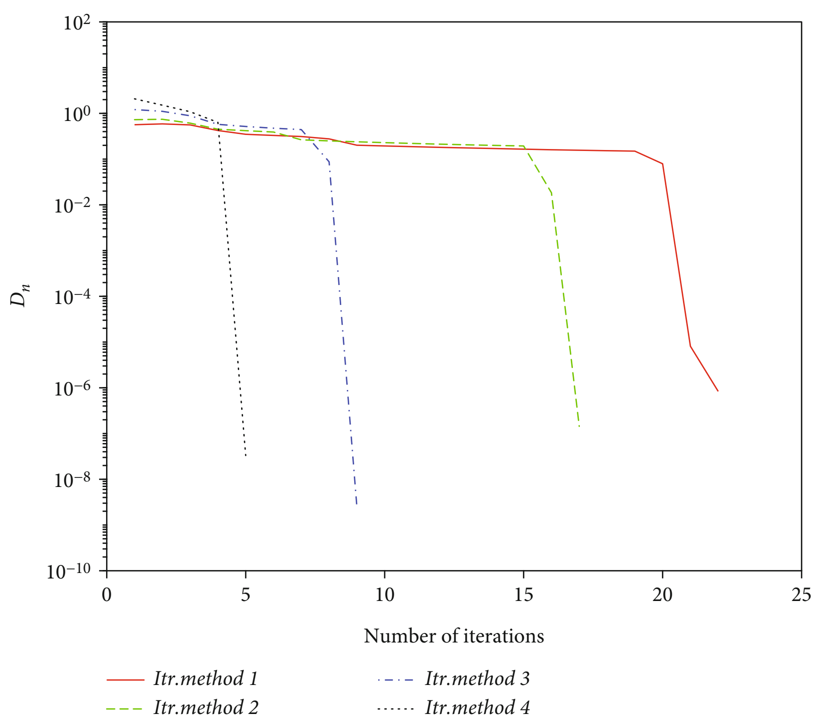


Figure 9: All methods are compared computationally while $x_0 = (2, 3, 2, 5, 2)^T$.

The step size rule for the next iteration is evaluated as follows:

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{J}_2, \Xi)$.

Corollary 19. *Assume that $\mathfrak{J}_2 : \Xi \longrightarrow \Pi$ is a κ-strict pseudo-contraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{J}_2, \Xi) \neq \varnothing$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0, 1), \varkappa \in (0, 1)$, $\phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0, +\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{96}$$

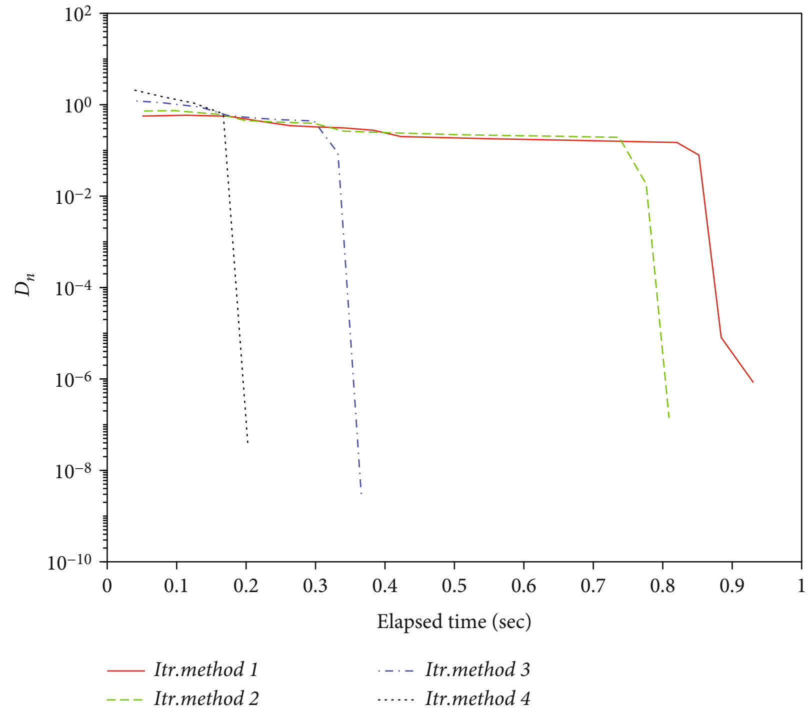


FIGURE 10: All methods are compared computationally while $x_0 = (2, 3, 2, 5, 2)^T$.

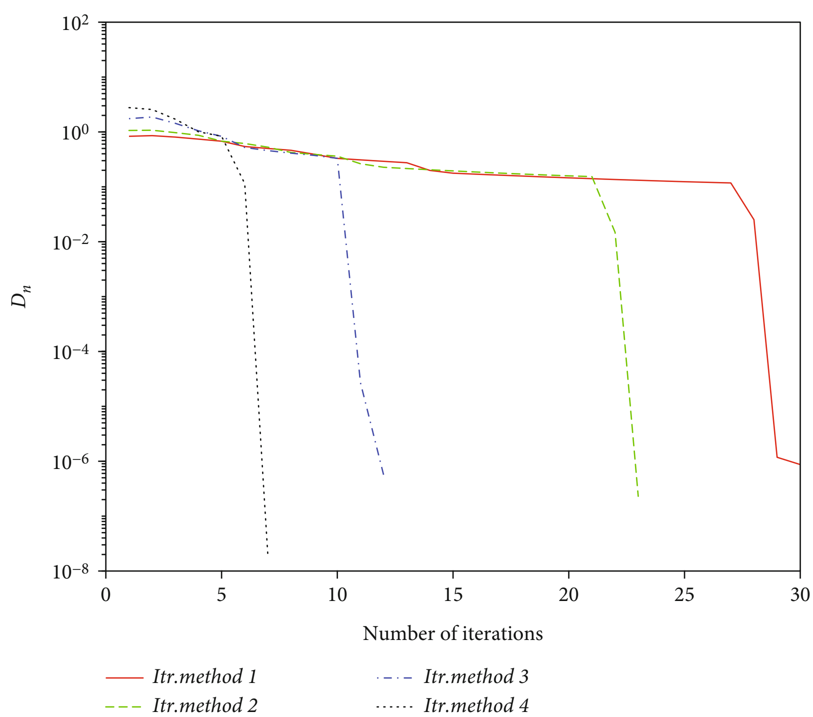


FIGURE 11: All methods are compared computationally while $x_0 = (1, 3, 5, 4, 7)^T$.

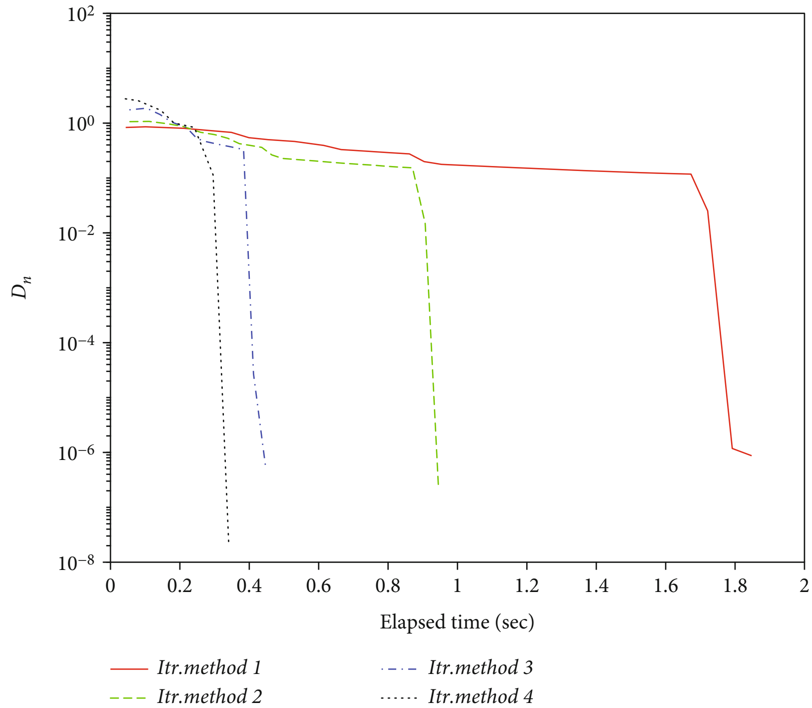


FIGURE 12: All methods are compared computationally while $x_0 = (1, 3, 5, 4, 7)^T$.

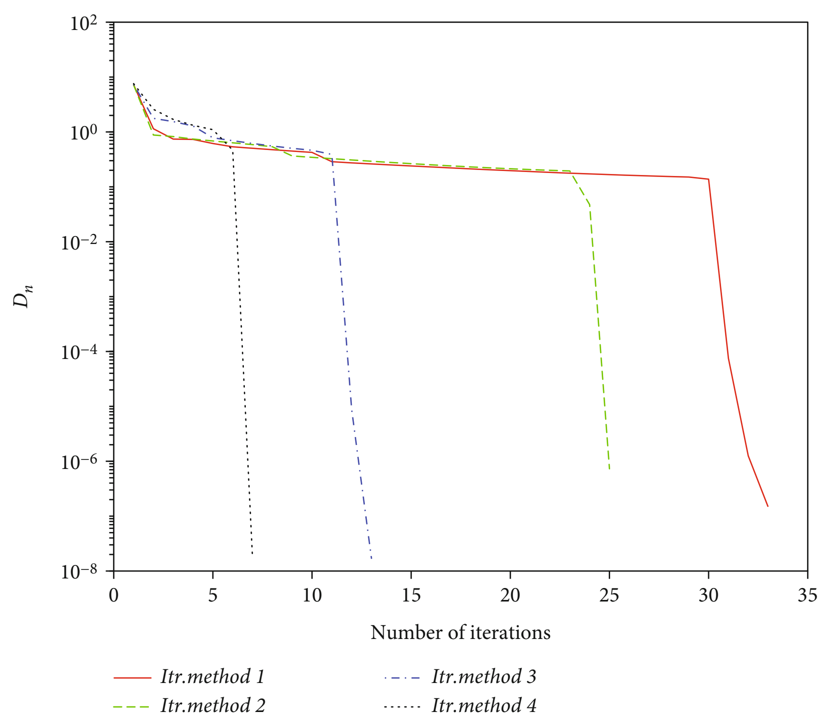


FIGURE 13: All methods are compared computationally while $x_0 = (2, -3, 5, 9, -5)^T$.

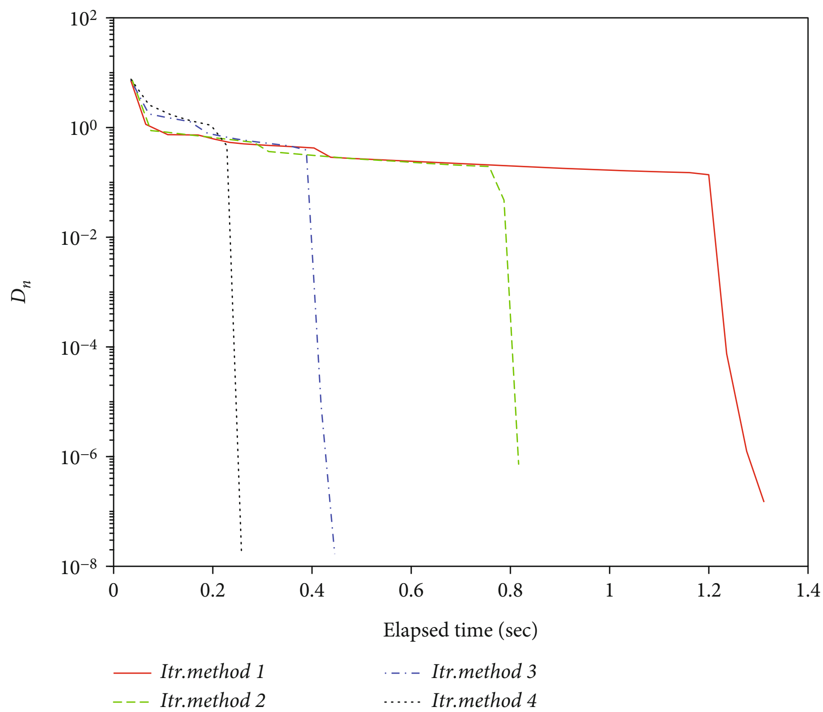


FIGURE 14: All methods are compared computationally while $x_0 = (2,-3,5,9,-5)^T$.

TABLE 3: All methods' numerical values for Figures 9–14.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1 in [22]	Algorithm 2 in [35]	Algorithm 1 in [22]	Algorithm 2 in [35]
$(2,3,2,5,2)^T$	22	17	0.9305202000	0.808993700
$(1,3,5,4,7)^T$	30	23	1.8477304000	0.945203900
$(2,-3,5,9,-5)^T$	33	25	1.3113005000	0.816565900

TABLE 4: All methods' numerical values for Figures 9–14.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
$(2,3,2,5,2)^T$	09	05	0.366167800000000	0.202759300000000
$(1,3,5,4,7)^T$	12	07	0.446752600000000	0.341142700000000
$(2,-3,5,9,-5)^T$	13	07	0.445763600000000	0.257909300000000

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \ne x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{97}$$

Compute

$$\begin{aligned} v_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi [v_k - \daleth_k (v_k - \mathfrak{F}_2 (v_k))], \\ x_{k+1} &= P_\Xi [v_k - \daleth_k (y_k - \mathfrak{F}_2 (y_k))]. \end{aligned} \tag{98}$$

The step size rule for the next iteration is evaluated as follows:

$$
{}_{k+1} = \begin{cases} \min\left\{ \beth_k + p_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|v_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle (v_k - y_k) - [\mathfrak{I}_2(v_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle} \right\}, & \text{if } \langle (v_k - y_k) - [\mathfrak{I}_2(v_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \beth_k + p_k, \quad \text{otherwise.} & \end{cases} \tag{99}
$$

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_2, \Xi)$.

5. Numerical Illustrations

This section describes a number of numerical experiments conducted to demonstrate the validity of the proposed methods. Some of these numerical experiments provide a thorough understanding of how to select effective control parameters. Some of them demonstrate the advantages of the proposed methods over existing ones in the literature. All MATLAB codes were run in MATLAB 9.5 (R2018b) on an Intel(R) Core(TM) i5-6200 Processor CPU @ 2.30 GHz 2.40 GHz, with 8.00 GB RAM.

Example 20. The first sample problem here is drawn from the Nash-Cournot oligopolistic equilibrium model in [16]. In this example, the bifunction Γ can be formulated as having

$$
\Gamma(x, y) = \langle Px + Qy + c, y - x\rangle, \tag{100}
$$

where P, Q, and vector c are defined by

$$
P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}. \tag{101}
$$

The eigenvalues of the matrix $Q - P$ are as follows: $-2.9050, -2.7808, -1.0000, -0.8950, -0.7192$. As a result, the matrices $Q - P$ and Q are symmetrically negative semidefinite and symmetrically positive semidefinite, respectively. Furthermore, the values for Lipschitz-like parameters are $c_1 = c_2 = 1/2\|P - Q\| = 1.4525$. The constraint set $\Xi \subset \mathbb{R}^M$ is regarded as

$$
\Xi \coloneqq \left\{ x \in \mathbb{R}^M : -2 \le x_i \le 5 \right\}. \tag{102}
$$

The beginning points for these numerical investigations vary, as does the error term $D_k = \|x_{k+1} - x_k\|$. Figures 1–8 and Tables 1 and 2 show several results for the error term 10^{-5}. Consider the following information regarding control settings:

(1) For Algorithm 1 in [22] (in short, *Itr.Method1*), we use

$$
\begin{aligned} \phi &= 0.45, \\ \beth &= \frac{1}{2c_2 + 8c_1} \end{aligned} \tag{103}
$$

(2) For Algorithm 2 in [41] (in short, *Itr.Method2*), we use

$$
\begin{aligned} \zeta_k &= 0.12, \\ \varkappa &= 0.11, \\ \beth_0 &= 1 \end{aligned} \tag{104}
$$

(3) For Algorithm 1 (in short, *Itr.Method3*), we use

$$
\begin{aligned} \beth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2} \end{aligned} \tag{105}
$$

(4) For Algorithm 2 (in short, *Itr.Method4*), we use

$$\begin{aligned} \beth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2}, \\ p_k &= \frac{100}{(1+k)^2}. \end{aligned} \tag{106}$$

Example 21. Consider that the possible set $\Xi \subset \mathbb{R}^N$ is defined as follows:

$$\Xi = \left\{u \in \mathbb{R}^N : Au \leq b\right\}, \tag{107}$$

where matrix A has an order $100 \times N$. Consider that $\Gamma : \Xi \times \Xi \longrightarrow \mathbb{R}$ is expressed by

$$\Gamma(u, y) = \langle \mathscr{L}(u), y - u \rangle, \quad \forall u, y \in \Xi, \tag{108}$$

where $\mathscr{L} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is an operator evaluated as $\mathscr{L}(u) = Pu + r$ with $r \in \mathbb{R}^N$ and $P = QQ^T + R + S$, where Q is an $N \times N$ matrix, R is an $N \times N$ skew-symmetric matrix, and S is an $N \times N$ positive definite diagonal matrix. It is simple to demonstrate that Γ is monotone and that the Lipschitz constants are $2c_1 = 2c_2 = \|M\|$ (for more information, see [42, 43]). The beginning points for these numerical investigations vary, as does the error term $D_k = \|x_{k+1} - x_k\|$. Figures 9–14 and Tables 3 and 4 show several results for the error term 10^{-5}. Consider the following information regarding control settings:

(1) For Algorithm 1 in [22] (in short, *Itr.Method1*), we use

$$\begin{aligned} \phi &= 0.45, \\ \beth &= \frac{1}{2c_2 + 8c_1} \end{aligned} \tag{109}$$

(2) For Algorithm 2 in [41] (in short, *Itr.Method2*), we use

$$\begin{aligned} \zeta_k &= 0.12, \\ \varkappa &= 0.11, \\ \beth_0 &= 1 \end{aligned} \tag{110}$$

(3) For Algorithm 1 (in short, *Itr.Method3*), we use

$$\begin{aligned} \beth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2} \end{aligned} \tag{111}$$

(4) For Algorithm 2 (in short, *Itr.Method4*), we use

$$\begin{aligned} \beth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2}, \\ p_k &= \frac{100}{(1+k)^2} \end{aligned} \tag{112}$$

6. Conclusion

The research proposed four explicit extragradient-like strategies for dealing with an equilibrium problem in a real Hilbert space involving a pseudomonotone and a Lipschitz-type bifunction. A novel step size rule that does not rely on Lipschitz-type constant information has been proposed. The convergence theorems and applications of the main results have been demonstrated. Several experiments are given to show the numerical behavior of our two algorithms and to compare them to other well-known algorithms in the literature.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

No potential conflict of interest was reported by the authors.

Acknowledgments

The first and second authors would like to thank Phetchabun Rajabhat University. This research was supported by Chiang Mai University and the NSRF via the program management unit for human resources & institutional development, research and innovation (grant number B05F640183).

References

[1] M. Bianchi and S. Schaible, "Generalized monotone bifunctions and equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 90, no. 1, pp. 31–43, 1996.

[2] E. Blum, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, pp. 123–145, 1994.

[3] G. Mastroeni, "On auxiliary principle for equilibrium problems," in *Nonconvex Optimization and Its Applications*, pp. 289–298, Springer, 2003.

[4] G. Bigi, M. Castellani, M. Pappalardo, and M. Passacantando, "Existence and solution methods for equilibria," *European Journal of Operational Research*, vol. 227, no. 1, pp. 1–11, 2013.

[5] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Berlin Springer Science & Business Media, 2002.

[6] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1996.

[7] I. Konnov, "Application of the proximal point method to nonmonotone equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 119, no. 2, pp. 317–333, 2003.

[8] I. Konnov, *Equilibrium Models and Variational Inequalities*, vol. 210, Elsevier, 2007.

[9] L. Muu and W. Oettli, "Convergence of an adaptive penalty scheme for finding constrained equilibria," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 18, no. 12, pp. 1159–1166, 1992.

[10] A. A. Cournot, *Recherches sur les principes math'ematiques de la th'eorie des richesses*, Hachette, Paris, France, 1838.

[11] K. J. Arrow and G. Debreu, "Existence of an equilibrium for a competitive economy," *Econometrica*, vol. 22, no. 3, p. 265, 1954.

[12] J. Nash, "Non-cooperative games," *Annals of Mathematics*, vol. 54, no. 2, p. 286, 1951.

[13] J. F. Nash Jr., "Equilibrium points in n-person games," *Proceedings of the National Academy of Sciences*, vol. 36, no. 1, pp. 48-49, 1950.

[14] F. Browder and W. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, no. 2, pp. 197–228, 1967.

[15] S. Wang, Y. Zhang, P. Ping, Y. Cho, and H. Guo, "New extragradient methods with non-convex combination for pseudomonotone equilibrium problems with applications in Hilbert spaces," *Univerzitet u Nišu*, vol. 33, no. 6, pp. 1677–1693, 2019.

[16] D. Q. Tran, M. L. Dung, and V. H. Nguyen, "Extragradient algorithms extended to equilibrium problems," *Optimization*, vol. 57, no. 6, pp. 749–776, 2008.

[17] B. Polyak, "Some methods of speeding up the convergence of iteration methods," *USSR Computational Mathematics and Mathematical Physics*, vol. 4, no. 5, pp. 1–17, 1964.

[18] F. A. H. Attouch, "An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping," *Set-Valued Analysis*, vol. 9, no. 3–11, p. 1, 2001.

[19] A. Beck and M. Teboulle, "A fast iterative shrinkage-thresholding algorithm for linear inverse problems," *SIAM Journal on Imaging Sciences*, vol. 2, no. 1, pp. 183–202, 2009.

[20] V. Dadashi, O. S. Iyiola, and Y. Shehu, "The subgradient extragradient method for pseudomonotone equilibrium problems," *Optimization*, vol. 69, no. 4, pp. 901–923, 2020.

[21] D. V. Hieu, "An inertial-like proximal algorithm for equilibrium problems," *Mathematical Methods of Operations Research*, vol. 88, no. 3, pp. 399–415, 2018.

[22] D. V. Hieu, Y. J. Cho, and Y. Bin Xiao, "Modified extragradient algorithms for solving equilibrium problems," *Optimization*, vol. 67, no. 11, pp. 2003–2029, 2018.

[23] A. N. Iusem and W. Sosa, "On the proximal point method for equilibrium problems in Hilbert spaces," *Optimization*, vol. 59, no. 8, pp. 1259–1274, 2010.

[24] T. D. Quoc, P. N. Anh, and L. D. Muu, "Dual extragradient algorithms extended to equilibrium problems," *Journal of Global Optimization*, vol. 52, no. 1, pp. 139–159, 2011.

[25] H. ur Rehman, P. Kumam, I. K. Argyros, M. Shutaywi, and Z. Shah, "Optimization based methods for solving the equilibrium problems with applications in variational inequality problems and solution of Nash equilibrium models," *Mathematics*, vol. 8, no. 5, p. 822, 2020.

[26] H. ur Rehman, P. Kumam, Y. Je Cho, Y. I. Suleiman, and W. Kumam, "Modified Popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems," *Optimization Methods and Software*, vol. 36, no. 1, pp. 82–113, 2021.

[27] H. U. Rehman, P. Kumam, Q. L. Dong, Y. Peng, and W. Deebani, "A new Popov's subgradient extragradient method for two classes of equilibrium programming in a real Hilbert space," *Optimization*, vol. 70, no. 12, pp. 2675–2710, 2021.

[28] H. ur Rehman, P. Kumam, I. K. Argyros, N. A. Alreshidi, W. Kumam, and W. Jirakitpuwapat, "A self-adaptive extragradient methods for a family of pseudomonotone equilibrium programming with application in different classes of variational inequality problems," *Symmetry*, vol. 12, no. 4, p. 523, 2020.

[29] P. Kumam, I. K. Argyros, W. Kumam, and M. Shutaywi, "The inertial iterative extragradient methods for solving pseudomonotone equilibrium programming in Hilbert spaces," *Journal of Inequalities and Applications*, vol. 2022, no. 1, 2022.

[30] H. ur Rehman, P. Kumam, W. Kumam, M. Shutaywi, and W. Jirakitpuwapat, "The inertial sub-gradient extragradient method for a class of pseudo-monotone equilibrium problems," *Symmetry*, vol. 12, no. 3, p. 463, 2020.

[31] H. ur Rehman, P. Kumam, M. Shutaywi, N. A. Alreshidi, and W. Kumam, "Inertial optimization based twostep methods for solving equilibrium problems with applications in variational inequality problems and growth control equilibrium models," *Energies*, vol. 13, no. 12, p. 3292, 2020.

[32] J. Yang, "The iterative methods for solving pseudomontone equilibrium problems," *Journal of Scientific Computing*, vol. 84, no. 3, 2020.

[33] J. Yang and H. Liu, "The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in hilbert space," *Optimization Letters*, vol. 14, no. 7, pp. 1803–1816, 2019.

[34] Y. Censor, A. Gibali, and S. Reich, "The subgradient extragradient method for solving variational inequalities in Hilbert space," *Journal of Optimization Theory and Applications*, vol. 148, no. 2, pp. 318–335, 2010.

[35] H. ur Rehman, P. Kumam, A. B. Abubakar, and Y. J. Cho, "The extragradient algorithm with inertial effects extended to

equilibrium problems," *Computational and Applied Mathematics*, vol. 39, no. 2, p. 3, 2020.

[36] N. T. Vinh and L. D. Muu, "Inertial extragradient algorithms for solving equilibrium problems," *Acta Mathematica Vietnamica*, vol. 44, no. 3, pp. 639–663, 2019.

[37] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics*, Springer International Publishing, 2nd edition, 2017.

[38] J. V. Tiel, *Convex Analysis: An Introductory Text*, Wiley, New York, NY, USA, 1984.

[39] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, no. 4, pp. 591–597, 1967.

[40] K. Tan and H. Xu, "Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.

[41] H. ur Rehman, P. Kumam, Y. J. Cho, and P. Yordsorn, "Weak convergence of explicit extragradient algorithms for solving equilibirum problems," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 2019.

[42] Q. L. Dong, Y. J. Cho, L. L. Zhong, and T. M. Rassias, "Inertial projection and contraction algorithms for variational inequalities," *Journal of Global Optimization*, vol. 70, no. 3, pp. 687–704, 2017.

[43] M. V. Solodov and B. F. Svaiter, "A new projection method for variational inequality problems," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 765–776, 1999.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2775733, 7 pages
https://doi.org/10.1155/2022/2775733

Research Article

Some Fixed-Circle Results with Different Auxiliary Functions

Elif Kaplan,[1] **Nabil Mlaiki**,[2] **Nihal Taş**,[3] **Salma Haque**,[2] **and Asma Karoui Souayah**[4,5]

[1]*Ondokuz Mayis University, Department of Mathematics, Samsun, Turkey*
[2]*Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia 11586*
[3]*Balikesir University, Department of Mathematics, 10145 Bal kesir, Turkey*
[4]*Department of Business Administration, College of Science and Humanities, Dhurma, Shaqra University, Saudi Arabia*
[5]*Institut préparatoire Aux études d'ingénieurs de Gafsa, Gafsa University, Tunisia*

Correspondence should be addressed to Salma Haque; shaque@psu.edu.sa

Received 15 April 2022; Accepted 19 May 2022; Published 9 June 2022

Academic Editor: Santosh Kumar

As the generalization of the fixed-point theory, the fixed-circle problems are interesting and notable geometric constructions. In this paper, we prove that some new necessary conditions are investigated for the existence of a fixed circle of a given self-mapping in $\mathbb{G}$-metric spaces. The well-known Braincari and Chatterjea contractive conditions are generalized for proving the uniqueness of obtained theorems. Finally, an application to parametric rectified linear unit activation functions are given to show the importance of studying the fixed-circle problem.

1. Introduction and Preliminaries

Recently, there has been a trend to work fixed-circle problems in both metric spaces and some generalized metric spaces [1–17]. For some self mappings, when the fixed point is not unique, it is an open question about the geometric shape and in some cases the set of fixed point form a circle. For example, in establishing some applicable areas such as neural networks, besides many others. This approach was initiated in [6, 7] to examine the geometry of the set of fixed-points when the number of the fixed-points of self-mappings is more than one on both metric and S-metric spaces. Fixed-circle theorems were proved and extended with various aspects and were applied to discontinuous activation functions (for example, see [18–20] and the references therein), to rectified linear units activation functions used in the neural networks [21].

In this paper, we establish various fixed-circle theorems in $\mathbb{G}$-metric spaces. Different examples and application to parametric rectified linear unit activation functions are considered to illustrate the usability of our obtained results.

Firstly, we recall the concept of a $\mathbb{G}$-metric space.

Definition 1.1 (see [22]). Consider the set $\mathfrak{F} \neq \emptyset$ and $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow \mathbb{R} \cup \{0\}$ such that, for all $\xi, \zeta, \varpi, \eta \in \mathfrak{F}$, the following conditions are satisfying:

($\mathbb{G}1$)$\mathbb{G}(\xi, \zeta, \varpi) = 0$ if and only if $\xi = \zeta = \varpi$;

($\mathbb{G}2$)$0 < \mathbb{G}(\xi, \xi, \zeta)$ for all $\xi, \zeta \in \mathfrak{F}$ with $\xi \neq \zeta$;

($\mathbb{G}3$)$\mathbb{G}(\xi, \xi, \zeta) \leq \mathbb{G}(\xi, \zeta, \varpi)$ for all $\xi, \zeta, \varpi \in \mathfrak{F}$ with $\eta \neq \varpi$;

($\mathbb{G}4$)$\mathbb{G}(\xi, \zeta, \varpi) = \mathbb{G}(\xi, \varpi, \zeta) = \mathbb{G}(\zeta, \varpi, \xi) = \cdots$, (symmetry in all three variables);

($\mathbb{G}5$)$\mathbb{G}(\xi, \zeta, \varpi) \leq \mathbb{G}(\xi, \eta, \eta) + \mathbb{G}(\eta, \zeta, \varpi)$ for all $\xi, \zeta, \varpi, \eta \in \mathfrak{F}$, (rectangle inequality).

Then, the function $\mathbb{G}$ is called a $\mathbb{G}$-metric on $\mathfrak{F}$.

Definition 1.2 (see [22]). A $\mathbb{G}$ -metric space $(\mathfrak{F}, \mathbb{G})$ is called be symmetric if

$$\mathbb{G}(\xi, \zeta, \zeta) = \mathbb{G}(\zeta, \xi, \xi), \tag{1}$$

for all $\xi, \zeta \in \mathfrak{F}$.

In [23], Kaplan and Tas introduced the notion of circle on a $\mathbb{G}$-metric space. More precisely, let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $\xi_0 \in \mathfrak{F}$, $r \in (0,\infty)$. The circle of center ξ_0 and radius $r > 0$ is defined as

$$C_{\mathbb{G}}(\xi_0, r) = \{\xi \in \mathfrak{F} : \mathbb{G}(\xi_0, \xi, \xi) = r\}. \tag{2}$$

Example 1.1. Let $\mathfrak{F} = \mathbb{R}$ and d be a metric space. Let the function $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \max \{d(\xi, \zeta), d(\zeta, \varpi), d(\varpi, \xi)\} \tag{3}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$ [22]. Then, $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space. Let us consider the function $d : \mathfrak{F} \times \mathfrak{F} \longrightarrow \mathbb{R}$ as

$$d(\xi, \zeta) = \left| e^{\xi} - e^{\zeta} \right| \tag{4}$$

for all $\xi, \zeta \in \mathfrak{F}$. Then, we get

$$C_{\mathbb{G}}(\ln 2, \ln 4) = \ln 6 \tag{5}$$

the circle of center ln 2 and radius ln 4.

They also introduced the notion of fixed circle on a $\mathbb{G}$-metric space [23]. Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\xi_0, r)$ be a circle. For a self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$, if $\mathfrak{T}\xi = \xi$ for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$ then, the circle $C_{\mathbb{G}}(\xi_0, r)$ is said to be a fixed circle of $\mathfrak{T}$.

2. Some New Existence Conditions for Fixed Circles with Auxiliary Functions

Now, we present some new existence theorems for fixed circles of self-mappings.

Theorem 2.1. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$. Consider $\mathbb{M}_r : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ as*

$$\mathbb{M}_r(\eta) = \begin{cases} \eta - r & if \eta > 0 \\ 0 & if \eta = 0 \end{cases}, \tag{6}$$

for all $\eta \in \mathbb{R}^+ \cup \{0\}$. If the self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ is a function such that, for all $\xi \in \mathfrak{F}$, the following conditions are fulfilled:

(1) $\mathbb{G}(\xi_0, \mathfrak{T}\xi, \mathfrak{T}\xi) = r$ for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$,

(2) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \mathfrak{T}\zeta) > r$ for all $\xi, \zeta \in C_{\mathbb{G}}(\xi_0, r)$ with $\xi \neq \zeta$,

(3) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \mathfrak{T}\zeta) \leq \mathbb{G}(\xi, \xi, \zeta) - \mathbb{M}_r(\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi))$ for all $\xi, \zeta \in C_{\mathbb{G}}(\xi_0, r)$.

Then, the circle $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Proof. Fix $\xi \in C_{\mathbb{G}}(\xi_0, r)$. By hypothesis (1), we have $\mathfrak{T}\xi \in C_{\mathbb{G}}(\xi_0, r)$ for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$. We claim that $\xi = \mathfrak{T}\xi$, that is, ξ is a fixed point of $\mathfrak{T}$. Now, let us suppose that $\xi \neq \mathfrak{T}\xi$. Firstly, using the condition (2), we obtain

$$\mathbb{G}\left(\mathfrak{T}^2\xi, \mathfrak{T}^2\xi, \mathfrak{T}\xi\right) > r. \tag{7}$$

Using the condition (3), we have

$$\begin{aligned} \mathbb{G}\left(\mathfrak{T}^2\xi, \mathfrak{T}^2\xi, \mathfrak{T}\xi\right) &\leq \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi) - \mathbb{M}_r(\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi)) \\ &= \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi) - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi) + r = r. \end{aligned} \tag{8}$$

Then, it follows from the inequalities (7) and (8), which is a contradiction. Hence, it should be $\xi = \mathfrak{T}\xi$. As a consequence, $\mathfrak{T}$ fixes the circle $C_{\mathbb{G}}(\xi_0, r)$. □

Remark 2.1.

(1) Note that, in Theorem 2.1, the center of $C_{\mathbb{G}}(\xi_0, r)$ need not to be fixed

(2) Theorem 2.1 generalizes Theorem 3 given in [9].

(3) Since the notion of a $\mathbb{G}$-metric and an S-metric are independent (see, [24] for more details), then Theorem 2.1 is independent from Theorem 4.1 given in [1].

Example 2.1. Let $\mathfrak{F} = [0,\infty)$ be the interval of nonnegative real numbers and let $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \begin{cases} 0 & if \xi = \zeta = \varpi \\ \max \{\xi, \zeta, \varpi\} & otherwise \end{cases} \tag{9}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$. Then, $\mathbb{G}$ is a $\mathbb{G}$-metric on $\mathfrak{F}$.

The circle $C_{\mathbb{G}}(1, 3)$ is obtained as follows:

$$C_{\mathbb{G}}(1, 3) = \{\xi \in \mathfrak{F} : \mathbb{G}(1, \xi, \xi) = 3\} = \{3\}. \tag{10}$$

If $\mathfrak{T}_1 : \mathfrak{F} \longrightarrow \mathfrak{F}$ is defined by

$$\mathfrak{T}_1\xi = \begin{cases} \kappa & if \xi = 1 \\ 3 & if \xi \neq 1 \end{cases}, \tag{11}$$

for all $\xi \in \mathfrak{F}$ and $\kappa \neq 1$, then $\mathfrak{T}_1$ satisfies all the hypotheses of Theorem 2.1 and the circle $C_{\mathbb{G}}(1, 3)$ is fixed by $\mathfrak{T}_1$. That is, the self-mapping $\mathfrak{T}_1$ has the unique fixed point $\xi = 3$. Notice that the center 1 of the circle $C_{\mathbb{G}}(1, 3)$ is not fixed by the self-mapping $\mathfrak{T}_1$.

Theorem 2.2. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space, $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$ and let define $\varphi : \mathfrak{F} \longrightarrow [0,\infty)$ by*

$$\varphi(\xi) = \mathbb{G}(\xi, \xi, \xi_0), \tag{12}$$

for $\xi \in \mathfrak{F}$. Suppose that the following conditions hold:

(1) $\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \varphi(\xi) + \varphi(\mathfrak{T}\xi) - 2r$,

(2) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \leq r$,

for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$ such that $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$. Then, $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Proof. Let $\xi_0 \in C_{\mathbb{G}}(\xi_0, r)$ be any arbitrary point. Together with (1), we obtain

$$\begin{aligned}\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &\leq \varphi(\xi) + \varphi(\mathfrak{T}\xi) - 2r \\ &\leq \mathbb{G}(\xi, \xi, \xi_0) + \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - 2r \\ &= \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0).\end{aligned} \tag{13}$$

From (2), the point $\mathfrak{T}\xi$ should lie on or interior of the circle $C_{\mathbb{G}}(\xi_0, r)$. If $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) < r$, which leads to a contradiction by the inequality (2.5). Therefore, it should be $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) = r$. If $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) < r$, then by the inequality (13) we have

$$\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - r = r - r = 0 \tag{14}$$

and we obtain $\mathfrak{T}\xi = \xi$. As a consequence, the circle $C_{\mathbb{G}}(\xi_0, r)$ is fixed circle of $\mathfrak{T}$. □

Remark 2.2. Notice that the condition (1) implies that $\mathfrak{T}\xi$ is not inside $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Similarly, (2) guarantees that $\mathfrak{T}\xi$ is not outside of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Thus, $\mathfrak{T}\xi \in C_{\mathbb{G}}(\xi_0, r)$ for any $\xi \in C_{\mathbb{G}}(\xi_0, r)$ and so we get $\mathfrak{T}(C_{\mathbb{G}}(\xi_0, r)) \subset C_{\mathbb{G}}(\xi_0, r)$.

(1) Theorem 2.2 generalizes Theorem 2.2 given in [7].

(2) Theorem 2.2 is independent from Theorem 3.11 given in [6].

Example 2.2. Let $\mathfrak{F} = \mathbb{R}$ and the mapping $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = |\xi - \zeta| + |\xi - \varpi| + |\zeta - \varpi|, \tag{15}$$

for each $\xi, \zeta, \varpi \in \mathfrak{F}$ [25]. Then, $(\mathfrak{F}, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us take the circle $C_{\mathbb{G}}(0, 6)$. If we define $\mathfrak{T}_2 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_2\xi = \frac{7\xi + 9\sqrt{3}}{\sqrt{3}\xi + 7}, \tag{16}$$

for all $\xi \in \mathfrak{F}$, then $\mathfrak{T}_2$ confirms that the conditions (1) and (2) in Theorem 2.2. Hence, the circle $C_{\mathbb{G}}(0, 6)$ is a fixed circle of $\mathfrak{T}_2$.

In the following example, we present an example of a self-mapping that satisfies the condition (1) and does not satisfy the condition (2).

Example 2.3. Let $\mathfrak{F} = \mathbb{R}$ and $(\mathfrak{F}, \mathbb{G})$ be the $\mathbb{G}$ -metric space defined in Example 2.2. Let us consider the circle $C_{\mathbb{G}}(-2, 4)$ and define the self-mapping $\mathfrak{T}_3 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_3\xi = \begin{cases} -5 & \xi = -4 \\ 5 & \xi = 0 \\ 10 & \text{otherwise} \end{cases}, \tag{17}$$

for all $\xi \in \mathfrak{F}$. Then, the self-mapping $\mathfrak{T}_3$ satisfies the condition (1) in Theorem 2.2 but does not satisfy the condition (2) in Theorem 2.2. Obviously, $\mathfrak{T}_3$ does not fix the circle $C_{\mathbb{G}}(-2, 4)$.

In the next example, we present an example of a self-mapping that satisfies the condition (2) and does not satisfy the condition (1).

Example 2.4. Let $\mathfrak{F} = \mathbb{R}$ and the mapping $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \max\{|\xi - \zeta|, |\xi - \varpi|, |\zeta - \varpi|\}, \tag{18}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$ [25]. Then, $(\mathfrak{F}, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us take the circle $C_{\mathbb{G}}(0, 1/2)$. If we define $\mathfrak{T}_4 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_4\xi = \begin{cases} -\frac{1}{2} & if\, \xi = -1 \\ \frac{1}{2} & if\, \xi = 1 \\ 3 & otherwise \end{cases}, \tag{19}$$

for all $\xi \in \mathfrak{F}$, then $\mathfrak{T}_4$ confirms that condition (2) in Theorem 2.2 but does not satisfy the condition (1) in Theorem 2.2. Clearly, $\mathfrak{T}_4$ does not fix the circle $C_{\mathbb{G}}(0, 1/2)$.

Now, we present the following theorem.

Theorem 2.3. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$. Let the mapping φ be defined as Theorem 2.1. If the self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ is a function such that for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$ and $k \in [0, 1)$, the following conditions are satisfied:*

(1) $\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \varphi(\xi) - \varphi(\mathfrak{T}\xi)$,

(2) $k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) + \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \geq r$,

then the circle $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Proof. Let $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Conversely, suppose that $\xi \neq \mathfrak{T}\xi$. Then, take into account the conditions (1) and (2), we conclude that

$$\begin{aligned}\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &\leq \varphi(\xi) - \varphi(\mathfrak{T}\xi) \\ &= \mathbb{G}(\xi, \xi, \xi_0) - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \\ &= r - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \leq k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \\ &\quad + \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \\ &= k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi),\end{aligned} \tag{20}$$

which is a contradiction $k \in 0, 1)$. As a result, we get $\xi = \mathfrak{T}\xi$ and $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$. □

Remark 2.3. Notice that the condition (1) guarantees that $\mathfrak{T}\xi$ is not in the exterior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Similarly, the condition (2) guarantees that $\mathfrak{T}\xi$ can lies on or exterior or interior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Hence $\mathfrak{T}\xi$ should lies on or interior of the circle $C_{\mathbb{G}}(\xi_0, r)$.

(1) Theorem 2.3 generalizes Theorem 2.3 given in [7].

(2) Theorem 2.3 is independent from Theorem 3.2 given in [8].

Now, we present some examples concerning with self-mappings which have a fixed circle.

Example 2.5. Let $\mathfrak{F} = \mathbb{R}$ and $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space defined in Example 2.4. Let us consider the circle $C_{\mathbb{G}}(1, 3) = 3$ and define the self-mapping $\mathfrak{T}_5 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_5\xi = \begin{cases} 2\xi - 3 & \xi = 3 \\ 5 & \text{otherwise} \end{cases}, \tag{21}$$

for all $\xi \in \mathfrak{F}$. Then, the self-mapping $\mathfrak{T}_5$ satisfies the condition (1) and (2) in Theorem 2.3. So, $C_{\mathbb{G}}(1, 3)$ is a fixed circle of $\mathfrak{T}_5$.

Example 2.6. Let $\mathfrak{F} = \mathbb{R}$ and the function $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \left|e^{\xi} - e^{\zeta}\right| + \left|e^{\zeta} - e^{\varpi}\right| + \left|e^{\xi} - e^{\varpi}\right|, \tag{22}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$. Then, it can be easily checked that $(\mathfrak{F}, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us consider the circle $C_{\mathbb{G}}(0, 2) = \{\ln 2\}$ and define the self-mapping $\mathfrak{T}_6 : \mathfrak{F} \longrightarrow \mathfrak{F}$ as

$$\mathfrak{T}_6\xi = \begin{cases} \xi & \xi \in C_{\mathbb{G}}(0, 2) \\ \ln 5 & \text{otherwise} \end{cases}, \tag{23}$$

for all $\xi \in \mathfrak{F}$. So, the self-mapping $\mathfrak{T}_6$ provides the condition (1) and (2) in Theorem 2.3. Hence, $C_{\mathbb{G}}(0, 2)$ is a fixed circle of $\mathfrak{T}_6$.

Next, we give an example of a self-mapping which provides the condition (1) and does not provide the condition (2).

Example 2.7. Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r)$ be a circle on $\mathfrak{F}$. If we take $\mathfrak{T}_7\xi = \xi_0$ as the self-mapping on $\mathfrak{F}$, then we deduce that the self-mapping $\mathfrak{T}_7$ satisfies the condition (1) in Theorem 2.3 but does not satisfy the condition (2) in Theorem 2.3. So, it can be easily shown that $\mathfrak{T}_7$ does not fix a circle $C_{\mathbb{G}}(\xi_0, r)$.

In the next example, we present an example of a self-mapping which satisfies the condition (2) and does not satisfy the condition (1).

Example 2.8. Let $\mathfrak{F} = \mathbb{R}$ and let the function $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \max\{|\xi - \zeta|, |\zeta - \varpi|, |\xi - \varpi|\}, \tag{24}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$ [25]. Let us consider the circle $C_{\mathbb{G}}(0, 5)$ and define the self-mapping $\mathfrak{T}_8 : \mathfrak{F} \longrightarrow \mathfrak{F}$ as $\mathfrak{T}_8\xi = 5$ for all $\xi \in \mathfrak{F}$. Then, the self-mapping $\mathfrak{T}_8$ provides the condition (2) in Theorem 2.3 but does not provide the condition (1) in Theorem 2.3. It can be easily shown that $\mathfrak{T}_8$ does not fix the circle $C_{\mathbb{G}}(0, 5)$.

Theorem 2.4. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$. Let the mapping φ be defined as Theorem 2.1. If the self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ is a function such that for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$ and $k \in [0, 1)$, the following conditions are satisfied:*

(1) $\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} - r$,

(2) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq r$,

then the circle $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Proof. Let $\xi \in C_{\mathbb{G}}(\xi_0, r)$ such that $\xi \neq \mathfrak{T}\xi$. We show $\xi = \mathfrak{T}\xi$ under the following two cases:

Case 1: Let $\max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} = \varphi(\xi)$. Then, we get

$$\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} - r = \varphi(\xi) - r = r - r = 0, \tag{25}$$

a contradiction. Hence, we get $\xi = \mathfrak{T}\xi$.

Case 2: Let $\max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} = \varphi(\mathfrak{T}\xi)$. Then, we obtain

$$\begin{aligned}\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &\leq \max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} - r = \varphi(\mathfrak{T}\xi) - r \\ &= \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - r \leq r + k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \\ &\quad - r = k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi),\end{aligned} \tag{26}$$

a contradiction with $k \in 0, 1)$. Therefore, we have $\xi = \mathfrak{T}\xi$. □

Consequently, the circle $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Remark 2.4. Notice that condition (1) guarantees that $\mathfrak{T}\xi$ is not in the interior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Similarly, the condition (2) guarantees that $\mathfrak{T}\xi$ is not the exterior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Hence $\mathfrak{T}\xi \in C_{\mathbb{G}}(\xi_0, r)$ for each $\xi \in C_{\mathbb{G}}(\xi_0, r)$ and so we get $\mathfrak{T}(C_{\mathbb{G}}(\xi_0, r)) \subset C_{\mathbb{G}}(\xi_0, r)$.

(1) Theorem 2.4 is independent from Theorem 4.2 given in [1].

(2) If we consider the self-mapping $\mathfrak{T}_5 : \mathfrak{F} \longrightarrow \mathfrak{F}$ defined in Example 2.5, then $\mathfrak{T}_5$ satisfies the conditions (1) and (2) in Theorem 2.4 and so $C_{\mathbb{G}}(1, 3)$ is a fixed circle of $\mathfrak{T}_5$.

Notice that the identity mapping $I_{\mathfrak{F}}$ defined as $I_{\mathfrak{F}}(\xi) = \xi$ for all $\xi \in \mathfrak{F}$ satisfies conditions (1) and (2) (resp., (1) and (2)) in Theorem 2.2 (resp., Theorem 2.3). Therefore, we need a condition which excludes the identity map in Theorem 2.2 (resp., Theorem 2.3). For this aim, we give in [23] the following theorem.

Theorem 2.5 (see [23]). *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space, $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping having a fixed circle $C_{\mathbb{G}}(\xi_0, r)$ and the mapping φ be defined as 2.2. The self-mapping $\mathfrak{T}$ satisfies the condition*

$$(I_{\mathbb{G}})\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq h[\phi(\xi) - \phi(\mathfrak{T}\xi)], \tag{27}$$

for all $\xi \in \mathfrak{F}$ and some $h \in [0, 1/4)$ if and only if $\mathfrak{T} = I_{\mathfrak{F}}$.

Now we give the another theorem which excludes the identity map using the auxiliary function ξ_r defined in (6).

Theorem 2.6. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space, $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping having a fixed circle $C_{\mathbb{G}}(\xi_0, r)$ and the mapping $\mathbb{M}_r$ defined in (6). The self-mapping $\mathfrak{T}$ satisfies the condition*

$$(I^*_{\mathbb{G}})\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) < \mathbb{M}_r(\mathbb{G}(\xi, \xi, \mathfrak{T}\xi)) + r, \tag{28}$$

for all $\xi \in \mathfrak{F}$ if and only if $\mathfrak{T} = I_{\mathfrak{F}}$.

Proof. Let $\xi \in \mathfrak{F}$ be any point such that $\xi \neq \mathfrak{T}\xi$. Using the inequality $(I^*_{\mathbb{G}})$, we get

$$\begin{aligned} \mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &< \mathbb{M}_r(\mathbb{G}(\xi, \xi, \mathfrak{T}\xi)) + r \\ &= \mathbb{G}(\xi, \xi, \mathfrak{T}\xi) - r + r = \mathbb{G}(\xi, \xi, \mathfrak{T}\xi), \end{aligned} \tag{29}$$

a contradiction. Hence we get $\xi = \mathfrak{T}\xi$ and so $\mathfrak{T} = I_{\mathfrak{F}}$. □

The converse statement is clear.

3. Some New Uniqueness Conditions for Fixed Circles with Integral Type Contractions

In [26], Braincari gave an integral contractive condition which was a generalization of Banach contraction in a metric space. By the Braincari type contractive condition, we obtain a uniqueness theorem as follows.

Theorem 3.1. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$. Let $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping satisfying the inequalities of Theorem 2.1 (resp., Theorem 2.2, Theorem 2.3 and Theorem 2.4). If the contractive condition*

$$\int_0^{\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \mathfrak{T}\zeta)} \omega(t)dt \leq c \int_0^{\mathbb{G}(\xi, \xi, \zeta)} \omega(t)dt \tag{30}$$

is satisfied for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$, $\zeta \in \mathfrak{F} - C_{\mathbb{G}}(\xi_0, r)$ where $c \in [0, 1)$ and $\omega : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesque measurable map which is summable (that is, with a finite integral) on each compact subset of $[0, \infty)$ such that $\int_0^{\varepsilon} \omega(t)dt > 0$ for each $\varepsilon > 0$, then $C_{\mathbb{G}}(\xi_0, r_0)$ is the unique fixed circle of $\mathfrak{T}$.

Proof. Suppose that the self-mapping $\mathfrak{T}$ has two different fixed circles $C_{\mathbb{G}}(\xi_0, r_0)$ and $C_{\mathbb{G}}(\xi_1, r_1)$. Let $u \in C_{\mathbb{G}}(\xi_0, r_0)$ and $v \in C_{\mathbb{G}}(\xi_1, r_1)$ be arbitrary points such that $u \neq v$. We show that $\mathbb{G}(u, u, v) = 0$ and hence $u = v$. By the contractive condition of $\mathfrak{T}$, that is, using the inequality (30), we have

$$\int_0^{\mathbb{G}(u,u,v)} \omega(t)dt = \int_0^{\mathbb{G}(\mathfrak{T}u, \mathfrak{T}u, \mathfrak{T}v)} \omega(t)dt \leq c \int_0^{\mathbb{G}(u,u,v)} \omega(t)dt \tag{31}$$

which is a contradiction $c \in [0, 1)$. Consequently, $C_{\mathbb{G}}(\xi_0, r_0)$ is the unique fixed circle of $\mathfrak{T}$. □

Taking into consideration that Chatterjea type contraction condition [27], we prove the following theorem.

Theorem 3.2. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r_0)$ be any circle on $\mathfrak{F}$. Let $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping satisfying the inequalities of Theorem 2.1 (resp., Theorem 2.2, Theorem 2.3 and Theorem 2.4). If the contractive condition*

$$\int_0^{\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \mathfrak{T}\zeta)} \omega(t)dt \leq \eta \left(\int_0^{\mathbb{G}(\xi, \xi, \mathfrak{T}\zeta)} \omega(t)dt + \int_0^{\mathbb{G}(\zeta, \zeta, \mathfrak{T}\xi)} \omega(t)dt \right) \tag{32}$$

is satisfied for all $\xi \in C_{\mathbb{G}}(\xi_0, r), \zeta \in \mathfrak{F} - C_{\mathbb{G}}(\xi_0, r)$ and $\eta \in [0, 1/2)$ where $\omega : [0,\infty) \longrightarrow 0,\infty)$ is a Lebesque measurable map which is summable (that is, with a finite integral) on each compact subset of $[0, \infty)$ such that $\int_0^{\varepsilon} \omega(t)dt > 0$ for each $\varepsilon > 0$, then the fixed circle of $\mathfrak{T}$ is unique.

Proof. Assume that there exist two different fixed-circles $C_{\mathbb{G}}(\xi_0, r_0)$ and $C_{\mathbb{G}}(\xi_1, r_1)$ of the self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$. Let $u \in C_{\mathbb{G}}(\xi_0, r_0)$ and $v \in C_{\mathbb{G}}(\xi_1, r_1)$ be arbitrary points such that $u \neq v$. Using the inequality (32) and the symmetric property of $\mathbb{G}$-metric, we obtain

$$\begin{aligned} \int_0^{\mathbb{G}(u,u,v)} \omega(t)dt &= \int_0^{\mathbb{G}(\mathfrak{T}u, \mathfrak{T}u, \mathfrak{T}v)} \omega(t)dt \\ &\leq \eta \left(\int_0^{\mathbb{G}(u,u,\mathfrak{T}v)} \omega(t)dt + \int_0^{\mathbb{G}(v,v,\mathfrak{T}u)} \omega(t)dt \right) \\ &= \eta \left(\int_0^{\mathbb{G}(u,u,v)} \omega(t)dt + \int_0^{\mathbb{G}(v,v,u)} \omega(t)dt \right) \\ &= 2\eta \int_0^{\mathbb{G}(u,u,v)} \omega(t)dt, \end{aligned} \tag{33}$$

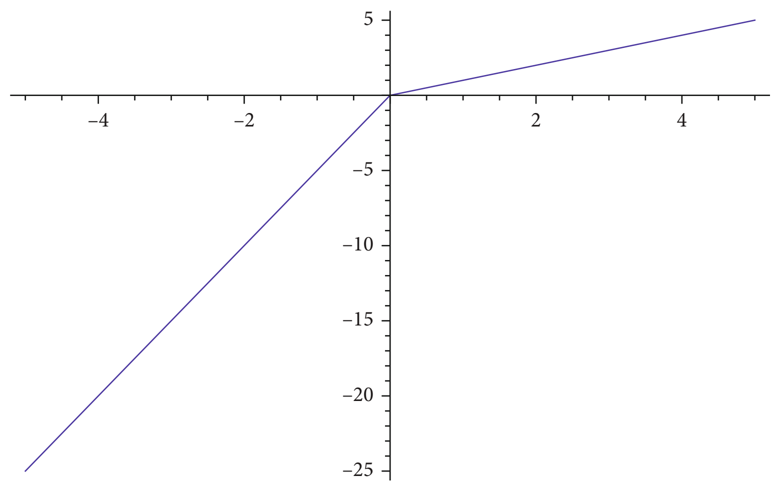


Figure 1: The activation function *PReLU*.

which is a contradiction. Consequently, it should be $u = v$ and thus $C_{\mathbb{G}}(\xi_0, r_0)$ is the unique fixed circle of $\mathfrak{T}$. □

Remark 3.1. The choice of used contractive condition in uniqueness theorem is not unique. Any contractive condition used to derive the fixed-point theorem can also be selected.

4. An Application to Parametric *ReLU*

In this section, we present a new application to "Parametric Rectified Linear Unit (*PReLU*)" using the obtained fixed-circle results. This activation function *PReLU* was defined to generalize the traditional rectified unit and it adaptively learns the parameters of the rectifiers (see [28] for more details). This activation function is defined by

$$P\,\mathrm{Re}\,LU(\xi) = \begin{cases} c\xi & if\xi < 0 \\ \xi & if\xi \geq 0 \end{cases}, \tag{34}$$

with parameter c. Let us take $\mathfrak{F} = [0,\infty)$ with the $\mathbb{G}$-metric defined as in Example 2.1 and $c = 5$. Then we have

$$P\,\mathrm{Re}\,LU(\xi) = \begin{cases} 5\xi & if\xi < 0 \\ \xi & if\xi \geq 0 \end{cases}, \tag{35}$$

for all $\xi \in 0,\infty)$ (see, Figure 1).

If we choose a circle $C_{\mathbb{G}}(0, 1) = \{1\}$, then *PReLU* satisfies the conditions of Theorem 2.1 (resp., Theorem 2.2, Theorem 2.3 and Theorem 2.4). Thereby, $C_{\mathbb{G}}(0, 1)$ is a fixed circle of *PReLU*. On the other hand, this activation function fixes all circles $C_{\mathbb{G}}(0, r)$ with $r > 0$, that is, the number of fixed circles of *PReLU* is infinite. In this case, it is important because it increases the learning capacity of the activation function.

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Funding

This research work did not receive any external funding.

Acknowledgments

The authors N. Mlaiki and S. Haque would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

References

[1] U. Çelik and N. Y. Özgür, "On the fixed-circle problem," *Facta Universitatis, Series: Mathematics and Informatics*, vol. 35, no. 5, pp. 1273–1290, 2021.

[2] N. Mlaiki, N. Y. Özgür, and N. Taş, "New fixed-circle results related to Fc-contractive and Fc-expanding mappings on metric spaces," 2021, https://arxiv.org/abs/2101.10770.

[3] N. Mlaiki, U. Çelik, N. Taş, N. Y. Özgür, and A. Mukheimer, "Wardowski Type Contractions and the Fixed-Circle Problem on -Metric Spaces," *Journal of Mathematics*, vol. 2018, Article ID 9127486, 9 pages, 2018.

[4] N. Mlaiki, N. Y. Özgür, and N. Tas, "New Fixed-Point Theorems on an $S-metric$ Space via Simulation Functions," *Mathematics*, vol. 7, no. 7, p. 583, 2019.

[5] N. Mlaiki, N. Taş, and N. Y. Özgür, "On the fixed-circle problem and khan type contractions," *Axioms*, vol. 7, no. 4, p. 80, 2018.

[6] N. Y. Özgür, N. Taş, and U. Çelik, "New fixed-circle results on S-metric spaces," *Bulletin of Mathematical Analysis and Applications*, vol. 9, no. 2, pp. 10–23, 2017.

[7] N. Y. Özgür and N. Taş, "Some fixed-circle theorems on metric spaces," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 4, pp. 1433–1449, 2019.

[8] N. Y. Özgür and N. Taş, "FIXED-CIRCLE problem on S-METRIC spaces with a geometric viewpoint," *Facta Universitatis,*

Series: Mathematics and Informatics, vol. 34, no. 3, pp. 459–472, 2019.

[9] N. Y. Özgür and N. Taş, "Some fixed-circle theorems and discontinuity at fixed circle," in *AIP conference proceedings*, vol. 1926, AIP publishing LLC, 2018no. 1, Article ID 020048.

[10] N. Y. Özgür and N. Tas, "On the geometry of fixed points of self-mappings on $S-metric$ spaces," *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*, vol. 69, no. 2, pp. 190–198, 2020.

[11] H. N. Saleh, S. Sessa, W. M. Alfaqih, M. Imdad, and N. Mlaiki, "Fixed Circle and Fixed Disc Results for New Types of Θc-Contractive Mappings in Metric Spaces," *Symmetry*, vol. 12, no. 11, p. 1825, 2020.

[12] N. Taş, N. Y. Özgür, and N. Mlaiki, "New types of FC-contractions and the fixed-circle problem," *Mathematics*, vol. 6, no. 10, p. 188, 2018.

[13] N. Taş, "Suzuki-Berinde type fixed-point and fixed-circle results on S-metric spaces," *Journal of Linear and Topological Algebra*, vol. 7, no. 3, pp. 233–244, 2018.

[14] N. Taş, "Various types of fixed-point theorems on S-metric spaces," *Journal of Balikesir University Institute of Science and Technology*, vol. 20, no. 2, pp. 211–223, 2018.

[15] A. Tomar, M. Joshi, and S. K. Padaliya, "Fixed point to fixed circle and activation function in partial metric space," *Journal of Applied Analysis*, 2021.

[16] M. Joshi, A. Tomar, and S. K. Padaliya, *Fixed point to fixed disc and application in partial metric spaces*, Chapter in a book Fixed Point Theory and its Applications to Real World Problem, Nova Science Publishers, New York, USA, 2021.

[17] M. Joshi and A. Tomar, "On Unique and Nonunique Fixed Points in Metric Spaces and Application to Chemical Sciences," *Journal of Function Spaces*, vol. 2021, Article ID 5525472, 11 pages, 2021.

[18] N. Y. Özgür and N. Taş, "New discontinuity results at fixed point on metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 23, no. 2, pp. 1–14, 2021.

[19] R. P. Pant, N. Y. Özgür, and N. Taş, "On discontinuity problem at fixed point," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 1, pp. 499–517, 2020.

[20] N. Taş and N. Y. Özgür, "A new contribution to discontinuity at fixed point," 2017, https://arxiv.org/abs/1705.03699.

[21] N. Taş, "Bilateral-type solutions to the fixed-circle problem with rectified linear units application," *Turkish Journal of Mathematics*, vol. 44, no. 4, pp. 1330–1344, 2020.

[22] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[23] E. Kaplan and N. Taş, "Non-Unique Fixed Points and Some Fixed-Circle Theorems on G-Metric Spaces," in *Submitted to Fixed Point Theory*, 2022.

[24] N. Van Dung, N. T. Hieu, and S. Radojević, "Fixed point theorems for $g-monotone$ maps on partially ordered $S-metric$ spaces," *Filomat*, vol. 28, no. 9, pp. 1885–1898, 2014.

[25] R. P. Agarwal, E. Karapınar, D. O'Regan, and A. F. Roldán-López-de-Hierro, *Fixed Point Theory in Metric Type Spaces*, Springer, Cham, 2015.

[26] A. Branciari, "A fixed point theorem for mappings satisfying a general contractive condition of integral type," *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 9, 536 pages, 2002.

[27] S. K. Chatterjea, "Fixed point theorems," *Comptes rendus de l'Académie bulgare des Sciences*, C. R. Acad, Ed., vol. 25, pp. 727–730, 1972.

[28] K. He, X. Zhang, S. Ren, and J. Sun, "Delving deep into rectifiers: surpassing human-level performance on image net classification," 2015, https://arxiv.org/abs/1502.01852.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 7197541, 9 pages
https://doi.org/10.1155/2022/7197541

Research Article

Fixed Points of Proinov Type Multivalued Mappings on Quasimetric Spaces

Erdal Karapinar,[1,2,3] **Andreea Fulga**,[4] **and Seher Sultan Yeşilkaya**[1]

[1] *Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam*
[2] *Department of Mathematics, Çankaya University, Etimesgut, Ankara, Turkey*
[3] *Department of Medical Research, China Medical University Hospital, China Medical University, 40402 Taichung, Taiwan*
[4] *Department of Mathematics and Computer Sciences, Transilvania University of Brasov, Brasov, Romania*

Correspondence should be addressed to Erdal Karapinar; erdalkarapinar@tdmu.edu.vn

Received 25 March 2022; Revised 26 April 2022; Accepted 6 May 2022; Published 25 May 2022

Academic Editor: Santosh Kumar

In this paper, we obtain new results which have not been encountered before in the literature, in multivalued quasimetric spaces, inspired by Proinov type contractions. We use admissible function as proving theorems. We also give an example that supports our theorems.

1. Introduction and Preliminaries

Fixed point theory has become an important research topic after the famous mathematician Banach's definition of the metric fixed point [1]. Many theoretical and applied studies have been done on fixed point theory. In the 21st century, the fixed point is still a popular and dynamic research topic. The concept of metric space, which forms the basis of the fixed point theory, is generalized by many researchers and new spaces (b-metric, quasimetric, partial metric, fuzzy metric, etc.) are introduced. One of the important generalizations is quasimetric space proved in 1931 as follows.

Definition 1 (see [2–4]). Let $\mathscr{X} \neq \varnothing$. A function $q : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R}_0^+$is a quasimetric on $\mathscr{X}$ if it satisfies the following:

$$\begin{aligned} q(t,u) = q(u,t) = 0 \Leftrightarrow t = u, \\ q(t,w) \leq q(t,u) + q(u,w), \end{aligned} \tag{1}$$

for all $t, u, w \in \mathscr{X}$ in this case, the pair $(\mathscr{X}, q)$ is a quasimetric space.

Let q be a quasimetric on $\mathscr{X}$, and the set $\mathfrak{B}_q(t,e) = \{w \in \mathscr{X} : q(t,w) < e\}$. Thus, the family $\{\mathfrak{B}_q(t,e) : t \in \mathscr{X}, e > 0\}$ forms a base for a T_0 topology τ_q on $\mathscr{X}$. Moreover, if A is a subset of $\mathscr{X}$, we denote by $cl_q(A)$ the closure of A with respect to T_0 topology; we say that the subset A is τ_q -closed if it is closed with respect to τ_q.

A sequence (t_r) in a quasimetric space converges to $t \in \mathscr{X}$, (in τ_q) if and only if $q(t, t_r) \longrightarrow 0$ as $r \longrightarrow \infty$. Moreover, we say that the sequence (t_r) is

(1) left-Cauchy if for every $e > 0$ there exists $r_e \in \mathbb{N}$ such that $q(t_r, t_m) < e$, whenever $r_e \leq r \leq m$

(2) right-Cauchy if for every $e > 0$ there exists $r_e \in \mathbb{N}$ such that $q(t_m, t_r) < e$, whenever $r_e \leq r \leq m$

Thereupon, a quasimetric space is called to be left (resp., right) complete if every left (resp., right) Cauchy sequence converges (to respect τ_q) (see, e.g., [5, 6, 40, 41]).

Nadler [7] is the first who introduced the framework for multivalued contraction mappings. The author proved the important theorem generalized Banach principle using the Hausdorff metric for multivalued mappings. After the proof of Nadler theorem, the theory of multivalued contraction mappings attracted great attention and is used in various branches of mathematics. Multivalent mappings in different spaces are introduced. One of them is multivalued mapping introduced in quasimetric-spaces by Shoaib [8] (see also [9, 10]).

Let $(\mathcal{X}, q)$ be a quasimetric space. We shall denote by $\mathcal{P}(\mathcal{X})$ the set of all nonempty subsets of $\mathcal{X}$, by $\mathcal{C}l_q(\mathcal{X})$ the set of all nonempty closed bounded subsets of $\mathcal{X}$, and let $\mathcal{K}_q(\mathcal{X})$ be the set of all compact subsets of $\mathcal{X}$.

Definition 2. Let $\mathcal{X} \neq \varnothing$ and $Z : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{X})$ be a multivalued map. A point $t \in \mathcal{X}$ is said to be a fixed point of Z if $t \in Z(t)$.

The set of the fixed point of a mapping Z is denoted by $\mathcal{F}(Z)$.

Lemma 3 is an important condition in the following main results.

Lemma 3 (see [8]). *Let A and B be nonempty closed bounded subsets of a quasimetric space $(\mathcal{X}, q)$ and let $\delta > 1$. Then, for all $t \in A$, there exists $u \in B$ such that $q(t, u) \leq \delta H_q(A, B)$.*

Nadler [7] stated that if $A, B \in K(\mathcal{X})$ in the metric spaces it is also provided for $\delta \geq 1$. With similarly thinking, the following lemma can be written.

Lemma 4. *Let A and B be nonempty compact subsets of a quasimetric space $(\mathcal{X}, q)$, and let $\delta \geq 1$. Then, for all $t \in A$, there exists $u \in B$ such that $q(t, u) \leq \delta H_q(A, B)$.*

Many researchers have stated different studies on well-known quasimetric spaces, see e.g., [11–13]. In recent years, Alqahtani et al. [14] introduced a new generalization in quasimetric spaces and defined Δ-symmetric quasimetric spaces. This definition is as follows.

Definition 5 (see [14]). Assume that $(\mathcal{X}, q)$ is a quasimetric space. If there exists a positive real number $\Delta > 0$ such that

$$q(t, u) \leq \Delta \cdot q(u, t), \tag{2}$$

for all $t, u \in \mathcal{X}$, then, the pair $(\mathcal{X}, \mathrm{q})$ is called a Δ -symmetric quasimetric space.

To simplify the notations, in the following, we will mark by $(\mathcal{X}, q)_\Delta$ a Δ-symmetric quasimetric space.

It is clear that if $\Delta = 1$, thus $(\mathcal{X}, \mathrm{q})_1$ becomes a metric space.

Definition 6 (see [8]). Let $(\mathcal{X}, q)_\Delta$ and $A, B \in \mathcal{P}(\mathcal{X})$. A function $H_q : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \longrightarrow [0,\infty)$, defined by

$$\mathrm{H_q}(A, B) = \max \left\{ \sup_{t \in A} \mathrm{q}(t, B), \sup_{u \in B} \mathrm{q}(A, u) \right\}, \tag{3}$$

where $q(t, A) = \inf_{u \in A} q(t, u)$ and $q(A, t) = \inf_{u \in A} q(u, t)$), satisfies all the axioms of quasimetric and is known as the Hausdorff quasimetric induced by the quasimetric q.

Example 7. Let $(\mathbb{R}, d)$ be a metric space and a function $q : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$, where

$$\mathrm{q}(t, u) = \begin{cases} 3d(t, u), & \text{if } t \geq u, \\ d(t, u), & \text{otherwise.} \end{cases} . \tag{4}$$

Then, $(\mathcal{X}, q)$ is a 3-symmetric quasimetric space, but it is not a metric space.

In the following, we shall collect some main properties of a Δ-symmetric quasimetric space.

Lemma 8 (see [15]). *Let $(\mathcal{X}, q)_\Delta$, $\{t_r\}$ be a sequence in $\mathcal{X}$ and $t \in \mathcal{X}$. Then,*

(i) $\{t_r\}$ is right-Cauchy $\Leftrightarrow \{t_r\}$ is left-Cauchy $\Leftrightarrow \{t_r\}$ is Cauchy

(ii) if $\{u_r\}$ is a sequence in $\mathcal{X}$ and $q(t_r, u_r) \longrightarrow 0$ then $q(u_r, t_r) \longrightarrow 0$

Recall the notion of α-admissibility introduced in [16, 17].

Definition 9. A map $Z : \mathcal{X} \longrightarrow \mathcal{X}$ is defined α-admissible if for every $t, u \in \mathcal{X}$, we have

$$\alpha(t, u) \geq 1 \Rightarrow \alpha(Zt, Zu) \geq 1, \tag{5}$$

where $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ is an offered function.

Some authors [18–21] introduced by slightly modifying this definition.

Definition 10. Let $(\mathcal{X}, q)_\Delta$ and $w : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$. A multivalued mapping $Z : \mathcal{X} \longrightarrow \mathcal{C}l_q(\mathcal{X})$ is called to be strictly $*$ -triangular-admissible on $\mathcal{X}$ if the following conditions are satisfied:

($\mathrm{w_t}$) for each $t, u, v \in \mathcal{X}$, $w(t, u) > 1$ and $w(u, v) > 1$ implies $w(t, v) > 1$

($\mathrm{w_a}$) for each $t, u \in \mathcal{X}$, $w(t, u) > 1$ implies $w^*(Zt, Zu) > 1$ where $w^*(\mathrm{Z}t, \mathrm{Z}u) = \inf \{w(x, y) \colon x \in \mathrm{Z}t, y \in \mathrm{Z}u\}$.

Definition 11. Let $(\mathcal{X}, q)$ be a Δ-symmetric quasimetric space, and let $w : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$. The space $(\mathcal{X}, q)$ is said to be strictly w^* -regular if for any sequence $\{t_r\} \subset \mathcal{X}$ such that $w(t_r, t_{r+1}) > 1$ for all $r \in \mathbb{N}$ and $t_r \longrightarrow t$ as $r \longrightarrow \infty$, we have $w(t_r, t) > 1$ for all $r \in \mathbb{N}$.

In recent years, researchers working on the fixed point theory seem to focus on introducing new contractions in known spaces. These new contractions are also accepted by many researchers and there are important studies, for example, F-contraction ([22–26]), θ-contraction [27], and interpolation contraction [28].

In 2020, Proinov [29] introduced new and interesting contractions in metric spaces. Proinov proved that several fixed point results (Wardowski [22]; Jleli and Samet [27]) observed in recent years are the result of Skof's fixed point theorem [30], and he introduced a very general fixed point theorem containing the main result of Skof.

Theorem 12 (see [29]). *Let $(\mathscr{X}, d)$ be a complete metric space and $Z : \mathscr{X} \longrightarrow \mathscr{X}$ a map which satisfies the contractive type condition:*

$$\psi(d(Zt, Zu)) \leq \varphi(d(t, u)) \text{ for all } t, u \in \mathscr{X} \text{ with } d(Zt, Zu) > 0, \tag{6}$$

where $\psi, \varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are two functions such that

(i) $\varphi(m) < \psi(m)$ for all $m > 0$

(ii) ψ is nondecreasing

(iii) $limsup_{m\longrightarrow\varepsilon+}\varphi(m) < \psi(\varepsilon+)$ for each $m > 0$

Hence, Z has a unique fixed point $w \in \mathscr{X}$ and $Z^r(t_0) \longrightarrow w$ for all $t_0 \in \mathscr{X}$, as $r \longrightarrow \infty$.

There are several studies using Proinov's contractions; some interesting ones are as follows: Alqahtani et al. [31] proposed the Proinov type mappings by involving certain rational expression in dislocated b-metrics. Alqahtani et al. [32] introduced the common fixed point of Proinov type contraction via simulation function. Roldán López de Hierro et al. [33] examined multiparametric contractions in b-metric spaces, inspired by Proinov results. Alghamdi et al. [34], on the other hand, introduced a new type of contraction using admissible mappings, inspired by Proinov and E-contraction.

Besides these, Karapnar et al. [35] combined contractions of Proinov [29] and Górnicki [36] in complete metric spaces and proved new fixed point theorems using admissible functions. Later, Ahmed and Fulga [37] generalized the Górnicki-Proinov type contraction to quasimetric spaces. Erdal et al. [38] published the notion of $(\alpha, \beta, \psi, \phi)$-interpolative contraction using a combine of interpolative contractions, Proinov type contractions, and ample spectrum contraction. Roldán López de Hierro et al. [39] proposed a new class of contractions in non-Archimedean fuzzy metric spaces, based on the Proinov fixed point results.

2. Main Results

Let us now give an important lemma that we will use in our main results.

Lemma 13 (see [37]). *Let $\{t_r\}$ be a sequence on $(\mathscr{X}, q)_\Delta$ such that $\lim_{r\longrightarrow\infty} q(t_r, t_{r+1}) = 0$. If the sequence $\{t_r\}$ is not left-Cauchy sequence thus there exists an $e > 0$ and two subsequences $\{t_{m_l}\}$, $\{t_{r_l}\}$ of $\{t_r\}$ such that*

$$\lim_{k\longrightarrow\infty} q\left(t_{r_{k+1}}, t_{m_{k+1}}\right) = \lim_{k\longrightarrow\infty} q\left(t_{r_k}, t_{m_k}\right) = e+. \tag{7}$$

Proof. Supposing that the sequence $\{t_r\}$ is not left-Cauchy, we can find $e > 0$ and the sequences of positive integers $\{n_l\}$, $\{r_l\}$, with $l \leq r_l < n_l$ for every $l \geq 0$, such that

$$q\left(t_{r_l}, t_{n_l}\right) > 2e. \tag{8}$$

□

On the other hand, since $\lim_{r\longrightarrow\infty} q(t_r, t_{r+1}) = 0$, we can find $r_0 \geq 1$ such that

$$q(t_r, t_{r+1}) < \frac{e}{2a}, \tag{9}$$

for every $r \geq r_0$, where $a = \max\{1, \Delta\}$. Moreover, since the space is supposed to be Δ symmetric,

$$q(t_{r+1}, t_r) \leq \Delta q(t_r, t_{r+1}) < \frac{e}{2}, \tag{10}$$

for every $r \geq r_0$. Therefore,

$$\begin{aligned} 2e &< q\left(t_{r_l}, t_{n_l}\right) \leq q\left(t_{r_l}, t_{r_l+1}\right) + q\left(t_{r_l+1}, t_{n_l}\right) \leq q\left(t_{r_l}, t_{r_l+1}\right) \\ &\quad + q\left(t_{r_l+1}, t_{n_l+1}\right) + q\left(t_{n_l+1}, t_{n_l}\right) < \frac{e}{2} + q\left(t_{r_l+1}, t_{n_l+1}\right) \\ &\quad + \frac{e}{2a} \leq e + q\left(t_{r_l+1}, t_{n_l+1}\right), \end{aligned} \tag{11}$$

for every $l \geq r_0$. Consequently, we have

$$q\left(t_{r_l+1}, t_{n_l+1}\right) > e, \tag{12}$$

for every $l \geq r_0$. Now, let m_l be the smallest positive integer, greater than n_l, such that

$$q\left(t_{r_l+1}, t_{m_l+1}\right) > e, q\left(t_{r_l}, t_{m_l}\right) > e. \tag{13}$$

Thus, we have either

$$q\left(t_{r_l}, t_{m_l-1}\right) \leq e, \tag{14}$$

$$\text{or } q\left(t_{r_l+1}, t_{m_l}\right) \leq e. \tag{15}$$

In the case of the first inequality holds,

$$e < q\left(t_{r_l}, t_{m_l}\right) \leq q\left(t_{r_l}, t_{m_l-1}\right) + q\left(t_{m_l-1}, t_{m_l}\right) \leq e + q\left(t_{m_l-1}, t_{m_l}\right), \tag{16}$$

and letting $l \longrightarrow \infty$, we get $\lim_{l\longrightarrow\infty} q(t_{r_l}, t_{m_l}) = e+$. Similarly, in case of the second inequality holds, we can consider

$$e < q\left(t_{r_l}, t_{m_l}\right) \leq q\left(t_{r_l}, t_{r_l+1}\right) + q\left(t_{r_l+1}, t_{m_l}\right) \leq q\left(t_{r_l}, t_{r_l+1}\right) + e, \tag{17}$$

so, we also obtain $\lim_{l\longrightarrow\infty} q(t_{r_l}, t_{m_l}) = e+$. Now, by the triangle inequality, and taking into account the above considerations, we have

$$\begin{aligned} e < q\left(t_{r_l+1}, t_{m_l+1}\right) &\leq q\left(t_{r_l}, t_{r_l+1}\right) + q\left(t_{r_l+1}, t_{m_l+1}\right) \\ &+ q\left(t_{m_l+1}, t_{m_l}\right) \leq q\left(t_{r_l}, t_{r_l+1}\right) + q\left(t_{r_l+1}, t_{m_l+1}\right) \\ &+ \Delta \cdot q\left(t_{m_l}, t_{m_l+1}\right), \end{aligned} \tag{18}$$

and as $l \longrightarrow \infty$, we get

$$\lim_{l\longrightarrow\infty} q\left(t_{r_l+1}, t_{m_l+1}\right) = e+. \tag{19}$$

We will give multivalued (w, ψ, φ)-contractive mappings.

Definition 14. Let $(\mathscr{X}, q)_\Delta$, be a Δ-symmetric quasimetric space, a mapping $w : \mathscr{X} \times \mathscr{X} \longrightarrow [0,\infty)$ and $Z : \mathscr{X} \longrightarrow CB(\mathscr{X})$ be a multivalued operator. We say that Z is a multivalued (w, ψ, φ)-contractive mapping if there exist two functions $\psi, \varphi : (0,\infty) \longrightarrow \mathbb{R}$ such that

$$\psi\left(w(t, u) H_q(Z(t), Z(u))\right) \leq \varphi(q(t, u)), \tag{20}$$

for every $t, u \in \mathscr{X}$ with $w(t, u) > 1$ and $H_q(Z(t), Z(u)) > 0$.

Theorem 15. *Let $(\mathscr{X}, q)_\Delta$ be a complete Δ-symmetric quasimetric space, and $Z : \mathscr{X} \longrightarrow CB(\mathscr{X})$ be a multivalued (w, ψ, φ)-contractive mapping. Assume that following conditions are satisfied:*

$(\mathscr{K}_1)$ Z is strictly $*$-admissible and there exist $t_0 \in \mathscr{X}$ and $t_1 \in Z(t_0)$ such that $w(t_0, t_1) > 1$

$(\mathscr{K}_2)$ if $\{t_r\}$ is a sequence in $\mathscr{X}$ such that $w(t_r, t_{r+1}) > 1$ for all $r \in \mathbb{N}$ and $t_r \longrightarrow t$ as $r \longrightarrow \infty$, we have $w(t_r, t) > 1$

$(\mathscr{K}_3)$ ψ is nondecreasing and $\varphi(v) < \psi(v)$ for all $v > 0$

$(\mathscr{K}_4)$ $\limsup_{v\longrightarrow j+} \varphi(v) < \psi(j+)$ for all $j > 0$

Therefore, Z has a fixed point in $\mathscr{X}$.

Proof. Let t_0 be an arbitrary point in $\mathscr{X}$ and $t_1 \in \mathscr{X}$ such that $q(t_0, Zt_0) = q(t_0, t_1)$ and $q(Zt_0, t_0) = q(t_1, t_0)$. Let now $t_2 \in Zt_1$ be such that $q(t_1, Zt_1) = q(t_1, t_2)$ and $q(Zt_1, t_1) = q(t_2, t_1)$. Continuing in this way, we can build the sequence $\{t_r\}$ of points in $\mathscr{X}$, such that $t_{r+1} \in Zt_r$, with $q(t_r, Zt_r) = q(t_r, t_{r+1})$ and $q(Zt_r, t_r) = q(t_{r+1}, t_r)$, for $r \in \mathbb{N}_0$. Moreover, by condition $(\mathscr{K}_1)$, we have that there exist $t_0 \in \mathscr{X}$ and $t_1 \in Z(t_0)$ such that $w(t_0, t_1) > 1$. Supposing that $r_0 \neq r_1$, if $r_1 \in Z r_1$, we get that t_1 is a fixed point of Z. Then, let $t_1 \notin Zt_1$. As Z is a strictly $*$-admissible map, we have that $*(Zt_0, Zt_1) > 1$. Thus, there exists $t_2 \in Z(t_1)$ such that $w(t_1, t_2) > 1$ which implies $*(Zt_1, Zt_2) > 1$. By continuing this process, we can construct a sequence $\{t_r\}$ in $\mathscr{X}$ such that $t_{r+1} \in Z(t_r)$ where $t_{r+1} \neq t_r$ for every $r \geq 0$ (as otherwise, if $t_r \in Z(t_r)$, thus t_r is a fixed point of Z) and $w(t_r, t_{r+1}) > 1$. Therefore, $H_q(Zt_{r-1}, Zt_r) > 0$. From Lemma 3 with $w(t_r, t_{r+1}) > 1$, we obtain

$$q(t_r, t_{r+1}) \leq w(t_{r-1}, t_r) H_q(Z(t_{r-1}), Z(t_r)), \tag{21}$$

for each $r \geq 1$. Keeping in mind $(\mathscr{K}_3)$ and (20) and we get

$$\psi(q(t_r, t_{r+1})) \leq \psi\left(w(t_{r-1}, t_r) H_q(Z(t_{r-1}), Z(t_r))\right) \leq \varphi(q(t_{r-1}, t_r)). \tag{22}$$

□

By hypothesis $(\mathscr{K}_3)$, we have

$$\psi(q(t_r, t_{r+1})) \leq \varphi(q(t_{r-1}, t_r)) < \psi(q(t_{r-1}, t_r)). \tag{23}$$

Thus, since ψ is a nondecreasing map, $q(t_r, t_{r+1}) < q(t_{r-1}, t_r)$ for each $r \geq 1$. So, the sequence $\{q(t_{r-1}, t_r)\}$ is positively decreasing. Then, there exists $G \geq 0$ such that $\lim_{r\longrightarrow\infty} q(t_{r-1}, t_r) = G+$.

Assuming that $G > 0$ on account of (23), we get a contradiction to supposition $(\mathscr{K}_4)$ as follows:

$$\psi(G+) = \lim_{r\longrightarrow\infty} \psi(q(t_r, t_{r+1})) \leq \lim_{r\longrightarrow\infty} \sup \varphi(q(t_{r-1}, t_r)) \leq \lim_{v\longrightarrow G+} \sup \varphi(v). \tag{24}$$

Therefore, $G = 0$, as a result, $\lim_{r\longrightarrow\infty} q(t_{r-1}, t_r) = 0$.

We prove that the sequence $\{t_r\}$ is left-Cauchy. Let us suppose by contradiction that the sequence $\{t_r\}$ is not left-Cauchy. Thus, by using Lemma 13, there exist $e > 0$ and two subsequences $\{t_{r_k}\}, \{t_{m_k}\}$, $(t_{m_k} > t_{r_k} \geq k,)$ of $\{t_r\}$ such that (7) is fulfilled. From (7), we conclude that $q(t_{r_k+1}, t_{m_k+1}) > \varepsilon$ and since the mapping Z is strictly triungular admissible, $w(t_{r_k}, t_{m_k}) > 1$ for every $k \geq 1$. Substituting $t = t_{r_k}$ and $u = t_{m_k}$ in (7), we obtain for each $k \geq 1$,

$$\psi\left(q\left(t_{r_k+1}, t_{m_k+1}\right)\right) \leq \psi\left(w\left(t_{r_k}, t_{m_k}\right) H_q\left(Zt_{r_k}, Zt_{m_k}\right)\right) \leq \varphi\left(q\left(t_{r_k}, t_{m_k}\right)\right), \tag{25}$$

then,

$$\psi\left(q\left(t_{r_k+1}, t_{m_k+1}\right)\right) \leq \varphi\left(q\left(t_{r_k}, t_{m_k}\right)\right) < \psi\left(q\left(t_{r_k}, t_{m_k}\right)\right), \tag{26}$$

for any $k \geq 1$, so that is $q(t_{r_k+1}, t_{m_k+1}) < q(t_{r_k}, t_{m_k})$ Because of $\lim_{k\longrightarrow\infty} q(t_{r_k+1}, t_{m_k+1}) = \varepsilon+$, we obtain $\lim_{k\longrightarrow\infty} q(t_{r_k}, t_{m_k}) = \varepsilon+$. Therefore, we can write

$$\psi(\varepsilon+) = \lim_{k\longrightarrow\infty} \psi\left(q\left(t_{r_k+1}, t_{m_k+1}\right)\right) \leq \lim_{k\longrightarrow\infty} sup \varphi\left(q\left(t_{r_k}, t_{m_k}\right)\right) \leq \lim_{\gamma\longrightarrow\varepsilon+} \varphi(\gamma), \tag{27}$$

which contradicts the supposition $(\mathscr{K}_4)$; then, $\{t_r\}$ is left-Cauchy sequence in $(\mathscr{X}, q)$, so that it is Cauchy sequence using Lemma 8. Therefore, the sequence $\{t_r\}$ is Cauchy in the complete Δ-symmetric quasimetric space and so converges to limit $t^* \in \mathcal{X}$. Now, we consider the following cases.

Case 1. If $q(t_{r+1}, Z(t^*)) = 0$ for some $r \in \mathbb{N}$, so by triangle inequality of Δ-symmetric quasimetric space

$$q(t^*, Z(t^*)) \leq q(t^*, t_{r+1}) + q(t_{r+1}, Z(t^*)) < q(t^*, t_{r+1}), \quad (28)$$

and thus, letting $r \longrightarrow \infty$, we conclude that $q(t^*, Z(t^*)) \leq 0$, that is,

$q(t^*, Z(t^*)) = 0$. As $Z(t^*)$ is closed, we obtain $t^* \in Z(t^*)$.

Case 2. On the contrary, if $q(t_{r+1}, Z(t^*)) > 0$ for every $r \in \mathbb{N}$ from $(\mathcal{K}_2)$, we have $w(t_r, t^*) > 1$ for all $r \in \mathbb{N}$. We claim that $q(t^*, Z(t^*)) = 0$. Supposing, on the contrary, $q(t^*, Z(t^*)) > 0$, there exists $r \in \mathbb{N}$ such that $q(t_r, Z(t^*)) > 0$. Therefore, we obtain

$$\begin{aligned} \psi(q(t_{r+1}, Z(t^*))) &\leq \psi\left(w(t_r, t^*) H_q(Z(t_r), Z(t^*))\right) \\ &\leq \varphi(q(t_r, t^*)) < \psi(q(t_r, t^*)). \end{aligned} \quad (29)$$

Taking into account the condition $(\mathcal{K}_3)$, we get $q(t_{r+1}, Z(t^*)) < q(t_r, t^*)$. Passing to limit as $r \longrightarrow \infty$, we obtain $q(t^*, Z(t^*)) < 0$. Therefore,

$q(t^*, Z(t^*)) = 0$, as $Z(t^*)$ is closed, $t^* \in Z(t^*)$.

Example 16. Let $\mathcal{X} = [0,\infty)$ be endowed with the 2-symmetric quasimetric $q : \mathcal{X} \times \mathcal{X} \longrightarrow [0,+\infty)$, where

$$q(t,u) = \begin{cases} 2(t-u), & \text{if } t \geq u, \\ u - t, & \text{otherwise,} \end{cases} \quad (30)$$

and a mapping $Z : \mathcal{X} \longrightarrow CB(\mathcal{X})$, defined as

$$Zt = \begin{cases} \left\{0, \dfrac{t}{8}\right\}, & \text{if } t \in [0,1], \\ \{2,3\}, & \text{otherwise.} \end{cases} \quad (31)$$

We choose two functions $\psi, \varphi : (0,\infty) \longrightarrow \mathbb{R}$ with ψ is nondecreasing, and $\varphi(m) < \psi(m)$ for all $m > 0$ where $\psi(m) = m$ and $\varphi(m) = m/2$. Let also

$$w(t,u) = \begin{cases} 2, & \text{if } t, u \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \quad (32)$$

We check that Z is a multivalued (w, ψ, φ)-contractive mapping of (20). Actually, if taking into account the way the function w is defined, we have consider the case $u, t \in [0, 1]$.

Let then, $t, u \in [0, 1]$, $u \geq t$. We get

$$q(0, Zu) = \inf\left\{0, \frac{u}{8}\right\} = 0, q(0, Zt) = \inf\left\{0, \frac{t}{8}\right\} = 0, \quad (33)$$

$$\begin{aligned} q\left(\frac{t}{8}, Zu\right) &= \inf_u \left\{2\left|0 - \frac{t}{8}\right|, 2\left|\frac{t}{8} - \frac{u}{8}\right|\right\}, q\left(\frac{u}{8}, Zt\right) \\ &= \inf_t \left\{2\left|0 - \frac{u}{8}\right|, 2\left|\frac{t}{8} - \frac{u}{8}\right|\right\}, \end{aligned}$$

$$\begin{aligned} H_q(Zt, Zu) &= \max\left\{\sup_{t \in Zt} q(t, Zu), \sup_{u \in Zu} q(u, Zt)\right\} \\ &= \max\left\{\sup_{t \in Zt} \inf_u \left\{\left|\frac{t}{4}\right|, \left|\frac{t}{4} - \frac{u}{4}\right|\right\}, \sup_{u \in Zu} \inf_t \right. \\ &\quad \left. \cdot \left\{\left|\frac{u}{4}\right|, \left|\frac{u}{4} - \frac{t}{4}\right|\right\}\right\} = \left|\frac{t}{4} - \frac{u}{4}\right|. \end{aligned} \quad (34)$$

So, we obtain

$$\psi\left(w(t,u) H_q(Z(t), Z(u))\right) = 2\left|\frac{t}{4} - \frac{u}{4}\right| = \left|\frac{t}{2} - \frac{u}{2}\right| \leq |t - u| = \varphi(q(t,u)). \quad (35)$$

Therefore, (20) fulfilled. Further, all other cases are satisfying, from $w(u, t) = 0$. Consequently, by Theorem 15, map Z has a fixed point, this being $t = 0$.

Definition 17. Let $(\mathcal{X}, q)_\Delta w : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ and $Z : \mathcal{X} \longrightarrow CB(\mathcal{X})$ be a multivalued operator. Z is said to be a multivalued C′iric′ type (w, ψ, φ)-contractive mapping if there exist two functions $\psi, \varphi : (0,\infty) \longrightarrow \mathbb{R}$ such that

$$\psi\left(w(t,u) H_q(Z(t), Z(u))\right) \leq \varphi(\Theta(t,u)), \quad (36)$$

for every $t, u \in \mathcal{X}$ with $w(t,u) > 1$ and $H_q(Z(t), Z(u)) > 0$ where

$$\Theta(t,u) = \max\left\{q(t,u), q(t,Zt), q(u,Zu), \frac{(q(t,Zu) + q(Zt,u))}{2}\right\}. \quad (37)$$

Theorem 18. *Let $(\mathcal{X}, q)$ be a complete Δ-symmetric quasimetric space, $w : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+ \setminus \{0\}$ and $Z : \mathcal{X} \longrightarrow K(\mathcal{X})$ be a multivalued C′iric′ type (w, ψ, φ)-contractive mapping. Assume that following conditions are satisfied:*

$(\mathcal{K}_1)$ Z is strictly $*$-triangular-admissible and there exists $t_0 \in \mathcal{X}$ and $t_1 \in Z(t_0)$ such that $w(t_0, t_1) > 1$

$(\mathcal{K}_2)$ if $\{t_r\}$ is a sequence in $\mathcal{X}$ such that $w(t_r, t_{r+1}) > 1$ for all $r \in \mathbb{N}$ and $t_r \longrightarrow t$ as $r \longrightarrow \infty$, we have $w(t_r, t) > 1$

$(\mathcal{K}_3)$ ψ is nondecreasing, and $\varphi(v) < \psi(v)$ for all $v > 0$

$(\mathcal{K}_4)$ $\limsup_{v \longrightarrow j+} \varphi(v) < \psi(j+)$ for all $j > 0$

Therefore, Z has a fixed point in $\mathcal{X}$.

Proof. By condition $(\mathcal{K}_1)$, and following the lines of the proof of the previous theorem, we have that $w(t_r, t_m) > 1$, for every $m > r \geq 1$. Moreover, $H_q(Zt_{r-1}, Zt_r) > 0$ and from Lemma 3 with $w(t_r, t_{r+1}) > 1$, we obtain

$$q(t_r, t_{r+1}) \leq w(t_{r-1}, t_r) H_q(Z(t_{r-1}), Z(t_r)), \quad (38)$$

for each $r \geq 1$. Keeping in mind $(\mathscr{K}_3)$ and (36), we get

$$\psi(q(t_r, t_{r+1})) \leq \psi(w(t_{r-1}, t_r)H_q(Z(t_{r-1}), Z(t_r))) \leq \varphi(\Theta(t_{r-1}, t_r)). \quad (39)$$

As $Z(t)$ is closed for every $t \in \mathscr{X}$, we get that $t_r \in Z(t_{r-1})$ such that $q(t_{r-1}, t_r) = q(t_{r-1}, Z(t_{r-1}))$,

$$\psi(q(t_r, t_{r+1})) \leq \varphi(\Theta(t_{r-1}, t_r)) = \varphi(\max\{q(t_{r-1}, t_r), q \cdot (t_{r-1}, Z(t_{r-1})), q(t_r, Z(t_r)), q(t_{r-1}, Z(t_r)) + \frac{q(Zt_{r-1}, t_r)}{2}\right) = \varphi(\max\{q(t_{r-1}, t_r), q(t_r, t_{r+1})\}), \quad (40)$$

for every $r \geq 1$. □

If $\max\{q(t_{r-1}, t_r), q(t_r, t_{r+1})\} = q(t_r, t_{r+1})$ so $\psi(q(t_r, t_{r+1})) \leq \varphi(q(t_r, t_{r+1}))$, from assumption $(\mathscr{K}_3)$, this is a contradiction. Hence, we obtain $q(t_{r-1}, t_r) > q(t_r, t_{r+1})$, and

$$\psi(q(t_r, t_{r+1})) \leq \varphi(q(t_{r-1}, t_r)). \quad (41)$$

Similarly, again using $(\mathscr{K}_3)$, we get

$$\psi(q(t_r, t_{r+1})) \leq \varphi(q(t_{r-1}, t_r)) < \psi(q(t_{r-1}, t_r)). \quad (42)$$

But, the function ψ is nondecreasing map, so that we get $q(t_r, t_{r+1}) < q(t_{r-1}, t_r)$ for all $r \geq 1$. Therefore, the sequence $\{q(t_{r-1}, t_r)\}$ is positively decreasing, and then, there exists $G \geq 0$ such that $\lim_{r \longrightarrow \infty} q(t_{r-1}, t_r) = G+$. If $G > 0$, from (42), we obtain

$$\begin{aligned}\psi(G+) &= \lim_{r \longrightarrow \infty} \psi(q(t_r, t_{r+1})) \leq \lim_{r \longrightarrow \infty} \sup \varphi(q(t_{r-1}, t_r)) \\ &\leq \lim_{\rho \longrightarrow G+} \sup \varphi(\rho),\end{aligned} \quad (43)$$

which contradictions $(\mathscr{K}_4)$. Therefore, $G = 0$ and, as a result,

$$\lim_{r \longrightarrow \infty} q(t_{r-1}, t_r) = 0. \quad (44)$$

We claim that $\{t_r\}$ is Cauchy sequence. Let us assume by contradiction that the sequence $\{t_r\}$ is not left-Cauchy. Then, by Lemma 13, we can find $e > 0$ and two subsequences $\{t_{r_k}\}, \{t_{m_k}\}$, (with $m_k > r_k \geq k$) of $\{t_r\}$ such that (7) holds. Thereupon, we have that $w(t_{r_k}, t_{m_k}) > 1$ for all $m_k > r_k > k \geq 1$. Letting $t = t_{r_k}$ and $u = t_{m_k}$ in (9), we get

$$\psi(q(t_{r_k+1}, t_{m_k+1})) \leq \psi(w(t_{r_k}, t_{m_k})H_q(Zt_{r_k}, Zt_{m_k})) \leq \varphi(\Theta(t_{r_k}, t_{m_k})), \quad (45)$$

for every $k \geq 1$, where

$$\Theta(t_{r_k}, t_{m_k}) = \max\left\{\begin{array}{c} q(t_{r_k}, t_{m_k}), q(t_{r_k}, Zt_{r_k}), q(t_{m_k}, Zt_{m_k}), \\ \dfrac{q(t_{r_k}, Zt_{m_k}) + q(Zt_{r_k}, t_{m_k})}{2} \end{array}\right\}. \quad (46)$$

Keeping in mind the way the sequence $\{t_r\}$ was define, let $t_{r_k+1} \in Zt_{r_k}$ and $t_{m_k+1} \in Zt_{m_k}$. Thus,

$$\begin{aligned} q(t_{r_k}, t_{m_k}) \leq \Theta(t_{r_k}, t_{m_k}) &= \max\left\{\begin{array}{c} q(t_{r_k}, t_{m_k}), q(t_{r_k}, t_{r_k+1}), q(t_{m_k}, t_{m_k+1}) \\ \dfrac{q(t_{r_k}, t_{m_k+1}) + q(t_{r_k+1}, t_{m_k})}{2} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} q(t_{r_k}, t_{m_k}), q(t_{r_k}, t_{r_k+1}), q(t_{m_k}, t_{m_k+1}), \\ \dfrac{q(t_{r_k}, t_{r_k+1}) + q(t_{r_k+1}, t_{m_k+1}) + q(t_{r_k+1}, t_{r_k}) + q(t_{r_k}, t_{m_k})}{2} \end{array}\right\}. \end{aligned} \quad (47)$$

Letting $k \longrightarrow \infty$ in the above inequality, and taking into account (44), respectively (7), we get

$$\lim_{k \longrightarrow \infty} \Theta(t_{r_k}, t_{m_k}) = e+. \quad (48)$$

Moreover, since the function ψ is nondecreasing, taking the limit superior when $k \longrightarrow \infty$ in (45) we get

$$\psi(\varepsilon+) = \lim_{k \longrightarrow \infty} \psi(q(t_{r_k+1}, t_{m_k+1})) \leq \limsup_{k \longrightarrow \infty} \varphi(\Theta(t_{r_k}, t_{m_k})) \leq \limsup_{\rho \longrightarrow \varepsilon+} \varphi(\rho), \quad (49)$$

which contradicts the supposition $(\mathscr{K}_4)$; then, $\{t_r\}$ is left Cauchy sequence in $(\mathscr{X}, q)$, so that it is Cauchy sequence using Lemma 8. Therefore, the sequence $\{t_r\}$ is Cauchy in the complete Δ-symmetric quasimetric space and so converges to a point $t^* \in \mathscr{X}$. Now, we consider following cases:

Case 1. If $q(t_{r+1}, Z(t^*)) = 0$ for some $r \in \mathbb{N}$, so by triangle inequality of Δ-symmetric quasimetric space

$$q(t^*, Z(t^*)) \leq q(t^*, t_{r+1}) + q(t_{r+1}, Z(t^*)) < q(t^*, t_{r+1}), \quad (50)$$

and thus, letting $r \longrightarrow \infty$, we conclude that $q(t^*, Z(t^*)) \leq 0$, that is,

$$q(t^*, Z(t^*)) = 0. As\, Z(t^*) \text{ is closed, we obtain } t^* \in Z(t^*). \quad (51)$$

Case 2. If we suppose the contrary, that is, $q(t_{r+1}, Z(t^*)) = 0$ for any r, from $(\mathscr{K}_2)$ we know that $w(t_r, t^*) > 1$ for all $r \in \mathbb{N}$. We assert that $q(t^*, Z(t^*)) = 0$. Suppose, on the contrary, $q(t^*, Z(t^*)) > 0$. Thus, there exists $r \in \mathbb{N}$ such that $q(t_r, Z(t^*)) > 0$ for every r. Using (36), we obtain

$$\begin{aligned}\psi(q(t_{r+1}, Z(t^*))) &\leq \psi(w(t_r, t^*)H_q(Z(t_r), Z(t^*))) \\ &\leq \varphi(\Theta(t_r, t^*)) < \psi(\Theta(t_r, t^*)),\end{aligned} \quad (52)$$

where

$$\Theta(t_r, t^*) = \max \left\{ \begin{array}{c} q(t_r, t^*)), q(t_r, Z(t_r)), q(t^*, Z(t^*)), \\ \dfrac{q(t_r, Z(t^*)) + q(Zt_r, t^*))}{2} \end{array} \right\} = \max \left\{ \begin{array}{c} q(t_r, t^*)), q(t_r, t_{r+1})), q(t^*, Z(t^*)), \\ \dfrac{q(t_r, Z(t^*)) + q(t_{r+1}, t^*))}{2} \end{array} \right\}. \tag{53}$$

Taking into account the condition $(\mathscr{K}_3)$, we get $q(t_{r+1}, Z(t^*)) < \Theta(t^*, Z(t^*))$. Passing to limit as $r \longrightarrow \infty$, we obtain $q(t^*, Z(t^*)) < q(t^*, Z(t^*))$ a contradiction, then $q(t^*, Z(t^*)) = 0$. As $Z(t^*)$ is compact, $t^* \in Z(t^*)$.

Corollary 19. *Let $(\mathscr{X}, q)$ be a Δ-symmetric quasimetric space and $Z : \mathscr{X} \longrightarrow K(\mathscr{X})$ be a multivalued mapping satisfying the condition:*

$$\psi\left(H_q(Z(t), Z(u))\right) < \varphi(q(t, u)), \tag{54}$$

for every $t, u \in \mathscr{X}$, where the functions $\psi, \varphi : (0,\infty) \longrightarrow \mathbb{R}$ and $H_q(Z(t), Z(u)) > 0$. The map Z admits a fixed point in $\mathscr{X}$ provided that following conditions hold:

(K_1) ψ is nondecreasing, and $\varphi(v) < \psi(v)$ for all $v > 0$

(K_2) $\limsup_{v\longrightarrow j+}\varphi(v) < \psi(j+)$ for all $j > 0$

Letting $\varphi(a) = \delta\psi(a)$, in Corollary 19, we obtain the following result.

Corollary 20. *Let $(\mathscr{X}, q)$ be a Δ-symmetric quasimetric space and $Z : \mathscr{X} \longrightarrow K(\mathscr{X})$ be a multivalued mapping satisfying the condition:*

$$\psi\left(H_q(Z(t), Z(u))\right) < \delta\psi(q(t, u)), \tag{55}$$

for every $t, u \in \mathscr{X}$, where the functions $\psi, \varphi : (0,\infty) \longrightarrow \mathbb{R}$ and $H_q(Z(t), Z(u)) > 0$. The map Z admits a fixed point in $\mathscr{X}$ provided that following conditions hold:

(K_1) ψ is nondecreasing, and $\varphi(v) < \psi(v)$ for all $v > 0$;

(K_2) $\limsup_{v\longrightarrow j+}\varphi(v) < \psi(j+)$ for all $j > 0$.

Corollary 21. *Let $(\mathscr{X}, q)$ be a Δ-symmetric quasimetric space and $Z : \mathscr{X} \longrightarrow K(\mathscr{X})$ be a multivalued mapping satisfying the condition:*

$$\psi\left(H_q(Z(t), Z(u))\right) < \varphi(\Theta(t, u)), \tag{56}$$

for every $t, u \in \mathscr{X}$ and $H_q(Z(t), Z(u)) > 0$, where the functions $\psi, \varphi : (0,\infty) \longrightarrow \mathbb{R}$ and

$$\Theta(t, u) = \max \left\{ q(t, u), q(t, Zt), q(u, Zu), \frac{(q(t, Zu) + q(Zt, u))}{2} \right\}. \tag{57}$$

The map Z admits a fixed point in $\mathscr{X}$ provided that following conditions:

(K_1) ψ is nondecreasing, and $\varphi(v) < \psi(v)$ for all $v > 0$

(K_2) $\limsup_{v\longrightarrow j+}\varphi(v) < \psi(j+)$ for all $j > 0$

Taking $\varphi(a) = \delta\psi(a)$, in Corollary 21, we get the following result.

Corollary 22. *Let $(\mathscr{X}, q)$ be a Δ-symmetric quasimetric space and $Z : \mathscr{X} \longrightarrow K(\mathscr{X})$ be a multivalued mapping satisfying the condition:*

$$\psi\left(w(t, u)H_q(Z(t), Z(u))\right) \leq \delta\psi(\Theta(t, u)), \tag{58}$$

for every $t, u \in \mathscr{X}$ and $H_q(Z(t), Z(u)) > 0$, where the functions $\psi, \varphi : (0,\infty) \longrightarrow \mathbb{R}$ and

$$\Theta(t, u) = \max \left\{ q(t, u), q(t, Zt), q(u, Zu), \frac{(q(t, Zu) + q(Zt, u))}{2} \right\}. \tag{59}$$

The map Z admits a fixed point in $\mathscr{X}$ provided that following conditions hold:

(K_1) ψ is nondecreasing, and $\varphi(v) < \psi(v)$ for all $v > 0$

(K_2) $\limsup_{v\longrightarrow j+}\varphi(v) < \psi(j+)$ for all $j > 0$

3. Conclusion

In this paper, we expand the very interesting results of Proinov [29] in several ways: First, we involve a more general form of the function by considering multivalued mapping. Secondly, we refine the structure of the considered abstract space with Δ-symmetric quasimetric space. Indeed, quasimetric space is one of the novel extensions of metric space. Besides, Δ-symmetric quasimetric space is more reasonable to work since almost all quasimetric space form Δ-symmetric quasimetric spaces. There are still rooms for the fixed point results in the context of Δ-symmetric quasimetric spaces.

Data Availability

No data are used.

Disclosure

The authors declare that the study was realized in collaboration with equal responsibility.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the final manuscript.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fund. math*, vol. 3, no. 1, pp. 133–181, 1922.

[2] H. P. A. Künzi, "A note on sequentially compact quasi-pseudo-metric spaces," *Monatshefte für Mathematik*, vol. 95, no. 3, pp. 219-220, 1983.

[3] N. A. Secelean, S. Mathew, and D. Wardowski, "New fixed point results in quasi-metric spaces and applications in fractals theory," *Advances in Difference Equations*, vol. 2019, no. 1, 2019.

[4] W. A. Wilson, "On quasi-metric spaces," *American Journal of Mathematics*, vol. 53, no. 3, pp. 675–684, 1931.

[5] S. B. Chen, S. P. Tian, and Z. Y. Mao, "On Caristi's fixed point theorem in quasi-metric spaces," in *Impulsive dynamical systems & applications*, vol. 3, pp. 1150–1157, Watam Press, Waterloo, 2005.

[6] H. P. Kunzi and M. P. Schellekens, "On the Yoneda completion of a quasi-metric space," *Theoretical Computer Science*, vol. 278, no. 1-2, pp. 159–194, 2002.

[7] S. B. Nadler, "Multivalued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[8] A. Shoaib, "Fixed point results for $\alpha * -\psi$-mutivalued mappings," *Bulletin of Mathematical Analysis Applications*, vol. 8, no. 4, pp. 43–55, 2016.

[9] M. Cvetkovic and V. Rakocevic, "α-Admissible contractions on quasi-metric-like space," *Advances in the Theory of Nonlinear Analysis and Its Application*, vol. 1, no. 2, pp. 113–124, 2017.

[10] B. E. Rhoades, "Some fixed point theorems in quasi-metric spaces," *Demonstratio Mathematica*, vol. 30, no. 2, pp. 301–306, 1997.

[11] S. Cobzas, "Completeness in quasi-metric spaces and Ekeland variational principle," *Topology and its Applications*, vol. 158, no. 8, pp. 1073–1084, 2011.

[12] T. L. Hicks, "Fixed point theorems for quasi-metric spaces," *Mathematica Japonica*, vol. 33, no. 2, pp. 231–236, 1988.

[13] A. M. Patriciu and V. Popa, "A general fixed point theorem of Ciric type in quasi-partial metric spaces," *Novi Sad Journal of Mathematics*, vol. 50, no. 2, pp. 1–6, 2020.

[14] B. Alqahtani, A. Fulga, and E. Karapınar, "Fixed point results on δ-symmetric quasi-metric space via simulation function with an application to Ulam stability," *Mathematics*, vol. 6, no. 10, p. 208, 2018.

[15] E. Karapınar, A. F. Roldan-Lopez-de-Hierro, and B. Samet, "Matkowski theorems in the context of quasi-metric spaces and consequences on G-metric spaces," *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, vol. 24, no. 1, pp. 309–333, 2016.

[16] E. Karapınar and B. Samet, "Generalized ?-? contractive type mappings and related fixed point theorems with applications," in *Abstract and Applied Analysis*, vol. 2012, Hindawi, 2012.

[17] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for $\alpha - \psi$-contractive type mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[18] M. U. Ali, T. Kamran, and E. Karapınar, "Discussion on α-strictly contractive nonself multivalued maps," *Vietnam Journal of Mathematics*, vol. 44, no. 2, pp. 441–447, 2016.

[19] C. M. Chen, E. Karapınar, and D. O'regan, "On (α− ϕ)-Meir-Keeler contractions on partial Hausdorff metric spaces," *University Politehnica Of Bucharest Scientific*, vol. 2018, pp. 101–110, 2018.

[20] E. Karapınar, K. Taş, and V. Rakočević, "Advances on fixed point results on partial metric spaces," in *Mathematical Methods in Engineering*, pp. 3–66, Springer, Cham, 2019.

[21] H. Monfared, M. Asadi, and A. Farajzadeh, "New generalization of Darbo's fixed point theorem via α-admissible simulation functions with application," *Sahand Communications in Mathematical Analysis*, vol. 17, no. 2, pp. 161–171, 2020.

[22] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed point theory and applications*, vol. 2012, no. 1, 2012.

[23] M. Eshraghisamani, S. Mansour Vaezpour, and M. Asadi, "Some results by quasi-contractive mappings in f-orbitally complete metric space," *Communications in Nonlinear Analysis*, vol. 4, 2017.

[24] M. Asadi, "Discontinuity of control function in the (F, φ, θ)-contraction in metric spaces," *Filomat*, vol. 31, no. 17, pp. 5427–5433, 2017.

[25] H. Monfared, M. Asadi, M. Azhini, and D. O'Regan, "F (ψ, φ) \$ F (\psi,\varphi) \$-contractions for α-admissible mappings on M-metric spaces," *Fixed Point Theory and Applications*, vol. 2018, no. 1, pp. 1–17, 2018.

[26] H. Monfared, M. Azhini, and M. Asadi, "\$ C \$-class and \$ F (psi, varphi) \$-contractions on \$ M \$-metric spaces," *International Journal of Nonlinear Analysis and Applications*, vol. 8, no. 1, pp. 209–224, 2017.

[27] M. Jleli and A. Samet, "A new generalization of the Banach contraction principle," *Journal of inequalities and applications*, vol. 38, 8 pages, 2014.

[28] E. Karapinar, "Revisiting the Kannan type contractions via interpolation," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 2, no. 2, pp. 85–87, 2018.

[29] P. D. Proinov, "Fixed point theorems for generalized contractive mappings in metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 22, no. 1, p. 21, 2020.

[30] F. Skof, "Theoremi di punto fisso per applicazioni negli spazi metrici," *Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 111, pp. 323–329, 1977.

[31] B. Alqahtani, S. S. Alzaid, A. Fulga, R. López, and A. F. de Hierro, "Proinov type contractions on dislocated b-metric spaces," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–16, 2021.

[32] B. Alqahtani, S. S. Alzaid, A. Fulga, and S. S. Yesilkaya, "Common fixed point theorem on Proinov type mappings via simulation function," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–17, 2021.

[33] A. F. R. L. de Hierro, E. Karapınar, and A. Fulga, "Multiparametric contractions and related Hardy-Roger type fixed point theorems," *Mathematics*, vol. 8, no. 6, p. 957, 2020.

[34] M. A. Alghamdi, S. Gulyaz-Ozyurt, and A. Fulga, "Fixed points of Proinov E-contractions," *Symmetry*, vol. 13, no. 6, p. 962, 2021.

[35] M. E Karapnar, De La Sen, and A. Fulga, "A note on the Gornicki-Proinov type contraction," *Journal of Function Spaces*, vol. 2021, Article ID 6686644, 8 pages, 2021.

[36] J. Górnicki, "Remarks on asymptotic regularity and fixed points," *Journal of Fixed Point Theory and Applications*, vol. 21, no. 1, p. 29, 2019.

[37] A. E. S. Ahmed and A. Fulga, “The Górnicki-Proinov type contraction on quasi-metric spaces,” *AIMS Mathematics*, vol. 6, no. 8, pp. 8815–8834, 2021.

[38] K. Erdal, F. Andreea, and R. L. D. H. A. Francisco, “Fixed point theory in the setting of $(\alpha, \beta, \psi, \phi)$-interpolative contractions,” *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–16, 2021.

[39] A. F. R. L. de Hierro, A. Fulga, E. Karapınar, and N. Shahzad, “Proinov-type fixed-point results in non-Archimedean fuzzy metric spaces,” *Mathematics*, vol. 9, no. 14, p. 1594, 2021.

[40] A. Fulga, “On (ψ, ϕ)-rational contractions,” *Symmetry*, vol. 12, no. 5, p. 723, 2020.

[41] S. Romaguera and P. Tirado, “The Meir-Keeler fixed point theorem for quasi-metric spaces and some consequences,” *Symmetry.*, vol. 11, no. 6, p. 741, 2019.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 6795978, 10 pages
https://doi.org/10.1155/2022/6795978

Research Article

Ulam-Hyers Stability Results of λ-Quadratic Functional Equation with Three Variables in Non-Archimedean Banach Space and Non-Archimedean Random Normed Space

Ly Van An,[1] Kandhasamy Tamilvanan,[2] R. Udhayakumar,[3] Masho Jima Kabeto,[4] and Ly Van Ngoc[5]

[1]*Faculty of Mathematics Teacher Education, Tay Ninh University, Ninh Trung, Ninh Son, Tay Ninh Province, Vietnam*
[2]*Department of Mathematics, School of Advanced Sciences, Kalasalingam Academy of Research and Education, Srivilliputhur, 626 126 Tamil Nadu, India*
[3]*Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, India*
[4]*Department of Mathematics, College of Natural Sciences, Jimma University, Ethiopia*
[5]*Faculty of Mathematics Teacher Education, Quang Nam Phan Chau Trinh University, Cam Thanh, Cam Ha, Quang Nam Province, Vietnam*

Correspondence should be addressed to Kandhasamy Tamilvanan; tamiltamilk7@gmail.com and Masho Jima Kabeto; masho.jima@ju.edu.et

Received 27 March 2022; Revised 20 April 2022; Accepted 22 April 2022; Published 20 May 2022

Academic Editor: Santosh Kumar

In this paper, we introduce the λ-quadratic functional equation with three variables and obtain its general solution. The main aim of this work is to examine the Ulam-Hyers stability of this functional equation in non-Archimedean Banach space by using direct and fixed point techniques and examine the stability results in non-Archimedean random normed space.

1. Introduction

One of the most important areas of research in mathematics is the investigation of stability issues for functional equations, which has its origins in concerns of applied mathematics. The first question about the stability of homomorphisms was given by Ulam [1] as follows.

Given a group $(M, *)$, a metric group $(M', \cdot)$ with the metric d, and a function ϕ from B and B', does there exists $\delta > 0$ satisfying

$$d(\phi(u * v), \phi(u) \cdot \phi(v)) \le \delta, \tag{1}$$

for all $u, v \in B$, then there exists a homomorphism $h : B \longrightarrow B'$ such that

$$d(\phi(u), h(u)) \le \varepsilon, \tag{2}$$

for all $u \in B$?

Ulam's question on Banach spaces was partially answered affirmatively by Hyers [2]. By assuming an infinite Cauchy difference, Aoki [3] expanded Hyers' and Rassias' theorems for additive and linear mappings, respectively. Using the same method as Rassias [4], Gajda [5] discovered a positive solution to the question $p > 1$. Rassias and Šemrl [6], as well as Gajda [5], have proved that a Rassias' type theorem cannot be formed for $p = 1$.

The functional equation

$$\phi(u + v) = \phi(u) + \phi(v) \tag{3}$$

is known as the Cauchy additive equation.

Since the function $\phi(u) = u$ is the solution of the functional equation (3), every solution of the additive functional equation (3) is called as an additive function. Every solution of the functional equation (3), in particular, is called as an additive function.

The functional equation

$$\phi(u+v) + \phi(u-v) = 2\phi(u) + 2\phi(v) \tag{4}$$

is known as the quadratic functional equation.

Since the function $\phi(u) = u^2$ is the solution of the functional equation (4), each solution of the functional equation (4) is called as a quadratic function. Every solution of functional equation (4), in particular, is called as a quadratic mapping.

Skof [7] established the stability of the quadratic functional equation for the function f between normed space and complete normed space. The authors [8–14] recently examined the Ulam-Hyers stability results for the following α-functional equation

$$2f(x) - 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)), \tag{5}$$

in non-Archimedean Banach spaces.

The Skof theorem still applies when the relevant domain B is replaced by an Abelian group, according to Cholewa [15]. See [15–21] for other functional equations. A survey of the Ulam-Hyers stability results of functional equations was conducted by Brillouët-Belluot [22]. Park and Kim [11] demonstrated the Ulam-Hyers stability of quadratic α -functional equation.

In this paper, the authors present a new λ-quadratic functional equation with three variables as

$$2\xi\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\xi(\vartheta_3) = \xi\left(\frac{\vartheta_1+\vartheta_2}{2} + \vartheta_3\right) + \lambda^{-2}\xi\left(\lambda\left(\frac{\vartheta_1+\vartheta_2}{2} - \vartheta_3\right)\right), \tag{6}$$

where λ is a fixed non-Archimedean number with $\lambda^{-2} \neq 3$, and its general solution was obtained. The motivation behind this study is to investigate the Ulam-Hyers stability results for the above functional equation (6) in non-Archimedean Banach space by using direct and fixed point methods and non-Archimedean random normed space.

The following is the structure of this paper: in Section 2, we recall some fundamental notions and definitions, in Section 3, we look at the general solution of the equation (6), where V and W are two vector spaces. We investigate the Ulam-Hyers stability in non-Archimedean Banach space by using fixed point method and direct method in Sections 4 and 5, where V is a non-Archimedean normed space, W is a non-Archimedean Banach space, and $|2| \neq 1$ is a non-Archimedean Banach space. In Section 6, we recall some fundamental notions and results and investigate the Ulam-Hyers stability in non-Archimedean random normed space.

2. Preliminaries

To reach our major results, we use certain fundamental notations in [8, 10, 11].

A map $|\cdot|: \mathbb{K} \longrightarrow [0,\infty)$ is a valuation such that zero is the only one element having the zero valuation, $|k_1 k_2| = |k_1||k_2|$, and the inequality of the triangle holds true, that is, $|k_1 + k_2| \leq |k_1| + |k_2|$, for all $k_1, k_2 \in \mathbb{K}$.

We call a field $\mathbb{K}$ valued if $\mathbb{K}$ holds a valuation. Examples of valuations include the typical absolute values of $\mathbb{R}$ and $\mathbb{C}$.

Consider a valuation that satisfies a criterion that is stronger than the triangle inequality. A $|\cdot|$ is called a *non-Archimedean valuation* if the triangle inequality is replaced by $|k_1 + k_2| \leq \max\{|k_1|, |k_2|\}$, for all $k_1, k_2 \in \mathbb{K}$, and a field is called a *non-Archimedean field*. Evidently, $|-1| = 1 = |1|$ and $|n|$ are greater than or equal to 1, for all n in $\mathbb{N}$. The map $|\cdot|$ takes everything except 0 for 1, and $|0| = 0$ is a basic example of a non-Archimedean valuation.

Definition 1. Let$V$ be a linear space over $\mathbb{K}$ with $|\cdot|$. A mapping $\|\cdot\|: V \longrightarrow [0,\infty)$ is known as a non-Archimedean norm if it satisfies

(i) $\|v\| = 0$ if and only if $v = 0$.

(ii) $\|rv\| = |r|\|v\|$, $v \in V$, and $r \in \mathbb{K}$.

(iii) the strong triangle inequality.

$$\|v_1 + v_2\| \leq \max\{\|v_1\|, \|v_2\|\}, \; v_1, v_2 \in V. \tag{7}$$

Then, $(V, \|\cdot\|)$ is called a non-Archimedean normed space. Every Cauchy sequence converges in a complete non-Archimedean normed space, which we call a complete non-Archimedean normed space.

Definition 2. Let V be a non-Archimedean normed space and a sequence $\{v_p\}$ in V. Then,

(1) a sequence $\{v_p\}_{p=1}^{\infty}$ in V is a Cauchy sequence if $\{v_{p+1} - v_p\}_{p=1}^{\infty}$ converges to 0.

(2) $\{v_p\}$ is called convergent if, for any $\varepsilon > 0$, there is an integer $p > 0$ in $\mathbb{N}$ and $v \in V$ satisfies

$$\|v_p - v\| \leq \varepsilon \text{ for all } p \geq \mathbb{N}, \tag{8}$$

for every $p, q \geq \mathbb{N}$. Then, we called as v is a limit of the sequence $\{v_p\}$ and denoted by $\lim_{p\longrightarrow\infty} v_p = v$.

(3) if every Cauchy sequence in a non-Archimedean normed space V converges, it is called a non-Archimedean Banach space.

Theorem 3 (alternative fixed point theorem). *Let (V, d) be a generalized complete metric space and a strictly contractive mapping $M : V \longrightarrow V$ with Lipschitz constant $0 < L < 1$.*

Then, for all $v_1 \in V$, either

$$d\left(M^m v_1, M^{m+1} v_1\right) = \infty, m \geq m_0, \tag{9}$$

or there exists a positive integer m_0 such that

(i) *$d(M^m v_1, M^{m+1} v_1) < \infty, m \geq m_0$.*

(ii) *the sequence $\{M^m v_1\}_{m \in \mathbb{N}}$ converges to a fixed point v_1^* of M.*

(iii) *v_1^* is the unique fixed point of M in $V^* = \{v_2 \in V | d(M^{m_0} v_1, v_2) < \infty\}$.*

(iv) *$d(v_2, v_1^*) \leq (1/1 - L) d(M v_2, v_2)$, for all $v_2 \in V^*$.*

3. Solution

Lemma 4. *If a mapping $\xi : V \longrightarrow W$ satisfies the functional equation (6) for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$, then the function ξ is quadratic.*

Proof. A mapping $\xi : V \longrightarrow W$ satisfies the functional equation (6). Replacing $(\vartheta_1, \vartheta_2, \vartheta_3)$ by $(0, 0, 0)$ in (6), we obtain

$$3\xi(0) = \lambda^{-2} \xi(0). \tag{10}$$

This implies that $\xi(0) = 0$. Replacing $(\vartheta_1, \vartheta_2, \vartheta_3)$ by $(\vartheta, \vartheta, 0)$ in (6), we obtain

$$\xi(\vartheta) = \lambda^{-2} \xi(\lambda(\vartheta)), \tag{11}$$

and so

$$\xi(\lambda v) = \lambda^2 \xi(\vartheta), \tag{12}$$

for all $\vartheta \in V$. Thus, equation (6) is reduced as

$$2\xi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + 2\xi(\vartheta_3) = \xi\left(\frac{\vartheta_1 + \vartheta_2}{2} + \vartheta_3\right) + \xi\left(\frac{\vartheta_1 + \vartheta_2}{2} - \vartheta_3\right), \tag{13}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$. Now, replacing $\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta$ in (13), we get

$$\xi(2\vartheta) = 2^2 \xi(\vartheta), \tag{14}$$

for all $\vartheta \in V$. Again, replacing ϑ by 2ϑ in (14), we have

$$\xi\left(2^2 \vartheta\right) = 2^4 \xi(\vartheta), \tag{15}$$

for all $\vartheta \in V$. From equalities (14) and (15), we can conclude that for any integer $p > 0$, we get

$$\xi(2^p \vartheta) = 2^{2p} \xi(\vartheta), \tag{16}$$

for all $\vartheta \in V$. Now, replacing $(\vartheta_1, \vartheta_2, \vartheta_3)$ by $(\vartheta_1, \vartheta_1, \vartheta_2)$ in (13), we reach (3) for all $\vartheta_1, \vartheta_2 \in V$. Hence, the function ξ is quadratic. □

For our notational simplicity, we use the following abbreviation:

$$\begin{aligned} \Delta\xi(\vartheta_1, \vartheta_2, \vartheta_3) = {} & 2\xi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + 2\xi(\vartheta_3) - \xi\left(\frac{\vartheta_1 + \vartheta_2}{2} + \vartheta_3\right) \\ & - \lambda^{-2} \xi\left(\lambda\left(\frac{\vartheta_1 + \vartheta_2}{2} - \vartheta_3\right)\right). \end{aligned} \tag{17}$$

4. Stability of (6) in Non-Archimedean Banach Space: Direct Method

Theorem 5. *Let $\rho : V^3 \longrightarrow [0, \infty)$ be a mapping and a mapping $\xi : V \longrightarrow W$ such that $\xi(0) = 0$ and*

$$\lim_{j \longrightarrow \infty} \left|2^2\right|^j \rho\left(2^{-j} \vartheta_1, 2^{-j} \vartheta_2, 2^{-j} \vartheta_3\right) = 0, \tag{18}$$

$$\|\Delta\xi(\vartheta_1, \vartheta_2, \vartheta_3)\| \leq \rho(\vartheta_1, \vartheta_2, \vartheta_3), \tag{19}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$. Then, there exists a unique quadratic mapping $Q : V \longrightarrow W$ satisfying

$$\|\xi(\vartheta) - Q(\vartheta)\| \leq \sup_{j \in \mathbb{N}} \left\{ \left|2^2\right|^{j-1} \rho\left(\frac{\vartheta_1}{2^j}, \frac{\vartheta_2}{2^j}, \frac{\vartheta_3}{2^j}\right) \right\}, \tag{20}$$

for all $\vartheta \in V$.

Proof. Setting $\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta$ in (19), we have

$$\left\|\xi(2\vartheta) - 2^2 \xi(\vartheta)\right\| \leq \rho(\vartheta, \vartheta, \vartheta), \vartheta \in V. \tag{21}$$

Thus, from inequality (21), it implies that

$$\left\|\xi(\vartheta) - 2^2 \xi\left(\frac{\vartheta}{2}\right)\right\| \leq \rho\left(\frac{\vartheta}{2}, \frac{\vartheta}{2}, \frac{\vartheta}{2}\right), \tag{22}$$

for all $\vartheta \in V$. Replacing ϑ by $\vartheta/2$ in (22), we obtain

$$\left\|2^2 \xi\left(\frac{\vartheta}{2}\right) - 2^4 \xi\left(\frac{\vartheta}{2^2}\right)\right\| \leq \left|2^2\right| \rho\left(\frac{\vartheta}{2^2}, \frac{\vartheta}{2^2}, \frac{\vartheta}{2^2}\right), \tag{23}$$

for all $\vartheta \in V$. Hence,

$$\begin{aligned} & \left\|2^{2l} \xi\left(\frac{\vartheta^l}{2}\right) - 2^{2m} \xi\left(\frac{\vartheta}{2^m}\right)\right\| \\ & \leq \max\left\{ \left\|2^{2l} \xi\left(\frac{\vartheta}{2^l} v\right) - 2^{2(l+1)} \xi\left(\frac{\vartheta}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{2(m-1)} \xi\left(\frac{\vartheta}{2^{m-1}}\right) - 2^{2m} \xi\left(\frac{\vartheta}{2^m}\right)\right\| \right\} \\ & \leq \max\left\{ \left|2^2\right|^l \left\|\xi\left(\frac{\vartheta}{2^l}\right) - 2^2 \xi\left(\frac{\vartheta}{2^{l+1}}\right)\right\|, \cdots, \left|2^2\right|^{m-1} \left\|\xi\left(\frac{\vartheta}{2^{m-1}}\right) - 2^2 \xi\left(\frac{\vartheta}{2^m}\right)\right\| \right\} \\ & \leq \sup_{j \in \{l, l+1, \cdots\}} \left\{ \left|2^2\right|^j \rho\left(\frac{\vartheta}{2^{j+1}}, \frac{\vartheta}{2^{j+1}}, \frac{\vartheta}{2^{j+1}}\right) \right\}, \end{aligned} \tag{24}$$

for all $m > l > 0$ and all $\vartheta \in V$. From inequality (24), the sequence $\{2^{2n}\xi(\vartheta/2^n)\}$ is a Cauchy sequence for all $\vartheta \in V$. Since W is complete, thus the sequence $\{2^{2n}\xi(\vartheta/2^n)\}$ is convergent. Now, we can define a mapping $Q : V \longrightarrow W$ by

$$Q(\vartheta) \coloneqq \lim_{l\longrightarrow\infty} 2^{2l}\xi\left(\frac{\vartheta}{2^l}\right),\ \vartheta \in V. \tag{25}$$

Taking $l = 0$ and passing the limit $m \longrightarrow \infty$ in (24), we obtain (20). From inequalities (18) and (19), we have

$$\begin{aligned} &\|\Delta Q(\vartheta_1, \vartheta_2, \vartheta_3)\| \\ &\quad= \lim_{j\longrightarrow\infty} \left|2^2\right|^j \|\Delta\xi\left(2^{-j}\vartheta_1, 2^{-j}\vartheta_2, 2^{-j}\vartheta_3\right)\| \\ &\quad\le \lim_{j\longrightarrow\infty} \left|2^2\right|^j \rho\left(2^{-j}\vartheta_1, 2^{-j}\vartheta_2, 2^{-j}\vartheta_3\right) = 0. \end{aligned} \tag{26}$$

From above, we conclude that $\Delta Q(\vartheta_1, \vartheta_2, \vartheta_3) = 0$ for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$. By using Lemma 4, the function Q is quadratic. Consider another quadratic mapping $T : V \longrightarrow W$ satisfying (20). Then, we have

$$\begin{aligned} &\|Q(\vartheta) - T(\vartheta)\| \\ &\quad= \left\|2^{2q}Q\left(\frac{\vartheta}{2^q}\right) - 2^{2q}T\left(\frac{\vartheta}{2^q}\right)\right\| \\ &\quad\le \max\left\{\left\|2^{2q}Q\left(\frac{\vartheta}{2^q}\right) - 2^{2q}\xi\left(\frac{\vartheta}{2^q}\right)\right\|, \left\|2^{2q}T\left(\frac{\vartheta}{2^q}\right) - 2^{2q}\xi\left(\frac{\vartheta}{2^q}\right)\right\|\right\} \\ &\quad\le \sup_{j\in\mathbb{N}}\left\{\left|2^2\right|^{q+j-1}\rho\left(\frac{\vartheta}{2^{j+1}}, \frac{\vartheta}{2^{j+1}}, \frac{\vartheta}{2^{j+1}}\right)\right\} \longrightarrow 0 \text{ as } q \longrightarrow \infty, \end{aligned} \tag{27}$$

for all $\vartheta \in V$. Thus, we can conclude that $T(\vartheta) = Q(\vartheta)$, $\vartheta \in V$. Hence, the function Q is unique. Thus, the unique quadratic mapping $Q : V \longrightarrow W$ satisfies (20). Hence, the proof of the theorem is now completed. □

Theorem 6. *Let $\rho : V^3 \longrightarrow [0,\infty)$ be a mapping and a mapping $\xi : V \longrightarrow W$ such that $\xi(0) = 0$ and*

$$\lim_{j\longrightarrow\infty}\left\{\frac{1}{\left|2^2\right|^j}\rho\left(2^{j-1}\vartheta_1, 2^{j-1}\vartheta_2, 2^{j-1}\vartheta_3\right)\right\} = 0, \tag{28}$$

and (19) for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$. Then, there exists a unique quadratic mapping $Q : V \longrightarrow W$ satisfying

$$\|\xi(\vartheta) - Q(\vartheta)\| \le Sup_{j\in\mathbb{N}}\left\{\frac{1}{\left|2^2\right|^{j-1}}\rho\left(2^{j-1}\vartheta_1, 2^{j-1}\vartheta_2, 2^{j-1}\vartheta_3\right)\right\}, \tag{29}$$

for all $\vartheta \in V$.

Proof. Setting $\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta$ in (19), we have

$$\left\|\xi(2\vartheta) - 2^2\xi(\vartheta)\right\| \le \rho(\vartheta, \vartheta, \vartheta),\ \vartheta \in V. \tag{30}$$

From inequality (30), we obtain

$$\left\|\xi(\vartheta) - \frac{1}{2^2}\xi(2\vartheta)\right\| \le \frac{1}{\left|2^2\right|}\rho(\vartheta, \vartheta, \vartheta),\ \vartheta \in V. \tag{31}$$

Replacing ϑ by 2ϑ in (31), we get

$$\left\|\frac{\xi(2\vartheta)}{2^2} - \frac{1}{2^4}\xi\left(2^2\vartheta\right)\right\| \le \frac{1}{\left|2^2\right|^2}\rho(2\vartheta, 2\vartheta, 2\vartheta), \tag{32}$$

for all $\vartheta \in V$. Hence,

$$\begin{aligned} &\left\|\frac{1}{2^{2l}}\xi\left(2^l\vartheta\right) - \frac{1}{2^{2m}}\xi(2^m\vartheta)\right\| \\ &\le \max\left\{\left\|\frac{1}{2^{2l}}\xi\left(2^l\vartheta\right) - \frac{1}{2^{2(l+1)}}\xi\left(2^{l+1}\vartheta\right)\right\|, \cdots, \left\|\frac{1}{2^{2(m-1)}}\xi\left(2^{m-1}\vartheta\right) - \frac{1}{2^{2m}}\xi(2^m\vartheta)\right\|\right\} \\ &\le \max\left\{\frac{1}{\left|2^2\right|^l}\left\|\xi\left(2^l\vartheta\right) - \frac{1}{\left|2^2\right|^{m-1}}\xi\left(2^{l+1}\vartheta\right)\right\|, \cdots, \frac{1}{\left|2^2\right|^{m-1}}\left\|\xi\left(2^{m-1}\vartheta\right) - \frac{1}{2^2}\xi(2^m\vartheta)\right\|\right\} \\ &\le \sup_{j\in\{l,l+1,\cdots\}}\left\{\frac{1}{\left|2^2\right|^{j+1}}\rho\left(2^j\vartheta_1, 2^j\vartheta_2, 2^j\vartheta_3\right)\right\}, \end{aligned} \tag{33}$$

for all $m > l > 0$ and all $\vartheta \in V$. From inequality (33), the sequence $\{(1/2^{2n})\xi(2^n\vartheta)\}$ is a Cauchy sequence for all $\vartheta \in V$. Since W is complete, the sequence $\{(1/2^{2n})\xi(2^n\vartheta)\}$ is convergent. Now, we can define a mapping $Q : V \longrightarrow W$ by

$$Q(\vartheta) \coloneqq \lim_{n\longrightarrow\infty}\frac{1}{2^{2n}}\xi(2^n\vartheta),\ \vartheta \in V. \tag{34}$$

The remaining proof is the same as the proof of Theorem 5. □

Corollary 7. *Let $\xi : V \longrightarrow W$ be a mapping such that $\xi(\vartheta) = 0$ and*

$$\|\Delta(\vartheta_1, \vartheta_2, \vartheta_3)\| \le \theta(\|\vartheta_1\|^r + \|\vartheta_2\|^r + \|\vartheta_3\|^r), \tag{35}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$, where r and θ are in $\mathbb{R}^+$ with $r < 2$. Then, there exists a unique quadratic mapping $Q : V \longrightarrow W$ satisfying

$$\|\xi(\vartheta) - Q(\vartheta)\| \le \frac{3\theta}{|2|^r}\|\vartheta\|^r, \tag{36}$$

for all $\vartheta \in V$.

Corollary 8. *Let $\xi : V \longrightarrow W$ be a mapping such that $\xi(\vartheta) = 0$ and*

$$\|\Delta(\vartheta_1, \vartheta_2, \vartheta_3)\| \le \theta(\|\vartheta_1\|^r + \|\vartheta_2\|^r + \|\vartheta_3\|^r), \tag{37}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$, where r and θ are in $\mathbb{R}^+$ with $r > 2$. Then, there exists a unique quadratic mapping $Q : V \longrightarrow$

W satisfying

$$\|\xi(\vartheta)-Q(\vartheta)\|\leq\frac{3\theta}{|2^2|}\|\vartheta\|^r, \tag{38}$$

for all $\vartheta\in V$.

5. Stability of (6) in Non-Archimedean Banach Space: Fixed Point Method

Theorem 9. *Let* $\rho : V^3 \longrightarrow [0,\infty)$ *be a mapping such that there exists* $L<1$ *with*

$$\rho\left(2^{-1}\vartheta_1,2^{-1}\vartheta_2,2^{-1}\vartheta_3\right)\leq\frac{L}{|4|}\rho\left(2^{-1}\vartheta_1,2^{-1}\vartheta_2,2^{-1}\vartheta_3\right), \tag{39}$$

for all $\vartheta_1,\vartheta_2,\vartheta_3\in V$. *If a mapping* $\xi : V\longrightarrow W$ *such that* $\xi(0)=0$ *and (19) for all* $\vartheta_1,\vartheta_2,\vartheta_3\in V$, *then there exists a unique quadratic mapping* $Q : V\longrightarrow W$ *satisfying*

$$\|\xi(\vartheta)-Q(\vartheta)\|\leq\frac{L}{|2^2|(1-L)}\rho(\vartheta,\vartheta,\vartheta), \tag{40}$$

for all $\vartheta\in V$.

Proof. Setting $\vartheta_1=\vartheta_2=\vartheta_3=\vartheta$ in (19), we obtain

$$\|\xi(2\vartheta)-4\xi(\vartheta)\|\leq\rho(\vartheta,\vartheta,\vartheta), \tag{41}$$

for all $\vartheta\in V$. Consider

$$S\coloneqq\{q : V\longrightarrow W, q(0)=0\}, \tag{42}$$

and the generalized metric d defined by

$$d(p,q)\coloneqq\inf\{\varepsilon\in\mathbb{R} : \|p(\vartheta)-q(\vartheta)\|\leq\varepsilon\rho(\vartheta,\vartheta,\vartheta),\forall\vartheta\in V\}, \tag{43}$$

here, as usual, $\inf\xi=+\infty$. Clearly, (S,q) is complete (see [23]). Next, consider a mapping $J : S\longrightarrow S$ defined by

$$Jp(\vartheta)\coloneqq 2^2p\left(\frac{\vartheta}{2}\right),\ \vartheta\in V \tag{44}$$

For all $p,q\in S$ such that $d(p,q)=\varepsilon$, then

$$\|p(\vartheta)-q(\vartheta)\|\leq\varepsilon\rho(\vartheta,\vartheta,\vartheta), \tag{45}$$

for all $\vartheta\in V$. Hence,

$$\begin{aligned}\|Jp(\vartheta)-Jq\xi(\vartheta)\|&=\left\|2^2p\left(2^{-1}\vartheta\right)-2^2q\xi\left(2^{-1}\vartheta\right)\right\|\\&\leq\left|2^2\right|\varepsilon\rho\left(2^{-1}\vartheta,2^{-1}\vartheta,2^{-1}v\right)\\&\leq\left|2^2\right|\varepsilon\frac{L}{|2^2|}\rho(\vartheta,\vartheta,\vartheta)\leq\left|2^2\right|L\varepsilon\rho(\vartheta,\vartheta,\vartheta),\end{aligned} \tag{46}$$

for all $\vartheta\in V$. Thus,

$$d(p,q)=\varepsilon\Rightarrow d(Jp,Jq)\leq L\varepsilon. \tag{47}$$

This concludes that

$$d(Jp,Jq)\leq Ld(p,q),\ p,q\in S. \tag{48}$$

From inequality (41),

$$\left\|\xi(\vartheta)-2^2\xi\left(\frac{\vartheta}{2}\right)\right\|\leq\rho\left(2^{-1}\vartheta,2^{-1}\vartheta,2^{-1}\vartheta\right)\leq\frac{L}{|2^2|}\rho(\vartheta,\vartheta,\vartheta),\ \vartheta\in V. \tag{49}$$

Therefore,

$$d(\xi,J\xi)\leq\left|\frac{1}{2^2}\right|L,\ \vartheta\in V. \tag{50}$$

By using Theorem 3, there exists a mapping $Q : V\longrightarrow W$ satisfying the following conditions:

(1) Q is a fixed point of J, i.e.,

$$Q(\vartheta)=2^2Q\left(2^{-1}\vartheta\right)\forall\vartheta\in V. \tag{51}$$

In the set below, the function Q is the unique fixed point J.

$$M=\{p\in S : d(\xi,p)<\infty\}. \tag{52}$$

This proves that the uniqueness of the function Q satisfies (51) such that there exists $\varepsilon\in[0,\infty)$ such that

$$\|\xi(\vartheta)-Q(\vartheta)\|\leq\varepsilon\rho(\vartheta,\vartheta,\vartheta),\ \vartheta\in V. \tag{53}$$

(2) $d(J^l\xi,Q)$ tends to 0 as taking the limit $l\longrightarrow\infty$. This implies

$$\lim_{l\longrightarrow\infty}4^n\xi(2^{-n}\vartheta)=Q(\vartheta),\text{ for all }\vartheta\in V. \tag{54}$$

(3) $d(\xi, Q) \le (1/1-L)d(\xi, J\xi)$, which implies

$$\|\xi(\vartheta) - Q(\vartheta)\| \le \frac{L}{|2^2|(1-L)} \rho(\vartheta, \vartheta, \vartheta), \text{for all } \vartheta \in V. \quad (55)$$

From (39) and (51),

$$\begin{aligned} \|\Delta Q(\vartheta_1, \vartheta_2, \vartheta_3)\| &= \lim_{j\longrightarrow\infty} |2^2|^j \|\Delta\xi(2^{-j}\vartheta_1, 2^{-j}\vartheta_2, 2^{-j}\vartheta_3)\| \\ &\le \lim_{j\longrightarrow\infty} |2^2|^j \rho(2^{-j}\vartheta_1, 2^{-j}\vartheta_2, 2^{-j}\vartheta_3) = 0, \end{aligned} \quad (56)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$. Thus,

$$\Delta Q(\vartheta_1, \vartheta_2, \vartheta_3) = 0, \quad (57)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$. By using Lemma 4, the function Q is quadratic. Hence, the proof of the theorem is now completed. □

Theorem 10. *Let $\rho : V^3 \longrightarrow [0,\infty)$ be a mapping such that there exists $L < 1$ with*

$$\rho(\vartheta_1, \vartheta_2, \vartheta_3) \le L|2^2|\rho(2^{-1}\vartheta_1, 2^{-1}\vartheta_2, 2^{-1}\vartheta_3),\ \vartheta_1, \vartheta_2, \vartheta_3 \in V. \quad (58)$$

If a mapping $\xi : V \longrightarrow W$ such that $\xi(0) = 0$ and (19) for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$, then there exists a unique quadratic mapping $Q : V \longrightarrow W$ satisfying

$$\|\xi(\vartheta) - Q(\vartheta)\| \le \frac{1}{|2^2|(1-L)} \rho(\vartheta, \vartheta, \vartheta), \quad (59)$$

for all $\vartheta \in V$.

Proof. Setting $\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta$ in (19), we have

$$\|\xi(2\vartheta) - 4\xi(\vartheta)\| \le \rho(\vartheta, \vartheta, \vartheta), \quad (60)$$

for all $\vartheta \in V$. From the inequality (60), we get

$$\left\|\xi(\vartheta) - \frac{1}{2^2}\xi(\vartheta)\right\| \le \frac{1}{|2^2|} \rho(\vartheta, \vartheta, \vartheta),\ \vartheta \in V. \quad (61)$$

The generalized metric space (S, d) is defined in the proof of Theorem 9. Consider a mapping $J : S \longrightarrow S$ defined by

$$Jp(\vartheta) \coloneqq \frac{1}{2^2} p(2\vartheta),\ \vartheta \in V. \quad (62)$$

From inequality (61),

$$d(\xi, J\xi) \le \frac{1}{|2^2|}. \quad (63)$$

Hence,

$$\|\xi(\vartheta) - Q(\vartheta)\| \le \frac{1}{|2^2|(1-L)} \rho(\vartheta, \vartheta, \vartheta),\ \vartheta \in V. \quad (64)$$

The remaining proof is the same as in the proof of Theorem 9. □

Corollary 11. *Let $\xi : V \longrightarrow W$ be a mapping such that $\xi(0) = 0$ and*

$$\|\Delta\xi(\vartheta_1, \vartheta_2, \vartheta_3)\| \le \theta\left(\sum_{i=1}^{3} \|\vartheta_i\|^r\right), \quad (65)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$, where r and θ are in $\mathbb{R}^+$ with $r < 2$; then there exists a unique quadratic mapping $Q : V \longrightarrow W$ satisfying

$$\|\xi(\vartheta) - Q(\vartheta)\| \le \frac{2\theta\|\vartheta\|^r}{|2|^r - |2^2|}, \quad (66)$$

for all $\vartheta \in V$.

Corollary 12. *Let $\xi : V \longrightarrow W$ be a mapping such that $\xi(0) = 0$ and*

$$\|\Delta\xi(\vartheta_1, \vartheta_2, \vartheta_3)\| \le \theta\left(\sum_{i=1}^{3} \|\vartheta_i\|^r\right), \quad (67)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$, where r and θ are in $\mathbb{R}^+$ with $r > 2$; then there exists a unique quadratic mapping $Q : V \longrightarrow W$ satisfying

$$\|\xi(\vartheta) - Q(\vartheta)\| \le \frac{2\theta}{|2^2|^r - |2|} \|\vartheta\|^r, \quad (68)$$

for all $\vartheta \in V$.

6. Stability of (6) in Non-Archimedean Random Normed Space

Definition 13 [24]. A random normed space is triple (V, μ, T), where V is a vector space, T is a continuous $t-$norm, and a mapping $\mu : V \longrightarrow D^+$ satisfies

(RN1) $\mu_\vartheta(t) = \varepsilon_0(t), \forall t > 0$ if and only if $\vartheta = 0$.

(RN2) $\mu_{\lambda\vartheta}(t) = \mu_\vartheta(t/|\lambda|)$ for all $\vartheta \in V$, $\lambda \ne 0$.

(RN3) $\mu_{\vartheta_1+\vartheta_2}(t_1 + t_2) \ge T(\mu_{\vartheta_1}(t_1), \mu_{\vartheta_2}(t_2))$ for all $\vartheta_1, \vartheta_2 \in V$ and $t_1, t_2 \ge 0$.

Definition 14 [25]. A random normed space (V, μ, T) is said to be non-Archimedean random normed space if it satisfies

(NAR1) $\mu_\vartheta(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $\vartheta = 0$.

(NAR2) $\mu_{\lambda\vartheta}(t) = \mu_\vartheta(1/|\lambda|)$ for all $\vartheta \in V$, $t > 0$, $\lambda \neq 0$.

(NAR3) $\mu_{\vartheta_1+\vartheta_2}(\max\{t_1, t_2\}) \geq T(\mu_{\vartheta_1}(t_1), \mu_{\vartheta_2}(t_2))$ for all $\vartheta_1, \vartheta_2 \in V$ and $t_1, t_2 \geq 0$.

It is clear that if (NAR3) holds, then so

$$(RN3)\mu_{\vartheta_1+\vartheta_2}(t+s) \geq T\left(\mu_{\vartheta_1}(t), \mu_{\vartheta_2}(s)\right). \tag{69}$$

Example 1 [25]. Let a non-Archimedean normed space $(V, \|\cdot\|)$ and we define

$$\mu_\vartheta(t) = \frac{t}{t+\|\vartheta\|}, \tag{70}$$

for all $\vartheta \in V$ and all $t > 0$. Then, the triple (V, μ, T_M) is a non-Archimedean random normed space.

Definition 15 [25]. Let (V, μ, T) be a non-Archimedean random normed space and a sequence $\{\vartheta_n\}$ in V. Then, the sequence $\{\vartheta_n\}$ is called as convergent if there exist $\vartheta \in V$ such that

$$\lim_{n\longrightarrow\infty} \mu_{n-\vartheta}(t) = 1, \tag{71}$$

for all $t > 0$. In particular, ϑ is called the limit of the sequence $\{\vartheta_n\}$.

Here, let V be a vector space over a non-Archimedean field $\mathbb{K}$ and (W, μ, T) be a non-Archimedean random Banach space over $\mathbb{K}$. And consider that $2 \neq 0$ in $\mathbb{K}$.

Next, we define a random approximately quadratic function. Let a distribution mapping $\psi : V \times V \longrightarrow [0,\infty)$ satisfies $\psi(\vartheta_1, \vartheta_2, \vartheta_3, \cdot)$ which is symmetric and nondecreasing and

$$\psi(\lambda\vartheta, \lambda\vartheta, \lambda\vartheta, t) \geq \psi\left(\vartheta, \vartheta, \vartheta, \frac{t}{|\lambda|}\right), \tag{72}$$

for all $\vartheta \in V$ and all $\lambda \neq 0$.

Definition 16. A function $\xi : V \longrightarrow W$ is called as a ψ-approximately quadratic if

$$\mu_{2\xi(\vartheta_1+\vartheta_2/2)+2\xi(\vartheta_3)-\xi((\vartheta_1+\vartheta_2/2)+\vartheta_3)-\lambda^{-2}\xi(\lambda((\vartheta_1+\vartheta_2/2)-\vartheta_3))} \geq \psi(\vartheta_1, \vartheta_2, \vartheta_3, t), \tag{73}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$ and $t > 0$.

Theorem 17. *Let a function $\xi : V \longrightarrow W$ be a ψ-approximately quadratic mapping. If for some real number $\alpha > 0$, and some integer k, $k > 1$ with $\alpha > |2^k|$,*

$$\psi\left(2^{-k}\vartheta_1, 2^{-k}\vartheta_2, 2^{-k}\vartheta_3, t\right) \geq \psi(\vartheta_1, \vartheta_2, \vartheta_3, \alpha t), \tag{74}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$ and $t > 0$, and

$$\lim_{n\longrightarrow\infty} T_{j=n}^{\infty} M\left(\vartheta, \frac{\alpha^j t}{|2|^{kj}}\right) = 1, \tag{75}$$

for all $\vartheta \in V$ and every $t > 0$; then there exists a unique quadratic mapping $Q : V \longrightarrow W$ such that

$$\mu_{\xi(\vartheta)-Q(\vartheta)}(t) \geq T_{i=1}^{\infty} M\left(\vartheta, \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \tag{76}$$

where

$$M(\vartheta, t) \coloneqq T\left(\psi(\vartheta, \vartheta, \vartheta, t)\psi(2\vartheta, 2\vartheta, 2\vartheta, t), \cdots, \psi\left(2^{k-1}\vartheta, 2^{k-1}\vartheta, 2^{k-1}\vartheta, t\right)\right), \tag{77}$$

for all $\vartheta \in V$ and all $t > 0$.

Proof. First, we demonstrate by induction on j that for all $\vartheta \in V$, $t > 0$ and $j > 0$,

$$\mu_{\xi(2^j\vartheta)-2^{2^j}\xi(\vartheta)}(t) \geq M_j(\vartheta, t) \coloneqq T\left(\psi(\vartheta, \vartheta, \vartheta, t), \cdots, \psi(2^{j-1}\vartheta, 2^{j-1}\vartheta, 2^{j-1}\vartheta, t)\right). \tag{78}$$

Setting $\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta$ in (73), we obtain

$$\mu_{\xi(2\vartheta)-2^2\xi(\vartheta)}(t) \geq \psi(\vartheta, \vartheta, \vartheta, t), \tag{79}$$

for all $\vartheta \in V$ and all $t > 0$. This proves that (78) for $j = 1$. Suppose that (78) holds for some $j > 0$. Replacing ϑ by $2^j\vartheta$ in (73), we get

$$\mu_{\xi(2^{j+1}\vartheta-2^2\xi(2^j\vartheta))}(t) \geq \psi\left(2^j\vartheta, 2^j\vartheta, 2^j\vartheta, t\right), \tag{80}$$

for all $\vartheta \in V$ and all $t > 0$. Since $|2^2| \leq 1$,

$$\begin{aligned}
&\mu_{\xi(2^{j+1}\vartheta)-2^{2(j+1)}\xi(\vartheta)}(t) \\
&\quad\geq T\left(\mu_{\xi(2^{j+1}\vartheta)-2^2\xi(2^j\vartheta)}(t), \mu_{2^2\xi(2^j\vartheta)-2^{2(j+1)}\xi(\vartheta)}(t)\right) \\
&\quad= T\left(\mu_{\xi(2^{j+1}\vartheta)-2^2\xi(2^j\vartheta)}(t), \mu_{\xi(2^j\vartheta)-2^{2j}\xi(\vartheta)}\left(\frac{t}{|2^2|}\right)\right)T \\
&\qquad\cdot\left(\mu_{\xi(2^{j+1}\vartheta)-2^2\xi(2^j\vartheta)}(t), \mu_{\xi(2^j\vartheta)-2^{2j}\xi(\vartheta)}(t)\right) \\
&\quad\geq T\left(\psi\left(2^j\vartheta, 2^j\vartheta, 2^j\vartheta, t\right), M_j(\vartheta, t)\right) = M_{j+1}(\vartheta, t),
\end{aligned} \tag{81}$$

for all $\vartheta \in V$. Thus, condition (78) holds for all $j > 0$. In particular,

$$\mu_{\xi(2^k\vartheta)-2^{2k}\xi(\vartheta)}(t) \geq M(\vartheta, t), \tag{82}$$

for all $\vartheta \in V$ and all $t > 0$. Replacing ϑ by $2^{-(k+kn)}\vartheta$ in (82) and using the inequality (74), we have

$$\mu_{\xi(\vartheta/2^{kn})-2^{2k}\xi(\vartheta/2^{kn+k})} \geq M\left(\frac{\vartheta}{2^{kn+k}}, t\right) \geq M\left(\vartheta, \alpha^{n+1}t\right); n = 0, 1, 2, \cdots, \tag{83}$$

for all $\vartheta \in V$ and all $t > 0$. Then,

$$\mu_{(2^{2k})^n\xi(\vartheta/(2^k)^n)-(2^{2k})^{n+1}\xi(\vartheta/(2^k)^{n+1})}(t) \geq M\left(\vartheta, \frac{\alpha^{n+1}}{|(2^{2k})^n|}t\right); n = 0, 1, 2, \cdots, \tag{84}$$

for all $\vartheta \in V$ and all $t > 0$. Hence,

$$\begin{aligned} &\mu_{(2^k)^n\xi(\vartheta/(2^k)^n)-(2^{2k})^{n+p}\xi(\vartheta/(2^k)^{n+p})}(t) \\ &\geq T_{j=n}^{n+p}\left(\mu_{(2^k)^j\xi(\vartheta/(2^k)^j)-(2^{2k})^{j+p}\xi(\vartheta/(2^k)^{j+p})}(t)\right) \\ &\geq T_{j=n}^{n+p}M\left(\vartheta, \frac{\alpha^{j+1}}{\left|(2^{2k})^j\right|}t\right) \geq T_{j=n}^{n+p}M\left(\vartheta, \frac{\alpha^{j+1}}{\left|(2^k)^j\right|}t\right). \end{aligned} \tag{85}$$

Since $\lim_{n\longrightarrow\infty} T_{j=n}^{\infty} M(\vartheta, (\alpha^{j+1}/|(2^k)^j|)t) = 1$ for all $\vartheta \in V$ and all $t > 0$, $\{(2^{2k})^n\xi(\vartheta/(2^k)^n)\}_{n\in N}$ is a Cauchy sequence in (W, μ, T). Hence, we can define a mapping $Q : V \longrightarrow W$ such that

$$\lim_{n\longrightarrow\infty}\mu_{(2^{2k})^n\xi(\vartheta/(2^k)^n)-Q(\vartheta)}(t) = 1, \tag{86}$$

for all $\vartheta \in V$ and all $t > 0$. Now, for all $n \geq 1$,

$$\begin{aligned} &\mu_{\xi(\vartheta)-(2^{2k})^n\xi(\vartheta/(2^k)^n)}(t) \\ &= \mu_{\sum_{i=0}^{n-1}\left(2^{2k}\right)^i\xi\left(\vartheta/\left(2^k\right)^i\right)-\left(2^{2k}\right)^{i+1}\xi\left(\vartheta/\left(2^k\right)^{i+1}\right)}(t) \\ &\geq T_{i=0}^{n-1}\left(\mu_{(2^{2k})^i\xi(\vartheta/(2^k)^i)-(2^{2k})^{i+1}\xi(\vartheta/(2^k)^{i+1})}(t)\right) \\ &\geq T_{i=0}^{n-1}M\left(\vartheta, \frac{\alpha^{i+1}t}{|2^{2k}|^i}\right), \end{aligned} \tag{87}$$

for all $\vartheta \in V$ and $t > 0$. Thus,

$$\begin{aligned} \mu_{\xi(\vartheta)-Q(\vartheta)}(t) &\geq T\left(\mu_{\xi(\vartheta)-(2^{2k})^n\xi(\vartheta/(2^k)^n)}, \mu_{(2^{2k})^n\xi(\vartheta/(2^k)^n)-Q(\vartheta)}(t)\right) \\ &\geq T\left(T_{i=0}^{n-1}M\left(\vartheta, \frac{\alpha^{i+1}t}{|2^{2k}|^i}\right), \mu_{(2^{2k})^n\xi(\vartheta/(2^k)^n)-Q(\vartheta)}(t)\right). \end{aligned} \tag{88}$$

By taking the limit $n \longrightarrow \infty$, we have

$$\mu_{\xi(\vartheta)-Q(\vartheta)}(t) \geq T_{i=1}^{\infty}M\left(\vartheta, \frac{\alpha^{i+1}t}{|2^k|^i}\right). \tag{89}$$

This shows that (76) holds. Since T is continuous, by a well-known result in probabilistic metric space (see, e.g., [[26], Chapter 12]), that

$$\lim_{n\longrightarrow\infty}\mu_{(2^k)^n\Delta\xi(2^{-kn}\vartheta_1,2^{-kn}\vartheta_2,2^{-kn}\vartheta_3)}(t) = \mu_{\Delta Q(\vartheta_1,\vartheta_2,\vartheta_3)}(t), \tag{90}$$

for all $t > 0$.

On the other hand, replacing $(\vartheta_1, \vartheta_2, \vartheta_3)$ by $(2^{-kn}\vartheta_1, 2^{-kn}\vartheta_2, 2^{-kn}\vartheta_3)$, respectively, in (73) and using (NAR2) and (74), we get

$$\begin{aligned} \mu_{(2^k)^n\Delta\xi(2^{-kn}\vartheta_1,2^{-kn}\vartheta_2,2^{-kn}\vartheta_3)}(t) &\geq \psi\left(2^{-kn}\vartheta_1, 2^{-kn}\vartheta_2, 2^{-kn}\vartheta_3, \frac{t}{|2^k|^n}\right) \\ &\geq \psi\left(\vartheta_1, \vartheta_2, \vartheta_3, \frac{\alpha^n t}{|2^k|^n}\right). \end{aligned} \tag{91}$$

Since $\lim_{n\longrightarrow\infty}\psi(\vartheta_1, \vartheta_2, \vartheta_3, \alpha^n t/|2^k|^n) = 1$, we can conclude that the function Q is quadratic. Consider another quadratic mapping $Q' : V \longrightarrow W$ such that $\mu_{Q'(\vartheta)-\xi(\vartheta)}(t) \geq M(\vartheta, t)$ for all $\vartheta \in V$ and all $t > 0$; then for all $n \in N$ and $\vartheta \in V$ and all $t > 0$,

$$\mu_{Q(\vartheta)-Q'(\vartheta)}(t) \geq T\left(\mu_{Q(\vartheta)-(2^{4k})^n\xi(\vartheta/(2^k)^n)}(t), \mu_{(2^{2k})^n\xi(\vartheta/(2^k)^n)-Q'(\vartheta)}(t), t\right). \tag{92}$$

From condition (86), we arrive at the conclusion that $Q = Q'$. □

Corollary 18. *Let a function $\xi : V \longrightarrow W$ be a ψ-approximately quadratic. If for some real number $\alpha > 0$ and some integer k, $k > 1$, with $|2^k| < \alpha$,*

$$\psi\left(2^{-k}\vartheta_1, 2^{-k}\vartheta_2, 2^{-k}\vartheta_3, t\right) \geq \psi(\vartheta_1, \vartheta_2, \vartheta_3, \alpha t), \tag{93}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$ and $t > 0$, then there exists a unique quadratic mapping $Q : V \longrightarrow W$ satisfying

$$\mu_{\xi(\vartheta)-Q(\vartheta)}(t) \geq T_{i=1}^{\infty}M\left(\vartheta, \frac{\alpha^{i+1}t}{|2|^{ki}}\right), \tag{94}$$

where

$$M(\vartheta, t) \coloneqq T(\psi(\vartheta, \vartheta, \vartheta, t), \psi(2\vartheta, 2\vartheta, 2\vartheta, t), \cdots, \psi\left(2^{k-1}\vartheta, 2^{k-1}\vartheta, 2^{k-1}\vartheta, t\right), \tag{95}$$

for all $\vartheta \in V$ and all $t > 0$.

Proof. Since

$$\lim_{n\longrightarrow\infty} M\left(\vartheta, \frac{\alpha^j t}{|2|^{kj}}\right) = 1, \tag{96}$$

for all $\vartheta \in V$ and all $t > 0$ and T is of Hadzic type, from Proposition 2.1 in [25], it follows that

$$\lim_{n\longrightarrow\infty} T_{j=n}^{\infty} M\left(\vartheta, \frac{\alpha^j t}{|2|^{kj}}\right), \tag{97}$$

for all $\vartheta \in V$ and $t > 0$. Now, we can obtain our needed result by using Theorem 17 □

Example 2. Let a non-Archimedean random normed space (V, μ, T_M), in which

$$\mu_\vartheta(t) = \frac{t}{t+\|\vartheta\|}, \tag{98}$$

for all $\vartheta \in V$ and every $t > 0$, and let (W, μ, T_M) be a complete non-Archimedean random normed space (see Example 1). Now, we can define

$$\psi(\vartheta_1, \vartheta_2, \vartheta_3, t) = \frac{t}{1+t}. \tag{99}$$

It is obvious that (74) holds for $\alpha = 1$. Furthermore,

$$M(\vartheta, t) = \frac{t}{1+t}. \tag{100}$$

We obtain

$$\begin{aligned} \lim_{n\longrightarrow\infty} T_{M,j=n}^{\infty} M\left(\vartheta, \frac{\alpha^j t}{|2|^{kj}}\right) &= \lim_{n\longrightarrow\infty}\left(\lim_{m\longrightarrow\infty} T_{M,j=n}^{m} M\left(\vartheta, \frac{t}{|2|^{kj}}\right)\right) \\ &= \lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty}\left(\frac{t}{t+\left|2^k\right|^n}\right) = 1, \end{aligned} \tag{101}$$

for all $\vartheta \in V$ and all $t > 0$.

7. Conclusion

In this paper, we introduced λ-quadratic functional equation and obtained its general solution. In Section 4 and Section 5, we investigated Ulam-Hyers stability of equation (6) by using direct method and fixed point method in non-Archimedean Banach space, and also in Section 6, we investigated the Ulam-Hyers stability results in non-Archimedean random normed space. The direct method requires us to find the Cauchy sequence and prove that every Cauchy sequence is convergent, as well as prove the uniqueness of the function; this method was introduced by Hyers [2], and the fixed point method requires us to use the Banach contraction principle and Lipschitz constant L to obtain the stability results of the functional equation; this method was introduced by Radu [27]. The fixed point method gives more accurate stability results when compared with the direct method. Finally, these stability results generalized the findings of [11].

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflict interests.

Authors' Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version of the manuscript.

References

[1] S. M. Ulam, *A Collection of Mathematical Problems, Interscience Tracts in Pure and Applied Mathematics, No. 8*, Interscience Publishers, New York, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, pp. 297–300, 1978.

[5] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, 434 pages, 1991.

[6] T. M. Rassias and P. Šemrl, "On the Hyers-Ulam stability of linear mappings," *Journal of Mathematical Analysis and Applications*, vol. 173, pp. 325–338, 1993.

[7] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.

[8] J. R. Lee, C. Park, and D. Y. Shin, "Additive and quadratic functional in equalities in non-Archimedean normed spaces," *International Journal of Mathematical Analysis*, vol. 8, pp. 1233–1247, 2014.

[9] C. Park, "Additive ρ-functional inequalities," *Journal of Mathematical Inequalities*, vol. 7, no. 5, pp. 296–310, 2014.

[10] C. Park, "Functional inequalities in non-Archimedean normed spaces," *Acta Mathematica Sinica, English Series*, vol. 31, no. 3, pp. 353–366, 2015.

[11] C. Park and S. O. Kim, "Quadratic α-functional equations," *International Journal of Nonlinear Analysis and Applications*, vol. 8, pp. 1–9, 2017.

[12] C. Park, S. O. Kim, J. R. Lee, and D. Y. Shin, "Quadratic ρ -functional inequalities in β-homogeneous normed spaces," *International Journal of Nonlinear Analysis and Applications*, vol. 6, pp. 21–26, 2015.

[13] K. Tamilvanan, A. M. Alanazi, M. G. Alshehri, and J. Kafle, "Hyers-Ulam stability of quadratic functional equation based on fixed point technique in Banach spaces and non-

Archimedean Banach spaces," *Mathematics*, vol. 9, no. 20, article 2575, p. 15, 2021.

[14] K. Tamilvanan, A. M. Alanazi, J. M. Rassias, and A. H. Alkhaldi, "Ulam stabilities and instabilities of Euler–Lagrange-Rassias quadratic functional equation in non-Archimedean IFN spaces," *Mathematics*, vol. 9, no. 23, article 3063, p. 16, 2021.

[15] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes mathematicae*, vol. 27, no. 1, pp. 76–86, 1984.

[16] A. Gilányi, "On a problem by K. Nikodem," *Mathematical Inequalities and Applications*, vol. 5, pp. 707–710, 2002.

[17] A. Gilányi, "Eine zur Parallelogrammgleichung äquivalente Ungleichung," *Aequationes Mathematicae*, vol. 62, no. 3, pp. 303–309, 2001.

[18] C. Park and A. Najati, "Generalized additive functional inequalities in Banach algebra," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 2, pp. 54–62, 2010.

[19] E. Guariglia and K. Tamilvanan, "On the stability of radical septic functional equations," *Mathematics*, vol. 8, no. 12, p. 2229, 2020.

[20] S. O. Kim and K. Tamilvanan, "Fuzzy stability results of generalized quartic functional equations," *Mathematics*, vol. 9, no. 2, p. 120, 2021.

[21] C. Park, K. Tamilvanan, B. Noori, M. B. Moghimi, and A. Najati, "Fuzzy normed spaces and stability of a generalized quadratic functional equation," *AIMS Mathematics*, vol. 5, no. 6, pp. 7161–7174, 2020.

[22] N. Brillouët-Belluot, J. Brzdęk, and K. Ciepliński, "On some recent developments in Ulam's type stability," *Abstract and applied analysis*, vol. 2012, Article ID 716936, 41 pages, 2012.

[23] D. Mihet and V. Radu, "On the stability of the additive Cauchy functional equation in random normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 567–572, 2008.

[24] A. N. Šerstnev, "On the concept of a stochastic normalized space," *Doklady Akademii Nauk SSSR*, vol. 149, pp. 280–283, 1963.

[25] J. M. Rassias, R. Saadati, G. Sadeghi, and J. Vahidi, "On nonlinear stability in various random normed spaces," *Journal of Inequalities and Applications*, vol. 2011, no. 1, Article ID 62, 2011.

[26] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathematics*, North-Holland Publishing Co., New York, 1983.

[27] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed point theory*, vol. 4, pp. 91–96, 2003.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2965300, 8 pages
https://doi.org/10.1155/2022/2965300

Research Article

Fixed Point Theorems of Superlinear Operators with Applications

Shaoyuan Xu [1] **and Yan Han** [2]

[1]School of Mathematics and Statistics, Hanshan Normal University, Chaozhou, China
[2]School of Mathematics and Statistics, Zhaotong University, Zhaotong, China

Correspondence should be addressed to Yan Han; hanyan702@126.com

Received 16 March 2022; Accepted 22 April 2022; Published 18 May 2022

Academic Editor: Santosh Kumar

In this paper, by using the partial order method and monotone iterative techniques, the existence and uniqueness of fixed points for a class of superlinear operators are studied, without requiring any compactness or continuity. As corollaries, the new fixed point theorems for α-convex operators $(\alpha > 1)$, e-convex operators, positive α homogeneous operator $(\alpha > 1)$, generalized e -convex operator, and convex operators are obtained. The results are applied to nonlinear integral equations and partial differential equations.

1. Introduction

Linear operators are a kind of operators with good properties and rich theoretical results, which have formed a classical branch in functional analysis. However, in order to solve the fixed point problems involving operators or equations in practical applications, we need a large number of nonlinear operators, including two classes of significant operators, namely, superlinear operators and sublinear operators. Since some of these operators have concavity or convexity, they bring convenience to the related research. The concepts of concave operators and convex operators were proposed in 1960s, which attracted people's great interest. Many authors obtained a lot of meaningful results, see [1–27]. Among them, α-convex operators $(\alpha > 1)$ [12, 17], e-convex operators [13], and generalized e-convex operators [16] are a very important class of convex operators. It has important applications in many fields. However, it was difficult to study the α-convex operators $(\alpha > 1)$ (including positive α-homogeneous operators) and e-convex operators because they had strong superlinear properties [13] and described nonlinear problems [12]. Until now, the results are still very few and not very ideal (see [7], P457). Therefore, under what conditions, these operators have a unique fixed point remains a very important and meaningful problem.

In [7], a fixed point theorem for a class of superlinear operators was obtained by topological degree method under the condition that there are inverse upward and downward solutions. In [17], using some results of δ-concave operator, the author transformed the positive α-homogeneous superlinear operator into δ-concave operator and studied the existence and uniqueness of the solutions of positive α -homogeneous superlinear operator equations. In [13], the existence of fixed points was investigated when the α-convex operators $(\alpha > 1)$ was a strict set contraction. In [16], Zhao and Du obtained the existence of fixed points of generalized e-concave operators and generalized e-convex operators. As an application, the singular boundary value problems for second order differential equations were discussed. In [10], according to the properties of totally ordered sets, the existence and uniqueness of new positive fixed points for a class of superlinear homogeneous operators were studied in abstract spaces. The results were applied to a class of superlinear Hammerstein-type integral equations.

In this paper, we study a class of superlinear operators without requiring any compactness or continuity and obtain some new fixed point theorems for superlinear operators by using the partial order and the monotone iteration which are different from those mentioned above in the literature. As corollaries, new fixed point theorems for α-convex operators

$(\alpha > 1)$, e-convex operators, positive α homogeneous operator $(\alpha > 1)$, generalized e-convex operator, and convex operators are obtained. The results are applied to nonlinear integral equations and partial differential equations.

2. Preliminaries

Let E be a real Banach space and P be a subset of E, θ denotes the zero element of E and $\mathrm{int}P$ denotes the interior of P. The subset P is called a cone if:

(i) $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$

(ii) $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x < y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in$ int P. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \tag{1}$$

The least positive number satisfying the above inequality is called the normal constant of P.

Let $D \subset E$, $A : D \longrightarrow E$ be an operator. If there exists a point $x \in D$ such that $Ax = x$, then x is called a fixed point of A in D. Let $u_0, v_0 \in E$, and $u_0 \leq v_0$, then

$$[u_0, v_0] = \{x \in E \mid u_0 \leq x \leq v_0\}, \tag{2}$$

is said to be an ordering interval. The operator $A : D \longrightarrow E$ is said to be increasing; if for any $x, y \in D$, $x \leq y$ implies $Ax \leq Ay$.

Throughout this paper, we always assume that E is a real Banach space and $\leq$ is a partial ordering with respect to P; θ denotes the null element of E.

Definition 1 (see [19]). Let $D \subset E$. D is called a star-shaped subset of the real Banach space E; if for any $x \in D$ and $0 < t < 1$, it holds that $tx \in D$.

Note that a convex set D in the real Banach space E with the null element $\theta \in D$ is a star-shaped subset of E. Especially, any cone P in the real Banach space E is a star-shaped subset of E.

Definition 2 (see [7]). Let D be a star-shaped subset of the real Banach space E and $A : D \longrightarrow D$ be an operator, then

(1) A is said to be sublinear, if for all $x \in D$ and $0 < t < 1$, $A(tx) \geq tAx$;

(2) A is said to be superlinear, if for all $x \in D$ and $0 < t < 1$, $A(tx) \leq tAx$.

Definition 3 (see [4, 7]). Let $e > \theta$. $A : P \longrightarrow P$ is called an e-concave operator, if

(i) A is e-positive, that is, $A(P - \{\theta\}) \subset P_e$, where

$$P_e = \{x \in E \mid \text{there exist } \lambda, \mu > 0, \text{ such that } \lambda e \leq x \leq \mu e\}. \tag{3}$$

(ii) For all $x \in P_e$ and $0 < t < 1$, there exists $\eta = \eta(t, x) > 0$ such that

$$A(tx) \geq (1 + \eta)tAx, \tag{4}$$

where $\eta = \eta(t, x)$ is called the characteristic function of A.

Similarly, if in the above definition, (ii) is replaced by the following (ii′):

(ii′) For all $x \in P_e$ and $0 < t < 1$, there exists $\eta = \eta(t, x) > 0$ such that

$$A(tx) \leq (1 - \eta)tAx, \tag{5}$$

where $\eta = \eta(t, x)$ is called the characteristic function of A; then, $A : P \longrightarrow P$ is called an e-convex operator.

Definition 4 (see [16]). Let $e > \theta$. $A : P \longrightarrow P$ is called a generalized e-concave operator, if

(i) $Ae \in P_e$, where

$$P_e = \{x \in E \mid \text{there exist } \lambda, \mu > 0, \text{ such that } \lambda e \leq x \leq \mu e\}. \tag{6}$$

(ii) For all $x \in P_e$ and $0 < t < 1$, there exists $\eta = \eta(t, x) > 0$ such that

$$A(tx) \geq (1 + \eta)tAx, \tag{7}$$

where $\eta = \eta(t, x)$ is called the characteristic function of A.

Similarly, if in the above definition, we replace (ii) by the following (ii′):

(ii′) For all $x \in P_e$ and $0 < t < 1$, there exists $\eta = \eta(t, x) > 0$ such that

$$A(tx) \leq ((1 + \eta)t)^{-1}Ax, \tag{8}$$

where $\eta = \eta(t, x)$ is called the characteristic function of A; then, $A : P \longrightarrow P$ is called a generalized e-convex operator.

Definition 5 (see [4, 17]). Let $A : P \longrightarrow P$ be an operator, $\alpha > 0$.

(1) A is said to be an α-concave operator, if for any $x \in P$ and $0 < t < 1$, $A(tx) \geq t^{\alpha}Ax$

(2) A is said to be an α-convex operator, if for any $x \in P$ and $0 < t < 1$, $A(tx) \leq t^{\alpha}Ax$

(3) A is said to be a positive α-homogeneous operator, if for any $x \in P$ and $t > 0$, $A(tx) = t^{\alpha}Ax$.

Remark 6 (see [9]). Any α-convex operator $(\alpha > 1)$ must be an e-convex operator, where the characteristic function $\eta(t, x) = 1 - t^{\alpha-1}$.

Remark 7. Clearly, any e -convex operator must be a superlinear operator. Thus, α -convex operators $(\alpha > 1)$ and e -convex operators are special superlinear operators.

Remark 8. Any generalized e -convex operator A must be a superlinear operator if $\eta(t, x) \geq 1/t^2$ for any $x \in P_e$ and $0 < t < 1$ where $\eta = \eta(t, x)$ is the characteristic function of A. Thus, generalized e-convex operators are special superlinear operators under suitable conditions.

Remark 9. Noting $A : P \longrightarrow P$ is called a convex operator if $A(tx + (1-t)y) \leq tAx + (1-t)Ay$ for all $x, y \in P$ and $0 < t < 1$; we can easily see that any convex operator $A : P \longrightarrow P$ satisfying $A\theta = \theta$ must be a superlinear operator.

3. Main Results

In [18], the author proved that there was no operator which was decreasing and e-convex, where $e > \theta$. Now, we give some important theorems of increasing superlinear operators, which generalize increasing e-convex operators.

Theorem 10. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing superlinear operator. If there exist $a \in (0, 1)$ and $u_0, v_0 \in P$, $u_0 < v_0$ such that $u_0 \leq Au_0, Av_0 \leq av_0$, then the operator A has a unique fixed point x^* in $[u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Proof. We firstly prove the existence of the fixed point. Let $u_n = Au_{n-1}, v_n = Av_{n-1}$. Since A is increasing, we have

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{9}$$

Take $v_0' = v_0, v_n' = a^{-1}Av_{n-1}' (n = 1, 2, \cdots)$, then

$$u_0 \leq v_n' \leq v_0 (n = 1, 2, \cdots), \tag{10}$$

$$v_n \leq a^n v_n' (n = 1, 2, \cdots). \tag{11}$$

Equation (10) can be proved by iteration. Indeed, for $n = 1$, we get

$$u_0 \leq Au_0 \leq a^{-1}Au_0 \leq a^{-1}Av_0' = a^{-1}Av_0' = v_1' \leq a^{-1}av_0 = v_0, \tag{12}$$

which means equation (10) holds when $n = 1$. Suppose that equation (10) holds for $n = k$, that is

$$u_0 \leq v_k' \leq v_0. \tag{13}$$

By the fact that A is increasing, we obtain $Au_0 \leq Av_k' \leq Av_0$, then

$$u_0 \leq Au_0 \leq a^{-1}Au_0 \leq a^{-1}Av_0 \leq a^{-1}Av_k' \leq a^{-1}Av_0 = v_1' \leq v_0, \tag{14}$$

which implies $u_0 \leq v_{k+1}' \leq v_0$. Thus, equation (10) holds for all $n \in \mathbb{N}$. Now, we prove that equation (11) is also true. Indeed, if $n = 1$, then

$$v_1 = Av_0 = aa^{-1}Av_0 = aa^{-1}Av_0' = av_1', \tag{15}$$

that is, (11) holds when $n = 1$. Suppose (11) holds for $n = k$, i.e.,

$$v_k \leq a^k v_k'. \tag{16}$$

It follows that $Av_k \leq a^k Av_k'$ since A is an increasing superlinear operator. Hence, we see that

$$v_{k+1} = Av_k \leq A\left(a^k v_k'\right) \leq a^k Av_k' = a^{k+1}a^{-1}Av_k' = a^{k+1}v_{k+1}', \tag{17}$$

which gives $v_{k+1} \leq a^{k+1}v_{k+1}'$. So, equation (11) holds for all $n \in \mathbb{N}$. □

Combining equations (9), (10), and (11), for any $p \geq 1$, we know

$$\theta \leq v_n - u_n \leq a^n v_n' - u_n \leq a^n v_0 - a^n u_0 = a^n (v_0 - u_0), \tag{18}$$

$$\theta \leq u_{n+p} - u_n \leq v_n - u_n, \theta \leq v_n - v_{n+p} \leq v_n - u_n. \tag{19}$$

By equations (18) and (19) and the normality of P, we can check that $v_n - u_n \longrightarrow 0 (n \longrightarrow \infty)$, which implies that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in E. Then, there exist u^*, $v^* \in [u_0, v_0]$ such that $u_n \longrightarrow u^*$, $v_n \longrightarrow v^* (n \longrightarrow \infty)$, and $u^* = v^*$. Denote $x^* = u^* = v^*$. We have $u_n \leq u^* \leq v^* \leq v_n$ by (9). Therefore,

$$u_{n+1} = Au_n \leq Au^* \leq Av^* \leq Av_n = v_{n+1}. \tag{20}$$

Let $n \longrightarrow \infty$ in (13), then $u^* \leq Au^* \leq Av^* \leq v^*$. This gives $u^* = Au^* = Av^* = v^*$; that is, the operator A has a fixed point x^* in $[u_0, v_0]$.

Next, we prove the uniqueness of the fixed point. If there exists $\bar{x} \in [u_0, v_0]$ such that $A\bar{x} = \bar{x}$, then $u_0 \leq \bar{x} \leq v_0$. By the monotonicity of A, we see $Au_0 \leq A\bar{x} \leq Av_0$, i.e., $u_1 \leq \bar{x} \leq v_1$. It is easy to deduce that $u_n \leq \bar{x} \leq v_n$, for any $n \geq 1$. So $\bar{x} = x^*$ as $n \longrightarrow \infty$.

At last, for any $x_0 \in [u_0, v_0]$, the sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$ satisfies

$$u_n \leq x_n \leq v_n (n = 1, 2, \cdots), \tag{21}$$

by iteration. Letting $n \longrightarrow \infty$, we know $x_n \longrightarrow x^*$ $(n \longrightarrow \infty)$.

Similarly, if the superlinear operator has an upward solution, we have the following result.

Theorem 11. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing superlinear operator. If there exist $a \in (0, 1)$ and $u_0, v_0 \in P$, $u_0 < v_0$ such that $au_0 \leq Au_0, Av_0 \leq v_0$, then the equation $Ax = ax$ has a unique fixed point x^* in $[u_0, v_0]$.*

For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.

Proof. Let $B = a^{-1}A$, then

$$\begin{aligned} Bu_0 = a^{-1}Au_0 \geq a^{-1}au_0 = u_0, \\ Bv_0 = a^{-1}Av_0 \leq a^{-1}v_0. \end{aligned} \tag{22}$$

For any $x \in P$ and $0 < t < 1$, we obtain

$$B(tx) = a^{-1}A(tx) \leq a^{-1}tAx = tBx. \tag{23}$$

□

Thus, B is a superlinear operator which satisfies all conditions of Theorem 10. The conclusions are true by Theorem 10.

Similar to Theorem 10, we immediately get the following result.

Theorem 12. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing superlinear operator. If there exists $\varepsilon \in (0, 1)$ such that $A\theta > \theta$, $A^3\theta \leq \varepsilon A^2\theta$, then the operator A has a unique fixed point x^* in $[A\theta, A^2\theta]$. For any $x_0 \in [A\theta, A^2\theta]$ and iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Proof. We use Theorem 10 to give the proof of Theorem 12. Set $u_0 = \theta, v_0 = A^2\theta$. Then, $u_0, v_0 \in P$. Since the operator A is increasing and $A\theta > \theta$, we have $A^2\theta \geq A\theta$. Obviously, we have $A^2\theta > A\theta$ (otherwise if $A^2\theta = A\theta$, then $A^3\theta = A^2\theta \leq \varepsilon A^2\theta$ $(0 < \varepsilon < 1)$, which implies that $A^2\theta = \theta$, so $A\theta = \theta$. This is a contradiction since $A\theta > \theta$.

Now letting $a = \varepsilon \in (0, 1)$, we see that

$$\begin{aligned} u_0 = A\theta \leq A^2\theta = Au_0, \\ Av_0 = A^3\theta \leq \varepsilon A^2\theta = av_0. \end{aligned} \tag{24}$$

So, all conditions of Theorem 10 are satisfied. By Theorem 10, we know that the conclusions of Theorem 12 hold true. □

Remark 13. Compared with ([7], Theorem 3.1), in order to obtain the existence and uniqueness of positive fixed points, the superlinear operator $A : P \longrightarrow P$ in Theorem 10 and Theorem 11 does not need any compactness or continuity. It is quite different from [7] (Theorem 3.1), which required that $A : P \longrightarrow P$ is a condensing operator.

Remark 14. Since superlinear operators include three classes of operators: generalized e-convex operators, e-convex operators, and α-convex operators, Theorem 10 and Theorem 11 improve or generalize lots of famous results in [5, 7, 9, 12–17].

Corollary 15. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing e-convex operator. If there exist $a \in (0, 1)$ and $w_0, v_0 \in P$, $w_0 < v_0$ such that $w_0 \leq Aw_0, Av_0 \leq av_0$, then the operator A has a unique fixed point x^* in $[w_0, v_0]$. For any $x_0 \in [w_0, v_0]$ and iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 16. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing e-convex operator. If there exist $a \in (1, \infty)$ and $w_0, v_0 \in P$, $w_0 < v_0$ such that $aw_0 \leq Aw_0, Av_0 \leq v_0$, then the equation $Ax = ax$ has a unique fixed point x^* in $[w_0, v_0]$. For any $x_0 \in [w_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 17. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing α-convex $(\alpha > 1)$ operator. If there exist $a \in (0, 1)$ and $u_0, v_0 \in P$, $u_0 < v_0$ such that $u_0 \leq Au_0, Av_0 \leq av_0$, then the operator A has a unique fixed point x^* in $[u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 18. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing α-convex $(\alpha > 1)$ operator. If there exist $a \in (1, \infty)$ and $u_0, v_0 \in P$, $u_0 < v_0$ such that $au_0 \leq Au_0, Av_0 \leq v_0$, then the equation $Ax = ax$ has a unique fixed point x^* in $[u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 19. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing positive $\alpha(\alpha > 1)$ homogeneous operator. If there exist $a \in (0, 1)$ and $u_0, v_0 \in P$, $u_0 < v_0$ such that $u_0 \leq A u_0, Av_0 \leq av_0$, then the operator A has a unique fixed point x^* in $[u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 20. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing positive $\alpha(\alpha > 1)$ homogeneous operator. If there exist $a \in (1, \infty)$ and $u_0, v_0 \in P$, $u_0 < v_0$ such that $au_0 \leq Au_0, Av_0 \leq v_0$, then the equation $Ax = ax$ has a unique fixed point x^* in $[u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0$ $(n \longrightarrow \infty)$.*

Corollary 21. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing generalized e-convex operator satisfying $\eta(t, x) \geq 1/t^2$ for any $x \in P_e$ and $0 < t < 1$ where $\eta = \eta(t, x)$ is the characteristic function of A. If there exist $a \in (0, 1)$ and $w_0, v_0 \in P$, $w_0 < v_0$ such that $w_0 \leq Aw_0, Av_0 \leq av_0$, then the operator A has a unique fixed point x^* in $[w_0, v_0]$. For any $x_0 \in [w_0, v_0]$ and iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 22. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing generalized e-convex operator satisfying $\eta(t, x) \geq 1/t^2$ for any $x \in P_e$ and $0 < t < 1$ where $\eta = \eta(t, x)$ is the characteristic function of A. If there exist $a \in (1, \infty)$ and $w_0, v_0 \in P$, $w_0 < v_0$ such that $aw_0 \leq Aw_0, Av_0 \leq v_0$, then the equation $Ax = ax$ has a unique fixed point x^* in $[w_0, v_0]$. For any $x_0 \in [w_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 23. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing convex operator satisfying $A\theta = \theta$. If there exist $a \in (0, 1)$ and $w_0, v_0 \in P$, $w_0 < v_0$ such that $w_0 \le Aw_0, Av_0 \le av_0$, then the operator A has a unique fixed point x^* in $[w_0, v_0]$. For any $x_0 \in [w_0, v_0]$ and iterated sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Corollary 24. *Let P be a normal cone in E and $A : P \longrightarrow P$ be an increasing convex operator satisfying $A\theta = \theta$. If there exist $a \in (1, \infty)$ and $w_0, v_0 \in P$, $w_0 < v_0$ such that $aw_0 \le Aw_0, Av_0 \le v_0$, then the equation $Ax = ax$ has a unique fixed point x^* in $[w_0, v_0]$. For any $x_0 \in [w_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - x^*\| \longrightarrow 0 (n \longrightarrow \infty)$.*

Remark 25. In Corollary 15 and Corollary 16, the existence and uniqueness of positive fixed points are proved, without appealing to the monotonicity or any compactness and continuity of the e-convex operator $A : P \longrightarrow P$. This is very different from [9] (Theorem 9), which required that there existed $M (>1)$ homogeneous increasing functional $F : P_e \longrightarrow (0, +\infty)$. In addition, Corollary 15 and Corollary 16 in the paper are quite different from [14] (Corollary 2.4), which only obtained the existence of positive fixed points while the condition required the strong condition of that there existed $\varepsilon_0 > 0$ such that

$$\begin{aligned} &Ax \ge \varepsilon_0 \|Ax\| e, \forall x \in P^+, \overline{\lim_{t \longrightarrow 0+}} \eta(x, t) \\ &> \max \left\{ 1 - \frac{\varepsilon_0 \|A(\varepsilon_0 e)\| \|e\|}{N^2}, 1 - \frac{1}{MN} \right\}, \\ &\quad \text{uniformly for } x \in C_e, \end{aligned} \tag{25}$$

with $M = \sup \{ \|Ax\| \mid x \in P, \|x\| = 1 \}$.

Remark 26. In Corollary 17 and Corollary 18, the existence and uniqueness of positive fixed points are proved, without appealing to the monotonicity of α-convex operator $(\alpha > 1)$ or any compactness and continuity of the operator $A : P \longrightarrow P$. This is very different from [12] (Theorem 9), [8] (Theorem 2), and [15] (Theorem 1.3), which required that there existed a linear operator $L : E \longrightarrow E$ which satisfied certain conditions, and the increasing α-convex operator $(\alpha > 1)$ was completely continuous, respectively.

Remark 27. In Corollary 19 and Corollary 20, the positive α-homogeneous operator $(\alpha > 1) A : P \longrightarrow P$ does not need to have any compactness or continuity, but Theorem 1 in [17] requested that the α-homogeneous operator $(\alpha > 1)$ $A : P \longrightarrow P$ can be decomposed into $A = FC$, where $F : P_e \longrightarrow (0, +\infty)$ was an increasing positive β functional and $C : P_e \longrightarrow P_e$ was an increasing operator in P_e $(e > \theta)$. Therefore, the methods and techniques of Corollary 19 and Corollary 20 are different from those of [17] (Theorem 1).

Remark 28. In this paper, we use the partial order and the monotone iteration to study the fixed point theorems of superlinear operators in Banach spaces. The methods and techniques are different from those used in the literature [7–10, 12, 14, 15, 17], but the existence and uniqueness of the fixed points and the convergence of the iterative sequences of superlinear operators are obtained.

4. Applications

Now, we give some examples to show the applications of our main results in nonlinear integral equations and partial differential equations.

Example 1. Let $\alpha > 1$. Consider Hammerstein integral equation

$$x(t) = (Ax)(t) = \int_{-\infty}^{+\infty} K(t, s)(x(s))^\alpha ds. \tag{26}$$

Conclusion 29. Let $K : R \times R \longrightarrow$ R be a nonnegative continuous function. If there exists a constant $0 < c < 1$ and two continuous functions $u = u_0(t)$, $v = v_0(t)$ satisfying $0 < u_0(t) \le v_0(t)$, $-\infty < t < +\infty$, and

$$u_0(t) \le \int_{-\infty}^{+\infty} K(t, s)(u_0(s))^\alpha ds, \int_{-\infty}^{+\infty} K(t, s)(v_0(s))^\alpha ds \le c v_0(t). \tag{27}$$

Then, equation (26) has a unique solution $x^*(t)$ satisfying $u_0 \le x^* \le v_0$. For any $x_0(t)$ which satisfies $u_0(t) \le x_0(t) \le v_0(t)$, the iterated sequence

$$x_n(t) = (Ax_{n-1})(t) = \int_{-\infty}^{+\infty} K(t, s)(x_{n-1}(s))^\alpha ds, \tag{28}$$

uniformly converges to $x^*(t)$ in $(-\infty, +\infty)$.

Proof. Let $E = C_B(R)$ be a bounded continuous function space in R^n. Define $\|x\| = \sup_{t \in R} |x(t)|$, then E is a Banach space. Let $P = C_B^+(R)$ denote all nonnegative continuous functions in E, then P is a normal cone in E. We claim that $A : P \longrightarrow P$ is a homogeneous operator. In fact, by equation (26), we have

$$\begin{aligned} (A\lambda x)(t) &= \int_{-\infty}^{+\infty} K(t, s)(\lambda x(s))^\alpha ds \\ &= \lambda^\alpha \int_{-\infty}^{+\infty} K(t, s)(\lambda x(s))^\alpha ds \le \lambda A x(t), \end{aligned} \tag{29}$$

which means $A : P \longrightarrow P$ is a homogeneous operator. It is clear that A satisfies all conditions of Theorem 10. The conclusion is true. □

Similarly, we also have the following.

Example 2. Let $\alpha > 1$. Consider Hammerstein integral equation (see the equation (9) in [10])

$$x(t) = (Ax)(t) = \int_0^1 K(t, s)(x(s))^\alpha ds. \tag{30}$$

Conclusion 30. Let $K : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be a nonnegative continuous function. If there exists a constant $0 < c < 1$ and two continuous functions $u = u_0(t)$, $v = v_0(t)$ satisfying $0 < u_0(t) \le v_0(t)$, $0 < t < 1$, and

$$u_0(t) \le \int_0^1 K(t,s)(u_0(s))^\alpha ds, \int_0^1 K(t,s)(v_0(s))^\alpha ds \le c v_0(t). \tag{31}$$

Then, equation (30) has a unique solution $x^*(t)$ satisfying $u_0 \le x^* \le v_0$. For any $x_0(t)$ which satisfies $u_0(t) \le x_0(t) \le v_0(t)$, the iterated sequence

$$x_n(t) = (Ax_{n-1})(t) = \int_0^1 K(t,s)(x_{n-1}(s))^\alpha ds, \tag{32}$$

uniformly converges to $x^*(t)$ in $(-\infty, +\infty)$.

Remark 31. In Example 2, we obtain the existence of positive solutions of the integral equation (30), without requiring that the integral kernel $K(t,s)$ can be decomposed into $K(t,s) = h(t)g(s)$ (see condition C1 in [10]). The methods and techniques used in this paper are different from those in [10].

Example 3. Let Ω be a bounded convex domain in $R^n (n \ge 2)$ whose boundary $\partial\Omega$ belongs to $C^{2+\mu}$ for some $0 < \mu < 1$. Consider the Dirichlet problem

$$\begin{cases} Lu = f(x,u), \\ u|_{\partial\Omega} = 0, \end{cases} \tag{33}$$

where $f(x,u)$ is nonnegative and continuous on $x \in \bar{\Omega}$ and $u \ge 0$ and

$$Lu = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \tag{34}$$

i.e., there exists a positive constant μ_0 such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \mu_0 |\xi|^2, \tag{35}$$

for any $x \in \bar{\Omega}$ and $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in R^n$, and $a_{ij}(x) = a_{ji}(x)$,$c(x) \ge 0$. Here, all functions $a_{ij}(x)$, $b_i(x)$, and $c(x)$ belong to $C^\mu(\bar{\Omega})$ (see [3]).

Finding the solution of the above problem is equivalent to finding the fixed point of the integral operator A:

$$Au(x) = \int_{\bar{\Omega}} G(x,y) f(y, u(y)) dy, \tag{36}$$

where $G(x,y)$ is the corresponding Green function, which satisfies

$$0 < G(x,y) < \begin{cases} K_0 |x-y|^{2-n}, & n > 2 \\ K_0 |\ln |x-y||, & n = 2 \end{cases} (x, y \in \Omega, x \ne y). \tag{37}$$

Hence (see Guo and Lakshmikantham [4]), the linear integral operator

$$Gv(x) = \int_\Omega G(x,y) v(y) dy, \tag{38}$$

is a completely continuous operator from $C(\bar{\Omega})$ into $C(\bar{\Omega})$, and therefore, operator A maps P into P and is completely continuous, where $P = \{u(x) \in C(\bar{\Omega}) \mid u(x) \ge 0, \forall x \in \bar{\Omega}]$ is a normal cone of space $C(\bar{\Omega})$.

Conclusion 32. Let the function $f(x, u(x))$ be increasing and satisfy

$$f(x, tu) < tf(x,u), \forall u > 0, x \in \Omega, 0 < t < 1. \tag{39}$$

If there exist $a \in (0,1)$ and $\theta < v_0 = v(x_0) \in P$, such that $\int_{\bar{\Omega}} G(x_0, y) f(y, v(y)) dy \le a v(x_0)$ for some $x_0 \in \Omega$, then the Dirichlet problem has a unique fixed point x^* in $[\theta, v(x_0)]$.

Proof. Firstly, we prove that the operator A is e-convex, where

$$e(x) = \int_{\bar{\Omega}} G(x,y) dy, \forall x \in \bar{\Omega}. \tag{40}$$

Here, we need to use a conclusion about integral operator (17), which can be found in Amann [2]: linear integral operator (17) is e-positive, i.e., for any $v > \theta$, there exist $\alpha = \alpha(v) > 0$ and $\beta = \beta(v) > 0$ such that $\alpha e \le Gv \le \beta e$, i.e.,

$$\alpha \int_{\bar{\Omega}} G(x,y) dy \le \int_{\bar{\Omega}} G(x,y) v(y) dy \le \beta \int_{\bar{\Omega}} G(x,y) dy, \forall x \in \bar{\Omega}. \tag{41}$$

Now, let $u > \theta$. Then, there exists an $x_1 \in \Omega$ such that $u(x_1) > 0$, and it follows from (39) that

$$0 \le f(x_1, 2u(x_1)) < 2f(x_1, u(x_1)). \tag{42}$$

Consequently, $fu > \theta$, where f denotes the Nemitskyi operator:

$$fu(x) = f(x, u(x)). \tag{43}$$

Thus, from (41), we know that there exist $\alpha > 0$ and $\beta > 0$ such that

$$\alpha e \le Au = Gfu \le \beta e, \tag{44}$$

i.e., A satisfies condition (i) of Definition 4.

Next, suppose $u \in P$ satisfying $\alpha_1 e \le u \le \beta_1 e$ $(\alpha_1 = \alpha_1(u) > 0, \beta_1 = \beta_1(u) > 0)$ and $0 < t < 1$. Since $e(x) > 0$ for any x

$\in \Omega$, we have by (39)

$$tf(x, u(x)) - f(x, tu(x)) > 0, \forall x \in \Omega, \tag{45}$$

and hence, by (41), there exists $\alpha_2 > 0$ such that

$$\int_{\bar{\Omega}} G(x, y)\{tf(y, u(y) - f(y, tu(y))\}dy \geq \alpha_2 e(x), \forall x \in \bar{\Omega}. \tag{46}$$

On the other hand, it is clear that

$$\int_{\bar{\Omega}} G(x, y)f(y, u(y))dy \leq Me(x), \forall x \in \bar{\Omega}, \tag{47}$$

where

$$M = \max_{x \in \bar{\Omega}} f(x, u(x)). \tag{48}$$

It follows therefore from (46) and (47) that

$$\begin{aligned} &\int_{\bar{\Omega}} G(x, y)f(y, tu(y))dy \\ &\leq t\left(1 - \frac{\alpha_2}{Mt}\right)\int_{\bar{\Omega}} G(x, y)f(y, u(y))dy, \forall x \in \bar{\Omega}, \end{aligned} \tag{49}$$

i.e., $A(tu) \leq t(1-\eta)Au$, where $\eta = \alpha_2/Mt > 0$. Thus, the operator A satisfies condition (ii) of Definition 4, and therefore, A is e-convex.

Take $w_0 = \theta$, then $w_0 < v_0$ and $w_0 \leq Aw_0$. Therefore, all conditions of Corollary 15 are satisfied. By Corollary 15, we see that the conclusion is true. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The research is partially supported by the Special Basic Cooperative Research Programs of Yunnan Provincial Undergraduate Universities' Association (No. 202101BA070001-045).

References

[1] H. Amann, "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces," *SIAM Review*, vol. 18, pp. 602–709, 1976.

[2] H. Amann, "On the number of solutions of nonlinear equations in ordered Banach spaces," *Journal of Functional Analysis*, vol. 11, no. 3, pp. 346–384, 1972.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.

[4] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[5] D. Guo, V. Lakshmikantham, and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, 1996.

[6] M. A. Krasnosel'skij and L. F. Boron, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, The Netherlands, 1964.

[7] F. Li, "Two point expansion fixed point theorem and solutions of convex concave operator equations and applications," *Acta Mathematica Sinica. Series A*, vol. 40, no. 1, pp. 457–464, 1997.

[8] J. Sun, "Some new fixed point theorems of increasing operators and applications," *Applicable Analysis*, vol. 42, no. 1-4, pp. 263–273, 1991.

[9] W. Wang and Z. Liang, "Fixed point theorems of a class of nonlinear operators and applications," *Acta Mathematica Sinica. Series A*, vol. 48, no. 4, pp. 789–800, 2005.

[10] W. Wang, F. Mi, and J. Wang, "The existence and uniqueness and applications of positive fixed points for a class of superlinear operators," *Journal of North University of China (Natural Science Edition)*, vol. 34, no. 5, pp. 524–527, 2013.

[11] S. Xu and B. Jia, "Fixed-point theorems of φ concave-$(-\psi)$ convex mixed monotone operators and applications," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 2, pp. 645–657, 2004.

[12] C. Zhai and C. Guo, "On α-convex operators," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 556–565, 2006.

[13] Z. Zhao, "Existence of fixed points for some convex operators and applications to multi- point boundary value problems," *Applied Mathematics and Computation*, vol. 215, no. 8, pp. 2971–2977, 2009.

[14] Z. Zhao, "Fixed points of τ-ϕ-convex operators and applications," *Applied Mathematics Letters*, vol. 23, no. 5, pp. 561–566, 2010.

[15] Z. Zhao, "Existence of positive fixed points for α-convex operators with applications," *Acta Mathematica Sinica. Series A*, vol. 49, no. 1, pp. 139–144, 2006.

[16] Z. Zhao and X. Du, "Fixed points of generalized e-concave (generalized e-convex) operators and their applications," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 1426–1438, 2007.

[17] Q. Zhang, "Existence and uniqueness of positive solutions for positive α-homogeneous superlinear operator equations and applications," *Journal of Systems Science and Mathematical Sciences*, vol. 19, no. 1, pp. 29–33, 1999.

[18] Z. Zhang, "Some new results about abstract cones and operators," *Nonlinear Analysis*, vol. 37, pp. 449–455, 1999.

[19] P. Takáč, "Asymptotic behavior of discrete-time semigroups of sublinear, strongly increasing mappings with applications to biology," *Nonlinear Analysis*, vol. 14, no. 1, pp. 35–42, 1990.

[20] M. Imdad, M. Asim, and R. Gubran, "Order-theoretic fixed point results for $(\psi, \varphi, \eta)_g$ generalized weakly contractive mappings," *Journal of Mathematical Analysis*, vol. 8, no. 6, pp. 169–179, 2017.

[21] M. Imdad, M. Asim, and R. Gubran, "Fixed point theorems for generalized weakly contractive mappings in metric spaces with applications," *Indian Journal of Mathematics*, vol. 2018, no. 1, pp. 85–105, 2018.

[22] M. Asim, M. Imdad, and S. Shukla, "Fixed point results for Geraghty-weak contractions in ordered partial rectangular b-metric spaces," *Afrika Matematika*, vol. 32, no. 5-6, pp. 811–827, 2021.

[23] T. Abdeljawad and E. Karapinar, "Common fixed point theorems in cone Banach spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 2, pp. 211–217, 2011.

[24] W. Shatanawi, E. Karapinar, and H. Aydi, "Coupled coincidence points in partially ordered cone metric spaces with acdistance," *Journal of Applied Mathematics*, vol. 2012, 15 pages, 2012.

[25] E. Karapinar, "Couple fixed point theorems for nonlinear contractions in cone metric spaces," *Computers and Mathematics with Applications*, vol. 59, no. 12, pp. 3656–3668, 2010.

[26] M. Asadi and E. Karapinar, "Coincidence point theorem on Hilbert spaces via weak Ekeland variational principle and application to boundary value problem," *Thai Journal of Mathematics*, vol. 19, no. 1, pp. 1–7, 2021.

[27] H. Aydi, M. Jleli, and B. Samet, "On positive solutions for a fractional thermostat model with a convex-concave source term via ψ-Caputo fractional derivative," *Mediterranean Journal of Mathematics*, vol. 17, no. 1, pp. 1–15, 2020.